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TRIGONOMETRY

PART II

HIGHER TRIGONOMETRY

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by

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PREFACE TO PART II

THIS part of our text-book on Trigonometry provides an introduction to analytical trigonometry and to the theory of the logarithmic and the exponential functions.

Chapters XIII and XIV contain a discussion of complex numbers and a proof of Demoivre's Theorem. Applications of that theorem are given in Chapter XV.

In the remaining chapters a knowledge of the Calculus is assumed, some important trigonometrical applications of the Calculus being discussed in Chapter XVI, while the theory of the logarithmic and the exponential functions, developed in Chapter XVII, is based on the definition of the logarithm as an integral.

As in Part I, numerous examples, with answers, are provided at the ends of the chapters.

The authors wish again to thank Mr. Albert Anderson and Dr. George Thomson for valuable help in the task of reading proofs and verifying examples.

T. M. M.

W. A.

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TRIGONOMETRY

PART II

HIGHER TRIGONOMETRY

CHAPTER XIII

COMPLEX NUMBERS: VECTORS

§ 1. The Imaginary Unit

IN the domain of real numbers there is no number which satisfies the equation $x^2 = -1$. In order to enlarge our conception of number in such a way that it may be possible to apply all the operations of algebra—addition, subtraction, multiplication, division and root extraction—to any number whatsoever, a new kind of number, denoted by i and known as *the imaginary unit*, is introduced. This number is defined as satisfying the fundamental laws of algebra—associative, commutative and distributive—and as being such that $i^2 = -1$. The two roots of the equation $x^2 = -1$ are then i and $-i$. This generalisation of the idea of number is a valid one, in that no deductions from it lead to contradictions. A number of the form pi , where p is real, is called an *imaginary number*.

§ 2. Complex Numbers

A number of the form $p + iq$, where p and q are real numbers, is called a *complex number*. If $q = 0$ the number is purely real; if $p = 0$ it is purely imaginary.

Example 1.—Express the number $(a + ib)/(c + id)$, where a, b, c and d are real, in the form $p + iq$.

[Multiply numerator and denominator by $c - id$; then $p = (ac + bd)/(c^2 + d^2)$, $q = (bc - ad)/(c^2 + d^2)$.]

Conjugate Complex Numbers.—The numbers $p + iq$ and $p - iq$ are called conjugate numbers.

Example 2.—Show that the roots of the equation $x^2 + x + 1 = 0$ are conjugate numbers.

Example 3.—Show that the product of two conjugate numbers is a positive real number.

Identities.—If

$$p + iq = r + is,$$

where p, q, r and s are real, then $p = r$ and $q = s$.

$$\begin{aligned} \text{For} & \quad p - r = -i(q - s), \\ \text{and therefore} & \quad (p - r)^2 = -(q - s)^2, \\ \text{or} & \quad (p - r)^2 + (q - s)^2 = 0. \end{aligned}$$

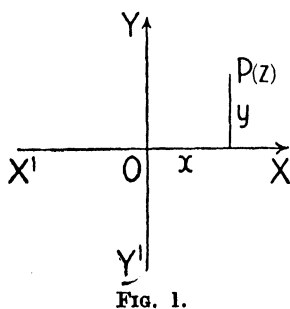
Now $(p - r)^2$ and $(q - s)^2$ cannot be negative; hence they must both be zero; therefore $p = r$ and $q = s$.

In particular, the complex number $p + iq$ cannot have the value zero unless p and q are both zero.

Notation.—If $p + iq$ is a complex number, p is called its *real part*, and is denoted by $R(p + iq)$; while q is called the *imaginary part*, and is denoted by $I(p + iq)$. If two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

§ 3. The Argand Diagram *

The following geometrical representation of a complex number is very useful. A system of rectangular co-ordinate



axes $X'OX$, $Y'OY$ (Fig. 1) is taken in a plane, and the complex number $z = x + iy$ is represented by the point P whose abscissa and ordinate are x and y respectively. P is called the point z , and z is said to be the *argument* of the point P . To each complex number z there corresponds one and only one point in the plane. In particular,

points on the x -axis correspond to real numbers, and points on the y -axis to purely imaginary numbers. The figure is known as the Argand Diagram.

The geometrical representation of complex numbers was suggested by J. R. Argand in 1806, and earlier by, among others, a Norwegian surveyor, Caspar Wessel, in 1797.

Example 1.—Show that the mid-point of the line joining the points z_1 and z_2 in the Argand Diagram is the point $\frac{1}{2}(z_1 + z_2)$.

* In this and the following chapters a knowledge of the elementary analytical geometry of the straight line and the circle will be assumed.

[Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. The mid-point of the join of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$; i.e., the point whose argument is $\frac{1}{2}(z_1 + z_2)$.]

Example 2.—Show that the straight line joining the points z_1 and z_2 in the Argand Diagram is divided in the ratio $m : n$ at the point $\frac{mz_2 + nz_1}{m + n}$.

Example 3.—Show that the centroid of the triangle whose vertices are the points z_1, z_2, z_3 is the point $\frac{1}{3}(z_1 + z_2 + z_3)$.

Example 4.—Prove that, if the arguments of two points are conjugate complex numbers, each point is the image of the other in the x -axis.

§ 4. The Modulus and the Amplitude of a Complex Number

P, the point z (Fig. 2) in the Argand Diagram, has polar co-ordinates r and θ , where r is the positive measure of the length of OP, and θ is the measure* of the angle XOP. The number r is called the *modulus* of z , and is written *mod* z or $|z|$; the number θ is the *amplitude* of z , and is written *amp* z . The amplitude has an infinite number of values differing from one another by multiples of 2π . That value which satisfies the inequalities

$$-\pi < \text{amp } z \leq \pi$$

is called the *principal value of amp* z . As a rule, when the amplitude is mentioned, it is to be understood that the principal value is referred to.

The Cartesian co-ordinates are given in terms of the polar co-ordinates by the equations

$$x = r \cos \theta, \quad y = r \sin \theta \quad . \quad . \quad (1)$$

* From here onward it will be assumed, unless the contrary is stated or implied, that angles are measured in radians.

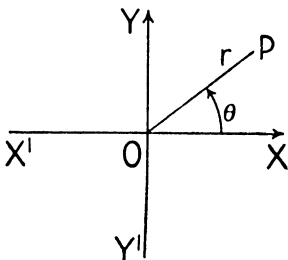


FIG. 2.

and the polars in terms of the Cartesians by the equations

$$r = \sqrt{(x^2 + y^2)}, \quad \theta = \tan^{-1} \frac{y}{x}, \quad . \quad . \quad (2)$$

where the quadrant in which θ lies is determined by means of equations (1).

Since $z = x + iy$, it follows from (1) that

$$z = r(\cos \theta + i \sin \theta), \quad . \quad . \quad . \quad (3)$$

a formula which expresses the complex number z in terms of its modulus and its amplitude.

Example 1.—Express the following complex numbers in the form (3). Mark their positions on the Argand Diagram.

- (i) $1 + i\sqrt{3}$, (ii) $-1 + i\sqrt{3}$, (iii) $1 - i\sqrt{3}$, (iv) $4 + 4i$,
 (v) $-2 - i\sqrt{12}$, (vi) -2 , (vii) 2 , (viii) $2i$.

Ans. (i) $2(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$, (ii) $2(\cos \frac{2}{3}\pi + i \sin \frac{2}{3}\pi)$,
 (iii) $2\{\cos(-\frac{1}{3}\pi) + i \sin(-\frac{1}{3}\pi)\}$, (iv) $4\sqrt{2}(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$,
 (v) $4\{\cos(-\frac{3}{4}\pi) + i \sin(-\frac{3}{4}\pi)\}$, (vi) $2(\cos \pi + i \sin \pi)$,
 (vii) $2(\cos 0 + i \sin 0)$, (viii) $2(\cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi)$.

Example 2.—Show that $|\cos \theta + i \sin \theta| = 1$.

Example 3.—If $z = x + iy$, show that $|x| \leq |z|$, $|y| \leq |z|$.

Example 4.—Prove that $\left| \frac{x - iy}{x + iy} \right| = 1$.

Example 5.—If $|z_1| = |z_2|$ and $\text{amp } z_1 + \text{amp } z_2 = 0$, show that z_1 and z_2 are conjugate complex numbers.

Example 6.—P and Q are the points $2 - 3i$ and $3 + 2i$ respectively, and PQ is produced to R, so that $PR : RQ = 3 : -2$. Find the modulus and the amplitude of the argument of R.

Ans. 13, $\tan^{-1}(12/5)$.

§ 5. Geometrical Construction for the Sum of two Complex Numbers

Let P_1 and P_2 (Fig. 3) be the points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ on the Argand Diagram. Complete the parallelogram $P_2OP_1P_3$. Then, if $z_3 = x_3 + iy_3$ is the argument of P_3 ,

$$z_3 = z_1 + z_2.$$

For $x_3 = \text{Proj. of } OP_3 \text{ on } X'OX$

$= \text{Proj. of } OP_1 \text{ on } X'OX + \text{Proj. of } P_1P_3 \text{ on } X'OX$

$= \text{Proj. of } OP_1 \text{ on } X'OX + \text{Proj. of } OP_2 \text{ on } X'OX$

$= x_1 + x_2.$

Similarly, by projecting on $Y'OY$ it can be shown that

$$y_3 = y_1 + y_2.$$

Hence $z_3 = x_3 + iy_3 = (x_1 + x_2) + i(y_1 + y_2) = z_1 + z_2$.

Alternative Proof.—The co-ordinates of the mid-point of P_1P_2 are $\frac{1}{2}(x_1 + x_2)$ and $\frac{1}{2}(y_1 + y_2)$. But this point is also the mid-point of OP_3 , whence it follows that P_3 is the point $(x_1 + x_2, y_1 + y_2)$. The result follows as before.

Difference of Two Complex Numbers.—Since

$$z_1 - z_2 = z_1 + (-z_2),$$

a subtraction may be regarded as an addition.

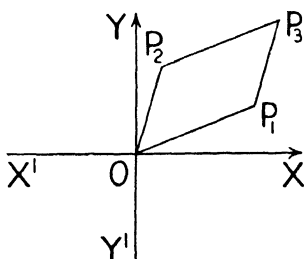


FIG. 3.

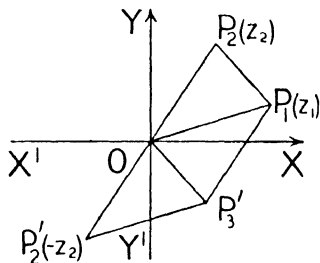


FIG. 4.

Let P_1 and P_2 (Fig. 4) be the points z_1 and z_2 . Draw the line P_2O , and produce it its own length to P_2' , the image of P_2 in O . Then, if $z_2' = x_2' + iy_2'$ is the argument of P_2' ,

$$z_2' = x_2' + iy_2' = -x_2 - iy_2 = -z_2.$$

Now complete the parallelogram $P_1OP_2'P_3'$; the point P_3' has then the argument

$$z_3' = z_1 + z_2' = z_1 - z_2.$$

§ 6. Vectors

A complex number z is completely defined if its modulus and its amplitude are given. This is clear from the formula $z = r(\cos \theta + i \sin \theta)$. Or, geometrically, if P (Fig. 5) is the point z , its position is fixed by the angle XOP , which is $\text{amp } z$, and the length OP , which is $|z|$. Thus we see that the magnitude and direction of the line OP suffice to

determine the number z ; the position of the line is immaterial.

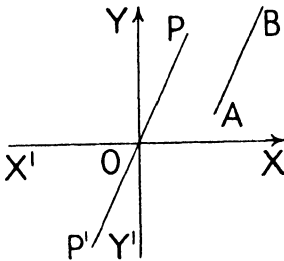


FIG. 5.

Any equivalent line or displacement AB (a line equal to, parallel to and in the same direction as OP) may equally well be used to represent the number z . Such a line is called a *vector*. The symbol \overline{AB} is employed to denote the vector AB . The vector symbol may be written in place of the complex number represented by

the vector. Thus $\overline{AB} = \overline{OP} = z$. It follows that

$$\text{mod } (\overline{AB}) = \text{mod } (\overline{OP}) = |z| = OP = AB,$$

where OP and AB denote the (positive) measures of these segments, and

$$\text{amp } (\overline{AB}) = \text{amp } (\overline{OP}) = \text{amp } z = \angle XOP.$$

If P' (Fig. 5) is the image of P in O , the argument of P' is $-z$. Hence $\overline{OP'} = -\overline{OP}$; but $\overline{AB} = \overline{OP}$ and $\overline{BA} = \overline{OP'}$. Therefore

$$\overline{BA} = -\overline{AB}. \quad . \quad . \quad . \quad (4)$$

Expression for the Vector joining two given Points in the Argand Diagram.—It was shown that, in Fig. 4,

$$\overline{OP_3'} = \overline{OP_1} - \overline{OP_2}.$$

But $\overline{P_2P_1} = \overline{OP_3'}$; hence

$$\overline{P_2P_1} = \overline{OP_1} - \overline{OP_2},$$

or

$$\overline{P_1P_2} = \overline{OP_2} - \overline{OP_1} = z_2 - z_1. \quad . \quad . \quad (5)$$

Thus the vector joining two points in the Argand Diagram represents the difference of the arguments of the points.

Addition of Vectors.—If A, B, C, \dots, M, N are any points in the Argand Diagram,

$$\overline{AB} + \overline{BC} + \dots + \overline{MN} = \overline{AN}. \quad . \quad . \quad (6)$$

For

$$\begin{aligned} \overline{AB} &= \overline{OB} - \overline{OA}, \\ \overline{BC} &= \overline{OC} - \overline{OB}, \\ &\dots \\ \overline{MN} &= \overline{ON} - \overline{OM}, \end{aligned}$$

and, on adding, it is found that

$$\overline{AB} + \overline{BC} + \dots + \overline{MN} = \overline{ON} - \overline{OA} = \overline{AN}.$$

It follows that

$$\overline{AB} + \overline{BC} + \overline{CA} = 0,$$

so that
$$\overline{AB} = \overline{CB} - \overline{CA}, \dots \dots \dots (7)$$

a formula of which (5) is a particular case.

Example.—P and Q are the points $1 + 2i$ and $5 - 6i$ respectively, and the points M and N divide PQ, so that

$$\frac{PM}{MQ} = - \frac{PN}{NQ} = \frac{3}{1}.$$

Determine the number represented by \overline{MN} , and find its modulus and its amplitude.

By Example 2, § 3,

$$\overline{OM} = \frac{3(5 - 6i) + 1(1 + 2i)}{3 + 1} = 4 - 4i,$$

and
$$\overline{ON} = \frac{3(5 - 6i) - 1(1 + 2i)}{3 - 1} = 7 - 10i.$$

Thus
$$\overline{MN} = \overline{ON} - \overline{OM} = 3 - 6i;$$

and therefore
$$|\overline{MN}| = \sqrt{9 + 36} = 3\sqrt{5},$$

and
$$\text{amp } (\overline{MN}) = \tan^{-1}(-2) = -\tan^{-1}(2),$$

the angle being in the fourth quadrant, since the cosine is positive and the sine is negative.

§ 7. Inequalities for the Moduli of Sums

The inequalities established in the two following theorems are of great importance.

THEOREM I.—The modulus of a sum is less than or equal to the sum of the moduli of the terms; *i.e.*, if z_1, z_2, \dots, z_n are any numbers, real or complex,

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|. \quad (8)$$

Let P_1, P_2, \dots, P_n (Fig. 6) be points in the Argand Diagram, such that the vectors $\overline{OP_1}, \overline{P_1P_2}, \dots, \overline{P_{n-1}P_n}$ represent the numbers z_1, z_2, \dots, z_n respectively; then $\overline{OP_n}$ represents the sum $z_1 + z_2 + \dots + z_n$, and

$OP_1 = |z_1|, P_1P_2 = |z_2|, \dots, P_{n-1}P_n = |z_n|$,
while $OP_n = |z_1 + z_2 + \dots + z_n|$.

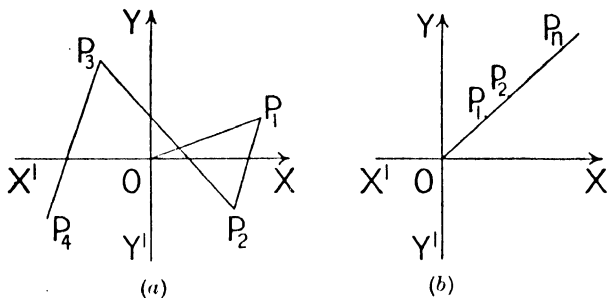


FIG. 6.

Now OP_n , being the shortest distance from O to P_n , is less than the sum of $OP_1, P_1P_2, \dots, P_{n-1}P_n$; unless, as in Fig. 6 (b), the points lie *in order* on a straight line from the origin, in which case OP_n is equal to the sum of $OP_1, P_1P_2, \dots, P_{n-1}P_n$. Thus the theorem has been proved.

THEOREM II.—The modulus of the sum of any two

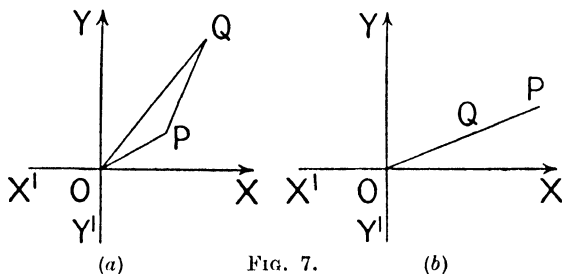


FIG. 7.

numbers is greater than or equal to the difference of their moduli; *i.e.*, if z_1 and z_2 are any two numbers, real or complex,

$$|z_1 + z_2| \geq |z_1| - |z_2|, \quad |z_1 + z_2| \geq |z_2| - |z_1|. \quad (9)$$

Let P and Q (Fig. 7) be points in the Argand Diagram such that the vectors \overline{OP} and \overline{PQ} represent the numbers z_1 and z_2 respectively: then \overline{OQ} represents the sum $z_1 + z_2$, and

$$OP = |z_1|, PQ = |z_2|, OQ = |z_1 + z_2|.$$

Now $OQ > OP - PQ$, unless, as in Fig. 7 (b), O, Q and P lie in order on a straight line, in which case $OQ = OP - PQ$. Also $OQ > PQ - OP$ unless P, O, Q lie in order on a straight line, in which case $OQ = PQ - OP$. Thus the theorem has been proved.

COROLLARY.—The modulus of the difference of any two numbers is greater than or equal to the difference of their moduli.

For

$$|z_1 - z_2| = |z_1 + (-z_2)| \geq |z_1| - |-z_2| = |z_1| - |z_2|.$$

Thus

$$|z_1 - z_2| \geq |z_1| - |z_2|, |z_1 - z_2| \geq |z_2| - |z_1|.$$

Example 1.—Give an algebraic proof of Theorem I.

For two numbers the inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

holds if

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \leq \{ \sqrt{(x_1^2 + y_1^2)} + \sqrt{(x_2^2 + y_2^2)} \}^2;$$

or if
$$x_1x_2 + y_1y_2 \leq \sqrt{(x_1^2 + y_1^2)} \cdot \sqrt{(x_2^2 + y_2^2)}.$$

This inequality holds if the L.H.S. is negative; if the L.H.S. is positive it holds if

$$(x_1x_2 + y_1y_2)^2 \leq (x_1^2 + y_1^2)(x_2^2 + y_2^2);$$

or if
$$0 \leq (x_1y_2 - x_2y_1)^2,$$

which is necessarily true.

Again, for three numbers,

$$|z_1 + z_2 + z_3| \leq |z_1| + |z_2 + z_3| \leq |z_1| + |z_2| + |z_3|;$$

and similarly the inequality can be shown to hold for any set of numbers.

Example 2.—Give an algebraic proof of Theorem II.

The inequality

$$|z_1 + z_2| \geq |z_1| - |z_2|$$

can be put in the form

$$\sqrt{\{(x_1 + x_2)^2 + (y_1 + y_2)^2\}} \geq \sqrt{(x_1^2 + y_1^2)} - \sqrt{(x_2^2 + y_2^2)}.$$

This holds if the R.H.S. is negative ; if the R.H.S. is positive the inequality is true if

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \geq \{\sqrt{(x_1^2 + y_1^2)} - \sqrt{(x_2^2 + y_2^2)}\}^2,$$

or if $x_1x_2 + y_1y_2 \geq -\sqrt{(x_1^2 + y_1^2)} \cdot \sqrt{(x_2^2 + y_2^2)}.$

If the L.H.S. is positive the inequality holds ; if the L.H.S. is negative the inequality holds if

$$(x_1x_2 + y_1y_2)^2 \leq (x_1^2 + y_1^2)(x_2^2 + y_2^2),$$

or if $0 \leq (x_1y_2 - x_2y_1)^2,$

which is necessarily true.

EXAMPLES XIII

1. If $c^2 + s^2 = 1$, show that

(i) $\frac{1 + c + is}{1 + c - is} = c + is$; (ii) $\frac{1 + s + ic}{1 + s - ic} = s + ic.$

2. Find the conjugate of the complex number

$$\frac{2 - i}{(1 - 2i)^2}.$$

Ans. $-(2 + 11i)/25.$

3. Express the number $(5 + i)/(2 + 3i)$ in the form $p + iq$, where p and q are real, and find its modulus and its amplitude.

Ans. $\sqrt{2}, -\frac{1}{4}\pi.$

4. A and B are the points $2 + i$ and $-3i$ respectively ; C is a point which moves on a circle with centre the origin and radius 6. Show that the centroid of the triangle ABC moves on a circle with centre at $\frac{2}{3}(1 - i)$ and radius 2.

5. Find the modulus and the amplitude of

$$(1 + i3\sqrt{3})/(\sqrt{3} + 2i).$$

Ans. $2, \frac{1}{6}\pi.$

6. If $w = z/(z + 3)$, where $w = u + iv$ and $z = x + iy$, and if z moves on the circle $(x + 3)^2 + y^2 = 1$, find the locus of the point w .

Ans. $(u - 1)^2 + v^2 = 9.$

7. Calculate the modulus and the amplitude (i) of the sum, (ii) of the product of the roots of the equation

$$(1 + 4i)z^2 + (5 + 3i)z + (8 - 2i) = 0.$$

Ans. (i) $\sqrt{2}, \frac{3}{4}\pi$; (ii) $2, -\frac{1}{2}\pi.$

8. Find the modulus and the amplitude of

(i) $\frac{5 + \sqrt{3}i}{4 - 2\sqrt{3}i}$; (ii) $(1 + i)\frac{2 + i}{3 - i}.$

Ans. (i) $1, \frac{1}{3}\pi$; (ii) $1, \frac{1}{2}\pi.$

9. If
$$x + iy = \frac{3}{2 + \cos \theta + i \sin \theta},$$

prove that $(x - 1)(x - 3) + y^2 = 0$.

10. Prove that the four points

$$1, \quad -1, \quad 3 + 4i, \quad \frac{1}{3 + 4i}$$

on the Argand Diagram lie on a circle of radius $\sqrt{10}$, and find the argument of the centre.

Ans. $3i$.

11. If $|(z - 1)/(z + 1)| = 2$, prove that the point z lies on the circle

$$x^2 + y^2 + \frac{10}{3}x + 1 = 0.$$

Show that the number $(-5 + 4i)/3$ satisfies both conditions, and illustrate by a diagram.

12. A, B, C have arguments $-2 - 3i$, $3 + 5i$, $2 + i$ respectively, and G is the centroid of the triangle ABC. Find the modulus and the amplitude of the argument of G.

Ans. $\sqrt{2}$, $\frac{1}{4}\pi$.

13. P and Q are the points $1 + 2i$ and $-9 + 7i$ respectively, and M and N divide PQ internally and externally, so that

$$\frac{PM}{MQ} = -\frac{PN}{NQ} = \frac{2}{3}.$$

Find the modulus and the amplitude of the vector MN.

Ans. $12\sqrt{5}$, $-\tan^{-1}\frac{1}{2}$.

14. A and B are the points $-4 + 3i$ and $5 + 12i$ respectively, and AB is divided internally at C and externally at D, so that

$$\frac{AC}{CB} = \frac{AD}{BD} = \frac{5}{13}.$$

Find the arguments of C and D.

Ans. $\frac{1}{2}(-3 + 11i)$, $-\frac{1}{3}(77 + 21i)$.

15. (PQ, RS) is a harmonic range, P, Q, R and S being collinear points in the Argand Diagram, whose arguments are p , q , r and s respectively. Show that

$$(p + q)(r + s) = 2pq + 2rs.$$

16. A, B, C are the points z_1 , z_2 , z_3 and G is the centroid of the triangle ABC. If $4z_1 + z_2 + z_3 = 0$, show that the origin is the mid-point of AG.

