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ADVANCED PRACTICAL PHYSICS FOR STUDENTS

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FOR STUDENTS

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PREFACE

THE course of Practical Physics described in this book is based upon that followed in King's College, London, by students who have completed their Intermediate Course, and who are proceeding to a Pass or Honours Degree. This has been extended, and it is hoped that the book will be useful to a wider circle of students of Physics than those immediately concerned with University Examinations.

A number of well-known Physicists have contributed to the development of the King's College course, amongst whom we may mention Professors H. A. Wilson, C. G. Barkla, H. S. Allen, and W. Wilson, who formerly worked here in the Wheatstone Laboratory, and Professor O. W. Richardson, the present occupant of the chair.

The general aim has been to provide with each experiment a short theoretical treatment which will enable the student to perform the experiment without immediate reference to theoretical treatises. To aid this scheme an introductory chapter in the Calculus has been included. This chapter is an innovation in a book of this type, but it is hoped that the student will find here a bridge over that period during which his Physics demands more advanced mathematics than his systematic study of that subject has yet given him.

We take this opportunity of expressing our gratitude to Professor O. W. Richardson, who has allowed us to make use of laboratory manuscripts and results of experiments. We are also greatly indebted to our colleagues and to Mr. G. Williamson, who have given us many suggestions, and to the Honours students of

the past session who have supplied us with numerical and graphical results. We have been greatly helped by the ready assistance on the part of The Cambridge and Paul Scientific Instrument Co., Messrs. Elliot Bros., Gambrel, Ltd., Adam Hilger, Ltd., W. G. Pye & Co., and the Weston Electric Co., who supplied us with the blocks for many of the illustrations.

B. L. W.
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March, 1923

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ADVANCED PRACTICAL PHYSICS FOR STUDENTS

INTRODUCTION TO DIFFERENTIAL AND INTEGRAL CALCULUS

The Differential Calculus

§1. ANY quantity x which may assume a series of values is called a variable quantity or simply a variable, and if its value does not depend on that of any other quantity it is called an independent variable.

On the other hand a quantity y, which bears a particular relation to x, assumes values which depend on the values of x, and for this reason is called a dependent variable. We may have for example:

$$y = 2x - 3.$$

Here y takes values which depend in a quite definite manner on those of x.

We may also have a dependence defined by the relations:

$$y = \sin x, \quad y = \log x, \quad \text{and} \quad y = e^x.$$

Such expressions as $2x - 3$, $\sin x$, $\log x$, etc., are called functions of x, and when we say that y is a function of x we mean that y depends on the values that x assumes.

In case we do not specify definitely how y and x are related we write

$$y = f(x).$$

$f(x)$ denotes any function of x.

It is often convenient in Physics to show by means of a diagram the relation between two variables y and x. For example, a record may be required of the atmospheric pressure at various times. Such a record is drawn automatically by a self-recording barometer so that it can be seen how the pressure and time are related. Here we have as independent variable the time and the dependent variable is the barometric pressure.

In fig. 1, the curve represents the relation between x and y and the shape depends on the way y and x are connected.

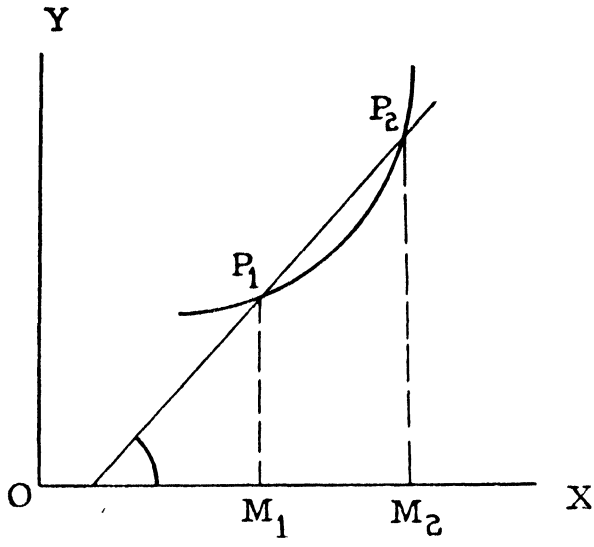


FIG. 1

If $y = 2x - 3$, the curve becomes a straight line, and if $y = \sin x$, we have the familiar sine curve, fig. 2.

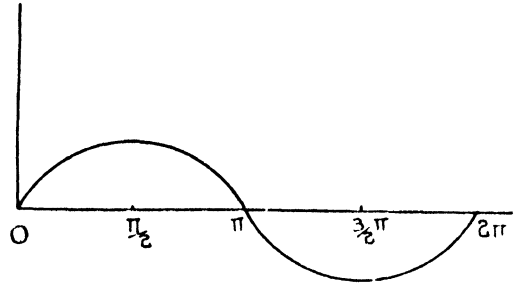


FIG 2

A function is said to be a continuous function of a variable when the graph representing it is a curve in which there is no sudden change in value of the ordinate at any point. In such a curve, if we approach a point where $x = a$, from left to right, we find a certain value for y, and if we approach the point from right to left we find the same value. In fig. 3 we have an example of a function which is discontinuous at $x = 0$. If we approach the origin from left to right the value of y is very great and

negative in sign, while in approaching from the right y is very large and positive. In nature we are chiefly concerned with continuously varying quantities. If a train is at rest at a station at a particular instant, and is observed to be moving with a velocity of ten miles per hour ten minutes later, it must have

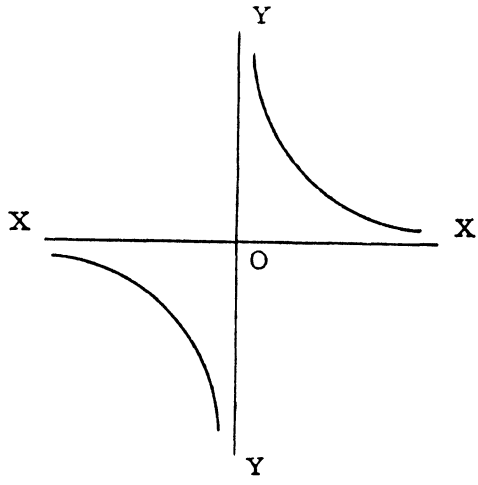


FIG. 3

possessed every possible velocity between zero and ten miles per hour during the interval.

The speed is continuous, and if it depends on the lapse of time from the start it is said to be a continuous function of the time.

We do not contemplate the possibility that the train could possess a speed of five miles per hour at one instant and at the next without any interval whatever a velocity of six miles per hour. If this were possible we should describe the speed as discontinuous, because it had no value between five and six. If this were so we should consider that our powers of observation were at fault, and we should describe the motion as changing very rapidly between five and six miles per hour ; so rapidly that we had failed to detect the lapse of time in which the change took place.

Discontinuous functions are of frequent occurrence in Mathematics. Consider as an example the case of $y = \frac{1}{x}$.

When x is a very small positive number, let us say $\frac{1}{10^6}$, y is large and has the value 10^6.

On the other hand if $x = -\frac{1}{10^6}$, y is large in magnitude but negative, it equals -10^6.

As x passes through the value zero y suddenly leaps from an enormously large negative value to a very great positive value, and has no value between.

This is represented in the diagram, fig. 3. The curve has two branches: they are the two parts of the rectangular hyperbola

$$xy = 1.$$

We shall not be concerned with such functions so we dismiss them briefly. It is to be borne in mind that our applications of the Calculus are to continuous functions only. The results we obtain must not be applied to discontinuous functions without closer examination.

It is important to understand the meaning of the limit of a function.

Suppose y depends on x, and that as x approaches the value a, y approaches the value b.

b is called the limit of y as x approaches a, and we write:

$$\underset{x \longrightarrow a}{\text{Lim.}}\, y \longrightarrow b$$

If reference be made to fig. 1, as x approaches the value OM_1 y approaches the value M_1P_1 and M_1P_1 is actually the value of y when $x = a$.

Cases occur in which the conception of a limit is not so simple. If we examine the curve

$$y = \frac{1}{x}$$

in the neighbourhood of the origin as $x \longrightarrow 0$, we obtain a different value of y according as we begin on the right or left of the origin.

On account of the discontinuity the limit of y as x approaches zero is not definite.

Another case occurs in which x may continue to increase to any extent while y continually approaches some particular value.

We may turn once more to the curve

$$y = \frac{1}{x}$$

As x gets larger and larger, y gets smaller and smaller approaching the value zero.

We may get as near zero as we please by making x larger. For example, we may make y as small as one-millionth by choosing $x = 10^6$.

This is a very important point in defining a limit. It must be possible to get as close to the limiting value as we please by choosing x properly, although it may not actually be possible to cause y to attain the limit. We have in our example a case in point. y is never zero however large x may be, but it is possible to make y nearer and nearer zero by increasing x.

The former definition of the limit of a function is not very satisfactory. A limit is accurately defined as follows:

The limit of a function of x is some number, b, such that as x approaches a particular value, a, the difference between b and the function may be made as small as we please by taking x sufficiently near a.

§2. In describing natural phenomena by means of equations, simplifications are often brought about by neglecting certain terms in comparison with other more important terms.

Suppose an equation is obtained which we may write:

$$A_1 + B_1 + C_2 + D_2 = E_1 + F_3.$$

The suffix numbers denote the order of importance of the terms; that is to say, 1 denotes that the term is to be regarded as of first importance, or it is of the first order of magnitude. The 2 and 3 denote that the terms are only of second and third degrees of importance, they are of the second and third orders.

If we wish to include terms of the first and second orders we omit F_3, while if only terms of the first order are to be considered the equation becomes:

$$A_1 + B_1 = E_1.$$

Great care has to be exercised in thus drawing up a scale of magnitude, and this leads to a short consideration of infinitesimals.

Suppose a quantity X is divided into 1000 equal parts, these again subdivided in the same way, and so on. We then have a series of values:

$$X, \frac{X}{10^3}, \frac{X}{10^6}, \text{ etc.},$$

which provides a scale of magnitude.

If circumstances do not permit of accurate observation of quantities less than those of the same order as X we regard $\frac{X}{10^3}, \frac{X}{10^6}$, etc., as negligible.

Generally, if f is a small fraction, i.e. small compared with unity:

$$fX, f^2X, f^3X, \text{ etc.}$$

are all small compared with X, and are said to be small quantities

of the first, second, third, etc., orders. If these small quantities have zero limits they are called infinitesimals.

In equations between infinitesimals only the terms of the lowest order are to be retained, i.e. the terms of greatest magnitude.

This is made clearer by an example which has important Physical applications.

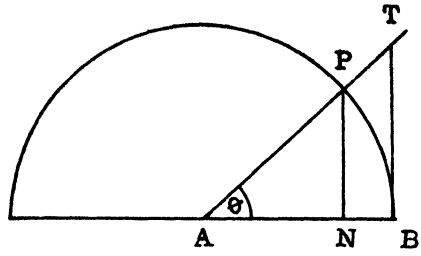


FIG. 4

In fig. 4, AB represents the radius of an arc, BP, of a circle which subtends an angle θ at A.

PN is normal to AB.

BT is also normal to AB cutting AP produced in T.

It may be regarded as an axiom that:

$$\text{PN} < \text{arc PB} < \text{BT}.$$

We shall examine the order of the differences between these quantities if θ be regarded as of the first order of small quantities.

By expansion of sin θ and cos θ in powers of θ we have:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\ldots\ldots \quad (1)$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\ldots\ldots \quad (2)$$

If only small quantities of the first order are retained:

$$\left.\begin{aligned}\sin\theta &= \theta.\\ \cos\theta &= 1.\end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (3)$$

$$\text{PN} = a\sin\theta.$$

$$\text{PB} = a\,\theta.$$

$$\therefore\ \text{PB} - \text{PN} = a(\theta - \sin\theta) = a\left\{\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \ldots\right\}$$

This difference is of the third order.

Thus up to considerations of magnitude of the third order:

$$\text{PB} = \text{PN}.$$

Again

$$BT = a \tan \theta.$$
$$= a\left(\theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots\right),$$

as may be shown by division of the expressions for $\sin\theta$ and $\cos\theta$.

Thus BT − arc PB = a quantity of the third order of magnitude.

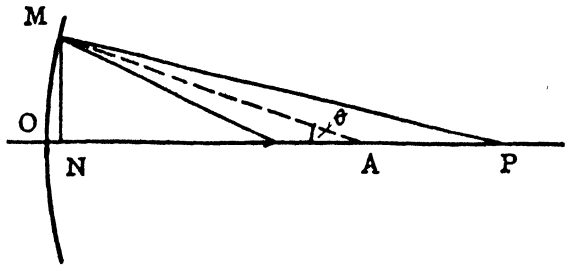


FIG. 5

$BN = a - AN = a(1 - \cos\theta)$ = a quantity of the second order of magnitude.

Thus, if we regard θ as of the first order and retain only this order in our equations we may write:

$$BN = 0, \quad PN = \text{arc } PB = BT.$$

and with the exception of BN = 0 this is true for the case when second order quantities are retained.

Extensive use is made of these relations in Geometrical Optics in the first study of reflection and refraction in mirrors and lenses.

In the case of a mirror, for example (see fig. 5), when the angle θ is small, i.e. when the rays from an object, P, strike the mirror at M not far from the pole, O, we establish certain formulæ by assuming that O and N may be regarded as being coincident.

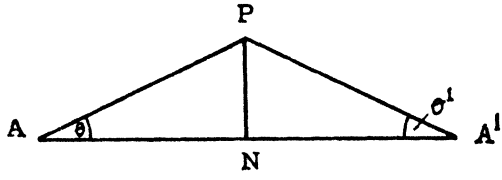


FIG. 6

This is because we do not retain quantities of an order higher than θ. Thus NO = 0 by the foregoing considerations.

Another important case is the calculation of the order of the difference between the sum of two sides of a triangle and the base when the base angles are of the first order of small quantities.

Referring to fig. 6, we have as for fig. 4,

PA − AN = a quantity of the second order in θ,

and similarly $PA^1 - A^1N$ is of the second order in θ^1

$\therefore$ PA + PA^1 − (AN + A^1N) = a quantity of the second order since θ and θ^1 are by the data small quantities of the first order.

$$\therefore PA + PA^1 = AN + A^1N = AA^1$$

to the first order.

This result is made use of in the establishment of Fermat's Law of the extreme path which plays a fundamental part in Optics. (See for example Houstoun's "Treatise on Light," p. 17.)

We consider as a final example the difference between a chord and an arc both subtending the same small angle θ at the centre of a circle.

Thus, again referring to fig. 4, we require the difference between chord BP and arc BP.

$$\text{arc BP} - \text{chord BP} = a\theta - 2a \sin\frac{\theta}{2}.$$

$$= a\left\{\theta - 2\left(\frac{\theta}{2} - \frac{1}{3!}\frac{\theta^3}{8} + \dots\right)\right\}$$

= a quantity of the third order.

We can thus regard the chord and arc as equal up to and including quantities of the second order.

It should be noted that the successive orders are vanishingly small with regard to the terms earlier in the scale, e.g. in comparing

$$a\theta, \quad b\theta^2, \quad c\theta^3,$$

if θ is of the first order, $\frac{b\theta^2}{a\theta} = \frac{b\theta}{a}$ so that as $\theta \longrightarrow 0$ $b\theta^2 \longrightarrow 0$ infinitely more rapidly than $a\theta$, and the same holds for any two consecutive terms in the scale.

The ratio of two quantities of the same order will be a finite quantity—not a vanishing or negligible quantity, but the ratio of two quantities of differing order (higher order ÷ lower order) is vanishingly small.

We are concerned with small variations of this kind in the differential Calculus.

§3. The Differential Coefficient

Let y be a function of x, and suppose x varies by a small quantity which we denote by δx. In consequence of this variation y will vary a small quantity, say δy.

The ultimate ratio $\frac{\delta y}{\delta x}$ when δx becomes very small is called the differential coefficient of y with respect to x. It is denoted by $\frac{d}{dx}y$ and written $\frac{dy}{dx}$ and sometimes denoted simply by Dy.

In accordance with our notation we may write :

$$\frac{dy}{dx} = \underset{\delta x \longrightarrow 0}{\text{Lim.}} \frac{\delta y}{\delta x}.$$

In general the quantities δy and δx are of the same order of magnitude, and $\frac{dy}{dx}$ is a finite quantity.

In order to illustrate this, consider the relation :

$$y = 2x - 3.$$

Let x become $x + \delta x$, then the new value of y is

$$2(x + \delta x) - 3$$

i.e.
$$\begin{aligned} y + \delta y &= 2(x + \delta x) - 3 \\ &= y + 2\delta x. \\ \therefore \ \delta y &= 2\delta x. \\ \therefore \ \frac{\delta y}{\delta x} &= 2. \end{aligned}$$

Now no matter how small δx becomes, the ratio is always 2, for δy is of the same order as δx, and their ratio is finite and equal to 2.

We have a simpler case still in the differential coefficient of a constant.

A constant is a number that does not depend on the variable.

Thus, if $y = A$ it does not matter how x varies, y still remains $= A$. Thus there is no change δy corresponding to a change δx.

Hence $\frac{dy}{dx} = 0$ if y is a constant.

It should be noted that $\frac{dy}{dx}$ does not mean $dy \div dx$. $\frac{dy}{dx}$ is a short notation for the operation of finding the ultimate ratio $\frac{\delta y}{\delta x}$. Nevertheless Physicists continually appear to use the coefficient as if it meant $dy \div dx$, and it is not a rare occurrence to find an equation :

$$\frac{dy}{dx} = x^2$$

written alternatively $dy = x^2\,dx$.

This is, in fact, a very convenient mode of expressing the result,

and it means that dy and dx now no longer retain the same significance. The second of these means :

$$\delta y = x^2 \delta x.$$

We have in the equation $\frac{dy}{dx} = x^2$ an expression of the rate of variation of y with respect to x at a particular point on the curve, which represents graphically the relation between y and x.

The alternative equation means that in the neighbourhood of this point we can calculate a small change δy corresponding to a small change δx. This point rarely causes difficulty in practice, and it is obviously inconvenient to change to and fro from d to δ, but to be strictly accurate we must bear the distinction in mind.

The definition of the differential coefficient gives the clue to its determination. We will not determine its value for more than one or two cases but be content with reference to a table of values of the important coefficients.

The Differential Coefficient for x^n where n is any Number.

Write :

$$y = x^n.$$

$$y + \delta y = (x + \delta x)^n = x^n \left(1 + \frac{\delta x}{x}\right)^n$$

$$= x^n \left\{1 + n\frac{\delta x}{x} + \frac{n(n-1)}{1 \cdot 2} \cdot \left(\frac{\delta x}{x}\right)^2\right\} + \ldots$$

$$= x^n + nx^{n-1} \cdot \delta x + \frac{n(n-1)}{1 \cdot 2} \cdot x^{n-2}(\delta x)^2 + \ldots$$

$$\therefore \quad \delta y = nx^{n-1}\delta x + \frac{n(n-1)}{1 \cdot 2} \cdot x^{n-2}(\delta x)^2 + \ldots$$

$$\therefore \quad \frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{1 \cdot 2} \cdot x^{n-2}\,\delta x + \text{higher powers of } \delta x.$$

δx is a quantity which we have called infinitesimal. In the next step of finding the limit of $\frac{\delta y}{\delta x}$ we shall suppose δx a quantity of the first order of magnitude. It is therefore infinitesimally small with regard to the finite quantity nx^{n-1}.

We thus neglect all quantities of higher order than nx^{n-1} and have :

$$\frac{dy}{dx} = \underset{\delta x \longrightarrow 0}{\text{Lim.}} \frac{\delta y}{\delta x} = nx^{n-1}.$$

Differential Coefficient of sin x.

$$y + \delta y = \sin(x + \delta x) = \sin x \cos \delta x + \cos x \sin \delta x.$$

We need retain only quantities of the first order on the right.

Thus: $\cos \delta x = 1$, $\sin \delta x = \delta x$, by equation (3).

$$\therefore \quad y + \delta y = \sin x + \cos x \cdot \delta x.$$

$$\therefore \quad \delta y = \cos x \cdot \delta x.$$

$$\text{or} \quad \frac{dy}{dx} = \cos x.$$

Similarly $\frac{d}{dx} \cos x = -\sin x.$

Differential Coefficient of log x.

$$y + \delta y = \log(x + \delta x) = \log x\left(1 + \frac{\delta x}{x}\right)$$

$$= \log x + \log\left(1 + \frac{\delta x}{x}\right).$$

$$= \log x + \left\{\frac{\delta x}{x} - \frac{1}{2}\left(\frac{\delta x}{x}\right)^2 + \ldots.\right\}$$

Retaining quantities of first order only:

$$\delta y = \delta x \cdot \frac{1}{x}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{x}$$

or $$\frac{d}{dx} \log x = \frac{1}{x}.$$

The same method of treatment can be applied to other cases. In the case of a function $f(x)$ we write:

$$\frac{dy}{dx} = \underset{\delta x \longrightarrow 0}{\text{Lim.}} \frac{f(x + \delta x) - f(x).}{\delta x}$$

The Differential Coefficient of the Sum of two Functions.

If $$y = \sin x + \cos x$$

we have $$\frac{dy}{dx} = \cos x - \sin x.$$

From the definition it follows that the differential coefficient is the sum of the differential coefficients of sin x and cos x.

In the general case if:

$$y = y_1 + y_2$$

where y_1 and y_2 are any two functions of x:

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}.$$

And similarly if $y = y_1 - y_2$

$$\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{dy_2}{dx}.$$

Differential Coefficient of a Product.

Let $y = y_1 y_2$

where y_1 and y_2 are any two functions:
e.g. we might have:

$$y = \sin x \times x^n,$$

sin x and x^n are two functions of x.

Suppose that x becomes $x + \delta x$ and in consequence y increases to $y + \delta y$, y_1 to $y_1 + \delta y_1$, y_2 to $y_2 + \delta y_2$.

Then

$$y + \delta y = (y_1 + \delta y_1)(y_2 + \delta y_2)$$
$$= y_1 y_2 + y_1 \delta y_2 + \delta y_1 y_2 + \delta y_1 \delta y_2.$$

Since $y = y_1 y_2$

$$\therefore \delta y = y_1 \delta y_2 + \delta y_1 y_2 + \delta y_1 \delta y_2.$$

$\delta y_1 \delta y_2$ is a term of the second order, and the other terms are of the first order.

Thus we need not retain it.

Dividing throughout by δx.

$$\frac{\delta y}{\delta x} = y_1 \frac{\delta y_2}{\delta x} + \frac{\delta y_1}{\delta x} \cdot y_2$$

Hence in the limit:

$$\frac{dy}{dx} = y_1 \frac{dy_2}{dx} + y_2 \frac{dy_1}{dx}.$$

In a product we differentiate one factor at a time, leaving the others unchanged, and add all the resulting expressions together.

This is true for any number of factors, as may be shown in the same way.

Thus, if

$$y = y_1 y_2 y_3 y_4.$$

$$\frac{dy}{dx} = \frac{dy_1}{dx} \cdot y_2 y_3 y_4 + y_1 \frac{dy_2}{dx} y_3 y_4 + y_1 y_2 \frac{dy_3}{dx} \cdot y_4 + y_1 y_2 y_3 \frac{dy_4}{dx}.$$

e.g. $y = \sin x \times x^n$

$$\frac{dy}{dx} = \sin x \cdot nx^{n-1} + \cos x \cdot x^n.$$

The Differential Coefficient of a Quotient.

We use the same notation as before and apply the same principles.

$$y = \frac{y_1}{y_2}$$

$$y + \delta y = \frac{y_1 + \delta y_1}{y_2 + \delta y_2}$$

$$\therefore \delta y = \frac{y_1 + \delta y_1}{y_2 + \delta y_2} - \frac{y_1}{y_2} = \frac{y_2\delta y_1 - y_1\delta y_2}{y_2^2\left(1 + \frac{\delta y_2}{y_2}\right)}$$

$$= \frac{(y_2\delta y_1 - y_1\delta y_2)}{y_2^2}\left(1 - \frac{\delta y_2}{y_2}\right)$$

$$= \frac{y_2\delta y_1 - y_1\delta y_2}{y_2^2} \text{ (retaining only terms of first order.}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{y_2\frac{\delta y_1}{\delta x} - y_1\frac{\delta y_2}{\delta x}}{y_2^2}$$

$$\therefore \frac{dy}{dx} = \frac{y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}}{y_2^2}$$

e.g. $$y = \tan x = \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \sec^2 x.$$

Differential Coefficient of a Function of a Function.

The expression :

$$y = a\sin^2 x + b\sin x + c$$

in which a, b, c are constant quantities, is a function of $\sin x$, $\sin x$ is itself a function of x.

Thus, y is a function of a function of x.

We proceed to determine the differential coefficient $\frac{dy}{dx}$ in this complex case.

Before attacking the general problem we will consider a special case.

Let $y = \log \sin x$.

and write $z = \sin x$.

Then $y = \log z$

and as above :

$$\delta y = \frac{1}{z} \cdot \delta z.$$

In this step we have regarded y as a function of z.

But $z = \sin x.$

$$\therefore \ \delta z = \cos x \cdot \delta x.$$

$$\therefore \ \delta y = \frac{1}{z} \cdot \cos x \cdot \delta x.$$

$$\therefore \ \frac{dy}{dx} = \frac{\cos x}{\sin x}.$$

Note that from $\delta y = \frac{1}{z} \delta z$ we have :

$$\frac{\delta y}{\delta x} = \frac{1}{z} \frac{\delta z}{\delta x}.$$

Hence $$\frac{dy}{dx} = \frac{1}{z} \frac{dz}{dx}.$$

Now $$\frac{dy}{dz} = \frac{1}{z}.$$

$$\therefore \ \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

The rule is :

Differentiate the whole function first as if the inner function (in this case sin x) were the independent variable, and thus obtain $\frac{dy}{dz}$, then multiply by the differential coefficient of the inner function.

This rule, which has been established in a special case, can readily be proved generally.

Let $y = F(z)$ where $z = f(x)$.

Then $$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x}.$$

In the limit : $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

e.g. $y = \sin^2 x = (\sin x)^2.$

Put $z = \sin x.$

then $y = z^2.$

$$\therefore \ \frac{dy}{dz} = 2z, \quad \frac{dz}{dx} = \cos x.$$

$$\therefore \ \frac{dy}{dx} = 2z \cos x = 2 \sin x \cos x.$$

It will be convenient at first to introduce z in this way, but with practice this intermediate step may be omitted.

§5. The Second Differential Coefficient

We have seen that when $y = \sin x$,

$$\frac{dy}{dx} = \cos x.$$

Thus $\frac{dy}{dx}$ is itself a function of x and will have a differential coefficient.

Let $\frac{dy}{dx}$ be denoted by z.

$$z = \cos x.$$

$$\therefore \frac{dz}{dx} = -\sin x.$$

Thus the differential coefficient of $\frac{dy}{dx}$ is $-\sin x$ in this case. This is called the second differential coefficient of y, and is written $\frac{d^2y}{dx^2}$ or D^2y.

Similarly we define higher coefficients $\frac{d^3y}{dx^3}$, etc., but we shall not be concerned with higher orders than the second.

As another example consider $y = x^n$.

$$\frac{dy}{dx} = nx^{n-1}.$$

$$\frac{d^2y}{dx^2} = n(n-1)\,x^{n-2}, \text{ etc.}$$

§6. Applications in Dynamics

When a particle describes a path under the action of a force or set of forces, its position will vary with the time. Thus if a stone is thrown vertically upward the position at any instant will depend on the time that has elapsed since the moment of projection.

If this distance is measured by y,

$$y = f(t).$$

Suppose the time at a particular instant is measured by t and the corresponding value of the distance is y. At a small time

δt later y will become $y + \delta y$. We may say the average velocity is $\frac{\delta y}{\delta t}$ in this short interval. The smaller we make the interval the more accurately will this ratio represent the velocity near the position denoted by y. Thus proceeding to the limit in which δt ultimately vanishes we have :

$$\text{velocity } v = \underset{\delta t \longrightarrow 0}{\text{Lim.}} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$

It is often convenient to measure the position of a point by its position measured from some point O along the arc it is describing (fig. 7).

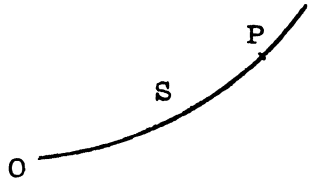


Fig. 7.

In this case $\frac{ds}{dt}$ is the velocity at P along the arc or the velocity in the direction of the tangent at P, where s measures the arc OP.

In the same way $\frac{dv}{dt}$ measures the acceleration of a particle moving with velocity v.

But $$\frac{dv}{dt} = \frac{d^2s}{dt^2} \quad \text{since} \quad v = \frac{ds}{dt}.$$

Thus the acceleration along the arc is measured by the second differential coefficient.

§7. A Geometrical Application

Referring to fig. 1, let P_1 denote the point $(x \cdot y)$ and P_2 the point $(x + \delta x \cdot y + \delta y)$.

Then $P_2N = \delta y$ and $P_1N = M_1M_2 = \delta x$.

The ratio $\frac{\delta y}{\delta x} = \frac{P_2N}{P_1N} = \tan P_2P_1N = \tan \psi$.

As the limit is approached the line P_2P_1 becomes closer and closer to the tangent at P_1 and finally actually coincides with it.

Thus $$\underset{\delta x \longrightarrow 0}{\text{Lim.}} \frac{\delta y}{\delta x} = \frac{dy}{dx} = \tan \psi$$

where ψ now denotes the inclination of the tangent at P_1 to the axis of x.

If a curve in the course of its extent is of the character shown in fig. 8, the turning points B_1B_3 are called maximum and B_2B_4 minimum values of the ordinate.

B_1 is a point such that it possesses the greatest value of y for points in its immediate neighbourhood. It is not necessarily the greatest ordinate of the curve.

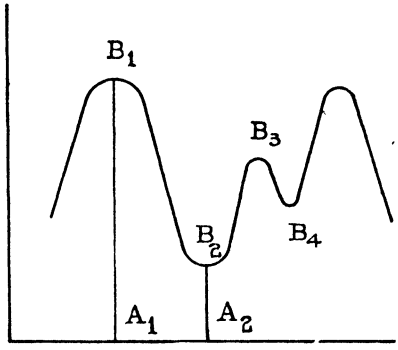


FIG. 8

A similar remark applies to the minimum values. At such extreme points the tangent is necessarily parallel to the axis of x. Thus $\psi = 0$ and $\tan \psi$ also vanishes.

Hence for maxima and minima:

$$\frac{dy}{dx} = 0.$$

This relation is also of great use in Physical problems, as in the case of the minimum deviation of a prism or in the theory of the formation of the rainbow.

§8 Integration

The process of Differentiation is to derive from a function its differential coefficient or rate of variation with respect to the independent variable.

We have the inverse problem in Integration, where from the differential coefficient we have to derive the function. This operation is more difficult and cannot be accomplished in every case.

There are many standard cases which can be readily recognized, e.g. we have seen that for $y = x^n$

$$\frac{dy}{dx} = nx^{n-1}$$

so that when we are given that

$$\frac{dy}{dx} = nx^{n-1}$$

we remember that y has the value x^n.

But if y has the value $x^n + A$, where A is some constant, and hence has a zero differential coefficient we still have :

$$\frac{dy}{dx} = nx^{n-1}$$

Thus, there is an uncertainty about the value of y to the extent of an unknown constant. Such a constant must always be taken

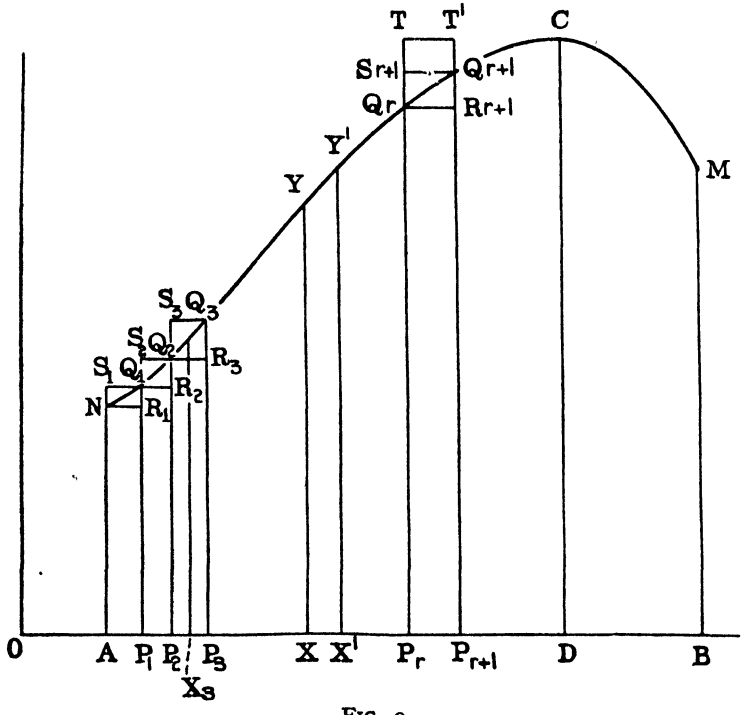


FIG. 9

into account in integration and is called the integration constant. In Physical problems this constant usually has some special value determined by the conditions of the problem.

We shall approach the Integral Calculus by a Geometrical consideration.

In fig. 9, NM represents a curve, $y = f(x)$. OA and OB are two fixed values of x and the area between the ordinates AN and BM, the curve NM and the axis of x is divided by drawing ordinates P_1Q_1, P_2Q_2, P_3Q_3, etc., between A and B, so that $AP_1 = \delta x_1$, $P_1P_2 = \delta x_2$, $P_2P_3 = \delta x_3$, etc.

All these elements are to be reduced and ultimately vanish.

Let x_1, x_2, x_3, etc., denote the values of the abscissæ for points lying in the strips AP_1, P_1P_2, P_2P_3, etc., respectively.

Of these the ordinate at X_3, which denotes the point with abscissa x_3, is a typical example:

Let the ordinates corresponding to these abscissæ be denoted by y_1, y_2, y_3, etc.

We shall suppose that between all the points such as NQ_1, Q_1Q_2, Q_2Q_3, etc., the curve is continuously increasing or decreasing. At every point where there is a maximum or minimum, as at C, we shall draw an ordinate. The curve is drawn with only one such turning point, and this is sufficient for our purpose since the argument may be extended to deal with cases where several such points occur.

From the points N, Q_1, Q_2, etc., draw perpendiculars NR_1, Q_1R_2, Q_2R_3, etc., to the ordinates P_1Q_1, P_2Q_2, P_3Q_3, etc., respectively, and from Q_1, Q_2, Q_3, etc., draw perpendiculars Q_1S_1, Q_2S_2, Q_3S_3, etc., to the ordinates AN, P_1Q_1, P_2Q_2, etc., respectively, as shown in fig. 9.

We have thus two step-like figures which we will call the outer and inner stepped figures.

Now consider the expression $y_3\,\delta x_3$.

This lies between $Q_3P_3 \times P_2P_3$ and $Q_2P_2 \times P_2P_3$.

It thus represents an area intermediate between that of the rectangle $P_2P_3Q_3S_3$ and $P_2P_3R_3Q_2$.

In the same way $y_1\delta x_1$ lies between $AP_1Q_1S_1$ and AP_1R_1N.

Thus, if we consider the sum:

$$y_1\delta x_1 + y_2\delta x_2 + y_3\delta x_3 + \ldots + y_n\delta x_n = \Sigma y\delta x,$$

where y_n represents an ordinate in the last strip which terminates at B and has a width δx_n, we know that it lies between the outer and inner stepped figures.

We have to inquire into the value of the sum when we make the number of divisions very large or, what is the same thing, the values δx_1, δx_2, δx_3, etc., very small.

In performing this summation we meet with the difficulty that although each term becomes small, the total number increases, and the question arises as to whether the sum remains finite under these circumstances. It seems almost axiomatic that the sum will in the limit prove to be the area between the curve, the extreme ordinates and the x-axis.

However small the values, δx, become the sum concerned always lies between the outer and inner stepped figures. The difference between these areas is the sum of the rectangles:

$$S_1Q_1R_1N_1, \quad S_2Q_2R_2Q_1, \text{ etc.}$$

There will always be one strip which has a width greater than that of any of the others, or at any rate its width will not be less than that of any other, though of course it is possible to have another strip of equal width. Let this be the strip lying between P_r and P_{r+1}, and draw the two rectangles as before, viz. $P_r P_{r+1} Q_{r+1} S_{r+1}$ and $P_r P_{r+1} R_{r+1} Q_r$.

In our figure the greatest ordinate is CD. Continue $P_{r+1} Q_{r+1}$ to T^1 and $P_r S_{r+1}$ to T so that $P_{r+1} T^1 = P_r T = CD$.

Now all the small rectangles S_1R_1, S_2R_2, S_3R_3, etc., lying between AN and the maximum ordinate at C could be removed and placed within the rectangle TP_{r+1}. Thus the difference between the two stepped figures lying on the left of C cannot be greater than the rectangle TP_{r+1}.

As the widths δx_1, δx_2, δx_3, etc., are diminished the area TP_{r+1} diminishes and vanishes in the limit.

Thus in the limit when the widths of the strips become infinitely small, the difference between the two stepped areas also diminishes indefinitely.

In other words, the two areas become equal in the limit, and each is then equal to the area between the curve, the ordinate, AN, the maximum ordinate and the x-axis.

The same process may be applied to the area between the maximum ordinate, BM, the curve and x-axis. Thus adding the two parts together we find that the total area is equal to the sum :

$$\Sigma y \delta x$$

in which δx tends to zero.

If there occur any finite number of maxima or minima in the curve, it may be divided at each and the same process applied to each partial area.

When the several areas are summed the result applies to the whole curve.

This sum is denoted by $\int_a^b y dx$ and it is called the integral of y with respect to x.

a and b denote the values OA and OB, and since these are definite abscissæ the integral is called a definite integral.

In some cases when we do not fix the limits we write simply $\int y dx$ and the integral is called indefinite. The lower limit is then tacitly assumed to be some convenient starting point, while the upper limit is any variable point which we denote by the variable x.

§9. Connection between Differentiation and Integration

If we consider the area included between the limits A and X where X is any point with co-ordinate X, the magnitude of the area depends on the positions of the points A and X, and the form of the curve. The form of the curve is fixed by its equation, and when X is changed the area will vary by an amount depending on the change in X.

The area thus depends on the value of X at which the summation is stopped. If the student feels any difficulty about this point he should draw a semicircle and choose the diameter as x— axis. Choose one end of the diameter for the point, O, and determine the area up to an ordinate drawn at a point, X, taken on the diameter. If OX is denoted by x it will be found that the area can be expressed as a function of x.

Thus we may write :

$$\int_a^x y\,dx = \text{a function of X, say, } f(\mathrm{X})$$

Write :

$$\mathrm{A} = \int_a^x y\,dx.$$

We are about to show what is the relation between A and y.

Proceed a step farther with the integration, up to the point, X^1. Denote XX^1 by $\delta\mathrm{X}$.

The area will alter by the amount, $\mathrm{XX}^1\mathrm{Y}^1\mathrm{Y}$, where YY^1 means the arc of the curve between Y and Y^1.

Write this change $= \delta\mathrm{A}$.

Then the new area is

$$\mathrm{A} + \delta\mathrm{A} = \int_a^{x^1} y\,dx.$$

$$= \text{area up to XY} + \mathrm{XX}^1\mathrm{Y}^1\mathrm{Y}.$$

But $\mathrm{A} = \text{area up to XY}.$

$$\therefore \delta\mathrm{A} = \mathrm{XX}^1\mathrm{Y}^1\mathrm{Y}.$$

$$= \mathrm{Y} \times \delta\mathrm{X} \text{ in the limit,}$$

where Y is an ordinate in the strip, XX^1, and in the limit we shall suppose the strip to possess a vanishingly small width.

$$\therefore \mathrm{Y} = \mathrm{Lim.}\,\frac{\delta\mathrm{A}}{\delta\mathrm{X}} = \frac{d\mathrm{A}}{d\mathrm{X}}$$

We have used X and Y to denote the variables for convenience, but we have not specified any particular values for them; they

denote any abscissa and corresponding ordinate so that we may just as well write :

$$y = \frac{dA}{dx}$$

Thus y is the differential coefficient of the integral which has been denoted by A.

The problem thus resolves itself into determining A if we know its differential coefficient.

The occurrence of the arbitrary constant is seen to be connected with the choice of the starting point.

We may summarize this by stating that if :

$$A = \int y dx$$

then

$$y = \frac{dA}{dx}$$

There is no general rule for passing from y to A, i.e. from a function to its integral.

We can only perform the operation by recognizing a standard form, and many devices have to be studied for the reduction of forms not directly recognizable, to more familiar ones.

For these the student is referred to textbooks on the Integral Calculus.

Reference should be made to tables in Mathematical textbooks for the important standard forms.

§10. Evaluation of a Definite Integral

Suppose that A has been determined in this way, and that it is now expressed as a function of x.

Then

$$A = \int_a^x y dx = f(x) + \text{constant.}$$

$$= f(x) + B \text{ (say).}$$

B is the constant which it is necessary to add for the reason explained above.

B depends on the arbitrary starting point so that there is some connexion between B and a.

Write $x = b$, i.e. perform the integration up to a fixed point b.

Then

$$\int_a^b y dx = f(b) + B.$$

If we write $x = a$ we have

$$\int_a^a y dx.$$

This must vanish, for it means the calculation of an integral

of no extent, or referring to our illustration it indicates an area of no width.

$$\therefore \int_a^a y dx = f(a) + B = 0.$$

$$\therefore B = -f(a).$$

This is the relation between B and a.

Hence : $$\int_a^b y dx = f(b) - f(a).$$

The rule is therefore : *Find the function of which y is the differential coefficient and express it as a function of x. Substitute in it the values of the upper and lower limits and subtract the results so obtained.*

e.g. $$\int_1^{10} x^2 dx.$$

x^2 is the differential coefficient of $\frac{x^3}{3}$,

$$\therefore \int_1^{10} x^2 dx = \frac{10^3}{3} - \frac{1^3}{3} = 33.$$

$$\int_0^{\frac{\pi}{2}} \sin x dx.$$

$\sin x$ is the differential coefficient of $-\cos x$.

$$\therefore \int_0^{\frac{\pi}{2}} \sin x dx = \left[-\cos x \right]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} - (-\cos 0) = 1.$$

This measures the area of the part of the sine curve from the origin up to its maximum ordinate at $\frac{\pi}{2}$, lying above the x axis.

§11. An important application of the Integral Calculus is to the calculation of moments of inertia. (Chapter II.)

A body is considered as made up of a large number of small masses δm. Let r denote the shortest distance of δm from an axis. Then the product $r^2 \delta m$ is called the moment of inertia of the particle δm about the axis. If all the particles of the body are taken into consideration and we make the summation $\Sigma r^2 \delta m$ for the whole body, we obtain the moment of inertia of the body. Thus $\Sigma r^2 \delta m$ or $\int r^2 dm$ is the moment of inertia of the body about the axis.

If we have an area A divided into small elements, δa, each of which has a corresponding distance r from some axis, we require sometimes the value $\int r^2 da$, and this may be called the moment of inertia (shortly M. of I.) of the area about the axis. Such an expression occurs in the treatment of bending of beams, and will be employed in the calculation of Young's Modulus by bending.

As an example of the calculation of a M. of I., consider the case of a thin rod of total mass M and length $2l$. Let its mass per unit length be m, so that $M = 2lm$.

Assume that the axis is through the c.g. and perpendicular to the rod (fig. 10).

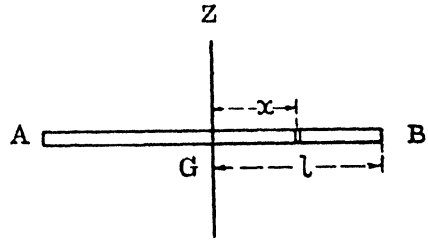


FIG. 10

The form of section is of no consequence, but the ends at A and B are perpendicular to the length.

The element of length dx at distance x from GZ will have a mass mdx and its moment of inertia about GZ is $mdx \cdot x^2$.

Thus the total M. of I. $= \int_{-l}^{l} mx^2 dx$ the limits indicating between what limits x extends.

Now $$\int_{-l}^{l} mx^2 dx = \left[\frac{mx^3}{3}\right]_{-l} = \frac{2}{3} ml^3 = \frac{1}{3} m(2l)l^2 = \frac{Ml^2}{3} \quad \ldots \quad (4)$$

The application of integration to the calculation of moments of inertia is extended in Chapter II.

§12. Oscillatory Motion

Many experiments make use of the fact that in a number of cases bodies slightly displaced from their position of equilibrium perform periodic isochronous vibrations about that position. Examples of this occur in the case of the simple pendulum, in cases of torsional oscillations, and movements of galvanometer needles.

We usually require the complete period of the oscillation in making calculations.

Such motion is best and most simply treated by means of the Differential Calculus.

A simple harmonic vibration is by definition one in which the body moves so that it is under the action of a force tending to restore it to the position of equilibrium, the magnitude of the force being proportional to the measure of its displacement.

Thus, if a point is moving to and fro in a straight line about a position O, and is performing S.H.M. it is always under a force directed towards O and proportional to the distance OP ($= x$).

But the acceleration is $\frac{d^2x}{dt^2}$, and if we write :

$$\text{force} = -kx$$

where k is constant and the negative sign denotes the direction of the force, we have by Newton's Second Law of Motion

$$m\frac{dx^2}{dt^2} = -kx.$$

It is usual to write this in the form :

$$\frac{dx^2}{dt^2} + p^2x = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $p^2 = \frac{k}{m}$.

This is an example of a differential equation.

We do not here consider any series of arguments leading logically to the solution of this equation, we merely state the solution and verify the truth of the statement.

This equation and another slightly more complicated are so important in Physics at an early stage that the solutions should be remembered and the student prepared to apply them with ease.

Consider $x = A \sin (pt + \alpha)$ .....(6)

A and α are arbitrary constants not occurring in the equation. They occur for the same reason that B occurred in § 10 ; in fact, we are proceeding from a differential coefficient to the function from which it is derived and so are performing an integration when solving the above equation. It should be remembered also that in the complete solution of a differential equation there must occur the same number of arbitrary constants as the number of the order of the highest differential coefficient in the equation.

In the present case the highest order is 2, and so we have A and α.

Now $\frac{dx}{dt} = p\text{A} \cos (pt + \alpha)$, $\frac{d^2x}{dt^2} = -p^2\text{A} \sin (pt + \alpha)$
$= -p^2x.$

Hence the value chosen for x satisfies the equation.

Since $x = \text{A} \sin (pt + \alpha)$ we shall have the same value of x occurring T secs. later when a complete period has elapsed.

$$\therefore x = \text{A} \sin [p(t + \text{T}) + \alpha] = \text{A} \sin (pt + \alpha).$$
$$\therefore \sin (pt + \alpha + p\text{T}) = \sin (pt + \alpha).$$

This is true if $p\text{T} = 2\pi, 4\pi$, etc.

Thus the first recurrence of the value of x is after an interval $\text{T} = 2\pi/p$, and this is the period of the S.H.M. described by our equation.

We note that $\text{T} = 2\pi \div$ *square root of the coefficient of* x *in the reduced equation.* .....(7)

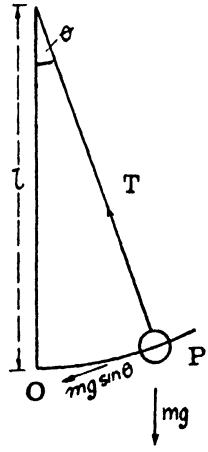


FIG. 11

Thus in considering any problem in which T is required, we have only to write down the equation of motion and put it in this form; we can then write down the value of T immediately.

We consider the case of the simple pendulum (fig. 11).

The displacement is measured by the angle θ.

The force along the tangent to the circle described by P is $mg \sin \theta$ or $mg\theta$ if θ is small. Note that the motion is only S.H.M. if θ is small, since the force is only then proportional to the displacement.

The acceleration in this direction is $\frac{d^2s}{dt^2}$ where $s =$ arc OP.

or since $s = l\theta$, it is $l\frac{d^2\theta}{dt^2}$

$$\therefore ml\frac{d^2\theta}{dt^2} = -mg\theta$$

$$\therefore \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

$$\therefore T = 2\pi \div \sqrt{\frac{g}{l}} = 2\pi\sqrt{\frac{l}{g}} \quad \ldots\ldots\ldots (8)$$

The more complicated case occurs when the effect of friction has to be included. The force of friction depends on the sum of a number of terms proportional to the powers of the velocities. In slow motions, such as those occurring in the movements of a galvanometer needle, the term of paramount importance is that containing the first power of the velocity. As this discussion is for the purpose of considering oscillations such as occur in galvanometers we consider this as a typical example.

The position of the needle or coil is defined by its angular displacement from a normal position.

For example, let OH denote the direction of a magnetic field in which the needle normally sets.

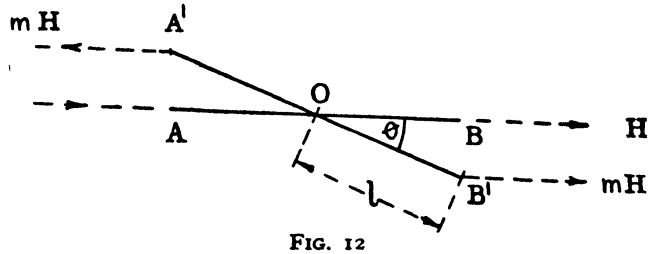


FIG. 12

In the new position, A^1B^1, let θ measure the angle BOB^1.

If θ is small, the restoring couple due to the field H is $2mHl \sin\theta = 2mHl\theta$, where l = half the length of the magnet. If the magnetic moment be M this couple may be written $MH\theta$. m denotes the pole strength. If I denotes the M. of I. of the needle, neglecting friction we have :

$$I\frac{d^2\theta}{dt^2} = -MH\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

and the motion is simple harmonic.

We assume in accordance with what has been stated above that there is a frictional force proportional to the angular velocity $\frac{d\theta}{dt}$ and write the force $= c\frac{d\theta}{dt}$, where c is constant.

The equation then becomes :

$$I\frac{d^2\theta}{dt^2} = -c\frac{d\theta}{dt} - MH\theta \quad \ldots\ldots\ldots\ldots (10)$$

and may be simplified by dividing throughout by I and changing the notation of the constants.

It will now be written :

$$\frac{d^2\theta}{dt^2} + K\frac{d\theta}{dt} + n^2\theta = 0. \quad \ldots\ldots\ldots\ldots (11)$$

The derivation of this result has been made by reference to a magnetic needle oscillating in a magnetic field, let us say, supported by a thread of negligible torsion.

The same result is obtained for such oscillations as occur in the case of bars vibrating at the end of stretched strings under torsion or for oscillations in the electrical discharge of a condenser.

We shall denote $\frac{d\theta}{dt}$ by $\dot{\theta}$ and $\frac{d^2\theta}{dt^2}$ by $\ddot{\theta}$.

Again, we do not attempt to deduce the solution of the equation, we merely verify a stated result.

The solution of such equations can be obtained by writing

$$\theta = Ae^{mt}$$

for on using this value of θ the equation is satisfied provided

$$m^2 + Km + n^2 = 0.$$

This is evident if we substitute $\dot{\theta} = m^2Ae^{mt}$, $\ddot{\theta} = mAe^{mt}$.

Hence $m = \dfrac{-K \pm \sqrt{K^2 - 4n^2}}{2} = m_1$ or m_2, say.

Thus the complete solution is : $\theta = Ae^{m_1 t} + Be^{m_2 t}$.

A and B are the two constants necessary in the complete solution.

If we substitute the values of m_1 and m_2 we find :

$$\theta = e^{-\frac{Kt}{2}}\left(Ae^{\sqrt{\frac{K^2}{4}-n^2}\cdot t} + Be^{-\sqrt{\frac{K^2}{4}-n^2}\cdot t}\right) \ldots\ldots\ldots (12)$$

When $\frac{K^2}{4} > n^2$ the indices of the exponentials in the bracket are real and θ is not periodic. θ merely changes in value with t according to the exponential law. Of course K and n depend on the particular problem considered. K measures the friction and n depends on the nature of the restoring force. If friction were not present the body would vibrate with period $\frac{2\pi}{n}$, so that this may be said to be the natural period of vibration if there is no friction.

The case with which we are concerned is when the motion is oscillatory, and this requires that the indices of the exponentials within the bracket should be imaginary,

i.e. $$\frac{K^2}{4} < n^2.$$

Write $i = \sqrt{-1}$, we then have for the terms in the bracket :

$$Ae^{i\sqrt{n^2-\frac{K^2}{4}}\cdot t} + Be^{-i\sqrt{n^2-\frac{K^2}{4}}\cdot t}.$$

By altering the constants we can put this in the form:

$$C \sin \sqrt{n^2 - \frac{K^2}{4}} \cdot t + D \cos \sqrt{n^2 - \frac{K^2}{4}} \cdot t,$$

or more simply:

$$E \sin \left(\sqrt{n^2 - \frac{K^2}{4}} \cdot t + \alpha \right)$$

Thus the solution is now to be written:

$$\theta = Ee^{-\frac{Kt}{2}} \sin \left(\sqrt{n^2 - \frac{K^2}{4}} \cdot t + \alpha \right) \ldots\ldots\ldots\ldots (13).$$

where E and α are the constants.

The student may, if he wishes, take θ as given by this value, substitute in the equation and verify that this satisfies it.

This form of the solution shows the similarity with the last case in which we had

$$\theta = A \sin (pt + \alpha).$$

The amplitude is now $Ee^{-\frac{Kt}{2}}$ instead of A. It thus varies with the time, and since K is positive, the value of $e^{-\frac{Kt}{2}}$ is less than unity, and the amplitude diminishes with the time. The motion is said to be damped, and the damping depends upon K.

θ does not maintain the same amplitude during the motion, as in the undamped case. The curve illustrating the motion is drawn in fig. 13.

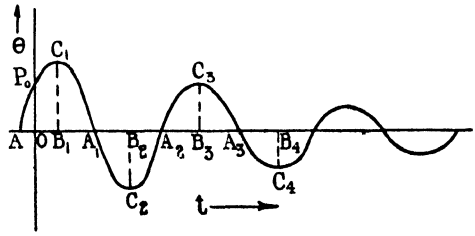


FIG. 13

When $t = 0$, $\theta_0 = E \sin \alpha$,

and when $t = \frac{\pi}{2} \div \sqrt{n^2 - \frac{K^2}{4}} - \alpha$, $\theta = Ee^{-\frac{Kt}{2}}$ where t has this particular value.

Other pairs of values (θ, t) may be obtained in the same way, and the curve drawn as above.

It will be noted that the curve crosses the t-axis at the points A_1A_2 . . . and that these points are equidistant.

The interval A_1A_3 is called the period of the motion.

It will be noticed that θ is zero when

$$\sin\left(\sqrt{n^2 - \frac{K^2}{4}}\,t + \alpha\right)$$

vanishes.

For simplicity write $q = \sqrt{n^2 - \frac{K^2}{4}}$

Then $\sin(qt + \alpha) = 0$ if

$qt + \alpha = 0,\ \pi,\ 2\pi,\ 3\pi,$ etc.

that is when $t = -\frac{\alpha}{q},\ \frac{\pi - \alpha}{q},\ \frac{2\pi - \alpha}{q},\ \frac{3\pi - \alpha}{q},$ etc.

On the curve these times correspond to the points A, A_1, A_2, A_3, etc.

In the case of point A_1 the oscillator is at the zero position, but is moving in a direction opposite to that at A or A_2.

The difference between the above values of t at A_1 and A_3 is $\frac{3\pi - \alpha}{q} - \frac{\pi - \alpha}{q} = \frac{2\pi}{q}$.

Thus the period is $\frac{2\pi}{\sqrt{n^2 - \frac{K^2}{4}}}$ .....(14)

Had there been no damping, the period would have been $\frac{2\pi}{n}$.

We shall call these T_1 and T_0 respectively.

At such points as B_1, B_2, B_3, etc., the value of $\sin(qt + \alpha)$ is numerically equal to unity, and the amplitude is then measured by $Ee^{-\frac{Kt}{2}}$ with the appropriate value of t.

At B_1 the value of t is $\frac{\pi}{2q} - \alpha$ and at B_2 half a period later the value is $\frac{\pi}{2q} - \alpha + \frac{1}{2}T_1 = \frac{3}{2}\frac{\pi}{q} - \alpha$.

The corresponding values of $\sin(qt + \alpha)$ are $+1$ and -1; this means of course that the displacement is on the opposite side of the mean position in the second case.

Thus $\frac{B_1C_1}{B_2C_2} = e^{+\frac{KT_1}{2}}$

In the same way

$$\frac{B_2C_2}{B_3C_3} = e^{\frac{KT_1}{2}} = \frac{B_3C_3}{B_4C_4} = \text{etc.} \quad \text{.....(15)}$$

Thus the ratios of successive maximum displacements are constant.

The value $\frac{KT_1}{2}$ is denoted by λ which is called the logarithmic decrement.

CHAPTER I

MEASUREMENT OF LENGTH, AREA, VOLUME, AND MASS.

THE measurement of the length of a body may be made by using one of the usual vernier or micrometer devices, such as the vernier calliper, the micrometer screw gauge, or the spherometer.

The measurement of small objects may also be carried out by use of the travelling microscope or the micrometer microscope.

In using either of these instruments care must be taken, when viewing the image of the ends of the object, that this image is in the same plane as the cross-hair or the small scale in the eyepiece, otherwise parallax errors may be introduced. The image seen should not show any relative movement with the cross-hairs when the eye is moved across the field of view.

When focussing the cross-hair, as a preliminary adjustment, the eye should be unstrained. The microscope is turned to a bright distant object and the adjustment of the eyepiece should be made so that the distant object viewed by the one eye is in focus when the cross-hair as viewed by the other eye is also clearly focussed.

The Comparator

When a length exceeds a few centimetres, the travelling microscope or micrometer microscope are not used individually as measurers, but are replaced by an arrangement of two such instruments arranged at a variable distance apart on a fixed graduated bed. Each microscope may be moved in the usual manner in a direction parallel to the length of the bed. Each microscope is provided with a scale or fine cross-hair in the focal plane of the eyepiece. Each eyepiece is adjusted so that the scale or cross-hair is in sharp focus for normal vision, and is replaced in the carriage.

An object to be measured is fastened rigidly along the bed, in the groove provided for it. The image of one end of the object as seen by the first microscope is brought into coincidence with the image of the intersection of the cross-hairs and the second microscope is moved until the image of the other end on the object coincides with cross-hair intersection in that microscope.

Thus the images of the two ends of the object as seen by the

two microscopes normal to its length are formed at the intersection of the two pairs of cross-hairs.

The object is removed and a standard rule substituted. The readings of the standard scale seen opposite the two cross-hairs obviously enables the length of the object to be ascertained by subtraction.

As an example of this *Substitution method* we will consider the experimental details of the following experiment.

The Comparison of the Yard and the Metre

Standard rules, engraved as finely as possible with inch and centimetre graduations, and of lengths one yard and one metre, are employed in this experiment.

Place the yard rule on the bed of the comparator and focus the one microscope on a scale division near one end of the yard, moving the latter and the microscope until this is accomplished. Then move the second microscope until a scale division near the other end of the rod is sharply focussed: the number of inches included between the two being noted—36 if the scale is sufficiently well graduated to allow of this. Then, taking care not to upset the arrangement of the microscopes in any way, remove the yard scale and substitute the metre, so that the graduations are in good focus. Move the scale so that the image of a division near one end is in coincidence with the cross-hair intersection in the one microscope. Under these circumstances the second microscope will not be opposite a division. Coincidence of the cross-hair and the image of a scale division is brought about by a movement of the microscope, parallel to the length of the bed, which is measured on the vernier scale attached to it.

The size of the gap between the two cross-hairs $\pm$ the movement of the microscope is then read off in cms.

Care is taken in noting the movement of the microscope to see exactly the unit used in these graduations. In this way we obtain two measurements, one in each system, for the same distance and may calculate the number of inches to the metre, or cms. to the yard.

The vernier scale movement on the microscope is often replaced by a micrometer screw capable of a much shorter range of movement. With such a screw traverse, the movement of the microscope may be readily measured to ·001 of a cm. Using such a comparator the values given below were obtained:

(1) 1 yard = 91·5 cms. + ·008 inch.
$\therefore$ 1 metre = 39·331 inches.

(2) 50 cms. = 19·6895 inches.
i.e. 1 metre = 39·379 inches.

(3) 1 metre = 39·375 inches + ·002 inch
= 39·377 inches

mean value

1 *metre* = 39·362 *inches.*

The Planimeter

The estimation of the area of a plane figure may be carried out by one of the many geometrical methods or by the use of a planimeter, an instrument designed to measure such areas directly.

Of this class of instrument the Amsler planimeter is generally used.

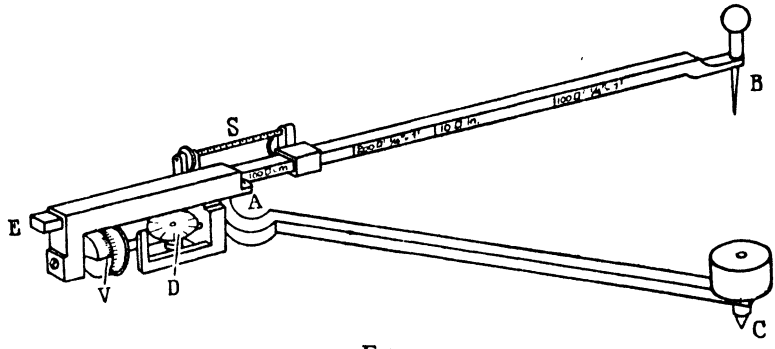


FIG. 14

It consists, essentially, of two arms AC and EB, hinged at A, fig. 14; AC is of fixed length and is provided with a needle point loaded above by a small weight as shown in the diagram. The second arm may be varied in length by sliding that portion of it which carries the tracer, B, into the slot provided in the other half, EA. By means of fine adjustment, S, the length BE may be set accurately at any division along the graduated face of BA.

In addition to the tracer, B, this arm is provided with a small wheel to which is attached a graduated drum which moves past a fixed vernier scale, V. By means of the graduated drum and vernier, the rotation of the wheel may be measured to $\frac{1}{100}$ of a complete turn. The axle of the wheel and drum is arranged parallel to the length of EB and is provided with a worm gear, which moves a horizontal indicator, D, one division per revolution.

When placed on a plane the instrument is supported at three points: the needle point C, the point of the tracer B, and the point of contact of the wheel with the plane.

To measure an area the needle C is placed at a point outside it, such that B may be moved round the boundary of the area. For the measurement of large areas this will be impossible, but the details for such a case will be seen later. Starting at any point on the boundary of the area, the tracer is moved carefully over its contour until it is finally in the starting position. During this operation the wheel will have rotated in general a definite number of revolutions plus a measurable fraction of a revolution. From this observation, the area of the figure may be calculated.

The method of calculating the area will be best understood by first considering the theory of the instrument.

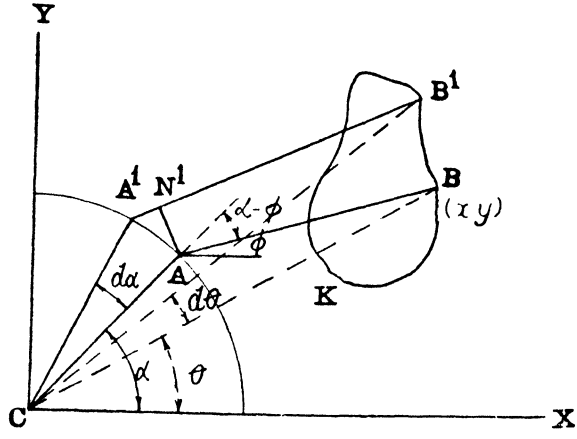


FIG. 15

When the needle C is fixed in the plane of the area to be measured, any movement of B along the boundary of the figure will result in a movement of A along the arc of a circle with C as centre and radius CA = a cms. (see fig. 15). Further, the wheel, W, will roll a distance equal to the total displacement when the movement of EB is at right angles to its length: movement parallel to the length causes no rotation, the forces acting on the wheel due to contact with the plane, under these latter circumstances only produces a couple tending to move the axle parallel to its length about the pivots.

So for any intermediate form of displacement, the rotation produced in the wheel will correspond to the component of the displacement at right angles to the length EB, and if n revolutions occur, $2\pi n$R will measure this normal displacement, R being the radius of the wheel.

Now it will be shown below that the area to be measured is equal to $2\pi n$R × AB: since the distance of the wheel from A

does not occur in this expression we must first show that this distance has no effect on the number of revolutions the wheel makes.

In fig. 15, let B and B^1 be two positions of the tracer a very small distance apart on the boundary of the area to be measured, B B^1 K ; A and A^1 being the corresponding positions of the hinge.

Let AB = b, and suppose that the centre of the wheel is at P (fig. 16). Draw AN^1 and PN normal to A^1B^1 from A and P and let $AN^1 = ds : PN = ds' : AP = c$.

AL being parallel to A^1B^1, let the angle LAP = $d\varphi$.

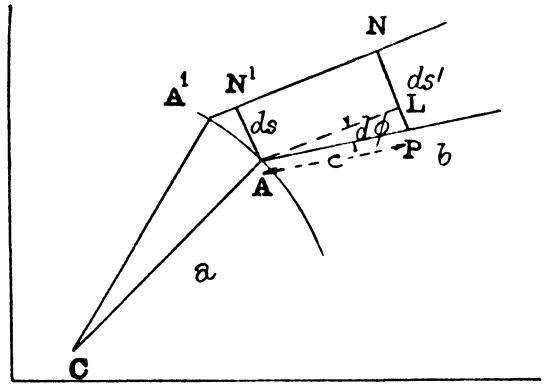


FIG. 16

Since BB^1 is a small distance, ds, ds', and $d\varphi$ are also small.

Now $PN = NL + LP$

or $ds' = ds + c \cdot d\varphi$.

The distance moved by P as B traces the boundary of the area BB^1K is $\Sigma\, ds'$

or $\Sigma\, ds' = \Sigma\, ds + \Sigma\, c d\varphi$.

B finally returns to the starting point, and therefore $\Sigma\, d\varphi = 0$

i.e. $\Sigma\, ds' = \Sigma ds$.

Thus a wheel placed at A would indicate the same movement as the one at P, or at any other point along EB as could be shown by the same process as above. *The position of the wheel on the arm* AB *does not, therefore, affect the reading of the instrument.*

To show that the area of a plane figure which does not include the needle point C is equal to $(2\pi n R)b$ let us refer the figures to rectangular axes with C as origin, fig. 15.

Let CAB be one position of the planimeter and CA^1B^1 a second position such that BB^1 is a small displacement of the tracer. Let the area to be measured be BB^1K.

Referred to these axes let x, y be the co-ordinates of B.

The area BB^1K may be conveniently referred to polar co-ordinate r and θ, i.e. let CB be r, $\angle BCX = \theta$ then if $\angle B^1CB = d\theta$, the arc $BB^1 \simeq r \cdot d\theta$ and the area $B^1CB = \frac{1}{2}r \cdot rd\theta = \frac{1}{2}r^2d\theta$, i.e.

$$\Sigma \tfrac{1}{2}r^2d\theta = \text{area of } BB^1K \quad \ldots\ldots\ldots (1)$$

For as the radius vector moves round the figure on the boundary remote from O, the small area contains, in turn, each element of the area to be determined, + the external triangle from the boundary of the figure on the side near to O; this latter area is deducted from the sum as the radius vector travels along this near boundary, for here $\delta\theta$ is negative.

Now $x = r\cos\theta$ $\qquad y = r\sin\theta$

$\therefore\ dx = -r\sin\theta \cdot d\theta + \cos\theta\, dr$, $\quad dy = r\cos\theta \cdot d\theta + \sin\theta \cdot dr$

whence $xdy - ydx = r^2\, d\theta$.

Or, the area of the figure, $\frac{1}{2}\Sigma r^2d\theta$ from (1)

$$= \tfrac{1}{2}\Sigma\,(xdy - ydx) \quad \ldots\ldots\ldots\ldots (2)$$

Let $\angle ACX = \alpha$; $A^1CA = d\alpha$; $AN^1A^1 = 90°$; $CA = CA^1 = a$; $AB = A^1B^1 = b$.

The co-ordinates x and y of the point B may be expressed as under:

$$x = a\cos\alpha + b\cos\varphi, \qquad y = a\sin\alpha + b\sin\varphi.$$

$$\therefore dx = -a\sin\alpha\, d\alpha - b\sin\varphi' d\varphi \quad dy = a\cos\alpha\, d\alpha + b\cos\varphi \cdot d\varphi.$$

$$\therefore x \cdot dy - ydx = (a\cos\alpha + b\cos\varphi)\,(a\cos\alpha \cdot d\alpha + b\cos\varphi \cdot d\varphi) + (a\sin\alpha + b\sin\varphi)\,(a\sin\alpha \cdot d\alpha + b\sin d + b\sin\varphi \cdot d\varphi).$$

$$= a^2d\alpha + b^2d\varphi + ab\cos(\alpha - \varphi)\, d\,(\alpha + \varphi) \quad \ldots\ldots (3)$$

But $(\alpha + \varphi) = 2\alpha - (\alpha - \varphi)$ or $d\,(\alpha + \varphi) = 2\,d\alpha - d\,(\alpha - \varphi)$.

$$\therefore ab\cos(\alpha - \varphi) \cdot d\,(\alpha + \varphi) = ab\cos(\alpha - \varphi)\,\{2\,d\alpha - d\,(\alpha - \varphi)\} = 2\,ab\cos(\alpha - \varphi)\, d\alpha - ab\cos(\alpha - \varphi) \cdot d\,(\alpha - \varphi)$$

and

$$AN^1 = ds = AA^1\cos A^1AN = a \cdot d\alpha\cos(\alpha - \varphi).$$

so that

$$ab\cos(\alpha - \varphi) \cdot d\,(\alpha + \varphi) = 2\,b \cdot ds - ab\cos(\alpha - \varphi) \cdot d\,(\alpha - \varphi),$$

and equation (3) becomes

$$x \cdot dy - y \cdot dx = a^2d\alpha + b^2d\varphi + 2b \cdot ds - ab\cos(\alpha - \varphi) \cdot d\,(\alpha - \varphi).$$

Now, when the trace moves round the curve, $\Sigma d\varphi = 0$, $\Sigma d\alpha = 0$ for the planimeter returns to the exact position of starting: also we have then

$$\Sigma\cos(\alpha - \varphi) \cdot d\,(\alpha - \varphi) = o, \text{ for } \Sigma\cos(\alpha - \varphi) \cdot d\,(\alpha - \varphi) = \Big[\sin(\alpha - \varphi)_r\Big]_1^2$$

But limits (1) and (2) are identical so that this = 0 (see page 22),

i.e. $\Sigma (x \cdot dy - y \cdot dx) = \Sigma\, 2bds = 2b\Sigma ds.$

Now $\Sigma ds = 2\pi n\mathrm{R}$, as already shown, and we saw in equation (2) above

$$\Sigma(xdy - ydx) = 2 \times \text{area enclosed in } \mathrm{BB^1K},$$

i.e. 2 area of figure $= 2b\ 2\pi n\mathrm{R}$.

$\therefore$ Area of figure $\mathrm{BB^1K} = b \cdot 2\pi n\mathrm{R}$.

Thus to measure any area sufficiently small, C is fixed in the paper at a point outside the area and B is taken round the boundary. The rotation of the wheel is measured, as is the distance from the hinge to B, and the area is thus equal to the product of BA and $2\pi n\mathrm{R}$.

To carry out the calculation, R and AB must be measured. The distance from the tracer point to the hinge may be estimated by holding the instrument parallel to a scale or squared paper, and estimating as nearly as possible the position of the axis of the hinge on the scale. The difficulty is of course in estimating the true position of this axis.

The value of R may be obtained by measurement with a screw gauge, which must be used with great care, as the edge of the wheel is easily damaged by screwing up the gauge unduly.

Another and safer way of finding R is to note drum-reading on the wheel, and move the wheel along a straight line ruled on paper, until the wheel has made several complete revolutions. The distance moved and the number of revolutions enable the value of $2\pi\mathrm{R}$ to be measured directly.

The two measurements described above cannot be made with very great precision, but are performed much more accurately in the construction of the instrument. The graduations on the arm BA, '100 □ cm,' etc., are made in the construction, and signify that when this graduation is adjusted to the fixed mark, one revolution of the wheel corresponds to an area of 100 sq. cms. Thus, if the graduations are to be trusted, i.e. unless the instrument has been subjected to rough handling, the area of the figure is equal to $n \times$ (the number on the graduated arm opposite the fixed mark.

The Case when the Needle is Inside the Figure.

Now let us consider the case when large areas are to be measured, e.g. $\mathrm{BE_1E_2E_3 \ldots E_{11}}$, fig. 17. The needle support is fixed at a central point C, and the tracer may then be made to trace the boundary line of the area.

But in this case it will be seen that $2\pi n\mathrm{R} \cdot b$ does *not* give the true value of the area of the figure; for in this case $\Sigma d\varphi = 2\pi$ and is not zero.

Take a point B on the boundary, such that BA is at right angles to CW, the line joining C to the centre of the wheel, W; and draw a circle with C as centre and CB as radius, shown in the broken line in fig. 17. Then if B is moved round this circle the wheel W will move round a second circle of radius CW; the relative positions of the two arms remaining constant so that the axle of wheel is always at right angle to the radius CW. Thus, while B is moved round the broken-line circle, W is moving always parallel to its axis around the second circle. So that during the whole revolution the wheel will not rotate. *That is to say, the circle* BF_1F_2, *etc., so drawn, is such that when traced by*

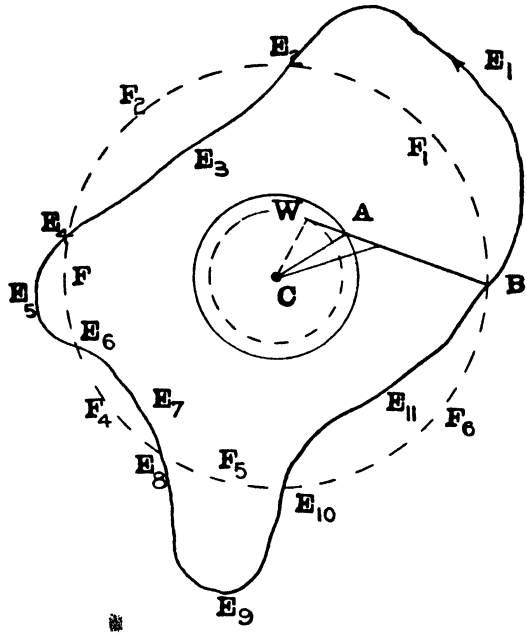


FIG. 17

the tracer, the wheel indicates zero movement. The area of the circle is $\pi(CB)^2$ and is quite definite in size, depending on the setting of the graduated arm. This is sometimes called the zero circle, or datum circle.

If now we consider our area $E_1E_2E_3 \ldots E_{11}$ and commence with the pointer at B: passing from B to E_2 via E_1, B is moved outside the circle. W will move towards C, and a definite rotation in one direction is made by the wheel. We could bring the tracer back to B along the path E_2F_1B without altering the reading of W. This reading corresponds, in the way previously considered, to the area E_2FBE_1.

However, having reached E_2, continue along the boundary

$E_2E_3E_4$. To do this the tracer moves inside the zero circle and W will therefore move outside its circle, i.e. in the opposite direction to the previous movement.

Similar movements occur round the figure. The planimeter therefore adds algebraically the area of the curve outside the zero circle. Having carefully noted the direction of rotation of W when B traverses such a part as E_1, we can tell from the final reading of the dial D whether the figure is of less or greater area of the zero circle. Suppose n revolutions of the wheel are indicated and the arm is set at the '100 □ cm.' mark. If the indication of the n revolutions is in the same direction as the indication of the wheel when moving outwards, e.g. along BE_1E_2, the area of the figure is

$$100 \cdot n + \pi CB^2 \text{ sq. cms. if CB is measured in cms.}$$

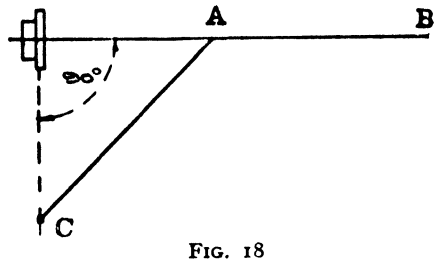


FIG. 18

The same difficulty as before is met with in finding CB. The instrument may be set along two lines drawn at right angles so that the point of contact of the wheel is at the intersection of the lines, and B and C are each on one line, fig. 18; CB is measured directly or WB and WC are measured and CB calculated. However, the value of this zero circle is inscribed on a second face of the arm BA. Adjacent to the '100 □ cm.' mark is a number which gives a value of the area of the zero circle, not usually in sq. cms., but in revolutions.

Thus, if in the case taken above, there are n revolutions indicated and the second scale gives m, as the equivalent area of the zero circle, the area of the figure is

$$100\ (n + m) \text{ sq. cms.}$$

To become acquainted with the instrument and familiar with the method of using, draw several small regular figures, calculate the areas, and then find them, using the planimeter at different graduated scale-settings.

Draw a circle of known radius, calculate the area. Move the tracer round the circumference when the movable arm is set at various graduations. Note the number of revolutions in each case. From the calculated area and the number of revolutions

observed find the area corresponding to one revolution, and so check the graduations. Repeat this process with several measured areas, and obtain a calibration of the instrument.

Measure b and R and again calculate a value for the area corresponding to one revolution. This is not so accurate as the above method, but shows the value of the construction as described.

Draw irregular figures on squared paper and find the areas by adding the squares. Compare these results with those obtained by the planimeter as calibrated.

Measure the radius for the zero circle at each setting, and check the value of the graduated scale, by calculating the area and dividing by $2\pi Rb$. The quotient should agree with the graduations on the scale.

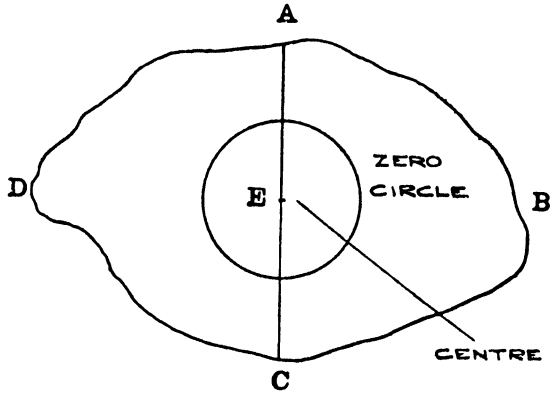


FIG. 19

The instrument may subsequently be used for area measurements when required, making use of the calibration if errors are found by these experiments.

Note.

When the boundary of the figure does not cut the zero circle the process is identical; for, suppose ABCD be such a figure, and the zero circle is completely inside the figure as shown.

The process is as before. Start at any point A, and, with the fixed point inside the figure, trace round the boundary in one direction, say along the path ADC. Join AC. Then, if the tracer be taken along CA the value of the area ADC may be calculated on the foregoing theory. If now the tracer be brought back along AC and thence via B to A, the wheel will indicate precisely the same as if the path ADCBA had been taken, for on reversing along AC the record on the wheel for the path CA will be neutralized. The foregoing theory shows that the sum of

the readings for the two paths record the number of revolutions corresponding to the two areas, and is equal to that for the boundary of the figure. Hence, starting at any point and tracing the complete boundary gives a record of the number of revolutions n, which when added to the zero circle number enables the area to be evaluated.

The Graduation and Calibration of a Tube

Graduation. The method of graduating a glass capillary tube described below is one which could be employed generally in the etching of scales on glass.

A length of glass capillary tube is coated with a thin layer of paraffin wax, by warming the glass and applying a small block of wax to the heated surface. The waxed tube is again gently heated, in the Bunsen flame, and rotated so that when cooled it is evenly coated with a thin wax layer.

The tube is then clamped to a board at such a height that the upper surface is approximately on the same plane as a metal scale which is fixed in line with the tube, and at the other end of the board.

A beam compass is arranged so that the two needle points are from 50 to 100 cms. apart, depending on the length of the glass tube. One needle point is placed in a cm. graduation of the scale and the other needle point is drawn across the wax coating of the tube, removing a straight line of wax.

The beam compass is moved 1 millimetre, and a short scratch again made on the wax coating. This process is repeated until the required length is marked in this way, the whole cm. marks and 5 mm. marks being made larger than the rest.

By means of a steel point the cm. divisions, 0, 1, 2, etc., are scratched on the wax coating.

To etch this scale on the glass, a swab of cotton wool, fastened at the end of a stick, is dipped into hydrofluoric acid, and then applied to the wax-coated tube. Where the scratches of the graduations and numbers have removed the wax, the glass is etched by the acid.

In this process care is taken, of course, to avoid any of the acid touching the skin or clothing.

One end of the tube is also given several scratches, and is covered with acid at the same time as the scale. After ten minutes or so examine one of these scratches near the end of the tube, by scraping away the wax: if the glass is sufficiently etched, the whole process is stopped; if not, leave for a few minutes and then examine a second test scratch near the end, and so on until the etching is complete. Wash off the acid with

tap water and remove the wax, the last traces may be removed with turpentine or xylol.

A little rouge, or lamp-black and shellac, rubbed over the etched scale brings out the markings.

Calibration of the Tube.

The inside of the tube is cleaned either by the usual method, using caustic soda, alcohol, ether and tap water, or by immersing it for about twelve hours in a solution of potassium bichromate and strong sulphuric acid (equal parts). The tube is finally washed well with tap water.

A thread of mercury, about one-third the length of the tube, is drawn into the bore, e.g. 10 cms. for a tube 30 cms. long. By means of a small length of rubber tubing attached to one end of the graduated tube, the position of the thread may be varied at will by altering the pressure within the tube.

The mercury is first adjusted so that one end comes near the zero graduation. By eye estimation the position of the other end may be determined to $\frac{1}{5}$ of a subdivision; hence the length of the thread when in this position is obtained. The mercury is then moved, by gently blowing through the rubber tube, until it occupies the central third of the tube, and its length is again estimated in scale divisions. The length taken by the mercury is finally measured in a similar manner in the remaining third of the tube. The volume of mercury is constant, so therefore the length indicated will depend on the average cross-section of that part of the tube filled with mercury. Thus this preliminary test will show the general form of the bore.

A short thread of mercury, about 1 cm. long, is next introduced into the tube, replacing the 10 cm. length. This is moved to occupy approximately the space between the 0 and 1 cm. graduation. The tube is arranged horizontally on a sheet of mirror glass on the platform of a travelling microscope. The readings on the scale engraven on the glass tube, corresponding to the two ends of the thread, are taken by means of this microscope. For this, the cross-hairs in the eyepiece of the microscope are turned so that one is parallel to, and the other at right angles to, the length of the tube. The intersection of the cross-hairs is brought into coincidence with the meniscus, so that one cross-hair appears tangential to it. The vernier reading of the microscope is noted. It is then moved towards the middle of the mercury thread until a graduation on the tube is seen. The difference between the vernier reading under these circumstances and the last readings gives the distance between the end of the thread and the glass tube scale reading. The distance between adjacent scale readings on the tube is measured in like manner.

From these readings the length of the mercury projecting beyond an engraved division on the tube may be calculated in terms of the graduation of the tube, and the length of the thread of mercury may be measured in these units.

The mercury is moved to occupy the space between the 1 and 2 cm. graduations, and again measured. This is repeated along the length of the tube, and the result entered as in column 2 of the table. The mean value of this length is obtained, and in the third column the difference between the observed and the mean value is tabulated for each part of the scale (p. 44).

From these observations we may calculate the correction to be applied at each part of the scale to convert the scale readings to the corresponding volume readings. Thus, if a tube is divided into 20 cms. the mean value of the thread length as calculated from column 2 gives the reading for all parts in a uniform tube which is 20 cms. long, i.e. the difference between the length of the thread between 0 and 1 as observed, and the mean value gives the correction to be applied to the scale reading to correct it to the true volume of thread equal to $\frac{1}{20}$ of the total volume of the bore.

For the bore between 1 and 2 cm. graduations, column 3 again gives the correction to be applied to this small part of the bore. To correct the total length from 0 to 2, to give $\frac{2}{20}$ of the volume, the sum of the first two terms in column 3 must be added algebraically. So, to find the correction for any scale reading, the sum of the third column must be taken up to and including the difference term for that reading.

This is done with the figures for a tube tested in this manner and the correction entered in the last column.

With the corrections so obtained a correction curve should be drawn, plotting the correction as ordinate and the scale-reading as abscissæ.

This kind of calibration is often required in thermometry, where equal volume expansion of mercury is employed. In such a case the scale-readings will not be uniform. In calibrating the thermometer the instrument maker fixes a few points along the stem by comparison with a standard thermometer. This method automatically makes some allowance for the change in the cross-section of the tube.

In the example taken the stem was divided into 1 cm. lengths for calibration: it is doubtful whether much is gained by such a small subdivision, but is used in the example above to bring out the principle involved. In practice, the use to which the tube is to be placed determines the number of points tested along the tube. With a thermometer 10 points along the length will be ample for most purposes, i.e. with a 0° to 100° thermometer

the thread detached should be the length of 10°C., or about 3 cms. This is described in greater detail later ("Thermometry").

POSITION OF THE THREAD IN THE TUBE	LENGTH OF THREAD	DIFFERENCE FROM MEAN	CORRECTION
0— 1	1·040	−0·044	− ·044
1— 2	1·000	− ·004	− ·048
2— 3	1·000	− ·004	− ·052
3— 4	0·975	+ ·021	− ·031
4— 5	0·950	+ ·046	+ ·015
5— 6	1·000	− ·004	+ ·011
6— 7	0·980	+ ·016	+ ·027
7— 8	1·000	− ·004	+ ·023
8— 9	1·000	− ·004	+ ·019
9—10	1·000	− ·004	+ ·016
10—11	1·000	− ·004	+ ·011
11—12	1·020	− ·024	− ·013
12—13	1·000	− ·004	− ·017
13—14	0·980	+ ·016	− ·001
14—15	1·010	− ·014	− ·015
15—16	0·975	+ ·021	+ ·006
16—17	1·000	− ·004	+ ·002
17—18	1·000	− ·004	− ·002
18—19	1·000	− ·004	− ·006
19—20	0·990	+ ·006	0·000
mean	0·996.		

The Balance

The balance, as seen in fig. 20, consists of two pans suspended by knife-edge supports, KK, from the ends of equal arms of the beam ST, which is pivoted on a central pair of knife-edges.

The central knife-edge is made of agate and rests on small plates of the same material, and KK support such plates. The free and sensitive movement of the beam depends upon the sharpness of the knife-edges. It is therefore important that they should support weight only when in use. To release the knife-edges the central pillar Q supports an "arrestment" A. The fixed arm A carries at each end two points which fit into a pair of cups on the upper agate plates at K. When the beam is lowered the weight is taken from the knife-edges by this means.

When releasing the beam, the latter should be in the horizontal position, i.e. the beam is only arrested when two outer knife-edges are opposite the supporting point of the arrestment. In such circumstances all three knife-edges are released with an absence of jolting.

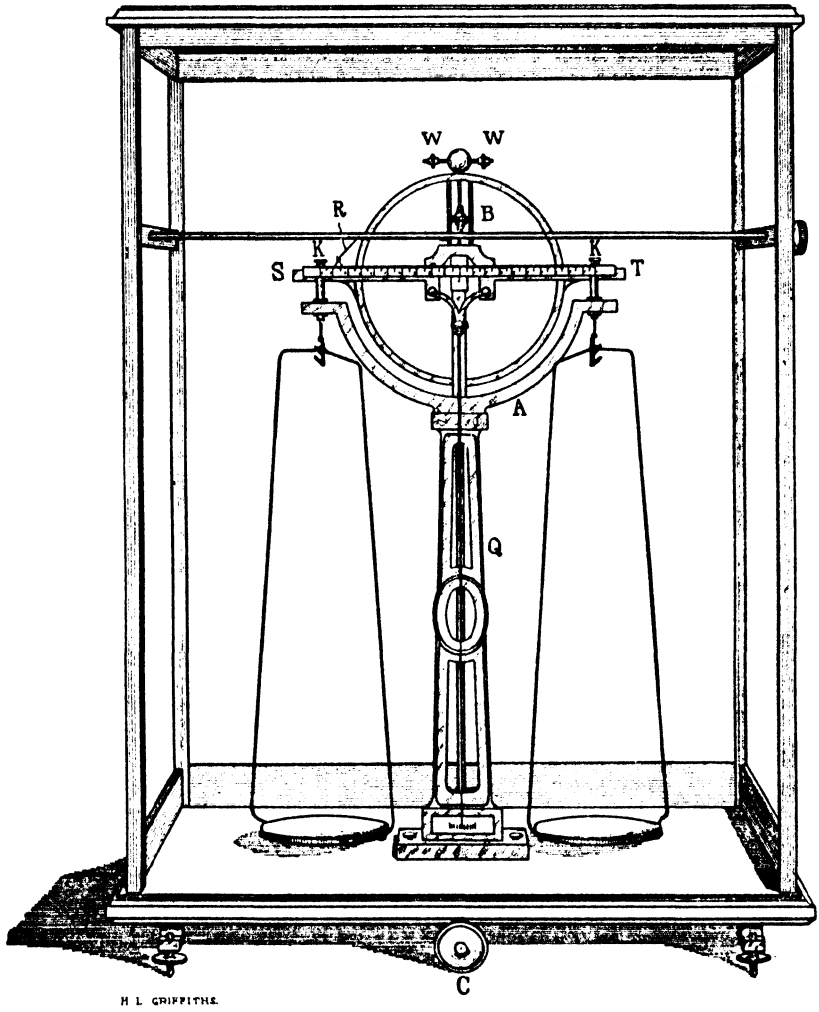


FIG. 20

In the ordinary way the centre of gravity of the beam, etc., is just under the line of support. The lower the centre of gravity the less sensitive the instrument. The position of the centre of gravity may be varied by an adjustment of the 'gravity bob' B.

The two masses W may be adjusted to change the rest positions of the pointer which moves over the small scale at the base of the pillar Q.

A fuller account of the balance will be found in " The Theory of the Physical Balance," by J. Walker.

In using the balance, several simple things should be remembered, viz., before using, dust the pans with a camel-hair or similar soft brush; arrest the beam before changing masses in the pans; use the rider to start oscillation. Never touch the pans or ' weights ' with the fingers, or place chemicals or wet vessels on the pans! Final observations should be performed with the case closed.

METHODS OF WEIGHING

The Oscillation Method of finding the Rest Position of the Pointer

The arrangement of knife-edge supports ensures that the friction is reduced to a minimum. Consequently, when the beam is oscillating it will have a long period of swing which is very slightly damped, i.e. the pointer moving just clear of the scale will take a long time in which to come to its final position of rest.

This rest position may be estimated by the method of oscillation as follows: Suppose the scale be graduated from left to right into, say, 20 graduations. The end of the pointer, being arranged as near to the scale as possible, is viewed by the eye or by the aid of a lens, taking care to avoid any slight parallax error. The position of the turning point should be estimated to at least $\frac{1}{5}$ of a small division—if possible to $\frac{1}{10}$ of a division. Seven such turning points should be obtained—three on one side and four on the other side of the zero position. The mean value of the three on the one side and also the mean of the other four should be obtained. Then the mean value of the two means gives the position of rest. (See example in the table below.)

For such damped oscillation the mean of an *even* number of observations on each side of the zero would give a value biased towards the side of the zero of the first observed turning point; so an *odd* number is taken as stated. For if we assume that the damping is small and that the first swing through the rest position is to the left, we have, from a consideration of such a lightly damped oscillation, the angular deflection, θ_L^1, the first swing to the left is after a time $\frac{T}{4}$ and is (see page 28),

$$\theta_L^1 = \theta_0 e^{-A\frac{T}{4}}$$

where A is a constant, θ_0 the undamped oscillation.

This is, under the conditions of very small damping,

$$\theta_L^1 = \theta_0\left(1 - A\frac{T}{4}\right) \text{ to the left.}$$

The deflections are therefore :

TO THE LEFT OF ZERO	TO THE RIGHT OF ZERO
$\theta_0\left(1 - A\frac{T}{4}\right)$	$\theta_0\left(1 - A\frac{3T}{4}\right)$
$\theta_0\left(1 - A\frac{5T}{4}\right)$	$\theta_0\left(1 - A\frac{7T}{4}\right)$
$\theta_0\left(1 - A\frac{9T}{4}\right)$	$\theta_0\left(1 - A\frac{11T}{4}\right)$
$\theta_0\left(1 - A\frac{13T}{4}\right)$	
mean $\theta_0\left(1 - A\frac{7T}{4}\right)$	$\theta_0\left(1 - A\frac{7T}{4}\right)$

i.e. the mean is o since the mean position left and right is an equal distance to each side of the zero.

Zero Position of the Unloaded Balance

As an example of the method of oscillation, the following determination of the zero position for the unloaded balance is taken.

The beam is released from the arrestment and given a slight oscillation over, say, about 5 divisions of the scale on each side of the zero. After a few swings the oscillation should be steady, and seven reading, are taken, as under :

READING TO THE LEFT		READING TO THE RIGHT	
1.	8·8	2.	12·0
3.	9·2	4.	11·16
5.	9·5	6.	11·2
7.	9·9		

Mean of left hand reading = 9·35
Mean of right hand reading = 11·6

$$\text{Zero positions unloaded} = \frac{9{\cdot}35 + 11{\cdot}6}{2} = 10{\cdot}475$$

A mean value of three such determinations gives the rest point. In general, call this rest position for no load x.

Sensitivity of the Balance

The sensitivity of the balance is defined as " the angle through which the beam will turn for one milligramme difference in load in the two pans."

It is usual in practice to measure the sensitivity in terms of the movement of the pointer over the scale.

This should be measured for *no load*, by adding one milligramme, by means of the rider, to one pan and noting the position of rest by the oscillation method as before.

Load each pan with increasing equal masses up to the limits specified for the balance, and for each mass find the sensitivity i.e. change of rest position for one milligramme on one pan.

If the beam were rigid and the knife-edges truly in the same line the sensitivity-load curve would be a straight line parallel to the load axis.

For increasing loads there may be, however, a slight depression of the knife-edges at the pan supports, and a change in sensitivity as a result.

Find the sensitivity-load curve for the balance and note the region, if any, of maximum sensitivity. The value of the load at which there is maximum sensitivity depends upon the use for which the balance is designed.

Method of Gauss or Double Weighing

As an example of this method we will consider a determination of the number of grammes equivalent to one ounce troy ; the process can naturally be repeated with any other unknown mass. Place the ounce troy in the left-hand pan and add ' weights ' (grammes) to the right-hand pan until the pointer remains on the scale when the beam is released ; the rest position of the oscillating pointer is estimated in the manner previously used. Let this be y on the scale.

Now find the sensitivity of the balance at this load, by adding one milligramme and proceeding as before. Suppose the sensibility for this load is s.

The mass of the ounce troy in grammes is the mass (to the nearest centigramme) in the scale pan $+ \left(\frac{y - x}{s}\right)$ milligrammes $= M_1$, say, when x is the zero reading for no load.

The ounce troy is then transferred to the right-hand pan and the process repeated by adding ' weights ' to the left until, to the nearest milligramme, M_2, the mass as so compared is obtained.

Suppose that the two arms of the balance are of slightly different length, the left-hand arm being a cms. and the right b.

Then W being the true mass of the ounce troy, neglecting buoyancy,

$$aW = M_1 b$$
$$aM_2 = Wb$$

Hence $$W = \sqrt{M_1 M_2}$$

The difference between M_1 M_2 and W will be very small, so that we may take as an approximation

$$W = \tfrac{1}{2}(M_1 + M_2).$$

Borda's Method—or the Substitution Method

A second method, quite as accurate as the double weighing, is a simple method of substitution. It eliminates equally well the errors due to unequal length of the arms, etc.

The ounce troy (the unknown mass) is placed on a scale pan and lead shot is used to counterpoise it. The position of rest when the counterpoise is complete is noted by the method of oscillation.

The ounce troy is now removed and replaced by standard masses until balance is again obtained. From the sensitivity of the balance for this load we may estimate very readily to a milligramme the mass which has exactly substituted the ounce troy. Thus the mass is obtained, avoiding errors due to faulty construction of the balance, etc.

Buoyancy Correction.

When discussing methods of weighing no account was taken of the buoyancy of the air on the ' weights ' and the mass to be compared with them.

Suppose, as before, we find that W is the mass of the body, by one of the methods above ; the true mass allowing for buoyancy we will denote by M.

Suppose that in the determination of W we used copies of standard masses made of a substance of density D.

Let ρ be the density of the ' unknown ' mass and σ the density of the air.

We have really compared (1) (the true mass of the body M — the buoyancy on the mass) with (2) (the mass W of the ' weight ' — the buoyancy on the ' weights '). These two quantities are equal,

i.e. $$\left(M - \frac{M}{\rho}\cdot\sigma\right) = W - \frac{W}{D}\cdot\sigma$$

i.e. $$M = \frac{W\left(1 - \frac{\sigma}{D}\right)}{1 - \frac{\sigma}{\rho}}$$

$$= W\left(1 + \frac{\sigma}{\rho} - \frac{\sigma}{D} \cdots\right)$$

neglecting $\frac{\sigma^2}{\rho D}$ cf. 1.

$$= W + W\left(\frac{1}{\rho} - \frac{1}{D}\right)\sigma$$

The observed value of W has therefore to be corrected by the factor $W\left(\frac{1}{\rho} - \frac{1}{D}\right)\sigma$.

This correction depends on the density of the 'weights' and the substance and the density of the air.

For most purposes the density of the air may be taken as ·0012, and for general use a table may be calculated giving the value of the correcting factor $\left(\frac{1}{\rho} - \frac{1}{D}\right)\sigma$ for the two common materials used in the manufacture of 'weights,' brass and aluminium, taking the density of brass = 8·4. Aluminium = 2·65. Thus:

DENSITY OF SUBSTANCE WEIGHED (ρ)	CORRECTION FOR BUOYANCY	
	BRASS 'WEIGHTS' D = 8·4 $\sigma\left(\frac{1}{\rho} - \frac{1}{8·4}\right)$	ALUMINIUM 'WEIGHTS' D = 2·65 $\sigma\left(\frac{1}{\rho} - \frac{1}{2·65}\right)$
·5	·00226	·00195
·55	etc.	etc.
·60	etc.	etc.
etc.		

CHAPTER II

MOMENTS OF INERTIA AND DETERMINATION OF 'g'

Kinetic Energy of a Body Rotating about an Axis

LET ABC (fig. 21) be a section of a body by a plane at right angles to the axis about which it is rotating, O being the point of intersection of this plane and the axis.

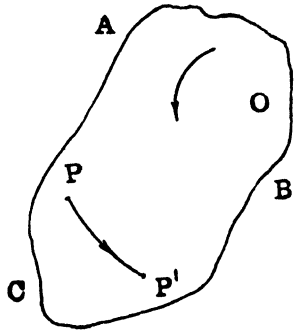


FIG. 21

If we imagine the body to be subdivided into a large number of very small particles of mass m_1, m_2, m_3, etc., distant r_1, r_2, r_3 cms. from the axis, it is evident that when the body rotates each particle will move with a velocity which depends on the distance r from the axis.

Consider one such particle at P, of mass m and distant r cms. from O. If the body rotate with a uniform angular velocity w, in the direction of the arrow, and if v is the velocity of P in the path, we have

$$\frac{v}{r} = w.$$

The kinetic energy of this particle is $\frac{1}{2}mv^2 = \frac{1}{2}mw^2r^2$. For all such particles the total kinetic energy of the body is therefore

$$\tfrac{1}{2}m_1w^2r^2 + \tfrac{1}{2}m_2w^2r_2^2 + \tfrac{1}{2}m_3w^2r_3^2 + \ldots.$$

i.e. $$\text{Kinetic Energy} = \Sigma\tfrac{1}{2}w^2\,mr^2$$
$$= \tfrac{1}{2}w^2\,\Sigma mr^2$$

The sum of such quantities as mr^2, taking every particle throughout the body, is defined as the *moment of inertia* of the

body *about the axis through* O. If we denote this by I_0, then the kinetic energy of the body is

$$\tfrac{1}{2} I_0 w^2 \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

thus I_0 replaces the mass, and the angular velocity replaces linear velocity in the corresponding case for linear motion where K.E. $= \frac{1}{2}mv^2$.

In the same way if the axis passed through the centre of gravity; I being the moment of inertia about an axis through the centre of gravity, K.E. $= \frac{1}{2}Iw^2$, I being of a different magnitude from I_0.

To express the moment of inertia of a body about an axis in terms of the *moment of inertia about a parallel axis* passing through the centre of gravity, we proceed in the following manner:

In fig. 22 let ABC be a section of the body at right angles to either axis. Let G be a section of the axis passing through the centre of gravity and O be corresponding point of intersection for a parallel axis, the distance OG being fixed and equal to a cms.

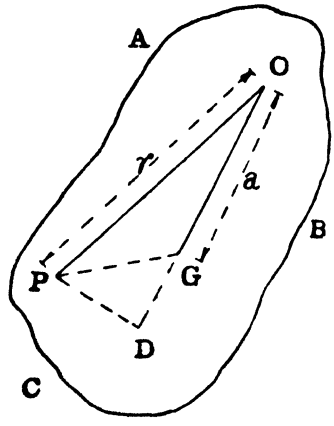


FIG. 22

Consider at any point P a small particle of mass m gm., OP being r cms.

The contribution of this particle to I_0 is mr^2. Now produce OG to D, and from P drop a line perpendicular to OD meeting it at D.

$$r^2 = OP^2 = OD^2 + PD^2 = PD^2 + DG^2 + GO^2 + 2OG \cdot GD$$
$$= PG^2 + a^2 + 2a \cdot GD$$

Thus $I_0 = \Sigma m\,(PG^2 + a^2 + 2a \cdot GD)$

$$= \Sigma mPG^2 + \Sigma ma^2 + \Sigma 2ma \cdot GD.$$

Now $\Sigma mPG^2 = I$, the moment of inertia about a parallel axis through the centre of gravity.

$$\Sigma ma^2 = a^2 \Sigma m = a^2 M,$$

where M is the total mass of the body. Further, the expression $2a\Sigma GD \cdot m = 0$, for by the definition of the centre of gravity the sum of the moments $(GD \cdot m)$ throughout the body, about an axis through G, is zero.

Thus $$I_0 = I + Ma^2 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Radius of Gyration.

We have defined the moment of inertia of a body about an axis as Σmr^2. Now if the whole of the mass of the body were concentrated at one point distant K from the axis, we should have

$$I = K^2M.$$

If the distance K were so chosen that

$$K^2M = \Sigma mr^2 = I$$

it is called the 'Radius of Gyration' of the body about the axis of rotation taken.

As with the moment of inertia K has different values depending upon the axis chosen.

Moment of External Forces.

Considering a rotating body as before, we may readily deduce an expression for the moment of the external forces applied to the body to impart a definite angular acceleration.

Imagine a force applied to such a body as shown in fig. 21. Suppose the body to be subdivided into small particles as before, of which one at P has mass m and is r cms. from O. Then r, the velocity of the particles, is given by

$$v = r \cdot \frac{d\theta}{dt} \text{ or } r\dot{\theta}$$

the acceleration of the particle in its path is

$$\frac{dv}{dt} = r\frac{d^2\theta}{dt^2} = r\ddot{\theta}$$

this is occasioned by a force $mr\ddot{\theta}$, whose moment about O is $mr^2\ddot{\theta}$. For the whole body to rotate with this angular acceleration the total external couple applied is thus

$$\Sigma mr^2\ddot{\theta} = \ddot{\theta}\Sigma mr^2 = I\ddot{\theta} \ldots\ldots\ldots\ldots (3)$$

Thus the moment of external forces applied to the body is $I\ddot{\theta}$.

Calculation of the Moment of Inertia for a Solid about any Axis.

The numerical value of the moment of inertia of a solid about any axis may be readily obtained by integration. Having calculated this value for an axis passing through the centre of gravity, the corresponding value for the case of the body suspended through a parallel axis may be obtained by adding the term Ma^2 as shown on page 52.

We will consider an example of such calculations which is often employed, especially in magnetism, and which illustrates the points already considered.

Calculation of the Moment of Inertia of a Rectangular Rod about an Axis at Right Angles to its Length and passing through the Centre of Gravity.

Let ABCD be the rectangular bar, with centre of gravity at G, and supported by an axis KK^1 passing through G, normally to the face AD, fig. 23. Let

M be the mass of bar (assumed to be uniform)
ρ the density of the material of the bar
$2l$ the length of the bar
$2b$ the breadth of the bar
$2d$ the depth of the bar.

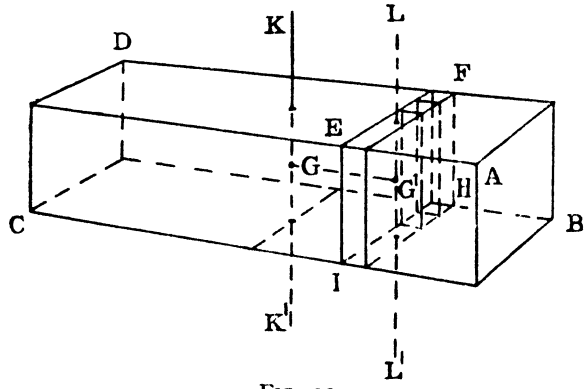


FIG. 23

Let G be the origin of a system of co-ordinate, the z axis coinciding with the axis of rotation : the x and y axes being at right angles to this, the x axis being parallel to the length, the y parallel to the breadth.

The most convenient method of finding I the moment of inertia about KK^1 is to consider firstly a very thin section of the bar cut at right angles to the x axis, and of thickness δx. Through the centre of mass G^1 of this section imagine an axis, LL^1, parallel to KK^1. The moment of inertia of this section about KK^1 is equal to the moment of inertia about LL^1 plus the product of the mass of the section and x^2, where x is the distance between G and G^1. Imagine a very small rectangular portion PQRS of EFHI, y cms. from the axis and of width dy (fig. 24).

The mass of this parallelepiped is $[dx\ dy\ 2d]\rho$. The moment of inertia of the section EFHI about LL^1 is therefore

$$\int_{-b}^{+b} 2ddxdy\ \rho y^2 = \frac{4b^3d\rho}{3} \cdot dx.$$

About KK^1 the moment of inertia is therefore

$$\frac{4db^3\rho dx}{3} + 2d2b\rho x^2 dx.$$

Hence the total moment of inertia of the whole body about the axis KK^1 is

$$\int_{-l}^{+l} \left(\frac{4b^3d}{3}\rho dx + 4\,bd\,x^2\rho dx\right)$$
$$= \frac{8bdl\rho}{3}(b^2 + l^2)$$
$$= \frac{M}{3}(b^2 + l^2).$$

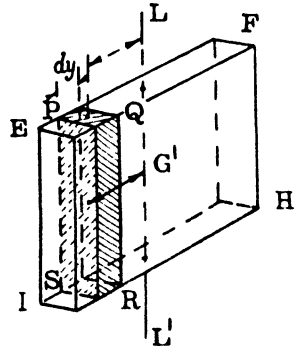


FIG. 24

EXPERIMENTAL MEASUREMENT OF MOMENT OF INERTIA.

When the moment of inertia of a body cannot be conveniently calculated it may be found experimentally by imparting to it a known amount of energy and observing the resulting rotation, or, if the body is small, the method of the moment of inertia table may be employed, whereby the change in moment of inertia in a given system, due to the body, may be directly calculated.

The following typical experiments will make these methods evident.

Moment of Inertia of a Fly-wheel

To find experimentally the moment of inertia of a fly-wheel about the fixed axis of rotation a mass is attached to the axle

of the fly-wheel by a cord which is wrapped several times round the axle. When the mass descends, it causes a rotation of the wheel. The mass in its descent loses a definite amount of potential energy. Neglecting friction for the moment, this loss is equated to the gain of kinetic energy of the mass and the fly-wheel, and an equation results in which all the terms are known or measurable except I, the moment of inertia of the wheel and axle about the fixed axis.

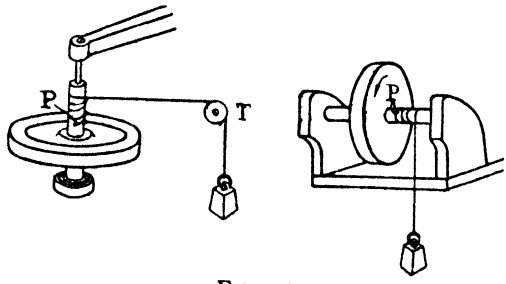


FIG. 25

The wheel might be supported on a horizontal or vertical axis. The two usual types met with are seen in fig. 25. The process to find I is the same, so we will consider one of them—the vertical axis type.

The mass m is attached to the axle at a point where there is either a hole or a pin. If there is a small hole in the axle as at P, then to the end of the cord a small 'pin' is attached. This can be made from a short length of suitable-sized brass wire to fit easily in the hole, or if the axle has a pin projecting, a loop is made at one end of the string. The length of the string is so adjusted that when m is on the floor, or whatever solid object is to arrest it in its descent, the other end of the string may be just attached to the axle. So that when the mass descends, the moment it is arrested, the string leaves the axle.

If w be the angular velocity imparted to the wheel, and r be the radius of the axle, the velocity of the mass m just before striking the floor is rw. So that, neglecting friction, we have

$$mgh = \tfrac{1}{2}mr^2w^2 + \tfrac{1}{2}Iw^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where h is the distance through which m has fallen.

To measure the angular velocity, a chalk-mark is made on the circumference of the wheel, in a position which can be seen the moment the mass touches the floor. The number of revolutions, n, made by the wheel after the mass becomes detached is counted by observing the chalk-mark. The time taken for the wheel to come to rest whilst completing the n revolutions is also observed (t secs.).

The wheel is finally brought to rest by the frictional forces acting against it. If this frictional force is constant, the wheel is uniformly retarded. It commences with a definite angular velocity and finishes with zero angular velocity, so that the initial velocity is double the average velocity.

Now the average angular velocity $= \frac{2\pi n}{t}$ radians per second,

i.e. $$w = \frac{4\pi n}{t}.$$

The linear velocity of the mass m is $\frac{4n\pi r}{t}$.

Another method, though inferior to the above, is to observe v directly by timing the descent of the mass. If the mass descends the distance h in t^1 secs, the average velocity is $\frac{h}{t^1}$, and the final velocity is $\frac{2h}{t^1}$.

As we have noticed above, the frictional forces are not always negligible, so that, for a more accurate determination of I, allowance must be made for the energy lost in overcoming friction.

Suppose during the descent of the mass the wheel makes n_1 revolutions, and that for one revolution f ergs are used in overcoming the frictional forces, the total energy used in overcoming this steady frictional force is $n_1 f$. We must amend our energy equation as follows:

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\mathrm{I}w^2 + n_1 f.$$

Now, we already know that the energy possessed by the rotating wheel, $\frac{1}{2}\mathrm{I}w^2$, is used up in overcoming friction, in n revolutions

$$\text{i.e. } fn = \tfrac{1}{2}\mathrm{I}w^2$$

$$f = \frac{\mathrm{I}}{n}(\tfrac{1}{2}\mathrm{I}w^2),$$

i.e. $$mgh = \tfrac{1}{2}mr^2w^2 + \tfrac{1}{2}\mathrm{I}w^2\left(1 + \frac{n^1}{n}\right) \quad \ldots\ldots\ldots (5)$$

Experimental Details.

Arrange the cord round the axle so that throughout the whole of the unwinding the cord from the axle to the pulley, T, is practically horizontal, or at right angles to the axle.

Bring the mass m so that the bottom of it is level with a fixed point, and the string of such a length that it fulfils the conditions already stated.

The distance h from the fixed point to the floor having been measured, the number of revolutions of the wheel whilst the mass is descending can be determined by making a chalk-mark on the

axle and allowing the mass to descend slowly, and counting the number of revolutions (n_1).

The mass is once more wound up and allowed to fall freely. When it is heard to strike the floor a stop-clock is started and the number of revolutions of the wheel before being brought finally to rest is counted, i.e. n and t are observed: m the mass is known and $w = \frac{4\pi n}{t}$, hence the value of I is calculated by aid of equation (5).

The experiment is repeated two or three times with the same mass and the mean value of $\frac{n}{t}$ taken.

I is further checked by repeating with two other masses m_1 and m_2.

The cord used should be of small diameter compared with the diameter of the axle, otherwise the value of r in equation (5) is the sum of the radii of the axle and cord.

Rolling Bodies

The two following experimental methods of finding the moment of inertia of a body about a given axis depend upon observations of rolling bodies.

The energy of a rolling body may be very simply obtained. Consider, for example, a cylinder rolling with a uniform linear velocity v cms. per second (fig. 26).

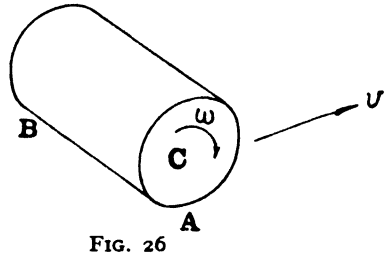


FIG. 26

AB, the line of contact of the cylinder and the plane on which it rolls may be regarded as the momentary axis of rotation of the cylinder.

The kinetic energy of the body is therefore given by

$\frac{1}{2}w^2$ (moment of inertia of the cylinder about the axis AB)

where w is the angular velocity of rotation.

Now, if a is the radius of the cylinder, m its mass, and I the moment of inertia about a parallel axis through the centre of gravity we have the kinetic energy is:

$$\frac{1}{2}w^2(I + ma^2)$$
$$= \frac{1}{2}Iw^2 + \frac{1}{2}mv^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

That is, the kinetic energy is equal to the sum of the kinetic energy of rotation and translation.

Wheel and Axle on an Inclined Plane

The moment of inertia of a wheel and axle about an axis passing through the centre of gravity and parallel to the axle may be obtained by observing its descent down an inclined plane, and applying equation (6).

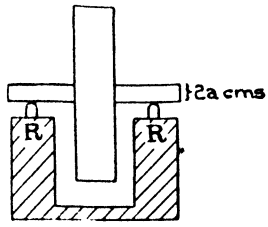


FIG. 27

The method of rotation will be apparent from a consideration of the section, fig. 27. R and R are rails supported on a hollowed inclined plane. The axle of the wheel rests upon the rails. The whole plane may be inclined at any angle to the horizontal. For each inclination the wheel and axle is allowed to roll down a measured length, l cms., of the plane, in a time which is measured by means of a stop-clock (t secs.).

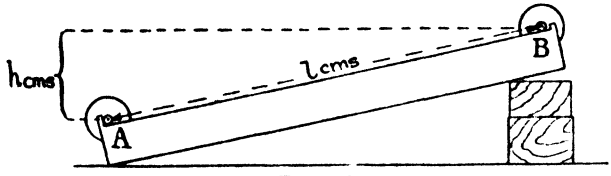


FIG. 28

If the vertical distance between the starting position and the finishing position be h cms., fig. 28, and v be the final velocity acquired in the descent, equating potential energy lost to kinetic energy gained, we have, according to (6) above,

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\mathrm{I}\left(\frac{v}{a}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where m is the mass of the wheel and axle, a the radius of the axle.

The body starts from rest and moves with a constant acceleration; the final velocity is therefore twice the average velocity. This latter is equal to $\frac{l}{t}$, i.e. $v = \frac{2l}{t}$.

The plane is adjusted by suitable means to one fixed inclination. The wheel and axle is placed at a convenient marked starting

point on the rails. The position of the centre of the axle is noted by means of a vertical-reading simple cathetometer. The position of the centre of the axle is also noted when the wheel is against the stop at the other end of the plane, hence h. The length l along the plane between these two points is measured directly. The value of the mean time of descent for three experiments is obtained, whence v. The mass m is also obtained by means of a spring balance or an ordinary balance which is capable of weighing such a mass.

a is obtained in the usual way by means of vernier callipers.

Hence, from (7), substituting the value $\frac{2l}{t}$ for v,

$$I = ma^2\left(\frac{ght^2}{2l^2} - 1\right)$$

The experiment is repeated for several values of h and the mean I is obtained.

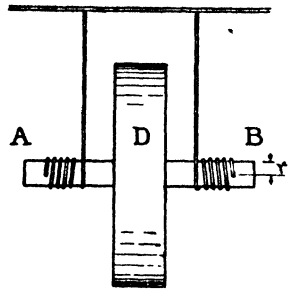


FIG. 29

Moment of Inertia of a Disc Supported on Strings

A disc, usually made of wood, is suspended by means of a metal axle on two strings, as shown in fig. 29. The string is wound evenly on the axle AB on both sides until as much string as possible is wound up. If now the axle and disc, of mass m grammes, is released, it will descend until the whole of the cord is unwound ; it will then rise again due to the string being wound on the axle in the other direction.

Suppose that from the starting point to the lowest point reached the distance the centre of the axle moves is h cms., and that the linear velocity at the moment when all the string is just unwound is v cms. per sec., then if r is the radius of the axle we have as the energy equation, equating potential energy lost to kinetic energy gained :

$$mgh = \tfrac{1}{2}Iw^2 + \tfrac{1}{2}mv^2,$$

I being the moment of inertia about at axis passing through the centre of gravity and parallel to the axle.

w the angular velocity at the lowest point is $= \frac{v}{r}$,

i.e. $$mgh = \frac{\frac{1}{2}Iv^2}{r^2} + \frac{1}{2}mv^2.$$

$$I = (mgh - \tfrac{1}{2}mv^2)\frac{2r^2}{v^2}.$$

If the time of descent of the disc is t seconds, the average velocity is $\frac{h}{t}$ cms. per sec.

v, the final velocity is therefore $\frac{2h}{t}$.

$$I = mr^2\left(\frac{gt^2}{2h} - 1\right)$$

Experimental Details.

Weigh the disc, then measure the distance between the position of the centre of the axle in the starting position and the final lower position.

The value of r, unless the cord has small radius compared with the radius of the axle, is equal to the sum of the radii of the axle and the cord which supports it. These radii are measured by means of a micrometer screw.

The cord is wound evenly on the axle until the disc is at the starting point. Care is taken to ensure that the axle is horizontal, otherwise the disc fouls the cords in descent.

The time of descent is measured several times by means of a stop-clock, and the mean value taken.

The Bifilar Suspension

In order to determine the moment of inertia of a body about an axis passing through its centre of gravity, we may make use of a bifilar suspension of known or measurable dimensions. The body is suspended with the axis of rotation vertical, and the time of vibration of the system, T, obtained by observing the time of 40 or 50 complete swings.

If, for example, we wish to determine the moment of inertia of a cylinder about an axis through the centre of gravity, the cylinder is supported in two wire stirrups CE and DF (fig. 30), which hang at the ends of two very thin wires AC and BD, which are fixed at A and B, AB being $2d$ cms. The distance between C and D remains fixed and equal to $2d^1$ cms.

When the body is displaced slightly in the horizontal plane,

it will perform oscillations whose periodic time T may be ascertained as already shown.

If m is the mass of the cylinder and l is the length of the wire AC or BD, we may readily see that

$$T = 2\pi\sqrt{\frac{Il}{mgdd^1}}.$$

For, let the tension in the string be t dynes and O be the mid-point between A and B.

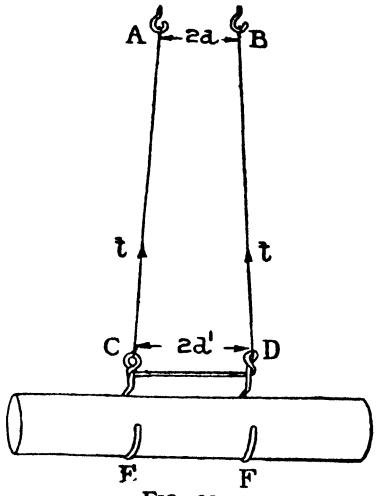


FIG. 30

When viewed from above, fig. 31 (a) represents the relative positions of the four ends of the wires. When displaced, the state of affairs is seen in fig. 31 (b). Where A^1 and B^1 are projections of A,B, fig. 30, on the horizontal plane through CD, and C^1, D^1 are the displaced positions of C, D.

Consider the forces at D. Due to the tension on the string there is a force t, which has a horizontal value $t \cos \alpha$, where α is the angle between the string BD^1 and the horizontal. Now $\cos \alpha = \frac{B^1D^1}{l}$, for, in reality, the point of suspension, B, is above B^1, BB^1D^1 being a right-angled triangle.

So that along D^1B^1 in the horizontal plane there is a force

$$f = t\frac{B^1D}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Of this force, the component at right angles to the displaced body, i.e. C^1D^1, is effective in restoring the body to its original position, the component along the direction OC having no turning moment.

From B^1 draw B^1E at right angles to C^1D^1. The component

normal to C^1D^1 is $f \sin ED^1B^1 = f\frac{EB^1}{B^1D^1} = f^1$, say;

from (8) $$f^1 = EB^1 \frac{t}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

A similar force acts at C^1, constituting a restoring couple of moment $f^1 \cdot C^1D^1$.

The restoring couple is, substituting value of f^1 from (9),

$$\frac{EB^1 t}{l} \cdot C^1D^1.$$

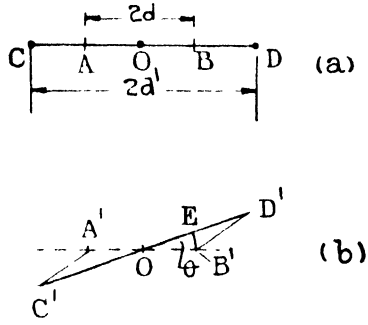


FIG. 31

If the small angle of displacement, $D^1OB^1 = \theta$.

$$EB^1 = OB \sin \theta = d \sin \theta.$$
$$C^1D^1 = 2d^1.$$

$$\therefore \text{ Moment of couple} = \frac{2dd^1}{l} t \sin \theta.$$

θ is usually made very small, so we have for the value of the restoring couple:

$$\frac{2dd^1}{l} \frac{mg}{2} \cdot \theta = \frac{dd^1 mg}{l} \cdot \theta \ldots\ldots\ldots\ldots (10)$$

for *when d is not very different from* d^1 $t \simeq \frac{mg}{2}$.

We have already seen (p. 53) that the moment of external forces acting on a suspended system is $I\frac{d^2\theta}{dt^2}$; this quantity is equal and opposite to the restoring couple,

i.e. $$I\frac{d^2\theta}{dt^2} = -\frac{mgdd^1}{l} \cdot \theta,$$

This will be recognized as the equation of simple harmonic motion (p. 25), as we have the angular acceleration $= -\frac{mgdd^1}{Il} \times$ the angular displacement, $\frac{mgdd^1}{Il}$ being a constant.

The time of vibration is therefore

$$= \sqrt{\frac{2\pi}{\frac{mgdd^1}{Il}}} \text{ or } T = 2\pi\sqrt{\frac{Il}{mgdd^1}} \quad \ldots\ldots\ldots(11)$$

It is interesting to note that for bodies of the same dimensions and of uniform density the value of T is the same. For, let ρ be the uniform density, then $I = \Sigma mr^2$. Consider the body divided into small volumes v, : $\Sigma mr^2 = \Sigma v\rho r^2 = \rho\Sigma vr^2$.

Also $$m = \Sigma\rho v = \rho\Sigma v$$

thus $T = 2\pi\sqrt{\frac{l}{gdd^1} \cdot \frac{\Sigma vr^2}{\Sigma v}}$, i.e. independent of the density.

A suitable form of body to use for the above experiment is a metal or wooden cylinder, or both.

If the dimensions of the wood and metal cylinder are practically identical, the time T will be found to be the same within small limits. Suspend each in turn and find T by timing 50 swings, and, from the formula above, calculate I.

For such a regular solid an independent calculation gives a second value of I, which should agree very nearly with that already obtained.

For such a cylinder, about the axis taken $I = M\left(\frac{L^2}{3} + \frac{r^2}{4}\right)$

where 2L is the length of the cylinder, r the radius.

Moment of Inertia Table

Fig. 32 shows the essential features of the moment of inertia table. AB is the table suspended by a fairly stout wire, K, from the overhead frame. The circular table is provided with a groove, concentric with the circumference; which supports three or more masses which just fit into it. The masses, W, may be moved round the groove into any position without altering the moment of inertia of the whole when the wire support is vertically above the centre of gravity. The masses are arranged so that the table lies horizontally. In that case the axis is through the centre of gravity and is normal to the surface of the table. If, now the table is given a slight twist, a restoring couple will be called into play in the wire, equal to say τ per unit angular displacement, and the result will be oscillations about the axis of support, whose periodic time T will be shown to be equal to

$$T = 2\pi\sqrt{\frac{I}{\tau}}$$

where I is the moment of inertia of the system about the axis

of rotation ; for suppose the table be twisted through an angle θ, the restoring couple due to torsion is $\tau\theta$. We have already seen, page 53, that the moment of external forces on such a rotating body is $I\ddot{\theta}$ where $\ddot{\theta}$ is $\frac{d^2\theta}{dt^2}$. This couple is *opposed* by $\tau\theta$: $\tau\theta$ and $I\ddot{\theta}$ are equal and opposite ; i.e.

$$I\ddot{\theta} = -\tau\theta$$

or

$$\ddot{\theta} = -\frac{\tau\theta}{I}.$$

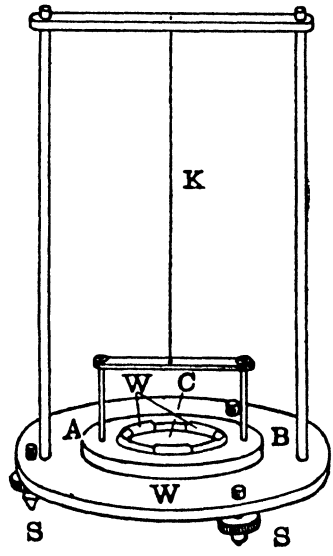


FIG. 32

This was shown in the introductory chapter (p. 25) to represent simple harmonic motion whose periodic time, T_1, is given by

$$T = \sqrt{\frac{2\pi}{\frac{\tau}{I}}} = 2\pi\sqrt{\frac{I}{\tau}}$$

If now a *regular* shaped body is placed symmetrically on the table at C, so that its centre of gravity is vertically above the centre of gravity of the table and therefore in the previous axis of rotation, we should have the time of oscillation for the loaded table

$$T_1 = 2\pi\sqrt{\frac{(I + k)}{\tau}},$$

where k is the moment of inertia of the regular body about the axis of oscillation : k may be calculated directly from the mass and dimensions of the body.

We have, therefore, from the two equations above :

$$T_1^2 = \frac{4\pi^2 (I + k)}{\tau} \quad ; \quad T^2 = \frac{4\pi^2 (I)}{\tau}.$$

Hence
$$\frac{T_1^2}{T^2} = \frac{I + k}{I} = 1 + \frac{k}{I},$$

or
$$I = k \frac{T^2}{T_1^2 - T^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

If, then, a body of unknown moment of inertia about a given axis is placed centrally on the table with the centre of gravity in the axis of oscillation, we may find the value of I_1, its moment of inertia about this vertical axis passing through the centre of gravity, by timing the oscillations of the table when loaded by the body. If T_2 is the time of complete swing,

$$T_2 = 2\pi \sqrt{\frac{I + I_1}{\tau}},$$

and we have

$$T = 2\pi \sqrt{\frac{I}{\tau}},$$

i.e. from these two equations :

$$I_1 = \frac{T_2^2 - T^2}{T^2} \cdot I \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Substituting, by equation (12) above,

$$I_1 = \frac{T_2^2 - T^2}{T^2} \cdot k \frac{T^2}{T_1^2 - T^2} = k \cdot \frac{T_2^2 - T^2}{T_1^2 - T^2} \quad \ldots\ldots (14)$$

Experimental Details.

The time T for a complete swing of the table is obtained by timing as many swings as possible, say 30 or 50.

To find the value of I, the moment of inertia of the table unloaded, about the wire as axis, a regular solid, such as a plain cylindrical ' weight ' is employed. Depending on the dimensions of the wire, etc., the mass should be fairly heavy so as to cause as big an alteration in T as possible. A two-kilogramme ' weight ' is of the order to employ with the apparatus described.

To ensure that this standardizing mass is arranged with its centre of gravity over that of the table, the lead weights W should be adjusted so that the table swings horizontally. Any alteration in the position of the masses, W, will not alter I, so long as the table is horizontal and the axis of oscillation is vertically through the centre of gravity.

Having obtained T and T_1, using a cylindrical regular ' weight,'

the value of k should be calculated. For such a flat cylindrical object, $k = \frac{Ma^2}{2}$ where a is the radius and M the mass of the 'weight.'

By equation (12) I is calculated and the table standardized.

Other regular solids may now be used and I, obtained by equations (13) or (14), and then by calculation a check is obtained on I_1 as previously obtained.

The following results were obtained in the above manner:

	Table unloaded T	Table loaded with 2000 grammes	Table loaded with unknown body
Time	9·375 secs.	9·7 secs.	15·5 secs.

radius of 2000 gm. 'weight' = 6·75 cms.

$$k = \frac{2000 \times 6{\cdot}75^2}{2} = 45{\cdot}562 \text{ gm. cm}^2.$$

$$I_1 = 45{\cdot}562 \times \frac{15{\cdot}5^2 - 9{\cdot}375^2}{9{\cdot}7^2 - 9{\cdot}375^2}$$

$$I_1 = 1{\cdot}11 \times 10^6 \text{ gm. cm}^2.$$

I, by approximate calculation, assuming a regular shape to the body $= 9{\cdot}86 \times 10^5$

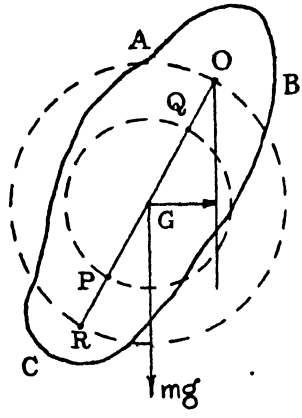


FIG. 33

The Compound Pendulum.

Let ABC, fig. 33, be a section of a body, passing through the centre of gravity, and at right angles to an axis about which it may turn, the point O being the intersection of the axis with this plane section.

The body will be at rest when the centre of gravity, G, is vertically under O.

If the body is given a *small* displacement so that GO makes a small angle θ with the vertical, then, m being the mass of the solid, there will be a *restoring* force which has a moment mg OG sin $\theta = mga \sin \theta$, putting OG $= a$.

We saw (p. 53) that in such a case the moment of such forces is equal to $I_0 \frac{d^2\theta}{dt^2}$,

i.e. $I_0 \frac{d^2\theta}{dt^2} = -mga \sin \theta = -mga\theta$ for small angular displacements.

This represents a simple harmonic motion (p. 25), whose periodic time, $T = 2\pi \sqrt{\frac{I_0}{mga}}$.

$I_0 = I + ma^2$, where I is the moment of inertia about a parallel axis through centre of gravity and is equal to mk^2 where k is the radius of gyration about this axis.

$$\text{Thus } T = 2\pi \sqrt{\frac{k^2m + a^2m}{mag}} = 2\pi \sqrt{\frac{\frac{k^2}{a} + a}{g}} \quad \ldots\ldots\ldots\ldots (15)$$

This result is similar to that obtained for a simple pendulum: in fact, a simple pendulum of length $l = \frac{k^2}{a} + a$ would have the same periodic time, T. Such a simple pendulum is called the 'Equivalent Simple Pendulum,'

In the case taken, if all the mass of the body were concentrated at a point, P, along OG, produced such that OP $= \frac{k^2}{a} + a$, we should have a simple pendulum with the same periodic time.

The point, P, is called the 'Centre of Oscillation,' O being called the 'Centre of Suspension.'

Now since

$$l = \frac{k^2}{a} + a$$

or

$$a^2 - al + k^2 = 0,$$

the length a is not the only value for OG, which has l as the equivalent simple pendulum, for the above equation has two roots, α_1 and α_2, such that,

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= l \\ \alpha_1 \alpha_2 &= k^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Since a is one value, α_1 say,

we have $\quad a + \alpha_2 = l \quad$ or $\quad \alpha_2 = l - a = \frac{k^2}{a}$.

Thus if the body were supported on a parallel axis through the

former centre of oscillation P it would oscillate with the same time T as when supported at O.

From what has been seen above it is evident that there are an infinite number of points distant, a and $\frac{k^2}{a}$ from G, for any point on a circle drawn from G as centre and radius a or $\frac{k^2}{a}$ will satisfy the condition given; so that any axis parallel to the normal at G on the curved surface of two cylinders, of which the dotted circles are sections, will be axes of suspension which give the same time.

If the body were supported by an axis through G, the time of oscillation would be infinite. From any other axis in the body the time is

$$T = 2\pi\sqrt{\frac{a^2 + k^2}{a\,g}}.$$

This has a minimum value when $\frac{a^2 + k^2}{a}$ is minimum.

Now $\frac{a^2 + k^2}{a} = \frac{a^2 - 2ak + k^2 + 2ak}{a^2} = \frac{(a - k)^2 + 2ak}{a}$.

This is a minimum when $a = k$.

The corresponding minimum T_1 is $T_1 = 2\pi\sqrt{\frac{2k}{g}}$ .....(17)

and will occur for a series of axes parallel to that through G, and on the surface of a cylinder whose axis is the axis through the centre of gravity and radius, k.

An experiment which brings out these facts may be performed by using as the body a rectangular rod of brass about 1 metre long, which may be suspended on a knife-edge at various points along its length. To facilitate such suspension it is convenient to have a series of holes drilled along the bar at about 2 cms. intervals (fig. 34).

Level the knife-edge, and suspend the bar at, say, every other hole in turn, and time 50 swings at each hole, which is a measured distance from the centre of gravity of the bar (which may be obtained by simple balancing). Or the holes may be measured from one end of the bar.

Having obtained a set of values for T, and the distance from the centre of gravity, plot a curve with the periodic times as ordinates and the distances as abscissæ. A curve such as shown in fig. 35 will be obtained.

The values of T near the minimum points, M M^1, should be further investigated by taking the time for vibrations in *every*

hole, three each side of the approximate position, and the graph completed.

Let the line CG, fig. 35, be drawn from C, which represents the centre of gravity of the bar.

Draw any line EABFD parallel to the axis. This cuts the curve in four points, which have the same periodic time, T = BC. It will be found that the lengths, FB, BA, and BE, BD, are equal; i.e. FB and BA correspond to radii GQ, GP, and BE and BD to radii GO and GR, in fig. 33.

Take either set in pairs, say BA, BD, these are corresponding lengths to α_1 and α_2 in equation (16), i.e. AB + BD = l, the length

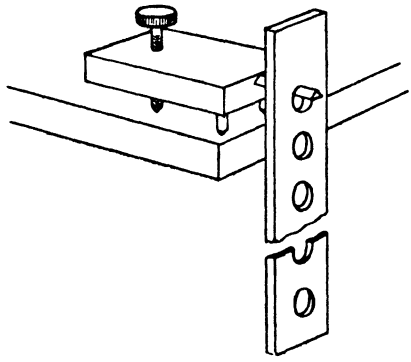


FIG. 34

of the equivalent simple pendulum. Its periodic time T is numerically equal to BC.

Hence from equation, $T = 2\pi\sqrt{\frac{l}{g}}$

all factors except g are known, whence g may be calculated.

If now a tangent is drawn to the curve, such as line LMM^1L^1, $HM = HM^1$ = radius of gyration about an axis through the centre of gravity: this may be measured directly.

Further, by equation (16), $k = \sqrt{\alpha_1\alpha_2} = \sqrt{AB \cdot BD}$, so a second value of k may be found. The corresponding periodic time is, numerically, the length of $HC = T_1$, say.

Hence, once more, by equation (17),

$$T_1 = 2\pi\sqrt{\frac{2k}{g}}.$$

g may be evaluated or, as the direct exact measurement of k (MH or M^1H) is difficult, as there is some doubt in the general case as to the exact location of M and M^1, this formula be used, assuming g, to calculate on third value of k,

$$k = \frac{T^2}{8\pi} \cdot g.$$

The mean value of k may be taken and the moment of inertia about a parallel axis through the centre of gravity calculated, for

$$I = k^2M,$$

where M is obtained by direct weighing.

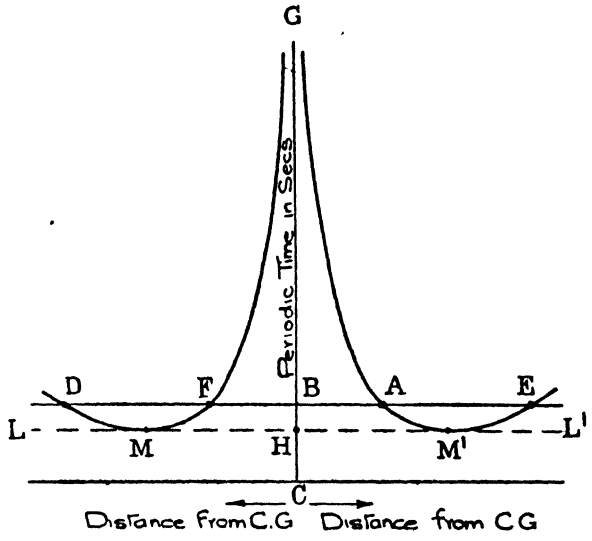


FIG. 35

Kater's Pendulum

From the preceding experiment it is obvious that if it were possible to obtain, for a rigid body, two parallel axes of suspension, along any line through the centre of gravity and on opposite sides of it, which have exactly the same time of swing, then the value of g could be very well determined by measuring the distance between such axes. This distance would be equal to the length of the equivalent simple pendulum, l. If the equal periodic time about these axes were T,

Then $$T = 2\pi\sqrt{\frac{l}{g}},$$

l and T being measured directly.

The Kater pendulum is one by means of which this may be realized in practice to a very close approximation.

It consists of a long rod which is provided with two fixed knife-edge supports, K and K, and terminates at each end in a 'bob,' B and B. Usually, the one bob, B, is made of brass and the other of wood.

M and m are two adjustable masses which may be fixed in any

position between the knife-edges. Their adjustment serves to move the centre of gravity to such a position that the time of swing is approximately the same from either knife-edge.

The pendulum is supported on knife-edge, K and K, in turn, and the approximate periodic time, T and T_1, is obtained by counting swings, timed by means of a stop-clock.

The large mass M is moved until these times are approximately the same. The small mass *m* serves as a fine adjustment to this purpose.

Having adjusted the masses so that the time for a complete vibration is very nearly the same from both knife-edges, it will be realized that to obtain exact agreement for T and T_1, the times from knife-edges K and K, would be a most tedious experiment.

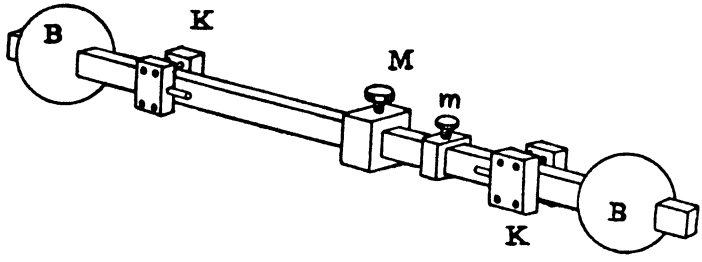


FIG. 36

However, we can see in the following way that such exact agreement is not essential.

Let a and a_1 be the distance from the centre of gravity of the pendulum to K and K.

$$T = 2\pi\sqrt{\frac{a^2 + k^2}{ag}} \qquad T_1 = 2\pi\sqrt{\frac{a_1^2 + k^2}{a_1 g}}$$

$$T^2 ag = 4\pi^2 (a^2 + k^2) \qquad T_1^2 a_1 g = 4\pi^2 (a_1^2 + k^2)$$

Subtracting

$$(Ta^2 - T_1^2 a_1)\, g = 4\pi^2 (a^2 - a_1^2)$$

$$\frac{4\pi^2}{g} = \frac{aT^2 - a_1 T_1^2}{a^2 - a_1^2} = \frac{1}{2}\left(\frac{T^2 + T_1^2}{a + a_1} + \frac{T^2 - T_1^2}{a - a_1}\right) \quad \ldots.(18)$$

a and a_1 may be made to differ by a fairly large amount by suitable adjustment of the masses, M and *m*. With a little care T and T_1 may be approached very nearly, and so the term $\frac{T^2 - T_1^2}{a - a_1}$ will be small. T and T_1 may be measured very accurately by the method of coincidence; $(a + a_1)$ may be measured directly as the distances between the knife-edges, by a comparison with a metal metre scale by means of a comparator

(p. 32). The important first term in equation (18), (R.H.S.) is thus carefully evaluated.

The second term is small and no serious error is involved if a and a_1 are measured from the knife-edges to a point at which the pendulum may be balanced horizontally on a knife-edge.

By this method a very reliable value of g may be obtained.

Method of Coincidences.

This method of timing a pendulum consists in hanging it by a knife-edge from a rigid support, in front of, say, a seconds pendulum the height of the support so arranged that the tails of both pendulums are on the same level. At rest, viewed by a telescope from in front, the Kater coincides with the seconds pendulum.

A scale is placed behind the seconds pendulum, and if both were of the same period and started oscillating together, when viewed through the telescope, would appear to move as one. If the periods are not the same, they will be seen to get 'out of step,' and at one point both will pass a fixed scale division together and going in the same direction. This will not again occur until one pendulum has gained or lost a whole swing.

Suppose the seconds pendulum makes n complete vibrations, each of period T (2 secs.), and the experimental pendulum makes $(n + 1)$ complete swings, period T_1,

Then
$$Tn = T_1 (n + 1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

$$\frac{T_1}{T} = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} = 1 - \frac{1}{n} + \frac{1}{n^2} \quad \ldots\ldots (20)$$

Suppose $n = 500$.

$$\frac{T_1}{T} = 1 - \frac{1}{500} + \frac{1}{250000}$$

certainly no more terms are required.

Hence T_1 is obtained in terms of T; which in the case taken is 2 secs.

Similarly the time about the other axis may be checked.

In the coincidence method it may be observed that one is never quite sure within a few (say m) passages of the pendulum which is the correct coincidence. We can easily see that the error introduced by this cause is not appreciable when n is fairly large. Thus we know that in equation (19) instead of n we may put $(n \pm m)$,

i.e.
$$T (n \pm m) = T_1 (n \pm m + 1).$$

$$\frac{T^1}{T} = \frac{1}{1 + \frac{1}{n \pm m}}$$

$$= 1 - \frac{1}{n \pm m} \text{ approx.}$$

$$= 1 - \frac{1}{n}\frac{1}{\left(1 \pm \frac{m}{n}\right)}$$

$$= 1 - \frac{1}{n} \pm \frac{m}{n^2} \cdots$$

In practice the coincidence may usually be limited to one of about 6, i.e. $m = 3$.
Small value for $n = 500$.

$$\therefore \quad \frac{T_1}{T} = 1 - \frac{1}{500} \pm \frac{3}{500^2};$$

i.e. an error of about 1 in 100000 is introduced, due to the uncertainty.

In practice a cross-hair in the focal plane of eyepiece of the telescope is a useful reference point in preference to a scale behind the pendulum.

Sphere on a Concave Mirror. An Approximate Method of Determining 'g', the Acceleration Due to Gravity

A concave mirror is arranged horizontally, facing upwards, so that a small steel ball may be allowed to perform oscillations on its surface, in a line through the lowest point.

The time of oscillation of the steel ball is obtained by timing as many oscillations as possible on the surface. The observation is repeated, and from these results a mean value of the periodic time T is calculated.

Then if

R is the radius of curvature of the upper face of the concave mirror, as measured by a spherometer,
m the mass of the sphere,
r its radius,
g the acceleration due to gravity,

it will be shown that

$$T = 2\pi\sqrt{\frac{\frac{7}{5}(R - r)}{g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

whence

$$g = \frac{28}{5} \cdot \frac{\pi^2}{T^2} \cdot (R - r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Consider the sphere in its position of equilibrium to be with its centre at B (fig. 37), and when displaced to the extreme position, with the centre at C.

We will consider the case of a mirror of large radius of curvature and the displacement BC to be small compared with R.

The potential energy of the sphere at C is $mg \cdot AB$. Now $AB = OB - OA = (R - r)(1 - \cos\theta)$, i.e. the potential energy is $(R - r)\, 2\sin^2\frac{\theta}{2} \cdot mg$, or when θ is small as specified:

$$\text{P.E.} = 2(R - r)\left(\frac{\theta}{2}\right)^2 mg = \frac{1}{2}(R - r)\,\theta^2\, mg.$$

If we measure the displacement from the position of rest by the distance in path between the centre of gravity at B and C, equal to BC cms., say, we have $\frac{BC}{R - r} = \theta$.

Hence the potential energy is

$$\frac{1}{2}(R - r)\frac{BC^2}{(R - r)^2}mg = \frac{1}{2}\frac{mg}{R - r} \cdot BC^2.$$

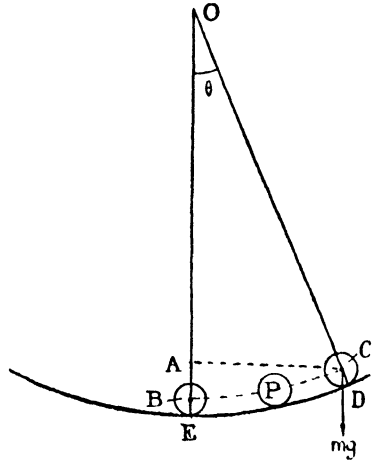


FIG. 37

At this point there is no kinetic energy.

At B the whole of the energy is kinetic, and equal to

$$\frac{1}{2}mv_m^2 + \frac{1}{2}Iw_m^2$$

where v_m is the maximum linear velocity of the centre of gravity and w_m the maximum angular velocity of rotation, and I is the moment of inertia of the ball about an axis through the centre of gravity, at right angles to the plane of the paper, i.e.

$$\text{K.E. at B is } \frac{1}{2}mv_m^2 + \frac{1}{2}I\frac{v_m^2}{r^2}.$$

At any intermediate point P distant x cms. from B along the

arc, the total energy is equal to either of these quantities and is therefore a constant,

i.e. $$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{I}{r^2}\dot{x}^2 + \frac{1}{2}mg\frac{x^2}{(R-r)} = \text{constant},$$

$\dot{x}$ being the velocity at that P along the path.

Differentiating we have :

$$m\dot{x}\ddot{x} + \frac{I}{r^2}\ddot{x}\cdot\dot{x} + \frac{mg}{(R-r)}x\cdot\dot{x} = 0.$$

Dividing by $\dot{x}$ and rearranging

$$\ddot{x} = -\frac{mg}{(R-r)\left(m+\frac{I}{r^2}\right)}\cdot x,$$

i.e., the acceleration is a constant times the displacement; this was shown (p. 25) to be S.H.M., whose periodic time T is given as under :

$$T = \frac{2\pi}{\sqrt{\frac{mg}{(R-r)\left(m+\frac{I}{r^2}\right)}}} = 2\pi\sqrt{\frac{(R-r)\left(m+\frac{I}{r^2}\right)}{mg}}$$

Now, I, the moment of inertia about the axis described, is equal to $\frac{2}{5}mr^2$.

Hence $$T = 2\pi\sqrt{\frac{(R-r)\cdot\frac{7}{5}}{g}} \quad \ldots\ldots\ldots\ldots\ldots(21)$$

T is obtained by observations as indicated above ; r is measured by means of a screw gauge, R, by means of a spherometer, *not* by an optical method, unless the front surface is silvered. Hence all the terms in (21) are known except g, which may be calculated.

Atwood's Machine

A modern form of Atwood's machine is illustrated in fig. 38. The two masses, M_1 and M_2, are equal, and are connected over the pulley, N, by a strip of white paper in the form of tape, while an equally long strip, M_1QM_2, connects the other ends of the weights, so that for all positions of M_1 and M_2 there is an equal mass of paper at each side of the pulley.

The vibrator, V, carries an inked brush, B, and as the paper passes below it a trace, somewhat in the form of a sine curve, is drawn on the tape. The complete period of the vibrator

is usually about one-fifth of a second ; but this should be carefully tested by means of a good stop-watch before beginning the experiment.

The curve then provides a time record. The apparatus is used to provide an exercise in the determination of the acceleration due to gravity ; but even in its best form the experiment has no claim to great accuracy ; it does, however, provide an instructive exercise in Mechanics.

M_1 stands on a platform, as shown in the diagram, and the mechanism of the apparatus provides for the release of the

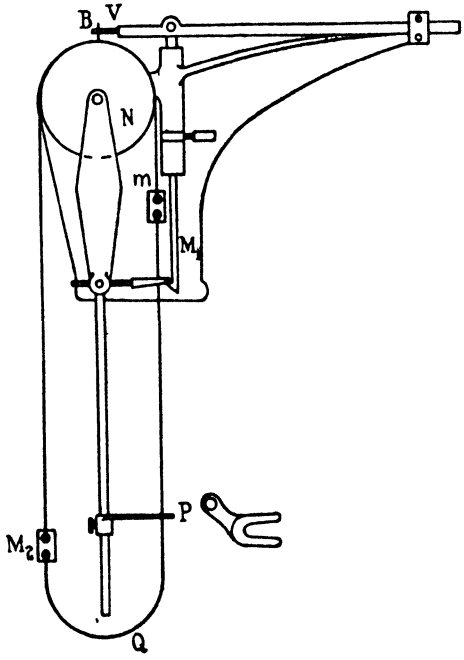


FIG. 38

vibrator and of M_1 simultaneously. Small weights are provided, which rest on the top of M_1, and on release of the platform cause an acceleration of the masses. The rider can be removed by a second platform, P, after a velocity has been acquired.

The trace on the tape records the acceleration and velocity beyond P, and from the former of these the value of g may be determined.

There is always a frictional resistance to be accounted for, although this is reduced by making the pulley light and mounting it on ball-bearings.

It is best to get rid of the retardation due to friction by making

loops of wire, which may be placed on the top of M_1 and remain when M_1 passes through P.

The necessary addition can be judged approximately by observing the fall of M_1 after it has been given a small velocity. If M_1 moves down uniformly without appreciable loss of speed, the frictional error is nearly corrected. A finer observation may then be made by allowing the brush to make a trace. If the line consists of uniformly spaced waves the velocity is uniform. When this has been adjusted the wire loops are left in position and are not taken into account in the calculations.

Suppose a rider of mass, m, lies on M_1 (in addition to the wire loops).

When M_1 is allowed to fall, suppose it does so with an acceleration, f.

Let T_1 denote the tension in the paper above M_1. The lower paper strip is loose and is not supposed to exert any force on the two masses.

Let I denote the moment of inertia of the pulley, and let T_2 denote the tension in the paper on the left of the pulley, i.e. the tension acting on M_2. Denote the radius of the pulley by a and its angular velocity at any instant by w. Then $\frac{dw}{dt}$ is the angular acceleration, and we have :

$$a\frac{dw}{dt} = f.$$

From the forces on $(M_1 + m)$ we have :

$$(m + M_1)f = (m + M_1)g - T_1$$

and from considering M_2

$$M_2 f = T_2 - M_2 g,$$

while the motion of the pulley is expressed by :

$$I\frac{dw}{dt} = (T_1 - T_2)a,$$

from which we have :

$$If = (T_1 - T_2)\,a^2.$$

We may therefore eliminate T_1 and T_2 and find :

$$f = \frac{gm}{m + M_1 + M_2 + \frac{I}{a^2}}$$

The quantity $\frac{I}{a^2}$ is called the ' equivalent mass ' of the pulley, and its magnitude in grammes is engraved on the pulley.

We may therefore find the value of g from this equation from observations which give the value of f.

FIG. 39

This may be determined from the trace. By removing P, the trace may be made long, M_1 being allowed an extended fall.

The line drawn by the brush will consist of waves which open out uniformly. Mark these off in groups of five, as at A, B, C, etc., beginning at a point A, where the trace is opened out sufficiently to be distinct. Measure carefully the distances AC, BD, CE, etc., and divide by the time interval which elapses between these points. This will give the average velocity over the strips measured, and this velocity is the velocity at the points B, C, D, etc. The differences between these velocities should all be the same, or very approximately so. Take the average of all the determinations and so obtain the average increase in velocity during five periods of the vibrator. Hence deduce the acceleration by dividing by the time of five vibrations.

This is the value of f.

Repeat the experiment with the various riders provided.

CHAPTER III

ELASTICITY

ALL bodies, when acted upon by forces, are deformed a certain amount. The magnitude of the deformation produced by a definite applied force enables a value of the elastic constant of the material used to be calculated.

We may, in a general manner, call the forces applied 'stresses,' and the deformations produced 'strains.' However, these two terms have, more often, a more precise meaning, depending on the mode of application of the forces. We shall recognize three ways of producing a deformation: (1) by uniform compression or extension, (2) by applying equal and opposite forces in one direction, i.e. stretching, (3) uniform shear. Deformation may be produced in any of these ways or by a combination of them.

(1) *Uniform Compression or Extension.*

If a body of volume V be subjected to a uniform pressure of p dynes per sq. cm., a contraction will ensue. This corresponds to a change in volume of δV, say. The fractional increase in volume is $\frac{\delta V}{V}$.

In this case the *stress* applied is p dynes per sq. cm., and the *strain* is $\frac{\delta V}{V}$ numerically.

(2) *Stretching.*

The most direct example of this type of deformation is seen in the case of a wire fixed at one end and supporting masses at the other end. In this case the force acting on the wire is the weight of the suspended masses and the reaction at the point of support. These are equal and opposite, acting in a direction which coincides with the length of the wire. Due to their action the wire will increase in length and at the same time will be reduced a very small amount in cross-section. The reduction in cross-section for a wire will not be of a sufficiently large amount to be readily measured.

