SOLID GEOMETRY

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PREFACE

This introduction to solid geometry is designed for a high-school or college student who has had one year each of elementary algebra and of plane geometry. A summary of important facts in plane geometry is contained in the Appendix. No previous knowledge of trigonometry is presupposed, but the cosine of an angle is introduced and used in connection with the study of projections.

A special feature of this text is the emphasis placed upon learning to visualize and draw three-dimensional figures, as well as to measure them and prove geometric theorems about them. The drawing is facilitated by a drawing triangle known as the trimetric ruler, which is enclosed in the pocket inside the back cover. With its aid, as explained briefly in Chap. 1 and more fully later, the student should be able to make quite satisfactory drawings of rectilinear three-dimensional figures in a short space of time.

Each of the four parts of the book contains ten chapters, the last of which is a review chapter. Each chapter is followed by a list of oral questions, which may be used for self-examination or for class discussion, and a list of written exercises for outside assignments. Answers to odd exercises are given whenever these exercises are computational problems.

It is expected that a college class of well-prepared students might study one chapter per lesson. For high-school classes it would generally be necessary to allow two lessons per chapter, except for the review chapters. Thus a 30-lesson course in college, including a test every sixth lesson, or a 50-lesson course in high school might reasonably be expected to cover the first 25 chapters. In a longer course it would be possible either to continue further in the book without skipping or to omit Chaps. 26 to 30 and pass on directly to the study of projections and maps in Part IV. In a college course of 50 lessons it should be possible to complete the text.

JAMES SUTHERLAND FRAME

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INTRODUCTION

We live and move in a three-dimensional world, and yet most of our impressions of this world through photographs and books are two-dimensional. By experience and intuition the mind of a sighted person can imagine a three-dimensional reality from the two slightly different two-dimensional pictures that he sees with his eyes. By the reflection of sound waves and what is sometimes called a sixth sense the blind man is aware of three-dimensional objects around him. But a precise knowledge of three-dimensional relationships and the ability to draw figures on a two-dimensional piece of paper so as to create a realistic impression of three dimensions come only as a result of study.

It is the purpose of this text in solid geometry to acquaint the student with the most important facts about the properties of simple three-dimensional figures and to supplement the theory and proofs by practice in drawing and visualizing three-dimensional figures.

J. S. F.

PART ONE

LINEAR AND ANGULAR MEASUREMENT IN SPACE

1

THE RECTANGULAR BOX

1·1 The Pythagorean theorem

Some theorems of solid geometry are surprisingly different from the theorems of plane geometry, while other theorems can be carried over from the plane to space with only slight changes of wording. Reserving detailed proofs and complicated terminology until later, we may cite as an example of the latter the theorem of Pythagoras, which is fundamental to many branches of mathematics. In plane geometry this theorem might be stated for the rectangle, instead of for the right triangle, as follows:

PYTHAGOREAN THEOREM FOR THE RECTANGLE. THEOREM 1A: *The square of the diagonal of a rectangle is equal to the sum of the squares of two adjacent sides.*

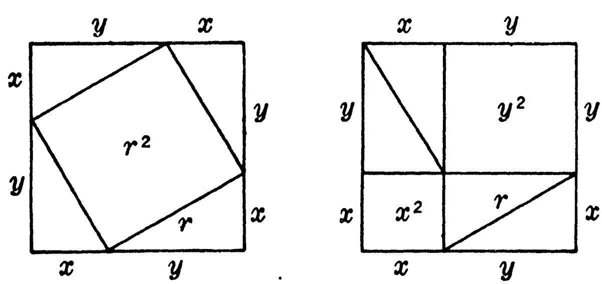

FIG. 1. Pythagorean theorem.

FORMULA: $r^2 = x^2 + y^2$. (See Fig. 1.)

PROOF: A proof of this theorem is easily obtained with the aid of

Fig. 1, which shows two large squares whose sides are each equal to the sum $x+y$ of two adjacent sides of the given rectangle. By subtracting the areas of four congruent right triangles, each of area $xy/2$, from the square area $(x+y)^2$, there is left in the one case a square whose side r is the diagonal of the given rectangle or in the other case two squares whose sides are, respectively, the two sides x and y of the given rectangle. Since these two remaining areas are equal, we have $r^2=x^2+y^2$.

1·2 The Pythagorean theorem in space

A formula, quite similar to the one just stated for the plane, applies to the computation of distances in space.

PYTHAGOREAN THEOREM FOR A RECTANGULAR BOX. THEOREM 1B: *The square of the diagonal of a rectangular box is equal to the sum of the squares of three adjacent edges that meet at a vertex.*

FORMULA: $r^2=x^2+y^2+z^2$. (See Fig. 2.)

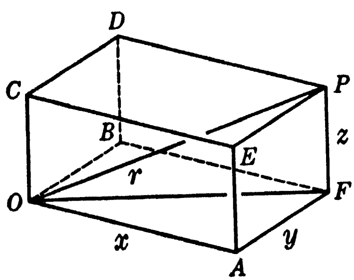

FIG. 2. Rectangular box.

PROOF: Referring to the diagonal $[OP]$ of a box lettered as in Fig. 2, the proof of this important theorem can be analyzed as follows:

1. Triangle $[OAF]$ is a right triangle (shown in horizontal plane, Fig. 2).
2. Hence, $\overline{OF}^2=\overline{OA}^2+\overline{AF}^2=x^2+y^2$.
3. Triangle $[OFP]$ is a right triangle (shown in vertical plane, Fig. 2.)
4. Hence, $\overline{OP}^2=\overline{OF}^2+\overline{FP}^2=(x^2+y^2)+z^2$.

The reason for statement 1 would be found in the definition of a rectangular box, which has not yet been stated explicitly (see Sec. 12·3); statements 2 and 4 follow from the Pythagorean theorem for a rectangle; but the reason for statement 3 must await Chap. 5. Exact proofs of theorems such as this can be given only after our assumptions about space have been clearly stated and our terms have been defined.

APPLICATION: Suppose that, in Fig. 2, the following lengths are given: $\overline{OA}=11$, $\overline{AF}=7$, $\overline{FP}=5$. Then we have $\overline{OP}^2=121+49+25=195$. Hence, $\overline{OP}=\sqrt{195}=13.96$.

1·3 Vertical and horizontal

Many of the space figures that are to be studied in this text are represented by plane drawings. In the plane drawings those lines which are parallel to the bound edge of the page will be called **vertical** lines, and those lines which (in space or in the plane) are perpendicular (at right angles) to the vertical lines will be called **horizontal** lines. For drawings of plane figures all horizontal lines on the page are parallel to each other. In three-dimensional space a vertical line may be described as one which goes up and down like a string on which a weight is hung. Vertical lines may be considered parallel to each other under this definition if the region of space considered is quite small compared with the size of the earth. However, since the earth is round, a vertical line at London is certainly not parallel to a vertical line at New York. The use of the word "perpendicular" to mean "vertical" is not good mathematical usage and should be avoided. Horizontal lines are perpendicular to vertical lines in space as in the plane, but *in three dimensions it is not true that all horizontal lines are parallel to each other.* In fact, it is always possible to pick a pair of intersecting horizontal lines that are perpendicular both to each other and to the vertical line through their point of intersection. This is well illustrated by the three edges of a matchbox (or of any rectangular box) which meet at one of its corners (vertices). The three edges $[OA]$, $[OB]$, and $[OC]$, meeting at the corner O of the box in Fig. 2, form a **rectangular reference system** for the drawing of the box, representing one vertical and two horizontal directions each perpendicular to the other two in space.

1·4 Figures in parallel projection

Until more complicated types of perspective figures are studied (Chaps. 9, 16, 22, 31, 32, 34, 36), fairly satisfactory figures may be obtained, without drawing instruments other than a ruler and pencil, as follows:

1. Draw parallel lines in space as parallel lines in the figure.
2. Draw vertical lines in space as vertical lines in the plane, measuring distances on a convenient uniform scale.
3. Select two convenient directions in the plane that are not ver-

tical directions to represent the two perpendicular horizontal directions in the rectangular reference system in space. The uniform scales to be used in measuring lengths along these lines or parallel lines are in general different from the vertical scale and will be described in detail in Chap. 9. It is usually best to choose directions as in Fig. 2 for which these two horizontal scales will be smaller than the vertical scale.

4. Draw unbroken lines to represent visible edges or lines of the space figure and dotted lines to represent those which are hidden.

1·5 The trimetric ruler

Enclosed in the back cover of this book is a triangular ruler that will greatly facilitate the drawing of space figures and will give exactly the scales to be used along the three directions of the reference system in the drawing. The rules for its use are simple.

1. *Always lay the trimetric ruler with its vertical edge exactly parallel to the vertical edges of the drawing paper*, and draw and measure all vertical lines along the vertical scale (for example, the lines $[OC]$, $[BD]$, $[AE]$, $[FP]$ in Fig. 2).

2. Choose units so that the figure is large enough to be seen clearly, but not too large to fit on the page; and *use the same number of subdivisions on each of the three scales to represent the same unit of length*, such as an inch or a foot, in the space figure.

3. Use the upper slanting edge of the ruler to draw and measure the "left to right" horizontal reference line $[OA]$ of the figure, and all lines parallel to it (for example, the lines $[OA]$, $[BF]$, $[CE]$, $[DP]$ in Fig. 2).

4. Use the lower slanting edge of the ruler to draw and measure the "front to back" horizontal reference line $[OB]$ of the figure and all lines parallel to it (for example, the lines $[OB]$, $[AF]$, $[CD]$, $[EP]$ in Fig. 2).

5. Draw unbroken lines to represent visible edges or lines of the space figure. Dotted lines may be used to represent parts of the figure that are hidden.

The principle of operation of the trimetric ruler is illustrated in Fig. 3. Equal segments $[O\overline{X}]$, $[O\overline{Y}]$, $[O\overline{Z}]$ on three mutually perpendicular lines are represented in the projection plane of the drawing by unequal segments [OX], [OY], and [OZ], drawn in such a way as to obtain correct parallel projection. The points X, Y, and Z would be found in a space model by dropping perpendiculars from points $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ to the projection plane. The directions to be

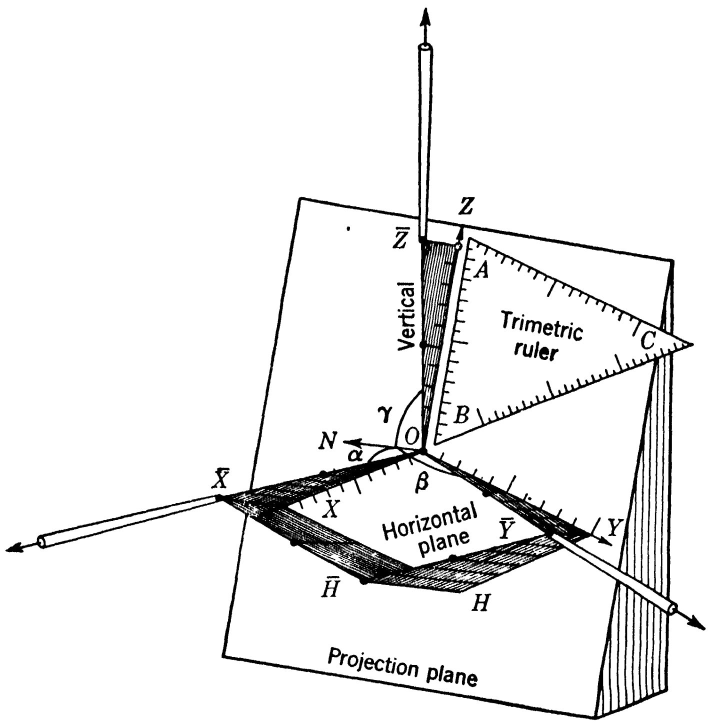

FIG. 3. The trimetric ruler.

used in drawing the segments [OX], [OY], and [OZ] and all lines parallel to them are those of the sides of the trimetric ruler (provided, of course, that one side is always kept vertical), and the scales to be used for measuring lengths in these directions are marked on the corresponding sides of the ruler.

1. ORAL QUESTIONS

A. If three adjacent edges of a rectangular box are 2 in., 2 in., and 1 in., respectively, what is the length of the diagonal?

B. If three adjacent edges of a rectangular box are 6 ft., 3 ft., and 2 ft., respectively, what is the length of the diagonal?

C. If three adjacent edges of a rectangular box are 2 ft., 1 ft., and 3 in., respectively, what is the length of the diagonal?

D. Would a log floating on a lake be represented by a vertical line, a horizontal line, or neither?

E. Vertical angles are studied in plane geometry. Is this use of the word vertical related in any way to vertical lines?

F. Can you place your thumb and two fingers so that each is perpendicular to the other two? How?

G. Could three lines in a plane be drawn so that each is perpendicular to the other two?

H. Would it be possible to place two telephone poles far enough apart so that they would each be vertical and yet each be perpendicular to the other? How?

I. In a plane, is it possible to have two intersecting lines that are both horizontal? Is this possible in space?

J. What are the rules to be followed in making a drawing in parallel projection?

1. WRITTEN EXERCISES

1. Find the length of the diagonal of a cube that measures 6 in. on each edge, rounding off the answer to the nearest hundredth of an inch.
2. A room has a rectangular floor 9 by 12 ft. and is 8 ft. high. What is the distance between diagonally opposite corners?
3. A brick measures 2 by 4 by 8 in. Find its diagonal to the nearest hundredth of an inch.
4. Find the diagonal of a box whose edges are $5\frac{1}{2}$, $3\frac{1}{2}$, and $2\frac{1}{2}$ units, respectively.
5. The lengths of the edges of a rectangular box are given by the literal expressions $x = 2ac$, $y = 2bc$, $z = a^2 + b^2 - c^2$. Express the length of the diagonal in terms of a, b, c.
6. Express in terms of a, b, c, d the length of the diagonal of a rectangular box if the lengths of its edges are given by the literal expressions $x = 2(ac - bd)$, $y = 2(bc + ad)$, $z = a^2 + b^2 - c^2 - d^2$.

7. Draw a figure showing the cube of Exercise 1 and its diagonal.
8. Draw a figure with the trimetric ruler showing the room in Exercise 2 and its diagonal. Take one subdivision to represent 1 ft.
9. Draw a figure with the trimetric ruler showing the brick in Exercise 3. Take two subdivisions to represent 1 in. NOTE: If drawn with the trimetric ruler so that the 4-in. dimension is from front to back (on the lower scale) and the 2-in. dimension is on the vertical scale, then the upper front edge of the brick should appear just below the lower back edge in the drawing.
10. Draw with the trimetric ruler in proper proportions the six different figures of the box in Exercise 4, which are obtained by assigning the three given edge lengths in all possible ways to the three directed scales. Use the same conveniently chosen number of subdivisions for a unit in each case.

2

ASSUMPTIONS ABOUT DISTANCE

2·1 Use of words

It is a common practice in writing fiction to plunge into the middle of the story and show the characters in action before tracing their histories, ancestries, and relationships to each other. This practice has been followed to a certain extent in Chap. 1. But before going further in our exploration of space it will be well to fill in some essential features in the background and define as carefully as possible the meanings of certain words that are fundamental to geometry.

Unfortunately, perhaps, the words **point, line, plane, distance** are commonly used in two somewhat different senses—the technical, or exact, sense, and the practical, or approximate, sense. When we refer to a pencil mark as a point, to a taut string as a line, or to a sheet of paper as a plane, we are using the words only in the approximate sense. All these have thickness, be it ever so small; but a theoretical point, line, or plane has no thickness whatever. When we speak of the distance between two cities, we are using the term distance in an approximate sense. Such a distance could not even be measured to the nearest foot, unless a specified point in each were first selected from which distances should be measured. Each city is then replaced by a point, and the approximate distance between cities on the earth is replaced by an exact distance between geometrical points on a sphere that is used to represent the earth. There are two worlds of measurement—the exact world of geometry, and the physical world in which the terms of geometry (point, line, plane, distance, etc.) are applied to physical objects which approximate them to within various degrees of precision. Numbers obtained from counting may be exact, but numbers obtained from the measurement of physical objects are always approximate. No numerical answer to a practical problem involving measurement can be determined exactly but should be rounded off to a degree of precision that is consistent with the precision of the given measurements. In this book, when distances are given in terms of *abstract units*, they will be *presumed to be geometrically exact;* but when distances are given in *denominate units*, such as inches, feet, yards, miles, or meters, the *answers should be rounded off* to three or four significant figures.

2·2 Significant figures

The first significant figure in a decimal number is the first nonzero digit that appears, reading from left to right. All the digits that follow are significant except zeros that precede the decimal point but do not precede any nonzero digit. These zeros may or may not be significant. To avoid ambiguity in the number of significant figures, the number 30,000 may be shown to three significant figures by writing 3.00×10^4. However, the number 0.0370, which may also be written 3.70×10^{-2}, is unambiguously given to three significant figures in either form. In rounding off a number like 87.349 to three significant figures we write 87.3, whereas in rounding off 87.351 to three significant figures we write 87.4, since the digits 49 represent less than half and the digits 51 more than half a unit in the third significant figure. The numbers 87.35 and 87.45 would both be rounded off to the three-figure number 87.4, since it is customary to round off to the even digit when the neglected fraction is exactly half.

Care should be taken to distinguish between the number of significant figures in a number, which is a measure of relative precision and does not depend on the units used, and the number of correct decimal places to which an answer is carried. For example, 0.0370 is given to four decimals and 87.4 to only one decimal place, but both are accurate to three significant figures. In multiplying or dividing these numbers it would be correct to keep three significant figures in the answer; but if these numbers represent approximate measurements in the same units, then their sum or difference should be rounded off to the least number of decimal places in either one.

2·3 Points in space

In the abstract sense of the word, a metric space of one, two, three, or more dimensions consists of an infinite set of undefined elements called **points,** each of which has no measurement by itself (such as length, breadth, or thickness), but which is related to each other point of the space by associating with each two distinct points of the space a measure called **distance.** We shall denote points by capital letters A, B, C, etc., and the distance between points A and B by the symbol $\overline{AB}$. In order that this space of geometric points should lead to practical results in the approximate measurement of objects in the physical world and at the same time should be sufficiently precise to be described by exact mathematical formulas, we must lay down certain assumptions, or postulates, concerning the nature and relationship of distances between geometrical points.

2·4 Properties of distance

The properties of space are based on the following properties that distance is assumed to possess.

ASSUMPTION 1: *Two points at zero distance are coincident;* that is, $\overline{AB}=0$ if and only if A and B are two different labels for the same point. NOTE: If $\overline{AB}$ is not zero, we say that A and B are **distinct points.**

ASSUMPTION 2: *Distance is symmetric;* that is, $\overline{AB}=\overline{BA}$ for any two points A and B.

ASSUMPTION 3: *The ratio of two distances is a nonnegative number.*[1] Given any four points A, B, C, D, of which C and D are distinct, then there is a uniquely determined nonnegative number $r=\overline{AB}/\overline{CD}$ known as the **ratio** of the distances $\overline{AB}$ and $\overline{CD}$. We may write $\overline{AB}=r\cdot\overline{CD}$.

ASSUMPTION 4: Given any three points A, B, C in space, the so-called **triangle inequality** holds for the distances between pairs of them: $\overline{AC}+\overline{CB}\geq\overline{AB}$.

[1] The word *number* in this book will be used for what is technically called a *real number.*

ASSUMPTION 5: Distances can be expressed in terms of any arbitrary unit of distance, defined by any selected pair of distinct points. Thus to any chosen pair of distinct points O and I may be assigned the number 1, so that $\overline{OI}=1$ unit. Each distance $\overline{AB}$ can then be written as the product of a number $d=\overline{AB}/\overline{OI}$ and a unit $\overline{OI}$.

It follows that, if a different pair of distinct reference points O' and I', such that $\overline{OI}/\overline{O'I'}=k$, are chosen to define a new unit, then the number of new units in the distance between any given pair of points is k times the number of old units:

$$\overline{AB}/\overline{O'I'}=(\overline{AB}/\overline{OI})\ (\overline{OI}/\overline{O'I'}).$$

2·5 Units of distance

For scientific purposes a unit of distance for the physical world is defined by two fine scratches on a metal bar kept in a vault at the U.S. Bureau of Standards under constant conditions of temperature and pressure. Other units may be defined by direct or indirect comparison with this standard, but in making such a comparison it is tacitly assumed that there are measuring sticks which do not change in length when they are moved from one place to another. This assumption is denied in the modern theory of relativity, where the measurement of distance is made to depend upon the measurement of time and where it is assumed that light appears to travel past an observer at a constant speed of about 186,000 miles per sec. in a vacuum, regardless of the motion of the source or of the observer.

The following units of distance are in common use:

ENGLISH SYSTEM

12	inches	= 1 foot (12″ = 1′)
3	feet	= 1 yard (3 ft. = 1 yd.)
16.5	feet	= 1 rod
5,280	feet	= 1 statute mile
6,080	feet	= 1 nautical mile

METRIC SYSTEM

10 millimeters	= 1 centimeter (10 mm. = 1 cm.)
100 centimeters	= 1 meter
1,000 meters	= 1 kilometer (1,000 m. = 1 km.)
10,000 kilometers	= 1 quadrant of a great circle on the earth approximately

Conversion from one system to the other is given by the definition 39.37 in. = 1 m. From this we obtain the approximate relations

$$1 \text{ in.} = 2.5400 \text{ cm.}$$
$$1 \text{ mile} = 1.609 \text{ km.}$$

2·6 The line segment

If A, B, and C are distinct points for which $\overline{AC}+\overline{CB}=\overline{AB}$, so that the equality holds in Assumption 4, then C is said to lie **between** A and B, and the three points A, B, and C are said to be **collinear** (Fig. 4).

FIG. 4.

DEFINITIONS: A **line segment**, or **segment**, consists of any two points A and B, called end points, together with all the points between them. The segment whose end points are A and B is denoted by $[AB]$ or $[BA]$, and the distance $\overline{AB}$ is called the **length** of the segment. The **extension** of a line segment $[AB]$ through A consists of all points C such that A lies between C and B.

From the triangle inequality (Assumption 4) it follows that a line segment is the shortest path between its two end points. For if C were any point not on the segment $[AB]$, then a path from A to B by way of C would be at least as long as $\overline{AC}+\overline{CB}$, and this is greater than $\overline{AB}$.

ASSUMPTION 6: *A line segment is a continuum;* that is, given any two points A and B and any real number r between 0 and 1, then there exists a unique point C between A and B such that $\overline{AC}/\overline{AB}=r$.

DEFINITION: The point M between A and B such that $\overline{AM}/\overline{AB}=\frac{1}{2}$ is called the **mid-point** of $[AB]$.

The student of Euclid will note that Euclid used the word **line** to denote what is here called a line segment. It is more common in modern geometry to use the word line in a more extended sense, to be defined in the next chapter.

2·7 Congruence

The concept of congruence is closely bound up with the notion of distance. It may be defined as follows:

DEFINITION: Two geometric figures are said to be **congruent** to each other (written $\cong$) if the points in the one figure can be made to correspond in a one-to-one manner with the points of the other figure so that the distance between any two points in the one figure is equal to the distance between corresponding points in the other figure.

The term congruence is used by some authors in a narrower sense which we shall call direct congruence. In an intuitive sense, two geometric figures are **directly congruent** if one can be moved rigidly so as to coincide with the other.[1]

Two figures which are directly congruent are commonly called **superposable.** Two figures which are congruent but not directly congruent are commonly called mirror images or reflections of each other; technically they are called **enantiomorphous.**

THEOREM 2: *Two line segments are congruent if and only if they have the same length* (see Exercise 10).

2. ORAL QUESTIONS

A. In geometry, a line segment is the shortest path between two points. What is meant by the shortest path between two cities A and B in traveling by automobile? Given a mileage chart of distances between principal cities, how could you determine whether or not a given city C is on the shortest path from A to B?

B. Two points A and B in a city are connected by a number of one-way streets. If the distance is measured by the reading of the speedometer on an automobile that travels from A to B and back, is it always true that $\overline{AB} = \overline{BA}$?

C. In mountain guidebooks the distance between two points is often given in hours of hiking time. Would this definition of distance satisfy the relation $\overline{AB} = \overline{BA}$?

[1] To avoid the undefined concept of a rigid motion we may substitute the following rigorous definition.

DEFINITION: Two congruent figures F_1 and F are said to be **directly congruent** if given any arbitrarily small distance d, it is possible to find a finite sequence of figures $F_1, F_2, \cdots, F_n = F$ each congruent to each other, and arranged in an order so that corresponding points in consecutive figures are within distance d of each other.

D. How do you interpret the expression "the distance from the earth to the sun"? Is the meaning exact in a geometric sense? Could it be made exact by introducing words such as "center"? Could such a distance be measured to the nearest foot? Must the time of measurement be specified?

E. In defining the number $\overline{AB}/\overline{CD}$ in Assumption 3, why is it necessary to demand that C and D be distinct points?

F. Is it possible to place four points in the plane so that each is at unit distance from each of the others? Is this possible in space?

G. What are the assumptions about distance and the line segment that were made in this chapter?

H. Which of the assumptions about distance and the line segment are satisfied if "space" should be made to consist of points on the surface of a spherical globe and "distance" should be measured by applying a tape measure along the arc of a great circle on the globe? Would there be a unique mid-point of the "segment" defined by any two points? Must more assumptions about points and distances be made in order to distinguish a flat surface from a curved surface?

I. When are two figures said to be congruent?

J. Can you give examples of figures in space which are enantiomorphous, that is, congruent but not directly congruent?

K. What is the result of rounding off the number 93,890,200 to four significant figures? To three significant figures?

L. What is the result of rounding off the number 0.009765 to three significant figures? To three decimal places?

2. WRITTEN EXERCISES

1. Round off each of the following numbers to four significant figures: (*a*) 6,999,530; (*b*) 651.947; (*c*) 651.951; (*d*) 0.014996.
2. Round off each of the following numbers to two decimal places: (*a*) 69.9953; (*b*) 651.947; (*c*) 651.951; (*d*) 0.014996.
3. Express 1 statute mile in each of the following units: (*a*) rods; (*b*) yards; (*c*) inches; (*d*) centimeters.
4. Express 180 cm. in each of the following units: (*a*) inches; (*b*) feet; (*c*) yards; (*d*) meters.

5. Sound travels at about 1,100 ft. per sec. Express this speed in miles per hour.
6. Using the assumptions about distance, show that if A, B, C are any three points and if we write $\overline{BC}=a$, $\overline{CA}=b$, $\overline{AB}=c$, $a+b+c=2s$, then none of the four quantities s, $s-a$, $s-b$, and $s-c$ can be negative.
7. Given the accompanying table of distances between points. Show that $\overline{AB}+\overline{BE}=\overline{AE}$. Find all other sets of three points from the table for which the equality in Assumption 4 holds. How are such sets of three points related to each other in space?

	B	C	D	E	F
A	63	119	119	105	51
B		70	70	42	30
C			112	56	68
D				56	100
E					64.6

8. Prove that if B lies between A and C, and C lies between A and D, then B lies between A and D. HINT: Show first that $\overline{AD}=\overline{AC}+\overline{CD}=\overline{AB}+\overline{BC}+\overline{CD}\geq\overline{AB}+\overline{BD}\geq\overline{AD}$. Show why the inequality is impossible.
9. Prove that if B lies between A and C and C lies between B and D, then $\overline{AB}+\overline{BD}=\overline{AC}+\overline{CD}$.
10. Prove that two line segments are congruent if and only if they have the same length.
11. Given three circles lying exterior to each other in a plane and denoted by A, B, C. Let the distance $\overline{AB}$ between circles A and B be defined as the length of the diameter of the smallest circle that touches both A and B. Does this definition of distance between circles always satisfy Assumption 4 (the triangle inequality)? Explain your answer.
12. Which of the first five assumptions made in this chapter about distances between points do not hold for distances between circles as defined in Exercise 11?

3

LINES AND PLANES IN SPACE

3·1 The infinite line

DEFINITION: The line (AB), determined by two distinct points A and B, consists of the points of the segment $[AB]$ and all points on the extensions of $[AB]$ through A and through B. The word line in this book means the same thing as straight line.

ASSUMPTION 7: *A line is unbounded in both directions.* That is, given any line segment $[AB]$, there exist points C and D in the extensions of $[AB]$ through A and B, respectively, for which the ratios $\overline{CB}/\overline{AB}$ and $\overline{AD}/\overline{AB}$ are arbitrarily large positive numbers.

ASSUMPTION 8: *A line is determined by any two of its points.*

It follows that any three points of a line are collinear. If all the points of a space are collinear, the space is said to be **one-dimensional.**

THEOREM 3A: *Each two lines are congruent.* The proof is left to the student.

3·2 Visual determination of a line

Under ordinary circumstances rays of light travel in straight lines.[1] It may then be possible to place the eye on the line of two points, but outside their segment, so that one point appears to cover the other point. It will then cover all points in their segment. This is the principle used in surveying, in aiming a rifle, and in many applications where the collinearity of points is determined visually. It may be used in checking the straightness of a stick of

[1] Exceptions occur, for instance, in the refraction of light in the atmosphere, in water, or by lenses.

wood or in finding a third point between two given points that are to be connected by drawing a line with a ruler that is too short to reach between them.

3·3 Representation of lines in a drawing

Since it is impossible to draw an infinite line on a finite piece of paper or even to draw any mark that has length but no breadth or thickness, lines and rays (see Sec. 4·2) are represented in drawings by thin marks of finite length drawn with a straightedge. Sometimes even thick "lines" are drawn to aid visualization. Lines that are hidden by other parts of a figure are often indicated by a series of collinear dots and dashes. It should be borne in mind that a line extends to infinity in both directions beyond the segment which represents it in the drawing.

3·4 Linear space

DEFINITION: A **linear space** is a set of points, such that if any two distinct points P and Q in the space are given, then every point of (PQ) is a point of the space.

If not all points of a linear space lie on the same line, then the space must contain more than one line through each of its points. Two distinct lines having a point in common are called **intersecting lines.**

3·5 The plane

DEFINITION: If two lines (OA) and (OB) have just one point O in common, then this point, together with the mid-points of the segments whose end points lie one on each of these lines, are said to form a **plane** (OAB).

The two intersecting lines are said to *determine* the plane. Since to each pair of points A' and B' on the lines (OA) and (OB), respectively, there corresponds a unique mid-point M of the segment $[A'B']$, and since, conversely, each point M determines a unique pair of points A' and B', the plane (which consists of the points M) is called **two-dimensional.**

ASSUMPTION 9: *A plane is a linear space.*

A line in a plane divides the plane into two **half planes,** each having the line as an **edge.** The segment joining two points both lying in one half plane, but not on the edge, lies entirely in the half plane and contains no point of the edge.

3·6 Representation of planes in a drawing

A plane is usually represented in a drawing by a parallelogram or other polygon lying in the plane, but it should be borne in mind that a plane is unlimited in extent, even though only a finite portion of it is represented in the drawing.

In representing a horizontal plane by a drawing with the trimetric ruler, it is convenient to draw a parallelogram, using the upper and lower slanting scales of the ruler ($OXHY$ in Fig. 3).

3·7 Coplanar and parallel lines

A set of lines or points or other geometric figures all lying in one plane are called **coplanar.** Two coplanar lines that do not have a point in common are called **parallel lines** (symbol $\|$). Two lines that have one point in common are **intersecting lines.** Two distinct lines cannot have two distinct points in common.

ASSUMPTION 10: EUCLID'S PARALLEL POSTULATE. In a plane, if a line and a point not on it are given, there is one and only one line passing through the point and parallel to the given line.

3·8 Determination of a plane

Four simple ways of determining a plane are as follows. A plane is determined

1. By two intersecting lines.
2. By a line and a point not on the line.
3. By three noncollinear points.
4. By two parallel lines.

The first of these statements follows from the definition of the plane. Proofs of the third and fourth are listed as Exercises 3 and 4, respectively. They should be written up in a manner similar to the proof of the second statement, which is given here as an illustration. First state the theorem in general terms. Then label specific points or lines in the hypothesis. Finally list the separate steps from hypothesis to conclusion, giving the reason for each step.

THEOREM 3B: *A unique plane is determined by a line and a point not on the line.*

HYPOTHESIS: Let A and B be two distinct points on a given line (AB), and let C be a point not on (AB) (Fig. 5).

PROOF: 1. A plane that contains the point C and the line (AB) contains the line (AC). (By Assumption 9.)

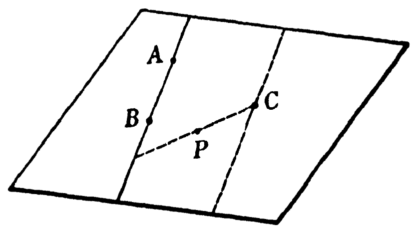

FIG. 5.

2. A unique plane is determined by the intersecting lines (AB) and (AC). (By the definition of a plane.)

3. All points P of this plane lie either on lines joining C to points of (AB) or on a line through C parallel to (AB). (By Assumption 10.)

Q.E.D.

THEOREM 3C: *Any two planes are congruent.*

The student should supply the proof of this theorem, after re-examining the definition of congruence.

3·9 Assumptions in three-dimensional space

Three assumptions, not all independent, will be made to fix the space of solid geometry as a space of exactly three dimensions.

ASSUMPTION 11: *Not all points of space are collinear.* (That is, space has more than one dimension.)

ASSUMPTION 12: *Not all points of space are coplanar.* (That is, space has more than two dimensions.)

ASSUMPTION 13: *Two planes having a point in common must have at least two distinct points in common.* (That is, space has not more than three dimensions.)

3·10 Parallel and intersecting planes

Two planes that have no points in common are called **parallel planes.** Two nonparallel planes have (by Assumption 13) at least two distinct points in common. Hence the line of these points lies wholly in each of the planes, and the planes have a line in common. No points not on this line can lie in both planes, or the planes would coincide. Hence two distinct nonparallel planes intersect in a line. A set of three or more planes having a line in common is called **coaxial** (or coaxal), and the common line is called the **axis.**

3·11 Skew lines

Two lines that are not coplanar are called **skew lines.** Since all the figures in the book are plane drawings, it is necessary to represent skew lines by coplanar lines in the figures. It should be borne in mind, however, that skew lines pass by each other without intersecting, even as a highway may pass over a railroad bridge without intersecting the tracks below, though on a map the highway and railroad may appear to intersect each other. To indicate this, the back line is broken at the point where it passes through the plane determined by the front line and the observer's eye. In Fig. 6 a set of coaxial planes is shown being intersected by a line skew to the axis.

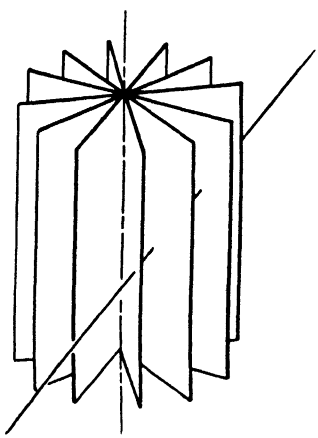

FIG. 6. Coaxial planes.

3·12 Lines and planes

If a line and a plane have no points in common, they are said to be **parallel.** If they have just one point in common, they are said to **intersect,** and the common point is called the **foot** of the line. If they have two points in common, the line lies wholly in the plane.

If a plane contains one, but not both, of two parallel lines, it is parallel to the other line. Just one plane can be drawn parallel to one of two skew lines and containing the other, namely, the plane that is determined by the second line and by a line drawn parallel to the first through any point of the second. It follows from this that a line may be parallel to a plane, yet not parallel to some line which lies in the plane.

The student will note that many of the statements of theorems of plane geometry apply immediately to space without rewording but that others must be modified somewhat before they are applicable.

3. ORAL QUESTIONS

A. What is the difference in meaning between the terms *line* and *line segment?*

B. In what sense is the term *line* used when people are asked to *line up?* What test can be used to see whether or not such a line is really a straight line? What are the points in such a line?

C. What is a linear space?

D. What is a plane? Can you prove that a plane is a linear space without using Assumption 9?

E. Are two intersecting lines always coplanar? Do two coplanar lines always intersect?

F. Is it true in space that two lines which do not intersect are parallel?

G. What are four ways of determining a plane?

H. Can you prove that two planes which have one point in common have at least two distinct points in common?

I. Given four points A, B, C, D not in the same plane, how many lines are there that join pairs of these points, and how many planes are determined each by three of these points?

J. Given the twelve edges of a box (Fig. 2). How many of the edges intersect, are parallel to, or are skew to a particular edge $[OA]$. (The student should examine the edges of a box such as a matchbox or should use a closed book, to aid in visualizing the figure.)

K. Given the same box (Fig. 2), how many planes are there that each contain at least two of the edges?

L. Given the same box (Fig. 2), show that 48 right triangles can be formed whose vertices are vertices of the box. HINT: How many are there in each of the planes of Question K?

M. Given two skew lines and two distinct planes parallel to each. Are the planes necessarily parallel to each other?

3. WRITTEN EXERCISES

1. With the trimetric ruler, draw a parallelogram representing a horizontal plane, and draw two intersecting lines each parallel to the plane.
2. Draw a figure showing three parallel lines in space that do not lie in the same plane. Indicate, by drawing parallelograms, the three planes that they determine.
3. Prove that three noncollinear points determine a plane.
4. Prove that two parallel lines determine a plane.
5. Prove that, if two parallel planes are both cut by a third plane, the lines of intersection are parallel. Draw a figure.
6. Prove that if a plane contains one but not both of two parallel lines it is parallel to the other. Draw a figure.
7. If $\overline{AC}+\overline{CB}>\overline{AB}$, does this necessarily imply that the points A, B, C do not lie on a line? Explain.
8. Prove that if a line is parallel to a plane, and a second plane is determined by the given line and any point in the given plane, then the line of intersection of the planes is parallel to the given line.
9. Prove that two planes each parallel to the same plane are parallel to each other.
10. In a plane, if A, B, C are points on one line and A', B', C' are points on another line (none of the six points being the point of intersection of the lines), and if each of the points A, B, C are joined by segments to each of the points A', B', C', then there must be at least one point P, other than the six given points, at which two of these segments intersect. Draw a figure for a case where there is only one such point P. Is there ever such an intersection point P if the two given lines are not coplanar?
11. Let A, B, C, D be four noncoplanar points in space, forming with the segments $[AB]$, $[BC]$, $[CD]$, $[DA]$ **a skew quadrilateral,** and let A_1, B_1, C_1, D_1 be mid-points of these segments, respectively. Prove that the plane $(A_1B_1C_1)$ contains D_1 and that the figure $A_1B_1C_1D_1$ is a parallelogram. HINT: First show that (A_1B_1), (AC), and (C_1D_1) are parallel.

12. Prove that, if (AB) and (CD) are skew lines and P a point not on either, there is a unique line through P which is coplanar with (AB) and also coplanar with (CD). Where is P situated if this line fails to intersect (AB)? HINT: Consider the line of intersection of the planes determined, respectively, by P and (AB) and by P and (CD).

4

PARALLEL LINES AND VECTORS IN SPACE

4·1 Parallel lines in space

From Euclid's parallel postulate for the plane we obtain the corresponding theorem for space.

THEOREM 4A: *There is one and only one line parallel to a given line through a given point in space not on the line.*

PROOF: 1. A parallel to the given line is coplanar with the given line. (By definition of parallels.)

2. The given line and the given point not on it determine a unique plane. (By Determination 2 of Sec. 3·8.)

3. In this plane there is a unique parallel to the given line through the given point. (By Assumption 10.) Q.E.D.

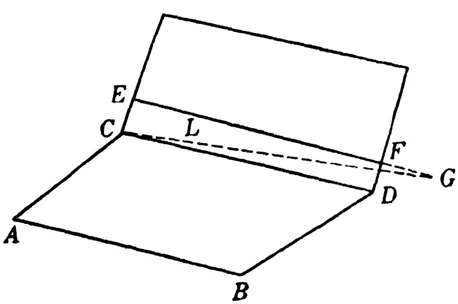

FIG. 7.

THEOREM 4B. *Two distinct lines, each parallel to the same line in space, are parallel to each other.*

DISCUSSION: If all three lines lie in the same plane, the theorem follows from a proposition in plane geometry. Let us assume that the three lines do not all lie in the same plane.

HYPOTHESIS: Let (AB) and (CD) be distinct lines each parallel to a third line (EF) not in the plane (ABC) (Fig. 7).

PROOF: 1. The plane (ABC) intersects the plane (CEF) in a line

L through C. (Two planes having a point in common have a line in common.)

2. Either this line L in plane (CEF) is (CD), or it intersects (EF) in a point G. (By Assumption 10.)

3. If G lies on (EF), the planes (ABE), (ABG), and (ABC) will coincide, contrary to hypothesis. [The lines (AB) and (EF), being parallel, are coplanar].

4. Then, since the plane (ABC) contains the points C, E, F, it contains (CD) parallel to (EF) through C. (By Assumption 10.)

5. Lines (AB) and (CD) are coplanar. (By 2, 3, 4.)

6. If a point P lies on each of the two lines (AB) and (CD), it is common to the two planes (ABE) and (CDE), and hence it lies on (EF). (Two intersecting planes have one line in common.)

7. No point P lies on each of the three lines (AB), (CD), and (EF). [Parallel lines (AB) and (EF) have no common point.]

8. Lines (AB) and (CD) are parallel. (Coplanar nonintersecting lines are parallel.)

4·2 Directed lines and rays

Given any two points A and B on a line, a **direction** on the line is defined by prescribing which of the points is counted first. If A is first and B is second, then the direction from A to B is called positive, while the direction from B to A is called negative, A is called the **initial point** and B the **terminal point**, and the directed line is denoted by $\uparrow (AB)$. In this case the directed length $\overrightarrow{AB}$ of the directed segment $\uparrow [AB]$ is the same as the distance $\overline{AB}$, but the directed distance $\overrightarrow{BA}$ is the negative of the directed distance $\overrightarrow{AB}$. For either choice of direction we have $\overrightarrow{AB}+\overrightarrow{BA}=0$. Two points A and B distinct from a given point O are in the **same direction** from O if either A lies on the segment $[OB]$ (Fig. 8) or B lies on $[OA]$. They are in **opposite directions** from O if O lies on $[AB]$. Given a point O and any other point A, the points on (OA) in the same direction from O as A are said to form a **ray**, or **half line**, with O as vertex, denoted by $\uparrow O(A)$ (Fig. 8). In the notation used here, *the parentheses ( ) enclose points*

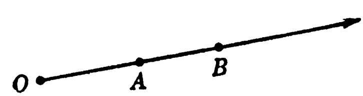

FIG. 8. Ray.

that serve to identify the figure but are not themselves identified by the figure, whereas the square brackets [ ] *enclose points that are identified by the figure.* The direction on a line, or ray, may be indicated by an arrowhead.

4·3 Directed parallels

Two directed parallel lines are said to have either the **same** or **opposite directions.** From two points A and B on a directed line $\uparrow(AB)$ let a pair of parallel lines (AC) and (BD) be drawn which intersect a second line parallel to (AB) in points C and D, respectively, forming a parallelogram $ABDC$ (Fig. 9).

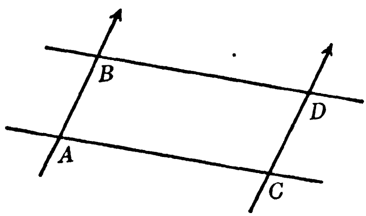

FIG. 9. Directed parallels.

Then the directed line $\uparrow(CD)$ is said to have the same direction as $\uparrow(AB)$, but the directed line $\uparrow(DC)$ is said to have the opposite direction from $\uparrow(AB)$.

4·4 Equivalent directed segments

Two directed segments are called **equivalent** if they have the same length and direction. From the theorem that two lines parallel to the same line are parallel to each other (Theorem 4B) it follows that *two directed segments, each equivalent to a third, are equivalent to each other.* Two equivalent directed segments will coincide if and only if they have the same initial point. They will then necessarily have the same terminal point.

4·5 Vector quantities

Quantities that have both direction and magnitude are important in many applications. They are called **vector quantities.** For example, in calculating the motion of an airplane it is necessary to know both its speed and the direction of its flight; and, in allowing for the effect of wind, both the speed and direction of the wind must be known. The vector quantities involved in this example are called **velocities.** The term velocity, properly used, involves not only the speed but the direction of motion as well. When velocities change, they may change in direction or in magnitude or in both respects. The change in velocity, which is a vector quantity and is not necessarily the same as the change in speed, is called the **vector of acceleration.** It is a fundamental law of physics that *the vector accelerations of material bodies are proportional to the forces which act upon them.* Hence forces are also vector quantities, having both direction and magnitude.

4·6 Vectors

The term vector, as used in this book, is characterized by the common property of direction and the number of units of length that is possessed by all the directed line segments that are equivalent to a given directed line segment. If a reference point O and a given unit of length have been specified and if $\uparrow O(A)$ is the directed segment through O that is equivalent to the given segment, then we shall denote the corresponding vector by the symbol $\uparrow A$ or by the boldface small letter **a**. In a drawing the vector labeled **a** may be represented by any one of the equivalent directed segments of the set with which it is associated. A **vector quantity** is to be thought of as a *vector multiplied by a dimensional unit* such as length, speed, or weight.

Sometimes the term *vector* is used in a sense which specifies either the initial point or the line along which representative line segments must lie. To avoid ambiguity it is better to use for these the terms **fixed vectors** and **sliding vectors,** respectively.

DEFINITIONS: A **unit vector** is a vector of unit length.

A **null vector** is a vector whose length is zero, obtained when the initial and terminal points of the corresponding directed segments are made to coincide. Since its direction is not defined, a null vector can be represented by the symbol **0**.

4·7 Addition and subtraction of vectors

Given the two vectors $\uparrow A$ and $\uparrow B$, represented by the directed segments $\uparrow [OA]$ and $\uparrow [OB]$, let $\uparrow [AS]$ be the directed segment from A equivalent to $\uparrow [OB]$. Then S is the fourth vertex of a parallelogram $AOBS$, having $[OA]$ and $[OB]$ as two adjacent sides, unless O, A, and B are collinear. The directed segment $\uparrow [OS]$ forms the diagonal of the parallelogram and is called the sum of the two directed segments $\uparrow [OA]$ and $\uparrow [OB]$. Furthermore, the vector $\uparrow S$ is defined to be the sum of the vectors $\uparrow A$ and $\uparrow B$. We write

$\uparrow S = \uparrow A + \uparrow B$ or $\mathbf{s} = \mathbf{a} + \mathbf{b}$.

Addition of three or more vectors is governed by two important rules, based on the properties of parallelograms.

RULE 1: $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$. (Commutative law)

RULE 2: $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$. (Associative law)

This last rule means that, if $\mathbf{s}=\mathbf{a}+\mathbf{b}$ and $\mathbf{t}=\mathbf{b}+\mathbf{c}$, then $\mathbf{s}+\mathbf{c}=\mathbf{a}+\mathbf{t}$.

The product of a vector **a** by a positive number n is defined to be the vector whose direction is the same as that of **a** but whose magnitude is n times the magnitude of **a**. It is written $n\mathbf{a}$. If two vectors are collinear, then one is a multiple of the other. The sum and difference of the two collinear vectors $m\mathbf{a}$ and $n\mathbf{a}$ are $(m+n)\mathbf{a}$ and $(m-n)\mathbf{a}$, respectively.

Two collinear vectors whose sum is the null vector **0** are called opposite vectors. If one of the vectors is denoted by **a**, the other is denoted by $-\mathbf{a}$. Subtraction of a vector is defined by adding the opposite vector. Thus $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$.

4. ORAL QUESTIONS

A. What are three of the most important facts about parallel lines in space?

B. What is the difference in meaning between the terms *directed line, ray, directed line segment, directed length?* How are they distinguished in the notation of this book?

C. If two points are moving along parallel lines, are they necessarily moving in the same direction? Explain.

D. When are two directed segments said to be equivalent?

E. In what ways do vectors and directed segments resemble each other, and in what ways are they different?

F. What is meant by the terms *null vector, unit vector, opposite vector?*

G. How may vector quantities be added?

H. What is the sum of a set of vectors, which are represented by the directed sides of a closed polygon, if each vertex is the initial point of one directed side and the terminal point of the preceding one?

I. What can be said of two vectors **a** and **b**, if the length of their sum **s** is the sum of the lengths of **a** and **b**?

J. What sorts of physical quantities are represented by vectors? Can you give any examples that were not mentioned in the text?

4. WRITTEN EXERCISES

1. If A, B, and C are any three distinct points on a directed line, show that $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$.
Consider separately each of the six possible orders of the points A, B, and C, in each case expressing the three directed lengths in terms of the distances $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.
2. Copy Fig. 2 (Chap. 1), and list all the directed segments in the figure that are equivalent to $\uparrow [OA]$.
3. Draw two directed segments representing vectors **a** and **b** in the plane such that **a** is twice as long as **b** but the vectors **a** + **b** and **a** − **b** have the same length. HINT: The diagonals of a rectangle are equal.
4. Draw two directed segments representing vectors **a** and **b** in the plane such that **a** and **b** have the same length as the vector **a** + **b**.
5. Draw three directed segments representing unit vectors whose sum is zero.
6. Draw a diagram to scale representing a vector 3 units long pointing north and a vector 4 units long pointing east. Draw the vector that is the sum of these, and measure its length.
7. Given any three vectors **a**, **b**, **c** in space. Show that the vectors **a** − **b**, **b** − **c**, and **c** − **a** are all parallel to the same plane. To help visualize the figure, let the vectors be represented by a thumb and two fingers. Describe how the required plane might be determined.
8. Prove that equal segments are cut off on a set of parallel lines by two planes which are parallel to each other but are not parallel to the given lines.
9. Prove that, if two lines intersect a set of three or more parallel planes, the segments cut off on these lines between pairs of planes are proportional.
10. Prove that, if two triangles $[ABC]$ and $[A'B'C']$ lie in parallel planes, the mid-points of $[AA']$, $[BB']$, $[CC']$ lie in a plane parallel to both.

5

PERPENDICULARS IN SPACE

5·1 Intersecting perpendiculars

Two intersecting lines are said to be **perpendicular, orthogonal,** or **normal** to each other if the four parts into which they cut their plane are congruent to each other. Each of the four parts of the plane is then called a **quadrant.** The figure formed by the two half lines that bound a quadrant is called a **right angle,** the half lines are called the **sides** of the right angle, and their intersection point is called the **vertex.** If the lines (AB) and (CD) are perpendicular, we write $(AB) \perp (CD)$.

Right angles in the plane of a drawing may be indicated in a figure by a small square with two sides along the sides of the right angle (Fig. 10). It should be noted, however, that a right angle in space does not appear as a right angle in a plane drawing (using the projection given by the trimetric ruler) except when one of the sides of the angle is parallel to the plane of the drawing. In such cases the small square in the corner of the quadrant in space is represented by a small parallelogram with two sides along the lines that represent the sides of the right angle in the projection (see Fig. 10).

5·2 Normal to a plane

A line that intersects a plane is defined to be perpendicular, or normal, to the plane if it is *perpendicular to every line in the plane* *drawn through its foot.* The plane is then likewise said to be perpendicular, or normal, to the line (Fig. 10).

That these definitions are not devoid of content will be evident when it is proved that there exists (in three-dimensional space) a unique plane perpendicular to a given line at a given point of the line and a unique line perpendicular to a plane at a given point of the plane. These statements are not true in spaces of more than three dimensions, but in three dimensions they follow as corollaries of the next theorem. It will also be proved later that there is a unique line perpendicular to a given plane and passing through any given point of space and, similarly, a unique plane perpendicular to a given line and passing through any given point of space.

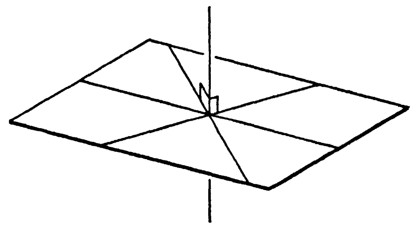

FIG. 10. Normal to a plane.

5·3 Mediator of a segment

THEOREM 5A: The locus of points equally distant from two given points in space is a plane perpendicular to their segment at its mid-point. (This plane is called the **mediator of the segment** joining the two points.)

HYPOTHESIS: Let $[AB]$ be a segment, M its mid-point. (Fig. 11.)

DISCUSSION: Consider any plane through the line (AB). In this plane the points of the perpendicular to (AB) at M and only these points lie on the locus. Since there is more than one plane containing (AB), the points of the locus cannot all lie on a line. Not all points of space lie on the locus. We shall show that, for each choice of two distinct points C and D of the locus, the line (CD) lies on the locus. Hence, since the locus is a linear space containing three noncollinear points, but not all of space, it must be a plane. It contains the set of perpendiculars to (AB) at M lying in each of the planes through (AB).

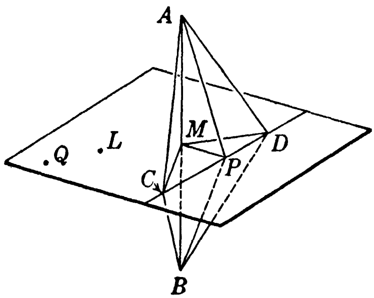

FIG. 11. Mediator of a segment.

PROOF: 1. There exist two distinct points C and D, not collinear with M, such that $\overline{AC}=\overline{BC}$, $\overline{AD}=\overline{BD}$. [Each plane through (AB) contains points of the locus.]

2. The triangles $[ACD]$ and $[BCD]$ are congruent. (Three sides.)

3. Hence, $\angle ACD = \angle BCD$. (Corresponding parts of congruent figures.)

4. If P is any arbitrarily chosen point on (CD), then the triangles $[ACP]$ and $[BCP]$ are congruent. (Side, angle, side.)

5. Hence, $\overline{AP}=\overline{BP}$. (Corresponding parts of congruent figures.)

6. Since C and D were arbitrary points of the locus and since an arbitrary point P on the line (CD) lies on the locus, the locus is a linear space. (Definition of linear space.)

7. Every point of the plane (CDM) lies on the locus. (A linear space containing three noncollinear points contains their plane.)

8. If L is any point of the locus other than M, then there exists a unique plane (LAM) containing L and (AM). (Plane is determined by a line and a point not on it.)

9. The planes (CDM) and (LAM), having the point M in common, must have a point Q in common distinct from M. (Assumption 13.)

10. All points of (MQ) lie on the locus. [Plane (CDM) is a linear space.]

11. In the plane (LAM) all points equidistant from A and B, including L, lie on the one line (MQ) through M. (Theorem in plane geometry.)

12. Each point of the locus lies in the plane (CDM). (If a plane contains a line, it contains each point of the line.)

13. Since the locus is a plane containing all the normals to (AB) at M, it is a plane perpendicular to (AB) at M. (Definition of a line perpendicular to a plane.)

The proofs of the following corollaries are listed later as exercises.

COROLLARY 5·1: A line perpendicular to each of two intersecting lines at their point of intersection is perpendicular to their plane.

COROLLARY 5·2: One and only one plane can be drawn perpendicular to a line at a given point on the line.

COROLLARY 5·3: One and only one line can be drawn perpendicular to a plane at a given point on the plane.

COROLLARY 5 4: Two lines each perpendicular to the same plane are parallel.

5·4 Skew perpendiculars

Two skew lines are said to be perpendicular to each other if there exists a line parallel to the one that intersects the other at right angles.

5·5 Construction of a normal to a plane

Let it be required to construct a line perpendicular to a given plane and through a given point not on the plane.

CONSTRUCTION: Let P be a given point and (AB) any line in the given plane (Fig. 12). Then in the plane (PAB) draw $[PM] \perp (AB)$, intersecting (AB) at M. In the given plane construct $(MN) \perp (AB)$ at M. The intersecting lines (PM) and (MN) determine a plane (PMN). In this plane let P_1 be the foot of the perpendicular from P on (MN). Then (PP_1) is the required perpendicular.

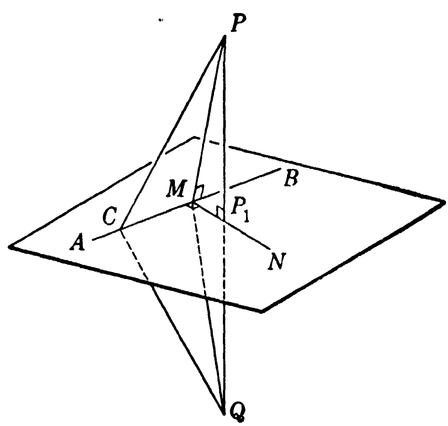

FIG. 12.

PROOF: To prove that (PP_1) is perpendicular to the given plane at P_1, let the point Q be chosen on (PP_1) so that P_1 is the mid-point of $[PQ]$. Then $\overline{PM} = \overline{QM}$, and each point C on (AB) is equidistant from P and Q. (Why?) Hence the given plane (ABP_1) is the mediator of $[PQ]$ and is perpendicular to $[PQ]$ at P_1.

5·6 Orthogonal and orthographic projection

The orthogonal projection of a point P on a given plane (or line) is defined to be the foot P_1 of the perpendicular from P to the plane (or line). The orthogonal projection on a given plane (or line) of a given line segment $[AB]$ not perpendicular to the plane (or line) is defined to be the segment $[A_1B_1]$ whose end points are the projections of A and B on the given line (or plane). Representing the points and lines of a space figure by means of their orthogonal projections on a picture plane is called **orthographic projection** (Sec. 31·3).

5·7 Perpendicular planes

DEFINITION: A plane is perpendicular to a second plane if it contains a line perpendicular to the second plane.

THEOREM 5B: If one plane is perpendicular to a second plane, then the second plane is perpendicular to the first.

THEOREM 5C: If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to the third plane.

5·8 Drawing of space figures involving perpendiculars

In making a drawing of a space figure that involves perpendicular lines and planes, it is easiest, whenever possible, to represent one line as a vertical line in the figure. The upper and lower sides of the trimetric ruler can then be used to draw projected lines representing two perpendicular lines in a horizontal plane. If a line segment in the figure is not parallel to one of these three mutually perpendicular reference lines in space, it can always be thought of as the diagonal of a box whose edges are parallel to these reference lines. Each edge of the box will be a segment equivalent to the projection of the given segment on the corresponding reference line.

5. ORAL QUESTIONS

A. What is meant by each of the terms *perpendicular*, *normal*, *orthogonal*, *mediator*?

B. Is it true without exception that, given a point and a line in space, there is one and only one line perpendicular to the line and passing through the point? If not, what are the exceptions?

C. Is it possible to prove that the mediator of a line segment is a plane, without making use of Assumption 13? Is the theorem true in a space of four dimensions?

D. What is the locus of points equally distant from three given distinct points in space? Does the locus ever fail to exist?

E. How many points are equally distant from four given points in space? Discuss all cases.
F. If a space figure is projected orthogonally onto a plane, which of the following parts in the figure satisfy the same description after projection: a line, two parallel lines, two perpendicular lines, a segment, two equal segments, a segment and its mid-point, a parallelogram, a rectangle, a circle?
G. Can a square be projected by orthogonal projection into anything except a line segment or a parallelogram?
H. Can an arbitrary parallelogram be obtained by the orthogonal projection of a square?
I. Given two triangles, is it always possible to project the first into a triangle similar to the second by an orthogonal projection?
J. If one triangle is projected orthogonally into a second, will the centroid of the first be projected into the centroid of the second? (The centroid of a triangle is the point of intersection of the medians. See Sec. 25·2.)
K. What are the steps to be taken in representing accurately the position of a line segment in space on a drawing in parallel perspective (with the trimetric ruler)?

5. WRITTEN EXERCISES

1. Using the trimetric ruler draw a figure of a square $ABCD$ whose sides are each 8 units, lying in a horizontal plane. At the mid-point M of the square draw $[MP]$ perpendicular to the plane of the square and 7 units long. Draw the segments $[PA]$, $[PB]$, $[PC]$, and $[PD]$, and find their lengths.
2. Using the trimetric ruler draw a tent floor in the form of a 10- by 12-ft. rectangle $ABCD$ in a horizontal plane. Draw two 8-ft. vertical posts $[ME]$ and $[NF]$ from the mid-points M and N of the 12-ft. sides $[AB]$ and $[DC]$. Draw a 10-ft. ridge-pole $[EF]$ parallel to $[AD]$ and $[BC]$. Draw the slanting edges $[AE]$, $[BE]$, $[CF]$ and $[DF]$. Use dotted lines for those which are not seen from in front of the tent. In a space model would (BF) and (EC) be perpendicular to each other? Are they perpendicular in the drawing?

3. Using the trimetric ruler draw (in parallel projection) a figure representing the two walls and floor of a room meeting in a corner. On each wall draw a window (with a horizontal line across the middle). On the floor draw a gray rectangular rug.
4. A window 2 ft. wide and 4 ft. high is 2 ft. above the floor and is centered on a wall that is 10 ft. wide and 8 ft. high. On the adjacent 10- by 8-ft. wall to the right there is a closed door $2\frac{1}{2}$ ft. wide and $6\frac{1}{2}$ ft. high that touches the floor and is 2 ft. from the intersection of the two walls. An electric cord is stretched tightly from a point P_1 under the lower left corner of the door to a hook P_2 on the left side of the window 1 ft. from the bottom. Draw a figure to scale in parallel projection with the trimetric ruler, showing the walls, floor, window, door, and electric cord.
5. Find the length of the electric cord between points P_1P_2 in Exercise 4.
6. Describe a method of constructing a line perpendicular to each of two skew lines. Draw a figure. HINT: First pass a plane through one of the lines parallel to the second line, then pass a plane through the second line perpendicular to the first plane.
7. Prove that the shortest distance from a point to a plane is measured along the normal to the plane through the point.
8. Prove that the shortest distance between two skew lines is measured along their common perpendicular. Draw a figure.
9. Prove that a line perpendicular to each of two intersecting lines at their point of intersection is perpendicular to their plane.
10. Prove that one and only one plane can be drawn perpendicular to a line at a given point on the line.
11. Prove that a line perpendicular to one of two parallel planes is perpendicular to the other also.

6

SENSED PLANES AND COORDINATES

6·1 Normal vectors and sensed planes

Since two lines each perpendicular to the same plane are parallel, it follows that *all lines perpendicular to a given plane are parallel.* Directed segments along normals to a plane have either the same or opposite directions. Those having unit length and the same direction determine a unit vector, called a **unit normal vector.** Each plane has two oppositely directed unit normal vectors. By arbitrarily choosing one of these unit normals in preference to the other it is possible to distinguish between the two sets of points into which the plane divides space, and the plane is then said to be **sensed** or **oriented.** A point may be said to be on the **positive side** or the **negative side** of a sensed plane according as the normal directed from the plane to the point has the same or the opposite direction as this chosen unit normal vector.

Even as a line can be sensed by assigning an order to two of its points, so a plane can be sensed by assigning an order to three noncollinear points. To illustrate, let three noncollinear points be marked on a floor within one step of each other. Assign an order to the points, and place the left foot on the first point and the right foot on the second. We may then define the upper side of the floor to be the positive side if the third point is in front of the line of the other two or the negative side if it is behind this line. Interchanging the order of two of the points would change the sense of the plane.

DEFINITION: The **directed distance** from a sensed plane to a point is a multiple of a unit distance by a number whose sign is positive or negative according as the point lies on the positive or negative side of the sensed plane and whose absolute value measures the perpendicular distance from the point to the plane.

6·2 Coordinates in space

The directed distances from three mutually perpendicular sensed planes to any given point P, divided by a chosen unit of length, are called **rectangular** or **Cartesian**[1] **coordinates** of the point P with

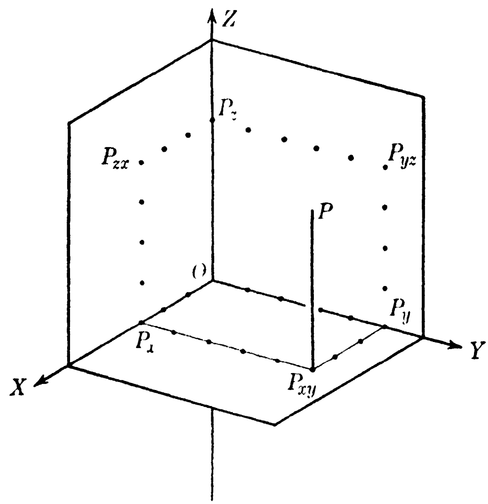

FIG. 13. Coordinate axes and planes.

respect to these planes. The three coordinates are commonly denoted by x, y, z. The three planes are called **coordinate planes** or **reference planes**, and the three lines in which these planes intersect by pairs are called **axes** or **reference lines**. Each axis is perpendicular to one of the coordinate planes and may be assigned the direction of the positive normal to this plane. The point of intersection of the three axes is called the **origin** and is usually denoted by O.

The three mutually perpendicular directed axes, given in a specified order, and the chosen unit of length define a **coordinate system.**

[1] After René Descartes.

The point O has all its three coordinates zero, other points on the axes have two coordinates zero, and points lying on just one of the coordinate planes have just one coordinate zero. In Fig. 13, the directed axes are $\uparrow O(X)$, $\uparrow O(Y)$, and $\uparrow O(Z)$, the projections of a point P on these axes are labeled P_x, P_y, P_z, respectively, and the projections of P on the coordinate planes (OYZ), (OZX), and (OXY) are labeled P_{yz}, P_{zx}, and P_{xy}, respectively. The directed distances $\overrightarrow{OP_x}$, $\overrightarrow{OP_y}$, and $\overrightarrow{OP_z}$ are equal, respectively, to the directed distances $\overrightarrow{P_{yz}P}$, $\overrightarrow{P_{zx}P}$, and $\overrightarrow{P_{xy}P}$. Dividing these distances by the arbitrarily selected unit of length we obtain the coordinates x, y, z of the point P referred to the given coordinate system. We write $P{:}(x,y,z)$ to indicate that P has the coordinates x, y, z.

6·3 Components of a vector

Given a coordinate system with origin O, let **i**, **j**, and **k** denote unit vectors whose directions are those of the coordinate axes, and let **p** be an arbitrary vector represented by the directed segment $\uparrow [OP]$. Then, by the rule for addition of directed line segments, we can write $\uparrow [OP]$ as the sum of three directed segments, one parallel to each of the axes.

$$\uparrow [OP] = \uparrow [OP_x] + \uparrow [OP_y] + \uparrow [OP_z] \tag{1}$$

The vectors $\uparrow P_x$, $\uparrow P_y$, $\uparrow P_z$, represented by $\uparrow [OP_x]$, $\uparrow [OP_y]$, and $\uparrow [OP_z]$, respectively, are the projections of $\uparrow P$ on the coordinate axes. If the coordinates of the point P are (x, y, z), then **p** may be expressed in terms of x, y, z as follows:

$$\uparrow P_x = x\mathbf{i}, \quad \uparrow P_y = y\mathbf{j}, \quad \uparrow P_z = z\mathbf{k} \tag{2}$$

$$\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3}$$

The coordinates x, y, z of P are called the components of the vector $\uparrow P$ (or **p**) with respect to the given reference system. Two vectors may be added or subtracted, by adding or subtracting the corresponding components. This fact is expressed by the following theorem:

THEOREM 6: *The projection on any directed line of the sum* (or difference) *of any two vectors is the sum* (or difference) *of their projections.*

PROOF: Let the vectors **a** and **b** whose sum (or difference) is **s** be represented by the directed segments $\uparrow [OA]$ and $\uparrow [AS]$ whose sum (or difference) is $\uparrow [OS]$ (Fig. 14). Let the projections of O, A, and S

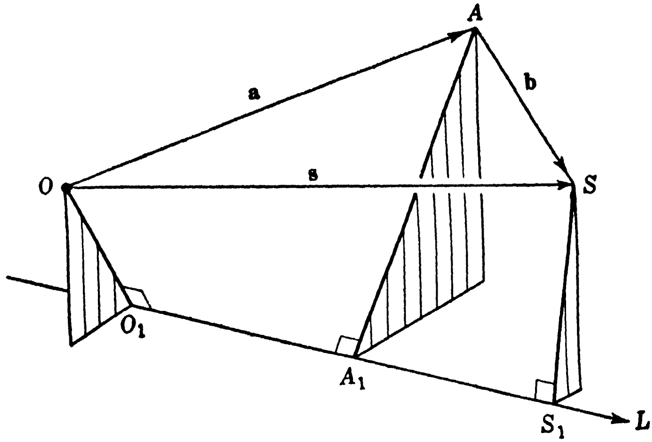

FIG. 14. Projection of a vector on a line.

on the given line L be O_1, A_1, and S_1. Then the projections of **a**, **b**, and **s** on L are represented by $\uparrow [O_1A_1]$, $\uparrow [A_1S_1]$ and $\uparrow [O_1S_1]$. For these we have $\overrightarrow{O_1A_1}+\overrightarrow{A_1S_1}=\overrightarrow{O_1S_1}$. Hence the theorem is proved.

6·4 Length of a vector

Given any point P and a rectangular coordinate system with origin O, the distance $\overline{OP}$ can be found by the method of Chap. 1, using the eight points O, P_x, P_y, P_z, P_{yz}, P_{zx}, P_{xy}, P of Fig. 13 as vertices of a rectangular box. From the Pythagorean theorem we then find

$$(4) \qquad \overline{OP}^2=\overline{OP_x}^2+\overline{OP_y}^2+\overline{OP_z}^2=(x^2+y^2+z^2)\overline{OI}^2$$

where $[OI]$ is a unit interval. Hence

$$(5) \qquad \frac{\overline{OP}}{\overline{OI}}=\sqrt{x^2+y^2+z^2}.$$

Thus, if x, y, z are the components of a vector **p** referred to a system of mutually perpendicular axes, the number of units in the length of the vector is $p=\sqrt{x^2+y^2+z^2}$.

Given any two points P' (with coordinates x', y', z') and P'' (with coordinates x'', y'', z''), the corresponding directed edges of a box whose diagonal is $[P'P'']$ and whose sides are parallel to the axes will be measured by the differences of the corresponding coordinates, namely $x''-x'$, $y''-y'$, $z''-z'$. In other words, the line segment $[P'P'']$ represents a vector whose components are $x''-x'$, $y''-y'$, $z''-z'$. Hence

$$(6) \qquad \frac{\overline{P'P''}}{\overline{OI}}=\sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2}$$

EXAMPLE: To find the distance between the points $(1,-2,5)$ and $(-1,4,2)$ we first find the differences between corresponding coordinates, namely, $-1-1=-2$, $4-(-2)=6$, $2-5=-3$, being careful to observe the usual algebraic rules for signs. Squaring and adding the quantities -2, 6, -3 we obtain $(-2)^2+(6)^2+(-3)^2=4+36+9=49$. Taking the positive square root we find that the distance between the given points is 7 units.