17. If $l^2 + m^2 + n^2 = 1$ and $m + in = (1 + l)z$, prove that

$$\frac{l + im}{1 + n} = \frac{1 + iz}{1 - iz}.$$

18. If $z = \cos \theta + i \sin \theta$, prove that

$$1 + z + z^2 = (1 + 2 \cos \theta)(\cos \theta + i \sin \theta).$$

19. Prove that

$$\frac{1 + \sin \theta + i \cos \theta}{1 - \sin \theta - i \cos \theta} = i(\tan \theta + \sec \theta).$$

20. Given that $3 - i$ is one root of the equation

$$x^4 - 8x^3 + 39x^2 - 122x + 170 = 0,$$

find the other roots.

$$\text{Ans. } 3 + i, \quad 1 \pm 4i.$$

21. If t is a real variable, and if $z = 4t + 3i(1 - t)$, show that the locus of z in the Argand Diagram is a straight line. Prove also that as t varies the minimum modulus of z is $2\frac{2}{3}$, and interpret this geometrically.

22. Prove the following construction to find on the Argand Diagram the points P and Q which represent the roots of the equation $z^2 - 2az + b^2 = 0$, where a and b are real and $0 < a < b$. With the origin as centre, describe a circle of radius b ; from the point A on OX at distance a from O draw a perpendicular to OX cutting the circle at P and Q.

23. If z_1, z_2, z_3 are three complex numbers such that

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3},$$

show that their representative points on the Argand Diagram lie on a circle through the origin.

24. Find the modulus and the amplitude of $12 - 5i$, given that $\tan 22^\circ 37' = 5/12$.

$$\text{Ans. } 13, \quad -22^\circ 37'.$$

25. If $a = \cos \alpha + i \sin \alpha$ and $b = \cos \beta + i \sin \beta$, express $a + b$ in the form $r(\cos \theta + i \sin \theta)$ and prove that

$$a^2 + b^2 = 2ab \cos(\alpha - \beta).$$

$$\text{Ans. } r = 2 \cos \frac{1}{2}(\alpha - \beta), \quad \theta = \frac{1}{2}(\alpha + \beta).$$

26. Find the modulus and the amplitude of

$$(-3 + 4i)(5 + 2i).$$

$$\text{Ans. } 5\sqrt{(29)}, \quad \pi - \tan^{-1}(14/23).$$

27. Calculate the modulus and the amplitude of

$$\sqrt{(\cot 35^\circ)} + i\sqrt{(\tan 35^\circ)},$$

giving the modulus correct to four significant figures.

$$\text{Ans. } 1.459, \quad 35^\circ.$$

28. Prove that if the modulus of $(z + 1)/(z - 1)$ is constant, the point z will lie on a circle, and that if the amplitude is constant the point z will lie on an arc of a second circle; show also that the two circles intersect orthogonally.

29. A, B, C are the points $-2 - 3i$, $7 + 2i$, $2 + 5i$ respectively, and a line DCE is drawn, with C as its mid-point, equal, parallel to, and in the same direction as AB; calculate the moduli and the amplitudes of the arguments of D and E.

Ans. $5/\sqrt{2}$, $\frac{3}{4}\pi$; $\frac{1}{2}\sqrt{394}$, $\tan^{-1}(15/13)$.

30. In the parallelogram OABC the side OA and the diagonal OB represent respectively the numbers $5 + i$ and $2 + 4i$. Find the complex numbers represented by the vectors OC and BC.

Ans. $-3 + 3i$, $-5 - i$.

31. Show that the numbers

$$5\left(1 - \cos \frac{2k\pi}{5} - i \sin \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4,$$

represent the vertices of a regular pentagon; find its centre and draw the diagram.

Ans. The point 5.

32. Represent on a diagram the vectors OA, OB, OC, OD when

$$\begin{aligned}\overline{OA} &= 12 + 5i, & \overline{OB} &= 8 + 15i, \\ \overline{OC} &= \overline{OA} + \overline{OB}, & \overline{OD} &= \overline{OA} - \overline{OB},\end{aligned}$$

and calculate the modulus and the amplitude of each vector.

Ans. 13, $\tan^{-1}(5/12)$; 17, $\tan^{-1}(15/8)$;
 $20\sqrt{2}$, $\frac{1}{4}\pi$; $2\sqrt{29}$, $-\tan^{-1}(5/2)$.

33. A and B are the points $-1 + 3i$, $2 + 5i$ respectively on the Argand Diagram. Find the modulus and the amplitude of the vector AB. If AB is revolved about A through the angle $+\frac{1}{4}\pi$ into the new position AC, its length remaining constant, find the complex number represented by the vector AC.

Ans. $\sqrt{13}$, $\tan^{-1}(2/3)$; $\frac{1}{\sqrt{2}}(1 + 5i)$.

34. The complex numbers z_1, z_2, z_3, z_4 are the arguments of the points A, B, C, D on the Argand Diagram. If $z_1 - z_2 = z_4 - z_3$, show that ABCD is a parallelogram and that the point $\frac{1}{4}(z_1 + z_2 + z_3 + z_4)$ is the centre of the parallelogram.

35. ABCD is a quadrilateral in the Argand Diagram, and L and M are the mid-points of AC and BD respectively. Prove

that the sum of the vectors \overline{AB} , \overline{AD} , \overline{CB} , \overline{CD} is four times the vector \overline{LM} .

36. \overline{OPQ} is a triangle in the Argand Diagram and $\overline{OR} = 2\overline{OP}$, $\overline{OS} = 3\overline{OQ}$. If $\overline{OT} = \overline{OR} + \overline{OS}$, and if OT cuts PQ at U , show that

$$\overline{OU} = \frac{1}{5}(2\overline{OP} + 3\overline{OQ}).$$

37. O is the origin of co-ordinates, \overline{OU} is the real unit vector and \overline{OP} represents $x + iy$; find the relation between x and y if the modulus of \overline{OP} is twice that of \overline{UP} and state what curve P describes when x and y vary.

Ans. The circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{3}{2}$.

38. The complex numbers $5 + 2i$, $7 + 5i$, $4 + 7i$, $2 + 4i$ are represented on the Argand Diagram by the points A , B , C , D respectively; show that the quadrilateral $ABCD$ is a parallelogram, and find the angles and the lengths of the sides of the parallelogram.

Ans. $\frac{1}{2}\pi$, $\sqrt{13}$.

39. Express the following numbers,

$$3 - \frac{4}{i}, \quad \frac{3}{2}(7i - 1)(1 - i), \quad \frac{5(i - 3)}{i + 1}, \quad \frac{2(i - 18)}{(1 + i)^2}$$

in the form $p + iq$, and show that they are the arguments of the vertices of a square on the Argand Diagram.

Ans. $3 + 4i$, $9 + 12i$, $-5 + 10i$, $1 + 18i$.

40. Reduce each of the numbers

$$\frac{3 + 5i}{-1 + i}, \quad \frac{5(5 + i)}{(2 + i)(1 - i)}, \quad -1 - \frac{10}{i}, \quad \frac{2 + 7i}{-i}$$

to the form $p + iq$. If they are the arguments of the points A , B , C , D respectively, show that $ABCD$ is a square.

Ans. $1 - 4i$, $7 + 4i$, $-1 + 10i$, $-7 + 2i$.

41. If $z = x + iy$, $w = u + iv$, and if $wz = 1$, show that, when the point z describes a straight line or a circle through the origin, the point w describes a straight line, while if the point z describes any other straight line or circle, the point w describes a circle.

CHAPTER XIV

PRODUCTS AND QUOTIENTS : DEMOIVRE'S
THEOREM

§ 1. Multiplication of Complex Numbers

CONSIDER the product

$$(\cos A + i \sin A)(\cos B + i \sin B)$$

where A and B are any angles. It is equal to
 $(\cos A \cos B - \sin A \sin B) + i(\sin A \cos B + \cos A \sin B)$,
 or $\cos(A + B) + i \sin(A + B)$,

so that

$$\begin{aligned} (\cos A + i \sin A)(\cos B + i \sin B) \\ = \cos(A + B) + i \sin(A + B). \end{aligned} \quad (1)$$

Now multiply both sides of this equation by

$$\cos C + i \sin C;$$

then

$$\begin{aligned} (\cos A + i \sin A)(\cos B + i \sin B)(\cos C + i \sin C) \\ = \{\cos(A + B) + i \sin(A + B)\}(\cos C + i \sin C) \\ = \cos(A + B + C) + i \sin(A + B + C), \text{ by (1).} \end{aligned}$$

Proceeding thus, we find that, for any number of factors,

$$\begin{aligned} (\cos A + i \sin A)(\cos B + i \sin B) \dots (\cos N + i \sin N) \\ = \cos(A + B + \dots + N) + i \sin(A + B + \dots + N). \end{aligned} \quad (2)$$

Now let n complex numbers be expressed in terms of their moduli and amplitudes as follows :

$$\begin{aligned} z_1 = r_1(\cos \theta_1 + i \sin \theta_1), z_2 = r_2(\cos \theta_2 + i \sin \theta_2), \dots \\ \dots, z_n = r_n(\cos \theta_n + i \sin \theta_n). \end{aligned}$$

Then it follows from (2) that their product is given by the formula

$$z_1 z_2 \dots z_n = r_1 r_2 \dots r_n \left\{ \begin{array}{l} \cos(\theta_1 + \theta_2 + \dots + \theta_n) \\ + i \sin(\theta_1 + \theta_2 + \dots + \theta_n) \end{array} \right\}. \quad (3)$$

Thus the modulus of a product is the product of the moduli of the factors ; while the sum of the amplitudes of the factors is one of the values of the amplitude of the product.

Example 1.—Find the modulus and the amplitude of the product $(1 + i)(1 + i\sqrt{3})$.

$$1 + i = \sqrt{2}(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi),$$

$$1 + i\sqrt{3} = 2(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi).$$

Hence the required modulus and amplitude are $2\sqrt{2}$ and $\frac{7}{12}\pi$.

Example 2.—If P and Q are the points in the Argand Diagram whose arguments are z and iz , show that (i) $OP = OQ$; (ii) $\angle POQ = \frac{1}{2}\pi$.

$$[\text{Since } i = \cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi,$$

$$|iz| = |z|, \text{ amp}(iz) = \text{amp } z + \frac{1}{2}\pi.]$$

Example 3.—If $z_1 z_2 = a^2$, where a is real, show that the points $z_1, z_2, a, -a$ are concyclic.

§ 2. Geometrical Construction for the Product of two Complex Numbers

Let P_1, P_2 and A (Fig. 1) be the points z_1, z_2 and 1 respectively. Join AP_1, OP_1 and OP_2 , and on OP_2 describe a

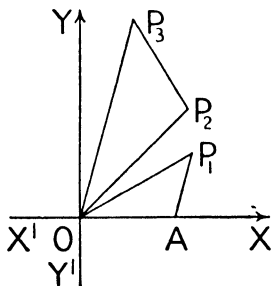


FIG. 1.

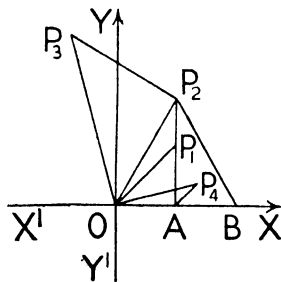


FIG. 2.

triangle OP_2P_3 directly * similar to the triangle OAP_1 . Then, if z_3 is the argument of the point P_3 ,

$$z_3 = z_1 z_2.$$

* Two similar triangles ABC and PQR are *directly* similar if they are both positive or both negative. A triangle is positive if its vertices are named in the counter-clockwise order, negative if they are named in the clockwise order.

For $OP_3 : OP_2 = OP_1 : OA$, and therefore

$$\frac{|z_3|}{|z_2|} = \frac{|z_1|}{1},$$

so that $|z_3| = |z_1| \cdot |z_2|$.

Also

$$\begin{aligned} \text{amp } z_3 = \angle XOP_3 &= \angle XOP_2 + \angle P_2OP_3 \\ &= \angle XOP_2 + \angle XOP_1 \\ &= \text{amp } z_2 + \text{amp } z_1. \end{aligned}$$

Hence, from (3),

$$z_3 = z_1 z_2.$$

Example.—Give a geometrical construction for the product $(1 + i)(1 + i\sqrt{3})$.

Let A and B (Fig. 2) be the points 1 and 2. On OB describe the equilateral triangle OBP_2 . Join AP_2 , and from AP_2 cut off AP_1 equal to OA. Then P_1 and P_2 are the points $1 + i$ and $1 + i\sqrt{3}$. Draw P_2P_3 perpendicular to and equal to OP_2 , so that $\angle OP_2P_3 = -\frac{1}{2}\pi$. Then the triangle OP_2P_3 is directly similar to the triangle OAP_1 ; and, in consequence, P_3 is the required point. [See Example 1, § 1.]

§ 3. Quotient of two Complex Numbers

From the identity

$$(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta) = \cos^2 \theta + \sin^2 \theta = 1,$$

it follows that

$$\frac{1}{\cos \theta + i \sin \theta} = \cos \theta - i \sin \theta,$$

or
$$\frac{1}{\cos \theta + i \sin \theta} = \cos (-\theta) + i \sin (-\theta). \quad (4)$$

Hence if

$$z_1 = r_1'(\cos \theta_1 + i \sin \theta_1), \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2),$$

$$\frac{z_1}{z_2} = \frac{r_1'}{r_2}(\cos \theta_1 + i \sin \theta_1)\{\cos (-\theta_2) + i \sin (-\theta_2)\},$$

and therefore, from (1),

$$\frac{z_1}{z_2} = \frac{r_1'}{r_2}\{\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)\}. \quad (5)$$

Thus the modulus of a quotient is the quotient of the moduli of the dividend and the divisor ; while the difference of the amplitudes of the dividend and the divisor is one of the values of the amplitude of the quotient.

In particular, for the reciprocal of z ,

$$\text{mod} \left(\frac{1}{z} \right) = \frac{1}{\text{mod } z}, \quad \text{amp} \left(\frac{1}{z} \right) = -\text{amp } z. \quad (6)$$

Example.—Show that the modulus and the amplitude of the quotient $(1 + i\sqrt{3})/(1 + i)$ are $\sqrt{2}$ and $\frac{1}{2}\pi$.

§ 4. Geometrical Construction for the Quotient of two Complex Numbers

Let P_1, P_2 and A (Fig. 3) be the points z_1, z_2 and 1 respectively. To find the point whose argument is z_1/z_2 , describe on OA

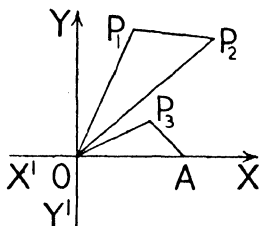


FIG. 3.

a triangle OAP_3 directly similar to the triangle OP_2P_1 . Then, as in § 2, if z_3 is the argument of P_3 , $z_1 = z_2 z_3$, so that P_3 is the required point.

Example 1.—Give a geometrical construction for the quotient

$$(1 + i\sqrt{3})/(1 + i).$$

In Fig. 2 draw a triangle OAP_4 directly similar to the triangle OP_1P_2 ; then P_4 is the required point. [See Example, § 3.]

Example 2.—If P_1, P_2, P_3, P_4 are the points z_1, z_2, z_3, z_4 respectively, and if $z_1 z_4 = z_2 z_3$, show that the triangles OP_1P_2 and OP_3P_4 are directly similar. Deduce a construction for the point $z_2 z_3 / z_1$ when the points z_1, z_2 and z_3 are given.

§ 5. Amplitude of the Quotient of two Vectors

If \overline{AB} and \overline{AC} (Fig. 4) are any two vectors drawn from a point A ,

$$\text{amp} (\overline{AC}/\overline{AB}) = \angle BAC. \quad (7)$$

Here the expression $\overline{AC}/\overline{AB}$ denotes the quotient of the complex numbers represented by the vectors AC and AB .

Let \overline{OP} and \overline{OQ} be vectors equal to \overline{AB} and \overline{AC} respectively; then

$$\begin{aligned} \text{amp} \left(\frac{\overline{AC}}{\overline{AB}} \right) &= \text{amp } \overline{AC} - \text{amp } \overline{AB} \\ &= \text{amp } \overline{OQ} - \text{amp } \overline{OP} \\ &= \angle XOQ - \angle XOP \\ &= \angle POQ = \angle BAC. \end{aligned}$$

COROLLARY.—If P_1, P_2, P_3 are the points z_1, z_2, z_3 respectively,

$$\text{amp} \left(\frac{z_1 - z_2}{z_3 - z_2} \right) = \angle P_3 P_2 P_1.$$

Example 1.—If

$$\text{amp} \left(\frac{z_2 - z_3}{z_1 - z_3} \right) = \text{amp} \left(\frac{z_2 - z_4}{z_1 - z_4} \right),$$

show that the points z_3 and z_4 are on the same side of the line joining z_1 and z_2 . Prove also that the points z_1, z_2, z_3, z_4 are concyclic.

If P_1, P_2, P_3, P_4 are the points whose arguments are z_1, z_2, z_3, z_4 , then $\angle P_1 P_3 P_2 = \angle P_1 P_4 P_2$; consequently, since these angles have the same sign, P_3 and P_4 lie on the same side of the line $P_1 P_2$. Also, since the angles are equal, the four points are concyclic.

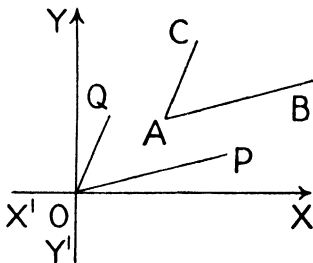


FIG. 4.

Example 2.—If

$$\text{amp} \left\{ \frac{(z_2 - z_3)(z_1 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \right\} = \pi,$$

prove that the points z_1, z_2, z_3, z_4 are concyclic.

Example 3.—If ABC is a triangle described positively, show that $\text{amp} (\overline{AC}/\overline{CB}) = \pi - C$.

Example 4.—Show that, if the triangles ABC and DEF in the Argand Diagram are directly similar,

$$\frac{\overline{AC}}{\overline{AB}} = \frac{\overline{DF}}{\overline{DE}}.$$

Since (Fig. 5) $\angle BAC = \angle EDF$,

$$\text{amp } \frac{\overline{AC}}{\overline{AB}} = \text{amp } \frac{\overline{DF}}{\overline{DE}}.$$

Also, since the triangles are similar,

$$\frac{AC}{AB} = \frac{DF}{DE} \quad \text{or} \quad \left| \frac{\overline{AC}}{\overline{AB}} \right| = \left| \frac{\overline{DF}}{\overline{DE}} \right|.$$

Therefore, as the ratios $\overline{AC}/\overline{AB}$ and $\overline{DF}/\overline{DE}$ have the same modulus and the same amplitude, they are equal.

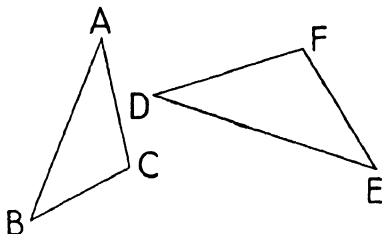


FIG. 5.

Example 5.—Show that if, in Fig. 5, the ratios $\overline{AC}/\overline{AB}$ and $\overline{DF}/\overline{DE}$ are equal, the triangles are directly similar.

$$[\angle BAC = \angle EDF \text{ and } AC/AB = DF/DE.]$$

Example 6.—If $P_1, P_2, P_3, \dots, P_6$ are the points whose arguments are $z_1, z_2, z_3, \dots, z_6$ respectively, and if the triangles $P_1P_2P_3$ and $P_4P_5P_6$ are directly similar, show that

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_4 - z_5}{z_6 - z_5}.$$

Conversely, if this equation holds, show that the triangles are directly similar.

§ 6. De Moivre's Theorem

If in formula (2) each of the n angles A, B, \dots, N is replaced by θ , the formula

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad . \quad . \quad (8)$$

is obtained. This is De Moivre's Theorem for positive integral values of the exponent n .

Again, if n is a negative integer, and $m = -n$, it follows from (4) that

$$\begin{aligned}
 (\cos \theta + i \sin \theta)^n &= \{\cos(-\theta) + i \sin(-\theta)\}^m \\
 &= \cos(-m\theta) + i \sin(-m\theta), \text{ by (8),} \\
 &= \cos n\theta + i \sin n\theta.
 \end{aligned}$$

Thus (8) also holds when n is a negative integer.

When $n = 0$ both sides of (8) have the value 1.

Example 1.—If n is an integer, prove that

$$(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta.$$

Example 2.—Establish the formula (8) for positive integral values of n by the method of induction.

Assume that (8) holds for the index n . Then

$$\begin{aligned}
 (\cos \theta + i \sin \theta)^{n+1} &= (\cos n\theta + i \sin n\theta)(\cos \theta + i \sin \theta) \\
 &= (\cos n\theta \cos \theta - \sin n\theta \sin \theta) + i(\sin n\theta \cos \theta + \cos n\theta \sin \theta) \\
 &= \cos(n+1)\theta + i \sin(n+1)\theta.
 \end{aligned}$$

Hence if (8) holds for the index n , it holds also for the index $n+1$. But it obviously holds when $n=1$. Hence it holds for all positive integral values of n .

In the next place, if p and q are integers,

$$\begin{aligned}
 \left(\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta\right)^q &= \cos p\theta + i \sin p\theta, \text{ by (8),} \\
 &= (\cos \theta + i \sin \theta)^p, \text{ by (8),}
 \end{aligned}$$

and therefore

$$\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta \text{ is one value of } (\cos \theta + i \sin \theta)^{p/q}.$$

But, if n is a rational fraction, it may be put in the form p/q . Hence $\cos n\theta + i \sin n\theta$ is one value of

$$(\cos \theta + i \sin \theta)^n.$$

Demoivre's Theorem may therefore be stated as follows.

If n is an integer, positive or negative,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta,$$

while, if n is a rational fraction, $\cos n\theta + i \sin n\theta$ is one of the values of $(\cos \theta + i \sin \theta)^n$.

Demoivre's Theorem for a positive integral index was given by Demoivre about 1730.

Example 3.—If n is a rational fraction, show that

$$\cos n\theta - i \sin n\theta \text{ is one value of } (\cos \theta - i \sin \theta)^n.$$

§ 7. Roots of Complex Numbers

If n is a positive integer, the n th roots of z are the solutions of the equation $w^n = z$, regarded as an equation in w .

Let $z = r(\cos \theta + i \sin \theta)$,
 and $w = \rho(\cos \phi + i \sin \phi)$.
 Then $\rho^n(\cos n\phi + i \sin n\phi) = r(\cos \theta + i \sin \theta)$;
 hence $\rho^n = r$,
 and $n\phi = \theta + 2k\pi, k = 0, \pm 1, \pm 2, \dots$

It follows that, since ρ and r are positive, ρ is equal to the ordinary arithmetical n th root of r , represented by $r^{1/n}$;

while $\phi = \frac{\theta + 2k\pi}{n}, k = 0, \pm 1, \pm 2, \dots$

Now, in the Argand Diagram describe the circle with the origin as centre and $r^{1/n}$ as radius (Fig. 6), and on it mark the points $w_0, w_1, w_2, \dots, w_{n-1}$, where

$$w_k = r^{1/n} \left(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right), \quad (9)$$

$k = 0, 1, 2, \dots, n - 1$.

Since the amplitudes of the successive points differ by $2\pi/n$, the n points are distinct, and consequently correspond to n distinct roots of the equation $w^n = z$. Other values of k simply give repetitions of these points, so that n and only n distinct roots of the equation can be obtained in this way. And, indeed, it is known from algebraic considerations that an equation of the n th degree cannot

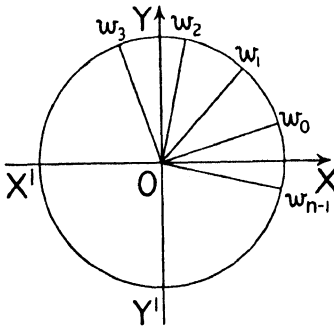


FIG. 6.

have more than n distinct roots. Thus the values of w given by (9) comprise all the n th roots of z .

Note.—Any other set of integral values of k can be taken in place of the set $0, 1, 2, \dots, n - 1$, provided that they

are such as to give distinct roots. For instance, it is often useful to take the set beginning 0, $\pm 1, \pm 2, \dots$

If $-\pi < \theta \leq \pi$, the root

$$w_0 = r^{1/n} \left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right)$$

is called the *principal value* of $z^{1/n}$.

Values of z^n when n is a Rational Fraction.—Let $n = p/q$, where p and q are integers with no common factor, and q is positive. Then

$$z^{p/q} = \{r^p (\cos p\theta + i \sin p\theta)\}^{1/q},$$

and therefore

$$z^{p/q} = r^{p/q} \left(\cos \frac{p\theta + 2k\pi}{q} + i \sin \frac{p\theta + 2k\pi}{q} \right), \quad (10)$$

where $k = 0, 1, 2, \dots, q - 1$.

This gives the q distinct values of $z^{p/q}$.