If δl is the increase in length of a specimen whose original length is l, the fractional increase in length, the *strain*, is $\frac{\delta l}{l}$.

The stress producing this strain is defined as the force acting on each unit of area normal to it, i.e. if a is the area of cross-section of the wire and the total mass applied is m grammes, the stress is $\frac{mg}{a}$ dynes per sq. cm.

(3) *Shear.*

Consider a cube of material ABCDEFGH (fig. 40) fixed at the lower face and acted upon by a tangential force F at the upper face. As a result of this force the cube will take up a position shown in an exaggerated manner by the broken lines in the figure, the vertical sides being sheared through an angle of φ radians from AH and BG to AH^1 and BG^1.

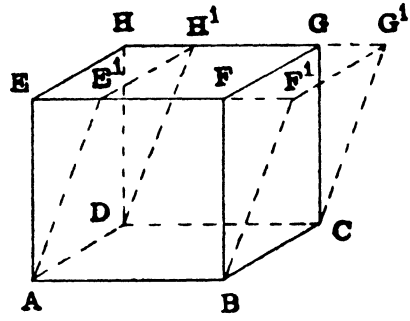


FIG. 40

For such a shear $\frac{F}{\text{area } EFGH}$ is the stress, i.e. the tangential force per unit area.

The strain is measured by φ, i.e. the ratio $\frac{EE^1}{AE}$ if EE^1 is small compared with AE^1.

In all the above cases the stress is measured as a force per unit area; in the c.g.s. system in dynes per square cm. The strain in each case is a ratio of like quantities, and has therefore no dimensions.

Hooke's Law.

If the stresses are below a certain limiting value which depends on the material of the body to which they are applied, the strain disappears when the stresses are removed. If the limiting value is exceeded, the material is strained beyond the elastic limit,

and such strain is permanent; as the stresses are still further increased the result is a fracture of the material.

For stresses below the elastic limit, it was established by Hooke that the strain produced is proportional to the stress applied, i.e. under such conditions we have

$$\frac{\text{stress}}{\text{strain}} = \text{constant.}$$

The constant has a definite value which depends on the material, and which, in the three cases taken, is called, (1) the 'bulk modulus of elasticity,' (2) 'Young's modulus,' and (3) the 'modulus of rigidity.'

Young's modulus is the most readily obtained *directly* by experiment.

The following notation will be used throughout in dealing with these elastic constants.

$$\text{(1) Bulk modulus} = \mathrm{K} = \frac{p}{\frac{\delta \mathrm{V}}{\mathrm{V}}}$$

$$\text{(2) Young's modulus} = \mathrm{Y} = \frac{\frac{\mathrm{F}}{\mathrm{A}}}{\frac{\delta l}{l}}$$

$$\text{(3) modulus of Rigidity, } n = \frac{\frac{\mathrm{F}}{\mathrm{A}}}{\varphi}$$

In addition to the above three elastic constants, we may add a fourth, which is concerned with stretching. We noticed that during stretching there is a lateral contraction of the specimen. The *fractional* lateral contraction produced is proportional to the longitudinal stress applied and the ratio of

$$\frac{\text{Fractional lateral contraction}}{\text{Fractional longitudinal extension}}$$

is called 'Poisson's Ratio' (μ). Thus, if the specimen is a cylinder of radius r and length l, and the changes produced in these dimensions are δr and δl, we have

$$\mu = \frac{\frac{\delta r}{r}}{\frac{\delta l}{l}}.$$

The following relations between the elastic constants may be

readily deduced (see, for example, Poynting and Thompson's " Properties of Matter ").

$$Y = \frac{9nK}{3K+n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\mu = \frac{(3K-2n)}{2(3K+n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Thus, if any two of the constants are found experimentally, the remaining two may be calculated from the above equations.

Determination of Young's Modulus for the Material of a Wire

A good method for finding Y directly is to support, vertically, a long length of the wire, load it with definite masses, and observe directly the extension produced.

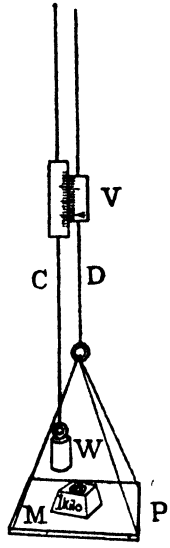


FIG. 41

It will usually be most convenient to obtain such a length that, when supported at the ceiling, the wire extends almost to the floor. A second wire, C, is supported in like manner from the same support, and carries a fixed load of sufficient magnitude to keep the wire taut. The wire D carries a platform P. At V a vernier scale is attached to the wires, one half fixed to one wire, the other half to the second wire.

The wire D, whose Young's modulus is to be determined, should be free from ' kinks ' and should carry sufficient load to make it taut, so that any further load added merely causes a stretching of the wire, and does not simply straighten out bends and kinks. If a heavy platform is employed at P, this weight

may be sufficient. Read the vernier; place one kilogramme on the platform and notice the extension. If this is due solely to the stretching of the taut wire, the vernier reading, on removing the kilogramme load, should be once more the same as at commencement; if this is not so increase the load until on adding a further kilogramme the readings on the vernier scale have a definite value, which is reduced to another definite value on removing the load.

Having obtained satisfactory repeats for this adjustment, the load on the scale pan P should be increased by equal increments and the vernier reading for each load noted. Having arrived at a safe maximum load, the latter is reduced by the same increments and the vernier readings again noted.

The values obtained may be tabulated as under.

LOAD	VERNIER READINGS.			EXTENSION FOR 6 KILOS
	LOAD INCREASING	LOAD DECREASING	MEAN	
0				
2				
4				
6				
8				
10				
12				
			mean extension for 6 kilos =	

The mean vernier reading for each load being taken and tabulated as shown, we may obtain several values of the extension of the wire for a definite load.

Thus, in the case taken, the loads were 0, 2, 4, 6, 8, 10, 12 kilos. The difference between the vernier readings for 0 and 6 kilo load gives the extension for 6 kilos. In the same way the difference between the readings for 2 and 8 kilo load, 4 and 10, 6 and 12, also gives the extension for 6 kilogramme increase in load. This is entered in the last column, the mean value, l say, of which is used in the calculation of Y.

The radius of the wire is measured in at least six places, using a micrometer screw, and the mean value taken, r cms. say. The original length of the wire, about 7 metres, is measured directly (L cms.).

Hence $$Y = \frac{\frac{6000 \times 981}{\pi r^2}}{\frac{l}{L}}$$ in the case taken.

An alternative way (due to G. F. C. Searle) in which to measure the extension of the wire is illustrated in fig. 42.

The standard wire terminates in a frame A which supports a mass M, sufficiently large to maintain the wire in a stretched state. The wire to be investigated is also fastened to a similar framework B. The two are fastened by cross-pieces C and D, which prevent relative rotation of the frames, but allow the frame B to be depressed relative to A, when masses are added to the scale pan S.

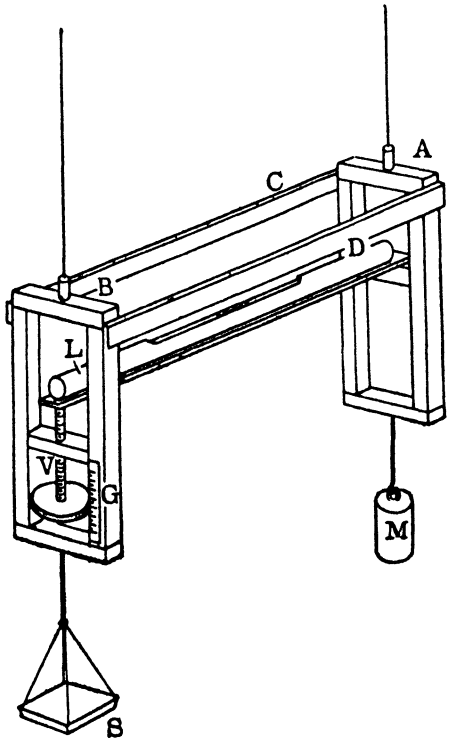


FIG. 42

A spirit-level L is supported at one end on a rigid cross-bar of the frame A, and at the other on the point of a micrometer screw V, which moves vertically through a rigid cross-bar. The micrometer screw has the usual circular division, which enables the movement of the head to be estimated to $\frac{1}{50}$ or $\frac{1}{100}$ of a complete turn, enabling the movement of the point of the screw (and hence the end of the spirit-level) to be measured to $\frac{1}{100}$ or $\frac{1}{100}$ of a millimetre.

The level is first adjusted, when the wire is suitably stretched free from 'kinks,' so that it is truly horizontal. The load of,

say, one kilogramme is added to the scale pan S. The micrometer screw is moved a suitable distance over to scale G, so that the spirit-level is once more horizontal.

The amount of movement required to bring this about is obviously equal to the elongation of the wire by the load added.

The results may be tabulated as in the former method and the value of Y calculated from the mean of a set of observations.

Bending of Beams

The value of Young's modulus may be found by less direct measurements for substances not in the form of a wire.

Consider a rod of any uniform cross-section, say rectangular,

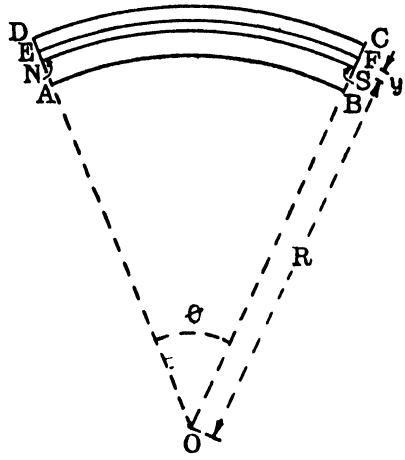


FIG. 43

bent into the form of a circular arc of fairly large radius. Take a section of the rod by a plane passing through the long axis of symmetry and parallel to it, and passing through the centre of curvature (i.e. the plane of bending). The layers of the material of the bar in the lower half will be compressed and the upper half extended. There will be one plane, therefore, at right angles to the plane of bending, which will remain of the same length as before the bending took place. This plane is called the neutral surface, and it will be shown to pass through the centre of gravity of the bar. It is represented in fig. 43 by NS.

If we imagine the bar to be made up of a number of filaments along the length, then such filaments, as stated above, will be extended or compressed according to their position above or below the neutral surface. One such as shown at EF, fig. 43, or at P in the section diagram (fig. 44) a distance y above the neutral surface is extended.

The strain in such a filament depends upon y, for, if R is the radius of curvature of the neutral surface, and θ the angle subtended at the centre O by NS, the unstretched length of the filament EF is the same as the present length of NS $= R\theta$
EF $= (R + y)\theta$.

Hence the elongation is $y\theta$ and the strain is therefore $\frac{y}{R}$.

Now if f is the force acting on the filament of cross-section a, we have from the definition of Young's modulus

$$Y = \frac{\frac{f}{a}}{\frac{y}{R}}$$

or
$$f = \frac{Ya}{R} \cdot y, \quad \text{i.e. } f \propto y.$$

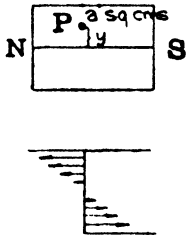


FIG. 44

Thus, the arrows in the lower part of fig. 44 show the type of forces acting on all such filaments into which we have subdivided the bar.

This system of forces on the bar must have an algebraic sum of zero.

i.e. $$\Sigma f = 0 \text{ or } \frac{Y}{R} \Sigma ay = 0.$$

Since $\frac{Y}{R}$ is not zero, $$\Sigma ay = 0.$$

Thus, *the neutral surface passes through the centre of gravity of the bar.*

The forces have a definite moment about the neutral surface. The moment for the single force is $fy = \frac{Y}{R} \cdot ay^2$.

For equilibrium the sum of such moments is equal and opposite to the external couple which set up the internal forces. If C is the external couple, we have

$$C = \Sigma \frac{Y}{R} \cdot ay^2 = \frac{Y}{R} i,$$

where $i = \Sigma ay^2$. From the similarity between this case and the corresponding sum in considering masses in connexion with moment of inertia, i is sometimes called ' the moment of inertia of cross-section.' It may be calculated in the same way as moment of inertia if area replaces mass.

We have therefore for such a bent beam

$$iY = CR \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Cantilever

Consider a light beam fixed horizontally at one end and loaded with a mass m at the other. If the mass of the beam is small compared with the load m, the whole depression may be taken as due to the load.

In fig. 45 let AC^1 be the unloaded position for the neutral surface, and AC the position taken when the load is applied at C.

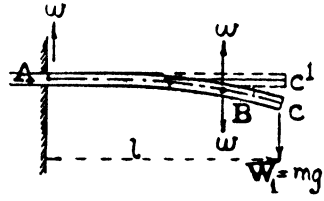


FIG. 45

To obtain an expression for the depression of the end in terms of the dimensions of the bar, m and Y, it is convenient to refer to a system of axes with the end A as origin ; AC^1 being the x axis, and a line at right angles to AC^1 from A in the plane of the paper the y axis.

We will assume that the curvature of the loaded rod is small, i.e. *the total depression at* C *is small.*

Consider a section at B, fig. 45, x cms. from A. As already seen, across the face of such a section a system of forces exist. These forces on the segment BC are extensions above the neutral surface, and compressions below, constituting a counter-clockwise couple $C = \frac{iY}{R}$ on BC.

At the same time the force mg at C has a clockwise moment equal to $mg(l - x)$ on BC.

For equilibrium these two opposite couples are equal in magnitude,

i.e. $$\frac{iY}{R} = mg(l - x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where R is the radius of curvature at B.

Now $$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

But in this case $\frac{dy}{dx}$ is small—for the total depression is assumed to be small—so $\left(\frac{dy}{dx}\right)^2$ is negligible compared with unity, and therefore we have for such small curvature

$$R = \frac{1}{\frac{d^2y}{dx^2}},$$

i.e. $$\frac{d^2y}{dx^2} = \frac{W}{iY}(l - x)$$ from equation (4) above.

The value of the total depression at the end of the bar may be obtained by integration, and is the value of y when $x = l$, after such a process.

Integrating we have

$$\left[\frac{dy}{dx}\right] = \frac{W}{iY}\left(lx - \frac{x^2}{2}\right) \quad \ldots\ldots\ldots\ldots (5)$$

the constant of integration being zero, for when $x = 0$, $\frac{dy}{dx} = 0$.

A second integration between the limits 0 and l gives

$$\left[y\right]_0^{y_0} = \frac{W}{iY}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l$$

where y_0 is the end depression,

i.e. $$y_0 = \frac{W}{iY}\frac{l^3}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

For a bar of rectangular cross-section, i about the neutral axis NS (fig. 44) is $\frac{bd^3}{12}$, where b is the breadth and d the depth of the bar.

Hence $$y_0 = \frac{12W}{bd^3Y} \cdot \frac{l^3}{3}$$

or $$Y = \frac{4\,mgl^3}{bd^3y_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6a)$$

If the mass of the beam is not negligible, see treatment on page 94.

Beam Supported at Two Knife-edges and Loaded in the Middle

If the beam is now supported at the two ends and loaded with a mass m at the mid-point we have a reaction $\frac{mg}{2}$ at each knife-edge. If l is the total length of the bar, the depression produced at the centre will be exactly the same as the depression at the end of a similar bar of length $\frac{l}{2}$ and loaded at the end with a mass $\frac{m}{2}$, i.e. if we imagine the bar clamped at the mid-point and a force $\frac{mg}{2}$ applied at one end. The movement of the end is precisely the same as the depression in the middle in the actual case taken.

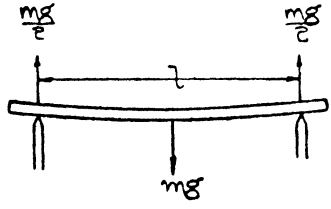


FIG. 46

Such a depression may be obtained by substituting the length $\frac{l}{2}$ and force $\frac{mg}{2}$ in equation (6*a*), giving for the depression

$$y_0 = \frac{W}{iY} \cdot \frac{l^3}{48}$$

or

$$Y = \frac{mgl^3}{4bd^3y_0} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The value of Young's modulus for the material of a beam may be obtained in this manner.

The beam, say of iron, and of about 1 cm. square cross-section, and about 1 metre long is supported on two knife-edges near its ends, and a load is applied at the mid-point by placing masses on a pan which is suspended from a knife-edge which rests on the bar at this place. The depression is measured on a vernier scale, one scale of which is fixed, the other moves with the beam.

The load is increased, and the vernier reading (y_0) for each load is tabulated as under. The value for $\frac{m}{y_0}$ is obtained in each case, the results being tabulated as under.

LOAD m	VERNIER READING	DEPRESSION (y_0).	$\frac{m}{y_0}$
0		0	0
2000 gms.			
3000 ,,			
4000 ,,			
5000 ,,			

$$\text{mean}\,\frac{m}{y_0}$$

A mean value of $\frac{m}{y}$ from the series of observations is taken.

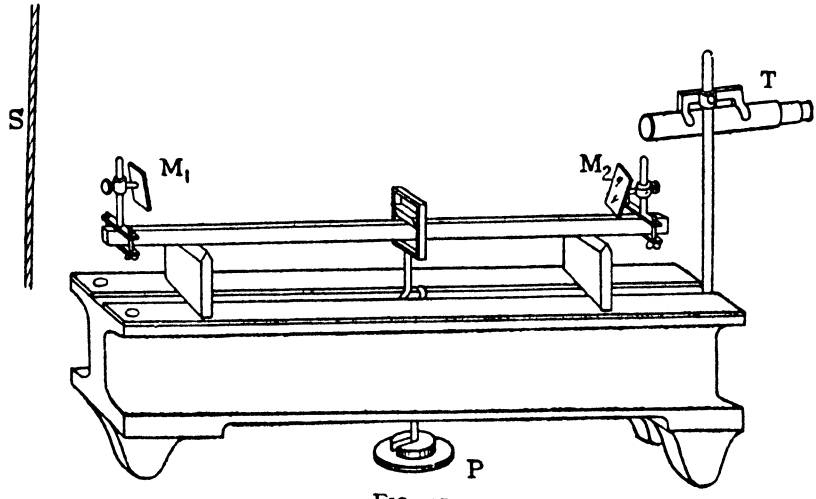


FIG. 47

The distance l between the knife-edges is measured directly, being of the order of 1 metre; this can be done with a good degree of precision.

Now d occurs in the third power, and is only a small quantity, therefore many observations must be taken and the mean value used. For d approx. 1 cm., an error of ·1 mm. means 1 per cent error, and this is magnified in d^3 to 3 per cent.

Substitute the values found, in equation (7)

$$Y = \left(\frac{m}{y}\right)\frac{g\,l^3}{4\,bd^3}$$

Another method, due to Koenig, of determining the value of Y for the material of a beam, is by means of the type of apparatus shown in fig. 47.

The bar carries a knife-edge which supports the load on a pan P.

At the ends of the bar are two mirrors, M_1 and M_2, almost normal to the bar, but slightly displaced to enable a scale S to be seen in the telescope T, the light from S having suffered two reflections.

The telescope carries a cross-hair in the eyepiece, and the apparatus is arranged so that a scale division, as seen in the telescope, coincides with the cross-hair.

If now the bar is loaded with, say, 1 or 2 kilogrammes, as a result of the depression produced the mirrors will be turned towards each other, and the scale division viewed in the telescope will be altered. This difference is noted $= x$ divisions, say.

Then we will show that

$$Y = \frac{3\,Wl^2(2D + m)}{2bd^3x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where $W = Mg$, M being the load in grms.,
l = distance between knife-edges,
D = distance between scale and the more remote mirror, M_2,
m = the distance between the mirrors,
b and d having the same values as before, b the breadth, d the depth of the bar.

For a bent cantilever we saw, equation (5)

$$\frac{dy}{dx} = \frac{W}{iY}\left(lx - \frac{x^2}{2}\right),$$

or between limits 0 and l,

$$\left[\frac{dy}{dx}\right]_0^l = \frac{W}{iY} \cdot \frac{l^2}{2}.$$

Now $\left[\frac{dy}{dx}\right]_0^l$ is tangent of the angle through which the beam has been bent. Let φ be this angle.

Then $$\tan \varphi = \frac{W}{iY} \cdot \frac{l^2}{2}.$$

Now for in the present case, l being the whole length of the bar supported by *two* knife-edges, we obtain the angle through which each end is turned by substituting $\frac{l}{2}$ for l and $\frac{W}{2}$ for W in the last equation,

i.e. in the present case

$$\tan \varphi = \frac{Wl^2}{16iY}.$$

For rectangular bar $i = \frac{bd^3}{12}$

$$\tan \varphi = \frac{3Wl^2}{4bd^3Y}$$

For small depressions the angle φ is very small and so $\tan \varphi = \varphi$.

$$\varphi = \frac{3Wl^2}{4bd^3Y} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Now a value of φ may be obtained from a consideration of the movement of the mirrors.

Let m_1 and m_2 be the original positions of the mirror (fig. 48). In the first case the image of the division at D is in coincidence with the cross-hair; when the mirrors move, each through an angle φ, F is then seen in coincidence with the cross-hair.

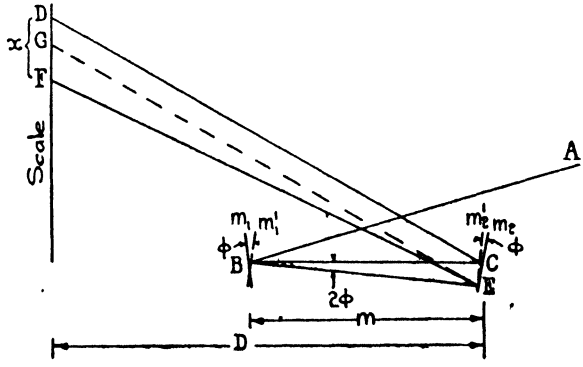


FIG. 48

Let us imagine the rays of light reversed, ABCD being the original path: when m_1 moves through φ and takes position m_1', BC is moved through 2φ striking m_2 at E.

Obviously m being the distance between the mirrors CE = $m \cdot 2\varphi$ very nearly.

EG is a line drawn parallel to CD. The ray EF is swung round through an angle 4φ, for in addition to BE having moved through 2φ, m_2 itself has rotated through φ.

i.e. since GEF = 4φ FG = $4\varphi \cdot$ D, very nearly.

But DG = CE = $2m\varphi$.

$\therefore$ x = DF = $2\varphi(m + 2D)$

$$\therefore \varphi = \frac{x}{2(m + 2D)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

from (9) and (10)

$$\frac{x}{2(m + 2D)} = \frac{3Wl^2}{4bd^3Y}$$

$$Y = \frac{3Wl^2(m + 2D)}{2bd^3x} \quad \ldots\ldots\ldots(8)$$

In performing this experiment a mean value of $\frac{W}{x}$ is obtained as in the last experiment, from observation of x corresponding to several loads and the other terms measured as before. Hence by substituting in (8) Y is obtained.

Determination of Young's Modulus of the Material of a Bar by the Vibration Method

We have already seen that the depression due to a load W at one end of a beam which is rigidly fixed at the other end is given by equation (6), page 89.

This strain sets up an equilibrating internal stress equal to $\frac{3iY}{l^3}y$ which acts as a restoring force.

If the beam be allowed to oscillate in a vertical plane the *restoring* force when depressed y cms. is therefore

$$\frac{3iY}{l^3} \cdot y$$

Expressing the load as W dynes (m grms.) we may write, if $\ddot{y}$ is the acceleration of the mass $m = \frac{W}{g}$ at the end of the bar,

$$\frac{W}{g}\ddot{y} = -\frac{3iY}{l^3}y$$

$$\text{or } \ddot{y} = -\frac{3iYg}{Wl^3} \cdot y$$

This is a periodic motion, whose period T is

$$T = 2\pi\sqrt{\frac{Wl^3}{3iYg}}$$

or, as the mass at the end of the rod is m grammes,

$$T = 2\pi\sqrt{\frac{ml^3}{3iY}} \quad \ldots\ldots\ldots\ldots\ldots(11)$$

A more complete treatment of this case may be seen under. In fig. 49 let O be the point of support and OA a horizontal through O. Suppose OB be a vertical section of the beam when depressed by a load m at the end, so that the end depression AB = z_0, and the depression of the centre of gravity shown

is u_0, whereas any point P on the bar is depressed a distance x_0, P being a distance s from O; the total length of the bar being l cms.

Now let us imagine the bar to be homogeneous, and of mass δ per unit length; so that its total mass is $M = l\delta$.

We have, if i is the 'moment of inertia of cross-section,' considering the length $(l - s)$, following the treatment of page 88,

$$iY = R\left[W(l - s) + (l - s)\delta \cdot g\frac{l - s}{2}\right],$$

taking into account the weight of the section of the bar concerned,

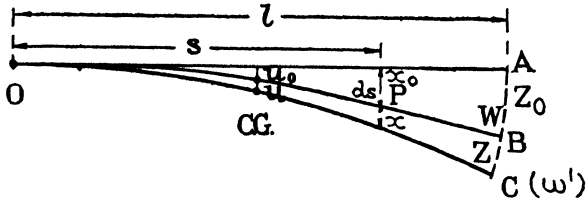


FIG. 49

which acts through the C.G. of the section. And since $\frac{1}{R} = \frac{d^2x}{ds^2}$ if the beam is not greatly bent,

$$iY\frac{d^2x}{ds^2} = W(l - s) + \frac{1}{2}(l - s)^2\ \delta \cdot g,$$

whence integrating

$$iY\frac{dx}{ds} = W\left(ls - \frac{s^2}{2}\right) + \frac{1}{2}\delta\left(l^2s - ls^2 + \frac{s^3}{3}\right)g,$$

for when $s = 0, x = 0$, i.e. the constant of integration is zero.

Integrating between the limits $s = 0, s = l, x = o$, to $x = z_0$

$$iYz_0 = W\frac{l^3}{3} + \frac{1}{2}\delta \cdot g\,\frac{3}{12}\,l^4$$

$$= l^3\left(\frac{W}{3} + \frac{Mg}{8}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

since $M = \delta \cdot l$.

This deals with a case of a steady depression. When the bar vibrates it is depressed below the position OB to some extreme position OC. To arrive at a value of the forces acting in the material of the bar in such a case let us assume that W is increased to W^1, causing the bar to be depressed into the position OC, where the extra depression of the end is z, of the centre of gravity of the projecting bar u, and of the point P is x.

Now $$iYz_0 = l^3\frac{W}{3} + \frac{Mgl^3}{8}$$

and $$iY(z_0 + z) = l^3\frac{W^1}{3} + \frac{Mgl^3}{8}$$

i.e. $$iYz = \frac{l^3}{3}(W^1 - W)$$

$$z = \frac{l^3}{3iY}(W^1 - W) \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

In this case $W' - W$ is just balanced by the internal stresses in the beam which are thus equal to

$$(W^1 - W) = \frac{3iY}{l^3} \cdot z \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

This gives the value of the *restoring* force, hence the equation of motion of the mass $\frac{W}{g}$, neglecting the mass of the beam, is

$$\frac{W}{g}\frac{d^2z}{dt^2} = -\frac{3iY}{l^3} \cdot z$$

or $$\ddot{z} = -\frac{3iYg}{l^3W} \cdot z$$

i.e. $$T = 2\pi\sqrt{\frac{Wl^3}{3iYg}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

This is identical with the result expressed in equation (11), which was deduced from very simple considerations. It does not, however, take into account the mass of the beam itself. This is most conveniently done by considering the energy of the system.

As the beam oscillates about OB, the steady deflected position of equilibrium, we will express the potential energy of the parts, using OB as the reference position of zero potential energy.

Thus, in the position OC the *Potential Energy* of the system is

$$-Wz - Mgu + (\text{Energy stored as strain in the beam}).$$

The last term is obviously equal to the work done in straining the beam, i.e. is

$$\int_0^z (\text{straining force})\, dz$$

$$= \int_0^z (W^1 - W)dz = \int_0^z \frac{3iYz}{l^3} \cdot dz \text{ by equation (13) above}$$

$$= \frac{3iYz^2}{2l^3}$$

$\therefore$ P.E. is

$$\frac{3}{2}\frac{iY}{l^3}z^2 - Wz - Mgu \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

If instead of obtaining equation (12) by integrating the previous equation between the limits o and l we had integrated between the limits o and $\frac{l}{2}$, we should have obtained the depression u_0 of the centre of gravity as under.

$$u_0 = \frac{W}{iY}\cdot\frac{5}{48}l^3 + \frac{17}{384}\cdot\frac{\delta g l^4}{iY}$$

u may be obtained in a similar way to z (p. 96) to be

$$u = \frac{W' - W}{iY}\cdot\frac{5}{48}l^3$$

and since by (13)

$$W' - W = \frac{3iY}{l^3}z$$

$$u = \frac{5}{16}z$$

So that in similar terms (16) becomes

$$\frac{3}{2}\frac{iY}{l^3}z^2 - Wz - \frac{5}{16}Mgz \ldots\ldots\ldots\ldots\ldots (17)$$

Now the kinetic energy of the system is the sum of the K.E. of the mass $W\left(=\frac{1}{2}\frac{W}{g}\dot{z}^2\right)$ and the total K.E. of the vibrating beam.

To obtain the K.E. of the beam consider an element of length ds at P. The K.E. of this element is

$$\tfrac{1}{2}\,\delta\cdot ds(\dot{x})^2$$

x may be obtained by precisely the same method as z and u.

$$x = \frac{1}{iY}\left(\frac{ls^2}{2} - \frac{s^3}{6}\right)W' - W$$

$$x = \frac{3}{l^3}\left(\frac{ls^2}{2} - \frac{s^3}{6}\right)z \text{ from (13)}$$

so that

$$x = \frac{3}{l^3}\left(\frac{ls^2}{2} - \frac{s^3}{6}\right)\dot{z}$$

The total K.E. of the bar is

$$\int_0^l \frac{1}{2}\,\delta(\dot{x})^2\cdot ds.$$

$$= \int_0^l \frac{1}{2}\,\delta\left\{\frac{3}{l^3}\left(\frac{ls^2}{2} - \frac{s^3}{6}\right)\right\}^2 ds\,\dot{z}^2$$

$$= \int_0^l \frac{1}{2}\delta\frac{9}{l^6}\left(\frac{l^2s^4}{4} - \frac{ls^5}{6} + \frac{s^6}{36}\right)ds \cdot \dot{z}^2$$

$$= \frac{\delta}{2}\cdot\frac{9}{l^6}\dot{z}^2\left[l^7\left(\frac{1}{20} - \frac{1}{36} + \frac{1}{252}\right)\right]$$

$$= \delta \cdot l\,\dot{z}^2 \;\frac{33}{280} = \frac{33}{280}\mathrm{M}\cdot\dot{z}^2$$

Thus the *total Kinetic Energy* in the system is

$$\frac{1}{2}\frac{\mathrm{W}}{g}\dot{z}^2 + \frac{33}{280}\mathrm{M}\cdot\dot{z}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

The sum of the Potential and Kinetic Energy is constant. Thus, adding (17) and (18) we obtain

$$\frac{1}{2}\left(\frac{\mathrm{W}}{g} + \frac{33}{140}\mathrm{M}\right)\dot{z}^2 + \frac{3}{2}\frac{i\mathrm{Y}}{l^3}z^2 - \mathrm{W}z - \frac{5}{16}\mathrm{M}gz = \text{constant.}$$

Differentiating

$$\frac{1}{2}\left(\frac{\mathrm{W}}{g} + \frac{33}{140}\mathrm{M}\right)z\dot{z}\ddot{z} + \frac{3}{2}\frac{i\mathrm{Y}}{l^3}\cdot 2z\dot{z} - \mathrm{W}\dot{z} - \frac{5}{16}\mathrm{M}g\dot{z} = 0$$

Dividing by $\dot{z}$ we have

$$\left(\frac{\mathrm{W}}{g} + \frac{33}{140}\mathrm{M}\right)\ddot{z} + 3\frac{i\mathrm{Y}}{l^3}z - \left(\mathrm{W} + \frac{5}{16}\mathrm{M}g\right) = 0 \ldots . (19)$$

This represents a harmonic medium whose period T is unaffected by the constant term $\mathrm{W} + \frac{5}{16}\mathrm{M}g$ as seen below.* The period is the same as that of

$$\left(\frac{\mathrm{W}}{g} + \frac{33}{140}\mathrm{M}\right) + \frac{3i\mathrm{Y}}{l^3}\cdot z = 0$$

or

$$\mathrm{T} = 2\pi\sqrt{\frac{\left(\frac{\mathrm{W}}{g} + \frac{33}{140}\mathrm{M}\right)l^3}{3i\mathrm{Y}}} \quad \ldots\ldots\ldots\ldots (20)$$

* We may write equation (19):

$$\left(\frac{\mathrm{W}}{g} + \frac{33}{140}\mathrm{M}\right)\ddot{z} + \frac{3i\mathrm{Y}}{l^3}\left\{z - \frac{l^3}{3i\mathrm{Y}}\left(\mathrm{W} + \frac{5}{16}\mathrm{M}g\right)\right\}$$

Put

$$p = z - \frac{l^3}{3i\mathrm{Y}}\left(\mathrm{W} + \frac{5}{16}\mathrm{M}g\right)$$

$$\therefore\ \dot{p} = \dot{z}, \qquad \ddot{p} = \ddot{z}$$

or

$$\left(\frac{\mathrm{W}}{g} + \frac{33}{140}\mathrm{M}\right)\ddot{p} + \frac{3i\mathrm{Y}}{l^3}p = 0.$$

$$\mathrm{T} = 2\pi\sqrt{\frac{\left(\frac{\mathrm{W}}{g} + \frac{33}{140}\mathrm{M}\right)l^3}{3i\mathrm{Y}}}$$

Now, i, the 'moment of inertia of cross-section,' depends, of course, on the form of the rod. For a rectangular rod, the 'moment' being taken about the neutral surface $i = \frac{bd^3}{12}$, where b is the breadth and d the depth of the cross-section in cms.

The Experiment may very well be carried out using an ordinary boxwood metre rule.

A definite length, l, of the rod is projected from the top of a table, to which it is rigidly clamped, as seen in fig. 50.

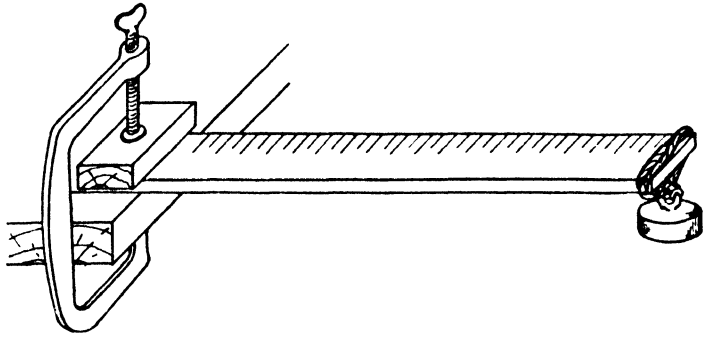


FIG. 50

A mass $\frac{W}{g}$ is *rigidly* attached to the end of the metre scale, so that it has no 'play,' i.e. there is only the one vibration, namely, that of the scale itself. The value of $\frac{W}{g}$ should be such as to cause but small depression. The rod is made to vibrate, and the time T is obtained by timing 50 vibrations in the usual manner. The length of the vibrating rod is altered, and the periodic time again determined. This is carried out for several lengths of rod, and also for several masses $\frac{W}{g}$ at the end of the scale.

It must be remembered that M *in the expression* (20) *for* T *is the mass of the vibrating part of the scale,* and must be obtained for each length l employed.

Of course, if m' is the total mass of the scale and L the total length, then $M = \frac{m'l}{L}$ for the uniform rod.

The results of the various experiments may be tabulated as under.

l	l^3.	$\frac{W}{g}$	M	$\frac{W}{g} + \frac{33}{140}$ M	T	T^2	$\frac{l^3}{T^2}\left(\frac{W}{g} + \frac{33M}{140}\right)$

$$\text{Mean } \frac{l^3}{T^2}\left(\frac{W}{g} + \frac{33M}{140}\right) =$$

$$Y = \frac{16\pi^2}{bd^3}\left\{\frac{l^3}{T^2}\left(\frac{W}{g} + \frac{33M}{140}\right)\right\}$$

NOTE. $\frac{W}{g}$ is the *mass* attached in grammes.

The method works equally well for such substances as wood, where the correction for the mass of the beam is small, and for brass, etc., where $\frac{33}{140}$ M is comparable with the values of W.

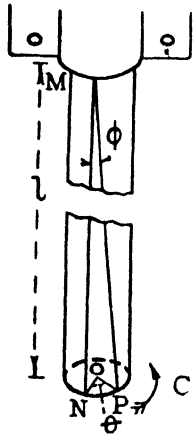


FIG. 51

RIGIDITY

The modulus of rigidity is determined by observation of the twist produced in a wire by a definite couple, either statically, or less directly by torsional oscillation.

Measurement of the Rigidity of a Wire by the Static Method

Let us consider a long wire fixed rigidly at the upper end and subjected to a couple, C, turning it in the direction shown in fig. 51.

We may consider a section of the wire (fig. 52), and of this section an annular ring as seen. Considering a small rectangular segment EFGHIJKL in the undisturbed wire, when the couple acts, this rectangular parallelepiped will shear as we have previously seen, the faces making an angle φ with the original vertical direction. This happens throughout the ring, and adjacent sections will behave in the same way throughout the length of the wire.

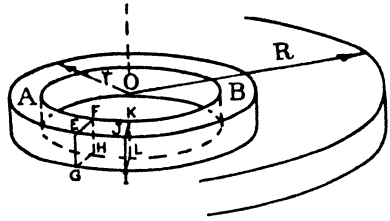


FIG. 52

A vertical line, MN, along the wire (fig. 51) is moved by each ring through an angle φ, and so finally takes up a position MP when MP is inclined at an angle φ to MN.

If l is the length of wire from the fixed end, and R is the radius of the wire, then since N is moved to P when the couple acts, the radius ON moving to OP, where the angle NOP $= \theta$ radians, we have

$$\text{arc NP} = l\varphi = \text{R}\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Considering again the annular ring of fig. 52, of radius r and width δr, we have the shear produced by the couple C.

Let f be the value of the shearing force over the area, $2\pi r \cdot \delta r$, of its upper and lower surfaces. From the definition of rigidity n

$$n = \frac{\frac{f}{2\pi r \cdot \delta r}}{\varphi}$$

or

$$f = 2\pi n \varphi r dr.$$

Since this force has a moment fr about the axis of the wire, we have, replacing φ by the more easily measured term θ, from (21),

$$fr = \frac{2\pi n \theta}{l} r^3 dr.$$

The total couple throughout the solid section is therefore :

$$\int_0^R fr = \int_0^R \frac{2\pi\theta n}{l} r^3 dr = \frac{\pi n \theta R^4}{2l}$$

a couple which is equal to the applied couple C for equilibrium, hence

$$C = \frac{\pi n \theta R^4}{2l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

This may be re-written :

$$\frac{C}{\theta} \cdot l = \frac{\pi R^4}{2} \cdot n.$$

$\frac{C}{\theta}$, the couple required to produce unit angular twist, is called the 'coefficient of torsion' $= \tau$, say.

We thus have :

$$\tau l = n \cdot \frac{\pi R^4}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

or $$\tau l = i \cdot n.$$

$\frac{\pi R^4}{2} = i$ the moment of inertia of cross-section for the circular wire, and θ in equation (22) is measured in radians.

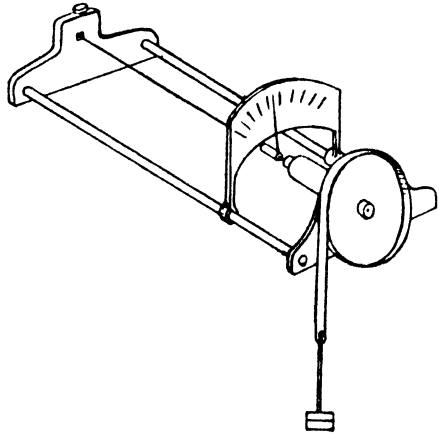


FIG. 53

Experimental Details.

The method of finding the coefficients is by fixing a wire specimen rigidly at one end, applying a measured couple at the other, and measuring the twist produced at a given distance l from the fixed end.

The two usual types of apparatus usually employed are shown in figs. 53 and 54.

Using the vertical wire type, pointers are fixed to the wires at different distances from the fixed end, and a couple is applied to the free end by adding weights to the scale pans S, S^1. The amounts of twist, θ_1, θ_2, θ_3, at the distances, l_1, l_2, l_3, from the clamped ends are observed on the circular scales shown. It will be found that $\frac{\theta_1}{l_1} = \frac{\theta_2}{l_2} = \frac{\theta_3}{l_3}$ as we may expect from equation (22).

Now consider a fixed length of wire l cms. from the support. If M is the total mass applied in S and S^1, and the diameter of the wheel at which the couple is applied is D, then $MgD = C$ in equation (22).

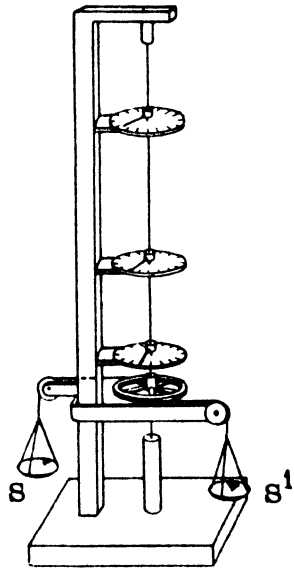


FIG. 54

A series of values for θ, corresponding to various masses on the pans, is obtained. These results may be tabulated as under.

MASS ON PANS M	ANGLE OF TWIST $\theta°$	$\frac{M}{\theta}$
0	0	—
100	15	6·6
200	30	6·6
300	46	6·52
400	61	6·55
500	77	6·50

The last column of the table shows the ratio $\frac{M}{\theta}$ to be a constant—or nearly so. For such a variation as the one shown in the table of results above, the mean value of $\frac{M}{\theta}$ is taken and substituted in equation (24). Re-writing (22) we have

$$Mg \cdot D = \frac{\pi n R^4}{2l} \cdot \theta \qquad \theta \text{ in radians,}$$

or

$$n = \frac{M}{\theta} \cdot \frac{2gDl}{\pi R^4} \qquad \theta \text{ in radians.}$$

The length of the wire is measured directly. The arm of the couple, D, is measured by means of callipers.

As R, the radius of the wire, occurs in the fourth power, especial care is taken to obtain its true value. Determinations of R are made with a micrometer screw gauge at several points along the length of the wire, and a mean value taken.

Substituting the mean value of $\frac{M}{\theta}$ from the table of results, all the unknowns of the equation are ascertained. However, as θ is usually measured on the circular scale in degrees, the value of the ratio $\frac{M}{\theta}$ must first be converted to radians.

Assuming therefore now that θ is measured in degrees, the end result which converts θ to radians, etc., is

$$n = \frac{M}{\theta} \cdot \frac{l}{R^4} \cdot \frac{g(360)D}{\pi^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

The same type of observations are necessary with the horizontal apparatus, which makes use of a shorter length of wire specimen.

Maxwell's Needle

If a bar, AB, is suspended horizontally by means of a wire whose modulus of rigidity n is to be determined, the value of n may be obtained in terms of I, the moment of inertia of the bar about the axis of suspension, and T, the time taken for the bar to make a complete horizontal oscillation.

Consider a small displacement θ from the position of rest of the bar. We have already seen (p. 102), that in such a case the couple called into play in the wire to equilibrate the displacing couple is $\tau = \frac{in}{l}$ per unit angular displacement, where $i = \frac{\pi R^4}{2}$, so that the couple exerted by the wire for an angular displacement θ is $\tau\theta$ for that particular wire.

It has been shown (p. 53) that the moment of such external forces on the oscillating bar is

$$I\frac{d^2\theta}{dt^2}$$

Hence $$I\frac{d^2\theta}{dt^2} = -\tau\theta$$

i.e. these couples are equal and opposite at any point.

$$\frac{d^2\theta}{dt^2} = -\frac{\tau}{I}\theta$$

This will cause vibrations whose periodic time, T, is given by (p. 25)

$$T = 2\pi\sqrt{\frac{I}{\tau}}$$

Hence, knowing T by timing a number of swings, if I can be ascertained τ may be evaluated and hence n, since $\tau = \frac{\pi R^4 n}{2 l}$.

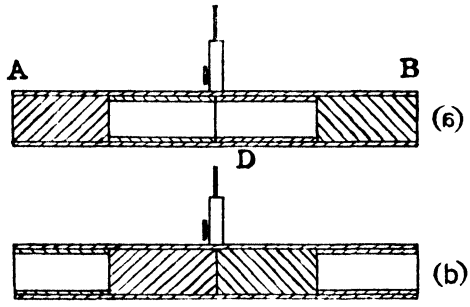


FIG. 55

In Maxwell's needle (fig. 55) the bar is replaced by a hollow tube, of length D cms. Fitting in the tube are four equal-sized cylinders, each of length $\left(\frac{D}{4}\right)$, two of solid brass, and two hollow.

Let I_0 be the moment of inertia of the hollow cylinder of length D about the wire as axis,

I_1 be the moment of inertia of the solid brass cylinder about a parallel axis through its centre of gravity,

I_2 be the similar moment of inertia for the hollow brass cylinder,

m_1 the mass of each solid brass cylinder,

m_2 the mass of each short hollow brass cylinder,

l the length of the wire in cms.

In the first case place the cylinders in the order shown in

fig. 55 (a) and find the periodic time T_1. Then arrange the cylinders as in (b), once more obtaining the periodic time by timing, say, 50 complete swings. Let this time be T_2.

Suppose that the moment of inertia of the complex bar in the first case is I′, and I″ in the second.

We have
$$T_1 = 2\pi \sqrt{\frac{I'}{\tau}}$$
$$T_2 = 2\pi \sqrt{\frac{I''}{\tau}}$$
$$T_1^2 - T_2^2 = \frac{4\pi}{\tau}\left(I' - I''\right)$$

Now we have
$$I'' = I_0 + 2I_1 + 2m_1\left(\frac{D}{8}\right)^2 + 2I_2 + 2m_2\left(\frac{3D}{8}\right)^2$$
for the moment of inertia of each of the four units is given by the ' law of parallel axes,'
$$I' = I_0 + 2I_2 + 2m_2\left(\frac{D}{8}\right)^2 + 2I_1 + 2m_1\left(\frac{3D}{8}\right)^2$$
whence
$$I' - I'' = 2m_2\left\{\left(\frac{D}{8}\right)^2 - \left(\frac{3D}{8}\right)^2\right\} + 2m_1\left\{\left(\frac{3D}{8}\right)^2 - \left(\frac{D}{8}\right)^2\right\}$$
$$= 2m_1\frac{D^2}{8} - 2m_2\frac{D^2}{8} = \left(m_1 - m_2\right)\frac{D^2}{4}$$

By such an arrangement $I' - I''$ may be evaluated and hence T may be calculated.

To enable an accurate measurement of T in both cases, the hollow frame carries a small mirror. A beam of light from a lamp is focussed on the mirror, and the reflected beam is directed on a scale. As the reflected beam, shown as a spot of light on the scale, passes a certain mark in one direction, a stop-clock is started, and, starting counting, 1, 2, 3, etc., as the spot *again* passes the scale reading *in the same direction*, the time for fifty complete oscillations is obtained, when the four short cylinders are arranged as in fig. 55 (a) and (b), whence T_1 and T_2 are obtained. The length l of the wire from the rigid support, and the length D are measured directly: m_1 and m_2 are obtained by weighing, and therefore all the factors for the determination of n are measured, for,
$$\tau = 4\pi\frac{I' - I''}{T_1^2 - T_2^2} = 4\pi\frac{m_1 - m_2}{T_1^2 - T_2^2}\cdot\frac{D^2}{4}$$
and
$$n = \frac{2\tau l}{\pi R^4}$$

Determination of the Modulus of Rigidity of the Material of a Flat Spiral Spring

In this experiment the value of n for the material of the wire of a flat spiral spring is deduced from a knowledge of the periodic time of vertical oscillations of the spring when loaded with a known mass, and the dimensions of the spring.

Let us consider a flat spiral spring, whose turns are closely wound with a wire whose radius r is small compared with the radius of the spring itself; such a spring may be made by winding the wire on a wooden cylinder of suitable diameter.

Let the ends of the spring be bent twice at right angles, the free ends, such as BC (fig. 56), being therefore along the axis of the cylinder.

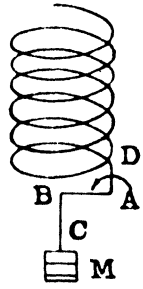


FIG. 56

If it is then clamped vertically at one end, and loaded with a mass, M, at the other end, as shown in the figure, the force, Mg, along the axis exerts a couple tending to twist the wire in the direction of the arrow.

It was shown on p. 102 that, under such circumstances, if l is the length of the wire from the fixed end, and R the radius of the wire,

$$C = \frac{\pi R^4}{2} \cdot \frac{n\theta}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

In the case of the spring we may apply this formula when l represents the total length of wire, and is equal to $2\pi RN$, where N is the number of turns.

If r = radius of the wire,

R = radius of the cylinder on which the spring is wound + $\frac{1}{2}r$,

l = $2\pi RN$,

M = the load in grammes,

then $C = \frac{\pi r^4}{2} \cdot \frac{n\theta}{2\pi RN}$ ..... (25*a*)

Take a section of the spring at A, shown in fig. 57 enlarged. Due to the couple, the arm AB will be twisted through the angle θ, given by equation (25*a*) above, and will take up the position AB^1.

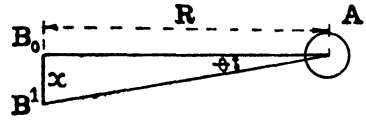


FIG. 57

The depression x, of the end B, is obviously equal to $R\theta$, approximately, when R is large,

i.e. $$x = R \cdot \frac{4\pi RNC}{\pi r^4 n} = \frac{4NR^2C}{r^4 \cdot n}.$$

If f be the restoring force on M due to the wire, the couple $C = fR$.

i.e. $$x \frac{r^4 n}{4NR^3} = f \quad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

The equation of motion of the moving mass is therefore

$$M \frac{d^2x}{dt^2} = -\frac{r^4 n}{4NR^3} \cdot x.$$

This was shown (p. 25) to represent S.H.M., whose period is given under.

$$T = 2\pi \sqrt{\frac{4MNR^3}{r^4 n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

From observation of the periodic time of the spring oscillating in a vertical manner, n, the rigidity coefficient of the material of the spring can therefore be obtained.

However, in the foregoing we have neglected the mass of the spring itself, and have also assumed that the total depression produced by M was due to twist alone.

We must now consider these factors.

Shear in the Spring

We can very easily see that the shear effect in the type of spring taken above is of a negligible order when the radius of the wire is small compared with the radius of the spring. For this purpose again neglect for the time the mass of the spring.

If f is the force acting along the axis, at any point A in the spring there is an equal and opposite force, f for equilibrium.

Thus, for the small segment AD, shown enlarged in fig. 58, at A there is a downward force f, and at D an upward force f, constituting a shearing couple which would cause a depression

of A to some lower position A^1, through an angle φ', say, where from the definition of rigidity

$$\varphi' = \frac{f}{An}.$$

The area A is the area over which the forces f act, which, in this case, is the area of cross-section of the wire, πr^2,

i.e. $$\varphi' = \frac{f}{\pi r^2 n}$$

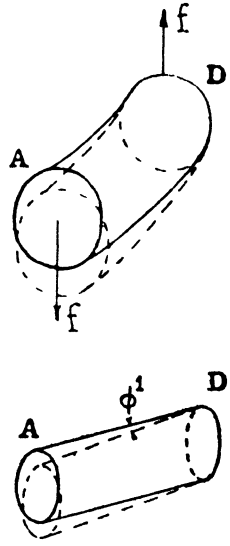


FIG. 58

The total depression due to this shear for a length l of the wire

$$= \varphi' l = \frac{fl}{\pi r^2 n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

Now, using the expression for the depression due to twist, given in equation (26), and substituting $l = 2\pi RN$ in the expression (28) above, we have

$$\frac{\text{Depression due to shear}}{\text{Depression due to twist}} = \frac{\dfrac{f 2\pi RN}{\pi r^2 n}}{\dfrac{f 4NR^3}{r^4 n}} = \frac{r^2}{2R^2}$$

Take an average case, $r = \cdot 05$ cm., $R = \cdot 8$ cm.

$$\frac{r^2}{2R^2} = \frac{1}{512}$$

i.e. depression due to shear is about ·2 per cent of the depression due to twist.

Thus, when r is small compared with R we may neglect the shearing effect.

The Mass of the Spring

The expression for the time of vibration is modified if account be taken of the mass of the spring itself.

The total mass moving is greater than M. We may regard the problem as being similar to the case of an ideal massless spring, loaded with $(M + m')$, where m' is an additional load which just has the same effect as the mass of the actual spring used; m^1 may be called the 'equivalent mass of the spring' and

$$T = 2\pi \sqrt{\frac{(M + m')\, 4NR^3}{r^4 n}}$$

Thus, using this formula, we may find n and m' by obtaining T for two loads, and solving for m' and n.

However, we may treat the problem in somewhat more detail, and deduce an expression for the equivalent mass m', in terms of the actual mass m of the spring.

It has been shown that the twist of the wire of the spring at the end is $\theta = \frac{2lMgR}{\pi r^4 n}$, since C, the couple, is equal to MgR.

The depression of the end due to this twist is $R\theta$.

If this depression is x,

$$x = \frac{2lMgR^2}{\pi r^4 n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

In the vertical oscillations, the mass M at the end of the spring is depressed further than this. To arrive at a value of the forces called into play due to such further straining of the spring, suppose a bigger mass, M^1, be applied, causing an increased depression z, i.e. total $x + z$.

Then
$$x + z = \frac{2lM^1gR^2}{\pi r^4 n}$$

$$z = \frac{2lR^2}{\pi r^4 n} \cdot (M^1 - M)g^* \quad \ldots\ldots\ldots (30)$$

To allow for m, we may consider the energy of the system with regard to the position of rest of the loaded spring.

*The internal forces *equilibrating* the force $(M^1 - M)g$, are therefore $= \frac{\pi r^4 n}{2lR^2}$

Therefore the equation of motion of the mass M is

$$M\ddot{z} = -\frac{\pi r^4 n}{2lR^2} z$$

or
$$T = 2\pi \sqrt{\frac{2MlR^2}{\pi r^4 n}}$$

as previously shown, equation (27), neglecting the mass m of the spring itself.

When displaced a distance z from the position of rest (which is x cms. below the unloaded end)

Potential Energy is

$-Mgz+mg$ (displacement of centre of gravity) + (energy stored as strains in the spring).

The centre of gravity is lowered a distance which is equal to half the depression of the end $= \frac{z}{2}$.

The energy of the spring is equal to the work done in straining the spring, i.e. since the internal forces are equal to $(M^1 - M)g$ $= \frac{\pi r^4 n}{2lR} \cdot z$, from (30), the work done in depressing the end a distance dz is $\left(\frac{\pi r^4 n}{2lR^2}\right) z dz$,

i.e. energy in the spring is

$$\int_0^z \frac{\pi r^4 n}{2lR^2} \cdot z \cdot dz = \frac{\pi r^4 n}{2lR^2} \cdot \frac{z^2}{2}$$

Total P.E. is therefore

$$\frac{\pi r^4 n}{4lR^2} z^2 - Mgz - \frac{1}{2} mgz.$$

The Kinetic Energy is

$\frac{1}{2} M(\dot{z})^2$ + K.E. of the moving spring.

If δ is the mass of unit length of the spring wire, the second term above is

$\int_0^l \frac{1}{2}$ (mass of a small element) $\times$ (velocity of the element)^2.

At a point P s cms. from the point of support measured along the length of the spring, consider a small element ds. The velocity of that element is $\frac{s}{l} \cdot \dot{z}$, its mass is δds,

i.e. K.E. of moving spring is

$$\frac{1}{2} \int_0^l \delta ds \cdot \frac{s^2}{l^2} \cdot \dot{z}^2$$

$$= \frac{1}{2} \frac{\dot{z}^2}{l^2} \delta \frac{l^3}{3} = \frac{1}{2} \left(\frac{m}{3}\right) \dot{z}^2 \qquad \text{since } m = l\delta.$$

The total energy of the system in the position considered, and which is constant by principle of conservation of energy, is therefore

$$\frac{\pi r^4 n}{4lR^2} z^2 - \left(Mg + \frac{mg}{2}\right) z + \frac{1}{2} \left(M + \frac{m}{3}\right) \dot{z}^2 = \text{constant}.$$

Differentiating and dividing by $\dot{z}$ we have

$$\frac{\pi r^4 n}{2lR^2} z - \left(Mg + \frac{mg}{2}\right) + \left(M + \frac{m}{3}\right)\ddot{z} = 0;$$

as shown on footnote, p. 98, such a system is a periodic motion whose periodic time T is the same as for

$$-\frac{\pi r^4 n}{2lR^2} \cdot z = \left(M + \frac{m}{3}\right)\ddot{z}$$

or

$$\ddot{z} = -\frac{\pi r^4 n}{2l\left(M + \frac{m}{3}\right)R^2} \cdot z$$

i.e.

$$T = 2\pi\sqrt{\frac{\left(M + \frac{m}{3}\right)2lR^2}{\pi r^4 n}} \quad \ldots\ldots\ldots (31)$$

The effect of the mass of the spring is therefore the same as though M were at the end of a massless spring together with a load equal to $\frac{1}{3}$ the mass of the spring.

Experimental Details.

A flat spiral spring is chosen, having a radius R which is fairly large compared with the radius of the wire. One end is clamped firmly in a heavy retort stand; a mass is attached to the lower end of the spring, and the time of vibration of the vertical oscillations is obtained by timing 50 vibrations with a stop-watch. This is repeated for various loads.

l, the length of the wire in the spring, may be obtained from a knowledge of N, the total number of turns in the spiral. If there are exactly N turns, the length is $2\pi NR$ where R is the average value of several observations of the mean diameter of the spiral, which is the value of the outside diameter of the wire spring, minus the diameter of the wire of which it is made.

r occurs in the fourth power, so should be measured with extreme care. A number of values are obtained with a screw gauge at points along the length of the spring, and the mean value taken.

l being equal to $2\pi RN$, we have from equation (31)

$$n = \frac{16\pi^2 R^3 N}{r^4} \cdot \frac{M + \frac{m}{3}}{T^2} \quad \ldots\ldots\ldots\ldots\ldots (32)$$

The results of the several experiments may be conveniently tabulated as over.

M.	$(M + m/3.)$	T.	T^2.	$\frac{M + m/3.}{T^2}$

$$\text{mean value of } \frac{M + \frac{m}{3}}{T^2}$$

The mean value of $\frac{M + \frac{m}{3}}{T^2}$ for the series of loads (M) taken is obtained and substituted in equation (32), thus giving the mean value of n for the complete set of observations

Determination of Young's Modulus of the Material of a Spring

The value of Young's modulus for the material of the wire of which a spring is made may be obtained by supporting the spring vertically and allowing a bar which is firmly fastened to the lower end to perform horizontal swings as in the case of Maxwell's needle (p. 102).

It is shown below that in such a case the time of a complete horizontal oscillation is

$$T = 2\pi\sqrt{\frac{8RNI}{r^4Y}}$$

where Y is Young's modulus for the wire,
R is the radius of the spiral, measured from the centre of the wire,
r the radius of the wire,
I the moment of inertia of the bar about the axis of suspension,
N the total number of turns in the spring.

The above assumes, as in the last experiment, that the spring is a flat spiral; the layers are very close together, and each may therefore be regarded as practically horizontal.

Let fig. 59 represent a horizontal section of a small length s of the spring, with NS the section of the neutral surface by the plane of the diagram. This neutral surface will be normal to the diagram. ABCD is the section of the element of the spring where the bar is in an undisturbed position.

When the bar is turned in a horizontal plane through any angle ψ, the section taken, in common with every other element of the spring, will become more curved, as shown in the broken lines.

Consider a single 'filament' of the material, as was done in the case of the bending of beams (p. 86). If a is the area of cross-section of the filament, and f is the force required to bend up the filament through a small angle

$$Y = \frac{\frac{f}{a}}{\text{Strain of the filament}}$$

Now, if R_0 is the normal radius of curvature of the neutral surface, and PQ is a distance x cms. beyond NS, the original

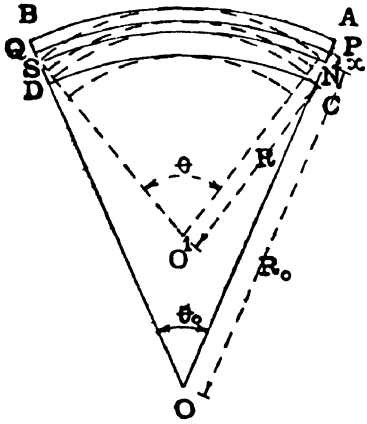


FIG. 59

length of PQ $= (R_0 + x)\theta_0$, where θ_0 is the angle between the radii from the original centre of curvature O to the ends AC and BD of the element considered.

Now, due to stresses similar to f, the element has a greater curvature, and consequently the neutral surface has a smaller radius of curvature R^1. If O^1 is the new centre of curvature, the new length of PQ is $(R + x)\theta$.

However, the neutral surface is of the same length as before, so that $R_0\theta_0 = R\theta$.

The strain is therefore $(R + x)\theta - (R_0 + x)\theta_0 \div$ original length $= \frac{x(\theta - \theta_0)}{S}$ approximately.

$$S = R_0\theta_0 = R\theta_1.$$

Strain is therefore

$$x\left(\frac{\theta}{R\theta} - \frac{\theta_0}{R_0\theta_0}\right) = x\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

$$\therefore Y = \frac{\frac{f}{a}}{x\left(\frac{1}{R} - \frac{1}{R_0}\right)}$$

$$f = Yax\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

The moment of this force on the filament about the neutral surface is fx. The total moment of the couple acting on the element considered is therefore

$$\Sigma fx = \int Yax^2\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

$$= Yi\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

Suppose in the case taken the angles subtended at the centre in each case by 1 cm. of the wire were φ and φ_0.

The couple is

$$C = Yi(\varphi - \varphi_0).$$

If l is the length of the wire in the spring, the total change of angle, i.e. the angle through which the inertia bar moves is $\psi = l(\varphi - \varphi_0)$

$$\text{i.e. } C = Yi\frac{\psi}{l}$$

This is the value of the couple due to the internal stresses, called into play by the strains in the wire,

i.e.
$$I\ddot{\psi} = -\frac{Yi}{l}\psi$$

The periodic time T is given by

$$T = 2\pi\sqrt{\frac{Il}{Yi}}$$

$i = \frac{\pi r^4}{4}$, $\quad l = 2\pi RN$

$$\therefore \quad T = 2\pi\sqrt{\frac{8INR}{r^4Y}}$$

or
$$Y = \frac{32\pi^2INR}{r^4T^2}$$

Experimental Details.

The spring is set up as in the last experiment, and a rod, say, of rectangular cross-section is clamped to the end of it, so that there is no free play between the end of the spring and the rod, and the centre of gravity of the rod is under the suspension.

The ' inertia rod ' is then given a displacement in the horizontal plane, and the subsequent horizontal oscillations are timed. T is obtained by timing 50 complete swings in the usual way.

r and R are carefully measured as previously described, and the total number of turns of wire in the spring, N, is counted.

The value of I may be calculated from a knowledge of the dimensions and the mass of the bar (see p. 54).

Determination of the Modulus of Rigidity and Young's Modulus for the Material of a Wire by Searle's Method

To find Y, Young's modulus, or n, the coefficient of rigidity, of the material of a wire specimen by this method, the wire is

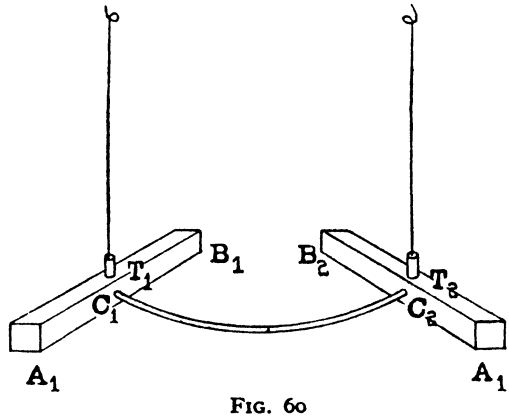


FIG. 60

fastened to two identical rods, A_1B_1, A_2B_2, at their mid-points, as shown in the diagram at C_1, C_2. These rods are usually square or circular in cross-section, and are supported by threads from points T_1 and T_2, such that the axis of suspension and the axis of the wire intersect at the centre of gravity, and the suspended rods are a distance apart, which allows the wire to be stretched in a straight line, the whole assuming an H formation.

If now the ends, B_1 and B_2, of the rods are drawn together and fastened by a loop of thin cotton, the wire will be bent into the arc of a circle, subtending an angle 2φ at the centre; each rod will make an angle φ with its original direction.

The suspension of the two rods being such that the torsion is negligible, the only couple acting on the bars is that due to

the bending of the specimen, C_1C_2. With this method of suspension we have a couple produced equal to $\frac{iY}{R}$ in the wire (p. 88), where i is the 'moment of inertia of cross section,' and R is the radius of curvature of the arc C_1C_2.

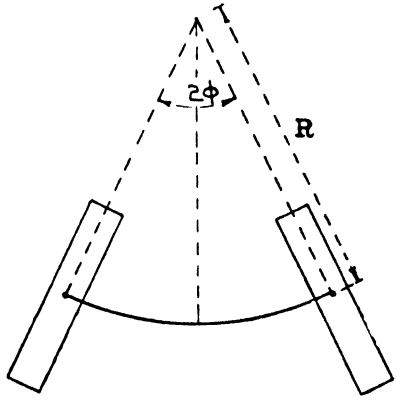


FIG. 61

Now, if l is the length of the wire between the rods,

$$R = \frac{l}{2\varphi}.$$

If r is the radius of the wire (assumed circular),

$$i = \frac{\pi r^4}{4},$$

whence the couple acting is

$$Y \frac{\pi r^4}{4} \cdot \frac{2\varphi}{l} = \frac{\pi Y r^4 \varphi}{2l}$$

Let I_1 be the moment of inertia of the rod, A_1B_1, about the axis of suspension, $\frac{d^2\varphi}{dt^2}$ the angular acceleration. The moment of the external force is therefore, by the theorem given on p. 53, $\frac{Id^2\varphi}{dt^2}$.

This is equal and opposite to the restoring couple exerted by the bent wire. The equation of motion for the rod is therefore

$$\frac{Id^2\varphi}{dt^2} = -\frac{\pi Y r^4}{2l} \cdot \varphi.$$

This is simple harmonic motion (p. 25), and

$$T = 2\pi \sqrt{\frac{2lI}{\pi Y r^4}}$$

If the rod A_1B_1 has a length 2L and width $2a$ and mass M grammes,

$$I = M\frac{L^2 + a^2}{3}$$

whence

$$Y = \frac{8\pi l}{r^4T^2}\cdot\frac{L^2 + a^2}{3}\cdot M.$$

To evaluate Y for a specimen of wire, the arrangement described is fitted up. The thin cotton loop holding the rods together at B_1B_2 is burnt, and the resulting oscillations are timed with a stop-watch, T being obtained by timing 50 complete oscillations. An alternative method of finding T, and which may be used in many experiments where T is required, is seen under.

A reference point having been chosen—say, a chalk-mark under the oscillating bar—the time is noted at which the counting of the swing is commenced. After 5 swings the time is again noted, and the times at the end of every 5 is recorded in a table as seen below.

(1) NO. OF OSCILLATIONS	(2) TIME	(3) NO. OF OSCILLATIONS	(4) TIME	(5) TIME FOR 30 OSCILLATIONS
0	u	30	a	$a - u$
5	v	35	b	$b - v$
10	w	40	c	$c - w$
15	x	45	d	$d - x$
20	y	50	e	$e - y$
25	z	55	f	$f - z$

mean time for 30 oscillations. =

Tabulating in the second column the time for the starting point, 5, 10, 15, 20, 25 oscillations, and in the fourth column the time for the 30, 35, 40, 45, 50, 55 oscillations, the difference in any line between the fourth and second column value will give the time for 30 complete oscillations. The mean value of the last column being taken as the time for 30, T may be evaluated.

The radius of the wire occurs in the fourth power. The value of the diameter is therefore measured in several places, say, six, and the mean radius, r, calculated. The other measurement for the evaluation of Y are straightforward.

It will be noticed that in this method the value of Y is obtained by timing, and not by observing deflections as in most of the

other methods. Another advantage of the method is that it requires only a small specimen of the material.

To find the modulus of rigidity of the specimen, the rod A_2B_2 is clamped rigidly in a horizontal plane and the wire acts as a support for the rod A_1B_1, which may therefore be made to perform horizontal torsional oscillations. The rod is displaced slightly from its position of rest causing a restoring couple to act on it, due to the twist of the wire.

For a displacement θ, the restoring couple is

$$\frac{\pi n r^4}{2l}\ \theta$$

as established on p. 102.

We therefore have as the equation of motions of the rod

$$I\frac{d^2\theta}{dt^2} = -\frac{\pi n r^4}{2l}\cdot\theta$$

This is once more a case of simple harmonic motion, whose periodic time, T_1, is given by

$$T_1 = 2\pi\sqrt{\frac{2lI}{\pi r^4 n}}$$

If the rod is of square or circular cross-section, the moment of inertia about the axis taken in this experiment is the same as the last case. If of rectangular section, the moment of inertia will be obtained by calculation as before.

T_1 may be found as described for the first experiment; the length l and the radius r are already known.

Hence n may be found by substitution in the formula below:

$$n = \frac{8\pi lI}{T_1^2 r^4}.$$

Determination of the Bulk Modulus for Glass

The bulk modulus, as shown on page 82, may be expressed as

$$k = \frac{p}{\frac{\delta V}{V}},$$

where p is an increase in pressure causing the small volume change δV.

In general, it will not be convenient to apply a uniform pressure to a body, but it is a simple matter in most cases to apply an extending force per unit area, or a pressure, in one direction only. If, as a result of such a pressure, the change in volume is δV^1, then $\delta V^1 = \frac{1}{3}\cdot\delta V$ where δV is the change in volume when the same pressure is applied uniformly over the body.

If a cylinder be supported at one end, and an extending force

be applied to the other, in a direction coincident with the axis of the cylinder, we have the extending force at one end and the equal reaction at the other. The change in volume in this case is of the type δV^1 above. So that if the change in volume of a body be observed in such a case, the value of k may be calculated.

The apparatus used to determine the bulk modulus of, say, glass is seen in fig. 62. A glass tube g is cemented to two caps of brass, A and B. The upper brass cap A is provided with two pegs, which act as a support for the whole apparatus. The lower cap B carries a hook to which a pan S is attached.

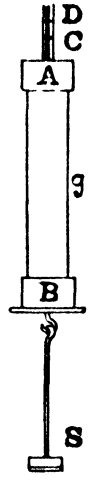


FIG. 62

Fitting in the upper end of the tube is a rubber cork, carrying a capillary tube, CD, which is graduated and calibrated in the manner described on page 41.

The tube g is filled with water. Care is taken to avoid trapping air, and the cork, etc., is placed in position, resulting in a little water rising in the capillary tube, i.e. the whole of the glass tube and part of the capillary are filled with water to a definite level. If now the pan S is loaded with, say, 5 kilogrammes the volume of the glass will increase by a small amount. The difference in the levels on the calibrated tube enables the value of the volume change to be determined if the temperature remains constant throughout the observation.

However, the apparatus is, by the construction, very much affected by small temperature changes ; it is a water thermometer with a very large bulb. To obtain a good approximation to the value of δV^1, the volume change, the reading on the tube is noted when S has no load ; 5 kilos are applied and the reading

noted, the load is removed, and once more the scale reading is taken and the process repeated with 5 kilos and no load for 10 observations, each one being taken at a regular time interval as under.

TIME FROM COMMENCEMENT	LOAD	READING	TIME FROM COMMENCEMENT	LOAD	READING
0 min.	0 kilos.		5 min.	0 kilos.	
½	5		5½	10	
1	0		6	0	
1½	5		6½	10	
2	0		7	0	
2½	5		7½	10	
3	0		8	0	
3½	5		8½	10	
4	0		9	0	
4½	5		9½	10	
			etc.		

A similar process is repeated with a load of 10, 15, and 20 kilos.

The results are plotted as a graph, using a fairly large scale for the ' scale readings ' as ordinates.

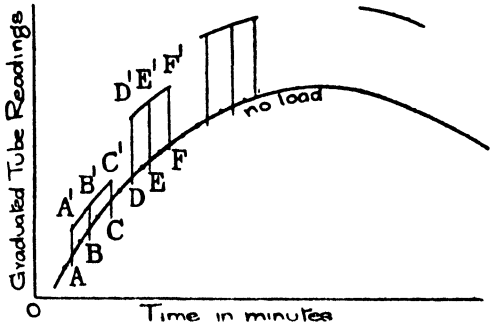


FIG. 63

The no-load curve will be a continuous line, and the short curves for 5, 10, etc., load will be at a vertical distance above it, depending on the load employed.

The form of curve obtained in a particular experiment, using one type of glass, is seen in fig. 63.

AA^1, BB^1, CC^1 are then drawn parallel to the axis of scale readings, and their values measured: the mean value of this

length gives the scale reading increase in volume of the glass tube *g*. The actual volume change is then obtained from the calibration curve of the capillary tube.

This process is repeated for the curves giving the results for each load, the mean of the three ordinates being taken in each case.

From each value of $\delta V'$ the increase in volume δV per kilo load may be calculated. A mean value for all loads therefore gives the mean for all the observations taken, for the value of δV per kilo.

The original volume of the glass tube may then be found by obtaining the product of the length between the brass caps and the area of cross-section.

To obtain the cross-section, a little water is poured into the empty tube so that the surface becomes visible above the lower cap B. A measured volume of water is then run into the tube so that the level is just below the upper cap. From a measurement of the increase in the level and the volume introduced from a graduated flask or measuring jar, the average cross-section may be calculated. Hence V is obtained.

To find the stress applied to the tube we must find the internal and external radii of the tube. The load applied at the end causes a stress, a force per unit area, equal to $\frac{mg}{\pi(R^2 - r^2)}$, where m is the mass, R the external and r the internal radius of the tube.

Now, πr^2 was determined above, when obtaining V. To find R, the external radius is measured with callipers at about 6 or 8 places, and the mean taken.

Thus, if δV is the mean value of the volume change per kilogramme load, i.e. for a force of 981×10^3 dynes in one direction, the change for a uniform stretching force would be $3\delta v$,

and
$$K = \frac{\frac{981 \times 10^3}{\pi R^2 - \pi r^2}}{\frac{3\delta V'}{V}} \text{ dynes/cm}^2.$$

Poisson's Ratio for India-rubber

A length of solid india-rubber is supported at one end and loaded at the other. The rubber, of circular cross-section, is marked at about six places along its length, and the diameter at these selected places is measured with a micrometer screw for each load placed in the scale pan at the free end of the india-rubber.

The length of the specimen is also measured for each load. Then, since Poisson's Ratio has been defined as

$$\mu = \frac{\dfrac{\text{Lateral contraction}}{\text{Original diameter}}}{\dfrac{\text{Longitudinal extension}}{\text{Original length}}}$$

μ may be calculated for each load, the mean value from the six measurements in the lateral direction being used.

For a rubber cord of about $\frac{1}{2}$-inch diameter, suitable loads would be 500, 1000, and 2000 grammes.

It will be found that, to some extent, the values obtained depend on the history of the specimen. The results when the load is increasing will be somewhat different from the values for the same load when decreasing.

In all observations the readings should not be taken until about ten minutes after the adjustment of the load.

CHAPTER IV

SURFACE TENSION

THE surface of a liquid acts in many respects in a manner analogous to a stretched membrane. The well-known example of mercury resting on a clean wooden surface shows the effect to a marked degree. The mercury takes the form of a globule, as if it were surrounded by a membrane supporting it in this form.

The examination of a water drop slowly formed at the end of a glass tube or tap from which it emerges provides another example of this phenomenon. The water in this case accumulates, as though it were collected in an invisible membrane, until of a definite size, when it is detached as a spherical drop.

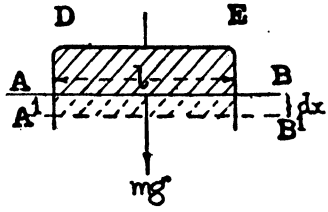


FIG. 64

These effects are due to forces existing in the surface of separation of the liquid from the air and the other media in contact with it. The effect is generally known as *surface tension.* This term may be defined in two ways, depending on the point of view taken.

If we imagine the surface of the liqiud to be cut by a plane, there is a definite force per cm. acting on the line of intersection, at right angles to its length and parallel to the surface. This force, expressed as dynes per cm., is defined as the surface tension of the liquid. The value of this force depends on the liquid and the surrounding medium, but, unless otherwise stated, will be taken throughout this book to represent the force per cm. when air is the medium.

For example, consider a frame ADEB, fig. 64, of width l cms., across which a film of liquid, whose surface tension is T dynes per cm., is stretched.

If the film terminates at the lower end on a light rod AB,

since the liquid film has two surfaces, it will exert an upward force of $2Tl$ dynes on the rod. If the rod has a mass of m grammes it will be in equilibrium when

$$2Tl = mg.$$

Now, suppose the rod be displaced a small distance δx, to A^1B^1, against the surface tension forces, the work done is $2Tl \cdot dx$, and the resulting increase in area of the surface is $2l \cdot \delta x$.

Thus, the potential energy acquired per sq. cm. of surface is

$$\frac{2Tl\delta x}{2l\delta x} = \text{T ergs per sq. cm.},$$

which may be taken as defining surface tension, i.e. the surface tension is the potential energy per sq. cm. of the surface. Further details of the two theoretical aspects of surface tension may be found in such a book as Poynting and Thompson's " Properties of Matter."

When a liquid is placed on a horizontal plane surface, the form it takes depends, for a given liquid, on the material of which the plane surface is made. Thus, if water is placed on a clean glass surface it spreads over it, whereas if the glass is greasy the water takes the form of globules.

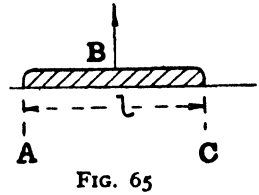


FIG. 65

The angle contained between the plane surface and the liquid surface is different in each case. If we measure this angle in the liquid we have a measure of the *angle of contact*. Thus, in fig. 75, showing a section of a mercury drop on a glass surface, θ is the angle of contact, whereas for a liquid like water which ' wets ' the glass the angle of contact is zero—the water spreads over the surface.

METHODS OF MEASURING THE SURFACE TENSION OF A LIQUID

(1) Wilhelmy's Method

To determine the approximate value of the surface tension of such a liquid as water, paraffin oil, or turpentine, the following method may be employed.

A clean wire, preferably platinum, is bent into the form shown in fig. 65, making three sides of a rectangle of breadth l cms.

It is then suspended from a beam of a balance by means of a *thin* wire, and counterpoised when the upper horizontal arm, B, of the frame is *almost* immersed in the liquid.

When a balance is obtained the frame is dipped under the surface of the liquid by lowering the beam of the balance, and then withdrawn again. A film of the liquid will be formed in the frame as seen in the shaded part of fig. 65.

Due to the downward surface tension pull on each side of the film of length l there will be an apparent increase in weight $= 2l\mathrm{T}$. This can be found experimentally by adding 'weights' to the other scale pan, until on raising the beam it remains horizontal.

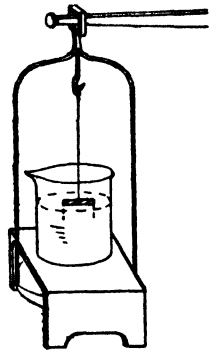


FIG. 66

To perform this approximate experiment clean the frame by holding it in a Bunsen flame until red hot.

Repeat the experiment as described above several times, and take the mean value, whence if m is the mean value of the added mass

$$2\mathrm{T}l = mg,$$

$$\mathrm{T} = \frac{mg}{2l} \text{ dynes per cm.}$$

Take care that the frame is the same height above the water surface when the balance is made, before and after immersing in water, to eliminate buoyancy errors. The surface tension effect on the two vertical limbs is eliminated, but the experiment may be regarded as one which gives an approximate value of T as described above.

(2) By Weighing Drops

The liquid whose surface tension is to be measured is allowed to form drops at the end of a narrow tube, C, fig. 67. If m is the mass of the drop, we have

$$mg = \mathrm{KT},$$

where K is a constant.

From a simple approximate investigation of the case we see that $mg = \pi r T$ as under.

When the drop is about to break away from the tube we will assume it has the cylindrical form shown in the firm lines, D. The broken line indicates one of the subsequent forms.

If r is the radius of the orifice, we have, at this level inside the drop, an excess of pressure equal to $\frac{T}{r}$ due to the cylindrical curvature of the liquid surface. This is equivalent to a downward *force* $\frac{T}{r} \cdot \pi r^2 = T\pi r$.

The weight of the drop being mg, the total downward force is

$$T\pi r + mg.$$

This is equal to the upward surface tension force over the circle of contact, i.e. $T \cdot 2\pi r$,

or $$T2\pi r = \pi T r + mg,$$

or $$T\pi r = mg.$$

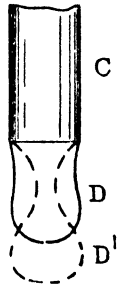


FIG. 67

This is deduced, assuming static conditions to hold in the actual case. To a closer approximation Lord Rayleigh showed that

$$mg = 3 \cdot 8rT$$

gives an expression of the relation between T and m.

In practice the uncertainty may be avoided by taking the expression $T = \frac{mg}{K}$ and finding the value of K for the particular tube used.

A tube is drawn out to a fine capillary of about ·5 mm. at the end. It is connected to a burette by means of a short length of rubber tubing. The burette and tube are of course first made thoroughly clean, as in all the surface tension determinations, by the method given on page 42. A liquid of known surface tension, say water, is placed in the burette and

the tap is opened to an extent which allows the drops to form at the rate of about one every second. A large number, say 100 or more, are collected in a weighed vessel and m per drop is obtained, hence, knowing T, K may be calculated. The same tube is now used for the liquid of unknown surface tension, say paraffin oil. Some of the liquid is run through the apparatus to remove all traces of water, and the burette is then filled with an uncontaminated sample of oil. The collecting and weighing is repeated, and knowing K, T is calculated.

Find by the above means the surface tension of alcohol, paraffin oil, benzine, etc.

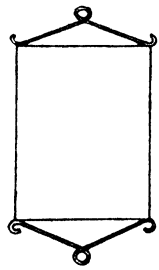


FIG 68

(3) Determination of the Value of the Surface Tension of a Solution in the Form of a Film

The value of the surface tension can be obtained for, say, a soap solution film in the following way.

Two pieces of copper wire, ABC and DEF, are bent as seen in figs. 68 and 69, so that AC = DF = about 4 or 5 cms. Through the loops at A, C, D, and F a length of cotton thread is arranged, so that when the whole arrangement is suspended at B the thread takes the form of a rectangle about 5 cms. by 10 cms. (fig. 68).

If now a film of soap solution be stretched across the thread, the latter will be pulled into the form shown in fig. 69, AD and CF taking the form of arcs of circles.

Let AC = DF = $2a$ cms., GH = $2b$ cms., AD = CF = $2h$ cms., and the mass of the lower copper wire, DEF = m grammes.

Let T be the value of the surface tension in dynes/cm. When stretched in this manner, there will be a tension in the thread equal to, say, f dynes.

Suppose α is the angle which is included between the horizontal and the threads as shown in fig. 69.

The film being two-sided, the vertical force upwards due to surface tension on the length $2a$ at the bottom of the film = $4a$T.

The resolved part of the tension f at D and F is $f \sin \alpha$ upwards, i.e. total upward force of $2f \sin \alpha$.

The downward forces are mg, the weight of the lower copper wire. We thus have for equilibrium :

$$mg = 2f \sin \alpha + 4a\mathrm{T}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Considering the equilibrium of half of the film as obtained by

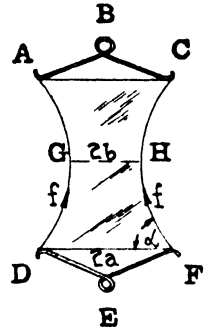


FIG. 69

a vertical dividing line through the mid-point of the film, we have the following equal and opposite forces acting :

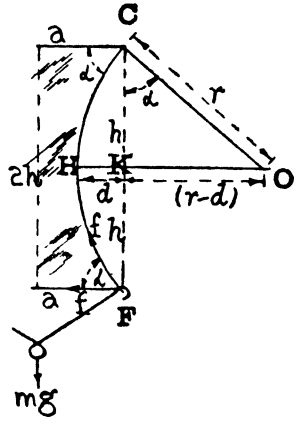


FIG. 70

(1) A force '$f \cos \alpha$' at the top and an equal one at the bottom
$= 2f \cos \alpha$.

(2) due to the two-sided film a surface tension effect
$= 4\mathrm{T}h$,

i.e. $$2f \cos \alpha = 4\mathrm{T}h. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2) we have $\tan \alpha = \dfrac{mg - 4a\mathrm{T}}{4\mathrm{T}h}$ $\ldots\ldots\ldots\ldots\ldots (3)$

From the geometry of the case (see fig. 70), if O is the centre of the circle of which CHF is arc, $\alpha = \angle OCK$.

So that $$\tan \alpha = \frac{KO}{CK},$$

and if $HK = d = a - b$,

$$\tan \alpha = \frac{r - d}{h}.$$

Eliminating r since $h^2 = (2r - d)d$,

$$r = \frac{h^2 + d^2}{2d},$$

We have $$\tan \alpha = \frac{h^2 - d^2}{2dh}$$

Equating this to the value given in (3) we get

$$\frac{mg - 4a\mathrm{T}}{4\mathrm{T}h} = \frac{h^2 - d^2}{2dh}$$

$$2d(mg) = 4\mathrm{T}(h^2 - d^2) + 4a\mathrm{T} \cdot 2d,$$

whence $$\mathrm{T} = \frac{mgd}{2(h^2 - d^2 + 2ad)},$$

Putting $d = (a - b)$

$$\mathrm{T} = \frac{mg}{2\left(a + b + \frac{h^2}{a - b}\right)} \quad \ldots\ldots\ldots\ldots (4)$$

Using the very simple apparatus described, a fair value of T can be obtained simply from a knowledge of a, b, h, and m.

A film is stretched across the string very easily by placing the frame horizontally in a flat dish containing the soap solution.

Care must be taken to avoid excess of soap solution spreading over the part of the apparatus other than the thread. None must be allowed to remain on the lower copper wire, for obvious reasons.

The measurement of the dimensions of the film can be made by means of ordinary dividers. The method does not justify the use of a travelling microscope. The measurements should be taken fairly rapidly so that they are all obtained before the condition of the film changes appreciably.

In an example $2a = 3{\cdot}5$ cms., $2b = 1{\cdot}2$ cms., $2h = 11{\cdot}7$ cms., $m = 1{\cdot}51$ grms., which gives T = 23 dynes/cm.

(4) By Measurement of the Rise of a Liquid in a Capillary Tube

If a clean, fine-bored capillary tube is depressed into a liquid which ' wets ' it, and is then clamped vertically, the lower end

of the tube being just below the surface of the liquid, it will be found that a column of the liquid remains in the tube, so that the surface in the latter is a height h cms. above the free surface of the liquid in the vessel which contains it.

Suppose r is the radius of the tube and ρ the density of the liquid.

The forces acting on the liquid in the tube are :

(1) *The weight of the liquid.* This is equal to (the volume of the liquid) $\times g\rho$.

Now the volume of the liquid is equal to $V = \pi r^2 h +$ (volume of the meniscus) for a uniform tube. If r is small the meniscus is practically hemispherical, hence

$$V = \pi r^2 h + \left\{(\pi r^2)\, r - \frac{2}{3}\pi r^3\right\}$$

$$= \pi r^2 \left(h + \frac{r}{3}\right),$$

i.e. the downward force is

$$\pi r^2 \left(h + \frac{r}{3}\right) g\rho. \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

(2) *The Upward Surface Tension Force*

The line of contact is the intersection of the glass wall and the liquid surface, i.e. a circle of radius r. If α is the angle of contact the upward force, from the definition of surface tension, is equal to

$$2\pi r \cdot T \cos \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

For equilibrium, the forces (5) and (6) are equal and opposite,

i.e. $$2\pi r T \cos \alpha = \pi r^2 \left(h + \frac{r}{3}\right) g\rho,$$

or $$T = \frac{g\rho r}{2 \cos \alpha} \left(h + \frac{r}{3}\right),$$

when the liquid wets the glass as in the case of water and clean glass $\alpha = 0$,

and $$T = \frac{g\rho r}{2} \left(h + \frac{r}{3}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

To find T, using a capillary tube, a glass tube is cleaned thoroughly. This may be done by using nitric acid (in which the tube is boiled) and caustic soda; the tube is washed in tap water and dried in alcohol and ether, or the glass tube is allowed to stand for several hours, overnight if possible, in a concentrated solution of sulphuric acid (one part) and potassium bichromate (one part). It is then washed in tap water and dried. It is not advisable to use distilled water.

The tube is then heated and drawn out to a capillary. A length of uniform bore is chosen and clamped vertically in a vessel which is brimful of the liquid whose surface is just above the top of the containing vessel, as shown in fig. 71. Care is taken to avoid touching the tube or the liquid in this adjustment, for even small traces of grease cause a large variation in the value of the surface tension.

The tube is viewed by means of a travelling microscope, provided with a vertical traverse, the lower end of the meniscus is focussed and the vernier reading of the microscope noted. The free surface of the liquid in the containing vessel is next focussed. If the liquid surface is just above the top of the vessel, this level may be viewed very readily, and from the vernier reading on the microscope scale in this position the value of h may be obtained.

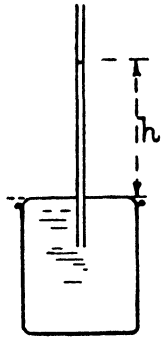


FIG. 71

To find r, the tube is broken at the point at which the meniscus rested, and viewed horizontally by the microscope.

By arranging the cross-hairs in the eyepiece to be tangential in turn to the two ends of a diameter, the internal radius may be measured on the vernier attached to the traverse.

Alternatively, a weighed amount of mercury (Density, D = 13·6) may be introduced into the tube and its length observed by means of the microscope, when the tube is horizontally on the bed of the microscope.

Hence r, the mean radius of the tube, may be found if l is the length occupied by the mercury of mass m, for

$$(\pi r^2) l D = m.$$

The experiment is repeated, using tubes of various diameter, and a mean value of T obtained.

An alternative method of measuring h, which also overcomes the difficulty of viewing the liquid through the glass beaker, is one which makes use of a pin bent twice at right angles so

that the point is displaced about one centimetre from its original position. The pin is attached to the capillary tube by means of a rubber band (see fig. 72). The point is in this way well removed from the curved surface of the liquid round the tube itself. It is adjusted to coincide with the free surface of the liquid.

The image of the upper end of the pin is brought into coincidence with the cross-hairs of a microscope, and the vernier reading on the vertical traverse is noted. The meniscus is next viewed, and the distance between it and the pinhead determined by the subtraction of the vernier readings. A subsequent measurement, by means of the microscope, of the vertical distance between the pin point and head enables the value of h to be obtained.

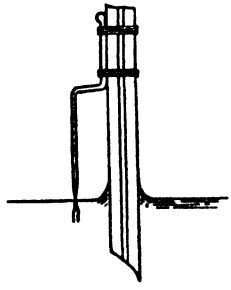


FIG. 72

(5) Surface Tension Determination from Measurements of Bubbles

This method is suitable for measurements of surface tension of soap solution.

Inside any curved film in equilibrium there is an excess of pressure over the outside, by an amount which depends on T, the surface tension, and R, the radius of curvature.*

Consider a bubble with excess pressure p inside: take a section through the centre; then there is an equivalent force over this section $= p \times$ (area) $= p\pi R^2$.

The opposite surface tension forces, since there are two surfaces to the bubble, is $2T \times$ (circumference of section)

$$= 2T \cdot 2\pi R.$$

* It can be readily shown that inside a cylindrical *film* (which has two surfaces) this pressure excess $= \frac{2T}{R}$.

If now the film has an equal curvature in the other direction at right angles—making spherical bubbles, there is a further pressure excess of $\frac{2T}{R}$, making a total $\frac{4T}{R} = p$ the excess of pressure.

For equilibrium these are equal and opposite,

i.e. $$4\pi TR = \pi R^2 p$$

$$p = \frac{4T}{R}$$

A suitable form of apparatus with which to obtain a measurement of p and R is seen in fig. 73.

This consists of a fairly wide U-tube, containing water, sealed to a T-piece, T. One arm of the T-piece is bent at right angles and terminates at B, where AB is parallel to the U-tube limbs. To the other arm at C is attached a piece of rubber tube, R, having a glass rod, P, which just fits it, and which can act as a piston.

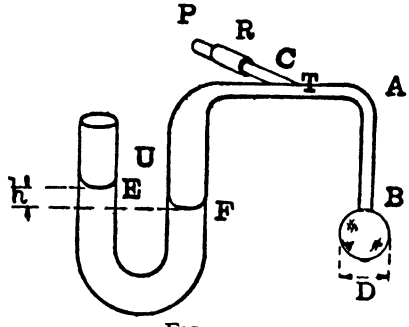


FIG. 73

The piston P is withdrawn and the end B immersed in the soap solution; then P is advanced slightly, causing a bubble to form at B. The excess of soap solution is drained off this bubble by touching any excess with a pencil, or the side of the dish which contains the solution. The rod is then further advanced, so that a bubble of, say, 1 to 2 cms. diameter is blown. It will be noticed that the water in the U-tube takes up a position as in the diagram indicating a pressure inside the bubble in excess of the atmospheric pressure.

This difference in level may be measured, using a microscope with horizontal and vertical traverse.

Having read this, the microscope is moved until the bubble is focussed, first the image of one side, and then the other being brought in coincidence with the cross-hair. The difference in reading of the vernier of the horizontal traverse giving the diameter D, of the bubble,

$p = hg$ dynes per sq. cm., for a *water* manometer.

$$R = \frac{D}{2}.$$

Hence $$T = \frac{hgD}{8}.$$

(6) Quincke's Method

This method is most readily followed for the determination of the surface tension of mercury. A large flat drop is formed on a horizontal platform, and from its dimension, as measured by means of a travelling microscope, the value of T, the surface tension, and θ, the angle of contact of mercury on the platform may be calculated.

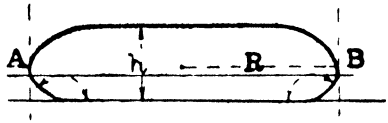


FIG. 74

We will assume that the radius of the flat drop, R, fig. 74, is large, so that the drop is truly flat, i.e. the pressure at a point just above and just below the upper surface is the same.

Consider a section as in fig. 75, obtained by cutting it by a vertical plane passing through the centre, LMRN, and two parallel vertical planes, AINL and ESRM normal to LMRN. Let ACDE be the horizontal plane of maximum area in the drop, so that a tangent plane at AE is vertical.

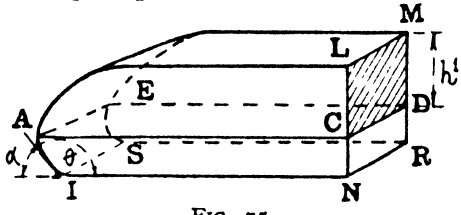


FIG. 75

By considering the forces on the upper portion, AECDLM, which is in equilibrium, a value of T may be obtained in terms of the dimensions.

The horizontal forces acting on this portion of the drop due to the remainder of the drop are

(α) T·LM, left to right,

(β) The hydrostatic *pressure* over the surface LMDC, from right to left varies from 0 at the upper surface to $g\rho h^1$ at CD, and is of a mean value $\frac{g\rho h^1}{2}$, i.e. total *force* is $\frac{g\rho h^1}{2}$ × area LMDC, where ρ is the density of mercury,

i.e. equating (α) and (β)

$$T \cdot LM = \frac{g\rho h^1}{2} \cdot LMh^1,$$

or

$$T = \frac{g\rho h^{1^2}}{2} \cdot \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Considering now the equilibrium of the whole slab, we have in addition to the corresponding terms, $T \cdot LM$ and $\frac{g\rho h^2}{2}$, where h is now the total height (fig. 74), a term $T \cdot IS \cdot \cos\alpha = -T \cdot LM \cos\theta$ (fig. 75), from left to right.

or
$$T \cdot LM(1 - \cos\theta) = \frac{g\rho h^2}{2} LM$$

$$2T \sin^2\left(\frac{\theta}{2}\right) = \frac{g\rho h^2}{2},$$

or, substituting the value of T from (8),

$$\sin\frac{\theta}{2} = \frac{h}{\sqrt{2}\,h^1} \qquad \ldots\ldots\ldots\ldots(9)$$

The *experimental details* necessary to carry out such a determination are as follows : A *clean* glass slab, provided with three levelling screws, is first of all arranged horizontally, by means of a spirit-level. Mercury is placed on its surface in the form of a circular drop (fig. 76).

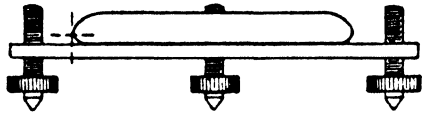


FIG. 76

N.B.—The mercury should be as free as possible from impurities. To ensure this it should, if possible, be submitted to one of the methods of purification described on page 532.

To obtain the values of h^1 and h, use is made of a travelling microscope, which is provided with a cross-hair. The cross-hairs are arranged so that one is vertical in the eyepiece ; this can be done by viewing a thin wire plumb line. The edge of the drop is then focussed. When the vertical cross-wire is tangential to the image of the side of the drop and the intersection of the cross-hairs is at the point of contact, such as A, figs. 74 and 76, the position on the vernier of the microscope is noted. Then by making use of the vertical movement the upper and lower surfaces of the drop can be focussed and the corresponding vernier reading will enable the values of h^1 and h to be ascertained.

Note

For liquids which wet the surface of the glass a simple modification enables the method to be utilized.

A concave lens of over one metre radius of curvature is supported concave side downwards, on three legs inside a glass box, having one plane glass side. The box is filled with the liquid, and an air bubble is blown under the concave surface, using apparatus similar to that shown on page 141. A narrow bent tube is connected to R in place of the U-tube, etc. shown in fig. 79. The slight concave surface enables this air bubble to be blown without much trouble. The section of such a submerged bubble would be similar to that shown in fig. 74, when inverted. Then, if h^1 is the distance between the lower surface of the large air bubble and the plane AB, and if h is the total thickness of the bubble as measured from the lower surface to the plane of contact with the lens, all measured through the plane glass window by means of a microscope, equations (8) and (9) will give T and θ for the liquid in the box.

(7) Rayleigh's Method

This method depends on a measurement of the wave length of ripples formed on the surface of the liquid whose surface tension is to be determined.

The velocity v of a harmonic disturbance on the surface of any liquid is given by*

$$v = \sqrt{\frac{\lambda}{2\pi}g + \frac{2\pi T}{\lambda\rho}}, \quad \ldots\ldots\ldots\ldots (10)$$

when λ is the length of the wave, T the surface tension, ρ the density of the liquid, that is

$$v = \sqrt{\frac{\lambda}{2\pi}\left(g + \frac{4\pi^2}{\lambda^2\rho}\cdot T\right)} \quad \ldots\ldots\ldots\ldots (10a)$$

The surface tension is therefore seen to increase the effective value of g. For large waves the term $\frac{4\pi^2}{\lambda^2\rho}\cdot T$ can be neglected, and the velocity of propagation is $\frac{\lambda g}{2\pi}$.

For waves of less than 1·5 cm. wave length, the value of the term involving T becomes more important, and when λ is sufficiently small the velocity becomes more nearly equal to

$$v = \sqrt{\frac{2\pi T}{\lambda\rho}},$$

i.e. the first term in equation (10) becomes negligible cf. the second when λ becomes very small.

Equation (10) shows that when $\lambda = 0$, $v = \infty$, and when

* See Poynting & Thompson's "Properties of Matter."

$\lambda = \infty$, $v = \infty$, between these values of λ there is a minimum value of v, corresponding to a value of λ, which is obtained when

$$\frac{\lambda}{2\pi} g = \frac{2\pi T}{\lambda \rho},$$

$$\lambda = 2\pi \sqrt{\frac{T}{g\rho}} \; ; \quad \ldots\ldots\ldots\ldots\ldots (11)$$

this minimum velocity is therefore

$$v_m = \left(\frac{Tg}{\rho}\right)^{\frac{1}{4}} \sqrt{2}$$

For water $\lambda_m = 1{\cdot}7$ cms. from equation (11), for $T = 75$, $\rho = 1$, $g = 981$, and hence

$$v_m = 23 \text{ cms. per sec.}$$

Waves having a smaller wave length than this critical value which corresponds to the minimum velocity are called ripples, and the more important term is the one involving T.

Such ripples cannot be viewed and λ measured directly, but the following method, first used by Lord Rayleigh, may be employed with success.

The liquid is placed in a large flat dish, say, a large porcelain developing dish. As in all surface tension experiments, the dish is cleaned thoroughly before introducing the liquid.

The ripples are made by having a tuning fork electrically maintained, arranged at one end of the dish as in fig. 77. Attached to one prong of the fork and dipping in the liquid is a thin sheet of aluminium foil, D. If the frequency of the fork is n, and λ the wave length of the ripples we have

$$v = n\lambda.$$

But due to the velocity, the ripples cannot be directly observed and measured. The phase change being rapid, a general illumination results.

If these ripples are viewed by intermittent light, the frequency of the flashes of illumination being the same as the source of the ripples, then they appear stationary and may be measured (Cf. stroboscope, p. 404) Alternatively, if the observer has intermittent views of the surface of the liquid, such views being of same number per second as the vibration number of the tuning fork which causes the ripples, between each view the ripples will have moved forward a distance equal to a wave length, and if illuminated by a constant source of light they will appear stationary.

To obtain such intermittent glimpses of the surface, the latter is observed through the prongs of a second tuning fork of the same frequency, and maintained in vibration by the same circuit Two thin pieces of aluminium foil, A and B, are attached to the

prongs of the second fork, so that direct vision is impossible when the fork is at rest, but is obtained when the prongs of the fork are at the position of extreme separation. Thus, for each complete vibration of the fork a view is obtained of the surface, and an apparently stationary train of waves is seen when viewed in this way. The wave length can be obtained by direct measurement of the longest possible number, m say, by dividers adjusted over the surface.

When performing this experiment with water, the measurement will be found to be somewhat complicated due to shadows cast by the water, and the exact setting of the dividers over the surface for the stretch of m waves will not be as easy as in the case of a more opaque liquid.

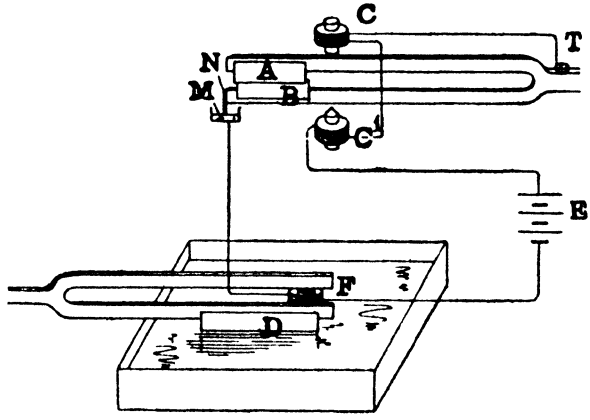


FIG. 77

A method which has been found to give satisfactory results is to suspend an incandescent bulb about two metres above the surface of the water. This casts a series of shadows on the bottom of the white porcelain dish. The dividers are arranged also to cast a shadow and adjusted so that the two ends of the dividers' shadow coincides with the corresponding parts of the first and last of the m waves. In such a case the magnification of the shadow of the waves due to the obliquities of the incident beam is compensated by a corresponding distortion of the shadow of the ripples, and the kind of thing shown in fig. 78 results: the length of, say, ten or more ripples may be obtained, whence λ.

Now, if n in the frequency of the fork,

$$v = n\lambda = \sqrt{\frac{\lambda}{2\pi}\left(g + \frac{4\pi^2}{\lambda^2\rho}T\right)},$$

or

$$T = \frac{\lambda^2\rho}{4\pi^2}(2n^2\lambda\pi - g).$$

A suitable experiment by such a method is to find the variation of surface tension of a salt solution with concentration.

Use first of all pure water. Make sure that the aluminium plate is clean, and that no soft wax, by which it is fastened to the tuning fork, adheres to it, and so contaminates the water.

Measure the value of λ and calculate T, n being known.

A suitable frequency for the two tuning forks is about 60.

Then repeat the experiment with a sodium chloride solution having $\frac{1}{2}$, 1, $1\frac{1}{2}$, to 5 grm. molecules per litre, and plot a graph showing the increase in the value of T with concentration of the solution.

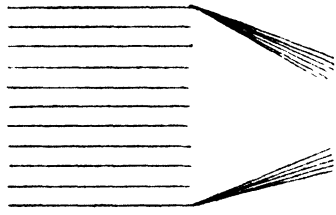


FIG. 78

(8) Jaeger's Method

In this method air bubbles are formed in the liquid under investigation. The bubbles in this case have one surface only, and therefore the excess of the internal pressure over the external assuming a hemispherical form is

$$p = \frac{2T}{R}.$$

The value of R is fixed, and equal to the radius of the orifice from which the bubbles emerge (approximately).

The following apparatus, as seen in fig. 79, is used.

A thin glass tube, C, drawn out to a fine capillary of less than ·5 mm. (·2 to ·5 mm. is satisfactory), and is fixed vertically in the liquid. It is connected either rigidly, as in the diagram, or by means of a piece of india-rubber tubing, to glass tube, AB, which terminates in another rubber tube, R, which is clamped in a screw pinch cock, T.

Sealed into AB is a glass T-joint which leads to a manometer U.

Air is sent in at D at a steady rate, and consequently, bubbles are formed in the liquid. To ensure that they are always formed at the same depth, h cms., below the surface of the liquid, a scratch is made on C, and this is adjusted to coincidence with the liquid surface.

When the bubble is of the same diameter as the orifice at C,

it becomes unstable and breaks away. During the formation of the bubble the manometer rises and the maximum difference in heights between the two columns, H cms., is observed for many bubbles, the pressure of the incoming air being constant.

When a good agreement for H for a series of bubbles is obtained, the tube C is taken out and the value of R determined by viewing the end with a microscope. If the capillary is very small, a microscope carrying a small scale in the eyepiece is used, and the value of the scale divisions in the eyepiece scale determined by comparing with the image of a small mm. scale viewed by the microscope.

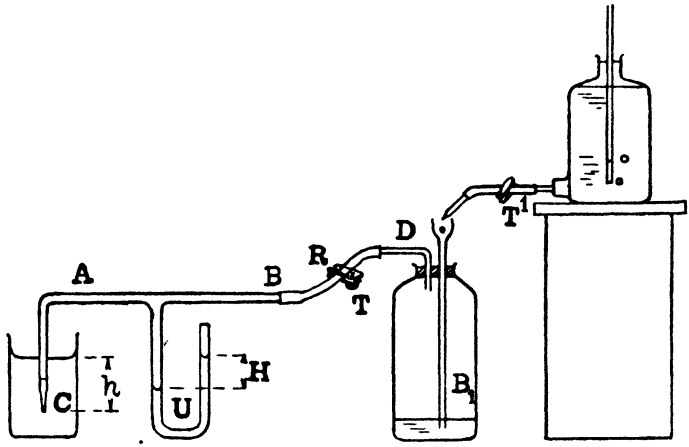


FIG. 79

The tube is rotated and other values of the 'diameter' taken. These measurements should be taken before commencing the experiment to ensure a regular tube.

A good method of producing the stream of air is to allow water to enter, one drop at several seconds interval, into a Winchester, closed at the top by a cork, which is waxed into the neck of the bottle and provided with an entrance and exit tube. The types shown in figs. 79 and 80 serve very well: the air is trapped inside the bottle by water in the bottom of the bottle in the form shown in 79, and in the U-tube entrance in the type illustrated in fig. 80.

In this way the pressure gradually rises until of sufficient magnitude to form a bubble of maximum diameter. This detaches and an interval then occurs, during which the pressure again rises, as is seen in the manometer U, and the whole process is repeated.

By *slowly* increasing the pressure, its maximum value corresponding to H cms. difference in the manometer may be easily

measured either directly on a scale or by means of a travelling microscope.

If only one large bottle is available, air is compressed into it and the outflow tube of rubber connected to B. By varying the compression on this tube by means of a screw clip the conditions described above may be obtained.

Let H be the maximum pressure difference established in the manometer, and h the depth at which the orifice is placed below the free surface of the liquid, then the pressure of the air inside the bubble, when the latter is about to leave the orifice, is $\pi + g\sigma H$, where π is the atmospheric pressure and σ the density of the liquid in the manometer.

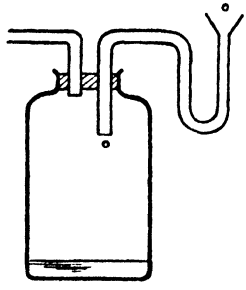


FIG. 80

The pressure outside the surface of the bubble is $\pi + g\rho h$, where ρ is the density of the liquid under investigation. Then, p being the excess of pressure inside the bubble over that on the outside

$$p = (\pi + g\sigma H) - (\pi + g\rho h) = \frac{2T}{R},$$

or* $$T = \frac{Rg}{2}(\sigma H - \rho h) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The values of R, H, and h, are obtained as already described, and T calculated.

The method is a good one for obtaining relative values of surface tension. As an example, find the variation of surface tension with concentration for a salt solution containing from 0 to 5 gramme molecules per litre.

* The result expressed in equation (12) above is developed assuming the bubble formed is hemispherical and of the radius of the tube.

Ferguson ("Phil. Mag.," No. 28, 1914, page 128 *et seq.*) has deduced an expression for T without making this assumption and arrives at the result:

$$T = g\,B + \left\{\frac{R^3\sqrt{3}}{12\sqrt{B}}\right\},$$

where $$B = \frac{R}{2}\left\{\sigma H - \rho\left(h + \frac{2R}{3}\right)\right\}.$$

By altering the temperature of, say, water in the beaker, the change of surface tension with temperature could be obtained.

(9) Capillary Tube Method (Sentis).

A capillary tube is drawn out to about ·5 mm. bore as in Jaeger's method. It is immersed in the liquid under investigation, and then withdrawn and clamped vertically. Some of the liquid will emerge at the lower end and form a drop as shown in fig. 81 (1), so that the distance from A, the meniscus in the tube, to B, the lowest point of the drop, is h_1 cms., and MN is $2r$.

If now the lower end of the tube is surrounded by a vessel, C, containing the liquid, the column will fall in general, but the meniscus may be brought to the original level by raising C until the free surface of the liquid in the beaker is h_2 cms. below the meniscus, fig. 81 (2).

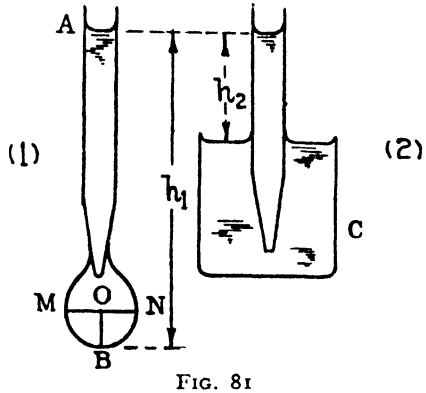


FIG. 81

From a knowledge of h_1, h_2, r, and ρ, the density of the liquid, T, may be calculated from the formula

$$T = \frac{g\rho}{2}\left\{r(h_1 - h_2) - \frac{r^2}{3}\right\}.$$

To establish this formula we may assume that the portion of the drop, shown in section as MONB, is hemispherical. This approximation is a safe one when the radius of the capillary tube is small, and r small compared with h_1.

Consider the forces acting below the horizontal plane of maximum area shown in section as MN.

The length OB $= r$, and hence the column from MN to the meniscus is $(h_1 - r)$; of this a length, h_2, is supported by the upper surface tension forces as shown in fig. 81 (2), and hence the pressure at MN due to the liquid column is $g\rho(h_1 - h_2 - r)$, which contributes a downward force $g\rho(h_1 - h_2 - r)\pi r^2$. The weight of the hemisphere $\frac{2}{3}\pi r^3 g\rho$ also acts downwards.

The surface tension acts vertically in the circle of section of the drop and plane MN, and has a value $T \cdot 2\pi r$; hence

$$T2\pi r = g\rho(h_1 - h_2 - r)\pi r^2 + \frac{2}{3}\pi r^3 g\rho,$$

or

$$T = \frac{g\rho}{2}\left\{(h_1 - h_2)r - \frac{r^2}{3}\right\}.$$

To make a determination of T for a liquid, the cleaned capillary about 25 cms. long is drawn out to a radius of about ·5 mm. at the one end, and is almost entirely submerged in the liquid, so that the latter fills the greater part of the bore. It is slowly withdrawn and is clamped vertically. By means of a travelling microscope, the length MN $= 2r$ is measured and the microscope is then focussed on the meniscus. A small glass beaker is raised under the tube, until the liquid in the beaker just touches the drop. The reading of the micrometer screw, which raises the platform carrying the beaker, is noted. The beaker is then further raised until the meniscus, as viewed by the fixed microscope, again acquires its original level. The micrometer screw reading is again noted. The difference between these two readings is equal to $(h_1 - h_2)$.

ρ is determined in the usual manner, whence T may be calculated from the formula deduced above.

If the form of adjustable table with micrometer or vernier attachment is not available, some simple convenient method may be devised for the measurement of $(h_1 - h_2)$. For example, two microscopes may be used. With the first the value of MN is observed, and then the point, B, is viewed and its image brought into coincidence with the cross-hairs; the second microscope is adjusted until, viewing the tube conveniently at right angles to the first, the image of the meniscus is in coincidence with the cross-hairs in the eyepiece.

The beaker, C, is then introduced and adjusted until the meniscus is again as before, producing an image in coincidence with the cross-hairs of the second stationary microscope. A pin is adjusted to coincidence with the free surface of the liquid in the beaker which is then removed. The first microscope is then moved a distance which is measured on the vernier scale attachment, until an image of the point of the pin is in coincidence with the cross-hairs, the vertical distance moved by the microscope is $(h_1 - h_2)$.

Care is, of course, taken that the capillary tube does not move during the experiment.

(10) Anderson and Bowen's Method

A method by which to determine the value of the surface tension of a liquid, and the angle of contact with glass was described in the " Philosophical Magazine," April, 1916. Another method not so readily adaptable to general laboratory use is seen in the same magazine, February, 1916.

A small rectangular sheet of thin cover glass is cleaned (by standing it in concentrated sulphuric acid and potassium bichromate, etc. as described previously) and dipped into the liquid whose surface tension is to be measured. It is withdrawn and clamped vertically. The liquid takes up the form shown in the diagram, fig. 82.*

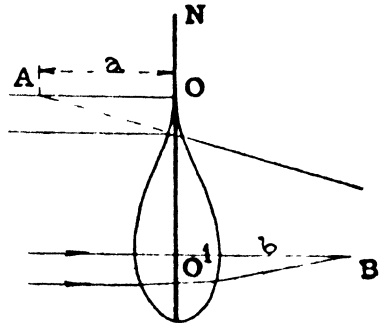


FIG. 82

The drop has two curvatures, making the equivalent of a cylindrical lens, concave at the upper half and convex below, the centres being at O and O^1.

The upper limit of the drop may be at O or any point, N, above. The drop is tangential to the glass plate at O.

If A is the focal point of the concave lens, OA the axis, f_1 the focal length, B the focal point of the convex lens, and O^1B the axis of the lens of focal length, f_2,

OO^1 = h cms.,

r_1 = the radius of curvature of the concave surfaces (assumed symmetrical and equal),

r_2 = the radius of curvature of the convex surfaces,

μ = the refractive index of the liquid,

ρ = the density of the liquid,

p_1 = the pressure in the liquid at O,

p_2 = the pressure in the liquid at O^1,

Π = atmospheric pressure,

T = surface tension of the liquid.

* It was established by the original experiment that for water, glycerine, olive oil, and turpentine, the angle of contact is zero, and hence the form of fig. 82 represents the section of the drop.

We have, as the lens is a thin one, using the lens formula $\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right)$, (page 278.)

$$\frac{1}{f_1} = (\mu - 1)\frac{2}{r_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$\frac{1}{f_2} = (\mu - 1)\frac{2}{r_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Since the pressure inside a cylindrical surface is greater than the pressure outside by $p = T\left(\frac{1}{R_1}\right)$, (see Poynting and Thompson's "Properties of Matter"), and here we have a cylinder of radius r, i.e. $R_1 = r$

$$p_1 = \Pi - \frac{T}{r_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

$$p_2 = \Pi + \frac{T}{r_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Now $p_2 - p_1 = g\rho h,$

also $p_2 - p_1 = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$ by (15) and (16) above.

Hence $T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = g\rho h.$

But by (13) and (14) above

$$\frac{1}{r_1} = \frac{1}{2f_1(\mu - 1)} \; ; \; \frac{1}{r_2} = \frac{1}{2f_2(\mu - 1)} \; ;$$

i.e, $$\frac{T}{2(\mu - 1)}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) = g\rho h,$$

or $$T = \frac{2g\rho h(\mu - 1)f_1 f_2}{(f_1 + f_2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

If one side only of the glass sheet is wet, using the same notation we have

$$T = \frac{g\rho h(\mu - 1)f_1 f_2}{(f_1 + f_2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

The apparatus used to obtain T, is a collimator illuminated by a sodium flame, and adjusted to give a parallel beam. The light passes through the cover glass and the liquid lens, and is viewed by a low-power microscope provided with a vertical traverse, and a *traverse parallel to the axis of the microscope.*

The usual type of travelling microsope will need a little modification to make this latter condition possible.

If a parallel beam of light be sent from the collimator from

left to right, and normal to the plate, a virtual image of the horizontal slit will be formed at A by the upper half of the liquid lens. The distance OA may be measured by using the low-power microscope, arranged with its axis parallel to the direction of the incident beam. The microscope is first focussed on the image at A, and then moved backwards a measured distance until the glass plate is in focus. The distance moved being $= OA = f_1$. In the same way OB and f_2 may be measured by noting the difference in reading of the microscope when the glass sheet is focussed and then when the image of the slit is coincident with the cross-hairs.

It will be found most satisfactory to use one side of the cover glass only, i.e., dry the other side before making the observation, and allow the incident beam to fall on the dry side.

As the incident beam is parallel, it will be found, of course, that the observing microscope must be moved in a vertical direction to enable a focus of first A and then B to be made. This distance, h, is noted.

The refractive index and the density may be obtained from tables, or by one of the many methods available.

Thus, having measured f_1, f_2, and h, knowing μ and ρ, T may be calculated for the liquid used.

Measure in this manner the surface tension of water and turpentine.

The Variation of the Surface Tension of a Liquid with Temperature

The variation of surface tension with temperature may be obtained by Jaeger's method, which enables a good comparison of the relative values of the surface tension at different temperatures to be made.

The details of the experiment are as described on page 140. The bubbles in this case are formed in the liquid at different steady temperatures, and H is determined for each.

A large beaker is filled with the liquid, say water, and heated to about 90°C., and then allowed to cool. The value of T being obtained every 10°C. The liquid is well stirred before each observation, and if a large volume is taken will remain sensibly at the same temperature throughout the observation.

A curve is plotted, showing the decrease of T with temperature.

Capillary Tube Method

The decrease of the value of the surface tension may also be investigated by a capillary tube method. Either form of apparatus shown in fig. 83 or fig. 84 may be employed. One form of apparatus is filled with the liquid and immersed in a water bath whose temperature can be regulated either by a

thermostat or by manipulation of the Bunsen burner and a stirrer. The bath is raised to boiling point and allowed to cool, so that at about 90–95°C. the whole of the apparatus and contents are at the temperature of the bath. The difference in level

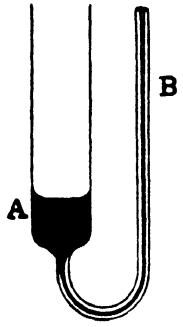


FIG. 83

between the two surfaces, A and B, is measured as quickly as possible with a travelling microscope. This process is repeated at different temperatures, say, every 10°C., and from a knowledge of this difference in level, h and the radius of the tube as measured

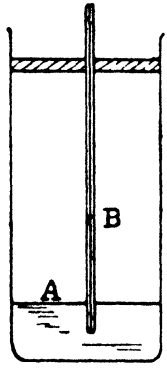


FIG. 84

by the method given on page 132, the value of T at each temperature may be calculated and a graph representing this relation plotted.

The value of the density (ρ) is obtained, for each temperature, experimentally, or from tables, and is used in the evaluation of T by the above methods.

The essential to success in this and every other surface tension experiment is that the glass, etc., is clean.

CHAPTER V

VISCOSITY

WHEN adjacent layers of a fluid move with a relative velocity, forces known as viscous forces are brought into play, tending to reduce this relative movement.

If we consider a fluid whose upper layer is moving with a velocity v in a fixed direction, the state of affairs shown in fig. 85 will be reached, where intermediate layers, between the upper layer AB, which has a velocity v, and the lower layer CD, which is at rest, have a velocity shown by the arrows.

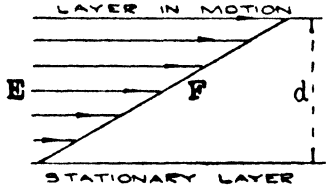


FIG. 85

The force, F, acting on any area in a plane at right angles to the diagram, and parallel to EF, is proportional to the area A; and to the velocity gradient, in the case taken $\frac{v}{d}$, i.e. at constant temperature.

$$F \propto A \times \text{(vel. gradient)}.$$

Taking the normal to EF, in the plane of the diagram, as a y axis we have

$$F = \eta A \frac{dv}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where η is a constant for the liquid and is called the coefficient of viscosity.

In the case of a liquid flowing down a tube, we have the axial stream moving with a definite velocity and the layers in contact with the walls of the tube at rest and, provided that the pressure difference which is causing the flow is not too great, we should have in this case the regular type of motion already considered.

If the pressure exceeds a certain limit, the liquid no longer proceeds in this regular manner, i.e. no definite stream-line flow takes place. The result in this case is called *turbulent motion.*

We will assume, in the experiments that follow, that the pressure applied is below this critical pressure, and that the motion is therefore regular.

The Determination of the Coefficient of Viscosity for a Liquid by Observation of the Flow of the Liquid through a Tube

The value of the coefficient of viscosity of a liquid, such as water, may be obtained by measuring the quantity passing per second through a tube of uniform radius, when a definite pressure difference exists between the ends of the tube.

Consider, as in fig. 86, a section of such a tube, whose radius is R cms., and imagine a thin tube of liquid in it of radius, r, and thickness δr; the area of cross-section of such a tube is $2\pi r \cdot \delta r$

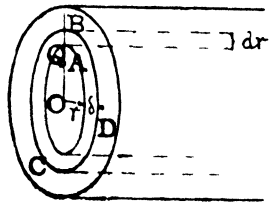


FIG. 86

If P be the pressure difference between the ends, then the force acting on the tube ABCD, due to this pressure, is $P \cdot 2\pi r \cdot \delta r$. Over the curved area of such a hollow cylinder there are viscous forces which are of a magnitude dependent on the value of the distance r from the axis of the tube.

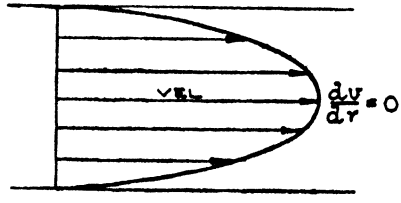


FIG. 87

Fig. 87 shows the form of distribution of velocity of the layers for various values of r; as shown later, equation (3), the curve of fig. 87 is a parabola, and hence $\frac{dv}{dr}$ is proportional to r, and therefore the viscous force, F, is also proportional to r by (1).

Let $\frac{dF}{dr}$ express the rate of change of this force with r, then

the value of the difference between its magnitudes at the two surfaces of the thin tube ABCD, since the thickness is δr, is $\frac{dF}{dr} \cdot \delta r$.

In steady flow we have, therefore, the equal and opposite forces, which may be expressed as :

$$-\left(\frac{dF}{dr} \cdot \delta r\right) = 2\pi r P \cdot \delta r,$$

$$-\frac{dF}{dr} = 2\pi r P \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now equation (1) gives

$$F = \eta \cdot A \cdot \frac{dv}{dr}.$$

The area A is the area over which F acts, i.e. is $2\pi r \cdot l$, l being the length of the tube. Equation (2) becomes

$$\frac{d}{dr}\left(-\eta \frac{dv}{dr} 2\pi r l\right) = 2\pi r P.$$

Integrating

$$-2\eta\pi r l \frac{dv}{dr} = 2\pi \frac{r^2}{2} \cdot P + C,$$

C being the constant of integration.

But when $r = 0, \frac{dv}{dr} = 0 \quad \therefore \quad C = 0,$

i.e.

$$-2\eta l \frac{dv}{dr} = rP.$$

Integrating once more

$$-\int 2\eta l dv = \int rP dr,$$

$$-2l\eta v = \frac{Pr^2}{2} + B, \qquad B \text{ a constant.}$$

When $r = R$, $v = 0$ $\therefore$ $B = -\frac{PR^2}{2}$

Hence $$v = \frac{P}{4l\eta}(R^2 - r^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In one second, down such a hollow tube, a column of liquid, v cms. long, and $2\pi r \cdot \delta r$ cross-section flows, i.e. volume passing per second $= \left\{\frac{P}{4l\eta}(R^2 - r^2)\right\} 2\pi r \cdot \delta r.$

Thus, for the whole tube of radius R, we shall have the sum of such expressions.

If Q is the total amount of liquid passing per second

$$Q = \int_0^R 2\pi \frac{P}{4l\eta} (R^2 r - r^3)\, \delta r$$

$$= \frac{1}{2} \cdot \frac{\pi}{\eta} \cdot \frac{P}{l} \cdot \left[\frac{R^2 r^2}{2} - \frac{r^4}{4} \right]_0^R$$

$$Q = \frac{\pi}{8} \frac{P}{l} \frac{R^4}{\eta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

A suitable form of apparatus to use in an experimental determination of η by this method is seen in fig. 88. The liquid, say water, is contained in a large bottle, B, standing a suitable distance above the level of the table. The water flows from

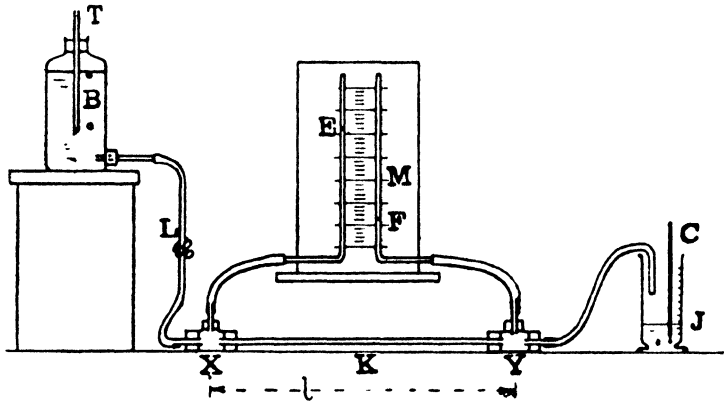


FIG 88

this reservoir to the union, X, thence through a capillary tube of known length to the union, Y, and so on, via a length of india-rubber tubing to a graduated jar, J, where the thermometer, C, measures the temperature of the emerging water.

From the unions, X and Y, two lengths of india-rubber tube make connection to the manometer M. The difference in the levels, E and F gives in cms. of water the value of the pressure difference between the ends of the experimental tube, K.

A pinch-cock L enables the flow of the liquid to be regulated.

In order to maintain a constant difference of pressure between the two ends, X and Y, whilst the water is flowing, the bottle B is closed by means of a tight-fitting india-rubber cork through which a glass tube passes, to a point well below the surface of the water. This end being open to the atmosphere allows the entrance of air bubbles as water flows through the tube. The lower end of the tube remains at atmospheric pressure, and so,

until the whole of the water above this point has passed through the tube, the manometer will remain with a uniform average difference of level.

The flow should be so arranged that the emergent water issues as a slow trickle or succession of drops, avoiding a rapid stream of water, which might cause the flow in the tube to become turbulent, in which case the formula which is developed, assuming a regular flow, will break down.

In adjusting the apparatus, the water passing is not collected. When everything is steady the tube is inserted into J, as seen in diagram, and the time is taken in seconds for a definite amount of water, say 500 cms., to pass. From this, Q is obtained.

If h is the difference in level in the water manometer, $P = g\rho h$, approximately, or as shown later, equation (11),

$$p = g\rho\left(h - \frac{2Q^2}{g\pi^2R^4}\right),$$

more exactly.

Before determining the remaining unknowns, l and R, several values of Q, corresponding to different values of P, should first be obtained, for we have

$$\eta = \frac{\pi}{8}\frac{R^4}{l}\cdot\frac{P}{Q}.$$

By varying P and Q, the mean value of $\frac{P}{Q}$ can be obtained, and this, *not* one value of P and corresponding Q, used for the computation of η, *provided that the temperature of the room remains constant* during the experiment.

The value of l may be obtained by direct measurement of K.

Now R, which occurs in the formula in the fourth power, must be obtained with as great accuracy as possible. A suitable method is the measurement of length and mass of a column of mercury in the dried tube as described on page 132.

Unless the liquid is passing through the tube with a very small velocity, equation (4) must be modified to allow for the kinetic energy imparted to the liquid. This reduces the effective pressure P.

To arrive at such a correction, consider the case of a liquid flowing through an irregular tube whose two ends are at different levels.

Consider two points, B and C, in the tube such that the cross-sections are A_1 and A_2, the velocity of the centre of the liquid, v_1 and v_2, and the pressure in the liquid, P_1 and P_2, whilst the distances of the centre at these places are L_1 and L_2 cms. below a fixed horizontal plane.

If the liquid moves a distance dx_1 at B, it will advance a distance dx_2 at C, such that

$$A_1 dx_1 = A_2 dx_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The work done on the liquid between B and C is therefore

$$P_1 A_1 dx_1 - P_2 A_2 dx_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

This work must be the equivalent of the energy gained in the liquid.

Gain of Kinetic Energy

The mass $(A_1 dx_1)\rho$ of liquid enters at B (ρ being the density of the liquid assumed constant). This has a K.E.,

$$\tfrac{1}{2} A_1 dx_1 \rho v_1^2.$$

Similarly, for the mass of liquid leaving, and since condition (5) holds, the gain of kinetic energy is

$$A_1 \delta x_1 (\tfrac{1}{2}\rho v_2^2 - \tfrac{1}{2}\rho v_1^2) \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

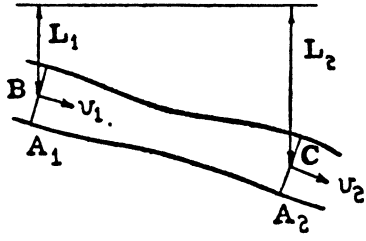


FIG. 89

Potential Energy

The P.E. of the mass of liquid at B is $gL_1(A_1 dx_1)\rho$, and at C is $gL_2 A_2 \delta x_2 \rho$.

For a *gain* of P.E., C must be lighter than B, $L_2 < L_1$, i.e. the gain of P.E. is

$$A_1 dx_1 (\rho g L_1 - \rho g L_2) \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Equating (6) to the sum of (7) and (8) we have

$$A_1 dx_1 (P_1 - P_2) = A_1 dx_1 (\tfrac{1}{2}\rho v_2^2 - \tfrac{1}{2}\rho v_1^2) + A_1 dx (\rho g L_1 - \rho g L_2),$$

or using (5) and re-arranging terms

$$P_1 + \tfrac{1}{2}\rho v_1^2 - g\rho L_1 = P_2 + \tfrac{1}{2}\rho v_2^2 - g\rho L_2,$$

i.e.
$$P + \tfrac{1}{2}\rho v^2 - g\rho L = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Applying to the case shown in fig. 90, where the liquid is contained in a wide tank, BC, and flows through a uniform tube, CD, with a constant maximum (central) velocity, v, we may find the effective pressure difference between the two ends of the tube using equation (9) above.

If Π is the atmospheric pressure we have, applying (9) to the point B and C, where C is just inside the tube and is at pressure p',

$$\Pi + \tfrac{1}{2}\rho v^2 - g\rho L_B = p' + \tfrac{1}{2}\rho v^2 - g\rho L_C$$

For such a wide cross-section as the tank we may assume the surface to be fixed. This may be further brought above by the device shown in fig. 88.

i.e. $$v_B = 0.$$

We will further use this surface as reference plane for measurements of L, i.e. $L_B = 0$, $L_C = L$ cms.

We therefore have

$$\Pi = p^1 + \tfrac{1}{2}\rho v^2 - g\rho L,$$

or $$p^1 = \Pi + g\rho L - \tfrac{1}{2}\rho v^2,$$

the pressure at the open end, D, being π, we have the pressure difference between the ends of the tube, CD, p, $= p^1 - \pi$,

i.e. $$p = g\rho L - \tfrac{1}{2}\rho v^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Thus, the effective ' head ' is reduced due to the kinetic energy gained by the liquid.

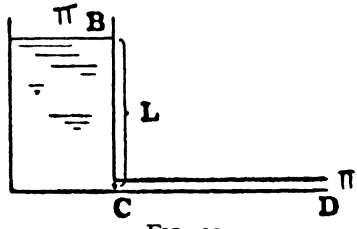


FIG. 90

It was seen (p. 151) that the velocity of the stream at a distance r from the axis of the tube, was

$$v = \frac{P}{2\eta l}\left(\frac{R^2 - r^2}{2}\right),$$

i.e. when $r = 0$, we obtain the value of v used in equation (10) above, i.e. the maximum velocity,

$$v = \frac{PR^2}{4\eta l}.$$

Now we saw (p. 152), approximately

$$Q = \frac{P}{l} \cdot \frac{\pi}{8} \cdot \frac{R^4}{\eta}$$

Hence $$v = \frac{2Q}{\pi R^2},$$

and the effective pressure, p, is given below,

$$p = g\rho L - \tfrac{1}{2}\rho\left(\frac{2Q}{\pi R^2}\right)^2$$

$$p = g\rho\left(L - \frac{2Q^2}{g\pi^2 R^4}\right) \quad \ldots\ldots\ldots\ldots\ldots (11)$$

An alternative apparatus for the determination of η is seen in fig. 91. The horizontal capillary, DE, is fixed in a cork which closes the lower end of the tube, CF, which has constrictions at A and B, on which are scratches on the glass. The whole vessel is filled with the liquid and the ground-glass stopper, S, placed in the neck above A. The volume between the scratches is calibrated, so that its capacity, V, is known. The stopper, S, is removed, and the time, t cms., is taken for this volume to flow

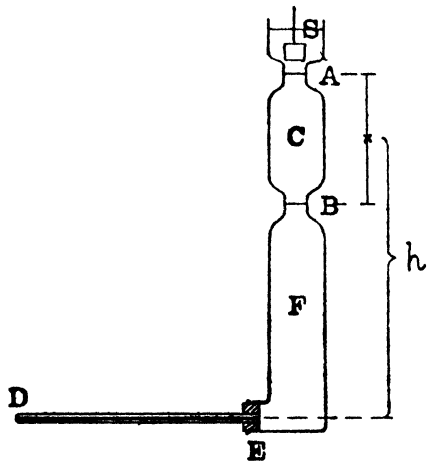


FIG. 91

through the tube, i.e. when the level reaches A a stop-clock is started, and stopped when the level reaches B, hence $Q = \frac{V}{t}$.

p, the pressure, is taken as due to the average height, h, of the liquid above the capillary tube level, i.e. $p = g\rho h$, or as shown on more precisely in equation (11),

$$p = g\rho\left(h - \frac{2Q}{g\pi^2 R^4}\right).$$

The other terms are measured as in the first form of apparatus described.

Determination of the Viscosity of a Liquid by the Coaxial Cylinder Method

The value of the coefficient of viscosity for a liquid such as glycerine may be obtained, using the apparatus shown diagrammatically in fig. 92. The liquid, say glycerine, is placed in the cylinder, AB, which may be rotated by hand or by a small motor. A belt driven by either means passes over the pulley, P, and the

rotation is imparted to the cylinder by the crown bevel wheels, W. The revolutions may be counted by the revolution counter, R. Hence, if a number of revolutions, n, be timed, the angular velocity, w, may be calculated.

Immersed in the glycerine is a second solid cylinder, CD, which is suspended on a phosphor-bronze suspension, which carries a mirror, M, and which is maintained central by the pivot S. Due to the viscous forces in the liquid, the inner cylinder will experience a couple, C, which turns it through an angle, θ, such that the torsional restoring couple just balances the turning moment due to the liquid.

If τ is the restoring couple per unit angular twist in the sup-

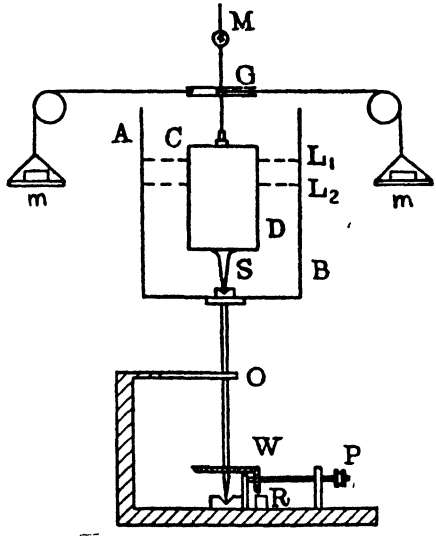


FIG. 92

porting wire, and θ be the constant deflection in radians, we have $\tau\theta$ = the restoring couple due to the torsion of the wire.

To obtain a value for the couple due to the viscous forces, we will first consider the co-axial cylindrical surfaces.

Let fig. 93 represent a section normal to the axis of these eylinders, of radius R_1 and R_2 as shown.

Take any thin cylindrical ring in the liquid having points, A and B, on the same radius. If the liquid rotated as a whole, there would be no velocity gradient—no relative motion. Thus, if AO = r and BO = $r + \delta r$, when A moves to A^1, for no relative motion, B, would move to C, where $AA^1 = rw$ and BC = $(r+\delta r)w$, w being the assumed constant angular velocity. Now, in the actual case, the outer cylinder moves with an angular velocity, Ω say, and the layer in contact with the inner cylinder is at

rest, i.e. the liquid does not have this constant angular velocity. Actually, the particle at B moves with some larger angular velocity $(w + \delta w)$, and B moves to B^1 where $BB^1 = (w + \delta w)(r + \delta r)$, thus having an excess over the no-relative motion velocity equal to $(w + \delta w)(r + \delta r) - w(r + \delta r)$, i.e. $= \delta w(r + \delta r)$ i.e. the velocity gradient is $\frac{\delta w(r + \delta r)}{\delta r}$, or since δr is negligible cf. r, the velocity gradient $\frac{dv}{dr} = r \cdot \frac{dw}{dr}$ in the limit.

Thus, considering the forces on the cylindrical shell, if l is the length of the inner cylinder from the surface of the liquid to

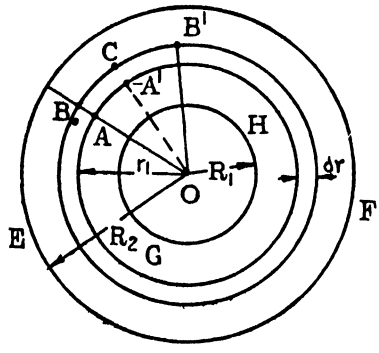


FIG. 93

the lower extremity, the area of the cylindrical shell is $(2\pi r)l$, and therefore from (1),

$$F = \eta(2\pi rl)\frac{dv}{dr},$$

where F is force due to the viscosity over the curved surface of such an imaginary shell. The moment of this about the common axis is

$$C = Fr = 2\pi r^2 l\eta r \frac{dw}{dr},$$

substituting the value for $\frac{dv}{dr}$.

Integrating we have

$$\int 2\pi l\eta dw = \int \frac{C dr}{r^3},$$

i.e.

$$2\pi l\eta w = -\frac{C}{2r^2} + B,$$

where B is a constant.

Now, when $r = R_1$, $w = 0$, $B = \frac{C}{2R_1^2}$,

$r = R_2$, $w = \Omega$,

i.e. $$2\pi l\eta\,\Omega = \frac{C}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_2^2}\right],$$

$$C = \frac{4\pi l\eta\,\Omega R_1^2 R_2^2}{R_2^2 - R_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Also we have

$$C = \tau\theta,$$

whence $$\eta = \frac{\tau(R_2^2 - R_1^2)}{4\pi l\,\Omega R_1^2 R_2^2}\,\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

In the actual case taken (12) must be modified to include the couple due to the viscous effect on the under surface of the inner cylinder. If this effect were truly due to the parallel circular plates, we might readily deduce an expression for it. But the stream-lines in the liquid will not be composed of two such regular patterns. The effect at the end of the cylinder, CD, will be a gradual transition from the one to the other. Some end correction is therefore needed. Let us assume that the couple exerted on the cylinder, CD, due to the under surface and the lower portion of the curved surface, and also the effect due to the support, S, which rests on AB, for an angular velocity, Ω, is $K\Omega$, then the total couple will be

$$C = \frac{4\pi\eta l' R_1^2 R_2^2}{(R_2^2 - R_1^2)}\,\Omega + K\Omega,$$

where l' is the length of the curved surface of the inner cylinder, measured from the liquid surface to the *lower limit of the regular stream-line between the cylinders;* the second term deals with the rest of the curved surface and the lower surface, etc., as stated.

If now we have the space between the cylinders filled to a level, such as L_1, we have, if l_1' is the corresponding length and θ_1 the deflection

$$\tau\theta_1 = \Omega\left[\frac{4\pi l_1'\eta R_1^2 R_2^2}{R_2^2 + R_1^2} + K\right] \quad \ldots\ldots\ldots(14)$$

For a second case, where the length of the cylinder immersed is l_2', corresponding to a level, L_2, in the figure, if θ_2 is the deflection

$$\tau\theta_2 = \Omega\left[\frac{4\pi l_2'\eta R_1^2 R_2^2}{R_2^2 - R_1^2} + K\right] \quad \ldots\ldots\ldots(15)$$

K may therefore be eliminated from (14) and (15) and, if the angular velocity, Ω, is the same for both determinations, we have by subtraction

$$\tau(\theta_1 - \theta_2) = \Omega\frac{4\pi\eta R_1^2 R_2^2}{R_2^2 - R_1^2}\left[l_1' - l_2'\right] \quad \ldots\ldots\ldots(16)$$

The length to be measured is therefore the difference between the two levels, L_1 and L_2.

Thus, by maintaining the speed of rotation constant, and observing the steady deflection corresponding to two levels of the liquid surfaces, knowing $L_1L_2 = l_1' - l_2'$, and the dimensions of the cylinders, η may be calculated in terms of τ, which itself may be evaluated by observation of the twist of the wire, when the liquid is not in the cylinders, and loads are applied at G, as indicated in the diagram.

For liquids whose viscosity is fairly large, the torsion control will not be sufficient. In this case the restoring couple is increased by adding masses, *m*, *m*, thereby increasing the restoring couple by *m* D, where D is the diameter of the wheel G.

In many cases it is advantageous to eliminate the torsional restoring couple entirely. This may obviously be done by adding masses, *m*, *m*, of such magnitude that the inner cylinder is brought to its rest position, as observed by the reflected beam from the mirror, M, on the scale, which was previously used to measure θ_1 and θ_2.

In that case, if masses of *total* value, M_1, are supported on the wheel, G, $C_1 = M_1 \cdot \frac{D}{2}$, and for the second case with the liquid at level, L_2, $C_2 = M_2\frac{D}{2}$.

Then equation (16) simplifies to

$$(M_1 - M_2)\frac{D}{2} = \Omega\frac{4\pi\eta R_1^2 R_2^2}{R_2^2 - R_1^2}(l_1' - l_2').$$

This latter method of working will be found most satisfactory for glycerine and similar liquid, whilst the former method, using the torsion of the fibre, will be satisfactory in the case of less viscous substances, such as water.

Viscosity of a Liquid (Oscillating Disc Method)

The viscosity of a liquid may be determined by timing the period of oscillation of a flat circular disc in air, and finding the logarithmic decrement in the liquid and in air.

O. E. Meyer has shown that with such a disc, the coefficient of viscosity for a liquid, η, is

$$\eta = \frac{16I^2}{\pi\rho T(r^4 - 2r^3d)^2}\left\{\left(\frac{\lambda - \lambda_0}{\pi}\right) + \left(\frac{\lambda - \lambda_0}{\pi}\right)^2\right\}^2, \ldots\ldots(17)$$

where I is the moment of inertia of the disc, and attachments about the axis of suspension,

ρ the density of the liquid,

T the time of a complete swing in air,

r the radius of the disc,
d the thickness of the disc,
λ the logarithmic decrement in the liquid,
λ_0 the logarithmic decrement in air.

The development of the above formula is beyond the scope of this book, and may be found in "Poggendorf Annalen," No. 113, page 55.

Without any discussion of the development of the result, we will use it as an empirical formula which agrees with determinations by other methods. It is an excellent method whereby to study the determination of the logarithmic decrement of an oscillating system.

Logarithmic Decrement

Consider a suspended body to oscillate about the suspension as axis in a simple harmonic manner. If I is the moment of inertia of the body about the axis of suspension, F the restoring force per unit angular displacement, we have the equation of motion,

$$I\ddot{\theta} + F\theta = 0.$$

If now a frictional resistance acts on the body so that K is the resulting opposing couple per unit angular velocity, the above equation is modified into the form

$$I\ddot{\theta} + K\dot{\theta} + F\theta = 0.$$

By the method given, p. 28, trying as solution $\theta = \theta_0 e^{\lambda}$, we arrive at the result,

$$\theta = \theta_0 \cdot e^{-\frac{K}{2I}t} \cos\left(\sqrt{\frac{F}{I} - \frac{K^2}{4I^2}} \cdot t + \alpha\right),$$

when θ_0 and α are arbitrary constants, θ being the angular displacement at any time t.

The period time, T, of such a motion is

$$T = \frac{2\pi}{\sqrt{\frac{F}{I} - \frac{K^2}{4I^2}}}$$

The amplitude is $\theta_0 e^{-\frac{K}{2I}}$, and decreases exponentially with the time. θ_0 is the amplitude when the friction is eliminated, i.e. when $K = 0$.

In the case where the friction resistance, K, is very small, the value of K^2 is correspondingly much smaller and $\frac{K^2}{I^2}$ becomes negligible compared with $\frac{F}{I}$.

The periodic time becomes

$$T = 2\pi \sqrt{\frac{I}{F}}.$$

The oscillating body therefore performs vibrations in equal times, but the amplitude gradually dies out (see p. 29).

Starting from the undisturbed position of the body, let α_1 be the angular displacement measured to the first turning point (measured to the right, say).

Let α_2 be the angular displacement following, on the other side of the zero (to the left, say), also measured from the zero position.

α_3 the next swing in the original direction measured from the zero.

The first deflection α_1 is after a time $t = \frac{T}{4}$; α_2 after time $\frac{3T}{4}$; $\alpha_3, \frac{5T}{4}$, and so on,

$$\text{To the right : } \alpha_1 = \theta_0 e^{-\frac{K}{2I} \cdot \frac{T}{4}}$$

$$\text{To the left : } \alpha_2 = \theta_0 e^{-\frac{K}{2I} \cdot \frac{3T}{4}}$$

$$\text{To the right : } \alpha_3 = \theta_0 e^{-\frac{K}{2I} \cdot \frac{5T}{4}}$$

$$\text{To the left : } \alpha_4 = \theta_0 e^{-\frac{K}{2I} \cdot \frac{7T}{4}}$$

Thus :

$$\frac{\alpha_1}{\alpha_2} = e^{\frac{K}{2I} \cdot \frac{T}{2}} = \frac{\alpha_2}{\alpha_3} = \frac{\alpha_3}{\alpha_4} = \frac{\alpha_4}{\alpha_5} \ldots = \frac{\alpha_n}{\alpha_{n+1}}$$

So that

$$\text{Log } \frac{\alpha_1}{\alpha_2} = \log \frac{\alpha_2}{\alpha_3} \ldots = \log \frac{\alpha_n}{\alpha_{n+1}} = \frac{K}{2I} \cdot \frac{T}{2} = \lambda, \text{ say, } \ldots\ldots (18)$$

$\lambda = \frac{K}{2I} \cdot \frac{T}{2}$ is called the *logarithmic decrement.*

Now we have

$$\log\left(\frac{\alpha_1}{\alpha_2} \cdot \frac{\alpha_2}{\alpha_3} \cdot \frac{\alpha_3}{\alpha_4} \ldots \times \frac{\alpha_n}{\alpha_{n+1}}\right) = \log \left(\frac{\alpha_1}{\alpha_2}\right)^n$$

Since

$$\frac{\alpha_2}{\alpha_3} = \frac{\alpha_3}{\alpha_4} \ldots \frac{\alpha_n}{\alpha_{n+1}} = \frac{\alpha_1}{\alpha_2}$$

$$\therefore \quad n \cdot \log \frac{\alpha_1}{\alpha_2} = \log \left(\frac{\alpha_1}{\alpha_{n+1}}\right)$$

$$\lambda = \frac{1}{n} \log\left(\frac{\alpha_1}{\alpha_{n+1}}\right) \quad \ldots . (19)$$

Again, since

$$\frac{\alpha_1}{\alpha_2} = \frac{\alpha_2}{\alpha_3} \quad : \quad \frac{\alpha_1}{\alpha_2} = \frac{\alpha_1 + \alpha_2}{\alpha_2 + \alpha_3}$$

$$\frac{\alpha_2}{\alpha_3} = \frac{\alpha_3}{\alpha_4} \qquad \frac{\alpha_2}{\alpha_3} = \frac{\alpha_2 + \alpha_3}{\alpha_3 + \alpha_4}$$

Hence

$$\log \frac{\alpha_1}{\alpha_2} = \frac{1}{n} \log \frac{\alpha_1 + \alpha_2}{\alpha_2 + \alpha_3} \cdot \frac{\alpha_2 + \alpha_3}{\alpha_3 + \alpha_4} \cdots \times \frac{\alpha_n + \alpha_{n+1}}{\alpha_{n+1} + \alpha_{n+2}}$$

$$\lambda = \frac{1}{n} \log \frac{\alpha_1 + \alpha_2}{\alpha_{n+1} + \alpha_{n+2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

(i.) The value of λ, the logarithmic decrement, may be obtained, as in (18), by observing the value of the first swing, say to the right, and the successive swing to the left, whence

$$\lambda = \log_e \frac{\alpha_1}{\alpha_2},$$

or if logarithms to base 10 are used

$$\lambda = 2{\cdot}303 \log_{10} \frac{\alpha_1}{\alpha_2}.$$

λ here depends on the observation of swings.

It will be seen that α_1 and α_2 cannot be *very* accurately observed, hence the method of equation (19) may be used.

(ii.) By observing the 1st and the $(n + 1)$th $\frac{1}{4}$ swings, the error is reduced to $\frac{1}{n}$ of that in the first method corresponding to equation (19).

(iii.) The third method of finding λ, set out in equation (20), depends on the observation of swing from left to right ($\frac{1}{2}$ periods). Here no knowledge of the zero reading is required. This last method is obviously the one to be recommended, for most purposes, of the three discussed above.

It will be seen that, if we replace $\alpha_1 + \alpha_2$ by β_1, $\alpha_2 + \alpha_3 = \beta_2 \ldots$ $\alpha_n + \alpha_{n+1} = \beta_n$, that equation (20) becomes

$$\lambda = \frac{1}{n} \log \frac{\beta_1}{\beta_{n+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Since T, the time for a complete swing, is constant when the damping is small, we may find λ from equation (21) by observing β_1 directly, and then allowing the system to oscillate a measured time until the amplitude is appreciably reduced, say, from half to quarter of the original, then measure β_{n+1}. The number of swings between the observations may be found from the time, for the number of *complete* swings may be obtained by dividing the time by T; hence $(n + 1)$ is known. β refers to half a

complete swing, so that twice the above quotient will give the number of swings between observations.

An application of the above may be seen in the following method of finding the logarithmic decrement, which is especially advantageous if there is uncertainty in measuring the ends of swings, and hence β_1.

Suppose we read an even number of successive left and right deflections, $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \ldots \alpha_m$.

Then
$$\beta_1 = \alpha_1 + \alpha_2,$$
$$\beta_3 = \alpha_3 + \alpha_4,$$
$$\beta_5 = \alpha_5 + \alpha_6, \text{ etc.},$$
$$\ldots\ldots\ldots\ldots$$

omitting β_2, β_4, etc., for convenience in tabulating.

After a timed interval, when the deflection is reduced to about half the original deflection, obtain the same number of left and right readings. From a knowledge of T, suppose the first of the second set of readings is α_n, and α_n, α_{n+1}, $\alpha_{n+2} \ldots$ are observed, we may obtain

$$\beta_n = \alpha_n + \alpha_{n+1},$$
$$\beta_{n+2} = \alpha_{n+2} + \alpha_{n+3}, \text{ etc.}$$

We have since $\frac{\beta_1}{\beta_2} = \frac{\beta_2}{\beta_3} \ldots\ldots = \frac{\beta_{n-1}}{\beta_n} = e^{\lambda}$,

$$\frac{\beta_1}{\beta_n} = \frac{\beta_1}{\beta_2}\cdot\frac{\beta_2}{\beta_3} \ldots \frac{\beta_{n-1}}{\beta_n} = \left(\frac{\beta_1}{\beta_2}\right)^{n-1} = e^{\lambda(n-1)},$$

and in same way for the other pairs giving

$$\frac{\beta_1}{\beta_n} = \frac{\beta_3}{\beta_{n+2}} = \frac{\beta_5}{\beta_{n+4}} \ldots \text{ etc.}$$

or
$$\frac{\beta_1}{\beta_n} = \frac{\beta_1 + \beta_3 + \beta_5 \ldots}{\beta_n + \beta_{n+2} \ldots \beta_{n+4}} = e^{\lambda(n-1)}.$$

Hence $(n-1)\lambda = \log_e \frac{\beta_1 + \beta_3 + \beta_5 \ldots}{\beta_n + \beta_{n+2} + \beta_{n+4} \ldots}$,

or
$$\lambda = \frac{2{\cdot}3026}{(n-1)} \log_{10} \frac{\beta_1 + \beta_3 + \beta_5 \ldots}{\beta_n + \beta_{n+2} + \beta_{n+4} \ldots} \quad \ldots(22)$$

The practical details of this are given below.