6·5 Drawing the coordinate system

A convenient and effective way of representing space figures in a plane drawing is to refer each of the important points P in the figure to a system of suitably chosen axes, thus locating the points P_x, P_y, P_z in space; then project the coordinate axes and the points P_x, P_y, P_z orthogonally onto the plane of the drawing; and finally reconstruct the projection of P_{xy} and P in the drawing by using the fact that the rectangles $OP_xP_{xy}P_y$ and $OP_{xy}PP_z$ project into parallelograms in the drawing (Fig. 13). Care should be taken, in relating the reference system to the plane of the drawing, that a line joining two important points in the figure is not perpendicular to the plane of the drawing, since these two points would then project into the same point. For most figures that are likely to occur in a first course in solid geometry this difficulty can be avoided by using the trimetric ruler, which specifies a reference system whose three axes are unequally inclined to the drawing plane and whose scales are incommensurable. As indicated in Chap. 1, *the shortest side of the ruler is always kept vertical on the page.* Line segments on or parallel to the $\uparrow O(Z)$ axis are drawn and measured with this vertical scale. Line segments parallel to $\uparrow O(X)$ are drawn with the lower edge of the triangle and measured on its own scale, and line segments parallel to $\uparrow O(Y)$ are drawn and measured with the upper scale.

6·6 Plotting points

For instance, let it be required to plot with the trimetric ruler a point whose coordinates are $(3,5,-2)$. First mark a point O, draw in projection a vertical axis $\uparrow O(Z)$ up from O with the vertical scale, draw the projection of the axis $\uparrow O(X)$ down to the left from O, using the lower scale, and mark the projection of the point P_x on it, 3 units from O. Then draw the projection of the axis $\uparrow O(Y)$ down to the right from O, using the upper scale, and mark the projection of the point P_y on it, 5 of these units from O. Using the same scale draw the projection of $\uparrow [P_xP_{xy}]$ parallel to the projection of $\uparrow [OP_y]$ and 5 units long. Then, using the vertical scale, go down 2 of these vertical units from the projection of P_{xy} to the projection of P.

6. ORAL QUESTIONS

A. How many unit normal vectors does a plane have? How are they related to each other? By what letters are the unit normal vectors of each of the three coordinate planes represented?

B. What is the meaning of the term *sensed plane?* Given three noncollinear points in a plane, how is it possible to give a sense to the plane by assigning an order to the points? Which of the six possible orders would define the same sense?

C. How many points are at a given distance from each of three planes? How many points are at a given directed distance from each of three sensed planes?

D. In what way are the coordinates of a point affected if the same axes are used but the unit of length is changed?

E. Are the coordinates of a point directed distances or numbers or both?

F. In defining a coordinate system, what must be given besides a unit of distance?

G. What are the components of a vector? Is it possible to associate a point with each vector so that the components of the vector are the coordinates of the point? If so, how?

H. Is it always true that the projection on a line of the difference of two vectors is the difference of their projections?

I. What is the length of the vector whose components are 2, -1, and -2?

J. What is the length of the line segment joining the points $(-6,-1,2)$ and $(2,0,-2)$?

K. If the projections of a point $P:(x,y,z)$ on the coordinate axes are P_x, P_y, and P_z, what are the coordinates of these points?

L. How can a projected coordinate system be used to locate the image point in a drawing of a point whose space coordinates are given?

6. WRITTEN EXERCISES

1. One of the unit normal vectors to a plane has components $(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$. What are the components of the other unit normal?
2. How long is the vector $7\mathbf{i}-4\mathbf{j}+4\mathbf{k}$? Find any other vector having the same length, and write it in terms of **i**, **j**, and **k**.
3. Given the four points $A:(1,1,1)$, $B:(-1,1,-1)$, $C:(-1,-1,1)$, $D:(1,-1,-1)$. Find the six distances between pairs of these points.
4. Using the trimetric ruler and a suitably large scale, draw the projections of each of the four points $A_1:(1,1,0)$, $B_1:(-1,1,0)$, $C_1:(-1,-1,0)$, $D_1:(1,-1,0)$. Then mark the points A, B, C, D of Exercise 3, and draw the six line segments that connect them in pairs.
5. Find to the nearest tenth of a unit the six distances between pairs of the following four points: $O:(0,0,0)$; $A:(8.7,5.0,0)$; $B:(0,10.0,0)$; $C:(2.9,5.0,8.2)$.
6. Using the trimetric ruler, draw a vertical axis $[OZ]$ 10 projected

units long, and draw the projections of $[OX]$ and $[OY]$ each 10 projected units in length along their appropriate scales. Take $\uparrow O(X)$ down to the left and $\uparrow O(Y)$ down to the right. Mark in the drawing the images of the points whose coordinates are as follows: O:(0,0,0); M:(0,5,0); A:(8.7,5,0); B:(0,10,0); F:(2.9,5,0); C:(2.9,5,8.2). Draw with a straightedge the images of each of the segments connecting O, A, B, C by pairs and the two segments $[MA]$ and $[FC]$. Note that in space $[MA]$ is perpendicular both to $[OA]$ and to $[FC]$.

7. An interesting configuration is formed by the 12 points A:(11,34,18), B:(−29,21,18), C:(−29,−21,18), D:(11,−34,18), E:(36,0,18), F:(0,0,40), A':(−11,−34,−18), B':(29,−21,−18), C':(29,21,−18), D':(−11,34,−18), E':(−36,0,−18), F':(0,0,−40). The point F is about 42 units from each of the points A, B, C, D, E, about 68 units from each of the points A', B', C', D', E', and 80 units from F'.
   (*a*) Find the approximate distances from A to each of the other 11 points.
   (*b*) Find the approximate distances from B to each of the other 11 points.
8. If in Exercise 7 the coordinates 29 and 11 are replaced by $20 \pm 4\sqrt{5}$, 18 and 36 by $8\sqrt{5}$ and $16\sqrt{5}$, and 34 and 21 by $\sqrt{800 \pm 160\sqrt{5}}$, respectively, then the 12 points are vertices of a space figure called a **regular icosahedron** (Sec. 12·4). The point F is exactly 80 units from one other of the 12, $2\sqrt{800+160\sqrt{5}}$ units from each of 5 others, and $2\sqrt{800-160\sqrt{5}}$ units from each of the remaining 5 points. Prove that the same is true for the distances from A to the other 11 points.
9. Make a full-page drawing of the configuration of 12 points given in Exercise 7, using the trimetric ruler to plot the points. Take the origin O at the center of the page. Draw in the 30 lines which connect pairs of points which are about 42 units apart, drawing in first those which are in front of the figure and then dotting in those which are behind. Note that the 3 points C, A', and E' are "behind" in the drawing, C', A, E "in front," and the other 6 points on the visible edges.

7

ANGLES AND DIHEDRAL ANGLES

7·1 Angle between rays

Two rays $\uparrow O(A)$ and $\uparrow O(B)$ drawn from the same point O form a figure called a **geometric angle** and written $\angle AOB$ (Fig. 15). The point O is called the **vertex** of the angle, and the rays $\uparrow O(A)$ and $\uparrow O(B)$ are called the **sides.** The portion of the plane included between the sides of the angle is called a **sector.** Two geometric angles are said to be equal if they are congruent.

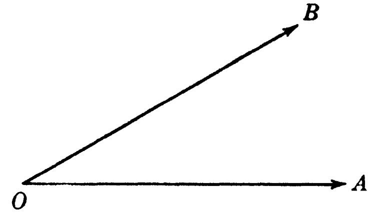

FIG. 15. Angle.

7·2 Three kinds of angles

The word angle, however, is commonly used with three different meanings, one geometric, one arithmetic, and one algebraic. Just as two points A and B determine a geometric figure $[AB]$ called a *line segment*, a denominate number $\overline{AB}$ called its *length*, and an algebraic signed quantity $\overrightarrow{AB}$ called its *directed length*, so two rays $\uparrow O(A)$ and $\uparrow O(B)$, emanating from a common vertex O, determine not only a **geometric figure** $\angle AOB$ but also both an **arithmetic quantity** $\measuredangle AOB$ and an **algebraic signed quantity** $\measuredangle AOB$, which measure the figure. The word angle, in the second sense, means the number of angular units, commonly between 0° and 180°, assigned as a *measure* of the sector formed, or of the circular arc intercepted, by the two rays. This is denoted by $\measuredangle AOB$. In the third sense, the word angle is used to denote a *signed measure* $\measuredangle AOB$, which may be positive, zero, or negative, without limit in size, and which, if $\uparrow O(A)$, $\uparrow O(B)$, $\uparrow O(C)$ are any three coplanar rays with common vertex, satisfies the equations

$$\measuredangle AOB = -\measuredangle BOA \qquad \measuredangle AOB + \measuredangle BOC = \measuredangle AOC$$

In this third sense an angle is not determined by its sides alone. An order must be assigned to the sides so that one is the **initial side** and the other the **terminal side,** a **positive direction of rotation** must be assigned in the plane of the angle, and an integer n, positive, negative, or zero, must be assigned such that the angle is greater than or equal to n complete revolutions of four right angles but less than $n+1$ complete revolutions of four right angles. In mathematical texts the usual method of assigning the positive direction of rotation in a sensed plane is to define it as the counterclockwise direction (opposite to that of the hands of a clock) when viewed from the positive side of the plane (Fig. 16).

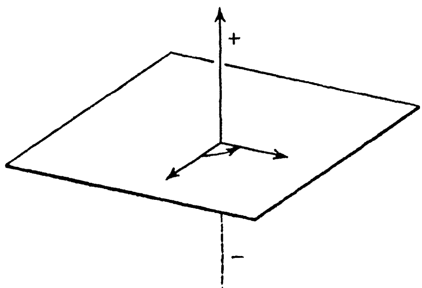

FIG. 16. Directed angle.

7·3 Degrees, minutes, and seconds

In both the second and third uses of the word angle, it is convenient to choose some particular geometric angle as a unit with which others may be compared. One possible unit is the **right angle,** defined in Chap. 5. Another is the **degree** (°), so defined that 90° is one right
* angle and so subdivided into **minutes** (′) and **seconds** (″) that $60' = 1°$, $60'' = 1'$. An angle of 63 degrees, 26 minutes, and 6 seconds, for example (which happens to be the measure of a base angle of an isosceles triangle of equal base and height) is written 63°26′06″.

7·4 Complementary and supplementary angles

Two angles are said to be **complementary,** or each to be the complement of the other, if their **sum is 90°,** or one right angle.

The complement of an angle A will be denoted by $\overline{A}$. Two angles are said to be **supplementary**, or each to be the supplement of the other, if their sum is 180°, or two right angles. If the angles are considered to be directed angles, then the sum indicated is the algebraic sum. For instance, the complement of 120° is −30°. A positive angle less than 90° is called **acute**, and its supplement, which is between 90° and 180°, is called **obtuse**. It is proved in plane geometry that *the sum of the three angles in a plane triangle is equal to 180°.*

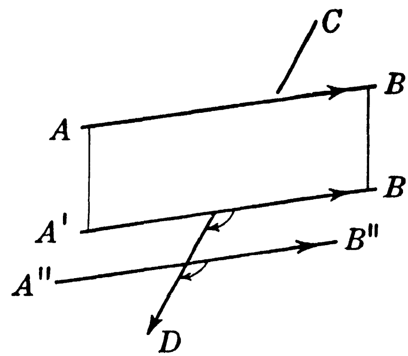

FIG. 17. Angle between skew lines.

7·5 Angle between lines

If two undirected lines (AB) and (CD) intersect in a point O, four undirected angles are formed, of which opposite pairs are equal, and adjacent pairs are supplementary. To remove the ambiguity as to which of two supplementary angles is intended, it is often best to use directed lines in defining angles. This may not be necessary when the two lines are perpendicular.

If a set of coplanar parallel directed lines have a common direction, then they make equal angles with any directed transversal line that intersects them. Hence we may define the **angle between two directed skew lines** to be the angle between one of them and any directed line drawn through a point of this line parallel to and in the same direction as the second line (Fig. 17).

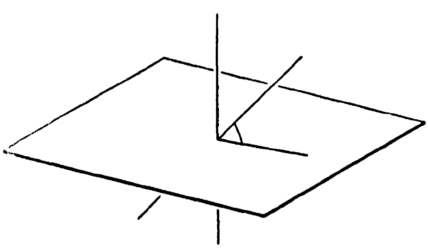
FIG. 18. Angle between line and plane.

7·6 Angle between line and plane

DEFINITION: The angle between a line and a plane not parallel to it is defined to be the complement of the angle that the line makes with the normal drawn to the plane at the foot of the line. If the line and normal are undirected, the acute angle between them is to be chosen (Fig. 18).

In Fig. 3 the directed reference lines $(O\overline{X})$, $(O\overline{Y})$, and $(O\overline{Z})$ in space make angles of α, β, γ, respectively, with the directed line $\uparrow (ON)$ normal to the projection plane. Hence they make angles $\bar{\alpha}=(90°-\alpha)$, $\bar{\beta}=(90°-\beta)$, and $\bar{\gamma}=(90°-\gamma)$, respectively, with the projection plane.

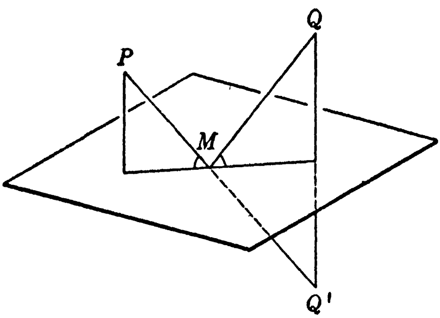

FIG. 19. Reflection from a plane.

7·7 Reflection in a plane mirror

If a ray of light travels from a given point P to a plane mirror, is reflected from the mirror at some point M, and then passes through a second given point Q (Fig. 19), we assume that, of all points in the mirror plane, M is that point for which $\overline{PM}+\overline{MQ}$ is least. We shall show that the plane (PMQ) is perpendicular to the mirror and that lines (PM) and (QM) make equal angles with the mirror.

PROOF: Let Q' be the image of Q in the mirror; that is, let the point Q' be such that the mirror plane is the mediator of $[QQ']$. Then $\overline{MQ}=\overline{MQ'}$. Now $\overline{PM}+\overline{MQ'}$ is least when M lies on the line (PQ'). The point M then lies in a plane (PQQ') perpendicular to the mirror. Also (MQ) and (MQ') make equal angles with the mirror, and (MQ') is the same line as (PM).

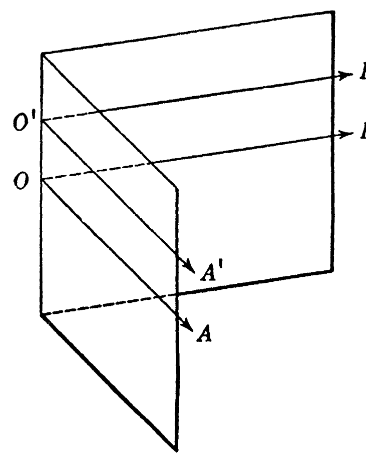

FIG. 20. Plane angles of a dihedron.

7·8 The dihedral angle

Two half planes, having a common edge but not coplanar, form a geometric figure called a **dihedron** (Fig. 20). The half planes are called the faces and their common line is called the edge of the dihedron. A plane perpendicular to the edge at a point O on the edge intersects the two faces in two rays, say $\uparrow O(A)$ and $\uparrow O(B)$, forming an angle, $\angle AOB$, called a **plane angle** of the dihedron. If O' is any point on the edge distinct from O, the corresponding plane angle, $\angle A'O'B'$, is equal to $\angle AOB$. The common magnitude of all plane angles of a dihedron we define to be the **dihedral angle,** written $\measuredangle A\text{-}OO'\text{-}B$ or $\sphericalangle A\text{-}OO'\text{-}B$ according as the plane angles are undirected or directed angles. Addition of the two dihedral angles of two dihedrons having a common edge and common face is defined by the addition of the corresponding plane angles.

7·9 Vertical and supplementary dihedrons

A pair of intersecting planes forms four dihedrons with a common edge. Adjacent ones are called **supplementary dihedrons,** since their plane angles are supplementary. Opposite ones are said to form a pair of **vertical dihedrons,** the word vertical being used in the sense of vertical angles, rather than in the sense of vertical lines.

7·10 Face normals of a dihedron

DEFINITION: A point P is said to be inside the dihedron if, in the plane through P normal to the edge, the point P lies in the sector formed by the plane angle.

THEOREM 7A: *If directed normals are drawn to the faces from a point inside a dihedron, the angle between these normals is the supplement of the dihedral angle.*

PROOF: The normals through P both lie in the plane through P perpendicular to the edge (Fig. 21). In this plane the normals and the sides of the plane angle form a quadrilateral having two right angles. The other two angles must be supplementary. They are the plane angle of the dihedron and the angle between the directed normals.

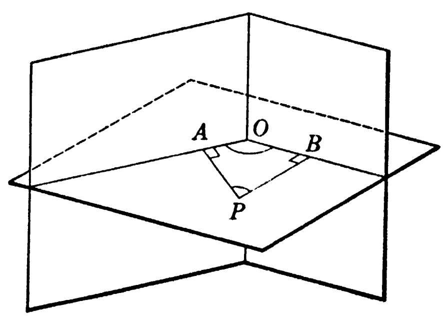

FIG. 21. Face normals of a dihedron.

7·11 Right dihedral angles

DEFINITION: Two planes are perpendicular or **normal** if their dihedral angle is a right angle.

THEOREM 7B: *Two planes are perpendicular if and only if one of them contains a line perpendicular to the other.*

THEOREM 7C: *If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to the third plane.*

The proofs of these theorems are listed as Exercises 10 and 11.

7. ORAL QUESTIONS

A. Angles remain unchanged under rigid motions. Can you think of any other transformations under which angles are preserved?

B. It is sometimes said that angles are preserved under reflection in a plane. Is this true for all three meanings of the word "angle"?

C. Turning the steering wheel of a car through 390° produces a different result from simply turning it through 30° or from turning it through 330° in the opposite direction. Illustrate by other examples some reasons for considering other angles than those which are positive and less than 180°.

D. A figure is projected orthogonally onto a plane. Under what circumstances are angles unchanged by the projection?

E. What is the complement of $63°26'06''$? What is its supplement?

F. What is the complement of $-20°$? What is its supplement?

G. Two angles in a triangle are 36° and 72°. What is the third angle?

H. What is meant by the angle between two skew lines?

I. How should you define supplementary or complementary dihedrons?

J. How many dihedrons are formed by three distinct planes that intersect by pairs in parallel lines? ("Three" is not the answer.)

K. What is the locus of points equidistant from the two faces of a dihedron?

7. WRITTEN EXERCISES

1. There are 1,600 **artillery mils** in a right angle. Express the following angles in mils: 45°; 1°; $10'$; $2'30''$.

2. A certain man's thumb nail held at arm's length subtends an angle of about 27 artillery mils at his eye, whereas the sun's diameter (in March) subtends an angle of about 9.5 artillery mils at the eye. Express these angles in degrees and minutes (see Exercise 1).
3. Another important measure of angles is the **radian,** defined so that 1 radian is the angle subtended at the center of a circle of unit radius by a unit length of arc. How many radians are there in one right angle? In 5°? In 1,000 mils?
4. Three mutually perpendicular lines in space make angles of $\alpha = 40°33'01''$, $\beta = 58°49'33''$, and $\gamma = 66°50'20''$, respectively, with the normal to the plane of a drawing. What angles do they make with the plane of the drawing? NOTE: *These angles are the ones used so that the projections of two of the axes in drawing will make angles of 105° and 120°, respectively, with the third as given by the trimetric ruler* (Fig. 3).
5. Two points P and Q are 10 in. apart. P is 2 in. and Q is 10 in. from the plane of a mirror. Find the shortest path $\overline{PM} + \overline{MQ}$, such that M is a point of the mirror.
6. Draw a figure showing a dihedral angle of 45°. Construct an equilateral triangle in one of the faces, and show its projection in the other face. Is the projection equilateral?
7. Draw a figure representing the two walls and floor of a room meeting in a corner. How big are the three dihedral angles in the figure?
8. Prove that the magnitude of the plane angle of a dihedral angle is the same no matter what point on the edge is chosen for the vertex.
9. If two planes intersect, prove that the planes which bisect the dihedral angles so formed are perpendicular to each other.
10. Prove that two planes are perpendicular if and only if one of them contains a line perpendicular to the other. Draw a figure.
11. Prove that, if two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to the third plane.

8

PROJECTIONS AND COSINES

8·1 The projection factor

If a directed line segment ↑ $[AB]$ on a directed line L (Fig. 22) is projected orthogonally into a directed segment ↑ $[A_1B_1]$ on a second directed line L_1, so that A_1 and B_1 are, respectively, the feet of the perpendiculars from A and B on L_1, then the ratio of the directed length $\overrightarrow{A_1B_1}$ to the directed length $\overrightarrow{AB}$ is a number between -1 and $+1$, which might be called the **projection factor** from L to L_1. This projection factor is $+1$ when the lines are parallel and have the same direction, it is positive but less than 1 when L and L_1 make an acute angle with each other, it is 0 when the lines are perpendicular, it is between 0 and -1 when the directed lines L and L_1 make an obtuse angle with each other, and it is -1 when the lines L and L_1 are parallel but oppositely directed. The projection factor is unchanged if ↑ $[AB]$ is replaced by any equivalent directed segment or if L_1 is replaced by any other parallel directed line with the same direction. Furthermore, if the directed length of ↑ $[AB]$ is multiplied by any positive or negative number, the directed length of its projection is multiplied by the same number, and thus the projection factor remains the same. Finally, upon replacing L_1 by a parallel Line L' through A, it can be shown by similar triangles in the plane

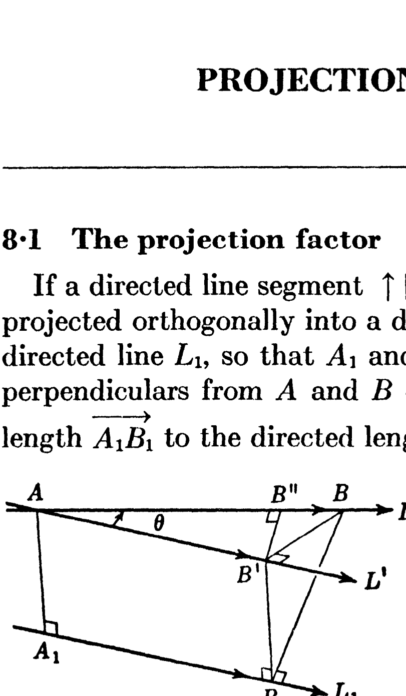

FIG. 22. Projection of a directed segment.

of L and L' that, if a point B on L projects orthogonally into a point B' on L' and B' projects orthogonally into a point B'' on L, then the projection factor $\overrightarrow{AB'}/\overrightarrow{AB}$ from L to L' is equal to the projection factor $\overrightarrow{AB''}/\overrightarrow{AB'}$ from L' to L. Thus *the roles of the two lines L and L_1 may be interchanged without changing the projection factor.* The one measurement that determines the projection factor uniquely is the undirected angle between the two given directed lines. Furthermore, a given projection factor determines a unique undirected angle between 0° and 180°. If this angle is denoted by the Greek letter θ (theta), then the projection factor is called the cosine of θ and is written $\cos\theta$. Thus the relationship between $\overrightarrow{AB}$ and its projection $\overrightarrow{A_1B_1}$ is

$$\overrightarrow{A_1B_1} = \overrightarrow{AB}\cos\theta$$

8·2 Properties of the cosine

The following four properties of the cosine are fundamental. The first three may be proved easily by referring to the properties of projection factors in Sec. 8·1.

PROPERTY 1: The cosine of an angle is the negative of the cosine of its supplement.

$$\cos(180° - \theta) = -\cos\theta$$

(The proof of this is obtained by reversing the sense of one of the given directed lines.)

PROPERTY 2: The cosines of acute, right, and obtuse angles are positive, zero, and negative, respectively.

PROPERTY 3: There is one and only one angle between 0° and 180° having a given number between 1 and -1 as its cosine.

PROPERTY 4: The cosine of a directed angle between two directed lines is defined to be the same as the cosine of the corresponding undirected angle. To an undirected angle θ will correspond the directed angles $\pm\theta$ and angles differing from these by multiples of 360°. Hence, if n is any integer,

$$\cos(-\theta) = \cos\theta$$

$$\cos(n360° \pm \theta) = \cos\theta$$

8·3 Cosines of 45°, 60°, and 30°

Numerical values for cos 45°, cos 60°, and cos 30° are obtained as follows:

Since the diagonal of a unit square has a length equal to $\sqrt{2}$ and makes an angle of 45° with an edge and since its projection on a side is equal to 1 (the length of the side), it follows that

$$\cos 45° = 1/\sqrt{2} \quad \text{or} \quad \sqrt{2}/2 \text{ (Fig. 23)}$$

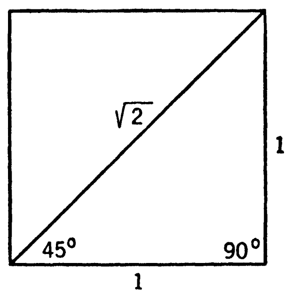

FIG. 23.

Since any side of an equilateral triangle whose sides are of length 2 units makes an angle of 60° with an adjacent side and since its projection on the adjacent side has length 1, it follows that

$$\cos 60° = \tfrac{1}{2} \text{ (Fig. 24)}$$

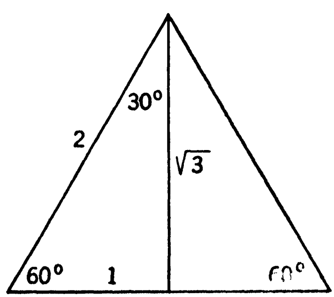

FIG. 24.

By the Pythagorean theorem, an altitude of this equilateral triangle has length $\sqrt{2^2-1^2}=\sqrt{3}$. Since the altitude makes an angle of 30° with an adjacent side, it follows that

$$\cos 30° = \sqrt{3}/2$$

8·4 Values of the cosine

Making use of the properties of the cosine already listed, the following table of values for cosines of several important angles may be completed:[1]

Angle θ	0°	±30°	±45°	±60°	±90°	±120°	±135°	±150°	±180°
$\cos\theta$ Exact value..	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	−1
Decimal value..	1.0000	0.8660	0.7071	0.5000	0.0000	−0.5000	−0.7071	−0.8660	−1.0000

8·5 The golden section

Even as the value of cos 45° is found by bisecting a square and the values of cos 60° and cos 30° can be found by bisecting an equilateral triangle, so the values of cos 36° and cos 72° can be found by examining the diagonals of a **regular pentagon.** A point is said to divide a segment in the golden section (sometimes called **extreme and mean ratio**) if one of the parts of the segment is a mean proportion between the other part and the whole segment. The following theorem is easily proved, and the student should supply the reasons for each step:

[1] Values of the cosines of other angles are given in tables of trigonometric functions. It is shown in the study of calculus that, if the angle is measured by the arc length θ intercepted on a unit circle, its cosine may be computed to any desired degree of accuracy by taking a sufficient number of terms of the series

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

where $2! = 2\cdot 1 = 2$, $4! = 4\cdot 3\cdot 2\cdot 1 = 24$, $6! = 720$, etc.
If the angle is less than 45°, so that the arc is less than $\pi/4$ radians, these four terms will suffice to give $\cos\theta$ to five decimals. The value of π to 15 decimals is $3.141592653589793\cdots$. This may be approximated to three figures by 22/7 $(=3.14286\cdots)$ or to seven figures by 355/113 $(=3.1415929\cdots)$. A good approximation for π^2 $(=9.8696044\cdots)$ is the fraction 227/23 $(=9.869565\cdots)$. To compute cos 36°, for example, we find that the arc is $\theta = \pi/5$. Hence

$$\begin{aligned}\cos 36° &= 1 - \pi^2/50 + \pi^4/15{,}000 - \pi^6/11{,}250{,}000 + \cdots \\ &= 1 - 0.197392 + 0.006494 - 0.000085 + = 0.80902\end{aligned}$$

THEOREM 8: The point of intersection of two diagonals of a regular pentagon divides each diagonal in the golden section.

PROOF: Let the diagonals [AC] and [BD] of the regular pentagon [$ABCDE$] intersect in the point P, and circumscribe a circle about the pentagon. Let the diagonals be of length d, and the sides of length s (Fig. 25).

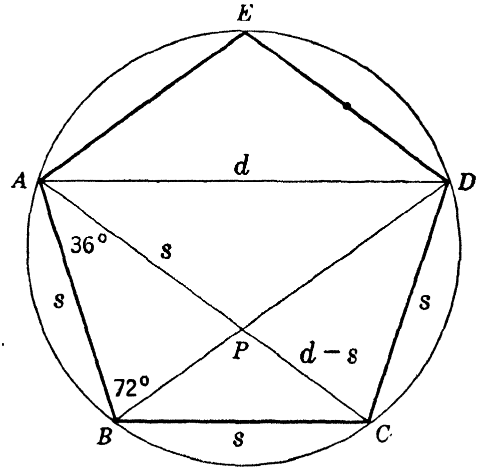

FIG. 25.

(1) $$\measuredangle BAP = \measuredangle CAD = \measuredangle ADB = \measuredangle CDP = 36°$$

(2) $$\measuredangle PBA = \measuredangle DCA = \measuredangle BAD = \measuredangle PCD = 72°$$

(3) Triangles [BAP], [CAD], [ADB], [CDP] are similar isosceles triangles with angles 36°, 72°, 72°.

(4) $$s = \overline{AB} = \overline{AP} = \overline{PD} = \overline{CD}$$

(5) $$\overline{BP} = \overline{CP} = d - s$$

(6) $$\frac{d-s}{s} = \frac{s}{d}$$

(7) P divides each of the segments [AC] and [BD] in golden section.

APPLICATION: Since in the pentagon [$ABCDE$] the projection of [AB] on [AC] is $\overline{AC}/2$, we have $\cos 36° = d/2s$. Also, since the projection of [AD] on [AB] is $\overline{AB}/2$, we have $\cos 72° = s/2d$. Solving the equation $(d-s)d = s^2$ obtained from (6), we have

$$4d^2-4sd+s^2=5s^2 \qquad 2d-s=s\sqrt{5}$$

$$\frac{d}{s}=\frac{\sqrt{5}+1}{2} \qquad \frac{s}{d}=\frac{d}{s}-1=\frac{\sqrt{5}-1}{2}$$

Since $\sqrt{5}=2.236068\cdots$, we have the values

$$\cos 36^\circ=\frac{\sqrt{5}+1}{4}=0.809017\cdots$$

$$\cos 72^\circ=\frac{\sqrt{5}-1}{4}=0.309017\cdots$$

8. ORAL QUESTIONS

A. What is the orthogonal projection of a line segment on a line perpendicular to it?

B. Is it always true that the projection factor is $+1$ when a directed line segment is projected onto a parallel line?

C. Under what circumstances is the projection factor negative?

D. How is $\cos 150^\circ$ obtained from $\cos 30^\circ$?

E. Is it possible for two different angles to have the same cosine if they lie between 0° and 180°?

F. Is it possible for two different angles to have the same cosine if they lie between -90° and $+90^\circ$? How?

G. Two acute angles A and B are such that $\cos A=0.600$, $\cos B=0.800$. Which is the larger angle?

H. What are the values of $\cos 30^\circ$, $\cos 45^\circ$, $\cos 60^\circ$?

I. What are the values of $\cos 120^\circ$, $\cos 135^\circ$, $\cos 150^\circ$?

J. What are the values of $\cos 0^\circ$, $\cos 90^\circ$, $\cos 180^\circ$?

K. What is meant by the golden section, and how is it related to the regular pentagon?

L. What are the values of $\cos 36^\circ$ and $\cos 72^\circ$, expressed in terms of $\sqrt{5}$?

8. WRITTEN EXERCISES

1. What are the angles whose cosines are $\sqrt{4}/2$, $\sqrt{3}/2$, $\sqrt{2}/2$, $\sqrt{1}/2$, $\sqrt{0}/2$, respectively?
2. In a circle of radius 2 units draw two diameters perpendicular to each other. Draw four chords, two of them perpendicular to each of the diameters, at 1 unit distance from the center.

Join to the center each of the eight points where these chords intersect the circle and each at the four points where the chords intersect each other. What arcs do these radii cut off on the circle?

3. A square has a diagonal 99 ft. long. What is the projection factor when the diagonal is projected orthogonally onto a side? What is the length of the side, measured to the nearest tenth of an inch?
4. A given equilateral triangle is 26 in. in height. What is the angle between an altitude and an adjacent side? What is the projection factor when the side is projected on the altitude? Find the length of the side to the nearest tenth of an inch.
5. If a rectangular box has edges of lengths 9, 6, and 2 units, respectively, find the length of the diagonal and the cosines of the angles between the diagonal and the respective edges.
6. If a rectangular box has edges of lengths 3 in., 4 in., and 12 in., respectively, find the length of the diagonal and the cosines of the angles between the diagonal and the respective edges.
7. A given regular pentagon measures 12 ft. on a side. Find the length of one of the diagonals to the nearest hundredth of an inch.
8. If an equilateral triangle in a horizontal plane has its center at the origin of a rectangular coordinate system and if one vertex is at the point (26,0,0), find the coordinates of the other two vertices.
9. Evaluate $\cos 0° + \cos 90° + \cos 180° + \cos 270°$.
10. Express $\cos 120°$ and $\cos 240°$ in terms of $\cos 60°$, and evaluate $\cos 0° + \cos 120° + \cos 240°$.
11. Express $\cos 144°$ and $\cos 216°$ in terms of $\cos 36°$, and express $\cos 288°$ in terms of $\cos 72°$. Prove that $\cos 0° + \cos 72° + \cos 144° + \cos 216° + \cos 288° = 0$.
12. A regular pentagon $[ABCDE]$ has its first vertex A at the origin, its second vertex B at the point (1,0), and its vertices C, D, E above the axis $\uparrow O(X)$. Write down the expressions for the projections on $\uparrow O(X)$ of the directed segments $\uparrow [AB]$, $\uparrow [BC]$, $\uparrow [CD]$, $\uparrow [DE]$, $\uparrow [EA]$. What is the algebraic sum of these projections?

9

DIRECTIONS IN SPACE

9·1 Projection ratios on the axes. Direction cosines

The direction of a directed line segment (or vector) in space is most easily described by reference to a set of three mutually perpendicular coordinate axes. The angles α, β, γ (alpha, beta, gamma) between the directed segment (or vector) and the axes $\uparrow O(X)$, $\uparrow O(Y)$, and $\uparrow O(Z)$, respectively, are known as the **direction angles** of the directed segment (or vector), and the cosines of these angles ($\cos\alpha$, $\cos\beta$, $\cos\gamma$) are known as **direction cosines.** The three direction angles (always chosen between 0° and 180°) or their cosines uniquely determine a direction in space. We define the direction angles and direction cosines of a sensed plane to be the direction angles and direction cosines, respectively, of its directed normal.

By definition in terms of projection factors, the cosines of the direction angles can be written as the components x, y, z, respectively, of the directed segment (or vector), each divided by the length r. Thus

(1) $$\cos\alpha=\frac{x}{r}\qquad \cos\beta=\frac{y}{r}\qquad \cos\gamma=\frac{z}{r}$$

Squaring and adding, we have

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=\frac{x^2+y^2+z^2}{r^2}=1$$

Hence, by substitution, writing $\cos^2\alpha$ for the square of $\cos\alpha$, etc., we have

(2) $$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

This is a fundamental identity satisfied by all sets of direction cosines.

In particular, if the directed segment lies in the plane (OXY), so that α and β are complementary angles and $\gamma=90°$, then the projection $r \cos \gamma$ is zero. Writing $\beta=\bar{\alpha}=90°-\alpha$, we obtain the important identity

$$\cos^2 \alpha+\cos^2 \bar{\alpha}=1 \qquad \text{where } \alpha+\bar{\alpha}=90° \tag{3}$$

9·2 Projection ratios on a plane

If the angles between the coordinate axes and a directed line are α, β, γ, then the angles between the coordinate axes and a plane normal to the line are the complements of these angles, which we denote by $\bar{\alpha}=90°-\alpha$, $\bar{\beta}=90°-\beta$, $\bar{\gamma}=90°-\gamma$. Unit segments on the coordinate axes project into segments of length $\cos \bar{\alpha}$, $\cos \bar{\beta}$, and $\cos \bar{\gamma}$, respectively, when the axes are projected orthogonally onto the plane. We have

$$\begin{aligned} \cos^2 \bar{\alpha}&=1-\cos^2 \alpha=\cos^2 \beta+\cos^2 \gamma \\ \cos^2 \bar{\beta}&=1-\cos^2 \beta=\cos^2 \gamma+\cos^2 \alpha \\ \cos^2 \bar{\gamma}&=1-\cos^2 \gamma=\cos^2 \alpha+\cos^2 \beta \end{aligned} \tag{4}$$

Adding, we derive the identity for line-to-plane projection.

$$\cos^2 \bar{\alpha}+\cos^2 \bar{\beta}+\cos^2 \bar{\gamma}=2 \tag{5}$$

9·3 Projection of one line on another

Let a directed line segment (or vector) of length r whose direction cosines are $\cos \alpha$, $\cos \beta$, $\cos \gamma$, respectively, be projected onto a second directed line L' whose direction cosines are $\cos \alpha'$, $\cos \beta'$, $\cos \gamma'$, and let the angle between the two directed lines be θ. Then if **i**, **j**, and **k** are unit vectors along the axes, their projections on L' are $\cos \alpha'$, $\cos \beta'$, $\cos \gamma'$. Furthermore, the given vector can be written in the form

$$(r \cos \alpha)\mathbf{i}+(r \cos \beta)\mathbf{j}+(r \cos \gamma)\mathbf{k}$$

Its projection on L', namely, $r \cos \theta$, can be written as the sum of the projections of its component vectors. Hence

$$r \cos \theta=(r \cos \alpha) \cos \alpha'+(r \cos \beta) \cos \beta'+(r \cos \gamma) \cos \gamma'$$

Dividing by r we obtain an expression for the angle between two given directions in space.

(6) $$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$$

APPLICATION: If the angles γ and γ' are both 90°, and $\alpha = 60°$, $\alpha' = 45°$, $\beta = 30°$, $\beta' = 45°$, then $\theta = 15°$ and we have

$$\cos 15° = \cos 60° \cos 45° + \cos 30° \cos 45° = \frac{1\cdot\sqrt{2} + \sqrt{3}\cdot\sqrt{2}}{2\cdot 2}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Similarly, if $\alpha = 120°$, $\alpha' = 45°$, $\beta = 30°$, $\beta' = 45°$, $\gamma = \gamma' = 90°$, then

$$\theta = 75° \qquad \text{and} \qquad \cos 75° = \frac{\sqrt{6} - \sqrt{2}}{4}$$

(7) $$\cos 15° = \frac{\sqrt{6} + \sqrt{2}}{4} = 0.965926\cdots$$
$$\cos 75° = \frac{\sqrt{6} - \sqrt{2}}{4} = 0.258819$$

9·4 Scalar product of two vectors

Given any two vectors in space, their **scalar product** is defined to be *the product of the directed length of the one by the directed length of the projection of the other upon it.* If two vectors **a** and **b** have lengths a and b, respectively, and make an angle θ with each other, then the projection of **b** on **a** is $b \cos \theta$, and thus the scalar product is equal to $ab \cos \theta$. A dot is used between two vectors to denote the scalar product. Thus we write

(8) $$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$$

For two identical vectors the scalar product is simply the square of the length.

If two vectors $\mathbf{r}$ and $\mathbf{r}'$ are given by their components (x,y,z) and (x',y',z'), respectively, and if their lengths are r and r', then we can express $\cos\theta$ in terms of the direction cosines x/r, y/r, z/r of the one and x'/r', y'/r', z'/r' of the other, using (6) and (1). Thus,

(9) $$\cos\theta = \frac{xx' + yy' + zz'}{rr'}$$

Clearing of fractions, we have a simple expression for the scalar product:

$$\mathbf{r}\cdot\mathbf{r}' = xx' + yy' + zz' \tag{10}$$

We note that if the two vectors are identical this formula gives the square of the length. **Work,** in physical science, is the energy produced by a force acting through a given distance. If the force and distance traveled are represented by nonparallel vectors, then the *work is the scalar product* of these vectors, since only the component of force in the direction of motion does any work, in the technical sense.

9·5 Perpendicular vectors

Since two vectors are perpendicular if and only if the projection of the one on the other is 0, the *condition for perpendicularity is the vanishing of the scalar product.*

THEOREM 9A: *The two vectors having components* (x,y,z) *and* (x',y',z'), *respectively, are perpendicular (orthogonal) if and only if*

$$xx' + yy' + zz' = 0 \tag{11}$$

This equation is sometimes called the **orthogonality condition.**

EXAMPLE: The three vectors with components $(-4,7,4)$, $(1,-4,8)$, $(8,4,1)$, respectively, are each perpendicular to the other two, since

$$\begin{aligned}(-4)(1) + 7(-4) + 4(8) &= 0\\ (-4)(8) + 7(4) + 4(1) &= 0\\ 1(8) + (-4)(4) + 8(1) &= 0\end{aligned}$$

The student should try to visualize these vectors in space by holding three pencils, each directed to represent one of the vectors, referred to a pair of axes on a horizontal sheet of paper and a third space axis going straight up from the table on which the paper rests.

9·6 Plane coordinates of projected points

Any space vector $\mathbf{p}$ can be written in the form $\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along the axes and x, y, z are the components of the vector referred to the space axes. The projection $\mathbf{p}'$ of this vector onto the plane of the drawing can then be expressed as $\mathbf{p}' = x\mathbf{i}' + y\mathbf{j}' + z\mathbf{k}'$, where $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ are the projections of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively. It is not necessary that the same units be used in the projection plane as in space, if a scale drawing is to be made. In fact, it will be convenient to choose a unit of length so that **k**′ is a unit vertical vector **v** in the plane. Then define **h** as a unit horizontal vector directed to the right (Fig. 26). Each of the vectors **i**′, **j**′, and **k**′ can be expressed in terms of **h** and **v**.

If the space axes make angles of α, β, γ, respectively, with the directed normal to the projection plane, then the angles they make with the plane are the complementary angles $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, respectively. The lengths of **i**′, **j**′ and **k**′ in space units are $\cos \bar{\alpha}$, $\cos \bar{\beta}$, and $\cos \bar{\gamma}$, respectively, but in the new projection units these lengths will be $\cos \bar{\alpha}/\cos \bar{\gamma}$, $\cos \bar{\beta}/\cos \bar{\gamma}$, and 1, respectively. Let us denote the angle in the projection plane between **j**′ and **k**′ by $180° - A$, the angle between **k**′ and **i**′ by $180° - B$, and the angle between **i**′ and **j**′ by $180° - C$, so that A,B,C are the acute angles between the sides of the trimetric ruler (Fig. 3). Then it can be shown (Sec. 32·7) that the quantities $\cos^2 \bar{\alpha}$, $\cos^2 \bar{\beta}$, and $\cos^2 \bar{\gamma}$ are proportional to the quantities $\cos A \cos \bar{A}$, $\cos B \cos \bar{B}$, and $\cos C \cos \bar{C}$.

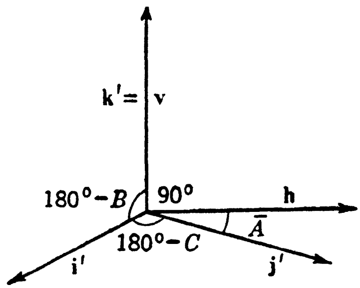

FIG. 26. Projections of **i**, **j**, **k**.

THEOREM 9B: *The lengths of the projected units* **i**′, **j**′, *and* **k**′ *on the scales of a trimetric ruler are proportional to the quantities*

$$(12) \qquad \sqrt{\cos A \cos \bar{A}} \qquad \sqrt{\cos B \cos \bar{B}} \qquad \sqrt{\cos C \cos \bar{C}}$$

For the trimetric ruler in this book we have

$$A = 75° \qquad \cos A = \frac{\sqrt{6} - \sqrt{2}}{4} \qquad \cos \bar{A} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\sqrt{\cos A \cos \bar{A}} = \frac{1}{2}$$

$$(13) \quad B = 60° \qquad \cos B = \frac{1}{2} \qquad \cos \bar{B} = \frac{\sqrt{3}}{2} \qquad \sqrt{\cos B \cos \bar{B}} = \frac{\sqrt[4]{3}}{2}$$

$$C = 45° \qquad \cos C = \frac{\sqrt{2}}{2} \qquad \cos \bar{C} = \frac{\sqrt{2}}{2} \qquad \sqrt{\cos C \cos \bar{C}} = \frac{\sqrt{2}}{2}$$

Hence the distances between scale divisions on the three sides of the trimetric ruler are proportional to 1, $\sqrt[4]{3}$, $\sqrt{2}$.

Now **i**′ makes angles of $90°+B$ and $180°-B$ with **h** and **v** respectively, and **j**′ makes angles of $90°-A$ and $180°-A$ with **h** and **v**, respectively. Hence, taking the lengths of **i**′, **j**′, and **k**′ to be $\sqrt{\frac{1}{2}}$, $\sqrt[4]{\frac{3}{4}}$ and 1, respectively, and expressing **i**′, **j**′, and **k**′ in terms of their **h** and **v** components, we have

$$\mathbf{i}'=\frac{\sqrt{2}}{2}\left[-(\cos \bar{B})\mathbf{h}-(\cos B)\mathbf{v}\right]=-\frac{\sqrt{6}}{4}\mathbf{h}-\frac{\sqrt{2}}{4}\mathbf{v}$$

$$(14)\qquad \mathbf{j}'=\frac{\sqrt[4]{3}}{\sqrt{2}}\left[(\cos \bar{A})\mathbf{h}-(\cos A)\mathbf{v}\right]=\frac{\sqrt[4]{27}+\sqrt[4]{3}}{4}\mathbf{h}-\frac{\sqrt[4]{27}-\sqrt[4]{3}}{4}\mathbf{v}$$

$$\mathbf{k}'=\mathbf{v}$$

Now $\sqrt{6}/4=0.6124\cdots$, $\sqrt{2}/4=0.3536\cdots$, $\sqrt[4]{27}/4=0.5699\cdots$, $\sqrt[3]{3}/4=0.3290\cdots$ approximately.

Hence, *the projection of the vector with components x,y,z is approximately*

$$(15)\qquad x\mathbf{i}'+y\mathbf{j}'+z\mathbf{k}'=(-0.6124x+0.8989y)\mathbf{h} +(-0.3536x-0.2409y+z)\mathbf{v}$$

This formula makes it possible to plot the orthogonal projection of any point with given space coordinates, on a projection plane in which the projected axes make angles of 105°, 120°, and 135° with each other. Since the numbers $\sqrt{6}$ and $\sqrt[4]{27}+\sqrt[4]{3}$ are incommensurable, *no two points with rational coordinates will project into exactly the same point in the projection plane.* This is an advantage of the trimetric projection as compared with the isometric projection in which the drawing plane is equally inclined to the space axes.

9. ORAL QUESTIONS

A. What are the direction angles of a directed line in space? What are its direction cosines?

B. What fundamental identity is satisfied by any set of direction cosines?

C. What is the formula for the cosine of the angle between two lines with direction angles α,β,γ and α',β',γ', respectively? To what does this formula reduce if the lines are parallel?

D. What is meant by the scalar product of two vectors? How is the scalar product of two unit vectors related to the angle between them?

E. How can the scalar product of two vectors be expressed in terms of their components?

F. How is the scalar product involved in computing the work done by a force?

G. What is the condition that the two vectors having components (x,y,z) and (x',y',z'), respectively, be perpendicular? Are the vectors $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ perpendicular to each other?

H. What are the direction cosines of the vector $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$?

I. If three unit vectors **i**, **j**, and **k** in space make angles of α,β,γ, respectively, with the directed normal to a plane, what angles do they make with the plane? What are the lengths of their projections on the plane?

J. How can the projection on a given plane of an arbitrary vector in space be constructed in terms of the projections of three mutually perpendicular unit vectors **i**,**j**,**k**?

9. WRITTEN EXERCISES

1. Find the direction cosines of a directed line that makes equal angles with the directed coordinate axes. Is there more than one solution? HINT: $\cos^2\alpha=\cos^2\beta=\cos^2\gamma$.
2. Find the direction cosines of a line that makes angles of 60° with each of the axes $\uparrow O(X)$ and $\uparrow O(Y)$. What angle does it make with $\uparrow O(Z)$? (Give both solutions.)
3. A space figure is formed by joining by pairs the four points $A{:}(6,1,-2)$, $B{:}(0,4,4)$, $C{:}(8,12,8)$, $D{:}(4,20,0)$. Find the direction components of each of the six lines in the figure. Show that (AB) and (CD) are mutually perpendicular skew lines. Identify the four right angles in the space figure.
4. Draw the figure of Exercise 3 in orthographic projection, using the trimetric ruler.

5. Find the scalar product of two vectors whose lengths are 3 and 8 units, respectively, if they make an angle of 60° with each other.
6. A box is dragged 3 ft. along a sidewalk by exerting a force of 8 lbs. on a rope making an angle of 60° with the horizontal. Find the work done. HINT: Compare with Exercise 5.
7. Find the direction components of a line which is perpendicular to each of the lines of which the components are $(1,-1,0)$ and $(0,1,-1)$, respectively.
8. Examine the formula for the cosine of the angle between two lines in the special case that $\gamma=\gamma'=90°$, $\theta=\alpha-\alpha'$. Express β and β' in terms of $\bar{\alpha}$ and $\bar{\alpha}'$, and derive a formula for $\cos(\alpha-\alpha')$.
9. Draw with a trimetric ruler the projection of a space figure having the six vertices A:(5,0,0), B:(0,5,0), C:(0,0,5), A':(−5,0,0), B': (0,−5,0), C':(0,0−5), joining each vertex to every other one.
10. Taking $-0.61x+0.90y$ and $-0.35x-0.24y+z$ as plane coordinates for the projection of the point whose space coordinates are (x,y,z), calculate the projected positions and plot each of the six points in Exercise 9 (and draw the same figure) on rectangular coordinate paper, without using the trimetric ruler.
11. Draw with a trimetric ruler the projection of a space figure having the six square faces $ABCD$, $AF'C'E$, $BED'F$ (in front) and $A'B'C'D'$, $A'FCE'$, $B'E'DF$ (in back), where the points A,B,C,D,E,F are A:(10,0,10), B:(0,10,10), C:(−10,0,10), D:(0,−10,10), E:(10,10,0), F:(−10,10,0) and where the segments $[AA']$, $[BB']$, $[CC']$, etc., are bisected by the origin. How many equilateral triangles are faces of this figure?
12. Taking $-0.61x+0.90y$ and $-0.35x\ -0.24y+z$ as plane coordinates for the projection of the point whose space coordinates are (x,y,z), calculate the projected positions and plot each of the 12 points in Exercise 11 (and draw the same figure) on rectangular coordinate paper, without using the trimetric ruler.

10

REVIEW OF LINES, PLANES, AND ANGLES

10·1 Points, distance, and the line

Our study of solid geometry is based on assuming the existence of certain undefined elements called points, with each pair of which is associated a measure called distance. A number of assumptions are made about distance, notable among which is the triangle inequality $\overline{AC}+\overline{CB}\geq\overline{AB}$. The points C for which the equality is obtained are defined to be the points of the segment $[AB]$. The concept of the line (AB) is derived by associating with the line segment its two extensions through A and B.

10·2 The plane and space

If each two points determine a line but not all points lie in the same line, then there must be at least two distinct lines through each point. Two distinct lines through a point determine a plane, each of whose points is the mid-point of a unique segment having its extremities on the two given lines. (This correspondence between mid-point and end points in itself gives a means of defining a coordinate system in the plane in terms of coordinate systems on the two lines.) The plane is a two-dimensional linear space, which contains with each two points their line. If not all points lie in a plane, then the space must have more than two dimensions. If the space is a linear space with three but not more than three dimensions, then it is the space in which the theorems of solid geometry apply. To restrict ourselves to three dimensions we assumed that, if two planes have a point in common, then they have two points (and hence a line) in common. This assumption was used in proving that the mediator of a segment is a plane.

10·3 Parallels and vectors

The notion of parallelism can be extended with slight changes in wording from the plane to space. In the plane, two lines (in the extended sense) that do not intersect are parallel. To be parallel in space, however, two lines not only must fail to intersect but must be coplanar. Two noncoplanar lines are called skew lines. With each set of parallel lines is associated a pair of opposite directions. The notion of directed distance is introduced to distinguish between measurements in opposite directions. The notion of a vector is introduced to divorce the notion of directed distance from any particular initial point from which the distance might be measured. Vectors are a useful device for combining such vector quantities as displacements, velocities, accelerations, forces, and rotary motions which arise in the physical sciences, and they are a powerful tool in some branches of geometry. Addition of two vectors is defined by the parallelogram law.

10·4 Perpendiculars and sensed planes

Theorems about perpendiculars in space differ from corresponding theorems in the plane in two important respects. In the plane it is not possible to pick more than two lines at a time, such that each is perpendicular to the others. In three dimensions the largest possible number of lines in a set of mutually perpendicular lines is three, and in a Euclidean space of more than three dimensions this number is the dimension number. Thus, if a line is perpendicular to each of two lines, it is perpendicular to their plane; and if a line is perpendicular to a plane, it is perpendicular to every line in the plane through its foot. Orthographic projection of a three-dimensional figure is based on assigning to each point of space an image that is the foot of the perpendicular from that point onto the plane of projection. A rectangular coordinate system is based on assigning to each point of space the coordinates of its three projections on a set of mutually perpendicular axes of reference. In defining coordinates on a line it is necessary to distinguish between opposite directions on the line by using the notion of directed distance. So in space the points on opposite sides of a plane may be distinguished by giving the plane a sense defined by one of its directed unit normal vectors. Two planes are perpendicular if and only if their normals are perpendicular. Two vectors, with components (x,y,z) and (x',y',z'), respectively, are perpendicular, or orthogc ıal, if and only if $xx'+yy'+zz'=0$.

10·5 Angles and their measurement

The use of the word angle in plane geometry is familiar both as a name for a geometric figure and as a name for its measure in degrees or right angles or other units. Not so familiar perhaps is the directed angle which is used in the algebraic sense to mean a signed number that measures the amount and sense of rotation, and which may be zero, positive, or negative without limit in size. Two angles, either undirected or directed, are supplementary if their sum is 180° and complementary if their sum is 90°. The complement of an angle A is denoted in this text by $\overline{A}$.

In space, angles may be defined between two lines, whether or not they intersect, between a line and a plane, or between two planes. The angle between two half planes is called a dihedral angle and is the supplement of the angle between a pair of normals to the two half planes that are directed inside the dihedron formed by the half planes.

The study of projections leads to an important measure of the angle between two lines, which may be called the *projection factor* or the *cosine* of the angle between the two lines. This is numerically equal to the length of the projection of a segment divided by the length of the given segment and is positive, zero, or negative according as the angle between the directed segment and its directed projection is acute, right, or obtuse.

Numerical values for the cosines of angles of 0°, 15°, 30°, 36°, 45°, 60°, 72°, 75°, 90° and the supplements of these angles can be expressed in terms of radicals by studying regular polygons of three, four, or five sides and by using the formula that expresses the angle between two vectors in terms of their direction cosines:

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$$

The direction cosines are the projections of unit vectors on the coordinate axes and may be used to specify a direction in space.

10·6 The trimetric ruler

The trimetric ruler is a flat drawing instrument having three edges on which are marked three appropriate scales, so that unit vectors along three mutually perpendicular axes in space are represented in an orthographic projection by units measured on the three scales and directed along the corresponding edges of the instrument. In plotting points with the trimetric ruler, one edge of the ruler is always kept vertical, and the three coordinates of the points are measured along the three scales of the ruler. With the aid of the trimetric ruler, fairly complicated figures may be drawn quickly and easily in correct parallel projection.

10. ORAL QUESTIONS

A. What is the length of the diagonal of a box whose edges are 2, 6, and 9 units, respectively?

B. What is meant by a unit of distance? How may the points of a segment be made to correspond to real numbers between 0 and 1?

C. Why is it not possible for a triangle to have sides measuring 1, 3, and 5, respectively?

D. What is meant by the terms "congruent" and "directly congruent"? What is the common property of two congruent line segments?

E. Can two lines be parallel but not coplanar? Can two lines be perpendicular but not coplanar?

F. What are some theorems in the plane that correspond to almost identical theorems in space?

G. What are some theorems that are true in the plane but are not true in space?

H. What are three important meanings of the word "angle"?

I. What are four important properties of the cosine of an angle?

J. Which of the fractions 22/7 and 355/113 is the closer approximation to π? By what fraction of a per cent is each in error?

K. What is the acute angle of intersection between the diagonals of a pentagon?

L. When projections are measured with a trimetric ruler having the angles 45°, 60°, and 75°, is it possible for two distinct points in space with rational coordinates to be represented by the same point in projection?

M. What assumptions have been stated in these chapters? Can you prove any of them without making more assumptions?

10. WRITTEN EXERCISES

1. Find the length of the edge of a cube whose diagonal is 97.00 cm. Draw a figure.
2. Draw a figure representing two skew lines and their common perpendicular. Show a plane containing one of the lines and parallel to the other.
3. Prove that one and only one line can be drawn perpendicular to a plane at a given point in the plane.
4. Prove that two lines each perpendicular to the same plane are parallel.
5. Prove that, if two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to the third plane.
6. Three unit vectors, each perpendicular to the other two, are projected onto a plane parallel to the third vector. Prove that the sum of the squares of the projections of the first two vectors is equal to the square of the projection of the third.
7. Given the vectors $\mathbf{a}=6\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $\mathbf{b}=3\mathbf{i}+8\mathbf{j}-5\mathbf{k}$. Find their lengths, their direction cosines, the cosine of the angle θ between them, and the magnitude of θ in degrees.
8. Draw a figure in parallel projection showing the two vectors of Exercise 7 and their difference $\mathbf{a}-\mathbf{b}$. Are any two of the three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}-\mathbf{b}$ orthogonal?
9. Draw a figure showing the vectors $\mathbf{i}-\mathbf{j}$ and $\mathbf{j}-\mathbf{k}$. Find the angle between them. How does the length of their sum compare with the sum of their lengths?
10. Two directed lines L and L' intersect in a point O, and the poir

A and B' are chosen on L and L', respectively, so that $\overrightarrow{OA} = \overrightarrow{OB'}$. If A' is the projection of A on L' and B is the projection of B' on L, prove that $\overrightarrow{OA'} = \overrightarrow{OB}$.

11. Draw the projections of the six points (1,0,0), (−1,0,0), (0,1,0), (0,−1,0), (0,0,1), (0,0,−1) and the segments joining each point to its four neighbors. This figure is called an octahedron.

12. Draw the projections of the ten points A:(47,34,76), B:(−18,55,76), C:(−58,0,76), D:(−18,−55,76), E:(47,−34,76), F:(76,55,18), G:(−29,89,18), H:(−94,0,18), I:(−29,−89,18), J:(76,−55,18) and the other ten points (A',B', etc.) obtained by changing the signs of all the coordinates. Join together each of the thirty pairs of points which are approximately 68 units apart, such as AB, BC,⋯,EA; AF, BG, CH⋯; FH', FI',⋯. This figure is called a dodecahedron.

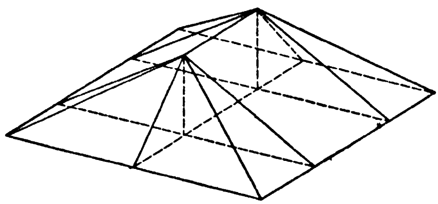

Fig. 27. Attic roof.

13. The roof planes covering a 24- by 24-ft. attic floor consist of two isosceles triangles whose planes make angles of 45° with the horizontal, and two isosceles trapezoids whose planes make angles of 30° with the horizontal and whose upper edges meet on the ridge (Fig. 27). Find the length of the ridge and its height above the floor.

14. A skeleton cube is made of wire segments placed along the edges of a cube. When such a skeleton cube is dipped in a suitable soap solution, a soap film may be formed as shown in Fig. 28. A small square of soap film is formed about the center, in a plane parallel to two faces of the cube; eight trapezoidal soap films each join an edge of the square to a parallel edge of the cube, and four triangular soap films each

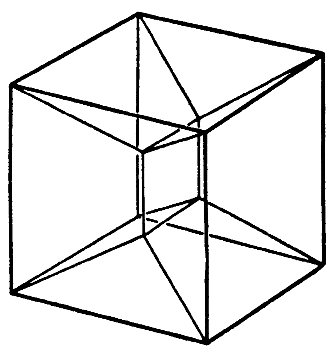

Fig. 28. Soap film on skeleton cube.

join a vertex of the square to the nearest cube edge that is perpendicular to the plane of the square. Wherever three soap films meet along a line they make dihedral angles of 120° with each other. Find the length of an edge of the small square if the edges of the cube are each 1 in. HINT: Compare the triangles and trapezoids with those in Fig. 27.

PART TWO

SOLID MENSURATION

11

THE TRIHEDRON

11·1 The trihedron

Three rays $\uparrow O(A)$, $\uparrow O(B)$, $\uparrow O(C)$ drawn from a common vertex O in space, and not all three coplanar, are the edges of a portion of space called **a trihedron** O-(ABC) (Fig. 29). The point O is called **the vertex** of the trihedron, the three angles **formed** by pairs of edges are called the **face angles,** the plane sectors included within these face angles are called the **faces,** and the dihedral angles between the faces are called the **dihedral angles** of the trihedron. Points that are inside each of the dihedrons of a trihedron are said to be inside the trihedron. The three face angles, denoted by a,b,c, respectively, and the dihedral angles α,β,γ of the three opposite dihedrons are called the six parts of the trihedron O-(ABC).

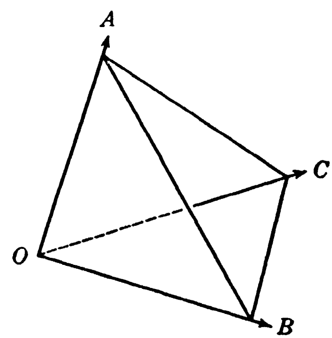

FIG. 29. Trihedron.

A trihedron in which the three edges are mutually perpendicular is called a **trirectangular trihedron.** All its face angles are right angles, and all its dihedral angles are right dihedral angles. The trirectangular trihedron is already familiar as the basis of a rectangular coordinate system.

11·2 Associated trihedrons

Two intersecting planes divide space into four dihedrons having a common edge, of which opposite pairs are congruent. If a third plane intersects the common edge in a point O, it divides each dihedron into two trihedrons called **codihedral trihedrons** (Fig. 30) and thus divides all space into eight trihedrons, called **associated trihedrons** (Fig. 31). Let T be one of a set of eight associated trihedrons, and let its dihedral angles be α, β, γ, respectively. Let T_α, T_β, T_γ denote the codihedral trihedrons that share one dihedral angle and two directed edges with T, choosing labels so that T and T_α together form a dihedron of dihedral angle α, etc. Finally let T^*, $T_\alpha{}^*$, $T_\beta{}^*$, $T_\gamma{}^*$ denote the other four associated trihedrons, so labeled that T^* is obtained from T by reversing the direction of each of the three edges, etc. Using "vertical" in the sense of "vertical angles," we might say that T and T^* are a pair of **vertical trihedrons.** Although vertical angles in the plane are directly congruent, it is not true that vertical trihedrons are directly congruent, except when at least two of the three dihedral angles α, β, γ are equal. The six parts of the one trihedron are, respectively, equal to the six parts of the other, but they follow in the opposite order around the trihedron. It is true, however, that vertical trihedrons are enantiomorphous (Sec. 2·7), and that they enclose equal portions of space.

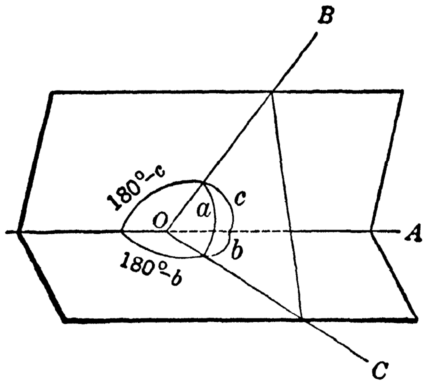

FIG. 30. Codihedral trihedrons.

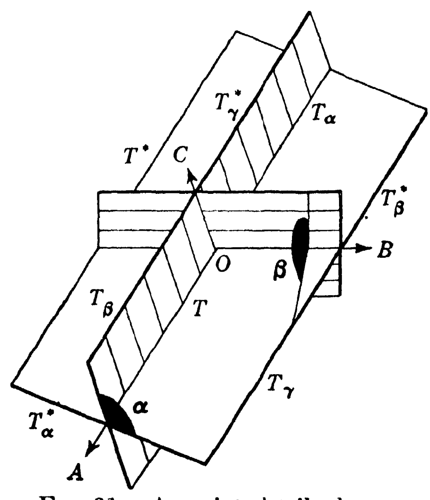

FIG. 31. Associated trihedrons.

11·3 The trihedral angle

The undirected trihedral angle, ∢O-(ABC) of a trihedron O-(ABC) will be defined as an additive measure of the portion of space within a trihedron, the same units being used in which the dihedral angle measures the portion of space within a dihedron. If we denote this trihedral angle for a trihedron T by the Greek letter σ (sigma), and the trihedral angles for the associated trihedrons T_α, T_β, T_γ by σ_α, σ_β, σ_γ, respectively, then

(1) $$\sigma + \sigma_\alpha = \alpha \qquad \sigma + \sigma_\beta = \beta \qquad \sigma + \sigma_\gamma = \gamma$$

Furthermore, the four trihedrons T, T_α, T_β, T_γ together enclose just half the space about a point, or as much space as a dihedron of 180°. Hence

(2) $$\sigma + \sigma_\alpha + \sigma_\beta + \sigma_\gamma = 180^\circ$$

or $$\sigma + \alpha - \sigma + \beta - \sigma + \gamma - \sigma = 180^\circ$$

or $$\alpha + \beta + \gamma - 180^\circ = 2\sigma$$

Hence, solving for σ, we have an expression for the trihedral angle σ in terms of the dihedral angles α, β, γ,

(3) $$\sigma = \tfrac{1}{2}(\alpha + \beta + \gamma - 180^\circ)$$

THEOREM 11A: *The sum of the dihedral angles in a trihedron is always greater than* 180° *and less than* 540°.

PROOF: We have $\alpha + \beta + \gamma = 180^\circ + 2\sigma$ and $\sigma > 0$. Hence $\alpha + \beta + \gamma > 180^\circ$. Since the three angles α, β, γ are each less than 180°, their sum is less than 540°.

11·4 The polar trihedron

Let each of the face planes of a trihedron T be sensed so that the edge not in a face lies on its positive side. Then the positively directed normals $\uparrow O(A')$, $\uparrow O(B')$, $\uparrow O(C')$ to the sensed planes (OBC), (OCA), and (OAB), respectively, are edges of a trihedron T' known as the polar trihedron of T (Fig. 32). The faces of T' are each perpendicular to the edges of T; therefore the polar trihedron of T' is the original trihedron T, and the two may be said to form a pair of **polar trihedrons.**

It has been shown (Sec. 7·10) that the angle between the directed normals to the faces of a dihedron is the supplement of the dihedral angle. From this the following theorem is obtained:

THEOREM 11B: *The dihedral angles* α, β, γ *of a given trihedron* T *are the supplements of the face angles* a', b', c' *of the polar trihedron* T'.

Similarly the face angles a, b, c of T are supplements of the dihedral angles α', β', γ' of T'.

(4) $$\alpha+a'=\beta+b'=\gamma+c'=\alpha'+a=\beta'+b=\gamma'+c=180'.$$

Now let half the sum of the face angles of T and of T' be denoted

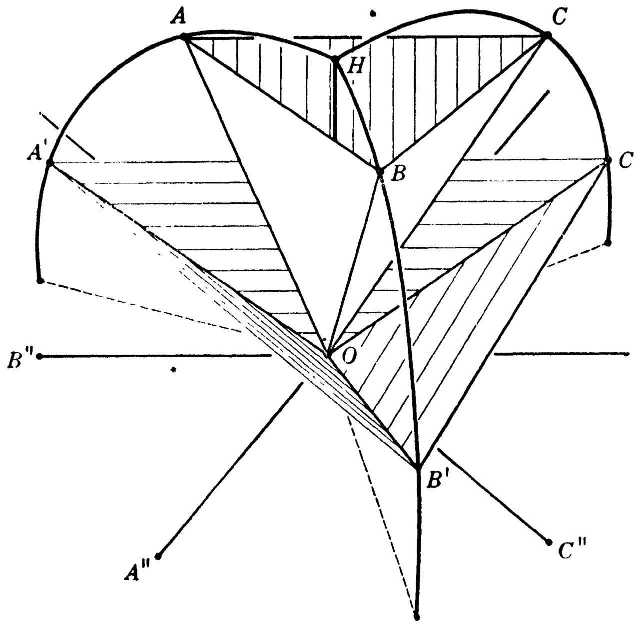

FIG. 32. Polar trihedrons.[1]

by s and s', respectively, and let the trihedral angles of T and T' be denoted by σ and σ', respectively. Then

$$\sigma=\tfrac{1}{2}(\alpha+\beta+\gamma-180°)=\tfrac{1}{2}(180°-a'+180°-b'+180°-c'-180°)$$
$$\sigma'=\tfrac{1}{2}(\alpha'+\beta'+\gamma'-180°)=\tfrac{1}{2}(180°-a+180°-b+180°-c-180°)$$

[1] It can be shown that the planes (OAA'), (OBB'), and (OCC') are coaxial in an axis (OH). In each of the two polar trihedrons T and T' these three planes are called altitude planes, since they are the planes through an edge perpendicular to the opposite face. The line (OH) is called the orthocentric axis, not only for the two polar trihedrons T and T', but also for all the associated trihedrons of each. Corresponding faces of the two trihedrons T and T' intersect in the lines (OA''), (OB''), and (OC''), each perpendicular to (OH). Each of the 10 lines in Fig. 32 is perpendicular to three others and is the orthocentric axis for a pair of polar trihedrons having the other six lines as edges. A section of this configuration by a plane not parallel to one of the 10 lines is a Desargues configuration for coplanar triangles (see Exercise 10).

Hence,

(5) $$\sigma=180°-s',\ \sigma'=180°-s$$

The trihedral angles of the associated trihedrons T_α, T_β, T_γ and T'_α, T'_β, T'_γ are, respectively,

(6)
$$\begin{aligned}\sigma_\alpha&=\alpha-\sigma=(180°-a')-(180°-s')=s'-a' & \sigma_\alpha'&=s-a\\ \sigma_\beta&=\beta-\sigma=(180°-b')-(180°-s')=s'-b' & \sigma_\beta'&=s-b\\ \sigma_\gamma&=\gamma-\sigma=(180°-c')-(180°-s')=s'-c' & \sigma_\gamma'&=s-c\end{aligned}$$

11·5 Restrictions on the parts of a trihedron

In addition to the fact that the sum of the dihedral angles of a trihedron lies between 180° and 540°, there are several other important restrictions on the magnitude of the parts.

THEOREM 11C: *The sum of the face angles in a trihedron is less than* 360°.

PROOF: The sum of the face angles is $2s$. Since $2\sigma'=360°-2s$ and σ' is positive, we have $360°-2s>0$. Hence $2s<360°$.

THEOREM 11D: *The sum of two face angles in a trihedron is greater than the third face angle.*

PROOF: The positive quantities $2\sigma_\alpha'$, $2\sigma_\beta'$, and $2\sigma_\gamma'$ may be expressed in the form $2s-2a$, $2s-2b$, and $2s-2c$, respectively, or in the equivalent form $b+c-a$, $a+c-b$, and $a+b-c$, respectively. Hence $b+c>a$, $a+c>b$, $a+b>c$.