Note.—The formula

$$z^{p/q} = r^{p/q} \left\{ \cos \frac{p}{q}(\theta + 2k\pi) + i \sin \frac{p}{q}(\theta + 2k\pi) \right\}, \quad (10a)$$

where $k = 0, 1, 2, \dots, q - 1$ also gives the q distinct values. For, if k_1 and k_2 are any two of these values of k , the corresponding values of $z^{p/q}$ cannot be equal unless

$$\frac{p}{q}(\theta + 2k_1\pi) - \frac{p}{q}(\theta + 2k_2\pi) = 2m\pi$$

for some integral value of m . If that were so, we would have

$$p(k_1 - k_2) = mq,$$

and therefore, as $|k_1 - k_2| < q$, q would have a factor in common with p , which is not the case.

Example 1.—Solve the equation $z^n = 1$, where n is a positive integer.

Here
$$z^n = \cos 2k\pi + i \sin 2k\pi,$$

and therefore

$$z = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k = 0, 1, 2, \dots, n - 1.$$

• Thus the n th roots of unity are 1, $\omega_n, \omega_n^2, \dots, \omega_n^{n-1}$,

where
$$\omega_n = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

Note.—It is sometimes advantageous in a case of this kind to write

$$z^n = \cos 2k\pi \pm i \sin 2k\pi,$$

so that
$$z = \cos \frac{2k\pi}{n} \pm i \sin \frac{2k\pi}{n}.$$

The complex roots are then given in conjugate pairs. Similarly, the equation $z^n = -1$ can be written

$$z^n = \cos (2k + 1)\pi \pm i \sin (2k + 1)\pi,$$

giving
$$z = \cos \frac{2k + 1}{n}\pi \pm i \sin \frac{2k + 1}{n}\pi.$$

Example 2.—Find all the values of $(-1 + i\sqrt{3})^{\frac{1}{4}}$. Since

$$-1 + i\sqrt{3} = 2 \left\{ \cos \left(\frac{2\pi}{3} + 2k\pi \right) + i \sin \left(\frac{2\pi}{3} + 2k\pi \right) \right\},$$

the required roots are

$$\sqrt[4]{2} \left\{ \cos \left(\frac{\pi}{6} + \frac{k\pi}{2} \right) + i \sin \left(\frac{\pi}{6} + \frac{k\pi}{2} \right) \right\}, \quad k = 0, 1, 2, 3;$$

or
$$\sqrt[4]{2} \left(\frac{\sqrt{3}}{2} + i \frac{1}{2} \right), \quad \sqrt[4]{2} \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right),$$

$$\sqrt[4]{2} \left(-\frac{\sqrt{3}}{2} - i \frac{1}{2} \right), \quad \sqrt[4]{2} \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right).$$

EXAMPLES XIV

1. The numbers $2 + 3i$, $8 + 11i$, $17i$ are the arguments of the points A, B, C respectively on the Argand Diagram. Show that A, B, C are three vertices of a square, and find the argument of the fourth vertex D.

Ans. $-6 + 9i$.

2. In the Argand Diagram OPQ is an equilateral triangle, named in the counter-clockwise order. If P is the point $r(\cos \theta + i \sin \theta)$, state the argument of the point Q, and show that

$$x_P + iy_P = (x_Q + iy_Q) \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right).$$

Deduce that, if P moves along the line $x + \sqrt{3} \cdot y + 2 = 0$, Q moves along the line $x - \sqrt{3} \cdot y - 2 = 0$.

3. P and Q are the points $-2 - 3i$ and $2 + i$ respectively, and PQR is an equilateral triangle described positively. Determine the modulus and the amplitude of the vector PQ, and the complex number represented by the vector PR.

Ans. $4\sqrt{2}$, $\frac{1}{4}\pi$, $2 - 2\sqrt{3} + i(2 + 2\sqrt{3})$.

4. Show that the vector ikz , where k is a real number, is at right angles to the vector z , and that the length of the former is k times that of the latter.

Deduce that if (x_1, y_1) and (x_2, y_2) are the rectangular coordinates of opposite corners of a square, the other corners are

$$\left\{ \frac{1}{2}(x_1 + x_2 - y_1 + y_2), \frac{1}{2}(x_1 - x_2 + y_1 + y_2) \right\}$$

and $\left\{ \frac{1}{2}(x_1 + x_2 + y_1 - y_2), \frac{1}{2}(-x_1 + x_2 + y_1 + y_2) \right\}$.

5. Show that the sum of the two numbers $\sqrt{3} + i$ and $-1 + i\sqrt{3}$ is $2\sqrt{2}(\cos \theta + i \sin \theta)$, where $\tan \theta = 2 + \sqrt{3}$; and that the two numbers can be represented on the Argand Diagram by two adjacent sides of a square. What are the length and the inclination to the real axis of the diagonal through their point of intersection?

Ans. $2\sqrt{2}, 75^\circ$.

6. ABCD is a square described positively in the Argand Diagram; show that $\overline{BA} = i\overline{BC}$.

If z_1, z_2, z_3, z_4 are the arguments of A, B, C, D respectively, deduce that

$$2z_2 = (1 + i)z_1 + (1 - i)z_3, \quad 2z_4 = (1 - i)z_1 + (1 + i)z_3.$$

7. Show that the vectors which represent the complex numbers $a(1 + i\sqrt{3})$ and $b(-\sqrt{3} + i)$, where a and b are real, form two adjacent sides of a rectangle, and find the ratio of b to a if the diagonal through their point of intersection makes an angle $\frac{3}{4}\pi$ with the real axis.

Ans. $\tan \frac{5}{2}\pi = 2 + \sqrt{3}$.

8. A is a point on the Argand Diagram; on the circle on OA as diameter are taken two points B and C so that $\angle AOB = \angle BOC = \phi$: if A, B, C represent the complex numbers z_1, z_2, z_3 respectively, show that

$$z_2^2 \cos 2\phi = z_1 z_3 \cos^2 \phi.$$

9. If z_1, z_2 are conjugate and z_3, z_4 are also conjugate, show that $\text{amp}(z_1/z_4) = \text{amp}(z_3/z_2)$.

10. A and B are the points $7 + i, 3 + 4i$, and M is the mid-point of AB. Find the point C which is such that the triangles AOB and MOC are directly similar.

Ans. $\frac{5}{4}(1 + 3i)$.

11. If z_1 and z_2 are the roots of the equation

$$z^2 - az + a^2 = 0,$$

where a is a complex number, show that the points z_1 and z_2 are the vertices of the equilateral triangles described on opposite sides of the line joining O to the point a .

12. A and B are the points $4 + 7i$ and $2 + 6i$, and AB is produced twice its own length to C. Find the argument of C, and the modulus and amplitude of $\overline{OC}/\overline{OB}$.

Ans. $-2 + 4i$, $\frac{1}{2}\sqrt{2}$, $\frac{1}{4}\pi$.

13. Give a geometrical construction for the point $1 + i\sqrt{3}$, and then apply the geometrical rule for obtaining the product of two complex numbers to find the point $(1 + i\sqrt{3})^2$; hence find the modulus and the amplitude of $(1 + i\sqrt{3})^2$.

Ans. 4, $\frac{2}{3}\pi$.

14. If Q is the point z_2 , show that the point z_3 , where $z_3 = (1 + ki)z_2$, k being real, lies on the perpendicular at Q to OQ. Find the value of k if $\text{amp } z_3 - \text{amp } z_2 = \frac{1}{3}\pi$.

Ans. $\sqrt{3}$.

15. A is the fixed point $a + ib$, and P a variable point $x + iy$. The point Q is given by the relation

$$\overline{OQ} = \frac{\overline{OP}}{\overline{OA}}.$$

Show that, as P moves on a circle with centre O and radius r , Q moves on the circle with centre O and radius $r/\sqrt{a^2 + b^2}$.

16. If $z_2^2 = z_1z_3$, where z_1, z_2, z_3 are the arguments of the points A, B, C respectively, show that the triangles OAB and OBC are directly similar. If P and Q are the mid-points of AB and BC respectively, prove that the triangle OPQ is also directly similar to the triangle OAB.

17. Calculate the modulus and the amplitude of each of the numbers 30, $36 + 27i$, $16 - 12i$, and show that, if these are the arguments of the points A, B, C respectively, the triangles AOB and COA are directly similar.

Ans. 30, 0; 45, $\tan^{-1} \frac{3}{4}$; 20, $-\tan^{-1} \frac{3}{4}$.

18. The triangles AOB and BOC in the Argand Diagram are directly similar. If BO is produced its own length to D, show that the triangle ADC is directly similar to each of these two triangles.

19. Given the point P, whose argument is z , on the Argand diagram, give a geometrical construction to obtain the points \sqrt{z} .

[Let A be the point 1; produce PO to B so that OB is of length 1. Through O draw a line parallel to AB to meet the circle PAB in Q and R, the required points.]

20. Given the points which represent the complex numbers z_1 and z_2 , find by construction those representing the two values of $\sqrt{z_1z_2}$.

21. If P is the point $a + ib$, and if c and d are positive, show that the following construction will give the point $(a + ib)(c + id)$. Produce OP to Q, making $OQ = c \cdot OP$; draw QR, making a positive right angle with OQ, and such that $QR = d \cdot OP$. Then R is the required point.

22. ABC is an equilateral triangle whose side is of unit length and whose centroid is the origin. Express the vectors OA, OB, OC by complex numbers, and show that their sum is zero for all orientations of the triangle.

$$\text{Ans. } z, z\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right), z\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right).$$

23. In the Argand Diagram P and Q are the points $3 - 2i$ and $2i - 1$ respectively. The vector PQ is turned through -30° to the position PR. Find the numbers represented by the vectors PR and OR.

$$\text{Ans. } -2(\sqrt{3} - 1) + 2(\sqrt{3} + 1)i, 5 - 2\sqrt{3} + 2\sqrt{3} \cdot i.$$

24. In the Argand Diagram ABCD is a square whose area is positive, A is the point $1 - 2i$, and the diagonals of the square intersect at the point $6 - i$. Find the numbers represented by the vectors AB and AD.

$$\text{Ans. } 6 - 4i, 4 + 6i.$$

25. If PQR is an equilateral triangle, prove that

$$(\text{Vector PQ})^2 + (\text{Vector QR})^2 + (\text{Vector RP})^2 = 0.$$

Show that, if the points z_1, z_2, z_3 are the vertices of an equilateral triangle

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

26. If $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, prove that

$$(i) \frac{x - y}{x + y} = i \tan \frac{\alpha - \beta}{2},$$

$$(ii) \frac{(x + y)(xy - 1)}{(x - y)(xy + 1)} = \frac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta}.$$

27. If $z = a(\cos \theta + i \sin \theta)$, where a and θ are real, show that $z/(z^2 + a^2)$ is purely real, and that $(a - iz)/(a + iz)$ is purely imaginary.

28. Find the complex number with modulus twice that of $4 + 3i$, and amplitude less than that of $4 + 3i$ by $\frac{1}{2}\pi$.

$$\text{Ans. } (7 - i)\sqrt{2}.$$

29. PQRS is a rhombus, the arguments of the vertices being z_1, z_2, z_3, z_4 respectively, and the angle RQP is equal to $\frac{1}{2}\pi$. Show that

$$(i) 2z_2 = z_1(1 + i\sqrt{3}) + z_3(1 - i\sqrt{3});$$

$$(ii) 2z_4 = z_1(1 - i\sqrt{3}) + z_3(1 + i\sqrt{3}).$$

30. The points A, B, C in the Argand Diagram have arguments $-6 + i$, $-2 + 5i$, $4 - i$ respectively: D is the mid-point of AB and E divides BC in the ratio 2 : 1. Find the arguments of D and E.

Evaluate $\overline{BC}/\overline{BA}$, and deduce the argument of F if the triangles ABC and DEF are directly similar.

Ans. $D(-4 + 3i)$, $E(2 + i)$, $\overline{BC}/\overline{BA} = \frac{3}{2}i$, $F(-1 - 8i)$.

31. ABCD is a cyclic quadrilateral in the Argand Diagram. Prove that

$$\frac{\overline{AD} \cdot \overline{CB}}{\overline{AB} \cdot \overline{CD}} + \frac{|\overline{AD}| \cdot |\overline{CB}|}{|\overline{AB}| \cdot |\overline{CD}|} = 0.$$

32. If A, B, C are the points $1 - 3i$, $3 + 3i$, $5 - i$ respectively, mark them on a diagram, and find the moduli and amplitudes of $\overline{BC}/\overline{AC}$ and $\overline{AB}/\overline{AC}$.

Ans. 1 , $-\frac{1}{2}\pi$; $\sqrt{2}$, $\frac{1}{4}\pi$.

33. If A, B, C, D are the points in the Argand Diagram whose arguments are z_1, z_2, z_3, z_4 respectively, and if

$$\frac{z_1 - z_4}{z_2 - z_4} = k \frac{z_1 - z_3}{z_2 - z_3},$$

where k is a positive real number, show that the four points are concyclic.

34. Draw the vectors OA, OB, OC, OD which correspond respectively to the numbers $3 + 3i$, $5 + 4i$, $4 + 5i$, $2 + 4i$, and show that ABCD is a parallelogram. What angle of the parallelogram represents the amplitude of $\overline{BA}/\overline{BC}$, and what are the size and the sign of this angle?

Ans. $\angle CBA$, $\tan^{-1}3$, positive.

35. The vectors OA, OB, OC are such that

$$\overline{OA} : \overline{OB} = \overline{OC} : \overline{OA}.$$

Draw the vectors, assuming that $|\overline{OA}/\overline{OB}| = 2$ and

$$\text{amp } (\overline{OA}/\overline{OB}) = 30^\circ.$$

If AO is produced to D so that OD = AO, prove that A, B, C and D are concyclic.

36. A, B, C, D, E are the points on the Argand Diagram whose arguments are z_1, z_2, z_3, z_4, z_5 respectively. If

$$(z_3 - z_2)z_4 = (z_1 - z_2)z_5,$$

prove that the triangles ABC, DOE are directly similar, where O is the origin.

If further $z_3 - z_2 = kiz_4$, where k is real and positive, and $\text{amp } (z_5/z_4) = \frac{1}{4}\pi$; prove that $\text{amp } (z_1 - z_2) = \text{amp } z_5$.

37. A, B, C are points in the Argand Diagram such that BC is parallel to AO, and $\angle OBA = \angle OCB$; prove that $\overline{AO}/\overline{AB}$ and $\overline{OB}/\overline{OC}$ represent conjugate complex numbers. If $\overline{OA} = -3 - i$, $\overline{OB} = -1 - 2i$, find the value of $\overline{AO}/\overline{AB}$, and deduce that of \overline{OC} .

$$\text{Ans. } 1 + i, \frac{1}{2} - \frac{3}{2}i.$$

38. Find the modulus and the amplitude of

$$\frac{(\sin x - i \cos x)^3}{(\cos x - i \sin x)^3}.$$

$$\text{Ans. } 1, 6x + \frac{1}{2}\pi.$$

39. Express in its simplest form

$$\frac{(1 + \cos \theta + i \sin \theta)^4}{(\cos \theta - i \sin \theta)^5}.$$

$$\text{Ans. } (2 \cos \frac{1}{2}\theta)^4 (\cos 7\theta + i \sin 7\theta).$$

40. Find the modulus and the amplitude of

$$\left(\frac{i \cos \theta + i - \sin \theta}{1 + \cos \theta - i \sin \theta} \right)^3.$$

$$\text{Ans. } 1, 3(\theta + \frac{1}{2}\pi).$$

41. Prove that

$$\begin{aligned} & [(\cos \alpha + i \sin \alpha) - (\cos \beta + i \sin \beta)]^4 \\ & + [(\cos \alpha - i \sin \alpha) - (\cos \beta - i \sin \beta)]^4 \\ & \quad = 2^5 \sin^4 \left(\frac{\alpha - \beta}{2} \right) \cos 2(\alpha + \beta). \end{aligned}$$

42. If $x = \cos \theta + i \sin \theta$, show that

$$i \tan n\theta = \frac{x^{2n} - 1}{x^{2n} + 1},$$

n being any integer.

43. Show that

$$\frac{1 + \sin \theta + i \cos \theta}{1 + \sin \theta - i \cos \theta} = \cos \left(\frac{1}{2}\pi - \theta \right) + i \sin \left(\frac{1}{2}\pi - \theta \right),$$

and deduce the value of

$$\left(\frac{1 + \sin \frac{1}{8}\pi + i \cos \frac{1}{8}\pi}{1 + \sin \frac{1}{8}\pi - i \cos \frac{1}{8}\pi} \right)^8.$$

$$\text{Ans. } -1.$$

44. Prove that $\left(\frac{-1 + i\sqrt{3}}{2} \right)^n + \left(\frac{-1 - i\sqrt{3}}{2} \right)^n$ has the

value -1 if $n = 3k \pm 1$, and the value 2 if $n = 3k$, where k is any integer.

45. Find the modulus and the amplitude of

$$\frac{1 - \cos \theta - i \sin \theta}{1 + \cos \theta - i \sin \theta},$$

where $0 < \theta < \pi$.

$$\text{Ans. } \tan \frac{1}{2}\theta, \theta - \frac{1}{2}\pi.$$

46. Reduce to its simplest form

$$[\cos A - \cos 2A + i(\sin A - \sin 2A)]^8 + [\cos A - \cos 2A - i(\sin A - \sin 2A)]^8.$$

$$\text{Ans. } (2 \sin \frac{1}{2}A)^8 \cdot 2 \cos 12A.$$

47. Show that

$$\begin{aligned} & [\cos 2A + \cos 2B + i(\sin 2A + \sin 2B)]^n \\ & + [\cos 2A + \cos 2B - i(\sin 2A + \sin 2B)]^n \\ & = 2^{n+1} \{\cos(A+B)\}^n \cos n(A+B). \end{aligned}$$

48. Find the modulus and the principal value of the amplitude of $(-1 + i\sqrt{3})^5$.

$$\text{Ans. } 32, -\frac{2}{3}\pi.$$

49. Find the modulus and the amplitude of

$$\left(\frac{1 + \cos A + i \sin A}{1 + \cos A - i \sin A} \right)^n,$$

where n is an integer.

$$\text{Ans. } 1, nA.$$

50. By expanding $(1+z)^n$ by the Binomial Theorem, and then putting $z = \cos \theta + i \sin \theta$, show that

$$2^n (\cos \frac{1}{2}\theta)^n (\cos \frac{1}{2}n\theta + i \sin \frac{1}{2}n\theta) = \sum_{r=0}^n {}^nC_r (\cos r\theta + i \sin r\theta).$$

Deduce that

$$\sum_{r=0}^n {}^nC_r \cos \frac{r\pi}{2n} = 2^{n-1} \left(\cos \frac{\pi}{4n} \right)^n.$$

51. If $x = \cos \theta + i \sin \theta$, $y = \cos \phi + i \sin \phi$, and m and n are integers, show that

$$\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2 \cos (m\theta - n\phi).$$

52. If $\alpha = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$, show that the sum

$$1 + \alpha^n + \alpha^{2n} + \alpha^{3n} + \alpha^{4n}$$

is 5 when n is 5 or a multiple of 5, but is 0 when n is any other integer. Then show that, if

$$f(x) = A_0 + \sum_{n=1}^{20} A_n x^n,$$

$$\begin{aligned} f(x) + f(\alpha x) + f(\alpha^2 x) + f(\alpha^3 x) + f(\alpha^4 x) \\ = 5(A_0 + A_5 x^5 + A_{10} x^{10} + A_{15} x^{15} + A_{20} x^{20}). \end{aligned}$$

53. Show that the real and imaginary parts of $(1 + xi)^n$, where x is real and n is a positive integer, will be equal if

$$x = \tan\left(\frac{4r + 1}{4n}\pi\right),$$

r being zero or any integer.

54. If $\alpha = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, where n is a positive integer, show that the sum

$$1 + \alpha^k + \alpha^{2k} + \alpha^{3k} + \dots + \alpha^{(n-1)k}$$

is equal to n if k is a multiple of n , but is zero for all other integral values of k .

55. If $a = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$,

and $b = a + a^2 + a^4$, $c = a^3 + a^5 + a^6$,

prove that

(i) $b + c = -1$; (ii) $bc = 2$; (iii) $\sum_{n=0}^6 a^{n^2} = 1 + 2b = i\sqrt{7}$.

56. Find the three cube roots of $2i - 2$.

Ans. $\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$, $\sqrt{2}\left(\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12}\right)$,
 $\sqrt{2}\left(\cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12}\right)$;

or $(1 + i)$, $-\frac{\sqrt{3} + 1}{2} + i\frac{\sqrt{3} - 1}{2}$, $\frac{\sqrt{3} - 1}{2} - i\frac{\sqrt{3} + 1}{2}$.

57. Calculate the modulus and the amplitude of

$$(2 - i)^2(4 + 3i);$$

then find the values of $(2 - i)^{2/5}(4 + 3i)^{1/5}$ in the form

$$r(\cos \theta + i \sin \theta),$$

expressing r correct to three significant figures and θ correct to the nearest minute.

Ans. 25 , $-16^\circ 16'$; $r = 1.90$, $\theta = -3^\circ 15'$, $68^\circ 45'$, $140^\circ 45'$,
 $212^\circ 45'$, $284^\circ 45'$.

58. Find all the values of $\{|2^{5/6}|(1 + i)\}^{3/4}$.

Ans.

$$2\left\{\cos\left(\frac{3\pi}{16} + \frac{3k\pi}{2}\right) + i \sin\left(\frac{3\pi}{16} + \frac{3k\pi}{2}\right)\right\}, k = 0, 1, 2, 3.$$

59. Find the cube roots of $i - \sqrt{3}$.

Ans. $2^{1/3}\left\{\cos\left(\frac{5\pi}{18} + \frac{2k\pi}{3}\right) + i \sin\left(\frac{5\pi}{18} + \frac{2k\pi}{3}\right)\right\}, k = 0, 1, 2.$

60. Find all the values of $(-\sqrt{3} - i)^{2/3}$.

Ans. $\sqrt[3]{4} \left\{ \cos \left(\frac{6k + 1}{9} \pi \right) + i \sin \left(\frac{6k + 1}{9} \pi \right) \right\}, k = 0, 1, 2.$

61. Show that one value of

$$(1 + i\sqrt{3})^{3/4} + (1 - i\sqrt{3})^{3/4}$$

is $\sqrt[4]{32}$.

62. Find all the values of $(1 - i)^{2/3}$.

Ans. $2^{-2/3}(\sqrt{3} - i), 2^{-2/3}(-\sqrt{3} - i), i\sqrt[3]{2}.$

63. Find all the values of $(\sqrt{3} - i)^{2/3}$.

Ans. $2^{2/3} \left\{ \cos \left(\frac{\pi}{9} + \frac{2k\pi}{3} \right) - i \sin \left(\frac{\pi}{9} + \frac{2k\pi}{3} \right) \right\}, k = 0, 1, 2.$

64. If ω is one of the complex cube roots of unity, show that

$$(i) \quad (x + y)^3 + (x\omega + y\omega^2)^3 + (x\omega^2 + y\omega)^3 = 3(x^3 + y^3),$$

$$(ii) \quad (x + y)^4 + (x\omega + y\omega^2)^4 + (x\omega^2 + y\omega)^4 = 18x^2y^2.$$

65. Find all the values of

$$\frac{(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)^{-4}}{(\cos \frac{1}{2}\pi - i \sin \frac{1}{2}\pi)^{2/3}},$$

Ans. $-1, \frac{1}{2}(1 \pm i\sqrt{3}).$

66. A, B, C, D, E are the vertices of a regular pentagon inscribed in a circle whose centre is O and radius r ; if OA makes the angle α with the axis OX, state the complex numbers which are represented by the vectors OA, OB, . . . , OE, and prove that these numbers are the roots of the equation

$$x^5 = r^5 (\cos 5\alpha + i \sin 5\alpha).$$

67. If ω is one of the complex cube roots of unity, show that

$$\frac{3}{x^3 - 1} = \frac{1}{x - 1} + \frac{1}{\omega x - 1} + \frac{1}{\omega^2 x - 1}.$$

Deduce the identity

$$3 \cot 3\theta = \cot \theta + \cot \left(\theta + \frac{1}{3}\pi \right) + \cot \left(\theta + \frac{2}{3}\pi \right)$$

by putting $x = \cos 2\theta + i \sin 2\theta$.

$$68. \text{ If } \cos \theta + \cos \phi + \cos \psi = 0$$

$$\text{and } \sin \theta + \sin \phi + \sin \psi = 0,$$

show that

$$\cos 3\theta + \cos 3\phi + \cos 3\psi - 3 \cos (\theta + \phi + \psi) = 0$$

$$\text{and } \sin 3\theta + \sin 3\phi + \sin 3\psi - 3 \sin (\theta + \phi + \psi) = 0.$$

[Put $a = \cos \theta + i \sin \theta$, $b = \cos \phi + i \sin \phi$,

$$c = \cos \psi + i \sin \psi :$$

then, if $a + b + c = 0$, $a^3 + b^3 + c^3 - 3abc = 0.$]

CHAPTER XV

APPLICATIONS OF DEMOIVRE'S THEOREM

§ 1. Solution of Equations

DEMOIVRE'S Theorem can be employed to obtain the solutions of certain types of equations. The most important of these are the following :

TYPE I.— $az^n + b = 0$.