Experimental Arrangements

A suitable form of apparatus with which to make a determination of η for a liquid is seen in fig. 94

The flat disc is suspended horizontally by a phosphor-bronze suspension which is attached to a rod rigidly fastened to the centre of the disc. This rod carries a cross-bar whose ends

have a screw thread, along which two masses may be screwed to balance the disc horizontally.

A small concave mirror is fastened to the rigid support.

The time of oscillation of this system is first obtained in air. This observation is carried out by aid of the usual lamp and scale arrangement. A beam of light from a lamp is directed on to the mirror, and is brought to a focus by the latter on a scale about a metre away. As the spot of light passes its rest position on the scale, during its oscillation, a stop-watch is started and stopped after 50 complete swings, hence T, the time for one swing (i.e. the interval between the spot of light passing the zero at consecutive times *in the same direction*).

Determination of I. The value of I may be obtained from the time of swing. If τ is the restoring couple per unit angular

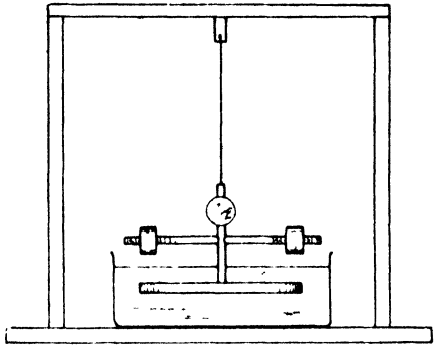


FIG. 94

displacement due to the suspension, we have, in air, where the logarithmic decrement is small

$$T = 2\pi \sqrt{\frac{I}{\tau}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

The value of T has been obtained as above.

A small ring of thick copper wire is now placed symmetrically on the disc and T^1, the time of a complete swing, is obtained from a determination of the time of 50 swings, for the loaded disc.

We have

$$T' = 2\pi \sqrt{\frac{I + I'}{\tau}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (24)$$

where I′ is the moment of inertia of the copper wire ring about the axis of suspension.

If a is the radius of the ring of wire, and if the centre were in the axis of suspension, $I_1' = ma^2$ where m is the mass of the wire ring.

Squaring (24) and (23) and dividing, we have

$$\frac{T'^2}{T^2} = \frac{I + I'}{I} = 1 + \frac{I'}{I} = 1 + \frac{ma^2}{I}.$$

Hence

$$I = \frac{T^2 \cdot ma^2}{T'^2 - T^2}.$$

r, d, and ρ may be readily found by the usual methods.

The logarithmic decrement is now found for the disc oscillating in air. Use is made of the method embodied in equation (22) above.

The disc is shielded from draught in an empty glass dish, as in the figure. A straight vertical wire is placed in front of the lamp and by the mirror, and a lens an image of the wire is focussed on the scale.

The disc is given a displacement, and the oscillations are observed by means of the light spot. As the spot passes the position of rest a stop-clock is started. The first turning point of the spot of light is read off, on the right, say, then one turning point on the left, and so on, tabulating as under 8 or 10 readings on each side. The disc is then allowed to perform the oscillation until the amplitude has decreased to half or quarter (depending on the time) of the original amplitude. At this stage the reading at the turning of the light spot at the right is observed, and the stop-clock stopped. The next reading to the left is taken, and so on for 8 or 10 on each side.

The time being t seconds between the starting of the clock and the first observation of the second set of swings, T being the periodic time of the system in air, there have been $\frac{t}{T}$ complete swings or $\frac{2t}{T}$ half periods, hence n as shown under. The above process is repeated in the liquid whose coefficient of viscosity is to be determined, say paraffin oil.

Thus, in the table opposite, which gives the observations for the disc in air and paraffin oil, we have, if T = 3·07 sec., n, the number of the $\frac{1}{2}$ swing at the commencement of the second set of observations for air $= 2\left(\frac{615}{3{\cdot}07}\right) = 400$,

i.e. for air

$$\lambda_0 = \frac{2{\cdot}3026}{(400-1)} \log_{10}\left(\frac{411{\cdot}2}{137{\cdot}1}\right)$$

$$= \frac{2{\cdot}3026}{399} (\cdot 4770)$$

$$= {\cdot}002752.$$

AIR			PARAFFIN OIL		
SCALE READING		AMPLITUDE β	SCALE READING		AMPLITUDE β
LEFT	RIGHT		LEFT	RIGHT	
2·4	45	42·6	5·1	47·2	42·1
2·5	45	42·5	6·0	46·6	40·6
2.8	44.8	42.0	6·5	46·0	39·5
3·0	44·5	41·5	7·2	45·2	38·0
3·1	44·3	41·2	7·8	44.5	36·7
3·3	44·2	40·9	8·5	43·8	35·3
3.3	44·0	40·7	9·1	43·2	34·1
3·6	43·8	40·2	9·7	42·6	32·9
3·7	43·7	40·0	10·2	42·0	31·8
4·0	43·6	39·6	11·0	41·5	30·5
		Total 411·2			Total 361·5
Time Interval = 615 secs.			Time Interval = 123 secs.		
16.5	30·4	13·9	20·7	31·3	10·6
16·5	30·4	13·9	20·9	31·2	10·3
16·5	30·4	13·9	21·0	31·0	10·0
16.5	30·4	13·9	21·2	30·9	9·7
16·55	30·3	13·7	21·4	30·7	9·3
16·6	30·2	13·6	21·5	30·6	9·1
16·7	30·2	13·5	21·6	30·5	8·9
16·7	30·2	13·5	21·8	30·2	8·4
16·6	30·2	13·6	22·0	30·1	8·1
16·6	30·2	13·6	22·5	30·0	7·5
		Total 137·1			Total 91·9

In the same way the value of the logarithmic decrement in paraffin may be calculated,

$$\lambda = \frac{2{\cdot}3026}{(n-1)} \log_{10} \frac{361{\cdot}5}{91{\cdot}9},$$

n being obtained as above from the time interval measured in that case from the commencement of the first set of observations to the commencement of the second set.

Thus, by the methods given, we have a value for I, ρ, T, r, d, λ and λ_0; hence substituting in equation (17) η may be calculated.

Repeat the experiment, using water and paraffin oil, and calculate the value of η for each.

VARIATIONS OF THE COEFFICIENT OF VISCOSITY OF WATER WITH TEMPERATURE

The variations of the coefficient of viscosity of water with temperature may be investigated by means of the apparatus shown in fig. 95.

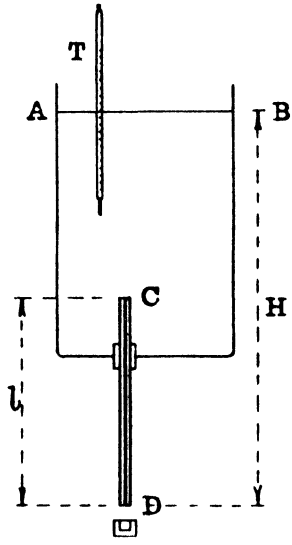


FIG. 95

AB is a copper cylindrical vessel into which the capillary tube, CD, is fitted by means of a rubber cork.

Water is heated in a boiler to a temperature 5 or 6 degrees above the one at which a determination is to be made, and is transferred to AB, the end, D, being closed by the rubber stopper shown. The water is stirred and the temperature noted by the thermometer, T. It is not very difficult to arrange that the water cools slowly and regularly about the desired temperature. The water is then allowed to flow into a graduated vessel and the time taken for a measured amount to pass. This process may be repeated, using water at various temperatures.

Care must be taken that the temperature, throughout the flow, is practically constant, otherwise unreliable results will be obtained.

The radius and the length of the tube may be measured as before.

Then, as shown under, we may obtain the value of η for each temperature from the following formula :—

$$\eta = \frac{\pi}{8}\frac{R^4}{Q}\frac{g\rho}{l}\left(H - \frac{2Q^2}{g\pi^2R^4}\right),$$

using the same notation as before.

In the case of a vertical tube, such as the one described, account must be taken of the gravity attraction on the liquid in the tube ; the direct application of the results of equation (4) dealing with a horizontal tube being thus inadmissible.

Consider a vertical tube of length, l cms., and in the liquid imagine a thin cylindrical tube of the liquid of radius r, and thickness δr.

Let p be the pressure difference between the two ends of the tube, then we have in the steady state of flow $p2\pi r \cdot \delta r + 2\pi r \cdot \delta r l \rho g$ the downward forces, opposed to $\frac{dF}{dr} \cdot \delta r$ due to viscosity

where $$F = \eta A \cdot \frac{dv}{dr},$$

i.e. $$2\pi r(p + g\rho l)dr = -d\left(\eta A\frac{dv}{dr}\right),$$

or integrating $$r(p + g\rho l)dr = -2\eta l dv,$$

since $$A = 2\pi r l.$$

Hence, once more integrating,

$$r = -\frac{r^2}{4\eta l} \cdot (lg\rho + p) + C,$$

where C, the constant of integration $= \frac{R^2}{4\eta l}(lg\rho + p)$, for when $r = R$, $v = 0$,

i.e. $$v = (R^2 - r^2)\frac{lg\rho + p}{4\eta l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

Through such a tube the quantity of liquid passing per second $= v(2\pi r \cdot \delta r)$,

i.e. the total quantity per second, Q, is

$$Q = \int_0^R 2\pi r \cdot v \cdot dr.$$

Substituting from (25) above and integrating,

$$Q = \frac{\pi}{8}\frac{R^4}{\eta}\frac{p + g\rho l}{l}.$$

In the case taken we may apply the result of equation (9), page 154, $p + \frac{1}{2}\rho v^2 - g\rho h = \text{constant}$.

If AB is the upper surface of the cylinder which contains the liquid (fig. 95), the cross-section being large, the velocity of

AB will be small compared with that at C. Taking AB as the reference plane for measurement of L, h being the distance between C and AB, we have,

$$\Pi = p' + \tfrac{1}{2}\rho v^2 - g\rho h,$$

i.e. the pressure, p', at C is

$$p' = \Pi + g\rho h - \tfrac{1}{2}\rho v^2,$$

where v is the velocity at the centre of C.

The pressure at D is Π, therefore $p = p^1 - \Pi$;

hence $$p = g\rho h - \tfrac{1}{2}\rho v^2,$$

and thus we have the value below

$$Q = \frac{\pi}{8} \cdot \frac{R^4}{\eta} \cdot \frac{g\rho h - \frac{1}{2}\rho v^2 + g\rho l}{l}$$

$$Q = \frac{\pi}{8} \frac{R^4}{\eta} \frac{g\rho H - \frac{1}{2}\rho v^2}{l} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

The correction $\frac{1}{2}\rho v^2$ may be obtained from (25) above, i.e.

$$v = (R^2 - r^2)\left(\frac{lg\rho + p}{4\eta l}\right)$$

To a first approximation, putting $p = g\rho h$,

$$v = \frac{R^2 - r^2}{4\eta l} \cdot g\rho(l + h),$$

or at the centre, where $r = 0$,

$$v = \frac{R^2 g\rho H}{4\eta l}.$$

To the same approximation

$$Q = \frac{\pi}{8} \cdot \frac{R^4}{\eta} \cdot \frac{g\rho H}{l},$$

i.e. $$v = \frac{2Q}{\pi R^2}.$$

Hence we amend equation (26)

$$Q = \frac{\pi}{8} \frac{R^4}{\eta} \frac{g\rho}{l}\left(H - \frac{2Q^2}{g\pi^2 R^4}\right) \qquad \ldots\ldots\ldots 26(a)$$

This equation is used to calculate η in the experiment already described.

Viscosity of Air

Consider two parallel circular plates, one suspended by a fibre, and the other rotating at a constant speed. If the space between the plates be filled with any gas a velocity gradient will be set up in the layers of the gas parallel to the rotating plate. The layer in contact with the rotating disc will move with the latter.

Due to the viscosity of the gas, the adjacent layer will also acquire a velocity comparable with the former. Thus throughout the space, the air strata will acquire a motion, just as in the case of a liquid flowing through a tube. The layer of the gas in contact with the suspended plate will therefore experience a force tending to rotate it in the same direction as the constantly rotating parallel plate. Due to the force, a couple will act on the plate, which will therefore turn through a definite angle of such a magnitude that the restoring couple due to the torsion in the suspension just balances the displacing couple of the viscous drag.

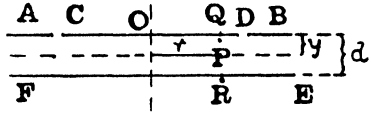


FIG. 96

It can be shown that each stratum of air moves as though it were a solid, i.e. it moves as a whole.

To obtain an expression for the deflection of the suspended plate in terms of η, the coefficient of viscosity of the gas, etc., let us assume that the edge effect is negligible, and the gas between the plates behaves as though the plates were of infinite dimensions, an assumption which is justified by using a guard ring round the suspended plate.

Let d be the distance between the discs,
w the angular velocity of rotation of the moving plate,
R the radius of the suspended plate.

Consider a stratum shown in fig. 96 by the horizontal broken line, between EF, rotating, and CD, which is suspended. (The parts, AB in the diagram represent a guard ring to eliminate the end effect, CD being of radius R.)

In the stratum considered take a point, P, on a circle of radius r cms. about the axis of rotation.

Points Q and R are the projections of P on CD and EF. The velocity of R is $r \cdot w$; the velocity of Q is zero. The velocity slope is therefore $\frac{wr}{d}$. If uniform, the value of the velocity of P is $\frac{y}{d} \cdot wr$, y being the distance of the stratum considered, below the plate CD.

We have seen that by the definition of viscosity

$$F = A\eta \frac{dv}{dr} = \eta \frac{wr}{d} A.$$

In the stratum considered, let a second circle of radius, $r+\delta r$, be drawn; between the two circles there is an annular ring of width, δr. The area of the ring is $2\pi r \cdot \delta r$. The viscous force acting on such a narrow ring is, from the equation above, equal to F, where

$$F = 2\pi r \cdot \delta r \cdot \frac{\eta w r}{d}.$$

The turning moment about the axis is

$$F \cdot r = 2\pi r^3 \frac{\eta w}{d} \cdot \delta r.$$

Such a moment acts on the suspended disc on the projection of this area.

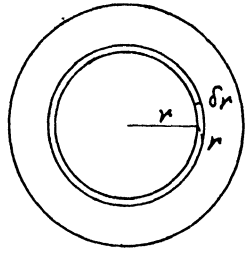


FIG. 97

The total couple is the sum of such couples taken over the entire area. Let this couple be C, then

$$C = \int_0^R 2\pi r^3 \frac{\eta w}{d} dr = \frac{2\pi\eta w}{d} \frac{R^4}{4} = \frac{\pi\eta R^4 w}{2d}$$

Let the suspended plate be turned through an angle, θ, due to this couple. The equilibrating couple due to the torsion of the suspension is $\tau\theta$, where τ is the restoring couple per unit angular displacement. This gives

$$\tau\theta = \frac{\pi\eta R^4 w}{2d} \quad \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

A suitable form of apparatus is essentially of the form shown diagrammatically in fig. 96. A brass plate is rigidly connected to a shaft which may be rotated by means of a belt drive on a pulley, which is on the shaft. A small counting indicator serves to record the revolutions of the disc, which is steadily turned by hand or by a small motor.

At a distance, which may be adjusted, above the plate is a mica disc, suspended by a phosphor-bronze suspension, and arranged inside a guard ring of brass. The suspension carries a small mirror which serves to measure the deflection by the

usual method of lamp and scale; the whole is enclosed in a brass case which serves as a shield. A preliminary experiment will give an indication of the most suitable speed with which to rotate the plate, for any given distance, d, between the plates.

The zero of the spot of light having been determined, the lower plate is rotated until a full scale deflection is obtained. The speed of rotation is maintained constant. When this is steady, the counting gear, having been read, is thrown into action, and a stop-watch started. Maintaining the spot constant by steady rotation for as long as possible (at least several minutes) the value of the deflection is noted. The counter is then thrown out of action and the stop-watch stopped. In that way the number of revolutions n, in a known time t, is obtained, hence

$$w = \frac{2\pi n}{t}.$$

As R occurs in the fourth power, several values of the diameter are obtained in all directions, as accurately as possible, and the mean value calculated.

d is measured by means of a cathetometer. From the value of the steady deflection and a knowledge of the distance between the mirror and the scale, θ may be calculated in radians. θ is half the value of the angle subtended at the mirror by the length of scale moved over by the spot of light, i.e. tan (2θ) may be obtained, knowing the linear deflection and the distance between the mirror and the scale.

Thus, from this experiment all the terms in equation (27) are known except τ and η.

To obtain τ the suspended plate is twisted from its equilibrium position, and the simple harmonic oscillations set up are timed. If T is the mean value of the periodic time for, say, 50 complete vibrations, we have (p. 105)

$$T = 2\pi\sqrt{\frac{I}{\tau}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

where I is the moment of inertia of the mica plate about the axis of suspension.

If now the mica plate is loaded by placing on it, symmetrically with regard to the suspension, a circle of brass wire of radius a cms. and mass m grammes, the moment of inertia has been increased to I′, where $I' = I + ma^2$.

The time of oscillation, T′, of the loaded plate is next determined by timing 50 oscillations, then

$$T^1 = 2\pi\sqrt{\frac{I'}{\tau}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

Squaring (2) and subtracting, we have

$$T'^2 - T^2 = \frac{4\pi(I' - I)}{\tau},$$

i.e.

$$\tau = \frac{4\pi ma^2}{T' - T^2}.$$

Thus, by equation (27)

$$\eta = \frac{2d}{\pi R^4} \cdot \frac{4\pi ma^2}{(T'^2 - T^2)} \cdot \frac{t}{2\pi n} \cdot \theta.$$

$$\eta = \frac{4ma^2t}{\pi R^4(T'^2 - T^2} \cdot \frac{\theta}{n}.$$

Having obtained the deflection in degrees, $\varphi°$ say, we have

$$\theta = \frac{\varphi\pi}{180},$$

so that

$$\eta = \frac{ma^2t\varphi}{45R^4(T'^2 - T^2)n}.$$

The Determination of the Viscosity of a Gas (by the Flow through a Capillary Tube)

A simple method for determining the viscosity of a gas which may be described as ' The Constant Volume Method,' has been described by Prof. A. Anderson (" Phil. Mag." Dec., 1921, pp. 1022–3). The apparatus is illustrated in fig. 98. It consists of a bulb, V, from which a tube, DO, projects downwards and is connected by rubber tubing to a glass tube, AB. These tubes contain mercury up to certain levels marked at A and O.

Just below the bulb a capillary tube leads from the tube, DO, as shown at D. This capillary tube is provided with a piece of rubber tubing which may be closed by the pinch-cock, P.

The whole apparatus is mounted on a stand and is of about the same size as the constant volume air thermometer.

The arm, AB, is mounted on a carriage which is readily adjusted by means of a rack and pinion which is regulated by the turn-screw, C.

The volume of the gas within V, down to some mark, O, and extending to the end of the capillary tube is determined. The pinch-cock is then opened to the air and the mercury brought well below O by properly adjusting AB. The pinch-cock is then closed, and the air compressed by raising AB until the mercury stands at O.

A few minutes' interval is allowed to elapse so that the temperature of the gas, which may have been disturbed in this

compression, once more attains the temperature of the room. The difference in level of the mercury in the two tubes is observed. The pinch-cock is opened for a definite accurately measured time and then closed. During this interval the level of AB is continually adjusted, so that the mercury always stands at O, and the gas consequently maintains a constant volume. At the end of the time the difference in level of the mercury is again noted. From the observations made, viz. from a knowledge of the volume, V, of the gas, the two pressure differences, p_1 and p_2 respectively, the time interval, t seconds, and the barometric pressure, P, the viscosity is determined from the formula :

$$\eta = \mathrm{P}t\,\frac{\pi \mathrm{R}^4}{8lV} \div \log_e \frac{(p_2 + \mathrm{P})\,(p_1 - \mathrm{P})}{(p_1 + \mathrm{P})\,(p_2 - \mathrm{P})},$$

where R = radius of capillary tube and l = its length.

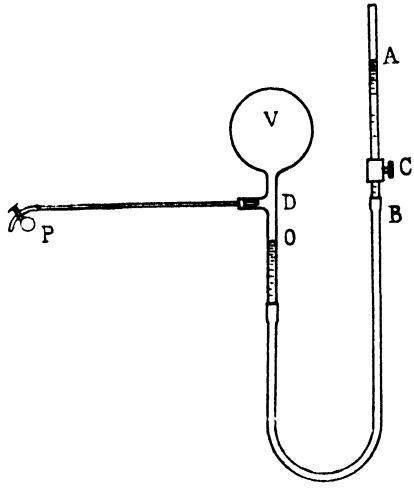


FIG 98

The above formula may be obtained from an extension of the result obtained for the flow of a liquid through a capillary tube.

Suppose, in the first case, that the gas enters the capillary at a fixed pressure P_1, and leaves at a pressure P_2. Let V_1 and V_2 be the volume entering and leaving *per second*; then $\mathrm{P}_1\mathrm{V}_1 = \mathrm{P}_2\mathrm{V}_2$. The volume of gas passing any point in the tube per second depends on the pressure at the point. The velocity is consequently variable along the axis of the tube, and therefore the method of the liquid flow cannot be applied to the full length of the tube.

Consider a small element, δx of the tube. Let p be the mean

pressure in the element and δp the difference in pressure at the ends.

Equation (4), page 152, for the liquid flow, is

$$Q_1 = \frac{P}{l} \cdot \frac{\pi}{8} \frac{R^4}{\eta}.$$

If q_1 is the volume of gas passing through the element, the above formula may be applied (δx is very small), and since in this case $\frac{P}{l} = -\frac{\delta p}{\delta x}$ (since the pressure decreases with increase in x),

i.e.
$$q_1 = -\frac{dp}{dx} \cdot \frac{\pi}{8} \cdot \frac{R^4}{\eta}.$$

Now, $P_1V_1 = P_2V_2 = q_1 p$, since the mass of gas passing any point is constant, i.e. $q_1 = \frac{P_1V_1}{p}$; therefore

$$\frac{P_1V_1}{p} = -\frac{dp}{dx} \cdot \frac{\pi}{8} \frac{R^4}{\eta},$$

$$P_1V_1 dx = -\frac{\pi}{8} \frac{R^4}{\eta} p dp.$$

Integrating over the limit of the tube

$$P_1V_1 \int_0^l dx = -\frac{\pi}{8} \frac{R^4}{\eta} \int_{P_1}^{P_2} p \cdot dp,$$

i.e.
$$P_1V_1 = \frac{P_1^2 - P_2^2}{l} \cdot \frac{\pi}{16} \cdot \frac{R^4}{\eta} \quad \ldots\ldots\ldots (30)$$

If V is the volume available to enter the tube at a pressure P_1, we may write

$$P_1V_1 = P_1 \cdot \frac{dV}{dt}.$$

We may apply Boyle's Law to the instantaneous values of P and V at entry,

i.e.
$$PV = \text{constant},$$

or
$$P\frac{dV}{dt} + V\frac{dP}{dt} = 0, \quad \text{i.e. } P\frac{dV}{dt} = -V\frac{dP}{dt},$$

whence from (30)

$$\frac{(P_1^2 - P_2^2)}{l} \frac{\pi}{16} \cdot \frac{R^4}{\eta} = -V\frac{dP}{dt} \quad \ldots\ldots\ldots (31)$$

where $\frac{dP}{dt}$ is the rate of change of pressure at the end of entry, and V the fixed volume.

P_2 in the experiment is the *constant* atmospheric pressure

at the end of the capillary, and P_1 has values, say, p_1 and p_2 at the beginning and ending of the period of observation of t seconds duration. Re-writing (31), we have

$$\frac{\pi R^4}{16 l \eta V} \cdot \int_0^t dt = + \frac{1}{2P_2} \int_{p_1}^{p_2} \left(\frac{1}{P_1 + P_2} - \frac{1}{P_2 - P_2} \right) dP,$$

$$\frac{\pi R^4 t}{16 \eta l V} = \frac{1}{2P_2} \left[\log \frac{P_1 + P_2}{P_1 - P_2} \right]_{p_1}^{p_2}$$

$$= \frac{1}{2P_2} \left\{ \log \frac{p_2 + P_2}{p_2 - P_2} - \log \left(\frac{p_1 + P_2}{p_1 - P_2} \right) \right\}$$

which becomes, on writing $P_2 = P$ = atmospheric pressure,

$$\eta = \left(\frac{\pi R^4}{8 l V} \right) \frac{Pt}{\log. \dfrac{(p_2 + P)}{(p_1 + P)} \cdot \dfrac{(p_1 - P)}{(p_2 - P)}}$$

The term $\frac{\pi R^4}{8 l V}$ may be calculated once for all for the apparatus and is the ' constant of the apparatus.'

The Constant Pressure Method

The determination of η may be made with constant pressure difference between the ends by maintaining the levels of the mercury columns in the two tubes at a fixed difference. The point, O, would be chosen at the commencement near the bottom of the tube, and the time taken for a volume corresponding to a length of tube between the original and final positions of O measured. If p is the total constant pressure inside V, we have

$$V_1 = \frac{(p^2 - P^2) \pi r^4}{16 \eta l p} \cdot \qquad \text{from (30),}$$

in which all terms except η are known.

The difficulty in this modification lies in keeping the mercury levels a fixed distance apart as the gas is driven out.

CHAPTER VI

THERMOMETRY AND THERMAL EXPANSION

The Comparison of a Thermometer by means of a Standard

IT is recommended that each student before beginning his experiments on heat should choose a thermometer, test its accuracy, and use it when required throughout his experiments. The method of comparison is to immerse it together with a standard thermometer in a bath and observe temperatures over a suitable range simultaneously by both instruments.

It is convenient for the following experiments to have two thermometers, one reading from 0° C. to 35° C. and the other from 0° C. to 100° C. Both should be calibrated in this way.

A large water or oil bath should be carefully heated over a Bunsen flame and constantly stirred. The thermometers should be placed in the liquid so that the mercury thread shows just above its surface, and with their bulbs close together.

A record of temperatures at intervals of 5° should be taken over the range, and a curve drawn with accurate temperatures as ordinates and with readings from the thermometer to be calibrated as abscissæ.

By means of this curve the readings of the thermometer in later experiments can be reduced to that of the standard.

A better method of heating the liquid is to place it in a bath standing in a box lined with cotton wool and to supply the heat by passing an electric current through a resistance coil immersed in the liquid. If the current is drawn from storage cells, and a variable resistance included in the circuit, it is possible to adjust the current so that the bath is maintained for a long time at a constant temperature. The most convenient form of stirrer is a small propeller driven by a small electric motor.

The Calibration of a Mercury Thermometer

It is impossible in practice to obtain a perfectly uniform bore in the stem of a mercury thermometer, so that it is not sufficient in an accurate instrument to divide the interval between the fixed points into a number of parts of equal lengths. The makers of thermometers usually attempt to make some correction for this lack of uniformity by adjusting the distance

between consecutive divisions to suit the bore of the stem at the various points. But, in spite of this, unless the thermometer is exceptionally carefully constructed, errors remain and a calibration has to be made if accurate observations are required.

In a given thermometer, as a rule, the divisions will be unequally spaced at different parts of the tube, and the bore will vary from point to point.

The first step is to divide the tube into segments consisting of five or ten degrees each, and over each to find the average distance between each division. We assume for the sake of definiteness that we are considering intervals of ten divisions. Measure each of these, beginning at 0° by means of a micrometer microscope, and deduce the average length per degree for each of the ten intervals up to 100°. We are assuming that the thermometer is divided into degrees Centigrade from 0° to 100°.

When this has been done a thread of mercury is broken off from the main column of length equal to that of about 10° on the thermometer scale.

This thread may be obtained by connecting a small jet, made by drawing a glass tube to a narrow bore, to a gas-pipe, and lighting the gas at the narrow end, adjusting the supply to produce a flame about half a centimetre long. If this flame be applied cautiously at the point of the thread where it is desired to sever it, the thread will divide. During the application of heat the thermometer must be rotated to avoid fracture due to unequal heating.

The thread is moved by gently jerking the thermometer until one end is at 0° and the other near 10°, and its length measured.

The thread is then moved so that one end lies at 10°, and the other near 20°, and so on up to 100°.

If it is difficult to get the detached thread down to zero, owing to the projection of mercury from the bulb past the zero mark, the bulb should be cooled by wrapping it in wool and moistening with ether. This will clear the tube and allow the thread to be moved down without its re-joining the main mass of mercury.

Denote the ten lengths of the thread by l_1, l_2, l_3, etc.

Let these be reduced to their equivalents in degrees. This is readily done since the average width of one degree is known in the different parts of the scale. Denote these equivalent lengths by t_1, t_2, t_3, etc., and the mean of these by t.

If the tube had a uniform bore of the same length as that of the actual instrument between 0° and 100°, the readings would be t, $2t$, $3t$, etc., instead of t_1, $t_1 + t_2$, $t_1 + t_2 + t_3$, etc.

Let the values added respectively to t_1, t_2, t_3, etc., to make them equal to t be δ_1, δ_2, δ_3, etc.

Then $t = t_1 + \delta_1$
$t = t_2 + \delta_2$, etc.

The corrections to be applied near 10°, 20°, 30°, etc., are therefore :

$$t - t_1, \quad 2t - t_1 - t_2, \quad 3t - t_1 - t_2 - t_3, \quad \text{etc.},$$

or $\delta_1, \quad \delta_1 + \delta_2, \quad \delta_1 + \delta_2 + \delta_3.$

The correct temperature corresponding to t_1 is $(t_1 + \delta_1)$, and corresponding to the point $(t_1 + t_2)$ it is $(t_1 + t_2 + \delta_1 + \delta_2)$, and so on.

We have assumed up to the present that the fixed points at 0° and 100° are correct.

A table is drawn up as shown below. The correction to be applied near 10° is − ·023, for the thermometer column is too long at this point, near 20° the correction to be applied is (− ·023 + ·032) or + ·009. The correction would have been + ·032 had the point near 10° been correct, but since that was not the case both errors come in. We correct similarly for other points by adding errors algebraically, and recording in the last column the amount to be added to the thermometer reading to obtain the correct temperature.

Note that in the example given the thread was not very near to the mean length of 10°. It is convenient to arrange this as closely as possible to 10°. Strictly, the error −·023 ought to be applied to the recorded temperature of 10·231, but the error will probably not vary very rapidly in the neighbourhood of any given point. Hence we take the corrections in the last column as applied at 10°, 20°, etc.

Mean length of thread, as deduced from column 4, 10·208.

REGION OF TUBE	MEAN LENGTH PER SCALE DIVISION	LENGTH OF MERCURY THREAD	EQUIV-ALENT OF THREAD IN DEGREES	DIFFERENCE FROM MEAN	CORREC-TION TO APPLY TO UPPER READING
0°– 10°	·2385	2·442	10·231	δ_1 −·023	−·023
10°– 20°	·2389	2·431	10·176	δ_2 +·032	+·009
20°– 30°	·2391	2·449	10·243	δ_3 −·035	−·026
30°– 40°	·2389	2·435	10·193	δ_4 +·015	−·011
40°– 50°	·2398	2·438	10·167	δ_5 +·041	+·030
50°– 60°	·2350	2·426	10·323	δ_6 −·115	−·085
60°– 70°	·2355	2·404	10·208	δ_7 ·000	−·085
70°– 80°	·2353	2·393	10·170	δ_8 +·038	−·047
80°– 90°	·2347	2·392	10·192	δ_9 +·016	−·031
90°–100°	·2367	2·408	10·173	δ_{10} +·035	+·004

Draw a curve with thermometer readings as abscissæ and the corrections to be applied to obtain the corrected readings as ordinates. The curve should pass through the x-axis at 0° and 100°, since these points have been assumed to be correct.

If, however, the fixed points are incorrectly placed, the errors must be found in the usual manner with ice and steam.

Suppose that the zero correction is δ_0, while that at 100° is δ_{100}.

The upper fixed point must be corrected for pressure, latitude, and height above sea-level.

The barometer must also be corrected, owing to the fact that it is probably not read at the temperature at which the instrument was standardized.

This last correction may be made by the following formula: If h_0 denotes the height at 0°, and h_t that read at t°,

$$h_0 = h_t(1 - \cdot 000162t).$$

For latitude λ and at a height d feet above sea-level, the length of the column which produces the standard pressure at 0° and at sea-level in the standard latitude of 45° is:

$$L = (760 + 1 \cdot 94562 \cos 2\lambda + \cdot 0000454 66d) \text{ mm.}$$

Under this pressure, L, the boiling point is 100° at the height and latitude of the place of observation.

Thus, h_0 is equivalent to $\frac{h_0}{L}$ atmospheres or $\frac{h_0}{L} \times 760$ mm. under normal conditions. From this and the following table the correction to the boiling point may be made.

MAXIMUM VAPOUR PRESSURE	CORRESPONDING TEMPERATURES
720 mm	98 493° C.
725	98 686
730	98 877
735	99 067
740	99 255
745	99·443
750	99 630
755	99·815
760	100·000
765	100 184
770	100·366
775	100·548
780	100·728

The zero on the scale actually records the reading $-\delta_0$, and the 100° records $(100 - \delta_{100})$.

Thus a correction is required for this.

Plot on the curve the two points (0,0), (100, $\delta_{100} - \delta_0$), and join them by a straight line. The ordinate of this line at $x°$ is $(\delta_{100} - \delta_0) \cdot \frac{x}{100}$, since each degree, even if correct as regards bore, would register only $\frac{100 - \delta_{100} + \delta_0}{100}$, we must add to it an amount, $\frac{\delta_0 - \delta_{100}}{100}$, on account of the errors at the fixed points.

To $x°$ we must add

$$x \cdot \frac{\delta_0 - \delta_{100}}{100}$$

Thus, if we draw on the same graph on which the first set of results was plotted this second curve the difference between the two ordinates taken algebraically will give the true reading corrected for errors due to bore and fixed points.

Re-draw a new curve, showing as abscissæ temperatures as recorded by the thermometer and as ordinates the difference of the ordinates of the two curves, and the resulting curve will give the amount to be added to any recorded temperature to give the true temperature.

Throughout the taking of measurements the temperature of the detached thread should remain constant; and in order to be sure that this condition holds, place another thermometer close by and observe whether it varies or not. Do not handle the thermometer under examination more than is necessary, and only do so by holding it at the tip away from the bulb.

Newton's Law of Cooling

The object of this experiment is to verify Newton's Law of Cooling, which states that the rate at which a body cools is proportional to the difference of temperature between itself and the enclosure in which it is placed. The constant of proportionality depends on the surface exposed and the thermal capacity of the exposed body, and the law is true for small differences of temperature only.

The apparatus required is a small metal thimble, into which water at about 80° C. can be placed, provided with a cork through which a thermometer passes for noting the temperature of the liquid.

The enclosure consists of two calorimeters, one inside the other, and containing water in the space between, to provide an enclosure at nearly constant temperature. A thermometer

placed in this water, which should be stirred occasionally, gives the temperature during the experiment.

Observe the temperature recorded by T_1 and T_2 (fig. 99), at

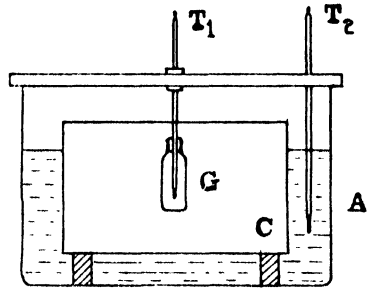


FIG. 99

intervals of half a minute during the initial stage of the fall, and, as the rate decreases, the interval between observations may be increased. The record of T_2 should not vary very much.

Tabulate the results thus :

TIME (MINS)	RECORD OF T_1	RECORD OF T_2
0	72·5° C.	12·2° C.
·5	71·3° C.	12·2° C.
1	70·1° C.	12·2° C.
1·5	69·0° C.	12·2° C.
2	67·8° C.	12·2° C.
2·5	66·7° C.	12·2° C.
3	65·6° C.	12·2° C.
3·5	64·5° C.	12·2° C.
4	63·5° C.	12·3° C.
4·5	62·5° C.	12·3° C.
—	—	—
—	—	—
—	—	—

Draw a curve showing the relation between the temperature T_1 and the corresponding time.

Make the temperatures the ordinates and times the abscissæ.

The rate of fall of temperature may be obtained from this curve by measuring the tangent of the angle of inclination of the tangent to the curve to the axis of t. As explained in the introductory chapter this measures the value of $\frac{dT_1}{dt}$ at the various points of the curve.

According to Newton's law these values should be proportional to the differences between T_1 and T_2. If T_2 does not vary very much its mean value may be regarded as the mean temperature of the enclosure.

If T_2 varies too much to permit this approximation, draw a curve showing the relation between T_2 and the time on the diagram which shows the relation between T_1 and the time.

Then at each time we can determine the value of $(T_1 - T_2)$ and the corresponding value of $\frac{dT_1}{dt}$ from the same graph.

Make another table containing two columns, one for values of $(T_1 - T_2)$ and the other for the corresponding values of $\frac{dT_1}{dt}$.

Draw a curve with the values of $(T_1 - T_2)$ as ordinates, and those of $\frac{dT_1}{dt}$ as abscissæ, when, if these two quantities are proportional, the result should be a straight line.

The Use of the Weight Thermometer

The weight thermometer consists of a glass bulb, B, drawn out at the upper end into a capillary stem, A.

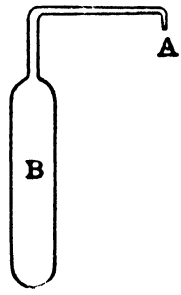


FIG. 100

A convenient size is obtained by making B about one and a half to two inches long, and between a quarter and half an inch wide. It forms a good exercise to make the apparatus from a piece of glass tubing. If this is done, care must be taken to get rid of any blob of glass likely to accumulate at the bottom of B, otherwise in the subsequent heating B will be very likely to crack.

The apparatus measures the expansion of any liquid placed in B relative to glass. In order to deduce the real expansion it is necessary to know that of B.

The coefficient of expansion of mercury has been found with great accuracy, and may be taken as ·0001818.

We may therefore use mercury to find the expansion of B, and then use B to determine the expansion of other liquids. We shall consider the determination of the expansion of water.

(1) *Determination of the Expansion of the Weight Thermometer*

Carefully weigh the apparatus.

Surround B by wire gauze and warm carefully in order to drive out air. Place the end A under clean mercury contained in a dish and allow B to cool so that a little mercury enters. Then, when sufficient has entered, boil the mercury so that the space above it may become full of mercury vapour. Once more place A below the mercury, when B will fill as the vapour condenses on cooling.

It is best to carry out the process gradually, heating and cooling B several times, and allowing a little mercury to enter at a time.

Warm up the mercury first to prevent the bulb from cracking when it enters.

Allow B to cool gradually, and finally surround it with ice, keeping A immersed all the time.

Bring up a small weighed, empty dish, and remove the mercury at A, replacing it by the empty dish. Remove the ice and allow B to acquire the temperature of the room. Mercury will, of course, flow over into the dish. As soon as the flow ceases, weigh the bulb, B, and the dish. Let the total weight of mercury within B at 0° be denoted by W_0.

Immerse B and as much as possible of the stem in boiling water, and catch the mercury that flows out in the little dish.

After the bulb has remained in the water for a quarter of an hour, to allow it to assume the temperature of the boiling water, remove it, dry and carefully weigh it. Suppose that w is the amount that has flowed out. As a check re-weigh the dish and again determine w.

Let ρ_0 denote the density of mercury at 0° C. and ρ_1 that at temperature T.

If V_0 denote the volume of the apparatus at 0° and V_1 that at T, while β denotes the coefficient of expansion of glass,

$$\frac{V_1}{V_0} = 1 + \beta T.$$

But
$$V_0 = \frac{W_0}{\rho_0}, \qquad V_1 = \frac{W_0 - w}{\rho_1},$$

$$\therefore \quad 1 + \beta T = \frac{W_0 - w}{W_0} \cdot \frac{\rho_0}{\rho_1} = \frac{W_0 - w}{W_0} \cdot (1 + \alpha T),$$

$$\therefore \quad \beta = \frac{W_0 - w}{W_0} \cdot \frac{1 + \alpha T}{T} - \frac{1}{1}$$

$$= \frac{W_0 - w}{W_0} \alpha - \frac{w}{W_0 T},$$

or $$\alpha = \frac{W_0}{W_0 - w} \beta + \frac{w}{W_0 - w} \cdot \frac{1}{T},$$

i.e $$\alpha - \beta = \frac{w}{W_0 - w} \beta + \frac{w}{W_0 - w} \cdot \frac{1}{T}.$$

The ratio $\frac{w}{W_0 - w}$ is small, and since β is also small we may often neglect the first term on the right and use simply

$$\alpha - \beta = \frac{w}{W_0 - w} \cdot \frac{1}{T}.$$

(2) *The Coefficient of Expansion of Water*

The value of this coefficient varies considerably throughout the range 0° to 100°. In this experiment the average value between two temperatures, say, 20° C. and 60° C. will be determined. If both temperatures are above that of the room the experimental difficulties are not so great. We assume this to be the case though the method is quite general, and exactly the same precautions have to be taken as in (1) if water flows out on moving B from the lower temperature enclosure to the balance. Carefully fill the weight thermometer with water that has been recently boiled to get rid of contained air, and place it completely immersed in water at temperature t_1. Remove it, dry, and weigh.

Place the thermometer in water at t_2 and repeat.

Use the formula given above to deduce α for water, making use of the value of β previously determined.

Determination of the Density of Water at Various Temperatures by means of a Glass Sinker

In this experiment a solid is weighed while totally immersed in water at different temperatures, so that by the principle of Archimedes the weights of the fluid displaced by the solid corresponding to the various temperatures are known.

Let V_0 denote the volume of the solid at 0° C. and α the coefficient of expansion of the solid, so that at temperature, $t°$, the volume is $V_0(1 + \alpha t)$. If ρ_t is the density of the water at temperature, t, the loss of weight due to immersion is $V_0(1 + \alpha t)\rho_t$. This value is observed by the balance; let it be W_t. Then

$$\rho_t = \frac{W_t}{V_0(1 + \alpha t)}.$$

In order to carry out the experiment, a long wire is attached to the scale-pan of a balance and passed through a hole in the base of the balance to support the solid, which hangs in the water. Since during some part of the experiment the water will be at a temperature considerably above that of the immediate surroundings of the balance, it is necessary to use a wire about 40 cms. long, so that convection currents may not disturb the equilibrium of the balance.

The wire should have a diameter not greater than $\frac{1}{10}$ mm., so that surface tension may not cause any appreciable effect where it enters the water. In practice thin copper wire is often employed, though it is preferable to use a short length of platinum wire, specially treated to diminish surface tension effect for immersion in the water.

A stirrer is necessary to keep the temperature of the liquid uniform, and as soon as the whirls, due to stirring, have died away, the balance is made and the temperature taken by a thermometer immersed in the water with its bulb as near as possible to the solid.

It is preferable to heat up the water to the highest desirable temperature and allow it to cool down. In this case the weights in the scale-pan will require to be continually diminished. The weights should be adjusted before an observation so that the solid appears a little too heavy. After a short interval the scale pointer will cross the zero position, and at this instant the temperature of the water should be observed.

In this way a series can be obtained very conveniently for temperatures above atmospheric.

The solid is first weighed in air so that the values, W_t, corresponding to different temperatures may be obtained by subtraction.

As a sinker it is usual to employ a glass bulb containing lead shot, and α may then be taken as ·000025.

When it is not possible to reduce the temperature right down to zero it may be taken down to some convenient low temperature, t_0. By finding the weight, W_{t_0}, of the submerged vessel at this temperature, the volume, V_{t_0}, may be deduced by the help of the density table appearing on p. 188, taken from Kohlrausch's " Physical Measurements."

For
$$V_{t_0} = \frac{W_{t_0}}{\rho_{t_0}}.$$

The densities at the other temperatures may then be deduced by the formula

$$\rho^t = \frac{W_t}{V_{t_0}\{1 + \alpha(t - t_0)\}}.$$

DENSITY OF WATER BETWEEN 0° AND 20° C. PER CC.

TEMP.	DENSITY	TEMP.	DENSITY
0°	0·99988	11°	0·99965
1°	0·99993	12°	0·99955
2°	0·99997	13°	0·99943
3°	0·99999	14°	0·99930
4°	1·00000	15°	0·99915
5°	0·99999	16°	0·99900
6°	0·99997	17°	0·99884
7°	0·99993	18°	0·99866
8°	0·99988	19°	0·99847
9°	0·99982	20°	0·99827
10°	0·99974		

In the laboratory it is convenient to begin at a temperature of about 80°, and make observations about every 10° C.

A curve should be drawn exhibiting the relation between temperature and density.

Alternatively the experiment may be made in order to determine the coefficient of expansion of water for varying intervals of temperature from the formula : $\rho_0 = \rho_t(1 + \beta t)$.

The Constant Pressure Air Thermometer

In this form of thermometer temperature is defined by means of the equation

$$V_t = V_0(1 + \alpha t);$$

t denotes the temperature, V_t and V_0 the volumes occupied of a certain mass of gas at two temperatures, the former at $t°$, and the latter at a convenient fixed point : the zero of the scale.

By choosing the melting point of ice as $t = 0°$, and the boiling point of water as $t = 100°$, under standard conditions we can find the value of α.

We may therefore say that the equation is assumed, and that t is defined by it,

or
$$t = \frac{1}{\alpha}\left(\frac{V_t}{V_0} - 1\right).$$

It is assumed that the pressure remains constant throughout.

The diagram shows a simple form of constant pressure thermometer.

The mercury reservoir is adjusted so that the mercury stands at the same level in the two tubes, CD and EF. When this is the case the pressure is equal to that of the atmosphere in both tubes.

Suppose it is desired to determine a certain temperature with this instrument, say the melting point of wax. First surround the bulb completely with powdered ice and turn the three-way tap, T, so that B and CD are open to the room, and adjust G so that the mercury stands at the zero division on the graduated scale of CD.

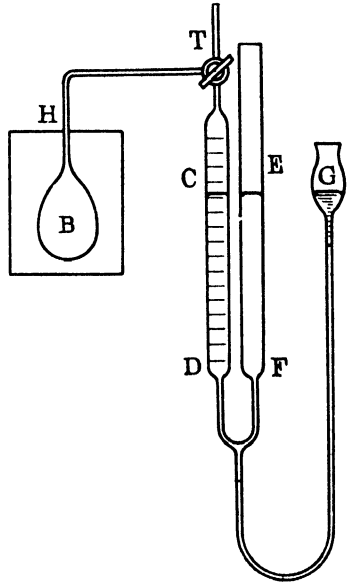


FIG. 101

Wait for about ten minutes to allow the bulb to cool exactly to 0° C., and turn the tap so that B and CD are connected to each other but cut off from the atmosphere. In this way the zero on the scale is made to correspond to 0° C.

Now immerse the bulb in boiling water, and lower G until the mercury stands at the same level in the two tubes, and observe the scale reading.

Read the barometric height, and deduce the boiling point of the water.

Surround the bulb with warm water and adjust its temperature to the melting point of the wax. To do this, put a small piece of the wax in a small test tube and immerse it in the water. Heat the water until the wax melts, and then let it cool a few degrees, and then warm up very slowly, keeping the water

stirred until the wax begins to melt again and then take the reading of the thermometer. Of course G must be adjusted so that the level of the mercury is the same in both tubes.

From the observations made we can deduce the melting point of the wax. Note the temperature also by means of an ordinary mercurial thermometer.

Theoretical Considerations

The fundamental equation of gas thermometry, whatever may be the form of thermometer, is simply:

Total mass of gas in the instrument = constant.

In practice it is not possible to maintain all the gas at the same temperature; some of it is necessarily remote from the point of application of the body examined. These remote regions are described by the term 'dead space.'

In our apparatus the dead space extends from H to the level of the mercury in C.

We shall suppose that the scale readings are in ccs., beginning at the zero and extending downwards.

Let the reading corresponding to the case when boiling water surrounds B be denoted by x_b, and let x_w be the reading when the wax is melting.

Let the volume from the top of the bulb at the point where it is immersed to the zero of the scale be denoted by v, with a suffix to indicate the temperature at which it is measured.

The temperature of the dead space, which has a total volume $(v + x)$, will vary from one end to the other; but we shall make our calculations by assuming that this temperature is uniform throughout and equal to that measured by placing a thermometer in a position approximately midway between the two ends of this space.

We shall denote this by the letter τ, and when the bulb temperature is t we shall write τ_t for the corresponding temperature of the dead space.

Let the volume of the bulb together with that part of the tube which is immersed be V_0 at the temperature zero, let β denote the coefficient of cubical expansion for glass, and ρ_0 the density of air at zero.

The mass of gas in terms of the quantities measured when the bulb is at 0° C. is:

$$V_0\rho_0 + v_{\tau_0}\rho_{\tau_0}.$$

The temperature of the dead space is, of course, not necessarily at 0°, it has some value τ_0.

Or we may write for this mass:

$$\rho_0 \left\{ V_0 + v_0 \frac{(1 + \beta\tau_0)}{(1 + \alpha\tau_0)} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

When the bulb is a temperature, t, let x_t denote the reading on the scale. This denotes the volume between the zero of the scale and the mark x_t at the temperature at which the apparatus was graduated. This is often done at about 15° C., and the laboratory temperature is usually in this neighbourhood. We shall not introduce any great error into our calculations if we regard this as measuring the true volume of this part of the apparatus under the conditions of the experiment.

The temperature of the dead space is now τ_t, and the total volume is therefore :

$$v_0(1 + \beta\tau_t) + x_t$$

The mass of gas in the dead space is :

$$\frac{v_0(1 + \beta\tau_t)\rho_0}{1 + \alpha\tau_t} + \frac{x_t\rho_0}{1 + \alpha\tau}$$

Hence the total mass is measured by :

$$\frac{V_0(1 + \beta t)\rho_0}{1 + \alpha t} + \frac{v_0(1 + \beta\tau_t)\rho_0}{1 + \alpha\tau_t} + \frac{x_t\rho_0}{1 + \alpha\tau_t} \quad \ldots\ldots\ldots (2)$$

If the expressions (1) and (2) be equated, since they denote the same quantity, it will be found that :

$$x_t = V_0(1 + \alpha\tau_t)\left[\frac{(\alpha - \beta)t}{1 + \alpha t} + \frac{v_0}{V_0}\left\{\frac{1 + \beta\tau_0}{1 + \alpha\tau_0} - \frac{1 + \beta\tau_t}{1 + \alpha\tau_t}\right\}\right] \ldots (3)$$

The second term in the square bracket may be written :

$$\frac{v_0(\alpha - \beta)(\tau_t - \tau_0)}{V_0(1 + \alpha\tau_0)(1 + \alpha\tau_t)},$$

where we have neglected the product $\alpha\beta$ in the numerator.

In practice the difference between τ_t and τ_0 is not very large, and the construction of the apparatus provides that $\frac{v_0}{V_0}$ is small.

We may thus neglect this term in comparison with the first term in the bracket without introducing any great error. The student is recommended to find an approximate value of the two terms, so that he may better appreciate the effect of this neglect of the second term.

We then have :

$$x_t = V_0(\alpha - \beta)t \cdot \frac{1 + \alpha\tau_t}{1 + \alpha t} \quad \ldots\ldots\ldots\ldots (4)$$

If we apply this to the case when the bulb is surrounded by boiling water, of which the temperature is b, corrected of course

for any variation of the barometric height from normal, we have from (4)

$$x_b = V_0(\alpha - \beta)b \cdot \frac{1 + \alpha\tau_b}{1 + \alpha b} \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Hence from (4) and (5)

$$\frac{x_t}{x_b} = \frac{1 + \alpha\tau_t}{1 + \alpha\tau_0} \cdot \frac{1 + \alpha b}{1 + \alpha t} \cdot \frac{t}{b}$$

$$= \frac{\frac{1}{\alpha} + \tau_t}{\frac{1}{\alpha} + \tau_b} \cdot \frac{\frac{1}{\alpha} + b}{\frac{1}{\alpha} + t} \cdot \frac{t}{b} \quad \ldots\ldots\ldots\ldots\ldots(6)$$

The value of $\frac{1}{\alpha}$ may be taken as 273·1.

τ_t and τ_e may be observed on a mercury thermometer, although this is introducing into the experiment the mercury scale.

Equation (6) is a linear equation in t, which is thus determined from the reading on the scale of the air thermometer.

This enables us to deduce the temperature of the melting wax.

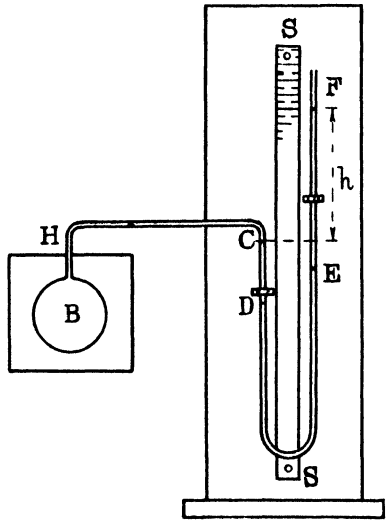


FIG. 102

The Constant Volume Air Thermometer

The diagram illustrates a common form of the apparatus.

The bulb, B, is connected by a capillary tube to rubber tubing, DE, and to the glass tube, EF.

EF slides against a scale, SS, by means of which the height between the mercury levels in the tubes, CD and EF, can be read.

EF can be clamped in any desired position, so that the level of the mercury in the tube CD stands always at a definite mark, C.

In this case the temperature is defined by assuming the relation:

$$p_t = p_0(1 + \alpha t), \quad \text{or} \quad t = \frac{1}{\alpha} \cdot \frac{p_t - p_0}{p_0}.$$

p_t denotes the pressure within B at a temperature t, and p_0 that at a standard zero position—the temperature of melting ice, while the volume of gas remains constant.

The pressures are measured by adding the atmospheric pressure to that due to the mercury column of length, h.

In practice the part of the apparatus containing the air expands with rise of temperature, and there is the dead space, HC, to be allowed for as in the last experiment.

Let V_0 denote the volume of the bulb and immersed portion of the apparatus, v_0 that of the dead space at 0°.

Suppose that the dead space remains at temperature τ_0 during the experiment when B is surrounded with ice, and τ_t when it is surrounded by steam.

We shall show how α may be determined experimentally.

If ρ_0 denote the density of air under normal conditions, and ρ that at pressure, p, and temperature, t,

$$\rho = \frac{\rho_0}{1 + \alpha t} \cdot \frac{p}{76}.$$

The mass of gas contained in the thermometer is:

$$\frac{p_0\rho_0}{76} \cdot V_0 + \frac{p_0\rho_0}{76} \cdot v_0 \frac{(1 + \beta\tau_0)}{(1 + \alpha\tau_0)} \quad \ldots\ldots\ldots\ldots (7)$$

expressed in terms of the conditions prevailing when B is at the temperature of melting ice.

Similarly the mass of gas expressed in terms of the conditions prevailing when B is at temperature, t, is:

$$\frac{V_0(1 + \beta t) \cdot p_t\rho_0}{76(1 + \alpha t)} + \frac{V_0(1 + \beta\tau_t)p_t\rho_0}{76(1 + \alpha\tau_t)} \quad \ldots\ldots\ldots (8)$$

Hence on equating (7) and (8):

$$p_0\left\{V_0 + \frac{v_0(1 + \beta\tau_0)}{(1 + \alpha\tau_0)}\right\} = p_t\left\{V_0 \cdot \frac{1 + \beta t}{1 + \alpha t} + v_0 \cdot \frac{1 + \beta\tau_t}{1 + \alpha\tau_t}\right\} \quad \ldots (9)$$

Again on account of the smallness of the ratio $\frac{v_0}{V_0}$ we have as an approximation:

$$p_0 = p_t \cdot \frac{1 + \beta t}{1 + \alpha t} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

When the bulb is at the temperature, b, of boiling water,

$$p_0 = p_b \frac{1 + \beta b}{1 + \alpha b}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

We may assume the value ·0000232 for β and thus calculate α from (11) by observing the pressures when the bulb is surrounded by melting ice and by steam respectively.

Equation (10) then enables us to deduce the temperature corresponding to any pressure, p_t.

Take a mercury thermometer and immerse it close to the bulb, observing its readings and the corresponding pressures. From the latter deduce the temperatures from (10) and draw up a table recording these in one column opposite to the records of the mercury thermometer in a second column.

Draw a graph with air temperatures as ordinates and mercury temperatures as abscissæ, exhibiting the deviations between the two temperature scales.

In order to calculate the value of $\frac{v_0}{V_0}$, if this is necessary, first adjust the mercury to the mark C, immersing B in water that has come to the room temperature.

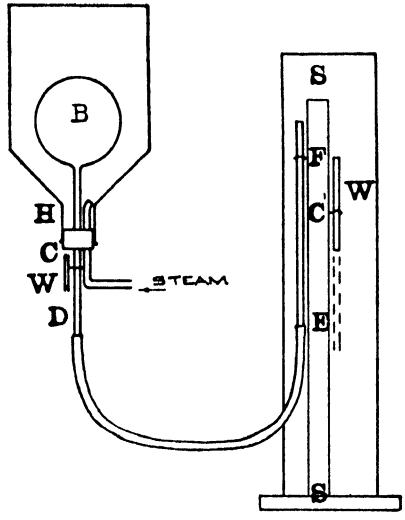


FIG. 103

Read off the pressure, P, to which the air is now subjected.

Carefully raise EF so that the mercury approaches the bend at H, and so fills nearly all the dead space. Let the pressure within the bulb be now P_2. Then, since the conditions are

isothermal and the whole volume, v_0, has been very nearly filled with mercury in the second case,

$$P_2V_0 = P_1(V_0 + v_0),$$

$$\therefore \frac{v_0}{V_0} = \frac{P_2 - P_1}{P_1}$$

We may read the temperatures τ_0 and τ_t by means of a mercury thermometer placed close to the dead space and obtain a closer approximation to the value of α.

In the second form of apparatus (fig. 103), into which the first may be readily converted, the volume of the dead space is made negligible.

In order to measure the difference of level between C and F, two tubes, WW, connected by a rubber tube containing water are adjusted so that the level on the left is the same as that at C, and consequently this is the same on the right at C^1.

The distance, C^1F is readily observed on SS.

CHAPTER VII

CALORIMETRY

The Specific Heat of a Solid by the Method of Mixture

THE student will be familiar with the principle of the method of mixture. The main object in this description is to give an account of the method of making a correction for the error arising from radiation.

If W is the water equivalent of the calorimeter and contents, m the mass of the solid, and s its specific heat, and if t_1 is the initial temperature of the calorimeter, t_2 the final temperature and T that of the solid initially, then if there has been no loss of heat we have :

$$ms\,(\mathrm{T} - t_2) = \mathrm{W}(t_2 - t_1).$$

In practice there is a loss or gain of heat from or to the calorimeter, which should be added on the right-hand side of this equation, since all the heat from the solid has not been retained in the calorimeter.

We may make the correction by the consideration that the final temperature, t_2, would have been $t_2 + \Delta t$, where Δt is an interval of temperature which must be small if the experiment is to be successful.

Hence the corrected equation is :

$$ms(\mathrm{T} - t_2) = \mathrm{W}(t_2 + \Delta t - t_1).$$

Δt will be small if during half the experiment the calorimeter gains heat, and in the other half loses heat. This can be arranged by adjusting the initial temperature so that the room temperature is approximately a mean between it and the final temperature.

A preliminary experiment is made to find out roughly the temperatures that will be attained during the experiment.

The same amounts of the materials are used in a second case, but the calorimeter is cooled down by adding small pieces of ice or warmed up, as may be necessary, so that t_1 and t_2 may lie at nearly equal temperature intervals below and above the temperature of the room.

The temperature of the calorimeter is noted immediately before immersing the hot body, and then at quarter or half minute intervals until the maximum temperature is attained, the

observations being continued beyond this point at definite intervals.

These results should be plotted on a graph (fig. 104).

The curve obtained will be similar to ABC. Had there been no losses or gains on account of radiation, the curve would have been similar to ADE, the final temperature remaining constant at the level DE.

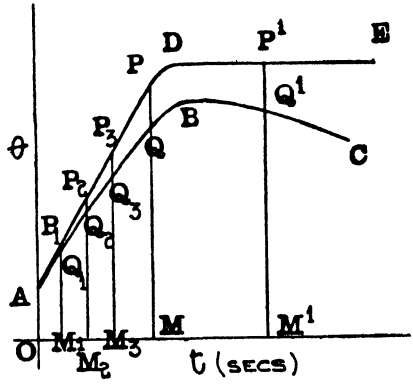


FIG. 104

We can, by applying Newton's Law of Cooling, derive the curve ADE from ABC.

Suppose the axis of t to be divided into small intervals, OM_1, M_1M_2, M_2M_3, etc., of magnitudes δt_1, δt_2, δt_3, etc., and let the average temperature during these be : θ_1, θ_2, θ_3, etc.

Then the loss by radiation in δt_1 is $k\theta_1\delta t_1$, where k is a constant.

$$\therefore\ P_1M_1 = Q_1M_1 + k\theta_1\delta t_1.$$

Similarly the loss during M_1M_2 is $k\theta_2\delta t_2$, and consequently

$$P_2M_2 = Q_2M_2 + k\theta_1\delta t_1 + k\theta_2\delta t_2.$$

This process may be continued to any extent, and if we make the intervals sufficiently short, Q_1M_1, Q_2M_2, etc., are not sensibly different from θ_1, θ_2, etc.

Thus, denoting by Θ the ordinate of the upper curve at a time, t, and by θ that of the lower at the same time :

$$\Theta_1 = \theta_1 + k\theta_1\delta t_1,$$
$$\Theta_2 = \theta_2 + k\theta_1\delta t_1 + k\theta_2\delta t_2;$$

and generally :

$$\Theta_n = \theta_n + k(\theta_1\delta t_1 + \theta_2\delta t_2 + \ldots + \theta_n\delta t_n)$$
$$= \theta_n + k(\text{area of lower curve from OA to ordinate } \theta_n),$$

i.e. $$\Theta = \theta + k\int_0^t \theta dt.$$

We can thus make the correction by carefully drawing the lower curve, calculating the area up to the point θ, which it is desired to correct, and add its product by k to θ.

We require in this experiment the ordinate corresponding to any point P^1 along DE, beyond the point B.

We therefore determine graphically OAQ^1M and apply the correction to Q^1M, thus obtaining P^1M.

In order to calculate k, find the rate of cooling along BC, corresponding to a mean temperature θ.

$$k = \frac{1}{\theta} \cdot \frac{d\theta}{dt} \qquad \text{(Newton's Law.)}$$

It will usually be convenient to note the fall for four or five minutes, and make the deduction from it.

Reynault's method of making the correction is actually to make the correction of the ordinates and draw the curve ADE.

For this purpose we require to know the rate of cooling at any particular temperature.

Find the rate at one period, as above, and plot $\frac{d\theta}{dt}$ against θ on a curve. In doing this it is assumed that the relation is linear in accordance with Newton's Law.

At the temperature of the room $\frac{d\theta}{dt}$ vanishes; for if the calorimeter and surroundings were at the same temperature then there would be no loss due to radiation.

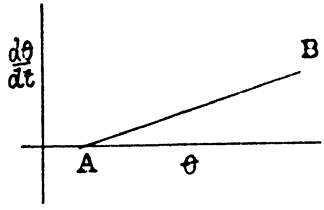


FIG. 105

We therefore have two points on the graph, and by joining these by a straight line we can determine $\frac{d\theta}{dt}$ at any value of θ (fig. 105.)

Divide ABC into sections very nearly straight, as AR, RQ, etc. Note the mean temperature over AR and from the graph for $\frac{d\theta}{dt}$ note the rate $\frac{d\theta}{dt}$ for this temperature. Multiply this by the

time, ON, and add the result to the ordinate, NR, thus obtaining NR^1. Let this correction be $\delta\theta_1$.

In the same way find the amount $\delta\theta_2$, lost during the interval, NM. Add the sum $(\delta\theta_1 + \delta\theta_2)$ to MQ and so obtain MQ^1.

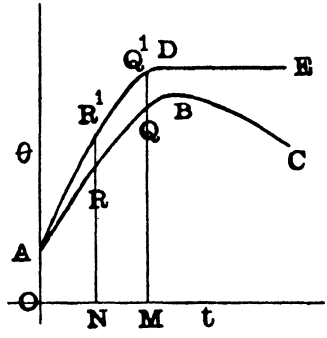


FIG. 106

Continue this process until the maximum ordinates along DE are attained.

The curve obtained in the experiment should attain this horizontal branch very nearly, and the ordinate is the quantity $t_2 + \Delta t_2$.

Specific Heat of a Liquid by the Method of Cooling

The rate of loss of heat of a body depends only on the temperature of the body and that of its surroundings, on the area, and on the nature of the surface exposed.

If the difference of temperature between the body and its surroundings is not large, the rate of emission of heat is proportional to the temperature difference. This is Newton's Law of Cooling.

Suppose a mass of liquid, M_1, is enclosed within a calorimeter of mass m_1, and let S_1 and s_1 denote the specific heats respectively. The thermal capacity of the system is $(M_1S_1 + m_1s_1)$. If the temperature fall from t_1 to t_2 in n_1 seconds, the average rate of loss of heat is $(M_1S_1 + m_1s_1) \cdot \dfrac{t_1 - t_2}{n_1}$.

In the case of the second liquid under the same conditions, let n_2 denote the number of seconds required for a fall of temperature from t_1 to t_2, and the loss of heat per second is

$$(M_2S_2 + m_1s_1)\left(\frac{t_1 - t_2}{n_2}\right).$$

We have by Newton's Law :

$$(M_1S_1 + m_1s_1)\frac{t_1 - t_2}{n_1} = (M_2S_2 + m_1s_1)\frac{t_1 - t_2}{n_2};$$

$$\therefore S_1 = \frac{n_1(M_2S_2 + m_1s_1)}{n_2M_1} - \frac{m_1s_1}{M_1} \quad \ldots\ldots(1)$$

The apparatus for carrying out the determination consists of a small calorimeter fitted with a rubber stopper through which a thermometer may pass (fig. 99.) A calorimeter of aluminium of about an inch diameter and three inches high serves the purpose very well. The calorimeter should be supported by threads, or should stand on a non-conductor within a double-walled enclosure, the thermometer passing through a cork in the lid of the enclosure.

In order to secure a uniform temperature, the space between the walls of the enclosure may be partly filled with water. In this case care must be taken that the inner box does not float, or it may happen that it will touch the calorimeter and there will be loss of heat by conduction.

The enclosure may consist of two calorimeters—an outer large one fitted with a lid, and an inner smaller one standing on blocks.

First fill the aluminium calorimeter about two-thirds full of water, and warm it to a temperature about 70° C. by immersing it in hot water. Place the apparatus in the position shown in the diagram, and take readings of the thermometer at intervals of half or whole minutes down to a temperature below 30° C.

Note from time to time the temperature of the enclosure, which should hardly vary during the experiment.

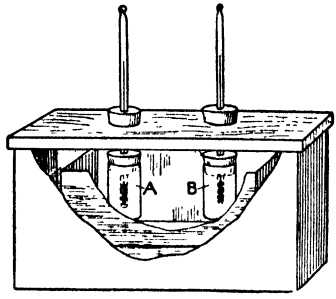


FIG. 107

Some time must of necessity elapse between the observations on one liquid and those on another, and although it is not difficult to maintain a constant enclosure temperature throughout each set of observations, it often happens that the mean temperatures recorded by the thermometer, T_2, are appreciably different in the two cases.

This difficulty may be avoided by using a second aluminium container similar to G, and suspending it by the side of G inside C. The records of the temperatures of the two liquids are then made almost together, and the enclosure temperature is the same for each.

Make up a table containing the liquid temperatures opposite the times of observation, and in a third column record the enclosure temperatures.

Draw on the same graph as illustrated in fig. 108 the curves, one for each liquid, with the differences of temperature between liquid and enclosure as ordinates, and with the times for abscissæ.

Let AB denote the curve for paraffin (say) and A_0B_0 that for water.

Draw the horizontal lines, T_1AA_0 and T_2BB_0 to cut the curves at A, A_0 and B, B_0 as shown.

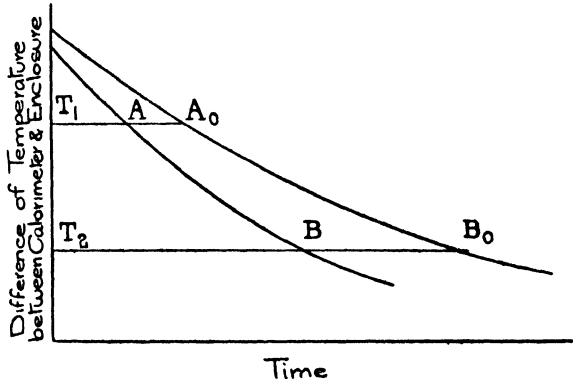


FIG. 108

Let t_1 denote the temperature of the liquids above that of the enclosure in the first case, and t_2 the corresponding temperature in the second.

Then in the time that elapses between the instant measured by T_1A until that measured by T_2B the paraffin cools down the interval $(t_1 - t_2)$, and the water cools down the same amount during the interval between T_1A_0 and T_2B_0. Denote the two periods of cooling by n_1 and n_2. In this case the value of S_2 is unity, and for aluminium the value of s_1 is ·219.

By weighing the liquids and calorimeters we obtain sufficient data to give the value of s_2 for paraffin by means of the formula (1).

Determination of the Specific Heat of a Solid by means of Joly's Steam Calorimeter

A metal jacket, J, enclosed in a casing of felt surrounds a platinum pan, P, suspended by means of a fine wire attached to one arm of a balance, whose base is shown at BB.

The upper end of J is closed by a light metal disc, D, through a hole in which the wire passes. This disc is free to move, and when oscillations occur in P it finally settles down so that the wire passes through the hole without contact with the disc. Just above D, a small coil of wire carrying a heating current round the suspension prevents condensation on it and also on D

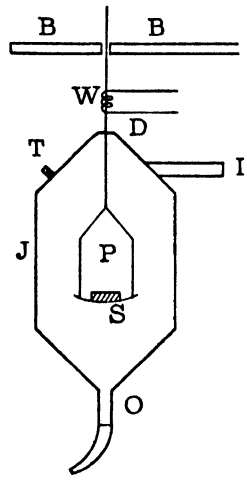


FIG. 109

In the first place, J is allowed to attain the temperature of the room, and the inlet and outlet pipes are then closed.

The pan is balanced in the usual way, and in the meantime water is boiled in a container ready to supply J with steam by means of I.

When a good supply is obtained O is opened and steam passed through I. When the steady state is attained it will be found that additional weights are required to counterpoise P on account of the condensation of steam on it. Suppose w grammes are condensed and let the initial temperature of P, which has been observed by a thermometer placed in T, be t_1, and that of steam t_2.

Once more allow the apparatus to dry and weigh the solid S. When steam is again passed into J with S in the pan, a greater amount of steam will be condensed on account of S. Let this now be W and suppose the initial temperature now is T_1.

Previously the scale-pan condensed w grammes of steam and rose in temperature through the interval $(t_2 - t_1)$. In the second case it rises from T_1 to t_2. The amount of condensation per degree rise of temperature is $\frac{w}{(t_2 - t_1)}$, hence the weight of steam condensed in the second case is : $\frac{w}{(t_2 - t_1)} \times (t_2 - T_1)$.

Denote the mass of the solid by m and the specific heat by s. The mass of steam condensed by the solid is :

$$W - \frac{w(t_2 - T_1)}{(t_2 - t_1)}.$$

Hence we have the equation :

$$ms(t_2 - T_1) = W - \left\{\frac{w(t_2 - T_1)}{(t_2 - t_1)}\right\} L,$$

by means of which the value of s may be calculated from a knowledge of the value of L or, conversely, L may be determined if the specific heat of the solid is known.

L denotes, as usual, the latent heat of steam.

A correction to account for the differences in apparent weights of the solid in air and steam has been neglected. The temperature of the solid is supposed equal to that of the apparatus and surrounding air. An interval of certainly not less than twenty minutes is required to allow the solid to acquire this temperature.

Bunsen's Ice Calorimeter

Description and Preparation for Use

A diagram of the apparatus is shown in fig. 110. The calorimeter is represented by ABCFED. It consists of a test tube, B,

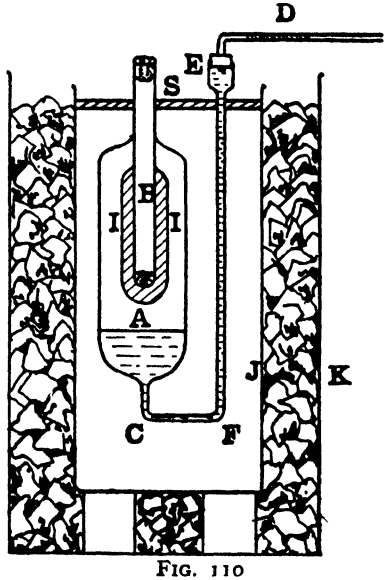


FIG. 110

fitted into the glass jacket, A, which is drawn out at its base into the tube, CFE. This tube ends in a cup, E, closed by a cork, through which passes the narrow glass tube, D.

A is filled partly with clean mercury and partly with distilled water containing no air.

The surrounding jacket, J, is a calorimeter closed with the cork or wooden stopper, S, which supports the apparatus.

In order to keep J and its contents at the freezing point, it is placed in a larger vessel, standing on non-conducting blocks and packed round with a mixture of ice and snow or with flaked ice.

By cooling the inner surface of B sufficiently a layer of ice may be formed round the outside, as indicated at I.

On melting one gramme of ice the volume diminishes by ·0907 c.c., so that if heat be added at B the amount may be determined by noting the change of volume as a result of the partial melting of I. The change of volume is observed by noting the movement of the end of the mercury column at D along the capillary tube. If this has been previously calibrated the change can be observed directly.

In order to fill the apparatus, remove the capillary tube, D, and the stopper in E, and introduce into A sufficient distilled water to fill it to about half. Invert the apparatus with the open end of the test tube downwards and carefully boil the water, continuing until A is about one-third full.

While this is proceeding, boil some distilled water in a large beaker, and towards the end of the evaporation of the water in A, place the end, E, well under the surface in the beaker.

Cease boiling the water in the calorimeter, and allow more to flow over from the beaker. In this way the inside of the calorimeter and the tube, CFE, become filled.

Clean mercury must now be passed in to lie below the water in A.

Introduce it gradually from a pipette held under the surface of the water in E, allowing displaced water to overflow. Take care that no air bubbles are introduced with the mercury, particularly when it becomes necessary to tilt the apparatus to allow water to pass over the mercury in A towards the tube. Fill up with mercury to E, place the stopper in position, and by carefully adjusting it make the end of the thread coincide with any desired position along D.

The apparatus should then be placed in a calorimeter containing water and ice to reduce the temperature as nearly as possible to zero.

This will probably take an hour at least, and the progress of the fall may be tested by placing a thermometer in B.

When the temperature is about 2° C. introduce cooled ether into B. The ether may be cooled by placing it in a cooled test tube, and standing it in the calorimeter with the apparatus.

Draw air through the ether and cause it to evaporate, continuing until a cap of ice surrounds B.

The solidification will cause D to move farther along the capillary, and enough ice should be formed to cause more expansion than is likely to be required in succeeding experiments with the apparatus.

The evaporation of the ether may be brought about by some such device as that indicated in fig. 111.

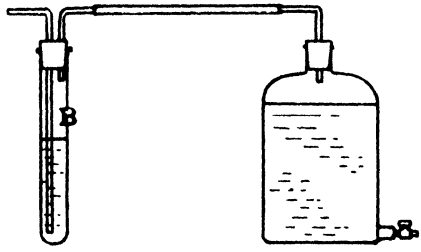


FIG. 111

When sufficient ice has been formed the remaining ether is evaporated, and a current of air further drawn through to remove all traces of it.

Now place the apparatus within the jacket, J, and stand it in the vessel, K, packing it round as described above with ice and snow.

Leave this standing for an hour or two until a steady state has been reached, and the movement in the capillary tube is only slight and steady.

It is not possible to maintain the end of the mercury thread quite steady with this arrangement, so that the slight motion must be accounted for in determining results from observations.

The capillary tube may be calibrated by the method described on p. 41.

Calibration of the Apparatus

By placing warm water within B of known mass and temperature, we may note the movement of D at various parts of the capillary tube for a known absorption of heat.

In performing the experiment it will be sufficient to calibrate the tube for one particular region, and in using the instrument again a slight pressure on the stopper or a slight easing of it will drive the end of the thread into the calibrated strip.

Heat up pure water to about 25° C., and transfer carefully to B. Allow the apparatus sufficient time to become steady, and note the displacement of the end of the mercury thread. Let this be l, and let the time be noted between the insertion of the water and the return to steady conditions. This will be denoted by t.

In order to correct for the small creep of the thread, observe the rate of motion just before adding the warm water, and also just after the absorption of heat.

All these measurements are to be made with a travelling microscope mounted conveniently opposite the capillary tube.

If the rates of creep are respectively p and p^1, then the average rate may be taken as $\frac{1}{2}(p + p^1)$ during the time t.

Thus the displacement of the thread due to absorption of heat from the water is $l - \frac{1}{2}(p + p^1)t$. We shall call this quantity L.

If m is the mass of water added and θ its initial temperature, then a motion of L units corresponds to the absorption of $m\theta$ calories.

Determination of the Specific Heat of a Substance

Let M grammes of the substance be heated to a temperature θ^1, and let s denote the specific heat.

The substance when placed in B will transfer to the ice $M\theta^1 s$ calories.

If the mercury thread moves a distance, L^1, in the calibrated region, the heat absorption is $\frac{m\theta}{L} \cdot L^1$ calories.

Thus
$$M\theta^1 s = \frac{m\theta L^1}{L},$$

or
$$s = \frac{m\theta}{M\theta^1} \cdot \frac{L^1}{L}.$$

The correction for creep must again be applied by observing the motion just before and just after the insertion of the mass. L^1 is the corrected length.

When a solid is put into B, a pad of cotton wool should be placed at the bottom of the tube to prevent breakage when it falls. In order to facilitate removal it is a good plan to tie a light thread round it. This will introduce only a slight error. During the absorption of heat, and generally while the apparatus is in use, the end of the tube, B, should be stopped with a plug of cotton wool.

In the case of determining the specific heat of a liquid the experiment is almost exactly a repetition of the calibration.

In order to dry the tube after liquid has been put in, a roll of clean blotting paper may be used.

The Determination of the Density of Ice

In this experiment we require to know the volume per unit length of the capillary tube. We have assumed that this has been previously calibrated. It thus remains to determine the

shrinkage due to absorption of a definite quantity of heat. The experiment may be performed in conjunction with the calibration just described.

Let L denote the latent heat of fusion of ice, and suppose warm water added to B imparts k calories.

The amount of ice melted is $\frac{k}{L}$ grammes.

Let δv denote the shrinkage as measured by the movement of the mercury thread.

Then $\frac{k}{L}$ grammes of ice have become $\frac{k}{L}$ grammes of water at o° C.

Let d_0 denote the density of water at this temperature.

The volume of water is $\frac{k}{d_0 L}$ c.c., and the volume of the ice is thus :

$$\left(\frac{k}{d_0 L} + \delta v\right) \text{c.c.}$$

Hence the density of ice is :

$$\frac{\frac{k}{L}}{\frac{k}{d_0 L} + \delta v} \text{ grammes per c.c.}$$

The Latent Heat of Fusion of Ice

It is assumed that the student is familiar with the principles of the determination of the latent heat of fusion and has carried out the experiment without making corrections for radiation. We are concerned in this description chiefly with an account of how this correction may be made. Care is taken, as in the determination of specific heat, to adjust the initial and final temperatures so that the room temperature is the mean of the two. In addition, care must be taken that the calorimeter is not cooled down so low that the dew point is reached, otherwise there will be a deposit of dew on the apparatus, and a liberation of latent heat in consequence.

We may make the correction for radiation as in the experiment on specific heat, but an alternative method will be described.

Note the temperature when the ice is placed in the calorimeter, and at intervals of half a minute until it is melted, and finally at intervals of one or two minutes during which the calorimeter is absorbing heat by radiation.

Let the temperatures observed in this second period be

$$T_1, T_2, T_3 \ldots T_{n+1}.$$

Then the change of temperature due to radiation during these intervals will be :

$$\delta\theta_1 = A[\tfrac{1}{2}(T_1 + T_2) - t_0]$$
$$\delta\theta_2 = A[\tfrac{1}{2}(T_2 + T_3) - t_0]$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\delta\theta_n = A[\tfrac{1}{2}(T_n + T_{n+1}) - t_0],$$

by Newton's Law of Cooling, where A is a constant depending on the calorimeter but not on its temperature, and t_0 is the temperature of the surroundings.

Thus the total change of temperature $\Delta\theta$ is given by :

$$\Delta\theta = A\left\{\frac{T_1 + T_{n+1}}{2} + (T_2 + T_3 + \ldots + T_n) - nt_0\right\}.$$

But $\Delta\theta = T_{n+1} - T_1$, so that A can be calculated.

In the first part of the experiment during the melting of the ice, let the observed temperatures be : $t_1, t_2, \ldots t_{n+1}$.

Then we have :

$$\Delta t = A\left\{\frac{t_1 + t_{n+1}}{2} + (t_2 + t_3 + \ldots + t_{n+1}) - nt_0\right\}.$$

Thus, since the observed minimum temperature is t_{n+1}, the corrected minimum is $(t_{n+1} + \Delta t)$.

In the above the $\delta\theta$'s and δt's are to be treated algebraically, for some will be negative and some positive if the initial temperature is adjusted in the manner described.

Thus, if W is the water equivalent of the calorimeter and contents, and m the mass of ice melted, we have :

$$mL + mt_{n+1} = W\{t_1 - (t_{n+1} + \Delta t)\}.$$

The Latent Heat of Vaporization. (Berthelot's Apparatus)

One form of this apparatus is depicted in fig. 112. Its essential feature is the condenser spiral, C, with the receptacle below. This is immersed in water in a calorimeter and the calorimeter shielded by packing it round with a non-conductor and placing it in a convenient vessel, or better still by standing it on non-conducting blocks in an empty larger calorimeter and enclosing within the larger vessel (see fig. 112).

The condenser is dried and weighed. It is placed in the water and allowed to stand until the temperature becomes steady as recorded by the thermometer, the uniformity of temperature

throughout the calorimeter being procured by means of a stirrer.

A quantity of vapour is introduced into the condenser, and the spiral provides a large area of contact with a cold surface, so that liquefaction takes place and the liquid collects in the receptacle. The water is constantly stirred, and its temperature observed by means of the thermometer.

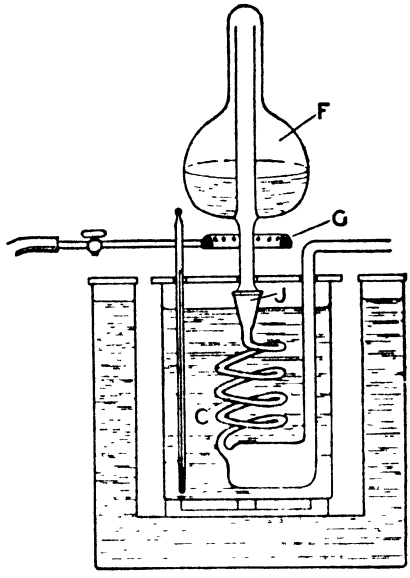


FIG. 112

When a suitable rise is obtained the supply of vapour is cut off and the thermometer watched until the maximum temperature is reached. In this interval, immediately after cutting off the supply, the end of the condenser outlet tube must be closed by a cork to avoid convection effects.

The condenser is removed, dried, and weighed, and the mass condensed thus determined. Denote this by m, so that if L denotes the latent heat the supply of heat on condensation is mL calories.

Let T_1 denote the temperature of vaporization, and T_2 the final temperature of the calorimeter and contents, while T_0 denotes the initial temperature of the calorimeter.

Let s denote the specific heat of the liquid, and W the water equivalent of the calorimeter, condenser, and remaining calorimeter contents. Then : $W(T_2 - T_0) = mL + m(T_1 - T_2)s$; so that L may be determined.

One of the weak points of this form of apparatus is the mode of introduction of the vapour.

As the diagram shows, the liquid is vaporized over a small gas ring in the reservoir. It thus easily happens that the vapour gets superheated, and, in addition, it is difficult to shield the calorimeter effectively from the heat of the flame. The screen, S, is introduced to reduce this effect.

It is preferable to use an electrical method of heating, and this is done in the more recent forms of the apparatus.

Kahlenberg's heater is illustrated in fig. 113. The liquid is contained in the vacuum flask, H, and is heated by passing an electric current through the platinum wire, PP.

For ordinary laboratory practice a test tube may take the place of H without introducing much difficulty in shielding the calorimeter.

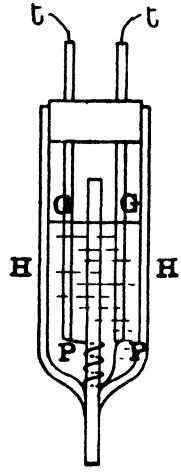


FIG. 113

Correction for loss due to radiation in the calorimeter, C, must be made by one of the usual methods (p. 197).

The Heat of Solution of a Salt

When a salt is dissolved by a liquid the solution is accompanied by an absorption or liberation of heat. The amount varies with the proportion of the salt to the solvent, i.e. with the resulting concentration, and with the amount of salt dissolved.

The number of calories absorbed or liberated when one gramme of a salt is dissolved in a certain amount of solvent is said to be the heat of solution for the particular concentration.

The experiment described below is designed to measure this quantity.

The apparatus necessary is illustrated in fig. 114. It consists of an outer protecting calorimeter of metal, C, which carries a cork through which the inner vessel, A, is supported.

This vessel also carries a cork through which pass a thermometer, stirrer, and thin test tube, B, into which the salt may be placed.

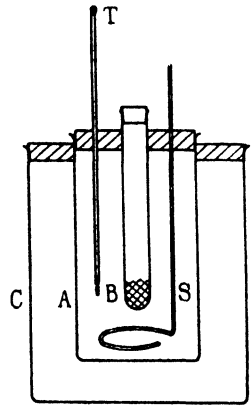


FIG. 114

The vessel, A, and stirrer, S, are usually of glass since many solutions attack copper. There is uncertainty about the specific heat of the glass, so that the water equivalent of the vessel, A, and its contents should be determined in a separate experiment. For many purposes, however, we may assume the specific heat of the glass to be ·16. The pure solvent is placed inside A, and it should surround the lower part of the tube, B, which contains the salt.

The apparatus is allowed to stand so that the salt may acquire the temperature of the solvent. Half an hour should be allowed for this, and during this interval the salt may be occasionally stirred by a clean dry glass rod.

The weight of solvent is obtained in the usual way before the insertion of B. Let this be denoted by W.

Suppose the water equivalent of A and its contents is w, and the specific heat of the solution s. This quantity must be determined later by the method of cooling (p. 199), or by any other convenient experiment.

It will be necessary, also, to make a correction for radiation by one of the methods previously described (p. 197).

Let the initial temperature of the salt and solvent be denoted by t_0, and the final temperature, corrected for radiation, by t_1,

and suppose that the weight of the salt dissolved is q. The heat of solution being Q, we have :

$$qQ = \{(W + q)s + w\}(t - t_0),$$

for $(W + q)$ is the weight of the solution.

The value of Q will be positive for the liberation of heat, and negative for absorption.

In order to mix the salt and solvent, the glass rod used above for stirring should remain in the salt until just before mixing. At this instant observe the temperature, t_0, recorded on the thermometer, and push the rod through the bottom of the thin tube, B.

This tube must, of course, be clean and dry, and the salt will then fall into the solvent and be dissolved. The rod should then be removed, and care taken not to carry with it any of the solvent. The solution is assisted by the stirrer, S, and observations of T taken every quarter or half minute in the initial stages, and later at longer intervals.

A graph is drawn showing the variation of temperature with time from the instant of mixing, and this curve corrected for radiation as explained above, or we may use the non-graphical method.

To find the Ratio of Specific Heats at Constant Pressure and Constant Volume for Air. (Clement and Desorme's Experiment)

Apparatus and Experimental Details

A glass reservoir, provided with a tap, T, giving a wide opening to the air is connected to an oil manometer, G, and to a pump;

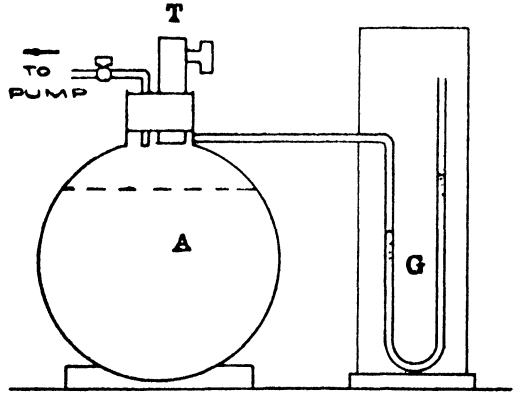


FIG. 115

an ordinary bicycle pump is convenient. A small excess pressure is applied to A, the difference between it and the atmospheric pressure being measured by the manometer (fig. 115).

In the first stage of the experiment the temperature under these conditions is allowed to become steady.

In the second the tap, T, is opened and closed suddenly by giving one half-turn.

By this means the pressure falls to that of the atmosphere in so short an interval that we may suppose there is no passage of heat to A during this expansion.

The condition of the expansion is therefore adiabatic.

Finally the temperature is allowed to return to that at the beginning of the experiment, during which process the pressure in A increases, though it does not recover its original value.

Theory.

Suppose that gas occupying the volume below the dotted line remains in the flask all the time.

Denote its volume by V_0 while that of the flask is V.

Let the initial pressure be P_0, and let that immediately after the adiabatic expansion be B.

Then $$BV^\gamma = P_0V_0^\gamma,$$

where γ = ratio of specific heats, the value of which we require.

The flask is open to the air so that B is the atmospheric pressure.

It is in order that we may ensure a fall of pressure from P_0 to B during a short interval that the tap is wide.

In the final stage let the pressure become P when the temperature has attained a steady value.

We have now passed from volume, V_0, and pressure, P_0, by an isothermal process to volume, V, and pressure, P.

$$\therefore \quad PV = P_0V_0.$$
$$\therefore \quad P'V^\gamma = P_0'V_0^\gamma.$$

Hence by dividing this by BV^γ and $P_0V_0^\gamma$, we have :

$$\frac{P^\gamma}{B} = P_0^{\gamma-1},$$

or

$$\left(\frac{P}{P_0}\right)^\gamma = \frac{B}{P_0};$$

$$\therefore \quad \gamma = \frac{\log \frac{B}{P_0}}{\log \frac{P}{P_0}}.$$

Let the difference in heights of the manometer be h_0 initially, and h finally.

Then if pressures be measured in terms of heights of the liquid columns :

$$P_0 = B + h_0, \quad P = B + h,$$

$$\therefore \quad r = \frac{\log\left(1 - \frac{h_0}{P_0}\right)}{\log\left(1 - \frac{h_0 - h}{P_0}\right)} = \frac{\frac{h}{P_0}}{\frac{h_0 - h}{P_0}} \text{ (approx.),}$$

by expansion in logarithmic series and neglecting higher powers of $\frac{h_0}{P_0}$ and $\frac{h_0 - h}{P_0}$ than the first.

This is permissible since the values of h used are only a few centimetres. 5 cms. is a convenient value for h_0.

Hence $$r = \frac{h_0}{h_0 - h}.$$

The result may also be obtained by another method.

In fig. 116 let AC denote any curve relating the pressure and volume of a gas. The elasticity is defined to be the ratio :

$$\frac{\text{stress}}{\text{strain}}.$$

The stress will be measured by a slight change in pressure, and the strain by the corresponding slight change in volume per unit volume. Let us consider the volume, v, represented by BC. Let a change of pressure, δp, denoted by FA, bring about the change in volume denoted by CF. We shall record this by δv, but $\delta v = -$ CF on account of the diminution of volume on the addition of pressure.

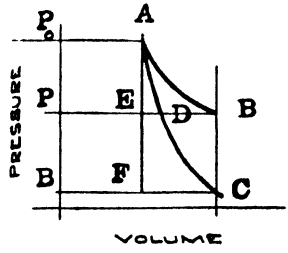


FIG. 116

Thus, the elasticity, E, is measured by AF $\div \frac{\text{CF}}{v}$, for $\frac{\text{CF}}{v}$ denotes the change of volume per unit volume.

$$\therefore \quad \text{E} = -v\frac{\delta p}{\delta v}.$$

When we proceed to the limit and make the changes very small we have :

$$\text{E} = -v\frac{dp}{dv}.$$

For a gas the value of E depends on how the change is made, and we shall consider two cases, first the case of an isothermal, and then that of an adiabatic change. Let AC denote the adiabatic curve and AB the isothermal; the former is steeper than the latter. Let E_ϕ denote the adiabatic and E_θ the isothermal elasticity.

From the formula for E we have:

$$E = -v \times \text{slope of curve.}$$

Thus $E_\phi = -v \times$ slope of adiabatic,

and $E_\theta = -v \times$ slope of isothermal.

If we start at A with a particular volume, v, measured for each curve by P_0A, we have:

$$\frac{E_\phi}{E_\theta} = \frac{\text{slope of adiabatic}}{\text{slope of isothermal}}.$$

The changes of pressure and volume in the experiment are small, so that the curves, AB and AC, are approximately straight, and the slopes can be measured by:

$$\frac{AF}{FC} \text{ and } \frac{AE}{EB} \text{ respectively.}$$

$$\therefore \quad \frac{E_\phi}{E_\theta} = \frac{AF}{AE}$$

Now, AF is the change in pressure during the adiabatic part of the expansion, viz., h_0, and AE is the change during the isothermal part. In our case the atmospheric pressure is that at the end of the adiabatic expansion, i.e. that at C, and since the point, B, on the curve represents the final state, CB denotes the pressure, h.

Thus $$AE = h_0 - h.$$

Thus $$\frac{E_\phi}{E_\theta} = \frac{h_0}{h_0 - h}.$$

For the adiabatic expansion we have:

$$pv^r = \text{constant},$$

$$\therefore \quad \log p + r \log v = 0.$$

Differentiating we have:

$$\frac{1}{p}\frac{dp}{dv} + \frac{r}{v} = 0,$$

i.e.

$$\frac{dp}{dv} = -\frac{p}{v}r.$$

In our case p and v denote the values of the co-ordinates at A, and $\frac{dp}{dv}$ is the tangent of inclination of the curve AC to the axis of x.

Similarly, for the isothermal case we have :

$$pv = \text{constant},$$

and

$$\frac{dp}{dv} = -\frac{p}{v}.$$

p and v are the co-ordinates of A, but the slope is now for AB. Thus the ratio of the slopes is γ.

$$\therefore \frac{E_\phi}{E_\theta} = \gamma = \frac{h_0}{h_0 - h}.$$

In carrying out the experiment make about six independent determinations, and increase the pressure cautiously so as not to expel oil from the manometer.

CHAPTER VIII

VAPOUR DENSITY AND THERMAL CONDUCTIVITY

Vapour Density. (Victor Meyer's Method)

THE vapour density of a substance can be found by measuring the volume of the vapour produced from a small quantity of the solid or liquid whose weight is known. In Victor Meyer's method this volume is found from the volume of air displaced by the vapour.

The apparatus (fig. 117) consists of a vertical glass tube provided with a bulb at the lower end, A, and a side tube, ST. The side tube dips under water in a beaker and a rubber cork closes the upper end of the tube.

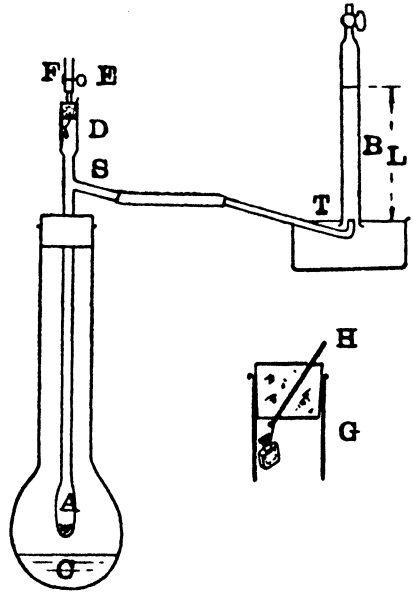


FIG. 117

As it is necessary, on introducing the vapour, to allow a small bottle, D, to fall the length of the tube, it is advisable to place a little asbestos at the bottom of A to prevent breakage.

The tube, A, is surrounded by a larger tube containing a liquid which boils at a higher temperature than the substance to be experimented on.

In the case of the determination of the vapour density of ether, water may be used in the bath.

The various parts of the inner tube should be kept at a constant temperature during the experiment, and in order to maintain this condition the outer tube is screened from draughts by surrounding it with a cylinder of asbestos or cardboard which fits it above the bulb.

Before beginning the experiment the inner tube must be quite dry. If necessary it should be warmed over a Bunsen flame while a current of air is blown through it.

The apparatus is set up as shown in the diagram, and heating is kept up until no more bubbles come out from the side tube.

The substance is weighed and enclosed in the small bottle, D, and suspended close to the upper end of the inner tube. When everything is quite steady the bottle is allowed to fall; vaporization takes place, and the cork of the bottle is blown out. Air

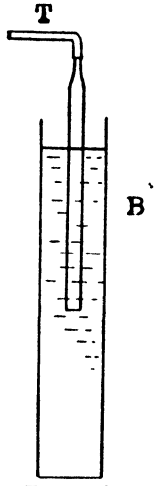


FIG. 118

passes over into the side tube and may be collected in the burette, B. It is better to collect it by the method illustrated in fig. 118, in which case the gas collected can be brought to atmospheric pressure by raising or lowering the burette. In the other case it is necessary to correct for the height of the water column, L.

In order to cause no disturbance on introducing the substance to be vaporized a piece of thread or thin wire should be passed through the cork and be held by a stop-cock, E, which pinches a piece of tubing, F. The bottle is allowed to fall by opening E, and then closing it immediately.

A better method is to use the apparatus shown at G. By turning the wire, H, through 180° the bottle will be caused to slip off the hook.

The air collected at B is over water at a temperature, T, and will be saturated with water vapour at this temperature. Let B be the saturation pressure at this temperature. If v is the measured volume of air, and H the total pressure, then v_0 the volume under normal conditions is given by

$$v_0 = v \times \frac{H - B}{76} \times \frac{273}{273 + T}.$$

If w = weight of substance enclosed in D, the density of the vapour is :

$$\frac{w}{v_0} = \frac{w}{v} \cdot \frac{76}{H - B} \cdot \frac{273 + T}{273}.$$

1 c.c. of hydrogen at 0° and 76 cm. pressure, weighs ·0000900 grammes, and its molecular weight is 2.

Hence the molecular weight of the substance examined is :

$$\frac{2}{\cdot 00009} \times \frac{w}{v} \cdot \frac{76}{H - B} \cdot \frac{273 + T}{273}$$

At the conclusion of the experiment remove the stopper from the end of the inner tube to prevent any sucking back of water from B into the bulb, A, as the apparatus cools. It is important to cause the bulb, A, to be heated by the steam from the bath, and it should be adjusted to prevent actual contact with the water in the bath, and to be out of reach of splashes when boiling takes place.

Vapour Density. (Dumas's Method)

A large flask is cleaned and dried, and fitted with a cork provided with a bent piece of glass tubing drawn to a fine point so that it can be easily sealed by the application of a Bunsen flame (fig. 119).

The flask is suspended from the arm of a balance and weighed. Let the observed weight be W_1, and let w_1 denote the weight of air displaced by the wall of the flask, while w_2 denotes the weight of air displaced by the closed flask, so that $w_2 - w_1$ denotes the weight of air within it. This will be denoted by w_a. Hence if W denote the real weight of the flask :

$$W_1 = W - w_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now introduce a small quantity (5–10 c.c.) of the liquid to be vaporized into the flask. The actual quantity will depend on the size of the flask and will be discovered by trial. The amount given is of the right order when a flask of capacity about a half litre is employed to find the vapour density of chloroform. This liquid is very suitable in the case of a laboratory exercise, since it has a high vapour density. The flask is placed within

an enclosure, J, over a sand bath, with the tube, T, projecting through the lid.

Heating is continued until the liquid is vaporized and no more issues from T. This may be tested by placing a polished surface near the end of T. It will become dimmed if vapour is still coming out.

When the steady state is reached, T is sealed off.

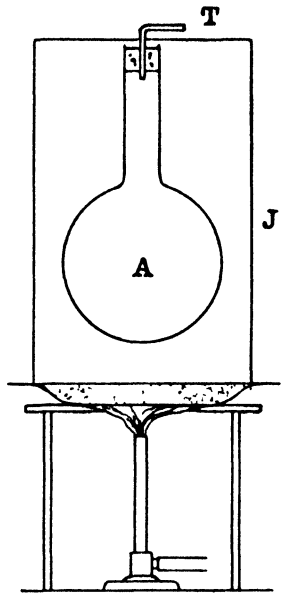


FIG. 119

Let this happen at a temperature, $t°$, measured by means of a thermometer hanging close to A, within the enclosure, J, and let the volume of the vapour be V_t, and density ρ_t. Allow the flask to cool, and weigh. Break off the end of T under water. The flask will fill with water, the space occupied by the condensed vapour becoming negligible under the new conditions.

Preserve the broken pieces from T, and after drying the outside of the tube re-weigh the flask and water, noting the temperature of the water, $t_0°$.

Let W_2 denote the weights in the scale-pan when the flask and vapour are weighed, and let w_v denote the weight of the vapour.

Then $$W_2 = W + w_v - w_2 ;$$

$$\therefore W_2 - W_1 = w_v - (w_2 - w_1) \quad \text{by equation (1)}$$

$$= w_v - w_a.$$

Let V_{t0} denote the volume of the flask obtained from the weight of water it contains at $t_0°$. Then if β denote the coefficient of cubical expansion of the glass, which may be taken as ·0000232,

$$V_t = V_{t_0} \{1 + \beta (t - t_0)\},$$

$$w_v = \rho_t V_t.$$

If the barometric pressure be P, we can reduce ρ_t to normal conditions by the formula:

$$\rho_0 = \frac{(273 + t)}{273} \cdot \frac{76}{P} \cdot \rho_t$$

$$= \frac{(273 + t)}{273} \cdot \frac{76}{P} \cdot \frac{w_v}{V_{t_0} \{1 + \beta (t - t_0)\}}.$$

w_a denotes the weight of air filling the flask at the temperature of the air within the balance. Let this be t^1° C.

Then if d_a denote the density of air under these circumstances,

$$w_a = d_a V_{t^1} = d_a V_{t_0} \{1 + \beta (t^1 - t_0)\}$$

$$= d_0 \cdot \frac{P}{76} \cdot \frac{273}{273 + t^1} \cdot V_{t_0} \{1 + \beta (t^1 - t_0)\},$$

d_0 = ·001293 gm. / c.c.

From these two formulæ, since $w_v = w_a + W_2 - W_1$, we may now calculate ρ_0.

One of the difficulties of the experiment is to drive out all the air from the flask and replace it by vapour. It frequently happens that on attempting to fill the flask with water, some air is left behind. More of the liquid is required for driving off the air in this case.

If the volume of air left over is small, we may apply a correction. At the temperature of the water, t_0, let the volume of air be v. This may be determined by filling up the flask with water from a measuring flask.

The total pressure to which the mixture of vapour and air is subjected is the sum of the partial pressures, P^1 and p^1, of the vapour and air respectively. The weight of the vapour is now w_v^1, obtained by subtracting the weight of air of volume, v, from w_v, determined as above:

$$w_v^1 = \rho^t V_t$$

$$= \rho_0 \cdot \frac{P^1}{76} \cdot \frac{273}{273 + t} \cdot V_{t_0} \{1 + \beta (t - t_0)\}.$$

V_{t_0} denotes as before the total volume of the flask, i.e. the volume of the water after the bubble of air has been replaced.

We can find P^1 by remembering that the air of volume, v, and at atmospheric pressure occupied a volume, V_t, under the partial pressure, p^1, the temperatures being respectively $t_0°$ and $t°$ C.

Thus

$$\frac{vP}{273+t_0} = \frac{V_t p^1}{273+t},$$

$$p^1 = P - P^1;$$

$$\therefore P^1 V_t = PV_t - \frac{273+t}{273+t_0} \cdot Pv,$$

$$w^1{}_v = \rho_0 \cdot \frac{273}{273+t} \cdot \frac{1}{76} \left\{ PV_t - \frac{273+t}{273+t_0} \cdot Pv \right\}$$

$$= \rho_0 \cdot \frac{273}{273+t} \cdot \frac{P}{76} \cdot V_{t_0}\{1+\beta(t-t_0)\} - \rho_0 \cdot \frac{P}{76} \cdot v \cdot \frac{273}{273+t_0},$$

or

$$\rho_0 = \frac{w^1{}_v}{\frac{273 \cdot P}{76} \cdot \left[\frac{V_{t_0}\{1+\beta(t-t_0)\}}{273+t} - \frac{v}{273+t_0} \right]}.$$

The term, $\frac{v}{273+t_0}$, will be small if the experiment is to be successful at all, so that it will only be necessary to change the value, w_v, to the actual weight of vapour, $w_v{}^1$.

It should also be remarked that the volume, v, is a mixture of air and water vapour, whereas it has been assumed to consist of air only. We shall not, however, further consider this correction, which is small and affects a term which must be already small. The discussion shows how the error affects our formula, and it would be sufficient to measure the volume of the bubble, multiply by the density, ·001293 gm./c.c., and subtract from w_v.

After a few attempts the bubble will usually be sufficiently small to be neglected altogether.

Conductivity of a Copper Bar

The apparatus consists of a bar of copper, CC (fig. 120), with two holes bored well into it to carry thermometers, T_1 and T_2, mercury being placed in the holes to ensure good thermal contact. At the ends of the bar are two metal boxes, through one of which, A, is passed steam, and through the other, B, a steady stream of water from a constant pressure head.

The shelves, LL, within B, serve to prevent any flow of water straight from inlet to outlet. The temperature of the water is taken just before it enters and leaves B by the thermometers, T_3 and T_4. The apparatus is allowed to attain a steady state, when all the thermometers will record steady temperatures.

It is usual to pack loosely round CC and the boxes some cotton wool, the whole being enclosed in a felt-lined wooden box, through which T_1 and T_2 project. T_3 and T_4 are kept as close

as possible to the box, and it is a good plan to wrap the T-pieces loosely with wool.

In this way the heat from A is transmitted by conduction to B, and the heat passing across in any time, t, is noted by collecting water as it leaves B and weighing it. If the mass collected is m the rate of transmission of heat is

$$\frac{m\,(T_4 - T_3)}{t}.$$

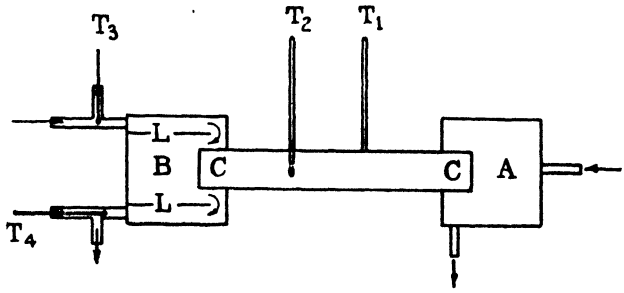


FIG. 120

T_3 and T_4 denote the initial and final temperatures of the water. Since this quantity of heat is transmitted from T_1 to T_2 —a distance, d, say, the amount is $k \cdot (T_1 - T_2) \cdot \frac{A}{d}$, where A is the area of section of the bar, and k the conductivity. A is measured by finding the diameter of the bar if it is cylindrical, or by measuring its breadth and height if rectangular.

We have, therefore :

$$k\,(T_1 - T_2) \cdot \frac{A}{d} = m\frac{(T_4 - T_3)}{t}.$$

In another form of apparatus the cold water is passed through a metal spiral wound round the end instead of through the metal box. Good thermal contact is made between the spiral and bar, and the temperature of the water measured at entrance and exit as before.

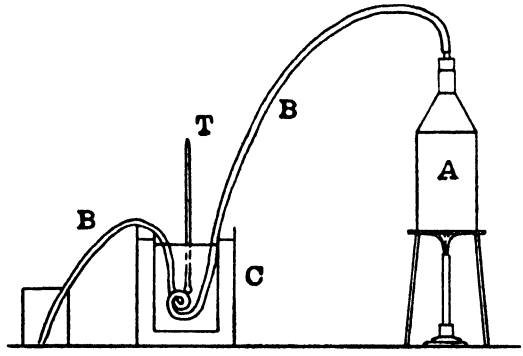
Thermal Conductivity of Rubber Tubing

The apparatus required consists of a length of rubber tubing, B, a copper heater, A, for producing steam, a calorimeter, C, thermometer, T, and a measuring glass, as illustrated in fig. 121.

The method of procedure is as follows :

A quantity of water is introduced into the calorimeter, C, and weighed.

Steam is passed through B until a rise of temperature of the calorimeter and water of from 10° to 20° C. has occurred. The initial and final temperatures are noted and also the time of passage of the steam. The tube is removed and the length that has been immersed is noted—l (say). Two pieces of cotton should be tied round the rubber at the points where it enters and leaves the water.



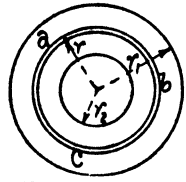


FIG. 121

Let the initial temperature be T and the final T^1. If there had been no loss of heat by radiation the final temperature would have been some other value, $T^1 + \Delta T$, and it is necessary to make the correction ΔT.

In order to do this, observe temperatures, at intervals of half a minute or some convenient period, beginning at T and ending at T^1, during the passage of the steam.

These will be denoted by $t_1, t_2, \ldots t_{n-1}$.

If the temperature of the room be t_0 we have for the change of temperature due to heat radiated or absorbed, as the case may be, according to Newton's Law of Cooling:

$$\delta\theta_1 = C\left\{\frac{T + t_1}{2} - t_0\right\},$$

$$\delta\theta_2 = C\left\{\frac{t_1 + t_2}{2} - t_0\right\},$$

$$\delta\theta_3 = C\left\{\frac{t_2 + t_3}{2} - t_0\right\},$$

. . . . . .

. . . . . .

$$\delta\theta_n = C\left\{\frac{t_{n-1} + T^1}{2} - t_0\right\},$$

where $\delta\theta$ denotes the change in any interval, the suffix denoting which interval.

Thus

$$\Delta T = \Sigma\delta\theta = C\left\{\frac{T + T^1}{2} - nt_0 + (t_1 + t_2 + \ldots + t_{n-1})\right\}.$$

C is a constant depending on the calorimeter and contents, and must be determined if Δt is to be calculated.

When the final temperature, T^1, has been attained, cut off the supply of steam and allow the calorimeter to cool, observing temperatures, T_1, T_2, . . . T_n, at equal intervals.

Then as before if δt denotes the loss of temperature in each interval we have :

$$\Sigma\delta t = C\left\{\frac{T_1 + T_n}{2} - (n-1)\,t_0 + (T_2 + T_3 + \ldots + T_{n-1})\right\}.$$

But $\Sigma\delta t = T_1 - T_n$, so that C can be calculated and may be employed in the above case.

The ranges of temperature in the two cases should be as nearly as possible the same.

If M denote the water equivalent of the calorimeter and contents the heat transmitted through the tubing is

$$M\,(T^1 + \Delta T - T).$$

We can connect this with the conductivity, k, in the following way :

Let the outer and inner radii of the tubing be r_1 and r_2, and consider a portion of unit length of the tube between radii, r and $r + \delta r$, at which the temperature is t.

The rate of change of temperature at the distance, r, is $-\frac{dt}{dr}$. The negative sign expresses the fact that the temperature diminishes as r increases.

Thus if Q denote the quantity of heat transmitted per unit length, i.e. across an area, $2\pi r$,

$$Q = -k \cdot \frac{dt}{dr} \cdot 2\pi r,$$

$$\therefore \int dt = -\frac{Q}{2\pi k} \cdot \int \frac{dr}{r},$$

where the integration is to be taken between the limits, r_1 and r_2 for r, and between the inner and outer temperatures of the tubing for t.

Let the steam temperature be τ, the outer temperature is taken as the mean of T and T^1,

$$\therefore \quad \tau - \tfrac{1}{2}(T + T^1) = \frac{Q}{2\pi k} \log_e \frac{r_1}{r_2},$$

or

$$k = \frac{Q}{2\pi \left\{ \tau - \dfrac{T + T^1}{2} \right\}} \log_e \frac{r_1}{r_2}.$$

But

$$Q = \frac{M}{l}(T^1 + \Delta T - T),$$

$$\therefore \quad k = \frac{M}{2\pi l} \times 2{\cdot}303 \times \frac{T^1 + \Delta T - T}{\tau - \dfrac{T + T^1}{2}} \cdot \log_{10} \frac{r_1}{r_2}.$$

(Change being made to logarithms to base 10.)

The value of r_1 may be determined by means of a screw gauge, and in order to find r_2, place a length of the tube of 5 to 10 cms. in water in a measuring glass and note the volume, v, displaced.

Then if L denote the length of tubing :

$$v = \pi L (r_1^2 - r_2^2).$$

All the quantities except r_2 are known, so that this value can be determined.

Another way of determining the radii is to cut the tube clean, normal to its length and use it as a rubber stamp, pressing it lightly on a clean sheet of paper.

The impress of the outer and inner circumferences will be distinct and the diameters may be measured by means of a travelling microscope.

The Conductivity of Glass

The conductivity of glass in the form of a tube may be found by the method described in the last experiment. A different arrangement of apparatus is required, but the theory is identical in both cases.

Steam is passed through a jacket, J, round the tube, B. Within B a stream of water is caused to flow from a supply which provides a constant head of pressure.

Within B is a spiral made of cord or rubber so that as it progresses up the tube the water is caused to traverse it spirally. This is important as the temperature at any cross-section of B

must be the same throughout the section. The rate of flow is adjusted to cause a difference of temperature of about 20° between T and T^1. The thermometers are enclosed in T-pieces as near as possible to the ends of B, the T-pieces being covered with felt or cotton wool to prevent loss of heat by radiation before the temperature is taken.

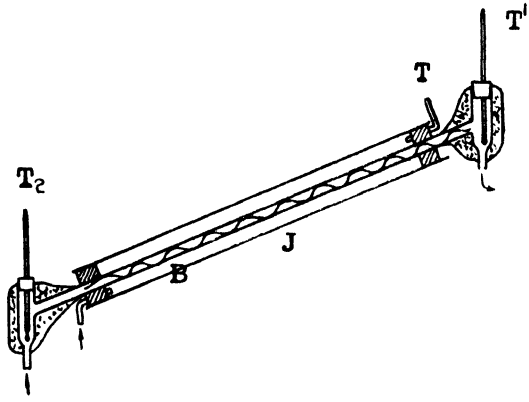


FIG. 122

In order to measure Q, water is collected on exit for a measured time.

Conductivity of Cardboard by the Method of Lees and Chorlton

The apparatus consists of a retort stand provided with a clamp (fig. 123) and metal ring, AB, from which hangs a cylindrical slab of copper or brass, DE, of diameter about 12·5 cms. On this rests a hollow cylinder, C, of the same diameter, provided with inlet and outlet tubes, G and H, through which steam may be passed.

Towards the base of C and into DE holes are bored so that the thermometers, T_1 and T_2, may be inserted and the temperatures read.

The two cylinders are nickel-plated in order to produce a surface of uniform emissive power.

Suppose a thin slab of material of the same diameter as the cylinders is placed between them, and let the loss of heat that is radiated from the edge of the slab be small enough to be neglected in the calculation. Then all the heat transmitted across the slab is radiated from DE during the steady state.

Let A denote the area of cross-section of the slab and *d* its thickness. Let the thermal capacity of DE be denoted by W, and let the thermometers, T_1 and T_2, record temperatures, T_1

and T_2 in the steady state, while the temperature of the surrounding air is T_0. The heat transmitted through the slab per sec. is :

$$k \cdot \frac{T_1 - T_2}{d} \cdot A,$$

where k is its thermal conductivity.

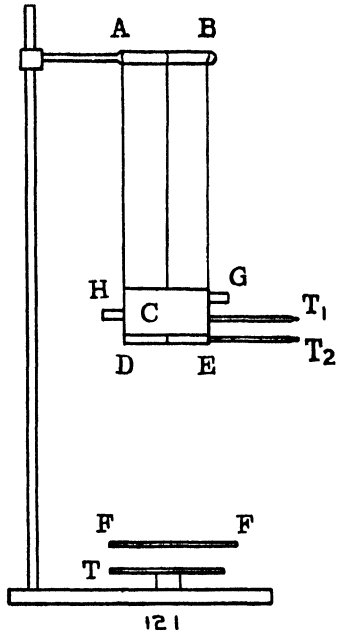


FIG. 123

The heat radiated per second from DE is

$$C \cdot (T_2 - T_0),$$

where C is a constant.

Thus $$k \cdot \frac{T_1 - T_2}{d} \cdot A = C\,(T_2 - T_0).$$

We may readily use the apparatus to give comparative results.

If two sheets be cut of the same diameter as DE, one of glass and the other of cardboard, about 1 mm. thick, and if they be inserted between the cylinders we may observe two sets of temperatures, one for each. Let the letters without dashes be used to describe the experiment with glass while those with dashes correspond to cardboard.

In the latter case we have :

$$k' \cdot \frac{T_1' - T_2'}{d'} \cdot A = C\,(T_2' - T_0').$$

On dividing we have :

$$\frac{k'}{k} = \frac{T_2' - T_0'}{T_2 - T_0} \cdot \frac{d'}{d} \cdot \frac{T_1 - T_2}{T_1' - T_2'} .$$

d and d^1 are measured by means of a screw gauge.

In order to find T_0 place the thermometer below a sheet of cardboard, FF, to protect it from direct radiation from DE, but it should be placed directly below DE, so as to give the temperature of the air which rises upwards to DE.

An absolute value may be found for k by determining the rate of fall of temperature of DE at the temperature, T_2.

This may be found by removing C and allowing a Bunsen flame to play on DE until T_2 registers a temperature about 10° above that recorded during the steady state.

Observe the temperature recorded by T_2 as the slab cools to about 10° below that recorded during the steady state. The observations should be made every half-minute, or more frequently if the change is rapid, and a graph drawn relating the time and temperature.

From the graph determine the value of the rate of change at various temperatures.

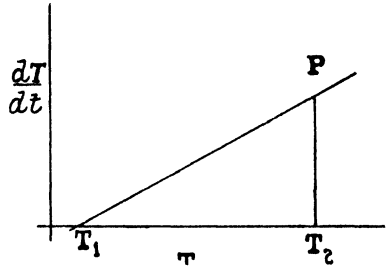


FIG. 124

Plot a second graph with the rates, $\frac{dT}{dt}$, as ordinates and the corresponding temperatures, T, as abscissæ (fig. 124). From this determine the particular rate for the temperature, T_2. Denote this by $\left(\frac{dT}{dt}\right)_2$.

Weigh the slab and determine its water equivalent, using for the specific heat of copper the value ·094, or of brass the value ·09. Denote this by W.

Hence the loss of radiation is also expressed by $W\left(\frac{dT}{dt}\right)_2$, and we have the equation :

$$k \cdot \frac{T_1 - T_2}{d} \cdot A = W \cdot \left(\frac{dT}{dt}\right)_2 ,$$

by means of which k is determined.

Make a table of the observations during cooling as follows :

TIME (MIN.)	TEMP. (T_2)	AVERAGE TEMP.	RATE $\frac{dT}{dt}$
0			
$\frac{1}{2}$			
1			
$1\frac{1}{2}$			
2			
$2\frac{1}{2}$			
3			
.			
.			
.			
.			
.			
.			

Forbes's Method of Determining the Conductivity of a Metal Bar

This experiment consists of two parts, in the first of which a metal bar is heated at one end until the steady state is reached.

In the second part a bar of the same material and cross-section, but shorter, is allowed to cool under similar external conditions.

In order to maintain constant external conditions the experiment should be performed in a part of the laboratory sheltered from draughts.

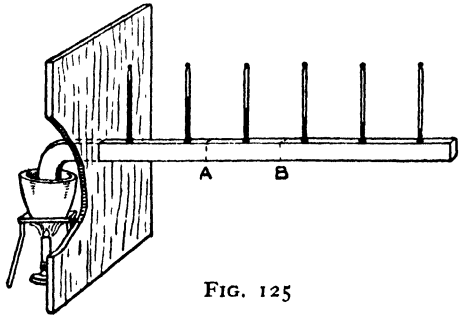


FIG. 125

The bar (fig. 125) usually has one end curved and dipping into a convenient molten metal contained in a vessel on the other side of a screen, which protects the bar from direct radiation from the source of heat.

The metal may conveniently be molten lead or solder. The bar is provided with a series of holes which lie regularly along its length into which thermometers fit.

If these are small and contain mercury or the molten liquid, it is found that the process of conduction is not disturbed by their presence. The holes near the hot end should contain the molten liquid and the remainder mercury.

By observation of the temperatures at various distances along the bar measured from the screen, the temperature slope may be found. This is best done by plotting a curve (fig. 126) and calculating the slope, $\frac{d\theta}{dx}$, from the inclination of the tangents at various points.

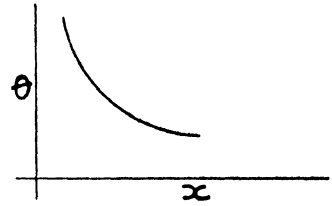


FIG. 126

A thermo-junction may be used alternatively to find the temperatures by dipping one junction successively into the holes. For the calibration of the junction the reader is referred to p. 545.

The importance of maintaining steady external conditions will be appreciated in this part of the experiment, and great care will be required to maintain the whole length of the bar simultaneously steady.

The bar is assumed to be sufficiently long that its end is at the external temperature, θ_0. A convenient length is one metre and its section may be about 2 cms. square.

In the steady state all the heat passing a section, B, escapes from the surface between B and the end.

The rate of flow at B is :

$$kA\frac{d\theta}{dx},$$

where k is the conductivity and A the area of section.

This rate of flow is calculated from the second part of the experiment by determining the rate of cooling of the portion of the bar beyond B. The equation thus obtained serves to find k.

The second bar may be conveniently 10 cms. long. It is heated to the temperature of the molten metal, but so that its surface is not damaged and remains similar to that of the first bar. To do this and at the same time to prevent sudden cooling of the molten mass, the rod is wrapped in several layers of paper and completely immersed.

It is provided also with a hole to carry a thermometer and is suspended under the same conditions as prevailed round AB, and its temperature observed at successive times so as to include in the range those values which prevailed along the bar.

A curve is drawn showing the relation between the temperature, θ, and the time, t, for the short bar.

This bar is similar to the long bar and cools under the same conditions, so that the rate of cooling for both is the same. From the curve we can deduce the rate of cooling, $\frac{d\theta}{dt}$, in the usual manner.

We require for the purpose of the calculation the rate of cooling for points along the bar in the first part of the experiment. We must correlate the values of x and $\frac{d\theta}{dt}$, x denoting distances measured from the screen.

This may be done from the curves.

Take a series of values of x from the curve illustrated in fig. 126 and observe the corresponding value of θ. From the curve relating θ and t take the values $\frac{d\theta}{dt}$ for these particular values of θ.

We then have the corresponding values x and $\frac{d\theta}{dt}$.

Plot these on a curve as illustrated in fig. 127.

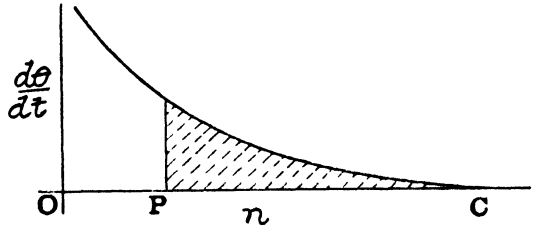


FIG. 127

This curve will cut the axis at a point, C, where $\frac{d\theta}{dt}$ vanishes, or where the temperature of the bar is equal to that of the surrounding air.

If s is the specific heat and ρ the density, the rate of loss of heat between two points separated by distance δx is

$$A\,\delta x \rho s \frac{d\theta}{dt}.$$

The total loss per second between B and the end of the bar is thus :

$$\int_B^{\text{end}} A\rho s \frac{d\theta}{dt}\, dx$$

$$= A\rho s \int_B^{\text{end}} \frac{d\theta}{dt}\, dx$$

$$= A\rho s \times \text{shaded area of fig. 127,}$$

where OP on the graph measures the distance from the hot end up to B.

This area may be found by the planimeter or calculated if carefully drawn on squared paper.

$$\therefore \quad k A \left(\frac{d\theta}{dx}\right)_B = A\rho s \times S.$$

S = shaded area and $\left(\frac{d\theta}{dx}\right)_B$ denotes the temperature slope at the point B.

$$\therefore k \cdot \left(\frac{d\theta}{dx}\right)_B = \rho s S.$$

We can thus determine k, the values of ρ and s being given as constants of the apparatus or determined in the usual way.

These values may be taken to be 8·93 gm. per c.c. and ·094 respectively when the bar is of copper.

The Determination of the Conductivity of a Bar of Metal by Ångström's Method

In this method heat is supplied to a long bar by alternately heating and cooling it at one region in regular periods.

In this way the temperatures at points along the bar fluctuate periodically, and on account of surface radiation the temperature

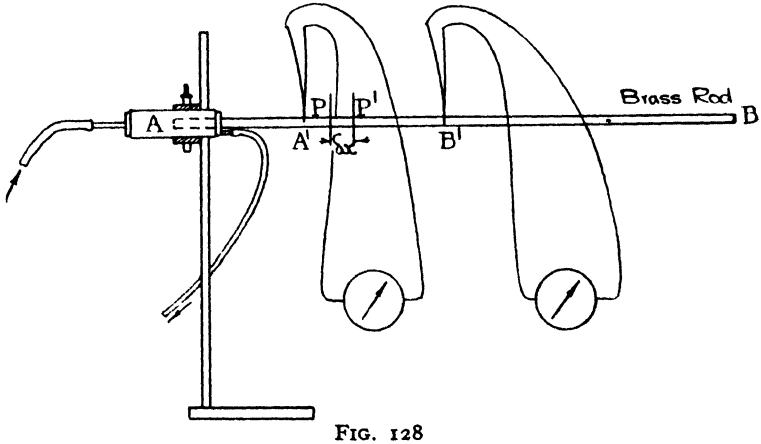


FIG. 128

amplitudes diminish as the distance from the region of supply increases, while the maximum and minimum values occur at later times with the increasing distance.

The bar must be chosen so that it is sufficiently long to allow

us to neglect the effect of the terminal faces. The fluctuations should die away at a short distance from the cooler end, which thus has the same temperature as the air surrounding it.

When the heating is continued long enough the periods develop themselves completely, in which case the mean temperature at any point of the bar preserves some constant value.

We first consider the theory of the experiment.

Let us consider a bar, AB (fig. 128), with one end, A, exposed for a definite time, T_1, to a current of steam and for a succeeding time, T_2, exposed to a current of cold water. Suppose that this process is continued regularly until the steady fluctuations throughout the bar are developed. These fluctuations will have a complete period of $(T_1 + T_2)$ which we may denote by T, and for the sake of convenience we shall write

$$p = \frac{2\pi}{T}.$$

Let the conductivity of the bar be, k, ρ its density, s its specific heat, A its cross-section, P its perimeter, and θ its temperature at a point, P, distant x from a convenient origin. We shall choose this to be at A.

Consider a point, P^1, distant δx from P, as in the diagram, and let its temperature be $(\theta + \delta\theta)$. The heat flowing into the element, PP^1, of the bar across a plane at P drawn normally to its length is equal to :

$$k \cdot \frac{\theta_P - \theta_{P^1}}{\delta x} \cdot A \text{ per sec.},$$

where the subscripts denote the points at which θ is measured, i.e. the heat flow amounts to :

$$-k \cdot \frac{\delta\theta}{\delta x} \cdot A$$

$$= -k \cdot \frac{d\theta}{dx} \cdot A = F \text{ (say)},$$

in the limit when δx is made infinitesimal.

Thus this expression denotes the flow from left to right at the point P, of the bar. Since θ depends on x, in the general case this flow will also depend on x, i.e. F depends on x.

Again consider the element PP^1. We have calculated the flow, F, into it at P, and the flow out at P^1 will be

$$F + \frac{dF}{dx} \cdot \delta x.$$

Hence the total flow of heat *into* PP^1

$$= -\frac{dF}{dx} \cdot \delta x$$

$$= k \cdot \frac{d^2\theta}{dx^2} \cdot A \cdot \delta x \text{ per sec.}$$

This heat is used up, partly in warming up the part of the bar concerned, and partly in radiation from the surface.

If the temperature is changing at the rate, $\frac{d\theta}{dt}$, t denoting the time, the first part amounts to :

$$A \cdot \delta x \cdot \rho \cdot s \cdot \frac{d\theta}{dt},$$

and the second to :

$$h \cdot P \cdot \delta x \, (\theta - \theta_0).$$

h denotes a constant for the surface and θ_0 is the external temperature.

We shall, however, suppose that θ is measured in degrees above the surrounding temperature, so that we may write the last expression simply :

$$h \cdot P \cdot \delta x \cdot \theta.$$

We, therefore, arrive at the equation :

$$kA\delta x \frac{d^2\theta}{dx^2} = h \cdot P \cdot \delta x \cdot \theta + A \cdot \rho s \delta x \frac{d\theta}{dt}.$$

or
$$\frac{d\theta}{dt} = K\frac{d^2\theta}{dx^2} - H\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where
$$K = \frac{k}{\rho s}, \qquad H = \frac{hP}{A\rho s} \cdot \ldots\ldots\ldots\ldots(3)$$

Any function which satisfies certain conditions can be expanded as a Fourier Series.

Students of Physics should make themselves acquainted with the Fourier Analysis and they may be referred to Carslaw's work on this subject. For the present purpose we need nothing more than the statement that the expansion is possible.

Fourier's Series is the following :

$$\left.\begin{aligned} f(t) = A_0 &+ A_1 \cos pt + A_2 \cos 2pt + A_3 \cos 3pt + \ldots \\ &+ B_1 \sin pt + B_2 \sin 2pt + B_3 \sin 3pt + \ldots \end{aligned}\right\} \quad (4)$$

Here $f(t)$ is any function existing over some interval from $t = a$ to $t = b$ and satisfying certain conditions with regard to continuity, etc., which we need not enter into here. We merely mention that the series is applicable to the function we shall use in our experiment.

With regard to the values of the A's and B's the rule is that :

$$A_0 = \frac{1}{b - a}\int_a^b f(u)\, du;$$

$$A_n = \frac{2}{b-a}\int_a^b f(u) \cos \frac{2n\pi}{b-a} u \, du;$$

$$B_n = \frac{2}{b-a}\int_b^a f(u) \sin \frac{2n\pi}{b-a} u. \, du;$$

n has the integral values, 1, 2, 3, etc.

The value of p in the above series is in this general case $\frac{2\pi}{b-a}$.

When $f(t)$ is periodic and has the complete period, T, we take the limits, a and b, at the ends of this period and write $a = 0$, $b = T$, so that the series represents $f(t)$ from $t = 0$ to $t = T$, and on account of its periodic character represents it also from T to 2T, etc. In this case :

$$p = \frac{2\pi}{T};$$

$$A_1 = \frac{1}{T}\int_0^T f(u) \, du;$$

$$A_n = \frac{2}{T}\int_0^T f(u). \cos npu \, du;$$

$$B_n = \frac{2}{T}\int_0^T f(u) \sin npu \cdot du.$$

The periodic function with which we are concerned is that which expresses the fact that from $t = 0$ to $t = T_1$ the temperature has some value, θ_1, and from $t = T_1$ to $t = T_2$ it has the value θ_2. Such a function can be represented by a Fourier Series and we shall assume that the particular expression is (4), with coefficients calculated according to the rule. These, however, are not required for the experiment.

It is shorter and convenient for our purpose to write (4) in the form :

$$a_0 + a_1 \sin (pt + r_1) + a_2 \sin (2pt + r_2) + \ldots\ldots\ldots (5)$$

And let us suppose that this is the expansion which expresses the temperature at A.

Now the temperature, θ_1, at the point whose distance is x_1 from A is a fluctuating function and so is that for any other point, x_2.

We may therefore write for θ_1, some such value as (5), viz.

$$\theta_1 = C_0' + C_1' \sin (pt + \delta_1') + C_2' \sin (2pt + \delta_2^1) + \ldots . (6)$$

and for the point, x_2

$$\theta_2 = C_0'' + C_1'' \sin (pt + \delta_1'') + C_2'' \sin (2pt + \delta_2'') + \ldots (7)$$

The quantities $C_0' C_1' \ldots$, $\delta_1' \delta_2' \ldots$, will differ from $C_0'' C_1'' \ldots$, $\delta_1'' \delta_2'' \ldots$, since they correspond to different values of x.

We have to solve the equation (2), and we have a clue from the experimental observations of diminishing amplitudes and lagging maxima and minima. Moreover, the value of θ is known at the point A where x is zero. This value is given by (5).

The solution is in fact :

$$\theta = a_0 e^{-\alpha_0 x} + a_1 e^{-\alpha_1 x} \sin(pt + \beta_1 x + r_1) + a_2 e^{-\alpha_2 x} \sin(2pt + \beta_2 x + r_2) + \text{etc.} \quad \ldots\ldots\ldots\ldots (8)$$

Where : $$\alpha_0^2 = \frac{H}{K}, \quad K(\alpha_n^2 - \beta_n^2) = H, \quad 2K\alpha_n\beta_n = -np \qquad \ldots\ldots\ldots (9)$$

When $x = 0$ this expression reduces to that of equation (5), so that the solution satisfies the end condition.

The reader may verify that the solution satisfies equation (2) by substitution. He will observe that the coefficients of the separate sine and cosine terms and the terms independent of trigonometrical functions all vanish if (9) holds.

Now if we consider the two points, x_1 and x_2, at a distance, l, apart, by equations (6), (7) and (9) we have :

$$C_1' = a_1 e^{-\alpha_1 x_1}, \qquad C_1'' = a_1 e^{-\alpha_1 x_2}, \qquad \delta_1'' - \delta_1' = \beta_1 l.$$

For $\delta_1' = \beta_1 x_1 + r_1$ and $\delta_1'' = \beta_1 x_2 + r_1$

Hence $$\alpha_1 l = \log \frac{C_1'}{C_1''} \quad \text{and} \quad \alpha_1\beta_1 = \frac{\delta_1'' - \delta_1'}{l^2} \log \frac{C_1'}{C_1''}.$$

But by the last of the conditions (9)

$$K = -\frac{p}{2\alpha_1\beta_1} = \frac{\pi l^2}{(\delta_1' - \delta_1'') \log \frac{C_1''}{C_1''}} \qquad \ldots\ldots\ldots (10)$$

and in the same way :

$$K = \frac{n\pi l^2}{T(\delta_n' - \delta_n'') \log \frac{C_n'}{C_n''}}, \qquad \ldots\ldots\ldots (11)$$

In this formula we note that the constant of radiation, h, does not appear and we are not troubled with the difficulties always associated with it. The conductivity, k, is given by $K\rho s$ so that if our experiment is performed carefully and as a consequence the quantities, C_n and δ_n, accurately known, we have to rely on the accuracy of the knowledge of the density and specific heat. Both these are accurately known. This is the reason of the importance of Ångström's Method. The student is recommended to refer to a translation of the original paper in the " Philosophical Magazine," series 4, vol. 25, p. 130, 1863.

From (11), we observe that each coefficient, C_n, gives rise to a value of K. As the carrying out of the experiment will show, the coefficient, C_1, gives the most reliable result but the value corre-

sponding to C_2 should also be worked out. It remains to describe the experiment and to show how to find the C's and δ's.

The bar, which may be of copper, iron or brass, has one end, A, inserted in a chamber so arranged that steam and cold water may be passed in alternately as described above. The total period is to be measured and care exercised to reproduce the conditions exactly in each successive period.

Steam from a conveniently large tin flask and water from the tap will set up the right conditions.

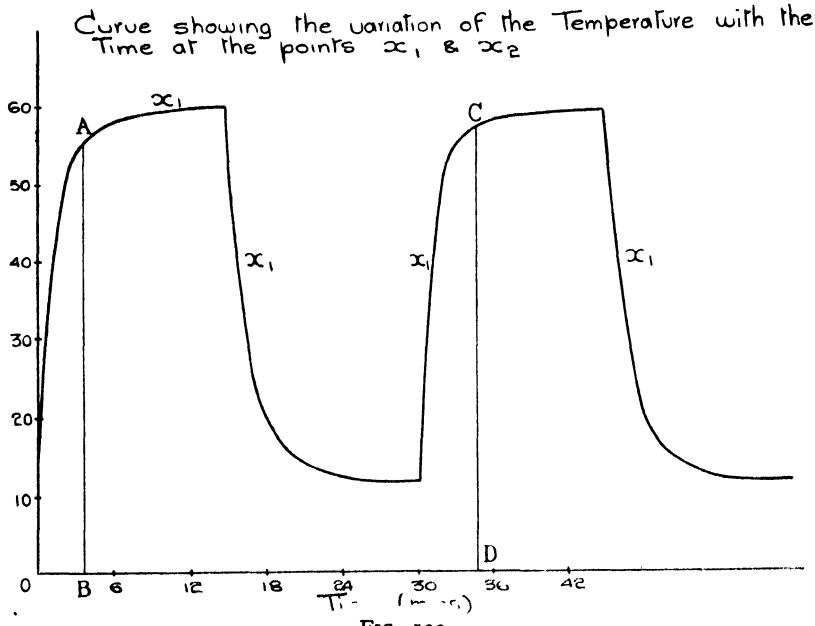


FIG. 129

The steady temperature fluctuations will be the more easily attained if the bar is sheltered to avoid draughts in its neighbourhood and consequent troublesome convection effects. The temperature at points on the bar should be observed by means of a thermo-junction placed in a small cavity in the bar. Two such cavities will be required and should contain mercury, or it will be sufficient to hold the junction in contact with the bar at the points concerned. The thermo-junction must be calibrated in the usual way (see p. 545).

When this is done read the temperature at convenient intervals by means of the galvanometer in the circuit and obtain as many readings as possible throughout two complete periods or more. Plot a curve for θ, against the time very carefully on squared paper as illustrated in fig. 129. Suppose that this refers to the point, x_1. Repeat the process for x_2.

We shall determine the necessary quantities from the graphs.

In order to see how to use the graphs multiply the equation (6) by sin pt and integrate both sides over a complete period from any arbitrary time, $t=t_1$ to $t=t_1+\mathrm{T}$, where, of course $\mathrm{T}=\frac{2\pi}{p}$

We thus have :

$$\int_{t_1}^{t_1+\mathrm{T}} \theta_1 \sin pt\, dt = \mathrm{C}_0' \int_{t_1}^{t_1+\mathrm{T}} \sin pt\, dt + \mathrm{C}_1' \int_{t_1}^{t_1+\mathrm{T}} \sin (pt + \delta_1') \sin pt\, dt,$$
$$+ \mathrm{C}_2' \int_{t_1}^{t_1+\mathrm{T}} \sin (2pt + \delta_2') \sin pt\, dt + \text{etc.} \quad \ldots\ldots(12)$$

In doing this it is well to remark that we are integrating the series on the right-hand side term by term, adding all the integrated terms and equating to the integrated function on the left. This process is not always legitimate, but on account of the properties of the particular series we may apply it in the present case.

On performing the integration the only term on the right-hand side which does not vanish is the second, and this has the value :

$$\tfrac{1}{2}\,\mathrm{TC}_1' \cos \delta_1'.$$

Denote the integral on the left by S_s.

Then $$\mathrm{S}_s = \tfrac{1}{2}\,\mathrm{TC}_1' \cos \delta_1'.$$

In the same way if S_c denote the value of

$$\int_{t_1}^{t_1+\mathrm{T}} \theta_1 \cos pt\, dt,$$

we find

$$\mathrm{S}_c = \tfrac{1}{2}\,\mathrm{TC}_1' \sin \delta_1'.$$

We can determine C_1' and δ_1' from these two equations, for

$$\tan \delta_1' = \frac{\mathrm{S}_c}{\mathrm{S}_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

and $$\mathrm{C}_1'^2 = \frac{4}{\mathrm{T}^2}(\mathrm{S}_s^2 + \mathrm{S}_c^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

In the same way if we multiply the series by cos $2pt$ and sin $2pt$ and integrate we find :

$$\mathrm{S}_{2c} = \int_{t_1}^{t_1+\mathrm{T}} \theta_1 \cos 2pt\, dt = \tfrac{1}{2}\,\mathrm{TC}_2' \sin \delta_2',$$

and as before we can express C_2' and δ_2' in terms of the quantities, S_{2c} and S_{2s}, the meaning of the latter being of course

$$\int_{t_1}^{t_1+\mathrm{T}} \theta_1 \sin 2pt\, dt = \tfrac{1}{2}\,\mathrm{TC}_2' \cos \delta_2'.$$

This suggests a graphical method of determining the necessary quantities.

Measure as many ordinates of the curve, fig. 129, as is conveniently possible and multiply each by sin pt or sin $\frac{2\pi t}{T}$.

T is the complete period of the periodic heating and t is the value of the time appropriate to each ordinate.

Plot a new curve with these new quantities as ordinates and times as abscissæ and thus obtain fig. 130.

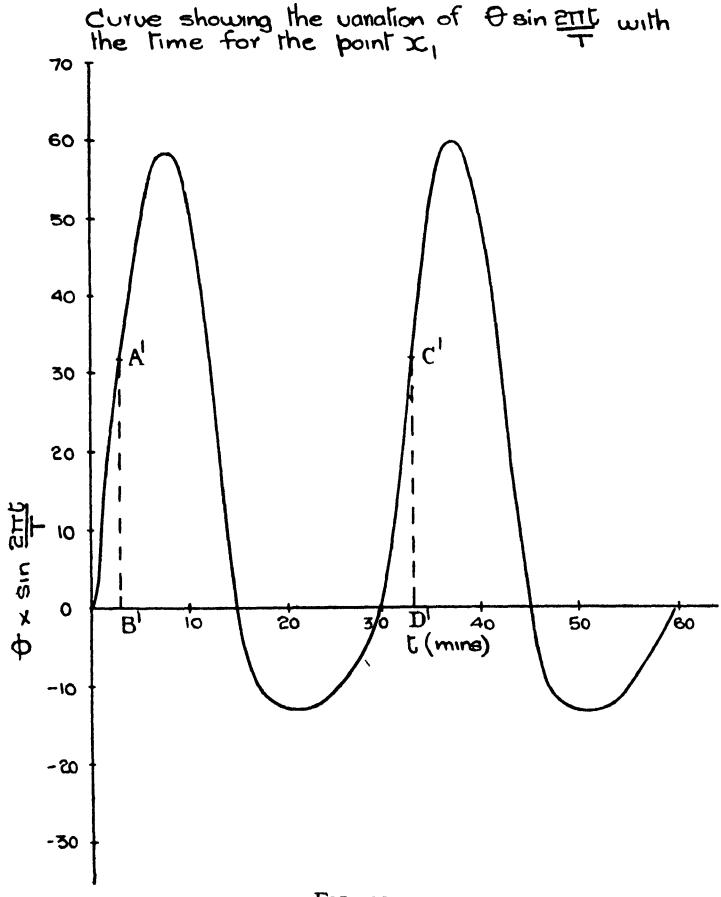


FIG. 130

Draw any ordinate A^1B^1 for some arbitrary time, t_1, and construct C^1D^1 the ordinate at one complete period later.

Measure the area between these ordinates, the curve and time axis carefully with a planimeter.

This gives S_s.

Go through the similar process to find S_0.

Multiply θ_1 by sin $2pt$ and cos $2pt$ to find S_{2s} and S_{2c}. All this has to be repeated for the second point, x_2, and we then have all the data necessary to deduce k from formulæ (10) and (11).

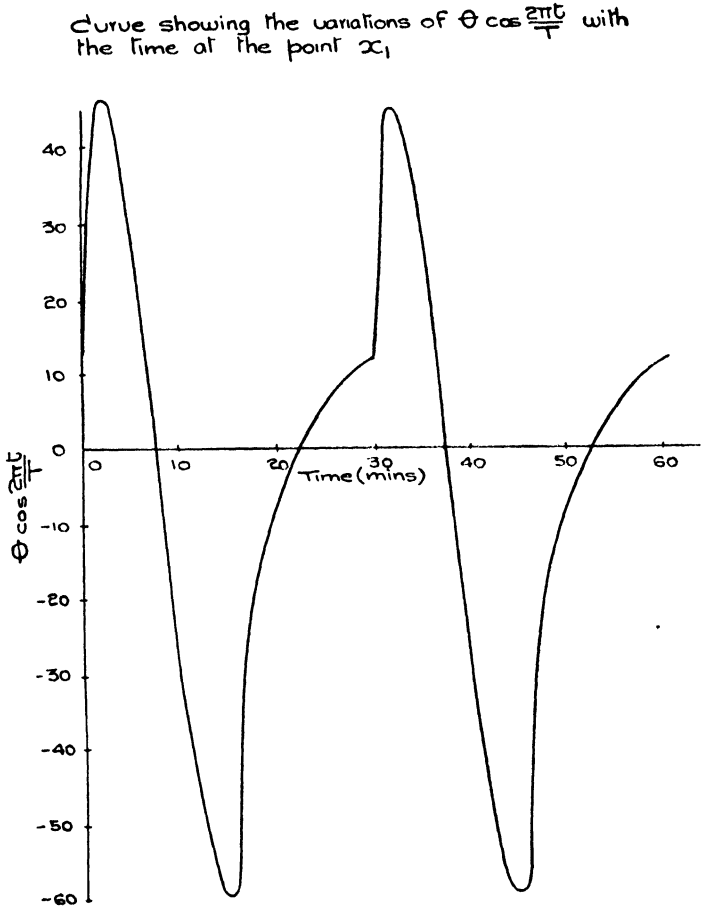


FIG. 131

Another way of making the calculation is to make two experiments with different periods, T′ and T″. If α_n' and α_n'' are the values of α_n in these cases respectively we find from (9) the result:

$$K = \frac{n}{2\alpha_n \alpha_n'} \sqrt{\frac{p'^2 \alpha_n'^2 - p''^2 \alpha_n^2}{\alpha_n'^2 - \alpha_n''^2}}.$$

α_n' and α_n'' are derived as before, and

$$p' = \frac{2\pi}{T'}, \quad p'' = \frac{2\pi}{T''}.$$

It is to be noted that we do not, in this case require the values δ.

The density and specific heat may be taken from tables or be measured by any of the usual methods.

In drawing the curves it is a good plan to have two complete periods shown to give a means of testing the accuracy of the areas, S_g and S_c.

Always test the end of the bar remote from the heat supply to determine if it remains at the air temperature. It will suffice to place a thermometer close to this end. A longer bar must be used if the fluctuations continue right to the end.

Ångström used a square bar of side 2·375 cms., and the length, l, between the two points, x_1 and x_2, was 5 cms.

It has been found, however, that a cylindrical bar of diameter from 1 to 2 cms., and its length about 60 cms., will suffice, with such a range of temperature as described above. In the original experiment Ångström heated his bar at the central region and, of course, had similar conditions on both sides

CHAPTER IX

MISCELLANEOUS EXPERIMENTS IN HEAT

Determination of the Radiation Constant. ("Phil. Mag.", ser. 6, 1905, p. 270)

By Stefan's Law the total radiation from a black body is proportional to the fourth power of the absolute temperature, or

$$R = \sigma T^4.$$

It is the object of this experiment to determine σ.

Apparatus

The diagram (fig. 132) illustrates the arrangement of apparatus which consists of a blackened hollow metal hemisphere, B, about ten inches in diameter, fitted into a wooden box, W, lined with tin.

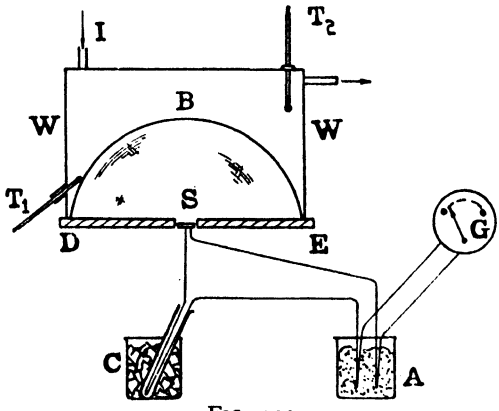


FIG. 132

This fits on to a table, of which the top, DE, is shown, containing a small hole at S, which lies at the centre of the hemisphere. B is heated to a uniform temperature measured by the thermometers, T_1 and T_2, by passing steam through the box above the hemisphere. The black surface of B is the radiator, and the heat is received by a small disc of silver placed at S, and blacked on the upper surface to prevent reflection. It is better to fit the disc in a vulcanite frame rather than to allow it to touch the table directly.

From the disc are lead away two wires, one of constantan, and the other of silver, to a galvanometer and second junction

S is thus one of the junctions of a thermo-electric couple, the other is placed in a tube containing oil standing in a calorimeter, C, containing water or ice.

The junctions to the galvanometer are kept in a canister, A, packed with cotton wool to prevent any electrical effect due to difference of temperature at these junctions.

It may be further necessary to include a resistance in the circuit to keep the deflection on the scale if the galvanometer is too sensitive.

A rise of temperature due to absorption by the disc is thus recorded on the galvanometer.

Theory

Let R_1 = radiation absorbed by the silver disc per unit area per second, and R that emitted. Let the temperature of the radiator be T_1, and of the disc T.

If the whole enclosure, the disc included, had temperature T_1, there would be equilibrium, and the disc would both emit and absorb R_1 in unit time. The energy absorbed would arise, of course, from B. This same energy falls on the disc when at the lower temperature and is absorbed, but the energy emitted is now R. Thus the gain of energy per sec.

$$= (R_1 - R)A,$$

where A denotes the area of the disc.

Let m denote the mass of the disc, s its specific heat, and $\frac{dT}{dt}$ its rate of change of temperature.

Then we have :

$$ms\frac{dT}{dt} = \frac{R_1 - R}{J} \cdot A = \frac{\sigma A}{J}(T_1^4 - T^4),$$

or

$$\sigma = \frac{Jms}{A(T_1^4 - T^4)} \cdot \frac{dT}{dt},$$

where J = Joule's equivalent ($4{\cdot}2 \times 10^7$ ergs).

All the quantities on the right are measured, and hence σ is calculated.

Experimental Details

It is first necessary to ascertain the relation between the readings of the galvanometer scale and the difference in temperature between the two junctions.

In order to make this comparison the disc is surrounded with cotton wool, and the cold radiator placed above it.

The calorimeter, C, is then heated and the difference in temperature between it and the disc recorded on a graph against the readings of the deflection.

We thus obtain : difference in temperature per scale division = $\frac{AB}{BC}$.

By measuring the temperature of C, we can then deduce from the galvanometer deflection the temperature of S from this graph.

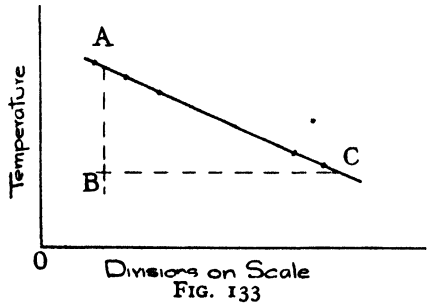


FIG. 133

Secondly, we require the rate of rise of the temperature of the disc when the radiator is put on.

As soon as the temperature in the enclosure, B, has become steady, the box is placed over S, keeping the latter at the centre, and the galvanometer is read at equal intervals of time ; these may conveniently be every 5 or 10 secs.

The table shows a record of observations made every 10 secs.

TIME	SCALE DIV.	TIME	SCALE DIV.
0	227	40	190
10	218	50	181
20	208	60	173
30	199	70	165

These results are plotted on a graph.

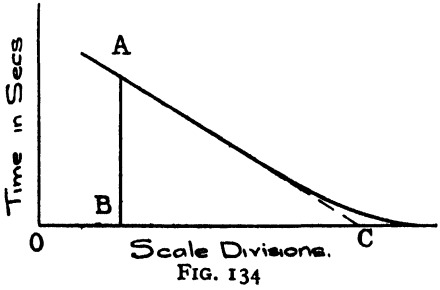


FIG. 134

Draw a tangent to the curve and measure the mean value of $\frac{dT}{dt}$ by the ratio $\frac{BC}{AB}$.

We thus calculate $\frac{dT}{dt}$ in the neighbourhood of the temperature of the disc at the instant when the radiator was put on.

m and s are found in the usual way. It will be convenient to regard these as constants of the apparatus, and to record them on the occasion of making the apparatus. It is inconvenient to make determinations at each experiment.

Or s may be measured by Regnault's Method from a piece of silver identical with that in the apparatus.

The Rise of Boiling Point of Solutions

The object of this experiment is to determine how the temperature at which a solution boils depends on the concentration, to determine the rise of the boiling point above that of the solvent, and to compare experimental results with those given by the thermo-dynamic formula.