The proofs of two other important restrictions on the size of the parts of a trihedron are listed as exercises.

THEOREM 11E: *If two dihedral angles of a trihedron are equal, the opposite face angles are equal, and conversely.*

Such a trihedron is called **isosceles.**

THEOREM 11F: *If two dihedral angles of a trihedron are unequal, the greater is opposite the greater face angle, and conversely.*

11·6 The convex polyhedral sector

An ordered set of three or more rays meeting in a point O are said to be the **edges** of a **convex polyhedral sector** $\angle O\text{-}(ABCD\cdots)$, if, given any pair of consecutive rays (including the last and first as a consecutive pair), all the other rays lie on the same side of the plane of these two. The plane sectors between consecutive pairs of edges are called **faces** of the polyhedral sector, the dihedral angles between consecutive face planes are the **dihedral angles** of the polyhedral sector, and the polyhedral sector itself consists of the points of all line segments whose end points are on the edges or face sectors.

The **polyhedral angle,** $\measuredangle O\text{-}(ABCD\cdots)$, of a convex polyhedral sector, $\angle O\text{-}(ABCD\cdots)$, which is defined to be a *measure in dihedral angle units* of the portion of space within the polyhedral sector, will be denoted by σ.

THEOREM 11G: *The supplement of the polyhedral angle of a convex polyhedral sector is one-half the sum of the supplements of its dihedral angles.*

PROOF: 1. A convex polyhedral sector with n edges can be subdivided into $n-2$ trihedrons by passing planes through the first edge and each of the other edges except the second and last. Let us denote the trihedral angles of these trihedrons by $\sigma_3, \sigma_4, \cdots, \sigma_n$.

2. The polyhedral angle σ is the sum of the trihedral angles $\sigma_3, \sigma_4, \cdots, \sigma_n$ of the $n-2$ trihedrons that form the polyhedral sector.

3. The sum of the dihedral angles of the convex polyhedral sector is equal to the sum of all the dihedral angles of the $n-2$ trihedrons. That is,

$$\begin{aligned}\alpha+\beta+\gamma+\delta+\cdots &= (2\sigma_3+180°)+(2\sigma_4+180°)+\cdots+(2\sigma_n+180°)\\ &= 2\sigma+(n-2)180°\end{aligned}$$

4. Subtracting each extreme member of the last equations from $n(180°)$, we have

$$(180°-\alpha)+(180°-\beta)+(180°-\gamma)+(180°-\delta)+\cdots=2(180°-\sigma)$$

5. Division by 2 gives the required result.

11. ORAL QUESTIONS

A. What is a trihedron? When is a trihedron isosceles? When is it trirectangular?

B. Can you suggest a definition of the term *equilateral trihedron?*

C. What trihedrons are *associated* with a given trihedron? How are their face angles and dihedral angles related to those of the given trihedron?

D. Two trihedrons are such that corresponding parts are equal. Are the two trihedrons congruent? Are they directly congruent?

E. Two of the face angles of a trihedron are 70° and 150°. What limitations are there to the size of the third face angle?

F. Two of the dihedral angles of a trihedron are each 120°. What limitations are there to the size of the third dihedral angle?

G. The dihedral angles of a trihedron are 80°, 100°, and 110°, respectively. Find the face angles of the polar trihedron.

H. The three dihedral angles of a trihedron are the same (except perhaps for order) as the three face angles of the polar trihedron. What are the angles of the trihedron?

I. What is the trihedral angle of a trirectangular trihedron?

J. The terms *dihedral angle* and *trihedral angle* are frequently used to mean the geometric figures that we have called the dihedron and trihedron. If the term **polyhedral angle** refers to a certain geometric figure, how would you define it? NOTE: Polyhedral means *having many faces*.

K. Is it true that the sum of the face angles in a polyhedral angle is less than 360°?

11. WRITTEN EXERCISES

1. Draw a trihedron O-(ABC) with the three edges cut off at equal lengths so as to rest on a horizontal plane like a tripod. HINT: Project the points $(1,\sqrt{3},0)$, $(1,-\sqrt{3},0)$, and $(-2,0,0)$ with the trimetric ruler. Connect these three points to any convenient point on the vertical axis. Then represent the horizontal base plane by a parallelogram, with sides parallel to the projected axes, large enough to contain the plotted points.
2. Two of the face angles of a trihedron are 80° and 120°. Which of the following angles are possible for the third face: 20°, 40°, 80°, 90°, 160°, 170°? Explain why.
3. Two of the dihedral angles of a trihedron are 40° and 80°. Which of the following angles are possible for the third dihedral angle: 30°, 60°, 90°, 120°, 135°, 150°? Explain why.

4. Given a trihedron with face angles $a=40°$, $b=70°$, $c=100°$. Find the dihedral angles of the polar trihedron. Which is the largest face angle in the polar trihedron?
5. Prove that, if two face angles of a trihedron are equal, the opposite dihedral angles are equal.
6. Prove that, if two face angles of a trihedron are unequal, the opposite dihedral angles are unequal in the same order.
7. Draw a figure representing a trihedron in which each of the three dihedral angles is 120°. Draw the polar trihedron in the same figure. HINT: Connect the three points $(1,\sqrt{3},\frac{1}{2}\sqrt{2})$, $(1,-\sqrt{3},\frac{1}{2}\sqrt{2})$ and $(-2,0,\frac{1}{2}\sqrt{2})$ to the origin and to each other, to represent one trihedron. For the polar trihedron, two of the edges pass through the points $(1,\sqrt{3},2\sqrt{2})$ and $(1,-\sqrt{3},2\sqrt{2})$.
8. Prove that, if two face angles of a trihedron are right angles, the opposite dihedral angles are each right dihedral angles and the third face angle is equal to its opposite dihedral angle.
9. The trihedral angles σ_α, σ_β, σ_γ of the three codihedral trihedrons of a trihedron are 30°, 40°, and 50°, respectively. Find the dihedral angles of the trihedron.
10. DESARGUES' THEOREM: A famous theorem in geometry states that, if the lines joining corresponding vertices of two triangles are concurrent, then the points of intersection of pairs of corresponding sides are collinear (except when the plane of one triangle is parallel to one of the sides of the other triangle). Draw a figure for this theorem, showing a trihedral angle O-(ABC) cut by two intersecting planes.
11. Prove Desargues' theorem (see Exercise 10), assuming that the two triangles do not lie in the same plane, nor in parallel planes.

12

POLYHEDRONS

12·1 Solids and their boundaries and sections

A geometric solid, usually called a **solid,** is a finite portion of three-dimensional space, continuously joined together, and separated from the rest of space by a set of points called its **boundary.** An **interior point** of the solid is a point I such that all points of space within a sufficiently small positive distance d from I are points of the solid. A **boundary point** is a point B such that, no matter how small a positive distance d is chosen, there are points of the solid and points not of the solid at a smaller distance than d from B. Two-dimensional portions of the boundary of a solid are called **bounding surfaces.** One solid is said to *include* or contain a second solid, if all points of the second are points of the first.

A solid (or plane region) is said to be **convex** if, given two arbitrary distinct points A and B of the solid (or plane region), all points of the segment $[AB]$ are points of this solid (or plane region).

A **section** of a solid by a given plane consists of the points of space that lie both in the solid and in the given plane. This plane is called the **plane of section.** A **bounding section** is one that contains at least one boundary point but no interior points of the solid. Two parallel bounding sections may be called **bases** (Fig. 33), and the distance between them is then called the corresponding **altitude** h. A **principal section** is defined to be a section parallel to the bases and lying between them. A **midsection** is a principal section halfway between the bases. If a solid has an axis, a **right section** is a section perpendicular to the axis.

A **segment** of a solid consists of those points of a solid included between two parallel sections. The distance between the parallel planes is the **altitude** of the segment, and the two sections are called its **bases.**

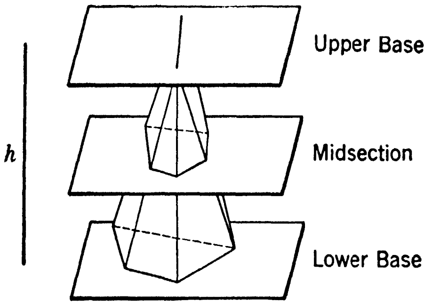

FIG. 33. Bases and midsection of a solid.

THEOREM 12A: *Each section (or segment) of a convex solid is convex.* The proof of Theorem 12A is given as Exercise 1.

12·2 The polyhedron

A solid bounded wholly by plane surfaces—at least 4 are necessary—is called a **polyhedron** (plural polyhedrons or polyhedra). In particular, a polyhedron having 4 faces is called a **tetrahedron,** one having 8 faces is an **octahedron,** one having 12 faces is a **dodecahedron,** and one having 20 faces an **icosahedron.** The plane-polygonal regions whose points are boundary points of a polyhedron are called the **faces** of the polyhedron, and the sides and vertices of these polygons are called the **edges** and **vertices** of the polyhedron, respectively. The following theorem about the sections of a polyhedron is given as Exercise 2.

THEOREM 12B: *Each section of a polyhedron is either a plane-polygonal region or a line segment or a point.*

A convex polyhedron has the following important property: Any closed curve consisting of interior points of the polyhedron can be continuously deformed and shrunk to a point without passing through any boundary points. It is a remarkable fact, a complete proof of which is beyond the scope of this book, that for any polyhedron having this property the number of its vertices V, the number of its edges E, and the number of its faces F are connected by the simple formula

$$V-E+F=2^{*} \tag{1}$$

* Some indication of the proof of this fact can be given, however. Suppose that the plane surfaces of a convex polyhedron are approximated by a model

The geometry of a polygon in the immediate neighborhood of a vertex can be described in terms of the angle formed by the two sides which meet at that vertex, or by the plane sector which they determine. At a vertex of a polyhedron three or more face planes and the same number of edges come together, and the geometry of the neighborhood of that vertex can be described in terms of the polyhedral sector determined by these edges. Two polyhedrons are congruent if and only if corresponding faces are congruent polygons, and the polyhedral sectors at corresponding vertices are also congruent.

12·3 The parallelepiped

A parallelepiped is a polyhedron having six faces that are parallel in pairs (Fig. 34). Each face is a parallelogram. At each vertex three edges meet, forming a trihedron, and the trihedrons at the eight vertices are congruent, respectively, to a set of eight associated trihedrons formed at a vertex by the planes of three adjacent faces. If these trihedrons are all congruent, and consequently trirectangular, the parallelepiped is called a **rectangular parallelepiped.** This is the figure commonly called a *rectangular box* (Fig. 2). Each face is a rectangle. The lengths of the three edges that meet at a vertex are called the **dimensions.** As we saw in Chap. 1, the sum of the squares of the three edges is equal to the square of the diagonal. A **cube** is a rectangular parallelepiped having equal edges or dimensions.

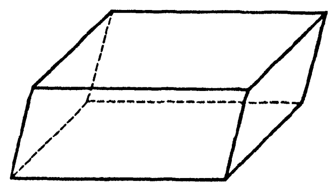

FIG. 34.

Any two parallel faces of a parallelepiped can be selected as **bases**

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made of thin rubber like a balloon and this model is then blown up until it assumes a spherical shape. We now have a configuration of V points joined by E curves, which divide the spherical surface into F regions. Let us suppress any curve joining two vertices at each of which at least three curves come together. In the new figure two regions are combined into one, and thus E and F are each decreased by one, but $V-E+F$ is the same as before. Continue this process until a vertex is left at which only two distinct curves come together. Combine these two curves into one by suppressing the vertex. This decreases E and V each by one but does not change $V-E+F$. By repeating both processes the figure is reduced to a figure of two points, connected to each other by two curves, which together separate the sphere into two regions. Hence $V-E+F=2$.

(usually thought of as upper and lower). The distance between the planes of the bases is the **altitude.**

12·4 Five regular polyhedrons

A **regular** polyhedron is a polyhedron all of whose faces are congruent regular polygonal regions and all of whose dihedral angles are equal (Fig. 35).

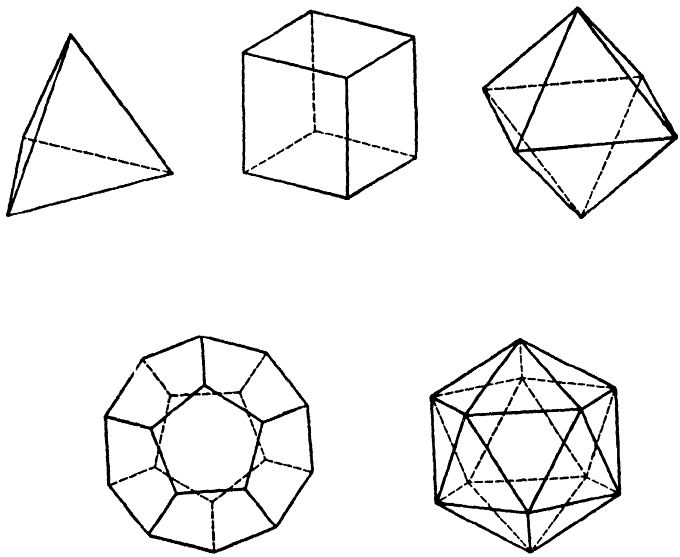


FIG. 35. The five regular polyhedrons.

We denote by m the number of edges at each vertex and by n the number of edges on each face. As before, we let V, E, and F denote the total number of vertices, edges, and faces, respectively.

Because of the symmetry of the regular polyhedron, a point equidistant from four of its faces that are not parallel to the same line is also equidistant from all of its faces. This point we call the **body center** (or three-dimensional center) and denote it by C_3. We call the center of any of the regular faces a **face center** (or two-dimensional center) and denote one of these by C_2 (Fig. 36). We call the mid-point of one of the edges of this face an **edge center** and denote it by C_1. The two vertices on this edge we denote by C_0 and C_0'. The two tetrahedrons $[C_0C_1C_2C_3]$ and $[C_0'C_1C_2C_3]$ are mirror images of one another. The edge $[C_iC_j]$ of $[C_0C_1C_2C_3]$ and the corresponding edge of $[C_0'C_1C_2C_3]$ have equal dihedral angles α_{ij} and equal lengths r_{ij}, but the dihedral angles occur in opposite cyclic order about corresponding vertices in the two tetrahedrons. If the whole polyhedron is subdivided into such elementary tetrahedrons having one vertex at C_3, one at a face center, one at an edge center on that face, and the fourth at a polyhedral vertex on that edge, then half these tetrahedrons will be directly congruent to $[C_0C_1C_2C_3]$ and half to $[C_0'C_1C_2C_3]$. About each polyhedral vertex there will be m of each type, on each polyhedral edge there will be 2 of each type, and touching each face center there will be n of each type. The total number of elementary tetrahedrons of each type can be counted in three ways:

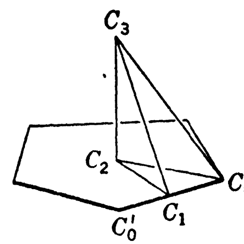


Fig. 36. Elementary tetrahedron.

(1) $mV = 2E = nF =$ number of elementary tetrahedrons of each type

Each of the plane faces of one of these elementary tetrahedrons is a right triangle. Hence

(2) $$r_{02}^2 = r_{01}^2 + r_{12}^2 \qquad r_{13}^2 = r_{12}^2 + r_{23}^2 \qquad r_{03}^2 = r_{01}^2 + r_{12}^2 + r_{23}^2$$

The dihedral angles α_{03}, α_{13}, and α_{23} can be found by counting the number of elementary tetrahedrons about C_0, C_1, and C_2, respectively. We have

(3) $$\alpha_{03} = \frac{180^\circ}{m} \qquad \alpha_{13} = 90^\circ \qquad \alpha_{23} = \frac{180^\circ}{n} \qquad \alpha_{02} = \alpha_{12} = 90^\circ$$

The trihedral angle σ at C_3 in each of the $4E$ elementary tetrahedrons that together fill the space about C_3 is given by Sec. 11·3 as follows:

(4) $$\sigma = \frac{360^\circ}{4E} = \frac{1}{2}\left(\frac{180^\circ}{m} + 90^\circ + \frac{180^\circ}{n} - 180^\circ\right)$$

Hence, dividing by 90°, we obtain

(5) $$\frac{1}{E} = \frac{1}{m} - \frac{1}{2} + \frac{1}{n}$$

Multiplying by $2mn$, we have

$$(6) \qquad 2mn/E = 2n - mn + 2m = 4 - (m-2)(n-2)$$

From (6) and (1) we can express V, E, F in terms of m, n thus:

$$(7) \qquad E = \frac{2mn}{4-(m-2)\,(n-2)}$$

$$(8) \qquad V = \frac{2E}{m} = \frac{4n}{4-(m-2)\,(n-2)} \qquad F = \frac{2E}{n} = \frac{4m}{4-(m-2)\,(n-2)}$$

THEOREM 12C: *There are just five distinct types of regular polyhedrons.*

PROOF: 1. At each vertex and on each face there must be at least three edges, and thus $m-2$ and $n-2$ are positive integers.

2. Since E is a positive integer, it follows by (7) that the product $(m-2)\,(n-2)$ is less than 4.

3. The only possible pairs of (m,n) are (3,3), (3,4), (4,3), (3,5), or (5,3). Q.E.D.

From Eqs. (7) and (8) we derive the number of vertices, edges, and faces for each type. Their descriptions and names are as follows:

Type	m	n	Kind of face	V	E	F	Name of polyhedron
1	3	3	Triangle	4	6	4	Regular tetrahedron
2	3	4	Square	8	12	6	Cube
3	4	3	Triangle	6	12	8	Regular octahedron
4	3	5	Pentagon	20	30	12	Regular dodecahedron
5	5	3	Triangle	12	30	20	Regular icosahedron

By far the most important of these is the cube, sometimes called a regular hexahedron.

12. ORAL QUESTIONS

A. How would you define a solid? What is a boundary point?

B. What is the difference between a section of a solid and a segment of a solid?

C. What are a principal section, a midsection, and a right section of a solid?

D. When is a solid said to be convex? What can be said of the sections of a convex solid?

E. What important relation connects the numbers of vertices, edges, and faces of any convex polyhedron?

F. How many faces does a convex polyhedron have if it has 12 vertices and 24 edges?

G. How many edges does a convex polyhedron have if it has eight vertices and six faces?

H. What is a rectangular parallelepiped? Describe some of its properties—angles, trihedrons, faces, diagonals, etc.

I. How are the eight trihedrons in a parallelepiped related to each other?

J. What are the five regular polyhedrons? What solid is obtained in each of the five cases if the centers of the faces of a regular polyhedron are taken as vertices of a new solid?

K. Why must a regular polyhedron have at least three and no more than five faces at a vertex?

L. Describe the surface that is formed by fitting regular hexagons together, three at a vertex. Have you ever seen such a surface on a tile floor?

12. WRITTEN EXERCISES

1. Prove that each section or segment of a convex solid is convex.
2. Prove that each section of a polyhedron is either a plane-polygonal region or a line segment or a point.
3. Prove that the sum of the two trihedral angles on an edge of a parallelepiped is equal to the dihedral angle on that edge and that the sum of the four trihedral angles on a face is equal to 180°.
4. Write out the reasons for each of the steps in the proof (of Theorem 12*C*) that there are only five types of regular polyhedrons.
5. If the three vertices of a face of a regular tetrahedron are joined to its center, show that the trihedral angle of the trihedron formed at the center is 90°. What are the corresponding trihedral angles for the regular octahedron and icosahedron?

6. Using Eqs. (7) and (8) of Sec. 12·4, prove that $V-E+F=2$ for any regular polyhedron.
7. Draw a regular tetrahedron in trimetric projection. HINT: Start with the projection of the equilateral triangle whose vertices are $(1,\sqrt{3},0)$, $(1,-\sqrt{3},0)$, and $(-2,0,0)$, and locate a fourth point on the vertical axis so that all six edges are equal.
8. Draw a regular octahedron in trimetric projection having its six vertices by pairs on the three axes.
9. Draw a regular octahedron with the equilateral triangle of Exercise 7 as one of its faces. HINT: The other three vertices are at $(-1,-\sqrt{3},z)$, $(-1,\sqrt{3},z)$, $(2,0,z)$, where z is so chosen that all the edges are of equal length.

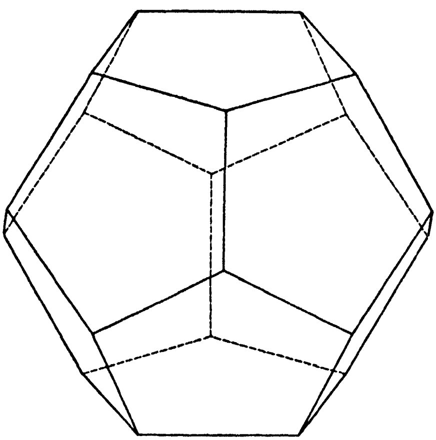


FIG. 37. Regular dodecahedron.

10. Show that the principal sections of the octahedron in Exercise 9 all have the same perimeter. What figure is the midsection?
11. A cuboctahedron is a solid whose 24 edges are obtained by joining together the mid-points of adjacent edges of a cube. Draw one in trimetric projection.
12. Draw a regular dodecahedron with its bottom face in a horizontal plane (Fig. 37).
13. Draw a regular icosahedron by first connecting the mid-points of the adjacent faces in Fig. 37 and then enlarging the figure.

13

VOLUME MEASUREMENT AND CAVALIERI'S THEOREM

13·1 Assumptions about volume

The volume V of a solid may be thought of as a measure of the amount of space contained in the solid. In this book there will be no occasion to consider the possibility of a solid that does not have a positive volume. The following assumptions are made in defining volume:

ASSUMPTION: 1 *The ratio of the volumes of two solids is a determinate number, not dependent on the method of measurement.*

ASSUMPTION 2: *Two congruent solids have the same volume.*

ASSUMPTION 3: *If one solid contains a second solid, the volume of the first is not less than the volume of the second.*

ASSUMPTION 4: *If a solid U is subdivided into two solid parts U_1 and U_2, such that U_1 and U_2 have no interior points in common but each is contained in U, and such that all points of U are either inside U_1 or U_2 or on one or both of their boundaries, then the volume of U is the sum of the volumes of U_1 and U_2.*

From Assumption 4 it follows that if a solid is subdivided into any finite number of solid parts, no two of which have interior points in common, then the volume of the whole solid is the sum of the volumes of its solid parts. Most of the solids we consider are convex.

13·2 The unit of volume

A convenient solid that is used as a standard in measuring volumes is the unit cube, that is, a cube whose edges are each a unit length.

Thus, according as the inch or foot or yard or centimeter is the chosen unit of length, the **cubic inch** (cu.in. or $in.^3$), the **cubic foot** (cu.ft. or $ft.^3$), the **cubic yard** (cu.yd. or $yd.^3$), or the **cubic centimeter** (cc. or $cm.^3$) is defined to be the **unit of volume.** Another common unit of volume is the **U.S. gallon,** which is equal to 231 *cu. in.* and is subdivided into four **quarts,** each equal to two **pints.**

13·3 Volumes of material objects

By the volume of a material object at a given time is meant the volume of the geometric solid that occupies the same portion of space at the given time. A given geometric solid has a definite constant volume. A material object may change its volume because of changes in temperature or pressure or for other causes. One way to find the volume of a solid material object which does not absorb or react with water is to fill with water a container large enough so that the object may be completely submerged in it and then measure the water displaced by causing it to overflow into a suitably graduated container. But since this method, although practical in some cases, is rather limited in its application, it is usually better to obtain measurements of an equivalent geometric solid and apply the theory of this chapter to it. The **weight** of a material object is defined to be the force of attraction of the earth on the object and the **density** of a substance to be the weight of a unit volume of the substance. If a material object has uniform density, its *weight equals the volume times the density* ($W = Vd$).

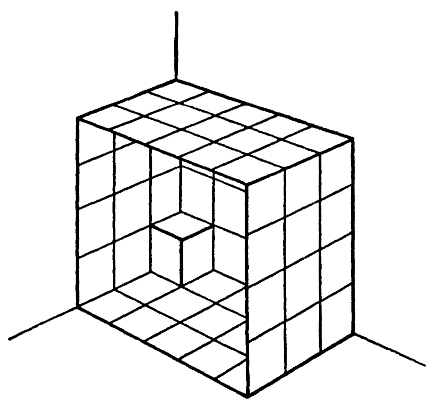


FIG. 38. Volume measurement.

13·4 Volumes of the rectangular parallelepiped and cube

Choosing any convenient length k, let points be marked along the three edges of the trirectangular trihedron that lies at one of the vertices of a rectangular parallelepiped (Fig. 38), at distances k, $2k$, $3k$, $4k$, $\cdots$ from the vertex, and let planes be passed through these points perpendicular to the respective edges. Then, given any positive integers a_1, b_1, c_1, the rectangular parallelepiped with edges $a = a_1k$, $b = b_1k$, $c = c_1k$ can be subdivided in this way into $a_1b_1c_1$ cubes, each with edge of length k. *Its volume is $a_1b_1c_1$ times the volume of the chosen cube.*

1. If k is chosen as the unit of length, we have

$$\begin{aligned} \text{Volume} &= a_1b_1c_1 = abc \\ &= \text{product of the three edges} \end{aligned}$$

2. If k is $1/n$ times the unit of length, with n any integer, then the unit cube contains n^3 of the small cubes with edge length k and thus the volume of each is $1/n^3$. The lengths of the edges are then $a = a_1/n$, $b = b_1/n$, $c = c_1/n$, and the volume is $(a_1b_1c_1)(1/n^3) = abc$, which is the product of the three edges.

3. If one or more of the edges are irrational with respect to the unit of length and if n, a_n, b_n, c_n are integers chosen so that $a_n/n \leq a \leq (a_n+1)/n$, $b_n/n \leq b \leq (b_n+1)/n$, $c_n/n \leq c \leq (c_n+1)/n$, then the volume lies between the volumes of an inside and an outside rectangular parallelepiped and we have

$$\frac{a_n}{n} \times \frac{b_n}{n} \times \frac{c_n}{n} \leq V \leq \frac{a_n+1}{n} \times \frac{b_n+1}{n} \times \frac{c_n+1}{n}$$

When n is taken larger and larger and a_n, b_n, c_n are always chosen as the greatest integers not less than an, bn, cn, respectively, the difference Δ_n between the left and right members becomes smaller and smaller and approaches zero[1]. Unless $V = abc$, a contradiction is obtained. For if either $V - abc$ or $abc - V$ were a positive quantity P, then n could be chosen such that $\Delta_n < \frac{1}{2}P$ and this would contradict the statement that both V and abc lie between the two members. Thus we have proved the following theorem:

THEOREM 13A: *The volume of a rectangular parallelepiped is equal to the product of its three edges.*

In particular for a cube, we have

(1) **Volume of cube = (edge)^3**

[1] The student may show that the difference Δ_n is less than

$$\frac{ab+bc+ca}{n} + \frac{a+b+c}{n^2} + \frac{1}{n^3}$$

Since the product of two adjacent edges of a rectangular parallelepiped is equal to the area of a base B and the third edge is the corresponding altitude h, we have

Volume of rectangular parallelepiped = area of base × altitude

or

(2) $$V = Bh$$

13·5 Cavalieri's theorem

The further analysis of volumes in this book will be based on an important theorem, named Cavalieri's theorem after the mathematician *Francesco Bonaventura Cavalieri* (1598?–1647), who also formulated an analogous theorem for plane figures.[1]

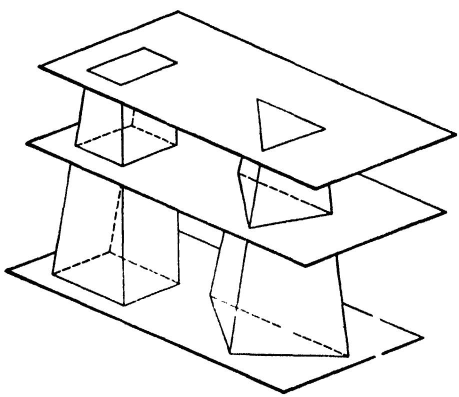


FIG 39. Cavalieri solids.

CAVALIERI'S THEOREM 13B: *If two solids can be placed so that they have equal altitudes and the same base planes and so that for every parallel plane between the bases the principal sections cut from the two solids have equal area, then the two solids have equal volumes* (see Fig. 39).

This theorem may appear obvious intuitively if both solids are sliced up into a pile of thin segments like cards (with beveled edges, perhaps). Corresponding segments have the same thickness, the same top area, and the same bottom area. An assumed difference in volume between two corresponding segments

[1] A discussion of Cavalieri's theorem is given by G. W. Evans, *Am. Math. Monthly*, vol. 24, p. 447, 1917. But the following alternate translation from the Latin is preferred by the author: *Plane figures of any sort constructed between the same parallels—in which, equidistant lines having been drawn parallel to these same lines, the included portions of these straight lines are equal—will be equal to each other; and solid figures of any sort constructed between the same parallel planes—in which, equidistant parallel planes having been drawn parallel to these planes, the plane figures in these planes included within the given solids are equal—will likewise be equal to each other.*

can be shown to lead to a contradiction, provided that the bounding surfaces are sufficiently regular. However, a detailed proof of this fact is beyond the scope of this book.

The student should be warned at this point that intuition may sometimes lead one astray. The following analogous statement about areas is not true:

FALSE STATEMENT: If two polyhedrons can be placed so that they have equal altitudes and the same base planes and so that for every parallel plane between the bases the principal sections cut from the two solids have the same perimeter, then their lateral surface areas are equal.

13·6 Solids for which $V = Bh$

THEOREM 13C: *If all principal sections of a solid, parallel to a suitably chosen base, have equal areas, the volume V of the solid is equal to the area of its base B times its altitude h* (see Fig. 40).

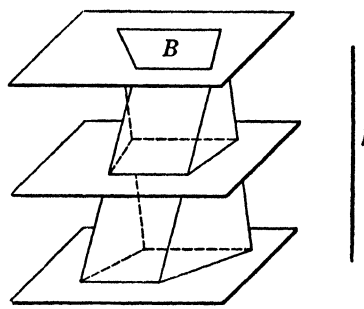


FIG. 40. Principal sections of equal area.

PROOF: The theorem follows from Cavalieri's theorem by comparing the volume of the given solid with that of a rectangular parallelepiped having the same base and altitude.

EXAMPLE: Take any uniform pack of cards, and lay it on the table. If the pack is twisted in any way, the volume of the solid so formed is the same as the original volume occupied by the pack.

13. ORAL QUESTIONS

A. What four assumptions were stated in defining volume?
B. What is meant by the volume of a material object?
C. What are meant by the weight and density of a material object?
D. If the density of water were exactly 62.5 lb. per cu. ft., how many cubic feet of water would there be in a ton? NOTE: The maximum density of water is about 62.4 lb. per cu. ft.
E. How many cubic inches are there in a gallon? In a pint?

F. What is the formula for the volume of a rectangular parallelepiped? For the volume of a cube?

G. A box in the form of a rectangular parallelepiped measures 11 by 7 by 3 in. inside dimensions. How many gallons of liquid will it hold?

H. What is Cavalieri's theorem?

I. What analogous statement about areas is not true? Can you give an example?

J. What are some solids whose volume is given by the formula $V = Bh$?

13. WRITTEN EXERCISES

1. How many cubes 1 ft. on an edge and, again, how many cubes 1 in. on an edge are contained in a cube 1 yd. on an edge?
2. If ice weighs 57.5 lb. per cu. ft., find the volume in cubic feet and the weight in pounds of a block of ice 16 by 18 by 20 in.
3. How many bricks 2 by 4 by 8 in. are required to build a wall 1 ft. thick, 10 ft. high, and 150 ft. long?
4. A pile of closely stacked bricks is 6 ft. high, 12 ft. wide, and 24 ft. long. If the bricks measure 2 by 4 by 8 in., how many bricks are in the pile?
5. A cube displaces 2 qt. of water. Find the length of its edge.
6. What length of lumber 1 in. thick and 4 in. wide is needed to occupy a volume of 1,000 cu. ft?
7. If water weighs 62.5 lb. per cu. ft. and gold weighs 19.3 times as much as water, how much does 1 cu. ft. of gold weigh?
8. If gravel weighs 2.5 times as much as water, what does 1 cu. yd. of gravel weigh?
9. If there are 1,000 cc. (cubic centimeters) in 1 liter and if there are 100 cm. in a meter, how many liters are there in 1 cu. m.?
10. If 231 cu. in. = 4 qts and 2.54 cm. = 1 in., how many quarts are there in 1 liter (1,000 cc.)?

14

THE PRISM AND CYLINDER

14·1 The prism

A prism is a polyhedron of which two faces are congruent polygons in parallel planes and the other faces are parallelograms (Fig. 41). The **bases** are the congruent polygons, the **lateral faces** are the parallelograms, the **lateral edges** are the edges not lying in the bases, and the perpendicular distance between the bases is the **altitude.**

The lateral edges are parallel segments intercepted between parallel planes. They are equal in length, and their common length will be called l.

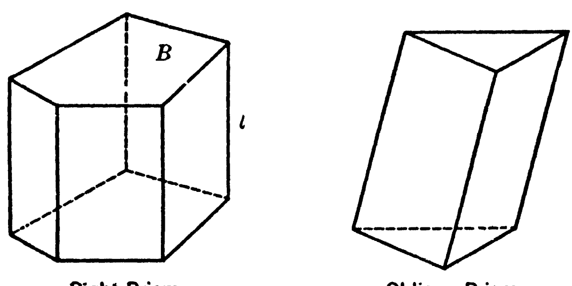


Fig. 41. Prisms.

THEOREM 14A: *Every principal section of a prism is a polygon congruent to each of the bases.*

The proof of this theorem is listed as an exercise.

THEOREM 14B: *The volume V of a prism is equal to the area of its base B times its altitude h.*

(1) $$\textbf{Prism: } V = Bh$$

PROOF: Since all principal sections of a prism are congruent, they have equal areas. By an earlier theorem the prism belongs to the class of solids for which $V = Bh$.

DEFINITIONS: A prism is said to be a **triangular prism,** quadrangular prism, pentagonal prism, etc., according as its bases are triangles, quadrilaterals, pentagons, etc. A prism whose bases are parallelograms is called a **parallelepiped** (Fig. 34).

14·2 Right and oblique prisms

A prism whose lateral edges are perpendicular to the bases is called a **right prism** (Fig. 41). A prism not a right prism is called an **oblique prism.** A right prism whose bases are regular polygons is said to be a **regular prism.** A **right section** of a prism is a section by a plane perpendicular to the edges, and cutting all the edges, prolonged if necessary. The area and perimeter of a right section are denoted by B_R and p_R, respectively. A **longitudinal section** of a prism is a section by a plane, not a lateral face, that contains two of the lateral edges.

THEOREM 14C: *A prism has the same volume and the same lateral surface as a right prism whose bases are congruent to right sections of the given prism and whose altitude is equal to a lateral edge of the given prism.*

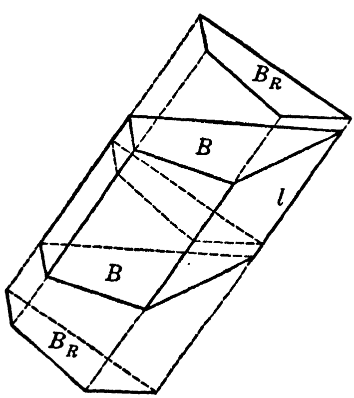


FIG. 42. Volume of oblique prism.

PROOF: Extend the lateral edges far enough on each side of the bases so that two planes can be drawn each perpendicular to all the prolonged edges in such manner that the given prism lies entirely between the two planes (Fig. 42). These planes are bases of a right prism whose altitude we denote by h_R. The bases, each with area B_R, are right sections of the given prism. The volume of this right prism is $V_R = B_R h_R$. Now the bases of the given prism cut this right prism into three pieces. When the middle piece is removed, the two end pieces can be fitted together to form a right prism with volume $V_R - V$, base B_R, and altitude $h_R - l$. Hence

(2) $$V_R - V = B_R\,(h_R - l)$$

(3) $$V = B_R l$$

Similarly, the lateral surface of a prism is equivalent to the difference of the lateral surfaces of two right prisms having equal bases and altitudes h_R and $h_R - l$ and is therefore equal to the lateral surface of a right prism whose base is a right section and whose altitude is equal to a lateral edge of the given prism.

14·3 Drawing of prisms

Since two bases of a prism are congruent polygons in parallel planes, the images of these bases in an orthographic projection will also be congruent polygons. Furthermore, the lateral edges will all be represented by parallel segments of equal length in the projection. It is usually convenient to think of one of the bases as lying in one of the coordinate planes and to plot its vertices with the trimetric ruler. Then it is necessary to plot one of the vertices of the other base and draw the lateral edge through this vertex. The parallelograms that form the lateral faces can then be completed by simply drawing parallels to this lateral edge and to the sides of the base. When finished, the polygons that represent the two bases should be congruent. Care should be taken to draw as dotted lines those edges which lie behind the solid.

14·4 Cylindrical surfaces

A cylindrical surface (Fig. 43) is a surface generated by a straight line which moves so that it always intersects a given plane curve and remains parallel to a fixed line not parallel to the plane of the curve. The curve is called the **directrix**, and the line in each of its positions is called an **element** of the cylindrical surface.

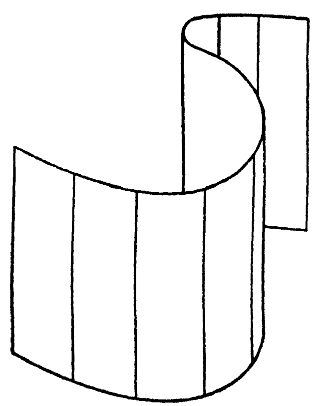


FIG. 43. Cylindrical surface.

We call attention to the fact that in modern mathematics the word **curve** includes a straight line or a broken line as a special case. By our definition, the lateral surface of a prism is a section of a cylindrical surface, having a closed broken line as directrix.

If the directrix is a closed curve, the cylindrical surface is called a **closed cylindrical surface.**

14·5 The cylinder

A cylinder is a solid[1] that is bounded by a closed cylindrical surface and two parallel planes. The cylindrical surface is called the lateral surface, and the plane boundaries are called the bases. The bounding curve of either base may be taken as directrix. The distance between the bases is called the altitude (h), and the common length of all the elements is called the slant height (l).

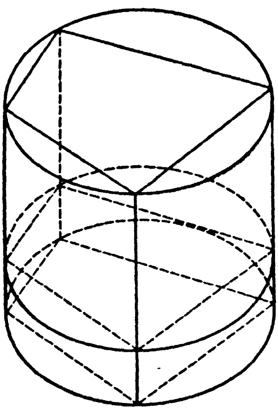


FIG. 44. Principal section of a cylinder.

THEOREM 14D: *The two bases of a cylinder and all the principal sections of the cylinder are congruent plane figures* (see Fig. 44).

PROOF: Pick out as lateral edges of a prism an arbitrary number (>2) of fixed elements of the cylinder and one element that is to be allowed to vary over all remaining elements of the cylinder. The two bases and all principal sections of this prism are congruent polygons. When one of these polygons is moved rigidly so that it coincides with the other for one position of the variable vertex, it will also coincide with the second for all other positions of the variable vertex and the two principal sections or bases of the cylinder will coincide throughout.

Since all principal sections of a cylinder are congruent, the cylinder belongs to the class of solids for which $V = Bh$.

(4) **Cylinder: $V = Bh$**

14·6 Types of cylinders

DEFINITIONS: A cylinder whose elements are perpendicular to the bases is called a **right** cylinder. A cylinder that is not a right cylinder is called an **oblique** cylinder. A section of a cylinder by a

[1] In analytic geometry the word cylinder is used to denote what is here called a cylindrical surface. So also the words cone and conoid are used in analytic geometry to mean surfaces rather than solids.

plane perpendicular to the elements, and cutting all the elements, produced if necessary, is called a **right section** of the cylinder.

THEOREM 14E: *The volume V of a cylinder is equal to the area of a right section B_R times the length of an element.*

INDICATION OF PROOF: With slight changes of wording the same proof can be used as that which was used for the analogous theorem about a prism.

Although it was standard practice in antiquity to define a circular cylinder to be one whose bases are circles, a more useful definition, given by some recent authors, is the one we adopt here. A **circular cylinder** (or a circular cylindrical surface) is a cylinder (or a circular cylindrical surface) whose right section is a circle. A **right circular cylinder** (Fig. 45) is defined without ambiguity to be a circular cylinder that is also a right cylinder. In common speech, the word cylinder is generally used to mean a right circular cylinder. A cylindrical surface (or cylinder) not a circular cylindrical surface (or cylinder), having a circular cross section in some plane oblique to the elements, is called an **elliptic cylindrical surface** (or cylinder) (see Fig. 104). It can be shown that an elliptic cylinder has two sets of parallel sections which are circles and that the planes of all these sections make equal angles with any one of the elements.

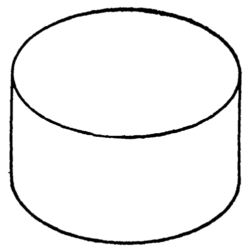


FIG. 45. Right circular cylinder.

14·7 Drawing of cylinders

The two bases of a cylinder project into congruent closed curves of which corresponding points are connected by parallel and equal line segments which represent the elements of the cylindrical surface. By drawing one of the base curves in its true shape on rectangular-coordinate paper the coordinates of a number of its points can be read off. These points can then be transferred into the projection plane by measuring the coordinate distances with two of the sides of the trimetric ruler. One point in the other base must then be located and joined to the corresponding point in the first base. Equal parallel segments drawn through each of the plotted points of the first base will represent elements of the cylinder and will locate the corresponding points in the second base. Each base can then be drawn in as a smooth curve. Care should be taken to dot in or omit those portions of the one base, and the corresponding elements, which lie behind the solid.

14. ORAL QUESTIONS

A. What is the formula for the volume of a prism?
B. Two prisms have equal altitudes and equal volumes. Do their corresponding principal sections all have the same area?
C. What is a cylindrical surface? Does every cylindrical surface have a circular section?
D. How are the following polyhedrons defined: prism; parallelepiped; right prism; regular prism?
E. How are the following solids defined: cylinder; right circular cylinder; elliptic cylinder?
F. What is meant by each of the following terms: directrix; element; base; right section of a cylinder?
G. For what cylinders is every longitudinal section a rectangle?
H. A pentagon can be obtained as a section of a triangular prism. How is this possible?
I. A car is driving at constant speed on a straight road, and raindrops are falling vertically, also with constant speed. In what region of space do all the raindrops lie that will hit the windshield in the next second?
J. An object in a horizontal plane casts a shadow on a parallel plane below. What geometric solid is formed by the portion of space in the shadow?

14. WRITTEN EXERCISES

1. Find the volume in cubic feet of a fireplace 36 in. high whose horizontal sections are trapezoids 12 in. deep with bases 32 in. and 24 in., respectively. Draw a figure.
2. In a swimming pool (with rectangular corners) 25 yd. long and 40 ft. wide, the water is 9 ft. deep at one end and 3 ft. deep at the other. Assuming that the bottom of the pool is a plane and the sides vertical, find the volume of the water. If 32 cu. ft.

of water weighs a ton, find the weight of the water in the pool. HINT: Consider the sides of the pool to be bases of a prism.

3. A house 60 ft. long and 40 ft. wide is covered by a roof consisting of two equally inclined roof planes that meet on a ridge 60 ft. long and 36 ft. above the ground floor. The sides of the house parallel to the ridge are rectangles 60 ft. long and 24 ft. high. Neglecting the cellar, find the volume of the house. What sort of prism is it?
4. A crystal has the form of a regular hexagonal prism 2 cm. in height, whose bases have sides 1 cm. long. Find the volume (Fig. 46).

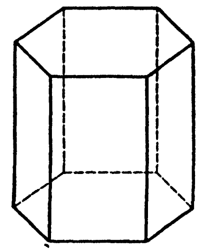


FIG. 46. Regular hexagonal prism.

5. A unit cube is placed with one diagonal vertical. A regular hexagonal prism is then constructed with this diagonal as axis, so that the six lateral edges each pass through the mid-point of an edge of the cube not intersecting this diagonal and the two bases each pass through the mid-points of three edges of the cube which intersect at one of the ends of the diagonal. Examine a cube, draw a figure, and find the altitude and volume of the prism.
6. Find the volume in U.S. gallons of a right circular cylinder of height 6 in. whose base has a diameter of 7 in. (Use 22/7 for π.)
7. Seven congruent solid right circular cylinders of radius r and height h are fitted into a hollow cylinder of inner radius $3r$ and the same height h. Show that the empty space inside the big cylinder is twice the volume of one of the small cylinders.
8. A section of metal pipe 10 ft. long has an inner radius of 1 in. and an outer radius of 1.5 in. Find the volume of the metal in cubic inches, assuming that the inner and outer surfaces are right circular cylindrical surfaces.
9. Prove that every principal section of a prism is a polygon congruent to each of the bases.
10. Prove that the volume of a cylinder is equal to the area of a right section times the length of an element.
11. Draw a figure showing an oblique cylinder with circular bases. Also, draw a circular section of this cylinder that is not parallel to the base.

15

THE PYRAMID

15·1 Solids for which $V = \frac{1}{3}Bh$

There is an important class of solids whose volumes are equal to one-third the product of the base times the altitude. Among these are the pyramids and cones.

15·2 The pyramid

A pyramid is a polyhedron of which one face is called the base and the other faces, called lateral faces, are triangles having a common vertex. This vertex is called the **vertex** of the pyramid, and the distance from the vertex to the base is called the **altitude** of the pyramid. A pyramid can be formed by joining the sides and vertices of any plane polygon (the base) to a point (the vertex) not in the same plane. It can also be formed by cutting off a polyhedral angle by a plane not passing through its vertex. If the base polygon is a triangle, the pyramid is called a triangular pyramid or **tetrahedron.** If the base is a square or any regular polygon and the lateral edges are equal, the pyramid is called a square pyramid or a regular pyramid.

15·3 Principal sections of a pyramid

THEOREM 15A: *Sections of a pyramid by planes parallel to the base are polygons similar to the base polygon. Corresponding lengths in two sections are proportional to the distances of the sections from the vertex, and corresponding areas are proportional to the squares of these distances.*

PROOF: 1. Let O be the vertex and h the altitude of a pyramid $O\text{-}ABC\cdots$ whose base is the polygon $[ABC\cdots]$ (Fig. 47); and let a plane parallel to this base, at a distance h' from O, cut the lateral edges of the pyramid at points $A', B', C', \cdots$, forming a polygon $[A'B'C'\cdots]$ (Fig. 47). This second polygon is the base of a second pyramid with vertex at O, which can be shown to be similar to the given one.

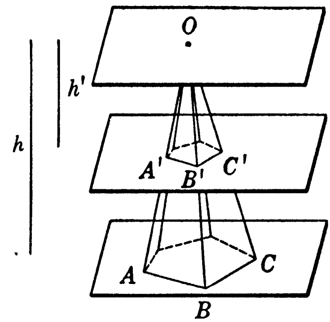


FIG. 47. Pyramid.

2. Since each pair of corresponding sides of the two bases are lines cut from two parallel planes by a third plane, they are parallel and we obtain successively the following relations between the two figures:

$$(A'B') \parallel (AB) \qquad (B'C') \parallel (BC), \text{ etc.}$$

3. Corresponding **edges** of the two pyramids are proportional to the altitudes.

$$\frac{h'}{h} = \frac{\overline{OA'}}{\overline{OA}} = \frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{OB'}}{\overline{OB}} = \frac{\overline{B'C'}}{\overline{BC}}, \text{ etc.}$$

4. Corresponding **angles** in the lateral faces are equal.

$$\angle OB'A' = \angle OBA \qquad \angle OB'C' = \angle OBC, \text{ etc.}$$

5. Corresponding **dihedral angles** on the lateral edges are equal.

$$\measuredangle A'\text{-}OB'\text{-}C' = \measuredangle A\text{-}OB\text{-}C, \text{ etc.}$$

6. Corresponding **trihedrons** at the vertices of the bases are congruent.

$$B'\text{-}(A'O'C') = B\text{-}(AOC), \text{ etc.}$$

7. Corresponding **angles** in the two bases are equal.

$$\angle A'B'C' = \angle ABC, \text{ etc.}$$

8. Since their corresponding angles are equal and their corresponding sides are proportional, the two base polygons are similar. Their areas are proportional to the squares of corresponding lengths and hence to the squares of their distances from the vertex O.

Q.E.D.

15·4 Volume of a tetrahedron

Given a tetrahedron [$D'ACD$] it is possible to construct a triangular prism having the triangle [ACD] as a base and the line segment [$D'D$] as a lateral edge (Fig. 48). Let the other two lateral edges be [$A'A$] and [$C'C$] and let the other base be the triangle [$A'C'D'$], which is congruent to [ACD]. The plane ($D'A'C$) and the face [$D'AC$] of the tetrahedron [$D'ACD$] cut the triangular prism into three tetrahedrons [$D'ACD$], [$D'ACA'$], and [$D'C'CA'$]. The first of these two have congruent bases [$D'DA$] and [$AA'D'$], in the same plane, and have a common vertex C. Principal sections of the two tetrahedrons by planes parallel to the plane ($D'ADA'$) are congruent triangles, similar to the bases. Hence, by Cavalieri's theorem, the two tetrahedrons have the same volume. Likewise, the second and third tetrahedrons [$D'ACA'$] and [$D'C'CA'$] have congruent bases [$A'AC$] and [$CC'A'$] in the same plane and have a common vertex D', and thus they also have equal volumes by the same argument. Since the prism is divided into three tetrahedrons all having the same volume, the volume of each tetrahedron is equal to one-third of the volume of the prism. Since the given tetrahedron has the same base and altitude as the prism, its volume must be one-third the product of its base and altitude.

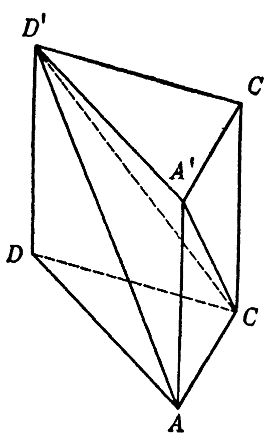


FIG. 48.

THEOREM 15B: *The volume of a tetrahedron is equal to one-third the area of its base times its altitude.*

Tetrahedron: $V = \frac{1}{3}Bh$

15·5 Volume of a pyramid

Any convex pyramid can be subdivided into tetrahedrons having a common altitude equal to the altitude of the pyramid, by first subdividing the convex base polygon into triangles, and then passing planes through the new lines of the base plane and the vertex of the pyramid. The sum of the areas of the triangular bases is equal to the area of the base of the convex pyramid. From this construction we obtain an important volume theorem. For a nonconvex pyramid, the argument requires a slight modification, but the relation still holds.

THEOREM 15E: *The volume of a pyramid is equal to one-third the area of its base times its altitude.*

$$\textbf{Pyramid: } V = \tfrac{1}{3}Bh$$

15·6 Drawing of pyramids

A pyramid can be drawn in projection in much the same manner as a prism. It is most convenient to take the base of the pyramid in one of the coordinate planes and to take the vertex of the pyramid on the axis normal to that plane. Vertices of the base polygon are first plotted, and then the vertex of the pyramid. The visible edges should be drawn first, and then the back edges should be dotted in.

EXAMPLE: To draw a regular pentagonal prism whose lateral faces are equilateral triangles, we first locate the coordinates of the vertices A, B, C, D, E of a regular pentagon in the horizontal plane. Let $\uparrow [OE]$ be taken as a unit segment along the axis $\uparrow O(X)$. Then, by Sec. 8·5, the projection of $\uparrow [OA]$ and of $\uparrow [OD]$ on $\uparrow O(X)$ is $\cos 72°$, or $(\sqrt{5}-1)/4$, or 0.309, whereas the projections of $\uparrow [OB]$ and $\uparrow [OC]$ on $\uparrow O(X)$ are $\cos 144°$, or $-\cos 36°$, or $-(\sqrt{5}+1)/4$, or -0.809. To find the y coordinates of A and D we have

$$y^2 = 1-(0.309)^2 = 0.9045 \qquad y = \pm 0.951$$

Similarly, to find the y coordinates of B and C we have

$$y^2 = 1-(0.809)^2 = (1.809)(0.191) = (0.603)(0.573)$$

$$y = \pm 0.588$$ [1]

Thus the coordinates of A, B, C, and D are A: (0.309,0.951,0), B: (−0.809,0.588,0), C: (−0.809,−0.588,0), D: (0.309,−0.951,0), while E is the point (1,0,0).

Let the vertex of the pyramid be the point F: $(0,0,z)$, where z is to be determined so that $\overline{EF} = \overline{BC}$. Then

$$\overline{EF}^2 = 1 + z^2 = \overline{BC}^2 = 4\left[1 - \left(\frac{\sqrt{5}+1}{4}\right)^2\right] = 4 - \left(\frac{\sqrt{5}+1}{2}\right)^2 = \frac{10-2\sqrt{5}}{4}$$

$$z^2 = \frac{6-2\sqrt{5}}{4} = \left(\frac{\sqrt{5}-1}{2}\right)^2$$

$$z = \frac{\sqrt{5}-1}{2} = 0.618$$

NOTE: If these points are correctly plotted with the trimetric ruler the projection of C will be just to the left of the vertical axis, in fact about one-sixth as far to the left as the length of the projection $[CF]$. This can be determined by using the components of a projected vector given in Eq. (15) of Sec. 9·6.

15. ORAL QUESTIONS

A. What are some of the most important properties of pyramids?

B. What can be said about corresponding lengths and areas in two principal sections of a pyramid?

C. When would two polyhedrons be called similar? How do lengths, areas, and volumes in two similar polyhedrons compare?

D. Is it true that any two parallel sections of a pyramid are similar polygons?

E. Is it possible to have two nonparallel sections of a pyramid that are similar? If so, how?

[1] The computation was facilitated first by factoring the difference of two squares as the product of the sum 1.809 and the difference 0.191. Then to make these factors nearly equal we divide the first by 3 and multiply the second by 3. Finally, the square root of the product of two nearly equal numbers is barely less than half their sum.

F. The base of a certain pyramid is congruent to each of its lateral faces. How many lateral faces are there?

G. In how many ways can a triangular prism be cut up into three tetrahedrons of equal volume?

H. Is the volume of a tetrahedron changed if one of its vertices is moved in a line parallel to one of the edges of the opposite face?

I. How can a trirectangular tetrahedron be constructed that has the same volume as a given tetrahedron?

J. What sort of surface is formed by the rays of light that join the observer's eye to a polygon in a plane not passing through the eye?

15. WRITTEN EXERCISES

1. Find the volume of a pyramid whose altitude is 9 in. and whose base is 64 sq. in.
2. Find the volume of a tetrahedron having a base whose sides are 3,4,5 and a corresponding altitude equal to 6.
3. A square hole is dug into a level sandy beach, but the sand slides back into the hole, leaving an opening in the form of an inverted square pyramid whose lateral faces make dihedral angles of 30° with its horizontal square base. Find the depth of the hole and the area of the base if the volume of the hole is 108 cu. ft.
4. Find the altitude and volume of a regular tetrahedron having unit edges.
5. Find the lengths of the edges of a regular tetrahedron having unit volume.
6. By cutting up a regular polyhedron into pyramids with their vertices at the center of the polyhedron, show that the volume of a regular polyhedron is one-third the product of the inradius times the surface area. Check this formula for the cube. (The **inradius** is the distance from the center to a face.)
7. Find the volume of a regular pentagonal pyramid all of whose 10 edges are of unit length.
8. A regular hexagonal pyramid has lateral faces that are isosceles triangles whose sides are 4,5,5. Find its altitude and volume.
9. The Great Pyramid of Khufu (Greek, *Cheops*) in Egypt was

originally 481 ft. 4 in. high and its base was a square 765 ft. 9 in. on an edge. What was its volume in cubic feet?

10. The dihedral angles at the base of a square pyramid are each equal to 45°. If its volume is 36,000 cu. ft., find the altitude.

11. By using cubic blocks of stone, 1 ft. on an edge, a solid is formed with a square base 10 by 10 ft., a second square layer, 9 by 9 ft., piled on top of this, and then a third, 8 by 8 ft., etc. The top layer consists of one block. The square layers have their sides parallel and their centers on the same vertical line. (*a*) What is the volume of the solid? (*b*) What volume of concrete would have to be added to make a pyramid 11 ft. high with an 11-ft.-square base? (*c*) Compare the volume of the solid of (*a*) with that of a pyramid 10.5 ft. high with base 10.5 ft. square.

16

THE CONE

16·1 Conical surfaces

A conical surface (Fig. 49) is a surface generated by a straight line which moves so that it always intersects a given plane curve and passes through a fixed point not in the plane of the curve. The curve is called the **directrix,** and the line in each of its positions is called a **generator** or **element** of the conical surface. The fixed point is called the **vertex** of the surface. If the directrix is closed, the conical surface is said to be **closed.**

A **circular conical surface** is a closed conical surface all of whose elements make equal angles with a fixed line through the vertex, called the axis. The common magnitude of these equal angles is called the **semivertical angle.**

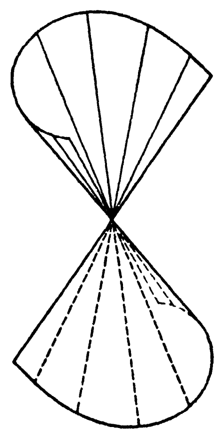


FIG. 49. Conical surface.

16·2 The cone

A cone is a solid[1] bounded by a conical surface (the **lateral surface**), of which the directrix is a closed curve, and a plane (the **base**) which cuts all the elements. A pyramid is a cone of which the directrix is a polygon.

The **altitude** of a cone is the perpendicular from the vertex to the base, or the length of that perpendicular segment.

A **longitudinal section** of a cone is a section by a plane through

[1] See footnote, p. 104.

the vertex that contains two distinct elements of the cone. If the directrix is a convex curve, each longitudinal section is a triangle.

A **cross section** of a cone is a section by a plane cutting all the elements.

A **circular cone** is defined here to be a cone whose lateral surface is a circular conical surface. Its **axis** is the axis of the conical surface.

A cone not a circular cone that has at least one circular cross section is called an **elliptic cone.** Many authors define a circular cone to be a cone with a circular base; such a cone is called an elliptic cone in this book, unless it is a right circular cone.

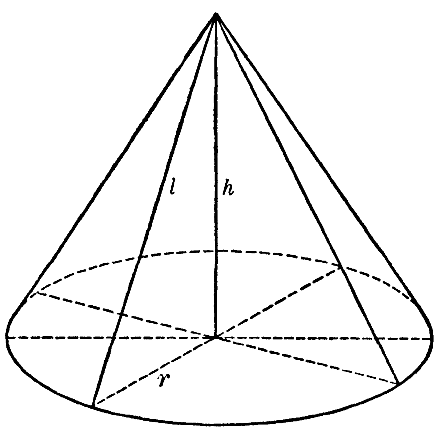


FIG. 50. Right circular cone.

A **right circular cone,** or **cone of revolution,** is a circular cone whose axis is perpendicular to the base (Fig. 50). The base and all principal sections of a right circular cone are circles.

The elements of a right circular cone are equal, and their common length is called the *slant height* l. The altitude is denoted by h and the radius of the circular base by r. A right circular cone can be generated by revolving a right triangle, having legs r and h and hypotenuse l, about one leg h. Hence $h^2+r^2=l^2$.

In common speech, the term cone is usually applied to a right circular cone.

16·3 Principal sections of a cone

THEOREM 16A: *Sections of a cone by planes parallel to the base are similar to the base. Line segments in two sections between corresponding pairs of points are proportional to the distances of the sections from the vertex, and corresponding areas are proportional to the squares of these distances.*

PROOF: Pick out as lateral edges of a pyramid an arbitrary number (>2) of fixed elements of the cone (Fig. 51) and one element that is to be allowed to vary over all remaining elements of the cone.

The base and all principal sections of this pyramid are similar polygons with the same ratio of proportionality for all positions of the variable element. Hence, corresponding angles in the two sections are equal, and corresponding lengths are proportional. The factor of proportionality is the ratio of the distances of the two sections from the vertex. Corresponding areas, whether bounded by straight lines or curves, are to each other as the products of two lengths and hence as the squares of the distances of the sections from the vertex.

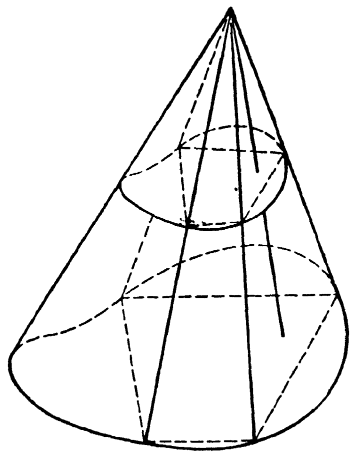


FIG. 51. Principal section of a cone.

16·4 Volume of a cone

THEOREM 16B: *The volume of a cone is equal to one-third the area of its base times the altitude.*

$$\text{Cone: } V = \tfrac{1}{3}Bh$$

PROOF: Construct a pyramid having the same altitude as the given cone and having a base equal in area to the base of the cone and lying in the same plane. Then the principal sections of the two solids by any plane parallel to their bases have equal area. Hence, by Cavalieri's theorem, the solids have equal volume.

16·5 Conelike solids

Any solid which has the property that the areas of its principal sections are proportional to the squares of their distances from one of the base planes is a conelike solid in the sense that its volume is equal to one-third the area of its base times its altitude.

16·6 Drawing of a right circular cone

To draw a right circular cone in orthographic projection it is most convenient to let the axis of the cone be vertical. The circular base in the horizontal plane will project, not into a circle, but into a symmetric curve called an **ellipse**, which can be visualized by looking at any circular object like a coin or a disk from a point not on the normal to its plane at its center. A diameter of the circle parallel to projection plane preserves its true length under orthographic projection, whereas the diameter perpendicular to this, which makes an angle γ with the projection plane, has its length diminished by the factor $\cos \gamma$ in the projection. However, in order to represent a unit vertical vector in space by a unit vertical vector in the plane we enlarged the scale in the plane in all directions by the factor $1/\cos \bar{\gamma}$ (Sec. 9·6). Hence the ellipse that represents a unit circular base in the projection will cut the vertical axis in the projection plane at a distance $\cos \gamma/\cos \bar{\gamma}$ ($=0.4279$, or nearly $\frac{3}{7}$) from O and will cut the horizontal axis at a distance $1/\cos \bar{\gamma}$ ($=1.0877$) from O. Since the difference of the squares of these projections is equal to 1, the length 1.0877 can be most easily approximated by taking the hypotenuse of the right triangle whose legs are a horizontal segment of unit length and a vertical segment of length $\frac{3}{7}$ in the projection plane. If the given circle does not have unit radius, each dimension can be multiplied by the given radius r. Having determined the points where the ellipse cuts the horizontal and vertical lines through O in the projection plane, we can draw short vertical and horizontal tangents at these points, respectively. Then with the upper and lower scales of the trimetric ruler we can mark off r units from O in each direction along the projected space axes and at the marked points on the one axis draw tangents parallel to the other projected axis. Because the ellipse is symmetric, it is even possible to get four more points on it by turning the trimetric ruler face down and using it again, just so long as the vertical side is kept vertical. A number of points on the ellipse having been plotted, these points are to be connected by a smooth curve and the portion that is on the back side of the cone is to be dotted in.

To check the accuracy of the drawing, take a coin, and hold it far enough from the eye so that one of its diameters just covers the long axis of the ellipse in the drawing. Then rotate it about this diameter until its perpendicular diameter covers the short vertical axis of the ellipse. The rim of the coin should just cover the curve in the drawing.

With the circular base projected, the rest is easy. Mark the vertex of the cone on the vertical axis at a distance from O equal to the height of the cone. Draw tangents from this vertex to the ellipse that represents the base of the cone. If it seems desirable, connect the vertex to the points where the projected axes intersect the ellipse, to help in giving the three-dimensional effect (Fig. 50).

16·7 Tangent lines, rulings, and normals to a curved surface

Consider any plane section of a surface not a plane surface, at a given point P on the surface. If the curve of section has a tangent line at P, then the line is called a **tangent line** to the surface, except when this tangent lies in the surface. It is then called a **ruling.** A ruling of a surface is a line all of whose points lie on the surface. *The elements of cylindrical and conical surfaces are rulings.* If even one curve of section at P has no tangent line at P, then P is called a **singular point** of the surface. For example, the vertex of a conical surface is a singular point. At a point P not a singular point, the line perpendicular to two distinct tangent lines or to a tangent line and a ruling is a **normal** to the surface (Fig. 52). All tangent lines to the surface at P lie in a single plane, called the **tangent plane** at P, that is perpendicular to the normal at P. Two surfaces having the same tangent plane at a point P are said to be tangent at P.

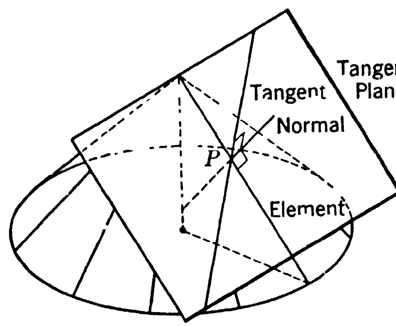


FIG. 52. Tangent and normal to a conical surface.

THEOREM 16C: *The normal to a cylindrical surface at a point P lies in the plane of a right section through the point P.*

THEOREM 16D: *The normal to a circular conical surface at a point P not the vertex lies in the plane determined by the element through P and the axis of the conical surface* (Fig. 52).

THEOREM 16E: *The normals to a circular conical surface drawn at the points of the circumference of a right section all pass through a common point on the axis.*

The proofs of these three theorems are listed as exercises for the student.

THEOREM 16F: *The tangent planes to a right circular cone make equal dihedral angles with the base.*

PROOF: The angle between a tangent plane and the base is equal to the angle between the corresponding normal and the axis of the cone. But this is the complement of the semivertical angle of the cone and is the same for all normals.

16. ORAL QUESTIONS

A. What is a conical surface? What can be said of two parallel sections of a conical surface?

B. What sort of surface is formed by the rays of light that join the eye to a curve that is being observed? Does a photograph of the curve reproduce a plane section of this surface?

C. At the time of a total eclipse of the sun, part of the earth is in the shadow of the moon. What geometric solid is formed by all the points in space that are in the moon's shadow?

D. What are some of the most important properties of a cone?

E. What cones have circular cross sections? If one section of a cone is a circle, is every section a circle?

F. What solid is formed by revolving a right triangle about one of its legs?

G. If the altitude of a right circular cone is 4 in. and the slant height is 5 in., what is the area of the base?

H. Into what figure does a circle project under an orthographic projection? Is this curve symmetric in any of its diameters?

I. How can a right circular cone be drawn in orthographic projection with the aid of the trimetric ruler?

J. What is a ruling of a curved surface?

K. How is the tangent plane to a curved surface at a given point defined? Are there any points on a surface at which no tangent plane is defined?

L. What is meant by the normal to a curved surface at a point? What surfaces have the property that all their normals are parallel to a fixed plane?

M. State at least two properties of the normals to a right circular cone.

16. WRITTEN EXERCISES

1. Find the volume of a right circular cone if the radius of the base is 8 in. and the altitude is 6 in.

2. Draw a figure to scale in orthographic projection, showing the cone in Exercise 1.
3. A dipper in the form of a hollow right circular cone is to contain 1 pt. of liquid. If the diameter of the base is 6 in., find the the depth of the dipper. (Use 22/7 for π.)
4. How deep is a cone whose volume is 1 gal., if it is similar to the cone of Exercise 3.
5. A dipper in the form of a right circular cone (vertex down) is 4.5 in. deep, and contains 1 qt. of liquid. Find the radius of its base, using 22/7 for π.
6. If the dipper in Exercise 5 is filled to a depth of 3.5 in., does it contain more or less than a pint?
7. A circular piece of filter paper 6 in. in diameter is folded along two radii so as to form a right circular cone (base up, vertex down) with slant height 3 in. Calculate the volume of the cone for each of the following altitudes: $h = 1$ in.; 1.5 in.; $\sqrt{3}$ in.; 2 in. Which gives the largest volume?
8. Draw each of the cones in Exercise 7 in orthographic projection. Choose such a scale that each takes about one-quarter of the page.
9. If a triangle is circumscribed about the base of a right circular cone, the pyramid having this triangle as base and the vertex of the cone as vertex is said to be *circumscribed* about the cone. Show that the ratio of the volumes of the two solids is equal to the ratio of the perimeters of their bases.
10. If a square is circumscribed about the base of a right circular cone, the pyramid having this square as a base and the vertex of the cone as vertex is said to be circumscribed about the cone. Show that the ratio of the volumes of the two solids is equal to the ratio of the perimeters of their bases.
11. Prove that the normal to a circular conical surface, at a point P not the vertex, lies in the plane containing the element through P and the axis of the conical surface.
12. Prove that the normals to a circular conical surface, drawn at the points of the circumference of a right section, all pass through a common point on the axis.

17

WEDGES, CONOIDS, FRUSTUMS, AND DECOMPOSABLE SOLIDS

17·1 The wedge

The term **wedge** in common speech refers to a solid such as an axhead whose two principal surface boundaries meet on an edge with a small dihedral angle. We shall define a wedge, in a technical sense, to mean a **triangular right prism,** of which one of the lateral edges is called the **edge** of the wedge, the opposite rectangular lateral face is called the **base** of the wedge, and the distance from edge to base is called the **altitude** (Fig. 53). The triangular faces that were called bases of the prism will be called the **ends** of the wedge.

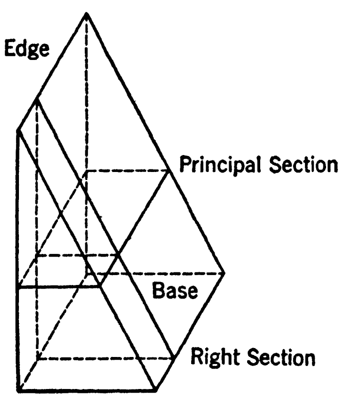


FIG. 53. Wedge.

Right sections of the wedge are the triangular sections perpendicular to the edge and parallel to the ends. Principal sections of the wedge are rectangles parallel to the base. Two important properties of the wedge are the following:

PROPERTY 1: *The areas of two principal sections of a wedge are directly proportional to their distances from the edge.*

PROOF: 1. One side of a rectangular principal section is equal to the edge of the wedge.

2. The areas of different principal sections vary as the lengths of their other sides.

3. These other sides appear in the triangular end of the wedge as line segments parallel to the base of a triangle.

4. Hence their lengths are proportional to their respective distances from the vertex of the triangle.

5. These distances are equal to the distances from the respective principal sections to the edge.

PROPERTY 2: *The volume of a wedge is equal to one-half the area of the base times the altitude.*

$$\textbf{Wedge: } V=\tfrac{1}{2}Bh$$

PROOF: Two congruent wedges can be fitted together to form a parallelepiped having the same base and altitude, by fitting a pair of corresponding triangular ends together to form a parallelogram. The volume of each wedge is half the volume of the parallelepiped.