Here $z^n = -\frac{b}{a}$,

and the roots of the equation are the n values of $\sqrt[n]{-\frac{b}{a}}$, obtained by the method described in Chapter XIV, § 7.

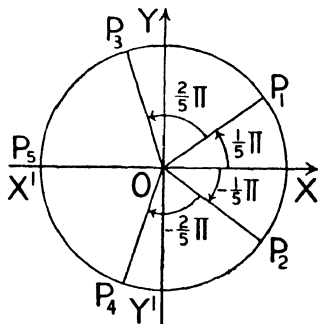


FIG. 1.

Example 1.—Solve the equation $z^5 + 1 = 0$, and mark the positions of the roots on the Argand Diagram.

Here

$$z^5 = -1 = \cos(2k + 1)\pi \pm i \sin(2k + 1)\pi,$$

where k is an integer. Thus

$$z = \cos \frac{2k + 1}{5} \pi \pm i \sin \frac{2k + 1}{5} \pi.$$

On giving k the values 0, 1, 2 the roots are found to be

$$\cos \frac{\pi}{5} \pm i \sin \frac{\pi}{5}, \quad \cos \frac{3\pi}{5} \pm i \sin \frac{3\pi}{5}, \quad -1.$$

The corresponding points P_1, P_2, P_3, P_4, P_5 on the Argand Diagram are marked in Fig. 1; they all lie on the unit circle.

TYPE II.— $az^{2n} + bz^n + c = 0$.

If this equation is treated as a quadratic in z^n , it gives rise to two equations, say,

$$z^n = p, \quad z^n = q,$$

each of which is of Type I. The roots of these equations are the $2n$ solutions of the given equation.

Example 2.—Find the roots of the equation

$$z^6 + z^3 + 1 = 0.$$

On solving this equation as a quadratic in z^3 it is found that

$$z^3 = \frac{-1 \pm \sqrt{-3}}{2}.$$

Hence $z^3 = \cos(\frac{2}{3}\pi + 2k\pi) \pm i \sin(\frac{2}{3}\pi + 2k\pi)$,
and therefore

$$z = \cos(\frac{2}{9}\pi + \frac{2}{3}k\pi) \pm i \sin(\frac{2}{9}\pi + \frac{2}{3}k\pi).$$

On giving k the values 0, -1 , 1 , the following values for the roots are obtained :

$$\cos \frac{2\pi}{9} \pm i \sin \frac{2\pi}{9}, \quad \cos \frac{4\pi}{9} \pm i \sin \frac{4\pi}{9}, \quad \cos \frac{8\pi}{9} \pm i \sin \frac{8\pi}{9}.$$

The last two of these can be put in the form

$$-\cos \frac{\pi}{9} \pm i \sin \frac{\pi}{9}.$$

TYPE III.— $p(az + b)^n + q(cz + d)^n = 0$.

The substitution $w = \frac{az + b}{cz + d}$

reduces this equation to the form

$$pw^n + q = 0,$$

which is of Type I. If w_r is one of the n roots of the latter equation, the corresponding root z_r of the given equation is the solution of

$$w_r = \frac{az_r + b}{cz_r + d}.$$

Example 3.—Solve the equation

$$(z + 1)^5 + (z - 1)^5 = 0.$$

Let $w = \frac{z + 1}{z - 1}$;

then $w^5 + 1 = 0$,

so that $w^5 = \cos(2k + 1)\pi + i \sin(2k + 1)\pi$,

and $w = \cos \frac{2k + 1}{5}\pi + i \sin \frac{2k + 1}{5}\pi$,

where k is an integer.

Now $z(w - 1) = w + 1$,
and therefore

$$\begin{aligned} z &= \frac{\cos \frac{2k+1}{5}\pi + i \sin \frac{2k+1}{5}\pi + 1}{\cos \frac{2k+1}{5}\pi + i \sin \frac{2k+1}{5}\pi - 1} \\ &= \frac{2 \cos^2 \frac{2k+1}{10}\pi + 2i \sin \frac{2k+1}{10}\pi \cos \frac{2k+1}{10}\pi}{-2 \sin^2 \frac{2k+1}{10}\pi + 2i \sin \frac{2k+1}{10}\pi \cos \frac{2k+1}{10}\pi} \\ &= \frac{2 \cos \frac{2k+1}{10}\pi \left(\cos \frac{2k+1}{10}\pi + i \sin \frac{2k+1}{10}\pi \right)}{2i \sin \frac{2k+1}{10}\pi \left(\cos \frac{2k+1}{10}\pi + i \sin \frac{2k+1}{10}\pi \right)} \\ &= -i \cot \frac{2k+1}{10}\pi. \end{aligned}$$

On giving to k the values $0, \pm 1, \pm 2$ the following roots are obtained :

$$\pm i \cot \frac{\pi}{10}, \pm i \cot \frac{3\pi}{10}, 0.$$

Example 4.—Solve the equation

$$x^{12} - x^{11} + x^{10} - \dots - x + 1 = 0.$$

On multiplying this equation by $x + 1$ the equation $x^{13} + 1 = 0$ is obtained. All the roots of the latter equation, with the exception of -1 , are roots of the given equation. They are

$$\cos \frac{2k+1}{13}\pi \pm i \sin \frac{2k+1}{13}\pi,$$

where $k = 0, 1, 2, 3, 4, 5, \dots$.

§ 2. Real Quadratic Factors of Polynomials

If a polynomial in x with real coefficients is equated to zero, and the resulting equation solved, the factors of the polynomial can be written down. For, if α is a root of the equation, $x - \alpha$ is a factor of the polynomial. In text-books on Algebra it is shown that complex roots must occur in conjugate pairs; and that, consequently, if the corresponding factors are multiplied together, they give real

quadratic factors of the polynomial; that is, quadratic factors in which the coefficients are real.

Example 1.—Find the real factors of $x^5 + 1$.

From Example 1, § 1,

$$x^5 + 1 = (x + 1) \left(x - \cos \frac{\pi}{5} - i \sin \frac{\pi}{5} \right) \left(x - \cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \right) \\ \times \left(x - \cos \frac{3\pi}{5} - i \sin \frac{3\pi}{5} \right) \left(x - \cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} \right).$$

The product of the second and third factors is

$$\left(x - \cos \frac{\pi}{5} \right)^2 + \sin^2 \frac{\pi}{5} = x^2 - 2x \cos \frac{\pi}{5} + \cos^2 \frac{\pi}{5} + \sin^2 \frac{\pi}{5} \\ = x^2 - 2x \cos \frac{\pi}{5} + 1.$$

Similarly, the product of the fourth and fifth factors is

$$x^2 - 2x \cos \frac{3\pi}{5} + 1 = x^2 + 2x \cos \frac{2\pi}{5} + 1.$$

Hence

$$x^5 + 1 = (x + 1) \left(x^2 - 2x \cos \frac{\pi}{5} + 1 \right) \left(x^2 + 2x \cos \frac{2\pi}{5} + 1 \right).$$

This equation is an identity in x ; therefore the coefficients of the powers of x on both sides are equal.

Example 2.—Find the real factors of

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$$

On multiplying by $x - 1$ and equating the product to zero, the equation

$$x^7 - 1 = 0$$

is obtained. The roots of this equation are

$$\cos \frac{2k\pi}{7} \pm i \sin \frac{2k\pi}{7},$$

where $k = 0, 1, 2, 3$. The first of these values gives the extra factor $x - 1$; while the other factors are

$$x - \cos \frac{2\pi}{7} \mp i \sin \frac{2\pi}{7}, \quad x - \cos \frac{4\pi}{7} \mp i \sin \frac{4\pi}{7}, \\ x - \cos \frac{6\pi}{7} \mp i \sin \frac{6\pi}{7}.$$

Hence, on combining the conjugate factors, the required expression

$$\begin{aligned}
 x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
 \equiv \left(x^2 - 2x \cos \frac{2\pi}{7} + 1\right) \left(x^2 - 2x \cos \frac{4\pi}{7} + 1\right) \\
 \times \left(x^2 - 2x \cos \frac{6\pi}{7} + 1\right)
 \end{aligned}$$

is obtained.

Identities.—By equating the coefficients in identical equations such as those obtained in Examples 1 and 2, or by giving particular values to x in these equations, various other identities can be derived.

Example 3.—Prove that

$$\begin{aligned}
 \text{(i)} \quad & \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}, \\
 \text{(ii)} \quad & \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \cos \frac{6\pi}{7} = \frac{1}{8}, \\
 \text{(iii)} \quad & \sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7} = \frac{\sqrt{7}}{8}, \\
 \text{(iv)} \quad & \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} = \frac{1}{8}.
 \end{aligned}$$

For (i) equate the coefficients of x^5 in the identity at the end of Example 2. For (ii) equate the coefficients of x^3 , and make use of (i). For (iii) put $x = 1$, and get

$$7 = 4 \sin^2 \frac{\pi}{7} \cdot 4 \sin^2 \frac{2\pi}{7} \cdot 4 \sin^2 \frac{3\pi}{7}.$$

Now take the square root; the positive sign is taken since all the angles are in the first quadrant. For (iv) put $x = -1$.

Notation for Products.—The symbol \prod is used to indicate a product, just as the symbol \sum is employed to indicate a sum. Thus the product $f_1 f_2 \dots f_n$ can be represented by $\prod_{r=1}^n f_r$, or simply by $\prod_1^n f_r$.

Example 4.—Prove that, if n is a positive integer,

$$x^{2n} - x^n + 1 = \prod_{r=0}^{n-1} \left\{ x^2 - 2x \cos \left(\frac{6r+1}{3n} \pi \right) + 1 \right\}.$$

Example 5.—Resolve $x^n - 1$, where n is a positive integer, into real factors, and show that

$$2^{\frac{n-1}{2}} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \dots \sin \frac{(n-2)\pi}{2n} = \sqrt{n}, \quad n \text{ even,}$$

$$2^{\frac{n-1}{2}} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \dots \sin \frac{(n-1)\pi}{2n} = \sqrt{n}, \quad n \text{ odd.}$$

On solving the equation $x^n - 1 = 0$ it is found that the factors of $x^n - 1$ are of the forms

$$x - \cos \frac{2k\pi}{n} \mp i \sin \frac{2k\pi}{n},$$

where k is an integer.

CASE I.— n even. Let $n = 2m$; then

$$x^{2m} - 1 = (x - 1)(x + 1) \prod_{k=1}^{m-1} \left(x^2 - 2x \cos \frac{k\pi}{m} + 1 \right),$$

the first two factors being given by $k = 0$ and $k = m$.

Now divide by $x - 1$; then *

$$x^{2m-1} + x^{2m-2} + \dots + 1 = (x + 1) \prod_{k=1}^{m-1} \left(x^2 - 2x \cos \frac{k\pi}{m} + 1 \right).$$

In this identity put $x = 1$, and get

$$2m = 2 \prod_{k=1}^{m-1} 4 \sin^2 \frac{k\pi}{2m}.$$

Hence, on taking the square root, it is found that

$$\sqrt{(2m)} = 2^{\frac{2m-1}{2}} \prod_{k=1}^{m-1} \sin \frac{k\pi}{2m},$$

which is the required formula when n is even.

CASE II.— n odd. Let $n = 2m + 1$; then

$$x^{2m+1} - 1 = (x - 1) \prod_{k=1}^m \left(x^2 - 2x \cos \frac{2k\pi}{2m+1} + 1 \right).$$

* The equation is an identity; that is, the expressions on both sides are the same polynomial in different forms. Hence if both sides are divided by the same factor the quotients must be identically equal. For an alternative method see § 6, Example 5.

Now as before divide by $x - 1$, and put $x = 1$. Then

$$2m + 1 = \prod_{k=1}^m 4 \sin^2 \frac{k\pi}{2m + 1},$$

and therefore

$$\sqrt{(2m + 1)} = 2^m \prod_{k=1}^m \sin \frac{k\pi}{2m + 1},$$

which is the required formula when n is odd. In both cases the positive sign is taken because all the angles are in the first quadrant.

Example 6.—If $\alpha = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, show that

$$x^{n-1} + x^{n-2} + \dots + 1 = (x - \alpha)(x - \alpha^2) \dots (x - \alpha^{n-1}),$$

and deduce that

$$\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \dots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

[Put $x = 1$ and equate the moduli.]

§ 3. Powers and Products of Sines and Cosines in Terms of Multiple Angles

Demoivre's Theorem may be employed to express a product of the form $\cos^m \theta \sin^n \theta$, where m and n are positive integers, in terms of sines and cosines of multiples of θ .

Let $z = \cos \theta + i \sin \theta, \dots \dots \dots (1a)$

so that $\frac{1}{z} = \cos \theta - i \sin \theta; \dots \dots \dots (1b)$

then $2 \cos \theta = z + \frac{1}{z}, \quad 2i \sin \theta = z - \frac{1}{z}. \dots \dots \dots (2)$

Again, from (1, a, b), by Demoivre's Theorem, if p is an integer,

$$z^p = \cos p\theta + i \sin p\theta, \quad \frac{1}{z^p} = \cos p\theta - i \sin p\theta, \quad (3)$$

and therefore

$$2 \cos p\theta = z^p + \frac{1}{z^p}, \quad 2i \sin p\theta = z^p - \frac{1}{z^p}. \dots \dots \dots (4)$$

Now

$$(2 \cos \theta)^m (2i \sin \theta)^n = \left(z + \frac{1}{z}\right)^m \left(z - \frac{1}{z}\right)^n.$$

If the R.H.S. of this equation is expanded in powers of z , and those terms whose indices are equal but opposite in sign taken together, the resulting sum is made up, if n is even, of terms of the form $(z^p + z^{-p})$, and, if n is odd, of terms of the form $(z^p - z^{-p})$. From (4) it follows that these terms are equal respectively to $2 \cos p\theta$ and $2i \sin p\theta$.

Example 1.—Express $\sin^4 \theta \cos^2 \theta$ in terms of cosines of multiples of θ .

$$\begin{aligned} (2i \sin \theta)^4 (2 \cos \theta)^2 &= \left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2 \\ &= \left(z - \frac{1}{z}\right)^2 \left(z^2 - \frac{1}{z^2}\right)^2 \\ &= \left(z^2 - 2 + \frac{1}{z^2}\right) \left(z^4 - 2 + \frac{1}{z^4}\right) \\ &= \left(z^6 + \frac{1}{z^6}\right) - 2 \left(z^4 + \frac{1}{z^4}\right) - \left(z^2 + \frac{1}{z^2}\right) + 4 \\ &= 2 \cos 6\theta - 4 \cos 4\theta - 2 \cos 2\theta + 4. \end{aligned}$$

Therefore

$$\sin^4 \theta \cos^2 \theta = \frac{1}{32} (\cos 6\theta - 2 \cos 4\theta - \cos 2\theta + 2).$$

[See Chapter XVI, § 1, Example 1.]

Example 2.—Show that

$$\begin{aligned} 2^{11} \sin^5 \theta \cos^7 \theta &= \sin 12\theta + 2 \sin 10\theta - 4 \sin 8\theta \\ &\quad - 10 \sin 6\theta + 5 \sin 4\theta + 20 \sin 2\theta. \end{aligned}$$

§ 4. Expansions of Circular Functions of Multiple Angles

The formulæ (24), (25) and (26) of Chapter VI can be established very easily by means of Demoivre's Theorem. Denote $\cos \theta$, $\sin \theta$ and $\tan \theta$ by c , s and t respectively. Then, if n is a positive integer,

$$\begin{aligned} \cos n\theta + i \sin n\theta &= (c + is)^n \\ &= c^n + {}^n C_1 c^{n-1} (is) + {}^n C_2 c^{n-2} (is)^2 + \dots + {}^n C_n (is)^n \\ &= \{c^n - {}^n C_2 c^{n-2} s^2 + {}^n C_4 c^{n-4} s^4 - \dots\} \\ &\quad + i \{{}^n C_1 c^{n-1} s - {}^n C_3 c^{n-3} s^3 + \dots\}. \end{aligned}$$

On equating the real and the imaginary parts in this equation it is found that

$$\cos n\theta = c^n - {}^nC_2c^{n-2}s^2 + {}^nC_4c^{n-4}s^4 - \dots, \quad (5a)$$

or $\cos n\theta = c^n\{1 - {}^nC_2t^2 + {}^nC_4t^4 - \dots\}, \quad (5)$

and $\sin n\theta = {}^nC_1c^{n-1}s - {}^nC_3c^{n-3}s^3 + \dots, \quad (6a)$

or $\sin n\theta = c^n\{{}^nC_1t - {}^nC_3t^3 + {}^nC_5t^5 - \dots\}. \quad (6)$

Hence, on division,

$$\tan n\theta = \frac{{}^nC_1t - {}^nC_3t^3 + {}^nC_5t^5 - \dots}{1 - {}^nC_2t^2 + {}^nC_4t^4 - \dots}. \quad (7)$$

Formulae (5), (6) and (7) are formulae (24), (25) and (26) of Chapter VI, respectively.

Example 1.—Find expressions for $\cos 5\theta$, $\sin 5\theta$ and $\tan 5\theta$ in terms of $\cos \theta$, $\sin \theta$ and $\tan \theta$ respectively.

If c , s and t denote $\cos \theta$, $\sin \theta$ and $\tan \theta$ respectively,

$$\begin{aligned} \cos 5\theta + i \sin 5\theta &= (c + is)^5 \\ &= c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5. \end{aligned}$$

Thus

$$\begin{aligned} \cos 5\theta &= c^5 - 10c^3s^2 + 5cs^4 = c^5 + 10c^3(c^2 - 1) + 5c(c^2 - 1)^2 \\ &= 16c^5 - 20c^3 + 5c; \end{aligned}$$

and

$$\begin{aligned} \sin 5\theta &= 5c^4s - 10c^2s^3 + s^5 = 5(s^2 - 1)^2s + 10(s^2 - 1)s^3 + s^5 \\ &= 16s^5 - 20s^3 + 5s. \end{aligned}$$

$$\text{Also } \tan 5\theta = \frac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4} = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}.$$

Example 2.—From the last formula in Example 1 deduce that the roots of the equation $x^4 - 10x^2 + 5 = 0$ are

$$\pm \tan (r\pi/5),$$

where $r = 1, 2$. Hence show that

$$\tan^2 \frac{\pi}{5} + \tan^2 \frac{2\pi}{5} = 10.$$

Example 3.—Solve the equation

$$x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 = 0.$$

Ans. $x = \tan \frac{(4r+1)\pi}{20}, \quad r = 0, 1, 2, 3, 4.$

Formulae (20), (21) and (22) of Chapter VI may be deduced in a similar manner from formula (2) of Chapter XIV. For, if

$$\begin{aligned}
 c_r &= \cos \theta_r, s_r = \sin \theta_r, t_r = \tan \theta_r, r = 1, 2, 3, \dots, n, \\
 \cos (\theta_1 + \theta_2 + \dots + \theta_n) &+ i \sin (\theta_1 + \theta_2 + \dots + \theta_n) \\
 &= (c_1 + is_1)(c_2 + is_2) \dots (c_n + is_n) \\
 &= c_1 c_2 \dots c_n (1 + it_1)(1 + it_2) \dots (1 + it_n) \\
 &= c_1 c_2 \dots c_n (1 + iT_1 - T_2 - iT_3 + T_4 + \dots),
 \end{aligned}$$

where T_r denotes the sum of the products of t_1, t_2, \dots, t_n taken r at a time. On equating the real and the imaginary parts, it is found that

$$\begin{aligned}
 \cos (\theta_1 + \theta_2 + \dots + \theta_n) \\
 = c_1 c_2 \dots c_n (1 - T_2 + T_4 - \dots) \quad (8)
 \end{aligned}$$

and

$$\begin{aligned}
 \sin (\theta_1 + \theta_2 + \dots + \theta_n) \\
 = c_1 c_2 \dots c_n (T_1 - T_3 + T_5 - \dots). \quad (9)
 \end{aligned}$$

Hence, on division of (9) by (8),

$$\tan (\theta_1 + \theta_2 + \dots + \theta_n) = \frac{T_1 - T_3 + T_5 - \dots}{1 - T_2 + T_4 - \dots}. \quad (10)$$

Formulae (8), (9) and (10) are formulae (20), (21) and (22) of Chapter VI, respectively.

§ 5. Factorisation

Formula (5a) may be written

$$\cos n\theta = c^n + {}^n C_2 c^{n-2} (c^2 - 1) + {}^n C_4 c^{n-4} (c^2 - 1)^2 + \dots;$$

so that

$$\cos n\theta = A_0 c^n + A_2 c^{n-2} + A_4 c^{n-4} + \dots,$$

where A_0, A_2, A_4, \dots are constants and

$$A_0 = 1 + {}^n C_2 + {}^n C_4 + \dots$$

Now, by the binomial theorem,

$$(1 + 1)^n = 1 + {}^n C_1 + {}^n C_2 + {}^n C_3 + \dots,$$

and $(1 - 1)^n = 1 - {}^n C_1 + {}^n C_2 - {}^n C_3 + \dots;$

therefore, on addition,

$$2^n = 2(1 + {}^n C_2 + {}^n C_4 + \dots),$$

and consequently $A_0 = 2^{n-1}.$

Thus, if n is even,

$$\cos n\theta = 2^{n-1}(c^2 - k_1^2)(c^2 - k_2^2) \dots$$

to $\frac{1}{2}n$ factors, while, if n is odd,

$$\cos n\theta / \cos \theta = 2^{n-1}(c^2 - k_1^2)(c^2 - k_2^2) \dots$$

to $\frac{1}{2}(n - 1)$ factors, k_1, k_2, \dots being constants.

If now $\theta = r\pi/(2n)$, where r is an odd integer, $\cos n\theta$ has the value zero, and consequently one of the factors on the right must vanish ; thus one of the quantities $\pm k_1, \pm k_2, \dots$ must have the value $\cos \frac{r\pi}{2n}$, so that $(\cos^2 \theta - \cos^2 \frac{r\pi}{2n})$ is a factor of $\cos n\theta$.

Hence, if n is even,

$$\begin{aligned} \cos n\theta = 2^{n-1} & \left(\cos^2 \theta - \cos^2 \frac{\pi}{2n} \right) \left(\cos^2 \theta - \cos^2 \frac{3\pi}{2n} \right) \dots \\ & \dots \left(\cos^2 \theta - \cos^2 \frac{(n-1)\pi}{2n} \right), \end{aligned} \quad (11)$$

while, if n is odd,

$$\begin{aligned} \cos n\theta = 2^{n-1} \cos \theta & \left(\cos^2 \theta - \cos^2 \frac{\pi}{2n} \right) \left(\cos^2 \theta - \cos^2 \frac{3\pi}{2n} \right) \dots \\ & \dots \left(\cos^2 \theta - \cos^2 \frac{(n-2)\pi}{2n} \right), \end{aligned} \quad (12)$$

since, in both cases, all the factors on the right are different.

Formulae (11) and (12) may also be put in the forms

$$\begin{aligned} \cos n\theta = 2^{n-1} & \left(\sin^2 \frac{\pi}{2n} - \sin^2 \theta \right) \left(\sin^2 \frac{3\pi}{2n} - \sin^2 \theta \right) \dots \\ & \dots \left(\sin^2 \frac{(n-1)\pi}{2n} - \sin^2 \theta \right), \end{aligned} \quad (13)$$

where n is even ; and

$$\begin{aligned} \cos n\theta = 2^{n-1} \cos \theta & \left(\sin^2 \frac{\pi}{2n} - \sin^2 \theta \right) \left(\sin^2 \frac{3\pi}{2n} - \sin^2 \theta \right) \dots \\ & \dots \left(\sin^2 \frac{(n-2)\pi}{2n} - \sin^2 \theta \right), \end{aligned} \quad (14)$$

where n is odd.

In (13) and (14) put $\theta = 0$, and take the square roots ; then

$$2^{\frac{n-1}{2}} \sin \frac{\pi}{2n} \sin \frac{3\pi}{2n} \dots \sin \frac{(n-1)\pi}{2n} = 1, \quad (n \text{ even}), \quad (15)$$

$$2^{\frac{n-1}{2}} \sin \frac{\pi}{2n} \sin \frac{3\pi}{2n} \dots \sin \frac{(n-2)\pi}{2n} = 1, \quad (n \text{ odd}), \quad (16)$$

the positive sign being taken since all the angles are in the first quadrant.

On dividing the expressions on the right of (13) and (14) by the squares of the expressions on the left of (15) and (16), it is seen that

$$\begin{aligned} \cos n\theta = & \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi}{2n}}\right) \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{3\pi}{2n}}\right) \dots \\ & \dots \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{(n-1)\pi}{2n}}\right), \quad (17) \end{aligned}$$

where n is even ; and

$$\begin{aligned} \cos n\theta = & \cos \theta \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi}{2n}}\right) \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{3\pi}{2n}}\right) \dots \\ & \dots \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{(n-2)\pi}{2n}}\right), \quad (18) \end{aligned}$$

where n is odd.