If P denote the osmotic pressure of a solution, i.e. the pressure due to dissolving one gramme molecule of the substance in one litre of the solvent, S, the molecular weight of a salt, and s, the number of grammes in 1c.c. of the liquid, we have:

$$\frac{P}{s} = \frac{2R\theta}{S},$$

where R is the value of the gas constant per gramme molecule, and θ is the temperature of the solution, i.e. for any particular salt and solvent and for a temperature θ,

$$P \propto s.$$

All this is true for dilute solutions only.

A calculation based on the second law of thermo-dynamics shows that the rise of temperature of the boiling point, T, is given by the formula:

$$T = \frac{P\theta}{L\rho}$$

where P is the osmotic pressure, L = latent heat of vaporization of the solvent, ρ = its density, and θ = temperature of boiling of the solvent.

For purposes of the calculation it may be assumed for the case of water, $\rho = \frac{1}{1{\cdot}003}$ at boiling point.

$P = 22{\cdot}3 \times 10^6$ dynes per gramme molecule per litre at 0°C.

If θ = B.P., the approximate value for P at temperature θ is:

$$\frac{\theta}{273} \times 22{\cdot}3 \times 10^6 \text{ dynes.}$$

θ is in absolute units, and the value for any particular concentration may be deduced by simple proportion.

L, for water at ordinary pressures, may be taken as $537 \times 4{\cdot}2 \times 10^7$ ergs, in which units it must be expressed for the equation

Suppose w grammes of solute are dissolved in W grammes of solvent.

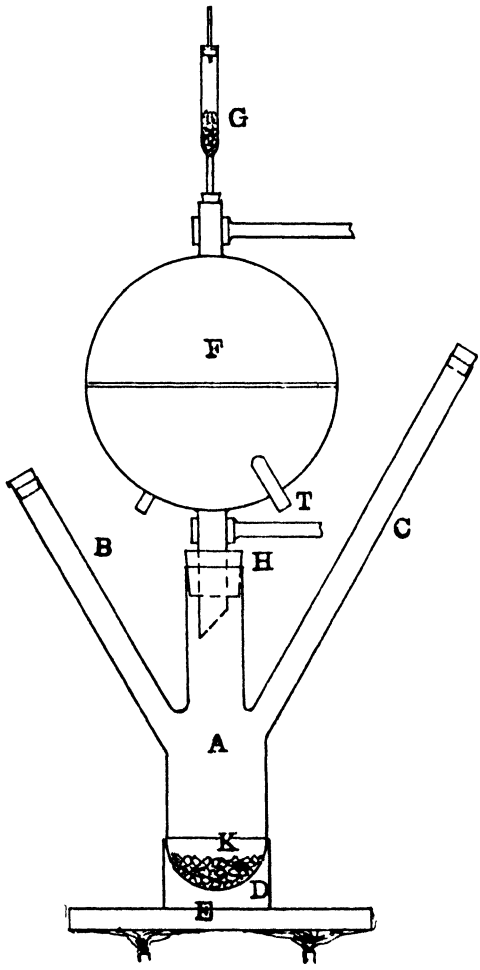


FIG. 135

The volume of the solvent is $\frac{W}{\rho}$ c.c. or $\frac{W}{1000\rho}$ litres.

If M denote the molecular weight of the solute, we have $\frac{w}{M}$ gramme molecules in $\frac{W}{1000\rho}$ litres, i.e. $\frac{1000\rho w}{MW}$ gramme molecules per litre.

Thus the depression according to the theory should be :

$$\frac{1000\rho w}{MW} \cdot \frac{P\theta}{L\rho} = \frac{1000wP\theta}{MWL}.$$

Verify this expression by taking several values of w.

A modern form of apparatus for carrying out this experiment will be described. It is represented diagrammatically in the figure (135). A is a glass tube which holds the solvent from which proceed two tubes, B and C, provided with corks.

The thermometer passes down B, and the bulb is immersed in the solution or pure solvent in A.

Weighed quantities of solute may be added through C.

Through the cork, H, passes the metal tube of the condenser, F, which is cooled by passing in water by one of the tubes, T, and allowing it to flow out by the other.

The small tube, G, carries garnets, which are placed at the bottom of A, and which together with a short wire through the base of A, assist the commencement of boiling and tend to prevent superheating. A stands on a cylinder of asbestos, D, which rests on a plate, E, of pipeclay, and the liquid is heated by two burners placed below E.

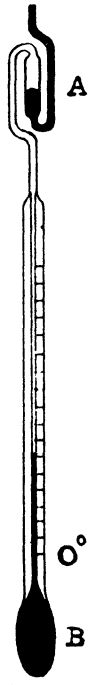


FIG. 136

The thermometer is set and its reading taken when the pure solvent is boiling, and the difference noted when the solute is added and the solution boils.

We have first to weigh the vessel, A, when detached from F, and not carrying the thermometer. Add a weighed amount of solute, w, to a convenient quantity of solvent. After solution, and after taking the boiling point of the solution, F is again removed and also the thermometer, and A is again weighed.

Since w is known, we can obtain W, the weight of solvent. Care must be taken not to remove any of the liquid with removal of the thermometer. The thermometer bulb must be placed in the solution ; if it is above it attains the temperature of the steam coming off from the solution, and this is not at the temperature of the solution, but at the temperature of steam appropriate to the prevailing pressure.

The Beckmann Thermometer

The magnitude of the change of temperature is not very great, and great care is necessary to measure it accurately. The most convenient form of thermometer for this purpose is one of the Beckmann type.

A common form of this thermometer is made of Jena glass, having an apparent coefficient of expansion for mercury of magnitude $\frac{1}{6370}$. It is illustrated in fig. 136.

Its special feature is the large bulb, B, containing a comparatively large quantity of mercury from which leads a fine capillary tube, bent as shown at A, where the capillary is widened.

The mercury in the bulb being of considerable volume gives a large additional volume on expansion, and this causes a large movement in the fine capillary tube when the change of temperature round the bulb is only slight.

If, however, the quantity of mercury in the bulb were fixed the range of temperature for which the instrument could be used would be very small unless the capillary were inconveniently long.

The thermometer is used to measure a small change in temperature above or below some particular temperature. Suppose that we require to measure a small change of temperature above T_0.

It would be best to arrange so that the mercury extended from the bulb up to the zero of the scale when at the temperature, T_0. A slight increase in temperature would then drive the mercury along the capillary, which is marked in hundredths of degrees at intervals of sufficient width to enable an experimenter to estimate to thousandths.

Suppose now that it is required to read temperatures slightly above another temperature, T_1, which is greater than T_0. If we could withdraw sufficient mercury from the bulb it would be possible to arrange that the mercury extended up to the zero mark at the temperature, T_1, and slight increases would drive the thread along the fine tube as before.

Of course, the expansion per degree rise of temperature in the second case is not strictly the same as in the first, since the initial volumes are not equal in the two cases.

In order to see how this effects the observations, let us suppose that the graduations on the stem are correct at $t_0°$ C., i.e. with mercury filling the bulb and extending to the zero mark, when the surrounding temperature is $t_0°$, a rise of temperature of 1° C. would cause an expansion to the first degree mark above the zero.

Let the volume of mercury at $t_0°$ under these conditions be denoted by V_0, and let the coefficient of apparent expansion of the mercury in glass be α.

Then the volume of the capillary per 1° is $V_0\alpha$.

Now suppose that mercury is drawn off until at temperature, t, the mercury fills the bulb and extends to zero. The volume drawn off is $V_0\alpha(t - t_0)$.

The remaining mercury had a volume at $t_0°$ of

$$V_0\{1 - \alpha(t - t_0)\},$$

and on further heating of one degree would expand by an amount

$$V_0\alpha\{1 - \alpha(t - t_0)\}.$$

Since the volume of the capillary per 1° is $V_0\alpha$ this expansion will be registered as $\{1 - \alpha(t - t_0)\}$ of a degree.

The degrees on the scale are too large for the temperature t. A correction could be made by dividing the scale readings by $\{1 - \alpha(t - t_0)\}$, and it would be sufficient to add to the observations an amount $\alpha(t - t_0)$ per degree registered on the scale to obtain the corrected rise.

As has been stated α has the value $\frac{1}{6370}$, so that if we suppose the thermometer correct at 90° C. and we record a rise just above 100° C. we should add to each degree an amount

$$\frac{10}{6370}° = \cdot0016°.$$

This is an appreciable amount since we can record up to thousandths of a degree.

The correction can be neglected when we are recording to hundredths of a degree.

The widening of the capillary at A enables the change in the quantity of mercury in the bulb to be made.

Suppose it is necessary to read a temperature between 100° and 101° C. The thermometer is placed into a bath at a temperature of about 102° C. to 103° C., and so that the mercury extends from the bulb up to the widened part when at this temperature.

If a quick jerk is made the thread will break at the point, where the capillary enters the wider part.

When cooled to about 100° C., i.e. when held in steam, the temperature of which may be determined by observing the atmospheric pressure, and referring to a book of tables, the top of the thread should sink to the lower end of the capillary. It will not lie exactly at zero, and the position occupied by the end at this known temperature is recorded. It may be necessary to vary the upper temperature when the thread is continuous before the end will sink conveniently towards the zero at the temperature of steam.

When this condition has been attained the small rise of temperature is readily recorded.

It is not necessary to bring the mercury thread exactly to zero before reading the small temperature change, and if the correction is to be applied it may be made by giving the value to t which corresponds to the temperature for which the apparatus is set, just as if it had been set at the zero

If it is more convenient to observe a diminution of temperature below a certain point, it is only necessary to bring the top of the mercury thread at this temperature to a point high up on the scale. A slight depression will cause the top of the column to sink, and the interval may be measured as before.

In such accurate temperature measurements it is necessary to make a correction for the emergent column. This is done by reference to Grützmacher's tables taken from the "Zeitschrift für Instrumentenkunde" (1896), p. 220.

A reference to this table will show how important such a correction is.

GRÜTZMACHER'S TABLE

TEMPERATURE INTERVAL.	VALUE OF ONE SCALE DIVISION IN DEGREES CENT. WITH THREAD TOTALLY IMMERSED.	WITH THE THREAD ALL OUTSIDE AND AT MEAN TEMPERATURES BELOW	CORRESPONDING VALUE IN DEG. CENT. PER SCALE DIVISION.
° C.	° C.	° C.	° C.
−35 to −30	·982	0	·977
0 to 5	·997	15	·995
45 to 50	1·011	26	1·015
95 to 100	1·021	32	1·032
145 to 150	1·027	38	1·045
195 to 200	1·028	44	1·053
245 to 250	1·024	50	1·055

The second column gives the correction due to the variation of mercury within the bulb, and it shows that the thermometer reads correctly at a temperature between 5° and 45°, for in this region lies the point where the value of one scale division is 1° C.

The fourth column gives, in addition, the correction for the fact that part of the mercury is at a different temperature from that in the bulb.

In order to make a correction, choose the part of the table which describes most nearly the condition of the experiment, e.g., at a temperature in the neighbourhood of 100° with a mean thread temperature about 30°, one degree recorded by the apparatus is approximately 1·032°.

Molecular Weight by Depression of Freezing Point

The thermo-dynamic formula for the depression of freezing point is

$$T = \frac{P\theta}{L\rho},$$

where L is the latent heat of fusion of the solvent, ρ its density, and θ is the temperature of fusion.

P is the osmotic pressure.

Let W denote the weight of solvent used, and let w grammes of solute be added of molecular weight, M.

Let P denote the pressure for one gramme molecule per litre, i.e. $22{\cdot}3 \times 10^6$ dynes/sq. cm. By using this value and $\rho = 1$, $L = 80 \times 4{\cdot}2 \times 10^7$ ergs, $\theta = 273°$ Absolute, calculate the corresponding depression T_0.

In the present case the volume of solvent is $\frac{W}{1000\rho}$ litres, and the number of gramme molecules dissolved is $\frac{w}{M}$.

The corresponding depression is T^1, where

$$T^1 = \frac{1000\rho}{W} \cdot \frac{w}{M} \cdot \frac{P\theta}{L\rho} = \frac{w}{WM} \cdot 1000 \cdot \frac{P\theta}{L}.$$

It will be the object of the experiment to calculate M.

The vessel, D, with the stirrer, S, is weighed and a quantity of the solvent introduced, and the vessel again weighed.

This gives W.

The thermometer is placed with its bulb in melting ice, and the top of the mercury thread adjusted so that it stands near the upper end of the scale. This temperature is 0° C. and small differences from this point may be read from the thermometer graduations. These may be corrected by Grützmacher's Table.

A weighed quantity of the solute is introduced into the solvent through the side tube, C, and solution brought about by help of the stirrer.

The vessel is placed in an enclosure, E, and surrounded by a freezing mixture, F.

The thermometer, B, must be kept in ice until D has cooled down to 0°, and then the transference must be quickly made to D. Otherwise the thread will rise beyond the scale, and mercury will flow into A.

During solidification the mercury will stand at a definite mark if the solvent is pure, and the difference, T^1, can be measured.

In order to set the thermometer ready for use in the experiment it must be placed in water cooled to within a degree or two of the freezing point. The thread of mercury which must be continuous from B to A is broken by a quick jerk at the top of A, when the temperature of the water is attained.

In this way the thread will extend nearly to the top of the scale, when the bulb is at the temperature of melting ice.

For the temperature at which the solution freezes the fall below the freezing point of water is obtained by subtracting the second reading with the bulb in the solution from that with the bulb in ice. The degrees thus read will require correction on account of the emergent column.

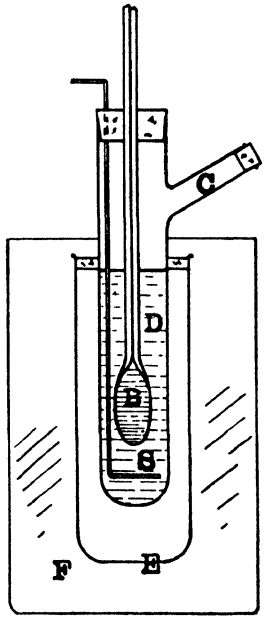


FIG. 137

In order to make the correction, use Grützmacher's Table, e.g., if a temperature difference recorded is 1·5° C. with an average room temperature 15° C., the degree as read on the scale is really only ·995°. Thus the difference is :

$$1{\cdot}5 \times {\cdot}995 = 1{\cdot}492°.$$

The thermometer reads to $\frac{1}{100}°$, so that we can estimate $\frac{1}{1000}°$ and it is therefore necessary to take account of such an error as that arising above, viz. of ·008° C.

Determination of the Mechanical Equivalent of Heat by Friction Cones

The apparatus consists of two vessels of gun-metal of the shape of truncated cones, one fitting within another. The inner cone contains water, and is held in position by a measured

couple applied by means of a weight hanging over a pulley and connected to a disc attached to the cone.

The outer cone is rotated by means of a cord to a large wheel rotated by hand.

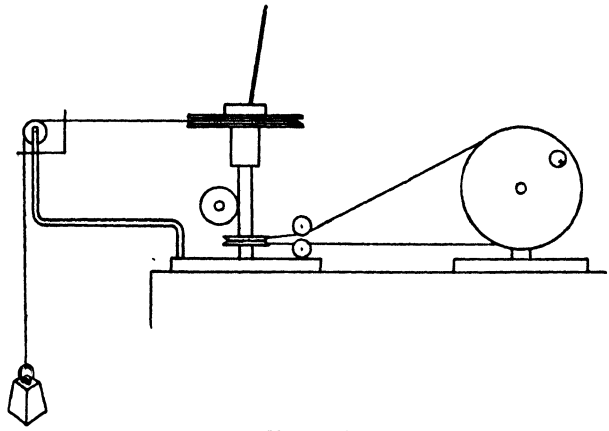


FIG. 138

It is possible by steady turning of the handle to cause the weight to hang almost steady, so that a steady couple is applied to the inner cone by friction, and its amount is measured by multiplying the force and the diameter of the wheel to which the cord is attached.

A counting device is fixed to the outer cone so that the number of rotations is recorded.

If n is the number of rotations, w the weight in dynes applied to the disc, and a its arm, the work done is

$$2\pi nwa \text{ ergs.}$$

The rise in temperature of the water is measured by means of a thermometer, so that if M represents the total water equivalent of the cone and contained water, and if the rise in temperature is T, the heat developed is MT calories.

If no heat is lost by radiation and conduction, the mechanical equivalent, J, or the number of ergs necessary to produce one calorie is given by:

$$J = \frac{2\pi nwa}{MT}.$$

A common form of apparatus in use in laboratories is shown in the figure, it was designed by Dr. G. F. C. Searle.

An improved design is described in the " Phil. Mag.," Sept. 1920, by H. P. Waran.

In order to prevent radiation losses it is a good plan to cause a rise in temperature of not more than 10° C., and to cool the water about 5° below the room temperature initially. The two

cones are mounted in a metal case lined with cork to diminish conduction losses.

The heat is conducted through the walls of the inner cone to the water, and it is necessary to stir the water during the experiment to assist the flow of heat throughout its extent. This may be done by means of the thermometer.

One of the features of Waran's improvement is the automatic stirring of the liquid which consists of an oil with good conductivity and known specific heat.

The Determination of Joule's Equivalent of Heat by Callendar and Barnes's Electrical Method

The principle of the experiment is to supply electrical energy to a wire surrounded by water, and to measure the heat developed in the water by noting the rise of temperature.

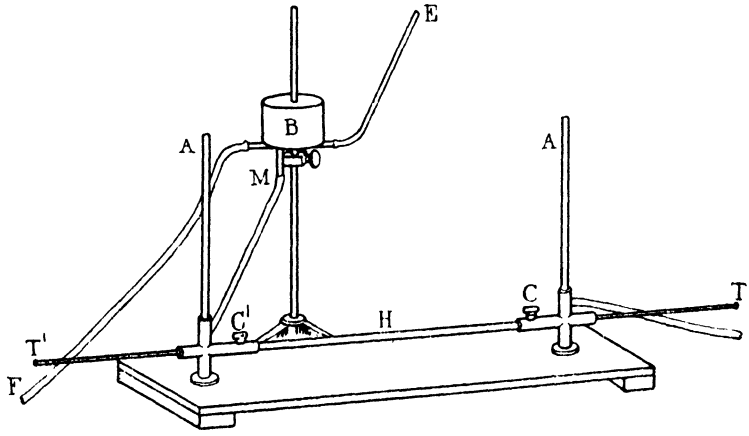


FIG. 139

If we express the electrical energy in ergs or joules we can thus deduce that required to generate one calorie.

Inside the glass tube, H (fig. 139), is fixed a helix of manganin wire of resistance about 9 ohms. The ends of the wire are joined to the terminals, C and C^1.

Water from the tank, B, enters H by a side inlet tube at one end and after flowing round the wire comes out from a similar outlet tube at the other. Thermometers, which enter the ends of H as shown, penetrate the inflowing and outflowing water, and serve to record the temperatures of the water before and after it has received heat from the wire.

The vertical tubes, AA, allow air bubbles which may flow in with the water, to escape. The level of B is variable for the tank is movable up and down its stand.

The makers of the apparatus recommend that the rate of flow

should lie between 55 c.c. and 65 c.c. per min., or about 1 c.c. per sec. B should be adjusted to produce this rate.

The tank consists of an inner and outer chamber.

Through the tube, E, water enters the outer chamber from the supply, and comes out through M to enter H.

The water flows over from the outer chamber to the inner and escapes to the drain by means of F. Thus water is supplied at a constant pressure to H, and the flow is made steady. The outflowing water may be collected in a measuring glass, or, of course, it may be collected in a weighed beaker and the mass per second flowing out deduced.

The apparatus gives best results for a current of approximately 2 amperes, so that a voltage of from 20 to 25 should be applied. To obtain a steady source ten or twelve accumulators should be used and connected through an adjustable resistance to C and C^1.

In this case, with the rate of flow indicated, a difference of temperature of the ingoing and outgoing water of about 8° C. is maintained.

If possible the mean temperature of the water should be that of the room, to avoid errors due to radiation.

When it is not possible to arrange this exactly, it is necessary to apply a correction. Let the temperature of the room round the apparatus be t_0, and the mean temperature of the water at entrance and exit be t.

Then the number of calories lost per second in radiation is

$$m(t - t_0) \times \cdot 05,$$

where m is the outflow per second. Or we may make the correction by adding $\cdot 05(t - t_0)$ to the difference of temperature recorded by T and T^1. This correction has been obtained experimentally by the designer.

Measure the resistance of the spiral by means of a Post Office Box.

Connect the resistance, R (fig. 140), in series with the supply of current and the ammeter, A, which has a range up to 3 amperes.

If a voltmeter is available there is no need to measure the resistance of the helix, for the difference of potential between C and C^1 may be measured by connecting it through the key, K, in parallel as in the diagram.

Adjust the current to about 2 amperes, and if necessary make slight variations from this to adjust the temperatures so that radiation losses are made as small as possible.

Let m denote the rate of outflow per second, C the current in amperes, E the voltage supplied between C and C^1, and T the difference in temperature of the two thermometers, corrected if necessary according to the rule given above.

The supply of electrical energy is EC joules per second, and the heat developed is mT calories per second.

Thus, J, the heat equivalent, is given by:

$$J m T = EC,$$

or

$$J = \frac{EC}{mT} \text{ joules}$$

$$= \frac{EC}{mT} \times 10^7 \text{ ergs.}$$

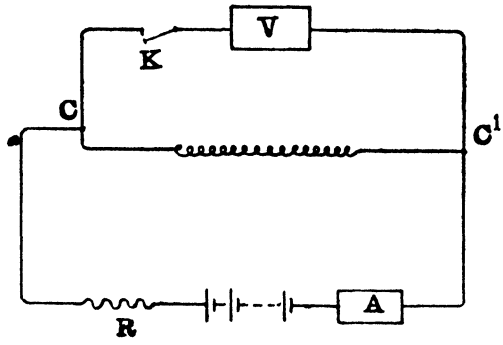


FIG. 140

If the resistance of the wire is measured, instead of EC we must write C^2R. The accuracy to be expected is between one-half and one per cent.

The following is an example of an experiment carried out in the laboratory, and indicates the order of the quantities:

Temperature of room, 17·3° C.
Temperature of water at inlet, 16·8° C.
Temperature of water at exit, 25·35° C.
Mean temperature of water, 21·07° C.
Temperature difference of water in the two cases, 8·55° C.
Radiation correction = ·05 × (21·07 − 17·3) = ·19° C.
Corrected temperature difference, 8·74° C.
Volume of water flowing out in 2 mins., 115 c.c.

$$m = \frac{115}{120} \text{ gms. per sec.}$$

Current, 2 amperes. Resistance of wire, 8·92 ohms.

$$\therefore J = \frac{4 \times 8{\cdot}92}{\frac{115}{120} \times 8{\cdot}74} = 4{\cdot}260 \text{ joules.}$$

CHAPTER X

REFLECTION

The Sextant

The instrument consists of a graduated arc, SS (fig. 141), with two radial arms, A and C.

A third arm, B, moves about an axis through one of its ends at right angles to the plane of SS. It is fitted with a clamp and tangent screw, so that it can be accurately adjusted, and carries a vernier at its end which moves over the scale of SS.

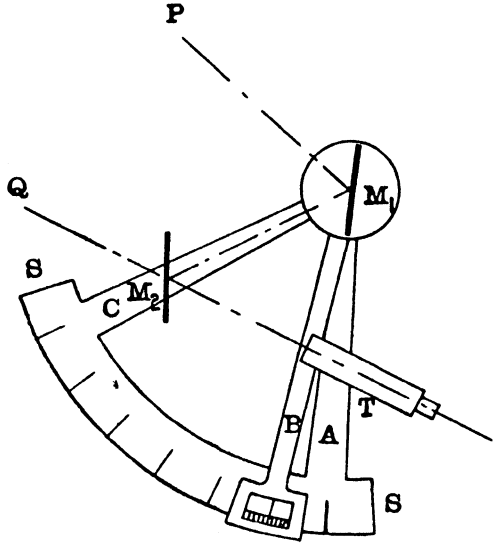


FIG. 141

A plane mirror, M_1, is attached to B, lying with its surface in the direction of B, and in a plane normal to that of the scale. The axis about which B turns lies in the surface of this mirror, which is called the index glass.

The second mirror, M_2, fixed to the arm, C, is called the horizon glass, and its plane must also be perpendicular to the scale. It consists of a plate of glass only one-half of which is silvered.

At T, on the arm, A, is fixed a telescope with its axis parallel to the plane of SS and passing through the centre of M_2.

Suppose the movable arm is turned so that the mirrors are parallel, and T directed towards a distant object, on which it is focussed.

Only one image will be seen of the object, for the light from M_1 and M_2 is brought into the telescope in the same direction, M_2T. The rays by the two reflections may not lie along the same line as those seen directly through M_2, but since they are parallel there is only one image formed by the telescope (see fig. 142).

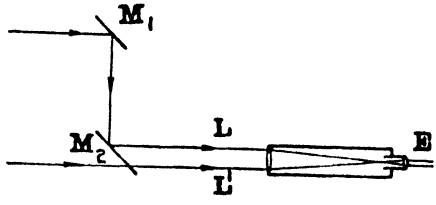


FIG. 142

If the index glass be rotated through an angle, A, by turning the arm, B, then the rays reflected by it into the telescope no longer come from the same object as that which supplies the ray, QM_2. Two superposed images are seen in the telescope, and the angle between PM_1 and QM_2 is 2A, since on turning a reflector through any angle a beam of light is rotated through double this angle. Thus every degree on the scale, SS, corresponds to a difference of direction of two degrees. The scale is marked to give directly the angle between the rays, PM_1 and QM_1, i.e. to measure the angle subtended at the instrument by two distant objects.

In order to measure this angle one of the objects is observed directly through M_2, while the other is made to produce a superposed image on that of the first by rotating M_1 into the proper position. The angle through which the index glass is turned from the parallel position is then one-half of the angle subtended. Before making any measurement the instrument must be tested to see that it satisfies the following conditions:

(1) The plane of the index glass must be normal to the plane of the scale,
(2) The axis of the telescope must be parallel to the plane of the scale, and
(3) The index and horizon glasses should be parallel, and at the same time the vernier should read zero.

It will not be necessary, as a rule, to adjust for the first of these, but in order to see that the instrument is satisfactory in this respect look at the image of the scale in M_1. Since M_1 passes through the centre of this scale the latter and its image will appear to intersect at the edge of the mirror and, if the adjustment is satisfactory, to lie in the same plane.

Both M_1 and M_2 are attached to frames which can be turned

through small angles by means of screws. If necessary M_1 may be adjusted until the test is satisfied.

The second condition is tested by observing two objects and causing them to coincide at the centre of the field of view. The axis of the telescope is a line joining the centre of the object glass to the centre of the eyepiece, or to the centre of the field of view. Perpendicular to this axis lies one of the cross-wires. Tilt the instrument until the images lie near the edge of the field, and note if they still coincide. Then tilt it so that they lie near the opposite edge. If coincidence persists the axis is correctly adjusted. If this is not the case the telescope can be adjusted by means of the screws.

Observe an object through the telescope and make its image appear in the field of view by reflection in M_1. If it is possible to cause them to coincide, the two mirrors are parallel, and on account of the first adjustment this will mean that the third condition is partly satisfied. By means of the screw attached to M_2 the two images can be made to coincide if they do not do so at first.

When these conditions are satisfactorily arranged it will probably happen that the pointer does not read zero when a distant object is viewed. To correct for this it is only necessary to note the zero error and apply it in all the observations. Coloured glasses are provided for diminishing the brightness of any object such as the sun. These can be made to intercept the light immediately before falling on the mirrors.

Experiment 1

Place two candles at as great a distance as is convenient, and measure the angle they subtend at the instrument.

Also find the angle by measuring the distance to each candle and their distance apart.

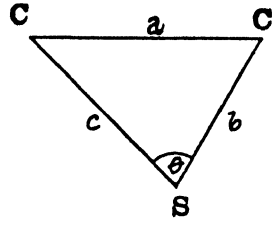


FIG. 143

Let the candles be C_1 and C_2, and let S denote the sextant.

The distances to be measured are a, b, and c, and if $s = \frac{1}{2}(a + b + c)$

$$\tan \frac{\theta}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Check the values of θ obtained by the two means.

Experiment 2

Let a trough of mercury or a carefully levelled mirror be placed so that the image of a lamp can be seen directly, and also by reflection, and measure the angle subtended at the sextant by the object and its image by holding the plane of the instrument vertically, at as large a distance from the lamp as possible.

The elevation of the lamp is half this angle.

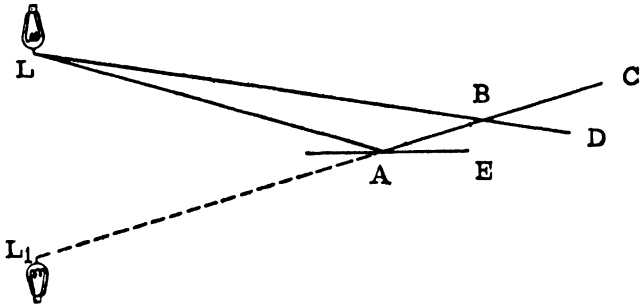


FIG. 144

The diagram (fig. 144), illustrates that the angle, CBD, is measured since the instrument is of necessity above the surface, AE.

Actually, we require the angle, LAL^1, but since we use a distant lamp, the two angles do not appreciably differ.

Thus, the elevation may be measured by half the angle CBD.

Measure the horizontal distance between L and A, and deduce the height of L above the floor.

Check the result by actual measurement.

Measurement of the Angles of Crystals by Wollaston's Goniometer

The goniometer is a convenient instrument for measuring accurately the angles between the faces of small crystals which are too small to be examined by means of a spectrometer.

It consists of a circular circle, S, which may be rotated by the large milled head, B, and its position read off by means of a fixed vernier, V.

The crystal is fixed by soft wax to a plate, P, carried by an adjustable support, D, which may be rotated by the smaller head, A.

The edge of the crystal formed by the two faces between which it is desired to measure the angle is adjusted so that it lies parallel to the axis of the circle.

This adjustment is first made approximately by eye. In order to make the adjustment accurately, view the upper corner of a distant window in both faces. On rotation the images will move in a vertical plane if the edge is parallel to the axis. This may be tested by noting if each image moves in a direction parallel to the edge of the window as seen directly by the eye.

Place the instrument so that the axis is parallel to a tall, distant window, and turn the screw-head, B, until the graduated circle comes against the stop.

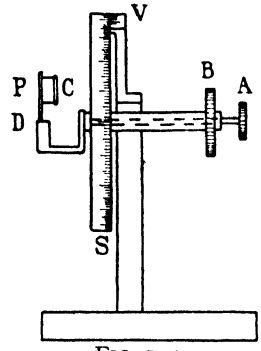


FIG. 145

The eye is placed close to the crystal so that an image can be seen by reflection in one of the two faces.

The axis is then rotated by the smaller screw-head, A, until the top of the window, as seen by reflection, appears to coincide with the bottom, as seen directly.

When the adjustment has been made the angular position of the circle is noted.

By means of B, the circle is now turned until the top of the window, as seen by reflection in the other face, coincides with the bottom, as seen directly. The second face now occupies a position parallel to the first, and if θ is the angle between them, the circle has been rotated through its supplement. This angle $(180-\theta)$ is read off from the circle and θ deduced.

It should be noted that the crystal must lie as close to the axis of the goniometer as possible, for the motion of the crystal from the first position with reflection in one face to the second position, with reflection in the adjacent face, consists both of translation and rotation, unless the crystal is on the axis. The amount of translation may be sufficient to cause an error in the angular measurement.

If the window is a long way away the error is only small.

The Determination of the Radii of Curvature of Spherical Mirrors

(A) *Concave Mirrors*

The most convenient method of determining the radius of curvature of a concave mirror is to place a pin point in front of it and to locate the position in which the image of the pin appears to coincide with the pin itself. The method of parallax is employed to ascertain when coincidence is attained.

The rays from the point of the pin falling on the mirror are reflected back from the surface along their original paths and must therefore strike it normally; consequently, the pin point lies at the centre of curvature of the surface.

Another method consists in locating a series of pairs of conjugate points for the surface and using the formula:

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

A pin is set up as object and another pin adjusted until the image of the first coincides with it. We can then measure a pair of values, u and v.

Several pairs of values are obtained, and the above formula then gives r. Take the average of four or six observations.

(B) *Convex Mirrors*

Method 1

In the case of a convex mirror the image is virtual, and it is not convenient to locate it by a pin placed in a particular position since the image lies behind the mirror.

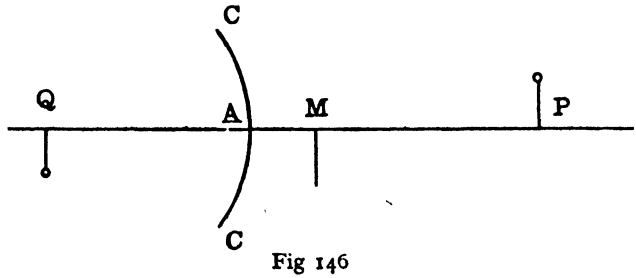


Fig 146

A pin, P, is set up in front of the mirror, CC, and in between them is placed a plane mirror, M, so that the image of P in both can be observed. The mirror is adjusted until the two images coincide (fig. 146.)

By the simple law of reflection in the plane mirror, M, we know that the image of P in M, say, Q, lies at the same distance from M as P does, but on the other side of it.

We can thus calculate the distance, AQ, for

$$AQ = MQ - MA = MP - MA.$$

Then $v = -AQ$, adopting the usual sign convention, viz. directions measured from A towards the object are positive, and in the opposite direction they are negative.

Thus, using the formula (1) we can again deduce r by measuring AP and AM.

Several pairs of values should be obtained, and they should give the same value of r.

Method 2

Another method is to set up a pin and form a real image of it by a convex lens. The image is located by placing a second pin, Q, so that there is no parallax between it and the image. Then place the convex surface between the lens and second pin, and move the surface until an image is formed coincident with the first pin, P. The rays after passing through the lens are directed to the point, Q, but strike the surface normally, and are therefore reflected back along their path. The radius of curvature of the convex surface is MQ (see fig. 147).

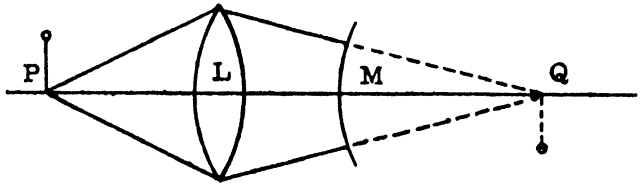


Fig 147

Method 3. *By means of a telescope, metre rule, and small millimetre scale*

The diagrams (figs. 148, 149, 150,) show the arrangement of apparatus. S is a small scale placed horizontally in contact with the surface of a convex mirror, along a line dividing it into two equal parts.

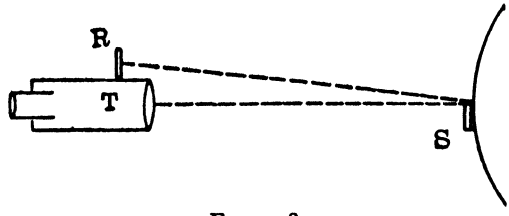


FIG. 148

The scale, RR^1, is mounted at some convenient distance away, usually about 60 or 70 cms., and below its middle point is fixed a telescope, T, focussed on the image of the scale, RR^1, in the mirror.

The apparent length of the image is read off by means of SS^1, which will be sufficiently well focussed to make this possible.

The distance from the centre of the mirror, P, to the middle

of RR^1, is measured, say, d, and from these two measurements, together with the length of RR^1, $2l$, it is possible to calculate the radius of curvature of the surface of the mirror.

Let rays from R and R^1 strike the mirror at L and L^1, and be

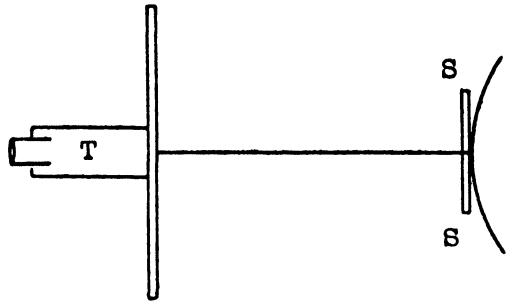


FIG 149

reflected down the telescope at O. Then LL^1 will denote the extent of the image, and the point, B, at which these two lines meet OP will be conjugate to O for reflection in the mirror. We may say that a point source at O will have a point image at B, so that if $PB = x$, we have :

$$-\frac{1}{x}+\frac{1}{d}=-\frac{2}{r},$$

r denoting the numerical value of the radius of the mirror.

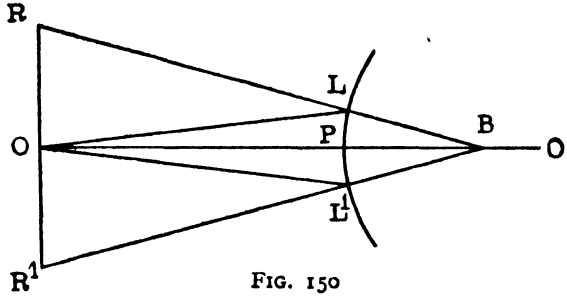


FIG. 150

But if we take LL^1 as approximately straight, since the image is of small dimensions, and denoting LL^1 by $2c$

$$\frac{l}{c}=\frac{d+x}{x}.$$

Hence
$$\frac{1}{x}=\frac{1}{d}\left(\frac{l}{c}-1\right);$$

so that
$$\frac{2}{r}=\frac{1}{d}\left(\frac{l}{c}-1\right)-\frac{1}{d},$$

or
$$r=\frac{2dc}{l-2c}.$$

The result may be verified by means of a spherometer.

The Focal Lines formed by a Concave Mirror

When light diverges from a point and falls on a mirror, it is supposed in the elementary theory that after reflection the rays pass through a single point or appear to proceed from a point. This is approximately true if the dimensions of the mirror are small compared with the distance from the source. A closer approximation to the truth is that the rays after reflection pass through two lines or appear to come from two lines, situated in parallel planes and perpendicular to one another.

These are the focal lines and it will be shown how to calculate their positions theoretically, while it will be the object of experiment to verify the result obtained.

We shall take the case of light falling obliquely on a concave mirror.

If the point source lies on the axis of the mirror we have symmetry about this axis, and the two lines degenerate into a point or circle through which the rays pass.

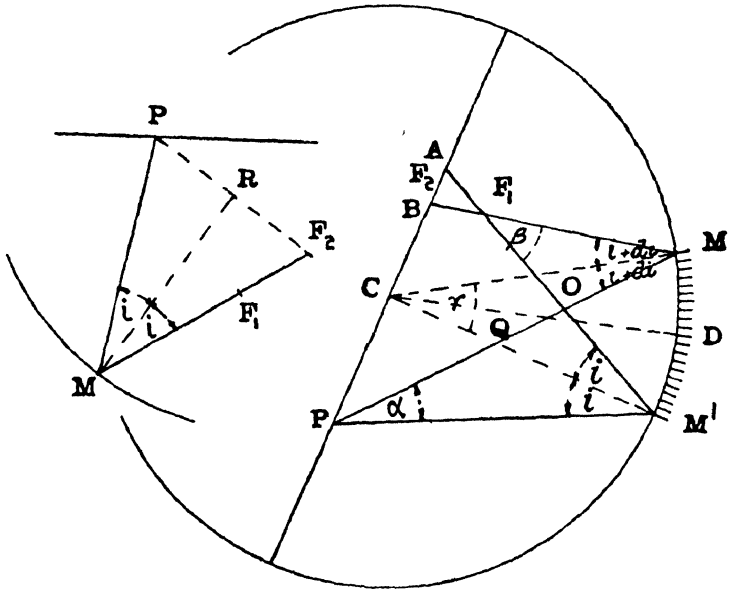


FIG. 151

In the diagram, MM^1 denotes the concave mirror, and C its centre of curvature. The complete circle of which the section, MM^1, forms a part is drawn for convenience.

P denotes the position of the point source of light, and the diameter is drawn through P.

The extreme rays, PM and PM^1, are drawn and the reflected rays, MB, M^1A, are drawn intersecting at F_1, and cutting off from the diameter the strip, AB.

The mirror is a part of a sphere, so that rays falling on the mirror from P, whether in the plane of the figure or not will pass through AB. AB is thus a focal line, and is denoted by F_2.

If we imagine the figure to be rotated about the diameter, the rays, M^1A and MB, will still intersect at a point but now out of the plane of the diagram. For a small rotation the point of intersection would be on a line through F_1 normal to the figure.

Thus, a second focal line is through F_1 perpendicular to the plane of the figure.

Let the angle of incidence at the point M^1 of the mirror be i, and let $PD = \rho$.

Denote the distances of the focal lines from D by ρ_1 and ρ_2, i.e. $DF_1 = \rho_1$, and $DF_2 = \rho_2$.

Let the radius of curvature, CD, be R.

Let the mirror subtend an angle, γ, at C, α at P, and β at F_1. We shall regard its dimensions as small in comparison with R, ρ and ρ_1, so that these three angles will be small and may be measured by drawing perpendiculars from M on to the corresponding lines, and dividing this perpendicular by the distance from the point concerned,

e.g.
$$\alpha = \frac{MM^1 \cos i}{\rho},$$

since the normal from M, on the line, PM^1, makes an angle very nearly equal to i with MM^1, and MM^1 is small and is regarded as straight.

In the same way

$$\beta = \frac{MM^1 \cos i}{\rho_1}$$

(for $\cos i = \cos (i + \delta i)$ when δi is small),

while
$$\gamma = \frac{MM^1}{R}.$$

From the triangles OMF_1 and POM^1

$$\angle OPM^1 + \angle OM^1P = \angle OF_1M + \angle OMF_1$$
$$\therefore \quad \alpha + 2i = 2(i + \delta i) + \beta,$$

or
$$2\delta i = \alpha - \beta;$$

and in the same way from triangles, CQM and PQM^1,

$$\delta i = \alpha - \gamma,$$
$$\therefore \quad 2\gamma = \alpha + \beta,$$

or
$$\frac{2MM^1}{R} = \cos i \cdot MM^1\left(\frac{1}{\rho} + \frac{1}{\rho_1}\right),$$

i.e.
$$\left(\frac{1}{\rho} + \frac{1}{\rho_1}\right)\cos i = \frac{2}{R}.$$

This is the formula concerning the position of the first focal line.

In order to determine ρ_2, we note that :

$$\Delta \text{PM}^1\text{A} = \Delta \text{PM}^1\text{C} + \Delta \text{CM}^1\text{A},$$

i.e. $$\tfrac{1}{2} \cdot \rho\rho_2 \sin 2i = \tfrac{1}{2} \rho R \sin i + \tfrac{1}{2} R\rho_2 \sin i,$$

which may be rewritten as :

$$\frac{2 \cos i}{R} = \frac{1}{\rho} + \frac{1}{\rho_2}.$$

So long as i remains constant, ρ_1 and ρ_2 vary with ρ just as u and v vary together in the formula :

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}.$$

When i is made zero we obtain the usual formula,

$$\frac{1}{\rho} + \frac{1}{\rho_1} = \frac{2}{R},$$

for both ρ_1 and ρ_2, so that the two lines coincide.

To find ρ_1 and ρ_2 experimentally, use a small hole in a screen, with a lamp behind as a source of light.

Find R first by adjusting the mirror so that an image is formed on the screen at the side of the hole as in the experiment on concave mirrors, page 263. The distance from screen to mirror will then be R.

Allow the light to fall on the mirror at angles of about 20°, 30°, and 40°, and find the positions of the focal lines by means of a sheet or card or white paper held in a clamp.

Measure the distances from the mirror to these lines, thus obtaining ρ_1 and ρ_2.

In order to find i, it is convenient to mount M on a stand carrying a pointer moving over a scale of degrees.

Read off the mark against the pointer when the image of the hole is thrown back near the object, and turn from this position to any required incidence.

If no scale of this kind is provided, measure the distances, PF_1 and PF_2, say, a and b.

We then know three sides of each of the triangles, PMF_1 and PMB, and can determine i from the usual trigonometrical formula for the tangent of half the angle of a triangle.

Compare the values calculated in this way with those deduced from the formula derived from the above theoretical considerations.

Searle's Methods of Determining Optical Constants

Accurate methods for the measurement of radii of curvature of polished surfaces, of focal lengths, and for the localization of the cardinal points, have been described by Dr. G. F. C. Searle.

For the original accounts the reader is referred to the "Philosophical Magazine," Feb., 1911, pp. 218–224, or to the "Proceedings of the Optical Convention, 1912," pp. 161–172.

The Determination of the Curvature of Spherical Surfaces

For this purpose a table is mounted on a tripod stand (fig. 152), two of the feet of which carry screws for levelling. The table is horizontal, and can rotate truly about a vertical axis. It carries a millimetre scale on the top which can be clamped in any position, and a carriage bearing the surface slides along it. The figure shows a lens system in the place where the carriage slides.

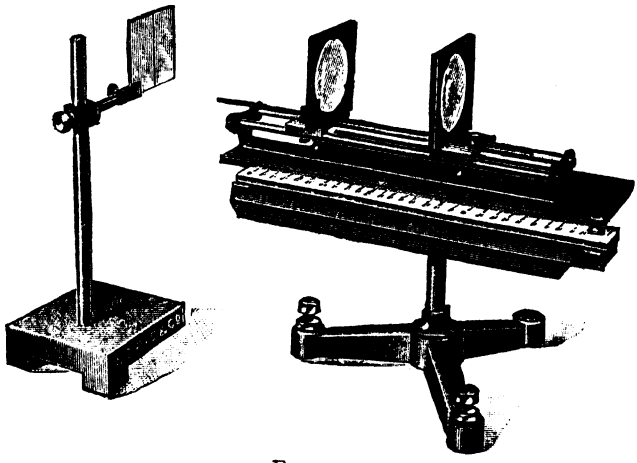


FIG. 152

It is essential that, as the carriage slides along the scale, the centre of curvature of the surface should move along a line which intersects the axis of rotation of the table.

In the figure the scale is shown, and also the wooden slider which acts as the carriage. On the slider is a metal mount which may be screwed to the carriage. This mount carries a horizontal spindle, each end of which is turned to a conical point. One of these ends is provided with a screw thread, so that it will fit a brass plate carrying three screws.

The screws fit into a second brass plate to which the lens or mirror to be examined is fixed by a little wax. The arrangement thus provides a convenient means of adjusting the surface.

The edge of the carrier is smooth and straight, so that it can slide along the scale, and an index mark serves to record the position.

In preparing the apparatus for use the spindle is first set accurately parallel to the edge of the carrier.

The tip of a pin is held just in contact with one of the conical

points of the spindle, from which the brass plate is removed. The carrier is taken from the table top and replaced so that the other conical point lies near to the pin.

If it is possible to bring this point and the pin into contact, the spindle is parallel to the edge of the board and scale.

The spindle is rotated until this adjustment is possible, and the mount then firmly clamped to the carrier by means of the screw. It is also necessary that the axis of the spindle should intersect the axis about which the table turns.

The scale is adjusted and the spindle moved along it until one point lies as nearly on the axis as can be judged by eye.

A microscope is then brought up and focussed on the point, with the scale lying normally to the axis of the microscope, The table top is turned through 180°, so that if the axis of the table passes through A (fig. 153), and P denotes the first position of the spindle point, the second position will lie at P′, where PM = M′P′. The slider has to be moved through the distance, P′P,″ in order to bring the point once more on to the cross-wires. If this distance is recorded and the slides moved back half this amount, the point will be at M′, the foot of the perpendicular from the axis to the line P′P″.

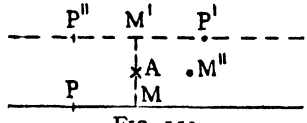


FIG. 153

The microscope is then traversed so that it is focussed on the point. Its axis is thus directed along MA. Turn the table through 90°, so that M′ moves to M″; and move the scale along a direction perpendicular to itself until the point once more comes on the microscope cross-wire. The point then lies on the axis of rotation of the table, and the axis of the spindle will intersect that of the table as it slides along the scale.

Mount the surface to be examined on the brass plate, and attach it to the spindle. If the surface is one of the faces of a lens, the side not under examination should be blacked by covering it with vaseline and lamp black to absorb rays striking it.

Set up an object in some convenient position and view its image in the surface. If on rotating the spindle about its horizontal axis the image does not move, the centre of curvature lies on this axis, and the preliminary adjustments are complete. Since the image remains stationary the rotation of the surface serves to bring up fresh parts of the sphere exactly into the place occupied by the part moved away, i.e. the centre lies on the axis of rotation.

The object may conveniently be a fine brightly illuminated line drawn on a piece of ground glass.

Set up this object again so that rays fall nearly normally on the surface, and examine the image formed by reflection in it by means of a microscope. By moving the carriage a position can be found in which a rotation of the table to and fro produces no displacement of the image. This occurs when the centre of curvature of the surface lies on the axis of revolution of the table, for rotation then has the effect of replacing one part of the surface by another, and reflection occurs as before.

The position of the index mark is noted.

If the lens is moved until the vertical axis of the table is a tangent to the surface, a rotation to and fro will not displace the centre of the lens. Place in the centre a few grains of lycopodium powder, and focus the microscope on one of these. The lens is moved and the focussing repeated until the movement ceases. The index reading is again noted. The difference of the two gives the radius of curvature.

Preliminary observations may be made by eye until the motion appears to cease in the two cases, and only the final exact adjustments need be made by the aid of the microscope.

CHAPTER XI

REFRACTION

Introductory Remarks

Rays of light in passing from one medium to another usually undergo a deviation from their course in the first medium.

On the wave theory this is accounted for by the fact that the light travels with different velocities in the two media. The refractive index with respect to two media is defined to be the velocity of light in the first divided by the velocity in the second, and we denote this by ${}_1\mu_2$.

When light travels from air to another medium, say glass or water, we shall write ${}_a\mu_g$ or ${}_a\mu_w$ unless there is no doubt that we are dealing with air and some other stated medium when we may write simply μ.

We have by definition,

$$ {}_1\mu_3 = {}_1\mu_2 \times {}_2\mu_3, $$

and

$$ {}_1\mu_3 = \frac{1}{{}_3\mu_1}. $$

In particular

$$ {}_g\mu_w = \frac{{}_a\mu_w}{{}_a\mu_g}, $$

a result which will shortly be found useful.

In many experiments in this and following chapters it will be necessary to furnish a bright source of monochromatic light. The most convenient way to provide such a source is to use a Mecca burner, which consists of a Bunsen burner rather larger than the ordinary type of burner provided with a wide end over which is stretched a gauze with a wide mesh.

If a small bead of soda glass is placed on this gauze and the Bunsen made to roar as much as possible a quite satisfactory yellow flame will be produced.

It is a great advantage that there is no crackling in the flame as in the case of the use of common salt, when small pieces of hot salt are thrown about falling on the bench and on the slits of spectrometers. In the case of the latter serious damage to their shape may result.

The slit may be illuminated directly, or better still, an image of the brightest part of the flame may be thrown on to it by a short focus convex lens.

The student will do well to pay attention to the small point of illumination of the slit. It is important always, but assumes greater importance in the case of experiments on interference and diffraction which will be described in the next chapter. The difficulty of discovering Newton's rings, interference fringes, and diffraction bands is almost always due to a lack of care to obtain the best possible illumination of the slit or whatever may be used as a source of light.

Determination of the Refractive Index of a Plate by its Apparent Thickness

The apparatus necessary is a good travelling microscope and a plane cover glass.

Set the microscope with its axis vertical and focus it on the metal platform.

Note the reading on the vertical scale.

Insert the cover glass over the point on which the instrument was focussed, again focus on the metal and observe the scale reading.

Usually the metal surface, though dark, is easy to observe, but if desired a thin sheet of white paper may be placed over it and the surface of the paper used instead of the metal surface.

Raise the microscope until the upper surface of the cover glass is sharply focussed. There will usually be specks of dust on the surface to assist this setting of the microscope, but if any difficulty arises place a small drop of ink on the surface and focus the extreme edge of the drop. The drop need be no larger than that made by a sharply pointed pen.

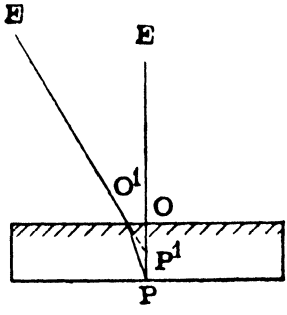


FIG. 154

Note the scale reading when this third adjustment is made. If we describe the scale readings by (1), (2), and (3) respectively; the difference between (1) and (3) gives the actual thickness of the glass and that between (2) and (3) the apparent thickness.

Consider a point, P (fig. 154), situated at the lower surface of

the glass and from which rays of light originate. Only those making small angles with the normal to the surface will enter an eye of microscope placed above P.

If one of these rays, PO^1E^1, makes an angle, i, with the normal in the air and an angle, r, in the glass,

$$\mu = \frac{\sin i}{\sin r} = \frac{OO^1}{O^1P^1} \cdot \frac{O^1P}{OO^1} = \frac{O^1P}{O^1P^1}.$$

For small values of i and r we may write :

$$O^1P^1 = OP^1, \quad O^1P = OP,$$

so that

$$\mu = \frac{OP}{OP^1} = \frac{\text{thickness of glass}}{\text{apparent thickness}},$$

for all rays inclined at such small angles appear to come from P^1, so that OP^1 is the apparent thickness.

Determination of the Refractive Index of Liquids by Total Reflection

When a ray of light passes from a medium of refractive index, μ_1, to another of refractive index, μ_2, with an angle of incidence i and of refraction, r, we have :

$$_1\mu_2 = \frac{\mu_2}{\mu_1} = \frac{\sin i}{\sin r}.$$

We may not always find a corresponding value of r for a given i unless $\frac{\mu_1}{\mu_2}$ is less than unity.

In the case when $\frac{\mu_1}{\mu_2} > 1$, the value $\frac{\mu_1}{\mu_2} \sin i$ must not exceed unity. In the limiting case when

$$\sin i = \frac{\mu_2}{\mu_1},$$

the corresponding value of r is 90°, and i then measures the critical angle. For values of i greater than this critical value the surface acts as a perfect reflector.

If i is slowly increased a value is finally attained when the refracted ray suddenly disappears.

In this case if the second medium is air so that $\mu_2 = 1$ we have :

$$\frac{1}{\mu_1} = \sin i.$$

This formula may be used to determine μ, by mounting a small rectangular trough, CD (fig. 155), with sides of plane glass on the table of a spectrometer, so that a parallel beam of light may be passed through it from a collimator, GH, and received in the

telescope, AB. The light is suddenly cut off when the air cell, EF, consisting of two plates of glass mounted parallel to one another and cemented together with a thin air space between, is turned so that the light falls on the air at the critical angle.

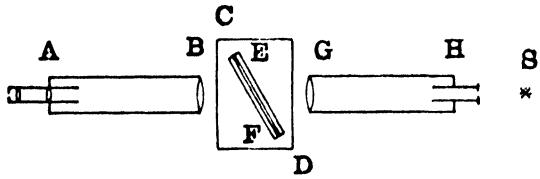


FIG. 155

It will be noted that the critical angle is that for air and glass, but the apparatus is used to determine the refractive index of the liquid in the trough.

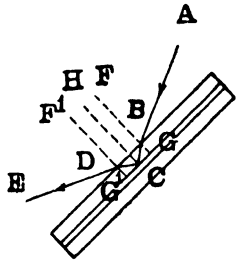


FIG 156

Let the ray, ABCDE, be incident from the water on the glass and be totally reflected at the glass-air surface at the critical angle. Then if FBG and F^1DG^1 be the normals to the glass at B and D, we have :

$$\frac{1}{{}_a\mu_g} = \sin \angle \text{BCH},$$

where CH is the normal at C and ${}_a\mu_g$ denotes the refractive index from air to glass.

Also $${}_w\mu_g = \frac{{}_a\mu_g}{{}_a\mu_w} = \frac{\sin \text{ABF}}{\sin \text{CBG}} = \frac{\sin \text{ABF}}{\sin \text{BCH}},$$

$$\therefore \quad {}_a\mu_w = \frac{1}{\sin \text{ABF}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We have thus to measure the angle ABF, and from it we deduce the value ${}_a\mu_w$, the refractive index from air to water.

As a source of light use a Bunsen flame containing sodium, and illuminate the slit of the collimator, which must be adjusted for parallel light (p. 279). Focus the telescope on the slit and turn EF (fig. 155), until the light just appears. EF is attached

to a pointer which moves over a scale of degrees. Note the position of the pointer. Turn EF from this position into another where the image again disappears.

Let AA^1 denote the axis of the telescope and EF the first position. The second will be E^1F^1 if $\angle AOE = \angle AOE^1$. The

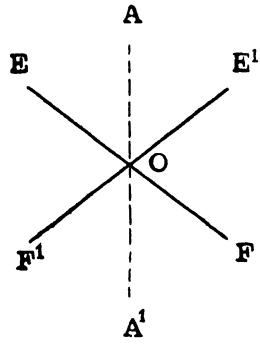


FIG. 157

second position is reached either by turning through the angle, EOE^1 or EOF^1, and $\angle EOF^1 = \pi - \angle EOE^1$.

When the apparatus is used there is no uncertainty concerning which angle is measured.

The angle we require is that between the ray and normal to EF or E^1F^1, i.e. the angle between OA and the normal to EF or E^1F^1. This angle is half the angle between the two normals, and this is the same as half the angle E^1OF.

Thus we have to note the angle through which EF turns and take one half of it to find the angle ABF of formula (1).

To Find the Refractive Index of a Liquid, using a Lens and a Plane Mirror

In the experiment a plano-concave lens is formed of the liquid under examination (fig. 158), and its focal length found experimentally. The refractive index and radii of curvature of the two surfaces enter into the formula for the focal length, so that it is possible to deduce the index, μ, from a determination of the focal length, f, and the radius of the curved surface of the lens. The liquid lens is made by placing a drop of the liquid on a plane mirror and laying a convex lens of from 10 to 20 cms. focal length on the drop. The liquid is squeezed into the space between the mirror and lens and we have a combination of two lenses—one of glass and the other of liquid, giving a combined focal length of, say, F.

If the convex lens has a focal length, f', then :

$$\frac{1}{F} = \frac{1}{f'} + \frac{1}{f}.$$

We may therefore deduce f from a knowledge of F and f'. To determine f', place the lens horizontally on the mirror and adjust a pin, held in a stand above the lens, until the inverted

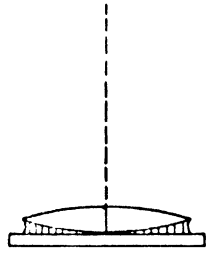


FIG. 158

image and object appear together and there is no parallax between them. The distance from lens to object gives f' (compare p. 263). Now place the liquid and lens on the mirror and again find the position of coincidence of object and image.

This gives F, so that we now have f.

For the purpose of substituting in the above formula, F and f' must be given their appropriate negative sings, and f will turn out to be positive.

If r is the radius of curvative of the curved liquid surface, and μ the refractive index of the medium,

$$\frac{1}{f} = \frac{\mu - 1}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

since the second surface has zero curvature.

r can be found by measuring the radius of the surface of the lens in contact with the liquid by means of a spherometer, or by any of the methods described in the last chapter, so that μ can be calculated from the equation (2).

If it is preferred, the determination of r may be avoided.

If water be used and its index regarded as known and having the value, 1·33, we may make a water lens as above, and calculate its focal length, f''.

But $$\frac{1}{f''} = \frac{\cdot 33}{r},$$

$$\therefore f = \frac{\cdot 33}{\mu - 1} \cdot f'',$$

or $$\mu = 1 + \frac{\cdot 33 f''}{f},$$

where μ is the refractive index of a liquid other than water.

A convenient liquid to use for the experiment is aniline. Care must be taken to prevent it from getting at the back of the mirror since it dissolves the varnish protecting the silvering.

Determination of the Refractive Index of a Lens by Boys' Method

This method of finding the refractive index of a lens consists in measuring its focal length and the radii of curvature of both surfaces. The value of μ may then be determined from the equation :

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

where f denotes the focal length of the lens and r and s the radii of curvature of its surfaces, with the usual convention regarding the signs of the quantities in the formula. In a convex lens let r_1 and r_2 denote the radii of curvature numerically and F the numerical value of the focal length. Then :

$$\frac{1}{F} = (\mu - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

Determine F by any of the methods described below, p. 293. In order to find r_1, set up a pin with its point on a level with the centre of the lens. Two images will be seen by reflection in the faces of the lens, one erect from the front surface which acts as a convex mirror and one inverted by the concave back surface. It is the latter which is required for the experiment. Move the pin until its inverted image is coincident with it, as judged by the method of parallax. When this is the case the rays must strike the back surface normally and be returned along their incident course.

If an eye be placed on the side of the lens remote from P (fig. 159), it will receive the transmitted part of the ray, PB, and will see the image of P in the direction, CBQ.

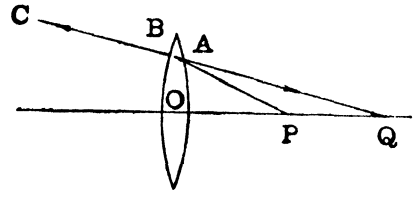


FIG. 159

Q will thus be the image of P in the lens, and OP and OQ are conjugate distances. Thus, writing $u = \text{OP}$ and $v = \text{OQ}$, we have :

$$\frac{1}{\text{OQ}} - \frac{1}{\text{OP}} = -\frac{1}{\text{F}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

This enables us to determine OQ from OP and F.

Q is the centre of curvature of the left-hand surface of the lens since QAB is normal to this surface. Thus OQ gives the value of the radius, say, r_1.

Turn the lens round and repeat the process to obtain r_2, the radius of curvature of the other face of the lens.

Let OP $= d_1$ in the first case, and let d_2 denote the corresponding distance in the second. Then from (3) :

$$\frac{1}{r_1} = \frac{1}{d_1} - \frac{1}{F}, \quad \frac{1}{r_2} = \frac{1}{d_2} - \frac{1}{F}$$

Hence
$$\frac{1}{F} = (\mu - 1)\left(\frac{1}{d_1} + \frac{1}{d_2} - \frac{2}{F}\right),$$

so that we may calculate μ from the experimental determination of F, d_1, and d_2.

It is sometimes difficult to see the image by reflection at the back surface of the lens, but by holding it over the surface of mercury or floating it in the mercury, the image may be made to stand out and be easily located.

The Spectrometer

The spectrometer consists essentially of a telescope and collimator. The latter is a system of lenses mounted in a telescopic tube with an adjustable slit at one end, and it serves the purpose of rendering rays from the illuminated slit parallel on emergence. Both are mounted on a rigid stand, the collimator being fixed, and the telescope rigidly attached to an arm which rotates about the centre of the stand. Both are mounted horizontally with their axes in the same plane.

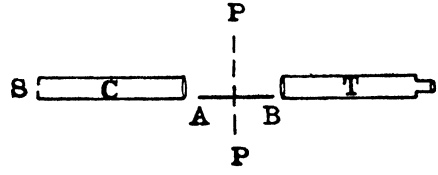


FIG. 160

The centre of the instrument is occupied by a table provided with screws for levelling. Underneath the table is a metal scale of degrees on which can be read off the positions of the telescope and of the table. In the diagram, T denotes the telescope, AB the table, and C is the collimator. PP denotes the axis of the instrument and the table and telescope rotate round it. The telescope is fitted with a Ramsden eyepiece carrying cross-wires.

Before using the spectrometer for any experiment certain adjustments have to be made in addition to those made in the construction of the apparatus by the maker.

In the first place, the eyepiece of the telescope is adjusted so that the cross-wires are distinctly focussed. The telescope should then be taken to an open window and focussed on a distant object such as a distant telegraph post, care being taken that there is no parallax between the image and the cross-wires.

When this is done the telescope must not be readjusted again during an experiment or it will be necessary to repeat this process. It may happen that a second observer whose sight differs from that of the first is unable to focus the cross-wires easily. He may readjust the eyepiece provided that he does not alter any other part of the telescope, for then he is not causing it to be out of focus for parallel rays. He merely gives himself convenience in focussing easily and leaves the cross-wires and image without parallax.

The telescope is now turned towards the collimator, the slit being made vertical and illuminated with monochromatic light, and the collimator is adjusted until a distinct image of the slit falls on the cross-wires.

The apparatus is now adjusted so that parallel rays pass from collimator to telescope.

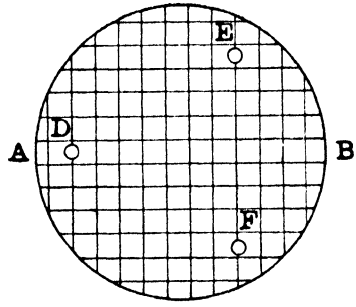


FIG. 161

When a prism is used in the spectrometer it is necessary to adjust it so that its refracting edge is vertical.

The screws on the table enable this to be done. They are shown at D, E, and F, placed at the corners of an equilateral triangle, and the table is usually ruled with lines as shown (fig. 161), to assist in setting the prism with one face normal to one of the sides of the triangle, for example EF.

The table should be made as nearly horizontal as possible by the use of a spirit-level, and the prism then placed with one face perpendicular to EF.

Now suppose light from the collimator falls on this face and is reflected into the telescope. If this face is vertical the slit will now appear to lie in the same part of the field of view of the telescope as when it is seen directly.

The three screws should be adjusted to restore the image to its direct position if necessary.

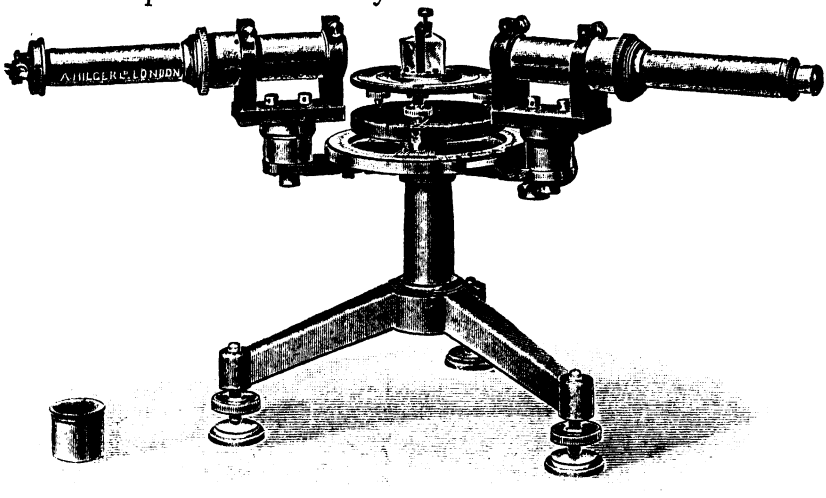


FIG. 162

Let the light be reflected into the telescope by the other face bounding the refracting edge. If further adjustment is necessary it must be done by the screw, D, for this will not disturb the previous adjustment, since it does not turn the face perpendicular to EF out of its vertical plane.

The two faces are now vertical and the instrument is adjusted.

It is sometimes necessary to arrange one face of a prism so that it lies normal to the collimator or telescope.

This may be done by turning the telescope from the position in which the slit is seen directly without the prism, through a right angle, so that the axes of the telescope and collimator are perpendicular to each other.

The prism is now placed on the table of the instrument and the table rotated until, by reflection in the face concerned, an image of the slit is thrown on the cross-wires. The face now lies at 45° to the axes of the collimator and telescope and a further rotation of 45° will bring it either perpendicular to the collimator or telescope.

In making measurements with the spectrometer the slit should, as a rule, be narrow, and the cross-wire should lie accurately down the centre of the image of the slit before the position of the telescope or table is noted on the metal scale.

Tangent screws help in the accurate final setting of the telescope and table, and verniers on this scale serve to measure the angles to an accuracy of one minute of arc.

A modern form of the apparatus is illustrated in fig. 162.

The Refractive Index of a Prism by the Method of Minimum Deviation

When a prism with a refracting angle, A, causes a minimum deviation, D, in light passing through it the refractive index is measured by the formula :

$$\mu = \frac{\sin \frac{1}{2}(A + D)}{\sin \frac{1}{2} A}.$$

Thus the determination of μ consists in measuring A and D. There are two methods in common use for measuring an angle of a prism.

One of these consists in allowing the light from a narrow slit to fall partly on one of the faces of the prism bounding the angle and partly on the other. An image of the slit can be seen in the telescope when the latter lies on either side of the prism.

Thus in the case illustrated (fig. 163), the telescope will receive rays from the direction AO on one side and from AK on the other. The telescope has to be turned through the angle KAO in turning from one direction to the other.

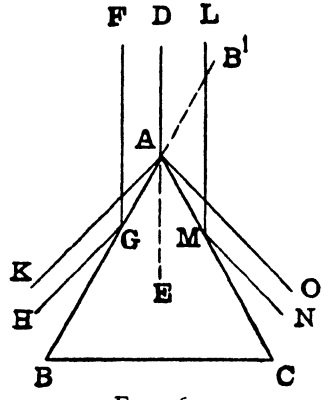


FIG. 163

Since DA and AO make equal angles with AC, we have :

$$< EAC = \angle CAO,$$

and similarly, $\angle EAB = \angle BAK.$

Thus $\angle OAK = 2 \angle BAC.$

To measure A we need only focus the image of the slit on the cross-wire in the two positions and halve the angle through which it is rotated.

The other method is to set the prism conveniently so that an image of the slit is reflected into the telescope. The latter is fixed in position while the position of the table is read off on the scale.

Suppose AB is the face which reflects the light into the telescope along AK. If the prism is rotated until the face, AC, now lies parallel to AB the rays will once more be reflected in the direction, AK, but now by AC. Some of these will enter the telescope if there is a sufficiently broad pencil of them. But the prism has been rotated so that AC moves round through the angle CAB[1].

Thus we measure, by means of the table, the angle ($180° - A$), and A can be deduced.

Measure the angle of the prism by both methods.

It now remains to find D.

Set the prism so that with A as refracting angle, light is refracted through it and received in T.

It will be found that as the prism is rotated the telescope has to be rotated to keep the image of the slit in the field of view. Rotate the prism until T is as close to the position directly opposite the collimator as possible, with the slit in the field of view. When this is the case the angle between the telescope and direct position is as small as possible.

Note the position of T and then remove the prism and observe the slit directly. Again note T's position, and hence find D.

It is, of course, necessary to adjust the spectrometer and edge of prism in the way described in the section on the adjustments of the spectrometer.

The final movements of the telescope or table must be made carefully with the tangent screws, and these must have their final rotations *always in the same direction* to avoid errors due to backlash in the threads.

The Dispersive Power of a Prism

The dispersive power of a medium is measured by :

$$w = \frac{\mu_B - \mu_R}{\mu - 1}.$$

μ_B and μ_R are the refractive indices for blue and red rays and μ has the value, $\frac{1}{2}(\mu_B + \mu_R)$.

The refractive indices may be found by the method of minimum deviation.

As a source of blue and red rays a discharge tube containing hydrogen may be used.

The tube should be held vertically, and the slit illuminated by it directly, or an image of a bright part of the tube thrown on to it by means of a short focus lens.

Three well-marked lines can readily be seen, one in the red, a second of blue-green colour, and a third in the violet. Use the first and third of these—they are known as the C and H_γ lines respectively, while the second is the F line.

The Refractive Index of a Liquid by Total Internal Reflection within a Glass Prism

A glass prism with one face unpolished is mounted on the table of a spectrometer. The table is levelled and the edge between polished surfaces set normal to its plane (see p. 281). The angle between these faces is measured in the usual way by allowing light from a wide collimator slit to fall on both faces and by measuring the angle between the two reflected beams.

Light from a sodium flame is allowed to fall on the unpolished surface, or an image of the flame is thrown on it by means of a lens to cover the whole matt surface.

It is first necessary to find the refractive index of the glass prism and then to coat one of the bright faces with a thin layer of the liquid. Glycerine is a very convenient substance with which to carry out the experiment.

The theory of both parts of the experiment is the same.

Theory

ABC denotes the prism, of which AB is the unpolished side This side acts as a collection of point sources of light of which S denotes one. Rays from it strike AC and are reflected and refracted there. Those like SD falling at an angle of incidence less than the critical angle are partially reflected and partially refracted so that the ray, EF, issuing from BC is less intense than the incident ray, SD.

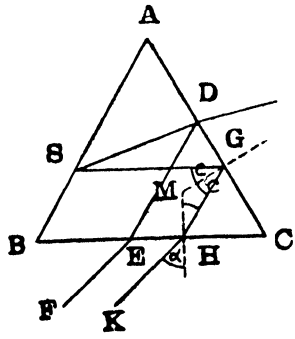


FIG. 164

Rays falling at angles greater than the critical angle suffer no refraction at AC, so that the emergent ray from BC is scarcely less bright than the incident ray. In any case there will be a

marked difference between the former and latter group. The ray, SG, is drawn for critical incidence so that the direction of the emergent ray, HK, stands between those for the faint and bright rays.

We get a similar state of things for any other point in AB, and for each point the rays striking AC at critical incidence give rise to rays parallel to HK on emergence.

Such a group of rays is brought to a focus in the focal plane of a telescope placed to receive them.

Similarly, a group of rays, parallel to EF, will correspond to all rays falling on AC parallel to SD, and these will fall on the telescope in a direction different from that of HK, and will form a line in the focal plane not coincident with the former. This is true for all the directions of rays from BC. We shall thus have a multitude of parallel lines in the field of view divided into two groups of different intensity by the critical direction, HK. The effect will produce a field sharply divided into bright and dark halves by the direction, HK, making α with the normal to BC.

If the telescope is turned to face AC the field will be similarly illuminated on account of the reflections that have taken place on BC. The issuing critical rays will make an angle, α, with the normal to AC.

If the cross-wires of the telescope are set on the dividing line of the field when it is directed towards BC, and again when towards CA, we can deduce the value of α by observing the angle through which the telescope has been turned, since the telescope is rotated through

$$180° - C + 2\alpha = \theta \text{ (say)}.$$

Thus
$$\alpha = \frac{\theta + C - 180°}{2},$$

where it is assumed that the telescope is moved from side, BC, towards CA in a counter-clockwise direction as seen in the diagram.

From C and α we can calculate μ from the formula given below.

Suppose that AC is coated with a substance of refractive index, μ_1, and that the index of the prism is μ.

We have on referring to the diagram :

$$\mu_1 = \mu \sin c,$$

$$\beta + c = C, \text{ where } \beta = \angle \text{MHG}$$

since the points C, G, M, H, are concyclic,

$$\sin \alpha = \mu \sin \beta,$$

$$\therefore \quad \mu_1 = \mu \sin (C - \beta) = \mu \sin C (1 - \sin^2 \beta)^{\frac{1}{2}} - \cos C \sin \alpha$$
$$= \sin C (\mu^2 - \sin^2 \alpha)^{\frac{1}{2}} - \cos C \sin \alpha.$$

When $\mu_1 = 1$, i.e. when there is no layer on face, AC, we have :

$$\mu^2 = \left(\frac{1 + \sin\alpha \cos C}{\sin C}\right)^2 + \sin^2\alpha.$$

This is the formula from which μ may be calculated.

Fig. 165 shows the state of affairs when the critical angle is large. This will be the case when the media on either side of AC are nearly of the same optical density.

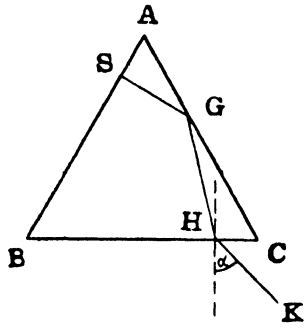


FIG. 165

The formula above will still apply if the value of α has its sign reversed.

The above method of measuring α can be used in finding μ, but when one side is coated it would be necessary to clean that side and coat the other when the telescope is turned. This would make it difficult to keep the prism fixed and would be inconvenient.

It is therefore best to use an eyepiece, such as the Gauss eyepiece with cross-wires that can be illuminated.

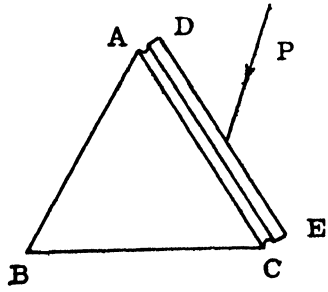


FIG. 166

The light from these may then be reflected in the face opposite the telescope and the image made to coincide with the object. When this is the case the telescope stands perpendicularly to the face and the angle between this direction and that in which the division of the field is viewed measures α.

After finding μ for the glass, place a few drops of the liquid on one of the polished faces and press over it a thin plate. This ensures that the face is covered with liquid.

It is better for the sake of definition of the two halves of the field to allow light to fall at grazing incidence on the prism surface, say, AC. Then the rays entering the prism make angles less than c with the normal so that the field is now only half illuminated and the edge corresponds to the direction, HK. AB should be kept dark by covering with a sheet of dark paper.

If the light is incident externally on the liquid film, it must enter by the edge, AD (fig. 166), any ray such as P would not reach AC at grazing incidence.

Calibration of a Spectroscope

The spectroscope already described had only one prism, but it is an advantage sometimes to use two mounted together on the spectrometer table. The experiment may be carried out with one only, but a wider spectrum is obtained with two.

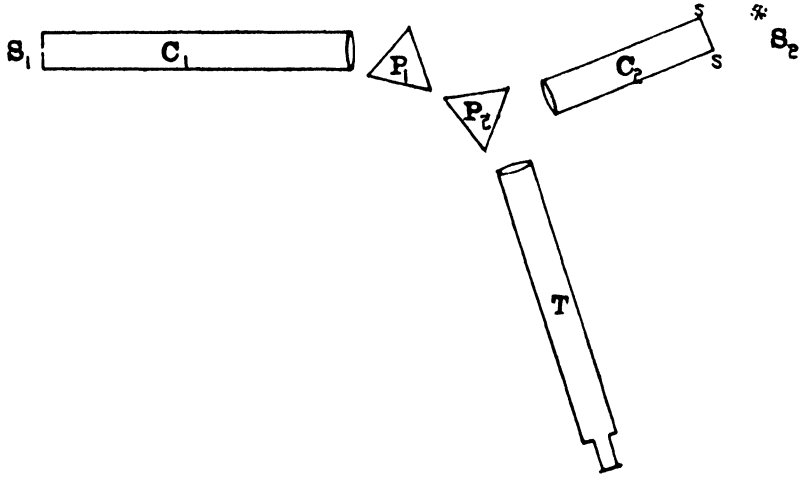


FIG. 167

If the prisms, P_1 and P_2, are used to produce a series of spectral lines due to some source, S_1, the telescope, T_1, will have a definite position on its scale corresponding to each particular wave length. To calibrate a telescope is to find the values of the wave lengths corresponding to the different parts of the scale. If a curve is drawn whose co-ordinates are the wave lengths and corresponding scale readings, such a curve is a ' calibration curve,' and may be used to determine any wave length from the division of the telescope scale at which it is seen.

First make the usual adjustments for parallel light. Place one prism, P_1, in the position corresponding to minimum deviation and then put in P_2 also in the position of minimum deviation.

Examine the spectrum of a Bunsen flame containing sodium light. Fix the telescope so that the sodium line lies on the cross-wire and note the position on the metal scale of the spectrometer.

A graduated scale is provided consisting of close rulings cut on an opaque screen so that the lines are transparent.

It is fitted at the end of a second collimator, C_2, which is adjusted to direct rays down the telescope after reflection on one face of P_2.

The telescope is already set for parallel rays so that by adjusting C_2 so that an image of the scale, ss, appears at the cross-wires, it is ensured that parallel rays emerge from C_2.

The position of the sodium line is taken as a point of reference and scale divisions are read so many to the right or left. If the scale, ss, is moved accidentally no error is then caused and the calibration curve will still be of use.

In a good spectroscope the sodium lines will be separated and will appear as two very close together. Use one of these as the reference line.

Measure the scale positions of a series of lines of known wave length extending over the visible spectrum.

Use only sharply defined lines.

The table gives the wave lengths of lines which may be produced conveniently in the laboratory.

In order to produce the lines from the metals use a spark between poles made of these metals connecting them to opposite terminals of a Leyden jar charged by an induction coil.

The salts are volatilized in a Bunsen flame as with sodium. Examine the inner cone of a Bunsen flame; it is due to carbon monoxide, and contains several series of lines. The spectra of light from discharge tubes containing various gases should be examined as well.

Plot the curve on a large scale on squared paper.

The Auto-Collimating Spectrometer

In this instrument the telescope acts also as the collimator. The apparatus is almost identical with the ordinary spectrometer, except that there is no separate collimator and the telescope is modified.

On looking into the eyepiece of the instrument, the field is seen to be divided into two parts, the lower half is dark and the upper bright, but crossed by a pin which extends from the upper edge downwards across about half of the bright part of the field.

TABLE OF WAVE LENGTHS

SALT OR METAL	DESCRIPTION OF LINE	WAVE LENGTH (TENTH METRES
Lithium Chloride or any Salt of Lithium	Bright Red Line	6706
Any Salt of Sodium	Double Yellow Line	5892
Salt of Potassium	Very deep Red Line	7680
	Violet Line	4045
Strontium Chloride	Sharp Blue Line	4607
Thallium Chloride	Green Line	5349
Lead		6656
		5607
		5372
		4386
		4246
Zinc		6362
		6102
		4923
		4911
		4809
		4721
		4679
Hydrogen	Red (C)	6530
	Blue-Green (F)	4861
	Violet (H_r)	4341

Near the eyepiece end the tube of the telescope is provided with an opening by means of which a slit just within may be illuminated.

Just below the dividing line of the two halves of the field of view a small right-angled prism is placed which deviates the light from the slit down the centre of the telescope tube. Its position is denoted by the dotted rectangle, P, in fig. 168.

If a polished surface such as the face of a prism is placed in front of the object glass and set at right angles to the axis of the telescope we may, by focussing, make the rays leaving the object glass parallel, so that they will be returned by reflection at the face of the prism and come to a focus in the focal plane of the telescope.

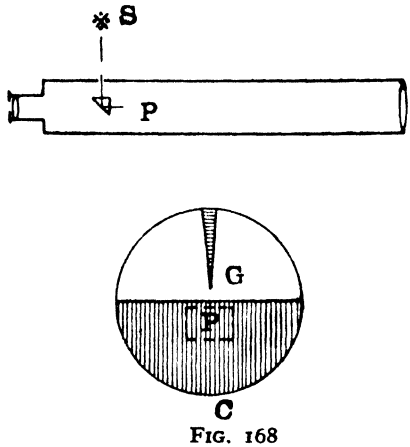


FIG. 168

Focus the eyepiece carefully on the pin. Illuminate the slit by means of a small electric lamp—a four-volt lamp, supplied by two accumulators, is convenient for the purpose. The slit will be observed to be of the shape of an inverted T. The image must be brought so that the horizontal bar lies along the edge bounding the two halves of the field, when the end of the vertical bar will coincide with the point of the pin. In order to obtain this result the face of the exterior prism must be accurately normal to the emergent rays. The table is provided with three screws in order to level to the prism, and the process described on p. 280 must be followed.

The positions of the slit and pin are such that when the image of the slit lies at the same distance from the object glass as the pin, the latter is at the principal focus.

The prism is slightly tilted to throw the image of the slit on to the upper half of the field of view.

To Find the Refractive Index of a Prism by means of the Auto-collimating Spectrometer

Place the prism on the table and adjust the faces bounding the refracting angle as described above so that the light reflected normally by both throws an image of the slit into the field of view just below the pin.

When this is so the faces are vertical and consequently so also is the edge of the prism.

Use monochromatic light by placing a sheet of yellow glass between the bulb of the electric lamp and the slit.

Observe the position of the table when the face, AB, reflects the light normally. Rotate the table until the light falling on AB and refracted there strikes AC normally and is returned along

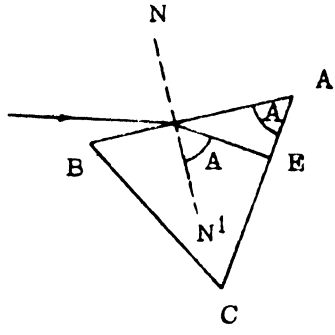


FIG. 169

its path. The table has been turned so that the face, AB, has turned from a direction normal to the rays from the telescope into the position at which refraction takes place, i.e. it has turned through the angle of incidence, i. The angle of refraction in this case is the same as the angle, A, and is so marked in the diagram.

This angle may be measured by setting the face, AB, normal to the rays and reading off the position of the table. Then by rotating the table until the rays strike AC normally we turn the table through an angle $(180 - A)^0$.

We may therefore calculate μ, since we know the angle of incidence and refraction in a particular case and

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin i}{\sin A}.$$

The Pulfrich Refractometer

This apparatus is shown in figure 170. It is designed to measure refractive indices of solids and liquids to an accuracy of about $\frac{1}{100}$ per cent.

The principal part of the apparatus consists of a prism having two plane polished faces at right angles to each other. One of these is horizontal and the other vertical. On the horizontal face is placed the substance whose refractive index is required. If this substance is a solid it must have two faces cut perpendicularly to one another, both of which are cut accurately plane, so that one may rest on the horizontal surface of the prism and the other stand vertically. Optical contact is brought about by placing a few drops of a liquid on the horizontal surface of the

prism, which has a refractive index higher than that of the solid to be experimented upon, and standing the solid on it. The makers recommend monobromonaphthalene for this purpose. In the case of a liquid, it is contained in a glass cell cemented to the prism.

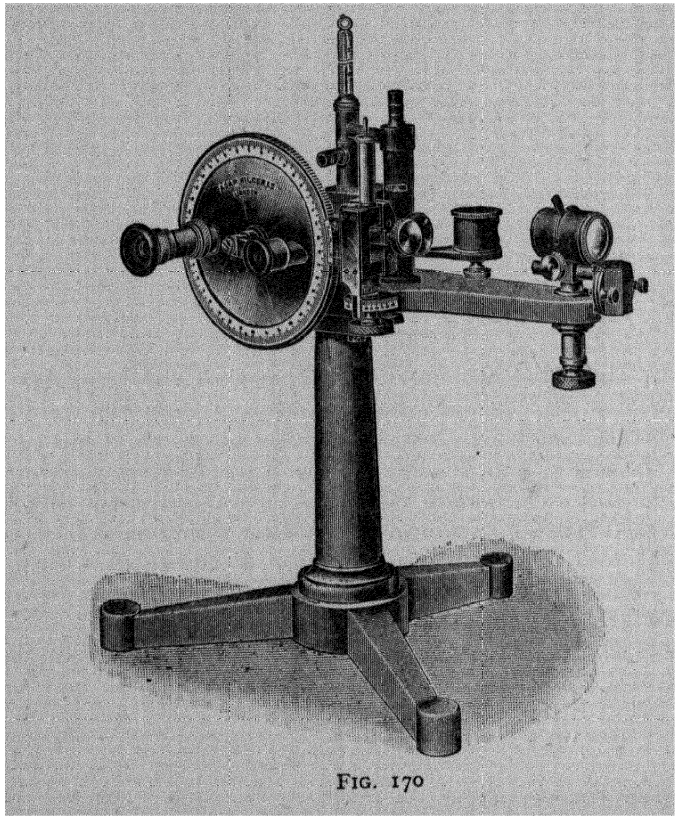


FIG. 170

Light is directed into the liquid in a direction almost parallel to the horizontal surface so that light entering the prism makes the critical angle, c, with the normal.

Let it emerge from the prism at an angle, i. (Cf. fig. 171.) Suppose that the refractive index of the substance to be examined is μ and of the material of the prism, μ_0.

Then
$$\sin c = \frac{\mu}{\mu_0},$$
$$\frac{\sin i}{\sin\left(\frac{\pi}{2} - c\right)} = \mu_0,$$

i.e.
$$\cos c = \frac{\sin i}{\mu_0},$$
$$\sin^2 c + \cos^2 c = 1 = \frac{\mu^2}{\mu_0^2} + \frac{\sin^2 i}{\mu_0^2},$$
$$\mu = \sqrt{\mu_0^2 - \sin^2 i}.$$

The substance must have a refractive index less than that of the material of the prism if it is to be examined by this method.

The apparatus measures the angle, i. For this purpose a telescope is attached to a circular scale and the rays are received by it. Since no rays within the prism make an angle greater

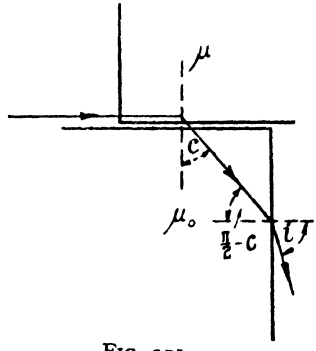


FIG. 171

than c with the normal on entrance to the prism, the angle, $\left(\frac{\pi}{2} - c\right)$, measures the minimum angle at which rays strike the vertical face. Corresponding to this, i measures the minimum inclination of the emergent rays to the normal. Thus in the telescope the rays emerging in this direction bound the field of view.

The apparatus is arranged so that the rays are deflected down the telescope, and when its cross-wires lie on the dark edge of the field the scale reads off the angle, i, to an accuracy of one minute of arc. μ_0, of course, is an instrument constant and has the value 1·74.

In addition a microscope screw is divided so that the value of the refractive index may be measured for different wave-lengths to a still higher degree of accuracy.

The prism and specimen are surrounded by a metal water jacket and thermometers are provided for reading the temperature. A table is supplied with the instrument giving the value of μ corresponding to different values of i.

Determination of the Focal Lengths of Thin Lenses by means of Pins

Convex Lens. Method 1

Support the lens vertically and place a vertical pin behind it so that its point lies on a level with the centre of the lens and on the axis of the lens.

The adjustments are rendered easier by placing a sheet of white paper behind the pin

Set up another pin so that it coincides with the image. When this is the case the image and second pin will not appear to move relatively to each other when the eye is moved horizontally from side to side.

Arrange the lens and first pin so that the image is not at the same distance from the lens as the object. Then without moving either pin displace the lens until a second position is found at which the pins occupy conjugate points.

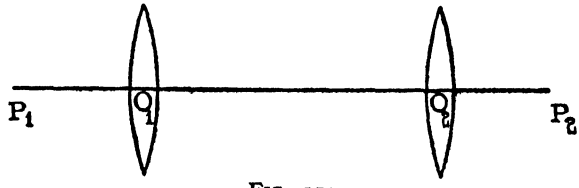


FIG. 172

Let the lens have been moved a distance, d cms., while the distance between the pins is D cms.

In the figure we have evidently :

$$P_1O_1 = u = \frac{D - d}{2},$$

and

$$O_1P_2 = v = \frac{D + d}{2} \text{ (numerically)}$$

Thus from the equation :

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

we have :

$$f = -\frac{D^2 - d^2}{4D},$$

since in the formula we have to write :

$$v = -\frac{D + d}{2}.$$

Determine several different values of D and d, and calculate the value of f from each ; the result should give a constant.

Method 2

Put up a plane mirror immediately behind the lens and parallel to it. Then place a pin in front of the lens as in the first case and adjust it to make it coincide with the image formed by refraction in the lens and reflection in the mirror. The distance from the pin to the lens is equal to the focal length of the lens,

for the rays are reflected back along their path and must therefore strike the mirror normally. They leave the lens as parallel rays, and must therefore originate from the principal focus.

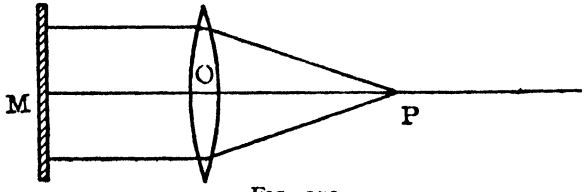


FIG. 173

Method 3

Set up a plane mirror, P^1Q^1, at some distance from the lens, and a pin on the other side of the lens. The pin should be mounted so that its centre is on a level with the centre of the lens and should be adjusted until the position of the image of the pin is made to coincide with the pin. The way this is brought about may be seen from the diagram. The image formed by the lens evidently lies on the surface of the mirror.

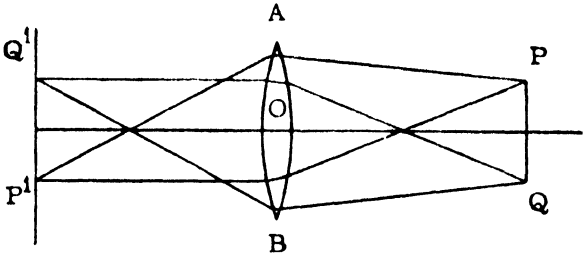


FIG. 174

If the distances be measured we have the positions of a pair of conjugate points.

Make several observations for different distances and calculate f.

In making experiments with a convex lens it is useful to know the focal length approximately before making an accurate determination of it. Sometimes time is wasted in trying to locate a real image when the object is so placed that a virtual one is formed. It should be noted that if the object is at a distance from a lens, which is less than the focal length, the image is virtual.

Focus a distant object—a lamp or window, if not too close, is suitable—and measure the distance from the lens to a well-defined image thrown on to a sheet of paper. Unless the lens has a very long focal length this distance will be approximately f.

Concave Lens

Method 1

Put up the lens and place a convex lens of known focal length in contact with it so that their axes coincide.

Note if the combination will form a real image of a distant object, and measure approximately the distance from the lens to image so formed. This will indicate whether the combination acts as a convex lens. If it does not the convex lens is too weak, and a stronger must be chosen.

Determine the focal length, F, of the combination by any of the above methods, and let f denote that of the convex lens, f^1 that of the concave.

Calculate f^1 from the formula :

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f^1}.$$

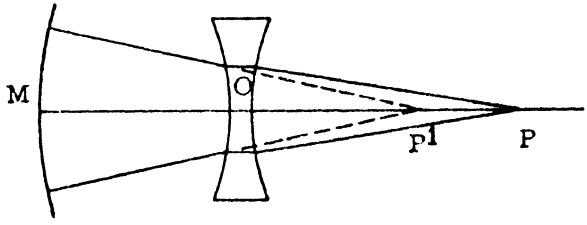


FIG. 175

Method 2

Set up a concave mirror behind the lens and a pin in front. An inverted image of the pin will be seen on looking through the lens, and it may be made to coincide with the pin by suitably adjusting the mirror.

The diagram illustrates that this is brought about by the normal incidence of the rays emerging from the lens on the mirror.

The virtual image of P is thus at P^1, the centre of curvature of the mirror.

Remove the lens after noting its position and the distance of P from it, and again adjust the pin until it coincides with its image formed by the mirror. We thus locate P^1 and can measure v, the distance from the position of the lens.

On substitution in the formula ·

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

we calculate f.

Repeat this for several different cases.

The Focal Length of a Lens Combination

In the case of a thin lens, in which no account of the separation of the two surfaces is taken, there are two points on either side of it called principal focal points at equal distances from the lens if it is situated in a medium which is uniform all round it.

These two points are distinguished by calling one the first and the other the second focal points. The former is the position of an object corresponding to an image at infinity while the second is the position of an image corresponding to an object at infinity.

A plane perpendicular to the axis at the position of the lens is called the principal plane and if f denote the focal distance, we have the relation :

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

connecting the distances of the conjugate points from the principal plane.

If the principal plane and focal points are known it is a matter of simple geometrical construction to determine the image of any object.

When the lens is not thin, or when it is necessary to deal with a system of lenses, Gauss has shown that the formula is the same but that in this case there are two principal planes, the first and second, separated by a finite distance.

The distance of the first principal plane to the first principal focus is called the first focal length, and the distance to the second principal focus from the second principal plane is the second focal length.

If the lens system is situated in a medium the same on both sides the two focal lengths are the same and we shall denote either by f.

When the object is at a distance, u, from the first principal plane and the image is at a distance, v, from the second, the formula is still :

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

A very important property of the principal planes is that if a ray is incident on the system towards a point on the first principal plane, the emergent ray is directed from a point in the second principal plane which lies on the same side of the axis and at the same distance from it as the first.

It is now possible by a geometrical construction to determine the position of an image corresponding to any object. This is illustrated in the diagram (fig. 176).

The ray, BL_1, parallel to the axis must pass on emergence through the second focus, F_2, and it appears to emerge from L_2, where $P_1L_1 = P_2L_2$.

P_1 and P_2 denote the principal points and P_1L_1 and P_2L_2 are the principal planes.

Again the ray through F_1 emerges parallel to the axis from M_2, where $P_1M_1 = P_2M_2$.

We thus find B^1 and can draw in the image, A^1B^1.

If the media on either side are different, P_1F_1 is not equal to P_2F_2.

Two other important points for the lens system are the nodal points. They are defined to be points on the axis such that a ray directed to the first nodal point emerges as a parallel ray directed from the second.

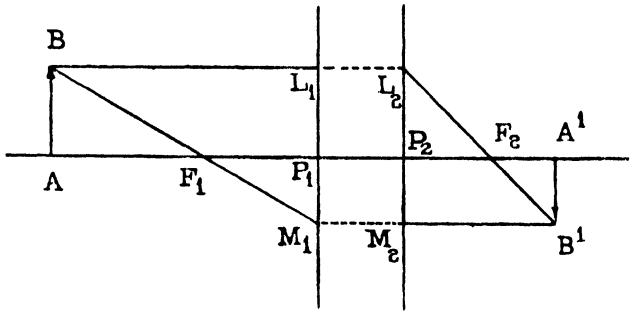


FIG 176

When the medium is the same on both sides of the system the first and second nodal points coincide with the first and second principal points.

We have, therefore, the line, BP_1, parallel to P_2B^1, and the magnification, m, is

$$\frac{A^1B^1}{AB} = \frac{A^1P_2}{AP_1} = \frac{v}{u},$$

just as in the case of a thin lens.

A direct determination of f requires the location of a pair of conjugate points, but we do not know the position of the principal points so that it is not possible to find v and u for substitution in the formula :

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

We may, however, determine f by measuring the magnification of an object by the lens for two different positions.

From this formula we have :

$$\frac{u}{v} = 1 + \frac{u}{f}.$$

Thus if u_1 and m_1 denote the distance of an object from the first principal plane and the corresponding magnification, we have :

$$\frac{1}{m_1} = 1 + \frac{u_1}{f},$$

and for a second position :

$$\frac{1}{m_2} = 1 + \frac{u_2}{f}.$$

$$\therefore f = \frac{u_1 - u_2}{\frac{1}{m_1} - \frac{1}{m_2}}.$$

Thus the actual lengths, u, are not required for the calculation, but only the distance between the two positions of the object, which is given by the difference between u_1 and u_2.

If several values of $\frac{1}{m}$ are observed and positions corresponding to u noted on the optical bench, a graph may be plotted of $\frac{1}{m}$ against u.

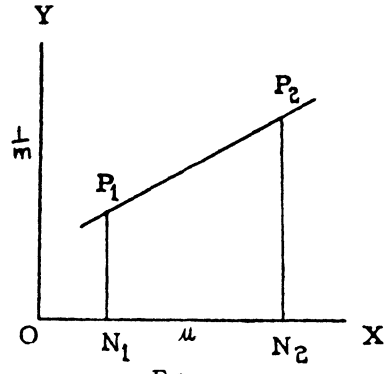


FIG. 177

The relation is linear, and if two points, P_1 and P_2, be chosen on the line as far apart as is convenient :

$$f = \frac{P_2N_2 - P_1N_1}{N_2N_1}.$$

It is convenient to use a transparent glass scale as object and to illuminate it by monochromatic light. A similar glass scale is used as a screen so that the apparent length of a certain number of divisions of the first scale seen in the image can be read off at once on the second scale.

The screen is adjusted as usual by noting when there is no parallax between the image and scale.

For accurate work it is necessary to use a low-power microscope to measure the image.

Another method which is instructive is to adjust the positions to obtain a magnification of magnitudes 1 and 2.

When the object and image are equal in size their distances from the corresponding nodal points are both $2f$.

This may be verified from a diagram or from the formula. Now keep the screen fixed and move the object and lens until the magnification is 2.

It may be verified that in this case the object is $\frac{3}{2}f$ from the first nodal point, and the image is $3f$ from the second. The lens system has thus been moved through a distance, f, and this may be measured directly.

Suppose the diagram (fig. 176), denotes the case for a magnification unity.

In this case AP_1 and A^1P_2 are both equal to $2f$. We know the value of f, so that since A is a definite known point we can locate P_1 and similarly P_2 is known from the point, A^1.

The lens system is sometimes encased in a tube, and in that case the distances from P_1 and P_2 to the ends of the tube should be recorded.

If the lens is merely a thick lens of glass the distances of the principal points from the faces of the lens should be recorded.

Determination of the Principal Planes of a Thick Lens

Let DAEB denote a thick lens and let C and C^1 be the centres of curvature of the faces, DBE and DAE respectively.

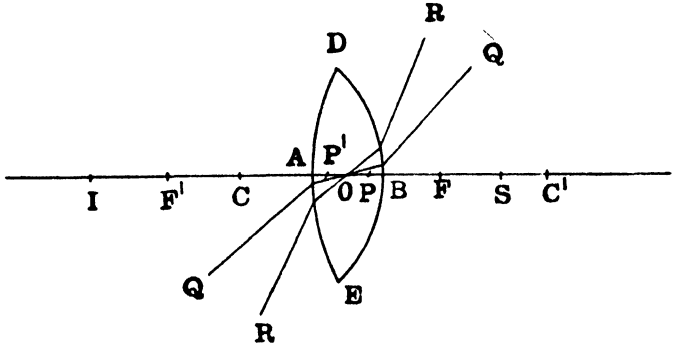


FIG. 178

There exists a point, O, such that all the rays which pass through it emerge from the lens parallel to their original direction. This point lies on the axis, CC^1, and is called the centre of the lens.

The rays passing through O after their first refraction, were directed originally towards a point, P. Thus P and O are conjugate points with respect to the first surface.

The rays after the second refraction appear to come from P^1. so that P^1 and O are conjugate points for the second surface, These points, O, P and P^1, are fixed points for the lens and P and, P^1 are the principal points.

If the distance of an object is measured from P and denoted by u, while the image is measured from P^1 and denoted by v, we have the relation :

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

as in the last experiment.

The quantities, v and u, are to be given positive signs if measurement is made towards the source of light, and negative if it is made in the opposite direction.

Parallel light incident on the lens on the right is brought to a focus at F^1 where $P^1F^1 = f$.

Light originating at F is parallel on emergence where $PF = -f$.

Let a source of light be located at a point, S, where $FS = x$. x is to be regarded as a number of cms., and is not given a sign. The focal distance is likewise measured by the number F. The lens, when convex, has a negative focal length, i.e. in the formula $f = -F$.

The image will be formed at I, say at distance, y, from F^1.

Then $u = (F + x)$, and $v = -(F + y)$.

Thus $$-\frac{1}{F + y} - \frac{1}{F + x} = -\frac{1}{F},$$

or $$xy = F^2.$$

The disposition of the conjugate points shown in the diagram is such as occurs in the case of a convex lens. The object and image both lie within the focal distance or without it. In this case x is taken positive when measured from F towards the right, and y is positive when measured from F^1 towards the left. This relation enables us to find F, for although it is not easy to locate P and P^1, which are required if the first formula is used, it is easy to locate F and F^1, and hence to measure the distance of object and image from them.

To locate F, set up a plane mirror on the left of the lens and put up a pin in such a position that its image coincides with it. The rays emerging from the lens must in this case be parallel and strike the mirror normally so that object and image lie at F.

Repeat this for the point, F^1, leaving the former pin in position.

Leave pins at F and F^1 and do not move the lens. Place a third pin at a point such as S, farther from the lens than F, and locate its image, I.

The distances, x and y, are now determined and we can calculate F.

Hence since P and P^1 lie at distance, F, from F and F^1 respectively, the principal points are located.

Repeat this for several values of x and y.

Plano-Convex Lens

In this case any ray, R, striking the lens at P in the figure, where P is the point of intersection of the axis and curved surface, is not deviated but gives rise to the parallel emergent ray, R^1, for it is just as if refraction took place in a slab of glass bounded by the face, A, and the tangent at P. R^1 appears to come from P^1, so that P and P^1 are the principal points. These may be located as above.

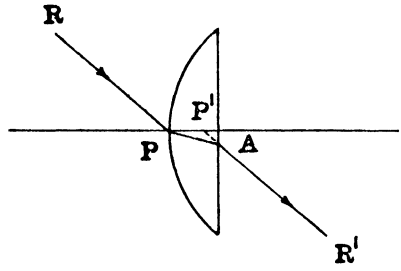


FIG. 179

We may, also, calculate the distance AP^1.

For AP^1 is the apparent thickness of the widest portion of the lens,

$$\therefore \quad \mu = \frac{AP}{AP^1} = \frac{t}{d}.$$

This gives d if μ and t are known.

It forms a useful exercise to determine P and P^1 by the method described above, and then to deduce μ from the observations.

If $PP^1 = \delta$,

$$\mu = \frac{t}{t - \delta}.$$

Determination of the Principal Points by Rotation of the Lens

If the lens is mounted on a stand which can be rotated about a vertical axis we may use the property of the nodal points to locate P and P'.

It is convenient to place the lens in a holder which can slide along a scale fixed to the rotating stand. The arrangement of the apparatus is illustrated in fig. 180.

A mirror is fixed on one side of the lens normally to the axis.

On the other side a pin is set up and moved until its image, by two refractions through the lens and a normal reflection at the mirror, coincides with the object.

The pin then lies at a principal focus.

The lens is moved along the scale and the pin adjusted so that image and object coincide until a position is found, when slight rotations of the stand fail to cause displacement between the pin and its image. When this is the case the rotation takes .place about a vertical axis through the nodal point nearer the pin

The figure explains this, for if P′ denote the other nodal point in the symmetrical position and P″ the position of this nodal point in a slightly displaced position a ray from F falling on P, first passes along PP′M and is reflected back along its path after incidence on the mirror at M.

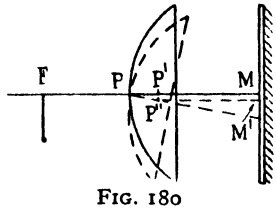


FIG. 180

When the lens is rotated into the position indicated by the dotted lines the ray, FP, is refracted and emerges from the lens parallel to its original direction, and directed from P′, i.e. it takes the course, P′M′, and is reflected back to P′, emerging once more along PF. The image of the point therefore keeps the same position.

The position of the axis thus fixes the nodal point, and the axis is usually clearly indicated on the stand. This finds the point, P, and, by turning the lens, P′ may be found in the same way.

The Measurement of the Focal Length of an Optical System by means of a Goniometer (Searle's Method). (Proc. Optical Convention," 1912, p. 165.)

A simple form of goniometer devised by Dr. G. F. C. Searle, in conjunction with Messrs. W. G. Pye and Co., provides a very instructive method of determining the focal length of an optical system by means of the properties of the nodal points.

The goniometer is illustrated in fig. 181, and consists of a wooden base provided at one end, with a spherical pivot, consisting of a ball of phosphor-bronze, and carrying at the other a scale marked in millimetres.

A movable arm rotating about the pivot carries an achromatic lens of focal length about 35 cms., a vertical adjustable frame, across which a vertical wire is tightly stretched, and a fine wire passing across an opening which serves as a scale index.

The ball is adjusted at a distance of 40 cms. from the scale, which can be read to an accuracy of one-tenth of a millimetre, so that a rotation of the arm through one-seventieth of a degree is able to be measured. This is about $\frac{1}{4000}$ radian.

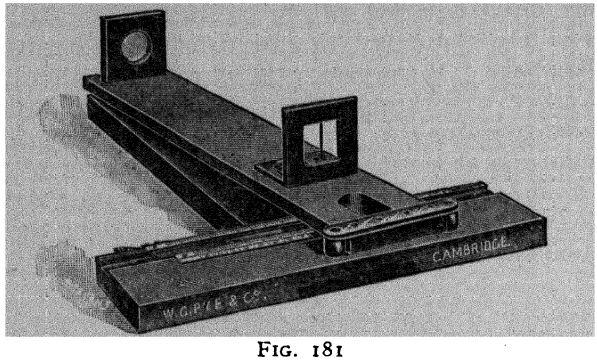


FIG. 181

In fig. 182, let H_1B and H_2B^1 denote the principal planes of an optical system, the points, H_1 and H_2, denoting the first and second principal points.

In practice it usually happens that the medium on each side of the system is the same, viz. air. As the general case presents no additional difficulties and can be examined also by this method, we shall write μ_1 for the refractive index of the medium on the right of the system and μ_3 that of the medium on the left. Then, if U is measured from H_1 and V from H_2 we have the relation,

$$\frac{\mu_3}{V} - \frac{\mu_1}{U} = \frac{1}{F}$$

between the object distance, U, and the image distance, V. F is a constant for the system, and if the value of U is F_1 for emergent parallel light, i.e. F_1 denotes the first focal distance,

$$F_1 = -\mu_1 F,$$

and similarly the second focal distance, F_2, is given by :

$$F_2 = \mu_3 F.$$

The usual sign convention is adopted in applying these formulæ, directions from the lens towards the source are positive and those in the opposite direction are negative. In fig. 182, F_1 and F_2 denote the first and second focal points respectively.

N_1 and N_2 are the nodal points, and with the same medium on either side of the system they coincide respectively with H_1 and H_2.

In the general case the first nodal point lies at the distance $(F_1 + F_2)$ from H_1, and N_2 is at the same distance from H_2.

Thus $$N_1F_1 = H_1F_1 - H_1N_1$$
$$= F_1 - (F_1 + F_2) = -F_2,$$

and similarly:

$$N_2F_2 = H_2F_2 + H_2N_2 = F_1.$$

As drawn in the diagram the value, $F_1 = F_1H_1$, is positive, since it is measured from H_1, while F_2 is negative.

Any ray incident on the system in a direction passing through N_1 gives rise to an emergent ray in a parallel direction passing through N_2.

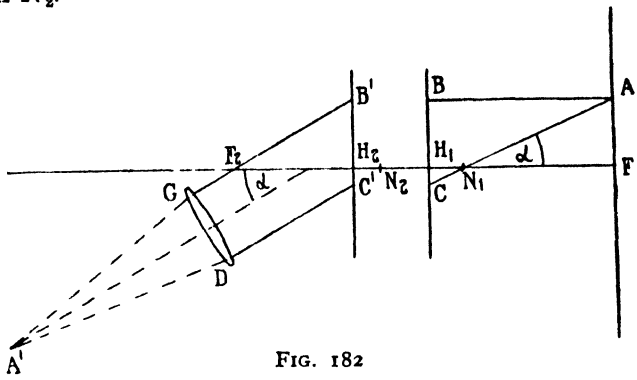


FIG. 182

Thus the ray, AN_1, gives rise to C'D, and C'D if produced would pass through N_2.

If the point, A, lies in the focal plane a divergent pencil from it gives rise to a parallel beam.

Two rays, AB and AN_1, are drawn and the corresponding rays are B^1F_2 and C'D.

Since C'D is parallel to AN_1 the angle between the axis and the emergent beam is equal to angle, AN_1F_1.

Thus if we measure the distance, AF_1, denoting it by d, we can find the length, N_1F_1, provided that we measure the angle between the axis and the parallel beam.

If this is denoted by α we have:

$$N_1F_1 = d \cot \alpha;$$

or when α is small, $$N_1F_1 = \frac{d}{\alpha} = F_2 \text{ (numerically)}.$$

The goniometer is a very convenient apparatus for carrying out this measurement.

It is first adjusted so that the vertical wire lies in the focal plane of the lens. This may be done by adjusting the wire so that the image of a distant object seen through the lens falls on the wire. Or a plane mirror may be placed near the lens on the side opposite to that on which the wire lies and the latter adjusted

until there is no parallax between the wire itself and its image. The frame is then clamped.

The optical system is now brought up so that its axis lies collinear with that of the goniometer lens when this lens lies so that the horizontal wire index is central.

A glass scale is placed in the focal plane of the optical system and may be represented by AF_1. A denotes one of the marks on the scale. The scale may be adjusted in the focal plane by looking through the goniometer lens and the system and by placing it so that its image is seen coincident with the vertical goniometer wire. The method of parallax is used to obtain an accurate adjustment. When this is made as closely as possible by means of the unaided eye a lens should be used as a magnifying glass to make it still more exact.

Note the reading on the glass scale when the apparatus is in the symmetrical position, so that the point corresponding to F_1 is observed. Note also the indication of the goniometer scale.

Turn the goniometer through some convenient small angle and note the mark of the scale seen coincident with the vertical wire. Again note the goniometer reading. We thus have the values of AF_1 and of the angle, α, and can consequently deduce F_2. By turning the system end for end, so that H_2 lies to the right of H_1, we find F_1.

We can thus locate the positions of N_1 and N_2 with respect to the outer surfaces of the system. These should be recorded, and since $H_1N_1 = F_1 + F_2 = H_2N_2$ we may also locate the principal points. Record these also.

In order to carry out this experiment it is convenient to employ two thin lenses situated at a known distance apart. These form an optical system of the type described, and having the same medium on each side.

Hence N_1 and N_2 are both nodal and principal points, and F_1 and F_2 are numerically equal but of opposite sign.

Find the focal length and the position of the principal points for this case.

It is shown in treatises on Optics (see, for example, Houstoun's "Treatise on Light," p. 45), that for two lenses separated by a distance, d, the position of the first nodal point is at a distance from the lens on which light is incident and whose focal length is f_1, given by

$$\frac{-f_1 d}{f_1 + f_2 + d},$$

while the second nodal point is at a distance :

$$\frac{f_2 d}{f_1 + f_2 + d}$$

from the second lens, f_2, denoting its focal length, while the focal length of the system is :

$$F = \frac{f_1 f_2}{f_1 + f_2 + d}.$$

Measure f_1, f_2, and d, and verify these results.

f_1 and f_2 may be readily measured by the goniometer by replacing the optical system by the lenses in turn.

Another property of this system, which is of importance in the theory of Optics, concerns the lens equivalent to this system. This lens is such that it produces an image of the same size, but not in the same position, as that produced by the system, when situated at a point at a distance from the first lens equal to $-\frac{Fd}{f_2}$, i.e. at the first nodal point.

Fig. 182, shows the relative position of the nodal points in the case of two convex lenses.

The Focal Length of a Microscope Objective

The goniometer provides a useful means of measuring a short focal length, such as that of a microscope objective.

The screen, AF_1, is replaced by a micrometer slide provided with a magnifying reading glass. The slide is provided with a small engraved scale with divisions at each tenth of a millimetre. The objective is placed to receive parallel rays emerging from the goniometer lens and focusses them on the scale. Thus an image of the goniometer vertical wire is received and is viewed by the magnifying glass.

The position is read for the symmetrical position and again after a slight displacement of the goniometer arm. The image is now focussed at a new point corresponding to A (fig. 182). The distance between the two images is read off, and from a knowledge of the angle of rotation of the arm, the focal length is calculated as before.

The Determination of the Focal Length and Principal Points of an Optical System by means of the Revolving Table

The principle of the method is the same as that described on page 302, for it depends on the same property of the nodal points. The revolving table affords, however, a much more convenient and accurate means of carrying out the experiment. (See figs. 152 and 183.)

Each nodal point is determined by the principle that when a small rotation is made about a vertical axis through the second nodal point, the image produced of a distant object is not displaced. Thus the position of the axis of rotation of the table locates the second nodal point.

The lens system is mounted on a slider with straight edges. and is placed on the table so that the edge slides along the scale, An index mark on the slider serves to locate positions with respect to the scale.

The design secures that there is no side shaking.

It is essential in this experiment as in the former (p. 268), to make the axis of the system intersect the axis of revolution.

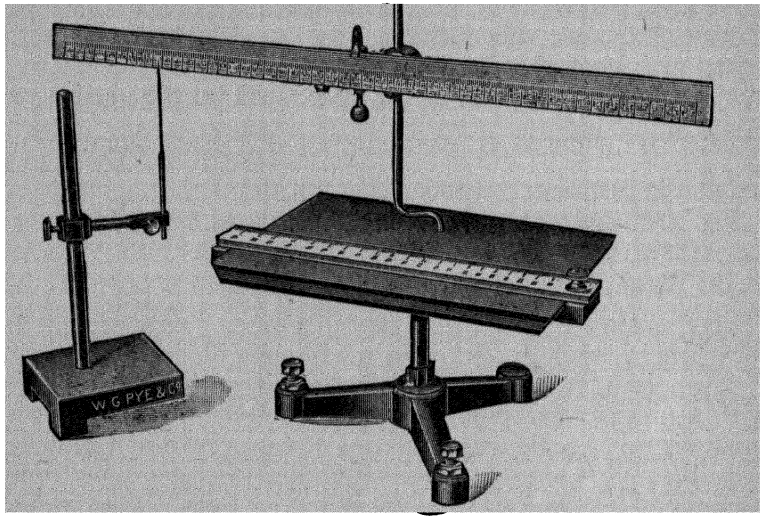


FIG. 183

In order to arrange this the image of a distant object is made to coincide with a pin held in a clamp and the system is moved on the table until, with this focussing exactly made, a slight rotation of the table causes no displacement between the image and pin.

Suppose that the axis of the system is represented by N_1N_2 (fig. 184), and that S denotes the distant source, while I is the image, S is so far distant that it may be supposed to lie on either of the lines, IN_1, $N_1{}^1L$, I^1N_1.

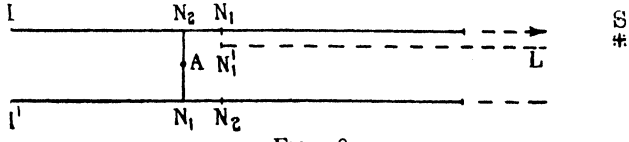


FIG. 184

Let A denote the point of intersection of the axis of revolution and the horizontal plane.

If AN_2 is normal to the direction in which the light is travelling a slight rotation does not displace N_2 at right angles to the axis. Thus the light incident at the first nodal point still emerges along N_2I, and I is not displaced.

N_1^1 denotes the displaced position of N_1 and the dotted line LN_1^1 denotes an incident ray from S.

Thus with this setting the nodal point, N_2, lies in a plane through the axis of revolution and normal to the direction of the rays.

The table top is now turned through two right angles, and the lens system moved along the scale until the distant object is focussed on a second pin as before.

In general this pin will not coincide with the first, but will be displaced in a direction normal to the direction of the light. This is illustrated in the figure. N_1 in the second case become the second nodal point.

If I^1 denotes the second pin it is clear that the axis lies on a line drawn parallel to the light and passing midway between the pins.

The scale on the table top is unclamped and moved through half the distance between I and I^1, so as to carry the axis of the system into the correct position.

The apparatus is now once more carefully adjusted so that the image does not move for slight rotations.

The axis then passes through the nodal point. The system is then turned end for end and the second nodal point found.

In order to find the focal length readily, a clamp is fixed to the table which carries a scale (fig. 183).

The pin is left in position and the table turned until the scale just touches it and the scale reading is noted. The table is turned so that the other end of the scale just touches the pin and the scale reading is noted for this case also. The distance between the two marks on the scale is determined by subtraction ; one-half of this gives the focal length.

A diagram is then made showing to a convenient scale the positions of the cardinal points with respect to the first and final surfaces of the optical system.

The apparatus may be employed to examine such a system as that described in the previous experiment with the goniometer. In this case the apparatus is not very convenient for the measurement of short focal lengths. It is better to use the goniometer for this purpose.

When a distant object is not available in the above experiment it is only necessary to place a plane mirror on the side of the lens system away from the pin and adjust the pin until image and object coincide.

The focal length of a thick lens, the radii of the surfaces being r and r^1, and the thickness t, is given by :

$$F = \frac{\mu r r^1}{(\mu - 1)\{\mu(r^1 - r) - (\mu - 1)t\}},$$

Here r and r^1 must be given their proper signs.

It is instructive to determine the value of the refractive index of the glass forming a thick convex lens.

Determine r and r^1 by the revolving table method (p. 269), and F by the method just described, while t may be measured by means of callipers.

Solve the resulting equation for μ.

The experiment is intended to provide practice in working with a thick lens and becoming familiar with the formula for F.

The distance between the principal points is given by

$$\frac{t(\mu - 1)(r^1 - r - t)}{\mu(r^1 - r) - (\mu - 1)t}.$$

This should be verified by the results of the experiment which locates the position of the nodal points, which are the same as the principal points in this case.

The student should verify the two expressions given above.

Spherical Aberration with a Thick Lens

In the simple theory of lenses it is assumed that all the rays starting out from a point on the axis are brought to a focus after refraction through the lens to another point on the axis.

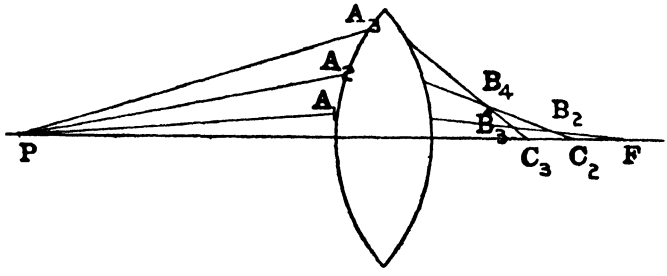


FIG. 185

This is only approximately true for rays lying very close to the axis. A ray, PA_1 (fig. 185), will be made to cross the axis at F, and if PA_1 makes a small angle with the axis, F will be the focus for all such rays. But a ray, PA_2, will cut the axis at C_2, and PA_3 at C_3, after refraction, these points lying closer to the lens as the incident ray is more inclined to the axis.

The distances, FC_2 and FC_3, are called the longitudinal aberrations of the corresponding rays.

The distance, FC_2, depends on the inclination of the emergent rays to the axis. If this distance is x and the tangent of the acute angle at C_2 is m, we may say that x is some function of m. We do not know the way in which x depends on m, but whatever be the form of this dependence we may write x in powers of m, or

$$x = a + bm + cm^2 + dm^3 + \ldots,$$

where a, b, c, d, etc., are constants.

Since the lens is symmetrical about the axis, FC_2 is the same for positive and negative values of m.

Thus if we write $-m$ instead of m in the above equation we have the same value of x, so that x depends only on even powers of m.

$$\therefore \quad x = a + cm^2 + em^4 + \ldots$$

When the emergent ray has a very small value of m, the point, C_2, is at F, or x vanishes. Thus a vanishes and we have :

$$x = cm^2 + em^4 + \ldots$$

If m is not very big, m^4 and higher powers of m, will be very small, so that we may write :

$$x = cm^2.$$

Let F be taken as origin and FP as axis of x.

Let y denote distances measured at right angles to FP.

The line, $B_4\,C_2$, has an inclination to the x axis whose tangent is m, and it passes through the point, C_2, with co-ordinates $(cm^2, 0)$.

So that its equation is :

$$y = m\,(x - cm^2).$$

The emergent rays cross one another at points B_2, B_3, B_4, etc., and the points at which consecutive rays cross lie on a curve known as the caustic of the lens.

Suppose that to a ray consecutive to that through C_2 the corresponding tangent is m^1.

Its equation will be :

$$y = m^1\,(x - cm^{1\,2}).$$

If we solve these two equations we can find the co-ordinates (x, y) of the point common to both lines. We do not actually require these but only the relation between these two co-ordinates, which will give a locus of the intersections of consecutive rays or the equation to the caustic.

Subtracting one equation from the other :

$$\begin{aligned} x\,(m^1 - m) &= c\,(m^{1\,3} - m^3), \\ x &= c\,(m^{1\,2} + mm^1 + m^2) \\ &= 3cm^2, \end{aligned}$$

since m and m^1 are so nearly equal ;

$$\therefore \quad m = \left(\frac{x}{3c}\right)^{\frac{1}{2}};$$

$$\therefore \quad y = \frac{x^{\frac{3}{2}}}{(3c)^{\frac{1}{2}}} - c \cdot \left(\frac{x}{3c}\right)^{\frac{3}{2}} = \frac{2}{3} \cdot \frac{x^{\frac{3}{2}}}{(3c)^{\frac{1}{2}}},$$

or $$27cy^2 = 4x^3.$$

Thus for values of m, such that m^4 is negligible the form of the caustic is a semi-cubical parabola.

In the figure, DCE^1 and D^1CE are the extreme rays from the lens and DEF, D^1E^1F the two branches of the caustic curve.

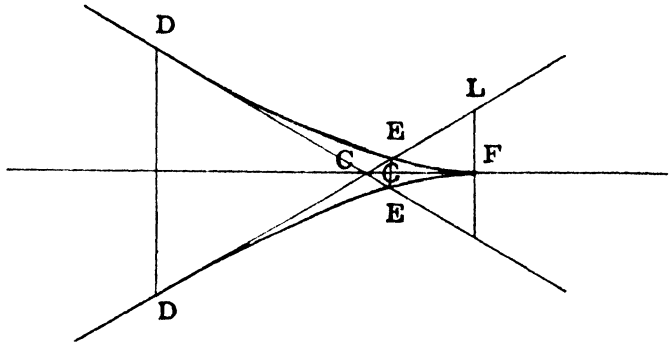


FIG 186

All rays inside the extreme rays touch this curve at points which are nearer to F, the smaller the inclinations of the rays to the axis. The rays are consequently crowded together just inside the caustic curve, and a screen placed between C and DD^1 will show a circular patch of light with a brighter illumination round the circumference. At points between C and EE^1, a brighter patch appears in the centre which increases to EE^1 where the illuminated area is a minimum. EE^1 is called the circle of least aberration.

Beyond EE^1 the first circle diminishes until at F it becomes a point.

Set up a large lens on the optical bench with a small hole illuminated by a Bunsen flame containing common salt as a source of light. The light may be concentrated on the hole by means of a small condensing lens.

It is best to have the distance from the hole to the large lens about twice the focal length of the lens.

Examine the image by means of a micrometer eyepiece, when the appearances described will be seen if the source lies on the axis of the lens.

Find the position F, and read off the position of the eyepiece on the bench scale. Move in towards the lens and measure the diameters of the circular rings at a series of distances from F. Plot the results on squared paper and verify the equation of the caustic curve.

The radius of the circle of least aberration is $r = \frac{1}{4} cm_0^3$, where c is constant in the equation and m_0 is the value of m for the extreme rays.

m_0 is nearly proportional to the aperture of the lens. Vary the aperture by putting diaphragms over the lens and measure the corresponding diameters of the least circles. Verify that the diameter is proportional to the cube of the aperture.

The above value of r is readily calculated by remembering that it is the value of the ordinate of the caustic at the point where it is cut by the extreme ray.

It is therefore necessary to solve the equations:

$$y = m_0(x - cm_0^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$
$$27cy^2 = 4x^3.$$

The solutions are

$$x = \tfrac{3}{4}cm_0^2, \qquad y = \pm \tfrac{1}{4}cm_0^3.$$

In order to find the value of FL, we note that by equation (4), which is the equation of the line, CL, the value of y corresponding to $x = 0$ is $-cm_0^3$.

This negative sign occurs because we have measured the angle of which m is the tangent from the x axis in a clockwise direction.

In the case of the line, CL, the value of m_0 is negative since the angle is obtuse.

The lateral aberrations of the extreme ray, i.e. the length of FL, is thus $-cm_0^3$.

But the radius of the circle of least aberration is $\frac{1}{4}cm_0^3$ (numerically).

Thus the ratio $\dfrac{\text{FL}}{\text{GE}} = 4$.

Verify this result.

CHAPTER XII

INTERFERENCE, DIFFRACTION AND POLARIZATION

Introduction

The theory that light consists of waves in the ether leads us to expect, by the principles of superposition, that class of phenomena described by the term 'interference.' Particles of a medium, when simultaneously displaced by the arrival of several disturbances, have a resultant displacement obtained by adding together the vectors representing individual displacements. In particular, if a particle is subject to displacements in directly opposite directions, it will be displaced a smaller amount than if it were subject to either separately, and if both the displacements are equal, but oppositely directed, there will be no displacement of the particle. On the other hand, displacements arriving at a particle in the same direction will cause it to be moved a distance equal to the sum of the separate displacements.

The arrival of oscillatory disturbances at any point may, therefore, cause larger or smaller displacements of an ether particle than would occur as a result of each separately.

The intensity of illumination at any point of a medium is proportional to the square of the amplitude of the vibrations executed by the ether particle at that point.

It can thus be seen how it is possible to produce in the ether places of large or small intensity as a result of the arrival of two trains of waves. It may even be possible that there will

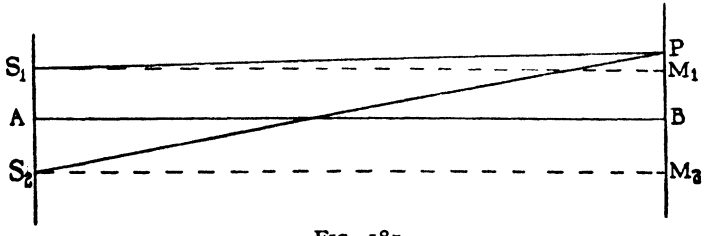


FIG. 187

be darkness at certain points since it is possible that as a consequence of adding vectorially the displacements we get no net effect.

This point will be considered in detail.

Let S_1 and S_2 denote two sources of monochromatic light. From each is thus emitted a train of waves of a particular wave

length, the same for each. In addition, suppose that the disturbance starting from S_1 is in the same phase as that starting from S_2. By this we mean that the ether particles at S_1 and S_2 are in exactly the same state of vibration, they are moving in the same direction, and are displaced the same amount from their central positions. These disturbances are propagated with a particular velocity in the ether, and at points equidistant from S_1 and S_2 the displacements will be in the same phase when they arrive there. In other cases where points are at different distances from S_1 and S_2, it may happen that the displacements are in different directions on reaching the point. Draw AB at right angles to the middle point of S_1S_2 and let it cut a screen, M_1M_2, normally at B.

We shall inquire as to the illumination at different points on the screen. The disturbances will reach B in the same phase, the displacements will be in the same direction, and will thus unite in increasing the illumination at B, which will always be bright.

Of course, we think of the waves as transverse, i.e. the motion of the particles is normal to the direction of propagation.

Take any other point, P, distant x_n from B.

The light from S_2 travels over the path, S_2P, and that from S_1 over S_1P. Thus those disturbances, which reach P simultaneously, started at different times from their sources.

The vibrations will thus be in different phases on reaching P. If, however, the difference between S_2P and S_1P is a whole number of wave lengths, the disturbance from S_2 left that point a complete number of periods before the disturbance from S_1 set out towards P. The displacements are thus in the same direction, and will unite at P to give brightness, just as they do at B.

If the difference of path is an odd number of half wave-lengths, the displacements on reaching P will be directed oppositely, and will tend to destroy the disturbance at P. If the amplitudes are equal there will be no displacement, and consequently no illumination at P. In any case there will be a marked falling off in brightness. We ought, therefore, to be able to distinguish on either side of B as we proceed outwards alternate bright and dark places.

Suppose the distance $AS_1 = AS_2 = d$, while $AB = D$.

Then $$PM_1 = PB - M_1B$$
$$= x_n - d,$$

and similarly $$PM_2 = x_n + d.$$

(M_1 and M_2 are the feet of the perpendiculars from S_1 and S_2 on to the screen.)

$$\therefore S_2P = \{D^2 + (x_n + d)^2\}^{\frac{1}{2}} = D\left\{1 + \frac{(x_n+d)^2}{2D^2}\right\} \ldots\ldots\ldots(1)$$

in which we neglect higher powers of $\frac{x_n+d}{D}$ than the square.

This means that we suppose that x_n and d are of small magnitude relatively to D. This will evidently be the case if we confine our attention to a few only of the alternations in intensity about B, for the distances apart of adjacent bright points on the screen are of a magnitude comparable with the wave lengths of the light, and this is very small.

In the same way

$$S_1P = D\left\{1 + \frac{(x_n - d)^2}{2D^2}\right\}; \ldots\ldots\ldots\ldots(2)$$

$$\therefore S_2P - S_1P = 2 \cdot \frac{d}{D} \cdot x_n.$$

Thus for a bright point :

$$\frac{2d}{D} x_n = n\lambda, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and for darkness :

$$\frac{2d}{D} x_n = \frac{2n+1}{2} \cdot \lambda; \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

or writing $\delta = 2d$, i.e. $S_1S_2 = \delta$,

we have $$x_n \cdot \frac{\delta}{D} = n\lambda \text{ or } (n + \tfrac{1}{2})\lambda, \ldots\ldots\ldots\ldots\ldots(5)$$

according as P is bright or dark.

In the experiments to be described (5) is of fundamental importance.

The first three experiments are three examples of obtaining sources S_1 and S_2 of the kind described.

It would be useless to set up two slits and illuminate them by a sodium flame, for we have in such a flame a multitude of sources in different phases.

The method adopted is to cause light from a single source to travel to the points of which P is typical by two different routes, and to unite on arrival. We thus have the equivalent of two sources emitting vibrations of the same phase.

In making experiments on interference and diffraction accurate measurements have to be made ; for this reason the optical bench is used. It consists of a strong, rigid metal frame, provided with levelling screws on which it stands The frame consists of two metal rails, one of which is graduated accurately in millimetres. Along the rails slide metal uprights, each of which is attached to a vernier at its lower end so that its position on the bench may be accurately read off.

The uprights serve to carry a slit, micrometer, microscope, lens, or whatever piece of apparatus is necessary. If it is necessary to move any piece of apparatus transversely across the bench, it is placed in an upright fitted in a support provided with a transverse micrometer screw.

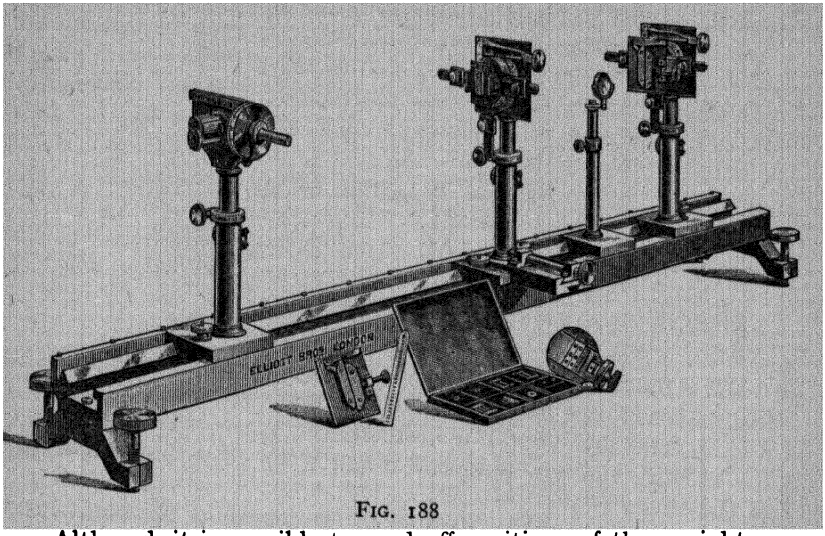


FIG. 188

Although it is possible to read off positions of the uprights on the bed of the frame, since the slit or cross-wire may not be exactly above the indicator mark we cannot read off directly the distance from slit to cross-wire. A correction must always be applied. In order to find this correction a stand, carrying a carefully measured rod is placed on the rails, one end of the rod is placed in contact with the slit while the other end is viewed in the micrometer. Let the length of the rod be l, and suppose the distance between the slit and micrometer uprights is l^1, as observed on the scale. Then to convert the readings as obtained from the upright to the distances required we must add to the observed readings the quantity $(l - l^1)$.

Determination of the Wave Length of Sodium Light by means of Lloyd's Single Mirror

The simplest way of obtaining interference bands is by means of a mirror silvered on the front surface or blackened at the back in order to avoid multiple reflections.

The diagram illustrates how the interference is brought about.

Light from a slit, S_1, travels directly to a screen, PB, and also by the alternative path after reflection at the mirror, MM. For example, a ray may reach P by the direct path, S_1P, or by the path, S_1CP. The latter ray produces the same effect at P

as would arise from a ray, S_2P, S_2 denoting the position of the image of S_1 in the mirror. We thus have the equivalent of two sources, S_1 and S_2, emitting vibrations in the same phase, except for any change of phase that may occur on reflection at MM.

MM is mounted vertically on an upright of an optical bench, and the slit, S_1, is carried vertically on another upright.

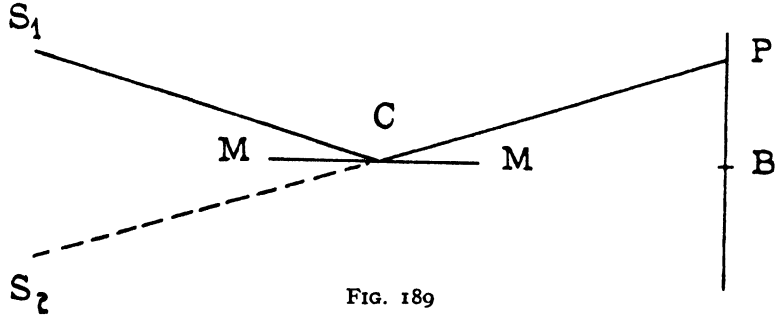


FIG. 189

MM is mounted as accurately parallel to S_1 as possible, and the fringes are looked for by the micrometer microscope. They will usually come into view, if not already there, after slightly rotating the mirror, and are rendered distinct by adjusting the width of the slit.

The mirror should be parallel to the length of the bench, and we shall see how to make this adjustment accurately shortly.

When the fringes appear as definite and bright as possible, it is necessary before proceeding further to make sure that the distance, S_1S_2, can be found.

The method adopted is to use a lens and obtain the magnified and diminished images of S_1,S_2, but we require a lens whose focal length is less than a quarter of the distance between S_1S_2 and PB.

In Chapter II, p. 293, we described a method of finding the focal length of a lens in which the object and screen are kept fixed while the lens is moved. In the diagram in connexion with this experiment, suppose that P_1 denotes the position of the object, S_1S_2, and O_1, one of the positions of the lens, while P_2 denotes the position of the image of S_1S_2.

By moving the lens to O_2, the image is formed at the same place, but it now differs from the former in magnitude.

Let d_1, d_2 denote the respective distances between the lines S_1 and S_2 in the image, while δ is the actual distance, S_1S_2, between the slits.

Then $$\frac{d_1}{\delta} = \frac{P_2O_1}{P_1O_1}, \qquad \frac{d_2}{\delta} = \frac{P_2O_2}{O_2P_1}$$

But $$P_1O_1 = P_2O_2 \text{ and } P_1O_2 = P_2O_1;$$

$$\therefore \quad \delta^2 = d_1d_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Thus, by measuring d_1 and d_2 we may deduce δ.

If MM is too close to PB it may be impossible to obtain both images with the lens between the mirror and screen. The mirror must be placed so that this is possible. This is the reason for performing this part of the experiment first since it is useless to measure the distance between consecutive fringes unless we can find the distance, S_1S_2. The distance apart of the fringes does not depend on the position of the mirror, but if the mirror has to be replaced in another position it is easy to throw out its adjustment. Both images of S_1 and S_2 will be in focus in the plane of the cross-wires only provided that MM is normal to this plane, i.e. to PB.

It is important that MM should be normal to PB. We have, therefore, to rotate MM slightly until the two are accurately in focus, the lens being adjusted so that its centre lies on a level with the centre of S_1 and opposite the edge of the mirror.

On removing the lens, MM should be rotated slightly about a horizontal axis to get the position where the fringes are brightest, when S_1 and MM are parallel.

Set the cross-wire accurately down the centre of the first bright fringe, and move it always in one direction by means of the screws giving the transverse motion, stopping at every three or four fringes to note the position. From the observations deduce the distance between consecutive fringes.

Finally measure the distance from S_1 to PB.

From (5) we have :

$$x_n = n\lambda \frac{D}{\delta},$$

and proceeding to the next bright band :

$$x_{n+1} = (n + 1)\,\lambda \frac{D}{\delta};$$

$$\therefore\ x_{n+1} - x_n = \frac{\lambda D}{\delta} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (6)$$

Let s denote the distance between consecutive fringes, and we then have :

$$\lambda = \frac{s}{D}\sqrt{d_1 d_2}.$$

As a rule it will be best to make use of the full length of the bench, so that D is large, also δ should be as small as convenient. At the same time d_1 or d_2 must not be too small or it will be difficult to make the determination of δ accurately.

In recording the results of this and the two following experiments it is a good plan to begin, as mentioned, at one edge of the field, and make out a table as shown below.

We can then readily obtain a series of independent readings for the determination of the separation of the bands.

In this experiment the first fringe does not lie at the point corresponding to B, fig. 187. This is because the reflected ray undergoes an abrupt change of phase on reflection at the mirror. This has the effect of displacing all the fringes towards B by a certain distance leaving the separation between successive fringes unaltered.

NO. OF BAND	MICROMETER READING (*a*)	NO. OF BAND	MICROMETER READING (*b*)	SEPARATION FOR 15 BANDS (*b*) − (*a*)
1		16		.....
4		19		.....
7		22		.....
10		25		.....
13		28		.....
				.....
Mean Separation for 15 Bands				
Mean Separation for 1 Band				

Interference by means of Fresnel's Double Mirrors

Interference fringes may also be formed by the use of two mirrors silvered on the front surface or blackened at the back, very slightly inclined to one another.

The diagram illustrates the arrangement. OM_1 and OM_2 denote the two mirrors which may be mounted on the optical bench and placed accurately vertical.

S denotes the source of illumination and may most conveniently be a Bunsen flame coloured by a sodium salt.

The fringes are again observed by means of a micrometer, and AB denotes the position of its cross-wires.

The screen, CD, protects AB from direct light from S.

The source is placed on one side of the bench as the diagram indicates, and may be the slit of the optical bench supported in a clamp. S_1 and S_2 are the two virtual images in the mirrors. S_1S_2 and S lie on a circle with centre at O.

The calculation, mode of measurement, and adjustment, are identical with those of the previous experiment.

The shaded area of fig. 190 shows where rays from both mirrors interfere, and the section of this region with the plane of the cross-wires of the microscope is the position within which the fringes lie.

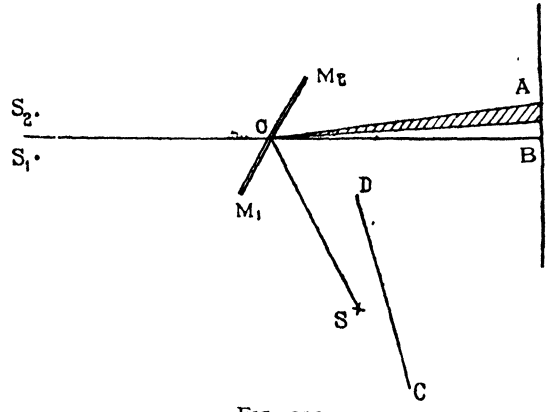


FIG. 190

Fresnel's Biprism

The biprism is a prism with one of its angles only a little less than two right angles, and with two equal small base angles. The figure illustrates its action. The biprism is represented by CDEF, and it acts like two prisms placed base to base.

Rays from a slit, S, are deviated in each part of the prism and unite on the screen, as in the case of the rays, SGP and SHP. We have virtually two sources, S_1 and S_2.

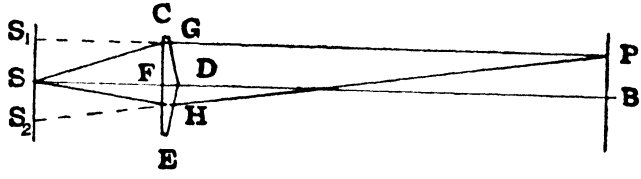


FIG. 191

The details of the experiment are very similar to those described in the experiment with Lloyd's mirror.

The biprism is held in a stand which can rotate about a horizontal axis parallel to the central line of the bench. By this means the edge, D, of the prism can be rotated to bring it parallel with the slit.

The adjustments are first made roughly by the eye, and usually the fringes will be observed even with the rough adjustment. Slight rotation of the biprism will, as a rule, improve the appear-

ance of the bands, and still better definition will be obtained by narrowing the slit.

The distance, S_1S_2, is measured with the aid of a lens as before.

In adjusting D parallel to S the following device is of great assistance. If the eye is placed on the side of the prism away from the slit, and moved across the bench, the slit will appear to cross from one side of the prism to the other. As it crosses D, unless the slit is parallel to the edge of the prism, the top or bottom will cross the edge first, while if parallel it will appear to make the transition suddenly. The prism must be rotated until this sudden jump occurs.

Be careful after arranging the apparatus to give good fringes, to make the determination of the length, S_1S_2, before measuring the separation of the fringes.

The Determination of the Radius of Curvature of the Face of a Convex Lens by means of Newton's Rings

Newton's Rings are formed as a result of interference between the incident and reflected rays from a source of monochromatic light on the air film between a plane glass plate and a convex lens in contact with it.

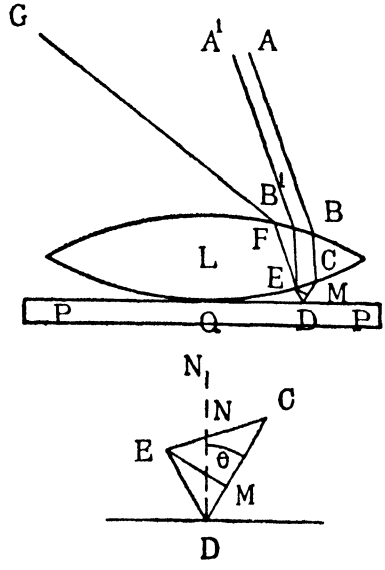


FIG. 192

The diagram shows the lens, L, and the plate, P, in contact at the point, Q.

Two incident rays are drawn, AB and A^1B^1. The first is partially reflected and refracted at the points, B, C, D, E, and F.

A ray is drawn which is represented as suffering refraction, except at D, where it is reflected. The neighbouring ray, A^1B^1, which follows the course, A^1B^1EFG is shown, and these two rays being brought together along EFG will interfere and may produce greater or less illumination than each separately, according to their phase difference. In calculating this it is to be remembered that on account of the reflection at D, at a medium optically denser than that in which the ray travels before reflection, a phase change of half a wave length is imparted to it.

The total phase difference between the two united rays is thus : (MD + DE + $\frac{1}{2}\lambda$) as reckoned in path length ; for had the rays both left the lens, the wave front would have been EM, so that the phases are the same at E and M.

Each ray, such as AB, has a corresponding ray, such as A_1B_1, arising from the same point in the source of light with which it can interfere, so that an extended source may be used and a Bunsen flame containing sodium acts very well ; in fact, an extended source is necessary in order to obtain a large area containing rings.

The rays entering the eye or an optical instrument will be contained in a small cone, and for the rays in this pencil the change in path difference is very small, so that all the rays from the small region of the lens will be naturally reinforced or caused to interfere.

There will be brightness or darkness according as :

$$\mathrm{MD} + \mathrm{DE} + \tfrac{1}{2}\lambda = n\lambda \text{ or } (n + \tfrac{1}{2})\lambda,$$

where n is a whole number, i.e. according as MD + DE = an odd or even number of half-waves.

Let CD fall on the plate at incidence θ.

Let t be the thickness of the film at this point.

The diagram is drawn with t large, or D a long way from Q, for convenience, but in the formation of the rings the film is very thin, and the part of it with which we are concerned is very close to Q.

Just near D we may regard the film as an element with parallel faces separated a distance, t, as in the enlarged element.

$$\begin{aligned}\mathrm{MD} + \mathrm{DE} &= 2\mathrm{CD} - \mathrm{CM} \\ &= \frac{2\ \mathrm{CN}}{\sin\theta} - 2\ \mathrm{CN} \sin\theta = 2\ \mathrm{CN}\ \frac{\cos^2\theta}{\sin\theta} = 2t\cos\theta.\end{aligned}$$

Thus the condition for brightness is :

$$2t\cos\theta = (2n + 1)\frac{\lambda}{2},$$

$$2t = \frac{2n + 1}{2} \cdot \frac{\lambda}{\cos\theta}.$$

The angle, θ, may be measured by the angle between the normal to the plate and the incident rays before striking the lens, since near Q the lens acts as a parallel plate of glass very approximately.

It is usual to cause the rays to fall normally on the lens, when we have :

$$2t = (n + \tfrac{1}{2})\lambda.$$

If we consider a plane normal to that of the paper through Q and D, we can see that the circumference of a circle with centre, Q, and radius, QD, will pass through points at which the air film has a uniform thickness, and consequently all round this circle rays incident vertically will undergo the same phase change, and thus alternate bright and dark rings are formed about Q.

Let the radius of the ring be $r_n = QD$, and let the ring be the $(n + 1)$th from the centre.

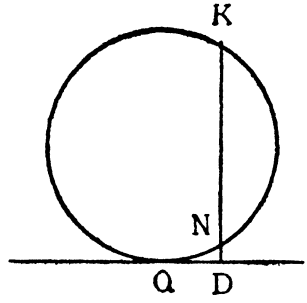


FIG. 193

The centre itself will be black, for the air film is infinitely thin at this point. If this is not the case at first it is because some dust particles lie between the surfaces, and these should be removed.

$$DQ^2 = DN \cdot DK = t(2R - t), \qquad \text{(fig. 193)}$$

R = radius of lower surface of the lens,

$$\therefore\ DQ^2 = r_n^2 = 2Rt,$$

approximately, since t^2 may be neglected.

Thus for brightness :

$$r_n^2 = R(n + \tfrac{1}{2})\lambda.$$

The first ring corresponds to $n = 0$; the second for $n = 1$, and so on.

Thus the radii are proportional to $\sqrt{1}$, $\sqrt{3}$, $\sqrt{5}$, etc.

For the purpose of the experiment a convex spectacle lens of about 100 cms. radius of curvature is suitable, and the light

from a sodium flame is reflected down on to it by means of a sheet of plane glass held at 45° to the vertical (see fig. 194).

The rings are viewed by means of a travelling microscope.

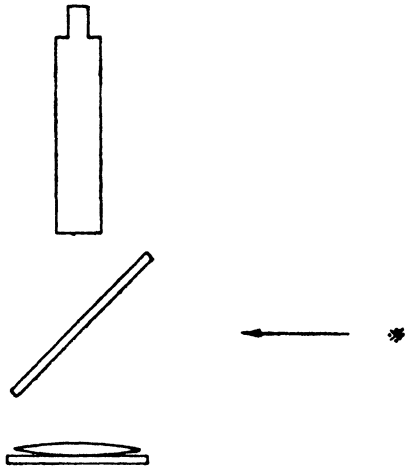


FIG. 194

In order to focus quickly on the rings, remove the lens and focus on the top of the glass plate. On replacing the lens, and adjusting the microscope over the point of contact the rings which lie in the air film should be distinct. A good bright sodium flame is necessary, and often the difficulties disappear if this point is attended to.

RING	MICROMETER READING (L)	MICROMETER READING (R)	DIAM.	(DIAM.)^2
20				
19				
18				
..				
..				
..				
5				
4				
3				
2				
1				

(L) denotes the reading on the left of the centre, (R), that on the right.

Move out the micrometer to about the twentieth ring from the centre, and then, moving back again, turning the screw always one way to avoid any errors due to backlash, set the cross-wire carefully down the centre of each bright ring and observe the micrometer reading. In this way, by observing the rings on both sides of the central dark patch, the various diameters are determined.

Make a table as shown above, and carefully arrange the cross-wire to lie perpendicular to the diameter of each ring.

Draw a curve with the square of the diameter as ordinate, and the number of the ring as abscissa. The graph should be a straight line.

Then if $$P_2N_2 = (D_{n_2})^2,$$
$$P_1N_1 = (D_{n_1})^2,$$

Since $$D_n^2 = R\lambda(n + \tfrac{1}{2}),$$

$$\frac{P_2N_2 - P_1N_1}{N_1N_2} = \frac{(D_{n_2})^2 - (D_{n_1})^2}{n_2 - n_1} = 4R\lambda.$$

If sodium light is used, λ may be taken as 5890×10^{-8} cms.

In this description we have not taken account of the fact that the rings are seen, not directly, but after refraction through the lens. They are formed in the air film in the space between the lens and plate of glass. This difficulty is avoided by placing the plate above the lens and in contact with it, for in that case we view the rings through a plane sheet of glass.

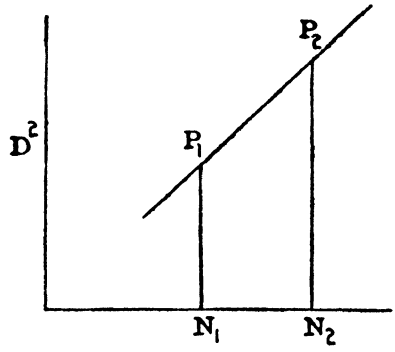


FIG. 195

It is, of course, more convenient to place the glass below the lens, for reasons of ease in keeping them steady. The error in this case is not great if a thin lens be used, for then the object —the rings—is at the surface of the lens and consequently at its principal plane. The image is in the second principal plane, and of the same size as the object. For a thin lens these planes and lens surface are nearly coincident. In practice we have an

image of magnification slightly differing from unity, or the diameters in the formula above are to be multiplied by such a fraction. For the purpose of the experiment this factor is omitted.

Jamin's Interferometer

The apparatus consists essentially of two glass plates, AB and A^1B^1, of the same dimensions and optical character. The plates are very carefully worked and are of the best optical glass. They are mounted parallel to one another, standing on tables on an optical bench at a distance apart of about one metre.

The first, AB, is set at 45° to the bench with its surfaces vertical, and it is illuminated by rays from a sodium flame.