17·2 The conoid

A conoid (Fig. 54) is here defined to be a solid whose boundary consists of a plane region (the **base**), bounded by a closed curve (the **directrix**), and a **lateral conoidal surface**, generated by a variable straight-line segment (the **generator**), which moves so that one end lies on the directrix and the other end meets at right angles a fixed line (the edge) parallel to the plane of the base.

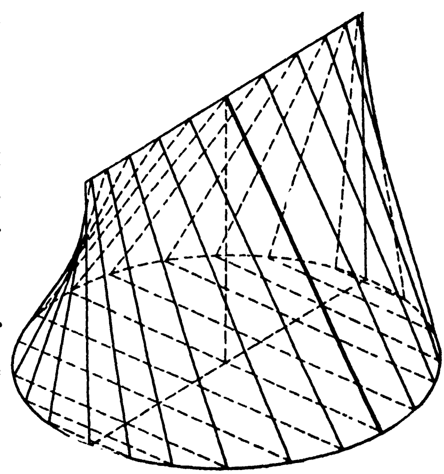


FIG. 54. Right circular conoid.

The generator in each of its positions is called an **element** of the conoid. The distance from the edge of the base is called the **altitude** of the conoid. The rectangular section through the edge perpendicular to the base of the conoid is called the **altitude section.** Sections of a conoid perpendicular to the edge are called **right sections.** Sections parallel to the base are called **principal sections.** If all right sections are isosceles triangles, the conoid is called a **right conoid.** A right conoid with a circular base is called a **right circular conoid.**

A conoid can be sliced by right sections into a number of solids that are approximately wedges having the same altitude as the given conoid. It can be shown that the areas of two principal sections are proportional to their distances from the edge. In comparing the conoid with a wedge having the same altitude and a base of equal area, it follows from Cavalieri's theorem that the conoid has Property 2 as well as Property 1, namely:

THEOREM 17: *The volume of a conoid is equal to one-half the area of its base times its altitude.*

Conoid: $V=\frac{1}{2}Bh$

17·3 Frustums of pyramids and cones

A frustum of a pyramid (or cone) is a segment of a pyramid (Fig. 55) (or cone) included between two parallel planes called the **bases of the frustum.** The distance between the bases is called the **altitude** h of the frustum. The areas of the lower and upper bases are denoted by B_L and B_U, respectively, and the area of the midsection by B_M.

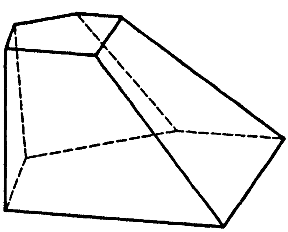


FIG. 55. Frustum of a pyramid.

The volume V of a frustum can be computed as the difference between the volumes of two similar pyramids (or cones), as follows: Assuming that the frustum is turned so that the larger base is the lower base, let h_L, h_U, h_M denote the distance from the vertex of the pyramid (or cone) (Fig. 56) to the planes of the lower and upper bases and the midsection, respectively. Then

(1) $$h=h_L-h_U \qquad h_M=\tfrac{1}{2}(h_L+h_U)$$

Since the bases B_L, B_M, and B_U are proportional to the squares of the distances h_L, h_M, and h_U, we have

(2) $$B_L=kh_L^2 \qquad B_M=kh_M^2 \qquad B_U=kh_U^2$$

where k is the same constant in all three equations.

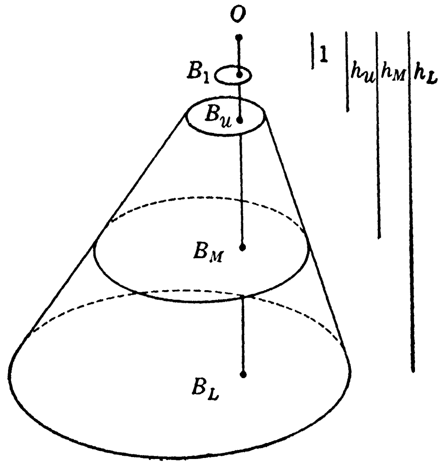


FIG. 56. Frustum of a cone.

For the volume V of the frustum we have

$$(3) \qquad V=\tfrac{1}{3}B_Lh_L-\tfrac{1}{3}B_Uh_U=\tfrac{1}{3}k(h_L^3-h_U^3)$$

Factoring h_L-h_U from the difference of the two cubes we have

$$(4) \qquad V=\tfrac{1}{3}kh(h_L^2+h_Lh_U+h_U^2)$$

Eliminating h_L and h_U from (4) by (2), we have

$$(5) \qquad \textbf{Frustum: } V=\tfrac{1}{3}h(B_L+\sqrt{B_LB_U}+B_U)$$

If $B_U=0$, so that the frustum becomes the whole pyramid (or cone), the formula reduces to $V=\frac{1}{3}hB_L$, which was already obtained for pyramids and cones. In the case of the frustum of a right circular cone whose bases have radii R and r, if $B_L=\pi R^2$ and $B_U=\pi r^2$, then $\sqrt{B_LB_U}=\pi Rr$, and we have

$$(6) \qquad \textbf{Frustum of right circular cone: } V=\frac{\pi}{3}h(R^2+Rr+r^2)$$

Another formula for the volume of a frustum, which has the principal advantage that it also is valid for a much larger variety of solids (including wedges, spherical segments, and many others), involves the area B_M of the midsection. Formula (4) may be written

(4′) $V=\frac{1}{6}kh[h_L^2+(h_L+h_U)^2+h_U^2)=\frac{1}{6}kh(h_L^2+4h_M^2+h_U^2)$

Eliminating h_L, h_M, and h_U by (2) we derive the following very useful volume formula:

(7) **Frustums and other solids:** $\boldsymbol{V=\frac{1}{6}h(B_L+4B_M+B_U)}$

This formula is known as the **prismoidal formula** and will be discussed at length in the next chapter.

17·4 Decomposable solids

By a decomposable solid we mean a solid that can be subdivided into a number of such simpler types of solids as prisms, pyramids, and wedges or cylinders, cones, and conoids, whose volumes we have learned to compute exactly. Such solids occur very frequently in practical measurement. Only the simplest types of houses can be thought of as simple prisms for the purposes of volume measurement. Many houses have wings, dormer windows or bay windows, or roofs more complicated than a single dihedron. They can be treated as decomposable solids. A milk bottle can be approximated as a solid composed of a large cylinder below, with a small cylinder on top, and a frustum in between. A table with round legs may be composed of a combination of prisms and cylinders.

To compute the volume of such solids it is generally necessary to compute individually the volumes of the component parts and add results.

17·5 Volumes of the regular polyhedrons

As was shown in Sec. 12·4, each one of the regular polyhedrons can be decomposed into $4E$ elementary tetrahedrons (Fig. 36) of equal size, where E is the number of edges of the solid. In our previous notation (Sec. 12·4) the base of such a tetrahedron is a right triangle whose area is $\frac{1}{2}r_{01}r_{12}$ and whose altitude is r_{23}. Hence the volume of each elementary tetrahedron is $\frac{1}{6}r_{01}r_{12}r_{23}$, and that of the regular polyhedron is

Regular polyhedron: $\boldsymbol{V=\frac{2}{3}Er_{01}r_{12}r_{23}}$

17. ORAL QUESTIONS

A. What is a wedge, and what are its principal sections and right sections?

B. What sorts of polygons besides triangles and rectangles can be obtained as plane sections of a wedge?

C. What solids do you know that have volumes equal to one-half the base times the altitude?

D. In what senses is a conoid a generalization of a wedge?

E. Can a solid plug be made that will fit a round hole, a square hole, and a triangular hole?

F. What are the shapes of the projections of a right circular conoid on three mutually perpendicular planes one of which contains the base and another the edge?

G. What is a frustum of a pyramid or cone?

H. Can you suggest some examples of frustums in nature, architecture, or household furnishings?

I. What is the formula for the volume of a frustum of a right circular cone expressed in terms of its altitude and the radii of the bases?

J. What is the formula for the volume of a frustum expressed in terms of the bases, midsection, and altitude?

K. How can a regular octahedron be decomposed into simpler solids for volume analysis?

17. WRITTEN EXERCISES

1. Find the volume of a wedge whose base is 1.5 by 4 in. and whose altitude is 6 in.
2. Given a right circular conoid whose altitude h is equal to the diameter of its base, show that this solid will fit into a round hole or a square hole or a triangular hole. Draw a figure.
3. Find the volume of the conoid in the preceding exercise if its altitude is 7 in. (Use 22/7 for π.)
4. A frustum of a pyramid is 2 ft. high and has square bases whose sides are 3 ft. and 2 ft., respectively. Find its volume.

5. Cleopatra's Needle (an Egyptian obelisk) is a stone with square cross sections in the form of a frustum 61 ft. high surmounted by a pyramid 9 ft. high. Find its volume if the bases of the frustum have sides $7\frac{1}{2}$ ft. and 4 ft., respectively. Find its weight if the stone weighs 170 lb. per cu. ft.
6. Prove that the area of the midsection of a frustum is always less than half the sum of the bases.
7. A coffeepot holding 2 qt. has the shape of a frustum of a cone, with circular bases having diameters 5 in. and 3 in., respectively. Find its height (using 22/7 for π). Remember that 1 U.S. gallon = 231 cu. in.
8. A pole 30 ft. long made from a tree trunk has the shape of a frustum of a right circular cone. If the radii of the ends are 6 in. and 3 in., respectively, find the volume.
9. If the eight corners of a cube are cut off by planes through the mid-points of the edges, the solid that remains is called **a cuboctahedron** (Fig. 96). Find the volume of a cuboctahedron with edges 1 in. long as the difference between a cube and the eight tetrahedrons cut off.
10. If alternate vertices of a cube are connected, a regular tetrahedron is formed each of whose edges is a diagonal of a face of the cube. If each edge of the cube is 2, find the volume of the tetrahedron as the difference of the cube and the four corner tetrahedrons.
11. A bottle 9 in. tall has a lower circular base of 3.5 in. inside diameter and an upper circular base of 1.4 in. inside diameter. The lower $4\frac{1}{4}$ in. of height is a right circular cylinder, the next $2\frac{3}{4}$ in. of height is a frustum of a cone whose bases have diameters 3.5 and 1.4 in., and the top 2 in. of height is a right circular cylinder. Using 22/7 for π, show that the volume of the bottle is 1 qt.

18

THE PRISMOIDAL FORMULA AND SIMPSON'S RULE

18·1 Mean base of a solid

The mean base $\overline{B}$ of any solid is defined to be the volume divided by the altitude: $\overline{B} = V/h$. It is obvious from the definition then that the volume of any solid is equal to its mean base times its altitude.

$$V = \overline{B}h \tag{1}$$

This tells us nothing new but focuses attention on the computation of a mean base for a solid—some sort of average of the areas of the principal sections—that when multiplied by the altitude will give the volume. If a solid of altitude h can be decomposed into simpler parts having the same altitude h, the mean base of the whole is seen to be equal to the sum of the mean bases of the parts.

For a prism or cylinder the mean base is the same as the lower and upper bases, for a wedge or conoid the mean base is half the base of the wedge or conoid and is equal to the area of the midsection, whereas for a pyramid or cone the mean base is one-third of the base of the pyramid or cone. By using all three areas B_L (lower base), B_M (midsection), and B_U (upper base), some of which may turn out to be zero in a given case, it is possible to derive a much more inclusive formula valid not only for these types of solids individually but also for a great many solids that could be subdivided into solids of these types.

To find this formula we resort to a little algebra and try to find the three unknown numbers x, y, and z such that

(2) $$\overline{B} = xB_L + yB_M + zB_U$$

Dividing by $\overline{B}$ we may write Eq. (2) in the form

(3) $$x\left(\frac{B_L}{\overline{B}}\right) + y\left(\frac{B_M}{\overline{B}}\right) + z\left(\frac{B_U}{\overline{B}}\right) = 1$$

For the prism each of the three ratios $B_L/\overline{B}$, $B_M/\overline{B}$, $B_U/\overline{B}$ is 1, and thus

(3*a*) $$x + y + z = 1$$

For the wedge we have $B_L/\overline{B} = 2$, $B_M/\overline{B} = 1$, $B_U/\overline{B} = 0$, and thus

(3*b*) $$2x + y = 1$$

For the pyramid we have $3\overline{B} = B_L = 4B_M$, $B_U = 0$, and thus

(3*c*) $$3x + \tfrac{3}{4}y = 1$$

These three equations (3*a*), (3*b*), (3*c*) have the common solution

(4) $$x = \tfrac{1}{6} \qquad y = \tfrac{4}{6} \qquad z = \tfrac{1}{6}$$

Hence Eq. (2) may be written in the form

(5) **Mean base:** $$\overline{\boldsymbol{B}} = \tfrac{1}{6}(\boldsymbol{B}_L + 4\boldsymbol{B}_M + \boldsymbol{B}_U)$$

We have already seen (Sec. 17·3) that this formula holds for a frustum of a pyramid or cone, as well as for prisms, cylinders, wedges, conoids, pyramids, and cones. To generalize still further its applicability we have the following theorem:

THEOREM 18A: *The mean base of any solid which can be decomposed into the sum or difference of cylinders (or prisms), cones (or pyramids), and conoids (or wedges) having the same altitude and the same base planes is given by the formula* $\overline{B} = (B_L + 4B_M + B_U)/6$.

PROOF: Using primes to refer to the various parts, we have

$$B_L = B_L' \pm B_L'' \pm B_L''' \pm \cdots$$
$$B_M = B_M' \pm B_M'' \pm B_M''' \pm \cdots$$
$$B_U = B_U' \pm B_U'' \pm B_U''' \pm \cdots$$

Since the mean base of the whole is the sum of the mean bases of the parts,

$$\overline{B} = \overline{B'} \pm \overline{B''} \pm \overline{B'''} \pm \cdots$$

Substituting the values of $\overline{B'}$, $\overline{B''}$, $\overline{B'''}$, etc., from (5), we have

$$\overline{B} = \tfrac{1}{6}(B_L' + 4B_M' + B_U') + \tfrac{1}{6}(B_L'' + 4B_M'' + B_U'') + \tfrac{1}{6}(B_L''' + 4B_M''' + B_U'') + \cdots$$

$$= \tfrac{1}{6}(B_L' + B_L'' + B_L''' + \cdots) + \tfrac{4}{6}(B_M' + B_M'' + B_M''' + \cdots) + \tfrac{1}{6}(B_U' + B_U'' + B_U''' + \cdots)$$

$$= \tfrac{1}{6}(B_L + 4B_M + B_U)$$

18·2 Prismatoids and prismoids

A prismatoid (Fig. 57) is a polyhedron all of whose vertices lie in two parallel planes (the bases). If the two bases have the same number of sides, the prismatoid is called a **prismoid.** Any convex prismatoid can be subdivided into prisms, wedges, and pyramids.

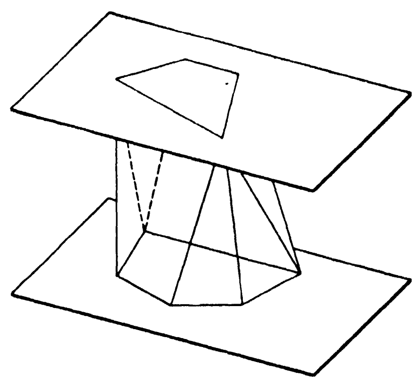


Fig. 57. Prismatoid.

Recalling Cavalieri's theorem, we shall find it convenient to define a **generalized prismoid** as any solid having two parallel base planes, whose areas of section by planes parallel to and between the bases are either equal to the corresponding areas of section of some one prismoid having the same base planes or equal to the differences of corresponding areas of section of two such prismoids. A generalized prismatoid may be characterized algebraically by saying that the *areas of its principal sections are quadratic functions of their distances from one base.*

THEOREM 18B: *The volume of any prismatoid or generalized prismoid or of any solid the areas of whose principal sections are quadratic* (or even cubic) *polynominal functions of their distances from one base, is given by*

(6) **The prismoidal formula:** $V = \dfrac{h}{6}(B_L + 4B_M + B_U)$

where V is the volume, h the altitude, and B_L, B_U, and B_M the areas of the lower and upper bases and midsection, respectively.

PROOF: Any section area that is a quadratic function of the distance from one base is equivalent to the algebraic sum that is, sum or difference) of a constant section area (of cylinder or prism), a section area proportional to the distance from the base (of conoid or wedge), and a section area proportional to the square of the distance from a base (of cone or pyramid). The prismoidal formula holds not only for each of these types separately but for their sum. Hence by Cavalieri's theorem it holds for solids of the type described. (The proof for cubic terms will not be given here in detail, except to point out that a cubic function which is zero on the midsection contributes as much to one half of a solid as it takes away from the other half and that the sum of its values for B_L and B_U is zero).

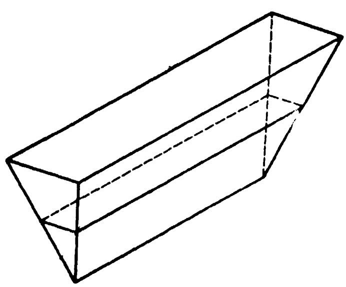


FIG. 58. Prismatoid trough.

EXAMPLE 1: A trough in the form of a prismatoid 12 in. deep has as its upper base a rectangle measuring 8 by 40 in. (Fig. 58). Its lower base consists of a line segment 30 in. long parallel to the long side of the upper base. Find its volume in cubic feet.

SOLUTION: The midsection is a rectangle whose dimensions are the average of those in the two bases, namely, 4 by 35 in. Hence,

$$B_U = 320 \text{ sq. in.} \qquad B_M = 140 \text{ sq. in.} \qquad B_L = 0$$

$$\overline{B} = \frac{320+4(140)+0}{6} \text{ sq. in.} = \frac{880}{6} \text{ sq. in.}$$

$$= \frac{880}{6(144)} \text{ sq. ft.} = \frac{55}{54} \text{ sq. ft.}$$

Since $h = 1$ ft.

$$V = 55/54 \text{ cu. ft., or } 1.02 \text{ cu. ft. approximately}$$

18·3 Simpson's rule for computing the volume of any solid

Although not every solid is the sum or difference of cylinders, conoids, and cones, any solid with "reasonably" smooth boundaries can be sliced by parallel planes into segments such that each segment can be closely approximated by a solid which is such a sum or difference. If the segments have equal altitudes, then, by applying the prismoidal formula to each segment, we obtain the approximate volume formula known as **Simpson's rule.** Let the areas of the two bases of the solid be B_0 and B_{2n}, let the areas of the intermediate bases of the segments be $B_2, B_4, \cdots, B_{2n-2}$, and the areas of the mid-sections be $B_1, B_3, \cdots, B_{2n-1}$ (Fig. 59). Then the approximate volume of the solid is

$$(7)\quad V=\frac{B_0+4B_1+2B_2+4B_3+2B_4+\cdots+4B_{2n-1}+B_{2n}}{6n}\cdot h \text{ approximately}$$

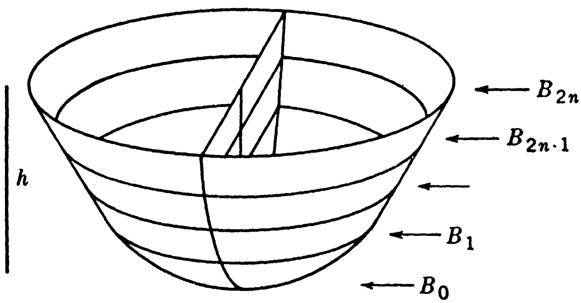


FIG. 59. Volume obtained from areas of slices.

For purposes of computation it is more convenient to write

$$(8)\qquad V\cong\frac{V_0+2V_1}{3}$$

where

$$(9)\qquad \begin{aligned} V_0&=(\tfrac{1}{2}B_0+B_2+B_4+\cdots+B_{2n-2}+\tfrac{1}{2}B_{2n})\left(\frac{h}{n}\right)\\ V_1&=(B_1+B_3+B_5+\cdots+B_{2n-3}+B_{2n-1})\left(\frac{h}{n}\right)\end{aligned}$$

EXAMPLE 2: Let it be required to find the volume in cubic inches included between two intersecting right circular cylindrical surfaces of radii $a=12$ and $b=4$ in., whose axes are skew lines perpendicular to each other but 2 in. apart (Fig. 60). The exact volume can be computed by elaborate computations involving advanced calculus, but a good approximation can be obtained by Simpson's rule. Let us assume that both cylinders have horizontal axes and that the axis of the inner (smaller) cylinder is above the axis of the larger one. Then the bases of the solid are line segments cut from the lower and upper elements of the inner cylinder, which are chords of a vertical circular section of the outer cylinder. A principal section of this solid at distance z in. above the lower base is a rectangle, in a horizontal plane, whose length is a chord of this same large circle $z+10$ in. from its bottom and $14-z$ in. from its top. Since half the chord is a mean proportion between these segments $z+10$ and $14-z$, the length l of the rectangle is $2\sqrt{(z+10)(14-z)}$ in. Similarly the width w

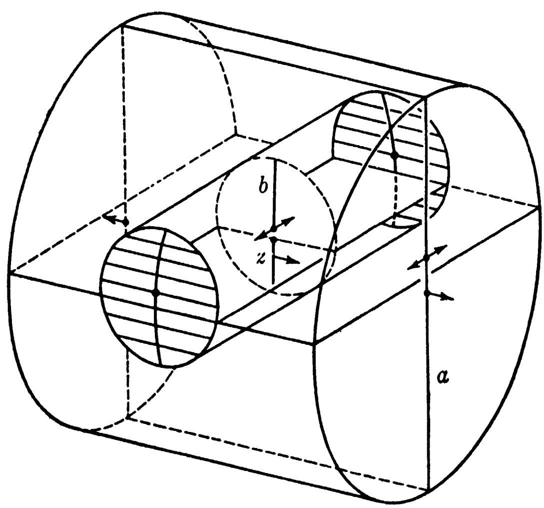


FIG. 60. Intersecting cylinders.

of the rectangular horizontal section of the solid is equal to a chord in a vertical circular section of the smaller cylinder. This chord is at distances of z in. and $8-z$ in., respectively, from the bottom and the top of this circle, and thus its length is equal to $2\sqrt{z(8-z)}$. Hence the area $B=lw$ of the rectangular section is $4\sqrt{(z+10)(14-z)z(8-z)}$. Upon taking $n=8$, the computation of the volume can be carried out systematically as follows:

z	l	w	B	
0	$2\sqrt{10\times 14}$	$2\sqrt{0\times 8}$	$0;\ \frac{1}{2}B_0 =$	0.0
1	$2\sqrt{11\times 13}$	$2\sqrt{1\times 7}$	$4\sqrt{1{,}001} =$	126.6
2	$2\sqrt{12\times 12}$	$2\sqrt{2\times 6}$	$96\sqrt{3} =$	166.3
3	$2\sqrt{13\times 11}$	$2\sqrt{3\times 5}$	$4\sqrt{2{,}145} =$	185.3
4	$2\sqrt{14\times 10}$	$2\sqrt{4\times 4}$	$32\sqrt{35} =$	189.3
5	$2\sqrt{15\times 9}$	$2\sqrt{5\times 3}$	$180 =$	180.0
6	$2\sqrt{16\times 8}$	$2\sqrt{6\times 2}$	$64\sqrt{6} =$	156.8
7	$2\sqrt{17\times 7}$	$2\sqrt{7\times 1}$	$28\sqrt{17} =$	115.4
8	$2\sqrt{18\times 6}$	$2\sqrt{8\times 0}$	$0;\ \frac{1}{2}B_{16} =$	0.0

$$\tfrac{1}{2}B_0+B_2+B_4+\cdots+B_{14}+\tfrac{1}{2}B_{16}=\overline{1{,}119.7}\text{ sq. in.}$$

z	l	w	B	
0.5	$\sqrt{21\times 27}$	$\sqrt{1\times 15}$	$9\sqrt{105} =$	92.2
1.5	$\sqrt{23\times 25}$	$\sqrt{3\times 13}$	$5\sqrt{897} =$	149.8
2.5	$\sqrt{25\times 23}$	$\sqrt{5\times 11}$	$5\sqrt{1{,}265} =$	177.8
3.5	$\sqrt{27\times 21}$	$\sqrt{7\times 9}$	$21\times 9 =$	189.0
4.5	$\sqrt{29\times 19}$	$\sqrt{9\times 7}$	$3\sqrt{3{,}857} =$	186.3
5.5	$\sqrt{31\times 17}$	$\sqrt{11\times 5}$	$\sqrt{28{,}985} =$	170.3
6.5	$\sqrt{33\times 15}$	$\sqrt{13\times 3}$	$3\sqrt{2{,}873} =$	138.9
7.5	$\sqrt{35\times 13}$	$\sqrt{15\times 1}$	$5\sqrt{273} =$	82.6

$$B_1+B_3+\cdots+B_{13}+B_{15}=\overline{1{,}186.9}\text{ sq. in.}$$

Since $h/n=1$ in.,

$$V_0=1{,}119.7\text{ cu. in.}\qquad V_1=1{,}186.9\text{ cu. in.}$$
$$2V_1=2{,}373.8\text{ cu. in.}$$
$$3V=\overline{3{,}493.5}\text{ cu. in.}$$
$$V=1{,}164+\text{ cu. in.}$$

approximately

A more exact computation of volume, based on advanced calculus, gives the value $V=1{,}171.7$ cu. in. It can be seen from this that Simpson's rule gives only approximate results but that the error of 7.2 cu. in. is small compared with the difference $V_1-V_0=67.2$ cu. in. Increased accuracy could have been had by doubling the value of n.[1] In fact for $n=16$, we should have $V_0=1{,}153.3$, $V_1=1{,}177.3$, $V=1{,}169.3$ cu. in.

EXAMPLE 3: *Volumes obtained from contour lines.* As one of many possible examples of the use of Simpson's rule, let it be required to estimate the number of gallons of water in a reservoir, given a map on which points at various selected levels are connected by contour lines. Suppose in particular that contour lines are given for every 10 ft. of depth and that the maximum depth is 60 ft. Then, by using an instrument called a planimeter, the areas on the map for depths of 0, 10, 20, 30, 40, 50, and 60 ft. can be measured directly. These areas, divided by the square of the fractional scale of the map, give the actual areas B_0, B_1, B_2, B_3, B_4, B_5, B_6 of principal sections of the lake. By applying Simpson's rule for $n=3$, the volume is obtained. From this the number of gallons can be computed.

18. ORAL QUESTIONS

A. What is meant by the mean base of a solid? Are there any solids such that the mean base is independent of the choice of base planes?

B. What is the prismoidal volume formula?

C. For what solids is the volume given exactly by the prismoidal formula?

D. What are the distinctions between a prismatoid, a prismoid, and a generalized prismoid?

E. Can two pyramids be cut off of a regular icosahedron so as to leave a prismatoid in the middle? Is this prismatoid also a prismoid?

F. A cube is cut in half by a plane through its center perpendicular to one of its diagonals. Is each half a prismoid?

[1] A close approximation to the volume between two intersecting cylinders of radii $a>b$, whose axes are perpendicular but at distance d apart, is given by the following approximate formula:

$$V \cong 2\pi ab^2 \frac{8a^2-8d^2-2b^2}{8a^2-4d^2-b^2}$$

This approximation works well if $a^2-d^2>2ab$. For $a=12$ in., $b=4$ in., $d=2$ in., it gives $V=1{,}171.90$ cu. in., with an error of only 0.2 cu. in.

G. Can you think of some examples of prismoids in nature, architecture, or household furnishings?

H. What is Simpson's rule for volumes?

I. How could Simpson's rule be applied to find the amount of water displaced by a ship?

J. Can you suggest ways in which Simpson's rule might be useful for miners in estimating the amount of coal or other ore in a vein?

K. Could Simpson's rule be used to estimate the amount of oil in a partly empty tank car when the depth of the oil is given?

L. What sort of solid is bounded by two intersecting cylindrical surfaces? What are its principal sections? Can its volume be computed exactly or be approximated by any method you know?

M. How could the volume of water in a reservoir be computed from a contour map of its bottom?

18. WRITTEN EXERCISES

1. A wastebasket in the form of a frustum of a right circular cone has bases 12 and 15 in. in diameter and is 15 in. tall. Find its volume in cubic inches and in gallons. Use the prismoidal formula.
2. Find the volume of an attic with a square floor space 24 by 24 ft., having one pair of opposite roof planes making an angle of 30° with the floor and the other pair making an angle of 45° with the floor (Fig. 27). HINT: First find the height, and show that the midsection is approximately 17 by 12 ft.
3. A sand pile has an oval base consisting of a rectangle 80 by 40 ft., rounded off by adding two semicircular ends of 20 ft. radius. The surface of the sand pile makes a constant angle with the horizontal called the angle of repose and is 10 ft. high along a central ridge. Find the volume of the pile.
4. A solid having circular principal sections is 6 ft. in altitude. If the radii of the two bases are each 3 in. and the radius of the midsection is $5\frac{1}{4}$ in., estimate the volume by the prismoidal formula. If the solid weighs 60 lb. per cu. ft., find its approximate weight. Compare your answer with the average weight of a 6-ft. man.

5. Measurements of the areas of principal sections of a solid 6 ft. high were made at intervals of 1 ft. from the base as given below. Estimate the volume by Simpson's rule.

Distance from base, ft.	0	1	2	3	4	5	6
Area, sq. ft.	72.0	71.7	67.0	60.3	51.2	36.2	0

6. Taking the data of Example 2 of Sec. 18·3, but taking $n=4$ instead of $n=8$, find the approximate volume in cubic inches of the solid bounded by two right circular cylindrical surfaces of radii 12 in. and 4 in., respectively, whose axes are mutually perpendicular skew lines 2 in. apart. Is this approximation more or less accurate than the one obtained from $n=8$? How does it compare with the average of the V_0 and V_1 of the example?
7. Find the approximate volume contained within two right circular cylindrical surfaces of radii 12 in. and 4 in., respectively, whose axes intersect at right angles. Take $n=8$, making use of symmetry to save computation.
8. Two right circular cylindrical surfaces of equal radii have axes that intersect at right angles. Show that the volume bounded by these surfaces is given exactly by the prismoidal formula, and express the volume in terms of the radius.
9. Two sewer pipes of diameters 6 ft. and 3 ft., respectively, have horizontal axes that are perpendicular to each other but do not intersect. The bottom elements of the two cylinders do intersect, whereas the top element of the smaller pipe intersects the axis of the larger pipe. Find approximately the volume of the portion of space inside both pipes. Take $n=6$.
10. Draw a figure representing the solid in Exercise 9.
11. A gasoline storage tank in the form of a right circular cylinder 10 ft. in length and 6 ft. in diameter lies with its axis horizontal. How many gallons are in the tank when the gasoline is 1 ft. deep? Use Simpson's rule with $n=2$, and assume 1 cu. ft. $=7.48$ gal.
12. Solve Exercise 11, using $n=4$.

19

AREAS AND THEIR PROJECTIONS

19·1 Areas of polygons

Formulas for the areas of parallelograms, triangles, and trapezoids are derived in plane geometry. They are as follows

$$
\begin{aligned}
&\text{Area of parallelogram} = bh = \text{base} \times \text{altitude}\\
(1)\quad &\text{Area of triangle} = \tfrac{1}{2}bh\\
&\text{Area of trapezoid} = \tfrac{1}{2}(b+B)h = \text{average base} \times \text{altitude}
\end{aligned}
$$

For a regular polygon of n sides with inradius[1] r and perimeter p the formula is

(2) $$\text{Area of regular polygon} = \tfrac{1}{2}rp$$

19·2 Area of a circle

If the number of sides n of the polygon is increased without limit while the inradius r is kept fixed, then the area and the perimeter of the polygon approach in value the area and the circumference of the inscribed circle, respectively. The number π is defined to be the ratio of the circumference of a circle to its diameter. Hence, as n increases without limit, p approaches $2\pi r$ and $\frac{1}{2}rp$ approaches πr^2. Hence

(3) $$\textbf{Area of circle} = \pi r^2$$

[1] The inradius and circumradius are, respectively, the radii of the inscribed and circumscribed circles.

The value of π has been computed to 707 decimal places by W. Shanks and recomputed by D. F. Ferguson and others.[1] To 20 places its value is

$$\pi = 3.14159\quad 26535\quad 89793\quad 23846\cdots \tag{4}$$

For three-place accuracy $22/7 = 3.1428\cdots$ is a good approximation. The fraction $355/113 = 3.1415929\cdots$ is somewhat better.

19·3 Surface areas of polyhedrons

The area of the surface of a polyhedron, called briefly the area of the polyhedron, is equal to the sum of the areas of its faces. For certain special polyhedrons some or all of the faces may be parallelograms or triangles, having a common base or a common altitude. This is the case for a prism or a regular pyramid. If l is the length of a lateral edge of a prism, p_R the perimeter of a right section, and B the area of a base, then the lateral area is the sum of the areas of parallelograms, each having a side equal to l, and such that the sum of the altitudes is p_R.

$$\text{Lateral area of prism} = lp_R \tag{5a}$$
$$\text{Total area of prism} = lp_R + 2B \tag{5b}$$

In a regular pyramid the n lateral faces are all equal triangles. If l and b are their altitudes and bases, respectively, and p, B, and r are the perimeter, area, and inradius of the base polygon, then

$$nb = p \qquad B = \tfrac{1}{2}pr$$
$$\text{Lateral area of pyramid} = n\cdot\tfrac{1}{2}lb = \tfrac{1}{2}lp \tag{6a}$$
$$\text{Total area of pyramid} = \tfrac{1}{2}lp + B = \tfrac{1}{2}(l+r)p \tag{6b}$$

19·4 Surface area of a cylinder

If the lateral surface of a right cylinder is slit along an element, it can be developed or laid flat on a plane so as to form a rectangle whose base p is the length of the closed directrix of the cylinder and whose height h is equal to the altitude of the cylinder (Fig. 61). Hence we obtain the formulas

[1] Ferguson claims that Shanks' result is in error at the 530th decimal place (*Nature*, March 16, 1946, p. 342).

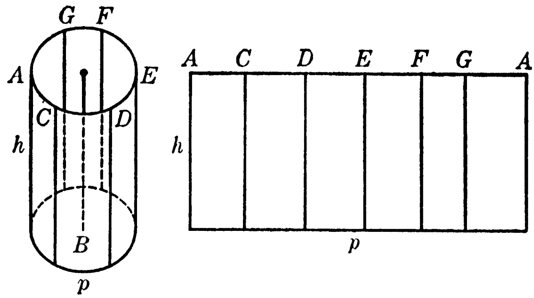


FIG. 61. Lateral surface of right cylinder.

(7a) Lateral area of right cylinder $= ph$
Lateral area of right circular cylinder $= 2\pi rh$
(7b) Total area of right circular cylinder $= 2\pi rh + 2\pi r^2$

If l is the slant height and p_R the perimeter of a right section, then, by the same method used to obtain the volume of a prism or cylinder as the difference of volumes of two right prisms or cylinders having right sections of the given solid as bases (Fig. 42) it can be shown that the lateral area of any cylinder is given by the formula

(8a) Lateral area of cylinder $= lp_R$

Adding the areas of the two bases, we obtain

(8b) Total area of cylinder $= lp_R + 2B$

19·5 Surface area of a right circular cone

The lateral area of a right circular cone can be obtained as the limit of the area of a regular pyramid with the same vertex, whose base is circumscribed about the circular base of the cone. If l is the slant height of the cone, then l is also the altitude of each lateral face of the pyramid. The limit of $\frac{1}{2}lp$ is $\frac{1}{2}l(2\pi r) = \pi rl$. Hence

(9a) Lateral area of right circular cone $= \pi rl$
(9b) Total area of right circular cone $= \pi rl + \pi r^2$

The lateral surface of a cone may also be developed, or laid flat, on a plane to form a sector of a circle of radius l and arc length $2\pi r$ (Fig. 62). The same formula is then obtained for its area.

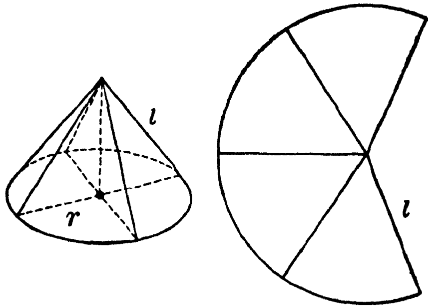


FIG. 62.

19·6 Areas of frustums

The lateral area of a frustum of a regular pyramid (or right circular cone) is the difference of the lateral areas of two regular pyramids (or right circular cones) (Fig. 63). If p_L, p_U, p_M denote the perimeters of the lower base, upper base, and midsection, respectively, l_L and l_U the slant heights of the large and small pyramids (or cones), B_L and B_U the areas of the bases, and $l = l_L - l_U$ the slant height of the frustum, then

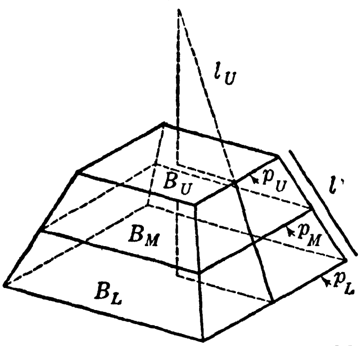


FIG. 63. Frustum of a pyramid.

$$\frac{l_L}{p_L} = \frac{l_U}{p_U} = \frac{l}{p_L - p_U}$$

$$\text{Lateral area} = \frac{1}{2}(l_L p_L - l_U p_U) = \frac{\frac{1}{2}(p_L^2 - p_U^2)l}{p_L - p_U}$$
$$= \tfrac{1}{2} l (p_L + p_U)$$

Since p_M is the average of p_L and p_U, we have

(10*a*) Lateral area of frustum $= l p_M$

(10*b*) Total area of frustum $= l p_M + B_L + B_U$

19·7 Orthogonal projections of polygons

If the sides of a plane polygon are projected orthogonally onto a plane not perpendicular to the plane of the polygon, a new polygon is formed that is called the *orthogonal projection* of the first. If the two polygons do not intersect each other, they are the bases of a truncated right prism. It will be shown that the ratio of their areas depends only on the angle between their planes (Fig. 64).

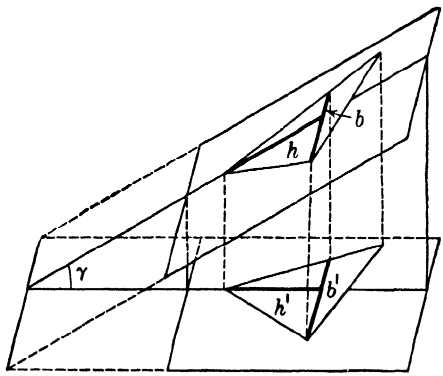


FIG. 64. Orthogonal projection of a triangle.

Consider a triangle having one side b parallel to the plane of projection, and let the angle between the plane of projection and the plane of the triangle be γ. Then a plane perpendicular to the given side b of the triangle and passing through the opposite vertex will be perpendicular to the plane of projection and will contain both the altitude h of the given triangle and its projection h'. The angle between these two is γ. Since the projection b' of b is equal to b, we have

$$(11) \quad \frac{\text{Area of orthogonal projection of triangle}}{\text{Area of given triangle}} = \frac{\frac{1}{2}b'h'}{\frac{1}{2}bh} = \frac{h'}{h} = \cos\gamma$$

A triangle not having one side parallel to the projection plane can be cut into two triangles by a line parallel to the projection plane as follows: Of the three planes, one through each vertex, that are parallel to the projection plane, no two coincide, and thus there must be a middle one. This middle one cuts the triangle in a line parallel to the projection plane. The line is a common base for the two triangular pieces of the given triangle. For each piece, and hence for the sum, the ratio of the projected area to the given area is $\cos\gamma$ (Fig. 64).

The formula holds equally well for the projection of any polygon, since the polygon can be cut up into triangular pieces. It also holds for the projection of any plane region, which can be approximated by a polygon with an arbitrarily small error that can be made to approach zero.

$$\text{(12)} \qquad \frac{\textbf{Area of projection}}{\textbf{Area of given plane region}} = \cos \gamma$$

EXAMPLE: Find the base dihedral angles of a regular square pyramid having eight equal edges, each of length e.

SOLUTION: The four lateral faces are each equilateral triangles having altitudes equal to $e \cos 30°$ and bases equal to e. Hence their areas are each $e^2\sqrt{3}/4$. Each projects into one-quarter of the base square, and thus the projected area is $e^2/4$. Hence $\cos \gamma = 1/\sqrt{3}$. To find γ in degrees it is necessary to use either trigonometric tables or some other approximate trigonometric formula for computing the angle.[1] The result is $\gamma = 54°44'$.

19. ORAL QUESTIONS

A. What are the formulas for the lateral area and total area of a prism or cylinder?

B. What are the formulas for the lateral area and total area of a pyramid or cone?

C. State a formula for the lateral area of a frustum. To what frustums does it apply?

D. Is 22/7 the exact value of π? If not, is the error more or less than four-hundredths of 1 per cent?

E. How are areas changed under orthogonal projection?

F. How does the area of one face of a regular tetrahedron compare with its orthogonal projection on another face? HINT: Project three faces on the fourth face.

G. How would you find the area of a regular pentagon, given the length of its edge?

[1] If $\cos \gamma = x < 0.7$, then the angle γ is given in degrees with an error not exceeding $4'$ by the close approximation $90° - \gamma = 172°x/(2 + \sqrt{1 - x^2})$. For $\cos \gamma = 1/\sqrt{3}$ this would give $90° - \gamma = 172°/(2\sqrt{3} + \sqrt{2}) = 17.2°(\sqrt{12} - \sqrt{2})$ $= 17.2°\ (2.050) = 35.26°$; $\gamma = 54°44'$.

H. How would you find the surface area of a regular dodecahedron, given the length of its edge?

I. What is the area of a regular octahedron if each edge is 2 units long?

J. Is it true that the areas of two sections of a right prism are equal if their planes make equal angles with the base plane?

19. WRITTEN EXERCISES

1. Find the surface area of a regular tetrahedron whose edges are each 2 in. long.
2. A right circular cylinder and a right circular cone both have the same volume and the same total area. If their bases are equal circles of radius 5 in., find their altitudes.
3. Find the total area of a regular pentagonal prism whose lateral faces are unit squares.
4. Find the lateral area and total area of a right circular cone if the altitude is 4 in. and the diameter of the base is 6 in.
5. Find the surface area of a roof that covers a ground area of 30 by 30 ft. if the two roof planes each make angles of 30° with the horizontal.
6. Find the surface area of a roof that covers a ground area of 30 by 30 ft. if the roof consists of four planes, one pair making an angle of 30° with the horizontal and meeting on the ridge, and the other pair making an angle of 45° with the horizontal (Fig. 27).
7. A lamp shade has the form of the surface of a frustum of a cone with slant height of 7 in. and with bases whose radii are 4 and 6 in. Find its area.
8. A regular hexagon with unit sides is revolved about a line in its plane perpendicular to a diagonal at one of the vertices. Find the surface area of the solid so formed.
9. A trirectangular trihedron O-(ABC) is cut by a plane (ABC) so chosen that $\overline{OA} = 1$, $\overline{OB} = 2$, $\overline{OC} = 3$. Find the areas of the orthogonal projections of the triangles $[OBC]$, $[OCA]$, $[OAB]$ onto the plane (ABC).
10. A saucepan in the form of a frustum of a right circular cone is

2 in. deep, its metal base has a radius of 3 in., and its open top has a radius of 4.5 in. Find the area of the metal. Assuming the pan to be of uniform thickness and to weigh 2 oz., find the weight of an equally thick sheet of the same metal 1 ft. square.

11. A prismoid of unit height has square bases of unit area. The diagonals of one base are parallel to the edges of the other base, and the lateral faces are isosceles triangles. Find its total area.

12. A regular octahedron is placed with two opposite faces as bases. Show that its principal sections all have the same perimeter. Find the combined area of the six lateral faces. Compare this area with that of a right prism having the same altitude and the same hexagonal midsection. Show that, although sections of the two solids have the same perimeter, the solids do not have the same lateral area.

13. A cuboctahedron (Fig. 96) is a semiregular solid with six square faces and eight triangular faces, all of whose 24 edges are equal. It may be formed by cutting off each vertex of a cube or octahedron by a plane through the mid-points of the edges adjacent to that vertex. Find the surface area of a cuboctahedron whose edges are 2 in. in length.

14. A rhombic dodecahedron is a semiregular solid whose 14 vertices lie at the centers of the faces of a cuboctahedron (see Exercise 13). Its 12 faces are equal rhombuses whose angles are 60° and 120°. Find its surface area if its edges are of unit length.

20

REVIEW OF SOLID MENSURATION

20·1 The trihedral angle

Even as the portion of space cut out by a dihedron is measured by its dihedral angle, so the portion of space cut out by a trihedron is measured by a trihedral angle, which is equal to one-half the amount by which the sum of its three dihedral angles exceeds 180°.

$$\sigma = \tfrac{1}{2}(\alpha + \beta + \gamma - 180°)$$

The trihedral angles of the three associated trihedrons that share a face with the given trihedron are $\alpha - \sigma$, $\beta - \sigma$, $\gamma - \sigma$, respectively. The directed normals to the face planes of the given trihedron are edges of a trihedron called the polar trihedron, whose dihedral angles are supplements of the face angles of the given trihedron, and whose trihedral angle $\sigma' = 180° - s$ is the supplement of one-half the sum of these face angles.

APPLICATION: Consider a regular polyhedron with regular triangular faces of which m meet at a vertex. Pass a plane sector, with vertex at the center of the solid, through each edge. Then the dihedral angles between adjacent planes at any vertex are $360°/m$. To each of the F faces of the solid corresponds a trihedron, with vertex at the center of the solid, whose trihedral angle is equal on the one hand to $\tfrac{1}{2}[(3 \times 360°/m) - 180°]$ and on the other hand to $360°/F$. Equating these expressions and dividing by 180°, we have

$$\frac{2}{F} = \frac{3}{m} - \frac{1}{2}$$

or

$$F = \frac{4m}{6-m}$$

Hence the only possible integral values of m (greater than 2) are 3, 4, or 5, and the corresponding values of F are 4, 8, and 20. These correspond to the regular tetrahedron, the regular octahedron, and the regular icosahedron.

20·2 Volume measurement and Cavalieri's theorem

Whereas the trihedron encloses an infinite portion of space, the polyhedron and other solids enclose finite portions of space. Measurements of the volumes of solids are usually expressed as some positive number (not necessarily a whole number) times the volume of some unit cube, such as the cubic foot (ft.^{3} or cu.ft.), cubic inch (in.^{3} or cu.in.), cubic centimeter (cm.^{3} or cc.). Complicated polyhedrons may be cut up into simpler polyhedrons, such as prisms, pyramids, and wedges, for volume analysis. These in turn may be compared directly with rectangular parallelepipeds—or as we might say "rectangular boxes"—whose volumes are the product of their three linear dimensions. The comparison of volumes of solids is greatly facilitated by Cavalieri's theorem (Sec. 13·5).

THEOREM 13B: *If two solids can be placed so that they have equal altitudes and the same base planes and so that for every parallel plane between the bases the principal sections cut from the two solids have equal area, then the two solids have equal volumes* (see Fig. 39).

20·3 Solids for which $V = Bh$

Prisms, cylinders, and all solids whose principal sections have equal areas (B) belong to a class of solids for which the volume is equal to the product of the base times the altitude.

20·4 Solids for which $V = \frac{1}{3}Bh$

Pyramids, cones, and all solids of which the principal sections have areas proportional to the square of the distance of the section from a point called the vertex of the solid belong to a class of solids for which the volume is equal to one-third the product of the base times the altitude. The altitude is the distance from the vertex to the base.

20·5 Solids for which $V=\frac{1}{6}(B_L+4B_M+B_U)h$

This volume formula, known as the **prismoidal formula,** applies to any solid of which the principal sections have areas which can be expressed either as a constant or as a linear, quadratic, or cubic polynomial function of the distance from one base. Such solids may be called **generalized prismoids.** This class of solids includes prisms, cylinders, pyramids, cones, and segments of these solids. We shall see later that it also includes segments of a sphere. The ratio of the volume to the height is known as the *mean base* and is computed as a weighted average of the two bases and the midsection, in which the midsection is counted four times and the two bases once each.

20·6 Simpson's rule

To approximate the volume of a solid that is not a generalized prismoid, we may slice the solid into smaller segments by planes parallel to a base and then approximate the volume of each segment by the prismoidal formula. This procedure is known as **Simpson's rule.** Its accuracy increases with the number of subdivisions, and the error can be made as small as desired by taking a sufficiently large number of subdivisions. Simpson's rule is useful for finding volumes in many practical problems where the given solid cannot be cut up exactly into prisms, pyramids, wedges, and the like.

20·7 Areas of surfaces

The area of the surface of a polyhedron can be written as the sum of the areas of its plane faces. These in turn can be computed by plane geometry. However, we note that for a prism or cylinder the lateral area is equal to the perimeter of a right section times the length of a lateral edge or element. For a regular pyramid or right circular cone, or frustum of one of these, the lateral area is equal to the perimeter of a midsection times the slant height. If a plane area A in one plane is projected orthogonally onto another plane, the projected area is $A \cos \gamma$, where γ is the angle between the planes.

20. ORAL QUESTIONS

A. What is a solid, and what is its boundary?
B. What assumptions were stated in defining volume?
C. How would you define the area of a curved surface?
D. What is Cavalieri's theorem?
E. For what solids is $V = Bh$?
F. For what solids is $V = \frac{1}{3}Bh$?
G. For what solids is $V = \frac{1}{2}Bh$?
H. What are prismatoids and prismoids?
I. What is the prismoidal formula, and when does it apply?
J. What is Simpson's rule for volumes?
K. How is the trihedral angle of a trihedron expressed in terms of its dihedral angles?
L. How many regular polyhedrons are there, and what are their names?
M. How is it possible to approximate the volume included between two intersecting cylinders?
N. What are the tangent plane and the normal to a curved surface at a point?
O. Is area ever increased under an orthogonal projection?

20. WRITTEN EXERCISES

1. The volumes of similar solids are to each other as the cubes of their linear dimensions. Find the weight of a man 6 ft. tall if he were built in the same proportions as a 12-lb. baby 2 ft. tall.
2. A cylinder 6 in. high displaces 1 qt. of water. Find the diameter of its base, using the approximate value of 22/7 for π.
3. Let V, E, F be the number of vertices, edges, and faces of a polyhedron. Make a table of values of V, E, F, and show that $V - E + F = 2$, for the following polyhedrons: cube; pyramid with n lateral faces; frustum of the same pyramid; wedge.
   There are polyhedrons for which the formula does not hold. Can you think of one? Does it hold for the solid of Exercise 14?
4. Find the volume of a square pyramid whose eight edges are all of unit length. How is this figure related to the regular octahedron?

5. The volume of a given cube is 2 cubic units. Show that it contains a cube whose edges are each 5/4 units. A rectangular parallelepiped of dimensions 5/4, 5/4, 32/25 has a volume equal to 2. What is the volume of a cube whose edge is the average of these three edges?
6. Complete the details of the proof (page 97) that the volume of a rectangular parallelepiped is given by the formula $V=abc$ when the edges a, b, c are irrational with respect to the unit of length. In particular show that Δ_n approaches zero as n becomes larger.
7. Show that the mediator of a diagonal of a cube cuts the cube in a regular hexagon.
8. Find the volume of the prismatoid, cut from a cube whose diagonal is 12 in. in length, by two planes perpendicular to the diameter at distances 4 and 6 in. from one extremity.
9. Find the surface area of a regular octahedron having edges of length 6 in.
10. A prismoid has two congruent regular hexagons for bases and equilateral triangles with unit edges for lateral faces. Find its surface area and its volume.
11. Find the area of the orthogonal projection on a horizontal plane of a right triangle with sides 5, 12, 13 if its plane makes an angle of 60° with the horizontal.
12. A canvas tent has a 12-ft.-square wooden base and a ridgepole 12 ft. long and 10 ft. above the floor. The side walls of the tent are 3 ft. high. Find the volume, and find the area of the canvas (see Fig. 113).
13. A canvas tent is bounded laterally by a cylindrical surface of radius 5 ft. and height 4 ft. and on top by a conical surface rising to 9 ft. above the ground at the middle and extending beyond the side walls to a height of 3 ft. above the ground. Find the interior volume of the tent and the area of the canvas.
14. Given a double frustum, consisting of two equal square frustums of bases 3 by 3 and 6 by 6 and altitude 4, welded together on the large bases. Cut out of the center a square prism of altitude 8 whose 3 by 3 square bases coincide with the bases of the double frustum. Find the volume of the remaining solid.

15. A regular decagon is revolved about one of its diagonals. Show that the surface area so generated is equal to $4\pi Rr$, where R and r are the radii of the circumscribed and inscribed circles of the decagon.
16. A tank in the form of a right circular cylinder of diameter 4 ft. and length 10 ft. lies with its axis horizontal. How many gallons of liquid are in the tank when it is filled to a depth of 1 ft.? Solve by Simpson's rule, taking $n=4$.
17. Prove that the normal at a point P of a cylindrical surface lies in the plane of a right section through the point P.

PART THREE

THE SPHERE AND SOLIDS OF REVOLUTION

21

CIRCLES AND LUNES ON THE SPHERE

21·1 The sphere, its center and radius

A sphere is a solid bounded by a closed spherical surface, all points of which are equally distant from a point within called the **center** O. A line segment from the center to a point on the surface is called a **radius** R, and a line segment passing through the center and having both end points on the surface is called a **diameter.**

All radii of a sphere are equal. Each diameter is equal to two radii. Two spheres with equal radii or diameters are congruent.

The earth is approximately a sphere of radius 3,959 statute miles. Although there is actually a difference of about 27 statute miles between the longest and shortest diameters of the earth, it is convenient for many purposes to consider the earth to be a sphere.

21·2 Great and small circles

A **great circle** of a sphere is a section of the surface made by a plane passing through the center of the sphere. A **small circle** is a section made by a plane not passing through the center of the sphere nor tangent to the sphere. It is readily seen that the radius of a great circle is equal to the radius of the sphere, whereas the radius of a small circle is less than the radius of the sphere. All great circles of the sphere are equal, since they have equal radii.

THEOREM 21: Every plane section of a sphere is either a circle or a point.

PROOF: Let M be the foot of the perpendicular from the center O

of the sphere to the plane of section (Fig. 65). Then, for any point Q on the curve of section, we have

$$(QM) \perp (OM)$$

if O and M are distinct. In any case we have

$$\overline{QM}^2 = \overline{OQ}^2 - \overline{OM}^2$$

Since the right-hand member is a constant independent of the position of Q on the curve of section, it follows that $\overline{QM}$ is constant. The locus of Q is a circle, except in the case when Q coincides with the point M.

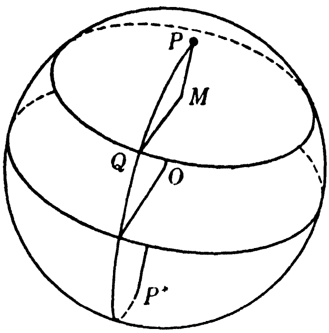


FIG. 65. Sections of a sphere.

21·3 Meridians and parallels

A sphere of radius R may be traced by revolving a semicircle of radius R through one revolution about its diameter as axis. Referred to this axis the extremities P and P^* are called **poles,** and the semicircular arc in each of its positions is called a **meridian.** Each point Q of the semicircular arc traces a circle on the spherical surface called a **latitude circle** or **parallel of latitude** or parallel circle. In particular, the mid-point of the semicircular arc traces a great circle referred to as the **equator.** The poles P and P^* of the axis are also called **poles** of these latitude circles.

To prove that each point Q on the semicircular arc traces a circle, let M be the projection of Q on the axis $[PP^*]$. Then in all its positions, (QM) is perpendicular to (PP^*), and therefore lies in a plane perpendicular to (PP^*) at M. This plane cuts the spherical surface in a circle of radius $\overline{QM}$ (Fig. 65).

21·4 Spherical distance

A spherical surface S of radius R is curved and cannot be mapped on a flat surface without distortion of lengths. The shortest path on the spherical surface S between two points A and B is a great-circle arc through A and B. If the two points do not lie on the same diameter, the plane (AOB) cuts the surface in a great circle, and the shortest arc $\widehat{AB}$ on this great circle is defined to be the spherical distance $\widehat{AB}$ between A and B. The arc $\widehat{AB}$ may be measured in units of length, but it is often more convenient to measure the arc in angular units by the angle it subtends at the center of the sphere. If A and B are opposite extremities of a diameter, then there are infinitely many great-circle arcs through A and B, and on each the arc $\widehat{AB}$ is 180°.

In the study of the sphere it is common to omit the word spherical in speaking of spherical distance $\widehat{AB}$. Although the great-circle distance $\widehat{AB}$ is not the same as the straight-line distance $\overline{AB}$ between two points A and B, for near-by points the difference becomes negligible. It is usually clear from the context in which sense the word distance is used.

The *polar distance* of a latitude circle is the common distance from each of its points to its nearer pole. The polar distance of a great circle is a quadrant arc, or 90°. The *meridian distance* between two latitude circles is the arc cut off between them on any meridian through their poles.

21·5 Radius of a circle of latitude

The latitude L of a point on the sphere is its meridian distance from the equator, measured positive to the north and negative to the south of the equator. Points having the same north latitude or the same south latitude lie on a small circle of radius $r = R \cos L$, where R is the radius of the sphere and L is the latitude of the circle. The proof of this fact is listed as Exercise 6.

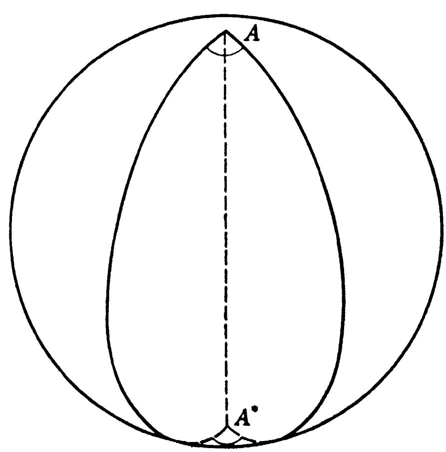


FIG. 66. Lune.

21·6 Angles and lunes

The angle between two curves on a spherical surface at a point of intersection of the curves is defined to be the angle between their tangents at the point. Two great circles on a sphere always intersect in diametrically opposite points and make equal angles with each other at the two points. The portion of a spherical surface included between two great semicircles having the same end points A and A^* is called a **lune** and the angle between the two curves at either point A or A^* is called the **angle** of the lune (Fig. 66).

The half planes determined by the two semicircles and having (AA^*) as a common edge form a dihedron whose dihedral angle is equal to the angle of the lune.

21·7 Tangents to a sphere from an external point

Let P be a point outside a sphere S with center at O, and let a tangent line (PT) touch the sphere at T (Fig. 67). Then in the plane (PTO) the line (PT) is tangent to a great circle of the sphere at T, and $\angle PTO$ is a right angle. By the Pythagorean theorem we have

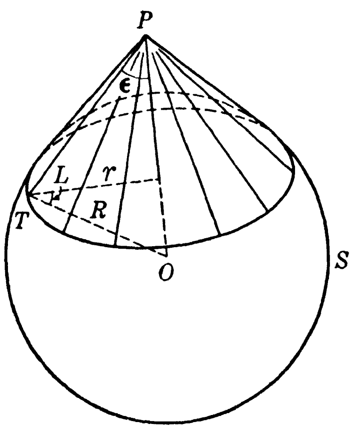


FIG. 67. Tangents to a sphere.

$$\overline{PT}^2 + \overline{TO}^2 = \overline{PO}^2$$

If we denote the distance $\overline{PO}$ by D and the radius of the sphere by R, then

$$\overline{PT}^2 = D^2 - R^2$$

It follows that all tangents to the sphere from P have equal lengths given by $\sqrt{D^2 - R^2}$. The points of tangency form a small circle of the sphere whose radius is $r = R \cos L = R \cos \epsilon$ (Fig. 67).

21. ORAL QUESTIONS

A. How would you define a spherical surface? What is the name of the solid bounded by a closed spherical surface?

B. What curve is obtained by cutting a spherical surface by a plane through its center?

C. What is meant by the poles of a circle on the sphere?
D. How are meridian circles defined? Are they all great circles?
E. How are latitude circles defined? Are they all small circles?
F. Given two points on a spherical surface, what is meant by the spherical distance between them? Is it uniquely defined?
G. Is there ever more than one great circle between two given points on a sphere?
H. If the polar distances of two latitude circles are 40° and 60°, respectively, what is the meridian distance between them? Consider both possible answers.
I. If the edge of a dihedron passes through the center of a sphere, what figure does it cut from the spherical surface?
J. Is the earth an exact sphere? If not, which is smaller, the polar diameter or the equatorial diameter, and by how much? What is the earth's mean radius? (See page 202.)
K. What can be said of the lengths of tangents to a sphere from an external point?

21. WRITTEN EXERCISES

1. Prove that the sections cut from a sphere by any two planes equally distant from the center of the sphere are congruent.
2. Prove that if two spherical surfaces intersect, their complete intersection is either a circle or a point. HINT: Pass any plane p through the line (OO') joining the centers O and O' of the two spheres. It cuts the spheres in two circles that intersect at at least one point P. What is the locus of P as p revolves about (OO')?
3. On a sphere 10 in. in diameter, find the radii of the small circles at distances 3 in. and 4 in., respectively, from the center.
4. On a sphere of radius 10 ft., what is the radius of a small circle 9.6 ft. from the center of the sphere?
5. On a sphere of radius 4,000 miles, what is the circumference of a latitude circle whose plane is 2,000 miles from the center of the sphere?
6. Prove that, on a sphere of radius R, the radius of a small circle

at latitude L is $R \cos L$. HINT: Let P be a point at latitude L, and project $[OP]$ orthogonally onto the plane of the small circle.

7. Find the circumference of a 45° latitude circle on a sphere of radius 3,960 miles.
8. If the circumference of a great circle on a sphere is 25,000 miles, find the radius.
9. A point Q on a semicircle S traces a circle C in space when the semicircle S is rotated about its diameter (PP^*). Prove that if the sphere generated by S and the circle C are projected orthographically onto a plane containing (PP^*) then the projection of C is a line segment perpendicular to (PP^*). HINT: Show that the plane of C is perpendicular to the projection plane.
10. If in Exercise 9 the point Q is at latitude L and the semicircle S is of radius 1, what is the length of the projection of C?
11. A tangent to a sphere of radius 4, from a point P on the polar axis extended, is 3 units in length. Find the radius of the circle of latitude through the contact point of the tangent.

22

DRAWING OF CIRCLES ON THE SPHERE

22·1 Orthographic projection of circles

Only circles whose planes are parallel to or coincide with the plane of projection are drawn as true circles in an orthographic projection. A circle whose plane is perpendicular to the projection plane projects into a line segment equivalent to that one of its diameters which is parallel to the projection plane. Every other circle, lying in a plane that is neither parallel nor perpendicular to the projection plane, is represented in a drawing by a symmetric oval curve called an **ellipse.** This curve has already been discussed (Sec. 16·6) in connection with the drawing of a right circular cone and will be studied in more detail later (Chap. 33). A few of its properties will be useful here in drawing a sphere and its great and small circles.

Let $[AA^*]$ be the projection of that diameter of a given circle which is parallel to the projection plane, and let $[BB^*]$ be the projection of

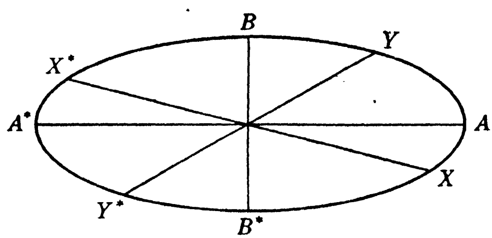


FIG. 68. Conjugate diameters of an ellipse.

the diameter perpendicular to $[AA^*]$ (Fig. 68). The two segments $[AA^*]$ and $[BB^*]$ are called the **major** and **minor axes**, respectively, of the ellipse. Assuming that the plane of the circle makes an angle θ with the plane of projection, we state here without proof the following properties of the projection:

PROPERTY 1: $\overline{BB^*} = \overline{AA^*} \cos \theta$.

PROPERTY 2: *The ellipse is symmetric in each of its axes.*

PROPERTY 3: *The tangents to the ellipse at A and A^* are parallel to (BB^*), and the tangents to the ellipse at B and B^* are parallel to (AA^*).*

More generally, let $[XX^*]$ and $[YY^*]$ be the orthographic projections of any pair of mutually perpendicular diameters of the circle. Then these segments are called **conjugate diameters** of the ellipse into which the circle projects.

Since chords of the circle parallel to one of these diameters are bisected by the other and since parallels and mid-points are preserved in orthographic projection, we obtain a corresponding property of the ellipse into which the circle projects.

PROPERTY 4: *Chords parallel to any diameter of an ellipse are bisected by the conjugate diameter.*

Since the tangents at the extremities of one of two mutually perpendicular diameters of a circle are parallel to the other diameter, we obtain a corresponding property of the ellipse.

PROPERTY 5: *Tangents to an ellipse at the extremities of one diameter are parallel to the conjugate diameter.*

One further property of importance concerns the lengths of two conjugate diameters of an ellipse.

PROPERTY 6: *The sum of the squares of two conjugate diameters of an ellipse is equal to the sum of the squares of its axes.*

To prove this last statement we express the square of each of the two conjugate diameters as the sum of the squares of its projections on the two axes of the ellipse. By referring back to the circle from which the ellipse was projected, it is seen that the sum of the squares of the two projections on the major axis is equal to the square of the major axis. Similarly, the sum of the squares of the two projections on the minor axis can be shown to equal the square of the minor axis of the ellipse.

22·2 Sketching an ellipse from two conjugate diameters

Consider a square C_1, D_1, C_1^*, D_1^* circumscribed about a circle, center at O. Denote the points of contact on the sides of the square by X, Y, X^*, Y^*, respectively, and denote the points in which the diagonals of the square intersect the circle by C, D, C^*, D^*, respectively, labeling the points so that X, C, Y, D, X^*, C^*, Y^*, D^*, X occur consecutively around the circle and so that C and C_1 lie on the same ray through O, etc. (Fig. 69). Then from plane geometry we derive the following properties of the figure:

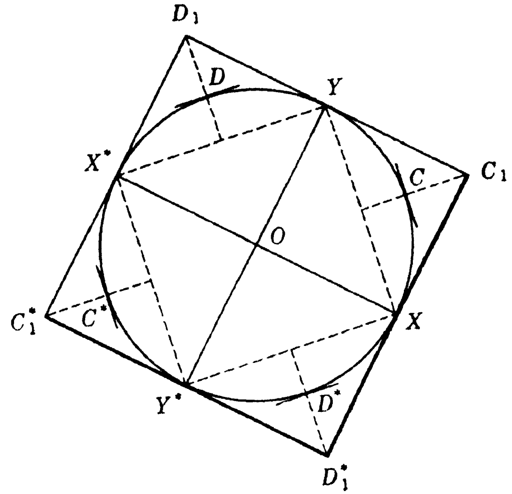


FIG. 69. Circle and related squares and tangents.

PROPERTY 1: $\overline{OC} = \overline{OC_1}/\sqrt{2} = \overline{OC^*} = \overline{OC_1}^*/\sqrt{2}$,
$\overline{OD} = \overline{OD_1}/\sqrt{2} = OD^* = OD_1^*/\sqrt{2}$.

PROPERTY 2: The tangent to the circle at any one of the eight points (such as D) is parallel to the line joining the two neighboring points (X^* and Y) and also to the line joining their neighbors beyond (C^* and C).

Now let the complete figure of circle and square and the eight points and tangents be projected orthographically onto a plane not parallel to the plane of the circle. Then the circle projects into an ellipse, the square into a parallelogram, the perpendicular diameters $[XX^*]$ and $[YY^*]$ of the circle into a pair of conjugate diameters of the ellipse, and the perpendicular diameters $[CC^*]$ and $[DD^*]$ into another pair of conjugate diameters of the ellipse. If the lettering used in the original figure (Fig. 69) is also used in the projected figure (Fig. 70), then Properties 1 and 2 still hold true on the ellipse, since parallelism and ratios of distances on the same line are preserved under this projection.

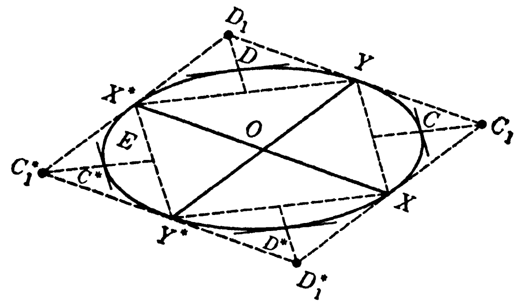


FIG. 70. Orthographic projection of circle and related squares and tangents.

Hence, starting with any two segments $[XX^*]$ and $[YY^*]$ that bisect each other at O we can sketch an ellipse having these segments as conjugate diameters as follows:

STEP 1: Draw tangents lightly in pencil at X and X^* parallel to $[YY^*]$ and at Y and Y^* parallel to $[XX^*]$, forming a parallelogram $C_1D_1C_1{}^*D_1{}^*$, lettered as in Fig. 70, so that points X, C_1, Y, D_1, X^*, $C_1{}^*, Y^*$, $D_1{}^*$ occur in this order around the parallelogram.

STEP 2: At about seven-tenths of the way from O to C_1 mark the point C, and draw a short tangent parallel to $[XY]$. Similarly, mark the points D, C^*, and D^* at seven-tenths of the way from O to D_1, $C_1{}^*$, and $D_1{}^*$, respectively, and draw short tangents parallel to $[YX^*]$ at D, to $[X^*Y^*]$ at C^*, and to $[Y^*X]$ at D^*.

STEP 3: Draw a smooth curve E tangent to one of these eight tangents at each of eight points X, C, Y, D, X^*, C^*, Y^*, D^*, keeping the hand on the concave side of the curve to facilitate the drawing.

STEP 4: Check to see that the longest and shortest diameters of the ellipse are perpendicular to each other and that the curve is symmetric in each of these lines. The use of a coin as described in Sec. 16·6 may help in locating irregularities.

22·3 Drawing of great circles on a sphere

In orthographic projection the outline of a sphere is a circle S. If the sphere is viewed so that its equatorial circle is represented in projection by an ellipse, not merely by a line segment, then the polar axis will not reach out to this outer circle S in the drawing. Instead, its projected length $\overline{ZZ^*}$ will be cos $\bar{\gamma}$ times the diameter of S, where γ is the angle between the plane of the circle and the plane of projection (see Sec. 9·6). It is easiest to start the figure by drawing first a vertical segment $[ZZ^*]$ to represent the polar axis and then any convenient segment $[XX^*]$, bisected at the mid-point O of $[ZZ^*]$, to represent the projection of one of the diameters of the equatorial circle of the sphere. Then the length of the diameter of the outer circle S is easily constructed as follows:

1. Mark the points F and F^* so that the segment $[FF^*]$ (not actually drawn) is perpendicular to $[ZZ^*]$ at O and so that $\overline{OF} = \overline{OF^*} = \overline{OZ}$ (Fig. 71).