In the same way it can be deduced from (6a) that, if n is even,

$$\begin{aligned} \sin n\theta = & 2^{n-1} \sin \theta \cos \theta \left(\cos^2 \theta - \cos^2 \frac{\pi}{n}\right) \left(\cos^2 \theta - \cos^2 \frac{2\pi}{n}\right) \\ & \dots \left(\cos^2 \theta - \cos^2 \frac{(n-2)\pi}{2n}\right), \quad (19) \end{aligned}$$

while, if n is odd,

$$\begin{aligned} \sin n\theta = & 2^{n-1} \sin \theta \left(\cos^2 \theta - \cos^2 \frac{\pi}{n}\right) \left(\cos^2 \theta - \cos^2 \frac{2\pi}{n}\right) \dots \\ & \dots \left(\cos^2 \theta - \cos^2 \frac{(n-1)\pi}{2n}\right). \quad (20) \end{aligned}$$

Hence, if n is even,

$$\sin n\theta = 2^{n-1} \sin \theta \cos \theta \left(\sin^2 \frac{\pi}{n} - \sin^2 \theta \right) \left(\sin^2 \frac{2\pi}{n} - \sin^2 \theta \right) \dots \left(\sin^2 \frac{(n-2)\pi}{2n} - \sin^2 \theta \right), \quad (21)$$

while, if n is odd,

$$\sin n\theta = 2^{n-1} \sin \theta \left(\sin^2 \frac{\pi}{n} - \sin^2 \theta \right) \left(\sin^2 \frac{2\pi}{n} - \sin^2 \theta \right) \dots \left(\sin^2 \frac{(n-1)\pi}{2n} - \sin^2 \theta \right). \quad (22)$$

But * (§ 2, Example 5),

$$\sqrt{n} = 2^{\frac{n-1}{2}} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \dots \sin \frac{(n-2)\pi}{2n}, \quad n \text{ even}, \quad (23)$$

$$\sqrt{n} = 2^{\frac{n-1}{2}} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \dots \sin \frac{(n-1)\pi}{2n}, \quad n \text{ odd}. \quad (24)$$

Hence, on dividing (21) by (23) squared and (22) by (24) squared, it is seen that

$$\sin n\theta = n \sin \theta \cos \theta \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi}{n}} \right) \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{2\pi}{n}} \right) \dots \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{(n-2)\pi}{2n}} \right), \quad (25)$$

where n is even; and

$$\sin n\theta = n \sin \theta \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi}{n}} \right) \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{2\pi}{n}} \right) \dots \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{(n-1)\pi}{2n}} \right), \quad (26)$$

where n is odd.

* For another method of obtaining formulæ (23) and (24), see § 7, note.

Note.—Formulae (11) and (12), and also (19) and (20), can be derived as follows :

Following the method of § 2, Example 5, it can be shown that

$$x^{2n} + 1 = \prod_{r=0}^{n-1} \left\{ x^2 - 2x \cos \left(\frac{2r+1}{2n} \pi \right) + 1 \right\},$$

$$\text{and } x^{2n} - 1 = (x^2 - 1) \prod_{r=1}^{n-1} \left(x^2 - 2x \cos \frac{r\pi}{n} + 1 \right).$$

In these equations divide by x^n and put $x = \cos \theta + i \sin \theta$; thus

$$\cos n\theta = 2^{n-1} \prod_{r=0}^{n-1} \left\{ \cos \theta - \cos \left(\frac{2r+1}{2n} \pi \right) \right\},$$

$$\text{and } \sin n\theta = 2^{n-1} \sin \theta \prod_{r=1}^{n-1} \left(\cos \theta - \cos \frac{r\pi}{n} \right).$$

From the first of these (11) and (12), and from the second (19) and (20), are easily deduced.

Example 1.—Show that, if n is a positive integer,

$$\cos n\theta - \cos n\phi = 2^{n-1} \prod_{r=0}^{n-1} \left\{ \cos \theta - \cos \left(\phi + \frac{2r\pi}{n} \right) \right\}.$$

Example 2.—Show that

$$(i) \cos 5\theta = 16 \cos \theta \left(\cos^2 \theta - \cos^2 \frac{\pi}{10} \right) \left(\cos^2 \theta - \cos^2 \frac{3\pi}{10} \right),$$

$$(ii) \cos 5\theta = \cos \theta \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi}{10}} \right) \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{3\pi}{10}} \right),$$

$$(iii) \sin 5\theta = 16 \sin \theta \left(\sin^2 \theta - \sin^2 \frac{\pi}{5} \right) \left(\sin^2 \theta - \sin^2 \frac{2\pi}{5} \right),$$

$$(iv) \sin 5\theta = 5 \sin \theta \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi}{5}} \right) \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{2\pi}{5}} \right).$$

Example 3.—Prove that

$$\begin{aligned} \cos 7\theta / \cos \theta &= 64 \cos^6 \theta - 112 \cos^4 \theta + 56 \cos^2 \theta - 7 \\ &= 64 \left(\cos^2 \theta - \cos^2 \frac{\pi}{14} \right) \left(\cos^2 \theta - \cos^2 \frac{3\pi}{14} \right) \\ &\quad \times \left(\cos^2 \theta - \cos^2 \frac{5\pi}{14} \right); \end{aligned}$$

and deduce that

$$\begin{aligned} \text{(i)} \quad \cos^2 \frac{\pi}{14} + \cos^2 \frac{3\pi}{14} + \cos^2 \frac{5\pi}{14} &= \frac{7}{4}, \\ \text{(ii)} \quad \cos^4 \frac{\pi}{14} + \cos^4 \frac{3\pi}{14} + \cos^4 \frac{5\pi}{14} &= \frac{21}{16}, \\ \text{(iii)} \quad \cos \frac{\pi}{14} \cos \frac{3\pi}{14} \cos \frac{5\pi}{14} &= \frac{\sqrt{7}}{8}. \end{aligned}$$

Example 4.—From (26) deduce that

$$\sum_{r=1}^m \operatorname{cosec}^2 \left(\frac{r\pi}{2m+1} \right) = \frac{2}{3} m(m+1).$$

Example 5.—Prove that, when n is odd,

$$\prod_{r=0}^{n-1} \left\{ 1 + \cos \left(\alpha + \frac{2r\pi}{n} \right) \right\} = \frac{1}{2^{n-1}} (1 + \cos n\alpha),$$

and find an expression for the product when n is even.

$$\text{Ans. } (1 - \cos n\alpha) / 2^{n-1}.$$

Example 6.—There are seven values of θ between α and $\alpha + 2\pi$ which satisfy the equation

$$\cos 7\theta = \cos 7\alpha.$$

Prove that

- (i) the sum of the cosines of these angles is zero;
- (ii) the arithmetic mean of their secants is $-\sec 7\alpha$.

§ 6. Limit of a Function

A single-valued function $f(x)$ is said to tend to the limit l as x tends to a if the difference between $f(x)$ and l can be made as small as we please by taking x sufficiently close to a (but x must not be put equal to a). This can be stated more precisely as follows.

The function $f(x)$ tends to the limit l as x tends to a if,

corresponding to any assigned positive number ϵ , however small, a positive number δ can be found such that

$$|f(x) - l| < \epsilon \quad \text{if } |x - a| \leq \delta, \quad x \neq a.$$

It is customary to express this by the symbolism

$$\lim_{x \rightarrow a} f(x) = l,$$

or by writing “ $f(x) \rightarrow l$ when $x \rightarrow a$.” The symbol \rightarrow is an abbreviation for “tends to.”

Note.—The inequality $|f(x) - l| < \epsilon$ is equivalent to the inequalities

$$l - \epsilon < f(x) < l + \epsilon.$$

If x tends to infinity instead of to a , the statement can be put in the form:—

The function $f(x)$ tends to the limit l when x tends to infinity if, corresponding to any assigned positive number ϵ , however small, a positive number N can be found such that $|f(x) - l| < \epsilon$ if $x \geq N$.

This is written

$$\lim_{x \rightarrow \infty} f(x) = l,$$

or “ $f(x) \rightarrow l$ when $x \rightarrow \infty$.”

The reader should write out the corresponding definition when $x \rightarrow -\infty$.

The function $f(x)$ is said to tend to infinity when x tends to a if, corresponding to any assigned positive number M , however large, a positive number δ can be found such that $f(x) > M$ if $|x - a| < \delta$, $x \neq a$.

The corresponding definitions for the cases when $f(x) \rightarrow -\infty$ and when $x \rightarrow \pm \infty$ are left as exercises for the reader.

Example 1.—If $\lim_{x \rightarrow a} f(x) = l$, show that

$$(i) \lim_{x \rightarrow a} \{-f(x)\} = -l; \quad (ii) \lim_{x \rightarrow a} \{cf(x)\} = cl,$$

where c is a constant.

Example 2.—If $\lim_{x \rightarrow a} f(x) = l$ and $\lim_{x \rightarrow a} \phi(x) = m$, show that

$$(i) \lim_{x \rightarrow a} \{f(x) + \phi(x)\} = l + m; \quad (ii) \lim_{x \rightarrow a} \{f(x)\phi(x)\} = lm,$$

$$(iii) \lim_{x \rightarrow a} \{1/\phi(x)\} = 1/m; \quad (iv) \lim_{x \rightarrow a} \{f(x)/\phi(x)\} = l/m,$$

provided that, in (iii) and (iv), $m \neq 0$.

[For (ii), $f(x)\phi(x) - lm = \{f(x) - l\}\{\phi(x) - m\} + m\{f(x) - l\} + l\{\phi(x) - m\}$, and each term on the R.H.S. $\rightarrow 0$ when $x \rightarrow a$.]

Example 3.—Prove that the limit of the sum of a finite number of functions is the sum of their limits.

Example 4.—Prove that the limit of the product of a finite number of functions is the product of their limits.

Example 5.—If n is a positive integer, show that

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

If $x \neq a$,

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1}.$$

Now, each term on the R.H.S. of this equation tends to a^{n-1} when $x \rightarrow a$, and there are n terms. Hence, when $x \rightarrow a$,

$$(x^n - a^n)/(x - a) \rightarrow na^{n-1}.$$

For instance, in § 2, Example 5, Case I, the result can be obtained as follows. In the equation

$$\frac{x^{2m} - 1}{x - 1} = (x + 1) \prod_{k=1}^{m-1} \left(x^2 - 2x \cos \frac{k\pi}{m} + 1 \right)$$

let $x \rightarrow 1$; then the L.H.S. $\rightarrow 2m$, and consequently

$$2m = 2 \prod_{k=1}^{m-1} 4 \sin^2 \frac{k\pi}{2m},$$

from which the result may be deduced as before. The result of Case II may be obtained in the same way.

Example 6.—Show that

$$\lim_{x \rightarrow 0} \frac{(a+x)^6 - (a-x)^6}{(a+x)^5 - (a-x)^5} = \frac{6}{5}a.$$

§ 7. An important Limit

The formula

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad . \quad . \quad . \quad (27)$$

is of fundamental importance in connection with the applications of the Calculus to Circular Functions. Here, as elsewhere in higher mathematics, the angles are measured in radians.

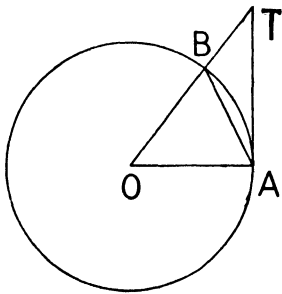


FIG. 2.

In the first place let it be assumed that θ is positive. Let the arc AB (Fig. 2) of a circle whose radius is r subtend the angle θ ($< \frac{1}{2}\pi$) at the centre O. Draw the chord AB and let the tangent at A meet OB produced at T. Then

$$\triangle OAB < \text{sector OAB} < \triangle OAT,$$

$$\text{or} \quad \frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2 \tan \theta,$$

since $AT = r \tan \theta$.

Thus $\sin \theta < \theta < \tan \theta$;

and, consequently, since $\sin \theta$ is positive,

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

$$\text{or} \quad 1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

But, when $\theta \rightarrow 0$, $\cos \theta \rightarrow 1$; hence $\sin \theta/\theta$ must also $\rightarrow 1$.

Next, if θ is negative, let $\theta = -\phi$; then

$$\frac{\sin \theta}{\theta} = \frac{\sin \phi}{\phi}.$$

But, when $\theta \rightarrow 0$, $\phi \rightarrow 0$; therefore

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \lim_{\phi \rightarrow 0} \frac{\sin \phi}{\phi} = 1.$$

COROLLARY 1.—If k is a constant,

$$\mathcal{L}_{\theta \rightarrow 0} \frac{\sin k\theta}{\theta} = k.$$

For, if $k\theta = \phi$,

$$\mathcal{L}_{\theta \rightarrow 0} \frac{\sin k\theta}{\theta} = \mathcal{L}_{\theta \rightarrow 0} k \frac{\sin \phi}{\phi} = k.$$

COROLLARY 2.—If k and l are constants,

$$\mathcal{L}_{\theta \rightarrow 0} \frac{\sin k\theta}{\sin l\theta} = \frac{k}{l}.$$

For

$$\mathcal{L}_{\theta \rightarrow 0} \frac{\sin k\theta}{\sin l\theta} = \mathcal{L}_{\theta \rightarrow 0} \left\{ \frac{\sin k\theta}{\theta} \middle/ \frac{\sin l\theta}{\theta} \right\} = \frac{k}{l}.$$

Note.—Formulae (23) and (24) may be derived from (21) and (22) by, in each case, dividing by $\sin \theta$ and letting $\theta \rightarrow 0$.

Example 1.—Show that

$$\mathcal{L}_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2} = \frac{1}{2}.$$

$$\left[\frac{1 - \cos \theta}{\theta^2} = 2 \left(\frac{\sin \frac{1}{2}\theta}{\theta} \right)^2 \text{ and } \frac{\sin \frac{1}{2}\theta}{\theta} \rightarrow \frac{1}{2} \text{ when } \theta \rightarrow 0. \right]$$

Example 2.—Show that

$$\begin{aligned} \frac{\sin 8\theta}{4 \sin 2\theta} &= 1 - 10 \sin^2 \theta + 24 \sin^4 \theta - 16 \sin^6 \theta \\ &= -16 \left(\sin^2 \theta - \sin^2 \frac{\pi}{8} \right) \left(\sin^2 \theta - \sin^2 \frac{\pi}{4} \right) \left(\sin^2 \theta - \sin^2 \frac{3\pi}{8} \right). \end{aligned}$$

Example 3.—Show that

$$\begin{aligned} \sin 7\theta &= 7 \sin \theta \prod_{r=1}^3 \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{r\pi}{7}} \right) \\ &= 64 \sin \theta \sin \left(\theta + \frac{\pi}{7} \right) \sin \left(\theta + \frac{2\pi}{7} \right) \dots \sin \left(\theta + \frac{6\pi}{7} \right). \end{aligned}$$

Deduce that

$$\sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7} \dots \sin \frac{6\pi}{7} = \frac{7}{64},$$

and that $\tan^2 \frac{\pi}{7} \tan^2 \frac{2\pi}{7} \tan^2 \frac{3\pi}{7} = 7.$

Example 4.—Show that

$$\begin{aligned} \text{(i)} \quad \int_{\theta \rightarrow 0} \frac{\cos 2\theta - \cos 3\theta}{1 - \cos 2\theta} &= \frac{5}{4}; & \text{(ii)} \quad \int_{\theta \rightarrow \frac{1}{2}\pi} \frac{\cot 2\theta + \tan 3\theta}{\sec \theta} &= -\frac{1}{8}; \\ \text{(iii)} \quad \int_{\theta \rightarrow 0} \frac{\cot 2\theta - \cot 3\theta}{\cot \theta - \cot 2\theta} &= \frac{1}{3}; & \text{(iv)} \quad \int_{\theta \rightarrow \frac{1}{2}\pi} \frac{\cos 2\theta + \cos 4\theta}{1 - \sin \theta} &= -12; \\ \text{(v)} \quad \int_{\theta \rightarrow 0} \frac{\sin \sqrt{\theta}}{\sqrt{(\sin \theta)}} &= 1; & \text{(vi)} \quad \int_{n \rightarrow \infty} \left(n \sin \frac{\theta}{n} \right) &= \theta; \\ \text{(vii)} \quad \int_{\theta \rightarrow 0} \frac{\sin^2 \theta + \sin^2 3\theta}{1 - \cos 2\theta} &= 5. \end{aligned}$$

Example 5.—Sum the series

$$\left(\sin \frac{\theta}{3} \right)^3 + 3 \left(\sin \frac{\theta}{3^2} \right)^3 + 3^2 \left(\sin \frac{\theta}{3^3} \right)^3 + \dots$$

to n terms and to infinity. [See p. 259, footnote.]

Ans. $\frac{1}{4}\{3^n \sin(3^{-n} \theta) - \sin \theta\}, \frac{1}{4}(\theta - \sin \theta).$

§ 8. Summation of Series

Demoivre's Theorem can be employed to obtain the sums of certain series : for instance, the series of sines and cosines of angles in arithmetical progression considered in Chapter VII, § 6.

Let $x = \cos A + i \sin A, \quad y = \cos B + i \sin B.$

Now $x + xy + xy^2 + \dots + xy^{n-1} = \frac{x - xy^n}{1 - y};$

therefore, since

$$\begin{aligned} xy^r &= (\cos A + i \sin A)(\cos B + i \sin B)^r \\ &= \cos(A + rB) + i \sin(A + rB), \end{aligned}$$

$C + iS$

$$= \frac{\cos A + i \sin A - \cos(A + nB) - i \sin(A + nB)}{1 - \cos B - i \sin B}, \quad (28)$$

where $C = \cos A + \cos (A + B) + \cos (A + 2B) + \dots$
 $\dots + \cos \{A + (n - 1)B\}$,
 and $S = \sin A + \sin (A + B) + \sin (A + 2B) + \dots$
 $\dots + \sin \{A + (n - 1)B\}$.

But the expression on the right of (28) is equal to

$$\begin{aligned} & \frac{2 \sin (A + \frac{1}{2}nB) \sin \frac{1}{2}nB - 2i \cos (A + \frac{1}{2}nB) \sin \frac{1}{2}nB}{2 \sin^2 \frac{1}{2}B - 2i \sin \frac{1}{2}B \cos \frac{1}{2}B} \\ = & \frac{-2i \sin \frac{1}{2}nB \{\cos (A + \frac{1}{2}nB) + i \sin (A + \frac{1}{2}nB)\}}{-2i \sin \frac{1}{2}B \{\cos \frac{1}{2}B + i \sin \frac{1}{2}B\}} \\ = & \frac{\sin \frac{1}{2}nB}{\sin \frac{1}{2}B} \left\{ \cos \left(A + \frac{n-1}{2}B \right) + i \sin \left(A + \frac{n-1}{2}B \right) \right\}. \end{aligned}$$

Hence, on equating real and imaginary parts,

$$C = \cos \left(A + \frac{n-1}{2}B \right) \frac{\sin \frac{1}{2}nB}{\sin \frac{1}{2}B},$$

and
$$S = \sin \left(A + \frac{n-1}{2}B \right) \frac{\sin \frac{1}{2}nB}{\sin \frac{1}{2}B}.$$

These are formulæ (9) and (10) of Chapter VII.

Example 1.—If

$$C = 1 + r \cos \theta + r^2 \cos 2\theta + \dots + r^{n-1} \cos (n-1)\theta,$$

and

$$S = r \sin \theta + r^2 \sin 2\theta + \dots + r^{n-1} \sin (n-1)\theta,$$

show that

$$(i) \quad C = \frac{1 - r \cos \theta - r^n \cos n\theta + r^{n+1} \cos (n-1)\theta}{1 - 2r \cos \theta + r^2};$$

$$(ii) \quad S = \frac{r \sin \theta - r^n \sin n\theta + r^{n+1} \sin (n-1)\theta}{1 - 2r \cos \theta + r^2}.$$

Let $z = r(\cos \theta + i \sin \theta)$; then

$$\begin{aligned} C + iS &= 1 + z + z^2 + \dots + z^{n-1} = \frac{1 - z^n}{1 - z} \\ &= \frac{1 - r^n \cos n\theta - ir^n \sin n\theta}{1 - r \cos \theta - ir \sin \theta} \\ &= \frac{(1 - r^n \cos n\theta - ir^n \sin n\theta)(1 - r \cos \theta + ir \sin \theta)}{(1 - r \cos \theta)^2 + (r \sin \theta)^2} \\ &= \frac{1 - r(\cos \theta - i \sin \theta) - r^n(\cos n\theta + i \sin n\theta) + r^{n+1}\{\cos (n-1)\theta + i \sin (n-1)\theta\}}{1 - 2r \cos \theta + r^2}. \end{aligned}$$

On equating real and imaginary parts, the required results are obtained.

Example 2.—Show that

$$\begin{aligned} \text{(i)} \quad & 1 + {}^nC_1 \cos \theta + {}^nC_2 \cos 2\theta + \dots + {}^nC_n \cos n\theta \\ & \qquad \qquad \qquad = (2 \cos \frac{1}{2}\theta)^n \cos \frac{1}{2}n\theta; \\ \text{(ii)} \quad & {}^nC_1 \sin \theta + {}^nC_2 \sin 2\theta + \dots + {}^nC_n \sin n\theta \\ & \qquad \qquad \qquad = (2 \cos \frac{1}{2}\theta)^n \sin \frac{1}{2}n\theta. \end{aligned}$$

§ 9. Convergent Sequences

The convergence of a sequence to a limit was explained in Chapter I, § 2. The condition for convergence will now be stated in a more precise form.

The sequence (u_n) converges to a limit l if, corresponding to any assigned positive number ϵ , however small, an integer m can be found such that, for $n \geq m$,

$$|u_n - l| < \epsilon, \quad \text{or} \quad l - \epsilon < u_n < l + \epsilon.$$

Then $u_n \rightarrow l$ when $n \rightarrow \infty$, or

$$\int_{n \rightarrow \infty} u_n = l.$$

When the limit is $+\infty$ the statement can be put in the form :

The sequence (u_n) diverges to $+\infty$ if, corresponding to any assigned positive number K , however large, a positive integer m can be found such that, for $n \geq m$, $u_n > K$.

The symbolism is then

$$\int_{n \rightarrow \infty} u_n = \infty,$$

or $u_n \rightarrow \infty$ when $n \rightarrow \infty$.

The necessary modifications when the limit is $-\infty$ are left as an exercise to the reader.

Example 1.—Show that, if $|r| < 1$, the sequence (r^n) converges to zero.

In the first place assume that $0 < r < 1$. Let $r = 1 - k$, so that $0 < k < 1$: then, since $1 - k^2 < 1$,

$$r = 1 - k < \frac{1}{1 + k}.$$

Hence $0 < r^n < \frac{1}{(1+k)^n} = \frac{1}{1+nk+\dots} < \frac{1}{nk}$,

since each of the terms in the expansion of $(1+k)^n$ is positive.

But, when $n \rightarrow \infty$, $1/(nk) \rightarrow 0$; therefore $r^n \rightarrow 0$.

Again, if $-1 < r < 0$, let $r = -t$; then

$$0 < t < 1, \text{ and } r^n = (-1)^n t^n.$$

Thus, when $n \rightarrow \infty$, $t^n \rightarrow 0$; therefore r^n also tends to zero. When $r = 0$ the result is obvious.

Example 2.—Show that, if $|r| < 1$, the sequence (nr^n) converges to zero.

Example 3.—Show that, if $|r| < 1$, the sequence $(n^k r^n)$, where k is a fixed number, converges to zero.

Example 4.—Show that the sequence $\left(\frac{r^n}{n!}\right)$, where r is any number, converges to zero.

Let $|r| = t$, and take a positive integer m greater than t ; then, if $n > m$,

$$\frac{t^n}{n!} = \frac{t^m}{m!} \frac{t^{n-m}}{(m+1)(m+2)\dots(n)} < \frac{t^m}{m!} \left(\frac{t}{m}\right)^{n-m}.$$

But, by Example 1, since $t/m < 1$, $(t/m)^{n-m} \rightarrow 0$ when $n \rightarrow \infty$. Hence $t^n/n!$ and consequently $r^n/n!$ tend to zero when $n \rightarrow \infty$.

Example 5.—If $-1 < r < 1$, prove that

$$(i) \quad 1 + r \cos \theta + r^2 \cos 2\theta + r^3 \cos 3\theta + \dots = \frac{1 - r \cos \theta}{1 - 2r \cos \theta + r^2};$$

$$(ii) \quad r \sin \theta + r^2 \sin 2\theta + r^3 \sin 3\theta + \dots = \frac{r \sin \theta}{1 - 2r \cos \theta + r^2}.$$

[In equations (i) and (ii) of Example 1, § 8, let $n \rightarrow \infty$; then $r^n \rightarrow 0$, and the sums to infinity* of the two series are obtained.]