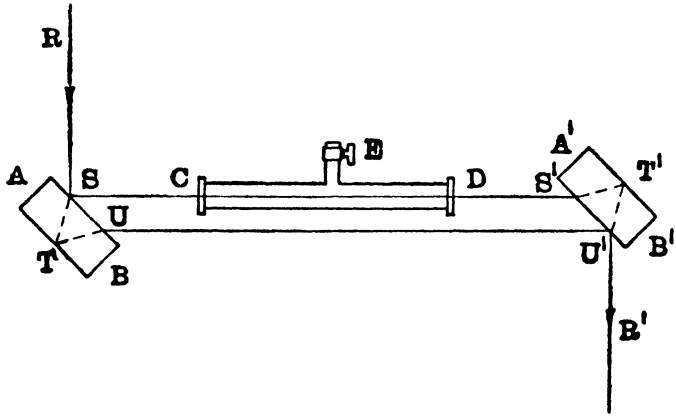


FIG. 196

In the diagram a ray, RS, is shown. It is partly reflected and refracted at S, and the refracted beam again partly reflected at T. The divided ray takes the paths, $RSS^1T^1U^1R^1$ and $RSTUU^1R^1$, so that it is reunited by the second glass plate. This plate is mounted parallel to the first, and set vertically. It rests on a table and may be slightly rotated about a vertical axis. If both plates are exactly similar and are parallel, the lengths of the two paths will be the same for all rays, but by slightly rotating one of the plates a difference in path may be introduced, differing for different directions so that alternate bright and dark bands will be obtained.

The tube, CD, is then placed in the path of one of the divided rays between the plates, the other being allowed to pass clear of CD.

CD is a hollow glass tube with ends of plane, optical glass. It is fitted with a tap, E, by means of which it may be exhausted or filled with gas.

The tube is filled with air at the atmospheric pressure, and the fringes found by eye. A telescope provided with cross-wires is then focussed on the fringes.

In order to ensure that the fringes arise from interference between rays that have passed, one through CD and the other outside it, we may cover up the end of CD, and note if they disappear as they should if due to this cause.

In order to be sure that the rays producing interference do not both pass down CD, it is necessary to intercept the light at the sides of the tube. If they are still present they will arise from rays passing down CD.

When the correct fringes are obtained the cross-wire of the telescope is focussed as accurately as possible down the centre of one bright band, after exhausting the tube as much as possible with an air-pump. The pressure should be brought down to as low as 1 cm. of mercury, and a manometer connected up to read the pressures. Air is now allowed to pass very slowly into the tube by the tap, and the fringes watched. They will appear to move across the field of view, and the number passing the cross-wire must be counted. Allow about 5 or 6 to pass, and then stop the inflow and read the manometer. Repeat this, step by step, until the tube is filled with air at atmospheric pressure. Draw a graph showing the relation between the pressure within CD, and the number of fringes that have passed from the initial stage. By an exterpolation deduce the number that pass between the limits of complete exhaustion and the attaining of atmospheric pressure.

Let this number be n.

This means that the difference of optical path in the tube, when completely exhausted and when filled with air at atmospheric pressure, is $n\lambda$.

If the tube is of length, l, and the refractive indices are μ and μ_0, when the pressure is atmospheric and when the tube is exhausted, respectively:

$$l(\mu - \mu_0) = n\lambda.$$

But $$\mu_0 = 1,$$

so that $$\mu = 1 + \frac{n\lambda}{l}.$$

A convenient length for the tube is 30 to 40 cms., when for the exhaustion produced by a good air-pump the value of n is of the order 200.

The compensator described in connexion with the experiment with Rayleigh's refractometer may also be used with Jamin's interferometer, and the calculation of the refractive index at normal temperature and pressure may be made with the help of Jamin's interferometer also.

A cylindrical lens with the cylindrical axis vertical is often placed just in front of the point S This has the effect of widening the source light in a horizontal direction, leaving it unchanged vertically. The source in this case is a slit.

The Refractive Index of Air by means of the Rayleigh Refractometer

A diagrammatic representation of the apparatus is given in fig. 197.

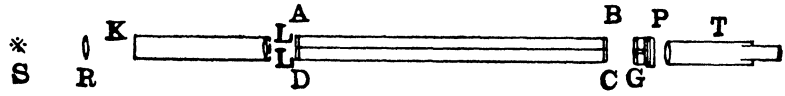


FIG. 197

ABCD is an airtight metal box, divided into two separate chambers, each of which may be connected to a manometer. The pressures in the chambers are varied and measured by means of the manometers. The chambers are closed at the ends by means of parallel plates of good glass.

The collimator is provided with a slit at K, and provides a parallel beam of light which falls on a screen, carrying two fine slits, LL, placed in front of the air chambers, so that one slit lies adjacent to the end of one, and the other slit adjacent to the second chamber

These slits are prolonged so as to extend higher than the top of the box, ABCD. Thus, light from the slits passes over the box as well as through it, and finally enters the telescope, T.

These two fine slits produce interference fringes in the focal plane of the telescope, and if the pressures in the chambers are equal, the set of fringes in the lower half of the field appear to be continuations of the fringes in the upper half which arise from rays that have passed over the top of the box.

In order to deviate the upper set of fringes down, to make comparison with the lower set easy, a prism, P, is provided which intercepts the upper set of rays, and deviates them downwards.

When white light is used, two sets of coloured fringes are obtained with a white central fringe.

If there is a difference of pressure between the two chambers, there is a displacement of the lower set of fringes owing to the resulting difference of optical path.

G consists of two plates of glass, inclined at a small angle, placed to intercept the lower set of rays. When it lies symmetrically with respect to the rays striking it, it introduces no additional path difference, but on rotating it the rays through one plate traverse a longer path within the glass than do those in the other.

In this way the lower central band can be moved about, and we may, by a rotation of G, bring back the fringe to its central position after it has been displaced on account of the difference in air density in the two chambers. G carries a pointer moving over a scale, and we may calibrate the scale so that the difference of path introduced by setting G is known. Thus, by altering the pressure within the chamber and then moving G to counteract the displacement of the central fringe, we can read directly from the calibrated scale to how many wave lengths the path difference in the two chambers amounts, and a graph is plotted with the pressure differences as ordinates and the number of wave lengths as abscissæ.

The calibration is performed by making the pressure in both parts of the chamber equal and illuminating K with monochromatic light, e.g. by a sodium flame.

The scale is set so that the bright bands of the upper set of fringes lie over those of the lower. The pointer is then moved over the scale until a certain number of bands pass a fixed point in the upper series. The following table will illustrate how this result should be recorded :

CALIBRATION OF SCALE IN WAVE LENGTHS

WAVE LENGTHS	SCALE READING	WAVE LENGTHS	SCALE READING	DIFFERENCE FOR 40 WAVE LENGTHS
0	1·48	40	4·50	3·02
5	1·85	45	4·89	3·04
10	2·24	50	5·24	3·00
15	2·62	55	5·61	2·99
20	3·01	60	5·99	2·98
25	3·38	65	6·37	2·99
30	3·74	70	6·75	3·01
35	4·12	75	7·12	3·00
				24·03
			Mean for 40 wave lengths	3·004
			Mean value of 1 wave length in scale divisions	·0752

After the calibration of the scale the collimator slit is again illuminated by white light, and the central fringes arranged one above the other.

The pressures in the chambers are varied and measured by the manometers.

The central fringes are kept one above the other by means of G, and the pressure differences recorded along with the position of the pointer on the scale.

Draw a graph with scale readings as ordinates and pressure differences as abscissæ (fig.) 198.

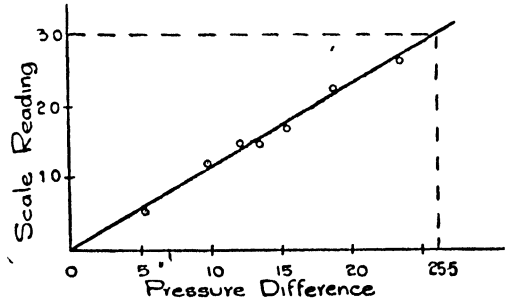


FIG. 198

White light is used in order to provide a definite central fringe. The position of this fringe is independent of colour while all other fringes have positions which depend on the wave length. If monochromatic light is used, all the fringes are alike, and the central fringe is indistinguishable from the others.

Since the screw attached to the compensating glasses has been calibrated in wave lengths of sodium light, the observations give the path differences in terms of so many wave lengths of yellow light, and the refractive index deduced will be that for this particular wave length.

Fig. 201 gives a general view of the apparatus.

From the slope of this graph may be deduced the difference in path in wave-lengths for a difference of pressure of 1 cm. of mercury.

Observe the temperature of the air in the tubes by placing a thermometer close to them and noting its indication throughout the course of the experiment. When the pressure is varied by means of the manometers, the changes should take place slowly, and time should be allowed for the air to take up atmospheric temperature.

Theory of the Experiment

For a gas the relation,

$$\frac{\mu - 1}{\rho} = \text{a constant},$$

is very approximately true. We assume it in this experiment μ is the refractive index, and ρ its density.

If T is the absolute temperature, we have also :

$$\frac{p}{\rho T} = \text{constant.}$$

Thus $$\frac{\mu - 1}{p} \cdot T \text{ is constant.}$$

If μ_0 is the refractive index of air at N.T.P.

$$\frac{\mu - 1}{p} \cdot T = \frac{\mu_0 - 1}{76} \times 273.$$

We shall see that μ_0 can be determined by observations in this experiment.

Suppose the length of the tubes containing the air to be l, and the refractive indices, μ_1 and μ_2, corresponding to pressures, p_1 and p_2. Let the wave lengths of monochromatic light be λ_1 and λ_2 respectively.

The excess of waves in one tube over those in the other is :

$$l\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) = \frac{l}{\lambda_0}\left(\frac{\lambda_0}{\lambda_2} - \frac{\lambda_0}{\lambda_1}\right) = \frac{l}{\lambda_0}(\mu_2 - \mu_1)$$

$$= \frac{l}{\lambda_0}\{(\mu_2 - 1) - (\mu_1 - 1)\} = \frac{l}{\lambda_0} \cdot \frac{\mu_0 - 1}{76} \cdot \frac{273}{T}(p_2 - p_1)$$

This number is deduced from the scale over which the pointer moves. Suppose it is m.

Then :

$$\mu_0 = 1 + m \cdot \frac{76 T \lambda_0}{273 l (p_2 - p_1)}.$$

The pressures are measured in centimetres of mercury, and the ratio, $\frac{m}{p_2 - p_1}$, is deduced from the graph as described, while the length of the tubes, which is usually about 25 cms., may be measured by means of a metre rule.

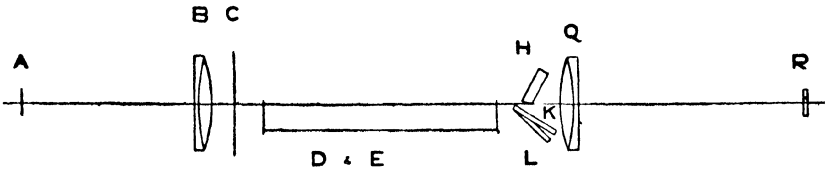


FIG. 199

In figs. 199 and 200 a modern form of apparatus is shown diagrammatically. The plates, L,K, correspond to G, and the prism, P, is here denoted by H, and its action shown in the fig. 200. It can be seen how such obstacles as the upper edge of

the tubes containing the gas, which would normally separate the lower interference bands from the comparison upper bands, produce no image in the field of view on account of the refraction in H. The two sets of fringes stand one immediately above the other.

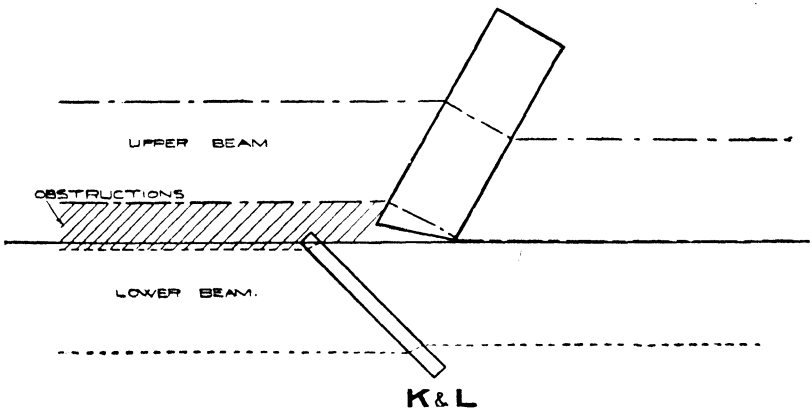


FIG. 200

They are focussed by the achromatic lens, Q, and examined by the cylindrical lens, R. This provides a large horizontal magnification of the fringes, which lie close together because

FIG. 201

the slits are of necessity rather wide apart (cf. formula 6 of this chapter, δ is large in the present case). The lens, however, does not give magnification in a vertical direction, so that the shadow cast by the obstacle is not broadened vertically, and there appears a sharp dividing line between the two sets of bands.

The Rayleigh Refractometer for Liquids

Fig. 202 shows a Hilger apparatus based on the foregoing principles for comparing the refractive indices of liquids.

The apparatus is shorter since the length of liquid traversed need not be so great as that required for a gas.

If a liquid is placed in one of the vessels, and the same liquid containing a solvent in the other, we may determine the effect of the solution on μ, or inversely, we may estimate the amount of solvent contained from its refractive index.

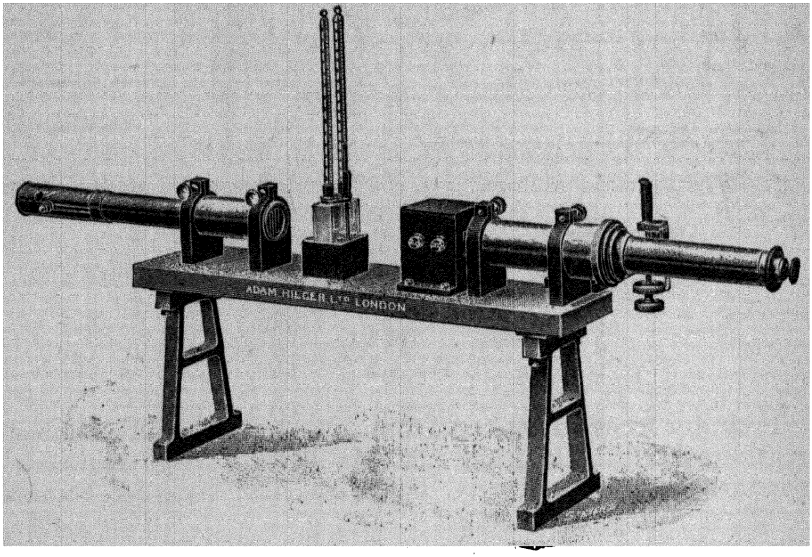


FIG. 202

In this instrument the movable plate is in the upper beam instead of the lower, but the method of use is otherwise similar to that of the previous experiment.

Two thermometers project into the liquid for recording their temperatures.

Examine the changes produced by adding small quantities of a salt to pure water, and placing in one cell the solution so obtained and water in the other.

We have, if μ and μ^1 denote the refractive indices of water and of the solution respectively, d the thickness traversed by the light, and λ the wave length :

$$(\mu \sim \mu^1)d = n\lambda,$$

where n is the number of displacements of the central fringe.

The experiment is performed with white light so as to have a definite central fringe, but λ is the wave length of the light used to calibrate the scale. By moving the pointer over the scale until the two central fringes are coincident, the scale reading

will give the number of wave lengths which one ray has fallen behind the other in traversing a path of different nature from that of the other.

The calibration is carried out in a preliminary experiment with both liquid cells containing water and with a sodium flame illuminations, as in the last experiment.

Michelson's Interferometer

The apparatus is illustrated diagrammatically in fig. 203. It consists of two plane mirrors, M_1 and M_2, silvered on their front surfaces and mounted vertically on a heavy, firm, rigid stand. The stand consists of a metal bed provided with a large micrometer screw of very fine pitch. Rotation of the screws causes M_1 to slide along the bed and its position may be read off at the screw-head. A general view of the apparatus is given in fig. 204.

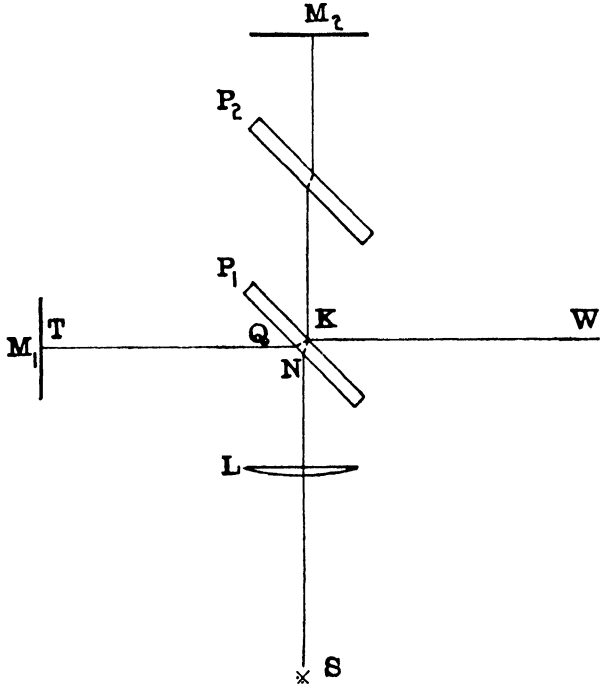


FIG. 203

The second mirror, M_2, is fixed at the end of a metal arm mounted at the end of the bed, and at right angles to it. This arm also carries the two sheets of plane optical glass, P_1 and P_2, which are equally thick, and are mounted at an angle of 45° to the arm.

In order to produce interference fringes, a source of light, e.g. a sodium flame, is placed at the focus of a lens, L, and a parallel beam of light thrown on to the glass, P_1, the direction of the beam being along the arm carrying M_2. One of the rays of such a beam is shown in the figure as SN. On striking P_1 it undergoes partial reflection and refraction at N, and the refracted part is again divided at K, and later further division takes place at P_2.

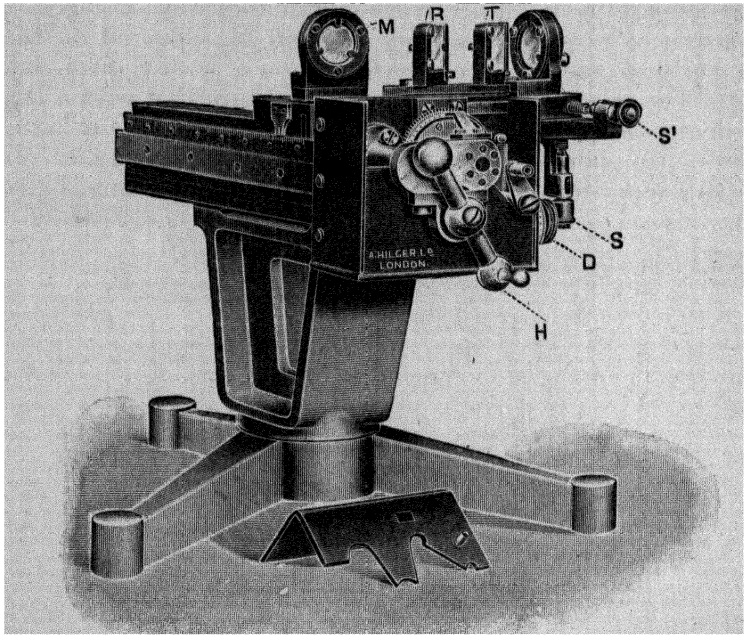


FIG. 204

We consider one of the ways in which division can take place, the recombination of two parts, which have started from one ray and have different optical paths, producing interference.

A marked difference in this case from that in some of the foregoing is the production of interference between rays differing in phase by very many wave lengths. In the case of Newton's rings or Fresnel's biprism, the path difference amounts to a few waves only.

The ray, SN, is refracted in P_1 and reaches K, where it is partly reflected so that it gives rise to KQT and partly refracted to give KM_2. These rays are both returned along their paths and reunite to produce KW. In the course of their journey each passes through the glass a distance equal to 3NK, and change in phase is brought about by the difference in the path in air. This may be varied by altering the position of M_1.

The ray, KW, may be observed by eye, and usually curved

interference bands similar to those observed in the experiment on Newton's rings will appear. The curves will not be seen closed, only parts of the closed curves can be seen.

We use here, as in the experiment on Newton's rings, an extended source of light, for the rays from each point of the source are divided and re-combination takes place between rays which originally belonged to the same ray. Rays falling on P_1 in the same direction all suffer the same phase change and emerge parallel to KW, so that an optical instrument will focus them together in its focal plane.

Other rays in a direction slightly inclined to SN, emerge as a set of parallel rays, slightly inclined to KW. These undergo a different phase change, and also lie in the focal plane displaced from the image due to rays parallel to KW.

It is necessary that the mirrors, M_1 and M_2, should both be vertical, and in order to allow this adjustment to be made, at the back of M_2 are three screws pressing against springs that cause it to rotate.

If a sheet of tin carrying a fine hole is placed in the path of the incident light, four images are seen as a rule, when the eye looks in the direction, WK. The reason for this is best seen by reference to a diagram (fig. 205).

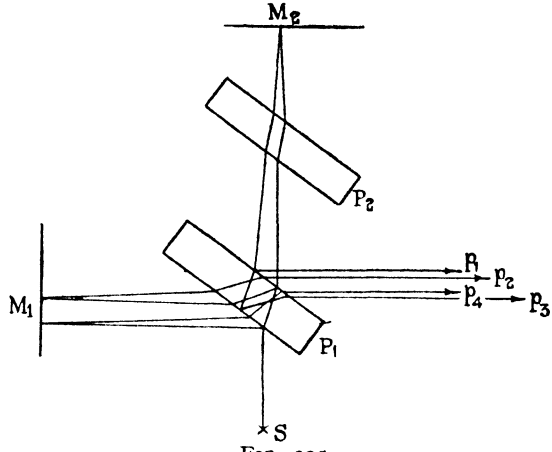


FIG. 205

The emergent rays are marked p_1, p_2, p_3 and p_4, and it will be noted that p_2 passes through the glass plates three times; so also does p_1.

Thus if the rays are made to coincide they will be in a condition to annul or reinforce each other according as the path difference is an even or odd number of half wave-lengths.

The rays, p_3 and p_4, traverse the plates five times and once respectively.

If a card is placed in front of M_1, the rays p_1 and p_3 still appear; and p_1 will be the brighter since it traverses the plates only three times, so that it can be distinguished.

By placing a card in front of M_2, the rays are cut out except p_2 and p_4, so that by this means the images can be distinguished, and those due to p_1 and p_2 caused to overlap by adjusting the screws behind M_1 and M_2.

When both mirrors are vertical the images coincide in pairs. One constituent of each double image lies behind the other. Those which produce interference lie along the direction, WK.

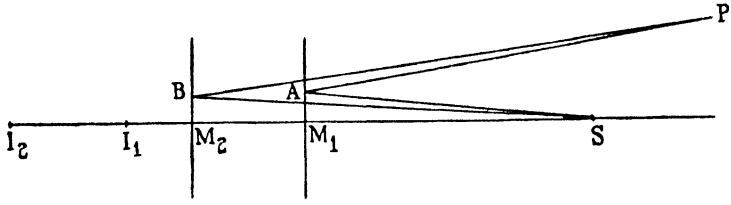


FIG 206

Fig. 206 illustrates the mode of production of the fringes from another point of view. The interferometer acts as if we had a source, S, from which rays could be reflected by two parallel mirrors, M_1 and M_2, in which they would produce images, I_1 and I_2. The distance, M_1M_2, is equal to the difference of the distances, NM_1 and NM_2, of fig. 203.

At any point, P, rays, SAP and SBP, would unite, and if their paths differed in length by $n\lambda$, a bright point would arise.

This is equivalent to stating that in this case:

$$PI_2 - PI_1 = n\lambda.$$

Thus we obtain the particular interference band at all points, P, for which this relation holds, i.e. P lies on a hyperboloid of revolution with SM_1M_2 or with WK, of fig. 203 as axis. In a plane perpendicular to that of the figure we have a circular section of the hyperboloid which explains why the fringes are circular as seen on looking along WK.

The Determination of the Frequency of Light from a Sodium Flame or any Monochromatic Source

A striking feature of the Michelson Interferometer is the screw which displaces M_1. In the apparatus illustrated in fig. 204 the screw has a length of 200 mm. and a pitch of 1 mm. The head of the screw is furnished with a scale divided into one hundred parts, each thus corresponding to one hundredth of a millimetre. The screw may be rotated by the handle seen in front of the apparatus. The small lever on the right of the front of the apparatus puts into action the slow motion screw, one turn of

which corresponds to one division on the head of the main screw. As the head of the slow motion screw is also divided into one hundred parts, it is possible to record a motion of the mirror of one ten-thousandth of a millimetre.

In order to obtain the fringes, set up the sodium flame at S, and place immediately in front of it a sheet of tin with a small hole in it just opposite the bright part of the flame, and adjust the flame and hole to lie on the level of the centre of the mirrors and plates.

Mount the lens, L, also so that its centre is at the same height as the hole, and place a piece of plane mirror between L and P_1, and move L until an image of the hole is thrown back on to the tin close to the hole. The light is then parallel as it leaves L. Remove the plane mirror and look in the direction, KW, when usually four images of the hole appear. Adjust the mirrors so that these images coincide, two by two, as indicated above.

Then remove the tin sheet; allow the light from the flame to fall through L on to the apparatus.

As a rule very slight movements of the mirrors will bring the fringes into view if they are not already to be seen. Sometimes a slight motion of M_1 along the bed of the apparatus helps to discover the fringes.

Set up in front of P_1 on the side towards W a sharp pin point to mark the position of the centre of one fringe. Rotate the screw slowly by the slow motion screw, and watch the movement of the fringes across the field of view, counting the number which seem to pass the point, and observing from the scales how far M_1 has moved.

When M_1 moves back a distance $\frac{1}{2}\lambda$ the path difference between the two rays which unite to interfere has been increased by λ, so that where a particular fringe originally appeared the neighbouring fringe now apparently lies.

Thus if M_1 moves a distance, l, the number of fringes which appear to move past the point is correspondingly $\frac{2l}{\lambda}$. If these are counted, since l can be measured, we can find λ.

Do this for such monochromatic extended sources as are available.

Note also that the intensity of the fringes appears to alternate as the distance of M_1 varies. We shall make use of this fact in the next experiment.

The Determination of the Difference of Wave Length for the Sodium D Lines.

Adjust the mirrors M_1 and M_2 so that their distances from N are equal, as nearly as can be judged by eye.

This adjustment may be brought about accurately by observing the images of the holes formed as above in the two mirrors, and adjusting M_1 until there is no parallax between them. In this case they are equally distant from the observer, and the two mirrors are consequently equidistant from N.

The light from a sodium flame, though for many purposes considered monochromatic, contains two fairly intense waves whose frequencies are close together.

In the present case both sets of fringes which arise from the two waves overlap, but if M_1 is slowly moved away there is a gradual separation of the two sets, and finally the bright band of the one lies over the dark band of the other. This happens when the distance that M_1 has moved from the first position contains one more quarter of a wave of the one than of the other, for then a difference of phase, corresponding to one-half wave length has been added to one more than to the other. Or, to put it otherwise, let l denote the distance moved by M_1, the additional air path added to each wave incident on M_1 is thus $2l$.

Suppose $2l$ contains n_1 waves of length λ_1, and n_2 of lengths λ_2. Then the difference between n_1 and n_2 is one-half.

Suppose for the sake of definiteness that $n_1 > n_2$ and consequently $\lambda_1 < \lambda_2$.

Then, since $n_1 = \frac{2l}{\lambda_1}$ and $n_2 = \frac{2l}{\lambda_2}$,

we have: $$2l\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) = \frac{1}{2},$$

or $$\frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1}{4l}.$$

If the two waves had equal intensity the field would become uniformly illuminated, and the fringes would disappear.

In this case, since one of the lines is more intense than the other, we get an alternation in distinctness, the brighter fringes still stand out in contrast with the adjacent less bright ones.

Note the positions of M_1 at the beginning and successively at positions where the fringes become least distinct and again distinct as M_1 goes further away and the path difference contains one complete wave more of one colour than of the other. Do this for as many cases as possible, and if d denote the distance between the positions of M_1 in which two successive distinct sets of fringes occur, we have

$$\frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1}{2d}.$$

Assume the shorter light in the sodium light to have a length

5890 × 10^{-8} cm., and deduce the difference between λ_1 and λ_2 in this case.

The Production of Coloured Fringes

These can be obtained when the mirrors are set for equal paths. Do this as in the last experiment with sodium light and then replace it by a white source. The fringes due to the different colours overlap in this position.

With sodium light, as M_1 is moved, it will be noticed that the circles change the direction of their curvature; the position required is just when the transition takes place. It is difficult to decide just when this occurs, and it is a good plan to note the two positions of M_1 in which the fringes are definitely curved in one way and then in the other. Put in white light when M_1 occupies the position corresponding to one of these directions of curvature, and then slowly move it back to the other position. The coloured fringes will appear during this movement.

M_1 must be moved slowly or the fringes change their places too rapidly to be noted.

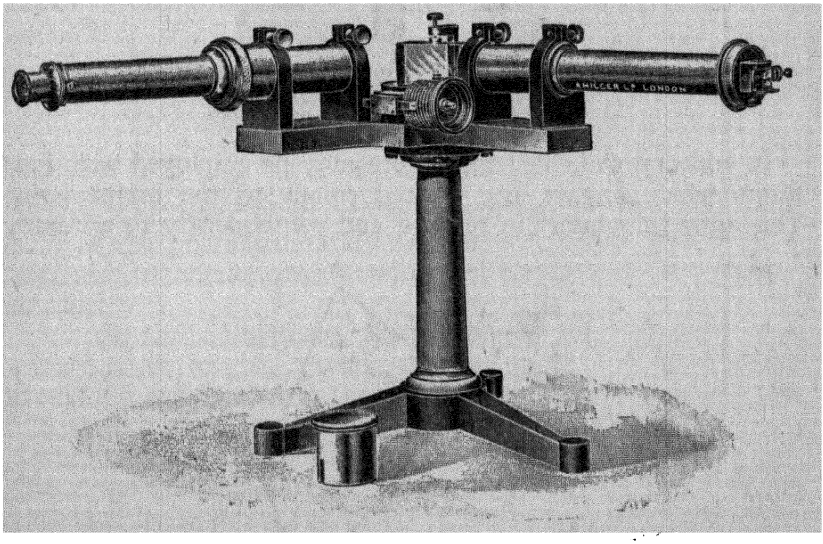


FIG. 207

The Hilger Wave Length Spectrometer

This apparatus, illustrated in fig. 207, consists of a cast-iron stand with two arms at right angles on which are held rigidly the telescope and collimator.

The telescope may be fitted with a high-power eyepiece with adjustable cross-webs or with a shutter eyepiece which can be adjusted to cut out any part of the field except that under particular examination with an adjustable metal pointer. The pointer has a brightly polished fine point and is illuminated by reflecting light from outside by means of the mirror shown in the figure in position above the eyepiece. Thus any point in the field may be taken as a reference point by setting the pointer to it (fig. 208).

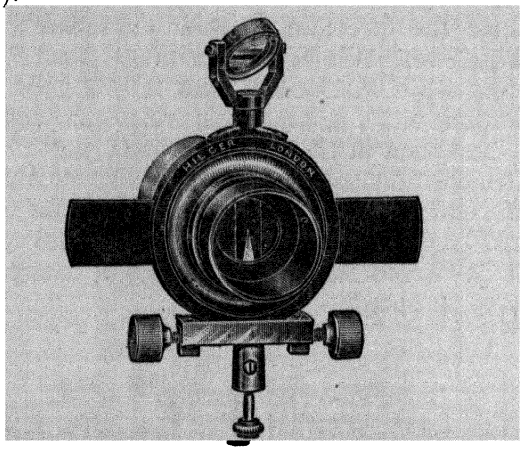


FIG. 208

In addition the shutter eyepiece may be employed with light filters which impart any desired colour to the bright point. This adds to comfort in reading and consequently to accuracy.

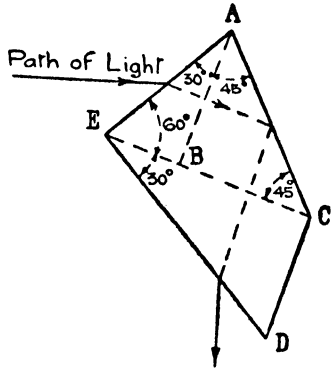


FIG. 209

The vertical collimator slit may also be reduced in length by means of a cross horizontal slit, so that a small rectangular source is obtained.

The principle of the apparatus is based on the constant deviation prism which is illustrated in fig. 209. The faces particularly concerned in the deviation of the ray are inclined at the angles marked in the figure, and total reflection occurs at the face AC.

The prism is mounted on a turntable in a position marked for it. The mean deviation of the rays is a right angle, and in order to pass through the spectrum the table is turned by means of a screw to which a drum is attached provided with a milled head (fig. 210). On the drum is a scale so that the wave length of any line under observation and appearing in the field of view of the telescope may be read off directly.

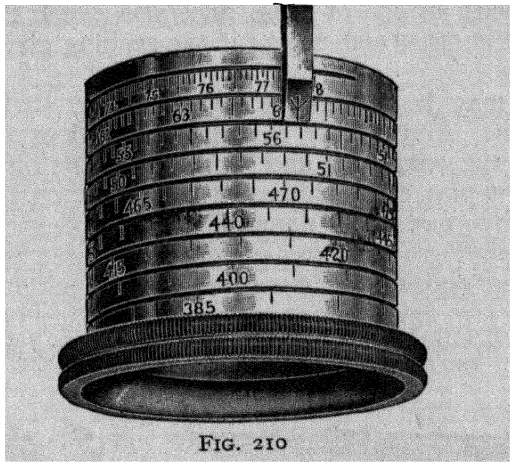


FIG. 210

Before taking any observations of wave length it is necessary to adjust the prism accurately so that the correct wave length is indicated when the corresponding line appears at the eyepiece indicator.

To make this adjustment, illuminate the slit with light of a standard wave length, set the drum so that the appropriate wave length is indicated at the index of the drum and adjust the prism so that the line appears under the eyepiece index.

Clamp the prism in position with the screw provided.

Other wave lengths of light illuminating the slit may then be determined by rotating the prism by means of the drum until the line appears at the eyepiece index and reading off the number against the drum index.

Fig. 209 indicates the course of a ray in the prism.

In the apparatus designed for use with certain accessories—the Lummer-Gehrcke Parallel Plate, the Fabry-Perot Etalon or the Michelson Echelon Grating, the collimator arm is of greater length than that illustrated in the figure to permit of interposing the accessory between the prism and collimator.

The method of fixing the collimator and telescope and of obtaining different parts of the spectrum by rotating the prism is very convenient and accurate; moreover, the drum can be rotated while looking through the eyepiece, and one is saved the inconvenience of moving round with a rotating telescope.

A suitable standardizing wave length is the red line of the helium spectrum, which has a wave length of frequency 6678·1 Ångström units (1 Ång. unit = 10^{-10} metre). It is the shorter of the two red helium waves. Or the sodium lines may be used. These are separated by the prism and have wave lengths 5890·2 and 5896·2 Ångström units respectively. It is a good plan to set the instrument on one of these lines and check the setting by turning to the other and noting if the reading gives the correct wave length.

The slit may be illuminated by throwing an image of the source on it. In the case of the sodium lines the source is obtained in the usual way, and for a helium line throw the image of the bright part of a helium discharge tube on the slit.

A protective metal cover for the prism table is provided.

When a photograph of the spectra is required the eyepiece is removed and replaced by a camera with a suitable focus lens. This is shown in fig. 211. It is capable of adjustment by tilting so that the whole spectrum can be photographed and a vertical

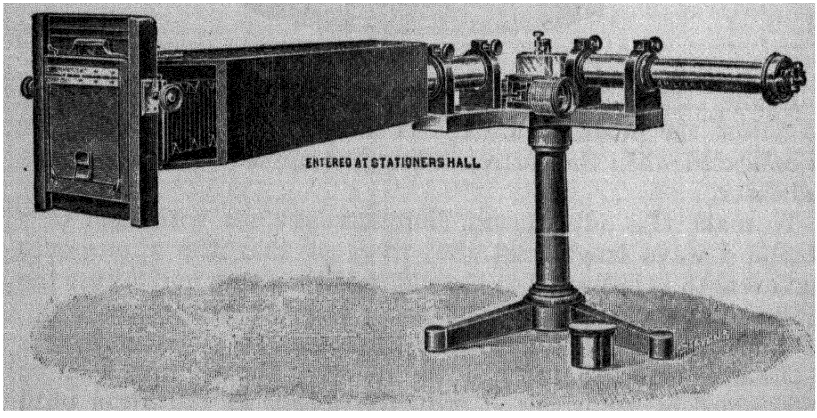


FIG. 211

displacement enables the same plate to be used giving photographs one above the other. There is, of course, also a shutter for exposure.

In addition to the usual vertical slit there is a horizontal adjustable slit attached to a hinge so that it can be swung out of the way or in a position covering the vertical one.

Thus we have crossed slits, and a small rectangular source can be obtained by closing down both the slits as much as is required.

The Lummer-Gehrcke Parallel Plate

This piece of apparatus is described in the "Annalen der Physik," vol. 10, 1903, p. 457, and a very complete account of its use and theory given.

The reader is recommended to refer to the original paper, but we shall give below as much as is necessary for our purpose

The plate can be used in connexion with the Hilger Constant Deviation spectrometer, and is of quartz of refractive index 1·544.

It has the following dimensions:

Length	130 mm.,
Width	15 mm.,
Thickness	4½ mm. (approximately).

Like the echelon grating it produces spectra of a high order, and has consequently a high resolving power. The approximate resolving power in the present case is about 200000.

It is therefore well adapted for the study of complex lines in the spectrum which in lower orders appear as single lines.

It may be used also to measure such small displacements as occur under the influence of magnetic fields, and in the section immediately following the present description we describe how it may be used to determine the value, $\frac{e}{m}$, from observations on the Zeeman effect.

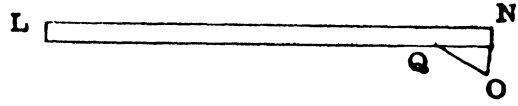


FIG. 212

The plate itself is shown in fig. 212 and mounted on its stand in position on the spectrometer in fig. 213. In this figure the screws permitting adjustment in the various directions are shown.

The action of the plate is illustrated in fig. 214. Here LM_1 denotes an incident ray of monochromatic light making an angle, i, with the normal to the plate. After refraction the angle is r.

The figure shows the production of two beams emerging from the plate on opposite sides. The rays, $M_1 1$, $M_2 2$, $M_3 3$, etc., and $N_1 1^1$, $N_2 2^1$, $N_3 3^1$, etc., on account of their different courses due to successive reflections and refractions, are in different phases when they reach the position denoted by 123 and $1^1 2^1 3^1$ respectively. These two traces mark out wave fronts and the emergent beams produce interference bands. The upper beam consists of a system of bands with alternating intensities, the maxima

having an intensity which may be measured by 2J, while the minima have intensity, J. Thus the effect is the same as is obtained by imparting to the whole field an intensity, J, and drawing bands across it of double the intensity. The transmitted system, however, consists of a set of maxima of intensity, J, with alternating minima of nearly zero intensity. For convenience in observation the second system is to be preferred, and, by using light at grazing incidence so that i is a right angle and r the critical angle for quartz the sharpness of the lines is increased.

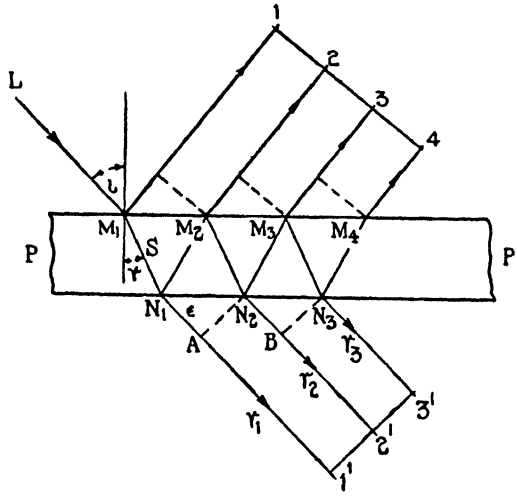


FIG. 214

It is not possible to give here the reasoning which leads to these statements; the discussion is given very clearly in the original paper.

These points find application in the Hilger pattern.

A slot will be noticed on the right of the carrier of the plate, fig. 213. Just opposite this is a prism of such dimensions that the light along the directions M_1N_1, M_2N_2, etc., is in the critical direction.

The prism lies on the under surface of the plate, so that the beam emerging from above is the transmitted beam.

The other beam is absorbed by the black lining of the stand on which the lower surface of the plate rests.

In fig. 212 the prism is represented at OQ.

The Theory of the Plate

Corresponding to every direction, i, of incidence there will be in the wave front, $1^1 2^1 3^1 \ldots$, a particular variation of phase from point to point on account of the different courses

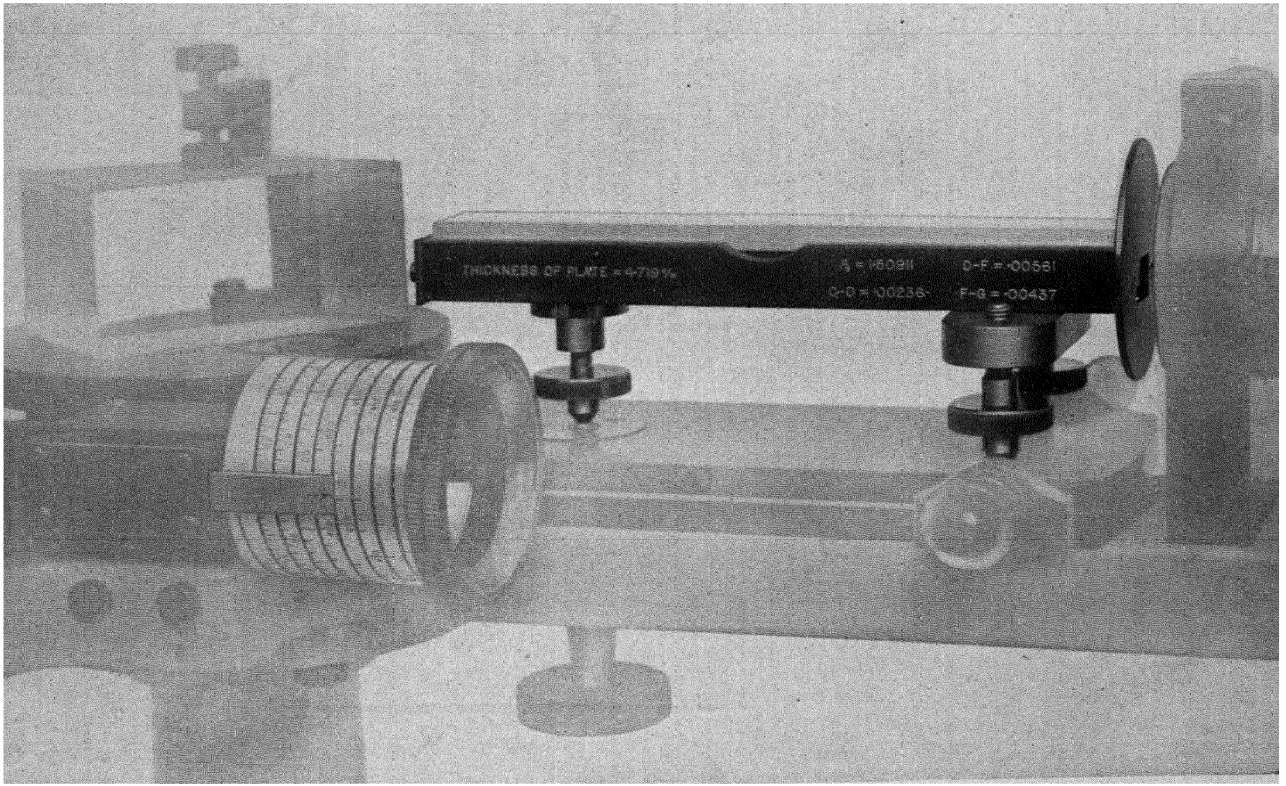

THICKNESS OF PLATE = 4·7191"
μ_3 = 1·60911
D-F = ·00581
C-D = ·00236
F-G = ·00437

FIG. 213

taken by the rays. These rays are received by some optical instrument, for example, the telescope of the spectrometer, and focussed in the focal plane. Corresponding to the waves drawn, we shall have a point image in the focal plane. These rays, however, are those lying in the plane represented by the paper. Above and below them lie rays coming from rays above and below LM_1, and parallel to LM_1, whose courses are exactly similar, so that in the focal plane above and below this point image lies a series of points forming a line. We shall work out the intensity corresponding to this line. If we consider a slightly different direction of incidence, $i + \delta i$, the emergent beam is also slightly different in direction, and in its course through the plate. The phase differences in the corresponding wave front will thus be different, and the corresponding line in the focal plane will have a different intensity.

A wave travelling along a direction denoted by r will at a time t, be represented by :

$$a \sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right).$$

Here T is the period of vibration in the wave, λ the wave-length, and a the amplitude. t is the time measured from some convenient instant, and r the distance from a convenient origin.

In our case the origin will be conveniently the point, N_1, and we shall consider the wave motion corresponding to the position, $1^1 2^1 3^1$, at which the time is measured by t.

On refraction into the plate there is a diminution of intensity, and since the intensity is proportional to the square of the amplitude of the wave motion we can express this by regarding the refraction as causing a diminution of amplitude, so that amplitude a in the air becomes sa in the quartz, where s is a fraction. On passing out at N_1, the amplitude is again reduced by a fraction, s^1. The amplitude for the ray $N_1 1^1$ is thus ss^1a.

We may not suppose that s and s^1 are the same, since in one case the refraction is from air to quartz, and in the other from quartz to air.

There is also a change of intensity on reflection at the points M_2, M_3, M_4, etc., and N_1, N_2, N_3, etc. We shall suppose this to diminish the amplitude each time by a fraction, σ.

Thus the ray, $N_2 2^1$, has undergone refraction at M_1 and N_2, and reflection at N_1 and M_2. The incident amplitude, a, is thus reduced to $\sigma^2 ss^1 a$; and similarly the amplitude of $N_3 3^1$ is $\sigma^4 ss^1 a$.

The disturbance which gives rise to the ray, $N_1 1^1$, starts out

from N_1 with an amplitude, ss^1a, and at 1^1, since it has travelled a distance, r_1, we may represent the displacement by :

$$ss^1a \sin 2\pi\left(\frac{t}{T} - \frac{r_1}{\lambda}\right).$$

Denote the foot of the perpendicular from N_2 on $N_1 1^1$ by A, and from N_3 on $N_2 2^1$ by B, and denote the equal distances, N_1A, N_2B, by ε. Let the distance, M_1N_1 be denoted by δ, and the refractive index of the quartz by μ. Then the distance traversed in quartz is equivalent to the distance, $\mu\delta$ in air.

Thus the point 2^1 is at an equivalent distance $(r_2 + 2\mu\delta)$ from N_1, and since $r_2 = r_1 - \varepsilon$ we may denote the displacement at 2^1 by :

$$\sigma^2 ss^1a \sin 2\pi\left(\frac{t}{T} - \frac{r_1 - \varepsilon + 2\mu\delta}{\lambda}\right);$$

and similarly the displacement at 3^1 is :

$$\sigma^4 ss^1a \sin 2\pi\left(\frac{t}{T} - \frac{r_1 - 2\varepsilon + 4\mu\delta}{\lambda}\right).$$

There is, of course, an indefinitely large number of such terms as the three given above, and since all these rays are focussed at a point in the focal plane of the telescope, the total effect at this point is obtained by adding together all the terms.

In practice we shall have a small parallel bundle of rays falling at M_1, giving rise to small parallel bundles at N_1, N_2, N_3, etc. Throughout the bundles there is, however, the same phase, as may be seen by considering a ray parallel to LM_1, falling at any angle, i, to the normal. Thus the total effect is merely multiplied by some constant factor on account of the incidence of more light than that we have supposed is represented by the ray LM_1.

Thus we have a total displacement, Y, where :

$$Y = ss^1a \sin 2\pi\left(\frac{t}{T} - \frac{r_1}{\lambda}\right) + ss^1a\sigma^2 \sin 2\pi\left(\frac{t}{T} - \frac{r_1 - \varepsilon + 2\mu\delta}{\lambda}\right)$$
$$+ ss^1a\sigma^4 \sin 2\pi\left(\frac{t}{T} - \frac{r_1 - 2\varepsilon + 4\mu\delta}{\lambda}\right) + \ldots$$

A typical term may be written :

$$ss^1a\sigma^{2p} \sin 2\pi\left\{\left(\frac{t}{T} - \frac{r_1}{\lambda}\right) + p \cdot \frac{\varepsilon - 2\mu\delta}{\lambda}\right\},$$

where p has the values, 0 to infinity,

and if we write $\alpha = 2\pi\left(\frac{t}{T} - \frac{r_1}{\lambda}\right)$, $\beta = \frac{2\pi}{\lambda}(2\mu\delta - \varepsilon)$,

we have :

$$Y = \Sigma ss^1a\{\sigma^{2p} \sin(\alpha - p\beta)\} \quad \ldots\ldots\ldots\ldots(7)$$

The sum of this series is finite since σ is a fraction, and has the value

$$\frac{\sin\alpha - \sigma^2 \sin(\alpha + \beta)}{1 - 2\sigma^2 \cos\beta + \sigma^4}.$$

For the proof of the summation the reader may be referred to Hobson's " Plane Trigonometry," §76, p. 91.

Thus the resultant displacement is :

$$Y = ss^1 a \cdot \frac{\sin\alpha - \sigma^2 \sin(\alpha + \beta)}{1 - 2\sigma^2 \cos\beta + \sigma^4}.$$

It will be noted that for any particular direction through the plate the quantities, ε and δ, are constant, so that β is a constant.

s, s^1 and a are also constant terms so that it is only α that contains the variable time, t.

$$\sin\alpha - \sigma^2 \sin(\alpha + \beta) = \sin\alpha(1 - \sigma^2 \cos\beta) - \cos\alpha \cdot \sigma^2 \sin\beta.$$

We may write this

$$= A \sin(\alpha - \phi)$$

where $$\tan\phi = \frac{\sigma^2 \sin\beta}{1 - \sigma^2 \cos\beta},$$

and $$A = \sqrt{1 - 2\sigma^2 \cos\beta + \sigma^4},$$

Thus $$Y = \frac{ss^1 a}{\sqrt{1 - 2\sigma^2 \cos\beta + \sigma^4}} \cdot \sin(\alpha - \phi)$$

$$= \frac{ss^1 a}{\sqrt{1 - 2\sigma^2 \cos\beta + \sigma^4}} \cdot \sin\left(\frac{2\pi t}{T} - \phi - \frac{2\pi r_1}{\lambda}\right).$$

This denotes a simple harmonic vibration of amplitude,

$$\frac{ss^1 a}{\sqrt{1 - 2\sigma^2 \cos\beta + \sigma^4}},$$

and it measures the amplitude on the line in the field of view of the telescope due to the reception of the emergent beam along the direction which makes an angle, i, with the normal to the plate. The intensity is proportional to the square of this amplitude, and is therefore a maximum when $(1 - 2\sigma^2 \cos\beta + \sigma^4)$ is a minimum, and a minimum when this expression is a maximum.

But $$1 - 2\sigma^2 \cos\beta + \sigma^4 = (1 - \sigma^2)^2 + 4\sigma^2 \sin^2 \tfrac{1}{2}\beta.$$

This has a minimum value when $\sin \frac{1}{2}\beta$ has the value zero, and a maximum when $\sin \frac{1}{2}\beta$ has the value unity.

Thus lines of maximum intensity correspond to the value :

$$\beta = 0,\ 2\pi,\ 4\pi, \text{ etc.},$$

and lines of minimum intensity correspond to :

$$\beta = \pi,\ 3\pi,\ 5\pi, \text{ etc.}$$

But $$\beta = \frac{2\pi}{\lambda}(2\mu\delta - \varepsilon).$$

Thus for a maximum,

$$2\mu\delta - \varepsilon = n\lambda,$$

and for a minimum,

$$2\mu\delta - \varepsilon = (n + \tfrac{1}{2})\lambda,$$

and n may have any integral value.

If d denote the thickness of the plate :

$$\delta = d \sec r,$$

and

$$\varepsilon = M_1M_2 \sin i$$
$$= 2d \tan r \sin i = 2d \tan r \cdot \mu \sin r.$$

Thus $2\mu\delta - \varepsilon = 2\mu d \cos r,$

and the bright bands lie in directions given by :

$$2\mu d \cos r = n\lambda.$$

By proceeding further with this discussion, Lummer and Gehrcke have drawn the conclusions, to which we have referred above, concerning the intensity of the bands.

Students who prefer it may substitute the following proof which enables the same formula to be derived shortly, but in a way that does not lead to any expression for the intensity of the lines.

The path difference between any two of the rays, say N_1I^1 and $N_2 2^1$ is equal to :

$$2\mu\delta - \varepsilon = 2\mu d \cos r.$$

We may take all the rays emerging from the plate in pairs which are separated by the same distance, equal to N_1N_2.

These therefore interfere and will produce darkness or brightness, according as this phase difference is equal to $n\lambda$ or $(n+\frac{1}{2})\lambda$, where n is a whole number as before.

Thus for bright lines :

$$2\mu d \cos r = n\lambda.$$

If we refer to fig. 213 and note the position of the plate it is clear that the different orders come out from the plate above one another. As the path difference increases the angle of emergence decreases, so that the higher orders will lie higher in the field of view than the lower in the case of a telescope with an erecting eyepiece.

In our case the mean angle, r, is the critical angle and there is grazing incidence. The different orders correspond to angles very slightly differing from grazing incidence.

By substituting the value, $\mu = 1{\cdot}544$, and the value for the critical angle, $r = 40°22'$, together with the value of d given above, we find that for $\lambda = 5890$ tenth metres (10^{-10} metre), the order is approximately 18000.

In order to set the instrument in position, the spectrometer

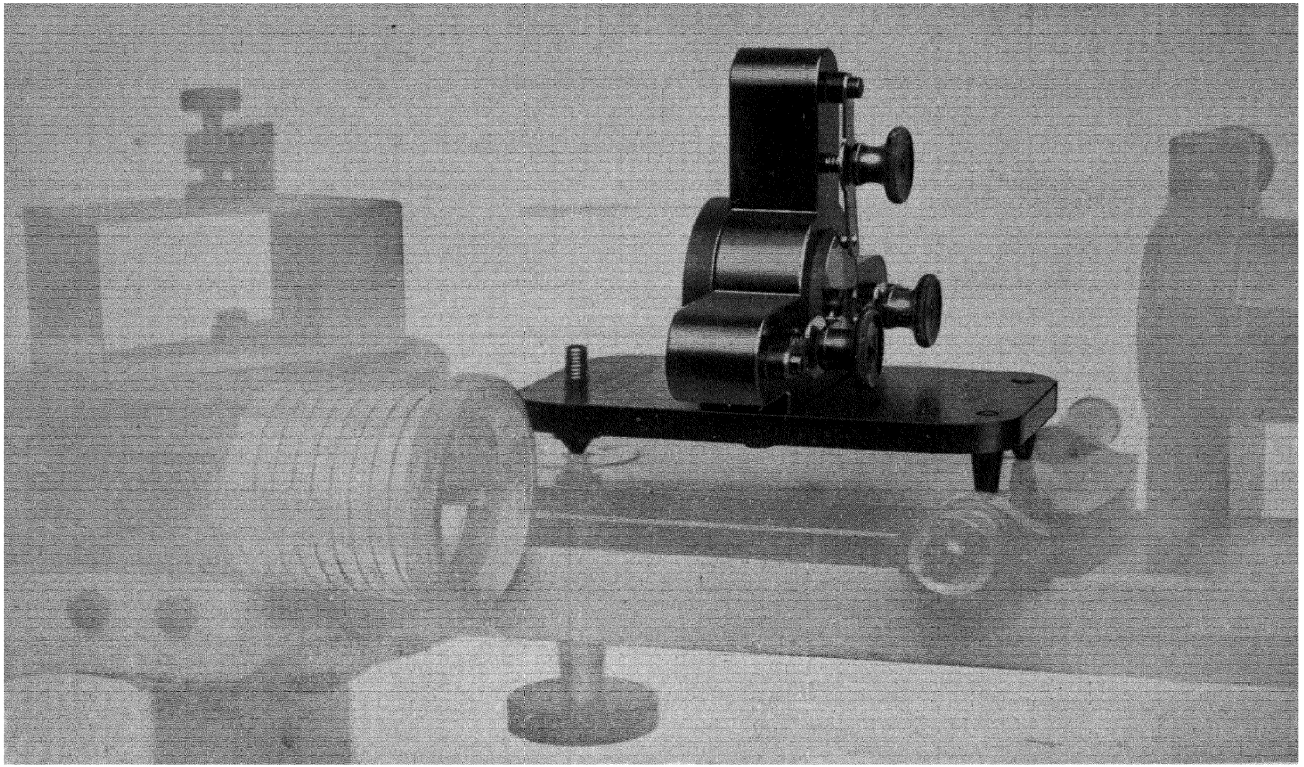


FIG. 215

is first set up in the manner previously described, and the slit illuminated by means of some convenient monochromatic light. The plate is placed in position and adjusted by the screws until the brightest image is obtained in the eyepiece. Since the orders lie one above another, a vertical slit cannot be used, for the different orders will appear overlapping the image of the slit. Thus the crossed slits must be used with a small rectangular source.

The images may then be viewed or photographed as desired. We have throughout disregarded the possibility of a change of phase on reflection at the two surfaces.

The Fabry-Perot Etalon. (*"Annales de Chimie et de Physique,"* 1897, *Ser.* 7, *p.* 459.)

This apparatus is another means of obtaining a high resolving power, and is in many respects similar to the Lummer-Gehrcke Plate. Fig. 215 shows the apparatus in position on the arm of a constant deviation spectrometer.

Fig. 216 shows the two plates, ABCD and FGHE, of which it is composed. These are placed accurately parallel, and are separated by a distance piece consisting of a hollow cylinder of fused silica. This substance has an extremely small coefficient of expansion, so that the distance between the plates may be regarded as independent of the temperature.

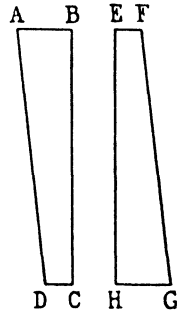


FIG. 216

The faces, BC and EH, are silvered by cathodic deposition in order to increase their reflecting power but so as to leave them partly transparent.

Light entering the plates therefore undergoes multiple reflections between the silvered faces and produces also partial transmission through the opposite face.

The faces, AD and FG, are inclined to CB and EH in order to avoid interference effects that would occur through multiple reflections and refractions if all were parallel, but AD and FG

are parallel, so that light incident on one side leaves the other undeviated.

Fig. 217 illustrates the action of the plate. The air space between the plates is shown, and the course of a ray as it is multiply reflected between the plates. The refracted portion leaving EH is shown, but the refraction at the other face is omitted for convenience. We consider an incident ray, LM_1, from the instant of its arrival at a point, M_1, within the first wedge and immediately before refraction into the air space.

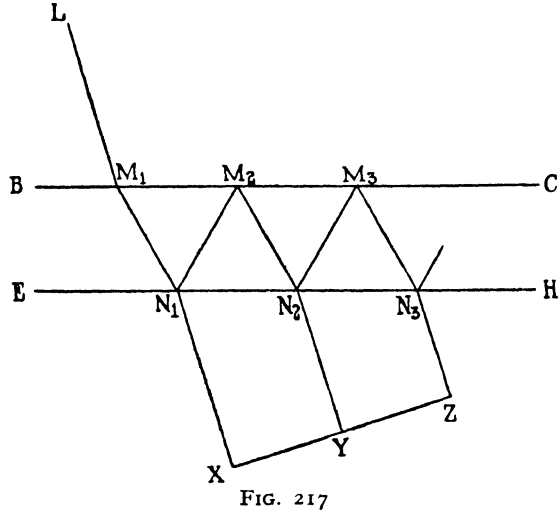


FIG. 217

The wave emerging within the second plate is shown, and XY denotes a wave front.

The wave is refracted out at the surface, FG, into the air, but as no further change of phase occurs after the light leaves the air film, we do not need to consider the wave beyond XYZ.

Let the wave on starting out from the point, M_1, have, in the air space, an amplitude, a, and let its period and wave length be T and λ, respectively. Whenever the wave passes from the air space into the second plate, let the amplitude be reduced in the ratio, θ, and on reflection at the partially silvered faces let the amplitude be reduced in the ratio, f. Both these quantities are positive proper fractions.

Thus the wave proceeding along M_1N_1 may be denoted by:

$$y = a \sin 2\pi \left(\frac{t}{T} - \frac{r}{\lambda}\right),$$

where r measures the distance from the point, M_1, at which the displacement, y, is considered.

Let the equivalent path difference between M_1 and the points, X, Y, and Z, differ successively by an amount, δ.

Then proceeding as in the theory of Lummer-Gehrcke Plate, we have :

$$\delta = 2e \cos i,$$

where e is the distance between the plates, and i the angle of incidence at the surface, EH.

Thus the displacement at X may be written :

$$\theta a \sin 2\pi \left(\frac{t}{T} - \frac{r_1}{\lambda}\right),$$

where r_1 denotes the equivalent air path between M_1 and X.

The displacement at Y, which after two reflections has been reduced by the ratio, f^2, and for which the equivalent path is $(r_1 + \delta)$, is given by :

$$\theta f^2 a \sin 2\pi \left(\frac{t}{T} - \frac{r_1 + \delta}{\lambda}\right)$$

and similarly at Z we have :

$$\theta f^4 a \, 2\pi \sin \left(\frac{t}{T} - \frac{r_1 + 2\delta}{\lambda}\right)$$

Of course there are many more points such as X, Y, Z ; and if we continue the series of terms until the contributions become negligible, and if we view the rays in a telescope, we have, for the total displacement in the field of view of the telescope, a quantity :

$$Y = \theta a \left\{\sin 2\pi \left(\frac{t}{T} - \frac{r_1}{\lambda}\right) + f^2 \sin 2\pi \left(\frac{t}{T} - \frac{r_1}{\lambda} - \Delta\right) + f^4 \sin 2\pi \left(\frac{t}{T} - \frac{r_1}{\lambda} - 2\Delta\right) + \ldots\right\}, \quad \ldots\ldots\ldots\ldots\ldots (8)$$

where Δ is written for $\frac{\delta}{\lambda}$.

Strictly, Δ ought to include a quantity due to change of phase on reflection at the air-glass surfaces ; if this is w then $\Delta = \frac{\delta}{\lambda} + w$.

In order to compare this with equation (7) above, we may note that the quantity, α, is the same for each, that θa takes the place of $ss^1 a$, and that f takes the place of σ, while $2\pi\Delta$ corresponds to β.

If, therefore, we write $\beta = 2\pi\Delta$, equation (8) may be written :

$$Y = \Sigma \, \theta a f^{2p} \sin (\alpha - p\beta), \quad \ldots\ldots\ldots\ldots (9)$$

where p has the values, 0 to infinity.

Thus as before, the value of Y is :

$$\frac{\theta a}{\sqrt{1 - 2f^2 \cos \beta + f^4}} \sin \left\{2\pi \left(\frac{t}{T} - \frac{r_1}{\lambda}\right) - \phi\right\} \ldots\ldots (10)$$

where :

$$\tan\phi = \frac{f^2 \sin\beta}{1 - f^2\cos\beta} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Thus the intensity of the line appearing in the focal plane of the telescope, corresponding to this particular direction, is proportional to the square of the amplitude of this expression, or we may measure the energy by the expression :

$$\frac{\theta^2 a^2}{(1 - 2f^2\cos\beta + f^4)}.$$

This may be written :

$$\frac{\dfrac{\theta^2 a^2}{(1-f^2)^2}}{1 + \dfrac{4f^2}{(1-f^2)^2}\sin^2\pi\Delta}.$$

Thus the intensity fluctuates between a maximum and minimum value, the former of magnitude :

$$\frac{\theta^2 a^2}{(1-f^2)^2}.$$

This may be large provided that the value of f is not far from unity.

Denote this by I_0.

Then we have for the intensity :

$$I = \frac{I_0}{1 + \dfrac{4f^2}{(1-f^2)^2}\sin^2\pi\Delta}.$$

The minimum of I corresponds to the value for Δ which makes

$$\sin^2\pi\Delta = 1,$$

and the magnitude is consequently :

$$\left(\frac{1-f^2}{1+f^2}\right)^2 I_0 \quad \ldots\ldots\ldots\ldots\ldots (A)$$

Now with f not very different from unity, this may be a very small fraction of I_0, so that there is a great contrast between the maxima and the minima.

Another point may most conveniently be brought out by a numerical example.

Suppose $f = \cdot 87 = \left(\frac{\sqrt{3}}{2}\right)$ — this value has been chosen for convenience.

Then

$$I = \frac{I_0}{1 + 48\sin^2\pi\Delta}.$$

A maximum occurs when $\Delta = m$, and the next when $\Delta = (m + 1)$, m denoting an integer.

Consider the case when $\Delta = m + \frac{1}{10}$,

i.e. when we have gone over of $\frac{1}{10}$ the interval between the two values of Δ.

In that case it follows that :

$$I = \frac{I_0}{1 + 48 \sin^2 \frac{\pi}{10}}$$

$$= \tfrac{1}{6} I_0, \text{ approximately.}$$

This means that the intensity falls off quickly as the maximum position is left, so that we have bright lines in the field of view separated by a comparatively long dark interval.

Thus, if there is a second ray in the field of slightly different wave-length, its lines will not overlap those of the other ray unless there is very little difference indeed between the two wave-lengths.

From the above theory we see the influence of the partial silvering in producing sharp bright lines on a background that is almost black. If a layer of air is used between two plates without any silvering, the reflecting power is small, and we have fringes produced and superposed on a field of uniform illumination with no comparative broad spaces between the fringes.

We can explain these points quantitatively by the aid of some numerical examples given in the original paper.

It is usual to speak of the reflecting power of a surface and not of the quantity we have denoted by f. But since the intensity of a ray is proportional to the square of the amplitude, and since the amplitude on reflection is reduced by f, we have, if the reflecting power is denoted by R,

$$R = f^2,$$

so that the expression denoted by (A) above gives for the ratio of the minimum to the maximum intensity :

$$\rho = \left(\frac{1 - R}{1 + R}\right)^2.$$

If

$$R = \cdot 042$$

$$\rho = \cdot 84,$$

so that for a small reflecting power the maxima and minima have nearly equal intensity.

If, however,

$$R = \cdot 74,$$

$$\rho = \cdot 02,$$

and the minima are very feeble.

The way the intensity falls off rapidly as the maximum is left has already been shown. The value for f taken above, corresponds to a value of R = ·75, approximately.

Fig. 218 shows graphically the difference in the two intensity curves corresponding to displacement in the field of view. The continuous line is the curve for the Fabry-Perot silvered plate, while the dotted curve is for plates of low reflecting power.

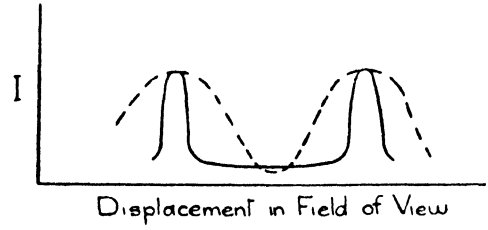


FIG. 218

Note in the one case the broad intervals of practically no intensity and contrast with the other in which the intensity falls off slowly, leaving comparatively small dark intervals.

Lines due to a second ray of slightly different wave length from that for which the second curve is drawn would produce crests over the dark intervals and leave the field nearly uniformly bright, and it would not be possible to distinguish the separate fringes.

The determination of the Ratio, $\frac{e}{m}$, for an Electron by means of the Zeeman Effect

When a monochromatic source of light is placed in a magnetic field, and rays are received in a direction parallel to the direction of the field, it is found that the normal frequency is changed, and two or more lines appear symmetrically displaced from the usual position in each direction.

The light in each case is circularly polarized; the component of higher frequency is polarized in the opposite direction of rotation to that for the lower.

Reference for the theory of this phenomenon should be made to "The Electron Theory of Matter" (O. W. Richardson).

If H denotes the intensity of the magnetic field, and its direction is parallel to that of the light, the increase and decrease in frequency of the two components is of magnitude $\frac{1}{2\pi} \cdot \frac{eH}{2m}$, and H is measured in 'gausses,' the name given to the electromagnetic unit of magnetic field.

In the experiment we measure the difference in wave length

of the two components, i.e. we measure $\delta\lambda$, corresponding to a change of frequency,

$$\delta\nu = \frac{1}{2\pi}\frac{e}{m} \cdot \mathrm{H}.$$

Now $$\delta\nu = \frac{c}{\lambda^2}\,\delta\lambda, \text{ (numerically)},$$

since $$c = \nu\lambda,$$

so that we measure $$\delta\lambda = \frac{1}{2\pi}\frac{\lambda^2}{c} \cdot \mathrm{H} \cdot \frac{e}{m}.$$

Hence : $$\frac{e}{m} = 2\pi \cdot \frac{c}{\mathrm{H}} \cdot \frac{\delta\lambda}{\lambda^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

e is measured in electromagnetic units.

An electromagnet with adjustable pole pieces and taking a current of three amperes gives good results with an ordinary vacuum tube.

A hole is drilled in the pole pieces so that the source may be observed along the direction of the magnetic field.

The field is measured by means of a fluxmeter, by which the total flux through an exploring coil placed between the pole pieces is measured. This will be read directly on the fluxmeter. Suppose it is B, Maxwell's. If A is the effective area of the coil and H the field, on placing the coil into the position, where H is required, the flux is HA, so that $\mathrm{H} = \frac{\mathrm{B}}{\mathrm{A}}$, and H is in gausses

(see experiment on Grassot Fluxmeter, p. 482.)

We have now to determine $\delta\lambda$ by means of the Lummer plate. First align the apparatus and drilled hole as nearly as possible. Place a sodium flame on the far side of the hole, and obtain the shorter sodium lines as sharply as possible with the drum set at 5890.

Place a helium tube or other convenient source in position between the pole pieces, and obtain the yellow line. The yellow line was used in an experiment and satisfactory photographs were obtained, but it would be preferable to use the blue line so that the exposure is not long, and ordinary photographic plates (special rapid) may be used. We shall, in what follows, consider the yellow line to have been employed, and the wave length is then 5875·6 Å.

Adjustments will be carried out as described above, and the best possible definition and illumination obtained.

On applying the magnetic field it will be observed that the lines broaden, and in some cases actual separation will occur, as in fig. 219.

The state of polarization may be tested by intercepting the light just before it enters the object glass of the telescope by a quarter-wave plate.

The plate must, of course, be suitable for the particular wavelength concerned—the usual plate met with in laboratories will be suitable to examine the yellow light. This part of the experiment is introduced so that the student may take the opportunity of verifying the character of the vibrations in the lines.

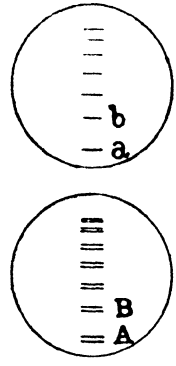


FIG. 219

The quarter-wave plate reduces the circular vibrations to two linear vibrations at right angles to one another, and on examining these with a Nicol it will be found that when one line is cut out the other is present in the field.

After carefully adjusting the apparatus so that the effect is seen by eye, attach the camera and photograph the spectrum without the magnetic field. By means of the rack and pinion raise the plate, put on the field and take a second photograph below the first.

The plates when developed will give lines as in fig. 219.

Separation of the lines is seen in the higher orders, e.g. at A and B. Let *a* denote the line when no field is applied to the source, and A the corresponding line with the field applied Measure the mean displacement of the components of A by means of a microscope, and measure the separation between the two successive orders *a* and *b*.

Denote these distances by *l* and L respectively.

We proceed to work out a formula, showing how $\delta\lambda$ may be derived from the ratio $\frac{l}{L}$.

By the equation for the Lummer-Gehrcke plate:

$$2\mu d \cos r = n\lambda, \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

we see that in proceeding to a neighbouring order $(n + 1)$, the angles of emergence and refraction are $(i + \delta I)$ and $(r + \delta R)$,

where : $2d\mu \cos (r + \delta R) = (n + 1)\lambda$ .....(14)

From (13) and (14) we have by subtraction :

$$- 2d \tan r \cos i \delta I = \lambda \quad \text{.....(15)}$$

In this step we use the relation : $\sin (i + \delta I) = \mu \sin (r + \delta R)$, and consequently $\cos i \delta I = \mu \cos r \delta R$ .....(16)

There is no variation in μ since the wave length remains constant ; we pass merely to a new angle of emergence.

δI represents the angle between the rays which produce two consecutive lines as *a* and *b* on the photographic plate (fig. 219).

Now consider the difference in direction, δi, for two lines in order, n, of wave lengths, λ and $\lambda + \delta\lambda$.

It is an angle, δi, which corresponds to the displacement between the components of A.

Since the plate is fixed in the spectrometer and the angles are small :

$$\frac{\delta i}{\delta I} = \frac{l}{L} \quad \text{.....(17)}$$

Referring once more to equation (13) we have in the order, n, a wave length, λ, corresponding to a direction measured by i, and a wave length, $\lambda + \delta\lambda$, corresponding to $i + \delta i$.

By differentiating (13) we find :

$$- 2d\mu \sin r \delta r = n\delta\lambda - 2d \cos r \frac{d\mu}{d\lambda} \delta\lambda \quad \text{.....(18)}$$

and from the equation,

$$\sin i = \mu \sin r,$$

$$\mu \cos r \delta r = \cos i \delta i - \sin r \frac{d\mu}{d\lambda} \delta\lambda \quad \text{.....(19)}$$

In this case it is necessary to take account of the variation of μ since the wave length changes.

By eliminating δr from (18) and (19) it is found that :

$$- 2d \tan r \cos i \ \delta i = \left(n - \frac{2d}{\cos r} \cdot \frac{d\mu}{d\lambda}\right) \delta\lambda ; \quad \text{.....(20)}$$

and on substituting for n from (13) :

$$- 2d \tan r \cos i \ \delta i = 2\left(\frac{\mu}{\lambda} \cos r - \frac{1}{\cos r} \cdot \frac{d\mu}{d\lambda}\right) \delta\lambda \quad \text{....(21)}$$

Thus from (15), (17), and (21) :

$$\frac{l}{L} = \frac{\delta i}{\delta I} = \frac{2d}{\lambda}\left(\frac{\mu}{\lambda} \cos r - \frac{1}{\cos r} \cdot \frac{d\mu}{d\lambda}\right) \delta\lambda \quad \text{.....(22)}$$

The value of r is the critical value for quartz, and it remains to determine $\frac{d\mu}{d\lambda}$ for the particular wave length used in the

experiment. The table below showing values of μ corresponding to different values of λ should be used to draw a graph from which $\frac{d\mu}{d\lambda}$ may be obtained by measuring the slope at any point where it is required.

From the equations (12) and (22) we find:

$$\frac{e}{m} = \frac{2\pi c}{H} \cdot \frac{l}{L} \cdot \frac{1}{2d\left(\mu \cos r - \frac{\lambda}{\cos r} \cdot \frac{d\mu}{d\lambda}\right)} \quad \ldots\ldots\ldots(23)$$

and the units are electromagnetic.

If possible take the ratio $\frac{l}{L}$ for more than one order—several orders will usually show sufficient separation for this purpose.

REFRACTIVE INDEX OF QUARTZ IN THE VISIBLE SPECTRUM AT 10°.

WAVE LENGTHS (IN $\mu\mu$)	REFRACTIVE INDEX.
396	1·5581
410	1·5565
434	1·5540
486	1·5497
508	1·5482
533	1·5468
589	1·5442
643	1·5423
686	1·5410
760	1·5392
768	1·5390

Measurement of Wave Length by Diffraction at a Straight Edge

The theory of this experiment will be found in Schuster's "Theory of Optics" in Chapter V.

In the first few sections of the chapter it is shown that on observing a straight edge illuminated by monochromatic light from a narrow slit a series of alternate bright and dark bands will be seen.

If S denotes the slit (fig. 220), EF the straight edge fixed parallel to it, and PQ a plane perpendicular to SE, we shall have the series of bands along PQ. Let Q be the nth bright band from P, where P is the point of intersection of SE and the plane.

Denote PQ by x_n, then it can be shown that :

$$x_n = \tfrac{1}{2}\sqrt{\frac{p\lambda(4n-1)(p+q)}{q}} \quad \ldots\ldots\ldots\ldots (24)$$

In this formula p is the distance, EP, and q is equal to SE, the distance between source and slit.

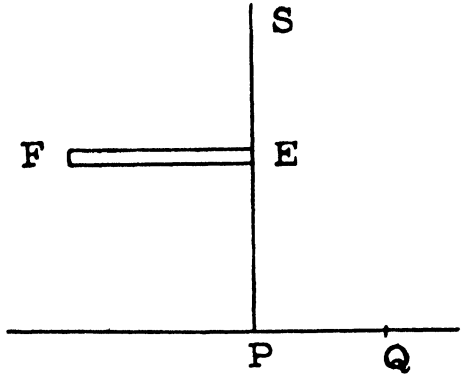


FIG. 220

The bands are not regularly spaced as are interference fringes.

To carry out the measurement set up in one of the stands of the optical bench a straight edge parallel to the slit which is illuminated by sodium light. First adjust the parallelism by eye, and finally make a slight rotation by the screw of the holder until the bands are most distinct ; then make the slit as narrow as is compatible with sufficient illumination. Observe the fringes with the travelling microscope, and measure the distance between the first or second bright band and one of the most distant that can be seen plainly.

If this is the nth and the first observed is the mth we have :

$$x_n - x_m = \tfrac{1}{2}\sqrt{\frac{p\lambda(p+q)}{q}}\left\{\sqrt{4n-1} - \sqrt{4m-1}\right\} \ldots . (25)$$

The student should take the opportunity of observing the diffraction fringes arising from the light passing a narrow wire, needle point and narrow slit.

In the case of the narrow wire diffraction bands unequally spaced will be seen outside the geometrical shadow. Within it a series of equally spaced bands will be observed. These may be described as interference fringes due to the two parts of the wave, one on either side of the wire. The effects of these are equal to those of the two half period zones which lie at the edges of the wire so that they act like two sources at a small distance apart.

Determination of the Wave Length of Light by means of a Plane Diffraction Grating

A diffraction grating is made by ruling a large number of equidistant parallel straight lines on glass. The lines are ruled by a diamond point moved by an automatic dividing engine containing a very fine micrometer screw which moves sideways between each stroke. A photographic replica of a plate made in this way is often used in its place.

The number of gratings to the inch is marked on the glass.

In handling the grating do not touch the faces of the glass, hold it between thumb and finger by the edges.

Adjust the collimator and telescope for parallel rays in the usual way and observe the direct image of the slit noting how it lies in the field of view.

Set up the grating with its face normal to one side, EF, of the triangle formed by the levelling screws (cf. fig. 161). Throw an image of the slit into the telescope by reflection from one face of the grating and adjust the screws to bring it into the same part of the field as that occupied by the direct image. This makes the faces vertical.

In order to set the grating at right angles to the rays adjust the collimator and telescope at right angles to each other, and turn the table until the slit is reflected on the cross-wires of the telescope. Then turn the table a further 45°.

It now remains to tilt the grating so that the lines are parallel to the slit.

View the first diffracted image, making the slit as narrow as is convenient, and adjust the screw D until the best image is obtained. The lines and slit are then parallel.

Find the diffracted images on both sides of the line of direct vision. It will be easy to observe two orders, and if a bright light is used and an image of the brightest part of the flame thrown on the slit by a short focus lens the third may be seen also.

If $(a + b)$ is the width of the grating element the formula is:

$$(a + b) \sin \theta = n\lambda,$$

for normal incidence, θ being the angle of diffraction, λ the wave length of the incident light, and n the order of the spectrum.

Obtain θ by taking half the angular distance between the corresponding images on each side.

$(a + b)$ is deduced from the number of rulings on the grating.

Use any source of monochromatic light or light giving well-marked lines as that from a discharge tube containing hydrogen which gives three well-marked lines, red, green, and violet, known as C, F and H_γ.

The above formula is obtained by considering rays in pairs passing through adjacent clear spaces of the grating

For example, consider the two spaces AB and CD.

We may consider the rays passing through them in pairs, taking together those rays which are symmetrically situated, as for example, QLQ^1 and TMT^1.

These reach the grating in the same phase, and the line ABCDE represents a section in the plane of the figure of the wave front incident on the grating.

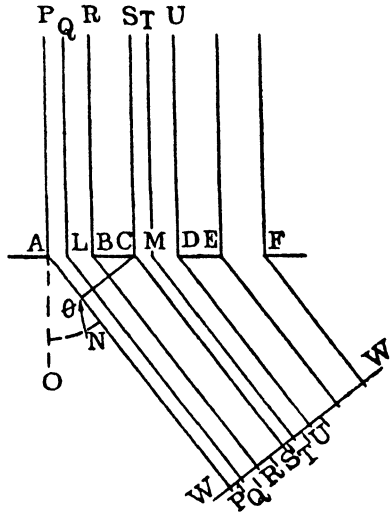


FIG. 221

Consider a wave front, WW, at a later interval, and suppose that it makes an angle, θ, with the plane of the grating.

Draw the rays AP^1, BR^1, etc., perpendicular to this wave front and making the angle, θ, with the normal, AO, to the grating. These rays are all received by a telescope and united in the focal plane. Their paths from the grating to the instrument differ, so that they reach the focal plane with different phases. This phase difference is due to the difference of path traversed after leaving the grating.

Take the case of the two rays, AP^1 and CS^1. Their path difference is AN, where N is the foot of the perpendicular from C on AP^1. Any other pair of rays, e.g. LQ^1 and MT^1, situated symmetrically in AB and CD have the same path difference. This holds for all pairs of rays symmetrically situated. Thus the phase difference on arrival in the apparatus that receives them is the same for each pair.

We shall thus have reinforcement if AN is equal to a whole number of wave lengths.

Now $$AN = AC \sin \theta.$$

If a is the width of a space and b that of a line,

$$AC = a + b,$$

$(a + b)$ is called a grating element. Thus a bright line occurs in a direction θ, provided that

$$(a + b) \sin \theta = n\lambda.$$

The Plane Reflection Grating

This reflection grating consists of a plain polished sheet of metal across which parallel lines are drawn very closely together as in the transmission grating.

It is necessary to mount the polished face vertically and this is done as in the previous case.

The light does not fall from the collimator normally on the grating, but at a measured angle, i.

In order to measure i, the grating is first set with the polished face normal to the rays from the collimator which, together with the telescope, has been adjusted for parallel rays.

The table carrying the grating is turned through a definite angle, say 10°, and is then fixed.

Suppose rays from any direction make θ with the normal to the grating and are received in the telescope.

Consider any three rays, MA, NB, and OC (fig. 222), incident on the grating element ABC. AB is polished while BC is the position of the line where the polish is scratched and where the incident rays are absorbed.

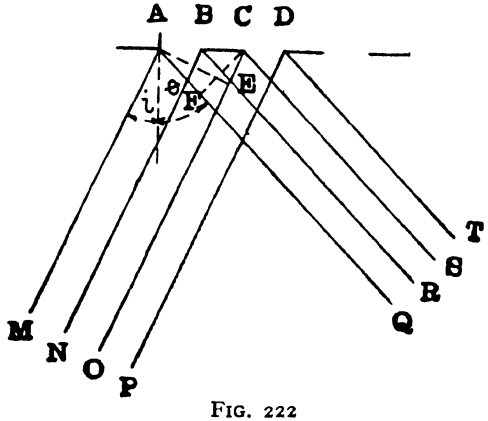


FIG. 222

The path difference between the extreme rays MAQ and OCS is AF−CE, where the dotted lines, AE and CF, denote incident and reflected wave fronts respectively. If AB = a, BC = b, this difference is equal to

$$(a + b)(\sin \theta - \sin i).$$

It is possible to divide up the bundle of rays falling on AB and CD into pairs, one from each bundle, having this same

difference of path. In order to do this it is only necessary to choose rays occupying the same relative positions in the two bundles.

If this path difference is a whole number of wave lengths, and the emergent parallel rays AQ, BR, CS, etc., are brought to a focus by a telescope, we shall get a bright image due to reinforcement of the rays. We thus have :

$$(a + b)(\sin\theta - \sin i) = n\lambda,$$

where n may have the values 1, 2, 3, etc. The corresponding values of θ, say, θ_1, θ_2, θ_3, etc., give the directions in which the different orders of diffracted images are seen.

If the observation is made from a direction on the same side of the normal as i, the formula is :

$$(a + b)(\sin i + \sin\theta) = n\lambda.$$

Obtain first the reading of the scale marking the position of the telescope when it is directed towards the slit with no grating intervening. As above, set the normal to the grating at some definite angle, i, to the incident rays. Move the telescope round to receive light in a direction corresponding to $\theta = i$ on the opposite side of the normal to i. This corresponds to order zero —it is the position where the ordinary reflected beam is received. Now move round to the next position where a distinct image can be found. This will correspond to $n = 1$. Carry out this process as long as it is possible to observe images at all. The higher orders get fainter but resolve the lines more than the lower.

The number of lines to the inch is given for the particular grating, so that it is possible to deduce $(a + b)$, which must be expressed in centimetres. From the values of i, θ and n, it is then possible to evaluate λ. It should be possible to obtain separation of the sodium lines in the second order, and the wave length for each constituent should be calculated.

Use three or four different values of i and compare the value of λ obtained, finally take the mean of the series.

Resolving Power of a Telescope

Theory

Let a parallel beam of monochromatic light fall on a slit, AB, and let us examine the intensity of the light along a direction CE, inclined at an angle, θ, to the normal to the slit. Let the light in this direction be brought to a focus at E. From A draw ADF perpendicular to the beam so that $\angle$ BAF is θ. The rays issuing from the slit on arrival at E will have different phases on account of the varying length of their paths from AB to E. We may measure these differences by such lengths as PM, varying in amount from zero at A to a maximum of BF for the extreme ray from B.

Divide the slit, AB, into a very large number of equal parts, such as PQ. If we make each of these small enough, all the rays in the element of which PQ is a section, may be supposed to be in the same phase on arrival at E. Suppose that the rays from each small element differ in phase by an amount, δ, from the rays of the adjacent element. Each element will be supposed to contribute an amplitude, R, to the whole beam so that if there were n slits, and all had the same phase on reaching E, the amplitude would be nR.

Let the disturbance at A be denoted by R sin pt.

This is an expression which represents the displacement in any simple harmonic vibration, and rays of light afford an example

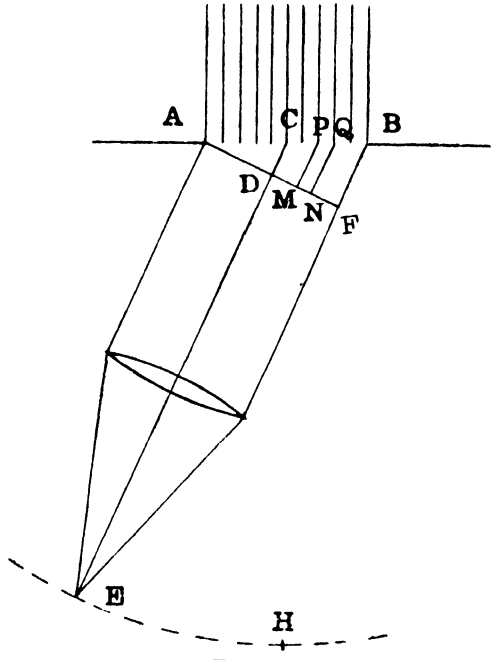


FIG. 223

of this type of vibration. The rays from the next element, differing in phase by δ must be represented by R sin $(pt + \delta)$, from the next by R sin $(pt + 2\delta)$, and so on for all the elements of number n.

Thus the total effect at E is obtained by adding up these separate elements.

Let T measure this effect so that :

$$T = A\{\sin pt + \sin(pt + \delta) + \ldots + \sin(pt + \overline{n-1}\,\delta)\}.$$

The maximum phase difference is measured by $2\pi\frac{BF}{\lambda}$ radians.

Let φ denote this phase difference; it is also measured by $n\delta$.

The above series for T can be expressed by the more convenient formula :

$$T = \frac{A \sin\left(pt + \frac{n-1}{2}\delta\right) \sin\frac{n}{2}\delta}{\sin\frac{\delta}{2}},$$

as is shown in textbooks on Trigonometry.

When δ is very small, $\sin\frac{1}{2}\delta = \frac{1}{2}\delta$;

$$\therefore T = \frac{nA \sin\frac{n\delta}{2}}{n\frac{1}{2}\delta} \sin\left(pt + \frac{n\delta}{2} - \frac{1}{2}\delta\right)$$

$$= \frac{nA \sin\frac{1}{2}\varphi}{\frac{1}{2}\varphi} \sin(pt + \tfrac{1}{2}\varphi),$$

in which $\frac{1}{2}\delta$ is neglected because of its smallness.

When φ is zero there is no difference of phase, and the direction CE is normal to AB. In that case the amplitude at E is nA, and we denote this by B. Thus the amplitude in any direction, EF, is $\frac{B \sin\frac{1}{2}\varphi}{\frac{1}{2}\varphi}$, the factor $\sin(pt + \frac{1}{2}\varphi)$ denoting the oscillating character of the disturbance.

The intensity is proportional to the square of the amplitude, or the intensity at E is measured by : $4B^2 \cdot \frac{\sin^2\frac{1}{2}\varphi}{\varphi^2}$.

This is a maximum when $\frac{\sin^2\frac{1}{2}\varphi}{\varphi^2}$ is a maximum.

If this function be examined for its maxima and minima by means of the differential coefficient, it will be found that the maxima occur at points where the values of $\frac{\varphi}{2}$ are :

$$0,\ 1{\cdot}43\pi,\ 2{\cdot}46\pi,\ 3{\cdot}47\pi, \text{ etc.},$$

and at minima

$$\pi,\ 2\pi,\ 3\pi,\ 4\pi, \text{ etc.}$$

At the latter values the intensity is zero.

The graph drawn in fig. 224 shows the relation between intensity and values of $\frac{1}{2}\varphi$.

When the light is examined at H the intensity corresponds to the ordinate at I, and on moving round we come to a place, E, say, where there is darkness corresponding to the zero ordinate at π.

In this case :

$$\tfrac{1}{2}\varphi = \pi = \tfrac{1}{2} \cdot 2\pi \cdot \frac{BF}{\lambda},$$

i.e. $$BF = \lambda.$$

Thus a minimum occurs in the direction in which BF = λ.

Now the angular distance between CE and CH is the same as angle BAF, and since this is small it is measured by $\frac{\lambda}{AB}$ or λ ÷ (breadth of incident beam).

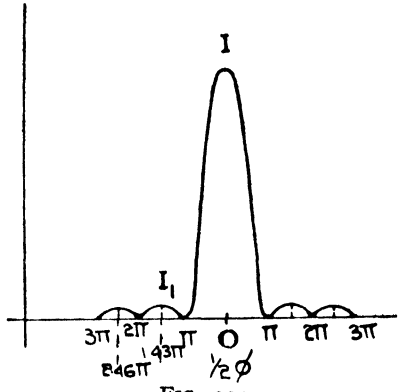


FIG. 224

If another beam were incident on AB in the direction along CE, there would be a maximum for this beam at E, and it would overlap the minimum of the former.

The intensity curves might then be represented on the same diagram as in fig. 225. The curve on the left represents the intensity curve for the second direction while the upper dotted curve shows the two compounded.

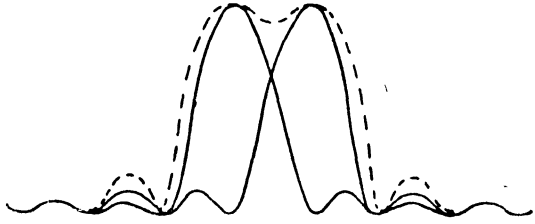


FIG. 225

The resultant has two pronounced maxima with an appreciable dip between. It will thus be possible by the aid of this falling off in intensity between two bright regions to distinguish the two beams or they will be 'resolved.' Moreover, the angle between the two directions is ECH or λ ÷ (width of beam). This is taken as a limiting case, and the resolving power is measured by this ratio. If the angle between the incident beams is less than this the two maxima cease to be distinguishable as two and blend into one.

In order to verify this theory a telescope is fitted with an adjustable slit which is placed as close as possible to the object

glass. The width of the slit is carefully measured by means of a micrometer microscope.

A distant gauze, illuminated by a sodium flame, is set up and viewed with the telescope, which is focussed on the gauze.

The slit is adjusted until the vertical wires of which the mesh is composed just cease to be separately distinguishable. The distance from the slit to the gauze is then measured.

A number of such observations are made at various distances from the gauze and the results tabulated as shown.

The distance between the gauze wires is measured also by means of a microscope, so that the angle between rays from adjacent vertical wires can be calculated and compared with the theoretical value, $\lambda \div$ (width of slit).

$\lambda = 5890 \times 10^{-8}$ cm.

EXPT.	DISTANCE GAUZE TO EYE-PIECE (D)	WIDTH OF SLIT (a)	DISTANCE BETWEEN GAUZE WIRE (d)	$\frac{d}{D}$	$\frac{\lambda}{a}$
1	111·7 cms.	·128 cms.	·0548 cms.	·000492	·000464
2	85·3 cms.	·102 cms.	,, ,,	·000632	·000578
3	96·5 cms.	·113 cms.	,, ,,	·000569	·000520

The angle subtended at the slit by two adjacent wires of the mesh measures the inclination between two sets of rays, one from one wire, and the other from its immediate neighbour, which are just indistinguishable as lying in separate directions, i.e. the angle measures θ.

Repeat the experiment with two point sources and circular apertures of different diameters.

In this case the field of view contains a bright small central circle with concentric alternate bright and dark rings. If two sources close together are such that the bright centre due to one falls on the first dark ring of the other, the two sources just cease to be distinguishable as separate.

Sir G. Airy has shown that if d is the diameter of the aperture, θ the angle subtended when separation ceases,

$$\theta = 1{\cdot}22 \frac{\lambda}{d}.$$

The point sources should be two fine holes in a sheet of tin. They should be illuminated by monochromatic light and placed at such a distance from the object glass that they just cease to be seen as two holes.

Place the telescope at different distances from the holes and focus it on them. Adjust the aperture diameter until the holes just cease to be distinguishable as two separate sources of light. Measure the distance between the centres of the holes by means of a micrometer microscope, and the distance between the plane of the holes and the aperture by a metre rule. From these measurements deduce θ and compare it with the theoretical value, $1{\cdot}22\frac{\lambda}{d}$, in each case.

Polarization by Reflection. Verification of Brewster's Law

When light is reflected from surfaces the reflected beam is partially polarized, that is to say, that the transverse vibrations constituting the light have, on the whole, a greater component along a particular direction than in any other.

Ordinary light is supposed to consist of a transverse vibration which changes its direction in space, though of course always in the wave front, so rapidly that on the average in any appreciable interval of time the component in one direction is the same as that in any other.

The reflected light has lost this property and is polarized so that it has a greater component in one direction. This direction is normal to the plane of incidence.

The transmitted light has a greater component in the plane of incidence.

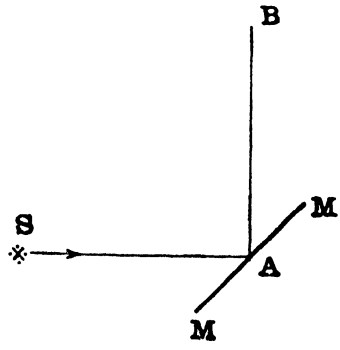


FIG. 226

In the diagram (fig. 226), a ray, SA, is represented as being reflected at a glass sheet, MM, so that AB is polarized. In order to test the polarization the ray is received in an analyser, which in some instruments consists of a Nicol prism. This is a prism of Iceland spar which is cut into two along a diagonal plane and cemented together with Canada balsam. Iceland spar has the property of dividing a ray of light into two rays refracted in different directions and one polarized perpendicularly to the other. The layer of Canada balsam serves to reflect one

of these rays to the side of the prism where it is absorbed by the blackened walls of a case, and the other ray is transmitted.

In this way the emergent beam is made to consist of light vibrations all in one direction. Incident light with its vibrations in this direction passes through the Nicol, while if the vibrations are perpendicular to this direction the light is unable to get through. Light with vibrations in any intermediate direction has only the components parallel to the direction of transmission passed on.

The vibrations transmitted are parallel to the shorter diagonal of the end of the prism.

On examining AB with this prism it will be found that as the prism is rotated there is a change of intensity in the transmitted light. This means that AB consists of vibrations with the components in one direction greater than in another or it is partially polarized.

On altering the inclination of MM the alternations can be varied in extent, and in one position the change between brightness and darkness is a maximum. The angle of incidence when this occurs should be noted by the help of the scale of angles attached to MM. Theoretically the light should be completely polarized in one position, so that for one particular

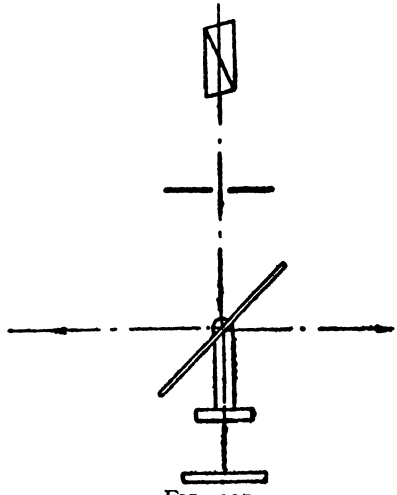


FIG. 227

setting of the Nicol darkness should be complete. In practice it will be found that the light is not quite all cut out, though with care a position will be found when this is very nearly true. The apparatus is represented diagrammatically in fig. 227.

Brewster's law is that for complete polarization :

$$\tan i = \mu,$$

where i is the angle of incidence and μ is the refractive index of the reflecting material.

This law should be verified. It will be found best to allow light from a window on the opposite side of the laboratory to fall on MM, and to adjust for the maximum effect, and afterwards to set up a sodium flame as at S, and make the exact adjustment.

In another form of apparatus the Nicol is replaced by a second glass plate which can be rotated about the direction AB as well as inclined at different angles to the horizontal.

When the mirrors are parallel the polarized light in AB is readily reflected by the second mirror, while on rotating it through a right angle from this position about AB this polarized light will not be reflected.

We shall thus obtain alternations in intensity on rotating the mirror about the vertical axis, and there will be a maximum effect for a particular inclination of MM, in which case $\tan i = \mu$.

Rotation of the Plane of Polarization. Laurent's Saccharimeter

The essential parts of the saccharimeter are two Nicol prisms, N_1 and N_2, illustrated in fig. 228, one of which serves to polarize a beam of light passing through it while the other analyses the transmitted beam and detects its plane of polarization. These Nicols are spoken of, respectively, as the polarizer and analyser.

FIG. 228

When N_1 has reduced the light vibrations to a particular direction, viz. parallel to the short diagonal at the end of the prism, all the light transmitted by N_1 can pass through N_2 if N_2 is oriented exactly in the same way as N_1, i.e. if its shorter diagonal lies parallel to that of N_1, and its length lies parallel to that of N_1. We are here neglecting the diminution of intensity due to absorption, which always goes on, since actual bodies are not perfectly transparent. We mean that no light is cut out in this case on account of polarizing effects of N_2.

In this position the Nicols are said to be parallel. If, however, N_2 is turned from this position through a right angle no light from N_1 can get through N_2, since N_2 is now so oriented that the light vibrations falling on it are in a direction perpendicular to its short diagonal, and such vibrations are not transmitted.

In this position the Nicols are said to be crossed.

Certain substances like quartz, and solutions like that of sugar,

possess the property of rotating the plane of vibration of light as it passes through them, so that if N_1 and N_2 are crossed when the active substance is not placed between them (in which case no light will get through N_2), on inserting the active material, on account of the change in direction of the vibration, some light will pass through N_2.

It is found that a rotation of N_2 in one direction or the other will bring it into a position when the light is once more stopped. This shows that the light is still polarized, but its vibrations have changed direction in traversing the medium. We ought, therefore, to be able to measure the amount of this rotation by measuring the angle through which N_2 is turned; but, unfortunately, N_2 can be turned through an appreciable angle when the light is cut out without any apparent return of the light. This lack of sensitivity is overcome in the saccharimeter by a special device.

Just in front of the polarizing Nicol on the side towards the analyser is placed a semicircular sheet of quartz cut parallel to the optic axis. The complete circle is made up by a semicircle of glass of such thickness that it absorbs the same amount of light as the quartz. The position of this circle is at H, and it covers the open end of N_1 completely.

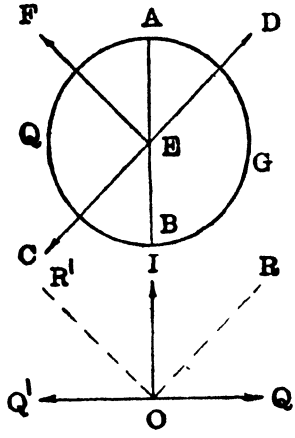


FIG. 229

When the light falls on the quartz it is separated into two components, polarized normally to one another, which travel through the quartz with different velocities. Let one component be represented by OI and the other by OQ just as the light reaches the quartz plate, these are the components of a vibration along OR. It is here supposed that the ether particle is at O but just moving in the direction OR, so that its component directions are OQ and OI. As the disturbance passes through the plate

there will be a gradual change of phase between these components on account of the differing velocities of transmission. After a time the disturbance will reach a point in the plate where one component displacement is along OI while the other component is along OQ^1. These combine to give a displacement OR^1.

The quartz plate is cut so that, as the disturbance just leaves the plate on the other side, this difference of phase exists between the components. The difference is one-half of a period, and the plate is called a half-wave plate or half shade.

Of course the light which traverses the glass side proceeds undisturbed, and its oscillations are still along the direction, ED, shown in the upper part of the diagram, parallel to OR.

If N_2 receives this light when its short diagonal is at right angles to OR^1 this component is not transmitted, and no light from the quartz side gets into the eye, while part of the component OR gets through and the glass side appears illuminated. Generally, light passes in from both sides of the plate, but both sides are not equally illuminated. When both sides present the same illumination the principal plane of the Nicol is either along AB or normal to it, for it is clear that in either of these positions the components transmitted are the same for both sides.

It happens that the eye can readily detect a change from the equality of illumination in both halves of the field, particularly if both halves are equally dark, i.e. when the Nicol is so placed that the smaller components are transmitted.

If the Nicol is set for equal illumination on both sides, and an active substance is interposed, it will be necessary to rotate the Nicol to find once more the position of equal intensities. The amount of rotation measures the angle of rotation of the plane of polarization.

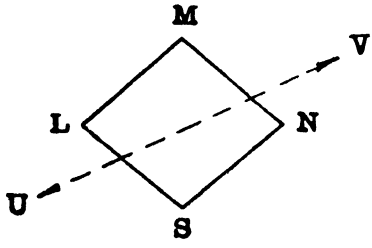


FIG. 230

Another common method of bringing about an increase in sensitivity is to use the biquartz. This consists of two semicircular discs of quartz fitted together to form a complete circle. One of these rotates the plane of polarization of the incident light in a clockwise, and the other in a counter-clockwise direction. The amount of rotation per unit thickness varies with the colour

of the light. For a particular thickness the rays from a sodium flame will be turned in opposite directions through a right angle, so that if the short diagonal of the Nicol lies parallel to this direction the yellow rays get through, and if the diagonal is perpendicular to this direction these rays are cut out.

When white light is used it is robbed of the yellow constituent when the Nicol lies in this latter position, and the colour observed is greyish and is called the tint of passage.

It is easy to detect a slight change from this uniform colour, for an appreciable change takes place to a partly blue and partly red field, one colour belonging to each side.

Let LMNS denote one end of the analysing Nicol, and let UV denote the direction of vibration of the light.

This light will be cut out if MS—the shorter diagonal—lies normally to UV. A small rotation of MS counter-clockwise will bring it into a position to cut out the light and so will a larger rotation in the opposite direction, the sum of these rotations being 180°.

It is thus not easy to decide which way the plane has been turned. But if two lengths of the rotating substance be used, one slightly longer than the other, the rotation for the longer must be greater than for the shorter.

The direction of rotation of MS which shows a larger angle in the case of the longer is the direction in which the rotation has taken place.

Tubes of glass with carefully worked end-pieces are used to carry the solution to be examined. The ends are held in position by metal caps screwed against them. It is necessary to have rubber washers between the glass ends and the tube to avoid strain when screwing up ; for a strained end will produce rotation.

Find the amount of rotation for a solution of sugar in water and deduce its specific rotation. This quantity is defined to be the amount of rotation produced by one decimetre of solution divided by the weight of dissolved substance in unit volume.

Let w grammes be dissolved in 100 c.c. and suppose a length, l cms., produces rotation, θ. The specific rotation is :

$$10\frac{\theta}{l} \div \frac{w}{100} = 1000\frac{\theta}{lw}.$$

Repeat for various strengths of solution, and for different lengths of tubes.

When several tubes are obtainable it is interesting to observe the effect of causing transmission through different lengths of a solution of a particular strength.

By this means it may be verified that the amount of rotation

is proportional to the distance traversed by the light in the solution.

Another instructive experiment is to make solutions of different known concentrations, which may be measured by the number of grammes of substance dissolved in 100 c. c. of solvent, and to measure the amount of rotation in traversing a particular distance through the solution. A curve showing the relation between the rotation and concentration should be plotted.

The Lippich Polarizing System

In modern polarimeters the half shade is replaced by a more convenient method of dividing the field. In the most recent polarimeter the field is divided into three parts (fig. 231), the two outer similarly illuminated for all positions of the analysing Nicol and the central portion which may be differently illuminated from the neighbouring regions and which has to be matched with the outer parts of the field.

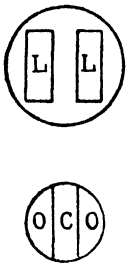


FIG. 231

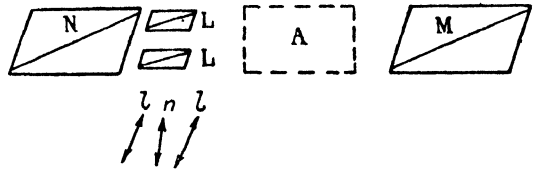


FIG. 232

The mode of producing the divided field is illustrated in fig. 232. N is the polarizing Nicol and LL are two small Nicols fixed in position and mounted in a brass cylinder in front of N. The directions of vibration of the light emerging from the cylinder and falling on the optically active substance are represented by the arrows, *l*, *n*, *l*. The central portion passes through the Nicol, N, only, while the outer parts pass through N and L.

When the analyser, M, is turned so that it transmits vibrations along a direction bisecting an angle between *n* and *l* the whole field is uniformly illuminated. For all other positions of M the field is not uniform.

Thus, to measure the amount of rotation of any substance placed at A, the analyser is first put into the position corresponding to uniformity of field. The substance to be examined is then put into position and M again rotated until the field is once more uniform. The angle of rotation measures the rotation due to the active substance.

The optical system by which the field is examined is not shown in the diagram. It lies to the right of M and is focussed on the plane through the right-hand ends of the Nicols, LL. This system has the advantage that it is suitable for the examination of all wave lengths, whereas the half shade has to be constructed for one wave length only. Fig. 231 shows at OCO how the field is divided into three parts.

The Half Shadow Angle

The various devices employed to enable accurate observations to be made in polarimetry, which have been described, produce two beams of polarized light with vibrations in directions inclined to one another.

In the Lippich system we have denoted the two directions by *l* and *n*.

In fig. 233 these directions are denoted by B*l* and B*n* respectively, and the angle between them is 2θ. This angle is called the ' half shadow angle,' and the magnitude of this angle has an important bearing on the question of sensitivity.

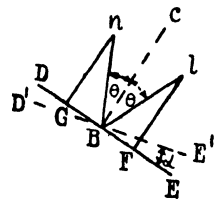


FIG. 233

Suppose that BC bisects the half shadow angle, and that DBE denotes the direction of vibration of the light which traverses the analyser.

When this lies at right angles to BC the intensities of the two beams are equal as seen through the analyser, for the components of the displacements transmitted, viz. BF and BG, are equal, and the intensities are in the ratio :

$$BF^2 : BG^2.$$

Suppose the analyser is turned through an angle, α.

Then the transmitted components are :

$$B\mathit{l} \cos \mathit{l}BE^1 \text{ and } B\mathit{n} \cos \mathit{n}BD^1,$$

or

$$B\mathit{l} \sin (\theta - \alpha) \text{ and } B\mathit{n} \sin (\theta + \alpha).$$

Thus the ratio of the intensities is :

$$\sin^2(\theta - \alpha) : \sin^2(\theta + \alpha).$$

In photometric work it is assumed that the eye can detect a difference of intensity of one per cent.

Thus, if we regard θ as a given angle, we may say that the

change in setting of the analyser of the amount α will just be detected when

$$\sin^2(\theta - \alpha) = \cdot 99 \sin^2(\theta + \alpha).$$

From this equation it follows that when α is a small angle, so that we may write :

$$\sin \alpha = \alpha, \text{ and } \cos \alpha = 1,$$
$$\alpha = \cdot 0025 \tan \theta.$$

If α has a small value the apparatus is sensitive, and it would appear that the sensitivity is improved by making θ as small as possible.

But as θ gets small difficulties arise on account of the fact that the light is never plane polarized, it is always elliptically polarized in practice.

θ may not be indefinitely diminished.

Polarimeters are usually fitted with a small movable arm projecting from the tube which carries the polarizer. This arm carries an index mark which moves over a scale. By means of it the polarizing Nicol can be rotated so that the half shadow angle can be adjusted within limits. A modern form of the apparatus is illustrated in fig. 234.

The sensitivity is thus to some extent under the control of the observer, who will discover as he becomes familiar with his instrument the best adjustment for sensitivity which suits him.

The reader may be referred for a more detailed and complete account of all these questions to the article on " Polarimetry " in the " Dictionary of Applied Physics."

Soleil's Compensator

Sometimes the saccharimeter is fitted with a piece of apparatus consisting of two quartz wedges. This is known as Soleil's Compensator. Fig. 235 illustrates the apparatus. The wedges are ABC and DEF, and these are mounted in metal holders which can be moved by means of a rack and pinion, so that the wedges are translated in either direction parallel to AC or DE.

Thus if a ray of light is passed perpendicularly to AC it is possible to place varying thicknesses of quartz in its path. The quartz wedges are cut so that the optic axis lies perpendicular to AC and DE, and polarized light passing through them suffers rotation of its plane of polarization.

The amount of rotation can be varied by moving the wedges by means of the rack and pinion.

The quartz wedge is placed just in front of the analyser.

An index mark moves along a scale as the wedges are displaced, so that a record can be made corresponding to each thickness of quartz interposed.

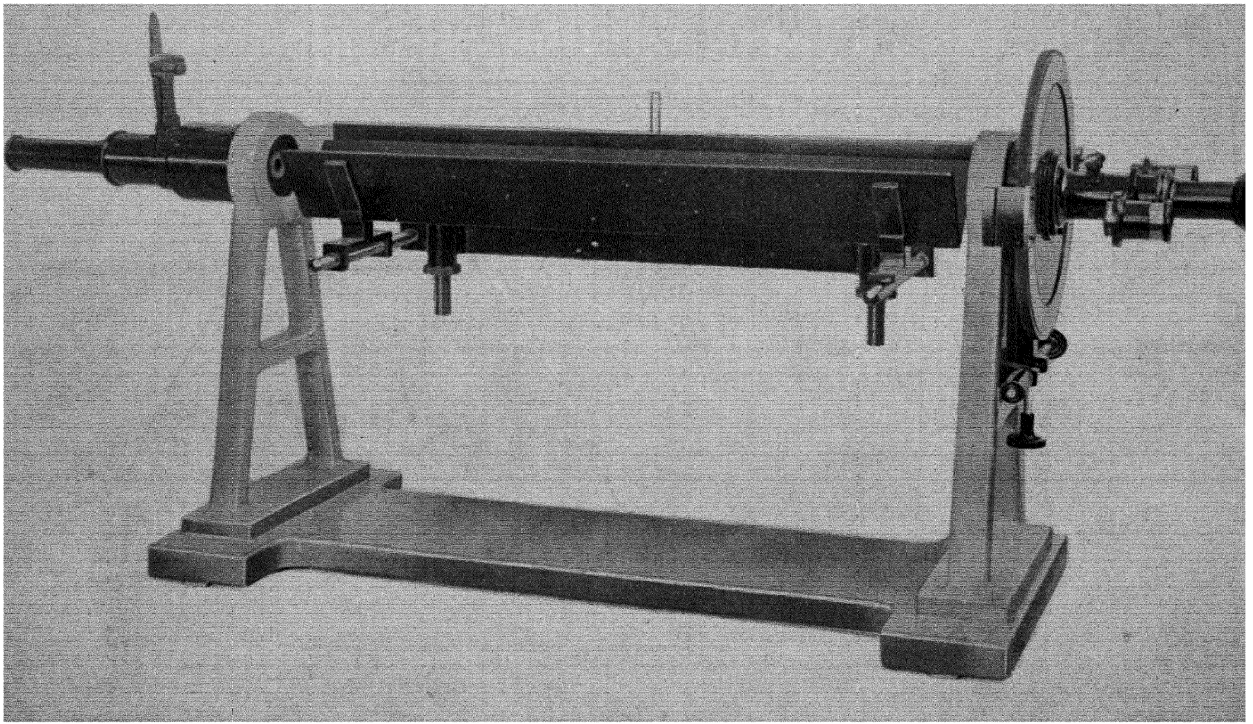


FIG. 234

The analyser is first rotated until with the quartz wedges occupying a convenient zero position the field is equally dark on both sides, supposing that the polarimeter is fitted with a half shade. The quartz wedges are then moved a small amount by the rack and pinion, and the amount of rotation of N_2 necessary to restore the uniformly dark field of view is recorded.

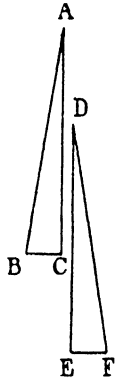


FIG. 235

By making a number of observations a curve can be plotted which shows the rotation corresponding to the various dispositions of the wedges.

When light has undergone a rotation before passing through the wedges, this rotation may be counteracted by interposing the correct thickness of quartz.

Thus suppose the Nicols and quartz occupy the zero position described above, and that an optically active material is interposed, the rotation resulting may be counteracted by displacing the quartz wedges either so as to increase the thickness traversed or to diminish it. When the appearance in the analyser is the same as that of the zero position, we know that the wedges have caused a rotation equal in magnitude, but opposite in direction to that of the active substance.

Thus by observing the record opposite the index mark we can deduce the amount of rotation due to the substance.

In plotting the graph, rotations in one direction will lie on one side of the origin up to 180°, while those in the other direction will lie on the other side up to 180°.

CHAPTER XIII

PHOTOMETRY

Introduction

The light emitted from a small source is absorbed very little by the air through which it passes, so that we may say that any surface, surrounding the source completely, will receive the same total amount of light.

Let this total amount be denoted by M.

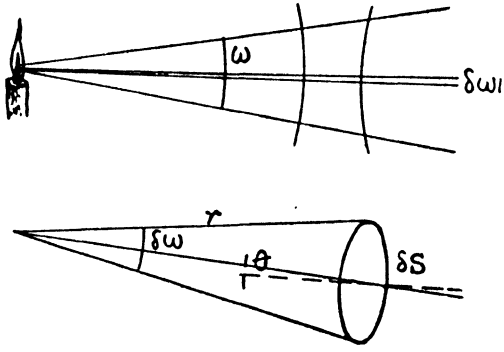


FIG. 236

Imagine a cone with its apex at a small source of light and let its solid angle be w. All surfaces receive the same amount of light on the parts lying within this cone.

If w is small, say δw, the cone may be regarded as defining a particular direction and the intensity will be regarded as the same for all rays within this cone. If L denote the total amount of light emitted within this cone per second, we write : $L = K\delta w$.

This equation defines K, which is sometimes called the candle power ' of the source.

In general, K is dependent upon the direction, but when K is the same for all directions

$$M = 4\pi K,$$

where M is the total amount of light emitted per second by the source.

Let the small cone cut a surface in the element, δS, and let the mean direction of this cone make an angle, θ, with the normal to δS.

Then $$\delta S = \frac{r^2 \delta w}{\cos\theta},$$

and $$L = \frac{K \cos\theta}{r^2} \delta S.$$

The intensity of illumination is defined to be the amount of light falling on unit area per second,

i.e. $$I = \frac{L}{\delta S} = \frac{K \cos\theta}{r^2}.$$

The intensity, I, varies inversely as r^2 and directly as $\cos\theta$. When I is the same for two similar surfaces they appear to the eye to be equally bright, and on this principle the use of photometers depends.

The efficiency of a source of light may be defined as the ratio of K to the material or energy consumed per second.

The average candle power, in the case when K is not constant, divided by the amount consumed per unit time is called the mean spherical efficiency. This term is employed because the average value of K is defined as $\frac{M}{4\pi}$, and this denotes the average amount of light falling on 1 sq. cm. of a unit sphere placed with its centre at the source.

The efficiency of a candle is :

$$K \div \text{weight of wax consumed per second,}$$

of a gas flame :

$$K \div \text{cubic feet of gas consumed per minute,}$$

and of an electric lamp :

$$K \div \text{watts supplied to it.}$$

The watts are measured by the product, volts $\times$ amperes, one watt denoting the rate of working when a current of one ampere falls through a potential difference of one volt.

The unit in which K is measured is the candle power of a standard candle in a horizontal direction when the flame is 50 mms. high.

If another source is used and compared with the standard the two are adjusted to the same height and placed at such distances, r_1, and r_2, from a conveniently placed screen that each makes it appear equally bright. In this case by what has been said above concerning intensity we have :

$$I_1 = \frac{K_1 \cos\theta}{r_1^2} = I_2 = \frac{K_2 \cos\theta}{r_2^2}.$$

since the angles are the same. If K_1 is the standard candle power its value is unity, and,

$$\frac{K_2}{r_2^2} = \frac{1}{r_1^2}.$$

It is assumed that the student is familiar with the simple forms of photometer such as Rumford's and Bunsen's. It is difficult to make accurate comparisons of the illuminating powers of sources with these instruments.

We shall be concerned chiefly with the more accurate types of instruments in this chapter. Even with these care and practice are required, but a skilled observer can obtain accurate results.

The Efficiency of Sources of Light

Take a gas-burner provided with an indicator registering the quantity of gas supplied, and adjust the flame until it stands at the same level as a candle flame and the grease spot of a Bunsen photometer.

The candle may be taken as the standard with the value of K unity, and it must be shielded from draughts and must burn steadily.

Determine K for the flame corresponding to different rates of supply of gas, by adjusting the distances between the sources, until the photometer screen appears alike on both sides.

Plot a curve, showing the relation between the efficiency and the supply per hour.

It will be found that the efficiency increases with the supply up to a maximum and then diminishes.

It is most economical to adjust the supply to the value appropriate to the maximum.

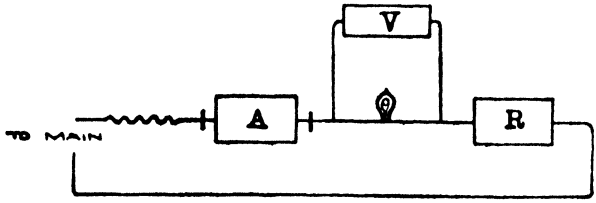


FIG. 237

We may similarly measure the efficiency of an electric lamp. Arrange the lamp, L, on the same level as the grease spot and candle flame, as before, and measure its candle power when the current is supplied at different voltages.

Fig. 237 illustrates the arrangement of apparatus.

V is a voltmeter joined to the terminals of L, and A measures the current in amperes. The resistance, R, is adjustable and is

used to vary the current supplied to the lamp. The power is obtained by connexion to the main through a plug. A convenient voltage is 100 volts.

Care must be taken not to short-circuit the mains.

Plot a curve, showing the variation of candle power, with rate of supply of energy as measured in watts.

The Flicker Photometer

One form of this instrument is illustrated in figs. 238 and 239. The essential part consists of a white wheel, W, of which the edge is about 1 cm. wide, and is cut to the shape of a ridge running spirally. This is shown at RR, in fig. 239. The wheel is mounted in a box, black on the inside, provided with a rod, so that it may be supported in the carrier of an optical bench.

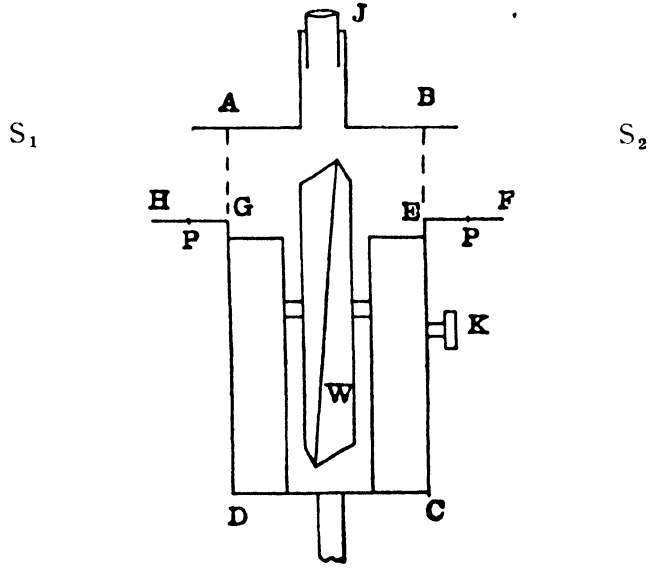


FIG. 238

The wheel is provided with a central axis and is rotated by a spring within the box, which is wound up by the key, K.

The two sources to be compared are placed one on each side of the apparatus, as at S_1 and S_2. These are carried in stands on the bench, so that the distances from the wheel are measured accurately and their heights are adjusted with the aid of two lenses fixed on the box.

The small doors EF and GH, are closed, and the lenses used to throw an image of the sources on marked points, P_1 and P_2, on the doors. When the images fall on these points the heights of the sources are correctly adjusted.

The edge of the wheel is observed by means of an eyepiece, J, and usually unequal parts of the edge are seen on the two sides. When the wheel rotates the widths of these vary, and unless the illumination is equal on both sides a flickering effect is observed. If the illumination is the same on each side the ridge character disappears.

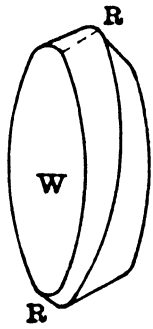


FIG. 239

S_1 and S_2 are moved to or from the apparatus until the flicker effect ceases, when the illuminations are equal, and if I_1 and I_2 represent the illuminating powers of the sources, and r_1 and r_2, their distances from the wheel, we have :

$$\frac{I_1}{r_1^2} = \frac{I_2}{r_2^2}.$$

The distances should be measured from the middle of the wheel. It is not easy to decide when the flicker ceases, and the degree of accuracy obtainable is not very high.

Compare in this way a standard candle and a lamp.

The Lummer-Brodhun Photometer

This photometer is one of the most accurate. It is illustrated in fig. 240, and consists of a box, LMNO, containing the prisms, P_1,P_2,A, and B, by means of which light is reflected and transmitted into the telescope, T, placed at 45° to the sides of the box, with its object glass in one corner.

The two sources to be compared are at S_1 and S_2, from which light falls on the slab, DD, which consists of magnesium carbonate. From the diffuse sides of the slab rays are scattered and absorbed by the sides of the box, except those that cut the sides of P_1 and P_2 normally. These are reflected by the hypotenuse faces into the two right-angled prisms, A and B.

These prisms are the principal part of the apparatus.

The hypotenuse of A is rounded off, except for a circular central portion which is placed in optical contact with the

larger face of B. The reason for this is, that rays of light may pass from A to B at this junction just as if the prisms formed one solid medium. Rays falling on other parts of the hypotenuse faces are totally reflected.

In this way rays from P_1 pass on through B, forming the central bundle of rays in the beam emerging from the right and entering T. The rays outside this circle are reflected and are absorbed by

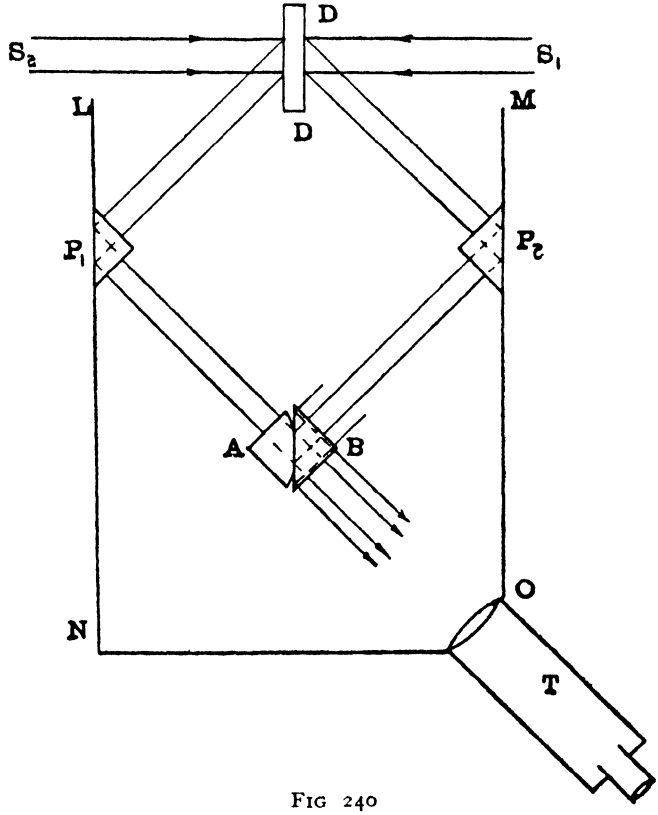


FIG 240

the sides of the box. In the same way, rays from P_2 are reflected outside the circle, while those falling on the circle are transmitted. Thus the field of view of the telescope is illuminated by a central circle of rays, originating at S_2, while the outer rays come from S_1. Generally these two parts will be of different brightness, and on moving S_1 or S_2, the two parts may be made equally bright. The eye can judge this easily and readily appreciate a slight deviation from equality. It is on this fact that the sensitiveness depends.

From DD both sets of rays follow similar paths and light is absorbed equally. When the field of T is uniform, the slab is

illuminated equally on both sides. If L_1 and L_2 denote the illuminating powers respectively, and d_1 and d_2 denote the respective distances from DD, we have :

$$\frac{L_1}{{d_1}^2} = \frac{L_2}{{d_2}^2}.$$

Thus any two sources may be compared with accuracy.

Compare two sources as in the last experiment.

In order to eliminate errors arising from inequality of the reflecting power of the two surfaces of the slab and a possible difference in optical paths, DP_1A and DP_2B, it is usual to mount the box, LNOM, on a central horizontal axis lying perpendicularly to S_1S_2. Readings are first taken with the telescope as in the figure, and then with the box turned through two right angles so that the telescope lies on the left.

The Nutting Photometer and its use for the Determination of the Absorption of a Solution

This form of photometer is the most accurate instrument of its kind. It is based upon principles described by P. G. Nutting, but has been modified by Messrs. Hilger and Co., whose apparatus is here described.

It is used in combination with the Constant Deviation Spectrometer (pp. 341 to 345), and the apparatus is illustrated in fig. 241, in the position ready for use.

The essential features of the photometer are illustrated in fig. 242.

The box, A, is of aluminium, blackened inside, provided with two small windows, Q and Q^1, by means of which light is admitted along the two directions, QR and Q^1S.

Light from a suitable source is deviated by two prisms, P and P^1, carried in the plate, C. On the plate a number is inscribed, indicating the distance that the source must be placed from it, in order that the necessary deviation may be produced. This distance is usually 19 cms.

Behind the window, Q^1, lies a Nicol prism, N_1, which polarizes the light entering at Q^1; but behind Q in the form of apparatus shown no Nicol prism is placed.

In another form a Nicol lies behind Q also, and its purpose is to counteract as much as possible the elliptic polarization that occurs on reflection at the surfaces marked R and S. It is found that this is reduced to a minimum by a particular orientation of the Nicol.

The prism, RS, is composed of three slabs, the two outer are alike and are cut at the ends, R and S, at an inclination of 45° to the length and to the incident light.

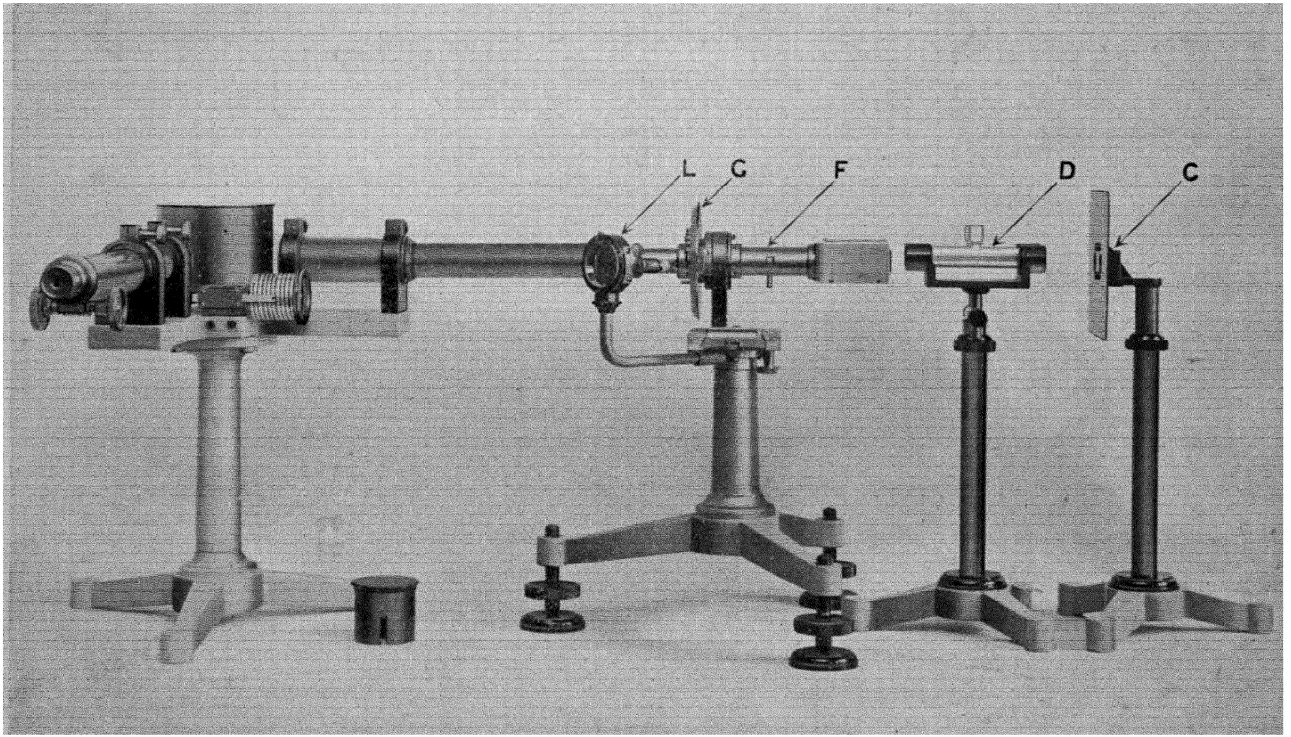


FIG. 241

Thus the beam, QRST, is totally reflected at R and S, by these two end faces. Between these slabs lies a central one of the same composition and thickness cut at R at 45° like the others, but cut square at the other end, and projecting as the diagram

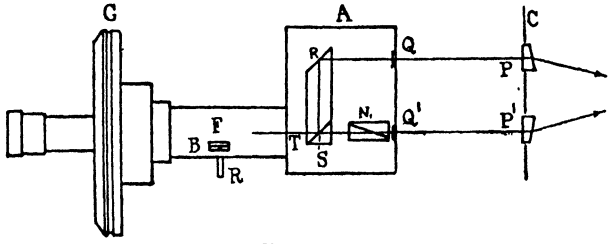


FIG. 242

indicates. Thus the central portion of the beam, QR, is totally reflected at R, and transmitted out at the other end, and is absorbed by the blackened walls of the box. The beam, Q^1S, is totally reflected at the end, S, by the upper and lower slabs, and the light absorbed by the walls of the box, but the central portion is transmitted in the direction, ST, by the central slab.

Thus a tripartite field is produced and may be observed from the end of the tube, F; the outer portions are illuminated by light which has entered at Q, and the central portion by light from Q^1.

The instrument is constructed and the prism and Nicol, N_1, chosen to cause the light within the box to suffer approximately the same absorption along the two paths, QRS and Q^1S.

The tube, F, carries a second Nicol which is not shown, which acts as an analyser and which can be rotated by means of the divided metal circle, G. This circle carries two scales, one marked in degrees and the other giving 'densities,' a term which will be explained below.

When the instrument has no zero error the degree scale reads zero when both Nicols are parallel. In this case the light already polarized by N_1 is transmitted by N_2, while the unpolarized outer portions of the field are polarized by N_2 in the same way as the central portion by N_1.

Thus when there is no absorbing medium between the source and one of the windows, the field appears under these circumstances of the same intensity in the outer and inner portions.

F also carries a condensing lens which may be adjusted by the rod, R, projecting downwards from the tube.

The purpose of this lens is to converge the light so that on exit all the light may enter the pupil of a normal eye. This is an important condition with which accurate photometric apparatus must comply in order that intensity comparisons may be of any use.

The rod carries a scale past an index mark in the slot, B, on which are engraved two sets of numbers which have to be associated in pairs. The upper scale records the distance of the source from the front of A in cms., and the number below this record on the lower scale denotes the maximum breadth of the source which is permissible if the photometric condition is to be satisfied.

Beyond the circle, G, the tube carries a lens system which focusses the light on to the slit of the spectrometer, and if it is desired to examine the field of view directly by eye an additional eyepiece is fitted into the tube.

The apparatus requires careful adjustment which may be carried out as follows :

Remove the prism from the spectrometer, illuminate the collimator slit and place on the prism table a piece of plane mirror or a right-angled totally reflecting prism, and adjust it until an image of the slit lies on the cross-wire of the telescope. Now place the source to be used with the photometer at the distance from the collimator slit at which it is to be situated during the experiment, having regard to the photometric condition which limits the distance to some extent on account of the size of the source. Adjust the source until an image of it lies in the centre of the field of view of the telescope, the collimator slit being now wide open and the eyepiece removed from the telescope, so that the source appears to lie at the centre of the object glass. Slide the photometer into position with the end of the tube, F, as nearly as possible 1·4 cms. from the slit, and with the window, Q^1, directly between the collimator slit and source. Cover up the window, Q.

Adjust the photometer by means of the three screws on which it stands, and by rotation about a vertical axis until the image of the source lies once more in the centre of the objective of the telescope.

Place the plate, C, into position at a distance of 19 cms. from the source. The distance between P and P^1 is 3·8 cms., so that by moving the source a distance 1·9 cms., it can be brought to lie opposite the middle point of PP^1 and it should then be in the correct position.

This may be judged first by observing if bright circular patches of light lie symmetrically round the windows, Q and Q^1.

Make sure that, with the rod, R, adjusted so that the index lies opposite the mark denoting the distance between the front of A and the source, the width denoted by the lower reading is greater than that of the source. If this is not the case the source has either to be displaced farther from A or diminished in size. Place the constant deviation prism and eyepiece of the telescope

in position and adjust the spectrometer correctly for sodium light as indicated on p. 343. In doing this a strongly coloured Bunsen flame may be placed just closer to C than the light source.

Make the line as sharp as possible by rotating slightly the milled head at the end of F.

Now remove the Bunsen flame and open the slit to let in a convenient quantity of light. It will be probable that the

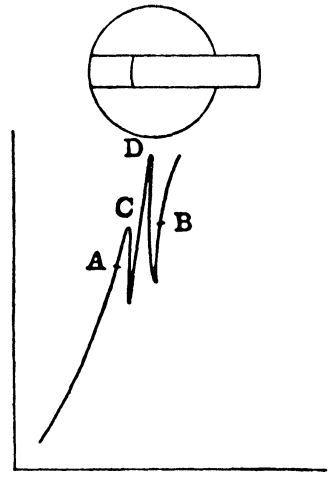


FIG. 243

central part of the field of view is slightly displaced with respect to the outer parts. This is exaggerated in the upper part of fig. 243. This may be corrected by a further small rotation of the photometer about a vertical axis. The three parts of the field should be separated by fine dark lines. If these are too wide they may be made narrow by adjusting the base screws of the apparatus.

Correction of Zero Error

In practice it is usually found that when the apparatus is set at zero the three parts of the field are not uniformly bright, and that the error is not the same for different wave lengths. In order to correct for this the readings of the apparatus are recorded when the field is uniform for different wave lengths. A shutter eyepiece is fixed to the telescope to cut down the light except over a narrow central strip, the wave length for which is recorded on the drum. When the central portion is the brighter, rotation of the analyser cuts it down, and the readings on both sides of the zero are observed and the mean taken for a series of different wave lengths and a curve plotted, showing the relation between error and wave length.

These readings are observed on the degree scale or density scale as may be required.

If the central part of the field is the darker it is not possible by merely turning the analyser to bring about uniformity. In this case a weak absorber, e.g. a plate of glass, is put into the path of the beam just before it enters Q. The thickness is chosen so that in the zero position the central field is just stronger, when the correction may be made as before. This should not often occur because the instrument is made so that the central portion is brighter than the outer, so that as the apparatus deteriorates with time the central portion may still be the brighter and the interposition of the weak absorber may be avoided.

The substance of which the absorption is required is placed in the path of the light before it enters Q.

In the case of a liquid it is necessary first to measure the absorption of the vessel containing it.

The observations consist of noting the readings on the scale, G, when the field is uniformly bright for both directions of rotation of G.

Let I_0 denote the intensity of the light entering at Q^1, and let I denote the intensity of that which enters at Q. The density scale records the values $\log \frac{I_0}{I}$, i.e. the logarithm of the ratio of the intensity of the light entering the medium to that transmitted by it.

If this number be divided by the thickness of the material traversed a quantity known as the 'extinction coefficient' is obtained.

Let a denote the amplitude of the polarized light transmitted by N_1 and suppose that the analyser is turned through an angle, θ, from the position in which it is parallel to N_1. The amplitude of the vibrations transmitted by the analyser is therefore $a \cos \theta$.

When the field is uniformly matched the intensity is I, and hence

$$\frac{I_0}{I} = \frac{a^2}{a^2 \cos^2 \theta};$$

$$\therefore \log \frac{I_0}{I} = \log \sec^2 \theta.$$

This shows the relation between the density scale and scale of degrees.

A solution which gives a characteristic absorption curve is one of eosin in alcohol. Eosin may be obtained from a bacteriological laboratory, and the solution must be very dilute, or so much absorption takes place that the transmitted light is very feeble.

Carry out the determination described above and plot on the same curve the values of the 'density,' $\log \frac{I_0}{I}$, and the corresponding wave lengths, for the zero error, for the vessel alone and for the vessel and solution.

From these it is possible to obtain the value, $\log \frac{I_0}{I}$ for the solution alone.

Fig. 244 shows the result of an experiment with eosin. Sometimes the variations occur very rapidly and a curve is obtained like that drawn in diagram 243. Unless frequent observations are made in the neighbourhood of AB the variations at C and D may be missed. Whenever the curve shows any sign of change the neighbourhood where this occurs must be examined with care.

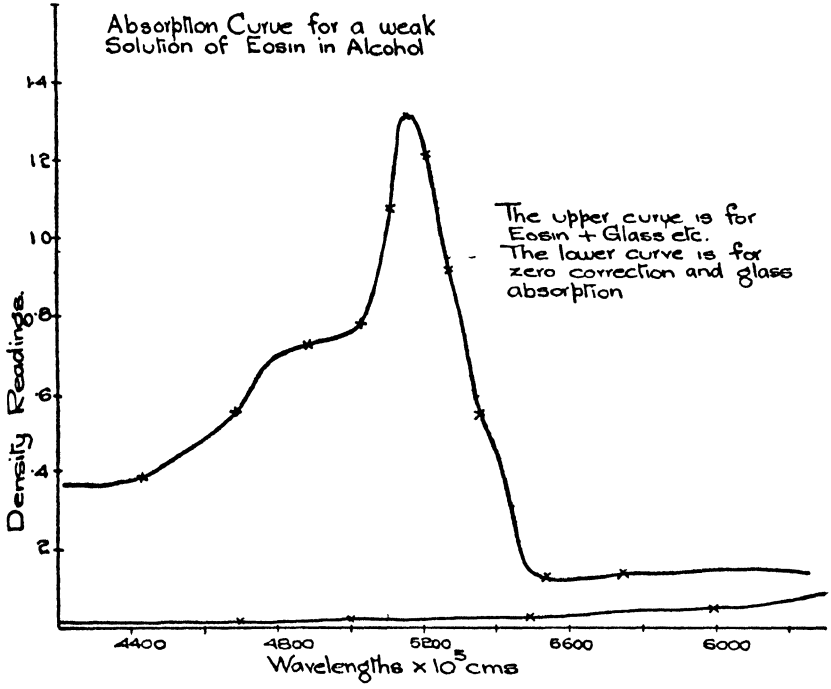


FIG. 244

It may not be necessary to determine exactly the value of $\log \frac{I_0}{I}$ for the solution, and in this case it is sufficient to make the zero correction and that for the containing vessel together, the whole appearing as a combined correction curve.

CHAPTER XIV

SOUND

To Find the Frequency of a Note by means of the Siren

In this instrument a musical note is produced by puffs of air following one another in rapid and regular succession. The series of puffs is produced by blowing air through a number of holes in a rapidly rotating plate.

The diagram (fig. 245) illustrates the instrument. It consists of a cylindrical metal chamber provided with a tap through which air can be blown. In the upper end is cut a series of holes lying regularly spaced on a circle with its centre on the axis of the

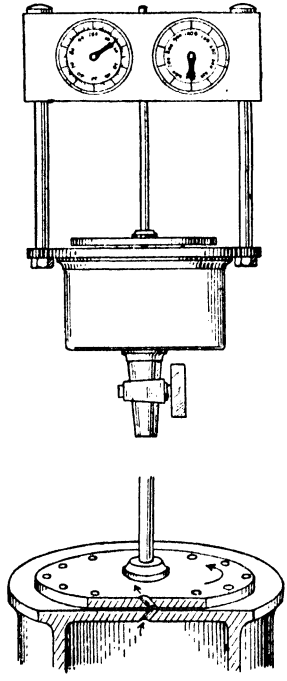


FIG. 245

cylinder. Above this lies a circular metal disc provided with a similar series of holes which fit above the former. The disc is mounted so that it can rotate about the axis of the cylinder and so alternately cover and expose the lower series of holes.

The two series of holes slant in opposite directions as the figure shows, and when a current of air is blown into the chamber and the disc given a slight rotation, the puffs of air on escaping produce a pressure which drives the disc.

By adjusting the influx of air the regularity and speed can be controlled so that notes of varying frequencies can be produced. If N is the number of revolutions made by the disc per second, and the number of holes is n, the frequency is Nn. In order to measure the number of revolutions the disc is provided with a metal bar, provided with a screw at one end, which works two dials, one registering units and tens, and the other hundreds of revolutions.

This form of apparatus is due to Cagniard de la Tour; but it has the disadvantage that the speed can only be increased with greater air pressure and a consequently louder note. It is also difficult to keep the speed uniform.

It is preferable to drive the disc with an electric motor, of which the speed may be regulated by including a resistance in the supply circuit, and the holes should be cut normally to the disc in order to avoid air pressure in the direction of the rotation.

The siren gives a large number of harmonics, and it is necessary carefully to single out the fundamental note.

Let it be required to find the pitch of a given note, as, for example, that produced by an open organ pipe.

Carefully adjust the speed of the siren until beats are heard between the note it gives and that of the organ pipe by blowing at a particular pressure from a bellows connected to the chamber. The blower should then endeavour by a slight change of pressure to produce from the siren a note giving no beats. To some extent the frequency may be controlled by the tap, but it is important to keep a steady pressure on the bellows.

At the same time a second observer should measure the speed of revolution by observing the number of revolutions recorded on the dials in a definite time (30 or 60 seconds).

Verify the result of the determination of frequency by measuring the length of the pipe and its diameter. For an open organ pipe emitting the fundamental the wave length is approximately twice the length.

The correction necessary to obtain a more accurate result is to add ·6 radius for the open end and 2·8 radius for the flute mouthpiece. If the length of the pipe be l, and the radius of the pipe, r, the half wave length is given by:

$$\frac{\lambda}{2} = l + 3{\cdot}4r,$$

or

$$\lambda = 2(l + 3{\cdot}4r).$$

The velocity of sound in air for ordinary temperature may be taken as V = 33,300 cms. per second, or make the accurate correction for temperature by formula (A), p. 416.

Thus the frequency is $\frac{V}{\lambda}$.

Determine the frequencies of several organ pipes theoretically and experimentally, and draw up a table recording the speed of rotation of the siren disc, and the observed and calculated frequencies in each case.

The Tonometer

In Scheibler's tonometer a number of tuning forks are arranged in ascending order of frequency, each of which gives the same number of beats with its neighbour. The forks thus form a series in which the frequency increases by equal steps, and they are arranged so that the highest frequency is twice that of the lowest.

In Appunn's tonometer the forks are replaced by reeds set in vibration by a blast of air from bellows of large capacity, and the apparatus has the appearance of a small harmonium provided with a series of stops by means of which any note may be sustained.

This form of apparatus is not so accurate as the original one, for Lord Rayleigh has shown that the frequency of a vibrating reed is to some extent affected by the vibrations of its neighbours. As it is necessary to vibrate two successive notes in the experiment we have no longer a constant register of frequency as in Scheibler's instrument.

It is first necessary to find the absolute frequency for each note on the instrument. Suppose there are $(k + 1)$ notes, and consequently k intervals between them, and that the frequencies are

$$N_1, N_2, \ldots N_{k+1},$$

beginning from the lowest.

If the number of beats be observed between all the successive notes, and be denoted by

$$n_1, n_2, \ldots n_k,$$

respectively, we have the following relations:

$$\begin{aligned} N_2 - N_1 &= n_1, \\ N_3 - N_2 &= n_2, \\ &\ldots\ldots\ldots \\ N_{k+1} - N_k &= n_k. \end{aligned}$$

Then adding both sides:

$$N_{k+1} - N_1 = n_1 + n_2 + \ldots + n_k.$$

But $$N_{k+1} = 2N_1;$$
$$\therefore \quad N_1 = n_1 + n_2 + \ldots + n_k.$$

We thus find N_1 by counting the successive numbers of beats, and we can then deduce from the equation the frequencies of all the other notes of the series. In practice, of course, the numbers of the beats will vary between the successive notes in different parts of the scale to a slight extent.

In order to determine the frequency of the lowest note, count the number of beats at five or six different parts of the range and deduce the average difference of frequency.

Let this be denoted by n. Then the frequency of the lowest note is $k \times n$.

After this determination has been made the frequency of any given note coming within the range may be determined. For example, suppose the frequency of a fork is required. Find by trial the note nearest to it in pitch and count the number of beats when the notes are sounded together.

If this number be denoted by x, and N_l is the frequency of the note nearest the unknown, the frequency of the latter is $N_l \pm x$.

The frequency of the note next above N_l is $N_l + n$, and on sounding this and the unknown note together the frequency of the beats will be less than n if its pitch is higher than that of N_l, and greater than n if its pitch is lower. This enables a distinction to be made between $\pm x$.

In making this last part of the experiment it is advisable not to rely on the accuracy of the average number of beats for each interval, but to measure separately the frequency of the beats of the notes immediately above and below N_l.

It will then be easy to decide on the exact position of the unknown pitch above or below N_l.

The Determination of the Frequency of a Tuning Fork by the Method of the Falling Plate

A smoked glass plate, P, is suspended vertically by a piece of thin string or thread over two nails, QQ, the thread being attached to the upper edge of the plate by means of sealing wax or by any other convenient method.

The fork is held in a clamp, H, and carries a light style of bristle or thin aluminium wire attached to one prong by as little wax as possible. The style is just in contact with the plate so that when the plate falls it removes some of the soot and leaves a trace.

In order to prevent breakage a padded wooden stand, AB, is placed just below the plate.

The fork is stroked gently by means of a violin bow and the

plate is allowed to fall straight down by burning the thread between the nails, QQ.

A wavy line is traced by the style similar to that shown in fig. 247, but with more waves, and usually of small amplitude.

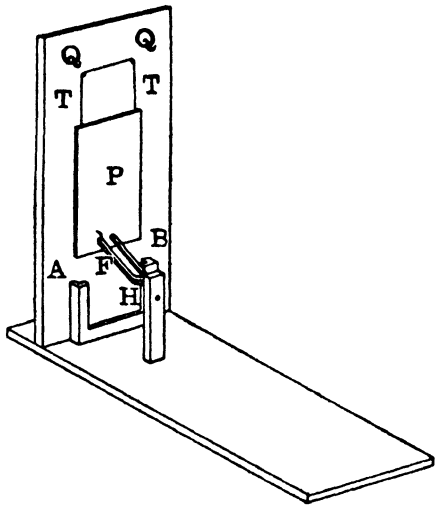


FIG. 246

A point, O, is chosen just clear of the indistinct portion drawn when the plate was moving down in the first stage of the motion, and consequently before its velocity had sufficiently increased

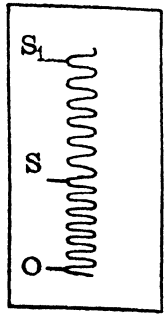


FIG. 247

to open out the waves, and from it is counted a number, n, of complete waves to the point S. Again, n waves are counted to the point S_1. Let the spaces, OS and SS_1 be of lengths, s and s_1, respectively. Then the time taken to fall over these two lengths is the same, let us say t.

If N denotes the frequency of the vibrations :

$$n = Nt.$$

Let u denote the velocity of the plate at the instant corresponding to the mark, O.

Then by the equation for space described under acceleration, g,

$$s = ut + \tfrac{1}{2}gt^2,$$

and

$$s + s_1 = 2ut + 2gt^2,$$

for the time of description of $s + s_1$ is $2t$.

Hence

$$s_1 - s = gt^2,$$

or

$$t = \sqrt{\frac{s_1 - s}{g}};$$

$$\therefore \quad \mathrm{N} = n \cdot \sqrt{\frac{g}{s_1 - s}}.$$

The distances, s and s_1, are measured carefully by means of a travelling microscope, and the value of N obtained is that for the fork vibrating with the load consisting of the style and wax and affected by the friction of the style against the plate.

This should be allowed for by taking a second fork of nearly the same frequency as that under examination, before attaching the style, but of slightly higher pitch. Carefully load this fork by adding wax until no beats are heard when both sound together.

Then when the first fork is loaded and has the style touching the plate as in the experiment, again sound the two together and count the beats. The number of these per second gives the number of periods lost per second on account of the loading and friction.

This number added to the value of N, determined in the experiment, gives the corrected frequency.

In marking off the points O, S, and S_1, be careful to choose them at corresponding points of the waves.

If O is at the summit of a crest, S and S_1 must lie in a similar position n and $2n$ waves later, respectively.

The plate may be conveniently blacked by holding it just over a turpentine flame which gives a good deposit of soot; the flame of a paraffin lamp gives also a satisfactory deposit.

Chronographic Methods of Determining the Frequency of a Fork

In both the methods to be described under this title a fork is set in vibration and electrically maintained, with a style attached to one of the prongs lightly touching the blackened surface of a cylinder which can be rotated about its axis.

On rotating the cylinder the track made by the style appears as a wavy line which can be opened out so that each vibration is distinctly separated from its neighbours by adjusting the speed of rotation.

The drum may be rotated by the handle, H, or by a falling weight not shown in the diagram, which rotates the drum at a convenient speed.

In this case the rotation is regulated by a governor, and by releasing a catch, the handle, H, can be employed for rapid movement of the drum. This also is not shown in the figure. The drum moves along an axis cut with a screw thread, so that the track drawn by the style forms a wavy helix round it.

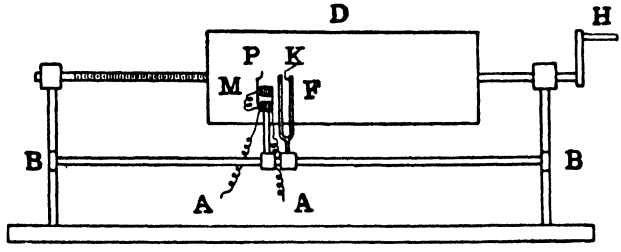


FIG. 248

It is then necessary to have some record of time with which to compare the vibrations. The two methods differ in the way the time is obtained.

In the first a stand is mounted conveniently near to the fork, as illustrated in fig. 248, which carries a pointer actuated by an electromagnet, M. The pointer, P, is carried on a lever one end of which consists of a strip of iron or steel, which is attracted by the electromagnet core when a current flows through the exciting coils. When the current is off the lever is held in a position with the strip a short distance from the magnet by means of a spring.

Thus, as the cylinder rotates, a line is drawn round the surface. On passing the current in the coils the pointer moves and makes a kink in the line. If the current is put on at regular and known intervals, a time record along the side of the waves is produced on the drum. Thus by counting the number of waves between consecutive kinks the frequency of the fork may be deduced.

The regular intervals are obtained by connecting the wires, AA, to a battery, and completing the circuit by means of the mercury cup, M, and the pendulum, P, as shown in fig. 249.

Mercury is poured into a hole cut in wax, W, so that it stands just above the wax surface, and the wax so placed that a strip of wire hanging down from the pendulum just touches the surface as it passes its lowest position.

The circuit is completed twice in each complete period, and at each instant the pointer makes a record on the drum.

It is improbable that the interval between successive records will be one half a period, since this would require exact coincidence of the point of contact with the mercury and the lowest point of

the swing. The time between alternate records will, however, measure the time of a complete period.

Thus, in counting up the vibrations, find the mean number between alternate records.

Obtain a long helix, begin at the first stroke of the pointer, and count the number of vibrations up to some later odd numbered stroke. Find the mean number per complete period. Repeat this, beginning with the second stroke and ending at some later even numbered stroke, and find the mean number again. The two values should agree, but if there is a slight variation take the average value of the two results.