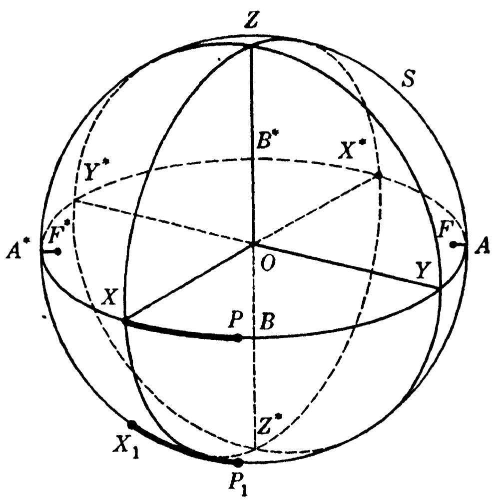


FIG. 71. Orthographic projection of sphere.

2. If R is the radius of outline circle S of the sphere, then by using the relation $\overline{FX} + \overline{XF^*} = 2R$, the length R can be laid off with a ruler or a pair of compasses. (The proof of this depends on the focal properties of the ellipse, which will be studied in Chap. 33.)

3. With R as a radius and O as center, draw the outline circle S of the sphere, and mark the points A and A^* where it crosses the line (FF^*) (extended).

4. Mark points B and B^* on $[ZZ^*]$ so that $\overline{FB} = \overline{FB^*} = R$.

5. Then the ellipse E_z having $[AA^*]$ as major axis and $[BB^*]$ as

minor axis will pass through the points X and X^* and will represent the equatorial circle of the sphere.

6. Using the method of Sec. 22·2, construct an ellipse E_y having $[XX^*]$ and $[ZZ^*]$ as conjugate diameters. This will represent a meridian circle on the sphere.

The trimetric ruler can be used to expedite some of this construction. With the vertical scale, mark points Z and Z^* at R vertical units above and below O. With the lower slanting scale, mark X and X^* at R of its units from O on each side. Finally, mark the points Y and Y^* with the upper slanting scale at R of its units from O on each side. Determine the true radius R of the circle S as before, but use tangents at the eight points A, A^*, B, B^*, X, X^*, and Y, Y^* in sketching the equatorial ellipse E_z. For the ellipse E_y through X, X^* and Z, Z^*, note that its minor axis is along (YY^*) and its major axis is perpendicular to (YY^*). This will either replace or supplement the use of the circumscribed parallelograms.

22·4 Measurement of arcs in orthographic projection

Two problems arise in connection with the measurement of arcs in orthographic projection. The first is to lay off an arc, having a given angular measure on the sphere, on an ellipse that represents a great circle in projection. The second is to measure the number of degrees represented by a given arc in a drawing.

Both problems are easily solved by means of the following theorem.

THEOREM 22: Let the ellipse E be the orthographic projection of a great circle of a sphere whose outline in projection is the circle S. Then two chords of S, perpendicular to the major axis of the ellipse E, intercept arcs on E and S that have equal angular measure on the great circles of the sphere that are represented by E and S in the projection.

PROOF: These chords are the projections of parallel circles of the sphere whose polar axis projects into the major axis of E. The ellipse E and the circle S are both projections of meridian circles through these poles. Equal arcs are cut from the two meridian circles by the two parallel circles.

To lay off an arc of 30° from the point X on the equatorial ellipse of Fig. 71, let the line perpendicular to the major axis $[AA^*]$ through

X meet the circle S in X_1. On S locate P_1 on the appropriate side of X_1 so that arc $\widehat{X_1P_1}=30°$ or so that $\measuredangle X_1OP_1=30°$. Then the perpendicular to $[AA^*]$ through P_1 meets the ellipse E in the required point P.

Similarly, to measure an arc $\widehat{XP}$ on the ellipse E, project both points X and P onto the points X_1 and P_1 of the circle S by perpendiculars to $[AA^*]$. The angle $\measuredangle X_1OP_1$ measures the arc $\widehat{XP}$.

22·5 Drawing of latitude circles on a sphere

To draw a small circle at a given latitude L, measure off the arc L from X in the manner just described on a meridian ellipse XZX^*Z^*, and thus locate the point P. Through P draw $[PP^*]$ parallel to $[XX^*]$ and bisected at the point M where it meets $[ZZ^*]$.

Through M draw a segment $[QQ^*]$ parallel to $[YY^*]$ and intersecting the meridian ellipse YZY^*Z^* in Q and Q^*. Then the small circle at latitude L is represented by an ellipse having $[PP^*]$ and $[QQ^*]$ as conjugate diameters, which is similar to the equatorial ellipse XYX^*Y^*. It will be tangent to S (Fig. 65) if L is smaller than the angle γ between the plane of the circle and the projection plane but will lie wholly inside S if L is greater than this angle.

22. ORAL QUESTIONS

A. Into what curves can a circle project under orthographic projection?

B. If the axis of a set of parallel circles is parallel to the projection plane, into what do the circles project?

C. How many of the great circles of a sphere project into circles under orthographic projection?

D. If the plane of a circle makes an angle θ with the projection plane, what is the ratio of the major and minor axes of its projection?

E. What are conjugate diameters of an ellipse? What are some of their properties?

F. Into what figure do a circle and circumscribed square project under orthographic projection? How can this figure be used in drawing ellipses?

G. Given the projections $[XX^*]$ and $[ZZ^*]$ of two mutually perpendicular diameters of a sphere, how can the diameter of the sphere be constructed?

H. How would you draw a sphere and three mutually perpendicular great circles in orthographic projection?

I. How can arcs be laid off in orthographic projection?

J. In orthographic projection how is the image of a small circle of a sphere related to the image of the great circle parallel to it?

22. WRITTEN EXERCISES

1. Sketch an ellipse having a major axis of 8 units length and a minor axis of 4 units length, using the eight-tangent method of Sec. 22·2.
2. Sketch an ellipse having two conjugate diameters of 8 and 4 units length, respectively, if the angle between them is 45°.
3. Sketch an ellipse having two conjugate diameters of equal length if the angle between them is 60°. Mark 12 points on the ellipse which represent equally spaced points on a circle of which it is the projection.
4. Draw a vertical line segment $[ZZ^*]$ 4 in. long with mid-point O. Then draw a segment $[XX^*]$ 3 in. long with mid-point at O, and such that $\measuredangle\ XOZ = 120°$. Construct an ellipse through X and X^* representing the equatorial great circle on a sphere whose poles project into Z and Z^*. Construct also the circle S that represents the sphere in projection.
5. Solve Exercise 4, using $\measuredangle\ XOZ = 150°$.
6. Using the trimetric ruler construct the projection of a sphere showing three mutually perpendicular great circles.
7. Construct the figure in Exercise 6, and add to it two latitude circles in the upper hemisphere, one at latitude 30° and the other at latitude 60°.
8. Look up the latitude and longitude of your own city in a geography book or encyclopedia, and draw an orthographic projection of the earth, showing the equator, the polar axis, the 0° meridian (Meridian of Greenwich), the meridian through your city, and the latitude circle through your city.

9. Prove Property 1 of Sec. 22·1, namely: *The ratio of the minor and major axes of the ellipse into which a circle projects is equal to the cosine of the angle between the plane of the circle and the projection plane.*
10. Prove in detail that the sum of the squares of two conjugate diameters of an ellipse is equal to the sum of the squares of its major and minor axes.

23

ZONES ON THE SPHERE

23·1 Zones and segments of a sphere

A zone is a portion of a spherical surface included between two parallel planes that intersect the sphere. The distance between the planes is called the **altitude** *h* of the zone, and the sections made by the planes are called the **bases** (Fig. 72). The division of the earth's surface into five zones—the Arctic, North Temperate, Torrid, South Temperate, and Antarctic Zones —is probably familiar to the reader from geography.

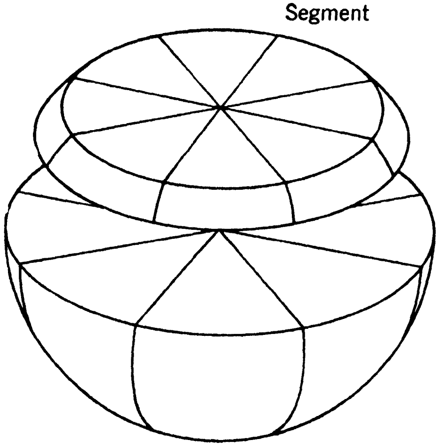


FIG. 72. Segments of a sphere.

A **segment** of a sphere (Fig. 72) is a solid portion of a sphere included between two parallel planes that intersect the sphere. The distance between the parallel planes is called the altitude *h* of the segment, and the sections made by the planes are called the bases. If one of the two planes of section is tangent to the sphere, then one of the two bases of the zone or the segment reduces to a point and the zone or segment is commonly called a **zone** or **segment of one base.** If both planes of section are tangent to the sphere, then the zone or the segment becomes the entire spherical surface or sphere.

In all other cases, the bases of a zone are circles, and the bases of a segment are circular regions.

The lateral surface of a segment is a zone. The axis of a segment or zone is the diameter of the sphere that is perpendicular to the planes of section.

23·2 Cylinder circumscribed about a sphere

A right circular cylinder whose base is equal to a great circle and whose altitude is equal to a diameter of a given sphere can be circumscribed about the sphere in such manner that the lateral surface of the cylinder is tangent to the sphere along a great circle and the bases are tangent to the sphere at the poles of this great circle (Fig. 73). Areas of zones and volumes of segments cut off on the sphere by planes parallel to the bases of the cylinder are closely related to the corresponding areas and volumes cut from the cylinder.

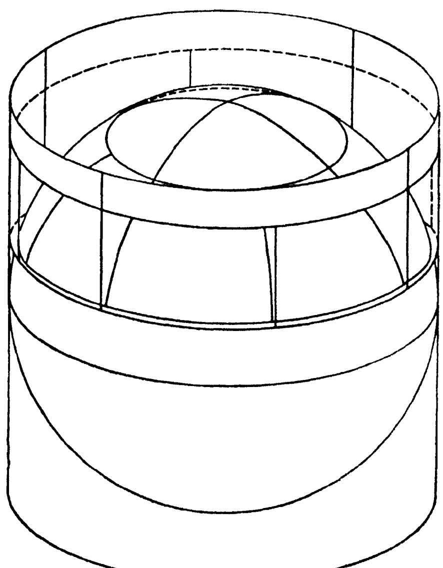


FIG. 73. Sphere and circumscribed cylinder.

23·3 Cylindrical projection of frustum and zone

The normals to the surface of a frustum of a right circular cone, drawn at the points of the circumference of its midsection, all pass through a point on its axis. A sphere may be constructed with its center at this point tangent to the frustum along its midcircle. A cylinder having the same axis as the frustum may then be circumscribed about the sphere as just described above. The base planes of the frustum cut off a zone on the sphere and a region on the cylindrical surface constituting the lateral surface of a circular cylinder. The latter surface is said to be the **cylindrical projection** of the zone and of the lateral surface of the frustum.

THEOREM 23A: *The lateral area of a frustum is equal to its cylindrical projection.*

PROOF: Let h be the height, l the slant height, and r the radius of the midsection of a frustum of a right circular cone, and let R be the radius of the sphere tangent to the frustum along the circumference of its midsection. Then, by similar triangles, we have

$$(1) \qquad \frac{h}{l}=\frac{r}{R} \qquad 2\pi rl=2\pi Rh$$

Since the lateral area of the frustum is $2\pi rl$, and since its cylindrical projection is $2\pi Rh$, these two areas are equal.

23·4 Area of a zone

THEOREM 23B: *The area Z of a zone of a sphere is equal to the altitude h times the circumference $2\pi R$ of a great circle of the sphere. It is equal to its cylindrical projection.*

$$(2) \qquad \textbf{Area of zone: } Z=2\pi Rh$$

PROOF: Let a meridian circle of the sphere cut the zone along an arc $\widehat{Q_0Q}$ (Fig. 74). Subdivide the arc into $N=2^n$ equal parts, by points $Q_1,Q_2,\cdots,Q_{N-1}$. Let the tangent lines to the arc at adjacent points intersect at $T_1,T_2,\cdots,T_N$, so that $Q_1,Q_2,\cdots,Q_{N-1}$ are the mid-points of the tangent segments $[T_1T_2]$, $[T_2T_3]$, $\cdots$, $[T_{N-1}T_N]$. When the plane of the meridian circle is revolved about its axis, these segments generate lateral surfaces of frustums tangent to the sphere along their midsection circles and the segments $[Q_0T_1]$ and $[T_NQ]$ likewise generate surfaces of tangent frustums. The lateral areas of all but the two end frustums are equal to their cylindrical projections. As n becomes larger, the two end frustums approach zero. The sum of the lateral areas of the other frustums approaches

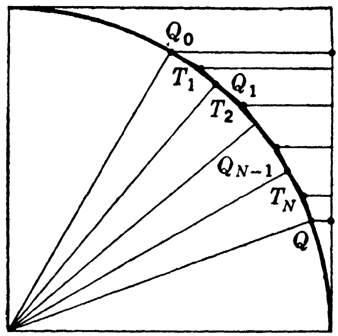


FIG. 74. Meridian sections of spherical zone, tangent frustums, and cylindrical projection.

a limit that is equal both to the area Z of the zone and to the area $2\pi Rh$ of its cylindrical projection. Hence these two areas are equal: $Z=2\pi Rh$.

23·5 Surface area of a sphere

When $h=2R$, we conclude from this formula that the area of the whole spherical surface is $4\pi R^2$.

THEOREM 23C: *The surface area of a sphere is equal to that of four great circles.*

$$\text{(3)} \qquad \textbf{Surface area of sphere} = 4\pi R^2$$

23·6 Zone visible from an airplane

At sea or in country where the horizon appears flat, the portion of the earth visible to an observer is a zone of one base, whose base is the horizon circle. If the observer is H miles above the earth's surface, the zone is very nearly of height H. The refraction of light in the atmosphere being neglected, the exact height of the zone would be $h=HR/(R+H)$, where R ($=3{,}960$ miles) is the radius of the earth. This is obtained from the proportion

$$\text{(4)} \qquad (R-h):R=R:(R+H)$$

which is readily derived from a study of similar triangles in a plane through the earth's center. If H is less than 10 miles, $R/(R+H)$ is so nearly 1 that no appreciable error is introduced by taking $H=h$. The area of the visible zone is therefore equal to $2\pi Rh$ sq. miles. For example, from a height of 1 mile ($H=1$) the visible area is about 25,000 sq. miles. Still assuming H to be small compared with R, the area of the zone is that of a circle of radius $\sqrt{2RH}$ miles. Hence, the refraction of light being neglected, the distance to the visible horizon is approximately $\sqrt{2RH}$ miles. If y denotes the height of the observer *in feet* above sea level, then since $2\times 3{,}960/5{,}280=3/2$, we have

(5) Distance to visible horizon $=\sqrt{\frac{3}{2}y}$ or $\sqrt{1.5y}$ statute miles

the refraction of light being neglected. It is found experimentally that under ordinary conditions the refraction of light increases the distance to the visible horizon by about 7.8 per cent, and thus we obtain, *allowing for refraction of light,*

(6) **Distance to visible horizon** $=1.32\sqrt{y}$ **statute miles**
$=1.146\sqrt{y}$ **nautical miles**

where y is the height of the observer in feet. For example, from an airplane 10,000 ft. above the sea, the horizon would be distant 132 statute miles, or 114.6 nautical miles (see Sec. 2·5).

23. ORAL QUESTIONS

A. How do you distinguish between a zone of a sphere and a segment of a sphere?
B. What is meant by a zone or segment of one base?
C. Do the normals to a zone all pass through its axis?
D. What is meant by the cylindrical projection of a frustum or zone? How does its area compare with that of the frustum or zone?
E. What is the formula for the area of a zone of height h?
F. What is the formula for the area of a hemisphere? For the area of a sphere?

G. Three tennis balls just fit into a cylindrical tin can that has a top and bottom. How does the total surface area of the three tennis balls compare with the total area of the can?

H. How does the area of a zone included between latitude circles 30° N. and 30° S. of the equator compare with the area of a hemisphere?

I. What are the five geographical zones on the earth's surface?

J. If the altitude of the Torrid Zone is about four-fifths of the earth's radius and the altitudes of the Arctic and Antarctic Zones are each about one-twelfth of the earth's radius, what percentage of the earth's surface lies in each of the five geographical zones (to the nearest per cent)?

K. Approximately how many square miles of the earth's surface are visible from a height of H miles?

L. If the horizon appears to be 20 miles away when seen by an observer 230 ft. above sea level, how high above sea level must an observer on the ocean be to see 10 miles in each direction?

23. WRITTEN EXERCISES

1. Find the area of a zone of height 2 in. on a sphere of radius 5 in.
2. Find the area of a zone of height 1,000 miles on a sphere of radius 3,960 miles.
3. A circle of 400-mile radius with its center near Paint Rock, Tex., will include the whole state of Texas. How high above Paint Rock would an observer have to rise on a cloudless day to see the whole state at once?
4. If the area of Texas is 265,900 sq. miles, what is the altitude of a zone of the earth having the same area as Texas?
5. Twelve gallons of paint is spread uniformly over a hemispherical dome 50 ft. in diameter. How thick is the paint?
6. Draw in orthographic projection a figure showing a hemisphere of radius 2 and next to it a cone of radius 2 and slant height 4. Compare their areas.
7. A right circular cone and a hemisphere have the same base and the same total surface area. Find the ratio of the height of the cone to the radius of its base.

8. Draw in orthographic projection a figure showing a sphere of radius 4 on which a zone is drawn between small circles whose polar distances are 30° and 60°, respectively.
9. Compute the area of the zone in Exercise 8.
10. Draw in orthographic projection a figure showing a sphere of radius 5 and the circumscribed cylinder and showing the sections of each made by a plane parallel to the base of the cylinder and 3 units above the center of the sphere.

24

VOLUMES OF SPHERICAL SEGMENTS AND SECTORS

24·1 Volume of a spherical segment

Let a cylinder having the same axis as a given spherical segment be circumscribed about the given sphere of radius R, and let a double cone be drawn with the same bases as the cylinder and with its

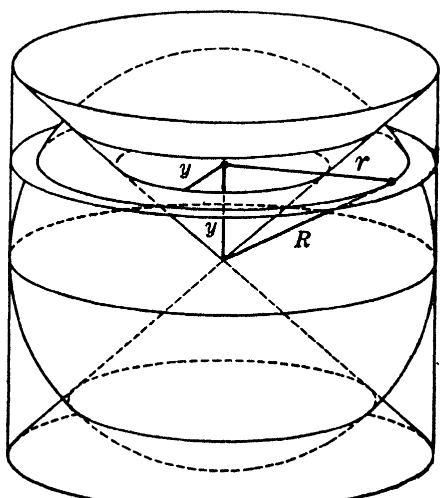


FIG. 75. Circles cut from sphere, related cone, and cylinder.

vertex at the center of the sphere (Fig. 75). Then, if $0 < y < R$, a plane parallel to the bases at distance y from the center cuts each of the three surfaces in circles. The radius of the cone circle is equal to the distance y from the center, since the elements of the cone make angles of 45° with the axis. The radius R of the circle cut from the cylinder is equal to the radius of the sphere. By the Pythagorean theorem, the radius of the small circle of the sphere is $r=\sqrt{R^2-y^2}$. Hence,

$$\pi r^2=\pi R^2-\pi y^2 \tag{1}$$

Thus the area of the section cut from the sphere is equal to the area of the section included between the cone and cylinder. Hence, by Cavalieri's theorem (Sec. 13·5) the volume of a spherical segment is equal to the volume of the ringlike solid bounded above and below by the base planes of the segment and laterally by the lateral surfaces of a cylinder circumscribed about the sphere and of a double cone inscribed in the cylinder. Since the prismoidal formula is valid for cylinders and frustums and is valid also for their difference, we obtain the following theorem:

THEOREM 24A: *The volume of a spherical segment is given by the prismoidal formula.*

If r_L, r_M, r_U are the radii of the lower base, midsection, and upper base, respectively, and h is the altitude, then we may write

$$\textbf{Volume of a spherical segment}=\tfrac{1}{6}\pi h(r_L^2+4r_M^2+r_U^2) \tag{2}$$

Two alternate formulas for the volume of a spherical segment can be derived that do not directly involve the radius of the midsection. If y_M, y_U, and y_L are directed distances from the center of the sphere to the midsection and bases of the segment, we have

$$y_M^2=R^2-r_M^2, \qquad y_U^2=R^2-r_U^2, \qquad y_L^2=R^2-r_L^2 \tag{3}$$

$$2y_M=y_U+y_L, \qquad h=|y_U-y_L|; \qquad 4y_M^2+h^2=2y_U^2+2y_L^2 \tag{4}$$

$$4(R^2-r_M^2)+h^2=2(R^2-r_L^2)+2(R^2-r_U^2) \tag{5}$$

Hence

$$4r_M^2=2r_L^2+2r_U^2+h^2 \tag{6}$$

$$\textbf{Volume of spherical segment}=\tfrac{1}{6}\pi h(3r_L^2+h^2+3r_U^2) \tag{7}$$

The first or last term of this expression is equal to one-half the volume of a cylinder of height h constructed on the lower or upper base, respectively. The middle term represents the volume of a sphere of diameter h, as will be shown below. Hence we obtain the following theorem:

THEOREM 24B: *The volume of a spherical segment of height h is equal to the average of the volumes of cylinders of height h constructed one on each of the bases of the segment, plus the volume of a sphere of diameter h.*

24·2 Volume of a sphere

Upon applying the prismoidal formula to the whole sphere, the two bases of the spherical segment reduce to zero, and the altitude h becomes $2R$. The expression $\frac{1}{6}h(4\pi R^2)$ reduces to $\frac{4}{3}\pi R^3$. Hence

(8) **Volume of sphere** $= \frac{1}{3}$ **radius** $\times$ **surface area** $= \frac{4}{3}\pi R^3$

24·3 Spherical sector

A sector of a sphere is a solid portion of a sphere bounded by a zone, called its base, and by one or two conical surfaces, having the

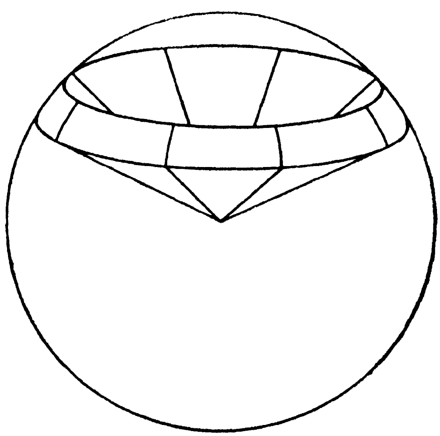


FIG. 76. Spherical sector.

center of the sphere as vertex, and the circular boundary (or boundaries) of the zone as directrix (or directrices) (Fig. 76). A sector of a sphere may be generated by revolving a sector of a circle about an axis in its plane, passing through its vertex but not through its interior.

THEOREM 24C: *The volume of a spherical sector is equal to one-third the area Z of its base times the radius R of the sphere.*

$$\text{(9)} \qquad \textbf{Volume of sector} = \tfrac{1}{3}ZR$$

PROOF: For a sector with one conical surface whose zone has an altitude h less than R and is bounded by a single circle of radius r, the sector is the sum of a segment of one base and a cone of altitude R-h and radius r. Hence by (7)

$$\text{(10)} \qquad V = \tfrac{1}{6}\pi h(3r^2 + h^2) + \tfrac{1}{3}\pi r^2(R-h)$$

Since

$$\text{(11)} \qquad r^2 = R^2 - (R-h)^2 = 2Rh - h^2 = h(2R-h)$$

we have

$$\text{(12)} \qquad V = \tfrac{1}{6}\pi h[6Rh - 3h^2 + h^2 + 2(2R-h)(R-h)] = \tfrac{1}{6}\pi h(4R^2)$$

$$\text{(13)} \qquad V = \tfrac{1}{3}(2\pi Rh)R = \tfrac{1}{3}ZR$$

Since the same formula holds for the whole sphere ($Z = 4\pi R^2$), it holds by subtraction for a sector with one conical surface having altitude $h > R$. Finally, it holds by substraction for a sector with two conical surfaces that is the difference of two sectors each having one conical surface.

$$\text{(14)} \qquad V = V_2 - V_1 = \tfrac{1}{3}Z_2R - \tfrac{1}{3}Z_1R = \tfrac{1}{3}(Z_2 - Z_1)R = \tfrac{1}{3}ZR$$

24. ORAL QUESTIONS

A. What is the difference between a spherical segment and a spherical sector?

B. How can the volume of a spherical segment be expressed in terms of its height and the areas of its bases and midsection? What is this formula called?

C. What other formulas can you remember for the volume of a spherical segment?

D. What is the volume of a hemisphere of radius R?

E. What is the formula for the volume of a spherical sector?

F. How do the volumes of a sphere and its circumscribed cylinder compare?

G. Given a sphere whose volume is 10 cubic units and a right circular cylinder and cone whose bases are equal in area to a great circle of the sphere and whose altitudes are equal to a diameter of the sphere. What are the volumes of the cylinder and cone?

H. A cube is circumscribed about a sphere. Is the volume of the sphere more or less than half the volume of the cube?

I. What is the ratio of the volume of a sphere to its surface area? How does this compare with the corresponding ratio for a cube circumscribed about the sphere?

J. If a sphere and a cube have the same surface area, which has the larger volume?

24. WRITTEN EXERCISES

1. Find the capacity (in cubic feet) of a spherical balloon 33 ft. in circumference. Use 22/7 for π.
2. Show that a hemispherical bowl of radius 3 in. would contain just less than 1 qt. when full and just over 1 pt. when filled to a depth of 2 in.
3. A sphere is sliced into six segments of equal altitude. What is the ratio of the volumes of the smallest and largest segments? What is the ratio of their spherical areas?
4. Draw a figure in orthographic projection to illustrate Exercise 3.
5. From an apple, assumed to be a sphere 3 in. in diameter, a cylindrical core 1 in. in diameter is cut out. Find the volume that remains, using the prismoidal formula.
6. A spherical segment of height 1 in. is cut from a sphere of radius 5 in. The difference of the radii of the two bases is 1 in. Find the volume of the segment.
7. A buoy is in the form of a hemisphere, whose base of radius R is turned upward, surmounted by a cone having the same base, whose altitude is equal to R. Find the volume.
8. From the middle of a segment of a sphere having equal bases B

and altitude h, a cylinder with the same bases and altitude is cut. Show that the volume of the remaining ring is equal to the volume of a sphere of diameter h.

9. A quadrant of a circle of radius 12 in. is revolved about a coplanar line passing through its vertex but not through the quadrant and forming an angle of 60° with one of the sides of the quadrant. Find the volume of the spherical sector generated.
10. Show that the volume of a segment of one base of a sphere of unit radius, having altitude equal to 0.653, is approximately one-fourth the volume of the sphere.
11. A sphere is inscribed in a regular hexagonal prism. Find the ratio of the volume of the sphere to the volume of the prism. This is the packing fraction if a large number of spheres are fitted in a single layer into a box, the small amount of waste space on the edges being neglected.
12. With the mid-points of the edges of a unit cube as centers, 12 spheres are constructed, each of radius $\frac{1}{4}\sqrt{2}$. One-quarter of each sphere is inside the cube. One more sphere of the same radius is constructed whose center is at the center of the cube. Show that this sphere is tangent to the other 12. How large a volume is in the cube, but not in any of the spheres? This is the minimum fraction of waste space in packing spheres together in space. (There is, of course, a little more waste space on the sides of the container.)

25

VOLUMES AND AREAS OF SOLIDS OF REVOLUTION

25·1 Solids and surfaces of revolution

A **solid of revolution** is a solid generated by revolving a connected plane region about a line in the plane not cutting the region. The area of the generating region will be denoted by A. A **surface of revolution** is a surface generated by revolving a plane curve about a line in its plane not cutting the curve. The generating curve may consist of one or more arcs or line segments connected together. Its length will be denoted by s. The curved portion of the boundary of a solid of revolution is a surface of revolution. The line about which the generating region or curve is revolved is called the **axis** of revolution or simply the axis (Figs. 77 and 79).

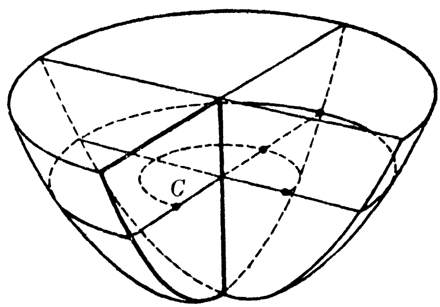


FIG. 77. Surface and solid of revolution.

Right circular cylinders and cones, spheres, and segments and sectors of spheres are examples of solids of revolution that have already been measured. Further analysis of volumes and areas of revolution depends on two theorems that bear the name of **Pappus of Alexandria,** who lived in the third century A.D. Both theorems involve the notion of a point called the **centroid,** or center of gravity, of the plane region or curve.

25·2 The centroid

For a plane region (or curve) having a center of symmetry C such that the region (or curve) is taken into itself by a rotation through 180° about C, the centroid is the center of symmetry (Fig. 78). For example, the mid-point of a line segment is its centroid, and the center of a circle or rectangle is the centroid both of the plane area and of the bounding curve.

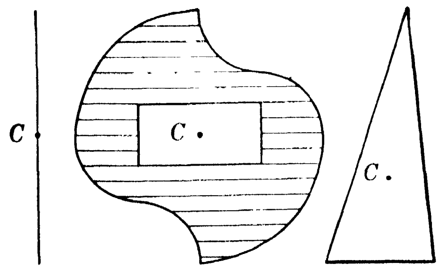


FIG. 78. Centroids.

The centroid of a connected plane region or curve not having a center of symmetry may be thought of as follows: Let a model of the region (or curve) be cut out of thin sheet metal (or wire) of uniform thickness. Then the centroid is a point C in the plane of the region (or curve) such that, if the model is placed horizontally over a knife-edge supporting the model along any line through C, the model will just balance on the knife-edge without tipping.

A metal triangle will just balance if the horizontal knife-edge is placed along one of the medians, since all line segments connecting two sides of a triangle and parallel to the third side are bisected by the median drawn to the third side. Hence the centroid of a triangle is the point of intersection of its medians (Fig. 78).

25·3 Theorem of Pappus on volumes of solids of revolution

THEOREM 25A: *The volume of the solid generated by revolving a connected plane region about an axis in its plane, but not cutting the region, is equal to the product of the area A of the generating region times the circumference $2\pi c$ of the circular path through which the centroid moves.*

(1) **Volume of solid of revolution: $V=2\pi cA$**

PROOF: Although the theorem is true for an arbitrary plane region p, we shall give a complete proof only for the case that p is convex and has a center of symmetry C at a distance c from the axis of revolution.

1. Each principal section of the solid consists of a ringlike area included between the circumferences of two concentric circles of radii r and R. (In particular, r may be 0 for some sections if the axis touches the generating region p.) The width w of the ring is $R-r$, and its midradius is $m=\frac{1}{2}(R+r)$. Its area $\pi R^2-\pi r^2$ may be factored and written in the form $2\pi mw$.

2. For two principal sections at the same distance from C but on opposite sides of C, the assumption of symmetry implies that the widths of the rings are equal and that the average of the two midradii m_1 and m_2 is equal to c.

3. Between the base planes of the given solid of revolution place a congruent solid turned upside down. Then the sum of the principal sections of the two congruent solids made by a cutting plane q parallel to the base is $2\pi m_1 w+2\pi m_2 w$, and the average of these two is $2\pi cw$.

4. A right cylinder (not necessarily circular), generated by moving the plane area p a distance $2\pi c$ perpendicular to its plane, has rectangular sections (by the same cutting plane q) of area $2\pi cw$, equal to the average of the principal sections of the two congruent solids of revolution. Its volume is $2\pi cA$.

5. By Cavalieri's theorem the solid of revolution has the same volume as the cylinder, namely, $V=2\pi cA$. Q.E.D.

25·4 Theorem of Pappus on areas of surfaces of revolution

THEOREM 25B: *The area S of the surface generated by revolving a connected plane curve about an axis in its plane, but not cutting the curve, is equal to the product of the arc length l of the generating curve times the circumference $2\pi c$ of the circular path through which the centroid moves.*

The proof of this theorem is beyond the scope of this text. The theorem can be made plausible, however, by considering the volume of a thin coat of paint of uniform thickness on a solid of revolution. The area of the surface is approximately the volume of the paint divided by its thickness, and the length of the curve that generates the surface is approximately equal to the area of p that is painted divided by the same thickness of the paint mark.

25·5 Applications of Pappus's theorems

EXAMPLE 1: **The torus.** The solid obtained by revolving a circle of radius b about an axis in its plane at a distance $c>b$ from its center is called a torus or anchor ring (Fig. 79). Its shape resembles that of a doughnut or the inner tube of a tire. Its generating circle has the area πb^2 and circumference $2\pi b$. Multiplying these in turn by $2\pi c$ we obtain the volume and surface area of a torus.

(2) **Torus:** $V=2\pi c\cdot\pi b^2=\mathbf{2\pi^2 b^2 c}$ $\qquad S=2\pi c\cdot 2\pi b=\mathbf{4\pi^2 bc}$

EXAMPLE 2: Lateral area of a **right circular frustum.** The lateral surface of such a frustum is generated by revolving a line segment of length l about an axis at a distance $c=r_M$ from its mid-point, where r_M is the radius of the midsection. Hence

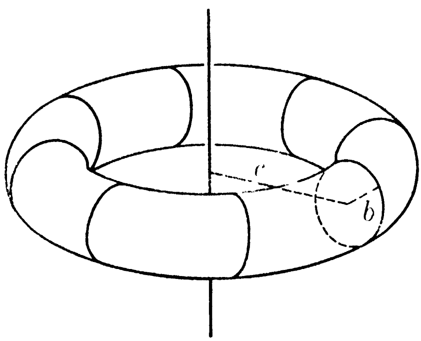


FIG. 79. Torus.

(3) $$S=2\pi r_M l$$

EXAMPLE 3: The **centroid of a semicircular area** of radius a. To find the distance of the centroid of the area of a semicircle from its bounding diameter, we may use Pappus's theorem in reverse. The semicircle of area $\pi a^2/2$ generates a sphere of radius a and volume $\frac{4}{3}\pi a^3$. Hence

$$\tfrac{4}{3}\pi a^3=2\pi c\left(\frac{\pi a^2}{2}\right)$$

(4) Centroid for semicircular area: $c=\dfrac{4a}{3\pi}=0.424a$

25. ORAL QUESTIONS

A. What is a solid of revolution, and what is meant by the area of the generating region?

B. What is a surface of revolution, and what is its axis?
C. What solid is generated by revolving a right triangle about one of its sides?
D. What solid is generated by revolving a rectangle about one of its sides?
E. What solid is generated by revolving a sector of a circle about an axis lying in its plane that meets the sector only at its vertex?
F. What is the generating curve that generates a zone of a sphere as a surface of revolution?
G. What is the centroid of a curve having a center of symmetry? Where is the centroid of a triangle?
H. What is Pappus's theorem for the volume of a solid of revolution?
I. What is Pappus's theorem for the area of a surface of revolution?
J. How can Pappus's theorem be used to find the centroid of a quadrant of a circle?
K. What is a torus, and what are the expressions for its volume and surface area?

25. WRITTEN EXERCISES

1. A circle of radius 3 in. is revolved about a line 6 in. from its center. Find the volume and surface area of the solid generated. What is the solid called?
2. Draw a figure showing the solid of Exercise 1.
3. A semicircular area of radius 3 in. is revolved about an axis in its plane, parallel to its diameter and 6 in. away. Two solids may be formed, depending on whether the semicircle does or does not lie between the axis and the diameter. Find the volume of each, and show that their sum is the volume of Exercise 1.
4. An equilateral triangle with 2-in. sides is revolved about a side. Compute the volume of the solid generated, first by using Pappus's theorem, and then by dividing the solid into two cones. Compare the answers.
5. A trapezoid having three 1-in. sides and one 2-in. side is revolved about its long side as an axis. Compute the volume by subdividing it into prisms and cones. Then find the centroid by Pappus's theorem. NOTE: The given trapezoid is half a regular

hexagon and may be subdivided into three equilateral triangles.

6. A square of side a is revolved about an axis parallel to a diagonal at distance a from the diagonal. Find the volume generated, using Pappus's theorem.
7. Find the area of the surface of the solid of Exercise 6. Is this area any different if the axis is at distance a from the center but is not parallel to the diagonal of the square?
8. Semicircles of diameter 1 in. are constructed on the sides of a 1-in. square, and the whole figure is revolved about an axis in its plane, 2 in. from its center. Find the volume generated.
9. Find the distance of the centroid of a semicircular arc of radius r from its center, using Pappus's theorem.
10. The distance from a reference axis to the centroid of an arc may be approximated by a formula similar to Simpson's rule for finding the mean base of a solid. Mark $2n+1$ points, equally spaced along the curve extending from one end to the other. Find the distances of these points from the axis. Multiply these by $1,4,2,4,\cdots, 2,4,1$, respectively, and divide by $6n$. The result is the required centroid distance. For a semicircular arc of radius 1, taking $n=2$, the distances to the five points from a diameter through the extremities of the arc are $\cos 90°$, $\cos 45°$, $\cos 0°$, $\cos(-45°)$, $\cos(-90°)$. The weighted average is $2(\cos 90°+4\cos 45°+\cos 0°)/12=(1+2\sqrt{2})/6$. Evaluate this to four decimals, and compare it with the exact centroid distance $2/\pi=0.6366$.

26

AREAS OF LUNES, SPHERICAL TRIANGLES, AND POLYGONS

26·1 Area of a lune

The area of a lune (Fig. 66) is proportional to the angle of the lune. For if any lune is divided by meridian semicircles through the vertices into any number of congruent parts, then its area and angle are each divided into the same number of equal parts and the ratio of the area to the angle remains the same for the parts as for the whole. A quadrant lune with angles of 90° is a quarter of the surface of the sphere, and thus its area is πR^2, where R is the radius of the sphere. Hence for any lune the ratio of area to angle is $\pi R^2/90°$.

THEOREM 26A: *The area L of a lune of angle $\theta°$ on a sphere of radius R is given by the formula*

$$L = (\theta°/90°)\pi R^2 \tag{1}$$

That is, *the area of a lune expressed in great circles is equal to the angle of the lune expressed in right angles.*

26·2 The sides and angles of a spherical triangle

Any three points A, B, C, on the surface of a sphere but not on the same great circle are vertices of a spherical triangle ABC (Fig. 80). The great-circle arcs $\widehat{BC}$, $\widehat{CA}$, $\widehat{AB}$, each less than 180°, which join the vertices in pairs, are sides of the spherical triangle and are denoted by a, b, c, respectively. The angles at the three vertices between the pairs of sides are called the angles of the spherical triangle and are denoted by α, β, γ.

A spherical triangle ABC on a sphere with center at O can be thought of as the intersection of the trihedron O-(ABC) with the surface of the sphere. The three face angles a, b, c of the trihedron are equal to the corresponding three sides of the triangle, when measured in angular measure, and the three dihedral angles α, β, γ of the trihedron are equal to the corresponding three angles of the triangle. In this manner there is a one-to-one correspondence between trihedrons with center at a point O and spherical triangles on a spherical surface S with center at O. The notation for the six parts a, b, c, α, β, γ of the spherical triangle is such that side a is opposite angle α, b is opposite β, and c is opposite γ.

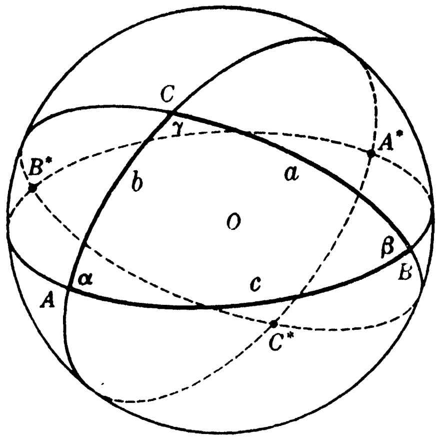


FIG. 80. Associated spherical triangles.

We also define the **semiperimeter** s of a spherical triangle to be

$$s=\tfrac{1}{2}(a+b+c) \tag{2}$$

26·3 Congruent, symmetric, and associated triangles

Two spherical triangles are said to be **congruent** if both have the same angles and sides and belong to spheres of equal radii. It is possible that both these conditions be satisfied and yet that the two spherical triangles cannot be made to coincide, because the equal parts occur in the opposite cyclic order around the triangle. Two such triangles are said to be **enantiomorphous.** One might be called the mirror image of the other.

The three great circles, arcs of which are sides of a spherical triangle ABC, divide the spherical surface S up into eight **associated spherical triangles** (Fig. 80). Adjacent pairs of triangles forming a lune on the sphere are called **colunar** triangles. **Opposite,** or antipodal, pairs of triangles can be shown to be enantiomorphous. In fact, let A^*, B^*, C^* be the points on S diametrically opposite to A, B, C, respectively. Then A, B, A^*, B^* lie on a great circle, and C and C^* are in opposite hemispheres. If the triangle $A^*B^*C^*$ is rotated through 180° about the axis of this great circle, then A^* will coincide with A and B^* with B, while C^* will be the mirror image of C in the plane of the great circle (Fig. 81). (The student may demonstrate this rotation on two halves of an orange.)

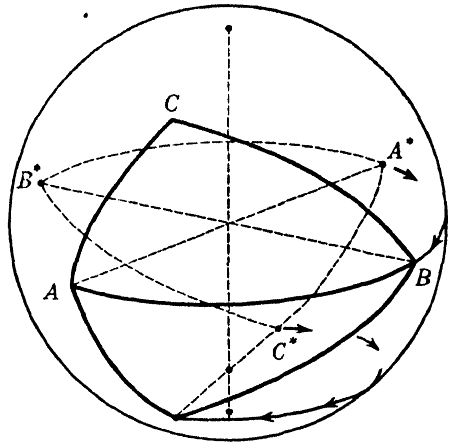


FIG. 81. Enantiomorphous triangles.

26·4 Area and semi-excess of a spherical triangle

The area of a spherical triangle is related to the area of a lune in the same way in which the trihedral angle of a trihedron is related to the dihedral angle of a dihedron. If σ is an angular measure of the area of a spherical triangle, in the same sense that the angle of a lune is an angular measure of the area of a lune, then the associated colunar triangles will have the measures $\alpha-\sigma$, $\beta-\sigma$, $\gamma-\sigma$, respectively. Adding the four, we obtain two quadrant lunes. Hence

(3) $$\sigma+(\alpha-\sigma)+(\beta-\sigma)+(\gamma-\sigma)=180°$$

Solving (3) for σ, we have

(4) $$\sigma=\tfrac{1}{2}(\alpha+\beta+\gamma-180°)$$

Since σ is one-half the amount by which the sum of the angles of a spherical triangle exceeds 180°, we call σ the **semi-excess** of the triangle.

THEOREM 26B: *The area of a spherical triangle is equal to the number of right angles contained in its semi-excess times the area of a great circle.*

PROOF: Since the spherical triangle has the same area as a lune of angle σ, its area is given by $(\sigma/90°)\pi R^2$.

26·5 Spherical polygons and pyramids

A pyramidal surface with vertex at the center of the sphere intersects the spherical surface in a **spherical polygon** (Fig. 82). The points where the edges intersect the sphere are the **vertices** of the spherical polygon, the great-circle arcs in which the faces of the pyramidal surface intersect the sphere are the **sides** of the spherical polygon, and the dihedral angles between consecutive faces of the pyramidal surface are the **angles** of the spherical polygon. The solid cut out from the sphere by the pyramidal surface is called a **spherical pyramid.**

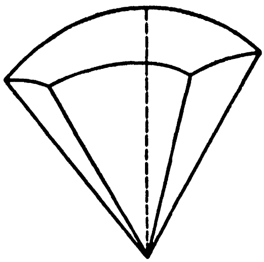


FIG. 82. Spherical pyramid.

26·6 Area of a convex spherical polygon

By drawing great-circle arcs from one vertex of a convex spherical polygon to all the $n-3$ nonadjacent vertices, the polygon can be cut up into $n-2$ spherical triangles. The area of the spherical polygon is equal to the sum of the areas of the $n-2$ spherical triangles. In angular measure this is $\sigma=\sigma_1+\sigma_2+\cdots+\sigma_{n-2}$, where σ_1, $\sigma_2,\cdots$ are the semi-excesses of the triangles. We define σ to be the semi-excess of the spherical polygon. It is equal to the angle of a lune of equal area.

THEOREM 26C: *The angle of a lune having an area equal to the area of a given spherical polygon is equal to the supplement of one-half the sum of its exterior angles.*

PROOF: 1. If the spherical polygon is cut into $n-2$ spherical triangles by passing great-circle arcs from one vertex to each of the nonadjacent vertices, the angle σ of the equal lune is given by

$$\sigma=\sigma_1+\sigma_2+\cdots+\sigma_{n-2}$$

where $\sigma_1,\sigma_2,\cdots$ are the semi-excesses of the $n-2$ triangles.

2. $2\sigma=$ sum of all the angles of the $n-2$ triangles $-(n-2)180°$.

3. $2\sigma=$ sum of interior angles of spherical polygon $-(n-2)180°$.

Adding the sum of the exterior angles to each side and noting that the interior and exterior angles at each vertex are supplementary, we have

4. $2\sigma +$ sum of exterior angles $= n(180°) - (n-2)180° = 2(180°)$.

5. $\sigma = 180° - \frac{1}{2}$ sum of exterior angles. Q.E.D.

26·7 Volume of a spherical pyramid

The spherical polygon that forms a part of the boundary of a spherical pyramid is called the **base** of the pyramid (Fig. 82). Its area will be denoted by B, the radius of the sphere by R, and the volume of the pyramid by V.

THEOREM 26D: *The volume of a spherical pyramid is equal to one-third the area of the base B times the radius R of the sphere.*

(5) **Spherical pyramid:** $V = \frac{1}{3}BR$

PROOF: If a number of points are chosen on the base of the spherical pyramid and tangent planes are drawn to the sphere at these points, then each point will be inside a polygon cut from its tangent plane by adjacent tangent planes or by the planes of the pyramidal surface. Upon taking these polygons as bases and the center of the sphere as vertex, a number of pyramids are formed whose altitudes are all equal to the radius of the sphere. Their total volume is equal to one-third the sum of their bases times the radius of the sphere. As the number of base polygons is increased by introducing new tangent planes to the spherical polygon in such manner that the area of the largest of these plane polygons approaches zero, the sum of the areas continually decreases and approaches the area of the spherical polygon as a limit; likewise, the sum of the volumes of the pyramids decreases and approaches the volume of the spherical pyramid. Hence in the limit we still have $V = \frac{1}{3}BR$.

THEOREM 26E: *The volume of a spherical pyramid is to the volume of the whole sphere as the semi-excess of its base is to* 360°.

The proof of this theorem is left to the student.

26. ORAL QUESTIONS

A. What are the six parts of a spherical triangle?

B. How is a spherical triangle related to a trihedron?

C. What are the relations between the corresponding parts of the eight associated triangles into which three great circles divide a spherical surface?

D. When are two spherical triangles enantiomorphous? When are they directly congruent?

E. What is the formula for the area of a lune?

F. What sized lune has the same area as a spherical triangle whose angles are 60°, 80°, and 100°?

G. What is the formula for the area of a spherical triangle? For the areas of the associated triangles?

H. What are spherical polygons and spherical pyramids?

I. If a cube is projected onto the circumscribed sphere, each face projects into a spherical polygon having four 120° angles. What is the angle of a lune having the same area as the polygon?

J. What is the volume, in a sphere of unit radius, of the spherical pyramid having the polygon of Question I as a base?

26. WRITTEN EXERCISES

1. Find the area of a 15° lune on a sphere of radius 6.
2. Find the area of a 45° lune on a sphere of radius 2.
3. Find the semi-excess and the area on a unit sphere of a spherical triangle whose angles are 100°, 120°, 150°. Find also the semi-excesses and the areas of the seven associated triangles whose sides lie on the same three great circles as the sides of the given triangle.
4. Find the semi-excess and the area on a unit sphere of a spherical triangle whose angles are 30°, 72°, 90°. Find also the semi-excesses and the areas of the seven associated triangles whose sides lie on the same three great circles as the sides of the given triangle.
5. Find the semi-excess and the area on a unit sphere of a spherical triangle whose angles are 60°, 90°, 135°. Find also the semi-excesses and the areas of the seven associated triangles whose sides lie on the same three great circles as the sides of the given triangle.
6. Given a spherical triangle ABC such that, if the perimeter is

traversed so that the six parts appear in the cyclic order b, α, c, β, a, γ, then the interior of the triangle is on the left as viewed from the outside of the sphere. Write down, in cyclic order, the parts of each of the other seven associated triangles, so that the interior of each triangle is on the left when the parts are traversed in that order.

7. A regular tetrahedron is projected onto the circumscribed sphere by connecting each two of its four vertices by great-circle arcs. What are the areas and angles of the four equilateral spherical triangles that are formed?
8. A regular dodecahedron is projected onto the circumscribed sphere by connecting each pair of adjacent vertices by great-circle arcs. What are the areas and angles of the 12 regular spherical pentagons which are formed?
9. What is the volume in a sphere of radius 3 of the spherical pyramid having one of the triangles of Exercise 7 as a base?
10. What is the volume in a sphere of unit radius of a spherical pyramid having one of the pentagons of Exercise 8 as a base?
11. Draw a figure showing a sphere in orthographic projection and showing a triangular spherical pyramid.

27

GEOMETRY OF THE SPHERICAL TRIANGLE

27·1 Isosceles, right, and quadrantal triangles

In general, no two of a set of eight associated spherical triangles are directly congruent. Opposite ones have equal area, but corresponding parts follow each other in the opposite sense of rotation. They will be directly congruent if and only if the triangle is isosceles.

An **isosceles spherical triangle** is one having two equal sides. The angles opposite the equal sides are also equal.

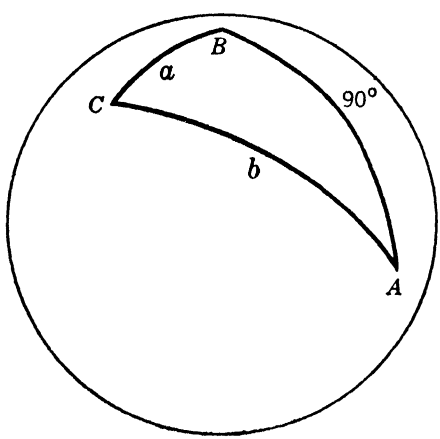


FIG. 83. Quadrantal triangle.

An **equilateral spherical triangle** is one having three equal sides and three equal angles.

A **right spherical triangle** is a spherical triangle of which at least one angle is 90°. On the sphere a triangle may have two or even three right angles. A lune bounded by two meridians is cut by the equator into two spherical triangles each having two 90° angles and two 90° sides. The remaining side and opposite angle are equal and may, in particular, be 90° also.

A **quadrantal triangle** (Fig. 83) is a spherical triangle having a side equal to a quadrant, or 90° arc, of a great circle. If a spherical triangle has two quadrant sides, then it must be a half a lune, having two right angles as well.

27·2 Polar triangles

Each side of a spherical triangle is a great circle having two poles. The pole of a side a that lies in the same hemisphere as the third vertex A will be denoted by A', and the points B' and C' will be defined in a similar manner as poles of b and c. The polar triangle $A'B'C'$ (Fig. 84) with sides a', b', c', respectively, defines a trihedron O-$(A'B'C')$ that is the polar of the trihedron O-(ABC). All the statements about a pair of polar trihedrons apply with minor changes of wording to the analogous properties of a pair of polar triangles. In particular, we have the following important relations between angles:

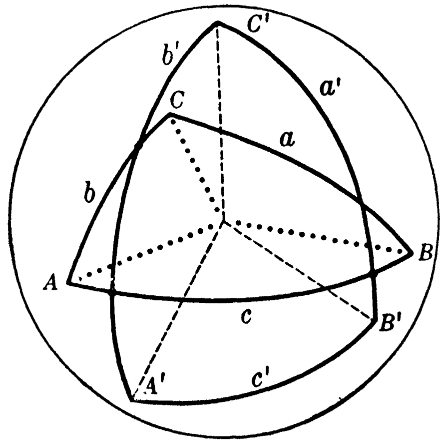


FIG. 84. Polar triangles.

THEOREM 27A: *If, of two given spherical triangles, one is the polar of the second, then the second is the polar of the first. The angles of a spherical triangle and the corresponding sides of the polar triangle are supplementary. The semi-excess of a spherical triangle and the semiperimeter of the polar triangle are supplementary.*

$$\alpha + a' = \beta + b' = \gamma + c' = \sigma + s' = 180^\circ$$
$$\alpha' + a = \beta' + b = \gamma' + c = \sigma' + s = 180^\circ$$

Since $\alpha' - \sigma' = s - a$, etc., the semi-excesses σ_α', σ_α', σ_β', σ_γ' of the four pairs of opposite spherical triangles associated with the polar triangle are as follows:

$$\sigma' = 180^\circ - s \qquad \sigma_\alpha' = s - a \qquad \sigma_\beta' = s - b \qquad \sigma_\gamma' = s - c$$

27·3 Elementary inequalities in a spherical triangle

For every theorem about the face angles and dihedral angles of a trihedron, there is an analogous theorem about the sides and angles of a spherical triangle, and conversely. Thus by arguments similar to those in Chap. 11 we derive immediately the following six inequalities:

INEQUALITY 1: *Each of the three sides a, b, c and each of the three angles α, β, γ of a spherical triangle are less than* 180°.

INEQUALITY 2: *In any spherical triangle, if two sides are equal they are opposite equal angles, and if two sides are unequal the respective opposite angles are unequal in the same order.*

INEQUALITY 3: *The sum of the sides of a spherical triangle is less than* 360°. *That is,* $0° < s < 180°$.

INEQUALITY 4: *The sum of two sides of a spherical triangle is greater than the third side.* This is equivalent to stating that $s-a$, $s-b$, $s-c$ are positive or that a, b, c are each less than s.

INEQUALITY 5: *The sum of the angles of a spherical triangle is greater than* 180° *but less than* 540°. This is equivalent to the inequality $0° < \sigma < 180°$ for the semi-excess σ.

INEQUALITY 6: *Each angle of a spherical triangle is greater than the semi-excess.* That is, $\alpha-\sigma$, $\beta-\sigma$, $\gamma-\sigma$ are positive angles. Or again, *the sum of any angle and the supplement of another angle is greater than the third angle.*

27·4 Parts nearest to right angles

Given a pair of polar triangles ABC and $A'B'C'$ on a sphere of unit radius (Fig. 84), let V be the volume of the parallelepiped P formed with three of its sides along the radii $[OA]$, $[OB]$, $[OC]$ and V' the volume of the parallelepiped P' constructed in a similar manner from $[OA']$, $[OB']$, $[OC']$. If the base of P is taken as the parallelogram in the plane (OBC), then the area of the base is $\cos \bar{a}$, since the angle between a side and adjacent altitude of the parallelogram is the complement of the angle a between the sides. The altitude of the solid P is $\cos \alpha^*$, where α^* is the angle, $\measuredangle AOA'$. Hence

$$(1) \qquad V = \cos \bar{a} \cos \alpha^* \qquad \bar{a} = 90° - a$$

Applying the same argument for each choice of base plane of the parallelepiped P and likewise for the parallelepiped P', we have

$$(2)\qquad \begin{cases} V = \cos \bar{a} \ \cos \alpha^* = \cos \bar{b} \ \cos \beta^* = \cos \bar{c} \ \cos \gamma^* \\ V' = \cos \overline{a'} \ \cos \alpha^* = \cos \overline{b'} \ \cos \beta^* = \cos \overline{c'} \ \cos \gamma^* \end{cases}$$

where β^* and γ^* are the angles, $\measuredangle BOB'$ and $\measuredangle COC'$, respectively. By division we have

$$(3)\qquad \frac{V}{V'} = \frac{\cos \bar{a}}{\cos \overline{a'}} = \frac{\cos \bar{b}}{\cos \overline{b'}} = \frac{\cos \bar{c}}{\cos \overline{c'}}$$

Since $\bar{\alpha} + \overline{a'} = 90° - \alpha + 90° - a' = 0$, we have $\cos \bar{\alpha} = \cos \overline{a'}$, etc. Hence

$$(4)\qquad \frac{V}{V'} = \frac{\cos \bar{a}}{\cos \bar{\alpha}} = \frac{\cos \bar{b}}{\cos \bar{\beta}} = \frac{\cos \bar{c}}{\cos \bar{\gamma}}$$

From this proportion we derive two important inequalities among the six parts of a spherical triangle. Note that for angles between 0° and 180°, the nearer an angle a is to 90°, the nearer $\bar{a}$ is to 0° and the larger $\cos \bar{a}$ will be.

INEQUALITY 7: *Of the two pairs of opposite parts a, α and b, β in a spherical triangle that is not isosceles, let a be the part nearest to 90°. Then both α and b are nearer to 90° than the part β adjacent to a.*

The proof of this follows from the proportion (4).

INEQUALITY 8: *Opposite parts in a spherical triangle are either both acute, both right, or both obtuse with a single exception: the side nearest 90° and the angle nearest 90° are opposite parts that need not obey this rule.*

27·5 Spherical trigonometry

The study of the exact relationships between the six parts of a spherical triangle that make it possible, given any three parts satisfying the conditions of Inequalities 1 to 8, to determine the other three parts, is called spherical trigonometry. The solution is unique except when two of the unknown parts are the pair of opposite parts nearest to 90°. If we define the **sine** of an angle to be the same as the cosine of the complementary angle,

$$\sin \theta = \cos (90° - \theta) \tag{5}$$

then the two most important relationships in spherical trigonometry can be written.

1. **Law of sines:** $\dfrac{\sin a}{\sin \alpha} = \dfrac{\sin b}{\sin \beta} = \dfrac{\sin c}{\sin \gamma}$

2. **Law of cosines:** $\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$

The first of these laws is a restatement of Eqs. (4) of Sec. 27·4. Five similar formulas are obtained from the second law, two of them by interchanging a and α with b and β or with c and γ, and three by considering the corresponding law for the polar triangle. Still other formulas can be derived from these, which are easier to use in actual numerical computations.

27. ORAL QUESTIONS

A. What can be said of the parts of a spherical triangle if two of its associated triangles are directly congruent? Discuss all possibilities.

B. What are polar triangles, and how are their angles related?

C. What are the limitations in size for the semiperimeter and semi-excess of a spherical triangle?

D. In what sense can each of the angles σ, $\alpha - \sigma$, $\beta - \sigma$, $\gamma - \sigma$, $180° - s$, $s - a$, $s - b$, $s - c$ be interpreted as a measure of area?

E. Why is it impossible to have a 30°, 60°, 90° spherical triangle?

F. What can be said of the size of the sum or difference of two sides of a spherical triangle?

G. What can be said of the size of the difference of two angles of a spherical triangle as compared with the supplement of the third angle? Is a 40°, 70°, 160° triangle possible?

H. Is it true without exception that an angle and opposite side of a spherical triangle are in the same quadrant? Explain.

I. Does it follow from Inequalities 7 and 8 that the average of two sides of a spherical triangle is acute if and only if the average of the opposite angles is acute?

J. What does the law of cosines become if angle $\alpha = 90°$?

27. WRITTEN EXERCISES

In each of the Exercises 1-16, three parts of an assumed spherical triangle are given. If such a triangle exists, find three parts of the polar triangle. If no such triangle exists, state which of the eight inequalities is contradicted.

1. $a = 188°$, $b = 62°$, $c = 130°$.
2. $a = 128°$, $b = 62°$, $c = 130°$.
3. $a = 170°$, $b = 160°$, $c = 110°$.
4. $a = 170°$, $b = 60°$, $c = 110°$.
5. $a = 170°$, $b = 60°$, $\gamma = 110°$.
6. $a = 30°$, $b = 70°$, $c = 80°$.
7. $a = 30°$, $b = 60°$, $c = 90°$.
8. $a = 40°$, $b = 50°$, $\alpha = 100°$.
9. $a = 40°$, $\beta = 50°$, $\alpha = 100°$.
10. $a = 130°$, $b = 50°$, $\alpha = 120°$.
11. $\alpha = 100°$, $\beta = 140°$, $\gamma = 150°$.
12. $\alpha = 120°$, $\beta = 140°$, $\gamma = 150°$.
13. $\alpha = 50°$, $\beta = 100°$, $\gamma = 150°$.
14. $\alpha = 40°$, $\beta = 80°$, $\gamma = 120°$.
15. $\alpha = 40°$, $\beta = 50°$, $\gamma = 80°$.
16. $\alpha = 40°$, $\beta = 50°$, $c = 80°$.
17. Prove Inequality 4.
18. Prove Inequality 8 by using Inequalities 2 and 7.

28

THE TERRESTRIAL SPHERE

28·1 The shape of the earth

The earth is approximately, but not exactly, a sphere of radius 3,958.8 statute miles. It revolves once every 24 hours sidereal time toward the east about a diameter called its **axis,** whose extremities P_n and P_s are the **north and south poles,** respectively. The **equator** is a great circle on the earth, midway between the poles, whose plane is perpendicular to the axis. Sections of the earth's surface by half planes having the axis as an edge are called **meridians.** Sections by planes perpendicular to the axis are called **parallels** or **latitude circles.**

Because of the centrifugal force due to its rotation, the earth bulges more at the equator than it does at the poles, so that its shape is nearly that of an oblate spheroid, generated by revolving an ellipse about its minor axis. Its polar diameter of 7,900 statute miles is about 27 statute miles shorter than its equatorial diameters, and its meridians are slightly elliptical rather than true semicircles. Mountains and valleys form, as we know, a surface that is not truly spherical. Yet these deviations from a true spherical surface are so small, relative to the size of the earth, that, if the earth were represented in exact proportions by a globe 3 ft. in diameter, sea level would be represented by a spheroidal surface whose polar diameter was only $\frac{1}{8}$ in. less than its equatorial diameters and the highest mountains would rise less than $\frac{1}{40}$ in. For this reason, results accurate enough for many purposes can be obtained by assuming that the earth is a sphere.

28·2 Nautical miles and knots

Angular measures of distance on the earth are easily converted into linear measures when the nautical mile (6,080.27 ft.), rather than the statute mile (5,280 ft.), is taken as the unit of length. By definition, one **nautical mile** is the **length** of one minute (1′) of arc of a great circle on an exact sphere having the same mean radius as the earth, and one **knot** is a **speed** of one nautical mile per hour. One degree (1°) on a great circle is equal to 60 nautical miles, or about 69.1 statute miles. One second (1″) of arc is a length of about 101 ft.

28·3 Spherical coordinates

Two appropriate angles serve as a pair of spherical coordinates to locate a point B on a spherical surface. One of these coordinates may be the distance, measured from 0° to 180°, from a fixed reference point C. Or it may be the complement of that distance, measured between +90° and −90°, which measures the directed distance from a great circle of which the reference point is a pole. The second coordinate of a point B gives the direction, or bearing, of B from C. It can be measured in three equivalent ways: (1) as an angle at C measured from a fixed prime direction; (2) as an arc along the great circle having C as a pole; (3) as the dihedral angle between two half planes that pass through C and through the center of the earth. In navigation this bearing is usually measured clockwise about C, from 0° at the prime direction and around to 360°; but it may also be measured in either sense from 0° to 180°, if the sense is specified by labeling the angle either + or − or E. (eastward) or W. (westward). The name of a bearing may be changed by changing its sign, or by subtracting it from 360°. Two bearings that differ by 360° are considered equivalent ($\cong$). For instance, 120°40′ W. is equivalent to −120°40′ E. or to 239°20′ E.

28·4 Latitude and longitude

When the north pole P_n is taken as a reference point for spherical cordinates, the distance is called polar distance p or colatitude $\overline{L}$, and its complement, the latitude L, is the angular distance north (N. or +)

or south (S. or −) of the equator. All points on a parallel circle have the same latitude (Fig. 85). The second spherical coordinate

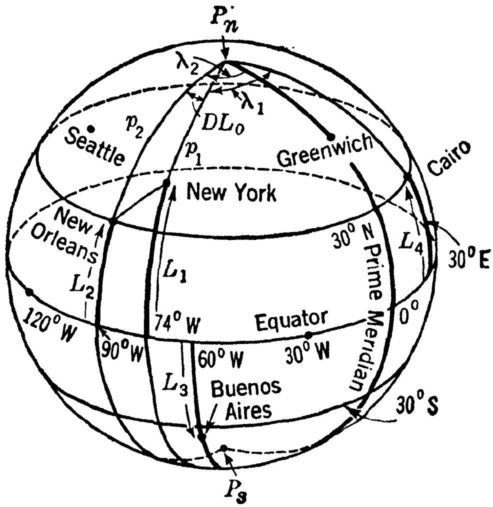


FIG. 85. Latitude and longitude.

in this system is called longitude (λ). It is the angle between the half plane of the meridian of a given point and that of a certain chosen prime meridian, usually taken to be the meridian of the transit instrument in the old observatory at Greenwich, England. Longitude is usually measured from 0° to 180° east (E.) or west (W.) of the prime meridian, but it is sometimes advantageous to use larger angles than 180° or to use negative angles, so that two longitudes which enter into a problem will have the same directional name. All points on a meridian have the same longitude.

The difference in longitude *DLo* between two points with longitudes λ_2 and λ_1 is the angle at the north pole or the arc of the equator between their meridians. It is equal to $|\lambda_2-\lambda_1|$ when this is less than 180° or to 360° $-|\lambda_2-\lambda_1|$ otherwise.

(1) $$DLo = |\lambda_2-\lambda_1| \quad \text{or} \quad 360° - |\lambda_2-\lambda_1|$$

If both longitudes have the same name, *DLo* is found by subtraction. But if two longitudes have opposite names, one of them should have its name changed either by using its negative ($-\lambda$) or by using its explement $(360° - \lambda)$, and then the subtraction can be performed according to the rules for signs in algebra. Two longitudes that differ by 360° are considered equivalent.

EXAMPLE: Given the following three places:

Dakar	Lat. 14°40′ N.	Long. 17°25′ W.
Buenos Aires	Lat. 34°36′ S.	Long. 58°52′ W.
Yokohama	Lat. 35°27′ N.	Long. 139°40′ E.

Find the colatitude of each place and the differences in longitude between each two.

SOLUTION: The colatitudes are

$$\bar{L}_1 = 90° - 14°14' = 75°20'$$
$$\bar{L}_2 = 90° + 34°36' = 124°36'$$
$$\bar{L}_3 = 90° - 35°27' = 54°33'$$

The differences in longitude are

From Dakar to Buenos Aires: 58°52′ W. − 17°25′ W. = 41°27′ W.
From Dakar to Yokohama: 139°40′ E. − 17°25′ W. = 157°5′ E.
From Yokohama to Buenos Aires: 58°52′ W. − 139°40′ E.
= (360° − 58°52′ − 139°40′) E.
= (301°8′ − 139°40′)E. = 161°28′ E.

The position of a point on the earth's surface is described by its latitude and longitude to within the limitations of accuracy specified. The coordinates (40°42′43″ N., 70°00′29″ W.) describe the position of the City Hall in New York City to the nearest second, or 101 ft., of latitude, and to the nearest second, or 77 ft., of longitude (at the given latitude).

28·5 Bearing and course

When spherical coordinates are referred to any point C other than the poles, the first coordinate may be taken as the great-circle distance from C and the second as a direction, or bearing, from C, referred to a suitable prime direction (Fig. 86). A bearing is called (1) a **true bearing** or (2) a **magnetic bearing** or (3) a **compass**

bearing or (4) a **relative bearing,** according as the prime direction at the point is (1) the north direction on the meridian at C or (2) the direction of the north magnetic pole or (3) the direction of the north-

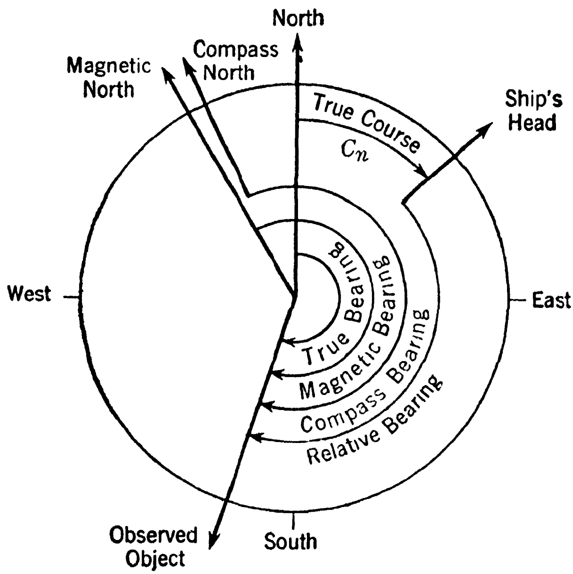


FIG. 86. Bearings.

seeking needle of a compass (which may be subject to local magnetism) or (4) any specified direction such as that of the ship's head. The true course of a ship is the true bearing of the ship's head. Bearings and courses are usually measured clockwise from the north toward the east from 0° to 360°. When so measured, the true course is denoted by C_n. If C_n exceeds 180°, we may use its explement $C=(360°-C_n)$W. instead. Thus $C_n=315°$ is equivalent to C = 45° W.

The system of coordinates based on the distance and bearing from a point is the basis for the **equidistant azimuthal** maps, sometimes seen in geography books and atlases, which will be discussed in Chap. 39. The term **azimuth** is another name for true bearing.

28·6 Parallel sailing and rhumb sailing

Navigation is concerned with determining the positions of ships at sea and of airplanes in flight and calculating their courses and their distances from known points. The easiest way to direct a ship is to sail on a constant course. If the course is 90° or 270°, the ship will sail east or west along a parallel of latitude. This is known as **parallel sailing,** and the distance in nautical miles, called **departure** (dep.), is given by the formula

(2) $$\textbf{Dep.} = (DLo) \cos L$$

since the ratio of departure to difference in longitude is equal to the ratio $r/R = \cos L$ of the radius r of the parallel circle to the radius R of the earth.

If the constant true course C_n is not a multiple of 90°, the ship will sail along a spiral curve called a **rhumb line,** which makes equal angles with all the meridians. On a Mercator chart[1] a rhumb line is mapped as a straight line, and a rhumb-line course between two points can easily be determined on such a chart with a straightedge. It is *not* a course of shortest distance between two points, like the great circle, but it approximates a great circle with sufficient accuracy to serve for practical purposes on short stretches. Distance on a rhumb line between points (L_1,λ_1) and (L_2,λ_2) is called the *Mercator distance* and is given exactly by the formula

(3) $$\text{M.dist.} = \left|\frac{L_2 - L_1}{\cos C}\right|$$

and approximately by the formula

(4) $$\text{M.dist.} = \sqrt{(L_2 - L_1)^2 + (\text{dep}_m)^2}$$

where the departure dep_m is computed for the mid-latitude $L_m = \frac{1}{2}(L_1 + L_2)$ or, better still, for a latitude slightly farther from the equator than mid-latitude. The first formula (3) is best when $\cos C$ is large, but formula (4) should be used when $\cos C$ is small.