Example 6.—Prove that, if $-1 < r < 1$,

$$1 + 2r \cos \theta + 2r^2 \cos 2\theta + 2r^3 \cos 3\theta + \dots = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.$$

[Multiply (i) of Example 5 by 2, and subtract 1 from both sides.]

* If, when $n \rightarrow \infty$, the sum of the first n terms of a series tends to a definite limit, that limit is the *sum to infinity* of the series or the *sum of the infinite series* (cf. Chap. XVI., § 3).

Example 7.—If $-1 < x < 1$, and if
 $u_n = (1 - x + x^2)(1 - x^2 + x^4)(1 - x^4 + x^8) \dots$
 $\dots(1 - x^{2^{n-1}} + x^{2^n})$,
 show that the sequence (u_n) converges to the limit

$$\frac{1}{1 + x + x^2}.$$

Example 8.—If

$$u_n = \frac{m(m-1)(m-2)\dots(m-n+1)}{(n-1)!} x^n,$$

where $|x| < 1$, show that the sequence (u_n) converges to zero.

$|u_{n+1}/u_n| \rightarrow |x|$ when $n \rightarrow \infty$. Let ρ be a number between $|x|$ and 1. Then a positive integer p can be found such that, if $n \geq p$,

$$\left| |u_{n+1}/u_n| - |x| \right| < \rho - |x|,$$

or

$$|u_{n+1}/u_n| < \rho.$$

Thus, if $n \geq p$,

$$u_n = \frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \dots \frac{u_{p+1}}{u_p} \cdot u_p,$$

so that

$$|u_n| < |u_p| \rho^{n-p}.$$

Therefore, by Example 1, $|u_n|$ tends to zero when $n \rightarrow \infty$. Hence (u_n) converges to zero.

§ 10. Demoivre's and Cotes' Properties of the Circle

Let A_1, A_2, \dots, A_n (Fig. 3) be the n vertices of a regular polygon inscribed in a circle

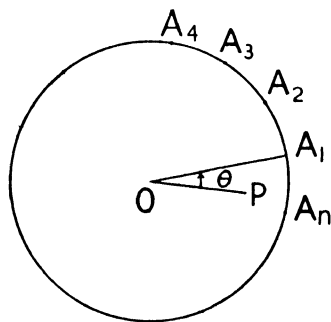


FIG. 3.

whose centre is O and radius r , and let P be a point such that $OP = a$, $\angle POA_1 = \theta$. Then

$$PA_1^2 \cdot PA_2^2 \dots PA_n^2 = a^{2n} - 2a^n r^n \cos n\theta + r^{2n}. \quad (29)$$

This is *Demoivre's Property of the Circle*.

In order to establish it, consider the equation

$$x^{2n} - 2x^n \cos n\theta + 1 = 0.$$

Regarded as a quadratic in x^n , it has solutions

$$x^n = \cos n\theta \pm i \sin n\theta = \cos(n\theta + 2k\pi) \pm i \sin(n\theta + 2k\pi),$$

where k is any integer. Thus the solutions of the given equation are

$$x = \cos \left(\theta + \frac{2k\pi}{n} \right) \pm i \sin \left(\theta + \frac{2k\pi}{n} \right),$$

where $k = 0, 1, 2, \dots, n - 1$. Hence

$$x^{2n} - 2x^n \cos n\theta + 1$$

$$= \prod_{k=0}^{n-1} \left\{ x - \cos \left(\theta + \frac{2k\pi}{n} \right) - i \sin \left(\theta + \frac{2k\pi}{n} \right) \right\} \\ \times \prod_{k=0}^{n-1} \left\{ x - \cos \left(\theta + \frac{2k\pi}{n} \right) + i \sin \left(\theta + \frac{2k\pi}{n} \right) \right\},$$

or

$$x^{2n} - 2x^n \cos n\theta + 1 = \prod_{k=0}^{n-1} \left\{ x^2 - 2x \cos \left(\theta + \frac{2k\pi}{n} \right) + 1 \right\}. \quad (30)$$

In this equation put $x = a/r$, and multiply by r^{2n} : then

$$a^{2n} - 2a^n r^n \cos n\theta + r^{2n} \\ = \prod_{k=0}^{n-1} \left\{ a^2 - 2ar \cos \left(\theta + \frac{2k\pi}{n} \right) + r^2 \right\}. \quad (31)$$

Now $\angle POA_{k+1} = \theta + \frac{2k\pi}{n}$,

so that $PA_{k+1}^2 = a^2 - 2ar \cos \left(\theta + \frac{2k\pi}{n} \right) + r^2$.

From this, with (31), formula (29) is obtained.

Cotes' Properties of the Circle.—If P lies on OA_1 , so that $\theta = 0$,

$$PA_1 \cdot PA_2 \dots PA_n = |a^n - r^n|; \quad (32)$$

while if OP bisects the angle $A_n OA_1$, so that $\theta = \pi/n$,

$$PA_1 \cdot PA_2 \dots PA_n = a^n + r^n. \quad (33)$$

Formulae (32) and (33) are *Cotes' Properties of the Circle*.

Example 1.—Show that, if P is a point on the circumference of the circle,

$$PA_1 \cdot PA_2 \dots PA_n = 2r^n \left| \sin \frac{1}{2}n\theta \right|$$

Example 2.—Show that

$$\cos n\phi - \cos n\theta = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cos \phi - \cos \left(\theta + \frac{2k\pi}{n} \right) \right\}.$$

If n is odd, deduce that

$$(i) \frac{\cos n\phi}{\cos \phi} = 2^{n-1} \prod_{k=1}^{n-1} \left(\cos \phi + \sin \frac{2k\pi}{n} \right);$$

$$(ii) \sin \frac{2\pi}{n} \sin \frac{4\pi}{n} \dots \sin \frac{2(n-1)\pi}{n} = (-1)^{\frac{n-1}{2}} \frac{n}{2^{n-1}}.$$

[In (30) divide by x^n and put $x = \cos \phi + i \sin \phi$, so that $x^n + x^{-n} = 2 \cos n\phi$.]

Example 3.—Show that

$$\sin n\phi = 2^{n-1} \prod_{k=0}^{n-1} \sin \left(\phi + \frac{k\pi}{n} \right).$$

In (30) put $x = 1$, $\theta = 2\phi$; then

$$4 \sin^2 n\phi = \prod_{k=0}^{n-1} 4 \sin^2 \left(\phi + \frac{k\pi}{n} \right),$$

so that
$$\sin n\phi = \pm 2^{n-1} \prod_{k=0}^{n-1} \sin \left(\phi + \frac{k\pi}{n} \right).$$

Now, if $0 < \phi < \pi/n$, all the angles in the equation lie in the first or second quadrant, and consequently the $+$ sign should be taken. If now ϕ varies continuously, $\sin n\phi$ changes sign whenever ϕ passes through a value $r\pi/n$, where r is any integer. But, at the same time, one and only one of the factors on the right changes sign. Hence the $+$ sign must always be taken.

Example 4.—From Example 3 deduce that

$$\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \dots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

Example 5.—From Example 3 deduce that

$$(i) \sin n\phi = n \sin \phi \cos^{n-1} \phi \prod_{k=1}^{n-1} \left(1 + \tan \phi / \tan \frac{k\pi}{n} \right);$$

$$(ii) \sin n\phi = n \sin \phi \cos^{n-1} \phi \prod_{k=1}^p \left(1 - \tan^2 \phi / \tan^2 \frac{k\pi}{n} \right),$$

where $p = \frac{1}{2}n - 1$ or $\frac{1}{2}(n - 1)$ according as n is even or odd.

Example 6.—Deduce from (30) that

$$\sin \frac{\pi}{2n} \sin \frac{3\pi}{2n} \dots \sin \frac{(2n-1)\pi}{2n} = \frac{1}{2^{n-1}}.$$

[Put $\theta = \pi/n$, $x = 1$ and take square roots.]

Example 7.—From Example 3 deduce that

$$\begin{aligned} \sin \left(\alpha + \frac{\pi}{n} \right) \sin \left(\alpha + \frac{3\pi}{n} \right) \sin \left(\alpha + \frac{5\pi}{n} \right) \dots \sin \left(\alpha + \frac{2n-1}{n} \pi \right) \\ = \frac{1}{2^{n-2}} \cos \frac{n\alpha}{2} \cos \frac{n(\alpha + \pi)}{2}. \end{aligned}$$

EXAMPLES XV

1. Find the real quadratic factors of $x^8 + 1$.

$$\text{Ans. } \prod_{r=0}^3 \left(x^2 - 2x \cos \frac{2r+1}{8} \pi + 1 \right).$$

2. Find the moduli and the amplitudes of the values of $\sqrt[3]{2+2i}$; and then give all the values of x which satisfy the equation

$$x^6 - 4x^3 + 8 = 0.$$

$$\text{Ans. } \sqrt[3]{2}; \frac{\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12};$$

$$\sqrt[3]{2} \left\{ \cos \left(\frac{\pi}{12} + \frac{2k\pi}{3} \right) \pm i \sin \left(\frac{\pi}{12} + \frac{2k\pi}{3} \right) \right\}, \text{ where } k = 0, 1, 2.$$

3. Find the real quadratic factors of (i) $x^{10} - 2x^5 \cos 5\theta + 1$;
(ii) $x^8 - 2x^4 \cos 4\theta + 1$.

$$\text{Ans. (i) } \prod_{k=0}^4 \left\{ x^2 - 2x \cos \left(\theta + \frac{2k\pi}{5} \right) + 1 \right\};$$

$$\text{(ii) } (x^2 - 2x \cos \theta + 1)(x^2 + 2x \cos \theta + 1)(x^2 - 2x \sin \theta + 1) \\ \times (x^2 + 2x \sin \theta + 1).$$

4. Solve the equation $x^{10} + 2x^5 + 2 = 0$.

$$\text{Ans. } \sqrt[10]{2} \left\{ \cos \left(\frac{8k+3}{20} \pi \right) \pm i \sin \left(\frac{8k+3}{20} \pi \right) \right\},$$

where $k = 0, 1, 2, 3, 4$.

5. Solve the equation $x^4 - 4x^2 + 16 = 0$.

$$\text{Ans. } \pm \sqrt{3} \pm i.$$

6. Find, in the form $r(\cos \theta + i \sin \theta)$, the values of $(\sqrt{3} + i)^{4/5}$. Show that the sum of these values is zero, and that their product is $8(i\sqrt{3} - 1)$.

$$\text{Ans. } 2^{4/5} \left\{ \cos \left(\frac{2\pi}{15} + \frac{2k\pi}{5} \right) + i \sin \left(\frac{2\pi}{15} + \frac{2k\pi}{5} \right) \right\},$$

where $k = 0, 1, 2, 3, 4$.

These are the roots of $x^5 = (\sqrt{3} + i)^4 = 8(i\sqrt{3} - 1)$.

7. Find all the roots of $x^3 + 1/x^3 = \sqrt{2}$.

$$\text{Ans. } \cos(8k + 1)\frac{\pi}{12} \pm i \sin(8k + 1)\frac{\pi}{12}, k = 0, 1, 2.$$

8. Find all the values of x which satisfy the equation $x^n + 1/x^n = 2 \cos \theta$.

$$\text{Ans. } \cos \frac{\theta + 2k\pi}{n} \pm i \sin \frac{\theta + 2k\pi}{n}, k = 0, 1, 2, \dots, n - 1.$$

9. Find all the values of $(\sqrt{3} + i)^{3/5}$, and show that their continued product is equal to $8i$.

$$\text{Ans. } 2^{3/5} \left(i \cos \frac{2k\pi}{5} - \sin \frac{2k\pi}{5} \right), k = 0, 1, 2, 3, 4.$$

10. Solve the equation $x^8 + 4x^4 + 16 = 0$, and mark the positions of the roots on an Argand Diagram. Also express $x^8 + 4x^4 + 16$ as a product of four real quadratic factors.

$$\text{Ans. } \sqrt{2} \left\{ \cos \left(\frac{3r + 1}{6} \pi \right) \pm i \sin \left(\frac{3r + 1}{6} \pi \right) \right\}, r = 0, 1, 2, 3.$$

$$x^8 + 4x^4 + 16 = (x^2 - \sqrt{6} \cdot x + 2)(x^2 - \sqrt{2} \cdot x + 2) \\ \times (x^2 + \sqrt{6} \cdot x + 2)(x^2 + \sqrt{2} \cdot x + 2).$$

11. Prove that the roots of the equation

$$(5 + x)^5 - (5 - x)^5 = 0$$

are given by $x = 5i \tan(k\pi/5)$, $k = 0, 1, 2, 3, 4$.

12. Solve the equation $(1 + x)^3 = i(1 - x)^3$, expressing the roots in their simplest forms.

$$\text{Ans. } x = i \tan \frac{(4k + 1)\pi}{12}, k = 0, 1, 2.$$

13. Solve the equation

$$32z^5 = (1 + z)^5.$$

$$\text{Ans. } z = \frac{2 \cos \frac{2k\pi}{5} - 1 - 2i \sin \frac{2k\pi}{5}}{5 - 4 \cos \frac{2k\pi}{5}}, k = 0, 1, 2, 3, 4.$$

14. Show that the roots of the equation

$$(z + 1)^6 + (z - 1)^6 = 0$$

can be written in the form $i \cot \frac{2k + 1}{12} \pi$, where k is an integer.

Deduce that

$$(z + 1)^6 + (z - 1)^6 = 2 \left(z^2 + \cot^2 \frac{\pi}{12} \right) \left(z^2 + \cot^2 \frac{3\pi}{12} \right) \left(z^2 + \cot^2 \frac{5\pi}{12} \right).$$

15. Write out the real quadratic factors of

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1,$$

and show that

$$\begin{aligned} \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} \\ = \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} + \cos \frac{4\pi}{7} \cos \frac{6\pi}{7} + \cos \frac{6\pi}{7} \cos \frac{2\pi}{7} = -\frac{1}{2}, \end{aligned}$$

and

$$\cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \cos \frac{6\pi}{7} = \frac{1}{8}.$$

Ans.

$$\left(x^2 - 2x \cos \frac{2\pi}{7} + 1 \right) \left(x^2 - 2x \cos \frac{4\pi}{7} + 1 \right) \left(x^2 - 2x \cos \frac{6\pi}{7} + 1 \right).$$

16. Verify by substitution that

$$x = - \left(\cos \frac{r\pi}{6} + 3i \sin \frac{r\pi}{6} \right) / \left(\cos \frac{r\pi}{6} - 3i \sin \frac{r\pi}{6} \right),$$

where r is an integer, is a solution of

$$(2x - 1)^6 - (x - 2)^6 = 0;$$

and deduce that the roots of this equation are

$$\pm 1, \quad \frac{1}{2} \pm i \frac{\sqrt{3}}{2}, \quad \frac{13}{14} \pm i \frac{3\sqrt{3}}{14}.$$

17. Solve the equation

$$z^3 = i(z - 1)^3,$$

and show that the points corresponding to the three roots are collinear.

Ans. $z = \frac{1}{2} - \frac{1}{2}i \tan \frac{4k + 1}{12} \pi$, $k = 0, 1, 2$; the points lie on the line $x = \frac{1}{2}$.

18. Show that the roots of the equation

$$(x + 1)^5 + x^5 = 0$$

are given by

$$x = -\frac{1}{2} \left(1 + i \cot \frac{m\pi}{10} \right), \quad m = 1, 3, 5, 7, 9.$$

19. Show that

$$x^6 - x^3 + 1 = \left(x^2 - 2x \cos \frac{\pi}{9} + 1\right) \left(x^2 - 2x \cos \frac{5\pi}{9} + 1\right) \\ \times \left(x^2 - 2x \cos \frac{7\pi}{9} + 1\right),$$

and deduce that

$$(i) \cos \frac{\pi}{9} + \cos \frac{5\pi}{9} + \cos \frac{7\pi}{9} = 0;$$

$$(ii) \cos \frac{\pi}{9} \cos \frac{5\pi}{9} \cos \frac{7\pi}{9} = \frac{1}{8};$$

$$(iii) \sin \frac{\pi}{9} \sin \frac{2\pi}{9} \sin \frac{4\pi}{9} = \frac{\sqrt{3}}{8}.$$

[Equate coefficients of x^5 and x^3 . For (iii) put $x = -1$.]

20. Solve the equation

$$(1 + iz)^3 = -1,$$

and show that the three roots are the arguments of the vertices of an equilateral triangle.

$$\text{Ans. } 2i, \frac{1}{2}(i \pm \sqrt{3}).$$

21. Show that the equation $(z + 1)^7 = (z - 1)^7$ is satisfied by

$$z = \pm i \cot \frac{k\pi}{7},$$

where k is an integer, and find all the distinct roots of the equation.

$$\text{Ans. } k = 1, 2, 3.$$

22. Verify by substituting that

$$x = \frac{2 \cos \frac{1}{4}r\pi - i \sin \frac{1}{4}r\pi}{2 \cos \frac{1}{4}r\pi + i \sin \frac{1}{4}r\pi}$$

is a solution of the equation $(3x - 1)^4 = (3 - x)^4$, where r is integral. Hence, show that the roots of the equation are

$$\pm 1, (3 \pm 4i)/5.$$

23. Verify by substitution that

$$x = \frac{1}{2} \left\{ 1 + i \tan \left(\frac{2r + 1}{8} \pi \right) \right\},$$

where r is an integer, is a solution of the equation

$$(x - 1)^4 + x^4 = 0.$$

Hence obtain the real quadratic factors of $(x - 1)^4 + x^4$.

$$\text{Ans. } 2(x^2 - x + \frac{1}{4} \sec^2 \frac{1}{8}\pi)(x^2 - x + \frac{1}{4} \sec^2 \frac{3}{8}\pi).$$

24. Write down, and represent in separate diagrams, the 6th roots of -1 and of i . Indicate the conjugate pairs in the first diagram, and explain their absence from the second.

$$\text{Ans. } \cos\left(\frac{2r+1}{6}\pi\right) + i \sin\left(\frac{2r+1}{6}\pi\right),$$

$$\cos\left(\frac{4r+1}{12}\pi\right) + i \sin\left(\frac{4r+1}{12}\pi\right), \quad r = 0, 1, 2, 3, 4, 5.$$

25. Show that the roots of the equation $x^6 - x^3 + 1 = 0$ can be written in the form $a, a^{-1}, -a^2, -a^{-2}, -a^4, -a^{-4}$, where $a = \cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi$, and hence find real quadratic factors for $x^6 - x^3 + 1$.

$$\text{Ans. } x^6 - x^3 + 1 = (x^2 - 2x \cos \frac{1}{3}\pi + 1)(x^2 + 2x \cos \frac{2}{3}\pi + 1) \\ \times (x^2 + 2x \cos \frac{4}{3}\pi + 1).$$

26. From the formula $2 \cos n\theta = z^n + z^{-n}$ (n integral), deduce that

$$2 \cos \theta \cos n\theta = \cos(n+1)\theta + \cos(n-1)\theta.$$

27. Express $\sin^3 \theta \cos^4 \theta$ as a sum of sines of multiples of θ .

$$\text{Ans. } \frac{1}{64}(3 \sin \theta + 3 \sin 3\theta - \sin 5\theta - \sin 7\theta).$$

28. If $\cos \theta + i \sin \theta$ is a root of $x^4 + x^2 + 1 = 0$, prove that $\cos \theta - i \sin \theta$ is also a root: solve the equation.

$$[\cos 2\theta = (x^2 + x^{-2})/2 = -\frac{1}{2}:$$

thus $2\theta = 2\pi/3$ or $4\pi/3$. Hence the roots are $\frac{1}{2} \pm i \sin \frac{1}{3}\pi$, $\cos \frac{2}{3}\pi \pm i \sin \frac{2}{3}\pi$, that is, $\pm \frac{1}{2}(1 \pm i\sqrt{3})$.]

29. Express $\cos^8 \theta$ in the form of a series of cosines of multiples of θ .

$$\text{Ans. } \frac{1}{128}(\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35).$$

30. Show that

$$(i) \quad 32 \cos^6 \theta = \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10;$$

$$(ii) \quad 64 \cos^7 \theta = \cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta.$$

31. Show that

$$(i) \quad 128 \sin^3 \theta \cos^5 \theta = 6 \sin 2\theta + 2 \sin 4\theta - 2 \sin 6\theta - \sin 8\theta;$$

$$(ii) \quad 128 \sin^2 \theta \cos^6 \theta = 5 + 4 \cos 2\theta - 4 \cos 4\theta \\ - 4 \cos 6\theta - \cos 8\theta.$$

32. From Example 31, (i) deduce that

$$128 \cos^3 \theta \sin^5 \theta = 6 \sin 2\theta - 2 \sin 4\theta - 2 \sin 6\theta + \sin 8\theta.$$

33. If $x + \frac{1}{x} = u$, prove that

$$x^5 + \frac{1}{x^5} = u^5 - 5u^3 + 5u;$$

then show that

$$1 + \cos 10\theta = 2(16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta)^2.$$

34. Solve the equation

$$(x + 1)^7 = x^7(\cos 7\alpha + i \sin 7\alpha).$$

Ans. $x = -\frac{1}{2} - \frac{1}{2}i \cot\left(\frac{1}{2}\alpha + k\pi/7\right)$, $k = 0, 1, 2, 3, 4, 5, 6$.

35. Solve the equation

$$x^4 - ix^3 - x^2 + ix + 1 = 0.$$

Ans. $x = \cos \theta + i \sin \theta$, where $\theta = -18^\circ, 54^\circ, 126^\circ, 198^\circ$.

36. Solve the equation $x^7 + 1 = 0$, and deduce that

$$(i) \quad x^7 + 1 = (x + 1) \prod_{r=0}^2 \left\{ x^2 - 2x \cos\left(\frac{2r+1}{7}\pi\right) + 1 \right\};$$

$$(ii) \quad \cos \frac{\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{5\pi}{7} = 4 \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} = \frac{1}{2}.$$

Ans. $x = \cos\left(\frac{2r+1}{7}\pi\right) \pm i \sin\left(\frac{2r+1}{7}\pi\right)$, $r = 0, 1, 2, 3$.

[For (ii) equate the coefficients of x^6 in (i); then put $x = 1$ in (i).]

37. Show that

$$x^{10} + 1 = \prod_{r=0}^4 \left\{ (x - 1)^2 + 4x \sin^2\left(\frac{2r+1}{20}\pi\right) \right\},$$

and deduce that

$$(i) \quad \sin \frac{\pi}{20} \sin \frac{3\pi}{20} \sin \frac{5\pi}{20} \sin \frac{7\pi}{20} \sin \frac{9\pi}{20} = \frac{\sqrt{2}}{32};$$

$$(ii) \quad \sin \frac{\pi}{10} \sin \frac{3\pi}{10} \sin \frac{5\pi}{10} \sin \frac{7\pi}{10} \sin \frac{9\pi}{10} = \frac{1}{16}.$$

[For (i) put $x = 1$ and extract the square root: for (ii) put $x = -1$, extract the square root, and multiply by (i).]

38. Find the real quadratic factors of $x^{16} + 1$, and deduce the value of

$$\sin \frac{\pi}{32} \cdot \sin \frac{3\pi}{32} \cdot \dots \cdot \sin \frac{15\pi}{32}.$$

$$\text{Ans.} \quad \prod_{r=0}^7 \left(x^2 - 2x \cos \frac{2r+1}{16}\pi + 1 \right), \quad \sqrt{2}/2^8.$$

39. Expand $(\cos \theta + i \sin \theta)^6$ by the Binomial Theorem, and hence express (i) $\cos 6\theta$ in terms of $\cos \theta$, (ii) $\tan 6\theta$ in terms of $\tan \theta$.

Ans. (i) $32c^6 - 48c^4 + 18c^2 - 1$, where $c = \cos \theta$;

(ii) $(6t - 20t^3 + 6t^5)/(1 - 15t^2 + 15t^4 - t^6)$, where $t = \tan \theta$.

40. Solve the equation

$$32x^3 + 48x^2 + 18x + 1 = 0.$$

Ans. $-\cos^2 \frac{\pi}{12}, -\cos^2 \frac{\pi}{4}, -\cos^2 \frac{5\pi}{12}$. [Cf. Example 39 (i).]

41. If $\cos 7\theta = 24 \cos^3 \theta - 23 \cos \theta$,
show that θ must be a multiple of $\frac{1}{2}\pi$. [Cf. § 5, Example 3.]

42. Show that

$\cos 7\theta / \cos \theta = 1 - 2(2 \cos 2\theta) - (2 \cos 2\theta)^2 + (2 \cos 2\theta)^3$,
and hence solve the equation

$$x^3 - x^2 - 2x + 1 = 0.$$

Ans. $x = 2 \cos \left(\frac{2k + 1}{7} \pi \right)$, where $k = 0, 1, 2$.