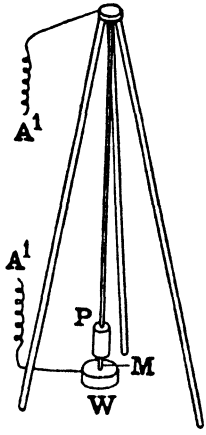


FIG. 249

To obtain a blackened surface, take a sheet of smooth white paper, and wrap it round the drum, one layer thick, holding it in position by means of gummed paper. To blacken it, rotate it over a turpentine flame, or coat it with camphor smoke.

The coating of soot should not be very thick, the style will then remove the soot and leave a white wavy trace.

The paper may be smoked again when once used and the track covered up.

The time of a complete period is measured in the usual way, by timing the pendulum.

The frequency obtained is, of course, subject to a correction similar to that of the last experiment, and this should be determined and applied in the manner described.

Fig. 250 represents a convenient method of carrying out the determination in an alternative way.

The cylinder is held vertically, and is rotated by a handle or string round a drum as shown.

The fork is maintained electrically as before, but the axle of the drum is connected to one secondary terminal of a small

induction coil, while the second terminal is connected to the fork. The primary circuit is completed through a pendulum and mercury cup as before. Each time the primary circuit is completed the induction coil is excited and a spark passes to the drum from the style, knocking off a little soot and leaving a white dot to record the instant of closing of the primary circuit. The same procedure and precautions are adopted as before. A thin, flexible copper wire will be found suitable for suspension of the pendulum, and with a small induction coil there is no inconvenience on account of shocks obtained when the apparatus is touched.

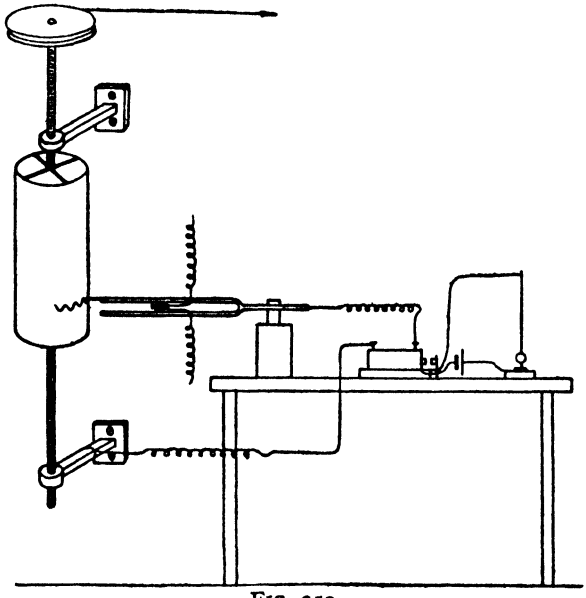


FIG. 250

This method was employed by A. M. Meyer. It should be noted that uniformity of motion of the drum is not necessary. The methods are both inferior to those in which an optical method is employed for determining frequency as in Rayleigh's method.

The Frequency of a Tuning Fork by the Stroboscopic Method

The fork is fitted with two very light plates fastened at the extremity of the prongs, one to each, and so that one may vibrate freely past the other. These plates may be of thin cardboard or aluminium, so that the loading affects the fork to the smallest possible extent, and they are stuck on to the fork by means of a little wax. A disc is taken provided with a number of dots placed at equal intervals round circles concentric with the disc. Each circle has its own interval length.

The fork is maintained electrically (p. 138), and the disc placed behind it with its dotted surface brightly illuminated. Each plate attached to the fork is provided with a slot, and when the fork is at rest the slots lie directly behind one another. Thus the disc can be seen through them.

The slots and one of the circles of dots are so placed that the dots can be seen by looking through the slot, as the disc slowly rotates. If the fork is vibrating it is possible to see through the slots twice in each complete period, and thus $2n$ times per second, where n denotes the frequency.

The disc is caused to rotate uniformly by means of an electric motor provided with a resistance in circuit to vary the speed.

The speed is gradually increased by adjusting the resistance until when the fork is vibrating the dots in one of the circles appear to be at rest.

In this case a dot moves up as the disc rotates, so that each time the slots are in line a dot is just in line with them. The eye sees apparently one stationary dot, and the effect of rotation is lost by looking through the slot.

In order to count the number of rotations per second made by the disc, a counting arrangement is attached and the time taken over a definite number of rotations, when the dots remain apparently steady, by means of a stop-watch.

Let there be N rotations per second in the case when the dots belong to a circle containing p of them.

In this case the time taken by a dot to take the place of the one preceding it is : $\frac{1}{Np}$ second. This is equal to one half the period of the fork.

Hence $$\frac{1}{Np} = \frac{1}{2} \cdot \frac{1}{n},$$

or $$n = \frac{1}{2} Np.$$

Now further increase the speed of rotation until once more the dots appear steady. In this case the speed of the disc is such as to cause a dot to take the place of the dot two intervals in front. The time taken to do this is $\frac{2}{N^1 p}$, where N^1 is the number of rotations per second.

From this we may calculate the value of n once more.

On further increasing the speed until a dot takes the place of another three intervals in front we can obtain a third calculation. Repeat this for several series of dots.

With the fork loaded it is necessary always to correct for the loss of pitch due to loading as before.

We may, however, avoid this by making one prong of the fork bright over a small area, and by rotating a disc provided with several series of concentric holes in front of it. If the fork is well illuminated and the disc carefully mounted, it will happen that for some particular speeds the fork appears stationary when seen through the holes.

The experimental details and the mode of deduction of the frequency of the fork from the observations are similar to the former.

The observations in stroboscopic experiments can nowadays be made much more conveniently by the use of a neon lamp than by the method described above.

This lamp has the property that it lights up immediately a voltage is supplied to it without any appreciable lag, and is extinguished immediately when the voltage is taken away.

The lamp consists of a flat piece of aluminium with a rod of the same metal lying a short distance from it and parallel to it in a bulb containing neon at low pressure.

When included in the secondary circuit of an induction coil, it lights up each time the current in the primary is broken. There is no effect at 'make,' because in the construction of the usual type of induction coil the effect at make is suppressed and that at break of the primary circuit intensified.

If the primary coil is put into the circuit which actuates the fork the primary current is made and broken once per vibration of the fork and the lamp flashes out once per complete vibration.

Of course, the make and break attached to the primary must not be allowed to work, the fork takes its place. All that is necessary is to make the connexions in the usual way and to screw the platinum-iridium point close up to the clapper to prevent separation and to place some object, e.g. a small block of wood, between the soft iron on the clapper and the end of the armature to prevent oscillation as the current fluctuates.

The lamp is used to illuminate the rotating disc provided with a series of dots on a white background and the speed adjusted until one row of dots appears stationary.

When this is the case one dot just moves up to take the place of a dot somewhere in front of it during the interval of darkness between the flashes of the lamp, i.e. during the period of vibration of the fork.

The calculation of frequency is made as before.

It is a great advantage to be able to avoid loading or marking the fork and to have the disc in any convenient position where it may easily be observed.

Instead of a series of concentric circles with regularly spaced dots, it is more convenient to draw on a circular disc a series of concentric regular polygons. A small triangle is drawn just

about the centre of the disc, about this a square, then a pentagon, and so on. The triangle may be coloured white, the space between it and the square blackened, the space between the square and pentagon coloured white, and so on alternately. When the speed is adjusted exactly one of these figures appears stationary and the frequency is easily calculated. By varying the speed the figures may be made to appear stationary in turn and several determinations of frequency made.

The Composition of Two Simple Harmonic Vibrations in the Same Direction (Beats)

An apparatus which will combine graphically two simple harmonic vibrations in the same direction has been invented by Koenig. It consists of a large fork mounted on a stand and provided with a clamp by means of which a strip of glass can be held horizontally and fastened to one of the prongs (fig. 251).

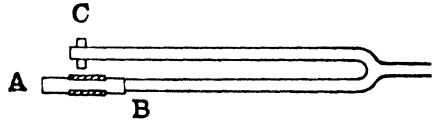


FIG. 251

AB denotes the strip and C a weight, attached to the other prong for the purpose of balancing. The glass is coated with a thin layer of lamp-black, by means of a smoky flame.

A second fork is mounted above the first and carries a light style adjusted so that it just touches the glass plate.

This fork is fixed to a sliding base by means of which the style can be drawn along the smoked plate.

If both style and plate are vibrating a curve can thus be traced which represents the motion of the upper fork relative to the lower.

Thus if at any instant the lower fork is displaced a distance, y, from the standard position, and the upper is displaced a distance, y^1, the displacement of the style over the plate is $(y^1 - y)$. The forks are made to vibrate with nearly equal amplitudes. This may be done by bowing or by using a strip of metal which is wide enough to open out the forks to a convenient extent; the metal is then quickly removed. The slider is drawn along, not too quickly, and the trace obtained examined.

When the forks have nearly the same frequency this will consist of a wavy line with waves of varying amplitude. In this case, with the amplitudes of the forks nearly equal, the smallest waves will have almost zero amplitude, and the fluctuation expresses graphically what the ear recognizes as beats.

The time between consecutive minima measures the interval between the beats.

If the glass plate is long enough three or four intervals extending between consecutive minima will be obtained. The speed of motion of the slider must be adjusted so that the individual vibrations are drawn out to an extent which enables them to be counted easily.

It is best to count the number of intervals on the plate and the total number of vibrations between the first and last minimum points. This enables the mean number of vibrations between consecutive beats to be obtained.

Let this number be x.

We shall suppose that the two vibrations have frequencies n and $(n + m)$ per second, so that they may be represented by:

$$y^1 = a \sin 2\pi (n + m) t$$

and

$$y = a \sin (2\pi nt + \alpha),$$

where α is included to take account of any difference in phase that may exist when the vibrations begin.

The resultant displacement recorded on the plate is:

$$Y = (y^1 - y) = a \{\sin 2\pi (n + m) t - \sin (2\pi nt + \alpha)\}$$

$$= 2a \cos \left\{2\pi \left(n + \frac{m}{2}\right) t + \frac{\alpha}{2}\right\} \sin \left(2\pi \frac{m}{2} t - \frac{\alpha}{2}\right).$$

In the cases when beats are heard, m is much smaller than n or $\left(n + \frac{m}{2}\right)$, so that the simplest way of regarding this expression is to consider it as a S.H.M. of amplitude:

$$2a \sin (2\pi \cdot \tfrac{1}{2}mt - \tfrac{1}{2}\alpha) = \text{A (say)},$$

and then

$$Y = A \cos \left\{2\pi \left(n + \frac{m}{2}\right) t + \tfrac{1}{2}\alpha\right\}$$

represents a S.H.M. of amplitude, A, and frequency, $\left(n + \frac{m}{2}\right)$.

The amplitude, A, attains a maximum value, $2a$, and sinks to zero alternately. It is zero for the values of t given by:

$$\pi mt - \tfrac{1}{2}\alpha = 0, \pi, 2\pi, \text{etc.},$$

i.e. for values of $t = \frac{\alpha}{2\pi m}, \frac{\alpha}{2\pi m} + \frac{1}{m}, \frac{\alpha}{2\pi m} + \frac{2}{m}$, etc.,

i.e. at intervals of $\frac{1}{m}$ secs., or m times per second.

Thus the number of beats is m per second, the same number as the difference between the frequencies of the two notes. The frequency of the note heard is $\left(n + \frac{m}{2}\right)$.

In the curve traced on the plate we shall find $\left(n + \frac{m}{2}\right)$ vibrations per second and m beats. The number of vibrations between two beats is :

$$\frac{1}{m}\left(n + \frac{m}{2}\right).$$

But this is counted on the plate and found to be x.

Thus $$\frac{1}{m}\left(n + \frac{m}{2}\right) = x,$$

or $$n = m\,(x - \tfrac{1}{2}).$$

The beats can be timed by means of a stop-watch.

When the notes are sounding, count them for as long as possible, and take the time of the interval. If the number of beats counted is N and the stop-watch is started on the first and stopped at the Nth, the interval between the beats is :

$$\frac{T}{N - 1},$$

where T is the time interval recorded on the stop-watch.

Thus $\frac{1}{m}$ is known, and the above equation gives n. The frequencies of the notes are thus, n and $(n + m)$.

The method must be regarded as an illustration of the phenomenon of beats—it is not an accurate method for the determination of frequency.

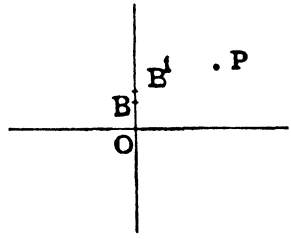


FIG. 252

The Composition of Two Simple Harmonic Vibrations Perpendicular to one another. (Lissajou's Figures)

Let the co-ordinates of a point, P (fig. 252), be (x, y) and let P move so that

$$x = a \sin pt$$
$$y = b \sin (p^1 t + \alpha).$$

The motion of P then consists of two simple harmonic motions along two perpendicular directions.

When t is zero, x is also zero ; but y has the value $b \sin \alpha$.
Thus the two S.H.M.'s are in different phases.
From the equations it follows that :

$$t = \frac{1}{p} \text{ arc sin } \frac{x}{a}$$

and

$$t = \frac{1}{p^1} \text{ arc sin } \frac{y}{b} - \frac{\alpha}{p^1}.$$

Hence between x and y there exists for all values of t the relation :

$$\frac{1}{p} \text{ arc sin } \frac{x}{a} = \frac{1}{p^1} \text{ arc sin } \frac{y}{b} - \frac{\alpha}{p^1}.$$

This represents the curve on which P lies.
When the relation between p and p^1 is simple, as for example :

$$p = p^1, \quad p = 2p^1, \quad 2p = 3p^1, \text{ etc.,}$$

the point, P, describes the curve completely in a short time, and afterwards retraces it.
The case, $p = p^1$, is very simple, for we then have :

$$x = a \sin pt,$$
$$y = b \sin (pt + \alpha).$$

In the general case this represents an ellipse with its centre at the origin.
If $\alpha = 0$, then :

$$\frac{x}{a} = \frac{y}{b},$$

and P describes a straight line.

If $\alpha = \frac{\pi}{2}$, P describes the curve :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is an ellipse whose principal axes lie along OX and OY.
In the case when the amplitudes of vibration are equal along both directions, $a = b$ and the locus of P is the circle :

$$x^2 + y^2 = a^2.$$

Thus when the periodic times are equal, in general P describes an ellipse ; special cases of this general case are the straight line and circle. All these curves are therefore appropriate to the case of equal frequencies.
If OB is equal to $b \sin \alpha$, B represents the position of P at the time $t = 0$, or at instants later by a complete period than this initial time.

If the frequencies are not quite equal, let us say that the frequency along OY is a little the greater, then when another period is complete the displacement along OY will be a little different from that in the first case, i.e. P will lie at some point, B^1.

If $OB^1 = b \sin \alpha^1$,

α^1 is slightly different from α,

and there has been a slight change in phase on the part of the OY motion. Thus α will continually vary, and will pass through all the values from 0 to 2π. This will cause a continual change in the shape of the curve described by P. It will sometimes be a straight line and sometimes an ellipse. If it happens that the two amplitudes are equal, we shall have a circle sometimes.

The closer p^1 is to p, the slower will this change take place, so that by watching the movement of P we can test the closeness of the two frequencies. If the figure is maintained steady, without change, the frequencies are equal.

The same argument can be applied to the cases where one of the other simple relations exists between p and p^1.

When the shape of the curves corresponding to these relations is known the approximate ratio of the frequencies can be recognized and the exactness of the ratio tested by observing the rate of change of shape throughout the series.

We can deduce the ratio of the frequencies by examining one of the curves, e.g. fig. 253. It cuts the Y axis in four points and the X axis in three, so that the frequencies are in the ratio, 4 : 3, for the vibrating point makes four vibrations parallel to OX in the same time that it makes three parallel to OY.

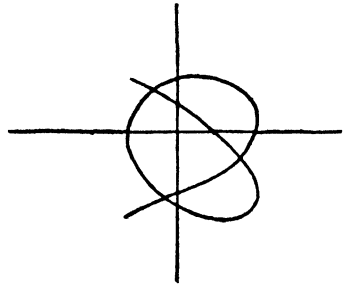


FIG. 253

This principle may be employed to investigate the way the period of vibration of a rod, fixed at one end, varies with the length of the rod. The apparatus consists of a vertical flat rod or spring with a lens fixed at its upper end and a horizontal spring carrying a screen with a small aperture (fig. 254).

Light from a source, S, is focussed by a lens, L_1, on to the aperture, A, while the lens, L_2, carried by the vertical spring, V, focusses an image of the aperture on the screen so that we have a bright point, P.

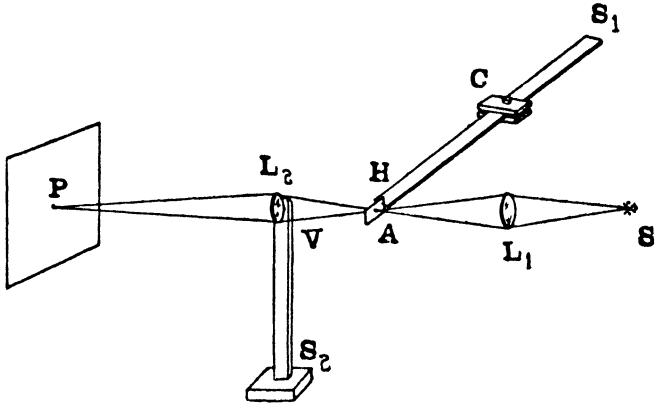


FIG. 254

Motion of the horizontal spring alone, causes P to trace a vertical line, and represents the S.H.M. of A.

Motion of the vertical spring alone, produces a horizontal S.H.M. on the screen. When both springs vibrate together, the path of P represents the combination of the two motions.

The vertical spring is of fixed length, but the horizontal spring can be clamped at various points, so that the vibrating length can be adjusted.

Make the first adjustment so that P describes an ellipse, straight line, or circle without change of form.

In this case the periods are the same.

Now change the length until another steady curve is obtained without change of form. Draw it carefully and deduce the ratio of the periods of vibration.

Make several determinations of the ratio of frequencies and corresponding lengths, and draw a curve showing the relation between the length and frequency, taking the vertical spring as a standard.

Fig. 255, shows the curves described by P for a few frequency ratios which will serve for reference.

The Vibration Microscope

The essential features of this apparatus are the same as those described in the last experiment. The two vibrating springs are replaced, one by a fork and the other by a vibrator, the frequency of which is to be compared with that of the fork.

The second vibrator is sometimes another fork or violin string.

A bright source of light such as a speck of chalk is attached to the second vibrator, while the fork carries, attached to one prong, a lens which forms the object glass of a small microscope.

If the lens alone vibrates, on looking through the eyepiece the motion of the chalk is simple harmonic, on account of the vibra-

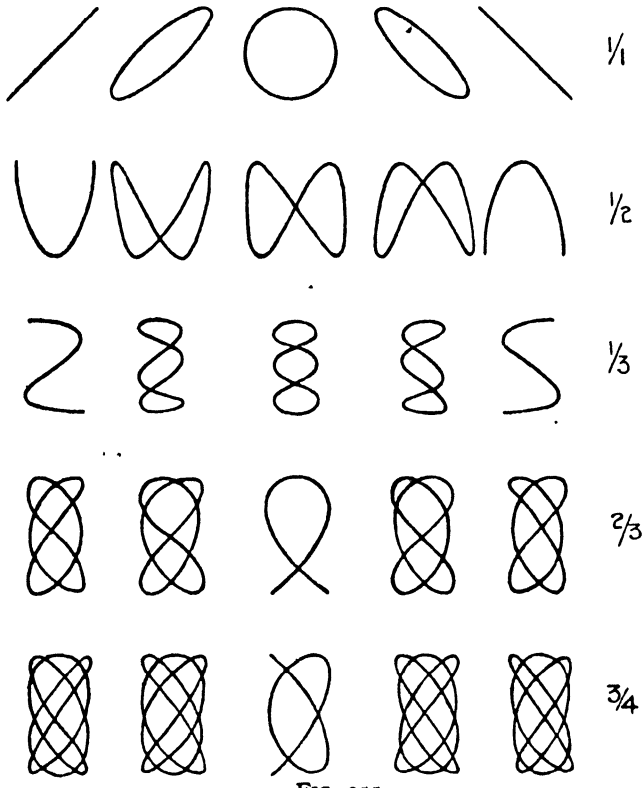


FIG. 255

tions of the lens. If the other vibrator is in motion and the lens is at rest, the motion observed is, of course, that of the vibrator alone. These two motions are arranged to take place in two perpendicular directions so that a figure of the type described in the last experiment is observed. It is steady if the frequencies are exactly adjusted, but goes through the appropriate series if the frequencies are not identical. The rate of progress through the series may be observed and determined by means of a stop-watch. After one completion of the cycle there has been a gain of a whole vibration by one vibrator over the other. If the frequencies be N and N^1, and the time for completion of the series is t seconds, then

$$N \sim N^1 = \frac{1}{t},$$

for Nt and N^1t are the numbers of vibrations made respectively, and these differ by one.

If the fork be slightly loaded we can find which is the greater of the frequencies by again observing the rate of progress through the cycle. If this time is shorter than before, the time of gaining a period is less than before and the frequency of the fork is the smaller.

We shall describe how the apparatus may be used to find out the character of the vibrations performed by a stretched string in the manner in which Helmholtz used the apparatus to examine the vibrations of a violin string.

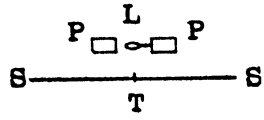


FIG. 256

The string, SS, is stretched below the prongs of the fork, PP, to one of which the lens, L, is attached. The part of the string to be examined is slightly blacked by ink, rubbed with wax, when dry, and powdered with starch or chalk. A few white particles will remain sticking to the string, and one of them is illuminated by a lamp and focussed by the microscope and its movements observed. The tension of the string is adjusted until the figure apparently described by the chalk, as seen in the microscope, remains steady. The vibrations of the fork are electrically maintained while the string may be bowed or plucked.

The frequency of the fork is known, so that, as its motion is simple harmonic, we can find the displacement due to it at any of the instants during the vibration. The string vibrates at right angles to the direction of vibration of the fork, so that

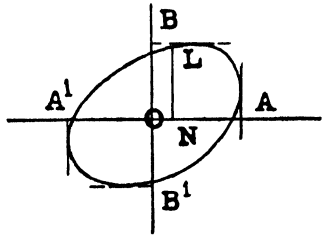


FIG. 257

from the curve obtained we can subtract the vibrations of the lens and draw a curve showing the displacement of the string at different times.

The white speck is first obtained in the centre of the field, and its mean position represented by the origin, O (fig. 257).

The curve is drawn to a convenient scale accurately from measurements observed by the scale in the eyepiece. If the microscope is not furnished with a scale, throw an image of an illuminated scale to lie coincident with the string and view the white speck and image together.

In the figure, OA denotes the amplitude of vibration of the fork, its motion being assumed to take place along AOA^1.

If its frequency is n, the motion is given by:

$$x = OA \sin 2\pi nt.$$

Thus for any time, t, we can find the displacement, ON, along the x-axis. At this instant the displacement along the other direction, i.e. due to the motion of the string, is NL. Thus by observing several values of the ordinates and the times corresponding, we can plot a second curve showing the time-displacement for the string. Its shape will indicate the character or quality of the note emitted.

Fig. 258 shows the observed curve and the time-displacement diagram for a stretched string when bowed. The first curve was observed while the string was being bowed. The bow is drawn slowly and regularly across the string. Slight fluctuations are liable to occur during this process, but they appear as slight variations of a figure remaining on the whole permanent. The dimensions of this figure were obtained.

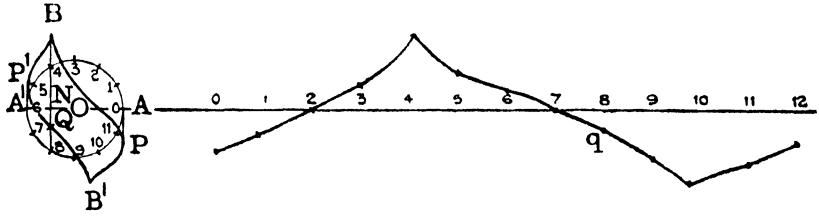


FIG. 258

The deduction of the diagram for the string is made in the following way:

Draw on squared paper the figure observed (fig. 258), and draw the extreme vertical tangents, AP and A^1P^1. We are assuming that the vibrations of the fork are executed along AA^1.

With the middle point of AA^1 as origin, describe a circle on AA^1 as diameter, and divide the circumference into a convenient number of equal parts. In the diagram the number is twelve.

If a perpendicular be drawn from any point on the circumference of this circle on to AA^1, the displacement of the foot of this perpendicular from O will represent the displacement of the prong of the fork from its mean position.

Along the line, A^1A, produced, beginning at 0, twelve equal

intervals are marked off, as shown, to represent intervals of time corresponding to the points on the circle. Beginning at A, draw ordinates to the curve passing through the points marked on the circle. For convenience in drawing only one of these, that through the point, 8, is drawn. Let this cut the curve at the point, Q, and the line, AA^1, in N. Then NQ denotes the displacement of the string from its central position in magnitude and direction.

This displacement is plotted at the point, 8, on AA^1 produced, and we thus obtain a point, *q*, on the displacement diagram for the string. This process is carried out for all the points on the circle, and the diagram plotted.

It will be noted that the ordinate, NQ, cuts the curve in a second point, B. There is no doubt as to which point is to be taken when actually drawing the curve, for we begin at A and pass along one branch of the curve and back along the other.

In this discussion we have associated the upper part of the curve from P to B and then to P^1, with the times corresponding to the points from 0 to 6.

Transverse Vibrations of Strings. (Melde's Experiment)

The object of the experiment is to verify the laws of vibration of a string under tension. In such a case a disturbance travels along the string with a velocity, *v*, given by :

$$v = \sqrt{\frac{T}{m}}.$$

T denotes the tension expressed in absolute units, i.e. poundals or dynes, and *m* is the mass per unit length.

When the string is fixed at both ends, there is a node at each end in its fundamental mode of vibration, with a loop or antinode in the middle. The corresponding wave length is twice that of the string. If this wave length is denoted by λ_1, we have :

$$n_1\lambda_1 = v = \sqrt{\frac{T}{m}},$$

$$n_1 = \frac{1}{\lambda_1}\sqrt{\frac{T}{m}},$$

n_1 denoting the frequency of this note.

This mode is illustrated in fig. 259. The string may also vibrate to produce the overtones or harmonics as illustrated in figs. 260 and 261.

In these cases, if the frequencies are n_2 and n_3, and the corresponding wave lengths λ_2 and λ_3, we have :

$$n_2 = \frac{1}{\lambda_2}\sqrt{\frac{T}{m}}, \text{ and } n_3 = \frac{1}{\lambda_3}\sqrt{\frac{T}{m}};$$

and if l denote the length of the string :

$$\lambda_1 = 2l, \ \lambda_2 = l, \ \lambda_3 = \frac{2}{3}l, \text{ and so on.}$$

Thus $n_1 : n_2 : n_3 : \ldots = 1 : 2 : 3 : \ldots$

FIG. 259

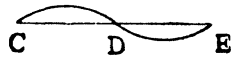


FIG. 260

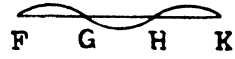


FIG. 261

In the experiment the string is set in resonant vibration by impulses having the same frequency as one of its modes of vibration. For this purpose ordinary string is unsuitable. It is not uniform and does not divide into equal segments; but it will be found that a length of fishing line is satisfactory, as a rule it is sufficiently uniform.

One end of the cord is attached to the prong of a tuning fork, by tying it to a small wire hook soldered on to the prong, or to a small screw which is held in a hole bored in the prong.

The other end passes over a small pulley and carries a weight which produces and measures the tension.

The vibrations of the fork are electrically maintained (see p. 138) and by properly adjusting the length and tension the string can

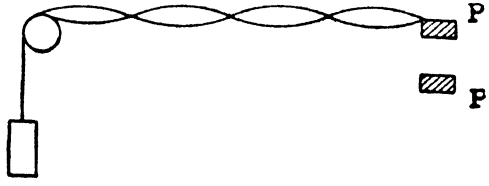


FIG 262

be made to break up into stationary undulation with well-defined nodes.

The fork may be placed so that the motion of the prongs is in the direction perpendicular to the string (fig. 262), or along it (fig. 263).

In the former case the frequency is the same as that of the fork, in the latter it is half as great.

For when in the second case the prong is in the extreme position on the left the string is slack in the first vibration, and when in

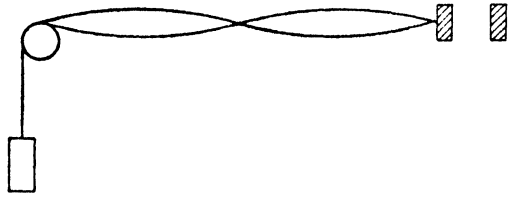


FIG. 263

the extreme position on the right it is horizontal and tight. The inertia of the string carries it onward so that when the prong returns to the extreme left position, and thus completes one vibration, the string completes a half vibration.

The student should examine the interesting effects produced when the prong moves in a direction between these two and thus produces in the string a combination of the two modes of vibration.

We shall consider the former case only in this description of the experiment.

The frequency of the fork is denoted by N, and when the string is in resonant vibration, this is also the frequency of the mode of vibration of the string, corresponding, let us say, to a wave length, λ.

Then :
$$N = \frac{1}{\lambda}\sqrt{\frac{T}{m'}}$$

or
$$\frac{\lambda^2}{T} = \frac{1}{mN^2} = \text{constant.}$$

Thus by varying the tension and consequently the wave length, we should find $\frac{\lambda^2}{T}$ constant.

To determine λ, measure the distance between the first well-defined node on the right of the string and the last on the left. Let this be d and suppose there are k loops between, then,

$$\lambda = \frac{2d}{k}.$$

Note that the ends of the string at the fork and the pulley should not be taken. There is a certain amount of movement at these two points.

Draw up a table showing values of λ, T, and $\frac{\lambda^2}{T}$.

KUNDT'S TUBE

(*a*) The Determination of the Velocity of Longitudinal Waves along Rods

Kundt's apparatus consists of a glass tube about a metre long, and of diameter about 3 cms., provided with an adjustable piston near one end. The tube is supported horizontally on a table by resting it on two wooden V-shaped stands.

Near the other end of the tube is a second piston, Q, attached to the end of a metal rod, DQ (fig. 264). This rod is clamped at its middle point, C.

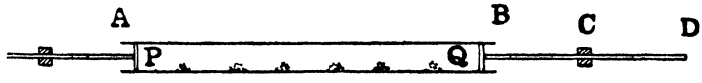


FIG. 264

For the purpose of the experiment the tube must be quite dry, and a light powder, such as lycopodium powder, is placed in a line at the bottom of the tube extending along its length between the two pistons. A convenient way of inserting the powder, is to spread it along a metre rule, place the rule in the tube and turn it upside down.

If the tube is not dry it must be warmed above a Bunsen flame, and a current of air blown through it.

The metal bar, which may be of brass with a diameter of about ·5 cm., can be set into longitudinal vibration by stroking it along CD with a piece of wash-leather and powdered resin.

In the fundamental mode of vibration the ends, D and Q, are antinodes, and the fixed point, C, is a node. The wave length is twice the length of the rod.

$$\lambda = 2l.$$

If the frequency of the note is n, and the velocity of the waves, v,

$$n\lambda = v.$$

If the distance between the pistons is L, the air between them will have a fundamental wave length, 2L, and overtones with corresponding wave lengths:

$$\mathrm{L},\ \tfrac{2}{3}\,\mathrm{L},\ \tfrac{1}{2}\,\mathrm{L},\ \text{etc.}$$

The corresponding frequencies are:

$$\frac{\mathrm{V}}{2\mathrm{L}},\ \frac{\mathrm{V}}{\mathrm{L}},\ \frac{\mathrm{V}}{\frac{2}{3}\mathrm{L}},\ \text{etc.},$$

where V denotes the velocity of sound in air.

If one of these frequencies is the same as that of the rod, the air will be set into strong resonant vibration, and will move the light powder. This will settle down at and near the nodes where

the air is least in motion. As many as possible of these should be used to find the average distance between two nodes. Choose as carefully as possible the position of a node at one end of the tube and locate the node nearest the other end. Measure the length between these two points, and divide by the number of spaces, such as NM (fig. 265).

FIG. 265

The pattern will be somewhat similar to that in this figure, and the longest line of each set marks the nodal position.

It will probably happen that there is not strong vibration of the air at first, but by slowly moving the piston, P, forward or backward, the length between the pistons may be adjusted so that resonance occurs.

Twice the distance between the nodes is the wave length of the sound in air. The velocity of sound at 0° in air is 33,060 cms. per second, and at a temperature, t:

$$V_t = 33060\left(1 + \frac{t}{273}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Thus the frequency, $n^1 = \frac{V_t}{\lambda^1}$, where λ^1 is the wave length in air. Since resonance occurs,

$$n = n^1;$$

$$\therefore \quad \frac{v}{2l} = \frac{V_t}{\lambda^1},$$

or

$$v = \frac{2l}{\lambda^1} V_t.$$

In carrying out the experiment it is a good plan for one observer to continue stroking the rod, while the other carefully adjusts the piston until the powder moves violently and settles down into the pattern of fig. 265.

(*b*) The Velocity of Torsional Vibrations in a Rod

If instead of stroking the rod longitudinally with the resined cloth, it is held near the end, D, and the cloth turned so that it slips over the surface in a direction that would cause the rod to rotate round its axis, a note is emitted of different frequency from that given when the rod is in longitudinal vibration. This note corresponds to torsional vibrations and will set the air in resonant motion as before.

Find in this way the velocity of these waves.

It is not easy to obtain a loud note by this method—the force applied should not be great, but, with a little practice, it should be possible to produce the note.

(*c*) The Determination of Young's Modulus and the Modulus of Rigidity

These constants may be determined from a knowledge of the velocity of longitudinal and torsional waves in the bar.

The formula for the former is :

$$v_l = \sqrt{\frac{E}{\rho}},$$

and for the latter :

$$v_t = \sqrt{\frac{n}{\rho}},$$

where E is Young's Modulus, n the modulus of rigidity, and ρ the density of the material of which the bar is made. Its value may be taken from a table of physical constants. When expressed in C.G.S. units, E and n are expressed in dynes per sq. cm.

Find the values of E and n from the determination of the velocities in the previous experiments.

(*d*) The Velocity Sound in Carbonic Acid Gas

Kundt's tube may also be used to determine the velocity of sound in gases. Suppose the tube to be filled with a gas in which the velocity is V_g and the wave length, λ_g.

Then the frequency is

$$\frac{V_g}{\lambda_g} = \frac{V}{\lambda^1};$$

$$\therefore \quad \frac{V_g}{V} = \frac{\lambda_g}{\lambda^1} = \frac{\text{distance between nodes in the gas}}{\text{distance between nodes in air}}.$$

The procedure is, therefore, first to obtain resonance between the rod and air column, and to find the mean distance between the nodes, then to drive out the air and fill the tube with the gas, and again obtain resonance, and measure the distance between the nodes in this case.

V_g is then found from the last equation.

The gas must be quite dry or the powder will stick to the glass and fail to respond to the motion of the gas when resounding. It may be necessary to pass it through drying tubes before filling the tube.

A slight modification of the apparatus is necessary for this purpose (fig. 266).

In the former case the adjustable piston may fit loosely, but for the present purpose it must be both adjustable and gas-tight.

It may consist of a cork round the outside of which a rubber band or piece of cloth is stretched.

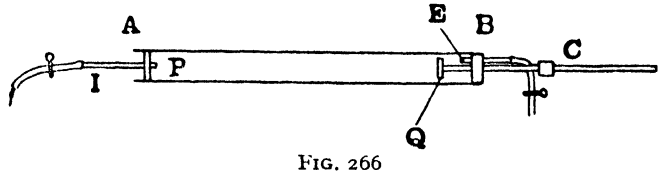


FIG. 266

The cork carries a tube to admit the gas and the piston may be adjusted by means of this tube.

The metal rod passes through a tightly fitting cork, also provided with a tube which can be opened and closed by means of a stop-cock.

We shall suppose that the velocity of sound in carbon dioxide is to be measured.

Connect the source of gas to the inlet tube, I, and open the exit tube, E, to the air. Allow the gas to flow in steadily so that it will fall to the bottom of the tube and the air will flow over at E, also, so as not to disturb too much the powder which we assume has been spread out along the bottom of the tube.

The experiment should be performed close to an open window or in a draught cupboard to prevent escape of the gas into the room.

Continue the passage of gas long enough to ensure that the air is driven out and then close the inlet and outlet tubes by means of stop-cocks, and proceed as in the last experiment.

Care must be taken that the temperature of the gas is the same as that of the air, unless the temperature of the gas is measured in some way. If the gas is delivered from a cylinder it will be colder than the air and it must be allowed to acquire the air temperature before closing the stop-cocks. Otherwise we shall be measuring the velocity in the air at one temperature, and that in the gas at another. Reduce the velocity to that at zero.

To do this, note that : $\frac{V_g^t}{V^t} = \frac{V_g^0}{V^0}$ = ratio of nodal distances,

where the affixes t and o denote temperatures.

In order to determine when the air is all driven out from the tube, collect a little of the gas issuing from E, in an inverted glass cylinder over mercury, and introduce on to top of the mercury column a little of a solution of caustic soda or potash. By noting how much of the gas is absorbed it can be seen if any air is left. The CO_2 is all absorbed by the solution and no gas should be left.

(e) To Calculate the Ratio of the Specific Heats of a Gas

From the result of the last experiment we may determine the constant, γ, for carbon dioxide, i.e. the ratio of the specific heat at constant pressure to that at constant volume.

For the velocity of sound in a gas at 0°, is given by the formula :

$$V_g{}^0 = \sqrt{\frac{\gamma p}{\rho_0}},$$

where p is the pressure and ρ_0 the density of the gas at 0° C. The value of p may be determined by the barometer since the tube has been filled at atmospheric pressure.

p must be expressed in dynes per sq. cm. To make the calculation, take the density of mercury as 13·60 gm. per c.c., and $\rho_0 = \cdot 001974$ gm. per c.c.

The measurement of the distances between nodes may be performed simply by the ordinary use of a metre scale ; but a slight addition to the apparatus will add to the accuracy.

A metre rule is fixed just below the tube and parallel to it, and sliding over the rule or along one of its edges is a wooden base, B, carrying a metal disc, D, with a hole, H, at the centre, and a frame, F, with cross-wires, C (fig. 267).

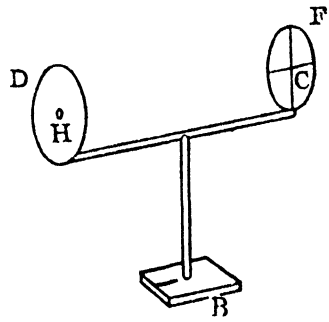


FIG. 267

H serves as an eyepiece and HC is aligned on the nodes marked out by the powder.

An index on the base indicates the position of the stand.

If the apparatus is aligned consecutively on the nodes, and the

positions of the index recorded, a table may be made out as follows :

NO. OF NODE	READING AT INDEX	NO. OF NODE	READING AT INDEX	LENGTH OF 5 HALF WAVES
1		6		
2		7		
3		8		
4		9		
5		10		
	Average Length of 5 Half Waves ..			
	Mean Half Wave Length			
	Mean Wave Length			

Chladni's Figures

The nodes of a vibrating stretched string are points of its length where, theoretically, there is no motion. On either side of the node the string moves simultaneously in opposite directions. In the case of a vibrating plate there exist nodal lines, i.e. lines in the plate where there is no motion. They divide the plate into segments so that the parts on either side of the nodal line at any instant are moving in opposite directions.

The point of support of a plate is necessarily on a nodal line. Round and square plates are usually supported centrally in the experiment, but interesting results arise when they are supported elsewhere.

By touching any point on the edge or surface of the plate with the finger nail, the plate is prevented from vibrating at the point, and if it lies on a nodal line the corresponding mode of vibration may be excited.

In the case of a circular plate clamped at the centre the nodal lines are radial, and the fundamental vibration gives two perpendicular diameters. These may be obtained by sprinkling white sand over the surface of the plate, touching two points on

the edge separated by one quarter of the circumference, and bowing vertically across the edge at a point one-eighth of the circumference from one of the fingers. A square plate clamped at the centre gives a large variety of figures. Some of these are shown in the figure in order to assist in producing them.

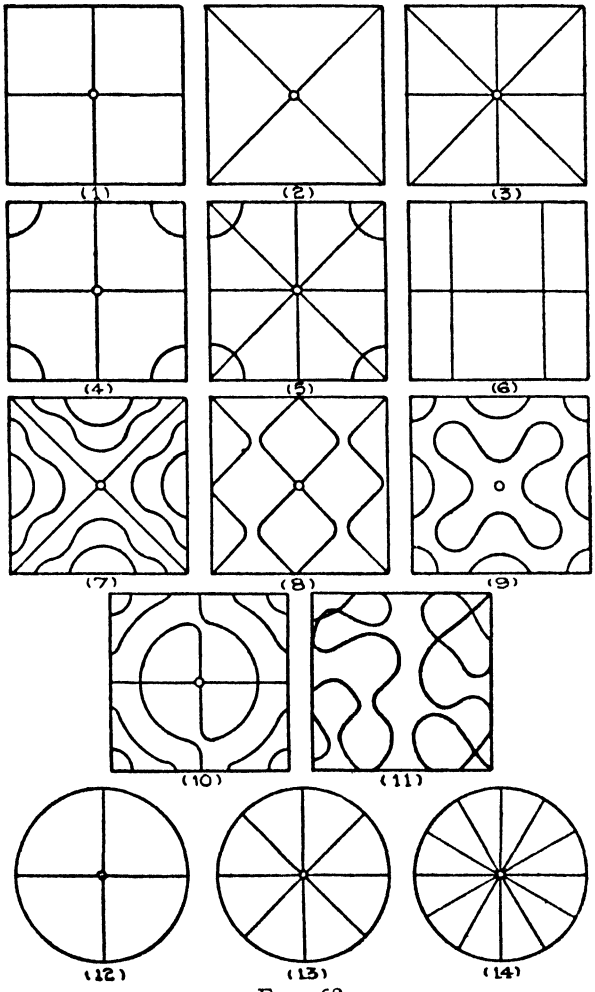


FIG. 268

Some of the nodal points on the edge and surface of the plate should be touched, and the plate bowed along the edge at one of the points midway between the nodes. The fundamental vibration gives two perpendicular lines through the centre parallel to the edges of the plate.

The simpler figures are easy to obtain, but skilful manipulation of the bow is required to produce the more complicated ones.

The student should obtain as many as possible, and clamp the plate at other points than the centre to investigate other modes of vibration.

The Relation between Pitch and Volume in the case of a Narrow-necked Resonator

Consider the case of a bottle containing air closed by a piston, P, without friction in the neck of the bottle.

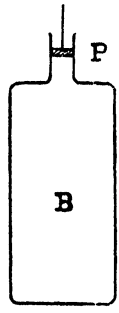


FIG. 269

Let v denote the value of the bottle below the piston, m the mass of the piston, and A its area of cross-section. Let the piston be originally in the position of equilibrium and let the pressure outside be p_0, while that inside is p. Then we have :

$$pA = p_0A + mg.$$

If now the piston be displaced downwards a distance, x, so quickly that the change may be regarded as adiabatic, a new pressure, p^1, will be generated, such that :

$$p^1 (v - Ax)_\gamma = pv^\gamma.$$

We shall suppose that x is small, so that we may write :

$$p^1 = p\left(1 - \frac{Ax}{v}\right)^{-\gamma}$$

$$= p + \frac{p\gamma A}{v} x.$$

The total force downward is now :

$$mg + p_0A - p^1A = (p - p^1)\,A$$

$$= -\frac{p\gamma A^2}{v} x.$$

Thus the equation of motion is :

$$m\frac{d^2x}{dt^2} = -\frac{p\gamma A^2}{v} x.$$

This is a simple harmonic motion and the time of vibration about the position of equilibrium is :

$$2\pi\sqrt{\frac{mv}{p\gamma A^2}} \qquad \text{(see p. 26)}$$

In the calculation we have assumed that the pressure is the same throughout the gas during the oscillations.

This is not true since time is required for the transmission of the pressure. The other assumption concerning the adiabatic character of the compression is very approximately true, especially as the neck is narrow and heat will not easily escape from the bottle or be transmitted to it. The piston will thus behave only roughly according to the formula, and will have an approximate period of oscillation given by the above value, i.e. it will have a frequency :

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{p\gamma A^2}{mv}}.$$

Thus we have approximately :

$$n^2 v = \text{constant}.$$

It is the object of this experiment to verify to what extent the gas behaves according to this formula.

The piston, in practice, is the layer of air in the neck of the bottle and by pouring water into it, resonance is produced between it and a series of forks of known frequency. A medicine bottle will be found convenient and must first be calibrated by pouring in water from a measuring flask to various depths and measuring the height of the water surface above the bottom of the bottle. This should be done for a series of intervals up to the base of the neck.

Make a table thus :

HEIGHT OF WATER	VOLUME POURED IN	VOL. OF AIR ABOVE HEIGHT IN COL. I.
h_1	v_1	$v_n - v_1$
h_2	v_2	$v_n - v_2$
h_3	v_3	$v_n - v_3$
.	.	.
.	.	.
.	.	.
h_n	v_n	0

v_n denotes the capacity of the bottle.

Draw a curve showing the relation between the heights of the water and the volumes of air above it, i.e. draw a curve with the values h as abscissæ and the volumes in column 3 as ordinates. By means of this curve we can deduce the volume of air in the bottle corresponding to various heights of water.

Forks of pitch varying from 256 to 512 should be used and a curve plotted with volumes as ordinates and values of $\frac{1}{n^2}$ as abscissæ. The resulting curve should be a straight line through the origin if the theory is correct, but in practice it will be found not to pass through the origin although the relation is very nearly linear. The curve will obey closely the law :

$$n^2 (v + c) = \text{constant.}$$

The value of c is a correction to be applied to v.

This may be regarded as a neck correction, and the ratio of c to the volume of the neck should be recorded.

The agreement of pitch between the fork and bottle should be tested by blowing across the neck and noting beats between the note obtained and that of the fork.

The Interference of Sound Waves. Determination of a highly pitched Note by means of a Sensitive Flame

When a continuous note is produced in front of a smooth wall the reflected and incident trains of waves produce stationary undulation in front of the wall, and consequently there exist nodes and antinodes in the air. The experiment consists in locating the positions of consecutive nodes from the wall outwards. Thus the wave length is determined, for the distance between two consecutive nodes is half a wave length. By noting the air temperature and deducing the velocity of sound in air appropriate to it (eq. (A), §10 of this chapter), we may thus deduce the frequency of the note by the relation :

$$V = n\lambda.$$

In order to locate the nodes a sensitive flame is used. This flame is produced by supplying the gas under pressure to a pin-hole burner. The requisite pressure can be obtained by leading the gas from the main into a large gas bag and placing a board on the top of the inflated bag to carry a weight.

From the bag a pipe leads the compressed gas to the burner. When the bag is full, turn off the main tap and put weights on the board so that a tall flame from one to two feet high is produced and there is no flaring. It will be found that in the sensitive state on making slight noises as, for example, by jingling

keys in the neighbourhood of the jet, the character of the flame changes. It flares and shortens, recovering its former state when the sound ceases. This is caused by the motion of the air as the sound wave passes.

Start with the flame close to the wall and sound a note from a highly pitched whistle, or other suitable source, and move the flame slowly outward from the wall.

It will be found that at certain points the flaring ceases and the flame increases in length. At these points the air is still, and the points are at nodes of the stationary wave motion.

If a smooth wall is not conveniently situated, set up a large sheet of glass or smooth board at the end of a table, and move the flame outward from the surface along the table. The flame will probably need adjusting before it will respond readily. To do this use the tap leading from the bag to the burner, and also vary the weights producing the pressure. It appears to be necessary to use a rather long gas tube to convey the gas from the bag to the jet.

In obtaining the most sensitive flame it is to be noted that the orientation of the flame is important and the burner should be turned about a vertical axis, so that different sides of the flame are presented towards the direction of the sound. The flame appears to have different degrees of sensitiveness on different sides.

CHAPTER XV

MISCELLANEOUS MAGNETIC EXPERIMENTS

Measurement of the Pole Strength of a Bar Magnet, using a Grassot Fluxmeter

The search coil of the fluxmeter (see p. 482) is placed on the bar magnet as shown in the figure so that it encircles the mid-point of the magnet. At this stage the reading of the fluxmeter is noted. Then, if the bar magnet is uniformly magnetized, the coil, when withdrawn, cuts all the lines due to the pole past which it moves.

The fluxmeter indicates the flux change in units which are specified; in the case of the instrument described on p. 482 each division corresponds to a change of flux equal to 10000 maxwells.

If there are x divisions change during the withdrawal of the search coil, and there are n turns of wire in the search coil, then since from a pole of strength, m, there are $4\pi m$ lines, we have :

$$4\pi m = \frac{10000x}{n},$$

or

$$m = \frac{10000x}{4\pi n}.$$

The experiment should be repeated, using search coils having different values for n.

This gives quite constant values for m as seen in the following experimental results.

Coil A : $n = 100$.

Initial reading of Fluxmeter	3
Final reading of Fluxmeter	46
Deflection, first experiment	49
Deflection, second experiment	50
Mean deflection	49·5

$$m = \frac{49{\cdot}5 \times 10000}{4\pi \times 100} = 394{\cdot}1.$$

Coil B : $n = 8$.

Deflection, 4 divisions.

$$m = \frac{10000 \times 4}{4\pi \times 8} = 394{\cdot}4$$

mean value of pole strength, 394·25.

Magnet was 1·58 cms. by ·75 cm., i.e. 1·23 sq. cms. in cross-section, i.e. intensity of magnetization, assumed uniform is :

$$\frac{394{\cdot}25}{1{\cdot}23} = 320{\cdot}5.$$

Distribution of Magnetism along a Bar Magnet

This may be determined by using the fluxmeter in a manner very similar to that of the last experiment. The magnet is marked off in centimetres along its length, and the search coil is placed at the mid-point, around the magnet. The coil is then advanced in centimetre steps and the deflection of the instrument noted, i.e. for 0 to 1 cm., 1 to 2 cms., 2 to 3 cms., etc. The deflection in each case is proportional to the magnetization in the interval moved over by the coil.

The variation of magnetization along the length is seen by plotting deflection against distance from the centre.

Gauss's Proof of the Law of Force

The most satisfactory proof that the force between two magnetic poles varies inversely as the *square* of the distance between them was first given by Gauss.

The method consists of a comparison of the magnetic force at a point on the axis of a magnet with the force at a point on a line drawn at right angles to it, at its mid-point.

Let us first calculate the value of the force at two such points, assuming that the force between poles varies inversely as the

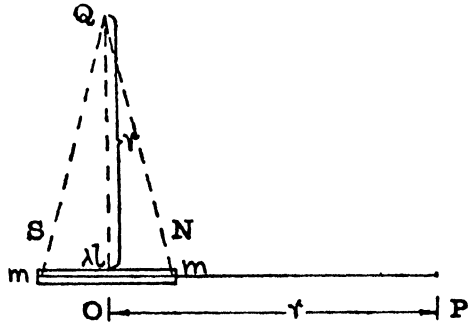


FIG. 270

nth power, so that the force between two poles of strengths, m and m', r cms. apart is :

$$F = \frac{mm'}{r^n}.$$

'End on' position ('A' position of Gauss)

Let NS be the magnet (fig. 270) and P a point along the axis produced, such that the distance from P to O, the centre of the

magnet, is r cms. If the length of the magnet is $2l$ cms., we have the magnetic force at P, which is the force on unit pole placed at P, is :

$$\frac{m}{(r-l)^n} - \frac{m}{(r+l)^n} = \mathrm{F}_A.$$

This is the net repulsion due to N and S on the unit pole at P, and is equal to :

$$\mathrm{F}_A = m\left\{\frac{1}{r^n\left(1-\frac{l}{r}\right)^n} - \frac{1}{r^n\left(1+\frac{l}{r}\right)^n}\right\},$$

$$\mathrm{F}_A = \frac{m}{r^n}\frac{\left(1+\frac{l}{r}\right)^n - \left(1-\frac{l}{r}\right)^n}{\left(1-\frac{l}{r}\right)^n\left(1+\frac{l}{r}\right)^n}.$$

If we use a bar magnet such that l is small compared with r, we have :

$$\mathrm{F} = \frac{m\left\{1+n\frac{l}{r}+\frac{n(n-1)}{2!}\left(\frac{l}{r}\right)^2\ldots-\left(1-\frac{nl}{r}+\frac{n(n-1)}{2!}\left(\frac{l}{r}\right)^2\ldots\right)\right\}}{r^n\left\{1-\left(\frac{l}{r}\right)^2\right\}^n},$$

expanding the two expressions.

Neglecting the powers of $\frac{l}{r}$ higher than the first, this expression becomes :

$$\mathrm{F}_A = \frac{2mnl}{r^{n+1}} = \frac{\mathrm{M}n}{r^{n+1}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where M is the magnetic moment of the magnet.

'Broadside on' Position ('B' Position of Gauss)

Again in fig. 270 let Q be a distance r cms. from O, measured along OQ, which is drawn at right angles to the magnet at the mid-point.

Due to the pole at N, there is a repulsion at Q along NQ equal to

$$\frac{m}{\{\sqrt{(r^2+l^2)}\}^n},$$

and a similar attraction due to S, along QS.

The resultant of these forces may be obtained by considering

the isosceles triangle, QNS, as triangle of forces. The sides QS and QN each representing a force $\frac{m}{(r^2 + l^2)^{\frac{n}{2}}}$ the resultant NS is $\frac{2ml}{\{r + l^2\}^n \{r^2 + l^2\}^{\frac{1}{2}}}$,

i.e. $$F_B = \frac{M}{\left\{r^2 + l^2\right\}^{\frac{n+1}{2}}} = \frac{M}{r^{n+1}\left\{1 + \left(\frac{l}{r}\right)^2\right\}^{\frac{n+1}{2}}},$$

or when l is small compared with r.

$$F_B = \frac{M}{r^{n+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Comparing equations (1) and (2) we see that for such a short magnet the magnetic force at the two points, P and Q, are as $n : 1$.

The numerical value of n is obtained by comparing the magnetic force at two such positions.

A magnetometer is employed. This consists of a small magnet fastened in a light frame which carries a small mirror, the whole being suspended by a thin silk fibre in a cylindrical brass case as shown in fig. 271. Fig. 272 shows the usual form of small suspended magnet.

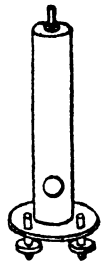


FIG. 271

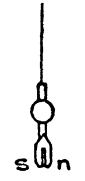


FIG. 272

The magnetometer magnet is allowed to come to rest, care being taken that all the torsion is removed from the silk suspension.

Two boxwood scales, C and D, are arranged as shown in fig. 273, one in the magnetic meridian, and one at right angles to it.

A lamp, L, and scale, S, are arranged in the usual way, as shown, so that an image of the lamp is reflected and focussed on the scale S. The distance of S from the magnetometer mirror should be about one metre.

When the magnetometer needle is arranged to swing freely in the centre of the case by means of the levelling screws shown (fig. 271), the position of the needle with respect to the ends of the boxwood scales can be very readily obtained, if the radius of the cylindrical case is measured.

If now a very small bar magnet is placed at any point, such as B, fig. 273, in an E. and W. position, a magnetic field will be set up at M, corresponding to the ' broadside on ' position. The distance, r, between the magnet centres being obtainable by

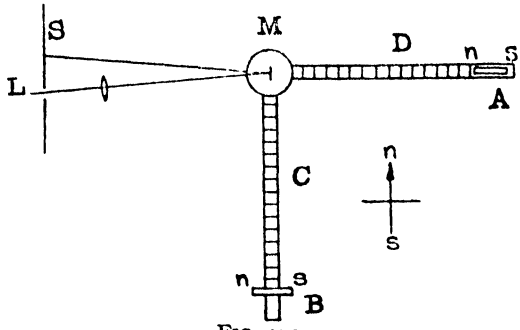


FIG. 273

direct measurement on the boxwood scale. The deflection of the magnetometer is found by observation of the movement of the reflected beam of light.

The magnet is then placed, again E. and W., at an equal distance from M, on the scale D (A, fig. 273), and the deflection again obtained.

It is desirable to eliminate the zero reading of the magnetometer; the double deflection, left and right, may easily be obtained by reversing the magnet at the two positions A and B.

The experiment is repeated for various values of r. For example, with a magnet 4 cms. long the values of the deflection at 50, 60, 70, 80 cms. may be obtained and tabulated as under.

DISTANCE OF MAGNET FROM CENTRE MAGNETOMETER IN CMS.	'END ON' (A) POSITION			'BROADSIDE ON' (B) POSITION			VALUE $\frac{A}{B} = n$
	SCALE READING MAG.		DOUBLE DEFLECTION A	SCALE READING MAGNET		DOUBLE READING B	
	DIRECT	REVERSED		DIRECT	REVERSED		
50 cms.	17·6	−16·7	34·3	8.2	−9.0	17.2	2.0
60	10·3	−10·1	20·4	4·6	−5·1	9·7	2.1
70	6·5	− 6·2	12·7	2·8	−3·3	6.1	2·08
80	4·4	− 4·1	8·2	1·8	−2·3	4·1	2·07
90	3·3	− 2·9	6·2	1·0	−2·2	3.2	1·94
						mean	2·04

Considering the small magnet of the magnetometer when the magnet is removed we have solely the horizontal component of the Earth's field acting as control field.

Due to the presence of the magnet at A or B, there is a field at right angles to this, equal to F_A or F_B, as previously calculated.

The magnetometer needle takes up a new position of rest at an angle $\theta°$ from the original, such that

$$H \sin \theta = F \cos \theta,$$

or

$$F = H \tan \theta.$$

So that for the same value for r

$$\frac{F_A}{F_B} = \frac{n}{1} = \frac{\tan \theta_1}{\tan \theta_2},$$

θ_1 being the deflection when the magnet is at A, θ_2 when the magnet is at B, in the 'broadside on' position.

If the distance from the scale S is fixed and the corresponding deflections are d_1 and d_2,

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{d_1}{d_2},$$

i.e.

$$n = \frac{d_1}{d_2}.$$

This ratio is tabulated in the last column of the table, and a mean value obtained.

The Variation of Residual Magnetism with Temperature

The object of this experiment is to investigate the behaviour of a magnetized carbon steel rod when subjected to temperature changes.

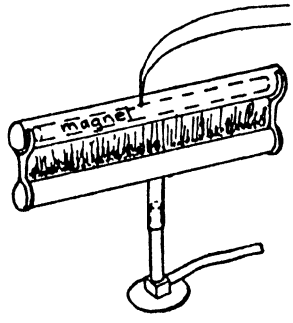


FIG. 274

A steel rod (carbon steel) of about 8 cms. in length and ·5 cms. in diameter is magnetized between the poles of an electromagnet. The magnet so formed is set up inside a copper or brass tube, which is of slightly larger diameter, as seen in the upper portion of fig. 274.

The magnet is drilled with a small hole which receives a thermo junction, as shown. This thermo-junction is calibrated as described on page 545 and serves to register the temperature of the magnet. Some simple heating device is arranged to vary the temperature of the magnet. In the diagram a second brass tube provided on the upper surface with a series of holes is shown, fitted on the end of a brass Bunsen burner. If a brass Bunsen burner is not available a simple heater may be made by taking such a second tube as the one shown, perforated with a series of holes, having one end closed and at the other a device for admitting an air-gas mixture similar to that of the common form of 'gas ring.'

The magnet is set up at right angles to the meridian, and at a distance d cms. from a mirror magnetometer, of the form shown in fig. 271; the distance, d, should be such that at room temperatures the magnet produces a full scale deflection. The magnitude of the deflection is noted, and the temperature is obtained from the thermo-junction balance point on the potentiometer.

The temperature of the rod is gradually raised by means of the heater, and the magnetometer deflection and the thermo-junction balance point are noted for every 20° C. rise, until the temperature is about 180° C. At this point the observations are made much more frequently as the critical part of the variation is being observed. At about 250° C. the readings may be again taken at about every 10° C. or 20° C. intervals, and observations continued until the temperature is at the maximum obtainable value for the heater employed.

The results are tabulated and the value of the temperature obtained for each point; the tangents of the corresponding angular deflections of the magnetometer are also tabulated.

The magnet is then allowed to cool slowly, and the observations are continued until finally the specimen is once more at room temperature.

The magnetometer is at a constant distance from the magnet, and in a fixed control field, H, hence the moment of the magnet is proportional to the tangent of the angle of deflection.

This quantity is plotted against temperature, and a curve somewhat similar to that of fig. 275 is obtained. The firm line shows the relation between tan α and $t°$ C. as the temperature rises, and the broken line the relationship as the temperature decreases.

For carbon steel in general, it will be found that at about 200° C. the magnetic moment is reduced to zero. An increase in temperature causes a reversal of the polarity as shown in the curve. This negative moment is a maximum at about 210° C. and then decreases to zero at about 800° C. In general such a high

temperature will not be available with the heater described, but the most interesting part of the curve may be obtained. On cooling, the negative moment is a maximum at D, at a temperature about 10° C. or 20° C. above the original position, and on regaining room temperature will have a small positive moment, corresponding to E in fig. 275.

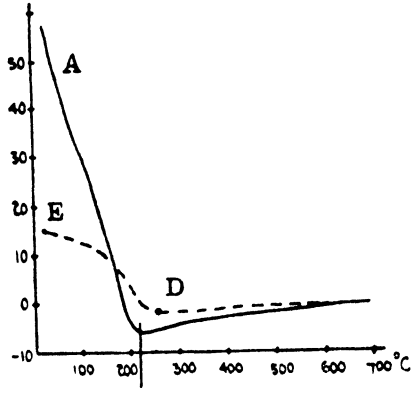


FIG. 275

The above experiment which demonstrates an interesting feature of the magnetization of such an annealed carbon steel rod is due to Prof. S. W. J. Smith and was described by him in the " Proceedings of the Physical Society of London," No. xxiv, 15 Aug., 1912. Reference should be made to that paper, which gives an explanation of the observed results in terms of the magnetization of the iron and iron carbide molecules which compose the bar.

Briefly, the iron carbide molecules are set in line on magnetization and exert a demagnetizing effect on the iron molecules which are reversed in the internal field due to the carbide. At about 210° the carbide is totally ineffective, but the reversed iron molecules being more retentive have a maximum external effect. As the temperature rises, the iron molecules approach their neutral temperature, and finally at about 800 entirely lose their magnetic effect. (See the original account in the place stated above.)

The Increase in Length of a Bar on Magnetization

To investigate any change in length produced in a rod of iron when magnetized, the following apparatus is set up. Fig. 276 shows a section of a brass case wrapped with two solenoids and provided with a water jacket, WW, through which water at a steady temperature may be circulated via tubes, T: the two coils, C, are wound together so that the magnetic effect of either

is the same at the axis of the cylinder. The free ends of each coil are connected to separate terminals, P, Q, R and S. The number of turns and mean diameter of the two coils are as nearly as possible the same, so that if a current were sent through from

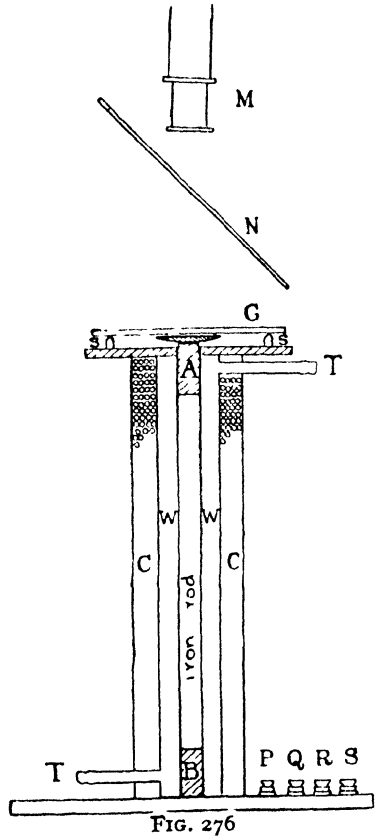


FIG. 276

P to Q, the magnetic effect is the same as when the same current is sent from R to S through the second coil, or if the two coils are connected in opposition there would be no magnetic field along the axis.

The method of winding and the resulting magnetic field should be tested at the outset by sending a current through the coils, and noting the magnetic effect on a compass needle held near the end of the solenoid.

At the top of the brass case is a circular brass end-plate to which are soldered rigidly, three pins, S, which terminate in points on which a sheet of plane glass, G, may rest.

The iron specimen to be investigated is first thoroughly demagnetized by heating, etc., or by reversals of a diminishing current through a solenoid surrounding it (as described on pages

449-451). The specimen should be a cylindrical rod with plane ends made to fit the cavity provided along the axis of the solenoids. The end effects of magnetizing solenoid are overcome by first placing inside the apparatus a cylindrical length of brass, B, with plane ends, the lower end resting firmly on the thick brass base of the instrument. Above the iron rod is placed a second brass cylinder, A, also with plane ends ; the upper end projecting slightly above the level of the upper surface of the end plate.

A long focus lens is fastened firmly to this brass cylinder by means of soft wax or plasticine. The length of the supports, S, is such that a small gap is left between the upper face of the lens and the lower surface of the glass sheet, G.

Above the parallel walled sheet of glass, G, is placed a second sheet, inclined at 45° to the normal. This reflects light from a sodium flame towards the lens and plane surface.

The reflected beams are viewed by a microscope M, and the Newton rings formed are observed in this way. The microscope contains a cross-hair which is used as a point of reference.

When a current is sent through the solenoids arranged in series, and to add their magnetic fields, the rod may be gradually magnetized by increasing the current strength. If any change in length occurs, there will be a movement of the Newton ring system, which may be observed in the microscope.

It will be seen (by reference to page 323), that if the rod changes in length by $\frac{\lambda}{4}$ where λ is the wave length of sodium light, that one particular part of the field which was formerly dark will now be bright or vice versa. The direction of movement of the fringes will show whether the rod increases or decreases in length.

Now, due to the current circulating in the coils, a certain amount of heat will be developed. Thus for a rod of 50 cms. length a rise in temperature of about ·03 will cause an increase in length more than sufficient to move fringes corresponding to a change in size of the gap equal to $\frac{\lambda}{4}$. It therefore becomes necessary to ensure that the rod is maintained constant in temperature throughout. Water at constant temperature is circulated through the jacket, WW, for some time before commencing observations and continuously during the experiment. Under these circumstances tap water from a supply removed from any hot water pipes will be admissible.

As a further check on the observations, to ensure that the results are not spurious due to the heating, each observation is preceded by one in which the current to be used is sent through

the two coils arranged in opposition, i.e. the heating which may affect the dimensions of the bar will be present, and any movement of the fringes is due to that cause, since there is no net magnetic effect.

The coils are rapidly changed over, and the same current causes a magnetization of the bar. The true effect on the length of the bar due to magnetization may therefore be obtained

To facilitate the rapid change in the direction of the current, a Pohl commutator may be employed: the ends of one coil, say, R and S, are connected to one pair of the terminals of the commutator; the other coil is connected in series with a battery and variable resistance to the middle terminals. Thus by throwing over the movable arm, the coils may be changed from a condition of producing fields in opposition to one in which their magnetic fields are in the same direction.

Carry out observations using currents up to a maximum value which is determined by the current-carrying capacity of the coils, CC.

CHAPTER XVI

TERRESTRIAL MAGNETISM

Determination of the Horizontal Components of the Earth's Magnetic Field (H)

The method due to Gauss, described below, is usually employed to determine the horizontal component of the earth's magnetic field ; it may also be used to measure any magnetic field which is uniform over a sufficiently large volume.

The method involves two experiments. In the first a magnet of known moment of inertia is suspended freely in the earth's field, at the place where H is to be found, and from observation of the time of swing the product, MH, where M is the moment of the magnet, may be calculated. In the second experiment the field due to the same magnet is compared with the earth's field by means of a magnetometer.

Determination of MH

Let the magnet be suspended in a light stirrup and perform oscillations whose periodic time is T seconds, it being supposed that the suspension has no initial twist. Then if I is the moment of inertia of the magnet about the axis of suspension, and i the moment of inertia of the suspending frame about the same axis, and if τ be the restoring couple per unit angular displacement due to torsion of the suspending fibre, we have :

$$T = 2\pi\sqrt{\frac{(I + i)}{(MH + \tau)}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

for $(I + i)$ is the moment of inertia of the system, and $(MH + \tau)$ is the total restoring couple per unit angular displacement, when small displacements are considered, whence :

$$MH = A, \text{ (say)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

To find $\frac{M}{H}$

A magnetometer is set up in the place at which MH was found, and the magnet placed with its centre d cms. from the centre of the magnetometer needle, east or west of it and lying east and west. If θ be the deflection of the magnetometer, and F the value of the field due to the magnet, we have :

$$F = H \tan \theta,$$

and

$$F = \frac{2Md}{(d^2 - l^2)^2},$$

where $2l$ is the distance between the poles of the magnet.

Alternatively, if the magnet were placed so as to be in the 'broadside on' position, i.e. E. and W. with its centre in the meridian through the centre of the magnetometer needle, we have F_1, the field produced at the magnetometer, in this case is

$$= \frac{M}{(d_1^2 + l^2)^{\frac{3}{2}}},$$

where d_1 is the distance between the centre of the magnet and the magnetometer in this case, and l is the distance between the *poles* of the magnet.

Whence for the 'end on' position

$$H \tan \theta = \frac{2Md}{(d^2 - l^2)^2} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

or for the 'broadside on' position

$$H \tan \theta_1 = \frac{M}{(d_1^2 + l^2)^{\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

in either case the values of $\frac{M}{H}$ may be found.

Let this value be B.

Whence from the two experiments:

$$H = \sqrt{\frac{A}{B}}.$$

In performing these experiments special forms of magnetometer, etc., are advisable. All iron is removed from the neighbourhood at the outset, and care taken to maintain the magnetic conditions the same throughout the two experiments.

The magnet used may very well be a true cylindrical bar magnet of not more than a few centimetres in length.

Let the length as determined to $\frac{1}{10}$ of a mm. be $2l_1$ cms., and the radius, measured with a micrometer screw, be r cms, the mass of the magnet, weighed to at least one in a thousand, being m grammes. I, its moment of inertia about an axis through the centre of gravity and normal to its length, is

$$I = m\left(\frac{l_1^2}{3} + \frac{r^2}{4}\right).$$

To find T, use is made of a light stirrup suspended by a fibre from a torsion head. A brass cylinder of the same mass as the magnet is first placed in the stirrup, and all the twist is then removed from the fibre. The torsion head is then turned so

that the brass rod is in the magnetic meridian. It is then replaced by the magnet.

The frame is provided with a small mirror, otherwise one end of the magnet is polished to act as a mirror, and a lamp and

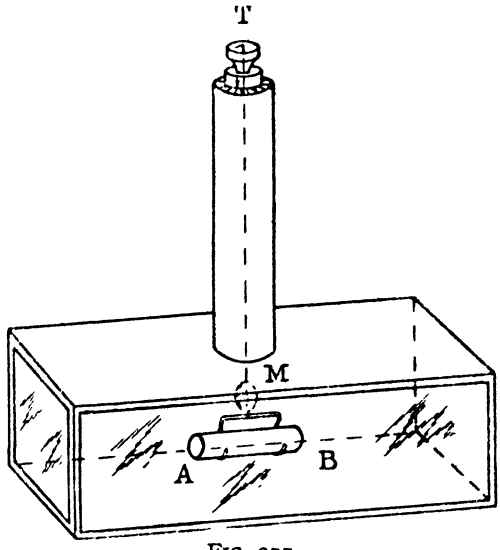


FIG. 277

scale is arranged so that a beam of light reflected from the mirror on to the scale enables the movement of the suspended system to be observed.

Set the magnet swinging through a very small angle, and obtain the time taken for 100 complete swings. Repeat this and take a mean value, from which T may be obtained.*

To find i in equation (1) weigh the stirrup and estimate the moment of inertia from the dimensions.

The restoring couple due to the torsion, τ, may be obtained by rotating the torsion head through, say, a complete turn. Note the resulting *angular* deflection of the magnet, ψ radians,

whence $$(2\pi - \psi)\tau = MH\psi,$$

or $$\tau = \frac{MH\psi}{(2\pi - \psi)}$$

It will be found that, if the fibre is thin and the stirrup a very light one, τ and i are negligible compared with MH and I. Including these corrections we have:

$$T = 2\pi \sqrt{\frac{I + i}{MH\left(1 + \frac{\psi}{2\pi - \psi}\right)}},$$

* Alternatively, the method of timing set out on page 118 may be used.

whence $$MH = \frac{4\pi^2}{T^2} \frac{(I + i)}{\frac{2\pi}{2\pi - \psi}}$$

The magnetometer experiment is performed, using a special form of magnetometer (Kew type magnetometer). This consists of several small magnets rigidly fastened in a metal frame, which also carries a small mirror. The whole is encased in a brass case, provided with suitably arranged glass windows so that a beam of light may be reflected from the mirror on to a scale S cms. away from it.

The magnetometer frame carries a long bar which has four pegs, P_1, P_2, P_3, P_4, so arranged that when a small carriage is placed on any of them, the cylindrical magnet, which the carriage supports, lies with its axis in the same plane as the magnetometer needle. Figs. 278 and 279 show diagrammatically the arrangement of the magnetometer.

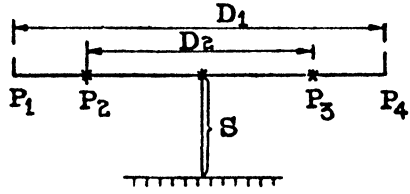


FIG 278

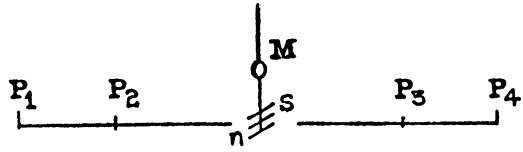


FIG. 279

The magnetometer and bar are first levelled (using a spirit-level) by means of the levelling screws on the base of the instrument.

P_1 and P_4, P_2 and P_3 are fixed, in the construction of the instrument, equidistant from the needle when the system is level. Measure $P_1P_4 = D_1$ cms., say ; measure $P_2P_3 = D_2$ cms.

The scale is then adjusted so that the reflected spot of light is at its centre when the magnetometer is under the influence of the earth's field alone. The arm which carries the pegs is set first of all in the direction of the magnetic meridian, and the small carriage in which the cylindrical magnet is carried is placed on P_1 and turned round so that the magnet is at right angles to the meridian. The true position for the magnet at right angles to the meridian is obtained by noting the deflection of the magnetometer. This will be a maximum when the normal position is acquired.

In this manner the deflection in cms. on the scale may be measured. The magnet is then rotated through 180° until a maximum deflection in the opposite direction is obtained in the magnetometer. Let these deflections be δ_1 and δ_1' cms., and the corresponding deflections when the magnet is supported on the other pegs be δ_2, δ_2', etc.

Obtain the mean value of $\delta_1\ \delta_1'$, $\delta_4\ \delta_4' = \delta$ say, that is the mean deflection for distance $\frac{1}{2}D_1 = d_1$,; also take the mean of $\delta_2\ \delta_2^1$, $\delta_3\ \delta_3^1 = \delta^1$, say, corresponding to a distance $\frac{1}{2}D_2 = d_2$.

If φ and φ' are the corresponding angular deflections of the magnetometer, we have :

$$\tan 2\varphi = \frac{\delta}{S}\ ;\ \tan 2\varphi' = \frac{\delta'}{S}.$$

Hence we may calculate φ and φ' from the observed deflections.

The cylindrical magnet was placed as described above so that the field at the magnetometer was that due to the 'broadside on' position ; thus from equation (3) above

$$H \tan \varphi = \frac{M}{(d^2 + l^2)^{\frac{3}{2}}},$$

i.e. $$\frac{M}{H} = (d_1^2 + l^2)^{\frac{3}{2}} \cdot \tan \varphi = (d_2^2 + l^2)^{\frac{3}{2}} \tan \varphi'.$$

Now $$(d^2 + l^2)^{\frac{3}{2}} = d^3\left(1 + \frac{3}{2}\frac{l^2}{d^2} - \ldots\right),$$

neglecting the fourth and higher powers of $\frac{l}{d}$,

i.e. $$\frac{M}{H} = d_1^3\left\{1 + \frac{3}{2}\left(\frac{l}{d_1}\right)^2\right\} \tan \varphi,$$

$$\frac{M}{H} = d_2^3\left\{1 + \frac{3}{2}\left(\frac{l}{d_2}\right)^2\right\} \tan \varphi'.$$

Hence l may be eliminated, giving

$$\frac{M}{H}\left(\frac{1}{d_1 \tan \varphi} - \frac{1}{d_2 \tan \varphi'}\right) = d_1^2 - d_2^2.$$

Hence $\frac{M}{H}$ is determined, and the value of H may be calculated using this and the previously obtained value of MH.

The bar carrying the four pegs may be alternatively arranged at right angles to the meridian, and the carriage containing the magnet again placed on the pegs with the magnet at right angles to the meridian, producing a field of force at the magnetometer corresponding to the 'end on' position. The process described above is repeated, and the average deflections obtained enable

the calculation of the mean deflections, θ_1 and θ_2, for the distances, d_1 and d_2.

Hence again :

$$H \tan \theta_1 = \frac{2M}{D_1^3} \times \frac{1}{\left\{1 - \frac{2l^2}{d_1^2} \cdots\right\}},$$

$$H \tan \theta_2 = \frac{2M}{D_2^3} \times \frac{1}{\left\{1 - \frac{2l^2}{d_2^2} \cdots\right\}},$$

to the same order as before.

Once more eliminating l, half the distance between the ***poles*** of the magnet, we have

$$2 \cdot \frac{M}{H}\left(\frac{1}{d_1 \tan \theta_1} - \frac{1}{d_2 \tan \theta_2}\right) = d_1^2 - d_2^2,$$

whence $\frac{M}{H}$ may be again calculated. This second method gives larger deflections and is therefore preferable to the first.

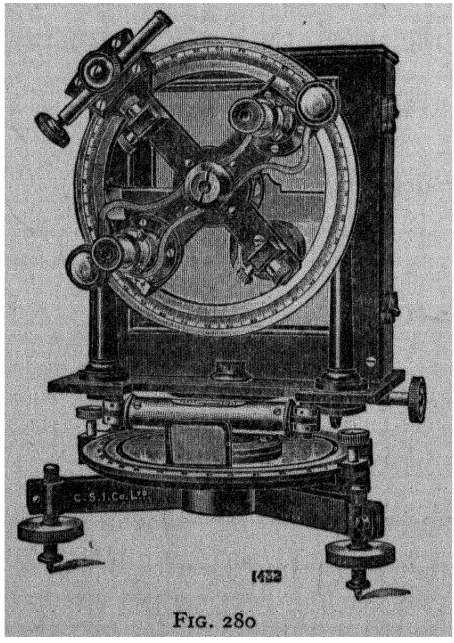


FIG. 280

The Dip Circle—Measurement of the Angle of Dip

The dip circle consists of a long magnet supported on a horizontal axis which passes (approximately) through its centre of gravity and at right angles to its length. The magnet is sup-

ported in wheel bearings or on agate edges, so that it may turn freely and with a minimum of friction. On the same base as the support for the knife-edges is a vertical graduated circle which enables the position of the ends of the needle to be obtained. The whole structure is supported on a circular table, which is capable of rotation around a vertical axis. This rotation may be measured on a horizontal circular scale about which the table may rotate about its axis, and, by means of a suitable vernier, may be measured to at least one minute (see fig. 280).

The position of the ends of the magnet may be read on the vertical scale by means of microscopes carrying verniers, which move round the scale. If the plane of the vertical scale be turned in the direction of the meridian, so that the horizontal axis of support of the magnet is at right angles to it, the magnet will set in the direction of the earth's line of magnetic force, and the angle included between the horizontal and the position of the needle would give the angle of dip.

However, there are many sources of error. The axis of rotation of the magnet may not quite coincide with the centre of the scale. One end, therefore, of the magnet would read too small, and the other too large a deflection.

The axis of rotation may not be truly through the centre of gravity, in which case a couple will be exerted, tending to turn the magnet so that the centre of gravity comes vertically under the axis of support.

The axis of the magnet does not usually coincide with the geometric axis of the needle, so that the centre of the ends of the magnet do not give a correct reading for the magnet when set along the earth's line. The possible errors are fully set out in any textbook of magnetism.

A consideration of the observations taken and described below will show that these errors may be eliminated.

To determine the angle of dip, the dip circle is set up and by means of the levelling screws on the base of the instrument, using a spirit-level, the ' horizontal ' scale is made truly horizontal.

The plane of the meridian must then be found, so that the magnet may set freely. To do this, the instrument is first turned so that the upper, S, pole is at the 90° mark, and the reading of the vernier on the horizontal scale is noted.

It will in general be found that the lower, north, pole of the magnet is not quite at the lower 90° scale reading. A slight turn of the screw which rotates the instrument about the vertical axis is then made until the north pole is at the 90° reading, and the horizontal vernier scale is once more read. The circle is next turned through 180°, and the two more observations are made. The needle is now reversed in its bearings and the four

observations repeated. The mean value of the horizontal scale readings for these eight positions is taken. This is the mean position for the plane of rotation at right angles to the meridian.

The case is then moved through an angle of 90° from this mean position so that the plane of the needle now coincides with the magnetic meridian.

The following 16 readings are then obtained :

(1) Read the position of each end of the magnet.

(2) Rotate the instrument through 180°, and once more read both ends of the magnet.

(3) Reverse the needle in its bearings and read both ends on the graduated circle.

(4) Turn the instrument through 180° and again read both ends

(5) Remagnetize the needle and repeat the above process.

The needle is remagnetized so that its poles are intercharged.

The north pole for the observations, 1 to 4, above now becomes a south pole. Under these new conditions eight more values are obtained. The mean value of the 16 readings gives the true value of the inclination. The observations may be conveniently tabulated as under.

MARKED SIDE OF MAGNET FACING	MARKED POLE DIPPING (NEEDLE READINGS)			MAGNET REMAGNETIZED (NEEDLE READINGS)		
	UPPER POLE	LOWER POLE	MEAN	UPPER POLE	LOWER POLE	MEAN
East			α			α'
West			β			β'
Needle reversed in bearings.						
East			γ			γ'
West			δ			δ'

$$\text{mean } \frac{\alpha+\beta+\gamma+\delta}{4} = A \qquad \text{mean } \frac{\alpha'+\beta'+\gamma'+\delta'}{4} = B$$

$$\text{Angle of Dip} = \frac{A+B}{2}.$$

NOTE —It may be necessary to obtain corresponding 16 values for the determination of the zero instead of the 8 described on pages 443-4

CHAPTER XVII

PERMEABILITY OF IRON AND STEEL

Measurement of Permeability by the Magnetometer Method

Consider a long thin specimen of iron wire placed in a magnetic field of uniform strength, H. Due to the field the soft iron specimen will become magnetized, having poles of strength, $\pm m$. Let

$2l$ be the distance between the magnetic poles,
s be the cross-section of the wire.

The number of lines coming out from the north pole end is $4\pi m$, and an equal number enter the south. Due to the poles developed, there are a number of lines of magnetic force passing from the south to the north *in the specimen*, equal to $4\pi m$.

This is equivalent to $\frac{4\pi m}{s}$ per sq. cm. of cross-section of the iron, assuming that the lines *enter and leave at the ends only*.

In addition to the above there are H lines per sq. cm., due to the field, making the total number per sq. cm. of $H + \frac{4\pi m}{s}$. This is the induction, B, in the specimen. Thus

$$B = H + 4\pi \frac{m}{s}.$$

The magnetic moment of the wire is $2lm$, its volume is $s \cdot 2l$. Hence the *intensity of magnetization*, which is the magnetic moment per unit volume, is

$$\frac{2lm}{2ls} = \frac{m}{s} = I.$$

Hence $$B = H + 4\pi I \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The *permeability* (μ) for any value of H is defined by the ratio $\frac{B}{H}$.

We therefore have :

$$\mu = 1 + 4\pi \frac{I}{H}.$$

The ratio $\frac{I}{H}$ is usually defined as the *susceptibility* or coefficient of induced magnetization, so we have, putting k for this quantity,

$$\mu = 1 + 4\pi k.$$

We have imagined the length of the magnet to be large. Otherwise there is an effect due to the poles at the end of the specimen to be considered, in expressing the value of H. For, due to the north and south poles developed in the specimen, there will be a magnetic field in the opposite direction to H, reducing the value to H^1, say. This demagnetizing field can be neglected, however, if $2l$ is made large compared with s. A suitable length is obtained when $2l$ is greater than 200 times the diameter of the wire.

If the specimen is arranged horizontally or vertically near a small magnet, suspended by an unspun silk fibre, the magnet will be deflected by the induced magnetism in the specimen. From a knowledge of the deflection and of H, the value of the constants, B, μ, I, k, may be calculated.

To find the above constants for a given wire specimen, the type of reflecting magnetometer described on page 429 is used.

The inducing magnetic field is obtained from a solenoid, through which a current of known magnitude may be passed, and the deflection of the magnetometer, etc., may be observed.

Iron Wire Specimen arranged Vertically

The arrangement of the apparatus is seen in fig. 281. L is a lamp and scale, M the magnetometer which is clamped to the wooden base shown, at a convenient distance from the specimen which is placed in a vertical solenoid, S. The wooden base is arranged in an east and west direction so that the magnetic effect of the specimen may be measured.

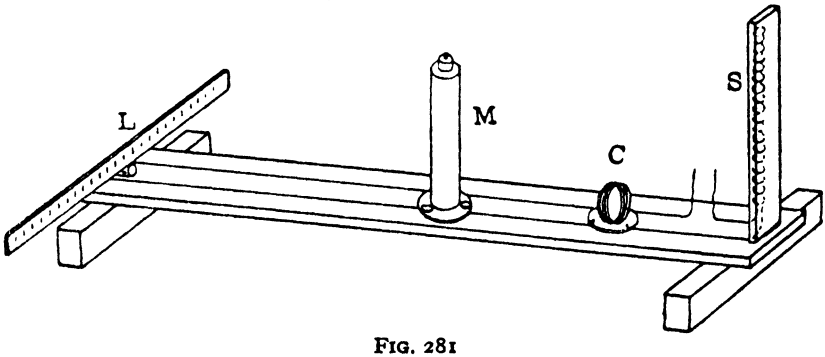


FIG. 281

Before introducing the specimen, the solenoid is connected in series with a small coil, C, and several accumulators, a variable resistance, an ammeter reading to 3 amperes, and a reversing switch.

The coil, C, is so arranged that the magnetic effect produced by it on the magnetometer is the opposite sign to that produced by the solenoid, S.

The position of C is adjusted so that whatever is the value of the current in the circuit there is no movement of the needle. In that way the whole movement of the magnetometer will be due to the magnetism induced in the iron wire.

The current is then switched off and, having noted the position of the reflected spot of light on the scale, the unmagnetized specimen is introduced. If there is any movement of the spot of light it may mean that there is a certain amount of magnetization in the specimen. It should be demagnetized by first of all heating to red heat and then hammering vigorously. Having once more replaced it in position in the solenoid, the current should be adjusted to about one ampere. The direction of the current should then be constantly reversed, whilst at the same time its magnitude should be reduced to zero. In this way any small residual magnetization can be neutralized.

A preliminary experiment on a second sample of the same iron wire enables a value of the maximum current to be determined. This corresponds to a field which causes a full scale deflection.

The specimen is then inserted in the solenoid, so that the lower end of the specimen is in the same horizontal plane as the axis of the magnetometer magnet.

Starting with the soft iron unmagnetized specimen, the zero position of the magnetometer on the scale is observed. The value of the current is then raised to, say, ·2 ampere and the deflection noted. Then without breaking the circuit the current strength is increased by increments of ·2 to a final value of about 3 amperes, a limit which depends upon the thickness of the wire in the solenoid winding, and also upon the result of the preliminary experiment, whichever limit is the smaller fixes the maximum current.

Having increased the current to a suitable maximum value, it is then decreased by similar steps, again taking care that the circuit is not broken in the process, until zero current circulates. The current is then reversed and increased to the same maximum in the other direction, when the current is again brought back in steps to zero; and having once more reversed the current it is raised to the maximum in the original direction.

Tabulating these results a connexion is found between the value of the current in amperes and the magnitude of the deflection—or value of the scale reading, which, when plotted, will show the general form of the curve below.

The results are then converted to the corresponding C.G.S. values for B and H.

In fig. 283 let $n\,s$ represent the magnetized specimen which will produce at P the position of the magnetometer, a field

strength, F, at right angles to the Earth's horizontal component, and which causes the deflection, θ, say.

We have $$F = H' \tan \theta,$$

where H′ is the magnitude of the Earth's horizontal component.

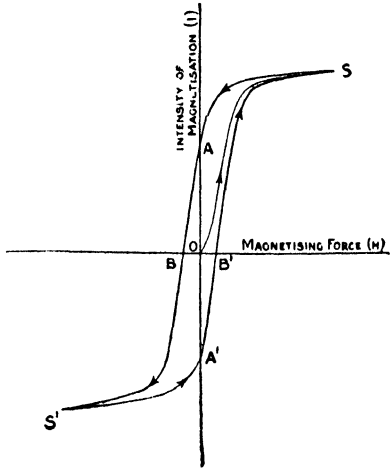


FIG. 282

Let m be the pole strength induced in the specimen,
l be the length in cms.,
r the distance from the lower end of specimen to the magnetometer in cms.;

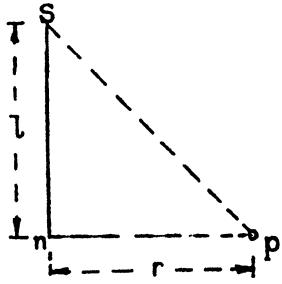


FIG. 283

Then :

$$F = \frac{m}{r^2} - \frac{m}{SP^2} \cos SPn$$

$$= \frac{m}{r^2} - \frac{m}{r^2 + l^2} \cdot \frac{r}{(r^2+l^2)^{\frac{1}{2}}} = m\left(\frac{1}{r^2} - \frac{r}{(r^2+l^2)^{\frac{3}{2}}}\right).$$

Whence :

$$m = \frac{H' \tan \theta}{\left(\frac{1}{r^2} - \frac{r}{(r^2 + l^2)^{\frac{3}{2}}}\right)};$$

and $I = \frac{m}{s}$, $B = H + 4\pi I$.

Further, using a solenoid of n turns per cm. having a current of c amperes, the uniform field $H = \frac{4\pi nc}{10}$ is effective on the specimen.

Hence, for each reading in the table of results, I, B and H, also μ and k, may be calculated. If the curve is drawn for I and H, the area enclosed is equal to the work done on the specimen in the cycle (i.e., the heat developed).

The values of H and B are in C.G.S. units, i.e. Gausses (see table on page 633).

(2) The B-H Curve for a Sample of Iron (using a Ballistic Galvanometer

This method is specially applicable to the determination of the B-H curve for a specimen in the form of an anchor ring, or a very short hollow cylinder. For such a specimen, magnetized by a magnetic force of, say, H gausses, no free poles will be developed, and therefore no demagnetizing field will be set up. The value of the induction, B, will therefore correspond to H, and not some smaller field, H', as in the previous case.

The experimental arrangements of this method, as shown in fig. 284, are such that a variable field, H, may be set up, by passing a current through a primary winding, P, which is closely wound on the anchor ring, and some means of measuring B in the specimen.

The method employed to measure B is to wrap a few turns of wire, S, round the anchor ring and primary winding and measure the quantity of electricity which passes through a ballistic galvanometer, BG, due to the change in the induction in the specimen for a known change in H. Since the ballistic galvanometer must be standardized, the secondary circuit is completed through a second small coil of a mutual inductance, M, used in the standardizing experiment. Thus the ballistic galvanometer is in a fixed resistance circuit in all measurements.

The current from an accumulator may be regulated by resistances R_1 and R_2 and measured by an ammeter, A. By means of the Pohl commutator the current may be sent directly or reversed through the primary windings, P, when K_2 is to the left, or through the standardizing mutual inductance, M, when the key K_2 is closed on the right.

By closing the key K_1, the resistance R_2 may be cut out.

The value of H may be calculated, as explained later, from the value of the current strength as obtained from the ammeter; and B may be calculated from the observed throw of the ballistic

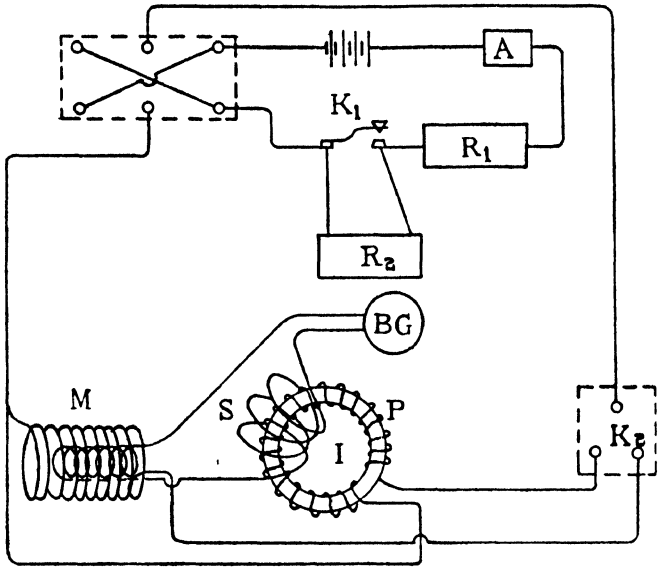


FIG. 284

galvanometer, BG. Since the galvanometer circuit is composed of a comparatively low resistance, a moving coil instrument may in general be too heavily damped to be serviceable (see page 481). A moving needle instrument is therefore generally advisable.

As a preliminary experiment, the key K_2 is closed to the left, and R_1 and R_2 decreased, until, on closing the Pohl commutator, the galvanometer gives a full scale deflection from the zero, which should be the central graduation of the scale. The current required to do this is noted and is used as a maximum value in the main experiment. Usually from 2 to 3 amperes will suffice, the sensitivity of the galvanometer being adjusted to measure the throw produced.

In general, the past history of the iron anchor ring will be such that residual magnetism in the specimen is almost certain. This must now be reduced to zero.

To do this the galvanometer circuit is broken and the resistances R_1 and R_2 reduced to a minimum. The current passing in P is then reversed many times and R_1 and R_2 gradually increased until finally the current which is reversed is very small. It should be noted that R_1 and R_2 are such that *changes in resistance* may be brought about *without breaking the circuit.*

Another method of demagnetizing the specimen is to pass an alternating current through P and a liquid resistance in series. The alternating current is gradually reduced to zero by withdrawing the electrodes of the resistance.

The galvanometer is once more put in circuit with S, etc., by closing the key in the circuit (not shown in the diagram): K_1 is closed, and R_1 given the value corresponding to the maximum current as determined by the preliminary experiment. The commutator is then closed *to the right*, and the throw of the ballistic galvanometer, δ, noted. The reading of the ammeter is also noted. The throw will correspond to an induction, B_1, and the ammeter reading to a magnetizing force, H_1, represented by some point such as S in fig. 285. We now use this as a point of reference. The galvanometer circuit is again broken, and the commutator is reversed rapidly some 20 to 25 times, and finally left on the right, i.e. the iron is taken several times round the cycle represented in fig. 285 and is said to be in the 'cyclic state.' BG is again put in circuit. When all is steady, R_2 is given a small value and K_1 is opened. The magnetizing field is thereby decreased and the throw of the galvanometer, δ_2, noted. This throw corresponds to a decrease in induction, $B_1 - B_2$. The value of the current corresponding to H_2 is noted on the ammeter

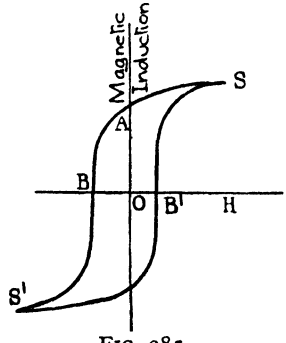


FIG. 285

K_1 is now closed, the galvanometer key opened, the Pohl commutator reversed 20 to 25 times, and finally left to the right. The galvanometer is put in circuit; R_2 is given a larger value; K_1 is then opened and the throw due to the change in the induction, say, $B_1 - B_3$, is noted; the ammeter is again read.

This process is repeated with the commutator to the right, until R_2 is infinite and consequently the current and H zero, i.e. the relation of B to H represented by the path, SA, of fig. 285, is investigated.

After each measurement, the iron, by the reversal of the maximum current, is returned to the state represented by S, which therefore becomes the reference point.

The key, K_1, being closed, etc., the commutator is reversed some twenty times once more and finally left to the right; R_2 is given a *large* value and the galvanometer is again put in circuit.

The commutator is then thrown over to the left, and *at the same time* K_1 is opened, i.e. the current is reversed and at the same time made of small value. This gives a point on the part AB of the curve. The same starting point is taken (S), the drop in induction measured, and the change in H from a maximum position to a small negative value obtained.

This process is repeated in many steps until finally $R_2 = 0$, i.e., until the change in field, H, and induction, B, corresponds to a reversal of the maximum positive to maximum negative values.

At this stage we may assume that the curve is symmetrical, and obtain the full hysteresis loop having obtained $SABS^1$ experimentally, or we may proceed to find $S^1A^1B^1S$ by experiment, using S^1 now as our reference point. The process is as before except that the commutator is now left on the left-hand side. K_1 is opened in turn to R_2 having values from 0 to infinity.

CURRENT C IN E.M. UNITS (1)	FIELD H IN E.M. UNITS (2)	THROW OF GALVO. (δ) (3)	CHANGE OF INDUCTION FROM S OR S^1 (4)	INDUCTION (B) ASSUMING SYMMETRY (5)
		FOR CURRENT CHANGE	Drop from S	
·24		0 to ·24 = 2·0 cms.	0	10000
·22		·24 to ·22 = 1·5 cms.	500	9500
·20		·24 to ·20 = 1·8 cms.	700	9300
·18		·24 to ·18 = 2·4 cms.	1000	9000
. . .		etc.		
. . .				
. . .				
+·02				
·0				
−·02				
. . .				
. . .				
. . .				
−·20				
−·22				
−·24		·24 to −·24 =	20000, say	−10000

At this stage (point A^1) R_2 is given values from infinity to 0, and K_1 opened as the commutator is thrown over to the right-hand side.

It should be found that the value of the throw for the reversal, when the full current is passing through the primary is identical for the change from S to S^1 and S^1 to S, and that either value is approximately twice the first recorded throw, δ_1.

The results may be tabulated as shown. The original direction of the current in the coil, P, is taken as positive and the current value may be recorded in column (1) and the corresponding throw in column (3).

To Calculate H

Imagine a unit magnetic pole to be taken round the axial circle of the solenoid. If H is the value of the magnetic field strength due to a current, C, in solenoid, the work done on the pole for such a complete circular path is $2\pi r \cdot H$, where r is the mean radius of the anchor ring.

If there are N turns in the winding, the unit pole is linked with each winding in the complete path, therefore doing work $4\pi C$ for each or a total of $4\pi NC$,

e.g. $$4\pi NC = 2\pi rH$$

$$H = \frac{2NC}{r},* \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where C is the current strength in E.M. units (*not* amperes).

The value of the current in E.M. units (not amperes) shown in column (1) may be converted to gausses (lines per sq. cm.) by multiplying by the factor $\frac{2N}{r}$ as seen in equation (1).

To calculate the induction, B, corresponding to the observed value of θ, an auxiliary experiment is necessary. The two-way switch, K_2, is closed on the right, so that a current may be sent through the long straight solenoid, M (about 40 cms. long and 4 cms. in diameter), which has m_1 turns per cm. Inside M is the second small solenoid which has been in series with S and the galvanometer throughout the preceding experiment.

For a current of C E.M. units flowing through M, since the solenoid is long, the field strength at the centre is $4\pi m_1 C$ gausses.

*The result of (1) may be obtained from the general formula giving H inside a solenoid of n_1 turns per unit length, i.e.

$$H = 4\pi n_1 C.$$

In our case $$n_1 = \frac{N}{2\pi r},$$

i.e. $$H = \frac{2NC}{r}.$$

Let the radius of the inner coil be r_2 cms., and N_2 the total number of turns in this coil, then the total flux in the inner coil is :

$$4\pi m_1 C(\pi r_2^2 N_2) \text{ maxwells.}$$

If now the current be reversed in M the change in induction in the central coil is :

$$8\pi^2 m_1 C r_2^2 N_2 \text{ lines.}$$

Now, if R is the total resistance in the galvanometer circuit since the electromotive force is numerically $\frac{dN}{dt}$,

$$cR = \frac{dN}{dt} \text{ or } \int c\,dt = \int \frac{dN}{R}.$$

$$Q = \frac{8\pi^2 m_1 N_2 r_2^2}{R} \cdot C \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

For an instrument of the moving-*needle* type :

$$Q = \frac{T}{\pi}\frac{H}{G} \sin\frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right),$$

or

$$Q = K_1 \sin\frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right),$$

where $K_1 = \frac{T}{\pi} \cdot \frac{H}{G}$ and is constant.

Now for small deflections, if d is the scale deflection in cms. $\sin\theta \propto d \propto \sin\frac{\theta}{2}$ and λ is constant in the present case, even using a moving coil instrument, for the circuit is constant,

i.e.

$$Q = K_2 d;$$

where K_2 is a constant (i.e. $Q \propto d$ for either type of galvanometer) substituting in (2) above

$$\frac{8\pi^2 m_1 N_2 r_2^2 C}{R} = K_2 d,$$

or

$$Kd = 8\pi^2 m_1 N_2 r_2^2 C,$$

where $K = K_2 R$ = a constant.

The last equation, which connects the deflection, d, with the total change in flux, enables the calculation of the value of B corresponding to each field H.

The change in the number of lines, threading the entire circuit, corresponding to 1 scale division deflection on the galvanometer scale is :

$$K = \frac{8\pi^2 m_1 N_2 r_2^2 C}{d}$$

The deflections δ_1, δ_2, etc., in column (3) therefore correspond to a change in flux equal to $K\delta_1$, $K\delta_2$, etc. This is a change in the total flux in the space of the secondary winding S. If r_1 is the radius of cross-section of the iron anchor ring, approximately that of the secondary winding, S, and m is the number of turns in the secondary, and $(B_1 - B_2)$ in the change in induction to be entered in column (4), we have, since B is the number of lines per square cm. of cross-section :

$$(B_1 - B_2)m \cdot \pi r_1^2 = K\delta,$$

$$B_1 - B_2 = \frac{K}{m\pi r_1^2} \cdot \delta$$

$$= \left\{\frac{8\pi}{d}\frac{m_1}{m}\frac{r_2^2}{r_1^2}N_2C\right\}\delta.$$

The term in the brackets is evaluated and used as a reduction factor, converting δ to induction for column (4).

To obtain column (5), assume a symmetrical form for the curve, i.e. assume that the induction for the maximum positive and negative current values are equally placed from the zero induction line, i.e. equal to half the induction for a full current reversal, which is the mean of the extreme values in (4). The method of calculation may be understood from the numbers shown in the suggested table.

Note

The B-H curve could also be obtained for an anchor ring as above, using a fluxmeter in place of the ballistic galvanometer. The calculation of H is as before. When the induction is large the number of turns in the secondary coil can be diminished, so that the fluxmeter reading is not excessive. The calculation of B from the observed deflection, θ, for any current change in the primary, is more direct than for the ballistic galvanometer. Unit scale deflection on the fluxmeter corresponds to a flux of 10000 maxwells. So that if the secondary coil used is of m turns, and the radius of each turn of the secondary is r_1, the induction change $(B_1 - B_2)$ corresponding to the deflection θ, is given by :

$$m \times (B_1 - B_2) \times \pi r_1^2 = 10000\theta,$$

$$B_1 - B_2 = \frac{10000\theta}{m\pi r_1^2}.$$

The experiment is carried out in a manner similar to that described above for the galvanometer as flux measurer.

CHAPTER XVIII

AMMETERS, VOLTMETERS AND GALVANOMETERS

THE most convenient form of current measurer is the ampère meter or ammeter, which indicates directly, on a graduated scale, the strength of the current passing through the circuit in which it is placed.

As a result of the method of construction the range of usefulness of the instrument is limited, in the usual form, to measurement of current not less than a milliampere. Recently, however, by improving the suspension, etc., the two-pivot instrument has been manufactured capable of measuring one microampere, for example, a Weston ammeter is made with an open scale division equal to one microampere.

Similar conditions hold for the voltmeter.

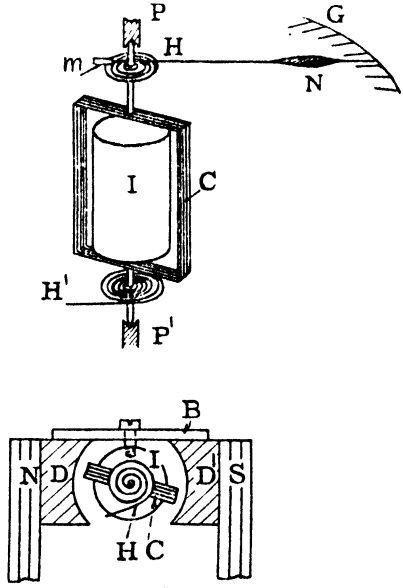


FIG. 286

The Voltmeter

This instrument consists essentially of a coil of thin copper wire, C, which is supported on an axis pivoted on jewelled pivots, P, P^1, and is free to move in the cylindrical gap between the soft iron pole piece, DD^1, of an aged steel permanent magnet, NS.

The field is strengthened and made approximately radial by the insertion of a soft iron cylinder, I, in the space inside the coil. This cylinder, as seen in the lower fig. 286, is fixed to a brass bar, B, which forms a bridge between the pole pieces. The air gap is thus reduced and the coil moves freely in this space.

To establish a restoring couple, the two hair-springs, H, H^1, are attached to the axis, and to points on the framework which are insulated from each other. These springs also serve as leads for the current, to and from the coil.

Attached near the upper spring is a light counterpoised pointer which moves over a scale G. The centre of gravity of the whole system is arranged to coincide with the axis of suspension.

When in use the instrument is placed in parallel with the points whose potential difference is to be measured. The internal resistance of the instrument must, therefore, be large in order to avoid any appreciable rearrangement of current and potential drop in the circuit. The current passing through the voltmeter is therefore very small for such a high internal resistance, and therefore the heating of the coil is not very great.

The internal resistance is not made up entirely of that of the copper coil. In most forms the greater part of the internal resistance consists of a resistance in series with it (fig. 287). The chief reason for this is to avoid any error due to heating in the moving coil.

Such heating may be due to (α) atmospheric rise in temperature, (β) the Joule (C^2R) effect. An increase in resistance of the moving part would occur in either case unless the temperature coefficient of the wire were small. Manganin has a small temperature coefficient, but a higher specific resistance, i.e. for the same coil resistance a less radiating surface is available for manganin than copper.

The effect of (α) in raising the temperature of the coil may be best eliminated by making the moving coil resistance fairly low, and hence the percentage change of the whole is reduced.

Hence the usual compromise is a copper moving coil of comparatively small resistance and a series manganin coil of high resistance. This series resistance is constructed of thicker wire than would be possible for the moving coil.

Having in this manner secured the best approximation to constancy for the internal resistance, it will be seen that the small current, C, passing through the instrument is proportional to the potential difference between the terminals. Now the deflection produced is proportional to the current for a large range of deflection, when the field is radial, and hence the deflec-

tion produced is proportional to the potential difference between the terminals.

The instrument is made ' dead beat ' by winding the moving coil on a copper frame.

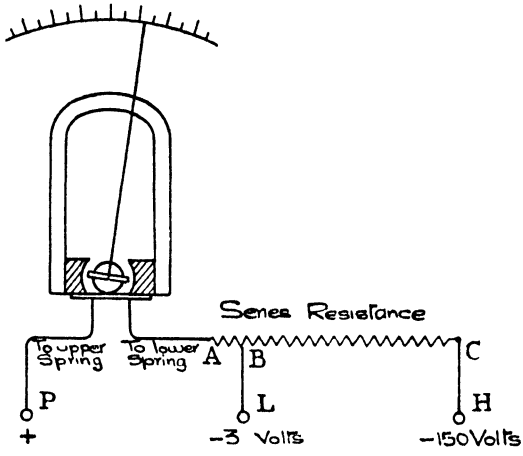


FIG 287

The same instrument may be used to measure different ranges of potential. This will be seen from fig. 287. Thus if a potential of 3 volts, when applied to PL, produces a full scale deflection in the instrument, whose internal resistance (AB + coil) is 345·5 ohms, it will be seen that the current in the coil is ·00865 ampere.

Now, if a higher potential, say, 150 volts, is to be the new value corresponding to a full scale deflection, this may be connected to H and P so that a bigger series resistance, BC, is included, such that the total internal resistance, R, is given by

$$\cdot 00865 = \frac{150}{R},$$

or R is 17270 ohms.

The same current will flow through the coil and hence the deflection will be again a full scale deflection.

The graduations on the scale will therefore subdivide the 0 to 150 into equal increments, and each division corresponds to fifty times the value which corresponds to the lower voltage applied to PL.

For the lower range voltmeter, measuring potential of the order of a millivolt, the value of the internal resistance is smaller, for the deflection is proportional to BC, where B is the magnetic flux in the gap and C the current. B is constant and so C to produce the deflection, when a low potential is applied, is obtained by decreasing R.

In such a moving-coil instrument the direction of deflection depends upon the directions of the current. The higher potential should be connected to P which is marked +. An accidental reversal of this order is apt to strain the needle.

Ammeters

To measure the current in a circuit, the measuring instrument used should be of low resistance, unless some account be taken (as in galvanometers) of the resistance introduced in this way. For example, if a small resistance of known magnitude, r, be included in the circuit, the current strength, c, may be found if the value of the potential drop, V, along r, be determined by means of a millivoltmeter, for $c = \frac{V}{r}$. This arrangement of a voltmeter shunted with a low resistance is utilized in the ammeter. The fixed-range ammeter usually contains the shunt inside the case. The value of the shunt resistance is small, and therefore the resistance of the whole instrument is of the same order. The moving coil of the ammeter is often, also, provided with a series resistance as in the voltmeter to minimize temperature variations as described above.

Many of the better forms of ammeter are not provided with fixed shunts but require an external one. The value of the shunt resistance determines the range of the instrument.

For bigger ranges the smaller is the resistance of the shunt. Suppose, as before, the maximum scale reading is obtained for a current, c, through the coil; this is proportional to the potential difference between the ends of the shunt. It will be obvious, that if the external current, C, be doubled the drop of potential along the shunt will be doubled, so that if a shunt of *half the original* resistance replace the first, the potential drop will be equal to that which is required to send a current, c, through the coil and series resistance of the instrument, and so produce a full scale deflection.

The shunts are made of manganin, which has a low temperature coefficient. The dimensions required in using a definite manganin strip may be calculated. If it is found that to produce no appreciable heating the shunt width has to be excessive, it is usual to construct the shunt of several strips in parallel.

The instrument should never be used without the appropriate shunt for the current to be measured.

The ammeters and voltmeters described above have the advantage of being direct reading on a calibrated scale; they are robust and do not require any adjustment. But as indicated at the outset, the general type of instrument is not sufficiently sensitive to measure currents of less than a milliampere or

potential less than one millivolt; but by more delicate construction and general improvement they can be made to measure to one micro-ampere and micro-volt. In such a case the extra sensitivity entails very precise work, and makes the cost of the instrument somewhat higher than for the ordinary range (i.e. one millivolt or ampere).

Some forms of instrument are available which combine the voltmeter and ammeter. The necessary shunts and series resistance are contained inside the case, and by connecting to the proper terminals, the instrument may be used either as an ammeter, or as a voltmeter of several ranges (see, for example, fig. 288).

Unipivot Instruments

To increase the sensitivity of the above types of instruments, a modification of the support of the moving part was introduced by R. W. Paul. The pivot friction was reduced very considerably by the use of the one-pivot method of suspension, and at the same time all the advantages of the form of double-pivot suspension were retained, so that a sensitivity corresponding to one subdivision per micro-ampere is obtainable for the uni-pivot instrument.

The construction is shown in figs. 289 and 290. A circular coil is suspended about a spherical core of soft iron between the poles of a permanent magnet.

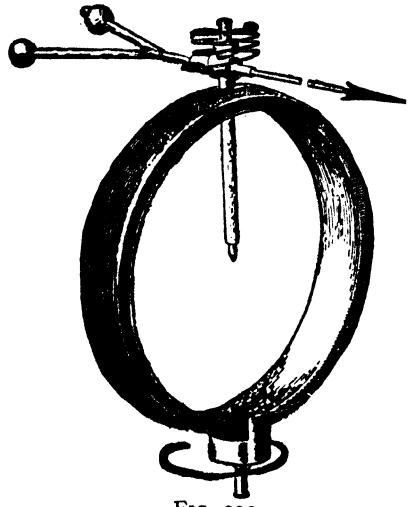


FIG. 290

Fig. 290 shows the detail of the coil support. A vertical spindle carries a light counterpoised pointer, and rests on a polished jewel at the bottom of a cylindrical hole drilled in the soft iron sphere.

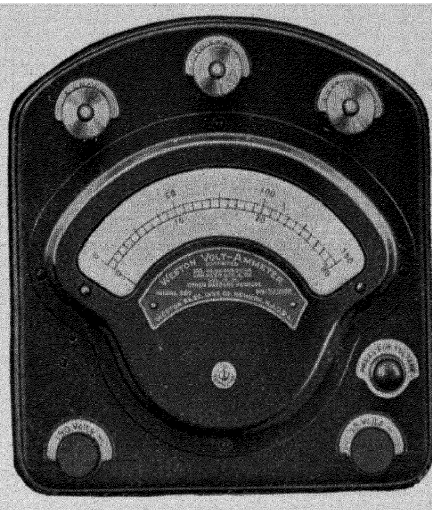


FIG. 288

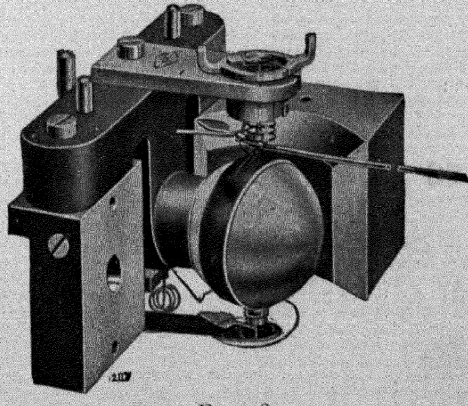


FIG. 289

The cylindrical spring at the upper end has a very slight lifting effect on the coil, and produces a restoring couple when the coil is deflected ; it also serves as a lead for the current to the coil. The current leaves the coil by the flexible wire shown at the lower extremity.

The centre of gravity of the moving part is at the point of support.

A simple device is included to raise the point off the jewel when the instrument is not in use. This is shown under the coil in fig. 289, which gives the general appearance of the instrument when one pole piece and the magnet are removed.

For many purposes it is necessary to be sure that the ammeter or voltmeter used is reliable to a fair degree of accuracy. The instrument may be calibrated in the laboratory (e.g. by potentiometer and standard cell), but it is here suggested that each laboratory be provided with one form of ammeter or voltmeter (or both) which has been tested at the National Physical Laboratory, and is provided with a correction certificate. Such instruments should be retained at laboratory standard, and the working instruments checked against such standard instruments.

Galvanometers

When smaller currents are to be measured, use must be made of some form of galvanometer, an instrument which is not as robust as the above, it must be levelled before use, and further the current must be calculated from the observed deflection produced by it.

Thus it is not as convenient and simple to use as the ammeter, but has a sensitivity which is impossible to attain in the latter.

The increase in sensitivity is largely produced by a more sensitive method of suspension. The friction of the pivot is entirely removed by the use of a fine suspension of silk or phosphor-bronze. The suspension carries a small concave mirror by means of which small movements of the moving part may be magnified. The two common methods of producing such magnification are by use of

(1) a lamp and scale,
(2) scale and telescope.

(1) *Lamp and Scale Method*

In this method a beam of light from an incandescent lamp or Nerst filament is directed by means of a lens on to the concave mirror, which reflects it to a scale some distance away. The greater the distance, D, between mirror and scale, the greater

the magnification produced. When the mirror rotates through an angle, θ, the reflected beam moves through twice that angle, causing a movement of the spot of light, say, d cms. on the scale.

Hence $$\tan 2\theta = \frac{d}{D}.$$

To measure such deflection it is essential that the mirror should produce a clear image. For this reason, with the size and character of the mirror available on such a suspension, the usual maximum value of D is one metre.

(2) *The Telescope and Scale Method*

A scale is set up horizontally at about one metre from the galvanometer mirror, and a telescope, usually under the central graduation of the scale, is turned towards the mirror. When the mirror, which should be a plane one for this method, is parallel to the scale, the latter may be seen in the telescope. The reading of the scale in coincidence with the cross-hair in the eyepiece of the telescope is noted. When the mirror moves a second scale reading will coincide with the cross-hairs. The difference for an angular rotation, θ, of the moving system may be accurately measured. If this is equal to d cms. then, once more, $\tan 2\theta = \frac{d}{D}$.

It is advisable, for simplicity of reading, to use either a telescope with an erecting lens or prism or to use a scale provided with 'inverted' graduations, so that the scale appears the right way up.

For many reasons it will be apparent that the instrument cannot be calibrated once for all. The scale distance is variable; the suspension may of necessity be replaced, and so on; there is, therefore, no permanent direct reading scale as in case of ammeters and voltmeters.

The usual method of converting scale readings to the corresponding currents is to obtain the 'sensitivity' of the instrument.

Sensitivity

The sensitivity of a galvanometer may be defined in many ways. In addition to being a 'reduction factor,' converting scale divisions to current, as indicated above, the sensitivity also gives an indication of the possibilities of the instrument. However for a comparison of two instruments another factor will be discussed later.

Current sensitivity may be defined in either of the two following ways:

(α) The number of *mms.* deflection produced on a scale one metre away by one *micro*-ampere, i.e. by 10^{-6} ampere, or

(β) the number of micro-amperes required to produce one *mm.* deflection on a scale one metre from the galvanometer mirror.

The second definition is perhaps from some points of view

the better, but (α) gives a value which varies directly with the property measured. It must be understood that a sensitivity of 1000 mms. per micro-ampere means only that a very small current produces a deflection which would correspond to 1000 mms. deflection when a micro-ampere passes, assuming the deflection to be still proportional to the angular movement of the coil. It is purely a theoretical mode of expression. The deflections measured never actually exceed from 5° to 8°.

Volt sensitivity may be similarly defined, substituting micro-volt for micro-ampere in the above.

It will be seen that for a fixed current strength the current sensitivity is greater for a bigger coil resistance, i.e. a bigger number of terms.

Volt sensitivity, which deals with a fixed potential applied to the galvanometer, is the greater the smaller the value of the resistance of the coils.

Determination of Current Sensitivity

The galvanometer whose resistance, G, has been determined, is connected in series with a megohm (10^6 ohms) and a cell of known electromotive force, E. The deflection on the scale one metre away is noted $= d_1$ *mms.*, say.

Now the current passing is $\frac{E \times 10^6}{(10^6 + G)}$ micro-amperes; from which the number of *mms.* deflection produced by one micro-ampere may be calculated.

An alternative method is shown in fig. 291.

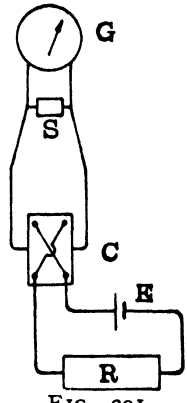


FIG. 291

A steady accumulator whose electromotive force, E volts, is known accurately—either by a potentiometer comparison with a standard cell or by measurement with a calibrated voltmeter—is connected in series with a high resistance, R ohms, through a commutator, C, to a low resistance, S, which is in parallel with the galvanometer of resistance G.

The mean deflection δ, say for both positions of the commutator is obtained.

The current causing the deflection is readily calculated, for the effective resistance of the shunt and galvanometer is $\frac{SG}{S+G}$; the main circuit current is therefore equal to

$$\frac{E}{R+B+\frac{SG}{S+G}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where B is the battery resistance which, if an accumulator is used, is negligible compared with R. Hence neglecting B in (1), the current through the galvanometer is

$$C_g = \frac{E}{R+\frac{SG}{(S+G)}} \times \frac{S}{(S+G)} \times 10^6 \text{ micro-amperes}$$

$$= \frac{E \cdot S \times 10^6}{R(S+G)+SG} \text{ micro-amperes} \quad \ldots\ldots\ldots\ldots (2)$$

Hence the sensitivity is $\frac{\delta}{C_g}$ mms. per micro-ampere.

Thus the sensitivity having been obtained the deflection produced in any case may be converted to the corresponding current values so long as the suspension remains the same and the scale is at the same distance, one metre, from the galvanometer mirror.

STEADY CURRENT MEASUREMENT

Galvanometers for the measurement of steady direct current may be subdivided into two general classes according to the nature of the moving part: (*a*) moving needle, (*b*) moving coil.

(a) Moving-needle Galvanometers

The student will be familiar with the simple tangent galvanometer. In this form a *small* magnet is suspended at the centre of a coil of wire of n turns. When a current of C, expressed in electromagnetic units passes through the coil, placed in the magnetic meridian, the magnetic field set up causes a deflection, θ, such that the restoring couple due to the horizontal component of the earth field, H, balances the couple due to the magnetic field of the current, and we have:

$$C = \frac{Hr}{2\pi n} \tan\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where r is the mean radius of the coil.

In general the assumptions upon which the formula is developed

are not justified, and corrections should be applied to allow for the width of the coil, etc. ; such corrections are but seldom used, and the galvanometer is not used as a small current measurer.

A modification, known as the Helmholtz galvanometer, consists of two similar coils placed on a common axis at a distance apart equal to the radius of either.

A short magnet is supported at the mid-point between the centres of the coils. As with the tangent galvanometer, the plane of the coils is placed in the magnetic meridian, then using the same notation as before,

$$C = \frac{5\sqrt{5}r^2H}{32\pi n} \tan\theta; \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

n is the number of turns in each coil.

These two forms are such that the absolute value of the current may be simply calculated, but they have no claim to great sensitivity.

Sensitive Current Detectors

The result expressed in (3) for a simple tangent galvanometer shows that, for this form of instrument, the deflection for a given current may be increased by

(α) decreasing H, the control field,
(β) decreasing r,
(γ) increasing n.

There is a limit to methods (β) and (γ) which is prescribed by the practical problem presented.

Further, as r is decreased and n increased, the conditions upon which (3) was developed no longer hold.

(α) may be best carried out by the use of an external control magnet which neutralizes the value of the horizontal component of the Earth's field. Further, by using an astatic pair of magnets or groups of magnets the control effect may be reduced without interfering with the magnitude of the deflecting couple.

The above conditions are embodied in the Thompson (or Kelvin) and Broca galvanometers.

The Thompson Galvanometer

The Thompson galvanometer is illustrated in fig. 292. The control magnets, EE, enable the galvanometer to be used in any position independently of the Earth's field, the suspended astatic system is shown at the right of the diagram. Each set of magnets is placed at the centre of a pair of coils, FF. The magnets are mounted on a light rod, the whole being supported from a torsion head by an unspun silk or a quartz fibre.

The magnitude and direction of the control field may be varied by alteration of the positions of EE along the vertical rod shown.

Sets of coils of different resistances are usually supplied. Using high-resistance coils a sensitivity of about 600 mms. per micro-ampere may be attained, i.e. the instrument will detect currents of the order of 10^{-9} ampere.

The movement of the magnet is detected, using the small mirror, B.

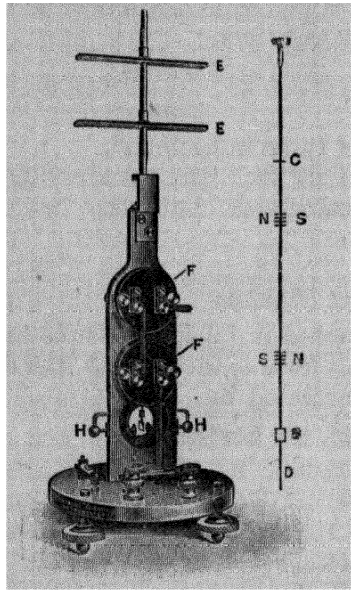


FIG. 292

The Broca Galvanometer

This instrument is shown in fig. 293. The moving 'astatic pair' is made up of two vertical magnets having consequent poles as shown. The control field is due to the magnet, B, which moves in a ball socket to any desired position.

The coils, EE, may be of any suitable resistance, say, 10, 100 or 1000 ohms as required for sensitivity in the circuit. They are connected in series, one at each side of the space occupied by the centre magnet, and arranged to produce a field in the same direction.

As in the Thompson galvanometer, an aluminium vane, G, moves between two parallel plates which may be adjusted by the rods terminating in the metal knobs, CC. The damping of the system may be altered by an adjustment of the distance between these plates. Currents of the order of 10^{-10} ampere may be detected.

The Thompson and Broca galvanometers are most advantageously employed as very sensitive current detectors whose sensitivity may be rapidly adjusted over a wide range by the control magnets.

Both forms *require levelling before using.*

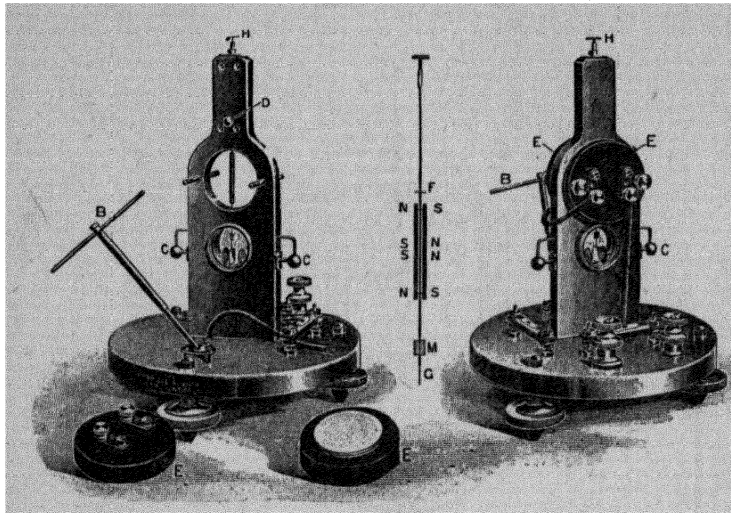


FIG. 293

(b) Moving-coil Galvanometer

The moving-coil galvanometer is constructed in a manner very similar to the ammeter already described. A phosphor-bronze strip, F, fig. 294, acts as suspension to the coil, C, which is free to move in the gap between the pole pieces of a permanent magnet, one of which is not shown in fig. 294, and a soft iron cylinder, I, which is screwed to a brass plate as seen in the lower figure.

In this case the magnetic field in the air gap is approximately radial, as shown by the broken lines in the lower figure (fig. 294).

The current enters the coil via the phosphor-bronze strip, F, and leaves at the under end by means of a helix of phosphor-bronze, S. The control is mainly due to the twist of F in such a case.

If B is the magnetic flux in the air gap, A the area of one turn, n the number of turns, and τ the restoring couple per unit angular twist of the suspension, F, we have, for the case of

a radial field, when a current, c, passes, causing a deflection, θ,

$$BnAc = \tau\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

or $$c = k\theta,$$

when k is a constant and equal to $\frac{\tau}{BnA}$.

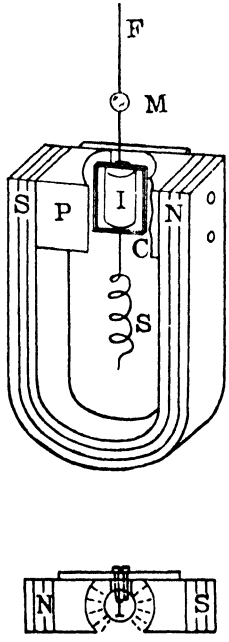


FIG. 294

For small deflections, $\theta = \tan\theta$, i e. θ is proportional to d, the deflection on the scale.

From (5) above it will be seen that to increase the sensitivity, τ must be made as small as possible, and B, n and A as large as possible.

To decrease τ we may decrease the cross-section of the strip or increase its length.

For a circular wire it was shown on page 102

$$\tau \propto \frac{R^4}{l}.$$

The most profitable method of decreasing τ is therefore to reduce the cross-section, i.e. use a fine suspension. This is limited by the fact that *very* fine suspensions are also very fragile and somewhat difficult to use. From this point of view

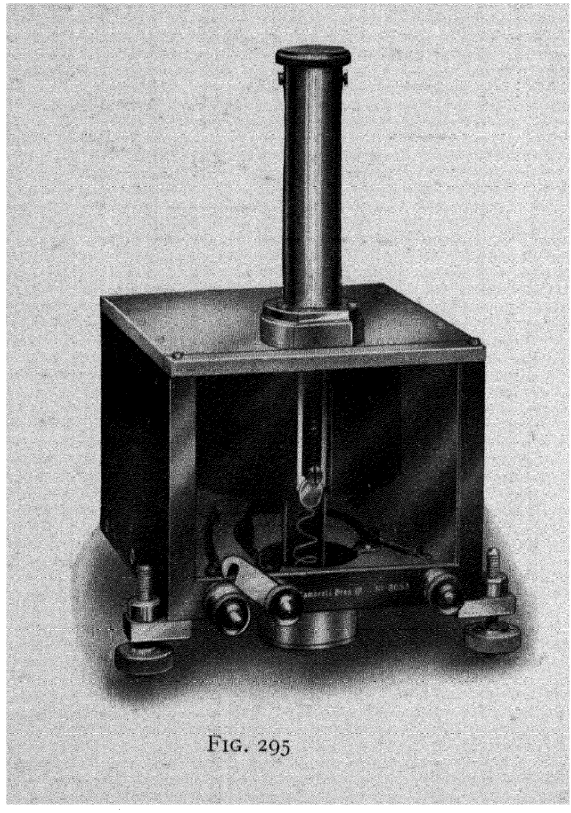


FIG. 295

phosphor-bronze is the most satisfactory material and is almost universally employed. The usual type of moving coil is supported by a fibre of the greatest convenient length.

The increase of B, n, and A must be considered together as the terms are interdependent.

Assuming the magnet to be fully saturated, the value of B in the air gap depends on the size of the gap. The smaller the air space, the larger is B. The effect of increasing nA is to increase the size of the coil, and consequently the air gap, and hence decrease B.

In practice it is usual to use a standard size of air gap which allows a frame of such size to be wound with wire, and move freely in the available space, that the best compromise between these two effects is obtained for a maximum value to the product $B \cdot n \cdot A$.

An example of galvanometers of this type is shown in fig. 295.

Such instruments may have a sensitivity as great as 1500 *mms.* per micro-ampere, i.e. the instrument will measure currents of the order 10^{-10} ampere.

Adjustment

The moving-coil instrument must be adjusted before use. The coil is first released, then the instrument is levelled by means of the levelling screws, so that the coil does not touch either the pole pieces or the iron core, and is thus able to swing freely.

The instrument may be used in any position, and the coil is turned by means of a torsion head, which carries the suspension, until the plane of the coil is approximately parallel to the sides of the magnet.

It is inadvisable to make this adjustment, unless the reflected beam does not fall on the scale, for there is a danger of breaking the suspension.

Onwood Moving-coil Instruments

The Onwood galvanometer differs somewhat from the above general types in that it requires no levelling, is not so fragile, and is more convenient to move.

The difference is in the method of suspension as seen in fig. 296. NS is the permanent magnet with a soft iron core, *d*, which is drilled to the centre. From this point, *h*, is a short suspension *c*, which is of sufficient length to support the coil, *a*, from the point, *g*, clear of the fixed parts. A small mass, *e*, at the end of the rod, *f*, keeps the system vertical.

The current is led through the frame to the iron core, and thence through the suspension, and the tube, *b*, to the coil which it leaves by means of a flexible ligament at the base of the coil, but not shown in the diagram.

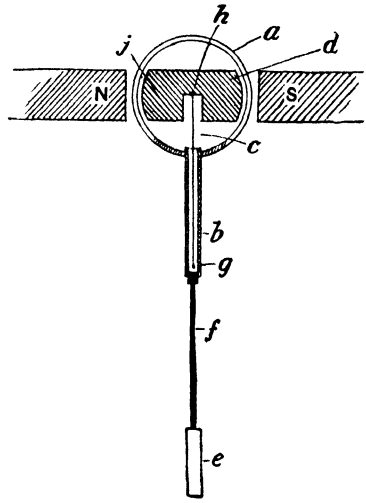


FIG. 296

Since the coil is supported at the centre it will be equidistant from pole pieces and iron core alike for all positions of the instrument not too far removed from the horizontal, i.e. the instrument does not require levelling.

A small spring, not shown, at the upper end of the suspension protects the latter against sudden shocks.

The galvanometer is a very small one and the sensitivity claimed ranges from 20 to 500 *mms.* per micro-ampere for different forms.

THE CHOICE OF A GALVANOMETER FOR THE MEASUREMENT OF A STEADY DIRECT CURRENT

To select the galvanometer most suitable for a particular experiment many factors must be considered. To decide firstly between the two general types described above, their relative advantages and disadvantages are discussed below.

Moving-needle Type

(1) The value of the control field is affected by the proximity of external magnetic fields. The instrument may register a deflection when no current circulates through the coils when

magnets are moved in the laboratory, or dynamos set in motion, or even by a passing electric tramway car. Thus there is some uncertainty unless such stray fields are eliminated.

(2) The moving magnet may have its moment altered by the field set up when a current passes through the coil.

(3) Unless the instrument is made 'dead beat' the needle takes a long time to come to rest and is therefore somewhat troublesome to use. This may be overcome to some extent by using an external damping circuit, consisting of a solenoid through which a current may pass when the circuit is closed. This solenoid is placed near the case of the instrument and the circuit closed momentarily as the needle swings, in such a way as to produce a field which opposes its motion.

(4) The damping of the instrument is independent of the external field.

(5) The needle may be supported by means of a quartz suspension which has the property of returning after deflection to its original position. However, the slight torsion in the fibre is usually neglected, and the expression for the current is therefore not strictly accurate.

Moving-coil Type

(1) The moving-coil galvanometer has a large permanent magnet, and is therefore practically unaffected by stray external fields.

(2) No demagnetizing effect is possible on the moving part.

(3) The movement of the coil may be very rapidly arrested, even when the instrument is not 'dead beat,' by connecting the ends of the coil together through a very low resistance as the coil passes its rest position.

(4) The electromagnetic damping varies with the value of the resistance of the external circuit. If used in series with a low resistance this damping is very high, and the coil *may take several minutes to attain the full deflection*, a fact which is very often overlooked when the instrument is used.

(5) The zero-keeping quality of the suspension depends upon the degree of sensitivity attained. The torsion at the fibre is taken into account in the expressions for the current.

Having decided, from the conditions of the experiment, the *type* of instrument most suitable, the next problem is, What is the most satisfactory galvanometer resistance?

The Order of Resistance of a Moving-coil Galvanometer which is most sensitive in a Given Circuit

In general terms we may state that if the current is to be of a fixed value, independently of the galvanometer, considering all

factors, the sensitivity will be proportional, approximately, to $\sqrt{G}$, i.e. will increase with increasing resistance.

If a fixed potential difference is to be measured the sensitivity will obviously be greater the smaller the resistance; approximately, the sensitivity is proportional to $\frac{1}{\sqrt{G}}$, i.e. high resistance for detecting current, and low for detecting small potential differences.

The problem usually presented is, given an external resistance, R ohms, what is the best value for G, the galvanometer resistance for measuring the current due to a fixed electromotive force.

We saw (pages 467-8, equation (5)) that the couple due to the current c is nABc, and if E is the electromotive force in the circuit

$$c = \frac{E}{R + G} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Further, it was shown on that page that in a galvanometer there is but a limited space between the pole pieces to obtain a maximum sensitivity. Fig. 297 shows a cross-section of the

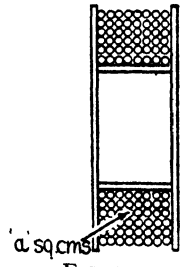


FIG. 297

frame which will just move freely in this space. Let the cross-section of the whole of the windings be a sq. cms. and assume that the windings entirely fill the space with copper. If p be the mean perimeter of the coil windings, we have length of wire used $= np$, cross-section of the wire $= \frac{a}{n}$.

Therefore $$G = \frac{\sigma np}{\frac{a}{n}} = \frac{n^2 p\sigma}{a}.$$

where σ is the specific resistance of the wire, say, copper,

i.e. $$n = \sqrt{\frac{Ga}{p\sigma}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Hence the couple due to the current

$$= BnAc$$

$$= BA\sqrt{\frac{Ga}{p\sigma}} \cdot \frac{E}{R + G}.$$

from (6) and (7).

The condition for the couple to be a maximum and therefore produce a maximum effect is that

$$\frac{BAE\sqrt{a}}{\sqrt{p\sigma}} \cdot \frac{\sqrt{G}}{R+G}$$

be a maximum.

We have seen that a and p are constant, so that the condition is

$$\frac{\sqrt{G}}{R+G} \text{ is to be a maximum,}$$

or that

$$\frac{R}{\sqrt{G}} + \sqrt{G} \text{ be a minimum,}$$

i.e.

$$R = G.$$

That is, under the circumstances stated, the maximum sensitivity is obtained *when the galvanometer resistance is equal to the total external resistance.*

It should be noted that the galvanometer resistance in this discussion refers to the copper coil resistance only. The resistance of the suspension should be included in the value R. This being so, it is apparent that there is a lower limit beyond which the resistance of a coil may not be reduced with any advantage.

If the galvanometer is to be chosen as a detector or measurer of small differences of potential the most suitable instrument will be one of low resistance (e.g. for thermo-couple work).

Other qualities of galvanometers to be considered when making a selection of galvanometers are :—

(1) Damping

The damping of the moving part in a galvanometer apart from external artificial agency may be considered due to two separate causes.

(*a*) *The damping due to the viscosity of the air.* This is present in moving coil and needle alike, and is approximately proportional to the angular velocity of the system. It is always present, but is usually small.

(*b*) *Electromagnetic damping.* In the case of a moving magnet the amount of damping due to this cause is very slight when the magnet is in a non-metallic case, e.g. when the coils are wound on wood or ebonite.

This is the reason for the long and troublesome wait which occurs before the needle returns to its rest position. This may be reduced as described under. However, in either case the amount of damping is obviously independent of the external circuit.

In the case of the moving-coil instrument, the suspended coil, when closed by an external circuit, is moving in a strong magnetic

field. Under such circumstances the electromotive force induced in the circuit sets up a reverse current in the closed circuit, which is therefore brought to rest.

The value of the damping current depends on the magnitude of the external resistance, and may become very great for a low series resistance.

For many purposes it is necessary or convenient to have a galvanometer such that the moving part very rapidly returns to the zero position after being deflected. A galvanometer having this property is said to be '**dead beat.**'

In both types of instrument this may be brought about by increasing the electromagnetic damping. In the moving-needle type this is accomplished by encasing the needle in a copper, or similar metal, case. The movement of the needle sets up eddy currents in the copper and the magnet is rapidly brought to rest.

The moving coil may be rendered dead beat by winding it on a metal 'former' or frame. This constitutes a closed metallic circuit, and the desired result is obtained. Alternatively, if the galvanometer is wound, for ballistic purposes, on a non-metallic frame, e.g. bamboo, the same result is obtained by facing the coil with a thin sheet of copper foil (cut into a 'picture frame' which is the same size as the edge of the coil).

(2) Period and Constancy of the Zero

A galvanometer which is very sensitive has a long time of swing; and also, due to a very fine suspension, some trouble may arise due to the 'creep' of the zero. However, the type of instrument used in experiments in this book will not be of the *extremely* sensitive order at which this trouble arises. The main cure for the trouble lies in the selection of suspension, and that really involves the selection of an instrument maker who will take the trouble to *minimize* this fault. Beyond this the correction of residual effect must be solved by the ingenuity of the experimenter as applied to the particular experiment involved.

Quartz for moving magnets and phosphor-bronze for moving coils cause least trouble in this respect.

For average work 5 to 10 seconds per complete swing will be best value for direct steady current measurement or for use in 'null' methods.

MEASUREMENT OF QUANTITY OF ELECTRICITY

The Ballistic Galvanometer

A galvanometer suitable for measuring a *quantity* of electricity is called a *ballistic galvanometer*, and has the following essential features :

(1) The periodic time of swing, T, of the moving part is fairly large.

(2) Damping of the moving part is very small.

The first condition is fulfilled by making the moment of inertia of the needle or coil which forms the moving part as large as practicable and by reducing the controlling forces, for

$$T = 2\pi\sqrt{\frac{I}{\tau}},$$

where I is the moment of inertia of moving part, and τ is the restoring couple per unit angular displacement.

Thus by increasing I and decreasing the restoring forces, T, the time for one complete swing, is increased.

The second factor, damping, is reduced in a way which depends on the instrument (needle or coil).

As seen when considering damping (page 473) the electromagnetic damping only may be reduced. The air damping is usually small.

(3) A third condition is, that when used to measure a quantity of electricity, the whole of the transient current shall pass before the needle or coil moves from the zero position. Should there arise a case in which the quantity of electricity to be measured takes longer time to traverse the instrument, due, for example, to inductance in the circuit, the time of swing of the needle must be increased, by loading it, so that this third condition is fulfilled.

As indicated above, the galvanometer may be of the moving-needle or moving-coil type. We shall develop a relation between the throw or angular deflection in either type, and the quantity of electricity which passes.

Moving-needle Type

This type of ballistic galvanometer consists of a needle suspended by a fine quartz or unspun silk fibre, at the centre of two coils, through which the quantity of electricity, Q, passes.

The control in this case is either the Earth's field or a control magnet. The needle in its zero position is arranged at right angles to the axis of the coils, so that when a current passes a field is set up at right angles to the control field.

Let G be the galvanometer constant, i.e. the field due to the coils for unit current circulating through them,
H the value of the control field strength,
I the moment of inertia of the magnet about the axis of suspension,
M the magnetic moment of the magnet.

Suppose a current of strength, c, to pass through the coils for a very small interval of time. Since the third condition above holds, the needle will be at right angles to a field of strength Gc, and will experience a turning moment, GcM.

This couple will produce an angular acceleration $\ddot{\theta}$ in the needle. Hence (page 53)

$$I\ddot{\theta} = GcM,$$

$$I\dot{\theta} = GM\int c\,dt = GMQ \quad \ldots\ldots\ldots\ldots\ldots(8)$$

At the end of the swing, the needle having turned through an angle θ_0, the kinetic energy of the moving needle, $\frac{1}{2}I\dot{\theta}^2$, has been reduced to zero in doing work against the magnetic force, Hm (m being the pole strength of the needle) at each pole.

The work done is $2Hm\left(\frac{l}{2} - \frac{l}{2}\cos\theta_0\right)$

$$= Hml\,(1 - \cos\theta_0)$$

$$= 2MH\sin^2\frac{\theta_0}{2},$$

where l is the distance between the poles.

Thus $$2MH\sin^2\frac{\theta_0}{2} = \frac{1}{2}I\dot{\theta}^2,$$

$$I\dot{\theta}^2 = 4MH\sin^2\frac{\theta_0}{2} \quad \ldots\ldots\ldots(9)$$

Hence squaring (8) and dividing by (9)

$$I = \frac{Q^2M^2G^2}{4HM\sin^2\frac{\theta_0}{2}} \quad \ldots\ldots\ldots\ldots\ldots(10)$$

We have also for the period, T, of the suspended needle in the control field, H,

$$T = 2\pi\sqrt{\frac{I}{MH}},$$

or $$I = \frac{T^2MH}{4\pi^2} \quad \ldots\ldots\ldots\ldots\ldots(11)$$

Equating (10) and (11):

$$\frac{Q^2MG^2}{4H\sin^2\frac{\theta_0}{2}} = \frac{T^2MH}{4\pi^2},$$

Hence:

$$Q = \frac{T}{\pi}\cdot\frac{H}{G}\cdot\sin\frac{\theta_0}{2} \quad \ldots\ldots\ldots\ldots\ldots(12)$$

In developing this result we have assumed that the original kinetic energy of the needle is wholly used in moving the magnet against the field, H, through an angle, θ_0. If, however, there are any frictional forces, i.e. the needle is slightly damped, some energy will be required to overcome this force, and the result will be that the observed angle of swing, say, α, is not truly of the magnitude given in the undamped case. The true value for Q is therefore dependent not on the observed value, α, of the angle, but on some slightly bigger angle θ_0.

It will be shown that $\theta_0 = \alpha\left(1 + \frac{\lambda}{2}\right)$, where λ is the *logarithmic decrement* of the suspended system,

$$Q = \frac{T}{\pi}\frac{H}{G}\sin\left[\frac{\alpha}{2}\left(1 + \frac{\lambda}{2}\right)\right] \quad \ldots\ldots\ldots\ldots(12a)$$

Moving-coil Ballistic Galvanometer

There are several disadvantages in the moving-needle instrument as in the case of measurement of steady direct current, e.g. (1) and (2), pages 470–1. These deflects are overcome in the moving-coil type of ballistic galvanometer.

The electromagnetic damping is reduced by using a coil wound on a bamboo frame ; T is increased by using a fine phosphor-bronze strip suspension.

To obtain the connexion between Q, a quantity of electricity discharged through the galvanometer, and θ_0, the first throw, we may first of all assume that there is no damping.

Suppose τ is the torsional couple exerted by the suspension per unit angular displacement.

Then $\tau\psi$ is the restoring couple when the coil is turned through any angle ψ.

To turn the coil through an additional small angle, $d\psi$, the work done is $\tau\psi \cdot d\psi$.

The total work required to twist the coil from its zero position to a final position, θ_0, is :

$$\int_0^{\theta_0} \tau\psi \cdot d\psi = \frac{\tau\theta_0^2}{2} \quad \ldots\ldots\ldots\ldots\ldots(13)$$

We now obtain an expression for this in terms of the work done by the current.

Let G be the galvanometer constant, i.e. the couple exerted on the coil when unit current passes through it.

For a current of strength c, we have W, the couple is

$$W = G \cdot c \quad \ldots\ldots\ldots\ldots\ldots(14)$$

If now a small quantity of electricity, dq, passes through the coil in a time, dt, we have :

$$dq = cdt = \frac{W}{G} \cdot dt \qquad \text{. (from (14))}$$

So that if a total quantity, Q, passes through the galvanometer

$$Q = \int dq = \int \frac{W}{G} \cdot dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

But if I is the moment of inertia of the suspended system, we have seen :

$$W = I\ddot{\theta}.$$

Thus (15) becomes

$$Q = \frac{I}{G}\int I\ddot{\theta}dt = \frac{I\dot{\theta}}{G} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Now from the condition that the discharge must pass through the galvanometer before any movement takes place in the coil, an angular velocity, $\dot{\theta}$, will be imparted to the coil which will therefore have a kinetic energy

$$= \tfrac{1}{2} I\dot{\theta}^2 \qquad \text{(page 52)}$$

From equation (16) above

$$\dot{\theta} = \frac{GQ}{I}.$$

So the K.E. is

$$\tfrac{1}{2} I \frac{G^2Q^2}{I^2} = \frac{G^2Q^2}{2I} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

This kinetic energy is reduced to zero in twisting the suspension. When the angle of deflection is θ, we have seen that work equal to $\frac{\tau\theta^2}{2}$ (equation (13)) is required, and if θ_0 is the extreme first deflection, the whole of the kinetic energy vanishes as such, and the work done $\frac{\tau\theta_0^2}{2}$ is equivalent to the P.E. stored as strain in the suspension,

i.e. $$\frac{\tau\theta_0^2}{2} = \frac{G^2Q^2}{2I} \qquad \text{(from (17))}$$

i.e. $$Q^2 = \frac{I\tau}{G^2}\theta_0^2$$

We may eliminate I from the above equation, for we have, if T be the time of swing of the suspended coil :

$$T = 2\pi\sqrt{\frac{I}{\tau}},$$

$$I = \frac{T^2\tau}{4\pi^2};$$

whence $$Q^2 = \frac{T^2\tau^2\theta_0^2}{\pi^2G^2 2^2},$$

or $$Q = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\theta_0}{2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

or if α is the observed first swing, and λ is the logarithmic decrement,

$$Q = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\alpha}{2}\left(1 + \frac{\lambda}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (18a)$$

Suppose now we send a *steady* current of known magnitude, c, through the galvanometer and observe the steady deflection, φ,

$$Gc = \tau\varphi$$

or $$\frac{\tau}{G} = \frac{c}{\varphi}$$

Hence $$Q = \frac{T}{\pi} \cdot \frac{c}{\varphi} \cdot \frac{\alpha}{2}\left(1 + \frac{\lambda}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

To correct the observed first swing (α) *for damping in either form of galvanometer.*

The observed first deflection, α, is reduced by damping forces which are proportional to the angular velocity.

The equation of motion, therefore, must be modified from the simple form :

$$I\theta + F\theta = 0,$$

where F may be τ or MH, according to the type of instrument, to include a term proportional to the angular velocity. Let this term be $K\theta$, then the equation of motion becomes :

$$I\theta + K\theta + F\theta = 0,$$

an equation similar to that already dealt with on page 27.

The amplitude of the system is therefore $\theta_0 e^{-\frac{K}{2I}t}$, using the notation of page 161.

Thus the first observed swing, $\alpha_1 = \theta_0 e^{-\frac{K}{2I}\cdot\frac{T}{4}}$.

Logarithmic decrement, $\lambda = \frac{K}{2I} \cdot \frac{T}{2}$,

i.e. $$\alpha_1 = \theta_0 e^{-\frac{\lambda}{2}},$$

or $$\theta_0 = \alpha_1 e^{\frac{\lambda}{2}}$$

$$= \alpha_1\left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{8} \ldots \text{etc.}\right)$$

In a *ballistic* galvanometer K is small and although T may be large, I is also large, and $\frac{KT}{4I}$ is small, i.e. λ^2 and higher terms may be neglected in comparison with unity,

i.e. $$\theta_0 = \alpha\left(1 + \frac{\lambda}{2}\right).$$

λ may be obtained by one of the methods of pages 162 to 164, and the correcting factor $\left(1 + \frac{\lambda}{2}\right)$ obtained. The *galvanometer must be in the circuit of the experiment when* λ *is obtained,* so that the damping is the same.

Another way of correcting for damping in the moving part does not involve an independent measurement of λ.

Suppose that in addition to noting the first swing, α_1, we observe the next swing on the other side of the zero, α_2, and the next swing on the same side of the zero, α_3, we have :

$$\alpha_1 = \theta_0 e^{-\frac{K}{2I}\cdot\frac{T}{4}};\ \alpha_2 = \theta_0 e^{-\frac{K}{2I}\cdot\frac{T}{2}};\ \alpha_3 = \theta_0 e^{-\frac{K}{2I}\cdot\frac{3T}{4}};\ \text{etc.};$$

or $$\frac{\alpha_1}{\alpha_2} = \frac{\alpha_2}{\alpha_3} = e^{-\frac{K}{2I}\frac{T}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

i.e. $$\alpha_1 = \theta_0 e^{-\frac{K}{2I}\frac{T}{4}} = \theta_0\left(e^{+\frac{K}{2I}\frac{T}{2}}\right)^{-\frac{1}{2}};$$

or by (20) $$\alpha_1 = \theta_0\left(\frac{\alpha_1}{\alpha_2}\cdot\frac{\alpha_2}{\alpha_3}\right)^{-\frac{1}{2}},$$

i.e. $$\theta_0 = \alpha_1\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Thus, suppose there is an error of one per cent in the determination of an angle, $\left(\frac{\alpha_1}{\alpha_3}\right)$ is liable to two per cent error or $\left(\frac{\alpha_1}{\alpha_3}\right)^1$ may be in error by one-half per cent.

The correction for damping obtained in this way will be sufficiently correct for many experiments.

Of course α_1, α_2, and α_3 are not measured as angles. The corresponding scale deflections, δ_1, δ_2, and δ_3, are measured.

Now $$\frac{\delta_1}{\delta_3} = \frac{\tan 2\alpha_1}{\tan 2\alpha_3} \text{ for small deflections,}$$

i.e. since under these circumstances the value of tan 2α is approximately the same as 2α,

$$\frac{\delta_1}{\delta_3} = \frac{\alpha_1}{\alpha_3}.$$

So that (19) for the moving-coil galvanometer becomes

$$Q = \frac{T}{\pi}\cdot\frac{c}{\varphi}\cdot\frac{\alpha_1}{2}\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots (22)$$

and (12*a*)

$$Q = \frac{T}{\pi}\frac{H}{G} \cdot \sin\left\{\alpha_1\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{2}}\right\} \quad \ldots\ldots\ldots\ldots(23)$$

In the above method the damping *under the conditions of the experiment* is obtained for the correction factor.

Equation (23) may be modified to conform with (22). For if a steady current, c, produces a deflection φ,

$$cG = H \tan \varphi,$$

i.e.

$$Q = \frac{T}{\pi}\frac{c}{\tan\varphi} \cdot \sin\left\{\alpha_1\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{2}}\right\} \quad \ldots\ldots\ldots\ldots(23a)$$

To Reduce the Excessive Damping in a Moving-coil Ballistic Galvanometer used in a Low-resistance Circuit

It was shown in the discussion of damping that the moving-coil galvanometer has a large amount of electromagnetic damping when in a closed circuit of low resistances. This would make the use of a moving-coil ballistic galvanometer inadmissible in many experiments unless some means were taken to reduce this.