28·7 Great-circle sailing

The great-circle track is the shortest distance between two points. For long journeys, especially in high latitudes, the great-circle track may

[1] See Chap. 39.

be considerably shorter than a single rhumb line. But if several points are first located on a great-circle track, the smaller segments of the track may then be approximated without great increase of distance by rhumb-line courses, which are easier to steer. A great-circle track may be plotted onto a Mercator chart with the aid of a so-called **gnomonic projection** on which great circles appear as straight lines, or it may be computed by solving spherical triangles by trigonometry. Except along the equator or along a meridian, the course on a great-circle track keeps changing. The points on a great circle farthest from the equator are called its **vertices**, and at these points the course is 90° or 270°. If the latitude of one vertex is L_v, then that of the other is $-L_v$ and the great circle crosses the equator at an angle L_v, with a bearing of $90° - L_v$ (Fig. 87).

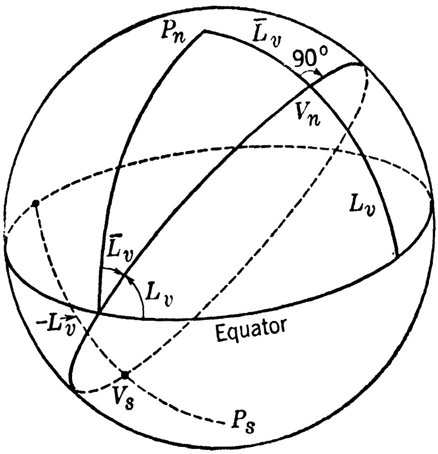


FIG. 87. Great circle track.

28·8 The terrestrial triangle

Any two points M_1 and M_2 on the earth with spherical coordinates (L_1,λ_1) and (L_2,λ_2) can be connected by a great-circle arc $\widehat{M_1M_2}$ whose length D is the distance between the points. If the points are on the same meridian, we have $D = |L_2 - L_1|$; and if the points are on meridians 180° apart, we have $D = 180° - (L_1 + L_2)$, where in both cases it is understood that north latitudes are positive and south latitudes neg-

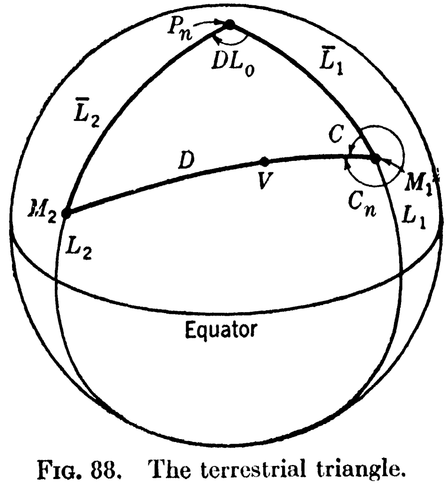


FIG. 88. The terrestrial triangle.

ative. If M_1 and M_2 are not in the same meridian plane, then M_1, M_2 and the north pole P_n are vertices of a spherical triangle $M_1P_nM_2$ called the **terrestrial triangle** (Fig. 88). The three sides of the triangle are

$$D, \bar{L}_2, \bar{L}_1 \tag{5}$$

where the angle included between $\bar{L}_2$ and $\bar{L}_1$ is *DLo*. The angle C at M_1 is called the initial course if M_2 lies to the east of M_1. Otherwise $360° - C$ is the initial course.

The exact computation of the course and great-circle distance between the two points M_1 and M_2 is based on spherical trigonometry and will not be considered in this text.

28. ORAL QUESTIONS

A. In what ways and by how much does the shape of the earth differ from that of an exact sphere?
B. Which diameter of the earth is shorter, an equatorial diameter or the polar diameter? By how much?
C. How is the nautical mile related to angular measure on the earth?
D. What is a knot? Why would it be incorrect to say that a ship was traveling at 10 knots per hr.?
E. What is the difference in longitude between two points if $\lambda_1 = 150°$ W. and $\lambda_2 = 140°$ E.?
F. What is the difference in latitude between two points if $L_1 = 41°$ N. and $L_2 = 34°$ S.?
G. What is a bearing? Name four kinds, and state the prime direction for each.
H. How do parallel and rhumb sailing differ from great-circle sailing? Which is used on short stretches? What are the advantages of each?
I. What is meant by the vertex of a great-circle track?
J. What is Mercator distance, and how can it be computed?
K. What are five of the parts of the terrestrial triangle?

28. WRITTEN EXERCISES

Latitudes and longitudes of eight places are given by the following table:

	Latitude	Longitude
Bergen, Norway	60°24′ N.	5°18′ E.
Boston, Mass.	42°20′ N.	70°53′ W.
Honolulu, Hawaii	21°19′ N.	157°52′ W.
Lisbon, Portugal	38°40′ N.	9°18′ W.
Recife, Brazil	22°54′ S.	43°10′ W.
Santiago, Chile	33°27′ S.	70°42′ W.
Seattle, Wash.	47°36′ N.	122°20′ W.
Wellington, New Zealand	41°17′ S.	174°47′ E.

1-4. Find the colatitudes of each of the following pairs of places and difference in longitude *DLo* between them.

1. Boston and Seattle.
2. Santiago and Wellington.
3. Recife and Bergen.
4. Honolulu and Lisbon.

5-8. Find the mid-latitude departure, difference in latitude, and approximate Mercator distance between each of the following, using the latitudes and longitudes given above and using the cosines of the mid-latitudes L_m as given. First convert degrees to nautical miles.

5. Boston and Bergen ($\cos L_m = 0.6243$).
6. Boston and Lisbon ($\cos L_m = 0.7604$).
7. Boston and Seattle ($\cos L_m = 0.7075$).
8. Seattle and Honolulu ($\cos L_m = 0.8245$).
9. Draw a sphere representing the earth in orthographic projection, and show on your figure the meridian of Greenwich, the equator, and the meridians and latitude circles for Boston and Bergen.
10. Draw a sphere representing the earth in orthographic projection, and show on your figure the equator, the 180th meridian, and the meridians and latitude circles for Honolulu and Seattle.

29

THE CELESTIAL SPHERE

29·1 The celestial sphere

If we look up at the heavens on a clear night, the stars appear to be points of light on a huge sphere that revolves to the westward about the polar axis of the earth at the rate of about 15° every hour. Because of the great distances of the stars from the earth, their positions relative to each other always appear the same—or almost the same—regardless of the motion of the stars themselves and regardless of the fact that the earth is journeying once a year around the sun on an orbit 186,000,000 miles in diameter. For this reason the stars are often referred to as the *fixed stars*, in contrast to the nearer bodies in the solar system such as the *sun*, the *moon*, and the *planets* (Greek, *wanderers*), whose positions relative to the distant stars appear to be changing continuously. For the purposes of navigation we are interested primarily, not in the distances to the stars and to other celestial bodies, but only in their directions with respect to an observer on the earth. For this reason we consider all celestial bodies to be projected onto a large sphere concentric with the earth, and we assign coordinates similar to latitude and longitude to each point of this sphere.

The **celestial sphere** is a spherical surface, concentric with the earth, whose radius is so large that the diameter of the earth's orbit is negligible by comparison. Parallel lines from all points within the earth's orbit meet the celestial sphere in a region so small compared with its distance from the earth that it appears to be a single point to an observer on the earth. If the celestial radius were chosen to be greater than a million times the diameter of the earth's orbit, then a region on the celestial sphere the size of the earth's orbit would subtend an angle of less than $\frac{1}{5}$ second of arc at the earth and could be considered a point for the purposes of navigation, since an accuracy of $\frac{1}{5}$ second on the earth's surface would locate an object within 20 ft. For theoretical purposes it is customary to consider the celestial radius to be infinite and to say that parallel lines meet the celestial sphere in the same point and parallel planes meet the celestial sphere in the same great circle. A distant celestial body B is considered to be projected into the same point M on the celestial sphere whether it is projected from the center of the earth or from an observer anywhere on the surface of the earth. But for bodies within the solar system a correction called **parallax** must be applied to allow for the small angular difference between the apparent position of the body as seen by an observer on the earth's surface and the apparent position as it would be projected from the center of the earth. The parallax of the nearer stars due to the motion of the earth in its orbit will not be considered here.

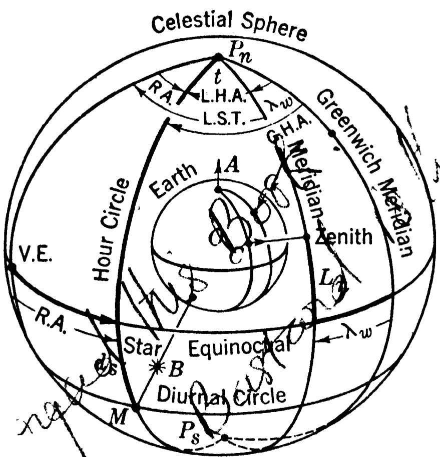


FIG. 89. The celestial sphere.

29·2 Declination and right ascension

The points where the polar axis of the earth pierces the celestial sphere are called the north and south celestial poles, P_n and P_s, respectively (Fig. 89). The great circle where the plane of the earth's equator meets the celestial sphere is called the **equinoctial,** or celestial equator, and is 90° distant from either pole. Great semicircles terminating at the north and south celestial poles and corresponding to meridians on the earth are called **hour circles.** Circles whose planes are parallel to the equinoctial are called **diurnal circles.** To latitude L on the earth corresponds **declination** d on the celestial sphere. The declination of a point M is its angular distance to the north or south of the equinoctial measured along its hour circle. The complement of the declination of M is an angle between 0° and 180° that measures the angular distance from P_n to the point M. It is called the **codeclination** $\bar{d}$ or **polar distance** p of M.

To east longitude $\lambda_{E.}$ on the earth corresponds right ascension (R.A.) on the celestial sphere. All points on a given hour circle have the same right ascension. Right ascension is measured eastward along the equinoctial from 0° to 360°, starting at a point called the vernal equinox (V.E.).

29·3 The equinoxes and the ecliptic

The **vernal equinox** is one of the two points where the equinoctial

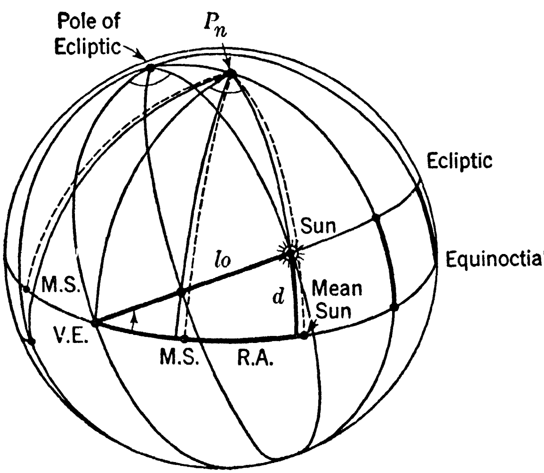


FIG. 90. Apparent annual motion of the sun and the mean sun.

cuts the plane of the earth's orbit. As the earth moves eastward around the sun once a year, the sun appears to an observer on the earth to move eastward around the celestial sphere once a year, following a great-circle path called the **ecliptic** (Fig. 90), which is the intersection of the celestial sphere with the plane of the earth's orbit. The planes of the equator and of the ecliptic make an angle with each other of 23°26′49″, called the **obliquity of the ecliptic.** During the winter months in the Northern Hemisphere, the sun appears south of the equinoctial, and it then has southern declination. At a certain instant in March (Mar. 20 or 21) when the sun crosses the equator, the lengths of the day and of the night become equal, (except for the effects of refraction of light by the atmosphere), and spring begins in the Northern Hemisphere. The sun is then at the vernal equinox V.E., and its right ascension and declination are both zero. For the next 6 months of spring and summer for the Northern Hemisphere, the sun will be in northern declination, reaching a maximum declination of 23°26′49″ N. at the Tropic of Cancer on the first day of summer (about June 21) and returning to zero declination at the autumnal equinox in September. The sun then goes south again, reaching the Tropic of Capricorn at 23°26′49″ S. about Dec. 21, and the Southern Hemisphere has its spring and summer while the Northern Hemisphere is having its fall and winter.

The sun does not advance quite uniformly in right ascension each day, partly because the earth's orbit about the sun is not quite circular, but chiefly because equal arcs on the ecliptic do not project into equal arcs on the equinoctial (Fig. 90). For reckoning mean solar time and determining the exact time of noon, a fictitious body called the **mean sun** is assumed to progress uniformly around the equinoctial in 1 year. A correction called the **equation of time**, which varies with the seasons and reaches a maximum of about 16 min.,[1] must be applied to observed times of sunrise, noon, and sunset to obtain the corresponding times for the mean sun. One effect of this is that in the Northern Hemisphere the earliest sunset comes about Dec. 7, the shortest period of daylight about Dec. 21, but the latest sunrise about Jan. 7.

29·4 Solar and sidereal time

Time is measured by the daily eastward rotation of the earth about its axis. This motion appears to an observer on the earth as a daily westward rotation of the celestial sphere and all celestial bodies. The sun's daily path across the sky depends on the season, as shown in Fig. 91 for an observer in the Northern Hemisphere. To avoid the necessity for correcting solar observations for seasonal variations

[1]According to our clocks the sun appears to be about 16 min. early near Nov. 5, 14 min. late near Feb. 11, 3 min. early near May 15, and 6 min. late near July 26.

such as the equation of time and the change in declination, it is common for astronomers and navigators to set their clocks by observations of the stars.

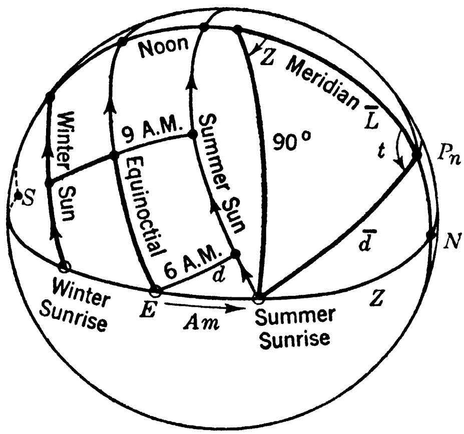


FIG. 91. Sun's daily apparent motion.

There are at least eight kinds of time used by navigators. Some of these are based on the sun (solar time), and some on the stars (sidereal time). One of several varieties of solar time, the **zone time** (Z.T.), is used in daily life. The length of a mean solar day is the average length of time between successive instants when the sun crosses the local meridian, and there are approximately 365.25 mean solar days in a year. During 1 year, however, each point of the celestial sphere has crossed the local meridian 366.25 times, so that there is 1 more sidereal day than solar day each year. Each solar day is about 4 min. longer than a sidereal day. For the earth, having completed one revolution with respect to the celestial sphere, requires 4 min. more to catch up to the sun, so to speak, since the sun has apparently moved forward in the meantime about 1° along the ecliptic in 24 hr. The sidereal day, then, is about 23 hr., 56 min., 4 sec. mean solar time, and this is equal to 24 sidereal hours.

Local sidereal time (L.S.T.) is defined to be the *right ascension of the local meridian*, and Greenwich sidereal time (G.S.T.) is defined to be the *right ascension of the meridian of Greenwich.* Both times may be measured in sidereal hours, minutes and seconds ($^{h.m.s.}$) from $0^h0^m0^s$ to $24^h0^m0^s$ or, in angular measure, from 0° to 360°. In most formulas involving time, two times that differ by 24^h or by 360° are to be considered equivalent. The dates will of course be different.

29·5 Local hour angle and longitude

The longitude of an observer may be found by computing the angle from the local meridian (L.M.) of the observer to the hour circle (H.C.) of a known star and noting the exact time of the observation. The angle measured westward from L.M. to H.C. is called the **local hour angle** of the star and is denoted by L.H.A.

The time of the observation determines the angle westward from the Greenwich meridian (G.M.) to the hour circle of the vernal equinox V.E. (or eastward from V.E. to G.M.). If the navigator's watch is set to Greenwich sidereal time (G.S.T.) then this can be converted directly to angular measure by changing each hour (1^h) to 15 degrees of arc (15°), each minute of time (1^m) to 15 minutes of arc (15′), and each second of time (1^s) to 15 seconds of arc (15″). Thus $13^h10^m3^s$ is equivalent to

$$(13\times 15)^\circ + (10\times 15)' + (3\times 15)'' = 195^\circ 150' 45'' = 197^\circ 30' 45''$$

Angles measured westward from the Greenwich meridian or local meridian to the observed body are called **Greenwich hour angle** (G.H.A.) or **local hour angle** (L.H.A.), respectively. The Greenwich hour angle and local hour angle of the vernal equinox, measured in time units are called, respectively, **Greenwich sidereal time** (G.S.T.) and **local sidereal time** (L.S.T.). Relationships between these four quantities, right ascension R.A., and longitude west λ_W. may be seen from the following table:

	G.M.	L.M.	H.C.	V.E.
Greenwich meridian......	0	λ_W	G.H.A.	G.S.T.
Local meridian..........	λ_E.	0	L.H.A.	L.S.T.
Star's hour circle.........	360° − G.H.A.	360° − L.H.A.	0	R.A.
Vernal equinox..........	24^h − G.S.T.	24^h − L.S.T.	24^h − R.A.	0

The angles are measured westward from the meridian or hour circle at the left of the row to the meridian or hour circle at the top of the column.

Thus from the star's H.C. west to the local meridian L.M. is 360°—L.H.A. If we pick any two rows and any two columns of this table, then, among the four angles so obtained, the sum of a pair that are in different rows and columns is equal to the sum of the other pair (or differs from it by 360° or 24^h). Thus, taking the first and third rows and the second and fourth columns.

$$\lambda_{w.} + \text{R.A.} = \text{G.S.T.} + (360° - \text{L.H.A.}) \pm \text{ a multiple of } 360°$$

EXAMPLE: The star Sirius in R.A. $6^h42^m48.4^s$ was observed at G.S.T. $4^h20^m40.0^s$, and its local hour angle was found to be L.H.A. $=272°42'33''$. Find the longitude of the observer.

SOLUTION: First find G.S.T. − R.A., after adding 24^h to G.S.T. to make the subtraction possible:

$$\begin{aligned} \text{G.S.T.} &= 28^h20^m40.0^s \\ \text{R.A.} &= \underline{\ 6^h42^m48.4^s} \\ \text{Diff.} &= 21^h37^m51.6^s = (21\times15)° + (37\times15)' + (51.6\times15)'' \\ &= 315° + 9°15' + 12'54'' = 324°27'54'' \end{aligned}$$

From this difference we subtract L.H.A. $= \underline{272°42'33''}$

$$\lambda_w = 51°45'21''$$

Thus the observer was in long. 51°45′21″ W.

29·6 Altitude and azimuth

A second system of spherical coordinates on the celestial sphere, called the **altitude-azimuth system** or the horizon system (Fig. 92), uses the **zenith** point Z as a reference point. The zenith Z is the point on the celestial sphere, vertically above the observer, and the **nadir** Na is the diametrically opposite invisible point below the observer. The great circle of which these two points are the poles is called the **horizon.** It is the circle in which the tangent plane to the earth at the point of observation meets the celestial sphere. Great circles through the zenith and nadir are called **vertical circles** and are perpendicular to the horizon. One of these coincides with the meridian and passes through the north and south points N. and S., respectively. On this meridian circle the pole P_n or P_s that appears above the horizon is called the **elevated pole.** The vertical circle perpendicular to the meridian is called the **prime vertical** and meets the horizon in the east and west points, E and W, respectively. The angular distance z of a body M from the zenith is called the **zenith distance** and is usually measured from 0° to 180°. (An exceptional case in which z may be considered negative when measured south along the meridian will be discussed later.) The complement of the zenith distance is called the **altitude** h of a body and measures angular distances above or below the horizon. The direction coordinate in this system is called the **azimuth** Z_n (an Arabic word for true bearing). It is measured clockwise about Z from 0° to 360°, starting with the north direction of the meridian through Z. The azimuths of the four **cardinal points** N, E, S, W on the horizon are 0°, 90°, 180°, 270°, respectively. Azimuth is often measured as an arc on the horizon or as a dihedral angle from the meridian plane to a vertical half plane through the observed point, instead of as an angle at Z. These definitions are obviously equivalent.

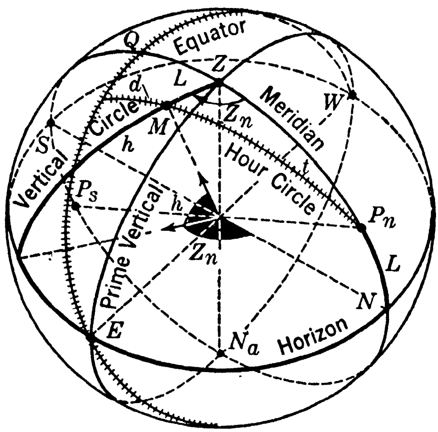


FIG. 92. Altitude and azimuth.

The zenith distance and azimuth on the celestial sphere correspond to the distance and course (or bearing) on the terrestrial sphere in the same way that codeclination and right ascension on the celestial sphere correspond to colatitude and east longitude on the terrestrial sphere. Important relations connecting two of these coordinate systems are these: *The latitude of the observer is equal to the declination of the zenith and is also equal to the altitude of the elevated pole* (that is, the pole which is above the horizon).

29·7 Calculation of latitude from meridian altitude

The meridian altitude of a celestial body is its altitude at the instant of its transit when it crosses the meridian of the observer.

The body is then said to be in **upper culmination.** The difference between the declination of the body and the declination of the

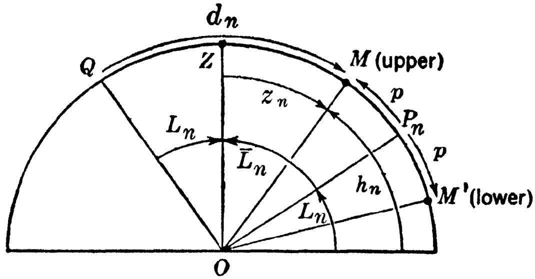


FIG. 93. Meridian altitude.

zenith is easily seen to be the zenith distance or coaltitude of the body (Fig. 93), provided that all these quantities are measured positively to the north and negatively to the south. For if Q is the point where the meridian crosses the equator, then we have, using directed angles,

(1) $$\measuredangle QOM = \measuredangle QOZ + \measuredangle ZOM$$

(2) $$d_{n.} = L_{n.} + z_{n.}$$

The subscript n. refers to the fact that these angles are to be measured positive to the north and negative to the south. If we write $\bar{h}_{n.}$ for $z_{n.}$, and solve for $L_{n.}$ we have,

(3) $$L_{n.} = d_{n.} - \bar{h}_{n.}$$

For bodies crossing the meridian north of the zenith we have

(4) $$L_{n.} = d_{n.} + h_{n.} - 90°$$

For bodies crossing the meridian south of the zenith we replace $-\bar{h}_{n.}$ by $+\bar{h}_{s.}$ and obtain

(5) $$L_{n.} = d_{n.} + 90° - h_{s.}$$

In each case $d_{n.}$ or $L_{n.}$ is to be considered negative for south declination or south latitude.

EXAMPLE 1: To find the latitude of the observer if the star Regulus with declination 12°16′ N. is observed in upper culmination with meridian altitude 64°25′ bearing north.

SOLUTION:

$$\bar{h} = 90° - 64°25' = 25°35'$$
$$L = +12°16' - 25°35' = -13°19' \text{ or } 13°19' \text{ S.}$$

EXAMPLE 2: To find the latitude of the observer if the star Fomalhaut with declination 29°57′ S. is observed in upper culmination with meridian altitude 18°13′ bearing south.

SOLUTION:

$$h_{n.} = -(90° - 18°13') = -71°47'$$
$$L = -29°57' + 71°47' = +41°50' \text{ N.}$$

29·8 Lower culmination and the midnight sun

A meridian was defined to be a great semicircle terminating at the poles. However, the other half of the same great circle is sometimes called the **lower branch** of the meridian, and the meridian semicircle itself is called the upper branch.

A body is said to be in lower culmination at the instant when it crosses (transits) the lower branch of the meridian. It will be visible in lower culmination only if it is within a small circle about the pole that is tangent to the horizon. This circle is known as the **circle of perpetual apparition.** The declinations of all points in this circle are of the same name as the latitude of the observer and exceed the polar distance of the observer. In lower culmination, the angle, $\measuredangle QOM$ in Eq. (1) is not the declination d but its supplement $\measuredangle QOM' = 180° - d$. To compute latitude by observing a body in lower culmination, subtract the complement of h from the supplement of d, and attach appropriate signs, + for north and − for south, not to d and h, but to $180° - d$ and $90° - h$. If the latitude comes out greater than 90°, then the observation was impossible. Otherwise, the sign of the latitude determines whether the observer is north or south of the equator.

EXAMPLE: Find the latitude of the observer if the star Arcturus, with declination 19°30′ N., is observed in lower culmination with meridian altitude 24°20′ bearing north.

SOLUTION:

$$180° - d = 160°30' \qquad \bar{h} = 65°40'$$
$$L = 160°30' - 65°40' = 94°50'$$

No solution is possible. The star could never have that altitude in lower culmination.

Midnight (local apparent time) is the time of the lower culmination of the sun. The region on the earth where the sun is sometimes visible in lower culmination is popularly known as the land of the **midnight sun.** The latitude of a place where the sun makes its lower transit on the horizon is equal to the codeclination of the sun at the given date. At the northern summer solstice about June 21, all points north of the Arctic Circle at lat. 66°33′ N. have 24 hr. of continuous daylight, while all points south of the Antarctic Circle at lat. 66°33′ S. have 24 hr. continuous darkness, except for the "sunset" glow that may appear near the circle itself. At the winter solstice for the Northern Hemisphere, about Dec. 21, the situation is reversed.

29·9 The celestial triangle

It is not often possible to measure the local hour angle of a star directly with any degree of accuracy. Instead, observations of altitude are made directly on one or more stars. Then the local hour angle can be computed by solving with the aid of trigonometry a spherical triangle on the celestial sphere that is known as the **celestial triangle.** Vertices of the celestial triangle are the projection M of the star, the celestial pole P_n (or P_s) and the zenith Z of the observer (Fig. 94). Sides opposite these vertices are $\bar{L}$, $\bar{h}$, and $\bar{d}$, the complements of the observer's latitude and the star's

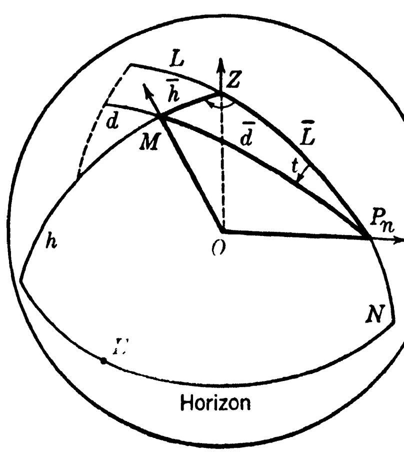


FIG. 94. The celestial triangle.

altitude and declination. Angles at P_n and Z are the local hour angle and the azimuth if these angles are less than 180°, or 360° minus the corresponding angle in the contrary case.

Tables have been published that reduce the solution of such triangles to a problem of interpolation.[1]

29. ORAL QUESTIONS

A. How are the following related to each other on the celestial sphere: hour circles; the equinoctial; diurnal circles; the poles?

B. How are right ascension and declination defined? Which corresponds to longitude on the earth and which to latitude?

C. At what point on the equinoctial is the right ascension equal to zero?

D. What is the ecliptic? What is the equation of time?

E. How many sidereal days are there in one year? Is the sidereal day longer or shorter than the solar day?

F. What name is given to the right ascension of the local meridian? To the right ascension of the Greenwich meridian?

G. What are the values in arc measure of each of the following sidereal times: 6^h; 1^h30^m 20^m; 15^s; 2^m30^s?

H. In west longitude which time is earlier (smaller) and which later (larger), local sidereal time or Greenwich sidereal time? Assume that the date is the same in both places.

I. If an observer in long. 45° E. observes a body with right ascension 23^h, at 17^h Greenwich sidereal time, what is the local hour angle?

J. What great circle has the zenith as one pole? What is its other pole?

K. For what point or points on the earth's surface are the altitudes of all stars equal to their declinations?

L. What are the azimuths corresponding to the following directions: east; west; southwest; northwest; south; west; northwest; north-by-east?

M. What is the relationship between the following quantities:

[1] Such tables are HO 214 and HO 218 published by the Hydrographic Office, Washington, D.C.

latitude of the observer; altitude of the elevated pole; declination of the zenith?

N. What are the meanings of the following terms: transit; upper culmination; lower culmination; lower branch of a meridian?

O. When do the summer and winter solstices occur? At summer solstice, what is the meridian altitude of the sun for an observer (*a*) at the equator? (*b*) At a point near the north pole? (*c*) At a point on the Antarctic Circle?

P. If the declination of a star is $-59°$, how far north of the equator is it visible?

29. WRITTEN EXERCISES

1- 8. In each of Exercises 1 to 8, three of the six quantities R.A., G.S.T., L.S.T., G.H.A., L.H.A., and λ are given. Find the other three.

1. R.A. $= 11^h0^m12.0^s$, G.S.T. $= 3^h10^m36.0^s$, L.H.A. $= 300°00'10''$.
2. R.A. $= 22^h54^m38.2^s$, L.S.T. $= 20^h32^m24.0^s$, G.H.A. $= 4°21'10''$.
3. R.A. $= 0^h5^m32.6^s$, G.H.A. $= 206°20'15''$, L.H.A. $= 105°20'00''$.
4. R.A. $= 18^h36^m5.4^s$, G.H.A. $= 95°54'45''$, L.H.A. $= 30°24'30''$.
5. R.A. $= 23^h2^m0^s$, G.S.T. $= 0^h6^m10.0^s$, $\lambda = 175°20'40''$ E.
6. R.A. $= 15^h32^m19.0^s$, L.S.T. $= 19^h42^m42.0^s$, $\lambda = 125°36'00''$ W.
7. R.A. $= 0^h40^m48.8^s$, G.H.A. $= 50°35'24''$, $\lambda = 32°15'54''$ E.
8. G.S.T. $= 8^h29^m36.8^s$, L.H.A. $= 315°18'54''$, $\lambda = 71°24'24''$ W.

9-16. In each of Exercises 9 to 16, find the latitude of the observer if a star with the given declination is observed at the given meridian altitude and bearing.

	d	*h*	*Bearing*
9.	28°47′ N.	71°28′	N.
10.	58°51′ N.	20°35′	N.
11.	18°17′ S.	28°32′	N.
12.	26°18′ S.	84°42′	N.
13.	10°52′ S.	32°48′	S.
14.	26°54′ N.	68°25′	S.
15.	57°31′ S.	45°50′	S.
16.	16°24′ N.	10°54′	S.

17-20. In Exercises 17 to 20 find the latitude of the observer if a star with the given declination is observed in lower culmination at the given meridian altitude and bearing.

	d	*h*	*Bearing*
17.	59°57′ N.	12°07′	N.
18.	62°47′ S.	30°52′	S.
19.	29°55′ S.	8°15′	S.
20.	45°05′ N.	34°20′	S.

30

REVIEW OF THE SPHERE

30·1 The sphere and its circles

The points in three-dimensional space that are at a fixed distance r from a point O form a spherical surface of radius r. The set of all points at distances less than r from O form the interior of a sphere of radius r. The intersection of a sphere with a plane is a great circle if the plane passes through the center of the sphere, a small circle if the plane contains interior points of the sphere but does not pass through the center, or a point if the plane is tangent to the sphere.

If a particular diameter of the sphere is selected as an axis, then the half planes through the axis cut the sphere in great semicircles called meridian circles (or hour circles on the celestial sphere), all meeting at two points called the poles. Planes perpendicular to the axis intersect the sphere in circles called parallel circles or latitude circles, which cut the meridians at right angles. One of these latitude circles is a great circle called the equator (or the equinoctial on the celestial sphere), and the others are small circles. Longitude on the earth (and right ascension on the celestial sphere) may be measured either as an arc on the equator (equinoctial) or as an angle between meridians (hour circles) at the poles.

On the celestial sphere a second system of intersecting circles is important, namely, the azimuth-altitude system. The poles in this system are the zenith Z (straight up) and the nadir (straight down). The horizon plays the role of the equator, and vertical circles play the role of meridians. To longitude corresponds azimuth, measured as an angle at the zenith or an arc on the horizon, starting from the north and increasing from north to east to south to west. To latitude corresponds the altitude, measured up from the horizon on a vertical circle.

30·2 Areas on the sphere

The area of the zone cut from a spherical surface by two parallel planes is the same as the area cut by these planes from a circumscribed cylindrical surface whose bases are parallel to these planes. This area is the product of the height by the circumference of a great circle.

$$Z = 2\pi Rh \tag{1}$$

Since the height of the whole sphere is $2R$, the area of the surface of a sphere is $4\pi R^2$, or four times the area of a great circle.

For surfaces of revolution in general, Pappus's theorem states that the area S is the product of the length l of the generating curve times the circumference of the circle through which its centroid moves as the surface is generated.

$$S = 2\pi c \cdot l \tag{2}$$

Areas of lunes on a sphere are proportional to their angles, the ratio being $\pi R^2/90°$. In the same ratio the areas of spherical triangles and polygons are proportional to their semi-excesses. The semi-excess σ of a spherical triangle is one-half the amount by which the sum of the angles $\alpha+\beta+\gamma$ exceeds 180°. The semi-excess of a spherical polygon of n sides is one-half the amount by which the sum of its angles exceeds $(n-2)$ 180°. (Note that, for $n=2$, the same formula gives the angle of a lune.)

30·3 Volumes of revolution

The volume of a segment of a sphere cut by parallel planes is the difference between the volumes of a cylinder and a frustum of a cone and is therefore given exactly by the prismoidal formula It can also be expressed as the average of the volumes of two cylinders of the same height h as the segment, each having one of the bases of the segment as its base, plus the volume of a sphere of diameter h. The volume of a sphere of radius R is

(3) $$V=\tfrac{4}{3}\pi R^3$$

The volumes of spherical sectors and pyramids are each equal to one-third the radius of the sphere times the area of the base.

For solids of revolution in general, Pappus's theorem states that the volume V is equal to the product of the area A of the generating plane region times the distance through which its centroid moves as the solid is generated.

(4) $$V=2\pi c\cdot A$$

30·4 Geometry of the spherical triangle

Three great circles not having a common diameter cut the sphere into eight associated spherical triangles of which opposite pairs have equal areas. The semi-excesses of the four pairs are σ, $\alpha-\sigma$, $\beta-\sigma$, $\gamma-\sigma$, respectively, where α, β, γ are the angles of the triangle, and all these quantities must lie between 0° and 180°. The poles of the three great circles are vertices of a set of eight associated polar triangles, opposite pairs of which have semi-excesses of $180°-s$, $s-a$, $s-b$, $s-c$, where a,b,c are the sides and s is the semiperimeter of the given triangle. These likewise must lie between 0° and 180°.

Further conditions on the angles of a spherical triangle are imposed by the fact that of two unequal sides the greater lies opposite the greater angle and the one nearer 90° lies opposite the angle nearer to 90°. The exact determination of three unknown parts of a spherical triangle when three parts are given is the subject of spherical trigonometry, which has important applications in navigation, astronomy, and geography.

30. ORAL QUESTIONS

A. Given a particular diameter of a sphere as axis, what circles of the sphere are coplanar with the axis, and what circles are in planes perpendicular to the axis?

B. How are distances measured on a sphere? Does the triangle inequality $\overline{AB}+\overline{BC}\geq\overline{AC}$ held for this measurement of distance?

C. When a sphere is drawn in orthographic projection, which of its circles are drawn as circles? What curves must be drawn to represent the other circles on the sphere?

D. What are spherical segments and zones, and how are they measured?

E. What is the altitude of a zone whose area is one-fourth the area of the sphere?

F. What is meant by the centroid of a plane area? Of a plane curve?

G. What are the theorems of Pappus?

H. What is the semi-excess of a spherical triangle having angles of 60°, 70°, 80°? What is its area on a sphere of unit radius?

I. What is the semi-excess of a spherical polygon having six angles each equal to 150°? How does its area compare with that of a great circle?

J. What are three inequalities that must be satisfied by the angles of a spherical triangle?

K. What is meant by *parallel sailing* on the earth? What are rhumb lines and Mercator distance?

L. How can the longitude of an observer be computed from the local hour angle of an observed star? What other data are required?

M. How can an observer compute his latitude by an observation of meridian altitude of a celestial body?

30. WRITTEN EXERCISES

1. Find the surface area of a spherical surface whose radius is 4,000 miles. If the earth's surface is 2 per cent smaller than this and if 29 per cent of it is land, find the total land area.
2. Find the total volume of the earth in billions of cubic miles, assuming that the volume is 3 per cent less than that of a sphere of 4,000-mile radius. Find the total mass of the earth if on the average a cubic mile of earth has a mass of 25,400,000,000 tons (1 ton = 2,000 lb.). (NOTE: This average density is about twice the density at the surface.)
3. Prove that tangents drawn to a sphere from an external point are equal.

4. Draw a figure in orthographic projection showing a sphere, a small circle of the sphere, and a cone of tangent lines making contact on the small circle.
5. Two sections of a sphere of radius 10 in. are made by parallel planes 2 in. apart that are 6 in. and 8 in., respectively, from the center of the sphere. Find the volume and total surface area of the solid bounded by the two planes and by the zone of the sphere between them.
6. Draw a figure illustrating Exercise 5.
7. A spherical pyramid whose base has four 135° angles has a radius of 2 ft. A spherical sector of the same radius and volume has a base that is a zone of the sphere. Find the height of this zone.
8. A great circle of a sphere and a diameter $[ZZ^*]$ of the sphere perpendicular to the plane of the circle are projected orthographically. Prove that the difference of the squares of the two axes of the ellipse into which the circle projects is equal to the square of the projection of the diameter $[ZZ^*]$.
9. A piece of metal pipe whose inner and outer diameters are 1 in. and 1.5 in., respectively, forms a quadrant bend that is one-quarter of the solid of revolution obtained by revolving the cross-section area of the pipe about an axis 8 in. from its center. Find the volume of the metal.
10. Find the total area of the pipe of Exercise 9, including the areas of the inner and outer surfaces of revolution and the two flat ends.
11. The angles of a spherical triangle are given as 100°, 120°, 140°. Find the semi-excesses of the given triangle and of each of the three colunar associated triangles and find the sides of the polar triangle.
12. If the New York City Hall is at lat. 40°42′43″ N. long. 74°00′29″ W. and the Paris Observatory is at lat. 48°50′14″ N. and long. 2°20′14″ E., find the difference in longitude between these points. Using an approximate mid-latitude of 45°, find the departure and the Mercator distance.

PART FOUR

PROJECTIONS AND MAPS

31

THE THEORY OF PARALLEL PROJECTIONS

31·1 The third dimension

No single two-dimensional picture can completely portray a three-dimensional object unless it is known that the object is regular to such an extent that the parts which are not depicted can be inferred from those which are. Various devices are used to create the impression of depth. In ordinary binocular vision the two eyes see two distinct pictures of the same scene, in which nearer objects appear slightly displaced relative to the background of distant objects, so that if the eyes are focused on distant objects, a double image will be seen of a near object such as a finger or a pencil that is placed in the line of sight.

The stereoscope is an effective means of creating the same impression, by causing the two eyes to look at two different pictures taken from points a short distance apart. The feeling of depth may even be greatly accentuated by taking the two pictures from points not just $2\frac{1}{2}$ in. apart, like a pair of eyes, but several hundred feet apart, as is done in some aerial photographic work. But stereoscopic pictures require elaborate viewing apparatus, they are not easily constructed mechanically, and they are not easily measured for actual numerical dimensions. Therefore other methods are used in engineering drawing, in blackboard drawings, and in textbook figures that do not have these particular disadvantages. However, in working with figures drawn by these other methods, the student must exert some extra mental effort to reconstruct in his own mind the feeling of depth that the stereoscope gives directly.

31·2 Perspective projections

Two-dimensional views of a three-dimensional object are called **projections** when they are constructed according to certain definite geometric principles. Perspective projections portray an object as it would appear from a certain point of view, called the **center of perspective.** The projection lines, which join the points of a three-dimensional object to the corresponding points in its plane of projection, are called **visual rays.** In perspective projections the visual rays are concurrent in the center of perspective. In parallel projections, the center of perspective is thought of as being at infinite distance, and the visual rays are parallel.

Drawings in parallel projection produce a satisfactory impression if viewed from a distance that is large in comparison with the greatest dimensions of the figure that is drawn. Drawings in true perspective produce a satisfactory impression if viewed from a distance that is approximately equal to the distance from the center of perspective to the plane of projection—the so-called "focal length" of the drawing. Some details of the theory of central perspective will be considered in Chap. 36. Since parallel projection is easier to draw than central perspective, it is generally used for blackboard drawings and textbook figures in solid geometry.

31·3 Oblique and orthographic projections

Parallel projections are classified as **oblique** or **orthographic** according as the projection lines (visual rays) are oblique or perpendicular to the plane of projection. Let the lines of a solid object be represented by a framework of wires, and let rays from the sun (or from a distant light) cast a shadow of this framework on a plane. Then if the rays (which are considered parallel) are oblique to the projection plane, the shadow forms an **oblique projection,** whereas if they are perpendicular to the projection plane the shadow forms an **orthographic projection.** Both types of parallel projection share the following important properties:

PROPERTY 1: *Parallel lines project into parallel lines, and intersecting* *lines project into intersecting lines, provided that none of the lines considered are parallel to the visual rays.*

PROPERTY 2: *Lines parallel to the visual rays project into points.*

PROPERTY 3: *The ratio of two segments on the same or on parallel lines is preserved under parallel projection.*

The orthographic projection is somewhat more pleasing to the eye than the oblique projection, since it represents a view looking directly at the object, whereas the oblique projection represents more nearly a view out of the corner of the eye, so to speak. With the aid of a device such as the trimetric ruler, drawings in orthographic projection can be made as easily as in oblique projection. Without the aid of such a device, however, the single-plane orthographic projection is so much more complicated to draw than the oblique projection that most solid-geometry teachers and textbook writers are accustomed to use the oblique projection in their drawings.

31·4 Axes of reference

We have seen in Chaps. 6 and 9 that three mutually perpendicular directed lines $\uparrow O(X)$, $\uparrow O(Y)$, and $\uparrow O(Z)$ in space can be chosen as axes of reference and that the position of any point P in space can be described in terms of its coordinates (x,y,z) referred to these axes. If P_x, P_y, and P_z are the orthogonal projections of P on the three axes and P_{yz}, P_{zx}, P_{xy} are its projections on the three coordinate planes (YOZ), (ZOX), and (XOY), respectively, then the coordinates (x,y,z) are given as follows in terms of the directed lengths of certain projections (Figs. 13 and 95):

$$(1)\qquad \begin{aligned} x&=\overrightarrow{OP_x}=\overrightarrow{P_yP_{xy}}=\overrightarrow{P_zP_{zx}}=\overrightarrow{P_{yz}P}\\ y&=\overrightarrow{OP_y}=\overrightarrow{P_zP_{yz}}=\overrightarrow{P_xP_{xy}}=\overrightarrow{P_{zx}P}\\ z&=\overrightarrow{OP_z}=\overrightarrow{P_xP_{zx}}=\overrightarrow{P_yP_{yz}}=\overrightarrow{P_{xy}P}\end{aligned}$$

Points having all three coordinates positive are said to lie in the **first octant.**

All this is true both for the oblique and for the orthographic projections. The difference between the two results from the manner of projecting the unit vectors **i**, **j**, and **k** along the axes.

31·5 Axes in oblique projection

In oblique projection the projection plane is chosen as one of the reference planes—usually the obverse vertical (or front vertical) plane (YOZ). All line segments in planes parallel to the projection plane are represented in their true proportions, and angles between them are preserved. Line segments parallel to the third axis, and therefore perpendicular to the projection plane, are drawn as oblique parallel lines with foreshortened lengths. Angles whose planes are not parallel to the projection plane are generally distorted.

Unit vectors **j** and **k** along $\uparrow O(Y)$ and $\uparrow O(Z)$ in space are represented as horizontal and vertical unit vectors, respectively, in the projection plane. The unit vector **i** along $\uparrow O(X)$ is projected into an oblique vector **i**′, which may be drawn down to the left with any convenient direction and length whatever. This line might be pictured as the shadow on a north-wall blackboard of a line perpendicular to the blackboard, cast by rays of the morning sun coming in a window from a southeasterly direction.[1]

If the direction components of the projection rays are taken as $(1,v,w)$, then the point with these coordinates projects into O; therefore we have

$$\mathbf{i}'+v\mathbf{j}+w\mathbf{k}=0 \qquad \text{or} \qquad \mathbf{i}'=-v\mathbf{j}-w\mathbf{k} \tag{2}$$

The length of a projected unit along $\uparrow O(X)$ will be denoted by u.

$$u=\sqrt{v^2+w^2} \tag{3}$$

If $90°+\gamma$ is the angle between the vectors **i**′ and **j**, then $\cos\gamma=w/u$.

The choice of v and w is arbitrary, except that the figure looks more realistic if u is less than 1. Furthermore a choice of v and w should be avoided for which $(1,v,w)$ are components of any line joining two vertices or important points in the figure. Three convenient types of oblique projection are the following:

Type A: $v=\frac{1}{2}$	$w=\frac{1}{2}$	$u=\frac{1}{2}\sqrt{2}$	$\gamma=45°$
Type B: $v=\frac{1}{4}$	$w=\frac{1}{2}$	$u=\frac{1}{4}\sqrt{5}$	$\gamma=26°34'$
Type C: $v=\frac{1}{4}$	$w=\frac{1}{4}\sqrt{3}$	$u=\frac{1}{2}$	$\gamma=30°$

[1] It is assumed that the illustration refers to a classroom in the North Temperate Zone of the earth. Otherwise, the sun might not be in the south.

31·6 Representation of points, lines, and planes in oblique projection

A point P can be represented in oblique projection by drawing the projections of the segments $[OP_x]$, $[P_xP_{xy}]$, and $[P_{xy}P]$ (Fig. 95). The first of these will be x oblique units or ux true units along the oblique axis from O, the second will be y true units parallel to $\uparrow (OY)$, and the third will be z true units parallel to $\uparrow (OZ)$.

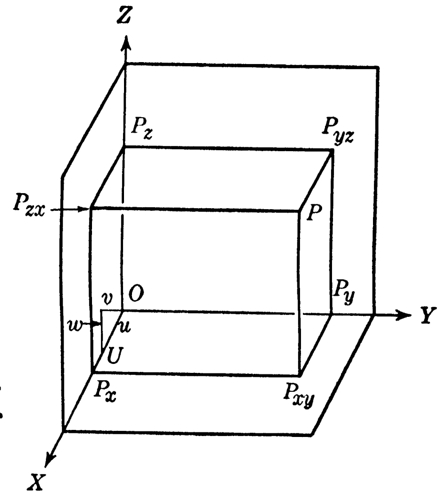


FIG. 95. Oblique projection.

Positive coordinates are measured in the same directions as the axes, negative coordinates in the opposite direction. Lines are represented by first representing any two points on the line and then connecting them by a line. It is often convenient to choose one or both of the two points in faces of the trihedron of reference.

A plane may be represented in parallel projection by the triangle in which it cuts the faces of the trihedron of reference, if it does, or by the projection of a parallelogram or polygon or curve lying in the plane.

Care should be taken in choosing the direction and scale of the oblique axis so that important lines in the object and lines joining important points are not parallel to the projection lines. Thus, in drawing a cube, type A should be avoided; in working with equilateral triangles or regular hexagons, type C should be avoided. Type B would be bad for plotting the point (4,2,1), since this point will cover the origin.

Finally, when the drawing has been completed, those lines in a figure which are supposed to be visible are drawn in heavily, whereas lines rendered invisible through shielding by the solid are indicated lightly by dotted lines. If two visible skew lines appear to intersect at a point in the drawing, the one that is behind should be broken at that point.

EXAMPLE: To draw a cuboctahedron in oblique projection (Fig. 96).

SOLUTION: A cube is to be drawn, the mid-points of its edges are then to be marked, and these are to be connected by lines. To avoid undesirable coincidences of points, projections of types A and B should be avoided. Type C is quite satisfactory. Having decided on the type and located the unit points on the axes, draw light lines in pencil through these points parallel to the axes and of unit lengths, complete the cube by drawing in the parallel edges, and mark the mid-points of the edges of the cube. Draw the 24 edges of the cuboctahedron through these 12 points, making sure that lines which are supposed to be parallel are drawn parallel. Then draw in heavily in ink those lines which are supposed to be seen, and dot in the lines which would be hidden behind the solid. The axes of reference and the edges of the cube can then be erased if so desired.

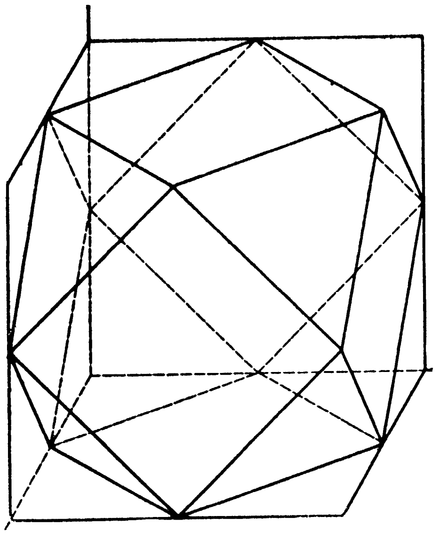


FIG. 96. Cuboctahedron in oblique projection.

31. ORAL QUESTIONS

A. What are some geometric means of producing the effect of depth in a picture? Do such things as shadows and color intensities supplement or alter the effects of geometric perspective?

B. What is meant by the focal length of a perspective drawing?

C. What is the difference between parallel projection and central projection?

D. What are three important properties of parallel projections?

E. How can rectangular coordinates for a point be defined?

F. Are there any reasons for not always using the same reference system in oblique projection?

G. In what sense may an oblique projection be likened to a shadow?
H. How can a reference system for oblique projection be constructed?
I. How can points and lines be located in oblique projection?
J. Are any plane figures left unchanged under oblique projection? If so, which?
K. In oblique projection, are ratios of segments preserved on parallel lines? On arbitrary nonparallel lines? On some pair of nonparallel lines?

31. WRITTEN EXERCISES

1. Draw an oblique reference system of type A, and locate on it the points whose coordinates are as follows: (1,1,0); (1,1,1); (2,4,−1); (−2,4,1); (2,1,1). Why does the last point cause difficulty?
2. In each of the three listed types A, B, C of oblique projection, draw a cube whose edges are each 2 in. long, and subdivide it into eight equal cubes by dotted lines. Which of these figures is preferable, and why?
3. Draw a 2-in. cube in oblique projection (type B), placing it with one of its diagonals along (OZ), and with four vertices in the plane (YOZ). Draw in the three sections of this solid by planes perpendicular to the given diagonal at its mid-point and at its quarter points.
4. Draw in oblique projection (type B) a regular tetrahedron with 3-in. edges, and locate its center.
5. Draw in oblique projection (type A) a tetrahedron whose base has edges of lengths 3,4,5 and whose other three equal slant edges have lengths of 6.5. Choose $\frac{1}{2}$ in. as a unit. HINT: First find the circumcenter of the base.
6. Draw in oblique projection (type C) a regular pyramid whose altitude is 4 in. and whose base is a 6-in. square.
7. Draw in oblique projection (type A) a figure showing a regular hexagonal pyramid cut by a plane parallel to the base. Let the sides of the base hexagon be each 1 in. and the altitude 3 in. Take the cutting plane at any convenient height.
8. Draw in oblique projection (type A) a regular octahedron with

2-in. edges, placing the figure so that one of the faces is in a horizontal plane and one edge lies along (OY).

9. Draw in oblique projection a figure for the fireplace of Exercise 1, Chap. 14.
10. Draw in oblique projection a figure for the house of Exercise 3, Chap. 14.
11. Draw in oblique projection a figure for the crystal of Exercise 4, Chap. 14.
12. Draw in oblique projection a figure for the cube and prism of Exercise 5, Chap. 14.

32

THE ORTHOGRAPHIC PROJECTIONS

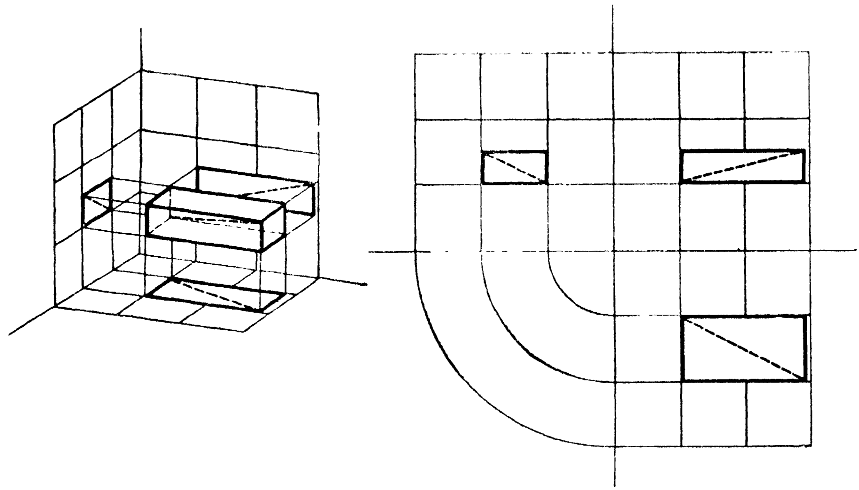
32·1 Types of orthographic projection

Orthographic projections are projections in which the visual rays are orthogonal to the projection plane. Of the two principal types the one represents an object by two or three projections on mutually perpendicular planes, one of which is horizontal and the others vertical (though these are occasionally supplemented by auxiliary projections in depicting complicated structures). The other, technically called the **axonometric projection,** is the one that represents an object by a single projection on a plane so chosen as not to be perpendicular to any of the important lines of the object. Its use is greatly facilitated by the trimetric ruler.

32·2 Three-plane orthographic projection

In the three-plane orthographic projection one of the views is a projection from the front or back onto the obverse vertical plane (YOZ) and is commonly called a **front elevation** (as in Fig. 113). A second vertical view is a projection from the right or left onto the profile plane (ZOX) and is called a **side elevation.** The third view, which is a projection from above or below onto the horizontal plane (XOY), is called a **plan.** In the case of a first octant projection (which is one of eight possible types) the views are from in front, from the right, and from above, onto the three plane quadrants that are the faces of the trihedron of reference (Fig. 97a). If the surface of the trihedron is slit along $\uparrow O(X)$ and opened out onto a plane.

then the three quadrants are hinged together along $\uparrow O(Y)$ (horizontally) and $\uparrow O(Z)$ (vertically), whereas $\uparrow O(X)$ is represented by the two sides of the lower left quadrant (Fig. 97b).



(a) (b)
FIG. 97. First octant orthographic projection.

A point P can be plotted in a three-plane orthographic projection by using two of its three coordinates to locate each of its images P_{yz}, P_{zx}, and P_{xy} in the three quadrant projections. A line segment is projected by connecting the projections of its end points. A plane may be represented either by projecting some polygon in the plane or else by drawing its traces on the three coordinate planes. The **trace** of one plane on another is its line of intersection.

EXAMPLE 1: To illustrate the theory of the three-plane orthographic projection, Fig. 97*a* shows a brick measuring 2 by 4 by 7.5 in. placed in the first octant, together with its projections onto the coordinate planes. A diagonal is drawn to help identify two of the vertices. The three coordinate planes of Fig. 97*a* are opened out in Fig. 97*b* to show the brick in the three-plane projection. Note that in each of the three views in Fig. 97*b* four of the faces of the brick are represented by line segments.

EXAMPLE 2: A second example is given by the three-plane first-octant projection of a regular triangular prism having 1-in. base edges and $\frac{3}{4}$-in. lateral edges. In Fig. 98 the plan is shown as an equilateral triangle having 1-in. sides (to scale), whereas the two elevations are rectangles, both with $\frac{3}{4}$-in. altitudes, but with bases

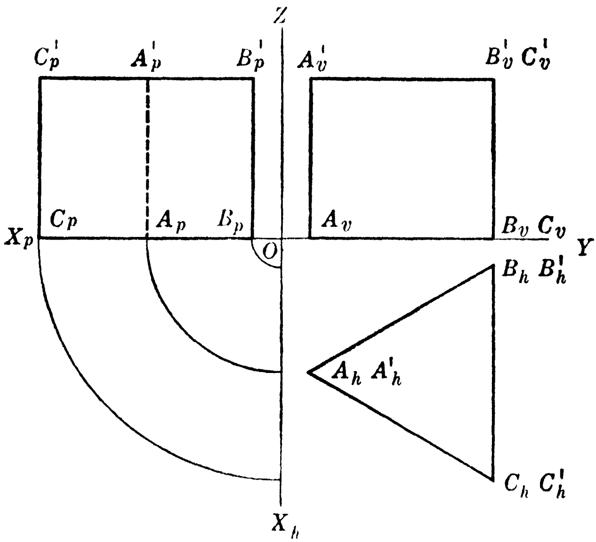


FIG. 98. Triangular prism in first octant projection.

of 1 in. and $\sqrt{3}/2$ in., respectively. The left-hand edge is invisible from the right, and therefore it is shown dotted in the side elevation.

The chief advantages of the three-plane projection are that measurements can be read off each of the views with the same scale and that measurements and angles of the figure in planes parallel to one of the coordinate planes will appear undistorted in lengths and angles in one of the three projections. Several of the eight possible combinations of top or bottom, right or left, and front or back views are in use in engineering drawing and descriptive geometry, but a detailed discussion of this type of projection is beyond the scope of this book.

32·3 Axonometric projection and the triangle of reference

An effective impression of depth is possible with a single-plane orthographic projection, technically known as the **axonometric projection,** which is more realistic than the oblique projection but somewhat harder to construct without the aid of special instruments.

The projection plane is chosen so as to intersect each of a set of three mutually perpendicular coordinate axes $\uparrow O(X)$, $\uparrow O(Y)$, and $\uparrow O(Z)$ to which the space figure is referred in points X, Y, Z distinct from O, and its orientation is so chosen that none of the important lines in the space figure are perpendicular to the projection plane. The triangle $[XYZ]$ in the projection plane is called the **triangle of reference,** and its sides are the traces on the projection plane of the faces of the trihedron of reference O-(XYZ) in space. Points in space are projected orthographically onto the projection plane in the axonometric projection. In particular, the projection of the origin O will be denoted by O', and the distance $\overline{OO'}$ from O to the projection plane will be denoted by d.

The orientation of the projection plane is described by the direction angles α,β,γ between $[OO']$ and the three space axes, respectively, or by their cosines l, m, n, defined by

$$(1)\quad l=\cos\alpha=\frac{\overline{OO'}}{\overline{OX}}\qquad m=\cos\beta=\frac{\overline{OO'}}{\overline{OY}}\qquad n=\cos\gamma=\frac{\overline{OO'}}{\overline{OZ}}$$

In terms of l, m, n, and d the intercepts of the projection plane on the space axes are[1]

$$(2)\qquad \overline{OX}=\frac{d}{l}\qquad \overline{OY}=\frac{d}{m}\qquad \overline{OZ}=\frac{d}{n}$$

The three planes containing (OO') and passing one through each of the points X, Y, Z are each perpendicular to the projection plane and also to one of the faces of the trihedron of reference. Hence they are perpendicular, respectively, to the sides $[YZ]$, $[ZX]$, and $[XY]$ of the triangle of reference, intersecting them in points that

[1] In the study of crystallography the orientation of a plane is given in terms of numbers called *Miller indices*, which are defined to be proportional to the reciprocals of the intercepts of the plane on the principal axes of the crystal. If these principal axes are perpendicular to each other, then from Eq. (2) these Miller indices must be simply a set of numbers proportional to the direction cosines l, m, n of the normal to the plane. In other words, the Miller indices of a crystal plane are direction components of the normal, whenever the principal axes of the crystal are perpendicular to each other.

we denote by X_0, Y_0, Z_0. This implies that the lines $[XX_0]$, $[YY_0]$ and $[ZZ_0]$ in which these three planes intersect the projection plane, respectively, are altitudes of the triangle of reference. Their point of intersection O' is the orthocenter (Fig. 99).

THEOREM 32A: *In axonometric projection, the projections of the edges of the trihedron of reference in space lie along the altitudes, and the projection of the origin is the orthocenter of the triangle of reference, in which the trihedron is cut by the projection plane.*

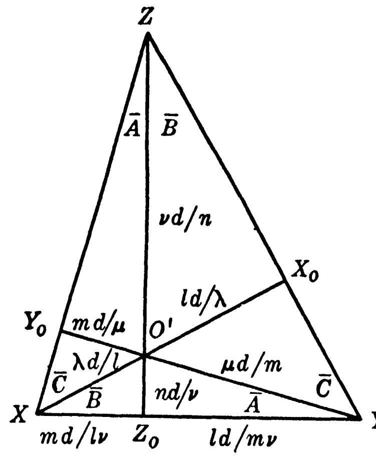


FIG. 99. Axonometric triangle of reference.

32·4 The drawing triangle and the trimetric ruler

A convenient aid in making drawings in a given type of axonometric projection is a **drawing triangle** $[ABC]$ obtained by rotating the reference triangle $[XYZ]$ through 90°. Its sides will be parallel to the projected axes, since they are parallel to the altitudes of the reference triangle. If one of the projected axes is vertical, as is usually the case, the drawing triangle must always be used with a particular side kept vertical on the drawing. Any lines in the drawing that represent lines parallel to one of the space axes can then be drawn with the appropriate side of the drawing triangle.

Next let scales called **axonometric scales** be marked on each side of the drawing triangle, with units proportional to the projected lengths of unit segments along the three space axes. Thus marked, the drawing triangle will be called a **trimetric ruler** (Fig. 3). With its aid, as we have seen, it is possible to construct without appreciable effort a segment of the proper direction and length to represent the projection of any segment parallel to one of the space axes. Furthermore, by considering an arbitrary segment in space to be the diagonal of a rectangular box (parallelepiped), having edges parallel to the coordinate axes and equal to the components of the segment, it is possible in three steps to project any segment by means of its components parallel to the projected axes.

32·5 Types of axonometric projection

It is a fact that *any acute-angled triangle can be chosen as the triangle of reference* for an axonometric projection and its altitudes will then be the reference lines for the projection. According as the triangle is equilateral, isosceles, or scalene, the projection is called **isometric, dimetric,** or **trimetric.** In all three types the lengths of segments on parallel lines are changed in the same ratio under projection: but in isometric projection the scale is the same for the projection of parallels to each of the three axes, in dimetric projection it is the same for two axes but different for the third (as in oblique projection), whereas for trimetric projection three different scales must be used on parallels to the three axes. The isometric projection has the advantage of simplicity, but the trimetric projection has the advantage that distinct lines of the object are less likely to project into the same line in the drawing. For many purposes the dimetric projection is a happy compromise and gives an effect that is better than the analogous oblique projection.

What remains unexplained at this point is: *How are the axonometric scales in dimetric or trimetric projection to be constructed or computed?* This may be done either geometrically, with a straightedge and compass construction, or algebraically, in a manner to be described later.

32·6 Geometric construction of axonometric scales

Starting with an arbitrary acute-angled triangle of reference $[XYZ]$ with its base $[XY]$ horizontal (Fig. 100), draw its altitudes (XX_0), (YY_0), (ZZ_0) meeting in the orthocenter O'. Then, by swinging three semicircular arcs on the sides of the triangle as diameters, locate the points O_x on (XX_0), O_y on (YY_0), O_z on (ZZ_0) such that $(O_xY) \perp (O_xZ)$, $(O_yZ) \perp (O_yX)$, $(O_zX) \perp (O_zY)$. The triangle $[XO_zY]$ so constructed in the projection plane is congruent to the triangle $[XOY]$, which is inclined to the projection plane, and may be thought of as obtained from the latter by revolving it about (XY) into the projection plane. If true unit lengths are measured on $\uparrow O_z(X)$ and $\uparrow O_z(Y)$ and these are projected onto $\uparrow O'(X)$ and $\uparrow O'(Y)$ by lines perpendicular to (XY), then the points U_x and U_y found on $(O'X)$ and $(O'Y)$ are the axonometric unit points. By using all three of the points O_x, O_y, O_z, each of the axonometric units is found twice, this serving as a check.

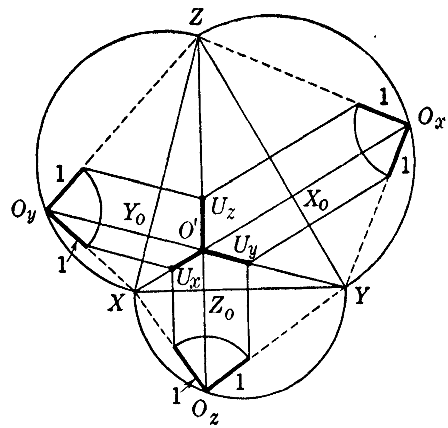


FIG. 100. Geometric construction of axonometric scales.

32·7 Measurements of the reference triangle

The axonometric ratios λ, μ, ν are measures of projections of unit segments from the space axes. Any lengths $\lambda\rho$, $\mu\rho$, $\nu\rho$ proportional to these can be taken as axonometric units. Since the angles $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ that the space axes make with the projection plane are the complements of the angles α, β, γ that they make with its normal $[OO']$, we have, by (5) of Sec. 9·2,

(3) Axonometric ratios: $\lambda = \cos\bar{\alpha} \quad \mu = \cos\bar{\beta} \quad \nu = \cos\bar{\gamma} \quad \lambda^2+\mu^2+\nu^2 = 2$

These are related to the direction cosines l, m, n of $[OO']$ by the Pythagorean relations

$$(4) \qquad \lambda^2 + l^2 = \mu^2 + m^2 = \nu^2 + n^2 = l^2 + m^2 + n^2 = 1$$

All the segments of the configuration formed by the reference triangle and its altitudes, as well as its three angles A, B, C (and their complements $\bar{A}$, $\bar{B}$, $\bar{C}$), can be expressed simply in terms of the six projection cosines λ, μ, ν, l, m, n, and the distance d from O to the projection plane. Conversely, given the angles A, B, C, it is possible to compute the projection ratios and thus construct a trimetric ruler.

Applying (1), (2), and (3) to the right triangles $[XOX_0]$, $[YOY_0]$, and $[ZOZ_0]$, perpendicular to the projection plane, and to the smaller right triangles in which they are cut by the altitude $[OO']$ of length d, we have

(5a) $\overline{OX}=\frac{d}{l}$ $\overline{OX_0}=\frac{d}{\lambda}$

(5b) $\overline{OY}=\frac{d}{m}$ $\overline{OY_0}=\frac{d}{\mu}$

(5c) $\overline{OZ}=\frac{d}{n}$ $\overline{OZ_0}=\frac{d}{\nu}$

(6a) $\overline{O'X}=\frac{\lambda d}{l}$ $\overline{O'X_0}=\frac{ld}{\lambda}$ $\overline{XX_0}=\frac{d}{l\lambda}$

(6b) $\overline{O'Y}=\frac{\mu d}{m}$ $\overline{O'Y_0}=\frac{md}{\mu}$ $\overline{YY_0}=\frac{d}{m\mu}$

(6c) $\overline{O'Z}=\frac{\nu d}{n}$ $\overline{O'Z_0}=\frac{nd}{\nu}$ $\overline{ZZ_0}=\frac{d}{n\nu}$

Applying (6a) to (6c) and the Pythagorean theorem to each of the six small right triangles into which the reference triangle is cut by its altitudes (Fig. 99), and then adding together the two segments which make up a side of the reference triangle, we have after simplifying by (4)

(7a) $\overline{YX_0}=\frac{nd}{m\lambda}$ $\overline{X_0Z}=\frac{md}{n\lambda}$ $\overline{YZ}=\frac{\lambda d}{mn}$

(7b) $\overline{ZY_0}=\frac{ld}{n\mu}$ $\overline{Y_0X}=\frac{nd}{l\mu}$ $\overline{ZX}=\frac{\mu d}{nl}$

(7c) $\overline{XZ_0}=\frac{md}{l\nu}$ $\overline{Z_0Y}=\frac{ld}{m\nu}$ $\overline{XY}=\frac{\nu d}{lm}$

Each of the triangles $[XY_0Z_0]$, $[X_0YZ_0]$, $[X_0Y_0Z]$ is similar to $[XYZ]$, since each has an angle in common with it and has its adjacent sides proportional to the correspondingly lettered sides (not the parallel sides) of $[XYZ]$. Thus we obtain the lengths of the sides of the so-called **pedal triangle** $[X_0Y_0Z_0]$, (not drawn in Fig. 99).

(8) $\overline{Y_0Z_0}=\frac{\lambda d}{\mu\nu}$ $\overline{Z_0X_0}=\frac{\mu d}{\nu\lambda}$ $\overline{X_0Y_0}=\frac{\nu d}{\lambda\mu}$

THEOREM 32*B*: *The axonometric ratios* λ, μ, ν *are proportional to the square roots of the sides of the pedal triangle, whose vertices are the feet of the altitudes of the triangle of reference.*

The proof of this follows immediately from Eqs. (8).

From Fig. 99 it is easily shown that the angles of the pedal triangle (not drawn) are $180°-2A$, $180°-2B$, $180°-2C$. It is proved in plane trigonometry that the sides of a triangle are proportional to the cosines of the complements of its angles.[1] These complements are $A-\overline{A}$, $B-\overline{B}$, $C-\overline{C}$, respectively since $2A-90°=A-\overline{A}$, etc. Thus from Theorem 32B we obtain the following theorem:

THEOREM 32C: *The axonometric ratios λ, μ, ν are proportional to the square roots of the cosines of the differences between the angles of the triangle of reference and their complements.*

(9) Formula: $\lambda:\mu:\nu=\sqrt{\cos(A-\overline{A})}:\sqrt{\cos(B-\overline{B})}:\sqrt{\cos(C-\overline{C})}$

An alternative expression for the ratios of λ, μ, ν in terms of the angles of the triangle of reference can be obtained by computing the cosines of A, B, C, $\overline{A}$, ,$\overline{B}$ $\overline{C}$ directly from Fig. 99 and Eqs. (6) and (7). Thus

$$(10)\qquad \begin{cases} \cos A=\dfrac{mn}{\mu\nu} & \cos B=\dfrac{nl}{\nu\lambda} & \cos C=\dfrac{lm}{\lambda\mu} \\[2ex] \cos\overline{A}=\dfrac{l}{\mu\nu} & \cos\overline{B}=\dfrac{m}{\nu\lambda} & \cos\overline{C}=\dfrac{n}{\lambda\mu} \end{cases}$$

From Eqs. (10) we obtain the proportions

$$(11)\qquad \frac{\cos A\cos\overline{A}}{\lambda^2}=\frac{\cos B\cos\overline{B}}{\mu^2}=\frac{\cos C\cos\overline{C}}{\nu^2}=\frac{lmn}{\lambda^2\mu^2\nu^2}$$

Hence λ, μ, ν are proportional to $\sqrt{\cos A\cos\overline{A}}$, $\sqrt{\cos B\cos\overline{B}}$, $\sqrt{\cos C\cos\overline{C}}$, as was already stated without proof in Theorem 9B, Sec. 9·6.