43. Show that

$\sin 7\theta / \sin \theta$

$$= 7 - 56s^2 + 112s^4 - 64s^6, \text{ where } s = \sin \theta,$$

$$= 64c^3 - 80c^4 + 24c^5 - 1, \text{ where } c = \cos \theta,$$

$$= 64 \left(\cos^2 \theta - \cos^2 \frac{\pi}{7} \right) \left(\cos^2 \theta - \cos^2 \frac{2\pi}{7} \right) \left(\cos^2 \theta - \cos^2 \frac{3\pi}{7} \right).$$

Deduce that

$$\cos^2 \frac{\pi}{7} + \cos^2 \frac{2\pi}{7} + \cos^2 \frac{3\pi}{7} = \frac{5}{4}.$$

44. Express $\cos 7\theta / \cos \theta$ in terms of $\sin \theta$, and prove that

$$(i) \sin^2 \frac{\pi}{14} + \sin^2 \frac{3\pi}{14} + \sin^2 \frac{5\pi}{14} = \frac{5}{4},$$

$$(ii) \cot^2 \frac{\pi}{14} + \cot^2 \frac{3\pi}{14} + \cot^2 \frac{5\pi}{14} = 21.$$

Ans. $1 - 24s^2 + 80s^4 - 64s^6$, where $s = \sin \theta$.

45. If $(1+x)/(1-x) = \cos 2\theta + i \sin 2\theta$, show that

$$x = i \tan \theta;$$

then prove that

$$(1+x)^{2m+1} - (1-x)^{2m+1} = 2x \prod_{r=1}^m \left(x^2 + \tan^2 \frac{r\pi}{2m+1} \right).$$

Deduce that

$$(i) \cos (2m+1)\theta = (\cos \theta)^{2m+1} \prod_{r=1}^m \left(1 - \tan^2 \theta \tan^2 \frac{r\pi}{2m+1} \right);$$

$$(ii) \sqrt{(2m+1)} = \prod_{r=1}^m \tan \frac{r\pi}{2m+1};$$

$$(iii) \quad m(2m - 1) = 3 \sum_{r=1}^m \cot^2 \frac{r\pi}{2m+1}.$$

[For (i) put $x = i \cot \theta$; for (ii) divide by x and let $x \rightarrow 0$.]

46. Show that

$$\begin{aligned} (1+x)^{2n} - (1-x)^{2n} &= 4nx \prod_{r=1}^{n-1} \left(1 + x^2 \cot^2 \frac{r\pi}{2n}\right) \\ &= 4nx \prod_{r=1}^{n-1} \left(x^2 + \tan^2 \frac{r\pi}{2n}\right); \end{aligned}$$

and deduce that

$$\begin{aligned} (i) \quad \sum_{r=1}^{n-1} \cot^2 \frac{r\pi}{2n} &= \frac{1}{3}(n-1)(2n-1); \\ (ii) \quad \sum_{r=1}^{n-1} \sec^2 \frac{r\pi}{2n} &= \frac{2}{3}(n^2-1); \\ (iii) \quad \prod_{r=1}^{n-1} \sin \frac{r\pi}{2n} &= \frac{\sqrt{n}}{2^{n-1}}; \quad (iv) \quad \prod_{r=1}^{n-1} \sin \frac{r\pi}{n} = \frac{n}{2^{n-1}}. \end{aligned}$$

47. If $z = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$, show that

$$1 + z + z^2 + \dots \text{ to } n \text{ terms} = 1 + i \cot \frac{\pi}{2n}.$$

48. Show that

$$\sin n\alpha + 2 \sum_{r=1}^{n-1} \sin(n-r)\alpha \cos r\theta = \sin \alpha \frac{\cos n\alpha - \cos n\theta}{\cos \alpha - \cos \theta},$$

and deduce an expression for $(1 - \cos n\theta)/(1 - \cos \theta)$ in cosines of multiples of θ .

$$\text{Ans. } n + 2 \sum_{r=1}^{n-1} (n-r) \cos r\theta.$$

49. If n is odd, show that

$$\tan \frac{\pi}{n} \tan \frac{2\pi}{n} \tan \frac{3\pi}{n} \dots \tan \frac{(n-1)\pi}{2n} = \sqrt{n}.$$

[In formula (26) put $\theta = \frac{1}{2}\pi$.]

50. Solve the equation

$$(2 + ix)^n - x^n = 0,$$

and show that the roots can be expressed in the form

$$x = i + \tan\left(\frac{\pi}{4} + \frac{k\pi}{n}\right), \quad k = 0, 1, 2, \dots, n-1.$$

Hence show that

$$(2 + ix)^{10} - x^{10} = -2 \prod_{k=0}^4 \left\{ (x - i)^2 - \tan^2 \frac{(2k+1)\pi}{20} \right\}.$$

Deduce that

$$\cos \frac{\pi}{20} \cos \frac{3\pi}{20} \cos \frac{5\pi}{20} \cos \frac{7\pi}{20} \cos \frac{9\pi}{20} = \frac{\sqrt{2}}{32}.$$

51. Prove that the sum of the first n terms of the series

$$\sin x + \frac{1}{3} \sin 2x + \frac{1}{3^2} \sin 3x + \dots$$

$$\text{is } \frac{\sin x - \frac{1}{3^n} \sin(n+1)x + \frac{1}{3^{n+1}} \sin nx}{\frac{1}{9} - \frac{2}{3} \cos x};$$

and deduce that the sum to infinity is

$$\frac{9 \sin x}{10 - 6 \cos x}.$$

52. If $x = \cos \theta + i \sin \theta$, show that

$$\frac{1}{1 - rx} + \frac{1}{1 - \frac{r}{x}} - 1 = \frac{1 - r^2}{1 - 2r \cos \theta + r^2},$$

and hence derive the expansion

$$\frac{1 - r^2}{1 - 2r \cos \theta + r^2} = 1 + 2r \cos \theta + 2r^2 \cos 2\theta + 2r^3 \cos 3\theta + \dots,$$

when $|r| < 1$.

53. Establish the following, stating the restrictions on θ and x necessary for the convergence of the series:

$$(i) \quad 1 + 2 \sum_{n=1}^{\infty} (-1)^n \tan^n \theta \cos n\phi = \frac{\cos 2\theta}{1 + \sin 2\theta \cos \phi};$$

$$(ii) \quad \cos \alpha + 2x \cos(\alpha + \beta) + 3x^2 \cos(\alpha + 2\beta) + 4x^3 \cos(\alpha + 3\beta) + \dots \\ = \frac{\cos \alpha - 2x \cos(\alpha - \beta) + x^2 \cos(\alpha - 2\beta)}{(1 - 2x \cos \beta + x^2)^2}.$$

[(i) $|\theta - r\pi| < \frac{1}{2}\pi$, r integral; (ii) $|x| < 1$.]

54. Prove that

$$\begin{aligned} 1 - {}^nC_2 t^2 + {}^nC_4 t^4 - \dots - \cot n\theta \{ {}^nC_1 t - {}^nC_3 t^3 + \dots \} \\ = (1 - t \cot \theta) \left\{ 1 - t \cot \left(\theta + \frac{\pi}{n} \right) \right\} \dots \\ \dots \left\{ 1 - t \cot \left(\theta + \frac{n-1}{n} \pi \right) \right\}. \end{aligned}$$

Deduce that

$$\begin{aligned} \operatorname{cosec}^2 \theta + \operatorname{cosec}^2 \left(\theta + \frac{\pi}{n} \right) + \dots + \operatorname{cosec}^2 \left(\theta + \frac{n-1}{n} \pi \right) \\ = n^2 \operatorname{cosec}^2 n\theta. \end{aligned}$$

55. If p is a positive integer, expand $\sin^{2p}\theta$ in terms of cosines of multiples of θ , and deduce that, if n is a positive integer which is not $2p$ or a submultiple of $2p$,

$$\sum_{r=0}^{n-1} \sin^{2p} \left(\theta + \frac{2r\pi}{n} \right) = \frac{n}{2^{2p}} \frac{(2p)!}{(p!)^2}.$$

56. Show that, if n is even,

$$\begin{aligned} \text{(i) } \cos n\theta = \cos^n \theta \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{\pi}{2n}} \right) \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{3\pi}{2n}} \right) \dots \\ \dots \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{(n-1)\pi}{2n}} \right); \end{aligned}$$

$$\begin{aligned} \text{(ii) } \sin n\theta = n \cos^n \theta \tan \theta \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{\pi}{n}} \right) \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{2\pi}{n}} \right) \dots \\ \dots \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{(n-2)\pi}{2n}} \right); \end{aligned}$$

while, if n is odd,

$$\begin{aligned} \text{(iii) } \cos n\theta = \cos^n \theta \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{\pi}{2n}} \right) \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{3\pi}{2n}} \right) \dots \\ \dots \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{(n-2)\pi}{2n}} \right); \end{aligned}$$

$$(iv) \sin n\theta = n \cos^n \theta \tan \theta \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{\pi}{n}}\right) \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{2\pi}{n}}\right) \dots$$

$$\dots \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{(n-1)\pi}{2n}}\right).$$

[Apply the identity

$$1 - \frac{\sin^2 \theta}{\sin^2 \alpha} = \cos^2 \theta \left(1 - \frac{\tan^2 \theta}{\tan^2 \alpha}\right)$$

to the formulæ (17), (25), (18), (26).]

57. Prove that

$$(i) \cos \alpha + x \cos (\alpha + \beta) + x^2 \cos (\alpha + 2\beta) + \dots$$

$$\dots + x^n \cos (\alpha + n\beta)$$

$$\cos \alpha - x \cos (\alpha - \beta) - x^{n+1} \cos \{\alpha + (n+1)\beta\}$$

$$+ x^{n+2} \cos (\alpha + n\beta);$$

$$= \frac{1 - 2x \cos \beta + x^2}{1 - 2x \cos \beta + x^2};$$

$$(ii) \sin \alpha + x \sin (\alpha + \beta) + x^2 \sin (\alpha + 2\beta) + \dots$$

$$\dots + x^n \sin (\alpha + n\beta)$$

$$\sin \alpha - x \sin (\alpha - \beta) - x^{n+1} \sin \{\alpha + (n+1)\beta\}$$

$$+ x^{n+2} \sin (\alpha + n\beta).$$

$$= \frac{1 - 2x \cos \beta + x^2}{1 - 2x \cos \beta + x^2}.$$

58. Show that

$$t^{2n+1} - {}^{2n+1}C_2 t^{2n-1} + \dots + (-1)^n {}^{2n+1}C_1 t$$

$$= t \prod_{r=1}^n \left(t^2 - \tan^2 \frac{r\pi}{2n+1}\right).$$

Deduce that

$$\sum_{r=1}^n \sec^2 \frac{r\pi}{2n+1} = 2n(n+1).$$

[Cf. formula (7).]

59. Prove that

$$\cos 9\theta = 256c^9 - 576c^7 + 432c^5 - 120c^3 + 9c,$$

where $c = \cos \theta$.

Without actual division, show that the equation

$$256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x - 1 = 0$$

possesses the roots $1, -\frac{1}{2}, -\frac{1}{2}$; and that its six remaining roots are equal in pairs.

Prove that

$$8 \cos \frac{2\pi}{9} \cos \frac{4\pi}{9} \cos \frac{8\pi}{9} = -1.$$

60. Prove that

$$\begin{aligned} \sin 9\theta &= 9s - 120s^3 + 432s^5 - 576s^7 + 256s^9 \\ &= 9s \left(1 - \frac{s^2}{\sin^2 \frac{\pi}{9}}\right) \left(1 - \frac{s^2}{\sin^2 \frac{2\pi}{9}}\right) \left(1 - \frac{s^2}{\sin^2 \frac{3\pi}{9}}\right) \left(1 - \frac{s^2}{\sin^2 \frac{4\pi}{9}}\right), \end{aligned}$$

where $s = \sin \theta$. Deduce that

$$\operatorname{cosec}^2 \frac{\pi}{9} + \operatorname{cosec}^2 \frac{2\pi}{9} + \operatorname{cosec}^2 \frac{4\pi}{9} = 12.$$

61. From Example 2, § 7, deduce the three roots of the equation

$$16x^3 - 24x^2 + 10x - 1 = 0,$$

and hence obtain, in surd form, the values of $\sin 22\frac{1}{2}^\circ$ and $\sin 67\frac{1}{2}^\circ$.

$$\text{Ans. } \sin^2 \frac{\pi}{8}, \sin^2 \frac{\pi}{4}, \sin^2 \frac{3\pi}{8}; \frac{1}{2}\sqrt{2 - \sqrt{2}}, \frac{1}{2}\sqrt{2 + \sqrt{2}}.$$

62. Prove that

$$\cos 8\theta = 128 \sin^8 \theta - 256 \sin^6 \theta + 160 \sin^4 \theta - 32 \sin^2 \theta + 1,$$

and then show that

$$(i) \cos 8\theta = 128 \prod_{k=1}^4 \left\{ \sin^2 \theta - \sin^2 \left(\frac{2k-1}{16} \pi \right) \right\};$$

$$(ii) \cos x = \prod_{k=1}^4 \left\{ 1 - \frac{\sin^2 \frac{1}{8}x}{\sin^2 \left(\frac{2k-1}{16} \pi \right)} \right\}.$$

63. Find the real factors of $x^9 - 1$, and deduce that

$$(i) \sin \frac{\pi}{9} \sin \frac{2\pi}{9} \sin \frac{3\pi}{9} \sin \frac{4\pi}{9} = \frac{3}{16};$$

$$(ii) \tan \frac{\pi}{9} \tan \frac{2\pi}{9} \tan \frac{3\pi}{9} \tan \frac{4\pi}{9} = 3;$$

$$(iii) 1 + 2 \sum_{n=1}^4 \cos n\theta = 16 \prod_{r=1}^4 \left(\cos \theta - \cos \frac{2r\pi}{9} \right).$$

$$\text{Ans. } (x-1) \prod_{r=1}^4 \left(x^2 - 2x \cos \frac{2r\pi}{9} + 1 \right).$$

[(i) Divide by $x-1$ and let $x \rightarrow 1$; (ii) put $x = -1$; (iii) divide by $x-1$, then by x^4 , and put $x = \cos \theta + i \sin \theta$.]

64. From Example 2, § 1, deduce that

$$z^3 + 1 + \frac{1}{z^3} = \prod_{r=0}^2 \left\{ z + \frac{1}{z} - 2 \cos \left(\frac{6r+2}{9} \pi \right) \right\};$$

and hence show that

$$(i) \sin \frac{\pi}{9} \sin \frac{4\pi}{9} \sin \frac{7\pi}{9} = \frac{\sqrt{3}}{8};$$

$$(ii) \cos \frac{2\pi}{9} \cos \frac{8\pi}{9} \cos \frac{14\pi}{9} = -\frac{1}{8};$$

$$(iii) \cos \frac{2\pi}{9} + \cos \frac{8\pi}{9} + \cos \frac{14\pi}{9} = 0;$$

$$(iv) 4 \cos^2 3\theta - 1 = 64 \prod_{r=0}^2 \left\{ \cos^2 \theta - \cos^2 \left(\frac{3r+1}{9} \pi \right) \right\}.$$

[For (i) put $z = 1$ and take square roots; for (ii) put $z = i$; for (iii) equate the coefficients of z^2 ; and for (iv) put $z = \cos 2\theta + i \sin 2\theta$.]

65. Show that

$\sin 5\theta$

$$= 16 \sin \theta \sin \left(\theta - \frac{\pi}{5} \right) \sin \left(\theta + \frac{\pi}{5} \right) \sin \left(\theta - \frac{2\pi}{5} \right) \sin \left(\theta + \frac{2\pi}{5} \right),$$

and deduce that

$$\sin^2 \frac{\pi}{5} \sin^2 \frac{2\pi}{5} = \frac{5}{16}.$$

[Cf. § 5, Example 2 (iii).]

66. If n is a positive integer, show that

$$\cos n\theta = 1 + \sum_{r=1}^n {}^nC_r \left(2 \sin \frac{\theta}{2} \right)^r \cos \frac{r\pi + r\theta}{2}.$$

67. Prove that, if n is a positive integer,

$$\begin{aligned} \cos \theta + {}^nC_1 \cos(\theta + \phi) + {}^nC_2 \cos(\theta + 2\phi) + \dots &+ \cos(\theta + n\phi) \\ &= (2 \cos \frac{1}{2}\phi)^n \cos(\theta + \frac{1}{2}n\phi), \\ \sin \theta + {}^nC_1 \sin(\theta + \phi) + {}^nC_2 \sin(\theta + 2\phi) + \dots &+ \sin(\theta + n\phi) \\ &= (2 \cos \frac{1}{2}\phi)^n \sin(\theta + \frac{1}{2}n\phi). \end{aligned}$$

68. Show that

$$\frac{5 \tan 5\theta}{\tan \theta} = \frac{(\tan^2 \theta - \tan^2 \frac{1}{5}\pi)(\tan^2 \theta - \tan^2 \frac{2}{5}\pi)}{(\tan^2 \theta - \tan^2 \frac{1}{10}\pi)(\tan^2 \theta - \tan^2 \frac{3}{10}\pi)}.$$

69. Show that, if n is an odd positive integer,

$$2^{2n-2}\{\cos^{2n}\theta + (-1)^n \sin^{2n}\theta\} \\ = \cos 2n\theta + {}^{2n}C_2 \cos (2n-4)\theta + \dots + {}^{2n}C_{n-1} \cos 2\theta,$$

and give the expansion when n is an even positive integer.

Ans. The last two terms are then ${}^{2n}C_{n-2} \cos 4\theta + \frac{1}{2}{}^{2n}C_n$.

70. Solve the equation $(x-1)^{2n} - 1 = 0$, and hence show that

$$(x-1)^{2n} - 1 = x(x-2) \prod_{r=1}^{n-1} \left\{ x^2 - 4(x-1) \cos^2 \frac{r\pi}{2n} \right\}.$$

Deduce that

$$4^{2n} - 3^{2n} = 7 \prod_{r=1}^{n-1} \left(1 + 48 \cos^2 \frac{r\pi}{2n} \right).$$

71. Sum to infinity

$1 + r \cos A \cos B + r^2 \cos 2A \cos 2B + r^3 \cos 3A \cos 3B + \dots$,
where $|r| < 1$.

$$\text{Ans. } \frac{1}{2} \frac{1 - r \cos(A+B)}{1 - 2r \cos(A+B) + r^2} + \frac{1}{2} \frac{1 - r \cos(A-B)}{1 - 2r \cos(A-B) + r^2}.$$

72. From the formula for $\tan 5\theta$ in § 4, Example 1, deduce the roots of the equation $x^4 - 10x^2 + 5 = 0$, and hence show that

$$\sum_{r=1}^4 \sec^2 \frac{r\pi}{5} = 24.$$

$$\text{Ans. } \pm \tan \frac{1}{5}\pi, \pm \tan \frac{2}{5}\pi.$$

73. Show that, in any triangle in which $b < c$,
 $\operatorname{cosec} A \cos B \sin C$

$$= 1 + \frac{b}{c} \cos A + \left(\frac{b}{c}\right)^2 \cos 2A + \left(\frac{b}{c}\right)^3 \cos 3A + \dots$$

74. Show that

$$(i) \lim_{\theta \rightarrow \alpha} \frac{\sin \theta - \sin \alpha}{\cos \theta - \cos \alpha} = -\cot \alpha;$$

$$(ii) \lim_{\theta \rightarrow 0} \frac{\cot \theta + \cot 2\theta}{\cot 3\theta} = \frac{9}{2}; \quad (iii) \lim_{\theta \rightarrow \frac{1}{2}\pi} \frac{\sin 3\theta + \cos 4\theta}{\sin \theta + \cos 2\theta} = -\frac{7}{3}.$$

75. Show that

$$(i) \lim_{x \rightarrow a} \frac{x^3 + ax^2 - 2a^3}{x^2 - a^2} = \frac{5}{2}a; \quad (ii) \lim_{x \rightarrow 2} \frac{x^3 - 3x^2 + 4}{x^3 - 2x^2 - 4x + 8} = \frac{3}{4}.$$

76. If A, B, C, D are the points in the Argand Diagram whose arguments are $1, z, z^2, z^3$, where $z = r(\cos \theta + i \sin \theta)$, show that $BC = rAB$, and that

$$AB + BC + CD = (r^2 + r + 1)\sqrt{(r^2 - 2r \cos \theta + 1)}.$$

77. Show that the points which correspond, in the Argand Diagram, to the roots of the equation

$$(z + 4)^5 = 32(z + 1)^5$$

are all at the same distance from the origin.

78. From the formula (§2, Example 5)

$$\frac{z^{2n} - 1}{z^2 - 1} = \prod_{r=1}^{n-1} \left(z^2 - 2z \cos \frac{r\pi}{n} + 1 \right)$$

deduce that

$$(i) \frac{\sin n\theta}{\sin \theta} = 2^{n-1} \prod_{r=1}^{n-1} \left(\cos \theta - \cos \frac{r\pi}{n} \right);$$

$$(ii) 2^{n-1} \sin \frac{\pi}{2n} \sin \frac{2\pi}{2n} \sin \frac{3\pi}{2n} \dots \sin \frac{(n-1)\pi}{2n} = \sqrt{n};$$

$$(iii) \cos \frac{\pi}{2m+1} \cos \frac{2\pi}{2m+1} \cos \frac{3\pi}{2m+1} \dots \cos \frac{2m\pi}{2m+1} = \frac{(-1)^m}{2^{2m}}.$$

79. Show that

$$x^{4n} + 1 = \prod_{r=0}^{2n-1} \left\{ x^2 - 2x \cos \frac{(2r+1)\pi}{4n} + 1 \right\},$$

and deduce that

$$\prod_{r=0}^{2n-1} \tan \frac{(2r+1)\pi}{4n} = (-1)^n.$$

Prove that the last result may also be obtained from the roots of the equation

$$(1 - xi)^{2n} + (1 + xi)^{2n} = 0.$$

80. Sum the series

$$\sec^2 x + \frac{1}{4} \sec^2 \frac{x}{2} + \frac{1}{16} \sec^2 \frac{x}{4} + \frac{1}{64} \sec^2 \frac{x}{8} + \dots$$

to n terms and to infinity.

$$\text{Ans. } 4 \operatorname{cosec}^2 2x - \frac{1}{4^{n-1}} \operatorname{cosec}^2 \left(\frac{x}{2^{n-1}} \right), \quad 4 \operatorname{cosec}^2 2x - \frac{1}{x^2}.$$

81. Solve the equation $z^6 - 1 = 0$, and show that its roots are represented by the vertices of a regular hexagon. If the vertices are P_1, P_2, \dots, P_6 , and if A is the point

$$a = \cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi,$$

prove, by applying the substitution $z = z' + a$ to the equation $z^6 - 1 = 0$, that

$$AP_1 \cdot AP_2 \cdot \dots \cdot AP_6 = 2.$$

82. If x is real, n a positive integer, and

$$u = x^n + nx^{n-1} \cos \alpha + \frac{n(n-1)}{2!} x^{n-2} \cos 2\alpha + \dots + \cos n\alpha,$$

show that u is the real part of $(x + a)^n$, where

$$a = \cos \alpha + i \sin \alpha,$$

and find all the values of x for which u is zero.

Ans. $x = -\cos \alpha + \sin \alpha \cot \left(\frac{2r+1}{2n} \pi \right)$, where
 $r = 0, 1, 2, \dots, n-1.$

83. Show that the sum of the series

$${}^n C_1 x \sin \alpha + {}^n C_2 x^2 \sin 2\alpha + {}^n C_3 x^3 \sin 3\alpha + \dots + x^n \sin n\alpha$$

is $x \sin n\alpha \prod_{k=1}^{n-1} \left\{ x - \sin \frac{k\pi}{n} \operatorname{cosec} \left(\alpha - \frac{k\pi}{n} \right) \right\}$,

and deduce that

$$(i) \prod_{k=1}^{n-1} \sin \left(\alpha - \frac{k\pi}{n} \right) \operatorname{cosec} \frac{k\pi}{n} = (-1)^{n-1} \frac{\sin n\alpha}{n \sin \alpha};$$

$$(ii) \sum_{k=1}^{n-1} \sin \left(\alpha - \frac{k\pi}{n} \right) \operatorname{cosec} \frac{k\pi}{n} = -(n-1) \cos \alpha.$$

84. Show that

$$\cos x + \sum_{r=1}^n {}^n C_r (2 \sin \frac{1}{2}\theta)^r \cos \left(x + \frac{1}{2}r\theta + \frac{1}{2}r\pi \right) = \cos (x + n\theta).$$

85. Show that the roots of the equation

$$x^n \sin n\alpha + {}^n C_1 x^{n-1} \sin (n\alpha + \beta) + {}^n C_2 x^{n-2} \sin (n\alpha + 2\beta) + \dots = 0$$

are given by

$$x = \sin (\alpha + \beta - r\pi/n) / \sin (r\pi/n - \alpha),$$

where $r = 0, 1, 2, \dots, n - 1$. Deduce that

$$(i) \prod_{r=0}^{n-1} \frac{\sin(\alpha + \beta - r\pi/n)}{\sin(\alpha - r\pi/n)} = \frac{\sin n(\alpha + \beta)}{\sin n\alpha};$$

$$(ii) \sum_{r=0}^{n-1} \frac{\sin(\alpha + \beta - r\pi/n)}{\sin(\alpha - r\pi/n)} = n \frac{\sin(n\alpha + \beta)}{\sin n\alpha}.$$

86. If

$$u_n = \frac{a}{n} \left(\sin \frac{a}{n} + \sin \frac{2a}{n} + \sin \frac{3a}{n} + \dots + \sin \frac{na}{n} \right),$$

show that the sequence (u_n) converges to the limit $2 \sin^2(\frac{1}{2}a)$.