The quantity of electricity passing through the instrument is, in general, due to some current change. If, therefore, the galvanometer is in circuit whilst such change takes place, and is then allowed to swing freely when it has received the impulse due to that cause, the damping is almost eliminated.

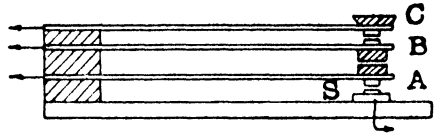


FIG. 298

By use of a compound key, such as that shown in fig. 298, this may be brought about. The three brass strips mounted on an ebonite block (shaded) are connected to separate terminals, as is the stop S. The battery circuit is connected to A and S, and the galvanometer to C and B. When the key is depressed, C and B, A and S are connected, but B and A remain insulated by the ebonite stops shown (shaded). When the key is released A and S are broken; impulse, due to induction or whatever cause is operative, is given to the galvanometer, and then C and B are disconnected by the upward move of the key. If the time interval between the break of the battery and galvanometer circuits is small compared with the period, T, of the coil of the galvanometer, the latter will swing an amount which is almost independent of electromagnetic damping due to the low resistance circuit.

If such a key is used for a moving-coil instrument the damping correction used should be obtained from observations of the coil when swinging freely in open circuit.

The Grassot Fluxmeter

The Grassot Fluxmeter is an instrument which performs the same function as a ballistic galvanometer, as for example in experiments on pages 426, 557, 582, 357, etc. However, the instrument is specially designed for measuring magnetic field strengths directly.

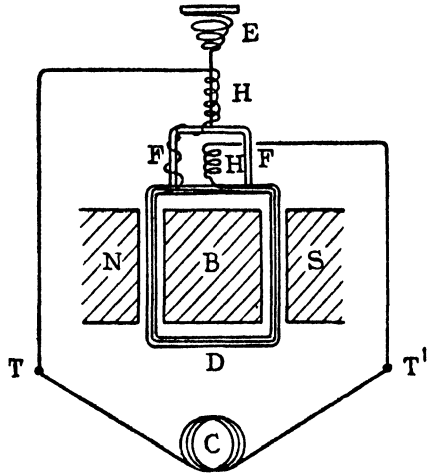


FIG. 299

It is a suspended coil instrument which depends entirely on the electromagnetic damping for control. The coil, D, is supported by a single cocoon silk fibre, which has a negligible torsional control, from a flat spiral, E (fig. 299), to eliminate the effect of shocks. The current enters and leaves the coil by the silver strip coils, H, as shown in the figure. The usual iron core, B, and permanent magnet pole pieces, N, S, supply a constant magnetic field in which the coil swings.

The light frame, FF, carries a pointer and a concave mirror, not shown in the figure. The points, T T′, correspond to terminals on the case of the instrument.

In general use a search coil, C, of known mean area and number of turns is connected to T,T′.

The search coil is placed into the magnetic field to be measured and the induced electromotive force causes a current to flow in the closed circuit and so produces a deflection which may be measured either by a lamp and scale arrangement, using the

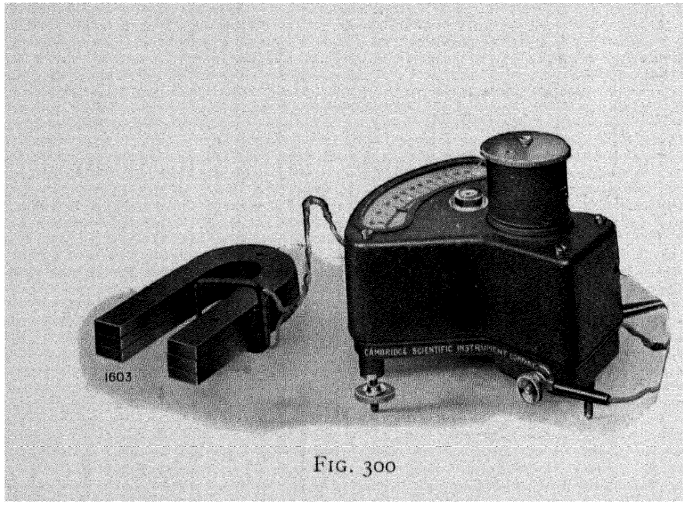


FIG. 300

concave mirror, or, if sufficiently large, by direct reading of the pointer over a scale.

The instrument therefore produces a deflection which is proportional to the total quantity of electricity which passes through it.

Fig. 300 shows the general appearance of the instrument in use to measure the magnetic distribution along a magnet.

The method of calibration of the graduated scale and general possibilities of the instrument will be apparent from a consideration of the theory of the instrument.

Let R = the resistance of the circuit, i.e. of the search coil, C, and leads, and the suspended coil, D,

L = the self-induction of the whole circuit,

I = the moment of inertia of the coil, D, about the axis of suspension,

E = the E.M.F. induced in C
c = the current in the circuit
ω = the angular velocity of the coil
} at any instant,

K = induced E.M.F. set up in D for unit angular velocity,

G = the galvonometer constant, i.e. the couple exerted on the coil due to unit current passing through it,

A = the couple due to air resistance for unit angular velocity (see damping, page 473.)

Expressing Ohm's Law for the circuit, we have :

$$cR = E - L\frac{dc}{dt} - K\omega, \quad \ldots\ldots\ldots\ldots\ldots (24)$$

i.e.
$$c = \frac{1}{R}\left(E - L\frac{dc}{dt} - K\omega\right).$$

The equation of motion of the coil is :

$$I\frac{d^2\theta}{dt^2} = Gc - A\omega \quad \ldots\ldots\ldots\ldots\ldots (25)$$

Substituting the above value for c and remembering that $\omega = \frac{d\theta}{dt}$,

$$I\frac{d\omega}{dt} = \frac{GE}{R} - \frac{LG}{R}\frac{dc}{dt} - \left(\frac{GK}{R} + A\right)\omega.$$

Now L, G, R, K, and A are constants ; further, if the coil starts from its rest position when no current passes, and becomes

deflected through an angle, α, by the quantity discharged through it, the coil is at rest at the end of the swing and no current passes through it.

Therefore, integrating the last equation over the whole swing with respect to t, we have

$$I\left[\omega\right]_0^0 = \frac{G}{R}\int E dt - \frac{LG}{R}\left[c\right]_0^0 - \left(\frac{GK}{R} + A\right)\left[\theta\right]_0^\alpha,$$

i.e. $$\int E dt = \frac{R}{G}\left(\frac{GK}{R} + A\right)\alpha,$$

or $$\int E dt = \left(\frac{AR}{G} + K\right)\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Thus it appears from (26) above that the value of the deflection, α, is determined by $\int E dt$, since the other terms are constant. For example, if the search coil is placed in a magnetic field, thereby setting up an induced E.M.F. in the circuit, the total deflection is independent of the speed of insertion or withdrawal of the search coil.

Now in general the value of A is small and the term $\frac{AR}{G}$ may be neglected in comparison with K. And hence we have :

$$\int E dt = K\alpha,$$

approximately.

With this approximation the deflection α is independent of the value of R, i.e. the search coil may be replaced by another, provided that R does not become very large for then the approximation is not justifiable.

For the purpose of this account the noteworthy feature is that the deflection is, as for the ballistic galvanometer, proportional to the quantity of electricity, $\int \frac{E}{R} \cdot dt$, which passes through the coils, i.e. α is proportional to the quantity.

The standard form of fluxmeter has a scale, graduated in maxwells ; this is graduated experimentally. The deflections corresponding to known flux change define points on the scale, and since α is proportional to $\int E dt$, which itself is proportional to B, the number of lines cut, the subdivision of the scale between such fixed points is a matter of dividing angles into equal parts.

The experimental arrangements for the instrument in measuring magnetic flux, quantity of electricity, are described in the chapters where such measurements find a place.

For many of the experiments for which the instrument is used the graduated scale is too coarse. Use is then made of the mirror, using a lamp and scale arrangement as in the ordinary galvanometer. The deflection on the scale is proportional to the quantity of electricity discharged through the instrument. A full scale deflection is produced in the instrument by a definite flux change in a search coil; the position of the reflected spot of light is noted, as is the movement of the pointer on the graduated scale. By repetition, the corresponding values are obtained. From such observation the value of the flux change, corresponding to 1 *mm.* scale deflection (at, say, 1 metre), is deduced from the graduated scale deflection.

MEASUREMENT OF VARYING CURRENT

The preceding account has included the method of measuring (1) steady direct current, (2) quantity of electricity.

We now consider briefly methods available for the measurement of varying currents. Such currents may be subdivided into (*a*) currents of short duration at regular or irregular intervals, (*b*) alternating currents.

(*a*) To detect currents of this kind, and also to measure the time interval between such short duration currents use may be made of

The Einthoven Galvanometer

The principle of this instrument may be understood from fig. 301. A 'string,' CC, of fine platinum or tungsten wire is supported vertically between the poles of an electromagnet. If a current be sent down the string, the latter will be deflected in a direction parallel to the face of the pole pieces. In the diagram, the direction of movement is shown by the arrow, *a*, for a field in the direction, SN.

To observe the movement, the pole pieces of the magnet are drilled, as shown by the broken lines, in a direction parallel to that of the magnetic field. Light from a bright point source is concentrated by a condenser, CF, placed in one hole, and the movement of the illuminated string is magnified by a telescope, DE, placed in the other.

The 'string' is usually attached to the ends of two small springs which keep it stretched. The tension may be altered to any desired amount by a micrometer adjustment at one end. The string and tension-varying device are supported on a frame which may be removed bodily from the gap between the pole pieces.

The deflection produced for a given current depends upon the tension on the string and the strength of the magnetic field. The former may be varied as indicated above, whilst the latter may be adjusted by regulation of the current flowing through

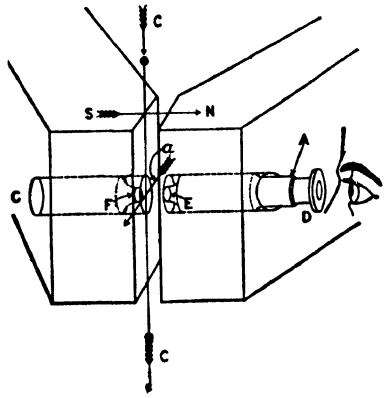


FIG. 301

the coils of the electromagnet. The sensitivity of the instrument may be therefore varied over a wide range quite simply. At a fixed sensitivity the deflection is proportional to the current.

The usual method of detecting deflections is to use a photographic arrangement. The beam of light emerging from ED if allowed to fall on a screen forms a shadow image of the string. This is reflected on to a cylindrical lens which forms an image of the central portion of the line, i.e. if the string is still and a photographic plate is moved vertically at the focus of the lens a straight white line is produced on the plate when developed. If a current passes, the string moves, and a shift of the shadow image results. The point focus of the cylindrical lens is thereby deflected. This results in a lateral displacement of the line image on the photographic plate. The magnitude of the displacement gives a measure of the current strength. The natural period of the string is small and it rapidly returns to the rest position when the current ceases to flow (not more than a few hundredths of a second is required). Thus if a succession of small currents pass in the circuit, the instrument detects them, even when but a few hundredths of a second interval occurs between successive currents, whereas an ordinary galvanometer would not distinguish the break between them.

To measure the interval of time between successive impulses, or the duration of one of them, a time scale is imprinted on the record of the current by means of a 'time marker' as shown in fig. 302. This consists of a device for intercepting the light at

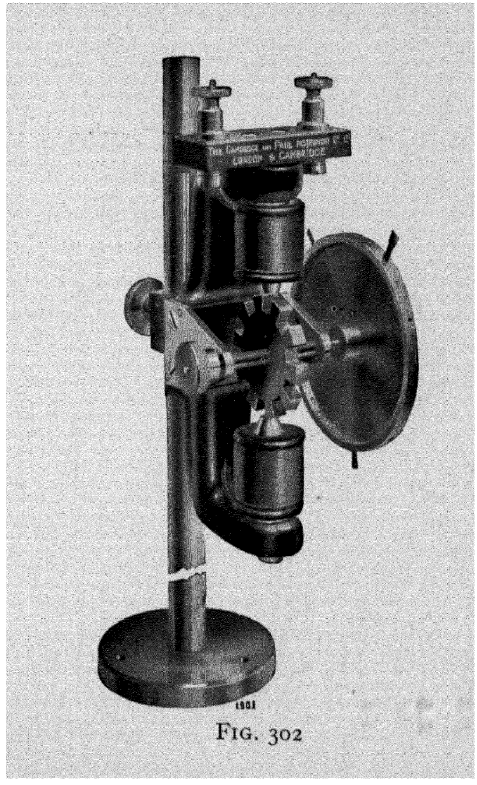


FIG. 302

regular intervals and thereby making transverse white lines across the photographic plate on which the current is recorded.

A small motor is made, as shown in the figure with a soft iron armature of, say, 10 teeth. Intermittent current is supplied to the electromagnets by connecting them in series with the circuit of an electrically maintained tuning fork, the impulses given to the motor are therefore regular. Suppose the fork vibrates 50 times per second, the synchronous motor will rotate 5 times per second. This drives a circular disc provided with projecting arms as shown in the figure. These arms are usually allowed to move across the beam of light illuminating the apparatus. The figure shows five such spokes, the fifth being broader than the others. A spoke intercepts the light 25 times per second and makes a time scale on the photographic plate of $\frac{1}{25}$ second.

By making 20 spokes to the wheel $\frac{1}{100}$ second graduations may be obtained. The width of a division may be varied by varying the speed of movement of the plate. For a continuous record a cinematograph film may be used.

Of course a number of strings may be used, and the image focussed simultaneously on the same film by placing a 45° right-angled prism in the path of each image. The whole is therefore concentrated on to the slit in the camera box. The arrangement of the parts is seen in fig. 303 which shows a plan of the apparatus.

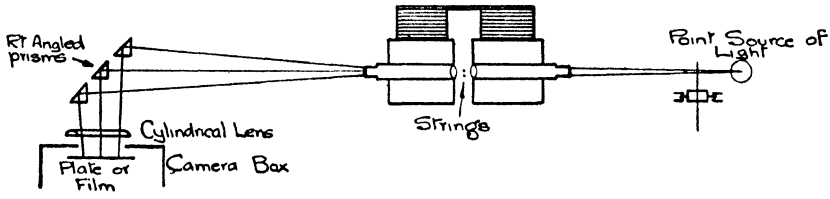


FIG. 303

(b) Alternating Current

Two types of instrument may be used for the measurement of alternating current:

(1) The oscillograph which gives the wave form of the current.

(2) Ammeters or dynamometers which give the effective current.

The Duddell Oscillograph

Fig. 304 shows the essential features of this form of instrument. A thin phosphor-bronze strip, *ss*, is supported over a small ivory bobbin, P, and fastened at the lower ends. The tension on the strip may be adjusted by regulation of the tension on the spring suspension of P.

If a current is passed through the loop, *ss*, the two strips will suffer a deflection in opposite direction, and consequently rotate a mirror, M, which is attached to both. If the current direction is reversed, the direction of rotation is also reversed.

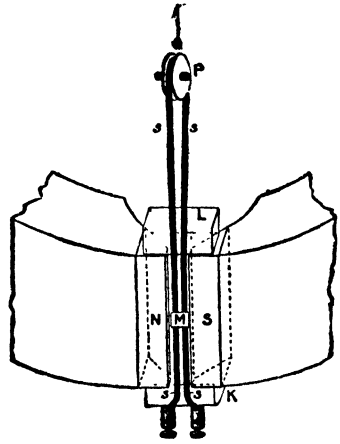


FIG. 304

Thus, for alternating current the mirror would rotate backwards and forwards, provided that the natural period of the loop is small compared with that of the alternating supply.

The image of a source of light reflected by the mirror on to a scale or screen by M would therefore be drawn out into a line for such an alternating current.

If the image were focussed on to a photographic plate in a camera, and the plate were moved at constant speed in a direction normal to the beam of light and the direction of vibration of the image, the trace on the plate would be approximately a sine curve. The form of the current time curve may then be investigated from the record.

To see the wave form on a screen the photographic arrangement could be dispensed with, and a mirror made to rotate and reflect the first beam on to a screen. The linear patch of light is again converted to wave form by thus adding a constant velocity normal to that produced by M.

The amplitude of the curve gives an indication of the maximum current strength, which is approximately proportional to it.

This instrument can also be employed for many of the purposes to which the Einthoven galvanometer may be used.

For a description of the other forms of alternating current measurers, attracted iron ammeters and dynamometers, the student is referred to any good text-book of electrical engineering; e.g., T. F. Wall: "Electrical Engineering." Methuen.

CHAPTER XIX

RESISTANCE MEASUREMENTS

The Wheatstone Bridge

When four resistances, P, Q, R, S (fig. 305), are arranged in the net shown, and adjusted so that no current passes through the galvanometer, G, i.e. so that the potential at B and D is the same, the relation between the resistance is expressed by:

$$\frac{P}{Q} = \frac{R}{S}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

as is shown in any good elementary textbook of physics.

The student will have applied this method, using the *Post Office Box* and the *Metre Bridge*, for the measurement of the resistance of wires, coils, etc.

Measurement of the Resistance of a Galvanometer

This may be done in several ways, of which one or two are given below.

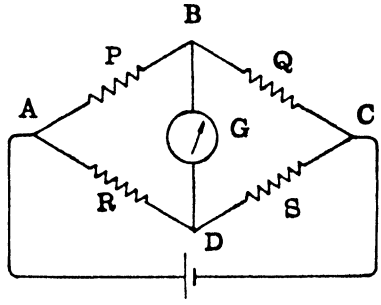


FIG. 305

Kelvin's Method

In this method the galvanometer acts as its own detector of balance in a Wheatstone net. The galvanometer is placed in the arm, DC (fig. 305), and the galvanometer of that figure is replaced by a single-way key, so that B and D may be connected together when the key is depressed.

When the battery circuit is completed, a steady current flows through the galvanometer. P, Q and R are adjusted until on joining

B to D through the key, no change is produced in that deflection, when $\frac{P}{Q} = \frac{R}{G}$, where G is the resistance of the galvanometer.

The usual difficulty with a sensitive galvanometer is that the steady current is too large. The galvanometer may not be shunted in this experiment, but the E.M.F. applied may be reduced, e.g. the cell may be connected through a high resistance, and leads from the end of a small fraction of the resistance may be taken to A and C instead of the battery directly applied. This, however, decreases the sensitivity of the method.

For a moving-magnet instrument it is better to apply the cell directly, and reduce the steady deflection to zero by adjustment of the control magnet, or, if that is not sufficiently strong, by the adjustment of an external bar magnet. The sensitivity is thereby retained.

For a moving-coil instrument no such adjustment is available, and the simplest course to follow is to clamp the coil of the instrument and find its resistance using another galvanometer as detector in the usual way.

The Carey Foster Bridge

The Carey Foster Bridge, shown in fig. 306, is a modification of the metre bridge. It is provided, as seen, with four gaps,

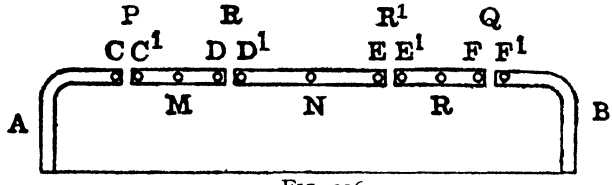


FIG. 306

CC^1, DD^1, EE^1 and FF^1, which may be closed by the insertion of resistances.

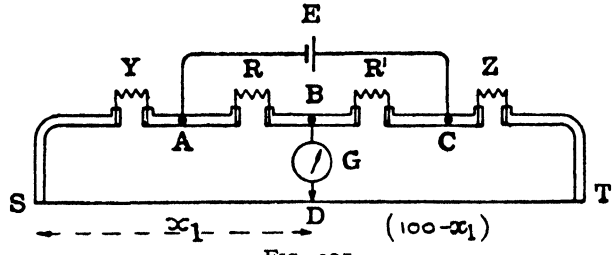


FIG. 307

Suppose the gaps be closed with resistance Y in CC^1, R in DD^1, R^1 in EE^1, Z in FF^1, as shown in fig. 307. The battery, E, and galvanometer, G, are arranged at points, A C and B D, which correspond to the same points in the theoretical net diagram, fig. 305.

If the point D is chosen such that no current passes through the galvanometer, we have, from (1):

$$\frac{R}{R^1} = \frac{Y + r_1 + x_1\rho}{Z + r_2 + (100 - x_1)\rho}, \quad \ldots\ldots\ldots(2)$$

where

ρ is the resistance per *cm.* of bridge wire,
x_1 the length, SD,
DT $= (100 - x_1)$ if ST is one metre,
r_1 is the value of the resistance at the soldered junction, S,
r_2 the resistance at T.

If the simple metre bridge were used to compare R and R^1, i.e. $Y = Z = 0$, we should have a balance at, say, l_1 *cms.*, such that, neglecting r_1 and r_2 for the moment,

$$\frac{R}{R^1} = \frac{l_1\rho}{(100 - l_1)\rho}.$$

Now suppose $Y = y\rho$ and $Z = z\rho$, equation (2) becomes:

$$\frac{R}{R^1} = \frac{(y + x_1)\rho}{\{(z + (100 - x_1)\}\rho},$$

if r_1 and r_2 be neglected.

It is obvious from a comparison of the two results that the Carey Foster bridge functions as though the length of the wire were increased, i.e. the same error in obtaining a balance point corresponds therefore to a less percentage error in the Carey Foster bridge determination.

Comparison of the British Association Ohm and the Legal Ohm

The comparison of two resistances, very nearly equal, serves to show a common use for this bridge. In the following method it will be seen that the end resistance is eliminated.

The resistances R and R^1 are made approximately equal to the values of Y and Z. Connecting as in the fig. 307 with, say, Z, the B.A. ohm, and Y, the legal ohm, whilst R and R^1 are each, say, 1 legal ohm, we should obtain a balance at a point, D, x_1 cms. from S, so that:

$$\frac{R}{R^1} = \frac{Y + r_1 + x_1\rho}{Z + r_2 + (100 - x_1)\rho}, \quad \ldots\ldots\ldots(3)$$

If now Y and Z are interchanged, Z being connected in the gap occupied in fig. 307 by Y, and Y replacing Z, a balance for such an arrangement could be obtained at a distance x_2 from S, from which we have:

$$\frac{R}{R^1} = \frac{Z + r_1 + x_2\rho}{Y + r_2 + (100 - x_2)\rho} \quad \ldots\ldots\ldots(4)$$

Equation (3) may be rewritten :

$$\frac{R}{R+R^1} = \frac{Y + r_1 + x_1\rho}{Y + Z + r_1 + r_2 + 100\rho},$$

and (4) similarly becomes :

$$\frac{R}{R+R^1} = \frac{Z + r_1 + x_2\rho}{Y + Z + r_1 + r_2 + 100\rho}.$$

Equating numerators of these equations we have :

$$Y + r_1 + x_1\rho = Z + r_1 + x_2\rho,$$
$$Y - Z = (x_2 - x_1)\rho, \quad \ldots\ldots\ldots\ldots\ldots (5)$$

i.e. the difference between Y and Z is equal to the resistance of the bridge wire between the two points of balance. It is independent of the value of r_1 and r_2, and of the total length of the bridge wire.

To obtain the value of the B.A. ohm in terms of the legal ohm, we have, Y being the legal ohm :

$$\text{B.A. ohm} = \{1 - (x_2 - x_1)\rho\} \text{ legal ohm.}$$

The value of the resistance of the bridge wire between the points of balance may be obtained by calibrating the bridge wire (see page 495), or, if the wire is uniform, the following simple method will serve.

Resistance of Unit Length of Bridge Wire (ρ)

The bridge connexions of the main experiment above remain as before. R and R^1 are approximately equal, and may very well be the same as above. Y and Z are replaced respectively by a fraction of an ohm, say, Y^1, and a stout strip of copper of negligible resistance, say $Z^1 = 0$.

Following the same procedure as before, a balance is obtained at a distance x_1^1 from S, when Y^1 and Z^1 are in the positions of Y and Z in fig. 307. When Y^1 and Z^1 are interchanged the balance will move to another point, x_2^1 from S^1.

Then by equation (5) :

$$Y^1 - Z^1 = (x_2^1 - x_1^1)\rho,$$

or since $Z^1 = 0$, and Y is known,

$$\rho = \frac{Y^1}{x_2^1 - x_1^1}.$$

Hence, using this value of ρ, the difference between Y and Z in the first case may be evaluated.

This method may obviously be applied to any similar case, and a comparison between two nearly equal resistances obtained.

Experimental Details.

The resistances and Y, Z, R_1 and R_2 are connected by means of stout copper strips. These will have practically zero resistance, and the small difference to be measured will be truly the difference in the resistance of the coils. It is also essential, of course, that all connexions be very tightly screwed, for the same reason.

In performing the second part of the experiment, i.e. to find ρ, two or three sets of observations should be made; the exact number will depend on the total resistance of the bridge wire.

For example, Y^1 should be made ·1, ·2, ·3, ·4 ohm successively, and ρ calculated in each case, from which a mean value is obtained. It may be found that, ·4 ohm acting as Y^1, no balance is obtainable. In that case the total resistance of the bridge wire is less than ·4 ohm.

The following set of observations shows the order of the result obtained in such an experiment.

Using Z = 1 B.A. ohm, Y = 1 legal ohm,

$x_2 = 51{\cdot}2$ cms., $x_1 = 48{\cdot}2$ cms,

The wire was not calibrated.

EXPERIMENTS TO DETERMINE ρ

Y^1 in ohms.	·1	·2	·3	·4
x_1^1 in cms.	36·35	23·76	11·2	no balance
x_2^1 in cms.	63·75	76·34	88·9	
ρ ohms per cm. $= \frac{Y^1}{x_2^1 - x_1^1}$	·00365	·0038	·00386	

Mean value of ρ = ·00377.

Hence since

$$Y - Z = (x_2 - x_1)\rho,$$

Legal ohm − B.A. ohm = (51·2 − 48·2) (·00377),

B.A. ohm = 1 − 3 × ·00377 = ·989 legal ohm.

Another equally simple method for finding the resistance per cm. of bridge wire may be used. Suppose Y, Z, R and R^1 (fig. 307) are all 1 ohm coils (of the same kind, e.g. B.A. ohms). R and R^1 are set up as before. To introduce a small difference between Y and Z, one of them is shunted with, say, 10 ohms, i.e. the net result is $\frac{10}{11}$ ohm. If now the process described above is carried out, interchanging the 1 ohm and effective $\frac{10}{11}$ ohm, we have two balance positions, l_1 and l_2 cms., say, and

$$Y - Z = (l_1 - l_2)\rho,$$

i.e.

$$\frac{1}{11} = (l_1 - l_2)\rho.$$

Hence ρ is determined.

To Construct a Resistance Coil of Known Magnitude

To construct, for example, a 1·ohm coil, a wire with small temperature coefficient between 10° and 20° C. is selected, say manganin, and the resistance per *cm.* of the specimen available is obtained by finding the resistance of about 100 *cms.*

The length of wire required to have a resistance not less than 1·1 ohms is calculated, and cut off. The insulation covering is removed from the ends and, using a non-corrosive flux (say, resin or 'fluxite'), the two ends are soldered to two stout copper wires which are soldered to flat copper forks, A and B (fig. 308).

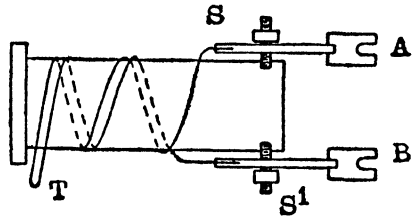


FIG. 308

A and B and the rods are then fastened to opposite sides of a wooden bobbin by means of terminals as shown, and the wire wrapped, as in the diagram, in a non-inductive manner round the bobbin. The middle of the wire T is freed from the silk cover. The resistance between A and B is obtained by the Carey Foster method. The middle, T, is twisted with pliers, cutting out the end loop until, compared with a standard 1 ohm coil by the method of page 491, the balance is in the centre of the bridge. T is then soldered in position, and the value of this copy of the standard ohm when completed is compared with the true standard as already described.

CALIBRATION OF A BRIDGE WIRE

The simple metre bridge, briefly referred to on page 489, also the Carey Foster Bridge, the potentiometer and similar instruments, depend upon measurement to a point of balance on a stretched wire. For simplicity it is often assumed that the wire is of uniform cross-section, and that its resistance per cm. is consequently constant throughout the length. Further, the soldered end and thick copper connecting strip is assumed to be of zero resistance.

In a practical measurement it is more satisfactory to determine the variation due to these causes by a preliminary calibration of the bridge.

To Determine the End Correction of the Bridge Wire

For this determination the outer gaps of the bridge (CC^1, FF^1 in fig. 306) are closed by short clean thick copper strips of negligible resistance, and in the inner gaps are two unequal resistances, say a 10 and a 1 ohm coil (R and R^1).

A balance is obtained at, say, x_1 *cms.* from A. The 10 ohm and 1 ohm coils are interchanged, and a second balance is obtained at x_2 *cms.* Now, if the 'end resistance' at A is equivalent to l_1 *cms.* of the bridge wire, and if the end resistance at B is equivalent to l_2 *cms.* of bridge wire, we have :

$$\frac{R}{R^1} = \frac{x_1 + l_1}{100 - x_1 + l_2},$$

and

$$\frac{R^1}{R} = \frac{x_2 + l_1}{100 - x_2 + l_2},$$

from which l_1 and l_2 may be calculated.

Calibration of the Wire

This may be done in many ways, of which we will consider two.

The object of these experiments is to find, at different points along the wire, lengths having the same resistance, usually equal to that of a gauge employed. Knowing the total resistance, S, of the wire, the mean value of the resistance of such a length is readily calculated, and hence the correction to be applied at each segment taken to reduce to the mean value may be determined. Alternatively, having obtained the lengths which have the fixed resistance, the method of analysis used in the calibration of a tube may be applied (page 43). Suppose we wish to test every 5 *cms.* of the wire, and this will be quite sufficient for most cases, the following methods may be used.

(1) Carey Foster's Method

We saw on page 492, that if two resistances, R and R^1, are fixed in the inner gaps of the Carey Foster Bridge, and resistances, Y and Z, are balanced, and then interchanged and once more balanced, that :

$$Y - Z = (x_2 - x_1)\rho,$$

where $x_2 - x_1$ is the difference in the balance points, and ρ the mean resistance per cm. between these balance points, i.e. the difference between the resistance of Y and Z is equal to the resistance of the wire included between the balance points, the end resistances being eliminated by this method.

If now we make Y − Z equal to the approximate resistance of 5 cms. of the bridge wire, and so arrange R and R^1 that the

values of x_2 and x_1 are made in turn to pass along the whole wire, the calibration is completed. This is done in the following manner.

The value of S, the total resistance of the bridge wire, is found either by using another measuring device or the same bridge.*

Knowing S, the mean value of 5 cms. of the bridge wire may be found. A length of wire, preferably of the same material as the bridge, is then taken; its resistance is measured and a length cut off 2 cms. in excess of that required to be of the same resistance as 5 cms. of the bridge wire.

This is then soldered to two stout copper lugs as shown in fig. 309. The two cms. excess being soldered to the lugs. This gauge is used as resistance Y. Z is composed of a thick copper connecting strip of practically zero resistance (fig. 310).

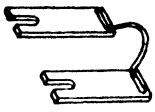


FIG. 309

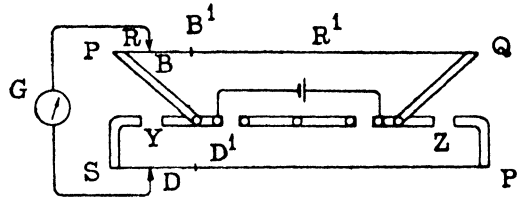


FIG. 310

Carrying out the process of balancing Y and Z, interchanging, etc., we have, if x_1 and x_2 are the two values of SD, as before,

$$Y - Z = \text{resistance of wire between } x_2 \text{ and } x_1.$$

By altering R and R^1 the part of the wire compared with $Y - Z$ may be varied. This process is best done by using another bridge wire as $(R + R^1)$ seen in fig. 310. PQ is the second bridge wire connected with thick copper strips to the bridge as shown. The galvanometer, G, is joined to B and D by means of variable contacts: the gauge is placed in the left-hand side and the zero resistance in the right, the point D is chosen near the end of the wire and B adjusted until the galvanometer when closed in the circuit shows no deflection. Y and Z are interchanged and, leaving B fixed (i.e. R and R^1 are fixed), D is adjusted to some position, D', where a balance is again obtained. The resistance of the length, DD', is then equal to the resistance of the gauge. Keeping the contact at D^1 fixed,

* The value of S may be found using the same bridge. Referring to fig. 307, the gap, Z, is closed with a stout copper strip of negligible resistance. The galvanometer is permanently corrected between points S ($x=0$) and B. R, R^1 and Y are adjusted for balance, the whole bridge wire making the fourth arm in a Wheatstone net. When balanced we have $S = \frac{R^1}{R} \cdot Y$. This gives S without allowing for the end corrections.

the gauge and the strip are returned to their original positions and B is adjusted to some position, B^1, where balance is obtained (i.e. a new ratio $R : R^1$ is obtained). Y and Z are interchanged, and the contact D′ is moved until again a balance is obtained, i.e. it is moved a further distance having a resistance equal to that of the gauge.

This method is carried out until the whole wire, SP, is covered in approximate 5 cm. steps. The length of wire $(x_2 - x_1)$ for the different steps are noted for each part of the scale, and tabulated as under.

REGION OF WIRE ON METRE SCALE	LENGTH OF WIRE OF SAME RESISTANCE AS GAUGE	DIFFERENCE FROM MEAN	CORRECTIONS TO REDUCE TO LENGTH OF MEAN ρ
0– 5	4·8	+·15	+0·15
5–10	5·1	−·15	0·00
10–15	5·2	−·25	−0·25
15–20	4·9	+·05	−0·20
20–25	5·0	−·05	−0·25
etc.	etc.	etc.	etc.
90–95			− ·05
95–100			0·0

Mean length = 4·95, say.

The correction is obtained by finding the mean of column (2), and finding what must be added or subtracted from the first value in column (2) to give the mean value.

For the 5–10 cm. range the correction is the algebraic sum of the first 0–5 cm. range, and the 5–10 cm., and so on.

A correction curve is drawn which will convert any length of the wire to the equivalent length of a uniform wire of the same length and resistance.

Thus, if in use in an experiment the bridge balances at 65 cms., and from the correction curve the correction, −0·2 say, is obtained, if the wire were uniform and of same length and resistance, 64·8 cms. would be the balance point, and the resistance ratios of the two segments of the wire is 64·8 : 35·2.

It will be seen that in this method the interchanges which are frequently made must be assumed to cause no variation in resistance. The gauge and the straight copper connecting strip are thoroughly cleaned and always screwed tightly in the gaps to avoid, as far as possible, such changes.

This method may be used to calibrate wire, PQ, since the contact, B, is moved a distance corresponding to equal increments

of resistance. It would therefore serve well for the calibration of a potentiometer wire.

Direct Calibration by a Potentiometer Method

The wire, ST, of the bridge is connected, as shown in fig. 311, in series with an adjustable resistance, R_2, and a 2 volt accumulator which is in good condition and well charged. A similar accumulator is connected in series with a second adjustable resistance, R_1, and a length of, say, 20 cms. of stretched wire of the same material and approximate cross-section as the bridge wire. G is a high-resistance galvanometer.

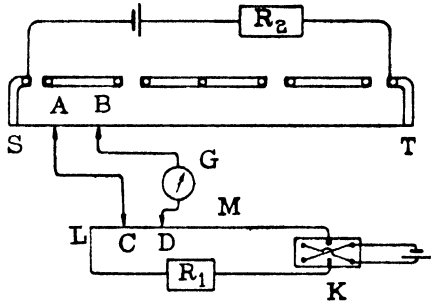


FIG. 311

C is soldered to the wire, M, and with A and B in contact with ST, say, between the 0 and 6 cm. readings, and about 5 cms. apart, D is adjusted until no deflection is given in the galvanometer. D is then soldered to M.

It should be noted that convenient resistances for R_1 and R_2 are of the order 20 to 50 ohms. The adjustment of these resistances is made until the currents are about equal.

Leaving B in contact with ST, reverse the current in M by means of the commutator, K, and move A to the other side of B; adjust till again no deflection is produced. Repeat this process over the length of ST.

If the latter is uniform, then of course the length AB will be the same. Variations of the cross-section will be made apparent by the different values of AB.

Tabulate the length of AB as before, and deduce the correction curve for the wire.

This method may also be used for any stretched wire, e.g. potentiometer, etc.

DETERMINATIONS OF THE VALUE OF A HIGH OR LOW RESISTANCE

Using the ordinary Post Office Box method of finding resistance is a very ready method, but for resistances of a million (10^6) ohms or more it is unreliable. The ordinary Post Office Box enables a magnification of 1000 to 10 only to be obtained, and in the adjustable arm 10000 ohms is the maximum resistance available, so, for a resistance greater than 10^6 ohms a special method is required.

Equal limitations for very small resistances make it necessary to employ special methods in this case.

Some of these special methods are described under.

Low Resistance

(1) The Direct Deflection Method

To find the resistance, r, of a low resistance wire, AB, the following simple method gives a fair approximation. AB is joined in series with a known low resistance, R, and an accumulator (2 volt), and an adjustable, fairly large resistance, S.

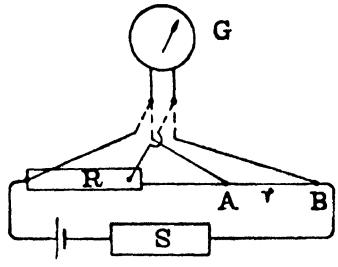


FIG. 312

Having by means of S adjusted a small current, c, through the circuit, there will be a drop of potential, cr, across AB, and cR between the ends of R. If a *high-resistance* galvanometer is connected to the ends of AB the current will be proportional to the potential, cr. The value of c will be least disturbed using such a galvanometer. If this causes a small deflection, d_1 cms., using the usual lamp and scale method, $cr \propto \theta$, approximately, and $\theta_1 \propto d_1$ approximately.

The same galvanometer will have deflection d_2 cms. (angle θ_2) when connected to the ends of R. Again $c\mathrm{R} \propto \theta_2$ and $\theta_2 \propto d_2$ approximately

i.e. $$\frac{\theta_1}{\theta_2} = \frac{d_1}{d_2} = \frac{cr}{c\mathrm{R}} = \frac{r}{\mathrm{R}},$$

Hence $$r = \frac{d_1}{d_2} \cdot \mathrm{R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

This method of observing deflections may be used *to find the specific resistance of copper.*

A length of copper wire is soldered to two terminals about 1 metre apart on a wooden base, and is connected in series with ·1 or ·01 ohm, and a constant source of potential such as a steady 2-volt accumulator, and a resistance box of 0 to 100 ohms. (It is not advisable to have less than 20 to 30 ohms in the circuit.)

The ends of the wire AB (r) and of the $\frac{1}{10}$ ohm (R) are connected to a double pole two-way switch (indicated by the broken lines in fig. 312), which is connected to a high-resistance galvanometer, G, the deflections of which may be observed by the usual lamp and scale method.

The deflection given when the potential difference between the ends of r and R are applied is measured by taking the reading of the deflected spot of light in each case. The battery is then reversed and the reading is obtained on the other side; half the difference in readings giving d_1 and d_2.

S is adjusted and the experiment repeated for 2 or 3 values of the current, and the mean value of r is obtained.

Hence, putting $R = \frac{1}{10}$ or $\frac{1}{100}$ in (6), r is evaluated.

If the length of the copper wire is l cms., and the radius is a cms., the specific resistance, s, is

$$s = \frac{R\pi a^2}{l};$$

a is the mean of several values along the wire, measured by the micrometer screw.

The following typical example of this experiment shows the order of the result obtainable.

S = 50 ohms, $d_1 = \dfrac{1{\cdot}85 + 2}{2} = 1{\cdot}925$ cms.,

R = $\frac{1}{10}$ ohms, $d_2 = \dfrac{13{\cdot}7 + 14{\cdot}3}{2} = 14{\cdot}0$ cms.,

$$r = \frac{1{\cdot}925 \times {\cdot}1}{14{\cdot}0} = {\cdot}0135 \text{ ohm};$$

S = 30 ohms, $d_1 = 3{\cdot}1$ cms.,
R = ·1 ohm, $d_2 = 23$ cms.,
$$r = \frac{23 \times {\cdot}1}{31} = {\cdot}0136 \text{ ohm};$$

S = 40 ohms, $d_1 = 2{\cdot}3$ cms.,
R = ·1 ohm, $d_2 = 16{\cdot}75$ cms.,
$$r = {\cdot}0137 \text{ ohm};$$

mean value of r = ·0136 ohm;

$l = 90$ cms.; $a = {\cdot}060$ cm.;

Hence
$$s = \frac{{\cdot}0136 \times \pi \times {\cdot}0036}{90},$$
$$= 1{\cdot}71 \times 10^{-6}.$$

The above method also enables an approximate comparison of the specific resistances of two materials in the form of wires. The wires are connected in series, and their resistances, r_1 and r_2, compared as above, i.e.

$$\frac{r_1}{r_2} = \frac{\theta_1}{\theta_2} = \frac{d_1}{d_2},$$

when θ_1 and θ_2, the angular deflections, are small—if this condition does not hold θ_1 and θ_2 may be calculated if D, the distance of the scale from the mirror of the galvanometer, is known (for tan $\theta_1 = d_1/D$, etc.).

If l_1 and l_2, a_1 and a_2 are the lengths and radii of the wires, and s_1 and s_2 their specific resistances,

$$\frac{r_1}{r^2} = \frac{s_1 l_1}{\pi a_1^2} \cdot \frac{\pi a_2^2}{s_2 l_2} = \frac{d_1}{d_2},$$

or

$$\frac{s_1}{s_2} = \frac{d_1 l_2 a_1^2}{d_2 l_1 a_2^2}.$$

(2) Kelvin Bridge

To obtain a more accurate comparison of two small resistances than may be obtained by the foregoing experiment, the Kelvin Bridge method is used. Fig. 313 shows the general arrangement of this network.

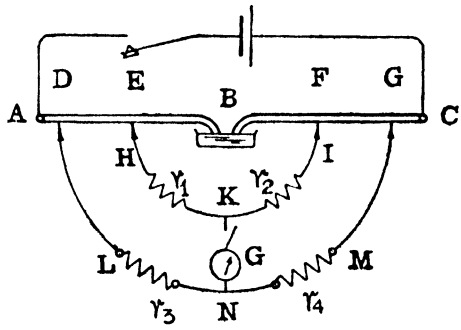


FIG. 313

r_1 and r_2 are equal resistances, as are r_3 and r_4. These are arranged as shown, so that the total resistance, D to N = NG, and resistances in the arm EK = resistance in KF.

The two resistances to be compared, if in the form of wires, are arranged as at AB and BC ; B being a mercury cup or other low-resistance junction.

The two resistances, r_1 and r_3, are connected to AB at points D and E, and the second low resistance is connected to r_2 and r_4 at F and G, the latter being an adjustable point..

The two pairs of equal resistances (about 1 or 2 ohms each) should be made of wire of the same material, or having similar temperature coefficient, and should be placed in close proximity to ensure a common temperature throughout the experiment. Further, when the current from the battery is sent through the circuits, its duration should be short to eliminate excessive heating in these resistances, and in the specimen AB and in BC; the latter, presumably, will have different coefficients, and the comparison might therefore be upset if such temperature variation are introduced.

The galvanometer between K and N should be of low resistance.

The points E and F *should be near* B and the point of contact, G, is adjusted until on closing the battery and galvanometer key no deflection is produced in the latter; then if the resistance DE $= r$, FG $= R$, we have

$$\frac{r}{R} = \frac{r_1}{r_2} = \frac{r_3}{r_4}.$$

Thus it is apparent that r_1, r_2, r_3, and r_4 are not of necessity equal, but the relation

$$\frac{r_1}{r_2} = \frac{r_3}{r_4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

must be satisfied. This condition is satisfied in the construction of the bridge which, like the Post Office Box, may be used with equal ratio arms or with factors 1 : 10, 1 : 100, 10 : 1, 100 : 1, in the usual form.

To establish the relation, $\frac{r}{R} = \frac{r_1}{r_2} = \frac{r_2}{r_3}$, let us put c = the current in DE (r), c_1 the current in EKF (when in the balanced condition *no current passes through the galvanometer*), c_2 the current through DNG.

Then, since the current $c - c_1$ passes along EBF, the current in FG (R) is c.

The above is the balanced condition of the net of conductors, i.e. when the potential at K is the same as the potential at N, i.e. when

$$\frac{cr + c_1r_1}{c_1r_2 + cR} = \frac{c_2r_3}{c_2r_4} = \frac{r_3}{r_4},$$

i.e. $$r_4(cr + c_1r_1) = r_3(c_1r_2 + cR),$$

or $$c(rr_4 - r_3R) = c_1(r_2r_3 - r_1r_4).$$

But the condition satisfied by the bridge, by construction, is that

$$\frac{r_1}{r_2} = \frac{r_3}{r_4} \text{ or } r_1r_4 = r_2r_3,$$

i.e. $$c(rr_4 - r_3R) = 0;$$

and since c is not equal to o :

$$rr_4 = r_3R,$$

or $$\frac{r}{R} = \frac{r_3}{r_4} = \frac{r_1}{r_2} \text{ from (7)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

With such an arrangement of resistances as already described the comparison of two nearly equal low resistances such as two wires, can be made ; then if a, l, and s, represent the radius, length and specific resistance of r, and similar terms apply to R, we have :

$$\frac{l_1 s_1 a_2^2}{l_2 s_2 a_1^2} = \frac{r_1}{r_2}, \text{whence } \frac{s_1}{s_2} \text{ may be calculated.}$$

To obtain the specific resistance of a metal in the form of a thick wire or rod, we may carry out the above process using a copper wire as a second low resistance, and assume s for copper (s for drawn copper at 18° C. $= 1{\cdot}78 \times 10^{-6}$).

Alternatively the copper rod may be replaced by ·01 ohm (r), and the unknown rod placed in BC (R). The length, FG, to balance may be found as above ; whence s for the material :

$$\frac{\cdot 01}{sl/a^2} = \frac{r_1}{r_2},$$

$$s = \cdot 01 \left(\frac{r_2}{r_1} \cdot \frac{a^2}{l}\right).$$

When a definite length of wire, or fixed small resistance, is to be measured it becomes advisable to have a variable value for r. This may be a variable length of copper wire of known cross-section for very small resistances (R), or a series of ·01 ohm coils, and a calibrated wire in series with the coils for larger resistances. This latter method is usually used in the ready-made Kelvin Bridge now on the market.

Of course it must be realised that when the values of R, the unknown resistance, is very small, the current passing through it must be correspondingly large if any appreciable potential is to be attained between the ends, and if the measurement is to be at all satisfactory, the reason for a sensitive *low*-resistance galvanometer will also be apparent.

As examples of the types of Kelvin Bridge on the market, consider figs. 314 and 315.

For measuring the specific resistance of a wire of low resistance, the special clamping contact arrangement shown at the back of fig. 314 is used. For other forms of low resistance, the terminals X of that figure and fig. 315 are used directly.

Such an apparatus has a wider range and is more adjustable than the simpler form already described but is, in principle, identical with the bridge illustrated in fig. 313. To emphasize this, fig. 315 has been drawn showing the internal arrangements of fig. 314, lettered to agree with fig. 313.

The ' ratio arms ' may be adjusted as indicated to give various ratios, viz. ·01, ·1, 1, 10, 100. The coils which make up r_1, r_2, r_3, r_4, are accurately adjusted to ensure the condition of (7) above.

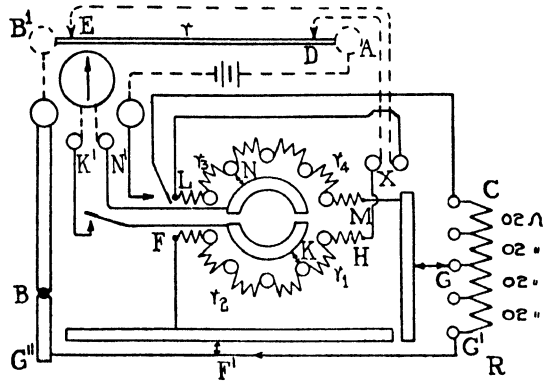


FIG. 315

As shown in the figure, r is the unknown resistance, and R may be one or more of the four coils each of ·02 ohms, together with a part, F'G', of the graduated and calibrated wire, G'G''. The length of the wire is about 450 mms. and each mm. has a resistance of ·00005 ohm, i.e. 40 cms. of the wire add ·02 ohms to the value taken from CG' (·04 in the diagram = GG'). The wire is graduated in fractionals of an ohm.

A comparison of figs. 315 and 313 will show the general arrangements.

For example, suppose an unknown resistance r were balanced when $\frac{r_1}{r_2}$ was made = ·01, by ·04 ohm from the coils, and 290 units of the slide wire, as shown approximately in fig. 315,

$$r = R\left(\frac{r_1}{r_2}\right) = (\cdot 04 + 290 \times \cdot 00005)\cdot 01$$
$$= (\cdot 04 + \cdot 0145)\cdot 01$$
$$= \cdot 00545 \text{ ohms.}$$

This instrument will measure resistances of 10 to ·00001 ohm.

Fig. 317 shows a second form of Kelvin Bridge made from the potentiometer illustrated in fig. 338, and in ' double ratio ' box (fig. 316). This enables a range of measurement from 1·5 ohms to ·00001 ohm, about.

The ' double ratio ' box contains resistances (fig 317), LN, NN', N'N'' and N''M, which are respectively $\frac{1}{2}$, $\frac{9}{22}$, $\frac{90}{1111}$ and $\frac{1}{101}$ of the total resistance of LM. A similar arrangement holds for

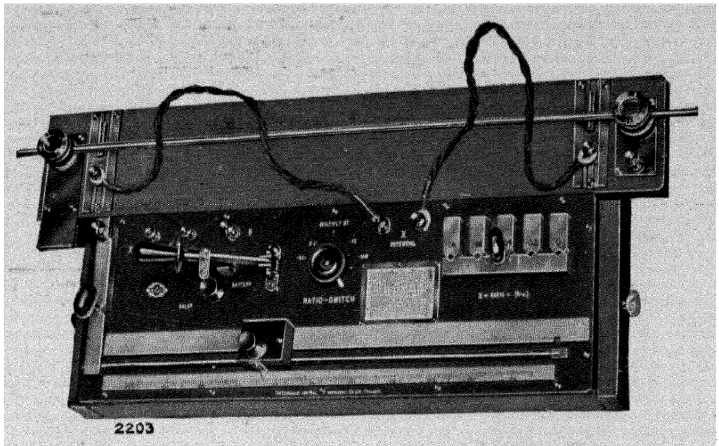


FIG. 314

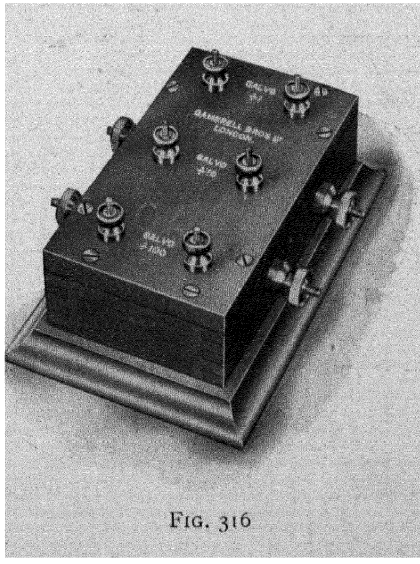


FIG. 316

the parts of HI, so that the resistances LN : NM = HK : KI (1 : 1) or LN' : N'M' = HK' : K'I (= 10 : 1), etc. The scheme of connexions is lettered to conform with the letters of fig. 313. A comparison with that figure shows the reason for this scheme : R, the unknown resistance, is equal to the resistance between E and D when the galvanometer is connected to NK as in the case taken, or is $\frac{1}{100}$ ED when the galvanometer is connected at N'K', etc.

The coils to which D is tapped are each ·1 ohm and the resistance of the slide wire is ·001 ohm per small division. Thus, using N″ K″ as galvanometer terminals, a resistance of ·00001 may be measured.

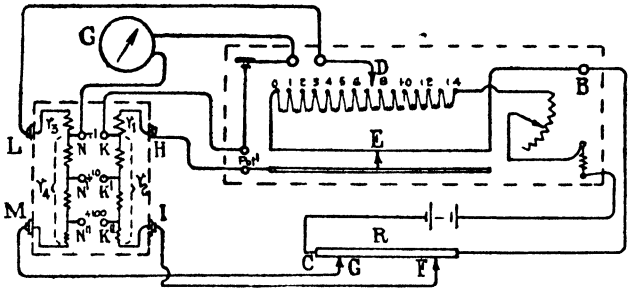


FIG. 317

High-resistance Measurement

As already stated, a special method is required to determine the resistance of greater value than 1 megohm (i.e. 10^6 ohms). One of the simplest ways of evaluating resistances of this order is the method of substitution. Fig. 291, page 463, shows the scheme of connexions.

A steady accumulator, E, is connected in series with the unknown resistance, R, and connected to a commutator, C, whence the current may be sent in either direction through a galvanometer, G, which is provided with a variable shunt, S.

Let the steady deflection due to the current be *d* cms. on a scale one metre away (corresponding to a movement of the suspended system of $\theta°$). A known adjustable resistance is substituted in place of R, and if on adjusting the known resistance a steady deflection *d* cms. is obtained again, the known resistance is of the same value as the unknown. If no variable known resistances of the order of the unknown resistance are available, the galvanometer is shunted so that one-thousandth part of the current is allowed to pass through it. When a large known resistance (e.g. 10000 to 20000 ohms from two P.O. boxes in series) is placed in series with it and the steady accumulator, a deflection of the same order as that given when the unknown resistance is placed in the circuit may be obtained.

Under these circumstances, suppose that d_1 is the deflection caused by the battery when in series with the unknown resistance, R ohms, and d_2 when in series with r, the known resistance.

Then if E is the E.M.F. of the accumulator, B the resistance of the battery, and G the resistance of the galvanometer (if the latter is a moving-coil galvanometer with a radial field), we have, where k is the galvanometer constant,

$$kd_1 = \frac{E}{B + G + R},$$

$$kd_2 = \frac{E}{\left\{B + \frac{SG}{S+G} + r\right\}} \frac{S}{S+G};$$

$$\therefore \frac{d_1}{d_2} = \frac{\{(r + B)(S + G) + SG\}}{\{(S + G)(B + G + R)\}} \frac{S+G}{S}$$

$$= \frac{(r + B)(S + G) + SG}{S(B + G + R)} = \frac{(r)\frac{S+G}{S} + G}{G + R}, \quad \ldots\ldots(9)$$

when, as is obviously the case, B may be neglected in comparison with r or R.

If further, G is negligible compared with r or R:

$$\frac{d_1}{d_2} = \frac{r\left(\frac{S+G}{S}\right)}{R},$$

i.e., if $\frac{1}{1000}$ part of the current goes through the shunted galvanometer, $\frac{S+G}{S} = 1000$,

i.e.
$$\frac{d_1}{d_2} = \frac{1000r}{R}.$$

The experiment may be carried out, using an adjustable known resistance, and shunting the galvanometer with a known shunt. The known resistance is adjusted until $d_2 = d_1$, i.e. *equal deflections* are obtained. Then, if G is small compared with R,

$$R = 1000r, \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

if the shunt is $\frac{1}{999}$.

Alternatively, if the known resistance is not sufficiently adjustable to cause equal deflections, the values of d_1 and d_2, both of the same order, are noted, then, neglecting G in comparison with R, we have:

$$R = \frac{d_2}{d_1}\frac{S+G}{S} \cdot r.$$

Again, if the shunt takes $\frac{999}{1000}$ of the current

$$R = \frac{d_2}{d_1} \cdot 1000r.$$

In both these cases, if G is not negligible compared with R, then the corresponding formulæ for the two cases is given by equation (9) above.

Determine by the above methods the value of the resistance of a leaky condenser. Another suitable high resistance to be measured by these methods is made by taking a sheet of ebonite, say 20 cms. by 5 cms. Two holes are drilled about 15 cms. apart, and the surface of the ebonite blackened, round the holes, with a black lead pencil. Two terminals are screwed down in the holes, and a fine black lead pencil line is ruled between them. The sheet is covered with a thin protecting second sheet of ebonite to prevent any accidental change in the dimensions of the line.

The apparatus should be set up as in fig. 291, and the values of R obtained.

For example, using a thin black lead pencil line,

$d_1 = 1{\cdot}95$ cms., $R = ?$ no shunt;
$d_2 = 9{\cdot}30$ cms., $r = 10000$ ohms, shunt $\frac{1}{999}$,

Hence :

$$R = \frac{9{\cdot}3}{1{\cdot}95} \times 10000 \times 1000 = 47{\cdot}7 \times 10^6 \text{ ohms.}$$

Using a thicker line on ebonite, i.e. smaller resistance, the first method (equation (10)) :

deflection 10 cms. when R was in circuit, no shunt.
deflection 10 cms. when 3640 ohms replaced R, shunt $\frac{1}{999}$.

Thus, $R = 3640 \times 1000 = 3{\cdot}64 \times 10^6$ ohms.

THE VARIATION OF RESISTANCE WITH TEMPERATURE

The value of a resistance of a specimen of wire, in most cases, increases with temperature.

The relation between R_0, the resistance at 0° C., and R_t, the resistance at another temperature, t° C., is :

$$R_t = R_0(1 + \alpha t + \beta t^2), \quad \ldots\ldots\ldots\ldots\ldots (11)$$

where α and β are constants.

The constant, β, is small, and therefore over small ranges of temperature the resistance is practically a linear function of the temperature, i.e.

$$R_t = R_0(1 + \alpha t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

expresses the relation, for small temperature ranges, between

resistance and temperature, (t should not exceed 100° C. for (12) to be valid.)

From (12) we have :

$$\alpha = \frac{R_t - R_0}{R_0 t},$$

and α may be called the '*coefficient of increase of resistance with temperature*' for the limited range taken.

To measure such resistance changes and determine the value of α, which is practically constant for all pure metals, we may make use of the Carey Foster method, since these resistance changes are small. This method is specially advantageous for such a determination of small differences, as has already been shown.

The theory of the method is almost identical with that already given, so that we may deal with the experiment as such.

Determination of the Resistance of a Wire at Temperature from 0° C. to 100° C.

A length of thin platinum wire of about 1 ohm resistance is mounted on a mica frame in the non-inductive manner shown in fig. 318. The ends of this wire are soldered to two thick copper wires whose resistance is negligible compared with that of the platinum. This arrangement is mounted rigidly in a glass tube, with mica or rubber supports as shown at M. The free ends of the copper wires are soldered to two long leads, and two identical leads, cut from the same wire, are joined (D) together. The four free ends of the leads are soldered to copper connecting strips, shown in fig. 322, P, P^1, C, and C^1. These ensure good contacts with terminals under which they are fixed.

The pair, P, P^1, are the ends of the leads from the platinum wire ; C, C^1 are the compensating leads.

The inner gaps in the Carey Foster bridge are occupied by two equal resistances of the same order as that of the platinum wire ; in the case taken these resistances, R_1 and R_2, are both made 1 ohm (fig. 319).

P and P^1 are connected in one of the outer gaps and the compensating leads in series with a resistance box, S, fill the fourth.

It must be remembered that R_1, R_2 and S should be connected to the bridge by means of copper strips, and every source of uncertain resistance, such as a bad or dirty connexion, must be eliminated.

All the resistances should be 'non-inductive,' for in this method it will be seen that the galvanometer is permanently connected in the circuit, and the battery takes the position in the sliding contact.

This is essential, for we must balance the resistance of the platinum at the temperature which is fixed by the surroundings, so that the current should not pass for any appreciable time and cause a heating in the spiral.

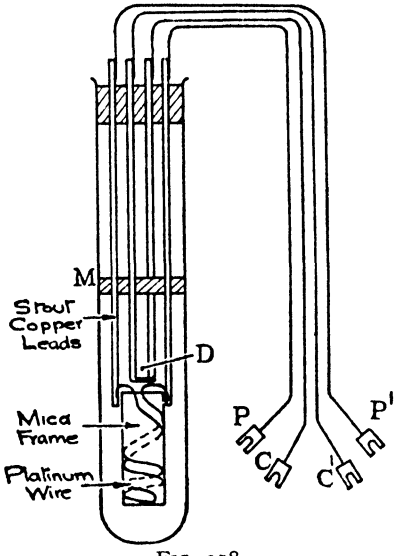


FIG. 318

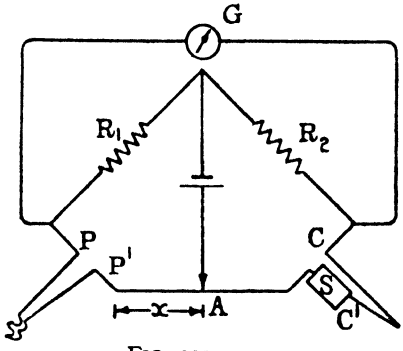


FIG. 319

S is a resistance box in which are all values from ·1 to 10 or more ohms.

The function of the compensating leads will be apparent. They are in opposite arms to the bridge to PP^1, and so, since the material and length of the leads are identical with those of the leads to the platinum, the value of the resistance of the last-named leads is eliminated ; and since the two pairs of leads are, in the main, side by side, variation in the resistance of this part of the circuit is also balanced.

In obtaining a balance at a point, A, x cms. from the end of the wire, the value of the resistance, S, is so adjusted that A is near the middle of the wire. Also the balance is obtained by having *no immediate deflection* of the galvanometer *when contact is made* with the bridge wire. If the current continues to flow, a change in the resistance of the platinum wire, due to the heating by the current, will cause a deflection in the galvanometer.

Suppose the glass tube containing the platinum wire specimen is immersed in a constant temperature bath and a balance is obtained at x cms. from the end of the bridge, we have, if R_t is the resistance of the platinum at this temperature $t°$,

$$\frac{R_1}{R_2} = \frac{R_t + r + x\rho}{r + S + (100 - x)\rho}.$$

where r is the resistance of either pair of leads, ρ is the resistance of 1 cm. of bridge wire.

Since $R_1 = R_2 = 1$ ohm,

$$R_t + r + x\rho = S + r + (100 - x)\rho,$$

$$R_t = S + (100 - 2x)\rho \quad \ldots\ldots\ldots\ldots (13)$$

Hence, if ρ be known, R_t may be calculated.

The value of R_0 is obtained by surrounding the platinum, etc., with melting ice. After twenty minutes or half an hour the whole of the immersed tube will have attained the temperature of 0°C.; the balance is obtained near the centre of the bridge by making S have a value S_0. A is x_0 cms. from the end. If after five minutes the balance is still at x_0, we may safely assume that the resistance is at 0° C.

The value of ρ may be obtained at this stage, by varying S_0 to S_0' and finding the new balance at x_0'. S_0' is again adjusted to a third value, S_0'', and the balance is now at x_0''. The platinum wire is meanwhile at 0° C. and therefore its resistance remains R_0,

i.e.
$$R_0 = S_0 + (100 - 2x_0)\rho,$$
$$R_0 = S_0' + (100 - 2x_0')\rho,$$
$$R_0 = S_0'' + (100 - 2x_0')\rho.$$

These taken in pairs yield three values of ρ, and further give by any one, preferably the value of S_0 which makes the balance in the neighbourhood of the centre of the wire, a value of R_0.

The glass tube and contents are then placed in a hypsometer, and a balance is obtained once more, when the temperature of the wire has acquired that of the steam which is made to pass round it in the hypsometer.

The value of the resistance at, say, 40° C., 60° C., 80° C. about, is obtained in the same way by immersing the container in a water bath maintained at steady temperatures near these points. In each case time is allowed for the platinum to acquire the temperature of the bath.

The values of R obtained are plotted against temperature. The result will be approximately a straight line. From the slope of the straight line drawn through the observed points calculate α as defined above.

In the theoretical account of the balance given above it was assumed that there were no end corrections to the bridge wire.

If l_1 and l_2 are the equivalent lengths obtained in the bridge calibration (see page 495), equation (13) would become

$$R_t + r + x\rho + l_1\rho = S + r + (100 - x)\rho + l_2\rho,$$

$$R_t = S + (100 - 2x)\rho + (l_2 - l_1)\rho.$$

Since l_1 and l_2 are known the effect produced may be allowed in this way throughout the determinations.

Further, the values of x and $(100 - x)$ should be corrected from a calibration curve as obtained in the manner given on pages 495-8.

The Platinum Resistance Thermometer. Callendar-Griffiths Bridge

The platinum resistance thermometer is a temperature measuring instrument which depends for its action on the variation of the resistance of a wire with temperature, as investigated in the last experiment.

The temperature of any enclosure is calculated from the observed value of the resistance of a calibrated specimen of platinum wire.

The instrument consists of a length of fine platinum wire, wound in a non-inductive manner on a mica frame. The ends of this thin wire are soldered to two thick leads which are connected at the other ends to two terminals on the cap of the porcelain tube which contains them.

Arranged parallel to these thick leads are a second pair, of identical material and size, which are likewise soldered to two more terminals on the cap. The other ends of this second pair are joined together as shown in fig. 320.

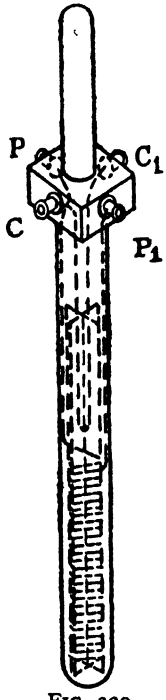


FIG. 320

The terminals are usually marked in a distinctive manner, P, P, C, C ; C, C being the ends of the compensating leads.

We have already seen (page 507) that the resistance, R_t, of a given specimen of wire, at a temperature $t°$ C., may be expressed in terms of the resistance at 0° C., R_0, and two constants α and β in the following manner :

$$R_t = R_0(1 + \alpha t + \beta t^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Thus, if the resistance of the platinum wire were measured at 0° and two other known temperatures, we should have two equations from which α and β could be calculated, and the wire would be standardized, so that if R_t were measured at an unknown temperature the latter could be calculated from equation (11) above ; or having R_0, α, and β, a calibration curve could be drawn showing the relation between R and t, and hence the temperature corresponding to any resistance could be obtained.

Alternatively we may define a new scale of temperature which expresses the relationship between temperature so defined and the value of the resistance in a very simple manner.

Let R_0 and R_{100} be the resistance of the platinum wire at 0° C. and 100° C.; $R_{100} - R_0$ being the increase in resistance for 100° rise in temperature, and is called the 'fundamental interval.'

The platinum scale makes the size of the degree such that each degree rise in temperature on this scale corresponds to an equal increase of resistance of the specimen of wire and is equal to

$$\frac{R_{100} - R_0}{100}.$$

The platinum scale so defined coincides therefore with the gas scale at 0° and 100°, but will differ at other points since equation (11) above expresses the true relation between R and t on the gas scale.

Let R be the resistance of the wire at any temperature, t° C., or t_p on the platinum scale; by definition:

$$t_p = 100 \frac{R - R_0}{R_{100} - R_0} \quad \ldots\ldots\ldots\ldots\ldots (14)$$

The difference between t_p and t has been found to be given by

$$t - t_p = k\left\{\frac{t}{100} - 1\right\}\frac{t}{100} \quad \ldots\ldots\ldots\ldots\ldots (14a)$$

where $k = 1{\cdot}5$ for pure platinum, but its value for any particular specimen may be obtained by obtaining the resistance R_0, R_{100}, and R_t at 0° C., 100° C. and a third standard temperature, say, the boiling point of sulphur corresponding to a known t° C.

The values of R_0, R_{100}, etc., may most conveniently be determined by using the Callendar-Griffiths Bridge.

The Callendar-Griffiths Bridge

The Callendar-Griffiths bridge is a compact form of the Carey Foster arrangement described previously in considering the variations of resistance with temperature. Fig. 321 shows the general appearance of a modern example of this type of bridge.

Reference to fig. 322 may make the principle of construction clear.

R_1 and R_2 are equal resistances. EF an adjustable resistance capable of giving 5, 10, 20, 40, 80, 160, 320, 640, 1280 arbitrary units of resistance. ML is a straight stretched wire chosen by reason of its uniformity; T a second parallel wire of the same material, which may be connected to ML by means of a short length of the same material wire. Thus, the possibility of thermo-electrical effect when the galvanometer is connected to the wire is eliminated.

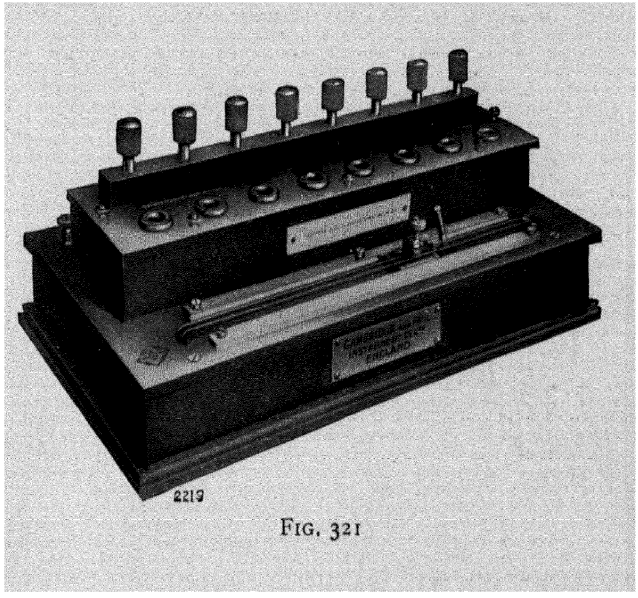


FIG. 321

Suppose the thermometer be balanced at a temperature t_p, against S units from EF, when the contact is made at O, the centre of ML, then using the same notation as before (page 509).

$$R_t + r + \text{Resistance LO} = S + r + \text{Resistance OM.}$$

or $$R_t = S.$$

If now at a second slightly different temperature t'_p the

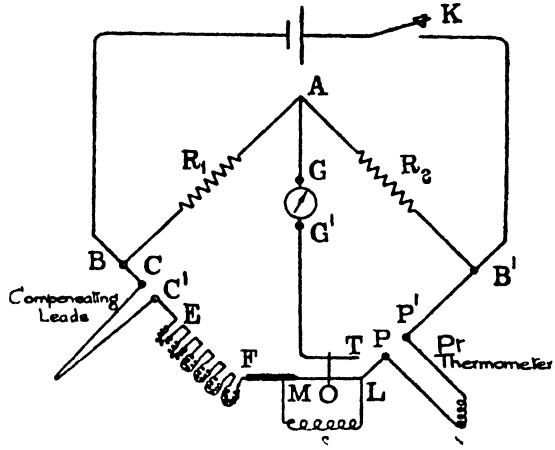


FIG. 322

thermometer has resistance R_1, and is balanced by moving the contact 1 unit of length to the right,

$R_1 + r +$ resistance of $(OL - 1) = S + r +$ resistance of $(OM + 1)$,

i.e. $R_1 = S + 2$ resistance of 1 cm. of LM.

If the resistance per cm. of ML is half an arbitrary unit, we have :

$$R_1 = S + 1 \text{ arbitrary units.}$$

The scale on which measurements of MO, etc., are made is usually inscribed in arbitrary units equivalent to the movement of the contact position.

If the balance is to the right of the mid-point of LM, the number of units must be added to S ; if to the left the number is taken from S.

The length of the wire, LM, is usually sufficient to allow about 15 units each side of the centre.

To arrange this simple relation between the length of the wire and the arbitrary units, a coil, *s*, of suitable resistance is shunted across the bridge wire. The coil is adjusted to give the desired value of resistance per cm. to the bridge wire.

The arbitrary unit of resistance often chosen is deduced from the value of the fundamental interval of the thermometer (i.e. the change in resistance for a change in temperature from 0° to 100° C.).

The platinum thermometer is constructed with such a resistance that its fundamental interval is 1 ohm and one hundredth part of this, i.e. ·01 ohm, is taken as the unit. The coils in S have, therefore, resistances of ·05, ·1, etc.

The resistance per cm. of bridge wire in the example taken would therefore be $\frac{1}{2}(\cdot 01)$ or ·005, the effect of 1 cm. change in balance being $2 \times \cdot 005 = \cdot 01$, i.e. a change of balance of 1 cm. corresponds to a change in resistance of the platinum wire of one unit.

The form of Callendar-Griffiths bridge illustrated in fig. 321, is provided with a scale for the slide wire bridge which is graduated from 0 to 15 units, i.e. the readings are continuous from one end of the wire to the other. The balance points give directly the number of units to be added to S to give the equivalent resistance of the thermometer. The half, MO, of the wire is obviously in this case within the instrument in the form of a coil.

A more recent feature of the bridge is the use of mercury cup contacts instead of the usual plug contacts in the adjustable arm, S. Fig. 323 shows an enlarged view of one of the contacts.

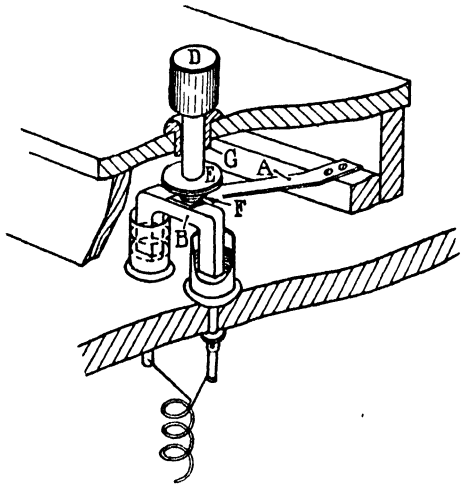


FIG 323

The coil is connected to two mercury cups inside a tightly fitting cover.

A plug, D, when inserted in the hole, G, corresponding to this resistance, strikes a thin circular sheet, E, which is covered with baize and is ordinarily held tightly against the hole by means of the spring, A, and a spiral spring, F, thus keeping dirt and dust from the mercury. When D is allowed to depress this arrangement, the copper connecting strip, B, which is amalgamated at the ends, dips into the cups and the resistance is thereby cut out.

The total value of the resistance, S, is therefore the sum of the numbers opposite the holes without plugs.

The makers suggest that when not in use all plugs should be in to maintain the amalgamation of the connecting strip, B.

The balance point on the bridge wire may be maintained by a rough movement of the slider, followed by a fine adjustment by means of the small lever attachment. The position of the balance is read on a vernier which enables one-tenth of the small scale division to be measured, i.e. if graduated in lengths corresponding to a unit and subdivided into tenths, one may read to $\frac{1}{100}$ of the unit.

Reference to fig. 324 may make clear the internal wiring of the bridge. The lettering in this diagram corresponds to that of fig. 322. A two-volt accumulator or Daniel cell is connected to BB^1 and, say, a Broca galvanometer to GG^1. PP^1 and CC^1 are gaps for the thermometer and compensating leads. It will be noticed that R_1 and R_2 are contained in the bridge.

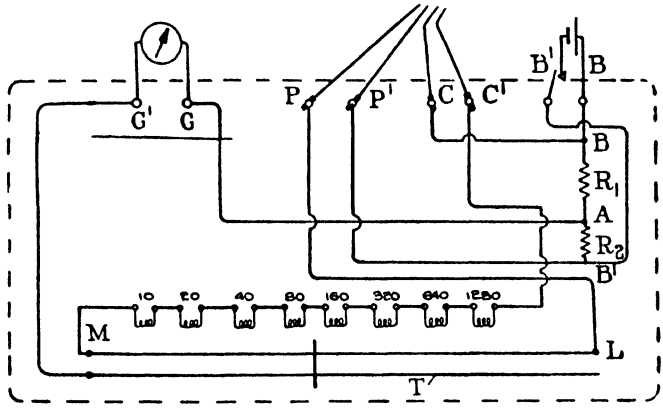


FIG. 324

The point A in fig. 324 is capable of slight adjustment in many forms of the bridge ; so that, if in error, R_1 may be made equal to R_2.

Before using the bridge with the thermometer it is advisable to check some of the points of construction which have already been described.

(*a*) See that the zero of the scale is truly the mid-point of the bridge. This is readily done by inserting equal resistances, e.g. two thick copper strips, in the gap provided for the thermometer and compensating leads. With S = 0, find the balance point on the wire. This is the mid-point and should coincide with the zero graduation ; if this is not so, then probably the values of R and R^1 may not be truly the same ; this may be further checked by using two practically equal resistances of

5 or 10 ohms in the gaps PP^1 and CC^1. A balance is obtained. The coils are then interchanged and a second balance obtained. The mean of the two readings should be the zero of the scale. This method cannot be very well used in the second form of bridge wire described above.

However, for the purpose of *comparing* resistances the slight error in equality will not affect the *comparison.*

(*b*) Verify the relation between arbitrary units of resistance in S and the resistance per unit length of bridge wire, and *see that the resistances in* S *are consistent within themselves.*

The gap CC^1 is closed by means of a copper connecting strip. and an external resistance or resistances are connected by copper strips to the gap PP^1.

The coil 1280 is unplugged and balanced against an external resistance of, say, 12·9 ohms. Suppose x_1 be the reading on the bridge at balance.

Replace the 1280 coil by the rest of the coils, i.e. 640 + 320 + ... 10 + 5, and again balance these coils (nominal value 1275 units) at a point, x_1^1, using the same external resistance. Let l_1 be the resistance corresponding to a change in balance $x_1^1 - x_1$.

Then coil 1280 − sum of the rest $= l_1$ .....(15)

Carry out this test using the 640 coil: balance against an external resistance making the point of balance at x_2, say. Cut out the 640 coil and, again leaving the external resistance the same, balance the rest of the coils (320 + 160 + 80 + 40 + 20 + 10 + 5) at a second point x_2^1 cms. Let the resistance corresponding to the change $x_2^1 - x_2 = l_2$,

then:

$$\text{Resistance (coil 640 − the rest)} = l_2 \quad \ldots\ldots\ldots (16)$$

Carry out this test with each coil in turn.

Finally:

$$\text{Resistance (coil 5 − 0)} = l_9.$$

Nine equations are obtained in this way for the bridge described.

The difference between the first and second equations, where each term represents the corresponding resistance is:

$$\text{coil 1280} - \text{coil (640 + 320 . . . +5)} - \{\text{coil 640} - \text{coil (320 + . . . +5)}\} = l_1 - l_2,$$

i.e. $$\text{coil 1280} - 2\ \text{coil 640} = l_1 - l_2$$

or $$\text{coil 640} = \frac{\text{coil 1280}}{2} + \frac{l_2 - l_1}{2} \quad \ldots\ldots (17)$$

In the same way, taking the third equation from the second:

$$\text{coil 320} = \frac{\text{coil 640}}{2} + \frac{l_3 - l_2}{2}, \quad \ldots\ldots\ldots\ldots (18)$$

and so on. Substituting in (18) the value of coil 640 in terms of 1280 from equation (17):

$$\text{coil } 320 = \frac{\text{coil } 1280}{4} + \frac{l_2 - l_1}{4} + \frac{l_3 - l_2}{2} \quad \ldots\ldots\ldots(19)$$

This process is carried on throughout the range. We may express each coil in terms of the largest one. Equation (17) and (19) give the corrections to be applied to each coil to make them consistent with the largest.

For a perfect set of coils $l_1 = l_2 = \ldots = l_8 = l_9 =$ the length of wire having 5 units of resistance.

It is important to note that the unit often taken is ·01 ohm,

EXTERNAL RESISTANCE	COIL	BRIDGE WIRE READING x units resistance	DIFFERENCE l	CORRECTION
	1280 640+320+...+5			
	640 320+160+...+5			
	320 160+80+...+5			
	160 80+40+...+5			
	80 (+1280) 40+20+10+5 (+1280)			
	40 (+1280) 20+10+5(+1280)			
	20 (+1280) 10+5 (+1280)			
	10 (+1280) 5 (+1280)			
	5 (+1280) 0 (+1280)			

so that for safety the 1280 coil should be out as a permanent addition when testing the 80, 40, 20, 10 and 5 unit coils.

The results may be tabulated as shown above, the last column giving the correction to each coil.

The above calibration gives the mean value per unit length of bridge wire.

For some purposes, and especially when the bridge has had considerable use, it may be necessary to calibrate the bridge wire. This may be done by a slight modification of the method given on page 498. But a new bridge wire should be uniform to within 0·3 per cent, and therefore will give readings correct to the second decimal place in degrees on the platinum scale.

Calibration and Use of the Platinum Thermometer

Having tested the bridge and calibrated the coil and wire as indicated above, the thermometer is placed in melting ice and allowed to remain there until the resistance remains constant. The value of this resistance is noted.

The thermometer is next placed in a hypsometer, in which water is boiled, and again the steady resistance, at the temperature of the steam, is obtained (after being in the steam for, say, 20 to 30 minutes).

The temperature of the steam corresponding to the atmospheric pressure is determined from tables ($= b°$ C. say) then, if R_b is the resistance obtained in the steam, and R_0 in ice at 0° C.,

$$\frac{R_b - R_0}{b}$$

is the change in resistance per degree centigrade.

The thermometer is next immersed in the *vapour* from boiling sulphur (see below for method), and from the balance of the bridge, when steady, the value of R, the resistance at the temperature of the sulphur vapour, is obtained.

The boiling point of the sulphur in degrees on the platinum scale is from (14) :

$$t_p = \frac{b(R - R_0)}{R_b - R_0}.$$

Now the value of the boiling point of the sulphur at 76 cms. of mercury pressure is 444·7° C., and the value at any other pressure, p mms., is :

$$t = 444{\cdot}7 + {\cdot}0904(p - 760) - {\cdot}000052(p - 760)^2.$$

The value of t, the boiling point at the pressure which obtains during the experiment is calculated and hence, substituting t_p and t in (14*a*), the value of k may be calculated.

It will be found to be very near 1·5 in most cases. The thermometer is now fully calibrated and may be used, for example, to find the B.P. of aniline first in degrees platinum from which, by (14*a*), the corresponding value in degrees centigrade may be found.

To Obtain the Boiling Point of Sulphur

The apparatus of fig. 325 is convenient. This is readily made by taking a length of iron tube about two inches diameter, and

about four or five inches longer than the thermometer. The lower end of the tube is closed by brazing on a circular disc of iron.

The lower end of the tube is covered completely with asbestos paper and then wound with nichrome wire of suitable gauge and length to produce sufficient heating. For example, the one

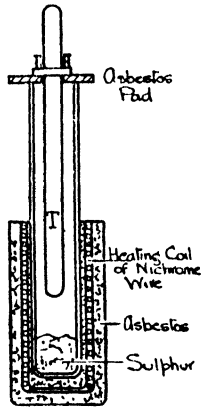


FIG. 325

shown in the figure was made by winding with nichrome wire of ·092 cm. diameter and six metres in length. This was satisfactory for use with a series resistance on 100-volt mains. The length of the same gauge wire may be varied to suit the potential of the mains available. It might be found advisable, in addition to the thick layer of asbestos around the nichrome winding, to surround the boiler with dry sand. The sulphur vapour condenses on the upper walls of the tube and runs back.

The boiling point under the pressure which obtains during the experiment, may be obtained from the formula given on page 518.

The Variation of the Resistance of a Bismuth Spiral in a Magnetic Field

In this experiment a bismuth spiral such as that shown in fig. 326 is placed in a variable magnetic field and the resistance

FIG. 326

is determined for each value of the field strength. The measurements involved are therefore (1) the strength of the field, (2) the resistance of the spiral.

The magnetic field should be of as large a range as possible, and may be obtained by using a large du Bois magnet as shown in

fig. 327. Current from the electricity mains is passed through a series of frame resistances, R, to the magnet, and an incandescent lamp, L, in parallel. The current passing through the windings of the *magnet* may be measured on a calibrated ammeter, A, and may be reversed by the commutator, C. The ammeter is placed at as large a distance from the magnet as is convenient.

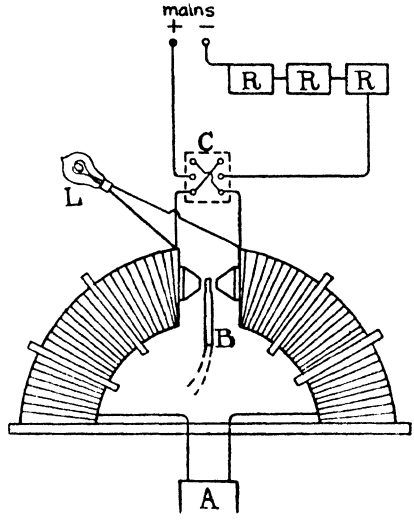


FIG. 327

By varying the frame resistance, the current passing may be of values from, say, 0 to 5 amperes.

The bismuth spiral is placed centrally between the poles of the magnet with the plane of the spiral normal to the magnetic field of the magnet and coinciding with the meridian. The resistance of the spiral is measured either by a Post Office Box, or by the Carey Foster method. In the latter case the scheme of connexions shown in fig. 319 may be used, where the gap, PP^1, is closed by the spiral and the resistance, S, may be of any value from 0 to 50 ohms. R_1 and R_2 may be given convenient values ; for a spiral of resistance 15 to 20 ohms make $R_1 = R_2 = 20$ ohms, say.

In either case the determination of the resistance, R_1, say, of the spiral will present no difficulties.

Its magnitude is determined for values of the field corresponding to currents in the electromagnet windings ranging from 0 to 5 amperes, by, say, ·3 ampere steps.

The initial value of the resistance will depend upon the past history of the specimen, but will increase with increasing magnetic fields. (The order of this increase will be 10 per cent of the original value, depending upon the field change which a current variation of 0 to 5 amperes creates.)

The values of the current and resistance are tabulated.

The next part of the experiment is an estimation of the magnetic field corresponding to the currents used.

This may be carried out in either of the following ways:

(1) *Fluxmeter Determination of the Field Strength* (H)

The fluxmeter is set up, away from the electromagnet, and a search coil of the same area as the bismuth spiral is made. This is connected to the fluxmeter and introduced between the poles of the magnet. The deflection of the suspended system is noted, either directly or on a scale one metre away (for the weaker field this method is essential). The scale readings are converted to maxwells by comparison with the graduated scale on the instrument as described on page 485.

If a is the area of the search coil (per turn) and there are n turns, the total flux recorded in maxwells (m maxwells, say) is

$$m = naH, \quad \text{or} \quad H = \frac{m}{na} \text{ gausses.}$$

As explained in dealing with the theory of the fluxmeter (page 482) the total deflection is independent of the speed of insertion of the search coil.

The field is found in this way for each of the current values used in the resistance determination.

(2) *Ballistic Galvanometer Method of Finding* H

In this method a similar search coil is made and connected to a ballistic galvanometer. This forms a low-resistance circuit, so that a moving-needle type would be least damped by such a low-resistance circuit. This, however, should be removed to a very great distance from the magnet and ammeter for obvious reasons.

It will generally be better to use a moving-coil instrument, also removed from the magnet, though not necessarily as far away as the needle type. In such a case a key such as described on page 481 should be used so that the circuit may be broken immediately after the passage of the discharge.

Under such circumstances, when the current is passed through the windings of the electromagnet, the search coil being in the gap between the pole pieces of the magnet, replacing the bismuth coil, a transient E.M.F. is set up in the galvanometer circuit through which a quantity of electricity, Q, will be discharged. The quantity, Q, as shown earlier, may be expressed by equation (19), page 479.

$$Q = \frac{T}{\pi} \cdot \frac{c}{\varphi} \cdot \frac{\alpha_1}{2} \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{2}},$$

where

T is the undamped periodic time (when the key described is used),

α_1 is the first deflection,

α_3 is the second deflection on the same side,

c is a small steady current,

φ is the corresponding steady deflection.

The correction for damping $\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}}$ will only be applicable for the moving-coil instrument when the circuit is broken, as described, as soon as the impulse is given to the galvanometer.

The value of φ corresponding to a current, c, may be obtained by passing a current from a 2 volt accumulator through one megohm, and the galvanometer shunted by a small resistance. The deflection is observed and the current, c, calculated, as on page 465. T, α_1, α_3 are observed in the usual way.

Now suppose that the field strength to be measured is H gausses, and let c^1 be the current passing through the coil at any instant, t, after the circuit is made. During a small interval of time, dt, let dN be the number of lines threading the circuit, then

$$d\mathrm{E} = \frac{d\mathrm{N}}{dt} \text{ (numerically)},$$

or

$$rc^1 = \frac{d\mathrm{N}}{dt},$$

where r is the total resistance of the galvanometer circuit (galvanometer, leads and search coil),

i.e.

$$\int rc\,dt = \int d\mathrm{N},$$

i.e.

$$\mathrm{Q}r = \Big[\mathrm{N}\Big]_0^{\mathrm{H}\cdot a\ n},$$

when a is the area of one turn ($= \pi \mathrm{R}^2$), and n is the number of turns, in the search coil,

i.e.

$$\mathrm{Q} = \frac{\mathrm{H}an}{r} \ *$$

i.e.

$$\mathrm{H} = \left\{\frac{\mathrm{T}}{\pi}\frac{c}{\varphi}\frac{r}{2an}\right\} \alpha_1 \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{4}}.$$

The bracket term is constant during the experiment and may be evaluated and used as a reduction factor throughout.

The observations described above under distinct headings may be carried out successively. The search coil is placed in the gap between the pole pieces; α_1, α_3 are observed. The bismuth

* Alternatively, if the search coil is placed in the gap between the magnets and the current reversed in the windings, $\mathrm{Q} = \frac{2an\mathrm{H}}{r}$.

spiral is then introduced and its resistance measured. The search coil is replaced and the current cut off. The values, α_1^1, α_3^1, are again noted. A new value of the current strength is obtained by an adjustment of the frame resistance, and the current switched on. The throws are observed and then the bismuth spiral resistance is again measured. The check value for the throws is again obtained when the current is switched off, and so on.

With either method of working compile a table showing the relation between the magnetic field strength and the resistance, and plot a curve showing this relation.

The form of the curve will suggest that the relation between R and H is of the form :

$$R_H = R_0 + aH + bH^2,$$

where R_H is the resistance in a field of strength, H,
R_0 the resistance in zero field,
a, b are constants.

By substituting values for two points, find a and b.

Calculate the expected value of R for another magnetic field, in the range of the experiment, as far removed from the first two points as convenient. The observed value will be found in close agreement with this value.

Determination of the Absolute Resistance of a Metal Rod. (Lorentz's Method)

This method of measuring resistance in absolute units is one in which a steady drop of potential at the ends of a rod (as determined by Ohm's Law) is balanced against one set up by induction. The measurements are reduced to those of length and time.

Fig. 328 shows the details ; the sketch shows a copper disc, mounted on a horizontal axis about which it may be rotated, either by hand or by a motor. A metal brush, C, makes contact at the rim of the disc, and a second contact to the disc is made at the axle.

At an adjustable distance, d, from the plane of the disc is a coil, D, of N turns of wire, of about the same radius as that of the disc (a cms).

By regulating the position of the movable base, S, the distance, d, may be made any length within the limits of the length of the base. Fig. 328 also shows, in a diagrammatic form, the connexions used in the experiment. AB is the metal rod, say, of brass. A single accumulator is connected in series with the coil, D, and the resistance, AB.

So that if R is the resistance of the rod in absolute E.M. units, the drop in potential at the ends of AB is cR, when c is the current in like units.

The ends of the rod are also connected by copper wires in series with the disc and a galvanometer.

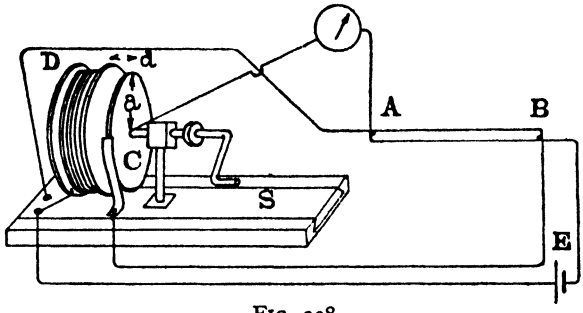


FIG. 328

When the current from the accumulator, E, circulates through the coil, a definite number of lines of magnetic force cut the disc. If M is the coefficient of mutual inductance of the disc and the coil, there will be Mc lines cutting the disc. If, therefore, the latter is made to rotate, an E.M.F. will be set up in the disc circuit, which will depend on the direction of rotation of the disc for its direction.

The direction of rotation is chosen such that the E.M.F. set up by induction opposes the E.M.F., Rc, due to the steady current of the coil circuit. By adjusting the speed of rotation, or by altering the distance, d (usually by both these methods), we may balance the potential due to the two causes. Suppose there are n revolutions per second when this balance occurs, as shown by no deflection in the sensitive galvanometer G.

In the disc circuit we have an E.M.F. due to the induction equal to

$$(\mathrm{M}c)n,$$

since Mc lines are cut by any radius for one revolution, there are n revolutions per second. Mcn is the number of lines cut per second, and opposing this is an E.M.F., Rc. For balance

$$\mathrm{M}cn = \mathrm{R}c,$$

or

$$\mathrm{R} = n\mathrm{M}.$$

The resistance, R, is therefore determined if M, the coefficient of mutual induction, is known, and n the number of revolutions per second is counted.

Maxwell's formula for the coefficient of mutual inductance in the case of two circuits of one turn each, of radius, a, and separated by a distance, d, is :

$$m = 4\pi a \left\{ \log_e \frac{8a}{d}\left(1 + \frac{3d^2}{16a^2}\right) - \left(2 + \frac{d^2}{16a^2}\right) \right\}.$$

The coefficient, M, above is thus Nm,

i.e. $$R = nmN,$$

or $$R = 4\pi aNn\left\{2{\cdot}303 \cdot \log_{10}\frac{8a}{d} \cdot \left(1 + \frac{3d^2}{16a^2}\right) - \left(2 + \frac{d^2}{16a^2}\right)\right\}.$$

The resistance in *ohms*, where the ohm is defined as 10^9 E.M.U. is, of course, $R \times 10^{-9}$.

Experimental Details

To avoid the complication of the Earth's field in the induction of the E.M.F. in the disc, the plane of latter is turned so that it is in the magnetic meridian.

The Broca, or other sensitive moving-magnet galvanometer, works very well in this experiment.

If no motor is available, a fair result is obtained by rotating the disc by hand. The speed of rotation which may be best maintained steady is found by a trial experiment, and the coil and disc placed at a convenient distance, d. When all is ready the disc is rotated, until the spot of light from the galvanometer is brought back to the zero reading. At this stage the rotation is maintained constant. The number of revolutions made is counted and timed with a stop-clock over as long a period as the light spot may be kept at zero.

From this a knowledge of n may be obtained.

d and a are measured in cms., and the value of R calculated.

If the length and cross-section of the rod are measurable, the value of the specific resistance may be also calculated.

The following is an experimental result for a determination made with such an apparatus.

400 revolutions of the disc in 172·5 seconds,

$\therefore$ $n = 2{\cdot}325$ revolutions per second,

$a = 11{\cdot}8$ cms.,

$d = 5{\cdot}0$ cms.,

N = 100 cms.,

$$R = 100 \times 2{\cdot}325 \times 148{\cdot}1 \left\{2{\cdot}303 \times 1{\cdot}2761 \left(1 + \frac{75}{2227{\cdot}8}\right) - \left(2 + \frac{25}{2227{\cdot}8}\right)\right\}$$

$$= 35372 \text{ absolute E.M. units.}$$

Specific Resistance :

Length of rod = 16·7 cms., area of cross-section = $\pi(.64)^2$,

$$\text{Specific Resistance} = \frac{35372 \times 1{\cdot}29}{16{\cdot}7}$$

$$= 2732 \cdot \text{E.M. units}$$

(or ·000002732 ohm).

CHAPTER XX

RESISTANCE OF ELECTROLYTES

Resistance of a Battery. (Mance's Method)

The resistance of a battery may be found by a modification of the Wheatstone net as shown in fig. 329. The resistances, P,Q,R, and the battery of resistance, B, form the network. The connecting wire and key, K, take the place of the battery in the usual form of the net.

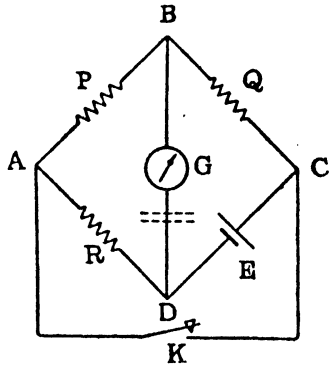


FIG. 329

It will be seen that for all values of P, Q, R, a current will pass through the galvanometer, G (i.e. in circuit, DCB). The resistances, P, Q and R are adjusted until, on depressing K, no change is produced in the steady deflection ; under such circumstances we have :

$$\frac{P}{Q} = \frac{R}{B} \text{ or } B = \frac{QR}{P}.$$

The main difficulty in such a determination is that the steady current in the galvanometer causes a deflection which is too large to keep the reflected beam of light on the scale ; or, if on the scale, the galvanometer is insensitive in detecting change in deflections, as seen by the fact that over a large range of resistance in the arms, R or Q, the above condition seems equally well fulfilled.

This is best overcome by placing a condenser of, say, one-third microfarad in series with the galvanometer (shown in broken lines in the figure).

The steady current in the galvanometer is therefore eliminated. When the key, K, is closed and the bridge is unbalanced, a throw of the galvanometer will be recorded. When balanced, no kick will be given by the galvanometer when K is closed.

Another method which may be employed to overcome the steady deflection, using a moving magnet galvanometer, is to alter the control field in such a way as to reduce the deflection to zero, and proceed as first described. This method, if possible with the control magnet on the instrument, gives good results as the galvanometer is sensitive to the variations produced by closing K, when in the zero position.

If the control magnet is too weak to restore the moving magnet to the original zero position, it may be restored by an external magnet.

Using a Kelvin (Thompson) galvanometer, the same result may be brought about by sending the current through one pair of coils only. The second pair of coils are connected to an independent circuit composed of an adjustable resistance and cell, and the current sent in such a direction as to produce a field in the opposite direction. The magnitude of this current is adjusted until the spot of light is at the zero position. The effect of tapping the key, K, will then be seen in the resulting deflection. The control magnet, in such a case, may be adjusted to give good sensitivity to the instrument.

Conductivity of Salt Solutions (Electrolytes)

When a direct current is passed through an electrolyte, the resulting polarization causes an increase in the resistance of the electrolyte. Further, for continued passage of current through such a liquid, the resulting decomposition also causes a change in the resistance, due to alteration of the concentration of the solution.

The usual special methods adopted to measure such resistances are designed to overcome these difficulties, and are either a potentiometer method, or one making use of an alternating current.

The second is the one most generally employed. If the alternating current is small, and the area of the electrodes large, the polarization effect is reduced to a negligible amount. The more completely is this brought about when very rapidly alternating current is used.

Thus, whereas it is impossible to obtain reliable values for

the resistance of salt solutions by the Wheatstone bridge method in the ordinary way, by using alternating current in conjunction with a Wheatstone net, the value of the resistance of the electrolyte may be found. Of course in such an arrangement an ordinary galvanometer is useless as detector; a telephone replaces it in the usual modification.

It is not advisable to introduce any inductance or capacity into the net, so for this reason the wire bridge is preferable to the post office box type, as the one or ten metres of wire have less self-inductance than the coils of the post office box.

The scheme of connexions is shown in fig. 330.

AB is a stretched wire of one or ten metres in length,
R an adjustable resistance,
S a vessel containing the solution,
T a telephone,
C a small induction coil.

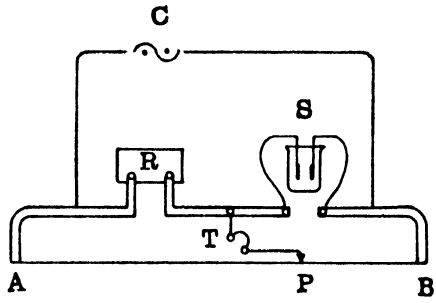


FIG 330

The alternating current is supplied by a small induction coil which will give but small potential. The induction coil is driven by a cell and the secondary is connected to the ends, A and B, of the bridge. If possible an induction coil without an iron core should be used; the coil with the greater number of turns should be used as primary, and the current supplied to it should be reversed by a rotating commutator driven by a small motor.

The telephone, T, should be a head-piece receiver.

The vessel, S, shown in fig. 330, should be surrounded by a water bath, and be provided with platinum electrodes. The temperature of the water jacket is maintained constant, as the resistance of the solution varies rapidly with temperature.

The platinum electrodes are coated with finely divided platinum to increase their effective area, and to decrease the back electromotive force due to polarization. If this has not already been

done, the electrode should be immersed in a solution of platinum chloride* and a smaller current passed first in one direction, and then the reverse. To prevent an absorption of salt from the solution, the platinized electrodes are then raised to dull red heat.

The solution of known concentration is placed in S, and R is adjusted so that a point P is obtained near the centre of the bridge, such that the sound in the telephone is entirely cut out or, as more often happens, until the sound is reduced to a minimum, when the usual Wheatstone result may be applied,

$$\frac{R}{S} = \frac{AP}{PB}.$$

It is usual to express the results in terms of the *specific conductivity* of the solution, i.e. the reciprocal of the resistance of one centimetre, one square *cm.* in cross-section.

If r is the resistance in ohms between the electrodes,
s the specific conductivity,

then $$s \propto \frac{1}{r},$$

or $$s = \frac{A}{r},$$

where A is a constant depending on the dimensions of the vessel used.

A may be obtained by finding r for a liquid of known specific conductivity. We may take potassium chloride as a standardizing solution, using the data in the table shown on page 530, where the specific equivalent conductivity is the quotient of the specific conductivity and the number of gramme molecules of the salt per litre.

It will be seen that for dilute solution this quantity is nearly independent of the concentration.

Make up a solution of KCl containing a definite number of gramme molecules per litre, and find the resistance, r, of the solution when filling the vessel, S. Dilute this solution so that the solution contains, say, half this number of gramme molecules. The value of the specific conductivity for these two strengths may be taken from the above table, and the mean value of A calculated.

Then, using this calibrated vessel, find the resistance of several solutions of N*a*Cl from a concentration of, say, 29·25 grammes

* The Platinum chloride solution to use is made by taking
1 part platinum chloride,
30 parts water,
·008 part lead acetate.

per litre $\left(\frac{N}{2}\right)$ to ·2925 grammes per litre, by diluting the concentrated solution first made.

Plot a curve showing the relation between specific conductivity and concentration, and specific equivalent conductivity and concentration.

The variation of the resistance of one of the solutions with temperature may also be investigated by heating the water baths surrounding the cell, S, and the value of the temperature coefficient may be calculated.

Using a container for the solution, of more measurable dimen-

PARTS BY WEIGHT OF KCl IN 100 PARTS OF SOLUTION	GRAMME MOL. PER LITRE	SPECIFIC CONDUCTIVITY AT 18°C.	SPECIFIC EQUIVALENT CONDUCTIVITY
·00746	·001	$0{\cdot}125 \times 10^{-3}$	125×10^{-3}
·0746	·01	1·206 ,,	120·6
·743	·1	11·300 ,,	113·0
1·48	·2	22·00 ,,	110·0
3·64	·5	50·60 ,,	101·6
7·43	1·0	99·10 ,,	99·1
13·7	2·0	188·8	94·4
19·7	3·0	274·2	91·4

sions, the specific conductivity, and hence equivalent conductivity may be calculated in the way usually employed in solids. For example, in fig. 331, a uniform tube connects two bottles in

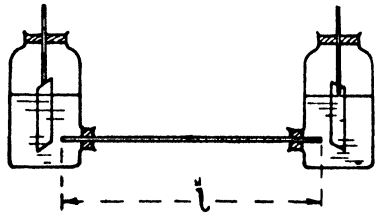


FIG. 331

each of which is immersed an electrode ; the column of liquid conveying the current is practically one coinciding with the tube, i.e. of length l cms. and cross-section equal to that of the internal cross-section of the tube. The resistance is measured as above ; the specific resistance is then directly calculated in the usual way, and the experiment proceeds as already described.

CHAPTER XXI

MEASUREMENT OF POTENTIAL

Standard Cells

The two standard cells in common use are the Weston or Cadmium cell and the Clark cell ; of these, the former is better for general service.

The two cells have each a *constant electromotive force* at one temperature, and also a *definite* temperature coefficient.

The Weston cell is shown in diagram form in fig. 332.

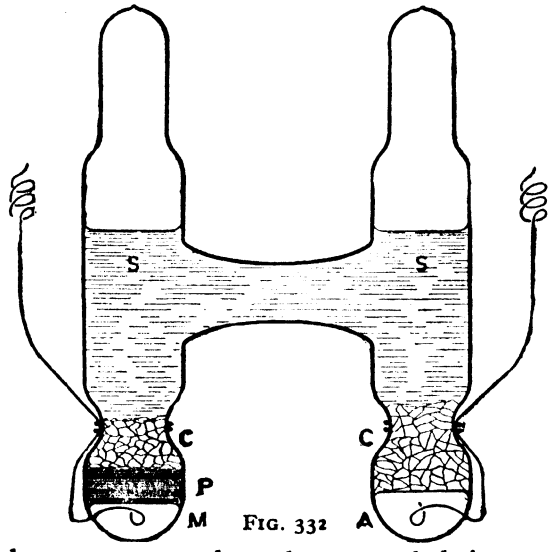


FIG. 332

Two tubes are arranged as shown, each being provided with an external lead which is in contact with the bottom layers. These layers consist of *pure* mercury, M, and an amalgam of pure mercury and cadmium, A, respectively. Above the pure mercury is a layer of a paste of mercurous sulphate, P, made as described later (shown by horizontal shading). Above this and the cadmium amalgam is a layer in each tube of pure cadmium sulphate crystals, CC. Finally, a layer of a saturated solution of pure cadmium sulphate occupies the upper parts of the tubes.

To make a Cadmium Cell

The following is a method of construction for the Weston cell ; for a more permanent and exact cell the specification given in the Report of the British Association Meeting, 1905, on page 98, by F. E. Smith, should be consulted.

(a) *Mercury* is at first obtained in as pure condition as possible commercially. It is then passed through a dilute solution of nitric acid, drop by drop. To bring this about take a tube of about three-quarters to one inch in diameter, and about 60 to 100 cms. long; draw out the end to a smaller diameter and bend this smaller tube so as to leave a short U-tube at the end; the shorter end is bent over as seen in fig. 333.

FIG. 333

A second length of the wide tube is drawn out at one end to a fine capillary of such diameter, that when the tube is filled with mercury the latter will just emerge as *very small* drops. This tube is inserted at the upper end of the first, which is filled with dilute nitric acid. The mercury is collected in the U-tube and passes over into a collecting vessel.

The process is repeated and the partially purified mercury is next distilled under reduced pressure, air being bubbled through it during this process. To carry out this distillation

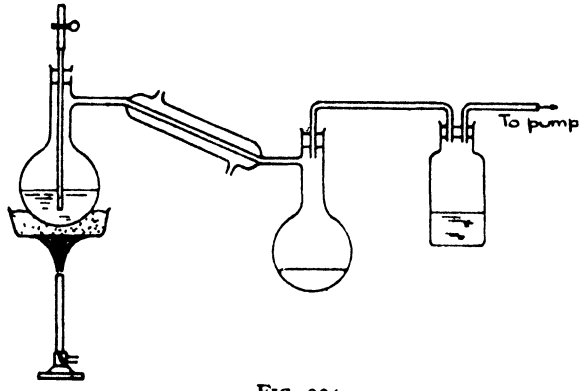


FIG. 334

the mercury is placed in a round-bottomed flask, provided with a side tube which leads through a condenser to a second round-bottomed flask, itself connected to a good water filter pump, as shown in fig. 334.

Into the first flask, through a tightly fitting cork, a narrow glass tube passes under the mercury. A clip regulates the inflow of air. When the pump has reduced the pressure inside the system, the mercury is heated on a sand bath, and the condensed mercury is collected in the second flask.

The air is allowed to enter through the mercury as a slow succession of bubbles. This oxidizes such impurities as zinc and minimizes the risk of their distillation with the mercury.

(b) *The amalgam* is made by dissolving pure cadmium in *pure* mercury, to make a 12 to 13 per cent cadmium amalgam (i.e. about in the proportion of 1 gramme of cadmium to 7 grammes of pure mercury).

(c) *The paste* is made in a mortar by grinding together *pure* mercurous sulphate, cadmium sulphate and purified mercury in the proportion of 8 : 4 : 1. The mixture is made into a thin paste by the addition of a solution of cadmium sulphate.

Before filling the cell the platinum wires must be amalgamated. In the form of tube shown in fig. 332, this is done by the electrolysis of mercuric nitrate in the glass tubes, using the wires as electrodes and reversing the current.

Another form of glass container for the cell consists of a double tube with open upper ends. Corks are selected to fit the open ends tightly, and through holes in the corks a narrow glass tube may be inserted in each side tube, to carry the wire through the cell content to the mercury or amalgam.

Such platinum wires could well be amalgamated by heating to red heat and dipping in mercury.

Having amalgamated the platinum wires, the tubes are next carefully filled as shown in fig. 332. If the open-tube type is used, the corks should be finally coated with marine glue or a mixture of beeswax and resin.

The E.M.F. of the cell so formed will be found constant with constant temperature. *On no account must a current of any appreciable magnitude be taken from the cell.*

The International Conference on Electrical Units and Standards, 1908, adopted the following formula as giving most accurately the E.M.F. of the cell :

$$E_t = 1{\cdot}0184 - 4{\cdot}06 \times 10^{-5}(t - 20) - 9{\cdot}5 \times 10^{-7}(t - 20)^2 + 10^{-8}(t-20)^3 \text{ volt}, \quad \ldots\ldots\ldots\ldots\ldots (1)$$

where t is expressed in degrees centigrade.

The temperature coefficient is therefore small.

The Clark Cell

This is made in a manner identical with that described above, with the exception that cadmium is replaced in this case

by zinc; cadmium sulphate by zinc sulphate, etc. Proceeding as above, again using pure salts and mercury, the standard cell so constructed has an electromotive force expressed by the formula:

$$E_t = 1{\cdot}4328 - 1{\cdot}19 \times 10^{-3}(t - 15) - 7 \times 10^{-6}(t-15)^2 \quad ..(2)$$

Thus this cell has a larger temperature coefficient than the Weston, a fact which explains the more general use of the former.

When using either form of cell in potentiometer work the device of fig. 337 (i.e. a large series resistance) is a useful one as a safeguard against damage to the cell when the 'balance point' is not approximately found.

Comparison of Electromotive Force

The potentiometer method of comparing two electromotive forces is the most satisfactory one. It is assumed that the reader is familiar with the direct comparison of two E.M.F.'s, using a stretched wire potentiometer.

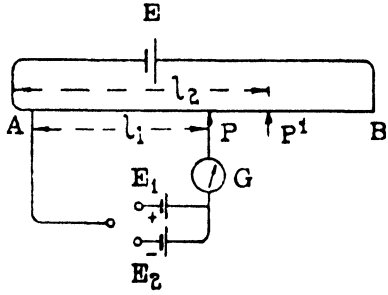


FIG. 335

In that method a steady accumulator is arranged as at E in fig. 335, and E_1 and E_2 whose E.M.F.'s are to be compared are in turn placed in series with the galvanometer, and a point of balance obtained at l_1 and l_2 cms. respectively; then, if the wire is of uniform resistance per cm.,

$$\frac{E_1}{E_2} = \frac{l_1}{l_2}.$$

Using one of the methods given on page 495, *et seq.*, the wire, AB, could be calibrated and then, more accurately, if l_1^1 and l_2^1 are the corrected lengths corresponding to l_1 and l_2

$$\frac{E_1}{E_2} = \frac{l_1^1}{l_2^1}.$$

For an absolute value of the E.M.F. of either cell a third balance could be obtained at, say, l_3 for a cadmium cell whose E.M.F. is known at the temperature of the experiment and is

approximately 1·0184 volts. Whence E_1 and E_2 may be obtained in volts.

For making such a comparison, the accuracy of the determination depends on the accuracy of obtaining the balance point. If instead of using a one-metre potentiometer a wire of ten metres (i.e. ten parallel wires in series, each one metre long) then each *cm.* of wire has a potential drop equal to one-tenth the drop in the simpler potentiometer, i.e. a movement of 1 *mm.* in the single wire bridge would correspond to 1 *cm.* movement in the ten-wire instrument; hence by using the ten-metre potentiometer the true balance point may be more nearly estimated.

It is often convenient to make such a potentiometer direct reading. To do this we arrange that between the ends of a definite length there is a fixed potential difference, say, 10^{-3} volt.

Using a ten-metre potentiometer the most convenient length to employ to correspond to 10^{-3} volt is 5 *mms.* This is brought about as follows:

A standard cadmium cell (E.M.F. = 1·0184, assuming temperature is 20° C.) is connected to A, and the jockey makes contact with the wire at 1018·4 units of length from the end.

We have chosen the unit for this purpose as 5 *mms.* i.e. P is fixed at 509·2 *cms.* from A as in fig. 336. R is now adjusted

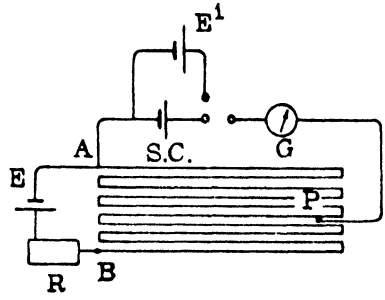


FIG. 336

until the galvanometer gives no deflection, i.e. the current in the wire AB, due to E, is such as to cause 10^{-3} volt drop per 5 mm. of the wire (assumed uniform). Leaving R fixed, any other E.M.F. may be formed by balancing on the potentiometer at, say, *l cms.* from A or $2l \times 5$ *mms.*, i.e. $2l \times 10^{-3}$ volts is the value of the balanced E.M.F. The above assumes that the accumulator E remains steady. This should be checked at intervals, if a series of comparisons are to be made, by reinserting the standard cell, SC, and adjusting R to bring a balance at 509·2 cms.

For many purposes it is convenient to replace the wire by variable standard resistances when comparing potentials. A suitable arrangement of apparatus using such a method is seen in fig. 337. R_1 and R_2 are two resistance boxes, each having a

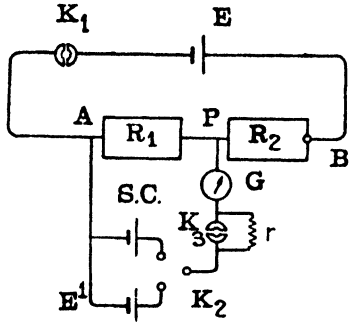


FIG. 337

resistance up to, say, 10,000 ohms (two post office boxes do very well).

E is a steady accumulator which is connected in series with R_1 and R_2. The cells to be compared are connected to A and P through a galvanometer, G.

For direct comparison the resistance $(R_1 + R_2)$ is kept constant, say 10,000 ohms, and R_1 and R_2 are varied until no deflection is obtained when K_2 is closed. The value of R_1 is noted.

The process is repeated with the second cell, say, a cadmium cell, a balance being obtained for a resistance R_1^1 in the box between AP.

Then, as before $$\frac{E_1}{E_2} = \frac{R_1}{R_1^1},$$

for the drop in potential between AP due to E is $\dfrac{E\,R_1}{R_1 + R_2 + B}$, where B is the resistance of the cell E. When a balance is obtained

$$E_1 = \frac{R_1}{R_1 + R_2 + B} \cdot E;$$

similarly

$$E_2 = \frac{R_1^1}{R_1^1 + R_2^1 + B} \cdot E,$$

and $$R_1 + R_2 = R_1^1 + R_2^1 = 10{,}000 \text{ ohms}.$$

Hence

$$\frac{E_1}{E_2} = \frac{R_1}{R_1^1}.$$

Another way of using the above form of potentiometer is similar to the direct reading method of using the wire potentiometer.

One of the cells to be balanced against E is a cadmium cell. If the temperature of the experiment is not far removed from 20° C., the electromotive force of such a cell is 1·0184.

R_1 is given the value 1018·4 ohms (a fraction ohm box *may* be included in series with R_1). Knowing the approximate electromotive force of the cell, E, the value of R_2 may be estimated such that the potential drop in AP $\simeq$ 1·0184 volts. The standard cell is placed in series with the galvanometer G; K_3 being open, *r*, of about 10^4 ohms is in series with SC, to avoid damaging it during the preliminary balancing. R_2 is now adjusted until no deflection is noted in G. K_3 is closed and the final balance verified. The total value of $R_1 + R_2$ under these conditions is noted and maintained constant throughout the comparison.

Now 1018·4 ohms have a drop of potential of 1·0184 volts, i.e. each ohm corresponds to a potential drop of 10^{-3} volt.

Therefore, when a second cell (E^1) is introduced, if R_1 has a new value, R_1^1 ohms at balance ($R_1^1 + R_2^1$ being equal to R_1 and R_2 as obtained in the first test), E^1 is $R_1^1 \times 10^{-3}$ volts.

The above methods of making the potentiometer direct reading are only suggested for those cases where several comparisons are to be made, for under such circumstances subsequent calculation is eliminated.

The experimental arrangements described above are most sensitive for comparison of electromotive forces of the order of one volt. If now a small difference of potential is to be determined it will be apparent that these arrangements are not sufficiently sensitive. In the case of a stretched wire, the sensitivity increases with increase in length; therefore, to measure a potential difference of the order of, say, 10^{-3} volts with fair accuracy the potentiometer wire would require an extension of several metres of wire, or, what is more convenient, the inclusion in the circuit of a resistance several times that of the wire (*r*). For example, if a potentiometer wire of one metre were of one ohm resistance, and the accumulator E had an E.M.F. of 2 volts, a potential difference of the order of 10 millivolts would balance at a distance of 5 mm. from the end of the wire. This could very easily be estimated at 4 *mm.*, or 20 per cent in error. Further, under such circumstances the calibration of the wire would be of greater importance.

If on the other hand 99 ohms were placed in series with the wire, the drop of potential in the wire would be 1/100 of 2 volts, and the true balance point would therefore be at 50 *cms.* An error in estimating the balance point of 1 *mm.* would only be ·2 per cent.

An example of the application of this method is seen in fig. 343, page 542.

Direct Reading Potentiometers

As described above, the potentiometer may be made ' direct reading.' Several forms of potentiometer are now on the market which are constructed on the above principles and are calibrated directly in potential.

Of the simpler forms of compact manufactured instruments, which measures potential of the order of a volt to one millivolt, we will describe the form illustrated in fig. 338. The internal wiring is indicated by white lines drawn on the case, and the instrument is an application of the form shown in fig. 336. The 2 volt accumulator which is connected to the terminals EF, sends a current through an adjustable rheostat, a series of coils provided with tappings to the studs shown, and through the slide wire. This constitutes the main circuit.

When the current is adjusted to the correct amount, the fall in potential along each of the resistances, connected to the studs, is ·1 volt, and along the slide wire, ·12 volt.

The wire is divided into 120 parts, each corresponding in adjustment to a fall of potential of 10^{-3} volts.

To adjust the current to this strength a standard cell is inserted in the gap marked ' potential,' and a galvanometer of low resistance in the gap marked ' galv.' The two adjustable contacts are set at points which correspond to the true E.M.F. of the cell at the temperature of the room, e.g. if the E.M.F. of the cell is 1·018, the sliding contact at the back of the instrument is set at 1·0 and the contact on the wire made at ·018.

The rheostat is then adjusted so that when the key is closed no current is indicated in the galvanometer.

If the rheostat is not sufficiently large to bring this about the battery should be connected to E and G, not EF. This introduces more resistance in the circuit as indicated in the figure, and, using a normal 2-volt accumulator, the balance for the standard cell will be attained.

To obtain the value of an unknown electromotive force, it is connected to replace the standard cell. Leaving the rheostat in the balanced position, the sliding contacts are adjusted until a balance is obtained, as indicated by no current in the galvanometer. The value of the electromotive force is then directly obtained on the calibrated scales.

The width of the small divisions is sufficiently large to allow eye estimation to $\frac{1}{5}$ of a division, but is not of suitable range for the measurement of the corresponding $\frac{1}{5}$ of a millivolt. Thus, whereas the instrument would measure a potential of 1·0182 with a good degree of accuracy, it cannot be used to measure ·0002 volt with any certainty, nor is it reasonable to expect

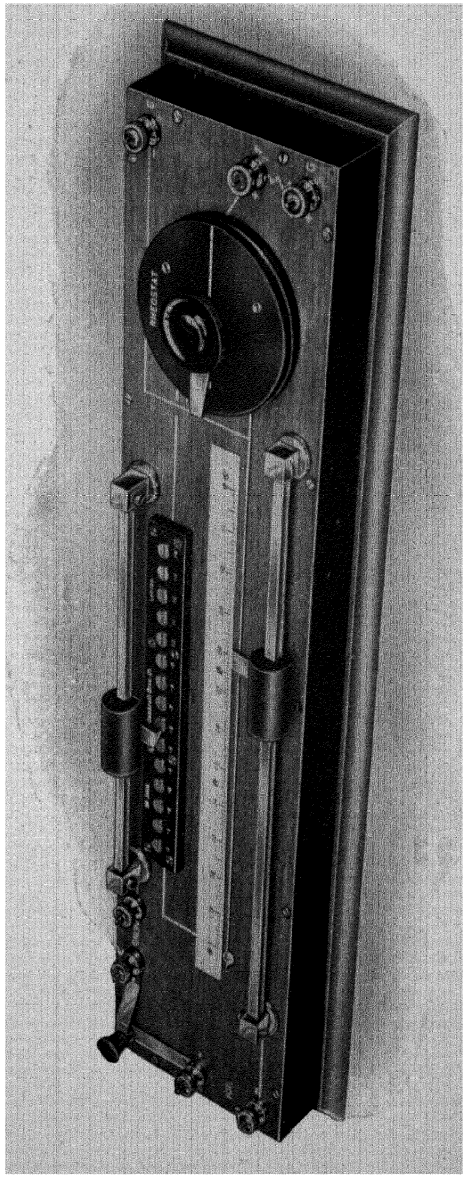


FIG. 338

+ 2V ACCD –
1D CELL –
+
X
–
0-30
30-60
60-90
STANDARD CELL SLIDE
MILLIVOLTS
FINE
STANDARDISING
RHEOSTATS
COARSE
1940

FIG. 339

such measurement with an instrument having a range 0 to 1·5 volts.*

For such measurements, as, say, for a thermo-junction, an instrument having a range of 0 to 50 millivolts is more suitable.

Potentiometers of such a range are manufactured by many firms, e.g. Nalder, Cambridge and Paul, Crompton, Gambrell, etc.

Figs. 339 and 340 show the general appearance and internal arrangements of such an instrument.

The current to main circuit is supplied by a 2 volt accumulator, B. As seen, this circuit consists of adjustable resistances, R_1 and R_2, fixed resistance E and F, MVC, D_1 and D_2, and the two stretched wires SS and VV.

The range of the instrument is 30 millivolts (0 to 30, or 30 to 60, or 60 to 90). The resistance MVC is made up of 29 similar coils of such a resistance that when the current is adjusted as described below, the potential difference between the ends of each coil is 1 millivolt. Also the wire, VV, is of a resistance such that for this adjustment the drop of potential along its length is 1·2 millivolts. By subdivision, a value of ·005 millivolts may be obtained.

Further, the value of the resistance, F, and the slide wire, SS, is such that the graduation along SS give the potential between M and N when adjusted.

To standardize the potentiometer, the standard cell, SC, is connected to the galvanometer, G, by means of the double pole switch. The point, N, is chosen equal to the potential of the standard cell, R_1 and R_2 are then adjusted until the low resistance galvanometer shows no deflection, i.e. the drop of potential along MN corresponds to the graduation value. Under such circumstances the potential difference per coil of MVC is one millivolt, etc.

To measure an unknown electromotive force between 0 to 30 millivolts, plugs are inserted at the points shown in fig. 340, and the unknown potential connected to X. The commutator is thrown over so that XX are connected to the galvanometer, and a balance is obtained by varying the point of contact, P, on MVC and Q on VV. If D_2 is of zero resistance the unknown potential corresponds to that between P and Q and is therefore obtained directly from the scale.

Suppose a bigger potential, say, 30 to 60 is to be measured, D_2 is now made of such resistance that there is a potential drop equal to 30 millivolts along its length, and therefore for the

* A modification of this form of instrument, having a range suitable for thermo-junction work, is now to be obtained

balanced position the potential is 30 + the readings of MVC and VV.

Similarly for 60 to 90 millivolts. D_2 is increased so that for the steady current in the main circuit the potential difference between the ends of D_2 = 60 millivolts.

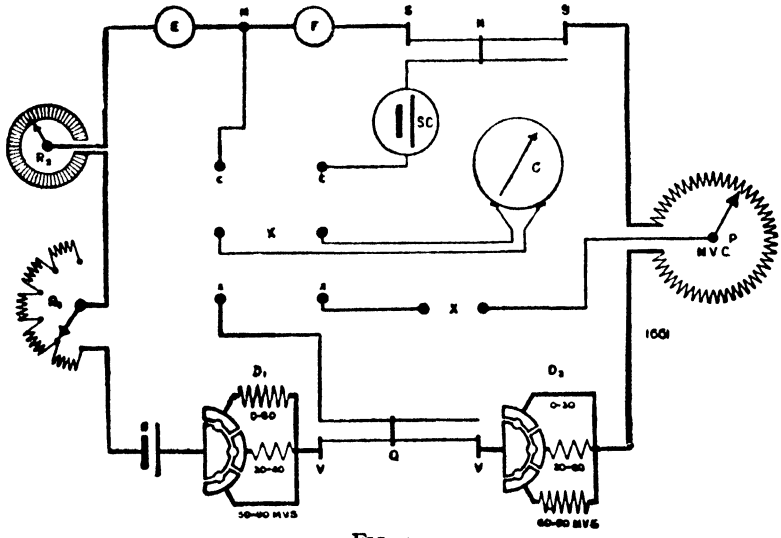


FIG. 340

To maintain the current in the main circuit at a fixed value, the resistance of D_1 must be decreased by the same amount as the increase in resistance of D_2. This is done in the manner shown in fig. 340.

Thermo-Electricity—Thermo-Junctions

When a circuit is composed of two dissimilar metals and the junctions of these metals are maintained at different temperatures an E.M.F. is set up in the circuit. This electromotive force varies with the difference in temperature between the junctions, and when one junction is maintained at 0° C. is given by

$$E_t = at + bt^2,$$

where a and b are constants and t expresses the temperature of the hot junction in degrees centigrade.

The direction of the electromotive force depends on the metals.

It is customary to express E_t for any metal with respect to a standard metal which is taken as one of the pair. The usual choice of standard metal is lead.

In drawing the curve giving the relation between the E.M.F. and temperature, the E.M.F. is taken as positive when the current tends to flow from lead to the metal at the hot junction.

Thus fig. 341 shows the form of these curves for Pb/Fe and Pb/Cu. At a temperature $t°$ C. AB represents the E.M.F. developed in a Pb/Cu junction and, according to the above rule, the electromotive force is from the lead to the metal at the hot junction. Similarly, AC is the magnitude of the electromotive force developed from lead to iron at the hot junction.

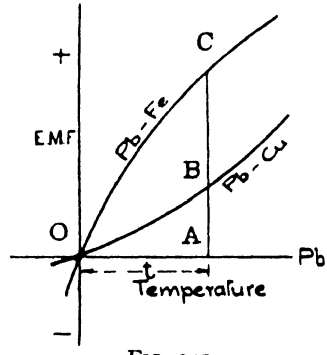


FIG. 341

The law of intermediate metals may now be applied to determine the value of the E.M.F. developed at a copper-iron junction at a temperature $t°$ C., the other junction being maintained at 0° C., for, according to this law the E.M.F. developed at the temperature $t°$, for a Cu/Fe junction, is expressed by

$$\begin{aligned} \text{E.M.F. Cu/Fe} &= \text{E.M.F.(Cu/Pb + Pb/Fe)} \\ &= -\text{AB} + \text{AC} \\ &= +\text{BC}, \end{aligned}$$

i.e. there is an electro-motive force of magnitude BC *from copper to iron at the hot junction* since the BC is positive.

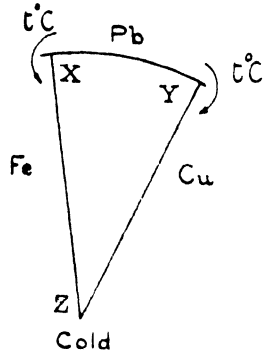


FIG. 342

Reference to fig. 342 will show this from another point of view. If junctions, X and Y, are maintained at temperature, $t°$, Z being maintained at 0° C., the arrows show the direction of the E.M.F.'s

of magnitude AC at X counter-clockwise and AB at Y clockwise—a net result BC counter-clockwise. This, according to the Law of Intermediate Metals, is the value of the E.M.F. if XY are brought together, the lead being removed. The result is an E.M.F., BC, in counter-clockwise direction, i.e. from the copper to the iron at the hot junction.

Experimental Determination of the Thermo-Electromotive Force—Temperature Diagram

The magnitude of the thermo-electric E.M.F. is of the order of a few millivolts. It is best investigated by means of a potentiometer of the form described on page 537. An instructive result is obtained using the three metals, copper, iron and lead. As will be seen from the above, it is only necessary to obtain the E.M.F. for two pairs of the metals, the third pair may be estimated, making use of the law of intermediate metals.

A uniform wire potentiometer of, say, 10 metres is employed (one of one or two metres, if the other form is not available). If the best results are to be obtained a preliminary calibration of the wire is advised.

A steady accumulator, C, is joined in series with R_1, R_2, and the potentiometer wire (R_1 and R_2 should contain resistances up to 10,000 ohms). A preliminary experiment gives the value of r the resistance of the ten metres of wire.

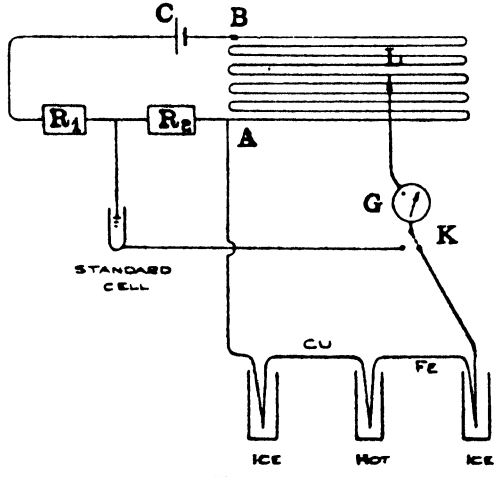


FIG. 343

A thermo-junction is constructed, using copper and iron wire. Care must, of course, be taken that the wires are in contact at the junction only. For this purpose a suitable form of junction is seen in fig. 344. The one wire, B, passes down

a thin glass tube, G, the other, A, is joined to it at J. To ensure good contact, J is dipped into mercury at the bottom of the test tube, T. This does not affect the E.M.F. If one or both of the metals are affected chemically by mercury, it must be dispensed with, and special care paid to the welding of the junctions.

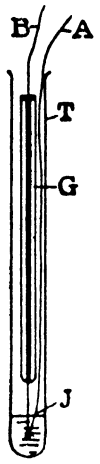


FIG. 344

Three such junctions are made. One, the hot junction, using iron and copper wires. The free ends of the copper and iron wires are each joined to connecting leads as illustrated. These two junctions are maintained at the same temperature, 0° C. in the experiment described. The use of such junctions is to eliminate thermal electromotive force between the metal of the connecting leads and the metals of the thermo-junction proper. In the particular case taken, the copper to copper connecting lead is not essential generally, but is a safeguard against the possibility of impurity in one wire, and is necessary for the Pb/Fe junction.

If the junctions to the connecting wires are maintained at 0° C. by surrounding with ice, and the 'hot junction' is placed in a water bath at a temperature t° C., an E.M.F. will be developed in the direction A to L via the junctions. C is therefore connected with the positive pole to B.

The maximum E.M.F. developed in the above couple is of the order of 1500 micro-volts: the rise in potential along AB should therefore be arranged not very much in excess of this. A voltmeter gives the approximate potential available from C, and r being known, the value of $R_1 + R_2$ to cause such a drop may be calculated,

i.e. $$\frac{\text{potential difference in AB}}{\text{E.M.F. of C}} = \frac{r}{r + R_1 + R_2},$$

the accumulator resistance being negligibly small.

Having fixed R_1 and R_2, the point, L, is made to coincide with A, and the standard cell, say a cadmium cell, is put in series with the galvanometer; R_2 is adjusted, keeping R_1 and R_2 at the value determined above, until no deflection is given in G when contact is made at A.

If the room temperature is approximately 20° C. the E.M.F. of the cadmium cell is 1·0184. Hence the drop of potential along AB is:

$$\frac{r}{R_2} \times 1{\cdot}0184,$$

whence the drop per cm. of the wire is:

$$\frac{r}{R_2} \times 1{\cdot}0184 \times 10^{-3} \text{ volt} \quad \ldots\ldots\ldots\ldots (3)$$

The thermo-junction is now placed in circuit and the hot junction is raised to about 95° C. in a water bath : the length of wire required for balance is obtained. The water is allowed to cool and balance points obtained for intervals of temperature of about 5° C.

The hot junction is placed in a hypsometer and again the E.M.F. is balanced : when the junction is at the temperature of the steam a *steady* balance point is obtained and noted.

The hot junction test tube is next placed in a boiling tube containing mercury. This is heated slowly. Balance points are obtained at intervals and the temperature noted on a special mercury thermometer which reads to 360° C. The process is carried on until the mercury boils.

The results are tabulated—temperature and lengths for balance. The lengths are converted to E.M.F. by multiplying by the reduction factor given in (3) above.

The process is repeated, using a lead wire in place of the iron, and the results are plotted as in fig. 341, E.M.F. in micro-volts (10^{-6} volt) against the temperature of the hot junction.

From these results obtain the corresponding curve for lead—iron.

It will be found that at a temperature of about 240° C. the Fe-Cu junction will give a maximum E.M.F. This temperature is called the neutral point.

The same results could be obtained directly, using a direct reading potentiometer. With a ten-metre instrument r would probably be just greater than 10 ohms. If a shunt, S, were placed between A and B (fig. 343) and S given a suitable value which can be calculated from a knowledge of r, the wire and shunt may be made to have exactly 10 ohms resistance.

As before, suppose the E.M.F. of the cadmium cell were 1·0184 at the temperature of the experiment (any other value can be treated in the same manner). Then to obtain a drop of 10^{-6} volt per cm. of bridge wire R_2 is given the value (1018·4 − 10) 1008·4 ohms. The cadmium cell is placed in series with the galvanometer, and the sliding contact, L, is made at B (fig. 343). R_1 is adjusted until a balance is obtained. Then the drop in potential from A to B $= \frac{10}{1018{\cdot}4}$ of 1·0184 or 10^{-2} volt, hence the drop per cm. is 10^{-6} volt.

The lengths for balance in the thermo-couple experiment now give the potential in micro-volts directly.

The same process applies in the case of a single-metre potentiometer. The value of r is found, and the shunt value, S, to make the value for the two in parallel 1 ohm, is calculated and R_2 made (1000 E − 1). In that case each millimetre of the wire corresponds to a potential drop of 10^{-6} volt.

If no standard cell is available the value of the potential per cm. of wire may be calculated from a knowledge of r, R_1 and R_2 and E, the E.M.F. of the cell, C, as determined by means of a high-resistance voltmeter.

Consider the curves obtained. If we take any two fixed temperatures and determine from the curve the value of the E.M.F. developed, *in micro-volts*, we shall be provided with two equations having two unknown constants, for we have seen that the relation between the E.M.F. developed E, and the temperature $t°$ C. is :

$$E = at + bt^2.$$

So by choosing two such pairs of temperatures and finding E on the P*b*-C*u* and P*b*-F*e* curves the values of a and b for iron and copper may be obtained.

Hence since $$\frac{dE}{dt} = a + 2bt,$$

we know the value of the thermo-electric power, $\frac{dE}{dt}$, at any temperature.

The thermo-electric power lines for C*u* and F*e* against the standard, lead, may thus be drawn and the point of intersection, which gives the neutral temperature for iron-copper, may be ascertained. This will be found, of course, to agree with the neutral temperature as found directly from the E.M.F. temperature curve for the iron-copper junction.

The Use of a Thermo-Electric Couple as a Thermometer

It will be seen from the curves obtained (fig. 341), that a couple such as copper-iron is not a suitable one to use as a thermometer. The neutral point is too near 0° C. for such use.

The ideal couple for such purpose is one with a neutral point well removed from 0° C., and which will therefore give an E.M.F. approximately proportional to the temperature difference. In many experiments in this book such a couple is required. A couple of copper and eureka or constantan serves well for this purpose.

When the thermo-junction is used in this way the form of potentiometer described on page 539, fig. 339 is a great convenience, but the form of direct reading potentiometer of ten metres of wire (page 544) when once adjusted serves quite well.

In either case the electromotive force-temperature curve of the junction should be obtained as described up to, say, 240° C. The value of the temperature corresponding to any other electromotive force may then be obtained from the standardizing curve so obtained.

Lippmann's Capillary Electrometer

The essential feature of the Lippmann Capillary Electrometer is a very fine capillary tube drawn from a thin glass tube, mounted so that a column of mercury of variable height may be placed over the meniscus of a liquid which rises in the capillary.

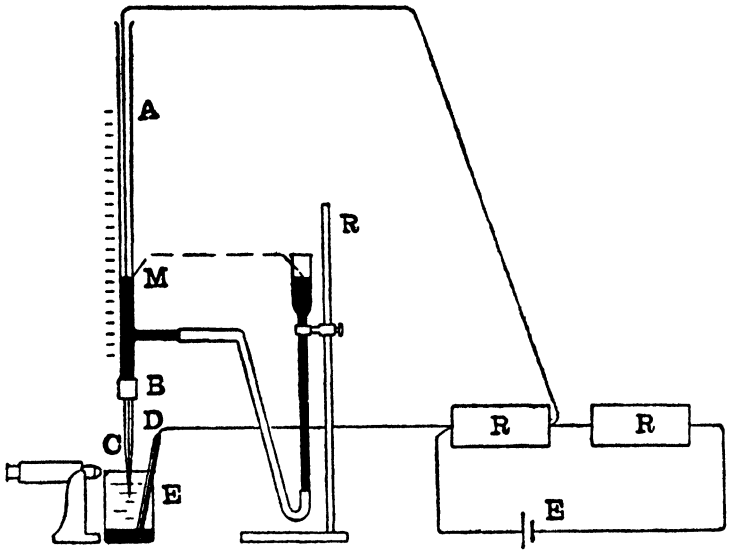


FIG. 345

Fig. 345 shows how this is usually arranged : C is the capillary attached to a vertical glass tube, AB, by means of a short length of pressure tube at B. The side tube is connected by pressure tubing to an adjustable reservoir, R, full of mercury. A scale in millimetres is placed behind either AB or the reservoir R, so that changes in level of the mercury may be measured.

The capillary tube, which dips into sulphuric acid, should be of sufficiently small diameter to prevent the acid from being driven from the tube by the mercury above it.

At the bottom of the beaker, E, which contains the acid, is a layer of mercury, into which a wire, passing down the centre of a glass tube, D, may be placed.

A second copper wire dips into the mercury above the meniscus, either by means of a platinum wire lead fused into the tube AB, or by inserting a wire into the mercury in AB or R. The two leads are connected to a potentiometer which consists of two Post Office boxes in series, R_1 and R_2, so that, maintaining $R_1 + R_2$ at 10,000 ohms, and adjusting R_1 and R_2, any fraction of the potential of the cell, E, may be applied to the junction of the upper mercury and the acid.

If the meniscus is focussed and made to coincide with the horizontal cross-hair in the focal plane of a high-power microscope, it will be found that when a potential is applied in one direction, the meniscus descends, and when applied in the other direction the meniscus ascends. The effective value of the surface tension of the acid mercury surface is altered by the applied voltage.

Suppose the reservoir, R, is lowered so that only a small pressure is applied to the surface, and the meniscus is focussed on the cross-hair of the observing microscope (which should be of high magnifying power), the observed level does not correspond to the true surface tension level.

At the junction of two liquids there is a contact difference of potential set up, and the value of the effective surface tension depends on this contact potential. In whichever direction this contact potential acts, the result is a decrease in the effect surface tension value. For two given liquids, i.e. sulphuric acid and mercury, the value is fixed in direction and magnitude.

We may regard the surface of separation of the liquids as a double layer functioning like a condenser, and having an energy per unit area of $\frac{1}{2}cV^2$, where c is the capacity and V the value of the contact potential.

Thus, if T is the effective surface tension, and T_0 the value of the surface tension if no potential exists, we may write:

$$T = T_0 - \tfrac{1}{2}cV^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

so that whatever sign be given to V the value of $T < T_0$.

If a potential be applied to the surface of separation in the same direction as the contact potential, then V increases and T becomes less, causing the column of acid to descend to a new equilibrium position, whereas, if the applied potential be of the opposite sign to the contact potential, we obtain a larger value for T, and consequently the acid rises in the tube.

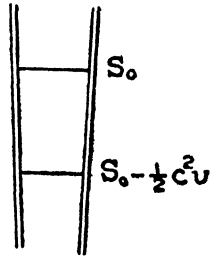


FIG. 346

The capillary when drawn out from the glass tube is always slightly conical, and the state of things showing two positions is seen in fig. 346.

If the applied E.M.F. opposing the contact potential increases

T becomes larger, until, when the applied potential is further increased, we have the double layer effect again coming into play, due to a net potential difference of the opposite sign.

For each value of the applied opposing E.M.F. the meniscus will take up a definite position, and it is obvious that when the contact potential is just neutralized, the meniscus will be at the highest point.

The conical shape of the capillary tube, with resultant change of diameter and of focus, renders unreliable the observations of the level of the meniscus. To be sure that the diameter of the tube at which the surface of separation lies is the same, the meniscus is always observed at one point.

For example, as the value of T increases the level is brought back to the previous one by raising the reservoir R and increasing the pressure on the surface. The microscope used for observing the level of the meniscus should be high-power and the reservoir readily adjustable, so that the level of the mercury in R or AB may be read for each value of the applied potential.

This process should be repeated for all values between 0 and 2 volts, or until electrolysis interferes with observation.

The values of the applied E.M.F. should be plotted as abscissae, and the pressure in cms. of mercury as ordinates.

The form of the curve as obtained from the results under, is seen in fig. 347. The maximum of the applied pressure corre-

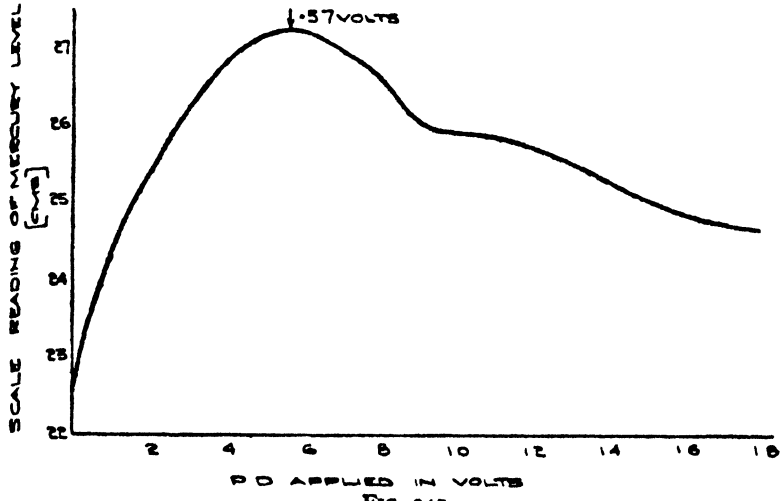


FIG. 347

sponds to a maximum, T, or if the value of T may be represented in some such form as (4) the maximum T is T_0 when the term depending on the contact potential is zero., i.e. it corresponds to an applied potential equal to the contact potential. From a

knowledge of the direction of the applied potential we may say at once which of the liquid is electro-positive and which electro-negative to the other.

In performing the experiment the first thing to do, having assembled the rest of the apparatus, is to draw out a suitable capillary. After one or two attempts a suitable one will be obtained, which is of sufficiently small dimensions to support the pressure.

It should be noted that the capillary tube should be drawn from a *clean* tube. This may be obtained by boiling the tube in nitric acid and rinsing in tap water, or by leaving the tubes to be drawn in a solution of potassium bichromate and sulphuric acid for 12 to 24 hours and then rinsing in tap water.

The mercury also should be cleaned, preferably redistilled.

The following is a record of an experiment where the above precautions were observed.

$$R_1 + R_2 = 10000 \text{ ohms.}$$

R_1	APPLIED POTENTIAL	APPLIED PRESSURE
ohms	*volts*	*cms. Hg*
0	0·0	22·7
1000	·2	25·3
2000	·4	26·8
3000	·6	27·2
4000	·8	26·7
5000	1·0	25·9
6000	1·2	25·8
7000	1·4	25·4
8000	1·6	24·9
9000	1·8	24·7

Contact potential from curve = ·57 (fig. 347). The negative terminal was connected to the upper mercury, i.e. in the experiment the mercury was electro-positive to the sulphuric acid.

CHAPTER XXII

MEASUREMENT OF CAPACITY AND INDUCTANCE

COMPARISON OF CAPACITIES OF CONDENSERS

(1) Deflection Method

Suppose C_1 and C_2 are the condensers, of capacity K_1 and K_2, and that to each is imparted a difference of potential equal to E volts. Let Q_1 and Q_2 be the respective charges on the plates. Then we have :

$$K_1 = \frac{Q_1}{E}, \quad K_2 = \frac{Q_2}{E};$$

or

$$\frac{K_1}{K_2} = \frac{Q_1}{Q_2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In the experiment described below the quantities, Q_1 and Q_2, are measured by discharging the condensers in turn through a moving-coil ballistic galvanometer. In such a case, if θ_1 and θ_2 are the angles of the first throw of the ballistic galvanometer, corresponding to movement of the spot of light, d_1 and d_2 cms., from the zero on the scale, we have (see page 479) :

$$Q_1 = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\theta_1}{2}\left(1 + \frac{\lambda}{2}\right),$$

$$Q_2 = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\theta_2}{2}\left(1 + \frac{\lambda}{2}\right),$$

or

$$\frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2},$$

If the deflections are small :

$$\frac{\theta_1}{\theta_2} = \frac{2 \tan \theta_1}{2 \tan \theta_2} = \frac{\tan 2\theta_1}{\tan 2\theta_2} = \frac{d_1}{d_2}.$$

whence

$$\frac{Q_1}{Q_2} = \frac{d_1}{d_2} = \frac{K_1}{K_2}, \qquad \text{from (1)}$$

i.e.

$$\frac{K_1}{K_1} = \frac{d_1}{d_2}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If the deflections are not small this approximation cannot be used. The values of θ_1 and θ_2 may easily be obtained, for $\tan 2\theta_1 = \frac{d_1}{L}$, where L is the distance from the scale to the mirror of the galvanometer; hence $2\theta_1$ and θ_1.

It often happens that with a steady source of potential, say, a steady 2 volt accumulator, that the throw θ_1 and θ_2 is too large, i.e. E is too great. Under such circumstances, instead of using E directly we may take any fraction of E by a potentiometer method.

Connect E in series with two resistance boxes, R_1 and R_2, making $R_1 + R_2 = 10000$ ohms. From the ends of R_1 connect up the leads which in the diagram (fig. 348) are shown directly connected with E.

By adjusting R_1, making $R_1 + R_2 = 10000$, we may obtain a suitable fraction of E, for the potential drop through the resistance, R_1, is $\frac{R_1}{R_1 + R_2} \cdot E$. These resistances are then kept constant and the throws are obtained.

The scheme of connexions is shown in fig. 348. K_1 and K_2 are condenser keys. K a single-way tapping key, useful in bringing the galvanometer to rest after observing the swings. BG a ballistic galvanometer of the moving-coil type.

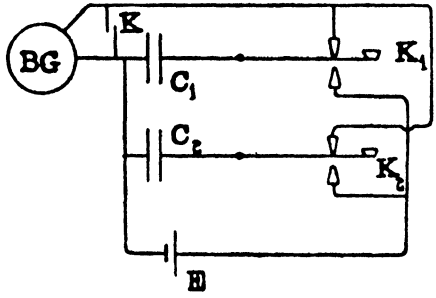


FIG. 348

K_1 is depressed and the condenser, C_1, is charged. K_1 is now raised. The charge, Q_1, is thus sent through BG, and θ_1 noted.

As soon as the galvanometer is brought to rest, the key K_2 is depressed for the same short time and then raised, θ_2 being noted. During the interval of the two depressions of the keys, the potential difference between the poles of the accumulator will remain sensibly constant.

Of course the experiment could be done using one condenser key, and placing C_1 and C_2 in turn in the single circuit.

The observations are repeated, say, three times, and the mean values taken.

(2) Null Method. (de Sauty's Method)

For this method the two condensers, C_1 and C_2, are arranged as two arms in a Wheatstone net; R_1 and R_2, two adjustable *high* resistances making the bridge complete. Fig. 349 gives the scheme of connexions. G is a high-resistance galvanometer and E a battery of cells having an E.M.F. of several volts (say, 4 or 6 Leclanché cells).

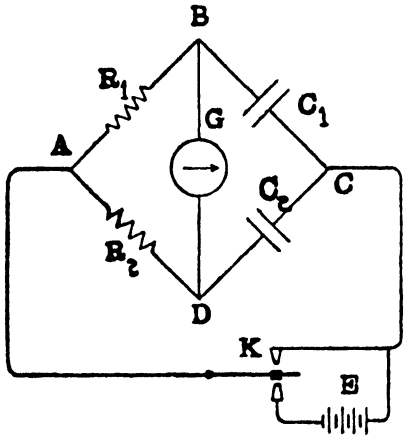


FIG. 349

The key, K, may very well be an ordinary condenser key, so that when depressed the condensers are charged, and when raised the condensers are discharged. The values of R_1 and R_2 are so arranged, that when the key K is moved up and down, charging and discharging the condensers, there is no movement of the galvanometer coil, i.e. in both processes the potential at B is the same as at D.

Let E^1 be the potential at A, and
v_1 be the potential at B at a time, t,
v_2 be the potential at D at a time, t.

The drop in potential along R_1 is $(E^1 - v_1)$ and along R_2 is $(E^1 - v_2)$. Thus, the currents in R_1 and R_2 are $\frac{E^1 - v_1}{R_1}$ and $\frac{E^1 - v_2}{R_2}$.

In a small interval of time, dt, the quantity of electricity passing along R_1 is $\left(\frac{E^1 - v_1}{R_1}\right) dt$, and along R_2 is $\left(\frac{E^1 - v_2}{R_2}\right) dt$.

In charging the condensers the quantity, $\int \frac{E^1 - v_1}{R_1} \cdot dt$, flows to C_1 and $\int \frac{E^1 - v_2}{R_2} \cdot dt$ to C_2.

Now when the condensers are charged, since the potential difference is E (that of the cells) we have :

Charge on C_1 is $K_1E = \int \frac{E^1 - v_1}{R_1} \cdot dt,$

and similarly $K_2E = \int \frac{E^1 - v_2}{R_2} \cdot dt,$

where K_1 and K_2 are the capacities of the two condensers;

or $K_1R_1E = \int (E^1 - v_1)dt,$

$K_2R_2E = \int (E^1 - v_2)dt,$

when no flow occurs through the galvanometer, i.e. $v_1 = v_2$ throughout the charging,

i.e. $\int (E^1 - v_1)dt = \int (E^1 - v_2)dt;$

or $K_1R_1E = K_2R_2E,$

i.e. $\frac{K_1}{K_2} = \frac{R_2}{R_1}$ .....(3)

In performing the experiment R_1 and R_2 may be obtained sufficiently high by using a Post Office Box for each, and following the scheme of connexions shown.

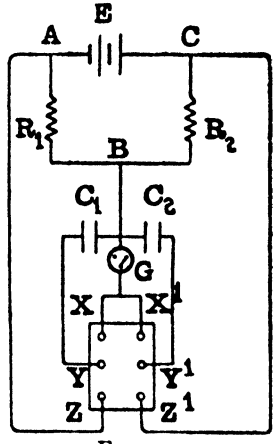


FIG. 350

(3) Method of Mixtures

Comparison of the capacity of two condensers may be made in one or two ways using the 'mixture' method. In this particular method the condensers are arranged to have equal charges and the potential required to do this is measured or compared by a potentiometer method. The theory of the method presents no special points which require separate treatment.

A battery of cells (two accumulators) is connected as in fig. 350 to send a current, c, through two high resistances, R_1 and R_2, which are adjustable and connected together at a point, B.

The potentials between A and B, B and C, are proportional to R_1 and R_2.

B is connected, as shown, to the two condensers whose other plates are connected to the central cups of a cleaned ebonite Pohl commutator from which the cross-connexions have been removed (or a slab of clean paraffin wax with six holes full of mercury will serve).

A galvanometer, G, is connected to the cups, X, X^1, of the commutator.

A and C are joined to Z and Z^1.

When connexion is made (by the rocker of the Pohl commutator) between Y and Z, Y^1 and Z^1, the condensers, C_1 and C_2, are charged to the potential differences of AB and BC, respectively

If Q_1 and Q_2 are the charges, and K_1 and K_2 the capacities, we have :

$$K_1 = \frac{Q_1}{R_1 c} \quad K_2 = \frac{Q_2}{R_2 c}, \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

c being the current through R_1 and R_2.

When the switch is thrown over so that Y and X, Y^1 and X^1 are connected, the condensers are discharged through G.

Suppose the current flow from A to B to C in the potentiometer circuit, then the inner plate of C_1 is at a lower potential than the outer, when charged, whereas for C_2 the inner plate is at a higher potential than the outer. So when the condensers discharge through G, C_2 will send its charge in opposition to C_1.