In constructing the axonometric scales, any quantities $\lambda\rho$, $\mu\rho$, $\nu\rho$ proportional to λ, μ, ν can be used as units. To obtain λ, μ, ν from $\lambda\rho$, $\mu\rho$, $\nu\rho$ we note that $2\rho^2=(\lambda\rho)^2+(\mu\rho)^2+(\nu\rho)^2$. First solve for ρ, then divide each of the given quantities $\lambda\rho$, $\mu\rho$, $\nu\rho$ by ρ.

[1] The cosine of the complement of an angle is known as the **sine** of the angle, and the relation in question is called the **law of sines**.

32·8 Axonometric projection of points, lines, and planes

Points, lines, and planes are located in axonometric projection in the same manner as was described for oblique projection, once the reference system has been set up and the axonometric units have been determined. The projection P' of a point P with rectangular coordinates (x,y,z) is located by drawing the directed segments $\uparrow [O'P_x']$ of directed length $\lambda\rho x$ true units along $\uparrow (O'X)$, then $\uparrow [P'_x P_{xy}']$ of directed length $\mu\rho y$ true units parallel to $\uparrow (O'Y)$, and finally $\uparrow [P'_{xy}P']$ of directed length $\nu\rho z$ true units parallel to $\uparrow (O'Z)$. The point P' so determined is the required projection of P, and the three segments so constructed represent three edges of a rectangular parallelepiped having $[OP]$ as diagonal. A line not perpendicular to the plane of projection is represented by joining the projections of two of its points. A plane not perpendicular to the plane of projection is represented by projecting its three traces in the trihedron of reference or by projecting some polygon or curve in the plane. Figures 2, 13, 27, 28, 37, 50, 60, 63, 97*a*, 101, etc., show some points, lines, and planes drawn in axonometric projection.

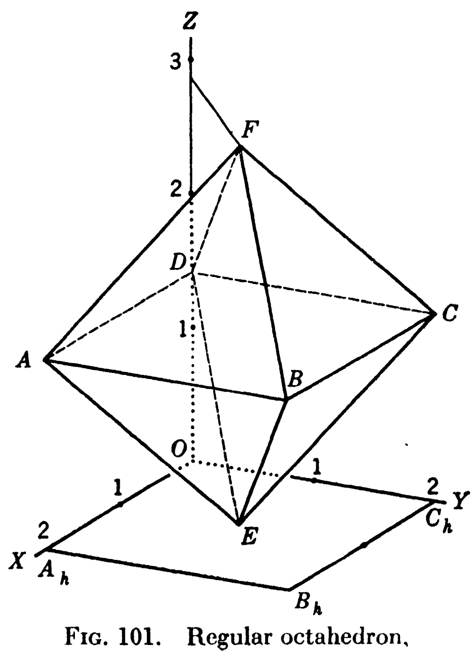


FIG. 101. Regular octahedron.

EXAMPLE: Draw a regular octahedron with edges 2 units long in trimetric axonometric projection, drawing one of the diagonals as a vertical line and choosing the following units on the three projected scales in millimeters (or in sixteenths of an inch if preferred):

$$\lambda\rho = 4\sqrt{6} = 9.8 \text{ mm.} \qquad \mu\rho = 4\sqrt{14} = 14.9 \text{ mm.} \qquad \nu\rho = 16 \text{ mm.}$$

SOLUTION: We have successively

$$(\lambda^2+\mu^2+\nu^2)\rho^2 = 96+224+256 = 576; \quad \rho^2 = 576/2 = 288 \quad \rho = 12\sqrt{2};$$
$$\lambda^2 = \tfrac{3}{9},\ \mu^2 = \tfrac{7}{9},\ \nu^2 = \tfrac{8}{9}; \quad l^2 = 1-\lambda^2 = \tfrac{6}{9},\ m^2 = \tfrac{2}{9},\ n^2 = \tfrac{1}{9}.$$

The diagonal of a square of side 2 units is $\sqrt{8}$ units, and thus the vertical diagonal of the octahedron should be $16\sqrt{8}$ mm. long (Fig. 101). In drawing the reference triangle it is convenient to multiply all its measurements by 16 mm. to enlarge the scale. Thus, writing $\sqrt{8} = 2\sqrt{2}$, taking $d = 16$, and using (6c) and (7c) we have (Fig. 99)

$$\overline{O'Z} = 32\sqrt{2} \text{ mm.} = 45.2 \text{ mm.}, \qquad \overline{Z_0Y} = 12\sqrt{6} \text{ mm.} = 29.4 \text{ mm.}$$
$$\overline{O'Z_0} = 4\sqrt{2} \text{ mm.} = 5.7 \text{ mm.}, \qquad \overline{Z_0X} = 4\sqrt{6} \text{ mm.} = 9.8 \text{ mm.}$$

On $\uparrow O(X)$ (Fig. 101) lay off $\overline{OA_h} = 19.6$ mm. (2 units), and on $\uparrow O(Y)$ lay off $\overline{OC_h} = 29.8$ mm. (2 units).

Let parallels to (OY) through A_h and to (OX) through C_h intersect in B_h, and let (OB_h) meet (A_hC_h) in E. Then the parallelogram $[A_hB_hC_hO]$ with its diagonals is the axonometric projection of a square, which is the plan of the octahedron. The vertex D is 22.5 mm. (or $\sqrt{2}$ units) above O on $\uparrow O(Z)$, and the vertices A, B, C are the same distances above A_h, B_h, and C_h, respectively. The sixth vertex F is twice this distance above E.

Having located the six vertices of the octahedron, join adjacent vertices by line segments, making sure that those edges of the octahedron which are supposed to be parallel are actually parallel. Draw in the visible edges with heavy lines, and draw in with dotted lines the edges that would be hidden behind the solid. Those portions of the reference lines which are behind the figure should also be dotted. Finally label the figure, and indicate the units on the three axes.

32. ORAL QUESTIONS

A. What is an orthographic projection?
B. How are parallel lines represented in an orthographic projection?
C. What advantage does an orthographic three-plane projection have over other methods of representing figures?
D. What advantages does the axonometric projection have over the three-plane projection?
E. How are points, lines and planes represented in three-plane orthographic projection?
F. How do you define the following terms: profile projection; obverse vertical projection; plan; elevation; trace of a plane?
G. What is the method of representing points and lines in an axonometric projection?
H. What triangles can be used as reference triangle for axonometric projections?
I. What is the difference between isometric, dimetric, and trimetric projections, and what are the advantages or disadvantages of each?
J. How are the projected axes related to the reference triangle in axonometric projection?
K. What, briefly, are the steps taken in constructing an axonometric reference system, including the location of the unit points of the axes?

32. WRITTEN EXERCISES

1. Draw the octahedron of Fig. 101 in three-plane orthographic projection.
2. Draw a regular tetrahedron in three-plane orthographic projection, placing one of the faces in a horizontal plane and one of the edges parallel to (OY).
3. Draw a regular tetrahedron in three-plane orthographic projection, placing one edge in the horizontal projection plane parallel to (OY), but placing the opposite edge parallel to (OX) in a second horizontal plane.

4. Draw a three-plane orthographic projection of a right circular conoid whose base is in a vertical plane, whose altitude is equal to the diameter of the base, and whose edge is parallel to (OZ) (see Fig. 54).
5. Draw a cube in three-plane orthographic projection, placing it so that a diagonal plane containing a pair of opposite edges is a vertical plane parallel to (YOZ).
6. Draw a cube in three-plane orthographic projection, placing it so that one of its diagonal lines is vertical and four of its edges are parallel to the plane (YOZ). HINT: The horizontal and profile views will both appear as regular hexagons, whereas the front view will be a rectangle.
7. Draw a cube in isometric axonometric projection, placing three of its edges along the axes, and compare this view with the views of the last exercise.
8. Draw a cube in dimetric axonometric projection, choosing $\mu^2 = \nu^2 = \frac{8}{9}$, and placing three of its edges along the axes.
9. Draw a cube in trimetric axonometric projection, choosing $l = \frac{6}{7}$, $m = \frac{3}{7}$, $n = \frac{2}{7}$, and placing three of its edges along the axes.
10. Draw a dimetric projection of a bookcase 45 in. tall, 30 in. wide, and 8 in. deep in outside dimensions, if there are four shelves and a top, each 1 in. thick with 10-in. spaces between, and if the ends and back are each 1 in. thick.
11. Show that rational values of λ, μ, ν in a dimetric projection may be obtained by choosing $\lambda = 4uv/(2u^2+v^2)$, $\mu = \nu$, where u and v are any integers for which $\lambda < 1$. Find a set of values in which $\lambda < \frac{1}{2}$.
12. Draw a trimetric axonometric projection of the frustum of a regular pyramid having square bases 6 by 6 and 15 by 15 and altitude 20. Use $\lambda = \sqrt{3}/3$, $\mu = \sqrt{7}/3$, $\nu = \sqrt{8}/3$.
13. Draw a cuboctahedron in trimetric projection by cutting the vertices off the octahedron of Fig. 101 by planes passing through the mid-points of the sets of four edges that meet at a vertex.

33

ELLIPTIC CYLINDRICAL SECTIONS

33·1 Parallel projection of curves

When a curve other than a straight line is projected by parallel projection, the projection lines through its points are elements of a cylindrical surface having the given curve as directrix, unless in particular they all lie in the plane of the curve. To study the parallel projection of circles, we must investigate cylindrical surfaces with circular sections.

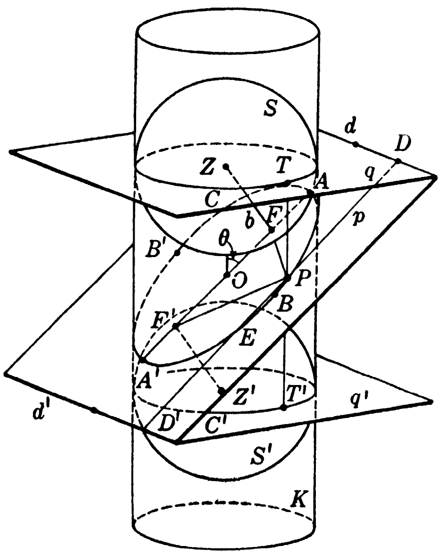


Fig. 102. Elliptic section of a right circular cylinder.

33·2 Sections of a circular cylindrical surface

By definition, a circular cylindrical surface is one whose right section is a circle. Let such a circular cylindrical surface K with radius b be cut along a curve E by any plane p that makes an oblique angle θ with the elements of K and meets the axis at a point O (Fig. 102). Two spheres S and S' with the same radius b and with centers at suitably chosen points Z and Z' on the axis of K can be inscribed in the surface K, one on each side of the plane p, so as to be tangent to p at points F and F', and tangent to K along the great circles C and

C', respectively, which are right sections of K. The planes of C and C' will be denoted by q and q'.

If A and A' are the points where the line (FF') meets the surface K, then the four distances $\overline{OZ}$, $\overline{OZ'}$, $\overline{OA}$, $\overline{OA'}$ are all equal and each will be denoted by a. The two equal distances $\overline{OF}$ and $\overline{OF'}$ will be denoted by c, and the ratio c/a by e. From the right triangle $[OFZ]$ we find

$$(1) \qquad e = \frac{c}{a} = \cos \theta$$

$$(2) \qquad c^2 = a^2 - b^2$$

Now let P be any point on the section curve E, and let the element of K through P meet the circles C and C' in the points T and T', respectively. Then $[PF]$ and $[PT]$ are both tangents to the sphere S and hence to the circle in which their plane cuts S. These two tangents have equal length, and the same is true of $[PF']$ and $[PT']$, which are both tangents to S'. Hence

$$(3) \qquad \overline{PF} + \overline{PF'} = \overline{PT} + \overline{PT'} = \overline{TT'} = \overline{ZZ'} = \overline{AA'} = 2a$$

33·3 The ellipse

An **ellipse** is a plane curve which is the locus of a point moving so that the sum of its distances from two fixed points is a constant. The fixed points F and F' are called the **foci** of the ellipse, the mid-point O of their segment is called the **center,** the segment $[AA']$ cut by the ellipse from the line [FF′] (when the foci are distinct) is called the **major axis,** and its extremities A and A' are called the **vertices** of the ellipse. The segments $[PF]$ and $[PF']$ drawn from a point P to the foci are called **focal radii,** and their sum is equal to the major axis.

The segment $[BB']$ cut by the ellipse from a line in its plane, through its center, and perpendicular to its major axis is called the **minor axis.** The lengths of the major and minor axes will be denoted by $2a$ and $2b$, respectively, and the distance $\overline{FF'}$ by $2c$. The ratio $e = c/a$ is called the **eccentricity** of the ellipse and is less than 1 since $c < a$. If the two foci F and F' coincide, the ellipse is a **circle** and $e = 0$. In general $e = c/a = \cos \theta$.

THEOREM 33A: *Any plane section of a circular cylindrical surface is an ellipse. It may, in particular, be a circle.*

The proof is included in the preceding discussion, which shows that $\overline{PF}+\overline{PF'}=\overline{AA'}$. It should be noted that any given ellipse can be obtained as a section of a right circular cylindrical surface whose diameter is equal to the minor axis.

EXAMPLE: A circular cylindrical surface of radius 3 in. is cut by a plane making an angle of 30° with the axis. Find the lengths of the major and minor axes, the distance between the foci, and the eccentricity of the section.

SOLUTION:

1. We have given $R=b=3$ in., $\theta=30°$.
2. Therefore, $e=\cos 30°=\sqrt{3}/2=c/a$, or $c=a\sqrt{3}/2$.
3. Since $c^2=a^2-b^2$, we have $3a^2/4=a^2-9$, $a^2=36$, $a=6$ in.
4. Hence the axes are $2a=12$ in., $2b=6$ in.; and we have $2c=6\sqrt{3}$ in.

33·4 Directrices of an ellipse

The plane of section p meets the planes q and q' of the circles C and C' in two lines d and d', called the **directrices** of the ellipse (Fig. 102). Let a line parallel to (AA') through the arbitrary point P on the ellipse E meet d and d' in the points D and D', respectively. Then

$$(4)\qquad \begin{aligned} \frac{\overline{PF}}{\overline{PD}}&=\frac{\overline{PT}}{\overline{PD}}=\cos\theta=e \\ \frac{\overline{PF'}}{\overline{PD'}}&=\frac{\overline{PT'}}{\overline{PD'}}=\cos\theta=e \end{aligned}$$

THEOREM 33B: *The ratio of the distance to the focus and the distance to the corresponding directrix is the same for all points on an ellipse and is equal to the eccentricity of the ellipse.*

This theorem is the basis for an alternate definition of the ellipse as a locus—the so-called "focus-directrix definition"—which will be discussed in Chap. 35.

33·5 Conjugate diameters of an ellipse

A **chord** of an ellipse is a line segment joining two distinct points of the ellipse. A **diameter** of an ellipse is a chord passing through the center. Two diameters of an ellipse are said to be **conjugate diameters** if each bisects all the chords parallel to the other (see Sec. 22·1). Two conjugate diameters of an ellipse are called **axes** if they are mutually perpendicular.

THEOREM 33C: *Two mutually perpendicular planes m and n through the axis of a circular cylindrical surface K cut an arbitrary plane of section p in a pair of lines which are conjugate diameters of the ellipse E in which the section plane p cuts the surface K.*

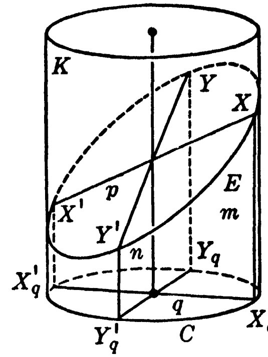


FIG. 103. Projection of two conjugate diameters of an ellipse.

PROOF: We assume the fact that the mid-points of a set of parallel chords of a circle lie on the diameter perpendicular to these chords. A plane q perpendicular to the axis of the cylindrical surface K cuts K in a circle C, whereas the given section plane p cuts K in an ellipse E (Fig. 103).

The given mutually perpendicular planes m and n through the axis each pass through the centers of E and C; and thus they cut the plane p in a pair of diameters $[X'X]$ and $[Y'Y]$ of E, and they cut the plane q in a pair of mutually perpendicular diameters $[X_q'X_q]$ and $[Y_q'Y_q]$ of C. These diameters of C are the orthogonal projections on the plane q of the pair of diameters of E. A set of chords of E parallel to $[X'X]$ projects orthogonally into a set of chords of C parallel to $[X_q'X_q]$, whose mid-points lie on $[Y_q'Y_q]$. Since mid-points are carried into mid-points by parallel projection, the mid-points of this set of chords of E must lie in the plane n, and hence on the diameter $[Y'Y]$ of E. Similarly, the diameter $[X'X]$ bisects the chords of E parallel to $[Y'Y]$. Hence $[X'X]$ and $[Y'Y]$ are conjugate diameters of the ellipse.

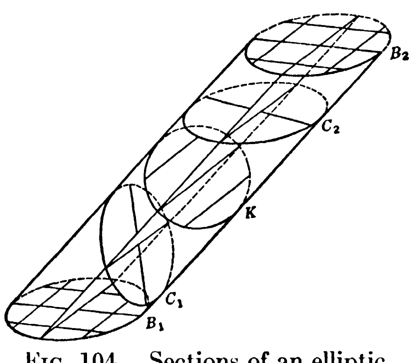


FIG. 104. Sections of an elliptic cylindrical surface.

33·6 Elliptic cylindrical surface

An elliptic cylindrical surface is one whose right section is an ellipse (Fig. 104). It is a fact, which we shall not prove in this book, that *every plane section of an elliptic cylindrical surface is an ellipse* (or in particular a circle).

33. ORAL QUESTIONS

A. What curves are sections of a circular cylindrical surface? Of an elliptic cylindrical surface?
B. How would you define an ellipse in terms of a plane locus? Is more than one definition possible?
C. What are the foci, center, vertices, major and minor axes, and eccentricity of an ellipse?
D. If a section plane makes an angle θ with the axis of a circular cylindrical surface, what is the eccentricity of the section?
E. What are the directrices of an ellipse, and how are they related to the cylindrical surface of which it is a section?
F. How can an ellipse be described in terms of a focus, a corresponding directrix, and the eccentricity?
G. What are conjugate diameters of an ellipse, and what chords do they bisect? Into what do they project when the ellipse is projected into a circle by parallel projection?
H. How may an ellipse be drawn, given a pair of conjugate diameters?
I. The sun casts a shadow of a tennis ball onto the court. What is the shape of the shadow? Does it vary with the time of day?
J. A circular pipe is cut by a plane not at right angles to the pipe. What is the curve of section?
K. A cutting tool, in the form of a rectangular bar from which a cylinder has been gouged out, has a semicircular cutting edge that is the intersection of the inner cylindrical surface and a plane making an angle of 45° with the elements. What is the shape of a right section of the inner surface?

33. WRITTEN EXERCISES

1. An ellipse is obtained as a plane section of a circular cylindrical surface of unit radius by a plane making an angle of 45° with the elements. Find the major and minor axes of the ellipse, the distance between the foci, and the eccentricity.
2. Solve Exercise 1 if the angle is 60°.

3. An ellipse whose major axis is 8 units long and whose minor axis is 4 units long is obtained as a plane section of a circular cylindrical surface of radius R by a plane p making an angle θ with the elements. Find R and θ.
4. Show that the area of an ellipse is equal to πab by projecting it onto a circular right section of the circular cylindrical surface of radius b of which it is a section.
5. In cutting down a certain tree whose trunk near the base was approximately a right circular cylinder 8 in. in diameter, the tree was first sawed horizontally as far as its center and then chopped with an axe so that a wedge-shaped cut was made at a 45° angle with the horizontal. Find the axes and eccentricity of the semi-ellipse which formed the upper face of the cut.
6. A circular section of an elliptic cylindrical surface is made by a plane making an angle θ with the elements. Show that the eccentricity of a right section is $e = \cos\theta$.
7. The foci of an ellipse with eccentricity $e = \frac{3}{5}$ are 18 in. apart. How far apart are the vertices, and how far apart are the directrices?
8. Show that the distance from the center of an ellipse to either directrix is a/c. Can a circle have directrices?
9. If the axes of two cylindrical pipes of equal radius intersect each other at right angles, the cylindrical surfaces of the pipes intersect in two ellipses whose planes make 45° angles with the axes. Find the eccentricities and the lengths of the axes of these ellipses.
10. Show that, if a point P is at distance $x > 0$ from the minor axis, its distance from the nearer focus is $a - ex$. What is the length of the other focal radius from P? HINT: Use the relations $\overline{PF} = \overline{PT} = a - \overline{OP_0}$, where P_0 is the orthogonal projection of P on the axis of the cylinder.
11. Prove the following theorem: *If the tangents to an ellipse from a point Q on the directrix touch the ellipse at P_1 and P_2, then the chord $[P_1P_2]$ passes through the focus.* HINT: Let the ellipse E be given as the intersection of a plane p and a circular cylindrical surface K, and let the sphere S be tangent to both. Show that the tangents drawn from Q to S meet S at points of a circle whose plane contains the focus F and the two elements of the cylindrical surface through P_1 and P_2. Where does this plane cut p?

34

CIRCULAR OBJECTS IN PARALLEL PROJECTION

34·1 Parallel projections of circles

The problem of drawing circles in projection has been considered earlier in this book in connection with the study of the cone (Chap. 16) and the sphere (Chap. 22). A more systematic study can now be made of both the oblique and the orthographic projections of circles, cylinders, cones, and spheres in the light of further knowledge about the elliptic sections of a cylindrical surface. The student should review Chap. 22 before proceeding with this discussion.

If a circle C is projected orthographically or obliquely onto a plane p by parallel projection lines, not lying in the plane of the circle, the parallel projection lines form an elliptic (or circular) cylindrical surface with the circle C as directrix. The circle and two mutually perpendicular diameters project into an ellipse and a pair of conjugate diameters. This ellipse will be a circle if and only if the plane of C is parallel to the projection plane.

A square circumscribed about the circle projects into a parallelogram circumscribed about the ellipse and tangent to the ellipse at the extremities of a pair of conjugate diameters. The diagonals of the parallelogram form a second pair of conjugate diameters of the ellipse. For each of these diagonals the ratio of the complete diagonal to the diameter cut off by the ellipse is $\sqrt{2}$ to 1, since this ratio of lengths is the same as for the circle with its circumscribed square.

The tangent to the ellipse at a point where it cuts one diagonal is parallel to the other diagonal of the parallelogram and also parallel to a line joining the nearer end points of the given conjugate diameters. One way to project a circle orthographically or obliquely is to project a pair of mutually perpendicular diameters by the methods already described for the parallel projection of line segments and then construct an ellipse having these projected lines as conjugate diameters.

34·2 Tangents perpendicular to a diameter of an ellipse

In more advanced mathematics the following theorem is proved:

THEOREM 34A: *The square of the diagonal of any rectangle circumscribed about an ellipse is equal to the sum of the squares of the axes.*

Referring to Sec. 22 and Theorem 34A, we then have also the following:

THEOREM 34B: *The square of the diagonal of any rectangle circumscribed about an ellipse is equal to the sum of the squares of any two conjugate diameters of the ellipse.*

Next let us assume that, of the pairs of opposite sides of a circumscribed rectangle, the first pair of length $2d$ is parallel and the second pair of length $2w$ is perpendicular to a given diameter (Fig. 105) of an ellipse. Let the lengths of the given diameter $[XX^*]$ and of its conjugate diameter $[YY^*]$ be $2r$ and $2r'$, respectively, and let the axes of the ellipse be $2a$ and $2b$, respectively. Then by Theorems 34A and B we have

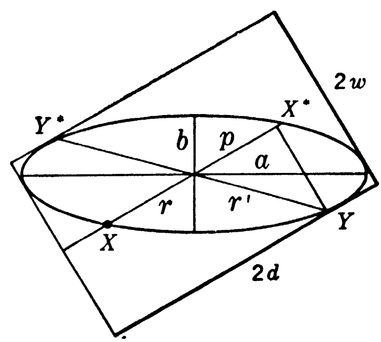


FIG. 105. Ellipse and circumscribed rectangle.

(1) $$d^2+w^2=r^2+r'^2=a^2+b^2$$

If $2p$ is the length of the orthogonal projection of $[YY^*]$ on $[XX^*]$, we have (see Fig. 105)

(2) $$p^2=r'^2-w^2$$

Hence, combining (1) and (2), we obtain

(3) $$d^2=r^2+p^2$$

This result may be expressed as follows:

THEOREM 34C: *The distance d from the center of an ellipse to the* *tangent perpendicular to a given diameter is equal to the length of the hypotenuse of a right triangle of which one side is the given semidiameter r and the other is the projection p upon it of the conjugate semidiameter.*

This theorem supplies added information which is helpful in constructing an ellipse from a pair of conjugate diameters. Suppose that one of these diameters is either vertical or horizontal. Then both the vertical and horizontal tangents to the ellipse can be constructed immediately, the one at the extremities of the conjugate diameter, and the other by Theorem 34C.

34·3 Parallel projection of circular cylinders and cones

To project a circular cylinder or cone or frustum by parallel projection, all that is necessary is to project a pair of mutually perpendicular longitudinal sections through the axis and then construct the elliptic projections of the circular bases by using the pairs of conjugate diameters that are the projections of the intersections of these longitudinal sections with the bases.

34·4 Oblique projection of the sphere

When a sphere is projected obliquely onto a plane, the great circle C, parallel to the projection plane (YOZ) is unchanged in the projection. The outline of the sphere itself, however, is not a circle but an ellipse E tangent to C at the extremities B and B' of its minor axis. The foci F and F' of this ellipse are the points X and X^* (Fig. 106), which mark the oblique projection of that diameter of the sphere which is perpendicular to the projection plane. This can be seen by examining Fig. 102, where the sphere S can be thought of as projected obliquely onto the plane of section p by rays that are parallel to the axis of the cylinder and oblique to p. The elements of the cylinder project the outline of the sphere into an ellipse, whereas the radius $[ZF]$ of the sphere perpendicular to the projection plane is projected into the segment $[OF]$.

To draw a sphere in oblique projection (Fig. 106) first project a set of three mutually perpendicular diameters $[XX^*]$, $[YY^*]$, $[ZZ^*]$ as described in Sec. 31·5. Draw the circle C, and find the points B and B' where it intersects the line perpendicular to $[XX^*]$ at O. Using the distance $\overline{BX} = a$ as the major axis of the outline ellipse E,

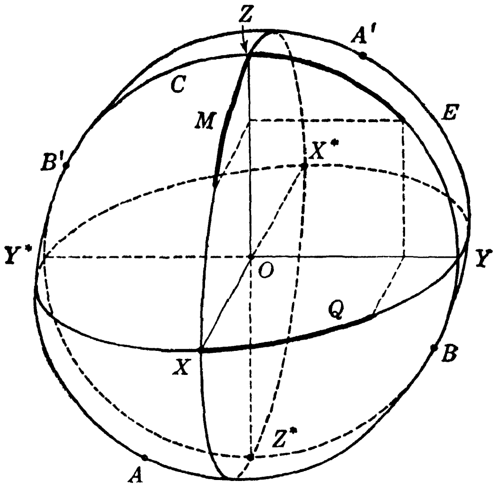


FIG. 106. Oblique projection of a sphere.

construct the vertices A and A' on (XX^*) so that $\overline{AO} = \overline{OA'} = a$, and sketch the outline ellipse E.

Next construct the meridian ellipse M, which has $[XX^*]$ and $[ZZ^*]$ as conjugate diameters, using Theorem 34C to locate its horizontal tangents, which are also tangent at the same points to the outline ellipse E. In like manner, construct the equatorial ellipse Q, which has $[XX^*]$ and $[YY^*]$ as conjugate diameters, locating by Theorem 34C its vertical tangents, which it shares with E.

To measure an arc on a meridian ellipse in oblique projection (Fig. 106), project the arc onto $[ZZ^*]$ by parallels to $[XX^*]$ and from there onto the circle C by parallels to $[YY^*]$. Similarly, to measure an arc on the equatorial ellipse, project it onto $[YY^*]$ by parallels to $[XX^*]$ and then onto C by parallels to $[ZZ^*]$. Arcs on the circle C can be measured directly. To construct a measured arc, project one end point onto C as described, measure the arc on C, and project back the other end point.

Small circles at latitude L are drawn as ellipses similar to the equatorial ellipse, with conjugate diameters shortened in the ratio cos L.

34·5 Axonometric projection of the sphere

Consider a sphere (Fig. 107), on which three mutually perpendicular great circles C_x, C_y, C_z have been drawn, to be projected

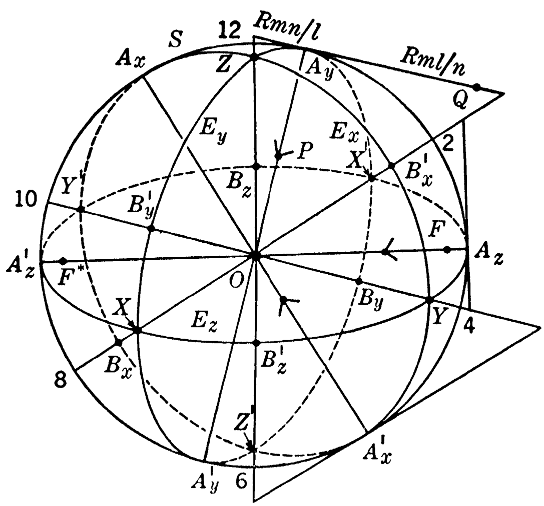


FIG. 107. Axonometric projection of a sphere.

axonometrically onto a plane p inclined to the planes of all three circles. The great circle parallel to the plane p will project into a circle S whose center O is the projection of the center of the sphere and whose radius R is equal to the radius of the sphere. The three diameters joining the points of intersection of the great circles will project into axonometric axes, and their lengths will be shortened to equal $2\lambda R$, $2\mu R$, $2\nu R$, respectively, where λ,μ,ν are the projection ratios (Sec. 32·7). Lines perpendicular to these axes in the projection plane will meet the circle S in the points A_x and A_x', A_y and A_y', A_z and A_z', respectively, where the elliptical projections E_x, E_y, E_z of the three given great circles are tangent to S. The minor axes $[B_xB_x']$, $B_yB_y']$, $[B_zB_z']$ of these three ellipses lie on the axonometric axes, and their lengths are $2lR$, $2mR$, $2nR$, respectively.

For each of the ellipses E_x, E_y, E_z, which are the projections of the given great circles, the major and minor axes and also another pair of conjugate diameters are known. Using the construction already given, we can find 16 tangents to each of these ellipses, 8 for each pair of conjugate diameters. From these each ellipse can be drawn in freehand or with a French curve.

Before the ellipses can be drawn, however, it is necessary to construct the three projected diameters. Although this could be done most easily with the trimetric ruler, the following construction can be made, with the aid of straightedge and compass only, for any desired choice of the directions of the projected axes:

Starting with three given angles $\overline{A}$, $\overline{B}$, $\overline{C}$, whose sum is 90°, draw a circle S of radius R, center at O', and lay off 12 consecutive arcs whose central angles are $\overline{A}$, C, $\overline{B}$, $\overline{A}$, $\overline{C}$, $\overline{B}$, etc., starting from the top and proceeding clockwise. Although of unequal length, these arcs may be likened to 12 hour arcs on a clock. They are somewhat regular, since the sum of three consecutive arcs is always 90°. Pairs of opposite even "hour points," 2 and 8, 4 and 10, 6 and 12, are joined through the center of S to give the axonometric axes $[X'X]$, $[YY']$, $[Z'Z]$. Pairs of odd hour points, 1 and 7, 3 and 9, 5 and 11, are the vertices of the ellipses E_x, E_y, E_z, and we label them in clockwise order as follows: A_y, A_z, A_x', A_y', A_z', A_x. These vertices are the points of tangency with S of the three ellipses (Fig. 107).

We now determine the semiminor axes of each ellipse and the positions of the points where the axonometric axes pierce the sphere, using the following geometric construction: Draw a line segment tangent to S at A_y, terminated by the axes (ZZ') and (XX') extended. The segments to the left and right of A_y are seen to be Rmn/l and Rml/n, respectively, since the tangent and two axes form a triangle similar to the triangle of reference of Fig. 99, with the radius $[OA_y]$ in Fig. 107 corresponding to the altitude $[YY_0]$ in Fig. 99. Mark the point P where a semicircle having this segment as a diameter meets the line (OA_y). Then since $\overline{PA_y}$ is a mean proportional between the two portions Rmn/l and Rml/n of the tangent segment, we have $\overline{PA_y} = Rm$. This is the required length $\overline{OB_y} = \overline{OB'_y}$ for the semimajor axis of the ellipse E_y, and it can be laid off on (YY'), on each side of O. Next let a circular arc of radius R, center at P, meet the tangent drawn at A_y in the point Q. Then $\overline{QA_y} = \sqrt{R^2 - R^2m^2} = R\mu$, and this is the length $\overline{OY} = \overline{OY'}$ by means of which the points Y and Y' (where the axis pierces the spherical surface) can be located on the line (YY') already drawn through 4 and 10. By similar steps the points B_x and B_x', X and X', B_z and B_z', Z and Z' can be located. Then the ellipses E_x, E_y, E_z should be drawn in as previously described, as many of the 16 available tangents being used in each case as may seem desirable. Shading the visible portions of the three circular sections of the sphere would help to make the figure stand out.

Any arc on one of the great circles C_x, C_y, or C_z can be measured by projecting the corresponding arc of the corresponding ellipse E_x, E_y, or E_z onto the circle S, projection lines perpendicular to the major axis of the ellipse being used. An arc of any given length can be constructed by projecting the initial point onto S in this manner, measuring the arc on S, and then projecting back the end point.

34. ORAL QUESTIONS

A. Into what curves can a circle project under parallel projection?

B. What diameters of an ellipse are the projections of two perpendicular diameters of a circle? Are they always, sometimes, or never perpendicular to each other?

C. What is the boundary of the projection of a sphere (*a*) in oblique projection; (*b*) in axonometric projection?

D. What can be said of the dimensions of a rectangle circumscribed about a given ellipse?

E. Given the lengths of the axes of an ellipse and that of one other diameter, how could you construct the length of the conjugate diameter?

F. What is an expression for the distance from the center of an ellipse to the tangent perpendicular to a given diameter of the ellipse?

G. In the oblique projection of a sphere, how would you locate the foci of the outline ellipse? How would you locate its horizontal and vertical tangents?

H. How can an arc of a given magnitude be measured on the ellipse that represents the meridian circle of a sphere in oblique projection?

I. In the axonometric projection of a sphere, what curve represents the outline of the sphere? How are the images of great circles on the sphere related to this outline curve?

J. If three mutually perpendicular great circles of a sphere are projected axonometrically, is it possible to choose arbitrarily the contact points of these ellipses with the outline circle? How then can the axonometric units be constructed?

34. WRITTEN EXERCISES

1-6. In each of Exercises 1 to 6 draw an ellipse having conjugate diameters of the given lengths with the given angle between them. Draw first the eight tangents obtained from conjugate diameters and the diagonals of the circumscribed parallelogram and then four or more tangents perpendicular to these. If possible, use different colors for the tangents and the curve.

1. 4 in., 4 in., 60°.
2. 4 in., 6 in., 60°.
3. 4 in., 6 in., 30°.
4. 3 in., 6 in., 30°.
5. 2 in., 4 in., 45°.
6. 4 in., 4 in., 45°.

7. Draw a sphere in oblique projection, constructing the outline ellipse E having minor axis 4 in. and the projections of three mutually perpendicular great circles. What is the eccentricity of E? What is the length of the major axis? Use the type A projection (Sec. 31·5).
8. The same as Exercise 7, but use the type B projection (Sec. 31·5). Add to the figure the meridians for every 30° of longitude.
9. Draw a sphere of radius 5 cm. (or 2 in.) in axonometric projection as described in Sec. 34·5, taking $\overline{A} = 30°$, $\overline{B} = 15°$, $\overline{C} = 45°$. Draw three mutually perpendicular great circles.
10. Draw a sphere as in Exercise 9, but add to the figure the parallels of latitude and meridians for every 30°.
11. Draw a sphere as in Exercise 9, but taking $\overline{A} = 7\frac{1}{2}°$, $\overline{B} = 22\frac{1}{2}°$, $\overline{C} = 60°$.
12. Draw a figure in oblique projection (type C), representing a circular cylinder of radius 2 in. cut by a plane perpendicular to the projection plane and inclined at an angle of 45° with the axis.
13. Draw a right circular cylinder of radius 3 cm. and height 4 cm. in dimetric axonometric projection, choosing $\lambda = \frac{16}{33}$, $\mu = \nu = \frac{31}{33}$. HINT: $\angle B = \angle C$.
14. Draw an axonometric isometric projection of a circular cylinder of unit radius cut by a plane making an angle of 60° with the axis. Show the elliptical section and the two spheres tangent to the cylindrical surface and to the plane of section.

35

SECTIONS OF A CONE

35·1 Tangent cone and sphere

Tangents to a sphere from an external point are equal in length, they are elements of a right circular conical surface, and they make equal angles ϵ with the line drawn from the point to the center of the sphere (Fig. 67). The complete conical surface consists, not merely of the points of these tangent segments, but of their whole infinite lines.

35·2 Nappes of a circular conical surface

A right circular conical surface can be generated, as we have seen, by revolving a line (the **generator**) about a second line (the **axis**), which intersects it in a point called the **vertex.** The vertex divides the generator into two half lines and divides the conical surface into two **nappes,** each generated by one of the half lines. Each position of the generator is called an **element** of the conical surface. Each point on the axis is equidistant from all the elements.

35·3 Types of conic sections

A plane section of a right circular conical surface is called a **conic section** or sometimes simply a **conic** (Figs. 108 to 110). We let θ denote the section angle between the plane of section p and the axis of the conical surface, and let ϵ be the so-called **semivertical angle** between the elements of the cone and the axis. If the plane p passes through the vertex, the conic is called **degenerate;** otherwise, it is called **nondegenerate.** Degenerate conic sections will be a single point if $\theta > \epsilon$, a line (element of the surface) if $\theta = \epsilon$, or a pair of intersecting lines if $\theta < \epsilon$. Nondegenerate sections can be shown to be ellipses if $\theta > \epsilon$ (Fig. 108); they are **parabolas** if $\theta = \epsilon$ (Fig. 109) and **hyperbolas** if $\theta < \epsilon$ (Fig. 110). Definitions of parabolas and hyperbolas will be given later. Elliptic sections are made by planes cutting all the elements on one nappe of the cone, parabolic sections by planes parallel to one of the elements, and hyperbolic sections by planes cutting the two nappes of the cone in two separated arcs or branches.

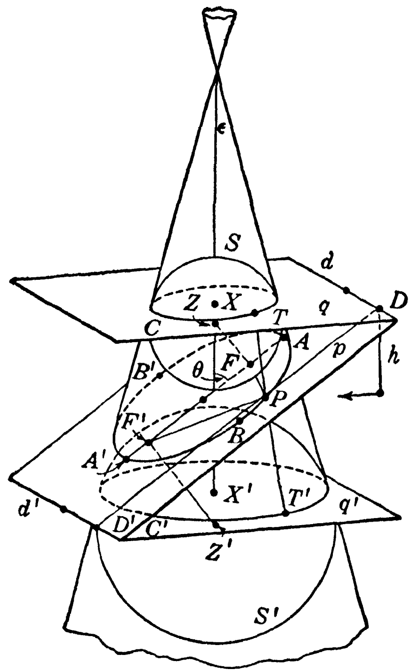


FIG. 108. Elliptic section of a circular conical surface.

A plane through the axis of the conical surface perpendicular to the plane of section cuts the latter in a line called the **major axis** of the conic section. The points A and A' (or A only, for a parabolic section), where the major axis meets the conic section, are called the **vertices** of the conic section (not to be confused with the vertex O of the cone).

35·4 Elliptic sections

As with sections of cylindrical surfaces, so also with elliptic conic sections, two spheres S and S' can be constructed with centers Z and Z' on the axis of the conical surface and on opposite sides of the plane of section p, tangent to the conical surface along the two circles C and C' (centers at X and X') in planes q and q', and tangent to p at the points F and F' (the foci) situated on the major axis of the section (Fig. 108). The lines d and d' in which p meets q and q', respectively, are called **directrices** of the conic section. Let P be any point on the curve of section, and let the element of the conical surface passing through P meet the circles C and C' in the points T and T', respectively. Then $[PF]$ and $[PT]$ are both tangents to the sphere S, and their lengths are equal. Similarly, $\overline{PF'} = \overline{PT'}$. Hence

$$(1) \qquad \overline{PF} + \overline{PF'} = \overline{PT} + \overline{PT'} = \overline{TT'} = \overline{ZZ'} \cos \epsilon = \frac{\overline{XX'}}{\cos \epsilon}$$

Taking for P the points A and A' in turn, we have

$$(2) \qquad 2\overline{ZZ'} \cos \epsilon = (\overline{AF} + \overline{AF'}) + (\overline{A'F} + \overline{A'F'})$$
$$= (\overline{AF} + \overline{A'F}) + (\overline{AF'} + \overline{A'F'}) = 2\overline{AA'}$$

Hence

$$(3) \qquad \overline{PF} + \overline{PF'} = \overline{AA'}$$

This proves that the elliptic section of a cone is indeed an ellipse according to our previous definition.

35·5 Focus-directrix property of conics

For any one of the three nondegenerate conic sections, let S be a sphere, tangent to the plane p and tangent to the conical surface on a circle C which is a plane section of the conical surface. Let F (the focus) be the point of tangency of S and p, and let the plane q through the circle of tangency C meet p in the line d (the directrix). Let P be an arbitrary point on the curve of the section, let the element of the conical surface through P meet C in T, let the foot of the perpendicular from P on d be D, and let the distance from P to q be h (Fig. 108). Then we have

$$(4) \qquad \overline{PF} = \overline{PT} = \frac{h}{\cos \epsilon}$$

$$(5) \qquad \overline{PD} = \frac{h}{\cos \theta}$$

Hence

$$(6) \qquad \frac{\overline{PF}}{\overline{PD}} = \frac{\cos \theta}{\cos \epsilon} = \text{constant} = e$$

We call the constant ratio the **eccentricity** of the conic section and denote it by e. We have proved the following theorem:

THEOREM 35: *The ratio of the distance to the focus and the distance to the corresponding directrix is the same for all points on a nondegenerate conic section and is equal to the eccentricity e. For an ellipse $e<1$ (since $\theta>\epsilon$), for a parabola $e=1$ (since $\theta=\epsilon$), and for a hyperbola $e>1$ (since $\theta<\epsilon$).*

35·6 The parabola

A special case of this theorem about conic sections is usually taken as the definition of a parabola (Fig. 109).

DEFINITION: *A parabola is the locus of a point whose distance from a fixed point (the focus) is equal to its distance from a fixed line (the directrix).*

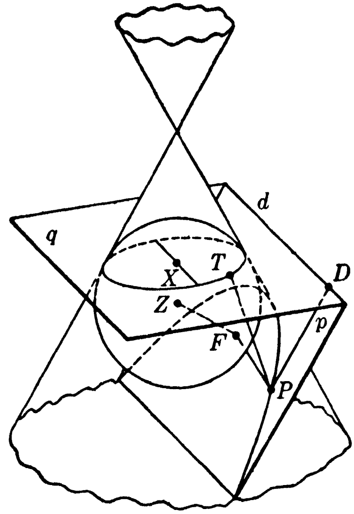


FIG. 109. Parabolic section of a circular conical surface.

The parabola is a curve that is familiar in such things as the path of a falling body (air resistance neglected); the cross section of the reflector on a searchlight, focusing flashlight, or automobile headlight; the arc of the cable on a suspension bridge (approximately); and the vertical section of the surface of a rotating liquid.

35·7 The hyperbola

When the plane of section p meets both nappes of the conical surface, the two spheres S and S', tangent to p and to the conical surface, lie in opposite nappes of the cone (Fig. 110). Using the same notation as with the elliptic section, if P is an arbitrary point on the branch of the curve of section that lies on the nappe of the conical surface corresponding to the focus F, we have

$$\overline{PF}=\overline{PT} \qquad \overline{PF'}=\overline{PT'}$$

$$(7) \qquad \overline{PF'}-\overline{PF}=\overline{PT'}-\overline{PT}=\overline{TT'}=\overline{ZZ'}\cos\epsilon=\frac{\overline{XX'}}{\cos\epsilon}$$

where X and X' are the centers of the circles of contact of the spheres S and S' with the conical surface.

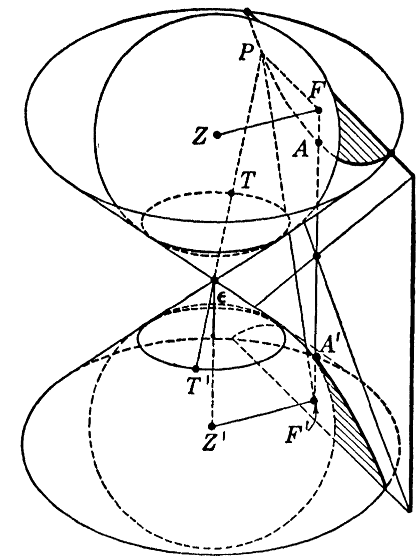


FIG. 110. Hyperbolic section of a circular conical surface.

For a point P' on the other branch we have

$\overline{P'F} - \overline{P'F'} = \overline{ZZ'}/\cos\epsilon$. Taking A for P and A' for P' we obtain

(8) $$2\overline{ZZ'}\cos\epsilon = (\overline{AF'} - \overline{AF}) + (\overline{A'F} - \overline{A'F'})$$
$$= (\overline{AF'} - \overline{A'F'}) + (\overline{A'F} - \overline{AF}) = 2\overline{AA'}$$

Hence

(9) $$\overline{PF'} - \overline{PF} = \overline{P'F} - \overline{P'F'} = \overline{AA'}$$

We have proved the following property of this type of conic section, which is usually taken to be the definition of a hyperbola:

DEFINITION: *A hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points* (*the foci*) *is a constant* (*the length of the major axis*).

Because of this property the hyperbola is important in a modern system of navigation called *loran* (long-range navigation). Suppose a pair of radio stations are 186 miles apart and each tuned to broadcast a radio pulse signal on the same frequency. If one station sends an initial pulse and the other rebroadcasts it when received 1/1,000 sec. later, all points on a branch of a hyperbola with these two stations as foci will receive the two pulses with the same time lag between them. This time lag is automatically recorded for the navigator on board an airplane and shows that he is on a certain hyperbola drawn on the loran chart. By tuning in on a second pair of stations at two other foci broadcasting with another frequency, a second hyperbola of position is determined whose intersection with the first hyberbola fixes the position of the plane. Of course, the distance between a pair of sending stations need not be exactly 186 miles, but the time lag depends on this distance in a simple way.

EXAMPLE: A circular conical surface with semivertical angle $\epsilon = 45°$ is cut by a plane parallel to the axis and k in. from the vertex. What is the curve of section, what is its eccentricity, how far apart are the foci, and how far apart are the vertices?

SOLUTION: Since the plane of section is parallel to the axis, we have $\cos\theta = 1$, $e = \cos\theta/\cos\epsilon = \sqrt{2}$. Since $e > 1$, the curve is a hyperbola. The major axis (AA') of the hyperbolic section makes angles of 45° with the elements (OA) and (OA') through the vertices. Hence $\overline{AA'} = 2a = 2k$ in. The distance between foci is $2ae$ or $2k\sqrt{2}$ in.

35·8 Asymptotes of a hyperbola

Given a nondegenerate hyperbolic section of a right circular conical surface, there is a degenerate section made by a parallel plane through the vertex, which consists of two intersecting lines. The lines through the center of the hyperbola, parallel to these two lines, respectively, are called the **asymptotes** of the hyperbola. It is a fact, which we shall not prove here, that the product of the distances to the two asymptotes is the same for all points on the hyperbola. As a point moves out on a hyperbola, its distance to one of the asymptotes approaches zero while it moves farther and farther from the other asymptote.

35·9 Symmetry of conic sections

Each of the three types of nondegenerate conic sections is symmetric in its major axis. This means that, if a perpendicular is dropped on the axis from any point of the curve and extended an equal distance beyond the axis, its other extremity will lie on the curve. That this is true is easily proved from the focus-directrix property of a conic section.

The ellipse and hyperbola both have two foci F and F'. These curves are called **central conics,** the mid-point of $[FF']$ is called the **center,** and the line perpendicular to the major axis through the center is called the **minor axis.** This minor axis is also a line of symmetry for the central conics, as is seen when these conics are defined as loci for which the sum or difference of distances from a point to the two foci is constant. Since chords through the center are bisected at the center, the central conics are said to be **symmetric** in their centers. The parabola is called a **noncentral conic,** since it has only one focus and no center.

35·10 Sections of an elliptic conical surface

Let a given ellipitic conical surface have a right section that is an ellipse, with eccentricity e and with major and minor axes $[AA']$ and $[BB']$ of lengths $2a$ and $2b$, respectively (Fig. 111). Let a right circular cone having semivertical angle ϵ be constructed with the same axis and with (OB) and (OB') as elements. Then the planes (OAA') and (OBB') are planes of symmetry for both cones. Consider the sections of both cones by a plane perpendicular to (OBB') and making an angle θ with the axis. The section of the circular cone is an ellipse if $\theta>\epsilon$. The section of the elliptic cone is a figure obtained from the other section by stretching all chords parallel to $[AA']$ in the ratio $a/b=(1-e^2)^{-\frac{1}{2}}$. For $\theta>\epsilon$ it is an ellipse the ratio of whose axes is a/b times the ratio for the section of the circular cone. It is a circle if $\cos\theta/\cos\epsilon=e$. Otherwise the curve of section is an ellipse, parabola, or hyperbola whose eccentricity depends only on the angle θ and the given values

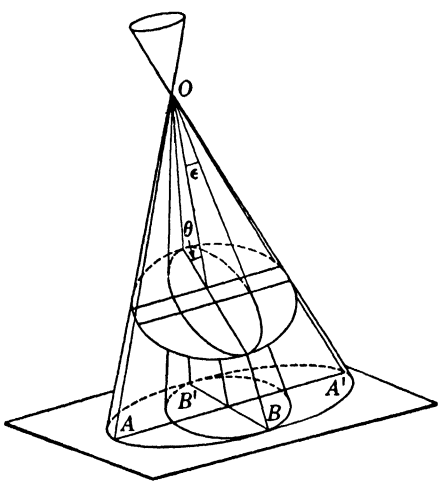


FIG. 111. Circular section of an elliptic conical surface.

of e and ϵ. Two plane sections cut by nonparallel planes each perpendicular to a symmetry plane are similar if and only if the planes make equal angles with the other symmetry plane.

35. ORAL QUESTIONS

A. Can two spheres always be drawn tangent to a given conical surface and to a given plane? What are the exceptions, if any?
B. What are the nappes of a conical surface? What curve of section is obtained in a plane cutting both nappes?
C. What curves are called nondegenerate conics? What is a degenerate conic?
D. What is the difference between the vertex of a cone and the vertex of a conic section?
E. What curve is obtained as a section of a right circular conical surface by a plane parallel to a single element of the surface?
F. What curve is obtained as a section of a right circular conical surface by a plane cutting all the elements of one nappe?
G. How can the eccentricity of a section of a right circular conical surface be expressed in terms of the angle of section and the semivertical angle?
H. What becomes of a right circular conical surface if a directrix circle is kept fixed and the vertex of the surface is made to recede indefinitely along the axis so that the semivertical angle approaches 0?
I. What is the focus-directrix property of conics? How does the type of curve depend on the eccentricity?
J. How are the foci of a conic section related to spheres inscribed in a right conical surface of which the curve is a section?
K. How is a hyperbola defined, and what are some of its properties?
L. What are the asymptotes of a hyperbola?
M. An electric light in the center of a cylindrical lamp shade with vertical axis gives out a cone of light having two nappes. What type of conic section is outlined by the light rays on a vertical wall or window shade?
N. What symmetries do conic sections possess?
O. What sections can be obtained from an elliptic conical surface?

35. WRITTEN EXERCISES

1. Draw a figure in oblique projection (type B) representing a cone of radius 2 in., altitude 4 in., cut by a plane parallel to the base and 1 in. from the vertex.
2. Draw a right circular cone of radius 3 cm. and height 5 cm. in isometric axonometric projection.
3. A right circular conical surface with a semivertical angle $\epsilon = 45°$ is cut by four planes making angles of 30°, 45°, 60°, and 90°, respectively, with the axis and not passing through the vertex. Describe the four curves of section, and find the eccentricity of each.
4. Draw a right circular cone of radius 1.5 in. and height 2 in. in oblique projection, showing an elliptic section made by a plane such that the distances from the vertex of the cone to the vertices of the section are 2.5 in. and 0.7 in., respectively.
5. A circular conical surface with semivertical angle $\epsilon = 30°$ is cut by a plane making an angle of 60° with the axis. If the major axis of the section is $2\sqrt{3}$, find the minor axis and the distance between the foci. Draw a figure in oblique projection.
6. A circular conical surface with semivertical angle $\epsilon = 60°$ is cut by a plane 2 in. from the vertex and making an angle $\theta = 30°$ with the axis. What is the curve of the section, what is its eccentricity, how far apart are the foci, and how far apart are the vertices?
7. Solve Exercise 6 if $\theta = 15°$.
8. If a sphere S is tangent to a circular conical surface K and to a plane of section p that is parallel to an element of K, show that the plane through the center of S perpendicular to the axis of K meets p in a line which is tangent to the parabolic section at its vertex.
9. Show that the length of a focal chord of a parabola drawn parallel to the directrix is equal to twice the distance from the focus to the directrix. This chord is called the **latus rectum.**
10. Using the definition of a parabola and the results of Exercise 9, plot a number of points on a parabola whose focus is 1 in. from

the directrix. Use ruler and compass to obtain at least seven points, and then sketch in the curve.

11. If P is a point on a parabola and Q is the point where the tangent at P meets the tangent t drawn at the vertex V of the parabola, then (PQ) is perpendicular to (QF). Assuming this, let F be any given point, and let the line t be a given line at distance $\frac{1}{2}$ in. from F. For at least seven different points Q on t, draw lines perpendicular to (QF). These lines are tangent to a parabola with focus at F, and the parabola can be sketched in tangent to these lines.
12. Draw both nappes of a right circular conical surface with semi-vertical angle 60°, placing the axis in a vertical position, and showing circular sections 2 in. above and below the vertex O. Draw also a hyperbolic section of this cone by a plane perpendicular to $\uparrow O(Y)$ and 1 in. from the vertex O. Use the dimetric projection of Exercise 13, Chapter 34.
13. Find the eccentricity and major axis of the hyperbola of Exercise 12.
14. Prove the following theorem: *If the tangents to a nondegenerate conic from a point Q on the directrix touch the conic at P_1 and P_2, then the chord (P_1P_2) passes through the focus.* HINT: Reword the hint of Exercise 11, Chap. 33, using "conical surface" in place of "cylindrical surface."

36

LINEAR ELEMENTS IN CENTRAL PERSPECTIVE

36·1 Perspectives of points, lines, and curves

The appearance of objects depends on the position from which they are viewed. Although true lengths and true angles are inherent in the object, apparent lengths and apparent angles may change with a changing point of view.

A projection in central, or true, perspective depicts objects on a plane exactly as they appear when viewed from a particular point O, called the **center of perspective,** and when the observer is focusing his eye in a particular direction, along a line (OC) called the **optical axis,** which passes through the center of perspective. The **picture plane** is a plane perpendicular to the optical axis at a chosen distance from the center of perspective called the **focal length** of the picture. The point C where the optical axis meets the picture plane is called the **optical center** of the picture. Lines through the center of perspective O are called **visual rays,** and planes through the center of perspective are called **visual planes.**

Points, lines, and curves (such as circles) in the object are depicted as follows: (1) A **projection line,** or visual ray, that joins a point of the object to the center of perspective meets the picture plane at a point called the **perspective** of the given object point. (2) The visual rays joining the center of perspective to the points of a line in the object lie in a plane, the **visual plane,** determined by the given line and the center of perspective; and this visual plane meets the picture plane in a line, called the perspective of the given line in the object. (3) The visual rays joining the center of perspective to the points of any curve in the object are elements of a conical surface, called the **visual cone,** that intersects the picture plane in a curve called the perspective of the given curve. If the object curve is an arc of a circle, ellipse, parabola, or hyperbola, the perspective of this curve will also be an arc of a conic section, but not necessarily one of the same type.

36·2 Importance of the point of view

It cannot be overemphasized that a perspective picture of any object depends on the position of the center of perspective and on the orientation of the picture plane and that in order to obtain the correct impression of a perspective picture, the picture must be placed at the correct focal distance from the eye and turned so as to have the correct orientation. A painting in an art gallery that has been drawn in true perspective must be viewed from a particular point in order to obtain the best effect. A photograph of an object is a perspective picture of the object, and it too must be viewed from a distance f equal to the focal length of the photograph in order to obtain an impression of true perspective. It is partly for this reason that an enlargement may be more pleasing to the eye than an original snapshot. The focal length is enlarged in the same ratio as the lines in the picture, and the new focal length may be a more convenient distance for the human eye to focus comfortably on the picture.

36·3 Objects parallel to the picture plane

In a perspective projection, plane objects parallel to the picture plane retain their true angles and shapes. The perspective of such a plane figure is similar to the given figure, and the ratio of corresponding lengths is equal to the ratio of the distances of the picture plane and object plane from the center of perspective. In particular, parallel lines are projected into parallel lines. These facts are an immediate consequence of the fact that parallel sections of a pyramidal or conical surface are similar plane figures (see Fig. 51).

36·4 Vanishing points and traces

In nonparallel perspective, the perspectives of a set of parallel lines that are not parallel to the picture plane are not parallel lines, however. Instead, they all meet in a point that we call the **vanishing point** of the set of parallel lines. This is the point V where the line through the center of perspective O parallel to these lines meets the picture plane (Fig. 112). The proof of this fact is simple. The

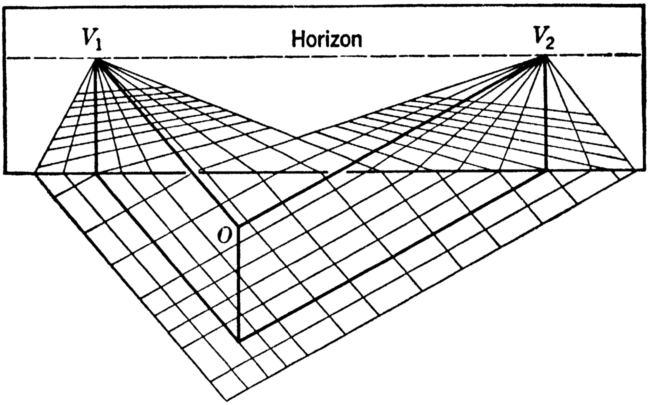


FIG. 112. Vanishing points and perspectives of parallel lines.

visual planes determined by each of the given set of parallel lines except the one through the center of perspective, passing as they do through the center of perspective, must each contain the line parallel to the given line through the center of perspective. Hence these visual planes are coaxial. If the picture plane is not parallel to this axis, it will intersect it in a point V, the so-called **vanishing point,** and this point will lie on the perspectives of all the given parallel lines.

If a given plane of the object is not parallel to the picture plane and does not pass through the center of perspective, then the line in which the picture plane is intersected by the visual plane parallel to the given plane is called the **vanishing trace** of the given plane. This vanishing trace is parallel to those lines of the given object plane which are parallel to the picture plane, and its points are the vanishing points for every other set of parallel lines in the given object plane.

The **horizon** is a line of especial importance when the picture plane is not horizontal. The horizon is the intersection of the picture plane with the horizontal visual plane, it is the vanishing trace of all horizontal planes, and it contains the vanishing points of all horizontal lines except those parallel to the picture plane. Since an actual picture cannot occupy the complete infinite picture plane, it may happen that the horizon line lies outside the finite region of the picture. For theoretical purposes, however, it is often convenient to extend the picture region, in a picture plane that is not itself horizontal, until it includes the horizon line.

36·5 Construction of a perspective of an object from a plan and elevation

Given two orthographic projections of a suitably simple object—one a plan in a horizontal plane and the other an elevation in some vertical plane—it is possible to construct from these a perspective picture of the object on any given plane as viewed from any given point. In this discussion we consider the picture plane to be vertical. Its position can be indicated by drawing its trace (the picture trace) on the plan of the object (Fig. 113).

The projection O_h of the center of perspective and the projection (O_hC_h) of the optical axis should also be located on the plan. This line may be called the **plan axis.** The height of the center of perspective is indicated by drawing the trace of the horizontal visual plane on the elevation. This will be called the **horizon line.** The plan and elevation are now placed on the same sheet of paper so that the plan axis is perpendicular to the horizon line. Their intersection point will be chosen as the optical center C of the perspective picture. The elevation should be moved far enough to the left (or right) and the plan far enough up so that there is room to draw the picture. The point C in the perspective picture, the point O_h in the plan, and the perspective center O in space all appear to coincide in the drawing of Fig. 113. The other points can be plotted as follows.

Let the line (O_hP_h) joining O_h to any important point P_h in the plan meet the picture trace in P_t. Take any line through P_h in the plan (preferably one that makes an angle of not less than 45° with the picture trace), and let it intersect the picture trace in Q_h, the plan projection of a point Q in the picture plane on a level with P. Furthermore, let the line through O_h parallel to (P_hQ_h) meet the picture trace in V_h, the plan projection of the vanishing point V of the line (PQ). The vanishing point V itself can be found at that

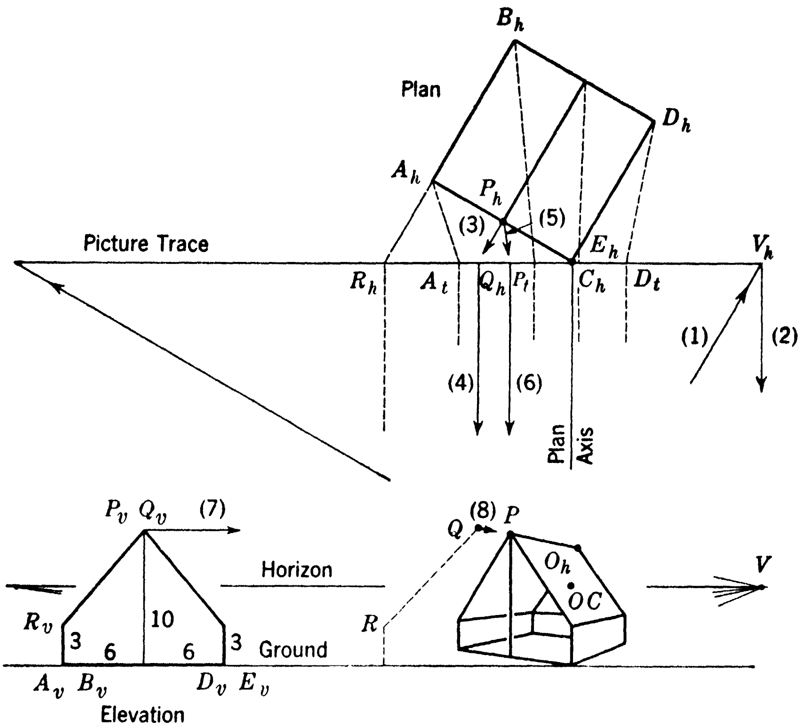


FIG. 113. Construction of a perspective drawing.

point where the horizon line is met by a vertical line through V_h in the drawing. Furthermore, since Q is in the picture plane, its apparent height will be the same as its true height, above or below the horizon line, which is the same as the height of the elevation P_v of the point P. The point Q is located in the picture as the intersection of a horizontal line through P_v and a vertical line through Q_h. The perspective of (QP) is the construction line (QV), and the required perspective of P lies on this line and on the vertical construction line through P_t.

To sum up the procedure, we connect a point P in the object with some point Q on the same level in the picture plane. Since distances of points in the picture plane are unchanged in the perspective, the point Q can be obtained directly from the plan and elevation. Having found the vanishing point V of (PQ) we can then draw the perspective of (PQ). To locate the perspective of P we find where the vertical visual plane through P meets (QV) as follows: We proceed from P_h toward O_h to P_t in the plan and then go vertically in the picture plane to the required point on (QV).

Other points are plotted in like manner, the same vanishing point being used as often as possible, to save work. The use of a second vanishing point on the other side may be useful as a check, however, even when not absolutely necessary. The points, having been plotted, the visible lines are drawn in heavily, and the invisible ones either as dotted lines or not at all.

EXAMPLE: In Fig. 113 the tent of Exercise 12, Chap. 20, is drawn in perspective as viewed from a point 24 ft. from a corner of the tent and 6 ft. above the tent floor. The floor measures 12 by 12 ft.; the ridgepole is 10 ft. above the floor, and the sides rise to a height of 3 ft. above the floor. The point P whose construction is given in detail is at the front of the ridgepole. This point, and each other point turn, is determined as the intersection of a vertical construction line and a construction line through a suitable vanishing point such as V. The vertical construction line for P is drawn through the trace point P_t where the line (O_hP_h) meets the picture trace in the plan. The construction line through V is the perspective of that one of a chosen convenient set of parallel lines which contains the required point. Each point is projected into the picture trace by parallels of this set.

Then, on verticals through these points of the picture trace (such as Q_h), the elevations (of Q, etc.) above and below the horizon line can be measured directly from the elevation drawing at the left. The points (such as Q) where the chosen set of parallels meet the picture plane are plotted directly from the plan and elevation. Then they are joined to the vanishing point V to give the perspectives of the chosen set of parallel lines. Where this perspective line for P crosses the vertical through P_t is the required point P. To obtain the effect of true perspective, the tent in Fig. 113 should be viewed with a magnifying glass or it should be copied using a scale six to eight times as large, since the human eye cannot easily focus at so short a focal length O_hC_h. This undesirably short focal length was chosen in order to show all the construction lines on a single page of the text.

36. ORAL QUESTIONS

A. How does central perspective differ from parallel projection?

B. Parallel projection is associated with sections of cylindrical surfaces; central perspective is associated with sections of conical surfaces. In what sense is this statement true?

C. How would a parabolic section of a right circular cone appear if viewed from the vertex of the cone looking along the axis?

D. What can be said of the projections of a set of parallel lines in central perspective (*a*) if they are parallel to the projection plane? (*b*) If they are not parallel to the projection plane?

E. How are vanishing points located in a perspective projection?

F. What is the vanishing trace of a set of parallel planes? Does every plane have a vanishing trace in the projection plane?

G. What is the focal length of a drawing?

H. If a 6-ft. man is represented by an image 3 in. tall, in a photograph taken from 12 ft., from what distance should the photograph be observed to get correct perspective?

I. What plane is determined by the optical axis and the horizon line? How are these lines represented in the orthographic plan and elevation of a given object?

J. How can the perspective of a point on a vertical picture plane be obtained from a plan and elevation of the point?

36. WRITTEN EXERCISES

1. Construct a perspective drawing of a 2-in. cube placed in the first octant with three edges along the edges of a trihedron of reference with respect to which the center of perspective has coordinates (4,3,1.2). Place the picture plane vertically through the nearest edge of the cube so that the plan axis goes through the vertex of the trihedron.
2. Using the perspective drawing of Exercise 1, mark the mid-points of the faces of the cube, connecting these points to form a regular octahedron.
3. Construct a perspective drawing of the cube of Exercise 1 if the center of perspective has coordinates (4,3,3).

4. Construct a perspective drawing of the Great Pyramid of Khufu, viewed from a convenient point situated on a level with its base at a distance of about 1,000 ft. from one corner, so that two faces are visible at different angles. See Exercise 9, Chap. 15.
5. Measure the perspective drawing in Fig. 113, and find the focal length. In what ratio would the figure have to be enlarged to appear in correct perspective at a distance of 1 foot? Make a copy of the final perspective figure enlarged in this ratio, omitting construction lines.
6. Show that the focal length of a vertical perspective drawing is never more than half the distance between the vanishing points of two mutually perpendicular horizontal lines. HINT: These vanishing points subtend a right angle at the center of perspective.
7. In a right triangle with sides a, b, c the bisector of the right angle divides the hypotenuse c into segments u and v that are proportional to a and b. If h is the altitude on the hypotenuse, show that $c/h=(u/v)+(v/u)$. Apply this equation to find the focal length of a vertical perspective drawing if the vanishing points of two sides of a square in a horizontal plane are 12 in. apart and if the vanishing point of one diagonal of the square divides this 12-in. segment into 4- and 8-in. segments.
8. Construct a perspective drawing of a right circular cylinder circumscribed about a sphere, assuming that the cylinder is semitransparent so that only its outlines show. Place the horizon plane slightly above the center of the sphere.
9. Construct a perspective drawing of a frustum of a right circular cone whose elevation is a trapezoid with bases 2 and 4 in. and height 3 in. Assume the horizon plane to be 1 in. above the top base. Place the center of perspective at a distance of 5 in. from the axis of the frustum.
10. Construct a perspective drawing of a card table whose top is 30 by 30 by 2 in. and whose four legs are 24 in. long. In the plan, let the plan axis pass through the mid-points of two adjacent sides of the table, and let the center of perspective be 45 in. and 60 in., respectively, from these sides produced. Let the horizon line be 12 in. above the table. Divide all dimensions by 12, and place the picture plane through the nearest point of the table.

11. Construct a perspective drawing of a double garage with a 20-ft.-square floor space, with walls 10 ft. high, and with a roof in the form of a square pyramid whose vertex is 16 ft. above floor level, viewed from a height of 6 ft. and at any conveniently chosen point about 40 ft. away from one corner, using a focal length of 4 in. Omit the doors, but show a partition between the two halves of the garage.

37

PERSPECTIVE AND MAPPING FROM PHOTOGRAPHS

37·1 Perspective in photographs

The representation of objects as seen in central perspective is common in photography. The novice who points his camera upward in taking a picture of a tall building discovers to his surprise that the sides of the building do not look parallel in his photograph but appear to converge to a point. And the person in the front row of a group photograph who sticks his foot too far forward will find his foot making more of an impression than his face in the finished photograph. The human eye refuses to focus simultaneously on objects that are near by and others that are far away, and thus the mind is not conscious in direct vision of some of the unpleasant effects of central perspective that may occur in photographs. We sometimes like to see things, not as they are, but as we are accustomed to think of them and as the mind can most easily interpret them.

Not only may photographs be used for creating a reproduction of an object that may be considered aesthetically pleasing or that may show certain features in relationship to others. Photographs may also be used for the geometrical purpose of making exact measurements of objects—a common practice in aerial photography. The difficulty here is that the distances in the photograph will not always bear a simple proportional relationship to distances in the object photographed; in fact, they will do so only when the object is a plane object that is photographed onto a plate or film parallel to its plane. After investigating the geometry of this simple case first, we shall consider the effect of tilt distortion and how a geometric construction may be devised to correct for it.