The above process is repeated, altering R_1 and R_2 between observations, until finally, a value of R_1 and R_2 is obtained, such that on discharging, no current passes through the galvanometer. Under such circumstances, $Q_1 = Q_2$.

From equations (4) above we have :

$$Q_1 = K_1 R_1 c, \qquad Q_2 = K_2 R_2 c.$$

In the adjusted position, since $Q_1 = Q_2$,

$$K_1 R_1 c = K_2 R_2 c,$$

or

$$\frac{K_1}{K_2} = \frac{R_2}{R_1}.$$

Determination of the Absolute Capacity of a Condenser

The value of the capacity of a condenser may be determined n any system of units if a measured potential, E, in these units,

applied to the condenser imparts Q units of charge in the same units ; for the capacity, K, of the condenser is defined by $Q = K \cdot E$,

or $$K = \frac{Q}{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

For example, if Q is expressed in coulombs and E in volts, K, the capacity, obtained in the equation above is expressed in farads. This can be transformed to micro-farads since this unit is 10^{-6} of the farad.

The usual method of finding K experimentally is to apply a known potential (in volts) to the condenser and measure the charge, Q, by discharging the condenser through a ballistic galvanometer.

The moving-coil ballistic galvanometer is the best form to use, as in most capacity experiments, whence if α_1 be the first *observed* throw of the instrument, due to the discharge, we have (see page 479) :

$$Q = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\alpha_1}{2}\left(1 + \frac{\lambda}{2}\right)$$

or, if the method of page 480 is used, the value of λ is not obtained so fully. If α_1 and α_3 are the first and second displacements of the spot of light *on the same side of the zero*, we have :

$$Q = \frac{T}{\pi} \cdot \frac{c}{\varphi} \cdot \frac{\alpha_1}{2}\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{2}},$$

when φ is the angular deflection produced by a steady current c.

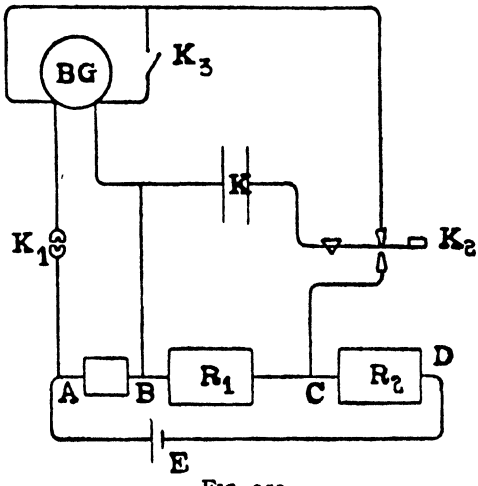


FIG. 351

For small displacements of δ, d_1, and d_3 cms. on the scale corresponding to φ, α_1, and α_3, we have :

$$Q = \frac{T}{\pi} \cdot \frac{c}{\delta} \cdot \frac{d_1}{2}\left(\frac{d_1}{d_3}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Fig. 351 shows the scheme of connexions for such an experiment. K is the condenser whose capacity is to be determined; R_1 and R_2 are resistance boxes, introducing a high resistance to the circuit; AB a resistance box of decimals of an ohm, r say; K_1 a single-way switch; K_3 a tapping key, whereby the galvanometer may be brought to rest; K_2 a condenser key; and BG a ballistic galvanometer of the moving-coil type.

The circuit, R_1, R_2, and the battery, E, serve in the first part of the experiment as a potentiometer. R_1 and R_2 have a constant sum of about, say, 10000 ohms. By adjusting these two values, keeping $(R_1 + R_2)$ constant the potential difference between B and C may be made any desired fraction of the potential, E, of the battery. This adjustment is carried out, and the deflection produced in the galvanometer when the condenser is afterwards discharged through it is noted.

Since $R_1 + R_2$ is high and r is never greater than 1 ohm, the

$$\text{potential between BC} = \frac{R_1}{R_1 + R_2} \cdot E = V \text{ volts, say.}$$

When K_2 is depressed, the condenser is charged, the potential applied being V. The charge, Q, will flow through the galvanometer causing a deflection, d_1 cms., when K_2 is raised.

This process is repeated, say three times, and a mean value of d_1 obtained for the fixed potential, V.

For each discharge, the reading corresponding to the second deflection, d_2, on the same side as d_1, is measured. This provides the data for the damping correction for the galvanometer under the identical conditions under which α_1 is measured.

The time of swing, T, is obtained by timing, say, 20 swings of the needle with a stop-watch. It is most convenient to use the position of rest of the spot of light as a reference point of such counting and timing of swings.

To standardize the galvanometer a steady current of known magnitude must now be sent through it and the scale deflection, δ, measured.

The key, K_1, is now closed. The current from the battery, E (the same 2 volt accumulator throughout), now passes through R_2 and R_1, and then through the galvanometer, shunted by the small resistance, r.

The resistance of the galvanometer and shunt is:

$$\frac{rG}{G + r},$$

where G is the resistance of the galvanometer.

Hence, if E is the potential difference in the circuit, the current in the main circuit, i, is :

$$\frac{E}{R_1+R_2+B+\frac{rG}{r+G}},$$

where B is the resistance of the battery. This may be neglected when the latter is an accumulator.

Of this current, the value of the part, c, through the galvanometer is :

$$c = \frac{E}{R_1+R_2+\frac{rG}{r+G}} \cdot \frac{r}{r+G},$$

$$c = \frac{Er}{(r+G)(R_1+R_2)+rG}.$$

We have also seen that the potential applied to the capacity is

$$V = \frac{R_1}{R_1+R_2} \cdot E.$$

Hence, substituting this value of V and Q from (6),

$$K = \frac{Q}{V}$$

$$= \frac{\frac{T}{2\pi} \frac{E \cdot r}{(r+G)(R_1+R_2)+Gr} \frac{d_1}{\delta}\left(\frac{d_1}{d_2}\right)^2}{\frac{R_1}{R_1+R_2} \cdot E}$$

$$= \frac{T}{2\pi} \frac{d_1}{\delta}\left(\frac{d_1}{d_2}\right)^2 \frac{r(R_1+R_2)}{R_1\{(r+G)(R_1+R_2)+rG\}},$$

a value independent of E, since E remains constant throughout.

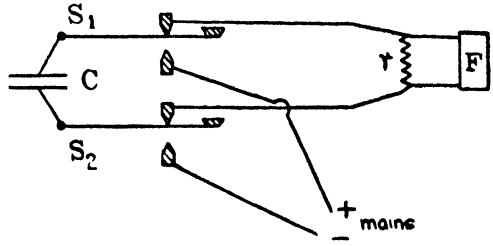


FIG. 352

Measurement of Capacity, using a Fluxmeter

A simple method of estimating the capacity of a condenser is illustrated in fig. 352.

The condenser, C, of capacity, K_1, is connected to two condenser keys, S_1 and S_2. The lower studs of the keys are connected to the mains of the electricity supply (i.e. to 100–200 volts). When the keys have been depressed the condenser is charged to that potential, V. On releasing the keys the charge on the condenser is discharged through a low resistance, r, which is shunted across the fluxmeter, F (the position shown in the diagram).

r is made small, about $\frac{1}{10}$ to $\frac{1}{100}$ ohm is suitable, and the shunt may therefore be regarded as taking the whole of the discharge, say, Q units.

The potential at the ends of r varies from a potential, V, to zero. i.e. the current through r at any instant when the potential is E is $\frac{E}{r}$ and the total quantity of electricity passing through r is therefore:

$$Q = \int \frac{E\,dt}{r}.$$

Now, if the resistance of the fluxmeter, usually about 30 ohms, is large compared with r, the instrument gives a deflection which is a measure of $\int E \cdot dt$ (see page 484). Let this deflection be x divisions or $x \times 10^4$ maxwells,

$$Q = \frac{x \times 10^4}{r}.$$

If now r is expressed in ohms, the units must be made consistent, i.e. $x \times 10^4$ maxwells should now be written $\frac{x \times 10^4}{10^8}$ practical units to obtain Q in coulombs,

i.e. $$Q \text{ (in coulombs)} = \frac{x\,10^{-4}}{r}.$$

If the potential applied to the condenser is measured by means of a voltmeter, the capacity in farads may be readily obtained.

The Effect of Inductance and Capacity in a Circuit Conveying an Alternating Current

In many of the subsequent methods of finding the self-inductance, mutual inductance, or capacity in a circuit, alternating current is employed and a system of non-inductive resistances, capacities, inductances are arranged to produce a balanced Wheatstone net, the usual galvanometer being replaced by a telephone as detector. In such circumstances it is not a difficult matter to find the relation between the values of the resistance, etc., when such a balance is obtained. The majority of cases may be solved by precisely the same method.

Consider an alternating potential $E_0 \cos pt$ to be applied to a circuit, of resistance R, self-inductance L, and containing a capacity K (fig. 353), (p is, of course, given by

$$2\pi n = p,$$

where n is the frequency of the alternation).

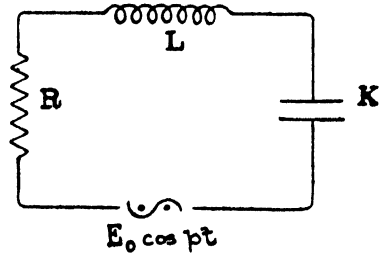


FIG. 353

Now $E_0 e^{ipt} = E_0(\cos pt + i \sin pt)$,

where $i = \sqrt{-1}$ and the *real* part of $E_0 e^{ipt}$ expresses, therefore, the alternating potential applied.

If c is the resulting current at any instant, the net applied potential $= Rc$,

i.e. $$Rc = E_0 \cos pt - \frac{L dc}{dt} - \frac{Q}{K},$$

when Q is the charge on the condenser at any time, t. Further, we have :

$$c = \frac{dQ}{dt};$$

$$\therefore \quad \frac{dc}{dt} = \frac{d^2Q}{dt^2}.$$

Hence : $$L \frac{d^2Q}{dt^2} + R \frac{dc}{dt} + \frac{1}{K} Q = \text{real of } E_0 e^{ipt}.$$

If $Q = A e^{ipt}$, i.e. $\frac{dQ}{dt} = ipAe^{ipt}$, $\frac{d^2Q}{dt^2} = -p^2 Ae^{ipt}$,

we have, where A is a constant,

$$-Lp^2 Ae^{ipt} + Rip\, Ae^{ipt} + \frac{1}{K} Ae^{ipt} = E_0 e^{ipt};$$

or $$A = \frac{E_0}{Rip - Lp^2 + \frac{1}{K}},$$

and $$Q = \frac{E_0 e^{ipt}}{Rip - Lp^2 + \frac{1}{K}};$$

or
$$c = \frac{dQ}{dt} = \frac{ipE_0e^{ipt}}{Rip - Lp^2 + \frac{1}{K}},$$

i.e.
$$c = \frac{E_0e^{ipt}}{R + i\left(Lp - \frac{1}{Kp}\right)} \quad \ldots\ldots\ldots\ldots(7)$$

$$c = \frac{E_0e^{ipt}\left\{R - i\left(Lp - \frac{1}{Kp}\right)\right\}}{R^2 + \left(Lp - \frac{1}{Kp}\right)^2}$$

$$= \frac{E_0e^{ipt}}{\left\{R^2 + \left(Lp - \frac{1}{Kp}\right)^2\right\}^{\frac{1}{2}}}\left[\frac{R}{\left\{R^2 + \left(Lp - \frac{1}{Kp}\right)^2\right\}^{\frac{1}{2}}} - i \cdot \frac{\left(Lp - \frac{1}{Kp}\right)}{\left\{R^2 + \left(Lp - \frac{1}{Kp}\right)^2\right\}^{\frac{1}{2}}}\right],$$

if
$$\frac{R}{\left\{R^2 + \left(Lp - \frac{1}{Kp}\right)^2\right\}^{\frac{1}{2}}} = \cos\alpha,$$

$$\frac{\left(Lp - \frac{1}{Kp}\right)}{\left\{R^2 + \left(Lp - \frac{1}{Kp}\right)^2\right\}^{\frac{1}{2}}} = \sin\alpha,$$

we have :

$$c = \frac{E_0e^{ipt}}{\left\{R^2 + \left(Lp - \frac{1}{Kp}\right)^2\right\}^{\frac{1}{2}}} \cdot e^{-i\alpha}$$

$$= \frac{E_0e^{i(pt-\alpha)}}{\left\{R^2 + \left(Lp - \frac{1}{Kp}\right)^2\right\}^{\frac{1}{2}}}.$$

The real part of this is

$$c = \frac{E_0\cos(pt - \alpha)}{\left\{R^2 + \left(Lp - \frac{1}{Kp}\right)^2\right\}^{\frac{1}{2}}}, \quad \ldots\ldots\ldots\ldots(8)$$

i.e. the current lags behind the applied E.M.F. unless $\alpha = 0$, i.e. $\cos\alpha = 1$,

i.e. $$R = \left\{ R^2 + \left(Lp - \frac{1}{Kp} \right)^2 \right\}^{\frac{1}{2}},$$

or $$Lp = \frac{1}{Kp}, \text{ or } p = \frac{1}{\sqrt{K \cdot L}}.$$

Thus we see that in the general case taken the method gives the value of the current at any time.

From equation (7) we see that the current C is equal to the applied electromotive force divided by $\left(R + Lip - \frac{i}{pK} \right)$. By analogy with the simple Ohm's Law equation $\left(R + Lip - \frac{i}{pK} \right)$ may be termed the '*quasi-resistance*' of the circuit, and be used in the same way as ordinary non-inductance resistance values, when alternating electromotive force is applied to a circuit.

For example, consider a Wheatstone net as illustrated in fig. 360. If a direct current were applied at AC and the net balanced we have :

$$\frac{\text{Resistance AB}}{\text{Resistance BC}} = \frac{\text{Resistance AD.}}{\text{Resistance DC}}$$

Now for alternating current applied using the quasi-resistances in the corresponding arms :

$$\frac{r_1}{r_2 + Lip} = \frac{-\frac{i}{K_1 p}}{R - \frac{i}{K_2 p}},$$

which gives on rearranging and equating real and imaginary quantities, thus :

(real) $$r_1 R = \frac{Lp}{K_1 p} \quad \text{or} \quad L = K_1 r_1 R,$$

(imaginary) $$\frac{ir_1}{K_2 p} = \frac{ir_2}{K_1 p} \quad \text{or} \quad K_1 r_1 = K_2 r_2,$$

a result which agrees with the fuller treatment given on page 575.

The above method of investigation assumes that a true sine E.M.F. is applied. Using a 'hummer' or the secondary of a small induction coil does not produce such a simple form of E.M.F., but would be the resultant of a series of such simple wave forms.

Such an E.M.F. would be satisfactory in those cases where the end result, as in the case taken above, is independent of the frequency or p.

In cases where p is involved in the result, a tuned telephone, which responds to one frequency only should be used. As an alternative, a vibration galvanometer such as that designed by A. Campbell ("Physical Society of London Proceedings," 1907, page 626). As such methods are not very much used, they are not included in this account.

MEASUREMENT OF SELF-INDUCTION

The measurement of the coefficient of self-induction of a coil may be carried out in several ways, of which the following methods are representative.

Some of these methods are very tedious as they involve a double balance of a bridge, for steady and variable currents. Perhaps the most unsatisfactory methods from this point of view are (1) the direct comparison with a standard self-inductance, (2) Maxwell's method. Rimington's method, too, is apt to be tedious. The methods which have been found to be most satisfactory in the laboratory are Rayleigh's, Anderson's, and Owen's. These methods are therefore used generally when a choice of methods is available.

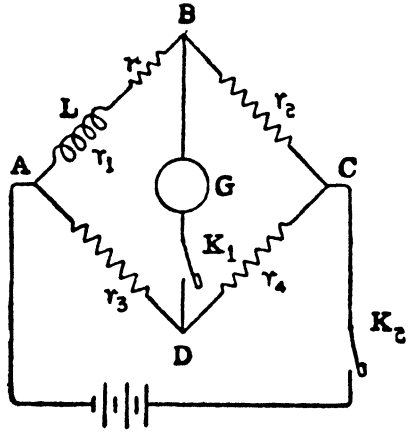


FIG. 354

(1) Rayleigh's Method

In this method the inductance, L, is placed in series with a small variable resistance, r, and arranged in the arm, AB, of the Wheatstone net, as shown in fig. 354. The remaining arms are of the *same order of magnitude* as the resistance, AB. G is a ballistic galvanometer; K_1 and K_2 are keys. The resistance, r, is reduced to zero and the network is balanced for steady currents by closing the key, K_2, before making the galvanometer circuit through K_1.

To obtain an accurate balance, it may be necessary, when the arms are equal, to introduce a smaller resistance than is available in the Post Office Box which provides the three resistances, r_2, r_3 and r_4. This may be effected by having a length of platinoid wire in series with L, and adjusting the length until an accurate balance is obtained.

If now a small E.M.F., E, is introduced in one arm, a current which depends on E will pass through each of the other resistances in the network.

Thus, if the battery key is closed, K_1 being meanwhile closed, an E.M.F. of magnitude $L\frac{dc}{dt}$ will be established in the arm, AB, resulting in a current in each of the arms in the net. Let the current in the galvanometer branch, BD, be $k \cdot \left(L\frac{dc}{dt}\right)$, where k is a constant which depends on the value of the resistances. Under such circumstances the total quantity of electricity which passes through the galvanometer due to this cause is:

$$Q = \int_0^t kL\frac{dc}{dt}\,dt = kLc_0, \quad \ldots\ldots\ldots\ldots\ldots (9)$$

where c_0 is the maximum steady current flowing through AB.

This quantity, Q, may be calculated from the observed throw in the ballistic galvanometer, and thus L is obtained in terms of the constants of the galvanometer, k and c_0.

Assuming that a moving-needle galvanometer is employed we have, page 477,

$$Q = \frac{T}{\pi} \cdot \frac{H}{G} \cdot \sin\frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right) = kLc_0, \quad \ldots\ldots\ldots (10)$$

where λ is the logarithmic decrement.

To eliminate k and c_0, a measurable small potential charge is introduced into the arm AB. This is brought about by adding a small resistance, r, to AB. (r should be not greater than $\frac{1}{500}r^1$, usually $\frac{1}{100}$ ohm does very well.)

Assuming that the current, c_0, will not be materially affected by this small charge, the potential introduced in the arm AB is c_0r: this causes a current, kc_0r, in the galvanometer, producing a steady deflection, θ_1.

Then G being, as above, the galvanometer constant, i.e. the field strength at the centre of the coil due to unit current, we have:

$$kc_0rG = H \tan\theta_1,$$

or

$$kc_0 = \frac{H}{G} \cdot \frac{\tan\theta_1}{r}.$$

Substituting this value in equation (10) above, we have :

$$L = r \cdot \frac{T}{\pi} \cdot \frac{\sin\left\{\frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right)\right\}}{\tan\theta_1}.$$

If r is expressed in ohms and T in seconds, L is in henrys (10^9 C.G.S. E.M. units).

It often happens that the moving-*coil* type of ballistic galvanometer must, of necessity, be employed ; the difficulty which may arise, when L has a small resistance, is to obtain a measurable deflection, as a result of the excessive damping set up, due to the fact that the galvanometer is in a closed circuit, i.e. is shunted by r_2 and r_4 or r_1 and r_3. Measurable and reliable results can be obtained if the galvanometer circuit is broken the moment the discharge has passed through it. This condition is most conveniently brought about by using a single key as shown in fig. 298 on page 481, for both battery and galvanometer circuits.

The three brass strips, A, B, and C, are insulated and connected to separate terminals. When C is depressed contact is made between C and B, then A and S, but not between B and A which are separated by an ebonite stop. S and A replace K_2 ; C and B replace K_1, when C is pressed down and a steady current flows in the circuit, the galvanometer which is in the circuit shows no deflection for a balanced bridge. On releasing C, the bottom contact is first broken (i.e. the battery circuit), and a very short interval of time afterwards C and B are separated. This interval is sufficient to allow the impulse to be given to the galvanometer coil which, due to the separation of B and C, swings without excessive damping due to induced currents.

If the moving-coil instrument is used in this way, the end result is slightly modified, for in this case (see page 479)

$$Q = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\theta}{2} = Lkc_0,$$

where θ is the corrected value of the first throw for no damping, i.e. is the observed throw $\times \left(\frac{1 + \lambda}{2}\right)$.

Now for a steady current equal to kc_0r the couple on the coil is Gkc_0r, so that

$$Gkc_0r = \tau\theta_1,$$

τ being, as above, the restoring couple in the suspension, per unit angular displacement. Combining the last two equations we see that

$$L = \frac{T}{\pi} \cdot \frac{r}{2} \cdot \frac{\theta}{\theta_1}.$$

(2) Comparison with a Standard Inductance

To effect a direct comparison of an unknown inductance, L_1, with a standard inductance, L_2, the arrangement of fig. 355 may be employed.

In series with L_1 is a non-inductive resistance of a variable

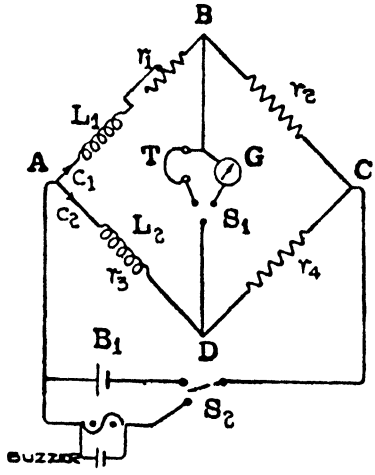


FIG. 355

magnitude, say a length of manganin wire, which is adjusted so that the total resistance included between A and B is r_1 ohms. By means of a key, S_1, the galvanometer, G, is included in the circuit to which a direct current from a 2 volt cell is supplied via key, S_2, and a balance is obtained for steady currents, i.e. the balance is obtained by closing the battery circuit first and when the current is established, the equality of potential at B and D is tested by closing the galvanometer circuit.

Having balanced the bridge for steady currents the balance is now tested for variable current. The first steady balance will not be one which also is in adjustment for variable currents. The resistance in series with L_1 is given another value and the steady current balance obtained once more for the net, which is once more tested with variable current. This process, which is often a lengthy one, is repeated until the resistances, r_1, r_2, r_3, r_4, are of such magnitude that the points, B and D, are at the same potential both with steady and varying currents.

It will be shown that under such circumstances

$$\frac{L_1}{L_2} = \frac{r_1}{r_3} = \frac{r_2}{r_4}.$$

The variable current may be obtained by breaking and making the battery circuit, or by switches, S_1 and S_2, a telephone ear-piece

may be made to replace the galvanometer, and an alternating current from a buzzer supplied instead of the direct current used in the steady balance experiment.

Assuming an alternating E.M.F. be applied to AC and the network adjusted so that the potential at B is the same as that at D, at all times we have, using in the Wheatstone relation the ' quasi-resistances ' of the arms :

$$\frac{L_1 ip + r_1}{L_2 ip + r_3} = \frac{r_2}{r_4}.$$

Equating the real parts we have :

$$r_1 r_4 = r_2 r_3, \text{ or } \frac{r_1}{r_2} = \frac{r_3}{r_4}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

a condition which was fulfilled by first balancing for steady currents.

Equating imaginary parts of the equation, we find

$$L_1 r_4 p = L_2 r_2 p,$$

or

$$\frac{L_1}{L_2} = \frac{r_2}{r_4} = \frac{r_1}{r_3}, \qquad \text{from (11).}$$

(3) Maxwell's Method

The arrangement of resistances and the inductance to be measured is shown in fig. 356. The method involves a comparison of L, with a capacity of known value. In the present method the resistances r_1, r_2, and r_3, together with L, are made into a balanced Wheatstone net, using a cell and galvanometer in the usual way. In parallel with r_1 is placed a condenser whose capacity, K, is known. During the process of balancing the resistances for steady currents, care is, of course, taken to ensure that the current is established before the galvanometer key is depressed to test the balance. The experiment may now be completed either with the aid of the galvanometer across BD and the cell connected through a switch to AC, or by means of a telephone ear-piece as detector and an alternating E.M.F. applied from a hummer.

If the galvanometer is retained, the current is made and broken whilst the galvanometer is permanently in the circuit, and r_2 and r_3 are adjusted until no deflection is obtained in the galvanometer.

If the alternating E.M.F. from a hummer is used, the resistances are adjusted until no noise is heard in the telephone ear-piece.

In either case the steady current balance will, in general, be upset. This must be again obtained and then the test repeated, until finally the balance is equally good for steady and varying currents.

When the network is balanced for steady and alternating currents the relation between the four 'quasi-resistances' may be applied.

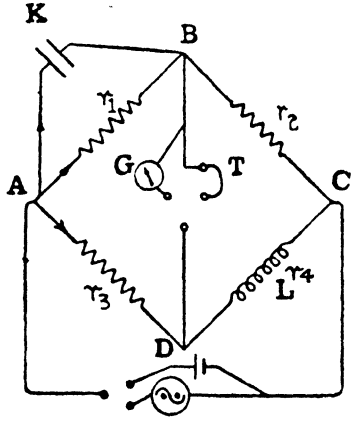


FIG. 356

Since the arm, AB, consists of $-\frac{i}{p\mathrm{K}}$ and r_1 in parallel, the effective 'quasi-resistance' is

$$\frac{r_1}{1 + ip\mathrm{K}r_1},$$

whence:

$$\frac{r_1}{r_2(1 + ip\mathrm{K}r_1)} = \frac{r_3}{r_4 + \mathrm{L}ip},$$

where r_4 is the total ohmic resistance in the arm, DC.

Equating real quantities in the above equation:

$$r_2r_3 = r_1r_4, \text{ or } \frac{r_1}{r_2} = \frac{r_3}{r_4}, \quad \ldots\ldots\ldots\ldots (12)$$

a condition which was experimentally established in the steady current balance.

Equating imaginary terms:

$$r_2r_3p\mathrm{K}r_1 = r_1\mathrm{L}p,$$

or $$\mathrm{L} = r_2r_3\mathrm{K},$$

or from (12) above $$= r_1r_4\mathrm{K},$$

from which L may be calculated.

(4) Rimington's Method

A method due to Rimington enables a determination of L to be made in terms of a known capacity, making use of a modification of the Maxwell arrangement. The self-inductance is placed

in the arm, DC (fig 357). The capacity, K, is placed in parallel with a variable portion, r, of the resistance r_1 (AB). The network is balanced for steady currents, and then the point, E, is chosen such that when the battery circuit is made and broken no deflection is produced in the galvanometer. This method has the advantage that in the second balance the adjustment of the resistances, r_1, r_2, r_3 and r_4, is not disturbed.

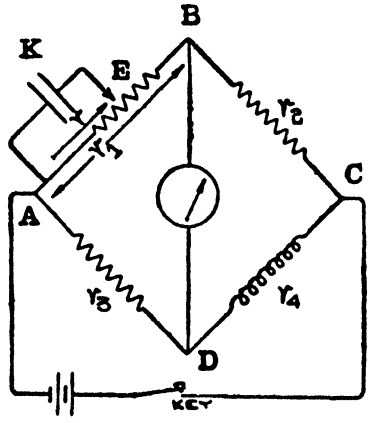


FIG. 357

When the above conditions are fulfilled we have the following relation between the physical quantities involved.

$$L = K\frac{r^2 r_4}{r_1} \text{ henrys.}$$

Fig. 357 shows the arrangement of apparatus. For this experiment it is not possible to use a telephone and alternating current, as will be seen by working out the case by the general method given; the conditions required for balance with such currents reduced the method to Maxwell's, i.e. $r_1 - r = 0$.

To establish the formula given above for the case of a balance with intermittent direct current, let c_1 be the steady current in AB (resistance, r_1) when the key is closed; c_4 the steady current in DC. Then when the circuit is broken, the quantity of electricity which passes through the inductance is $\frac{Lc_4}{R}$ i.e. $\int_{c_4}^{0} -\frac{L dc}{R}$, where R is the resistance between the points D and C (r_4) together with the resistance of the rest of the net, i.e.

$$R = r_4 + r_2 + \frac{G(r_3 + r_1)}{G + r_3 + r_1},$$

where G is the resistance of the galvanometer.

The fraction of this quantity which passes through the galvanometer is $\frac{r_1 + r_3}{r_1 + r_3 + G}$ of $\frac{Lc_4}{R}$,

i.e. is $$\frac{Lc_4(r_1 + r_3)}{\left\{r_2 + r_4 + \frac{G(r_3 + r_1)}{G + r_3 + r_1}\right\}(r_1 + r_3 + G)},$$

which reduces to

$$\frac{Lc_4 r_1}{r_2(r_1 + r_3) + G(r_1 + r_2)} \quad \ldots\ldots\ldots\ldots\ldots (13)$$

Also when the circuit is broken, the condenser discharges a quantity of electricity through the galvanometer, in the opposite direction to that due to the inductance.

Now the quantity of electricity on the condenser due to the steady current, c_1, in AB is $K(rc_1)$.

The part of this quantity passing through the resistances other than r is:

$$Krc_1 \cdot \frac{r}{r_1 + r_3 + \frac{G(r_2 + r_4)}{G + r_2 + r_4}}.$$

Of this amount, that passing through the galvanometer is:

$$\frac{Kr^2c_1}{r_1 + r_3 + \frac{G(r_3 + r_4)}{G + r_2 + r_4}} \times \frac{(r_2 + r_4)}{(G + r_2 + r_4)}.$$

This reduces to

$$\frac{Kr^2c_1 r_2}{r_2(r_1 + r_3) + G(r_1 + r_2)} \quad \ldots\ldots\ldots\ldots\ldots (14)$$

when substitutions are made using the fact that $r_1 r_4 = r_2 r_3$.

When the *quantities* represented by (13) and (14) are equal and opposite, the galvanometer will be unaffected by making or breaking the circuit by means of the key shown in the figure, i.e.

$$Kr^2c_1 r_2 = Lc_4 r_1,$$

or

$$L = \frac{Kr^2 r_2}{r_1} \cdot \frac{c_1}{c_4}.$$

We may further express $\frac{c_1}{c_4}$ in terms of the resistance for when the currents have acquired the steady values, c_1 and c_4, we have:

$$\frac{c_1}{c_4} = \frac{r_3 + r_4}{r_1 + r_2},$$

also

$$\frac{r_3}{r_1} = \frac{r_4}{r_2} = \frac{r_3 + r_4}{r_1 + r_2}, \text{ i.e. } \frac{c_1}{c_4} = \frac{r_4}{r_2}.$$

Hence

$$L = \frac{Kr^2 r_4}{r_1}$$

Experimental Details

The chief disadvantage to this experiment is that much time may very easily be spent in a fruitless effort to determine the value of an inductance, unless the order of suitable resistances, etc., to balance the network, is known. If, however, the approximate magnitude of L is known we may, as in the following example, obtain this information.

Example

An inductance (of 98·15 ohms resistance) is to be evaluated, using a capacity, one micro-farad.

Now from above we have :

$$L = K \cdot \frac{r^2 r_4}{r_1},$$

and the inductance is known to be of the order of one henry, i.e. r_4 is of the order 10^2 ohms.,

K is of the order 10^{-6} farad ;

so $$r_1 = \frac{10^{-6}}{1} \times 10^2 r^2 = 10^{-4} r^2.$$

r is a fraction of r_1, or $r = ar_1$, where a is not greater than 1,

i.e. $$\frac{r_1}{a^2 r_1^2} = 10^{-4},$$

or $$a^2 = \frac{10^4}{r_1};$$

or since $$a \not> 1$$

$$r_1 \not< 10^4.$$

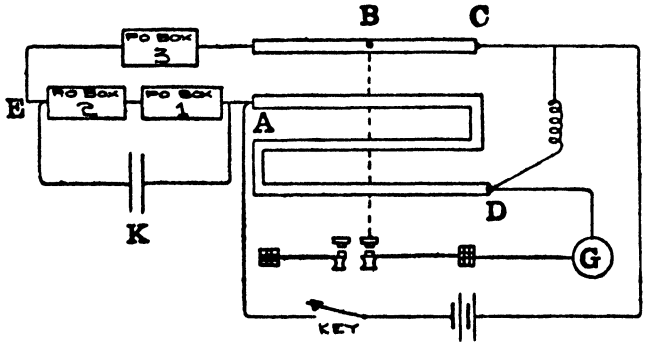


FIG. 358

The arm, AB, of the figure could be therefore two Post Office boxes in series. To obtain a subdivision as at E, however, it will be convenient in this case to use three Post Office boxes, or boxes having a similar range of resistances, for this arm of the net, as in fig. 358. The condenser, K, is placed in parallel with boxes 1 and 2, as shown.

The balance for steady currents having been obtained, the value of r is varied by taking out plugs from 1 and 2, and inserting the same value in 3, thus keeping r_1 constant and of the value required for a steady current balance, until r is finally obtained, such that the potential drop along AE is such as to cause a current through the galvanometer equal and opposite to that due to the inductance. Values of r too small will be shown by a movement of the galvanometer to the one side due to the inductance effect predominating, whereas when c is too large the capacity effect will be large and cause a deflection in the opposite direction.

This process was employed to find the value of L for the coil mentioned above, and the following values obtained,

$r_1 = 20000$ $\quad r_3 = 1963,$ $\quad r = 14500$
$r_2 = 1000,$ $\quad r_4 = 98{\cdot}15,$ $\quad K = 1$ micro-farad,

whence $L = \dfrac{10^{-6} \times 1{\cdot}45^2 \times 10^8 \times 98{\cdot}15}{20000} = 1{\cdot}03$ henrys.

The self-inductance was a nominal 1 henry.

(5) Anderson's Method

The scheme of connexions shown in fig. 359 illustrates a very convenient method of finding the value of an inductance in terms of capacity and resistance.

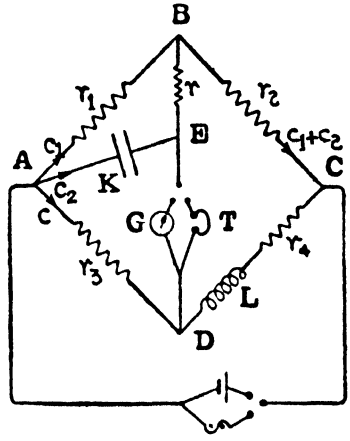


FIG. 359

The Wheatstone net, r_1, r_2, r_3, r_4 (r_4 being the *total* resistance of the arm, DC, including the self-inductance) is arranged as shown. The condenser, K, is connected from A to a variable point, E, in a resistance r (BE).

The network is first balanced for steady currents using a battery between AC and a galvanometer, G, between ED. For such steady currents K and r do not have any effect. When a balance is obtained, by first depressing the battery key and then the galvanometer key, we have :

$$\frac{r_1}{r_2} = \frac{r_3}{r_4}.$$

The network must now be balanced for varying currents. This may be done in one or two ways : either using the galvanometer and an intermittent direct current, or an alternating current and telephone receiver may be used. If the galvanometer is used as a detector, a tapping key is inserted in the battery circuit and r is so adjusted for the particular value of K, that on making and breaking the circuit no deflection of the galvanometer is produced.

When this condition holds, it may be shown, by the application of Kirchoff's Laws to the circuits, ABEDA, ABCDA, that

$$L = K\{r_2 r_3 + r(r_3 + r_4).\}$$

The more convenient way of proceeding with the experiment, however, is to replace the galvanometer by a telephone ; and the battery by a source of alternating current. In the general case when these changes have been carried out to conform with fig. 359 a note will be heard in the ear-piece when AE is closed.

By adjusting r (the resistance BE) this note may be reduced in intensity to zero.

We may readily find the relation which exists between the various resistances, L and K, under such circumstances. The condition satisfied is that the potential drop between A and E is the same *at all times* as the potential drop between A and D.

Let the currents in the arm, AB be c_1 ; in AE, c_2 ; and in AC, c. Then, when no current passes in either direction along ED, the magnitude of the current along BC is $c_1 + c_2$. Consider the circuit ADC, and let the applied potential be taken as the real part of $E_0 e^{ipt}$. The potential difference between A and D is $r_3 c$, where c is the solution of

$$E_0 e^{ipt} = r_3 c + L\frac{dc}{dt} + r_4 c.$$

Putting $c = Ae^{ipt}$ in the above equation we have :

$$A = \frac{E_0}{(r_3 + r_4) + Lip};$$

whence

$$c = \frac{E_0 e^{ipt}}{(r_3 + r_4) + Lip},$$

and potential difference between A and D is therefore

$$\frac{E_0 e^{ipt}}{r_3 + r_4 + Lip} \cdot r_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15$$

Now let Q be the charge on the condenser, K, at any instant. From a consideration of the circuit, ABC, we have :

$$E_0 e^{ipt} = r_1 c_1 + r_2(c_1 + c_2) \quad \ldots\ldots\ldots\ldots (16)$$

But the potential drop along AB is the same as the drop along AEB, or

$$c_1 r_1 = \frac{Q}{K} + c_2 r \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Substituting for c_1 (17) in (16) and putting $c_2 = \frac{dQ}{dt}$, we have :

$$E_0 e^{ipt} = \frac{dQ}{dt}\left(r_2 + \frac{rr_2}{r_1} + r\right) + Q\left(1 + \frac{r_2}{r_1}\right)\frac{1}{K},$$

whence as before :

$$Q = \frac{E_0 e^{ipt}}{\frac{1}{K}\left(1 + \frac{r_2}{r_1}\right) + ip\left(r_2 + \frac{rr_2}{r_1} + r\right)},$$

from which the potential difference between A and $E\left(\frac{Q}{K}\right)$ is

$$\frac{E_0 e^{ipt}}{\left(1 + \frac{r_2}{r_1}\right) + Kip\left(r_2 + \frac{rr_2}{r_1} + r\right)} \quad \ldots\ldots\ldots (18)$$

The condition specified above for no current through the telephone is that (15) and (18) are the same for all values of t.

Equating these expressions we have :

$$(r_3 + r_4 + Lip)\, E_0 e^{ipt} = r_3\left\{\left(1 + \frac{r_2}{r_1}\right) + \frac{Kip}{r_1}(r_1 r_2 + rr_1 + rr_2)\right\} E_0 e^{pt},$$

whence equating the real parts we have :

$$r_3 + r_4 = r_3\left(\frac{r_1 + r_2}{r_1}\right),$$

or

$$\frac{r_1}{r_2} = \frac{r_3}{r_4},$$

a condition which is assumed by the initial balance for steady currents.

Equating the imaginary quantities we have :

$$Lp = \frac{r_3}{r_1} pK(r_1 r_2 + rr_1 + rr_2),$$

or

$$L = K\{r_2 r_3 + r(r_3 + r_4)\} \quad \ldots\ldots\ldots (19)$$

This result could have been obtained using the quasi-resistance method already outlined. For we may consider the circuit taken

above and write down the complete 'Ohm's Law Equation' and find cc_1 and c_2 and hence the potential drops, AE and AD, and equate.

Thus, for circuit ADC:

$$E_0 e^{ipt} = (r_3 + r_4)\, c + Lipc.$$

Whence $$c = \frac{E_0 e^{ipt}}{r_3 + r_4 + Lip}$$ and

$$\text{Potential drop, AD} = \frac{r_3 E_0 e^{ipt}}{r_3 + r_4 + Lip} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Circuit ABC:

$$E_0 e^{ipt} = c_1 r_1 = (c_1 + c_2) r_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

By Kirchoff's Law since the E.M.F. in circuit, AEBA, is zero,

$$0 = c_1 r_1 - c_2 r - \left(-\frac{i}{pK}\right) c_2,$$

or $$G = \frac{c_2}{r_1}\left(r - \frac{i}{pK}\right).$$

Substituting in (21) above,

$$E_0 e^{ipt} = c_2\left[\left(r - \frac{i}{pK}\right) + r_2 \left\{1 + \frac{1}{r_1}\left(r - \frac{i}{pK}\right)\right\}\right],$$

whence $$c_2 = \frac{E_0 e^{ipt}}{r + r_2 + \frac{r_2 r}{r_1} - \frac{i}{pK}\left(1 + \frac{r_2}{r_1}\right)};$$

and since $\frac{dc_2}{dt} = Q$, we have, integrating, $Q = \frac{c_2}{ip}$, and the potential between AE, which is $\frac{Q}{K}$, is

$$\frac{E_0 e^{ipt}}{Kip\left\{r + r_2 + \frac{r_2 r}{r_1} - \frac{i}{pK}\left(1 + \frac{r_2}{r_1}\right)\right\}} \quad \ldots\ldots\ldots\ldots (22)$$

Equate to the value given in equation (20), we have, equating the real quantities:

$$\frac{r_1}{r_2} = \frac{r_3}{r_4},$$

and equating the imaginaries,

$$L = K\{r_2 r_3 + r(r_3 + r_4)\}.$$

(6) Owen's Method

The following method of finding the value of the self-inductance of a coil was described by Dr. D. Owen in the "Proceedings of the London Physical Society," 1914–15, vol. xxvii.

The coil whose inductance, L, is to be found is placed in one arm of a Wheatstone network, in series with a variable non-inductive resistance, so that the total resistance is r_2 ohms. The scheme of connexions is shown in the diagram (fig. 360).

K_1 is a standard capacity, K_2 a second capacity, and R an adjustable non-inductive resistance.

The bridge is first balanced for steady currents. To do this, the battery shown in fig. 360 is thrown in circuit, and r_1 and the adjustable portion of BC are adjusted so that no deflection is produced when the galvanometer key is closed (as in de Sauty's method). Under such circumstances

$$\frac{r_1}{r_2} = \frac{K_2}{K_1}.$$

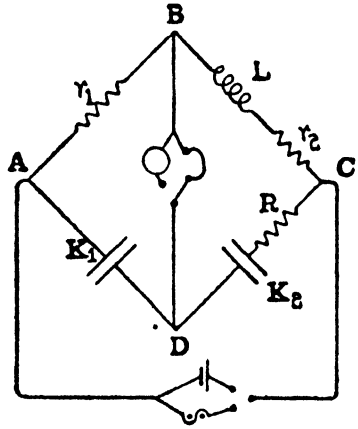


FIG. 360

The experiment may now proceed, as in Anderson's method, in either of two ways. The galvanometer may be retained and the battery circuit made and broken, and R adjusted so that the galvanometer which is permanently connected to BD shows no deflection for such current changes. Or the galvanometer may be replaced by a telephone ear-piece and the bridge supplied with an alternating current from a 'buzzer.' R is then adjusted until the hum in the ear-piece is reduced to zero.

As in Anderson's method, the two balance positions described are easily obtained, and the second balance does not upset the first one.

Now it was shown on page 561, as an example of the 'quasi-resistance' method of solving such problems, that when such a balance is obtained

$$L = K_1 r_1 R = K_2 r_2 R.$$

As a contrast with the method given on that page, we may show that the above expression gives the relation between the capacity and resistances used, from a fuller consideration. For let the applied E.M.F. be $E_0 e^{ipt}$; and, since no current flows through the galvanometer in the balanced state, let c_1 be the current in ABC.

For the circuit ABC we have :

$$E_0 e^{ipt} = c_1(r_1 + r_2) + L\frac{dc_1}{dt};$$

hence, putting $c_1 = Ae^{ipt}$ and substituting to find A, we have :

$$c_1 = \frac{E_0 e^{ipt}}{r_1 + r_2 + Lip}.$$

The potential difference between A and B is $r_1 c_1$ or

$$\frac{r_1 E_0 e^{ipt}}{r_1 + r_2 + Lip} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

Now, if c_2 is the current in ADC, we have for that circuit :

$$E_0 e^{ipt} = Rc_2 + Q\left(\frac{1}{K_1} + \frac{1}{K_2}\right),$$

or, since $\frac{dQ}{dt} = c_2$,

$$E_0 e^{ipt} = R\frac{dQ}{dt} + \left(\frac{1}{K_1} + \frac{1}{K_2}\right)Q.$$

Hence putting $Q = Be^{ipt}$ and $\frac{dQ}{dt} = Bipe^{ipt}$ we find B, which gives the following :

$$Q = \frac{E_0 e^{ipt}}{Rip + \left(\frac{1}{K_1} + \frac{1}{K_2}\right)}.$$

The potential difference, at any time, t, between A and D, is therefore $\frac{Q}{K_1}$

$$= \frac{1}{K_1} \cdot \frac{E_0 e^{ipt}}{\left\{Rip + \left(\frac{1}{K_1} + \frac{1}{K_2}\right)\right\}}.$$

The condition that there is no sound in the telephone is that at all times, t, the potential at B = potential at D, i.e. for all values of t :

$$\frac{r_1 E_0 e^{ipt}}{(r_1 + r_2) + Lip} = \frac{E_0 e^{ipt}}{K_1\left\{Rip + \left(\frac{1}{K_1} + \frac{1}{K_2}\right)\right\}},$$

i.e. $$r_1 K_1\left\{Rip + \left(\frac{1}{K_1} + \frac{1}{K_2}\right)\right\} = (r_1 + r_2) + Lip.$$

Equating real quantities :

$$r_1 K_1\left(\frac{1}{K_1} + \frac{1}{K_2}\right) = r_1 + r_2,$$

$$\frac{K_1 + K_2}{K_2} = \frac{r_1 + r_2}{r_1},$$

$$\frac{K_1}{K_2} = \frac{r_2}{r_1},$$

or $$r_1 K_1 = r_2 K_2$$

Equating imaginary quantities:

$$r_1 K_1 R i p = L i p,$$

or $$L = r_1 K_1 R \text{ or } r_2 K_2 R$$

MUTUAL INDUCTANCE

Direct Measurement with a Ballistic Galvanometer

The coefficient of mutual induction of two coils may be defined as the number of lines of magnetic force which pass through one coil when unit current circulates through the other.

Thus, if a current of maximum strength, c_0, passes through one of the coils, whose mutual induction is M, Mc_0 lines of magnetic force thread the second; and whilst the current grows in the primary, the number of lines of magnetic force threading the secondary is changing. Therefore an induced E.M.F. is set up in the secondary during the time of growth of the primary current. This E.M.F. is numerically equal to the rate of change of the number of lines of magnetic force in the secondary, i.e. $= \frac{d}{dt}(Mc)$ where c is the instantaneous current in the primary during the growth of that current.

If L is the coefficient of self-inductance of the secondary coil, and c^1 is the current in the secondary corresponding to c in the primary, we have a further E.M.F. in the secondary due to the self-inductance numerically equal to $\frac{L dc^1}{dt}$, i.e. if R is the total resistance of the secondary coil circuit, neglecting signs,

$$Rc^1 = L\frac{dc^1}{dt} + M\frac{dc}{dt}.$$

Now Q, the quantity of electricity passing through the secondary, is $\int c^1 dt$ where the integration is carried out over the whole time during which c rises to the steady value c_0.

$$Q = \int c^1 dt = \int \frac{L}{R} dc^1 + \int \frac{M}{R} dc.$$

The value of c^1 at the commencement and the end of this integration is zero, hence :

$$\int \frac{L}{R} dc^1 = 0 \text{ and}$$

$$Q = \int_0^{c_0} \frac{M}{R} \cdot dc = \frac{Mc_0}{R}.$$

If, therefore, the second coil is connected to a ballistic galvanometer and the throw is θ, due to the passage of this quantity of electricity, M may be calculated in terms of R, c_0 and the constants of the galvanometer, and θ.

In this case it will be well to calibrate the galvanometer in the circuit since R includes the resistance of the galvanometer and the second coil.

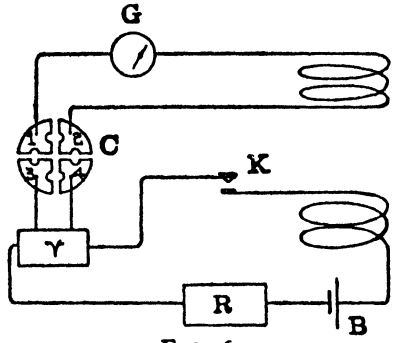


FIG 361.

Fig. 361 shows a convenient disposition of apparatus to carry out a direct measurement of M on these lines.

The current in the primary coil may be regulated to a suitable value by means of the resistance R. In series with R is a small resistance r ($\frac{1}{10}$ or $\frac{1}{100}$ Ω). C is a four-segment commutator, or may be a switch consisting of four mercury filled holes in a block of paraffin wax.

If 1 and 2 are connected together the ballistic galvanometer is in direct circuit with the secondary coil, and when K is depressed a deflection, θ, is obtained for the establishing of the steady current, c_0, in the primary. If a moving-needle ballistic galvanometer is used, we have :

$$Q = \frac{T}{\pi} \cdot \frac{H}{G} \sin \frac{\theta_0}{2} = \frac{Mc_0}{R}, \quad \ldots\ldots\ldots\ldots (25)$$

where θ_0 is the value of the first deflection corrected for damping, i.e. $\theta_0 = \theta\left(1 + \frac{\lambda}{2}\right)$ where λ is the logarithmic decrement and θ is the observed throw.*

* If a moving-coil instrument is used, the form of double key, described on page 481, should be used ; K is replaced by the lower pair of contacts in such a key and the upper pair act as a key in the galvanometer circuit.

If now C is arranged so that connexion is made between 1 and 3, 2 and 4 only, and a steady current, c_0, is passed in the primary circuit, the potential drop established at the ends of r is $c_0 r$, i.e. the current through the galvanometer is $\frac{c_0 r}{R}$, since r is very small compared with the resistance of the galvanometer. If this causes a steady deflection, θ_1, we have :

$$G \cdot \frac{c_0 r}{R} = H \tan \theta_1, \quad \ldots\ldots\ldots\ldots\ldots (26)$$

as shown on page 481.

Combining (25) and (26), we obtain the following value for M :

$$M = \frac{T}{\pi} \cdot r \cdot \frac{\sin \frac{\theta_0}{2}}{\tan \theta_1}.$$

If a moving-coil galvanometer must be used, since $Q = \frac{T}{\pi} \cdot \frac{\tau}{G} \cdot \frac{\theta_0}{2}$ the value for M becomes :

$$M = \frac{T}{\pi} \cdot \frac{r}{2} \cdot \frac{\theta_0}{\theta_1}.$$

It will be noted that the value of R is eliminated : r may well be a standard $\frac{1}{10}$ or $\frac{1}{100}$ ohm. T is obtained in the usual way by timing 20 or 30 swings, under the conditions of damping which obtain during the observations above ; $\theta_0 = \theta\left(1 + \frac{\lambda}{2}\right)$ The value of λ may be obtained by one of the methods given on page 480.

Comparisons of Mutual and Self-Inductance. (Maxwell's Method)

The one coil of self-inductance, L, is arranged in one arm of a Wheatstone network ; the other coil of the mutual inductance is connected, as shown in fig. 362, in series with the battery. For this method the resistance, r (AC), is not present. The coils are arranged by trial so that the *self and mutual induction effects in the arm* AB *are opposed.*

The resistances, r_1, r_2, r_3, r_4, are balanced for steady currents. The battery circuit is then made and broken, the galvanometer circuit being closed. r_2, r_3, r_4 are adjusted until the minimum effect is produced in the galvanometer for such current changes. The steady balance is again tested. By repetition the values r_1, r_3, r_4, may be finally arranged such that for steady and varying currents the galvanometer remains unaffected.

The steady current balance gives in the usual way the relations between the resistances,

i.e. $$\frac{r_1}{r_2} = \frac{r_3}{r_4}.$$

During the fall of the current at break, the self-inductance effect is an E.M.F. $L\frac{dc_1}{dt}$, where c_1 is the current in the arm ABC.

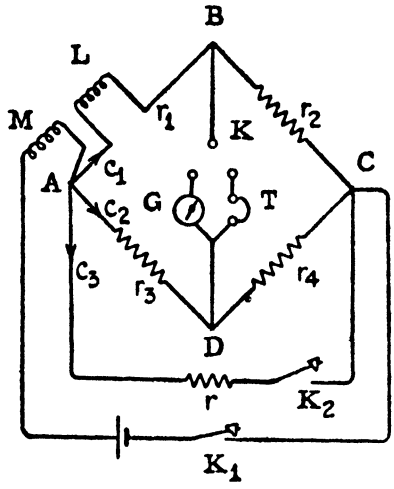


FIG. 362

In the opposite direction an E.M.F., $M\frac{dc}{dt}$, is set up in the same arm due to the mutual inductance. If these two E.M.F.'s are equal and opposite,

$$L\frac{dc_1}{dt} = M\frac{dc}{dt} \text{ (numerically)},$$

where $c = c_1 + c_2$,

$$Lc_1 = M(c_1 + c_2),$$

$$\frac{L}{M} = \frac{c_1 + c_2}{c_1}$$

$$= 1 + \frac{c_2}{c_1}$$

$$= 1 + \frac{r_1}{r_3}$$

$$= 1 + \frac{r_2}{r_4}.$$

It is apparent from the above equation that the experiment may only be performed if L is larger than M.

This method is open to the same criticism as the Maxwell and comparison methods of finding L, and is therefore seldom used.

A modification of this method is obtained by the insertion of the arm, AC, of resistance, r, as shown in fig. 362.

In this case the variable resistance, r, is placed in series with a key between A and C.

With K_2 *open* a steady current balance is first obtained. This having been accomplished, the resistances, r_1, r_2, r_3, r_4, are fixed; K_2 is now closed and r adjusted until the galvanometer is unaffected by intermittent current in the network, e.g. when the current is made and broken or reversed.

As before, no resultant E.M.F. is set up in AB during such changes of current, otherwise a current proportional to such E.M.F. would pass through the galvanometer. Thus we have again:

$$L\frac{dc_1}{dt} = M\frac{dc}{dt} \text{ (numerically)},$$

where c is the current passing through the coil M, i.e. is $(c_1 + c_2 + c_3)$.

Integrating as before, we have:

$$Lc_1 = Mc,$$

i.e.

$$\frac{L}{M} = \frac{c}{c_1}.$$

Now if R is the resistance of the whole net between A and C, we have:

$$cR = c_1(r_1 + r_2)$$

or

$$\frac{L}{M} = \frac{r_1 + r_2}{R},$$

$$= \frac{r(r_1 + r_2) + r(r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)}{r(r_3 + r_4)},$$

and

$$\frac{r_1}{r_2} = \frac{r_3}{r_4};$$

$$\therefore \frac{L}{M} = \frac{r\left\{1 + \frac{r_1}{r_3}\right\} + (r_1 + r_2)r_3}{r},$$

$$\frac{L}{M} = \frac{r(r_1 + r_3) + (r_1 + r_2)}{rr_3}.$$

This arrangement of resistance eliminates the troublesome double balancing of the last method. The steady balance is obtained with a cell and galvanometer in the usual way. The battery circuit is then made and broken or reversed; r is

adjusted until the galvanometer shows no deflection during that process. As before, care must be taken that the coils are so connected that the effect of the self and mutual induction oppose each other.

Alternatively a hummer may be used as a source of alternating E.M.F. and a telephone as detector, instead of making and breaking the battery circuit.

If the self-inductance is less than the mutual inductance a modification of the method above may be used. In this case the battery and the resistance, r, are interchanged, i.e. M is in series with r.

For balance, the effect of the induced E.M.F. in the arm, AB, due to L and M, must neutralize each other. As before, equating the *numerical* values of such E.M.F.'s we have, if c_3 is the current through r and M,

$$L\frac{dc_1}{dt} = M\frac{dc_3}{dt},$$

$$Lc_1 = Mc_3, \text{ and } c_1(r_1 + r_2) = c_3(r + x),$$

i.e.
$$\frac{M}{L} = \frac{c_1}{c_3} = \frac{r + x}{r_1 + r_2},$$

where x is the resistance of the coil, M. This method, however, cannot be performed using alternating current and telephone.

Determination of Mutual Inductance by the Fluxmeter

The mutual inductance of two coils may be very readily obtained, using a fluxmeter (which is described on page 482).

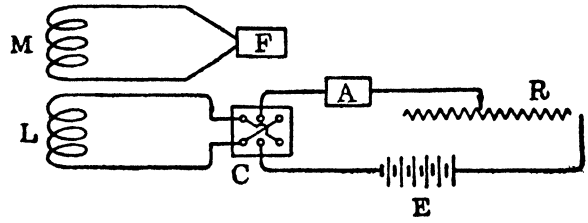


FIG. 363

One coil, M, is connected directly to the fluxmeter, F, fig. 363. The other coil, L, is connected through a commutator, C, to a circuit consisting of an accumulator, E, an ammeter, A, and a variable resistance, R.

The current in L is adjusted to some convenient amount, c amperes, and then reversed, causing a deflection of x divisions in the fluxmeter.

If each division of the fluxmeter (as usual) corresponds to 10000 maxwells, the change in the number of maxwells in the secondary, M, due to a reversal of c amperes in L is $10000x$, i.e. for $\frac{c}{10}$ E.M. units of current in L there are $5000x$ lines of magnetic force threading M.

Hence, since the coefficient of mutual inductance is defined as the number of lines in the one coil for unit current in the other, it has a value,

$$\frac{50000x}{c},$$

for the coils used.

The experiment is repeated using various values of c, and the mean value of $\frac{x}{c}$ is obtained, and hence the coefficient of mutual inductance in E.M. units may be calculated.

The mutual inductance in henrys is 10^{-9} times the above value, for two coils are said to have a coefficient of mutual inductance of 1 henry when a current change of 1 ampere per second in the primary causes an E.M.F. of 1 volt in the secondary, i.e. a flux of 10^8 lines for a current of 10^{-1} E.M.U., i.e. the henry is 10^9 E.M. units.

CHAPTER XXIII

THE QUADRANT ELECTROMETER

A MODERN development of the Kelvin Quadrant Electrometer is seen in fig. 364, and is due to Dr. F. Dolezalek. The four quadrants, QQ, etc. are supported on ambroid pillars, AA. As shown in the figure, two quadrants are mounted on a pivot, and may be swung aside to allow of the introduction of the 'needle.' Alternate quadrants are joined together to terminals

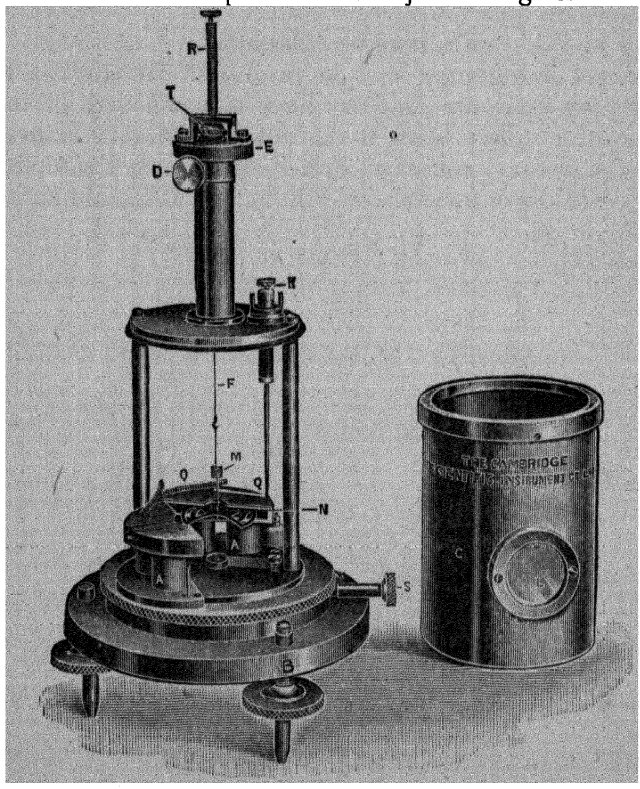


FIG. 364

under the base plate of the instrument, care being taken that such terminals are very well insulated from the case.

The needle may be either a light paper frame (fig. 365), coated with a metal to make it conducting, or as in fig. 366, a thin

mica sheet spluttered with silver. In some of the more recent instruments a very thin aluminium needle of the shape of the second form is used. The needle is attached to a light rod which carries a small mirror, M, and is supported by means of a thin quartz or phosphor-bronze strip from the torsion head, R.

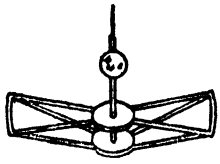


FIG. 365

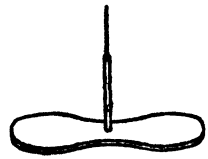


FIG. 366

The method of observing the deflection produced by a difference of potential on the quadrant consists of the use of the usual lamp and scale, as with galvanometers.

A beam of light from a lamp is directed on the mirror, M, which reflects the light on to a scale placed one metre away. If the mirror is concave, and of the correct focal length, a clearly defined image of the source will be obtained on the scale. If M is a plane mirror, a lens is required to produce a sharp image. Such a lens is placed in the path of the incident beam from a source of light below the scale; the reflected beam does not, therefore, pass through the lens.

The Suspension

(1) *Quartz Fibre*

If a quartz fibre is used as a suspension, the needle may be charged by means of the charging device shown at K. K is connected to the source of potential and the metal rod turned until contact is made between it and the rod which supports the mirror on the needle. K is then turned to its original position leaving the needle charged. Due to the very good insulating property of the quartz, this charge will be maintained. However, there is a danger of breaking the quartz suspension during this process.

For most purposes it is more convenient to avoid using the charging device. To make this possible the suspension is coated lightly with a calcium chloride solution. The hygroscopic properties of the latter ensures a conductivity which is sufficiently good to maintain the needle at the potential of the source which is permanently connected to R.

An alternative method is to splutter the quartz with silver, but this cannot as a rule be carried out conveniently in the laboratory.

Suitable quartz fibres are very simply made by the aid of a coal gas-oxygen flame. A burner for this mixture is shown in fig. 367. A piece of quartz rod is heated in the flame until thoroughly soft; when at this stage the ends are drawn apart. If this is done rapidly a fine quartz thread will result. If a still finer fibre is required draw out the quartz rod to about 1 mm. diameter, and then reintroduce into the flame. When the quartz becomes very soft the pressure in the flame itself will blow the quartz outwards into very fine fibres. It is advisable to have a black velvet cloth on the bench to receive these threads.

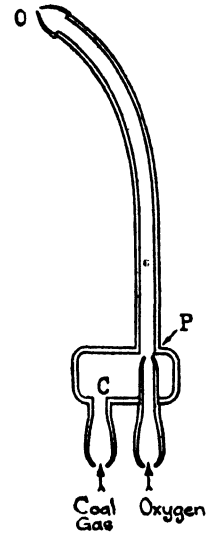


FIG. 367

Small hooks may be fastened to the ends of the fibre, when cut to the correct length, by means of a small globule of shellac (solution in methylated spirits); a hot iron is held over the globule, which is placed on the hook over the end of the quartz. The spirit evaporates and the fibre adheres.

Another method of fastening the hook to the fibre is to use indian ink. The end of the hook is dipped into indian ink which is allowed to become 'thick' by evaporating. The fibre is then placed on this plastic drop; when dry the two will be found to be very firmly held.

(2) *Phosphor-bronze Strip*

Of the metal suspensions phosphor-bronze strip is most satisfactory. Platinum and tungsten wires may be obtained with a smaller diameter, but usually have the disadvantage that the zero of the needle is not stable when very fine suspensions of these metals are used.

For the experiments described in this chapter, the finer phosphor-bronze strip obtainable commercially gives a sensitivity which satisfies the needs of the experiment. If, however, the phosphor-bronze strip available is too thick for the purpose, it may be treated as described below; which method enables a fair sensitivity to be attained.

A solution of one part nitric acid to four parts of water is taken, and the suspension immersed in it. The action of the acid causes an evolution of gas which adheres to the suspension, causing it in time to rise to the surface. The chemical action should be slow. If violent action takes place the solution should be further diluted. Of course, a preliminary test should be made with a small sample of the phosphor-bronze to be used in order to avoid undue waste.

The strip should be very well washed and dried, and its sensitivity tested in the instrument. The process is repeated until the required sensitivity is obtained. When this process has once been performed with the specimen of strip available, the time of immersion in the acid required to reduce cross-section to a suitable value may be very readily estimated, and the time spent in testing will be reduced, e.g. the strip may be sufficiently reduced by immersing four times in the acid, allowing it to come to the surface each time.

In this way a sensitivity of 700 to 1000 cms. per volt may be obtained with the instrument shown, but for the experiment in this chapter a sensitivity of about 25 cms. per volt will be found to be quite sufficient.

The above process is carried out using the wire without hooks. If the hooks are soldered on to the ends, the action of the acid will sever them. The small hooks are then soldered to the ends of the strip, using a very small pointed iron. Care should be taken that the ends and hooks are clean and the iron hot. Use soft solder and a trace of fluxite.

Adjustment

To prepare the instrument for use, the following adjustments must be made. Both pairs of quadrants are connected to earth, and are therefore at the same potential; R is also earth connected. Thus, the only forces acting on the needle are those due to the torsion of the suspension. The needle is raised or lowered by means of the screw, T, until it swings about the mid-plane within the quadrants. By means of the levelling screws, B, the whole instrument is then levelled, so that the suspension hangs centrally within the quadrants. To ensure this, the rod which supports the mirror, M, is sighted along the two diagonal spaces between

the four quadrants and a slight adjustment made if required to bring the rod truly central. The torsion head, E, is then turned until the needle appears symmetrical with respect to the quadrant.

A suitable high potential of, say, 100 volts is applied to the needle (in the manner described later, fig. 368). If it were adjusted to be precisely symmetrical no movement should result on making this change. Any error in the adjustment by eye for symmetry may now be corrected, by noting the direction of deflection of the needle. The potential is removed and the needle earthed. The torsion head is given a very slight turn in the direction of the movement previously observed. The needle is again charged and the process repeated, until on charging and earthing the needle alternately the deflection produced is of the order of a few cms. only. To reduce this small deflection to zero a slight adjustment of a levelling screw will be found sufficient—a portion of a turn of one of the levelling screws will be found to increase or decrease the deflection produced. When the potential is applied to the needle, the screw is turned in the direction required to reduce the deflection until it is finally eliminated.

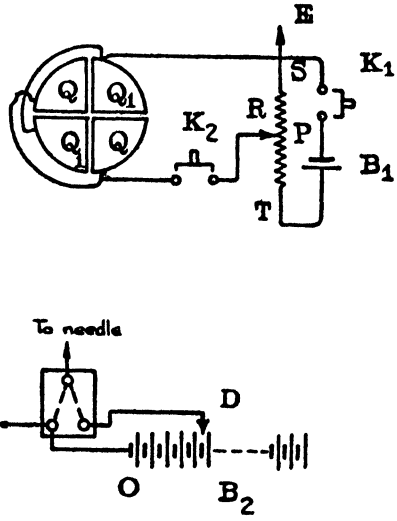


FIG. 368

With a fine suspension the above process is apt to be somewhat tedious until the student becomes familiar with the instrument; but it is essential to the success of any observation.

When this adjustment has been satisfactorily made the instrument should be tested for leak. To do this the arrangement of apparatus shown in fig. 368 is a convenient one, for by means

of this the adjustment already described may be also performed. Thus, when the two-way switch* is to the left, the needle is at zero potential: and if K_1* is open and K_2* closed, the quadrants are at zero potential. The needle is charged by closing the two-way switch to the right.

B_1 is a steady 2 volt accumulator; R a high resistance, provided with a slide contact, or two resistance boxes in series. Either arrangement serves as a potentiometer by means of which any fraction of the E.M.F. of B_1 may be obtained between S and P when K_1 is closed. When K_2 is closed this potential difference is applied to the quadrants. The needle, being maintained at a fixed potential, say 100 volts, by the battery, B_2, will be deflected, causing a movement of the spot of light of, say, d cms. K_2 is then opened; if the quadrants are fully insulated no movement will be given by the needle. If on the other hand a decrease in the deflection occurs, it indicates that the quadrants are losing charge, i.e. the insulating supports are faulty. The usual cause for this is dust or grease on the surface of the pillars, which should therefore be cleaned. When this has been carried out as thoroughly as possible the test is again made. In general one cannot remove the leak entirely, but it may be reduced to a very small amount. The rate of movement of the needle, when the one pair of quadrants is charged and, then insulated, is measured by timing, with a stop-clock, the movement of the spot of light. The number of divisions per second is called the *natural leak* of the instrument.

Note

The high potential is connected to the needle of the electrometer through a *water resistance.* This may be a small U-tube filled with water; the wire from the high potential is inserted in one limb, and the lead from the needle into the other. In the case of an accidental contact between the quadrant and the needle, the latter would be ruined, in the absence of a water (high) resistance. The potential on the needle is unaffected by such resistance, but unless measured by an electrostatic voltmeter there will appear to be a fall in potential. This is due to the fact that an ordinary voltmeter is not of a very high resistance compared with the water resistance. If only a moving-coil voltmeter is available to test this potential it should be applied to the point at which the high potential enters the water resistance.

A simple theory of the instrument (see, for example, Wheatham "Electricity and Magnetism") shows that if one pair of quadrants is maintained at a potential, V_1, and the other at a potential V_2, the needle, being maintained at potential V_n, will move,

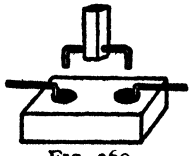


FIG. 369

* These switches are best made by boring small holes in a clean slab of paraffin wax. The holes are filled with clean mercury and connection is made by using a length of copper wire, bent twice at right-angles and mounted on a sealing-wax handle, as in figure 369.

through an angle, θ, which depends on the potential difference, $V_1 - V_2$, and

$$\theta = c(V_1 - V_2)\left(V_n + \frac{V_1 + V_2}{2}\right),$$

where c is a constant.

In general, V_1 and V_2 are small compared with V_n, and we may write :

$$\theta = c(V_1 - V_2)V_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Such a simple theory therefore leads to the conclusion that the deflection for a given potential difference on the quadrants is proportional to the potential of the needle. One would conclude, therefore, that the sensitivity is proportional to this potential. This, in fact, is not so, as may be seen from the following experimental test, using the arrangement of fig. 368.

Adjust the potential difference between P and S to such a value that the electrometer gives about 20 or 30 cms. deflection (K_1 and K_2 being closed) when a potential of about 100 volts is applied to the needle.

Note the deflection produced. Throw the switch over to the left and so earth the needle and reduce the deflection to zero ; when the spot of light is steady take the zero reading. Move D to some other point in the battery, and then put the switch back to the right-hand side ; again read the value of the deflection. Repeat this for a wide range of values of V_n. The value of V_n is, of course, taken for each setting, by means of a voltmeter which is connected to D and earth.

The zero readings between each deflected reading should be the same. Plot a curve showing the relation between the deflection produced for the fixed potential difference on the quadrants, and the potential on the needle. It will be found that the deflection produced increases with V_n, but not indefinitely, i.e. there is a potential above which there is no gain in sensitivity, this is usually between 70 and 120 volts for the type of instrument described.

Maintain the potential of the needle at this value and verify the fact that the deflection is proportional to $(V_1 - V_2)$ by applying different values to the quadrants. By adjusting P on the resistance R, the ratio, $\frac{\text{resistance PS}}{\text{resistance TS}}$, may be chosen to give, say, ·01, ·05, ·1, ·15 volt. For these values the deflections will be found to be as 1 : 5 : 10 : 15.

Thus, for a fixed potential on the needle the deflection is proportional to the difference of potential of the quadrants.

Thus, the equation (1) holds so long as the needle is maintained

at fixed potential, and the instrument may be used to compare potential.

A more complete theoretical treatment of the instrument may be found in " Phil. Mag." 1903, by G. W. Walker, or later (" Phil. Mag." 1912, p. 380) by Prof. A. Anderson. Here the variation of θ with V_n is more correctly stated. This leads to the expression :

$$\theta = \frac{2\alpha V_2\left(V_n - \frac{V_2}{2}\right) - 2\gamma K V^{n2}}{F + K V_n^2},$$

where F is the torsional couple due to the fibre per unit angular displacement, α, K and γ are constants depending on the particular instrument for their value.

Sensitivity

By the sensitivity we understand the number of cms. or mms. deflection obtained on a scale one metre away, when one volt potential difference is applied to opposite pairs of quadrants.

With the same instrument this factor may have vastly different values depending upon the dimensions of the suspending fibre. In most cases, especially in laboratory work, the fibre is conducting and very conveniently made of phosphor-bronze strip. The type of instrument with a fine phosphor-bronze strip may give, say, 60 cms. deflection per volt.

This factor may be found by use of the arrangement of fig. 368, as already set up for the previous experiments.

The needle being raised to the high potential already chosen, K_1 and K_2 are closed and P is adjusted so that a deflection of about 20 or 30 cms. is obtained. The value of the resistance in SP, r ohms say, is noted as is the total resistance, R, of ST.

A standard cell is then connected to S and P through a galvanometer, and the resistance, SP, is adjusted, keeping R the same as before, until no deflection is given in the galvanometer, i.e. the usual method of standardizing the potential is followed. If r_1 is the resistance, SP, under such circumstances, then the potential previously applied to the needle is obviously $\frac{r}{r_1}E$, when E is the E.M.F. of the standard cell.

From this the number of cms. deflection per volt may be calculated.

As already mentioned, a suitable sensitivity is about 25 to 50 divisions per volt. The suspension should be made of suitable size to give this value. The method has already been described.

Comparison of the E.M.F. of Two Cells

The adjusted quadrant electrometer is connected as shown in fig. 370. The needle is raised to the potential already found to be most satisfactory. The three-way paraffin wax switch is made and connected to the batteries B_1 and B_2 by mercury cups, 1 and 3. Cup No. 2 is connected to earth as are the negative poles of B_1 and B_2.

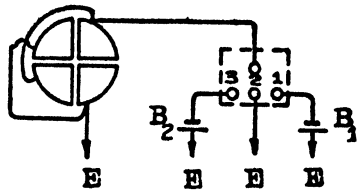


FIG. 370

By connecting to cup 2, all the quadrants are at zero potential. The battery, B_1, is then placed with its positive pole in contact with one pair of quadrants (via 1), the other pair being earthed. The deflection, d_1, is noted. The quadrants are then earthed, and B_2 is connected (via cup 3); the deflection, d_2, due to this also being observed.

Then
$$\frac{E_1}{E_2} = \frac{d_1}{d_2},$$

from the previous investigation, i.e. $(V_1 - V_2) \propto \theta$.

For large deflections d_1 and d_2 cms. must be replaced by the corresponding angles, θ_1 and θ_2; this may be readily done, for

$$\tan(2\theta_1) = \frac{d_1}{100}, \tan(2\theta_2) = \frac{d_2}{100},$$

whence:
$$\frac{E_1}{E_2} = \frac{\theta_1}{\theta_2}.$$

Verification of Ohm's Law

For this purpose a battery, B, an adjustable resistance, R, and a tangent galvanometer are connected in series with a resistance of fixed amount, r ohms, which may very well be a length of manganin or platinoid wire (fig. 371),

The end, C, is connected through a paraffin wax switch, K, to one pair of quadrants of an adjusted electrometer, and the end, D, is connected to earth and the other pair of quadrants.

The current flowing in the wire, CD, may be measured by means of the tangent galvanometer, for if θ is the deflection

produced in this instrument as measured by means of the lamp and scale method, we have :

$$c = \frac{Ha}{2\pi n} \cdot \tan\theta,$$

or

$$c \propto \frac{\tan\theta}{n},$$

where n is the number of turns used.

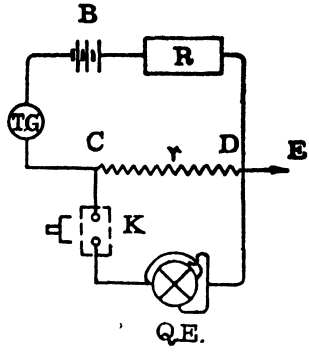


FIG. 371

Further, the potential difference between the ends, C, D, is proportional to φ, the deflection produced in the quadrant electrometer.

If φ is small, the corresponding scale deflections, d, may replace φ below.

For a range of values of c it might become necessary to alter the number of turns, n, used in the galvanometer. Vary R and tabulate the result as under.

$c \propto \frac{\tan\theta}{n}$	$E \propto \varphi$	$\frac{E}{c} \propto \frac{\varphi n}{\tan\theta}$

It will be found, if care has been taken to avoid large currents which would cause appreciable heating in the wire, that the value of $\frac{\varphi n}{\tan\theta}$ is constant.

If the sensitivity of the Q.E. has been found, the value of the relation, $\frac{\text{potential}}{\text{current}}$, may be found absolutely, i.e. for a

sensitivity of d cms. per volt, deflections, φ_1, φ_2, etc., being small, the potential drop along r is $\frac{d_1}{d}$, $\frac{d_2}{d}$, . . volts.

c may be calculated in amperes from the usual tangent galvanometer formula :

$$c = \frac{5\mathrm{H}a}{\pi n} \cdot \tan \theta,$$

where a is the mean radius of the coils, H the horizontal component of the Earth's magnetic field (about ·185 in London).

Whence tabulating absolute values the third column gives the constant value of the resistance, r, *at room temperature* in ohms. This should be tested independently.

Measurement of High Resistance

The following is a method of measuring a high resistance by finding the rate of leak of a charged condenser through the resistance.

A quadrant electrometer is set up and adjusted as previously described, care being taken to reduce the natural leak to a minimum. One pair of quadrants is connected to earth ; the other pair, as shown in fig. 372, is connected to a condenser, K, and through a switch, K_1 (see footnote, page 589), to a 2 volt cell, B, the other pole of which is earth connected. Connexion is also made to a two-way switch, K_2, by means of which this pair of quadrants may be connected to earth or to an earth-connected resistance whose value, R, is to be determined.

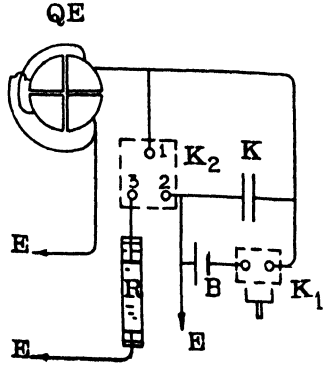


FIG. 372

With K_2 open, close K_1 ; in this way the condenser will be charged to a potential difference equal to that of the cell, B. Note the deflection, d_1, corresponding to an angular deflection, φ_1. This is due to a potential difference of V_1 volts,

say. Now close K_2 by joining 1 and 3; open K_1 and start a stop-watch. The charge on the condenser will slowly leak to earth through R. The potential of the quadrants will in consequence be reduced, i.e. d or φ will gradually decrease. If this process be allowed to continue for about two minutes or until the deflection is reduced to about 60 per cent of its original value (t seconds), the final deflection, d_2 (angular deflection φ_2), will be shown to be such that

$$R = \frac{t}{K \log \frac{\varphi_1}{\varphi_2}}.$$

For let

- Q and V be the charge and potential on the quadrants at any time,
- Q_1 V_1 the corresponding values at the moment K_1 is opened,
- Q_2 V_2 the values after t seconds,
- K the capacity of the condenser,

c, the current at any instant, is equal to the rate of decrease of the charge on the condenser,

or
$$C = -\frac{dQ}{dt}.$$

Also by Ohm's Law $c = \frac{V}{R} = \frac{Q}{KR}$, since $V = \frac{Q}{K}$,

i.e.
$$-\frac{dQ}{dt} = \frac{Q}{KR},$$

or
$$-\frac{dQ}{Q} = \frac{dt}{KR}.$$

Integrating over the limit indicated below

$$\int_{Q=Q_1}^{Q=Q_2} -\frac{dQ}{Q} = \int_0^t \frac{dt}{KR},$$

$$\log \frac{Q_1}{Q_2} = \frac{t}{KR},$$

$$R = \frac{t}{K \log \frac{Q_1}{Q_2}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and since
$$\frac{Q_1}{Q_2} = \frac{V_1}{V_2} = \frac{\varphi_1}{\varphi_2},$$

$$R = \frac{t}{K \log \frac{\varphi_1}{\varphi_2}}.$$

If φ_1 and φ_2 are small we may approximate and write :

$$R = \frac{t}{K \cdot \log\left(\frac{d_1}{d_2}\right)} = \frac{t}{2{\cdot}303\ K \log_{10}\left(\frac{d_1}{d_2}\right)} \quad \ldots\ldots(3)$$

In making an estimation of R several values of t should be taken—the range of values will be largely fixed by the value of R and the condenser used. When several values are obtained, the mean value of $\frac{t}{\log \frac{d_1}{d_2}}$ is taken and substituted in equation (3) above. When K is expressed in farads, t in seconds, R is in ohms.

The method outlined above may be very conveniently used for a determination of the specific resistance of, say, cadmium iodide solution in xylol. Such a solution is very useful when a high resistance is required for any purpose. The value of the specific resistance of solution of different concentrations should be measured. Pure xylol will be found to be practically without effect on the leak of the condenser.

A suitable container of liquids is made by selecting a length of glass tubing of uniform bore (about 2 cms. diameter) and about 80 cms. long. This is closed at both ends by corks and the tube clamped vertically (fig. 373). A disc of brass which just fits the tube rests on the upper surface of the lower cork and can be connected to an outside circuit by means of a thin brass rod soldered to the under surface of the disc and passing through the cork. This disc serves as one electrode.

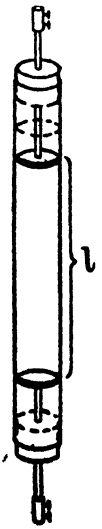


FIG. 373

A second brass disc of the same area is supported vertically above this by means of a brass rod which passes through the upper cork, as seen in fig. 373.

Using such a container a definite ' length of liquid,' l, with an area of cross-section practically equal to that of the area of the discs may be employed ; and hence the specific resistance may be calculated from the value of R measured in the experiment, l, the distance between the discs, and a, the radius of the discs.

Note

This experiment could also be performed using a ballistic galvanometer. The value of Q_1 at the commencement could be obtained by discharging the condenser through the ballistic galvanometer. The condenser is then recharged, allowed to leak through the resistance for a measured time, and then discharged through the galvanometer once more, thus Q_1 and Q_2 are measured ; and R is obtained from (2).

Measurement of the Capacity of the Quadrant Electrometer

Fig. 374 shows a simple arrangement of apparatus which will enable an estimation of the capacity of the quadrant electrometer to be made.

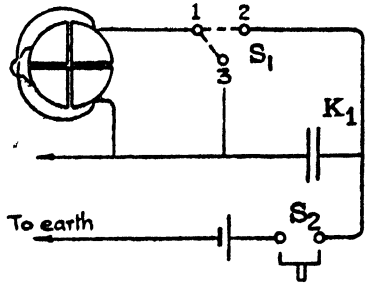


FIG. 374

K_1 is a small capcity of known size. When switch, S_1, is closed from 1 to 2, and S_2 is closed, the condenser becomes charged, and the electrometer is deflected an amount, φ_1 say, corresponding to a scale deflection of the spot of light of d_1 cms. If now switch, S_1, is closed from 1 to 3, the electrometer becomes discharged, and when the connecting strip is replaced to the position 1 to 2, S_2 now being open, the condenser, K_1, shares its charge with the electrometer. If K_1 is properly chosen of the same order of magnitude as the capacity, K, of the electrometer, the latter will show a marked drop in deflection compared with the former φ_1. Let this new deflection be φ_2.

The charge lost by the condenser, K_1, is equal to that gained by the electrometer, KV_2, where V_2 is the final potential corresponding to the deflection φ_2. This loss is also $K_1(V_1 - V_2)$ where V_1 is the original potential,

i.e. $$KV_2 = K_1(V_1 - V_2),$$

or $$K = K_1\left(\frac{V_1 - V_2}{V_2}\right) = K_1\frac{\varphi_1 - \varphi_2}{\varphi_2},$$

whence, if K_1 is known, K may be calculated.

If the known small condenser has a capacity which is large compared with the capacity of the electrometer, the value of φ_2 will not be very different from φ_1, and the value of K, as calculated from the above equation, will probably be inaccurate.

If no smaller capacity is available, the following slight modification of the method is very readily carried out.

The key, S_1, is moved from position 1 and 2 to 1 and 3, a definite number, n, times, i.e. the electrometer is charged and discharged n times, until the deflection is finally reduced a distinct readable amount.

At the end of the first sharing of charge we saw above that

$$KV_2 = K_1(V_1 - V_2),$$

i.e.
$$V_2 = \frac{K_1}{K_1 + K} \cdot V_1.$$

When the condenser, K_1, which is now at potential, V_2, is again connected to the discharged electrometer, a new common potential V_3, will be the result where

$$KV_3 = K_1(V_2 - V_3),$$

$$V_3 = \frac{K_1}{K_1 + K} \cdot V_2,$$

$$= \left(\frac{K_1}{K_1 + K}\right)^2 V_1;$$

so, after n repetitions

$$V_{n+1} = \left(\frac{K_1}{K_1 + K}\right)^n V_1,$$

or
$$\frac{V_{n+1}}{V_1} = \left(\frac{K_1}{K_1 + K}\right)^n;$$

of if φ_1 and φ_{n+1} are the corresponding deflections,

$$\left(\frac{K_1}{K_1 + K}\right)^n = \left(\frac{\varphi_{n+1}}{\varphi_1}\right),$$

whence
$$K = K_1\left\{\left(\frac{\varphi_1}{\varphi_{n+1}}\right)^{\frac{1}{n}} - 1\right\}.$$

A suitable form of measurable capacity to use in this and the following experiments, which require a known capacity, is a circular parallel plate air condenser having a guard ring.

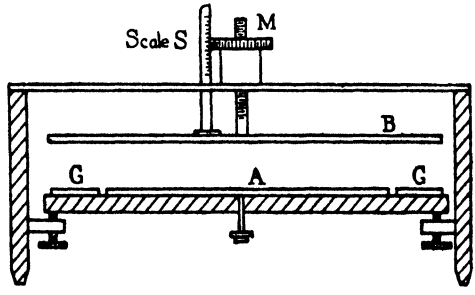


FIG. 375

The guard ring, GG (fig. 375), surrounds a central plate, A, which may be adjusted by a micrometer screw attachment, M, moving past a fixed scale, S, so that the distance, d, between the plate, may be measured, provided the zero reading on the scale

is known, whence, if plate, A, has an area A sq. cms., the capacity, K_1, in the above experiment is

$$\frac{A}{4\pi d} \text{ E.S. units}$$

or $$\frac{A}{9 \times 10^{20} 4\pi d} \text{ E.M. units} = \frac{A}{9 \times 10^{11} 4\pi d} \text{ farads.}$$

To obtain the zero reading of the instrument, the plate, A, is connected to a quadrant electrometer, and the system is given a charge, and then insulated. The plate, A, is moved towards the lower plate until the electrometer deflection is reduced to zero.

To allow for possible irregularity of plate surface or a slight inclination of one of the plates, the *effective* zero of the condenser may be obtained by a method which consists in finding the definite value of the deflection of the electrometer (φ) for each scale reading of the condenser (d). Several values of d and φ are taken as the plates approach each other. No observation of φ less than one-tenth of the original deflection need be observed. Plot φ against d. The point where this curve produced cuts the axis of d corresponds to the zero of the scale.

Another form of condenser suitable for the above experiments, and which is simply made, is described on page 605. This condenser has a fixed capacity.

Comparison of Small Capacities

Capacities such as small air condensers may be compared, using the method indicated in fig. 376. S_1 and S_2 are keys made as already described, and B is a 2-volt cell.

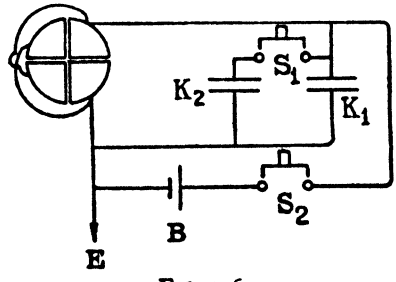


FIG. 376

With S_2 closed and S_1 open, K_1 and the quadrant electrometer are charged to a potential, V_1, equal to the E.M.F. of B.

If now S_2 is opened and S_1 closed, the charge on K_1 and the Q.E. is shared with K_2.

Let φ_1 be the deflection of the Q.E. before S_1 is closed, and φ_2 the deflection after the charge is shared, corresponding to a potential, V_2.

We have K_2V_2 = charge acquired by K_2, and $(K_1 + K)(V_1 - V_2)$ is the charge lost by the Q.E. and K_1, where K is the capacity of the electrometer,

i.e. $$K_2V_2 = (K_1 + K)(V_1 - V_2),$$

or $$\frac{V_2}{V_1 - V_2} = \frac{K_1 + K}{K_2} = \frac{\varphi_2}{\varphi_1 - \varphi_2},$$

or $$K_1 = K_2\left(\frac{\varphi_2}{\varphi_1 - \varphi_2}\right) - K$$

Whence by observing the deflections before and after the sharing of the charge, and knowing K, the relation between K_1 and K_2 may be obtained.

When φ_1 and φ_2 are small the corresponding scale deflections, d_1 and d_2 cms., may be used in the equation.

Determination of the Dielectric Constant

The value of k, the dielectric constant of a medium, may be obtained by measuring the change in capacity of a parallel plate air condenser, when a *parallel plane-faced slab* of the medium is introduced between the condenser plates.

The capacity of the condenser when the plates are entirely separated by air is

$$\frac{A}{4\pi d}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where A is the area of the plates and d the distance between them.

If the slab of thickness, t, and dielectric constant, k, is introduced between the plates of the air condenser, the capacity becomes

$$\frac{A}{4\pi\left\{d - t\left(1 - \frac{1}{k}\right)\right\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Thus the change produced in the capacity is numerically equal to the change produced when the distance between the plates is reduced by $t\left(1 - \frac{1}{k}\right)$, when air is the dielectric, for equation (3) would then become identical with (4).

Suppose the capacity be determined when the plates are a definite distance apart. Then let the slab be introduced; the capacity increases. If now the plates are separated until the capacity is restored to its original value, the distance, D, through

which the plates are moved is obviously equal to the equivalent movement produced by the introduction of the slab; thus

$$D = t\left(1 - \frac{1}{k}\right),$$

or

$$k = \frac{t}{t - D}.$$

To make a determination of k experimentally a convenient form of condenser is a circular parallel plate condenser, one plate of which is surrounded by a guard ring, as described on page 598, fig. 375.

Such a condenser is connected as shown at K_1 in fig. 374, the large plate, B, being earth connected and the central plate, A, inside the guard ring, connected to switches S_1 and S_2. The quadrant electrometer is adjusted, and the condenser, K_1, and one pair of quadrants are raised to the potential, V_1, of the cell, by closing S_1 and S_2, producing a deflection, φ_1 or d_1 divisions. The key, S_2, is then opened, and the slab of dielectric is introduced between the plates of K_1. The slab should be of uniform thickness and have an area not less than the plate, A, of the condenser.

Since the system has a fixed charge, the effect of the increase in the capacity of the condenser is evidenced by a drop in potential to V_2 (deflection φ_2). The reading of the micrometer adjustment of the condenser is taken and the movable plate is then withdrawn until the deflection of the quadrant electrometer is again φ_1 (d_1 cms.), i.e. the whole system has once more the original capacity. The movement of the condenser plate (D cms.) is known from the micrometer readings.

Hence, if t is measured in the usual way, and a mean value taken, k may be calculated.

Comparison of a Large Capacity with a Small

The method of comparing capacities given on page 599 is applicable when the capacities are of the same order. To compare a large capacity with a small one (both measured in electrostatic units) the method of repeated sharing of charges may be employed. This method was outlined for the case where the difference in the capacities was not very great on page 597. Now, when the order of the capacity of the two condensers is very different, as in the case of, say, $\frac{1}{3}$ micro-farad, and a simple parallel plate air condenser, n, the number of times the charge is shared, must be a large number to cause an appreciable change in the potential, or the deflection, φ. Some mechanical means must therefore be introduced to carry out rapidly the n steps. In fig. 377, which shows the arrangement of apparatus

for this determination, such a mechanical device, which is described later, is placed at M.

C_1 is the condenser of small capacity, K_1. A guard ring parallel plate air condenser as described on page 598 or page 605, would do very well. The distance between the plates is fixed and measured (d cms.); hence $K_1 = \frac{A}{4\pi d}$.

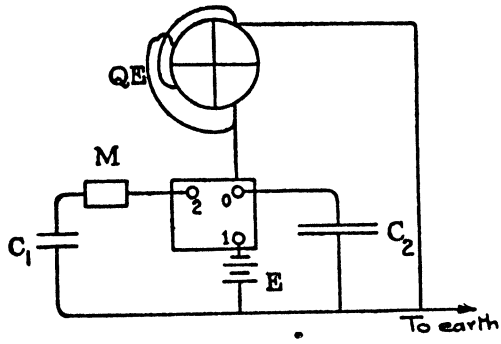


FIG. 377

The large capacity, C_2, of, say, one-third of a micro-farad (K_2) is connected to a paraffin wax key at cup 0.

Mercury cups 0 and 1 are connected together, thereby charging C_2 to the potential of the cell, E, and the deflection, φ_1, of the quadrant electrometer corresponding to this potential, V_1, is noted.

The mechanical sharing device is then set in vibration, and when regular, connexion between 0 and 2 is made (1 and 0 being separated). At the same time a stop-watch is started. The sharing of the charges is carried on for an exact number of seconds, t, until the deflection of the electrometer, φ_{n+1}, is from one-half to two-thirds of φ_1.

If the sharing is performed n times per second, N, the total number of times the charge is shared, is nt.

Now the capacity of the electrometer is small compared with that of C_2, and may be neglected in comparison with K_2, hence, as on page 598,

$$\left(\frac{K_2}{K_1+K_2}\right)^N = \frac{\varphi_{N+1}}{\varphi_1},$$

$$\frac{K_1}{K_2} = \left(\frac{\varphi_1}{\varphi_{N+1}}\right)^{\frac{1}{N}} - 1,$$

or

$$K_2 = \frac{K_1}{\left\{\left(\frac{\varphi_1}{\varphi_{N+1}}\right)^{\frac{1}{N}} - 1\right\}} \quad \ldots\ldots\ldots\ldots\ldots(5)$$

The mechanical device for sharing the charge of C_2 with C_1 is seen in fig. 378. (a) shows the end view and (b) shows one

form of vibrator which carries at one end a rectangular piece of copper wire, AB.

SS^1 is a flat steel strip (e.g. a steel foot-rule) fastened on an ebonite support at the end, S. A coil, C, is arranged over the strip, and a current from a cell, E, is arranged to pass round the circuit TKSC, causing a maintained vibration in SS^1. This apparatus could be very simply made from an old electric bell.

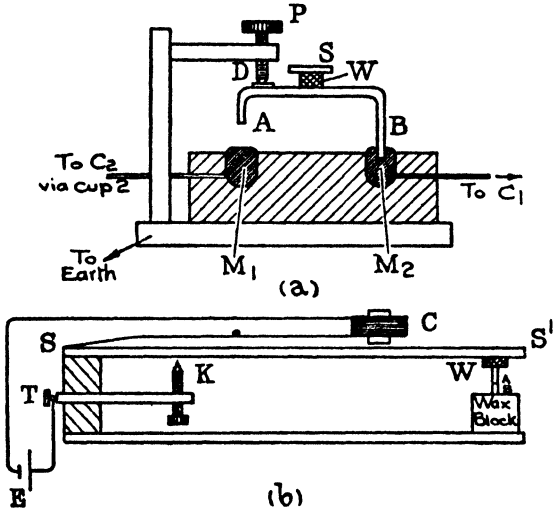


FIG. 378

Attached to the end of SS^1 by means of clean sealing wax, is the wire frame AB; A is shorter than B, as shown in the end view in (a). This is moved up and down by the vibrating spring, SS^1. The ends of A and B dip into mercury cups, M_1 and M_2, in a block of wax. B is of sufficient length to be always in the mercury, whilst A is first in M_1 and then lifted into contact with the small disc, D, at the end of the earth-connected screw, P.

Thus, if M_1 is connected to C_2, the large capacity, and C_1 to M_2, the charge on C_2 is shared with C_1 when the wire, AB, is depressed. C_1 is discharged when AB rises and makes contact with D. The continued vibration, thus shares the charges, and reduces the value of φ, the deflection produced in the quadrant electrometer.

A method to be followed is, therefore: charge C_2 by closing the cups 0 and 1, by means of a copper wire mounted on clean sealing wax. Note the deflection, φ_0, produced in the quadrant electrometer. Place the copper wire now from cup 0 to 2, having first adjusted K to a steady state of vibration; make a careful time estimate from the instant 0 and 2 are closed until the deflection, φ_1, is reduced to φ_{N+1} about one-half its original value.

Now if the value of the capacity of C_1 is calculated from the dimensions of the condenser, the capacity, K_2, of condenser, C_2, may be obtained if n is known.

If the vibrating bar is of such frequency that its vibrations may be counted, n may be estimated by counting the number of vibrations in several seconds. For quicker vibrations the frequency may be obtained either by matching the note emitted or preferably by attaching a light style to the end of the strip, SS^1, and finding n by examination of the trace produced on a chronograph.

Such an experiment gives all the terms on the right-hand side of equation (4), and therefore the value of the $\frac{1}{9}$ microfarad in E.S. units may be obtained.

The exact value of K_2 in E.M. units may be determined by the method given on page 555.

The capacity of the same condenser is therefore measurable in both systems of units.

Now:

$$\frac{\text{Capacity of } K_2 \text{ in E.S. units}}{\text{Capacity of } K_2 \text{ in E.M. units}} = \frac{\text{E.M. unit}}{\text{E.S. unit}} = v^2,$$

whence v may be calculated (see also the following pages).

CHAPTER XXIV

MISCELLANEOUS ELECTRICAL EXPERIMENTS

Measurement of a Small Capacity in Electromagnetic Units. Comparison of the E.S. and E.M. units

To compare the electrostatic (E.S.) and electromagnetic (E.M.) units of capacity, we may find the capacity of a small air condenser in the E.S. units by calculation from a knowledge of the dimensions of the condenser, and determine the value of the capacity in E.M. units by a special method. An alternative method is also given on page 601. We have then:

$$\frac{\text{E.S. unit}}{\text{E.M. unit}} = \frac{\text{Capacity of the condenser in the E.M. system of units}}{\text{Capacity of the same condenser in the E.S. system}} = \frac{1}{v^2}$$

when v is equal to the velocity of light (3×10^{10} cm./sec.) (see J. J. Thompson's "Elements of Magnetism and Electricity," Chapter XII).

The object of this experiment is to find v from the above equation.

A suitable condenser to use for this purpose is a parallel plate air condenser, such as the one shown in fig. 375, page 598. The zero having been found by one of the methods suggested in the account given, the plates are set at a distance, d cms., apart.

If the area of the plate is A sq. cms., the capacity in E.S. units is $K_s = \frac{A}{4\pi d}$, when air is the dielectric. The same formula gives the capacity for another simple form of parallel plate air condenser, which may be easily made as a substitute for the more elaborate condenser described.

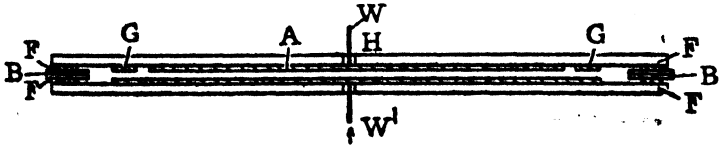


FIG. 379

This air condenser is made taking two large sheets of plane glass, fig. 379. The one sheet is coated with a circular sheet of tin-foil of about 50 cms. diameter. This may be attached to the glass by means of seccotine or shellac. The tin-foil is

connected to a wire, W^1, which passes through a small hole in glass; on the second sheet a smaller circular strip of tin-foil, A, is fastened and connected to a wire, W, through the hole, H. An annular circular strip of tin-foil is fastened outside A, shown at G G in the figure, to act as a guard ring. The radius of A is carefully measured. The two sheets of tin-foil are then set up as shown, at a fixed distance apart. This may be done by placing three or four stops, B B, between the plates. If B B are small pieces of ebonite or microscope slide glass cut from the same piece, the distance between the plates will be uniform. The thickness of the stops is measured before applying, and the stops are then covered above and below with pieces from the same tin-foil which was used in the construction of the condenser plates. In this way the distance, d, is equal to the thickness of the glass stops (·2 to 1 mm.).

Using either form of air condenser, K_a is calculated.

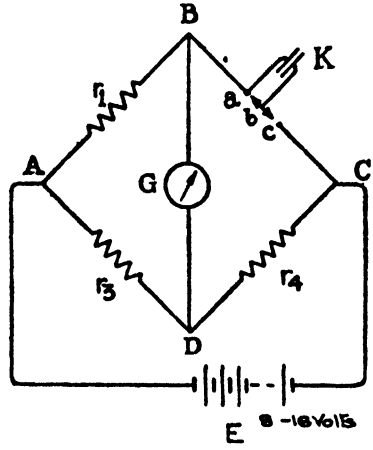


FIG. 380

To find K_m, the capacity in E.M. units, the Wheatstone bridge, shown in fig. 380, is set up with resistances, r_1, r_3, r_4; the condenser, K, is connected in the arm, BC, as shown. One plate of the condenser is connected to b which moves rapidly between a and c; so that when at c the condenser is charged, and when at a the condenser is discharged. By a mechanical device, described below, this process is carried out regularly n times per second. Under such circumstances the capacity may be balanced against r_1, r_3, r_4, in the ordinary way. When balanced, it may be shown that

$$\frac{1}{nK_m} = \frac{r_1(r_3 + r_4 + B) + Br_3}{r_3(r_4+B)+(G+r_1)(r_3+r_4+B)}\left\{G+r_4+\frac{r_4}{r_3}(G+r_1)\right\},$$

where G is the resistance of the galvanometer and B the resistance

of the battery (see, for example, J. J. Thompson's "Elements of Magnetism and Electricity").

The value of B is negligible compared with r_1, r_2 or r_3, and Br_3 negligible, c.f. r_1r_3 or r_1r_4,

i.e. $$\frac{1}{nK_m} = \frac{r_1(r_3+r_4)}{r_3r_4+(G+r_1)(r_3+r_4)} \cdot \left\{ G+r_4+\frac{r_4}{r_3}(G+r_1) \right\},$$

which reduces to

$$\frac{1}{nK_m} = \frac{r_1r_4}{r_3} \left\{ \frac{\left(1+\frac{r_3G}{r_4(r_1+r_3+G)}\right)}{\left(1-\frac{r_3^2}{(r_3+r_4)(r_1+r_3+G)}\right)} \right\}.$$

Now when K_m is small, as in the case taken, r_3 will be small compared with r_4 or r_1, and the above expression reduces to

$$\frac{1}{nK_m} = \frac{r_1r_4}{r_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

i.e. the condenser acts as though it were a resistance of $\frac{1}{nK_m}$ ohms.

To produce a rapid charge and discharge n times per second, the apparatus shown in fig. 381 may be used. R and R^1 are similar steel rods fastened in an ebonite block and provided with terminals as shown. A small soft iron rod is surrounded by a coil of wire, making a small electromagnet, C; from the end of R^1 a light metal style is arranged to just touch a metal plate which is connected to a separate accumulator, B. The connexions shown in the figure are made and the vibrations of the rod are maintained electrically.

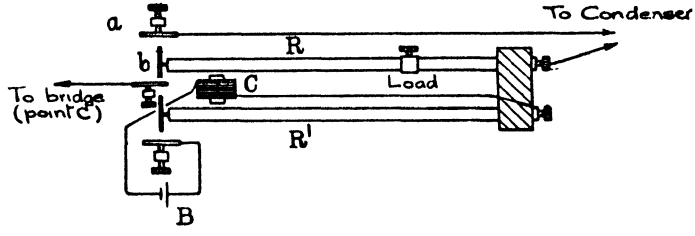


FIG. 381

The rod, R, is loaded, as shown, until it also vibrates with the same frequency, which is maintained by means of C. To the end of R is a second light metal style (*b*) which just touches the two plates, *a* and *c*, at the ends of the vibration.

The connexions to the condenser and the bridge are indicated, and the charge and discharge is thus brought about n times per second.

To find the value of n, the frequency of the note emitted by the fork is obtained by tuning with the note of a variable monochord. The frequency is further tested by use of the dropping smoked plate, and the chronograph (see page 397). The mean value obtained is taken as n in equation (1).

The bridge is then balanced and the values of r_1, r_3, r_4, found. It will be advisable to make these values as high as possible, since K_m is very small and $\frac{1}{nK_m}$ great.

r_1 and r_4 may each well be 1 megohm (10^6 ohms) and r_3 adjusted until a balance is obtained.

The value of the resistances must be expressed in E.M. units, i.e. if r_1, r_3, r_4 are in ohms, equation (1) becomes :

$$\frac{1}{nK_m} = \frac{r_1 r_4}{r_3} \times 10^9 \text{ E.M.U.}$$

Having obtained K_s and K_m by the methods indicated, we have

$$\frac{K_s}{K_m} = v^2,$$

which gives a value of v very nearly 3×10^{10}.

It should be noted that n should certainly be found by two methods. The comparison with a monochord alone is apt to give a value which corresponds to an overture of the note emitted, unless the observer is practised in the comparison of two different quality notes.

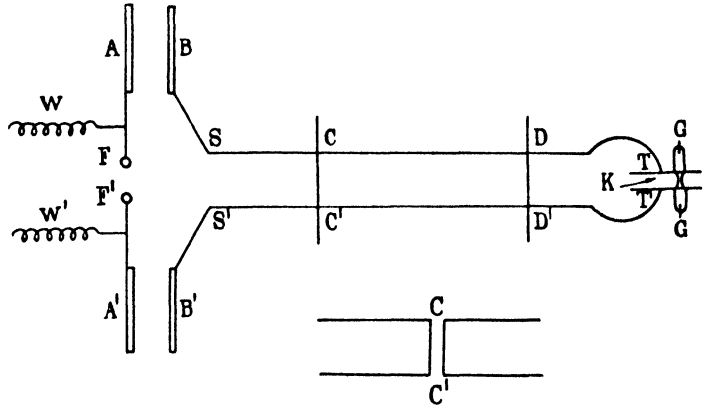


FIG. 382

Lecher's Experiment. The Determination of the Velocity of Electric Waves in Wires. "Ann. d. Physik." Band 41, 1890. p. 850.

The apparatus necessary for this experiment is illustrated in fig. 382. It consists of two square metal plates with sides about

40 cms. long, connected by wires about 50 cms. long to two spherical metal knobs, FF^1, of diameter 3 cms., separated by an air gap of ·75 cm.

They are connected by flexible leads, W and W^1, to a large induction coil, the primary current for which is supplied by accumulators or the supply mains.

The plates B and B^1 are plates similar to A and A^1 placed at about 4 cms. from them and thus forming two condensers with air as the dielectric. From B and B^1 are led away two long wires of length about 10 metres. For the greater part of their length they are parallel, and their distance apart may lie between 10 and 50 cms. This separation has to be arranged conveniently, as will be seen later. The ends of these wires are connected to the condenser, K, which consists of two parallel circular plates of radius, R, separated by air and at a distance, d cms., apart.

In Lecher's experiment R had the value, 8·96 cms., and d ·99 cm. It will be a good plan to keep to these dimensions.

When the induction coil is working electric vibrations are set up in the wires, the ends of which undergo rapid variations of potential. By analogy with the case of a closed organ pipe in which at the closed end there occur variations of pressure and at other points of the length places where the pressure remains unchanged we anticipate that at some point between B and K there will be no change of potential.

A discharge tube is placed close to the ends of the wires, TT^1, as illustrated by GG.

It is not necessary that there should be metallic connexion between the wires and tube. Lecher recommends that this should not be the case, and in his experiment glowing occurred in the tube when it was 10 cms. from the ends.

The tube may either be held a few centimetres from the ends of the wire, or may rest with the glass lying on the edges of the condenser plates as the figure illustrates.

If the wire be bridged by means of a wire carried on a wooden handle, as for example at CC^1, supposing for the present that DD^1 is absent, we expect that there would be a flow of current across CC^1 unless it happened that C and C^1 were always at the same potential. Thus, generally GG^1 would glow less brightly when the wire is bridged.

But the circuit, BCC^1B^1, has a definite frequency, and on moving CC^1 about the changes in frequency can be observed by the sounds from the sparking at FF^1.

Likewise CTT^1C^1 has also a definite frequency and we may obtain resonance between the two circuits, in which case the glowing continues although the wire is bridged.

It was found that the wires could be cut in two in this case

at C and C^1, and joined separately, as shown in the lower figure, and the glowing did not cease.

The inductive action arises chiefly from the bridge, CC^1, and if this is long the effect is great, while if it is short the effect is small and may even vanish altogether.

If CC^1 is too long it is not easy to place CC^1 in the position in which the glowing ceases because the energy from it even when only a small disturbance passes through it, is large enough to excite the tube. The tube remains bright for all positions of CC^1. When CC^1 is very short the tube remains dark for all positions of CC^1.

With the above dimensions it will be found possible to find two positions, CC^1 and DD^1, at which the wires may be simultaneously bridged and maximum brightness produced in the discharge tube. It must be ascertained that there are no intermediate positions between BB^1 and CC^1, between CC^1 and DD^1, or between DD^1 and TT^1 at which bridges may be also placed and leave the tube glowing. We may then regard the wires and condensers as forming three resonating circuits.

Since T and T^1 are always in opposite phases the fundamental vibration in DTT^1D^1D is that in which a half wave occupies the circuit while in DCC^1D^1D we have a whole wave.

If L denote the inductance of the former circuit, and K the capacity of the condenser, the period of the oscillation is:

$$T = 2\pi\sqrt{LK}.$$

In this case we have neglected the resistance of the wire, as we may, since fairly thick copper wires are employed.

The value of K should be large, but not large enough to make the circuit non-oscillatory. Lecher used a condenser of capacity, 20 electrostatic units.

The value of L may be deduced from Neumann's formula for self-induction. If l denote the length of wire in the circuit, TDD^1T^1T, including of course the bridge, and its radius of cross-section is r, we have:

$$L = 2l\left(\log_e \frac{2l}{r} - 1\right) \text{ electromagnetic units.}$$

The value of K expressed in this system of units is:

$$\frac{R^2}{4d} \times \frac{1}{9} \times 10^{-20}.$$

Thus
$$T = \frac{\pi R}{3} \times 10^{-10} \times \sqrt{\frac{2l}{d}\left(\log_e \frac{2l}{r} - 1\right)}.$$

The wave length is obtained from the circuit, CDD^{1}C^{1}C, and if CD is of length, a, and the bridge of length, b, we have:

$$\lambda = 2(a + b).$$

Hence the velocity, c, is given by:

$$c = \frac{\lambda}{T},$$

or

$$c = \frac{6 \times 10^{10}(a + b)}{\pi R} \sqrt{\frac{d}{2l\left(\log_e \frac{2l}{r} - 1\right)}}.$$

The experiment is performed by stretching two long wires not less than ten metres in extent along the laboratory and clear of metal pipes. The distance apart should be about 40 cms., and the capacities may all be formed of sheets of tin of the dimensions given above. The capacity, K, must be set up so that the plates are parallel, and its dimensions should be very carefully measured. Likewise the diameter of the wires must be accurately determined by means of a screw gauge.

If the wire is sufficiently long, a third bridge may be inserted simultaneously with the other two, and a second determination of wave length is then possible.

Lecher gives the following values observed in one of his experiments which are useful for comparison:

Total length of each wire, 1122 cms.
Distance between C and D, 939·6 cms.
Length of bridge, 42 cms.
Length of wire in DTT^{1}D^{1}D, 303·2 cms.
Diameter of wire, ·1 cm.

MEASUREMENT OF $\frac{e}{m}$ AND v FOR CATHODE RAYS

The measurement of $\frac{e}{m}$, and of v, the velocity of the electrons in a cathode stream, may be carried out, using the apparatus described later, by applying a magnetic and an electrostatic field of measurable magnitude and observing the deflections produced in the cathode stream.

We will first briefly consider the effect of such fields on a moving electron.

In fig. 383 let OX be the undeviated path of the electron, of mass, m, charge, e, and velocity; v.

If now a *uniform* magnetic field, H, be applied parallel to OZ, at right angles to the plane of the paper, the electrons will be deflected in the same manner as a current-conveying conductor in the magnetic field.

In this case the electron stream is equivalent to a current, ev, in a direction, X→O, since the electron has a *negative* charge; the force causing the deflection upwards is therefore Hev per electron, and is always normal to the direction of motion, and to the direction of the field, H. The path of the electron will, therefore, be circular. If ρ be the radius of curvature of the deflected path, we have:

$$\frac{mv^2}{\rho} = Hev \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

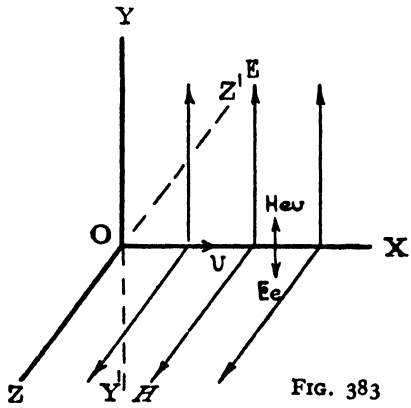


FIG. 383

In the case of a *uniform* electrostatic field the deflection produced is in the opposite direction to the field, for such a negative charge. The electron passing through the field of strength, E, and parallel to OY, is subjected to a force, Ee, in a direction parallel to YO.

If both fields are applied simultaneously the deflection will be the net effect of the two, and if suitably adjusted the fields may be made to produce deflections which neutralize each other, i.e. the force on the electron Hev upwards and Ee downwards become equal and opposite:

$$Hev - Ee = 0, \text{ or } v = \frac{E}{H} \quad \ldots\ldots\ldots\ldots (2)$$

Consider the magnetic deflection as expressed in (1). The curvature produced being small, by suitably adjusting H, $\frac{dy}{dx}$, referred to the axes shown in fig. 383, is small and $\left(\frac{dy}{dx}\right)^2$ negligible compared with unity, hence

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} = \frac{H}{v} \cdot \frac{e}{m}.$$

Whence

$$y = \int\left[\int \frac{H}{v} \cdot \frac{e}{m} dx\right] dx \quad \ldots\ldots\ldots\ldots (3)$$

where y is the displacement produced by the field H, the integration being carried from the point where the electron enters the field up to the point at which y is measured:

$$\frac{e}{m} = \frac{vy}{\int\left[\int H dx\right] dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

It is seen, therefore, that for known electric and magnetic fields, $\frac{e}{m}$ and v may be obtained.

Further discussion of the above results are postponed until the general experimental method has been described.

One form of apparatus, known as the Braun tube, is frequently used for this experiment. Fig. 384 shows its essential features. The long glass tube contains a plane-faced cathode, C, a ring anode, A, and is provided with a mica screen pierced with a small hole, M, at the centre. P, P^1 are two parallel plates which are long compared with the distance between them and which may be connected to an external source of potential. F is a fluorescent screen, very lightly graduated in millimetres. C and A are connected to a good induction coil, the commutator of which is arranged so that C is the negative terminal. The electron stream, leaving the cathode normally, passes through A, and a narrow pencil passes through M and strikes the screen, F, making a small luminous patch on the scale. The interrupter of the coil is adjusted so that a steady output is obtained.

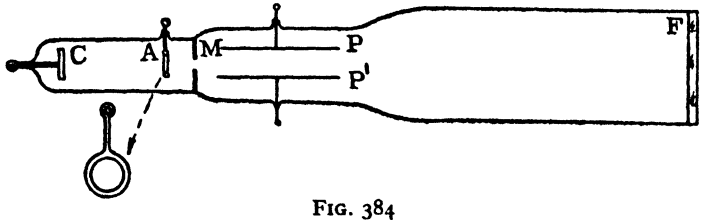


FIG. 384

To use this tube, it is firstly fixed in some definite direction relative to the permanent magnetic field, i.e. the Earth's field. If placed normal to the horizontal component of the Earth's field, the electron will suffer a slight initial deflection. This deflected position may be observed; and later the magnetic field, when applied, is measured independently of the Earth's field. The change in deflection due to this is noted. For many purposes this method is advisable. Another method is to place the tube *in the direction* of the *horizontal component* of the Earth's field.

The path of the electron thus coincides with the direction of the horizontal component and will consequently suffer no deflec-

tion due to this. The vertical component of the Earth's field will only cause a lateral displacement of the luminous spot, and is thus of no account, as we are measuring vertical displacement due to an applied horizontal field.

Two large bar magnets are now arranged normally to the tube, parallel to the face of the plates, P, P^1, and at such a distance from M that at the most only a very slight effect is produced on the electrons before passing through the hole. For if the magnetic field between A and M is large, the electron stream will be deflected before passing through M. Those electrons which pass through M do not do so along the axis of the tube and therefore the deflection measured on the screen, which we consider due to the field between M and F, will be too great. A trial experiment will show the order of deflection produced at F.

To introduce an electrostatic field the plates, P,P^1, are connected to a constant source of potential. A nest of small accumulators or a battery of small dry cells serve this purpose well. If these are not available the plates may be connected to the mains of the electricity supply, through a water resistance.

If the plates are not too far apart we may take the field, E, so produced as a uniform field and of an extent coinciding with the limit of the plates, i.e. neglect the end corrections if the plates satisfy the conditions already stated, i.e. are close together.

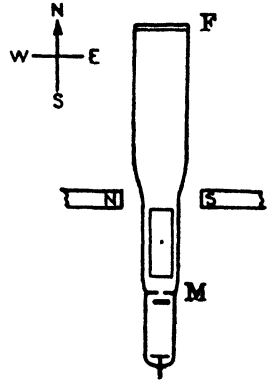


FIG. 385

The chief difficulty met with in most laboratories is the lack of a convenient, adjustable, steady potential difference up to 100 volts. Therefore if only a fixed voltage is available, apply this to P, P^1, and note the deflection; then introduce the bar magnets in the direction required to reduce this deflection and arrange the poles N and S (fig. 385) symmetrically and at such a distance apart that the field is of the sufficient strength to reduce the deflection to zero, i.e. fix E and adjust H to balance.

It is important that the leads from the coil to A and C should be well insulated from the rest of the apparatus, and that one plate, P, say, be connected to earth, otherwise troublesome leakages may occur.

Having obtained a balance, measure the potential difference between P and P^1 by means of a high-resistance voltmeter.

Remove the source of potential and note the deflection due to the magnetic field alone. Let this be y cms.

It will be apparent that the magnetic field used is not a uniform one, therefore the value $\int\left[\int H dx\right] dx$ of equation (4) must be found graphically.

The positions of M and F are marked on the bench (fig. 386), and the Braun tube is removed, care being taken that the magnets are not disturbed.

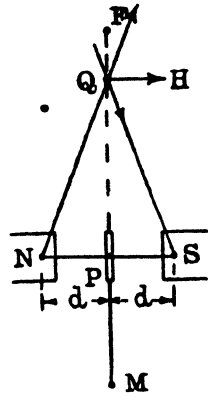


FIG. 386

A central line, MF, is drawn and subdivided by points, 1 or 2 cms. apart. The value of the field, H, at each point is then obtained. This may be done by means of the fluxmeter, or by means of a Searle magnet.

Determination of the Field by the Fluxmeter

The fluxmeter is set up and the mirror arranged to reflect a beam of light from a lamp on to a scale one metre from it. The search coil of area, A, and containing n turns is placed at P (fig. 386) at right angles to the field at its strongest point. The deflection is noted. The search coil is then reversed and the deflection again noted; the total deflection so obtained is halved, to give the value corresponding to the field, H_m. The mirror deflections are then standardized against the fluxmeter scale and the value of H_m is obtained by dividing the flux, corresponding to the throw observed, by the product, nA.

For points such as Q, away from the point, P, the magnetic field will be comparatively small. The deflection produced in the fluxmeter will be correspondingly reduced, and the value of the field, H_Q, at Q will probably be inaccurate. To increase the deflection produced, a search coil with a larger number of turns should be used.

Using a fluxmeter with a search coil placed with its plane normal to NS and parallel to FM, the throw observed corresponds to the net field normal to the axis of the tube (FM) due to both poles of each magnet; that is, the fluxmeter values are those effective in producing the deflection. The Earth's field, which produces no vertical deflection, also produces no effect on the fluxmeter since the plane of the search coil is in the magnetic meridian.

If the tube is placed in an east and west direction the value of the deflecting field will be equal to the observed field, plus or minus the Earth's field, for in this case the fluxmeter readings include the horizontal component of the Earth's field.

If the field magnets are *very long*, compared with the distance, MF, the values of the field, H, at points, Q, along MF may be calculated from the fluxmeter readings at P, and the distances, PQ = l and NS = $2d$. Under such conditions the effect of the distant poles are neglected.

The field at Q normal to FM is:

$$H = \frac{(m + m^1)d}{\{l^2 + d^2\}^{\frac{3}{2}}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where m and m^1 are the pole strengths of N and S.

Also the field H_m at P is:

$$\frac{m + m^1}{d^2},$$

i.e. (5) becomes:

$$H = \frac{H_m d^3}{\{l^2 + d^2\}^{\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

When such a method is justifiable, H at all points along FM is calculated from the observed maximum value at P.

Determination of H by means of an Oscillating Magnet

A small Searle needle is allowed to oscillate freely at each point marked on the line, MF, the axis of the needle being arranged at a height from the table corresponding to the axis of the tube when in position. The time of oscillation of such a needle is

$$T = 2\pi\sqrt{\frac{I}{MH}},$$

where I is the moment of inertia fo the needle about the axis of suspension, M the magnetic moment of the magnet, and H the field in which it is placed.

As a preliminary standardizing experiment the needle is allowed to oscillate freely in a position well removed from the magnets, i.e. in the Earth's field only. If H_0 is the known value of the Earth's field

$$T_0 = 2\pi\sqrt{\frac{I}{MH_0}},$$

or $$T_0^2 = \frac{k}{H_0}, \qquad \text{where } k = \frac{4\pi^2 I}{M}.$$

The value of T for each point is obtained, hence H at each point being $\frac{k}{T^2}$, we have :

$$H = \frac{T_0^2}{T^2} \cdot H_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

This gives the value of the deflecting magnetic field *so long as the Searle needle when at rest sets normal to* MF ; for points near the screen and aperture positions this condition will not hold as nearly as for points close to the magnets, due to the presence of the Earth's field.

The value of the field actually measured is R, the resultant of H and H_0.

If, therefore, the magnet is brought to rest and the inclination of its axis with FM, θ, be measured at each position, we have :

$$H = R \sin \theta.$$

Equation (7) should be written :

$$R = \frac{T_0^2}{T^2} H_0;$$

and therefore

$$H = \frac{T_0^2}{T_0} H_0 \sin \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Tabulate results somewhat as under :

$H_0 =$ $T_0^2 =$

DISTANCE FROM APERTURE (IN CMS.)	θ	SIN θ	TIME FOR 20 SWINGS	T	T^2	$H = \frac{T^2}{T^2} \cdot H_0 \sin \theta$
0						
2						
4						
etc.						

Now plot H against the corresponding distance from the aperture, as in fig. 387.

Obtain from such a curve the areas of small figure such as AB20, BC42, CD64, etc.

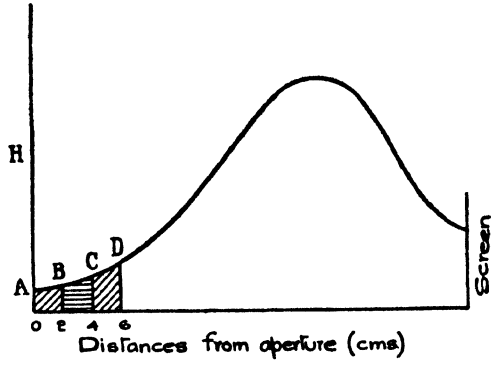


FIG. 387

Take care that the value of the areas are expressed in the proper units, i.e. if using the method of counting squares allow for the different scales in assessing the value of a square.

The values of the Hdx, as obtained for each small area, are conveniently tabulated and the true value of $\int Hdx$ for each point in the sum of such small areas. This is shown in the table below.

DISTANCE FROM APERTURE	Hdx FOR SMALL ELEMENT	$\int Hdx$
0		0
2	·7	·7
4	·9	1·6
etc.	etc.	etc.

A second curve is obtained by plotting $\int Hdx$, and the distance. The area of this curve gives $\int \left[\int Hdx \right] dx$.

The above treatment of the magnetic deflection has assumed that distances are measured in centimetres, H in gausses, and e in electromagnetic units (in establishing (1)). Further, the integration has been carried out from M to the fluorescent screen, F.

In the case of the electrostatic field the limits of the field are assumed to be the limits of the plate.

The field, $E = \frac{V}{t}$, where V is the potential in electromagnetic units (10^{-8} volts).

Now consider the deflection produced by the electric field. This must be evaluated separately and then equated to y, since H is not constant and equation (2) will not hold for such a case.

We will assume that the electrostatic field set up by the potential on the plates is E electromagnetic units. The force on the electron is therefore Ee parallel to the field (and not normal to the path as in the magnetic field). The equation of motion in this direction (using notation of fig. 383), is therefore

$$m\frac{d^2y}{dt^2} = Ee\ ;$$

and since v, the velocity of the electron in the path, is $\frac{dx}{dt}$, we have, by integration of the above equation,

$$\frac{dy}{dt} = E\frac{e}{m}t + B,$$

$$\frac{dy}{dx}\cdot v = E\frac{e}{m}t + B \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Now the value of $\frac{dy}{dx}$ at E (fig. 388) is obtained by putting t equal to the time the electron takes to travel the length of the plates,

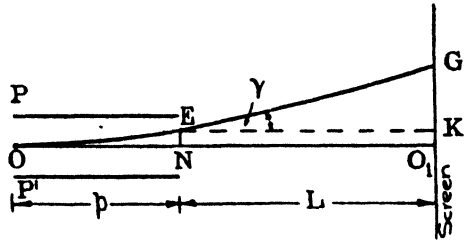


FIG. 388

and since when $t = 0$ and $\frac{dy}{dx} = 0$, $B = 0$ and $vt = p$, the length of the plates,

$$\frac{dy}{dx} = E\frac{e}{m}\frac{p}{v^2},$$

which is the value of the tangent at E of the path OE. On

leaving the electrostatic field the electron continues along the tangent EG, striking the screen at G. Putting angle GEK = γ,

$$\tan \gamma = \mathrm{E} \frac{e}{m} \frac{p}{v^2}.$$

Now from (9) we have, for any point, x cms. from the commencement of the electrostatic field, since $x = vt$,

$$\frac{dy}{dx} = \mathrm{E} \frac{e}{m} \frac{x}{v^2}$$

or $$\mathrm{EN} = \int_0^p \mathrm{E} \frac{e}{m} \frac{x}{v^2} \cdot dx = \frac{1}{2} \left[\mathrm{E} \frac{e}{m} \cdot \frac{x^2}{v^2} \right]_0^p = \frac{1}{2} \mathrm{E} \frac{e}{m} \frac{p^2}{v^2}.$$

Thus the total deflection on the screen is O_1G.

$$\begin{aligned} O_1G &= O_1K + GH \\ &= \frac{\mathrm{E}ep^2}{2mv^2} + \mathrm{L} \tan \gamma \\ &= \frac{\mathrm{E}ep}{mv^2} \left(\frac{p}{2} + \mathrm{L} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (10) \end{aligned}$$

Since $O_1G = y$, for when the two fields are acting no deflection is produced, we have numerically :

$$\frac{\mathrm{E}ep}{mv^2} \left(\frac{p}{2} + \mathrm{L} \right) = \frac{e}{m} \frac{1}{v} \int \left[\int \mathrm{H} dx \right] dx;$$

or $$v = \frac{\left(\frac{p}{2} + \mathrm{L} \right) p\mathrm{E},}{\int \left[\int \mathrm{H} dx \right] dx} \quad \ldots\ldots\ldots\ldots (11)$$

$$\frac{e}{m} = \frac{vy}{\int \left[\int \mathrm{H} dx \right] dx} \quad \ldots\ldots\ldots\ldots\ldots (12)$$

Thus every term is obtained for substitution in (11) and (12). Hence $\frac{e}{m}$ in E.M.U. and v in cms. per sec. are obtainable.

Repeat, using a higher potential difference between A and C ; note the increased v and same value for $\frac{e}{m}$.

Another way of finding $\frac{e}{m}$ and v (which involves a greater uncertainty than the above method) eliminates the electrostatic field.

The induction coil is adjusted as in the last method, so that the output is uniform, i.e. when the magnetic field is applied

the spot of luminescence is not drawn out into a very long patch, showing that the velocity of the electrons is approximately constant.

Suppose the potential applied *to the terminals* A *and* C be V volts, we have :

$$Ve = \tfrac{1}{2} mv^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Also by a magnetic deflection, as already described, we have from equation (4) :

$$y = \frac{e}{m}\frac{1}{v}\int\left[\int H dx\right] dx \quad \ldots\ldots\ldots\ldots (4a)$$

Hence, dividing (13) by (4*a*) :

$$v = \frac{2Vy}{\int\left[\int H dx\right] dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Hence :

$$\frac{e}{m} = \frac{2Vy^2}{\left\{\int\left[\int H dx\right] dx\right\}^2} \quad \ldots\ldots\ldots\ldots (15)$$

To find y and $\int\left[\int H dx\right] dx$, proceed as already described.

The determination of V may be made approximately by arranging an alternative spark gap in parallel with the terminals of the induction coil and A and C.

The spark gap should consist of two equal-sized *smooth* and *polished* spheres of known diameter, and their positions so adjusted that a spark will only occasionally pass when the tube is connected. The value of the potential is then obtained from tables, knowing the diameter and distance apart of the spheres.

For such tables see Kaye and Laby, " Physical Constants."

Thus for balls of 2·0 cms. diameter in air we have :

Spark gap in cms.	0·5	0·8	1·0	1·5	2·0
Potential $\times 10^{-3} v$ in volts	17·5	26·0	30·8	39.0	47·0

THE THREE-ELECTRODE VALVE

The three-electrode valve consists of an evacuated glass bulb containing (1) a *filament* usually made of tungsten wire which may be heated to incandescence by an external battery, (2) a *grid*, which may be a metal gauze or a helix of wire wrapped about the filament and (3) *an anode* or *plate* made of metal (often, nickel) foil.

In the French valve the filament is arranged centrally, and is surrounded by the grid in the form of a helix of wire and finally surrounded by the anode, as shown in fig. 389.

In some forms of the valve, the three elements are arranged as parallel planes, as shown diagrammatically in fig. 389, which also shows the method used in this account to represent the valve in subsequent connexion diagrams.

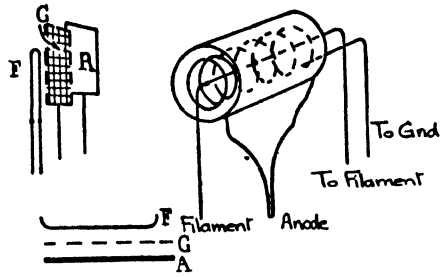


FIG. 389

The bulb is usually exhausted to as high a degree as is possible commercially, although in some cases a certain amount of residual gas is purposely retained.

The anode and grid are each provided with an external connecting lead so that each may be maintained at any desired potential.

When the filament is raised to incandescence it becomes a source of electrons.

The number of electrons emitted depends upon the temperature of the filament, which in turn depends upon the strength of the heating current. (The general relation between these factors is discussed in Chapter III in " Emission of Electricity from Hot Bodies," by O. W. Richardson.)

For a given heating current, the wire becomes a constant source of electrons, which may be directed to the anode by the application of a positive potential to the anode with respect to the filament. Also the current from the filament to the anode due to the electrons emitted by the heated wire is in general increased by raising the potential of the *grid*, i.e. increasing the field, and is decreased by applying a negative potential to the grid.

The current which flows in the valve to the anode is called the ' anode or plate current.' In the same way a smaller current passes to the grid and is called the '*grid current.*'

It is of importance to investigate the manner in which the anode current varies with the grid potential and also the anode potential, for a fixed filament current. The significance of these relations will be briefly discussed later.

The Characteristic Curves of the Three-electrode Valve

The general scheme of connexions shown in fig. 390 is one which may be used to find the characteristic curves of the valve. For some of the experiments slight modifications are required.

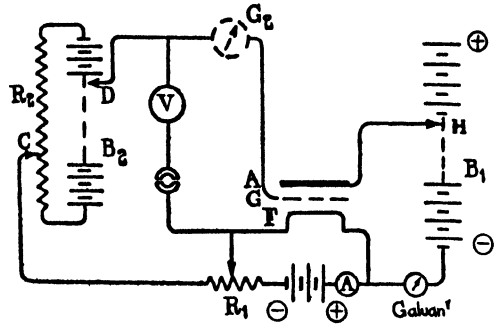


FIG. 390

The filament, F, may be heated by the current from the battery shown. This current may be regulated by the variable resistance, R_1. An ammeter, A, gives the value of the heating current, and should be sufficiently delicate to detect small current changes.

A battery of dry cells, B, or any other convenient source of potential from 0 to 100 volts, say, is connected through a galvanometer to the filament, and to the anode by means of a variable contact, H, which enables any positive potential from 0 to 100 volts (with respect to the filament) to be applied to the anode.

The grid is connected to the potentiometer arrangement shown. This latter consists of a battery of cells of, say, 30 volts. By suitably choosing the points, C and D, the grid may be maintained at any potential from + 30 to − 30 volts, with respect to the negative end of the filament. This potential is read directly on the high-resistance voltmeter, V.

(∝) *Total Emission from the Filament and the Filament Current*

To investigate the manner in which the total current from the filament varies with the temperature of the filament (which is dependent on the heating current), the grid and filament are connected together by adjusting C and D to coincidence.

The filament current is then given a small value and the thermionic current measured by means of the galvanometer. The heating current is given a slightly larger value and the current again measured. This process is repeated for gradually increased heating current values, care being, of course, taken not to exceed the safe limit of the filament. Since the temperature of the filament is not very readily measured in terms of

the filament current, the relation between the *thermionic current* and the *heating current* is obtained graphically from the observed results.

This relation is obtained and plotted for different values of the anode potential.

From such curves the value of the heating current suitable for a subsequent experiment may be estimated, for they show the maximum thermionic current available for each filament current, at a given anode potential.

(β) *Anode Potential and Anode Current*

To find the relation between anode current and anode potential, the grid is connected to the filament as in the last experiment, and a heating current is sent through the filament circuit.

The anode is given, successively, increasing positive potentials, and the thermionic current is measured for each potential, by means of the galvanometer.

If the heating current in the filament is as high as in the ordinary working conditions for the valve, the anode current in the above experiments will be of the order of a milliampere. The galvanometer must therefore be shunted. It may be necessary to use a very low-resistance shunt, say, length of copper wire. Alternatively, the value of the heating current may be reduced, but in this case the valve is not in the usual working condition.

The potential on the anode should be increased until saturation is produced. If the potential available is insufficient to produce saturation, the heating current should be reduced until this may be obtained.

The deflections of the galvanometer in all cases may be reduced to micro-amperes, by finding the current sensitivity for the instrument as adjusted for the experiment.

Fig. 391 shows the form of curve obtained in this experiment.

A similar test is then made, applying negative potentials to the anode.

(γ) *Anode Current and Grid Potential*

The relation between the anode current and *grid* potential is the one which is of importance when considering the valve as an amplifier. For such an investigation the arrangements of fig. 390 are set up as shown.

The anode is raised to a potential suitable for the particular valve (generally this is indicated on the valve), and the heating

current is adjusted so that the anode current is saturated with the anode potential. The heating current is maintained at this strength throughout the experiment; for this purpose it is necessary that the ammeter is capable of detecting small changes, and that the heating circuit rheostat is in good condition and capable of giving very good adjustment.

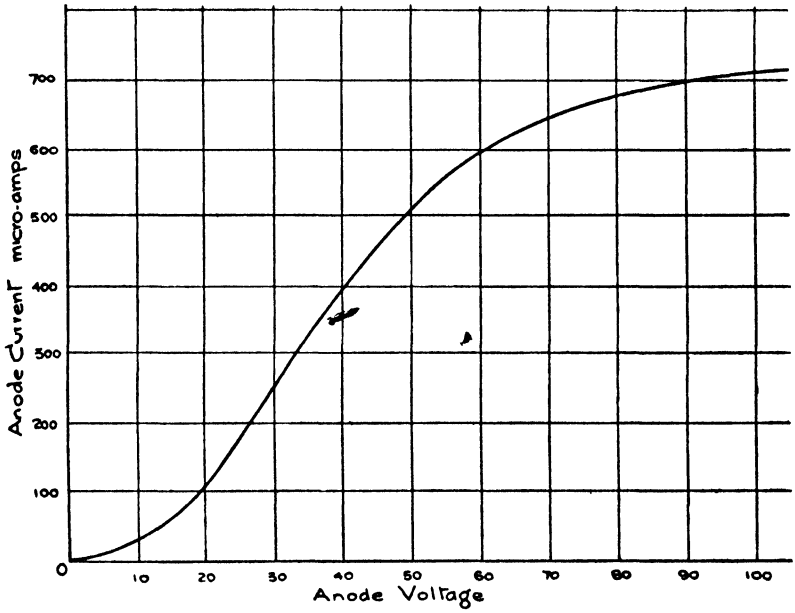


FIG. 391

By choosing the positions of C and D (fig. 390) the grid may be given successive values from, say, −20 to +30 volts (the exact value depend upon the valve in use—the upper positive limit should produce saturation).

The anode is then raised to different potentials, and for each the characteristic curve is obtained. The general form of these curves for a French valve is shown in fig. 392.

The galvanometer deflections are reduced to micro-amperes, as before, before plotting. It will probably be found that the characteristic curve corresponding to the marked anode potential is the one having the steep part approximately bisected by the current axis.

(δ) Grid Current and Grid Potential

The relation between the grid current and grid potential should be now investigated. To do this the galvanometer is placed at G_2, shown in broken lines in fig. 390. The anode is raised to a definite potential (30 volts in the case shown in fig. 392) and, by adjusting C and D, the potential of the grid is raised from zero by, say, 2 volt steps, and the corresponding grid current is measured by G_2. Fig. 392 shows the relative value of the grid and anode currents.

The same process should be tried for negative potentials on the grid.

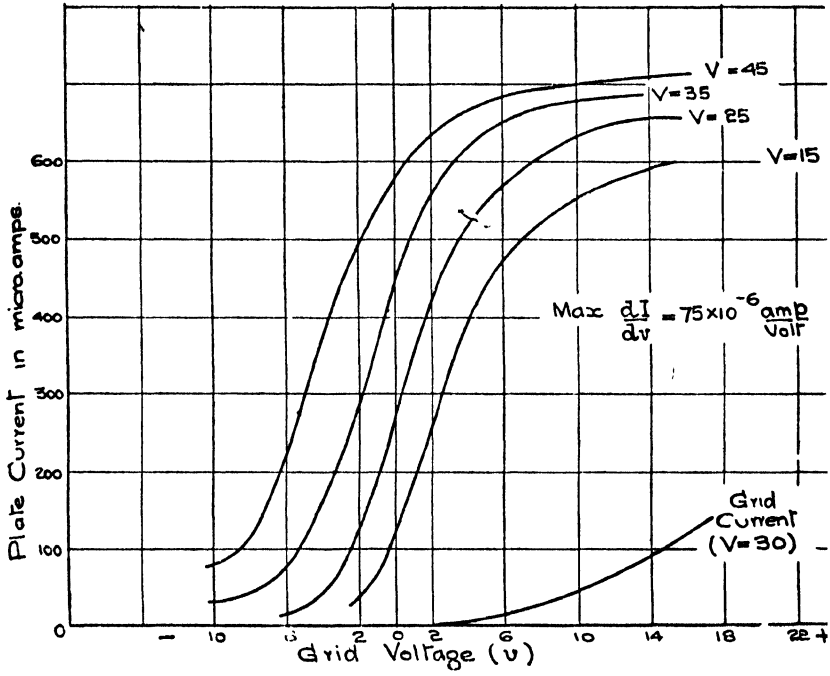


FIG. 392

Résumé of Results Obtained

The result of the first experiment (α) shows that the number of electrons emitted from the heated filament increases with rise of temperature of the filament (measured as an increased total thermionic current for an increased heating current), at first very gradually, and then very rapidly. At this stage a small increase in the filament current causes a comparatively large increase of the total thermionic current.

Experiment (β) shows the chief features of the *two*-electrode valve. In the absence of the heating current such a highly

evacuated bulb is incapable of passing current due to the comparatively low potentials applied. When the plate was raised to a positive potential a current could pass through the space carried by the electrons emitted from the filament. However, in the general form of valve, the application of negative potentials to the anode does not result in a current through it, unless the vacuum is impaired and a small amount of ionization is set up in the residual gas. In this way one function of the valve is seen to be that of rectifier; for the application of an alternating potential to the plate would result in a unidirectional current only in the valve. An example of this is seen in the experiment on page 628.

It should be noted, however, that for some purposes valves are made with a residual atmosphere of helium to purposely produce ionization (usually used in amplifying receiving in wireless telegraphy).

As a rectifier, too, it should be noted that for a given filament current there is a maximum thermionic current, corresponding to a definite anode potential. Any increase in the anode potential beyond this 'saturation potential' does not produce corresponding current increases.

The valve could therefore be used as a means of producing steady currents when there is an excess of potential available.

The result of experiment (γ) as expressed in curves similar to those in fig. 392 shows how the valve may be used as an *amplifier*.

Consider the curve shown for an anode potential of 35 volts. For a grid potential of −10 volts, an increase in potential of 1 volt (i.e. to −9) causes but small increase in the plate current; but when the grid is at a potential from −5 to +2 volts, a similar change in potential causes a very large change in the anode current. Thus, if a feeble potential charge be applied to the grid under such conditions the result is evidenced by a comparatively large anode current charge.

For such purpose, the valve, having an anode potential of 35 volts, would be maintained at $-1\frac{1}{2}$ volts, i.e. in the middle of the steep part of the curve.

Normally, the anode potential would be such that, when the filament current is adjusted, as already described, the middle of the steep part of the curve occurs for zero potential on the grid.

Working under such conditions the anode current follows very closely the changes of potential on the anode, however rapid these may be.

Many applications of the valve are to be found.

For a good account of the subject introduced here, see Van der Bilj, "The Thermionic Vacuum Tube and its Applications."

Generation of Undamped Electrical Oscillations by means of a Valve

Continuous (i.e. undamped) oscillations are most conveniently produced by means of a thermionic valve. The method to be described is only one of many, but is perhaps the simplest from a theoretical point of view. The essential feature is the introduction of a suitable amount of mutual induction between the grid and anode circuits.

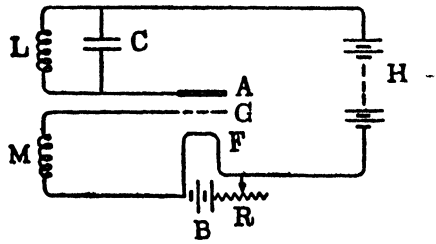


FIG. 393

In the diagram, A, G, and F are the anode, grid, and filament, respectively, B the filament battery, and R a rheostat of a few ohms resistance. The anode battery, H, will be of a voltage to suit the valve, and its positive terminal must be connected to the anode. LC is the oscillatory circuit, the inductance, L, being coupled to the inductance, M, in the grid circuit. It will be convenient to have the mutual inductance between L and M variable—a maximum coefficient of about 1 millihenry will in general be amply sufficient—and they may consist of two flat coils placed one above the other, variation being secured by sliding the upper one in its own plane.

The condenser, C, need not be inserted unless there is some special reason for so doing, as the 'self-capacity' of the coil is in general sufficient to give a definite periodicity to the circuit. If a condenser is used, however, it should be remembered that the mutual inductance, LM, required to maintain oscillations, is roughly proportional to the capacity of the oscillating circuit, and if the latter is large (e.g. of the order of 1 micro-farad) very 'close coupling' between L and M may be necessary. Briefly, the action is as follows:

If a small oscillation occurs in LC a voltage of the same frequency will be induced in M, and thereby impressed on the grid, G. In consequence of the controlling effect of the grid voltage on the anode current a variation of the latter will result, the frequency being the same as the natural frequency of LC. That is the supply of current to the oscillatory circuit will be synchronous with the oscillations already existing, and if the relative phases are favourable and the rate of supply of energy from H is equal to the rate of dissipation in LC, the oscillations

will be maintained with undiminished amplitude. The former condition requires the mutual inductance, LM, to be of the correct sign (ascertained by trial, reversing connexions of one if oscillations cannot be obtained) and the latter (for a given anode voltage and filament current) determines its magnitude.

The oscillations may be detected in very many ways, as for example by the aid of various types of wireless receiving apparatus. The simplest method, which is not, however, of entirely general applicability, is to include a milliammeter (or shunted galvanometer) in the anode circuit. It is usually found that the reading of such an instrument changes when oscillations set in, but under certain conditions this test may fail. A more certain, and at the same time an instructive, method is to connect a galvanometer in series with another valve across the coil, L, as shown in fig. 394. In the absence of any oscillations of voltage across L no current can flow through the galvanometer, since the anode of the valve in series with it is permanently negative with respect to the filament. When oscillations occur a fairly large alternating P.D. is set up in L, and during that half of the alternation for which the lower end of the coil is positive with respect to the upper, an electron current flows to the anode and is detected by the galvanometer. The deflection can be adjusted as desired by altering the filament current of the valve, which in this case acts simply as a rectifier.

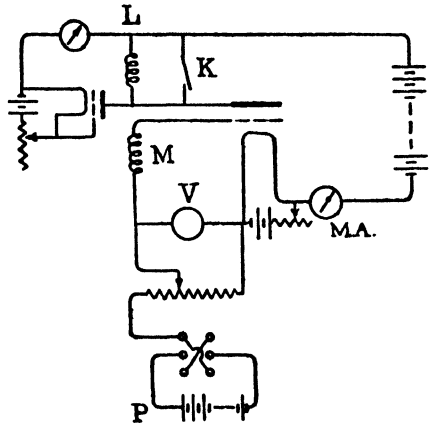


FIG. 394

With this arrangement a variety of experiments on oscillating valve circuits may be carried out. As an example we may take the determination of the grid voltages for which the production of oscillations is possible. If a variable voltage is applied to the grid it will be found that oscillations can only be obtained between certain limits. These may readily be determined, and

should be plotted on the anode current-grid voltage characteristic, which we will suppose to have been already obtained for the valve under identical conditions of anode voltage and filament current (fig. 392). They will in general be situated in the neighbourhood of the two ' bends ' in the characteristic, and at points where the slope of the curve is the same. This result, which can be established theoretically, should be verified by measurement of the two slopes.

The grid of the rectifying valve is maintained at a constant potential by connecting it to the filament.

The key, K, is for short-circuiting the coil, L, when the oscillations are required to be suppressed. Observations should be made by closing K and noting whether the galvanometer deflects on opening it, i.e. whether oscillations will commence. This procedure is necessary in order to obtain a definite limiting value of the grid voltage, as oscillations, once in progress, will often persist for grid voltages for which they will not recommence after being interrupted.

It is advisable to have a milliammeter (M.A. in diagram) or shunted galvanometer in the anode circuit for temporary use in detecting oscillations while the rectifying valve and galvanometer are being adjusted.

The voltage of the grid potentiometer battery, P, should be sufficient to take the grid voltage past the bends in the characteristic. V is a voltmeter for measuring the potential applied to the grid.

RADIOACTIVITY AND CONDUCTION OF ELECTRICITY THROUGH CASES

Saturation Curve

The object of this experiment is to determine the relation between the potential difference applied to an ionised gas, and the resulting current. Two metal plates are supported horizontally by ebonite rods, a fixed distance apart. The lower plate is coated thinly with a radio active substance, say uranium nitrate, and is connected to a battery of cells. The point of contact of this connecting wire is variable and the other end of the battery is earth connected, so that the applied potential may be any multiple of that of the single cell. The upper plate is connected to one pair of quadrants of an adjusted quadrant electrometer, and to one pool of mercury of a key of the type shown in the footnote on page 589. The second pair of quadrants is joined to the second mercury cup which is earthed. When the mercury key is closed the four quadrants are all at the same potential (zero).

To carry out the experiment, the potential of one cell, say two volts, is applied to the lower plate ; the mercury key is opened, and the resulting charge acquired per second by the quadrant is observed by timing the rate of movement of the reflected spot of light of the electrometer. The charging up of the electrometer should be regular, i.e., the time taken to traverse one centimetre of the scale, the same in all parts ; this is timed with a stop-watch over, say, 20 cms., and the number of divisions per second calculated. The key is then closed and all quadrants earthed ; the process is then repeated two or three times and a mean value obtained.

Similar observations are then made when the potential of two, three, four, etc. cells is applied in turn to the coated plate, until further potential produces no increase in the rate of movement of the electrometer needle.

The battery described on page 623 will serve very well for the above purpose ; if, however, the potential available from this source is not sufficiently large to produce saturation, careful use of the supply mains may be made. In this case the earth wire to the cells is replaced by a lead from the " positive " main, through a water resistance, and the potential of the main plus that of one, two, three or more of the cells is obtained. Of course the special conditions of the laboratory will determine the exact method of procedure in such a case.

Having, by the above method, obtained the rate of charging of the electrometer, the corresponding currents may be calculated from a knowledge of the sensitivity of the instrument and its capacity, as obtained by the methods given on pages 591 and 597. For example, suppose the sensitivity is s divisions per volt, the capacity k, and for one potential on the plate the number of divisions per second is D ; the increase of potential *per second* is $\frac{D}{s}$ volts, and the quantity of electricity given to the quadrants *per second* is, therefore, $\frac{kD}{s}$, where k, D and s are in the same units. As described above, k will be in electro-static units, and is therefore equal to $\frac{k}{9 \times 10^{20}}$ E.M. units, or $\frac{10^{-11}k}{9}$ farads; the current is therefore $\frac{10^{-11}kD}{9s}$ amperes. The results obtained are then represented graphically, the ionisation current being plotted as ordinates, and the applied potential as abscissal. It will be found that the general form of the curve is similar to that of fig. 391.

Repeat the above series of observations, for one or two values of distance between the plates and note the effect on the saturation current and the potential required to produce saturation.

Note.—It may be found that, when measuring the movement of the needle for the higher values of the potential, some uncertainty arises due to its rapidity. In such a case insert an air condenser in parallel with the electrometer and adjust the distance between the condenser plates to produce a slower movement of the needle. When calculating the current from such observations k in the above formula is the sum of the capacity of the condenser and the electrometer.

α φ β and γ Ray Activity of a Radioactive Substance

The apparatus of the last experiment is set up and the saturation potential is applied to the lower plate which is coated as before, on the upper surface, with the compound under investigation. Thin sheets of aluminium about .01 cm. thick are placed successively on the top of the compound and in each case the saturation current is measured. The effect will be to reduce the ions formed and consequently the value of the current. When all the α and β rays are cut off the further addition of thin foil will not reduce the current so greatly as before. By examination of the curve showing the current plotted against the thickness of aluminium, the amount of activity due to the α and β and to the δ rays may be obtained.

At the stage when γ rays alone are operative, the coefficient of absorption of the γ rays may be obtained in lead, by placing a thin sheet of lead over the aluminium and observing the saturation current once more.

If I_0 is the intensity of the γ rays on entering the lead (proportional to the saturation current before inserting the lead), and I the intensity after passing through the lead, μ, the coefficient of absorption in lead is defined by $I = Ie^{-\mu K}$ where K is the thickness of the lead sheet used. I and I_0 may be replaced by the corresponding *rates* of charging of the electrometer.

UNITS

LIGHT :

Wave length measurement—

1 Ångström (or Tenth Metre) $= 10^{-10}$ metre $= 10^{-8}$cm.
1 Micron (denoted by μ) - $= 10^{-6}$ cm.
1 $\mu\mu$ - - - $= 10^{-9}$ cm.

SOUND :

Velocity in air at 0° C. (V_0) = 33,129 cms. per sec.
Velocity at t° C. (V_t) - $= V_0 + 61\,t$.

ELECTRICITY :

E.M.U. denotes Electromagnetic unit.
E.S.U. denotes Electrostatic unit.
P.U. denotes Practical unit.

P.U.		E.M.U.	E.S.U.
Quantity.	Coulomb	10^{-1}	3×10^{10}
Current.	Ampère	10^{-1}	3×10^{10}
Potential.	Volt	10^{8}	$\frac{1}{3} \times 10^{-2}$
Resistance.	Ohm	10^{9}	$\frac{1}{9} \times 10^{-11}$
Capacity	Farad	10^{-9}	9×10^{11}
	Microfarad (10^{-6} Farad)		
Inductance	Henry	10^{9}	$\frac{1}{9} \times 10^{-11}$
	Centimetre	1	$\frac{1}{9} \times 10^{-20}$

The Electrochemical Equivalent of Hydrogen = ·00001037 grams per coulomb.

$\frac{e}{m}$ for an electron $= 1{\cdot}77 \times 10^{7}$ E.M.U. $= 5{\cdot}31 \times 10^{17}$ E.S.U.

e (charge on an electron) $= 1{\cdot}57 \times 10^{-20}$ E.M.U. $= 4{\cdot}71 \times 10^{-10}$ E.S.U.

MAGNETISM:

	P.U.	E.M.U.	E.S.U.
Intensity.	Gauss - -	1	3×10^{10}
Induction.	(Lines per sq. cm.) Gauss	1	$\frac{1}{3} \times 10^{-10}$
Flux.	(Total lines.) Maxwell	1	3×10^{10}

APPENDIX

Shuster's Method of Focussing a Spectrometer for Parallel Light

WHEN a distant object is not available for the purpose of focussing the collimator and telescope for parallel light, Shuster's method may be employed.

Illuminate the slit of the collimator with yellow light and without any focussing, place the prism approximately in the position of minimum deviation. Turn the prism slightly away from this position, bringing the refracting angle towards the telescope. Focus the telescope on the image as distinctly as possible, making the slight rotation of the telescope which may be necessary to keep the image in the field of view. Rotate the prism slightly to the other side of the minimum position and focus the collimator until on looking into the telescope the image is again as distinct as possible. Repeat this process of alternately focussing the collimator and telescope until the rotations of the prism do not cause the image to go out of focus. When this is the case the light entering and leaving the prism is parallel.

If the prism is first turned so that the refracting angle moves towards the collimator, then the first focussing must be made by means of the collimator. Should any mistake arise at this point the image will rapidly become more and more indistinct, and will call attention at once to the mistake. Usually only a few alternate focussings are necessary.

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