37·2 Mapping plane figures to scale

The simplest way a plane region S_o can be depicted is by mapping it to scale. The triangle formed by joining together any three points A_o, B_o, P_o in the object plane p_o is similar to the triangle formed by joining together the three corresponding points A', B', P' in the image, or mapping plane p'. The ratio of corresponding sides in the two triangles is called the representative fraction (R.F.) of the map. The same relation of similarity holds between corresponding polygons in the two planes. If the mapping plane is placed parallel to the object plane so that corresponding line segments in the two planes are parallel, then (by Desargues' theorem, Exercise 10, Chap. 11) the lines joining corresponding points in the two planes are concurrent in a point O, called the center of perspective (Fig. 114). The lines from O to the three selected points A_o, B_o, C_o in the object plane are edges of a trihedron that might be called the **visual trihedron.** If the eye were placed at O and the map S' were transparent, each point in the map would appear to cover the corresponding point in the region S_o of the object plane p_o.

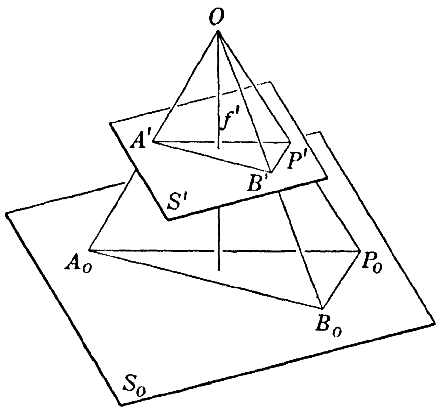


FIG. 114. Undistorted map on parallel plane.

In the case of photographic mapping, a triangle in an object plane that is being photographed, the corresponding triangle in the printed positive photograph, and its image in a map that is being constructed from the photograph can be considered as three plane sections from the same trihedron, and the corresponding triangle on the camera-plate negative is then a section of an associated oppositely directed trihedron with its edges parallel to the corresponding edges of the first trihedron (Fig. 115).

The two vertices of these trihedrons are called **nodal points and**

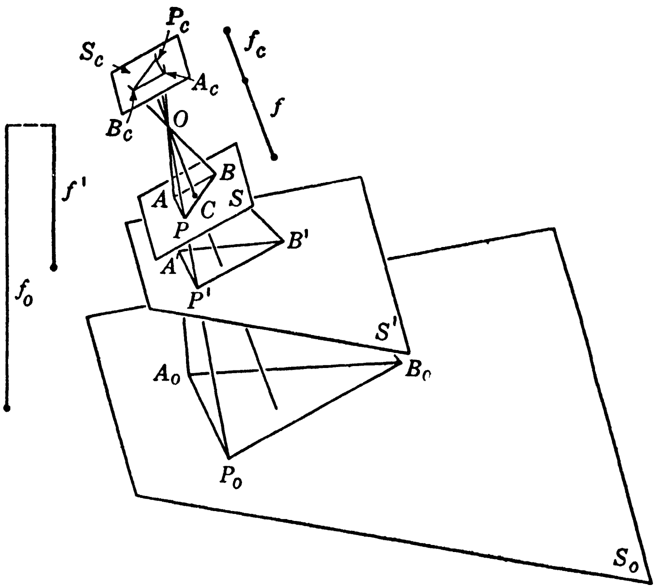


FIG. 115. Central projection of a plane region onto a parallel map and a tilted photograph.

are associated with the lens system of the camera. They lie close together on a line called the **optical axis,** which is perpendicular to the lenses and to the camera plate and passes through the center of each. The geometry of the figure is somewhat simplified if we consider the camera plate and its trihedron to be moved along the optical axis until the two nodal points coincide in a single center of perspective O. The two trihedrons then have the same vertex and the same lines as edges.

The line joining any object point P_o to the corresponding point P_c on the newly placed camera plate (or film) then passes through O if the objects photographed are clearly in focus. The distance f_c from the point O to the camera plate is the **focal length** of the camera, provided that the camera is focused at infinity (Fig. 115).

In this chapter we shall consider only a special case of the complicated problem of aerial photography, namely, the case where objects photographed are nearly enough on the same level so that it is appropriate to speak of a horizontal object plane p_o at the distance f_o from O. We assume that a region S_o of p_o is pictured on a rectangular region S_c of the camera plate p_c. The camera plate itself may be replaced by a positive photographic print, either of the same size or with lengths enlarged in the ratio f/f_c. The length f is defined to be the focal length of the photographic print. In the following discussion we assume that this *photo*, as we shall call the positive print, is placed in a plane p perpendicular to the optical axis at a distance f from O on the side toward the object plane and is so situated that its center C is on the optical axis and that each set of corresponding points—P_o in the object plane p_o, P_c on the camera plate p_c and P on the photo p—are all in the same line through O. That this is possible is due to the fact that corresponding figures on the camera plate and on the photo are similar. The photo itself will be a rectangular region S of the plane p. When it is possible to take a photograph with the camera plate parallel to the object plane, then the photograph itself gives an undistorted map of the given plane region S_o (Fig. 114).

37·3 Tilt distortion

However, if the camera plate is tilted at an angle of tilt τ with the vertical, there is a distortion of angles and relative lengths in the photo as compared with those in the object. This tilt distortion must be corrected in reconstructing from the photo an undistorted map of the object plane. The geometric relationship between photo and map can best be seen by using the same center of perspective O for both the photo and the map referred to the original object plane. The map will then be in a horizontal plane p' and the photo in a plane p that makes an angle τ with p'. The scale of the map is proportional to its distance f' from O. A convenient choice of scale f'/f_o is obtained when we take $f'=f$. The planes p and p' then intersect each other in a fixed line m, (DB), each of whose points is fixed in the mapping (Fig. 116).

Certain points, lines, and planes play an important role in the mapping when the angle of tilt τ is a positive angle less than 90°.

1. The **optical axis** (OC) pierces the rectangular photo p in the photo center C, $(\overline{OC}=f)$ but meets the mapping plane p' in a point C' that is not the center of the corresponding region S' of p' (Fig. 117).

2. The **vertical axis** (OV) through O meets p and p' in the vertical points V and V'.

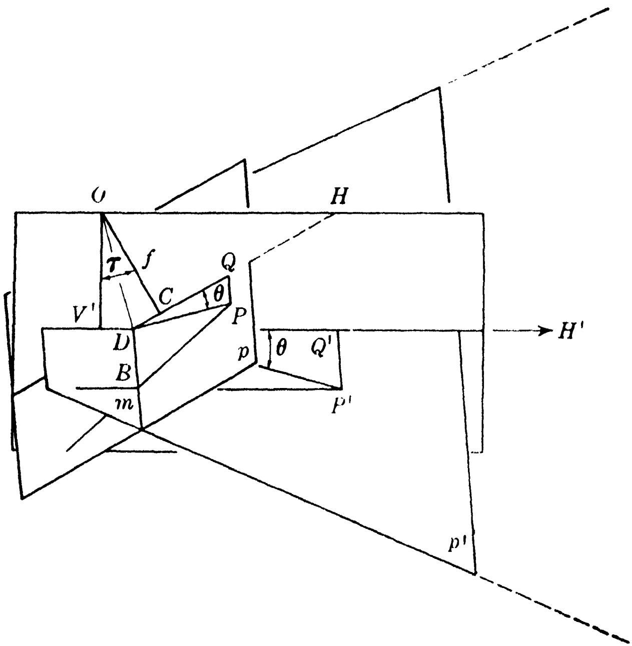


FIG. 116. Tilt distortion in central projection.

3. The vertical plane (VOC), which is called the principal plane, cuts the photo p and the map p' in their **principal lines** l and l', respectively.

4. The **true horizon line** (HK) is where the horizontal plane through O meets the photo plane p, extended if necessary beyond the actual photo. The point H where l meets (HK) is called the **principal horizon point.** The distance $\overline{OH}$ from O to the horizon line is denoted by h (Fig. 117).

5. The bisector (OD) of $\angle VOC$ meets the principal lines l and l' in their intersection point D, which is called the **distortion center.** Since the triangle $[DHO]$ is isosceles, the distance $\overline{DH}$ is also equal to h.

6. The two planes p and p' intersect in a line m called the **fixed line** (Fig. 116), which is perpendicular to the principal plane at D. All points B on m are fixed in the projection of p to p' from O.

7. The line (H_2K_2) in the photo plane p, which is halfway between the fixed line and the horizon line, will be called the **halfway line**

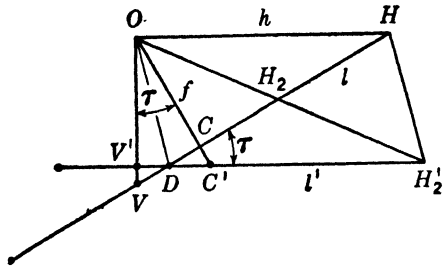


FIG. 117. Points in the principal plane.

(Fig. 118). Its image in p' is at distance h from the fixed line. The point H_2 is defined to be the mid-point of $[DH]$.

The positions of the important points O, V,D,C, H,H_2,H_2' in the principal plane are shown in Fig. 117. From the figure it is seen that $\measuredangle VOC = \measuredangle HDH_2' = \tau$, that $\measuredangle COH = 90° - \tau = \bar{\tau}$, and that $\measuredangle DOC = \tau/2$.

Since $\overline{OH} = \overline{DH} = h$, we have

$$(1) \qquad f = \overline{OC} = h \cos \tau \qquad \overline{CH} = h \cos \tau \qquad \overline{VH} = \frac{h}{\cos \tau}$$

$$(2) \qquad \overline{DC} = h - h \cos \tau = \frac{f(1 - \cos \tau)}{\cos \bar{\tau}}$$

If f and τ are known, the points C, H, D, and V can be located in the photograph. Given f, if any two of these points, such as V and C, can be found by inspection of the photograph, the rest of the measurements can be obtained. C, of course, is the center of the photograph, and V is the point where a vertical line (such as the intersection of two walls of a building) would appear as a point in the photograph. Once these points have been located, the actual geometric construction of an undistorted map from the photograph is easily described by the following properties:

PROPERTY 1: Lines parallel to the fixed line m in p are mapped into lines parallel to the fixed line in p'. This follows from the fact that planes through O parallel to m meet both the planes p and p' in lines parallel to m.

PROPERTY 2: Lines in p through the distortion center D are mapped into lines in p' through D that make the same angle with the fixed line (Fig. 116).

PROOF: Let (DP) be a line through D in p making an angle θ with the principal line l and an angle $90° - \theta$ with the fixed line m, and let (DP') be its image in p'. Then since p and p' are both perpendicular to the principal plane and are equally inclined to the edge

(OD) of the dihedral angle, $\angle H\text{-}OD\text{-}P$, a pair of trihedrons cut out from this dihedral at D by the planes p and p' are equal and the corresponding face angles are equal. Hence, $\measuredangle P'DH' = \measuredangle PDH = \theta$.

PROPERTY 3: Each line (BP) in p through a point B on the fixed line m is mapped into a line (BP') through B in p' (Fig. 116) such that, when the photo and map are superimposed so that their principal lines coincide and their fixed lines coincide (Fig. 118) then $\angle PBP'$ intercepts on the horizon line (HK) a directed segment $\uparrow [P_1P_2']$ equal to $\uparrow [DB]$. In particular, the line (BH) is mapped into the line parallel to l' through B.

PROOF: In the photo plane p, let P_2 be the point where (BP) meets the halfway line (H_2K_2), and let P_1 and P_2'' be the points in which (BP_2) and (DP_2), respectively, meet the horizon line (HK). (Note that P_2'' coincides with P_2' in Fig. 118.) Then triangles $[DP_2B]$ and $[P_2''P_2P_1]$ are congruent, with corresponding parts parallel and equal but oppositely directed. Hence $\overrightarrow{DB} = \overrightarrow{P_1P_2''}$.

In the mapping, D, B, and P_2 of the plane p are carried into D, B, and P_2' of the plane p', where P_2' is the point that occupies the same position relative to m and l' in the plane p' that P_2'' does in the plane p relative to m and l.

THEOREM 37: Given a point P in a tilted photo plane p with distortion center D, fixed line (DB), principal line (DH), horizon line (HK), and halfway line (H_2K_2). Then its image P' in an undistorted map (superimposed on p so as to have the same principal line and fixed line as p) is the point of intersection of (DP_2) and (BP_2'), where P_2 is the point in which the line (BP) through a fixed point B (other than D) meets (H_2K_2), and P_2' is the point where (DP_2) meets (HK).

A figure that appears as the trapezoid $PART$ of the plane p is shown in Fig. 118, transformed into the rectangle $P'A'R'T'$ of a map, by use of the method of this theorem for mapping each of the four vertices.

37·4 Mapping factors

DEFINITION: The ratio of the map length $\overline{P'T'}$ to the photo length $\overline{PT}$ of a segment parallel to the fixed line through a point P will be

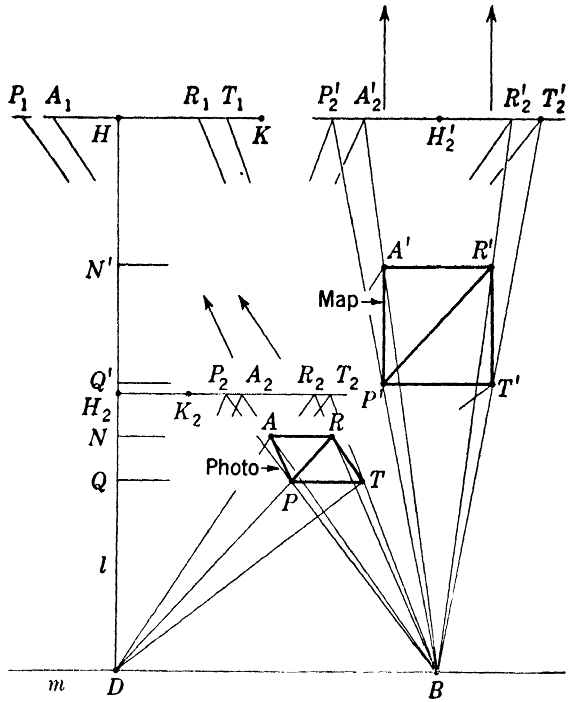


FIG. 118. Construction of map from tilt-distorted photo.

called the **local lateral mapping factor** at P. The limit of the corresponding ratio for short segments $\overline{P'R'}$ and $\overline{PR}$ on a line (DP) through the distortion point, as the point R approaches P, will be called the **local radial mapping factor** at P (Fig. 118).

NOTATION: Let P and R be distinct points on a radius through D in the plane p; Q and N their orthogonal projections on the principal line l; and P', R', Q', N' the corresponding points in p' obtained from P, R, Q, and N, respectively, by central projection from O. Denote certain directed distances as follows:

$$\overrightarrow{DQ}=x_1 \qquad \overrightarrow{DQ'}=x_1' \qquad \overrightarrow{DN}=x_2 \qquad \overrightarrow{DN'}=x_2'$$

(3) $$\overrightarrow{QP}=y_1 \qquad \overrightarrow{Q'P'}=y_1' \qquad \overrightarrow{NR}=y_2 \qquad \overrightarrow{N'R'}=y_2'$$

$$\overrightarrow{DP}=r_1 \qquad \overrightarrow{DP'}=r_1' \qquad \overrightarrow{DR}=r_2 \qquad \overrightarrow{DR'}=r_2'$$

PROPERTY 4: The local lateral mapping factor at a point P is equal to the ratio of the distances to the horizon line from the distortion center D and from the given point P, respectively.

$$\text{(4)} \qquad \text{Lateral mapping factor: } \rho_1 = \frac{h}{h-x_1}$$

PROOF: 1. By the definitions given above, the local lateral mapping factor ρ_1 at P is equal to y_1'/y_1.

2. Since triangles $[DQP]$ and $[DQ'P']$ are similar by Property 2, we have $x_1'/x_1 = y_1'/y_1 = r_1'/r_1$.

3. Since triangles $[Q'DQ]$ and $[OHQ]$ in the principal plane are similar (Fig. 116), we have $x_1'/x_1 = \overline{OH}/\overline{HQ} = h/(h-x_1)$.

4. Hence, by substitution of equals for equals, $\rho_1 = h/(h-x_1)$.

PROPERTY 5: Segments along radii through the distortion center have their lengths multiplied in the mapping by the product of the local lateral mapping factors at their end points.

PROOF: Let P and R be points on a line through D (Fig. 118), and let $\rho_1 = h/(h-x_1)$ and $\rho_2 = h/(h-x_2)$ be the local lateral mapping factors at P and R, respectively. Then

$$\text{(5)} \qquad \frac{\overline{P'R'}}{\overline{PR}} = \frac{r_2'-r_1'}{r_2-r_1} = \frac{\rho_2 r_2 - \rho_1 r_1}{r_2-r_1} = \rho_1\rho_2 \frac{(r_2/\rho_1)-(r_1/\rho_2)}{r_2-r_1}$$

But

$$\text{(6)} \qquad \frac{r_2}{\rho_1} - \frac{r_1}{\rho_2} = \frac{r_2(h-x_1)}{h} - \frac{r_1(h-x_2)}{h} = r_2 - r_1 + \frac{x_2 r_1 - x_1 r_2}{h} = r_2 - r_1$$

Hence

$$\text{(7)} \qquad \frac{\overline{P'R'}}{\overline{PR}} = \rho_1\rho_2$$

Q.E.D.

PROPERTY 6: The local radial mapping factor is equal to the square of the local lateral mapping factor.

PROOF: Let P approach R, and apply Property 5.

We note that the mapping factors ρ are greater than 1 on the "upper" side of the fixed line, toward the horizon, but less than 1 on the lower side. It is clear from Figs. 115 and 118 that the mapping we have described represents an enlargement on the side of the fixed line toward the horizon in the photo and a shrinking on the other side, which takes the rectangular photo into a trapezoidal map. The inverse transformation from map or object to photo takes a rectangle $P'A'R'T'$ of the map or object into a trapezoid $PART$ of the photo that has its narrow base on the side toward the horizon (Fig. 118).

37. ORAL QUESTIONS

A. Does a photograph show near-by objects in central perspective or in orthographic projection?

B. In order to obtain a picture of a plane object without distortion of angles and without the use of mirrors, how must the plane of the object be related to the plane of the image?

C. What is meant by the representative fraction of a map of a plane region that is drawn to scale?

D. How would you locate the vertical point in an aerial photograph of a city?

E. How is the distortion center related to the photo center and the vertical point in a tilted photograph of a plane horizontal region?

F. What is meant by each of the following terms with respect to a tilted photograph of a plane horizontal region: principal line; fixed line; horizon line; halfway line?

G. What are three properties of the mapping from a tilted photograph to an undistorted map of a plane horizontal region if the map and photograph have the same focal distance to the center of perspective?

H. In the mapping of Question G, which part of the photograph is enlarged, and which is made smaller in the mapping?

I. How does the local radial mapping factor compare with the local lateral mapping factor at a point?

J. How can the map image of a point P in a tilt-distorted photo be constructed if the position of the distortion center and horizon line are known?

37. WRITTEN EXERCISES

1. Draw and label a figure showing the important reference points and lines in a photograph (of a plane horizontal region) measuring 6 by 8 in. if the angle of tilt is $\tau = 30°$ and if $h = 4$ in. Let the horizon line be parallel to one of the short sides of the rectangle.
2. In the figure of Exercise 1, show that the horizon line should be just over $\frac{1}{2}$ in. from the edge of the photograph. What is the distance from the photo center to the vertical point? What is the focal length?
3. Draw to scale the trapezoidal boundary of an undistorted map of the same region represented by the rectangular photograph of Exercise 1. What are the lengths of its sides if the map and photograph have the same focal length?
4. Draw a rectangle in the map of Exercise 3, and construct its image in the photograph of Exercise 1, showing in detail the construction of one of its vertices.
5. Draw and label a figure similar to that of Exercise 1 if the angle of tilt is $\tau = 15°$. What is its focal length?
6. Describe how the horizon line can be located in the plane of a rectangular tilted photograph taken of a horizontal object, even when the horizon itself does not show in the picture region, provided that the position of the vertical point can be found first.
7. Describe a method of reconstructing a photo from a map. In particular, show how to draw a perspective photograph of the bottom face of a cube as viewed from the mid-point of an edge of the opposite face if the angle of tilt is $\tau = 45°$ and the optical axis passes through the center of the cube.
8. Draw a floor plan of 12- by 12-ft.-square floor, with an 8- by 10-ft. rug centered on the floor. Then draw a perspective photo as viewed from a point 6 ft. above the middle of one end of the rug, using an angle of tilt $\tau = 30°$, and taking the principal line along the middle of the rug. Use a scale of $\frac{1}{2}$ in. to a foot. HINT: The vertical point is given. Locate the center, the distortion point, the horizon, and the halfway line. Then locate the images of the four corners of the room and of the four corners of the rug.

38

THE STEREOGRAPHIC PROJECTION

38·1 Perspective projections of a spherical surface

The appearance of a sphere and of circles or other curves on its surface depends on the point of view of the observer. When the surface of the full moon is viewed through a telescope, the moon is so far away compared with its diameter that the rays of light coming from different parts of it are nearly parallel and one half of its surface appears as the interior of a circle, the other half being invisible. The view is essentially that of the orthographic projection that we have already studied in Chaps. 22 and 32. An observer in a rocket ship approaching the moon would gradually see less and less of its surface. The visible portion would be a zone including less and less of the moon's surface but occupying a larger and larger circle in the observer's field of vision. Drawings to illustrate these views should be done in central perspective. Circles on the sphere would generally be represented either as circles or ellipses, but those circles whose planes pass through the point of observation would be seen as straight lines.

When the plane of projection is perpendicular to the optical axis and the sphere is viewed from the outside, the outline of the sphere in projection will be a circle. If the plane of projection is oblique to the optical axis, the outline will be an ellipse, as it is in oblique parallel projection.

38·2 The stereographic projection

If, instead of viewing a solid sphere from the outside, we view a hollow spherical surface (like a planetarium dome or the Perisphere of the 1940 World's Fair in New York) from a point O inside, the points of a zone of the sphere occupy the whole field of vision, instead of just the interior of a circle. In particular, if O is the center of the sphere, the projection is the **gnomonic projection,** which will be discussed in the next chapter. Finally, if the point of observation O is on the surface of the sphere, there is a one-to-one correspondence between the points of a projection plane (placed perpendicular to the diameter of the sphere through the point of observation) and the points of the spherical surface, with the single exception of the point of observation itself. This perspective projection is called a **stereographic projection** of the sphere.

A stereographic projection is a perspective projection of the surface of a sphere S onto a plane p in which the center of perspective O is a point on the spherical surface, the optical axis lies along a diameter $[OC_0]$ of the sphere, and the plane p is perpendicular to the optical axis at a point C, called the **center** of the projection plane (Fig. 119). The great circle K_0, of which O and C_0 are poles, divides the sphere into two hemispheres. The hemisphere containing C_0 is projected into the interior of a circle K, center at C, called the **primitive circle,** and the other hemisphere containing O is projected into the exterior of the primitive circle K. The point O itself has no image on the projection, but points near O project into points at a great distance from K.

38·3 Three important properties of stereographic projection

PROPERTY 1: *The stereographic projections of circles through O on the sphere S are straight lines; conversely, straight lines in the projection are the image of circles through O on S.*

PROOF:

1. Any circle on S through O lies in a visual plane.
2. This plane intersects the projection plane p in a line.
3. Any line in p determines with O a visual plane.
4. This plane intersects the sphere S in a circle through O.

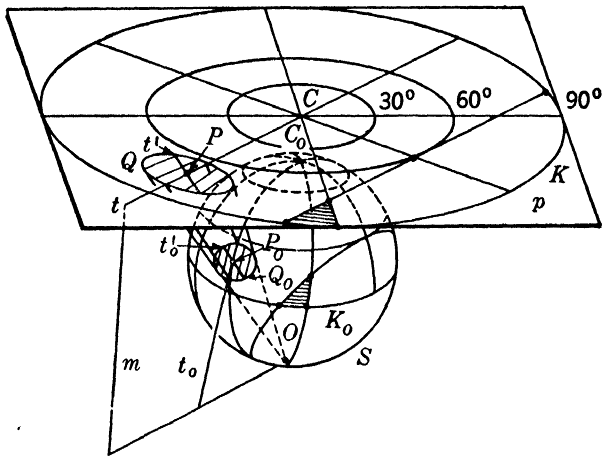


FIG. 119. Stereographic projection of a sphere.

PROPERTY 2: *The stereographic projections of circles on the sphere S that do not pass through O are circles.*

PROOF: 1. If O is a pole of the given circle Q_0, then the visual rays through O and the points of Q_0 form a right circular cone of which the axis is $[OC]$ and which intersects the projection plane p in a circle Q, center at C.

2. If O is not the pole of the given circle Q_0 in S, the visual rays through O and the points of Q_0 determine an elliptic conical surface E, having O as vertex and Q_0 as directrix (Fig. 119).

3. A meridian plane m, which contains the optical axis $[OC_0]$ and the center of the circle Q_0, is a plane of symmetry for the elliptic conical surface. If it cuts Q_0 in the points A_0 and B_0, then the two semicircles of Q_0 whose extremities are at A_0 and B_0 are mirror images of each other in the plane m. The plane q_0 of the circle Q_0 is perpendicular to m, and the corresponding halves of the conical surfaces are also mirror images of each other in m.

4. The bisector of the angle, $\angle A_0OB_0$, meets the sphere S in a pole P_0 of Q_0, and (OP_0) is the axis of the elliptic visual cone E.

5. The plane p_0, tangent to the sphere S at P_0, is parallel to the plane q_0.

6. The angle between the conical axis $[OP_0]$ and the plane p_0 is

the complement of the angle, $\measuredangle C_0OP_0$, between $[OP_0]$ and the optical axis $[OC_0]$.

7. Likewise, the angle between the conical axis $[OP_0]$ and the projection plane p is the complement of the angle, $\measuredangle C_0OP_0$.

8. The planes q_0 and p, being equally inclined to the axis of the elliptic cone and perpendicular to one of its symmetry planes, cut similar sections Q_0 and Q from the cone (see Sec. 35·10).

9. Since Q_0 is a circle, its image Q is a circle. Q.E.D.

DEFINITION: A projection of a surface S on a plane p is called **conformal** at a point P if P is the image of a point P_0 on S and if the angle between the tangents to any two curves on S through P_0 is equal to the angle between the tangents to the corresponding image curves through P in the projection.

PROPERTY 3: *Stereographic projection of a sphere is conformal at every point in the projection plane.*

PROOF: Let P_0 be a given point on the sphere S, t_0 the tangent at P_0 to the meridian circle through O and P_0, and t_0' the tangent to any other curve on S through P_0. Let p_0 be the plane containing t_0 and t_0', which is therefore tangent to S at P_0, and let m and p be the meridian plane and projection plane, respectively. Then the projections on p of t_0 and t_0' are the tangents t and t' to the two projected curves through P.

1. The visual plane m' determined by t_0' and t' forms with the visual plane m determined by t_0 and t a dihedron cut by planes p and p_0, each perpendicular to m and each equally inclined to the edge (OP_0).

2. The right trihedron with vertex at P_0 and edges along $\uparrow [P_0P]$, t_0, and t_0' is enantiomorphous to the right trihedron with vertex at P and edges along $\uparrow [PP_0]$, t, and t', since corresponding dihedral angles with edges t_0 and t are right dihedral angles, corresponding dihedral angles along $[PP_0]$ between m and m' are equal, and corresponding face angles in the plane m are equal; but the corresponding parts occur in opposite order.

3. The angles between t_0 and t_0' at P_0 and between t and t' at P are equal, since they are corresponding face angles of two congruent (enantiomorphous) trihedrons.

38·4 Arcs of great circles in stereographic projection

Each great circle, except the primitive circle K, intersects K at the extremities of a diameter of K (Fig. 120). The projection G of a

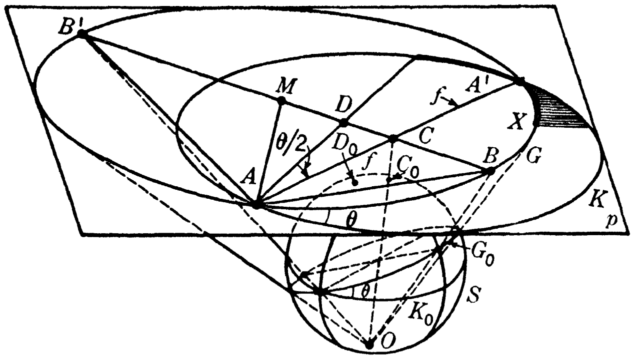


FIG. 120. Stereographic projection of great circles.

great circle G_0 can be determined if this diameter $[AA']$ and one other point X are known, since three points determine the circle G. If the plane of the great circle G_0 makes an angle θ with the projection plane, then G will make an angle θ with K at both the points A and A'. If M is the center of the projected circle G and C the center of the primitive circle K, then (MC) is the perpendicular bisector of $[AA']$, and the segment $[MC]$ subtends an angle θ at each of the points A and A'. Hence, if f is the radius of K, the radius of G will be $f/\cos\theta$, and $\overline{MC} = f \cos\bar{\theta}/\cos\theta$. The points B and B' where (MC) meets G will be at distances $(f/\cos\theta)(1 \pm \cos\bar{\theta})$ from C on opposite sides of C.

The image D of one of the poles D_0 of G_0 located at a distance $f(1-\cos\theta)/\cos\bar{\theta}$ from C on (CM) serves as a distortion center (Sec. 37·3) for projecting points from G onto K and thus laying off on G the equivalent of a given arc on K. In fact, any dihedron with (OD) as edge cuts off equal arcs on the two great circles G_0 and K_0 whose projections are G and K.

To locate the distortion center D, bisect either of the right angles, $\angle BAB'$ and $\angle BA'B'$, and mark the point where the bisector meets the line of centers of the circles G and K. Or construct $\angle BAD$ so as to intercept an arc of 90° on the circumference of K.

To measure an arc on a great circle G_0 from its stereographic projection G, locate its distortion center D, and project the arc of G from D onto K. Then measure the arc on K by a protractor.

38·5 Graphical solution of spherical triangles

When projected stereographically, a spherical triangle appears as a plane curvilinear triangle bounded by the arcs of three circles. The angles between these arcs are the angles of the spherical triangle, since the projection is conformal. The spherical measure of each arc can be obtained by locating its distortion center and projecting its end points onto the primitive circle K. In making a drawing involving a single spherical triangle in stereographic projection, it is usually easiest to take one of its sides along the primitive circle K so that it can be measured directly.

Given two sides and the included angle or two angles and the included side, a spherical triangle can be drawn to scale in stereographic projection, and its other parts can then be measured, the angles directly and the sides by projection on K from the appropriate distortion center.

38. ORAL QUESTIONS

A. What are three types of perspective projections of a sphere? How do they differ in the location of the center of perspective?

B. Into what curves do circles project in stereographic projection? Are there exceptions?

C. What are three important properties of stereographic projection?

D. One half of a great-circle arc in stereographic projection is represented by a circular arc inside the primitive circle and ending on the extremities of a diameter thereof. Where does the other half lie?

E. In the stereographic projection of a spherical surface, are there any points on the sphere that have no image in the projection plane? Which?

F. What is meant by a conformal mapping or projection?

G. What is the condition that a circle in a stereographic projection represents a great circle on the sphere?

H. Into what figure does a spherical triangle project in stereographic projection? Discuss all cases.

I. What is the relationship between the angles of a spherical triangle and the angles of its stereographic image? What can be said of the sides?

J. How can arcs be laid off to scale in stereographic projection?

38. WRITTEN EXERCISES

1. Prove that, in any perspective projection of a spherical surface, straight lines in the projection are images of circles on the spherical surface.
2. Prove Property 2 of Sec. 38·3, giving the reasons for each step.
3. Prove that the stereographic projection of a sphere is conformal, giving the reasons for each step.
4. Draw a figure showing a sphere, a circle on it, a plane of projection not parallel to the plane of the circle, and the stereographic projection of that circle on the plane.
5. Draw a stereographic projection of the Northern Hemisphere, using the south pole as center of perspective and drawing in the meridians and parallels for every 30°.
6. Draw a stereographic projection of a hemisphere of the earth, using a point on the equator as center of perspective. Draw in the projections of meridians spaced 30° apart and also that of the equator.
7. Lay off in stereographic projection a spherical triangle having one side (placed on the circle K) of length 60° and the adjacent angles equal to 60° and 45°, respectively. Measure the arcs on the other two sides by projecting them onto K.
8. Draw a figure as in Exercise 7, taking 90° for the side and 45° for each of the adjacent angles.
9. Draw a figure as in Exercise 7, taking 120° for the given side, 90° for an adjacent angle, and 45° for the side beyond the 90° angle.
10. Describe and illustrate a construction for finding the image of the pole of a great circle if its stereographic image and the primitive circle are given.

39

OTHER PROJECTIONS OF A SPHERICAL SURFACE

39·1 Distortion in spherical projections

It is impossible to represent a portion of a spherical surface on a plane without some distortion of relative lengths. If only a small portion of the surface is represented, the distortion may be so small that it cannot be detected or small enough so that for the purposes for which the projection is used the distortion may be neglected. In such cases the practical man, who is interested only in approximate measurements and not in mathematically exact measurements, will say there is no distortion. Just when the distortion which is always present becomes large enough to be significant depends on the scale of the map and the purposes for which it is used. Certainly, if a large portion of a spherical surface is represented on any single projection, there is bound to be some appreciable distortion of lengths and of angles or areas or both. The so-called **conformal projections** preserve angles exactly but fail to preserve the relative size of areas. The **equidistant azimuthal projections** preserve distances and angles measured at one particular point, which is the center of the map. The **equal-area projections** preserve the relative sizes of areas but sacrifice angles and lengths. Other projections commonly used are designed to minimize the distortion of lengths over a given region, so that a single scale can be made to serve for the whole projection with the smallest percentage of maximum error. Still others can be made to serve special purposes such as the plotting of great-circle arcs. The choice of which projection to use depends on the purposes for which it is intended.

39·2 Maps and charts

A representation of part or the whole of the earth's surface on a plane and with a reduced scale is called a **map** of the earth. If specially designed to meet the requirements of navigators, it is called a **chart.** Since the surface of the earth is approximately spherical, the construction of a map depends essentially upon some method of projecting onto a plane the meridians and latitude circles of a spherical surface. In charts to be used for navigation, slight corrections are applied to the spherical projection in order to allow for the flattening of the earth at the poles.

For projections of the polar regions the stereographic projection just described is quite suitable, since the parallels of latitude are mapped into concentric circles, center at the pole, with equally spaced meridians as equally spaced radii. Angles are preserved in this mapping, but straight lines not through the pole in the map are projections of small circles through the opposite pole, and not of great circles on the sphere. The scale factor at a point is inversely proportional to $1+\cos\theta$, where θ is the angular distance from the pole. The fractional deviation from the polar scale is $2/(1+\cos\theta)-1$ or $(1-\cos\theta)/(1+\cos\theta)$, which is less than 10 per cent for $\theta<35°$ but is 100 per cent for $\theta=90°$. Any zone less than a hemisphere can be represented fairly well by a stereographic projection; but the smaller the region, the more nearly does a uniform scale apply.

39·3 The gnomonic projection

Since the shortest distance between points on the earth's surface is measured along a great circle, it is often useful in navigation to have charts of portions of the earth's surface on which great-circle arcs are represented by straight lines. Intermediate points on a great circle are then located by a straight edge. This type of chart is called a **gnomonic projection.** It is a perspective projection of the sphere from its center, which is not conformal except at specialized points. The gnomonic projection is usually made upon several planes, such as the faces of a cube or octahedron or cuboctahedron or other regular or semiregular solid concentric with the sphere. Although a great-circle arc appears on each face as a straight-line segment, these consecutive segments representing the same great circle on two adjacent faces are not in the same straight line when the surface of the polyhedron is opened out on a plane. Instead, they make equal angles with the edge where they meet.

39·4 The Mercator projection

For navigation away from the polar regions, the conformal projection called the **Mercator projection** (Fig. 121) is a favorite projection with navigators, chiefly because of the fact that a curve on the earth called a **rhumb line,** which has the property that it crosses all the meridians at equal angles and is therefore the easiest curve to sail on by compass, is represented on the Mercator projection by a straight line (Fig. 121). The parallel circles of latitude

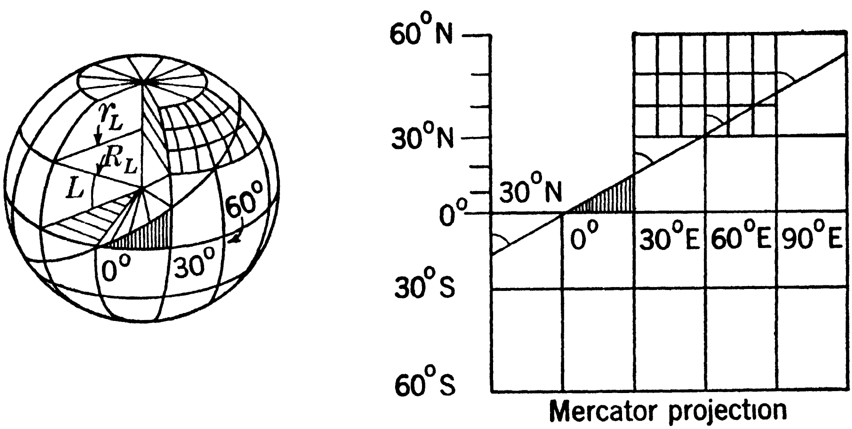


FIG. 121. Meridians, parallels, and a rhumb line in Mercator projection.

other than the equator are included as special types of rhumb lines; all others are loxodromic spirals, which wind around the poles of the earth. For short distances a rhumb line may closely approximate a great-circle arc, but for long distances the great-circle arc may be considerably shorter.[1]

The Mercator projection is not a perspective projection, but it is

[1] On an exact sphere the size of the earth the maximum difference between a course along a circle of latitude and the great-circle course between two points on the same latitude occurs when the points are at lat. 39°32′24″ and are 180° apart in longitude. The latitude-circle course is then 8,328 nautical miles as compared with a great-circle course of 6,055 nautical miles across the pole—a difference of 2,273 nautical miles.

conformal. Equally spaced meridians are projected into equally spaced vertical lines in the Mercator projection, but it is necessary to project equally spaced latitude circles into unequally spaced horizontal lines in order that the projection should have its important property of preserving angles (Fig. 121). To accomplish this, the latitude scale at latitude L must be enlarged in the ratio of the true length on the earth of a degree of latitude to the true length of a degree of longitude at latitude L. This ratio is R_L/r_L, where R_L is the radius of the curvature of the meridian and r_L is the radius of the latitude circle, both at latitude L. If the earth is assumed to be a sphere of radius R, then $R_L=R$, $r_L=R \cos L$. If account is taken of the flattening at the poles and each meridian is assumed to be an ellipse of eccentricity $e=0.082$, then it can be shown that the corrected formula for the scale factor is

$$\frac{R_L}{r_L}=\frac{1}{\cos L+k^2\cos^3 L} \quad \text{where } k^2=\frac{e^2}{1-e^2}=0.0068$$

which is equal to 0.9932 at $L=0$ and to 1 at $L=6°35.5'$. The distance M from the equator to the parallel at latitude L on the Mercator projection, expressed in minutes of arc at the equator, is known as the *meridional parts* for latitude L and is tabulated in navigation tables.[1] M is slightly less than L for $L<11°29'$ and is greater than L for $L>11°30'$.

In following a rhumb line on a Mercator chart, short distances may be measured by laying off on the meridian of the chart a segment equal to the chart distance between the two given points, so that the mid-points of the two segments are at about the same latitude, and measuring this segment on the latitude scale. For distances between more widely separated points, several intermediate points may be located on a gnomonic chart and transferred to the Mercator chart for measuring, and these shorter segments may be computed and added. Or some method for solving spherical triangles may be employed to find an exact solution.

[1] By methods of trigonometric calculus the following formula can be derived for M:

$$M=7{,}915.704' \log_{10}\tan\left(\tfrac{1}{2}L+45°\right)-23.115' \sin L-0.052' \sin^3 L$$

where the last two terms take the ellipticity of the earth into account.

39·5 Conical projections

Whereas the stereographic projection and the Mercator projection are conformal projections adapted to the polar regions and the equatorial regions, respectively, both suffer from considerable distortion of lengths in the Temperate Zones of the earth.

Widely used in air navigation in the continental United States, the **Lambert conical projection with two standard parallels** is a nonperspective conformal projection that has only a small variation in scale for as large a region as the United States. Equally spaced meridians are mapped as equally spaced concurrent lines, and the parallels of latitude are mapped as concentric circles perpendicular to these meridians. Two standard parallels having been drawn so that the scale on both of them is the same, the distance between them along any meridian line determines the size of the common scale, and the other circles are spaced so as to use this same scale on all meridians.

Another useful projection is the **polyconic projection,** for which circles of latitude are each unwrapped off a different tangent cone whose vertex is on the polar axis. Lengths along one meridian and along all latitude circles are measured with a uniform scale, but the other meridians are distorted into curves whose scale increases with the distance from the central meridian. The polyconic projection might be used to advantage in mapping the continent of Africa, which extends into both hemispheres.

39·6 Azimuthal and equal-area projections

The **equidistant azimuthal projection** plots correctly the angles and distances from a single point, the center, but shows increasing distortion farther out. It is an ideal projection for the use of a radio transmitting station or airplane field, since it gives accurate information about angles and distances measured from the central point.

There are several types of **equal-area projection,** of which the simplest (Fig. 122) represents the sphere by its cylindrical projection, as described in the section on the area of a zone (Chap. 23). These projections have the advantage that regions with equal area

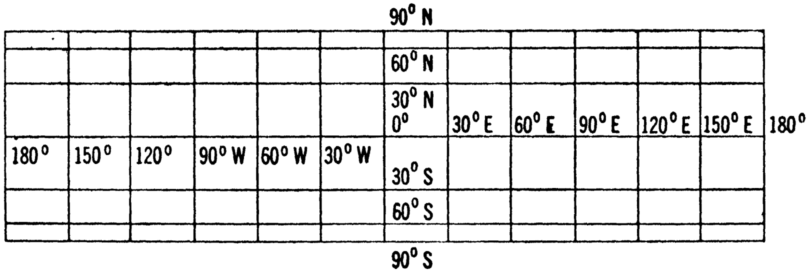


FIG. 122. Equal-area projection.

have equal areas on the map but the disadvantage that angles may be badly distorted.

The list of projections used by map makers and chart makers might be continued at considerable length. But the list is already long enough to show the student what efforts are made to overcome the difficulties created by the unfortunate fact that no spherical surface can be mapped on a plane without some distortion. Only on a spherical globe can the whole earth's surface be shown in true proportion.

39. ORAL QUESTIONS

A. Are there any projections of a spherical surface onto a plane that preserve distances?

B. How would you distinguish between a map and a chart?

C. What are some of the important characteristics of the gnomonic projection? When might it be used?

D. What are some important characteristics of the Mercator projection?

E. What is a rhumb line, and what are its uses in navigation?

F. How can a Mercator projection be used in conjunction with a gnomonic projection for laying out a route between distant points on the earth?

G. What is meant by meridional parts?

H. What are some important characteristics of the Lambert conical projection with two standard parallels? Along what curves in this projection is there an exact common scale of distances?

For what portions of the earth's surface might this projection be appropriate?

I. What projections would be advantageous to use for a map of the continent of Africa? Why?

J. What projection would be advantageous to use in connection with radio direction and range finding from a fixed station? Why?

39. WRITTEN EXERCISES

1. With the north pole as center, draw a gnomonic projection and a stereographic projection of the parallels of latitude at lat. 30°, 45°, 60°, 75° N., and draw the meridians for every 60°. Draw both figures on the same set of meridians, using different colors for the two sets of parallel circles, and using the same scale factor at the north pole.
2. A certain gnomonic projection of the sphere is made by projecting the points of the sphere from its center onto the faces of a circumscribed cube, tangent to the sphere at the poles. Prove that on the top and bottom faces of the cube the meridians project into concurrent lines and the latitude circles into concentric circles, whereas on the lateral faces of the cube the meridians project into parallel lines and the latitude circles into arcs of hyperbolas.
3. Make a list of the enlargement ratios for short arcs on the Mercator projection for lat. 15°, 30°, 45°, 60°, 75°, 85°, as compared with equal distances on the equator.
4. Using the following approximate construction for the spacing of parallels, lay off a strip of Mercator projection from lat. 60° S. to lat. 60° N. with meridians and parallels spaced at 15° intervals. CONSTRUCTION: With O as center (on middle left side of the paper) and with radius 2 in. (for a sheet of paper 10 in. high) draw a semicircle C with vertical diameter $[NS]$. Let $[OE]$ be a horizontal radius and t the vertical tangent at E. Let a line through O be drawn at a prescribed angle L (the latitude) with (OE), meeting C in P and t in T. Let the tangent at P meet (OE) in Q, and let the arc with Q as center and $\overline{PQ}$ as

radius meet t in Z. From Z lay off $\overline{ZM} = \frac{1}{3}\overline{TZ}$ toward E on t. Then $\overline{EM}$ is proportional to the meridional parts for latitude L (with an error less than 1 per cent for latitudes less than 60°). With small error the distance from the equator to lat. 15° can be used for spacing the meridians at 15° intervals.

5. Using the formula in the footnote (page 308), compute the meridional parts M for $L = 30°$, given that $\log_{10} \tan 60° = 0.23856$ and $\sin 30° = \frac{1}{2}$. By what per cent does this answer exceed the number of minutes in 30°?
6. Draw the latitude circles for $L = 25°, 30°, 35°, 40°, 45°, 50°$ N. and the meridians for $\lambda = 60°, 75°, 90°, 105°, 120°$ W. in a Lambert conical projection with the two standard parallels at latitudes $L_1 = 30°$ N., $L_2 = 45°$ N. Note the following facts: Taking a uniform vertical scale along the central meridian, the center C of the concentric latitude circles in the map is at a point corresponding to a latitude of $(L_2 \cos L_1 - L_1 \cos L_2)/(\cos L_1 - \cos L_2)$. For $\cos L_1 = \sqrt{3}/2$, $\cos L_2 = \sqrt{2}/2$, this is $(75 + 15\sqrt{6})°$, or 111.75°—somewhat beyond the north pole. The length of 1° of longitude along the standard circle at latitude L_1 is $\cos L_1$ times the length of 1° along the meridian. From this it follows that in the projection the meridians for 60° W. and 120° W. each makes angles of about 18.2° with the central 90° W. meridian at C.
7. Draw a Lambert projection as explained in Exercise 6, but from 20° N. to 70° N. and from 30° E. to 120° E., taking standard parallels at 30° N. and 60° N., respectively. At what latitude is the center C?
8. Consult an atlas, and see how many different types of projection you can recognize. Give references for at least four types, explaining if you can why the particular projection was chosen for the map.

40

REVIEW OF PROJECTIONS AND MAPS

40·1 Perspective drawings

The appearance of any object depends very much on the point of view of the observer. A perspective drawing constructed to be viewed from a certain center of perspective (in central perspective) or from a great distance (in orthographic parallel projection) will not look quite right if viewed from any other position. If viewed from the correct point, drawings in central perspective create a more realistic impression that those in parallel projection.

However, drawings in parallel projection are easier to construct and are more useful for making direct scale measurements of objects. Furthermore, if the focal distance of a central-perspective drawing is large compared with the dimensions of the drawing itself, then the lines of the drawing are so nearly in parallel projection that the latter is a good approximation. Thus a blackboard drawing in parallel projection looks much more realistic to the student in the back row than to the teacher at the blackboard.

40·2 Oblique parallel projection

Drawings in oblique parallel projection are commonly used by teachers and textbook writers since they are relatively easy to draw with no other aid than an ordinary ruler. Figures in planes parallel to the plane of the drawings are drawn to scale, the same units being used horizontally and vertically. An arbitrary scale and direction are chosen to represent line segments perpendicular to the projection plane. The unit point on the oblique axis is usually chosen below and to the left of the origin at a distance less than the horizontal and vertical unit. This point is one of the foci of the ellipse (center at the origin) that represents the outline of the projection of the unit sphere with center at the origin. In oblique projection, circles whose planes are parallel to the projection plane are represented as circles, those whose planes are parallel to the oblique-projection rays project into line segments, but most other circles project into ellipses. Such an ellipse can be constructed from a pair of conjugate diameters that are the projections of a pair of perpendicular diameters of the given circle.

40·3 Orthographic axonometric projection

In orthographic axonometric projection, a projection plane is chosen that makes acute angles with each of the three directed axes of a rectangular coordinate system in space, and the projected axes are altitudes of a reference triangle, cut from the first octant by a plane parallel to the projection plane. A triangle similar to this reference triangle and rotated through 90° will have its sides parallel to the projected axes. If scales are marked on the sides of this triangle, proportional to the square roots of the sines of twice the opposite angles, and if these angles are taken to be all different, then the triangle becomes a trimetric ruler. Segments parallel to each of the three space axes, respectively, can be drawn and measured with one of the three scales. Other segments can be drawn to scale in projection by first drawing their projections along the three axes and then adding the three projections by vector addition.

In orthographic axonometric projection a circle projects into an ellipse unless its plane is parallel or perpendicular to the projection plane. (The exceptional projections are circles and line segments, respectively.) In particular, the distance from the center of this ellipse to the focus is equal to the projected length of a segment equal to a radius of the circle, perpendicular in space to the plane of the circle at its center. The outline of a sphere in orthographic projection is a circle.

40·4 Conic sections

Plane sections of a right circular cone are called conic sections. If the section plane does not pass through the vertex of the cone, three types of nondegenerate conics may be obtained—ellipses (including circles as a special case), parabolas, and hyperbolas. If the plane does pass through a vertex, the resulting figure, consisting of a point or a line (doubly counted) or of a pair of lines, is called a degenerate conic. Plane sections of a right circular cylinder may be an ellipse (or circle) or a pair of parallel lines or a single line (doubly counted). An ellipse (hyperbola) has two foci and has the property that the sum (difference) of the distance from any of its points to the two foci is equal to the length of its major axis. Corresponding to each focus there is a line called a directrix such that the ratio e of the distances from any point of the conic to the focus and to the directrix is a constant called the eccentricity. For the ellipse, e is less than 1; for the hyperbola, e is greater than 1; for the parabola with its one focus and directrix, we have $e = 1$.

For the theory of parallel and central projection, an important fact is that every conic (and in particular a circle) projects into some kind of conic under any of these types of projection.

40·5 Central perspective and tilt distortion

Two important problems in connection with drawings and pictures in central perspective are (1) to create a drawing in central perspective as seen from a particular point by a geometric construction from a plan and elevation (of two other views in parallel projection) and (2) to obtain a map or a scale drawing from a photograph or drawing in central perspective.

For the first problem we may determine a horizon line and two or more vanishing points in the vertical picture plane. To these vanishing points will correspond directions in the given plan and elevation. The plan image P_h of a point P is projected parallel to one of these directions into the picture trace, dropped to the level of the elevation image of the point, and then moved in line with the corresponding vanishing point, until it lies under the point where OP_h meets the picture trace.

For the second problem, we assume a mapping plane placed at a distance from the observation point O equal to the focal length of the given photograph, and we call the line of intersection of the mapping plane and photoplane the fixed line m, since each of its points is fixed in the mapping. The plane through O perpendicular to m is called the principal plane and intersects the mapping and photoplanes in their principal lines, which in turn meet each other in the distortion center D. If the horizon line in the photoplane is at a distance h from D, then a line at a distance $h/2$ from D in the photoplane maps into a line at distance h from D in the mapping plane; and when the two planes are superimposed, the mapping appears to move the points on this "halfway line" radially from D to twice their former distance from D. This fact, together with the fact that lines go into lines and that points on the fixed line are fixed, is sufficient to indicate a geometric construction for the mapping of points, once the distortion center and horizon line have been determined.

40·6 Stereographic projection of the spherical surface

The stereographic projection of a spherical surface is a central perspective projection onto a plane from a point O on the spherical surface. Important among its properties are these:

Angles are preserved.

Circles through O project into lines and circles not through O into circles.

The great circle of which the pole is O projects into a circle which is intersected in diametrically opposite points by the projection of any other great circle.

The points of the spherical surface, O excepted, are in one-to-one correspondence with the points of the projection plane.

40·7 Nonperspective mappings

A variety of maps and charts of the earth's surface are in use that are not based on linear perspective, either central or parallel. These maps and charts are constructed to serve special purposes that are considered desirable by the user of the map. The Mercator projection (which is *NOT* a central projection of the sphere onto a circumscribed cylinder) has two advantages: it is conformal, and it maps meridians and parallels of latitude, and consequently, rhumb lines, into straight lines. Its chief disadvantages are that it maps great circles into rather complicated curves and distorts distances. The gnomonic projection remedies the first defect, mapping great circles into lines; but it is not conformal, and it distorts distances. The Lambert conical projection with two standard parallels minimizes the distortion of distances in such a region as the United States. The equidistant azimuthal projection is well adapted to radio direction finding from a fixed station. Other maps each have their special advantages, but no map is ideal from every point of view, since it is impossible to map a spherical surface on a plane without distortion.

40. ORAL QUESTIONS

A. What are the images of straight lines in perspective projections? Are there any exceptions?

B. What are the images of circles in perspective projections? Can every conic section be projected into a circle?

C. What are the chief differences between central and parallel projection?

D. What are the chief differences between oblique and orthographic parallel projection?

E. How can the image of a sphere with three orthogonal great circles be constructed in orthographic projection?

F. What is the easiest way to draw a box in trimetric orthographic projection? Can the same method be used on other figures?

G. What are some of the important properties of sections of circular cylinders and cones?

H. How can you locate the correct point from which to view a drawing in central perspective if the projection of a square appears in the figure?

I. What are the principal geometric facts that determine how to construct a map from a tilt-distorted aerial photograph?

J. What perspective projections of the sphere are in use, and what are some of their properties?

K. What are some of the principal nonperspective mappings of a spherical surface, and what are their advantages and disadvantages?

40. WRITTEN EXERCISES

1. Draw a figure of a cube and an inscribed right circular cylinder in oblique parallel projection.
2. Draw the figure of Exercise 1 in orthographic axonometric projection.
3. Draw a figure of a right circular cone in oblique parallel projection, showing a section of the cone by a plane cutting all the elements.
4. Write a description of several of the most important geometric properties of the plane curves obtained as sections of a right circular cone.
5. A right circular cone with a semivertical angle of 30° is cut by a plane perpendicular to one of its elements at 4 in. from the vertex of the cone. Find the lengths of the axes and the eccentricity of the curve of section.
6. Solve Exercise 5 if the semivertical angle is 60°.
7. Draw in central perspective a view of a right circular cylindrical tank 6 ft. in height and 6 ft. in diameter, viewed from a point 4 ft. above the base and 10 ft. from its axis. First project a circumscribed cube, and then sketch in the elliptic projections of the circular bases of the cylinder.
8. Describe briefly the effect of tilt distortion in the photograph of a plane object.
9. A cube is projected from its center onto the circumscribed sphere. The sphere is then projected stereographically from one of the cube's vertices onto a plane tangent to the sphere at the opposite vertex. Draw the edges of the cube as they appear in the final projection.
10. Describe the principal advantages and disadvantages of the Mercator projection and the Lambert conical projection with two standard parallels. Give for each a reference to two atlas maps in which the projection is used and your own reason why you think the particular projection was used.

APPENDIX

SYNOPSIS OF PLANE GEOMETRY

Congruence

1. Two geometric figures are said to be congruent to each other if the points in the one figure can be made to correspond in a one to one manner with the points of the other figure so that the distance between any two points in the one figure is equal to the distance between corresponding points in the other figure.

Lines

2. Any two lines (indefinitely extended) are congruent. Two lines having two distinct points in common are coincident.
3. Any two rays (or half lines) are congruent. They are coincident if they have in common the vertex and one other point.
4. Two line segments are congruent if and only if they have the same length.
5. Two lines (each extended indefinitely in both its directions) are parallel if and only if they lie in the same plane but do not intersect.
6. There is one and only one parallel to a given line through a given point not on the line. (Euclid's parallel postulate.)

Angles

7. An angle (in the geometric sense of the term) is a geometric figure formed by two rays that have a common vertex. The two rays are called the sides of the angle.

8. Two angles are equal if and only if they are congruent.
9. A straight angle is an angle whose sides lie in the same straight line, but do not coincide. All straight angles are equal.
10. Three rays having a common vertex form three angles. If one of these angles is a straight angle, the other two angles are called supplementary.
11. If two angles that are supplementary are also equal, each is said to be a right angle. The lines forming the two sides of a right angle are said to be perpendicular or orthogonal or normal to each other.
12. All right angles are equal.
13. If two lines intersect, the pairs of adjacent angles are supplementary and the pairs of vertical angles (opposite angles) are equal.
14. An angle in the arithmetic sense is a number that measures the size of a geometric angle. It is commonly expressed in degrees (°), minutes (′), and seconds (″). A right angle is 90°; a straight angle is 180°.
15. If three rays with a common vertex all lie in the same half plane, the largest of the three angles that they form is said to be the sum of the other two. If three rays with a common vertex lie in the same plane but do not all lie in the same half plane, the sum of the three angles that they form is equal to four right angles, or 360°.
16. Two angles are complementary when their sum is 90°. They are supplementary when their sum is 180°.
17. The total angular magnitude about a point is 360°.

Triangles

18. Two triangles are congruent if three sides of the one are equal, respectively, to three sides of the other.
19. Two triangles are congruent if two sides and the included angle of the one are equal, respectively, to two sides and the included angle of the other.
20. Two triangles are congruent if two sides and the angle opposite the longer given side of the one are equal, respectively, to the corresponding parts of the other.

21. Two triangles are congruent if two angles and a side of the one are equal, respectively, to the corresponding parts of the other.
22. The sum of the three angles in a triangle is equal to 180°.
23. If two sides of a triangle are equal, the angles opposite are equal, and conversely. Such a triangle is called isosceles.
24. The sum of any two sides of a triangle is greater than the third side, and the difference between any two sides is less than the third side.
25. In any triangle, if two sides are unequal, the greater side is opposite the greater angle, and conversely.
26. If two triangles have two sides of the one equal, respectively, to two sides of the other but the included angle of the first triangle greater than the included angle of the second, then the third side of the first is greater than the third side of the second, and conversely.

Parallels, Perpendiculars, and Bisectors

27. Two lines each perpendicular to the same line are parallel.
28. A line perpendicular to one of two parallel lines is perpendicular to the other also.
29. If two parallel lines are cut by a third line (called a transversal), the alternate interior angles are equal; conversely, when two lines in the same plane can be cut by a transversal so that the alternate interior angles are equal, the lines are parallel.
30. One and only one perpendicular can be drawn to a line through a given point (either on the line or off the line).
31. Of all the lines that can be drawn from a point to a given line not passing through the point, the perpendicular is the shortest. Its length is called the distance from the point to the line.
32. Two distinct points, each equidistant from the ends of a line segment, determine the perpendicular bisector of the segment.
33. The locus of a point equidistant from the ends of a line segment is a line perpendicular to the segment at its mid-point. It is called the perpendicular bisector of the segment.
34. The locus of a point equidistant from two given intersecting lines is a pair of lines bisecting the angles formed by those lines.

35. In any triangle, certain triples of lines are concurrent.
   a. The three perpendicular bisectors of the sides meet in a point P, the circumcenter, which is equally distant from the three vertices and is the center of the circumscribed circle.
   b. The three altitudes of a triangle (the perpendiculars from the vertices to the opposite sides) meet in a point H, the orthocenter.
   c. The three medians of a triangle (the lines drawn from the vertices to the mid-points of the opposite sides) meet in a point G, the centroid, which is two-thirds of the distance from each vertex to the mid-point of the opposite side and is also two-thirds of the distance from H to P on the line of these points.
   d. The bisectors of the interior angles of the triangle meet in a point I, the incenter, that is equally distant from the three sides and is the center of the inscribed circle.
   e. The bisectors of two of the exterior angles meet the bisector of the third interior angle in a point, one of the three excenters, that is equally distant from one of the sides and from the other two sides produced and is the center of an escribed circle.

Polygons

36. A quadrilateral is a portion of a plane bounded by four lines, called the sides. It is a trapezoid if just two of its sides are parallel. It is a rectangle if its angles are right angles.
37. A parallelogram is a quadrilateral (*a*) having opposite sides parallel (definition); (*b*) having opposite sides equal; (*c*) having two opposite sides equal and parallel.
38. A polygon is a portion of the plane bounded by a broken line. A regular polygon is a polygon that is both equiangular and equilateral.
39. The sum of the exterior angles of a convex polygon of n sides is 360°; the sum of its interior angles is $(n-2)\ 180°$.

Circles

40. A circle is a closed plane curve whose points are all equally distant from a point within called the center.

41. In the same circle or in equal circles,
    a. If two central angles are equal, they subtend equal arcs and equal chords, and conversely.
    b. If two central angles are unequal (each being assumed less than a straight angle), their respective arcs and chords are unequal in the same order, and conversely.
    c. If two chords are equal, they are equally distant from the center, and conversely.
42. A line from the center of a circle perpendicular to a chord bisects the chord and its subtended arcs, and conversely.
43. A line meeting a radius of a circle at its outer extremity is tangent to the circle if and only if it is perpendicular to the radius.
44. A central angle has the same measure as its intercepted arc. The length of arc intercepted on a circle of unit radius is called the radian measure of the angle.
45. An inscribed angle, or an angle formed by a tangent and a chord, has the same measure as half its intercepted arc.

Proportion and Similarity

46. A line parallel to one side of a triangle divides the other two sides proportionally, and conversely.
47. Two polygons are similar if corresponding angles are equal and corresponding sides are proportional.
48. Two triangles are similar (*a*) if corresponding angles are equal; (*b*) if two pairs of corresponding sides are proportional and the included angles are equal; (*c*) if corresponding sides are proportional.
49. Corresponding lengths in two similar polygons are proportional, and corresponding areas are proportional to the squares of these lengths.

Areas

50. The area of a rectangle, or of a parallelogram, is equal to the product of its base by its altitude.
51. The area of a triangle is equal to half the product of its base by its altitude.
52. The area of a trapezoid is equal to half the product of the sum of its bases by its altitude.

53. The area of a circular sector is equal to half the product of the radius by the arc length. The area of a circle of radius r is equal to πr^2 ($\pi = 3.1416\cdots$).

Pythagorean Theorem

54. The square of (or on) the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

ANSWERS TO ODD-NUMBERED EXERCISES THAT INVOLVE COMPUTATION

Chapter 1

1. 10.39 in.
3. 9.17 in.
5. $a^2+b^2+c^2$.

Chapter 2

1a. 7.000×10^6.
1b. 651.9.
3a. 320 rods.
5. 750 miles/hr.
7. *CED*, *AFC*, and one other set.
9. HINT: $\overline{AB}+\overline{BC}+\overline{CD}=$?
11. No.

Chapter 3

7. Not if A lies on $[BC]$ or B on $[AC]$.

Chapter 5

5. 7 ft.

Chapter 6

1. $(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$.
3. Each is $2\sqrt{2}$.
5. Each is 10.0.
7a. About 42 units from B, E, F, C', D'; 80 from A'; 68 from others.

Chapter 7

1. 800 mils; 17.8 mils.
3. $\pi/2$ or 1.5708; $\pi/36$; 0.982.
5. $6\sqrt{5}$ in. or 13.4 in.
7. 90°.

Chapter 8

1. 0°, 30°, 45°, 60°, 90°.
3. $\sqrt{2}/2$; 70.0 in.
5. 11 units; $\frac{9}{11}$, $\frac{6}{11}$, $\frac{2}{11}$.
7. 233.00 in.
9. 0.
11. $\cos 144° = \cos 216° = -\cos 36° = -(1+\sqrt{5})/4$.

Chapter 9

1. $\cos\alpha = \pm\sqrt{3}/3$
3. $\angle ABC$, $\angle ABD$, $\angle ACD$, $\angle BCD$ are right angles.
5. 12 units.
7. (1,1,1).
11. Eight triangles.

Chapter 10

1. 56.00 cm.
7. $a=7$, $b=7\sqrt{2}$, $\theta=45°$.
9. $\theta=120°$. Half.
13. 10.14 ft., 6.93 ft.

Chapter 11

3. 90°, 120°, 135°.
9. 90°, 100°, 110°.

Chapter 12

5. 45°; 18°.
7. Fourth point $(0,0,2\sqrt{2})$.
9. $z=2\sqrt{2}$.

Chapter 13

1. 27; 46,656.
3. 40,500 bricks.
5. 4.87 in.
7. 1,210 lb., approximately.
9. 1,000 liters.

Chapter 14

1. 7 cu. ft.
3. 72,000 cu. ft.
5. $h=2\sqrt{3}/3$; $V=1.5$.

Chapter 15

1. 192 cu. in.
3. 3 ft. deep.
5. $\sqrt[6]{72}=2.04$.
7. $(5+\sqrt{5})/24$.
9. 94,080,000 cu. ft.

11*a*. 385 cu. ft.

11*b*. 58.7 cu. ft.

11*c*. Diff. $=\frac{7}{8}$ cu. ft.

Chapter 16

1. 402 cu. in.
3. $3\frac{1}{16}$ in.
5. 3.5 in.
7. $(8/3)\pi$, $(27/8)\pi$, $2\sqrt{3}\ \pi$, $(10/3)\pi$.
9. Ratio is $3\sqrt{3}/\pi$.

Chapter 17

1. 18 cu. in.
3. 135 cu. in.
5. 2,127 cu. ft.; 181 tons.
7. 9 in.
9. $5\sqrt{2}/3$ cu. in.

Chapter 18

1. 2,160 cu. in.; 9.3 gal.
3. 20,200 cu. ft.
5. 327.1 cu. ft.
7. 1,180 cu. in.
9. 34.1 cu. ft.
11. 228 gal. (Error 3.5 gal.)

Chapter 19

1. $4\sqrt{3}$ sq. in.
3. 8.44 sq. units.
5. 1,040 sq. ft.
7. 220 sq. in.
9. $\frac{18}{7}$; $\frac{9}{14}$; $\frac{2}{7}$.
11. 6.085.
13. $24+8\sqrt{3}$.

Chapter 20

1. 324 lb.
3. Cube $V=8$, $E=12$, $F=6$.
5. 2.000376 cu. units.
9. $72\sqrt{3}$ sq. in.
11. 15.
13. $V=445$ cu. ft.; $S=285.7$ sq. ft.
15. Use $S=2\pi rh$.

Chapter 21

3. 4 in.; 3 in.
5. 21,770 miles.
7. 17,590 miles.
11. 2.4 units.

Chapter 23

1. 20π sq. in.
3. 17.4 miles.
5. 0.0049 in.
7. $\sqrt{3}$.
9. 36.8.

Chapter 24

1. 606 cu. ft.
3. Volumes 4:13; areas equal.
5. 11.8 cu. in.
7. πR^3.
9. 2.86 cu. ft.
11. $\pi/3\sqrt{3}$ or 0.605.

Chapter 25

1. $V=108\pi^2$ cu. in.; $S=72\pi^2$ sq. in.
3. $54\pi^2+36\pi$ cu. in.
5. $V=\pi$ cu. in.; $\bar{x}=2\sqrt{3}/9$ in.
7. $8\pi a^2$.
9. $\bar{x}=2r/\pi$.

Chapter 26

1. 6π.
3. $\sigma=95°$; area $=19\pi/18$.
5. $\sigma=52°30'$; area $=1.83$.
7. Area $=\pi R^2$; $\alpha=120°$.
9. $V=9\pi$.

Chapter 27

1. Inequality 1.
3. Inequality 3.
5. $\alpha' = 10°$, $\beta' = 120°$, $c' = 70°$.
7. Inequality 4.
9. $\alpha' = 140°$, $b' = 130°$, $a' = 80°$.
11. Inequality 6.
13. Inequality 6.
15. Inequality 5.

Chapter 28

1. $\overline{L}_1 = 47°40'$; $\overline{L}_2 = 42°24'$;
   $DL_0 = 51°27'$.
3. $\overline{L}_1 = 112°54'$; $\overline{L}_2 = 29°36'$;
   $DL_0 = 48°28'$.
5. $Dep. = 2{,}854$ nautical miles.
   $L_2 - L_1 = 1{,}081$ nautical miles.
   $D = 3{,}053$ nautical miles.
7. $Dep. = 2{,}184$ nautical miles.
   $L_2 - L_1 = 316$ nautical miles.
   $D = 2{,}207$ nautical miles.

Chapter 29

1. L.S.T. $= 7^h0^m12.7^s$.
   G.H.A. $= 242°36'0''$.
   $\lambda = 57°24'10''$ E.
3. G.S.T. $= 13^h50^m53.6^s$.
   L.S.T. $= 7^h6^m52.6^s$.
   $\lambda = 101°0'15''$ W.
5. L.S.T. $= 11^h47^m32.7^s$.
   G.H.A. $= 16°2'30''$.
   L.H.A. $= 191°23'10''$.
7. G.S.T. $= 4^h3^m10.1^s$.
   L.S.T. $= 6^h12^m14.0^s$.
   L.H.A. $= 82°51'18''$.
9. $L = 10°15'$ N.
11. $L = 79°45'$ S.
13. $L = 46°20'$ N.
15. $L = 13°21'$ S.
17. $L = 42°10'$ N.
19. $L = 68°20'$ S.

Chapter 30

1. 201 million sq. miles.
   57.1 million sq. miles.
5. $V = 318$ cu. in.; $S = 440$ sq. in.
7. 1 ft.
9. 12.34 cu. in.
11. $\sigma = 90°$, $\sigma_\alpha = 10°$, $\sigma_\beta = 30°$, $\sigma_\gamma = 50°$.
    $a' = 80°$, $b' = 60°$, $c' = 40°$.

Chapter 33

1. $a = \sqrt{2}$, $b = 1$, $2c = 2$, $e = \frac{1}{2}\sqrt{2}$.
3. $R = 4$, $\theta = 30°$.
5. $2a = 8\sqrt{2}$, $2b = 8$, $e = \frac{1}{2}\sqrt{2}$.
7. 30 in., 50 in.
9. $e = \frac{1}{2}\sqrt{2}$; $2r\sqrt{2}$, $2r$.

Chapter 35

3. Eccentricities $\frac{1}{2}\sqrt{6}$, 1, $\frac{1}{2}\sqrt{2}$, 0.
5. $2b = 2\sqrt{2}$, $2c = 2$.
7. Hyperbola; $e = \frac{1}{2}(\sqrt{6} + \sqrt{2})$;
   $2c = 2\sqrt{6}$ in.; $2a = (6 - 2\sqrt{3})$ in.
13. $e = 2$; $2a = \frac{2}{3}\sqrt{3}$ in.

Chapter 36

5. $f = 1.5$ in.; 8 to 1.
7. 4.8 in. focal length.

Chapter 37

5. Focal length $= (\sqrt{6} - \sqrt{2})$ in.

Chapter 39

3. 1.0288 at 15°, 1.1491 at 30°, 3.862 at 75°, 11.47 at 85°.
5. $M = 1{,}876.8$; 4.26%.
7. 101°.

Chapter 40

5. $2a = 4\sqrt{3}$; $2b = 4\sqrt{2}$. $e = \frac{1}{3}\sqrt{3}$.

INDEX

E

Q

R