87. If

$$u_n = \frac{\pi}{n} \left(\sin^2 \frac{\pi}{2n} + \sin^2 \frac{2\pi}{2n} + \sin^2 \frac{3\pi}{2n} + \dots + \sin^2 \frac{n\pi}{2n} \right),$$

show that the sequence (u_n) converges to the limit $\frac{1}{2}\pi$.

88. If

$$u_n = \frac{a}{n} \sum_{r=0}^{n-1} \cos^2 \frac{ra}{n},$$

show that the sequence (u_n) converges to the limit

$$\frac{1}{2}a + \frac{1}{4} \sin 2a.$$

89. If

$$u_n = \frac{1 + r^{3n}}{1 + r^{4n}},$$

discuss the convergence or divergence of the sequence (u_n) for

- (i) $-1 < r < 1$; (ii) $r > 1$; (iii) $r < -1$; (iv) $r = 1$;
 (v) $r = -1$.

Ans. (i) $r^n \rightarrow 0$ and sequence converges to 1; (ii) sequence converges to zero, as $u_n = r^{-n}(1 + r^{-3n})/(1 + r^{-4n})$, and $r^{-n} \rightarrow 0$; (iii) sequence again converges to zero, as $r^{-n} \rightarrow 0$; (iv) $u_n = 1$, and sequence converges to 1; (v) $u_n = 0$ or 1 according as n is odd or even. The sequence *oscillates* between 0 and 1.

90. If the sequence (u_n) converges to the limit l , prove that the sequence (v_n) , where

$$v_n = \frac{u_1 + u_2 + \dots + u_n}{n},$$

also converges to the limit l .

[Choose m so large that, for $n \geq m$, $|l - u_n| < \frac{1}{2}\epsilon$. Then

$$\begin{aligned}
 |v_n - l| &= \left| \frac{u_1 + u_2 + \dots + u_m - ml}{n} + \frac{u_{m+1} - l}{n} + \frac{u_{m+2} - l}{n} + \dots + \frac{u_n - l}{n} \right| \\
 &\leq \left| \frac{u_1 + u_2 + \dots + u_m - ml}{n} \right| + \left| \frac{u_{m+1} - l}{n} \right| + \dots + \left| \frac{u_n - l}{n} \right| \\
 &< \left| \frac{u_1 + u_2 + \dots + u_m - ml}{n} \right| + \frac{\epsilon}{2}.
 \end{aligned}$$

Now choose an integer $m' (\geq m)$ so large that, if $n \geq m'$,

$$\left| \frac{u_1 + u_2 + \dots + u_m - ml}{n} \right| < \frac{\epsilon}{2}.$$

Then, if $n \geq m'$, $|v_n - l| < \epsilon$, so that $v_n \rightarrow l$ when $n \rightarrow \infty$.]

91. If
$$u_n = \frac{1}{n} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right),$$

show that the sequence (u_n) converges to zero.

92. If

$$u_n = \frac{1}{n} \left(\sin \theta + 2 \sin \frac{\theta}{2} + 3 \sin \frac{\theta}{3} + \dots + n \sin \frac{\theta}{n} \right),$$

show that the sequence (u_n) converges to the limit θ .

93. Show that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{1}{n} \left(n \sin \theta + \frac{n-1}{2} \sin 2\theta + \frac{n-2}{2^2} \sin 3\theta + \dots + \frac{1}{2^{n-1}} \sin n\theta \right) \\
 = \frac{4 \sin \theta}{5 - 4 \cos \theta}.
 \end{aligned}$$

CHAPTER XVI

APPLICATIONS OF THE CALCULUS :
EXPANSIONS IN SERIES

§ 1. Introductory

IN this chapter a knowledge of the Calculus will be assumed.

Example 1.—Show that

$$\int \sin^4 \theta \cos^2 \theta d\theta = \frac{1}{32} (\frac{1}{8} \sin 6\theta - \frac{1}{2} \sin 4\theta - \frac{1}{2} \sin 2\theta + 2\theta + C).$$

[See Chapter XV, § 3, Example 1.]

Derivative of a Product.—From the formula for the derivative of a product it follows that, if u_1 and u_2 are functions of the same variable,

$$\frac{D(u_1 u_2)}{u_1 u_2} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2}.$$

It can be proved by induction that, for n such functions,

$$\frac{D(u_1 u_2 \dots u_n)}{u_1 u_2 \dots u_n} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \dots + \frac{u_n'}{u_n}. \quad (1)$$

For, if (1) holds for n functions,

$$\begin{aligned} & \frac{D(u_1 u_2 \dots u_n u_{n+1})}{u_1 u_2 \dots u_n u_{n+1}} \\ &= \frac{u_{n+1} D(u_1 u_2 \dots u_n) + u_1 u_2 \dots u_n D(u_{n+1})}{u_1 u_2 \dots u_n u_{n+1}} = \sum_{r=1}^{n+1} \frac{u_r'}{u_r}, \end{aligned}$$

so that (1) holds also for $n + 1$ functions.

But the formula holds when $n = 2$: hence it holds for all greater values of n .

An alternative proof will be found in Chapter XVII, § 2.

Example 2.—Show that

$$(i) \frac{nx^{n-1} \sin n\theta}{x^{2n} - 2x^n \cos n\theta + 1} = \sum_{k=0}^{n-1} \frac{\sin(\theta + 2k\pi/n)}{x^2 - 2x \cos(\theta + 2k\pi/n) + 1};$$

$$(ii) \frac{n \sin n\phi}{\cos n\phi - \cos n\theta} = \sum_{k=0}^{n-1} \frac{\sin \phi}{\cos \phi - \cos(\theta + 2k\pi/n)};$$

$$(iii) n \cot n\phi = \sum_{k=0}^{n-1} \cot(\phi + k\pi/n).$$

[For (i) differentiate formula (30), Chapter XV, with regard to θ ; for (ii) and (iii) differentiate the formulæ of Examples 2 and 3, § 10, Chapter XV, with regard to ϕ .]

§ 2. Partial Fractions

In finding partial fractions for a quotient of two polynomials, the methods of Chapter XV, §§ 1, 2 can sometimes be employed to obtain the factors of the denominator.

It is useful to note that if, in the quotient $f(x)/\phi(x)$, the denominator $\phi(x)$ has a simple factor $x - \alpha$, the corresponding partial fraction is $A/(x - \alpha)$, where

$$A = \frac{f(\alpha)}{\phi'(\alpha)}.$$

For, let

$$\frac{f(x)}{\phi(x)} = \frac{A}{x - \alpha} + \psi(x),$$

where $\psi(x)$ is the sum of the other partial fractions, and multiply this equation by $x - \alpha$. Then, since $\phi(\alpha) = 0$, the resulting equation may be written

$$\frac{f(x)}{\frac{\phi(x) - \phi(\alpha)}{x - \alpha}} = A + (x - \alpha)\psi(x).$$

But, from the definition of a derivative, when $x \rightarrow \alpha$,

$$\frac{\phi(x) - \phi(\alpha)}{x - \alpha} \rightarrow \phi'(\alpha).$$

Hence
$$\frac{f(\alpha)}{\phi'(\alpha)} = A,$$

which is the formula required.

Complex Functions of a Real Variable.—If $\phi(x)$ and $\psi(x)$ are real functions of the real variable x , the function $f(x) = \phi(x) + i\psi(x)$ is called a *complex function of the real variable* x .

The derivative and the indefinite integral of $f(x)$ are defined by the equations

$$D_x f(x) = D_x \phi(x) + i D_x \psi(x),$$

and
$$\int f(x) dx = \int \phi(x) dx + i \int \psi(x) dx,$$

and the definite integral with real limits a and b by the equation

$$\int_a^b f(x) dx = \int_a^b \phi(x) dx + i \int_a^b \psi(x) dx.$$

If h is a real increment of x

$$f(x+h) = \phi(x+h) + i\psi(x+h),$$

so that

$$\frac{f(x+h) - f(x)}{h} = \frac{\phi(x+h) - \phi(x)}{h} + i \frac{\psi(x+h) - \psi(x)}{h}.$$

Thus, if the derivatives $\phi'(x)$ and $\psi'(x)$ exist,

$$Df(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The rules for differentiating sums, products and quotients are just the same as for real functions.

Example 1.—Establish the formula

$$D(ax+b)^n = na(ax+b)^{n-1},$$

where a and b may be complex, (i) for n a positive integer, (ii) for n a negative integer.

Example 2.—Find partial fractions for

$$i \frac{(1+ix)^{2n+1} + (1-ix)^{2n+1}}{(1+ix)^{2n+1} - (1-ix)^{2n+1}},$$

where n is a positive integer.

It should be noted that the numerator is of lower degree than the denominator, so that it is unnecessary to divide out before proceeding to find the partial fractions.

The factors of the denominator are obtained by solving the equation

$$\left(\frac{1+ix}{1-ix}\right)^{2n+1} = 1 = \cos(2k\pi) + i \sin(2k\pi), \quad k = 0, \pm 1, \pm 2, \dots$$

This gives $\frac{1+ix}{1-ix} = \cos \frac{2k\pi}{2n+1} + i \sin \frac{2k\pi}{2n+1}$,

whence $ix = \frac{\cos \frac{2k\pi}{2n+1} + i \sin \frac{2k\pi}{2n+1} - 1}{\cos \frac{2k\pi}{2n+1} + i \sin \frac{2k\pi}{2n+1} + 1}$

$$= \frac{2i \sin \frac{k\pi}{2n+1} \left\{ \cos \frac{k\pi}{2n+1} + i \sin \frac{k\pi}{2n+1} \right\}}{2 \cos \frac{k\pi}{2n+1} \left\{ \cos \frac{k\pi}{2n+1} + i \sin \frac{k\pi}{2n+1} \right\}}$$

$$= i \tan \frac{k\pi}{2n+1}.$$

Hence

$$x = \tan \frac{k\pi}{2n+1}, \quad \text{where } k = 0, \pm 1, \pm 2, \dots, \pm n.$$

Thus

$$i \frac{(1+ix)^{2n+1} + (1-ix)^{2n+1}}{(1+ix)^{2n+1} - (1-ix)^{2n+1}} = \sum_{k=-n}^n \frac{A_k}{x - \tan \frac{k\pi}{2n+1}},$$

where

$$A_k = \left[i \frac{(1+ix)^{2n+1} + (1-ix)^{2n+1}}{D_x \{(1+ix)^{2n+1} - (1-ix)^{2n+1}\}} \right]_{x=\tan \frac{k\pi}{2n+1}}$$

$$= \left[\frac{(1+ix)^{2n+1} + (1-ix)^{2n+1}}{(2n+1)\{(1+ix)^{2n} + (1-ix)^{2n}\}} \right]_{x=\tan \frac{k\pi}{2n+1}}$$

$$= \frac{1}{(2n+1) \cos \frac{k\pi}{2n+1}}$$

$$\times \frac{\left(\cos \frac{k\pi}{2n+1} + i \sin \frac{k\pi}{2n+1} \right)^{2n+1} + \left(\cos \frac{k\pi}{2n+1} - i \sin \frac{k\pi}{2n+1} \right)^{2n+1}}{\left(\cos \frac{k\pi}{2n+1} + i \sin \frac{k\pi}{2n+1} \right)^{2n} + \left(\cos \frac{k\pi}{2n+1} - i \sin \frac{k\pi}{2n+1} \right)^{2n}}$$

$$= \frac{1}{\cos \frac{k\pi}{2n+1}} \frac{2 \cos k\pi}{(2n+1)2 \cos \frac{2nk\pi}{2n+1}}$$

$$= \frac{1}{\cos \frac{k\pi}{2n+1}} \frac{(-1)^k}{(2n+1) \cos \left(k\pi - \frac{k\pi}{2n+1}\right)} = \frac{1}{(2n+1) \cos^2 \frac{k\pi}{2n+1}}.$$

Hence

$$i \frac{(1+ix)^{2n+1} + (1-ix)^{2n+1}}{(1+ix)^{2n+1} - (1-ix)^{2n+1}}$$

$$= \sum_{k=-n}^n \frac{1}{(2n+1) \cos^2 \left(\frac{k\pi}{2n+1}\right) \left(x - \tan \frac{k\pi}{2n+1}\right)}.$$

Example 3.—Show that, if n is a positive integer,

$$\cot \theta = \sum_{k=-n}^n \frac{\cos \frac{\theta}{2n+1}}{(2n+1) \cos \frac{k\pi}{2n+1} \sin \frac{\theta - k\pi}{2n+1}}.$$

[In Example 2 put $x = \tan \frac{\theta}{2n+1}$.]

The following theorem is often found useful.

THEOREM.—If $f(a) = \phi(a) = 0$, while $\phi'(a) \neq 0$,

$$\lim_{x \rightarrow a} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}. \quad \dots \quad (K)$$

Since $f(a)$ and $\phi(a)$ are both zero,

$$\frac{f(x)}{\phi(x)} = \frac{f(x) - f(a)}{x - a} \cdot \frac{\phi(x) - \phi(a)}{x - a}.$$

Hence, from the definition of a derivative, when $x \rightarrow a$,

$$\frac{f(x)}{\phi(x)} \rightarrow \frac{f'(a)}{\phi'(a)}.$$

Example 4.—Show that

$$(i) \lim_{x \rightarrow a} \frac{x^{m+n} - a^{m+n}}{x^{r+s} - a^r x^s} = \frac{m}{r} a^{m+n-r-s};$$

$$(ii) \lim_{\theta \rightarrow \frac{1}{2}\pi} \frac{\cos 3\theta}{\cos \theta} = -3.$$

Example 5.—Show that

$$(i) \frac{n \sin n\theta}{\cos n\theta - \cos n\phi} = \sum_{r=0}^{n-1} \frac{\sin \theta}{\cos \theta - \cos \left(\phi + \frac{2r\pi}{n} \right)};$$

$$(ii) n^2 \operatorname{cosec}^2 n\phi = \sum_{r=0}^{n-1} \operatorname{cosec}^2 \left(\phi + \frac{r\pi}{n} \right).$$

From Chapter XV, § 10, Example 2, we have

$$\cos n\theta - \cos n\phi = 2^{n-1} \prod_{r=0}^{n-1} \left\{ \cos \theta - \cos \left(\phi + \frac{2r\pi}{n} \right) \right\},$$

and from formulæ (19) and (20) of Chapter XV it is evident that $\sin n\theta/\sin \theta$ is a polynomial in $\cos \theta$ of degree $n - 1$. Hence

$$\frac{n \sin n\theta}{\cos n\theta - \cos n\phi} = \sum_{r=0}^{n-1} \frac{A_r \sin \theta}{\cos \theta - \cos \left(\phi + \frac{2r\pi}{n} \right)}.$$

It follows that

$$\begin{aligned} A_r &= \mathcal{L}_{\theta \rightarrow \phi + \frac{2r\pi}{n}} \left\{ \frac{n \sin n\theta}{\sin \theta} \cdot \frac{\cos \theta - \cos \left(\phi + \frac{2r\pi}{n} \right)}{\cos n\theta - \cos n\phi} \right\} \\ &= \frac{n \sin n\phi}{\sin \left(\phi + \frac{2r\pi}{n} \right)} \mathcal{L}_{\theta \rightarrow \phi + \frac{2r\pi}{n}} \left\{ \frac{\cos \theta - \cos \left(\phi + \frac{2r\pi}{n} \right)}{\cos n\theta - \cos n\phi} \right\}. \end{aligned}$$

Now differentiate the numerator and the denominator of the fraction in the bracket; then from the theorem (K) above it follows that $A_r = 1$.

For (ii) divide both sides of (i) by $\sin \theta$, let $\theta \rightarrow 0$, and replace ϕ by 2ϕ .

Functions of a Complex Variable.—If x denotes the combination $\xi + i\eta$, where ξ and η are real variables, x is called a *complex variable*. A polynomial $f(x)$ in x is a simple type of function of a complex variable. The ratio of two such polynomials is another, for values of x for which the denominator does not vanish. The definition of a limit given in Chapter XV, § 6 holds for such functions, it being understood that δ and l are independent of the amplitude of $x - a$. The formula of Chapter XV, § 6, Example 5,

also holds; and, consequently, if the derivative be defined, as when x is real, by the formula

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

where h may be complex, the resulting formula for the derivative of a polynomial is of the same form as when x is real. The rules for differentiating sums, products and quotients of polynomials hold as before, as does also the theorem (K) on limits.

Example 6.—Find partial fractions for

$$\frac{2x^5 - 1}{x^{10} - x^5 + 1}.$$

The solutions of the equation $x^{10} - x^5 + 1 = 0$ are

$$x = \cos \alpha_r \pm i \sin \alpha_r,$$

where $\alpha_r = (6r + 1)\pi/15$, $r = 0, 1, 2, 3, 4$.

Therefore $\frac{2x^5 - 1}{x^{10} - x^5 + 1}$

$$= \sum_{r=0}^4 \left(\frac{A_r}{x - \cos \alpha_r - i \sin \alpha_r} + \frac{B_r}{x - \cos \alpha_r + i \sin \alpha_r} \right).$$

Here $A_r = \lim_{x \rightarrow z} \left\{ \frac{2x^5 - 1}{x^{10} - x^5 + 1} (x - z) \right\}$,

where $z = \cos \alpha_r + i \sin \alpha_r$, a complex number.

$$\begin{aligned} \text{Thus } A_r &= (2z^5 - 1) \lim_{x \rightarrow z} \frac{x - z}{x^{10} - x^5 + 1} \\ &= (2z^5 - 1) \frac{1}{10z^9 - 5z^4}, \text{ by (K)} \\ &= \frac{1}{5z^4} = \frac{1}{5}(\cos 4\alpha_r - i \sin 4\alpha_r). \end{aligned}$$

Similarly, $B_r = \frac{1}{5}(\cos 4\alpha_r + i \sin 4\alpha_r)$.

It follows, on adding the fractions in the bracket, that

$$\frac{2x^5 - 1}{x^{10} - x^5 + 1} = \frac{2}{5} \sum_{r=0}^4 \frac{x \cos 4\alpha_r - \cos 5\alpha_r}{x^2 - 2x \cos \alpha_r + 1}.$$

§ 3. Infinite Series

Let S_n denote the sum of the first n terms of the infinite series

$$u_1 + u_2 + u_3 + \dots + u_n + \dots$$

Then if the sequence (S_n) converges to a definite limit S as $n \rightarrow \infty$, the series is said to be *convergent*. The limit is called the *sum to infinity* or simply the *sum* of the series.

Example 1.—Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

Here $u_n = \frac{1}{n} - \frac{1}{n+1}$, and consequently $S_n = 1 - \frac{1}{n+1}$. Hence $S = 1$.

If $S_n \rightarrow +\infty$ or $-\infty$, the series is said to be *divergent*.

Example 2.—Show that the series $1 + 2 + 3 + \dots$ is divergent.

$S_n = \frac{1}{2}n(n+1)$, which tends to ∞ when $n \rightarrow \infty$.

If S_n is always finite, but does not tend to a definite limit, the series is said to *oscillate*.

Example 3.—Show that the series $1 - 1 + 1 - 1 + \dots$ is an oscillating series.

S_n has the value 1 or 0 according as n is odd or even.

Example 4.—Show that the series $\sum_{n=1}^{\infty} \cos n\theta$ oscillates if $\theta \neq 2r\pi$, where r is an integer, and diverges if $\theta = 2r\pi$.

Here

$$S_n = \cos \left(\frac{n+1}{2} \theta \right) \frac{\sin \frac{1}{2} n \theta}{\sin \frac{1}{2} \theta} = \frac{\sin (n + \frac{1}{2}) \theta}{2 \sin \frac{1}{2} \theta} - \frac{1}{2}.$$

If $\theta \neq 2r\pi$, S_n has definite values for all values of n , while $\sin (n + \frac{1}{2}) \theta$ does not tend to a definite limit as $n \rightarrow \infty$, but is never greater than 1 or less than -1 . If $\theta = 2r\pi$, the series becomes $1 + 1 + 1 + \dots$, which is divergent.

Example 5.—Show that the series $\sum_{n=1}^{\infty} \sin n\theta$ oscillates unless

$\theta = 2r\pi$, when it has the value zero.

Example 6.—For the series $1 - 2 + 3 - 4 + \dots$, S_n has the value $\frac{1}{2}(n + 1)$ if n is odd, and the value $-\frac{1}{2}n$ if n is even. Thus the sequence (S_{2n+1}) diverges to $+\infty$, and the sequence (S_{2n}) to $-\infty$. Such a series is said to *oscillate infinitely*.

§ 4. Expansions of Functions in Series

If, for all positive integral values of n , a function $f(x)$ can be expressed in the form

$$f(x) = S_n(x) + R_n(x),$$

where $S_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$,

then, when $n \rightarrow \infty$, $S_n(x)$ will tend to $f(x)$, provided that $R_n(x)$ tends to zero. ($R_n(x)$ is called the remainder after n terms.) In that case the function can be expanded in the form of a convergent infinite series

$$f(x) = u_1(x) + u_2(x) + u_3(x) + \dots = \sum_{n=1}^{\infty} u_n(x).$$

Example 1.—Show that, if $|x| < 1$,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

Since $\frac{1-y^n}{1-y} = 1 + y + y^2 + \dots + y^{n-1}$,

it follows that, if $y = -x$,

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^{n-1}x^{n-1} + R_n(x),$$

where $R_n(x) = (-1)^n \frac{x^n}{1+x}$.

But, if $-1 < x < 1$, $1/(1+x)$ has a definite value, and $x^n \rightarrow 0$ when $n \rightarrow \infty$. Hence $R_n(x)$ also tends to zero, and consequently the expansion has been established.

Example 2.—Show that, if $|x| < 1$, the series

$$a + ax + ax^2 + ax^3 + \dots$$

converges to $\frac{a}{1-x}$.

Thus

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad (3)$$

Note.—As $\cos x$ and $\sin x$ are even and odd respectively, it is natural that the powers of x on the right of (2) should be even, and that those on the right of (3) should be odd.

Two important assumptions have been made in the discussion of Maclaurin's Series, and consequently this method of obtaining formulæ (2) and (3) cannot be regarded as satisfactory. It was the method employed by Maclaurin in his *Treatise on Fluxions*, and it is clearly valid when $f(x)$ is a polynomial. It can, for instance, be used to obtain the Binomial Expansion for $(a + x)^n$ when n is a positive integer.

The following proof of Maclaurin's Expansion is completely valid, and suffices to establish the expansions (2) and (3) for all real values of x .

Maclaurin's Theorem.—If $f(x)$ and its first n derivatives are continuous for values of the argument between 0 and x ,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \dots \\ \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n(x), \quad (4)$$

where $R_n(x) = \frac{x^n}{n!} \int_0^1 n(1-t)^{n-1} f^{(n)}(xt) dt.$ (5)

For $\int_0^1 f'(xt) dt = \frac{f(x) - f(0)}{x}.$

Hence

$$f(x) = f(0) + x \int_0^1 f'(xt) dt \\ = f(0) + \frac{x}{1!} \left[- (1-t)f'(xt) \right]_0^1 + \frac{x}{1!} \int_0^1 (1-t)xf''(xt) dt \\ = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!} \int_0^1 2(1-t)f''(xt) dt.$$

If this procedure of integrating by parts is repeated $(n - 2)$ times, formula (4) is obtained.

From (4) it follows that, if $R_n(x) \rightarrow 0$ when $n \rightarrow \infty$,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots \quad (6)$$

This is *Maclaurin's Series* for $f(x)$.

Now let $f(x) = \cos x$; then

$$R_n(x) = \frac{x^n}{n!} \int_0^1 n(1-t)^{n-1} \cos(xt + \frac{1}{2}n\pi) dt,$$

and therefore, since $|\cos(xt + \frac{1}{2}n\pi)| \leq 1$,

$$|R_n(x)| \leq \left| \frac{x^n}{n!} \right| \int_0^1 n(1-t)^{n-1} dt = \left| \frac{x^n}{n!} \right|.$$

But, for all values of x , $x^n/n! \rightarrow 0$ when $n \rightarrow \infty$ (Chap. XV, § 9, Example 4). Hence the expansion (2) holds for all real values of x . Similarly it can be shown that the expansion (3) holds for all real values of x .

Alternative Forms for the Remainder.—Since $f^{(n)}(x)$ is continuous, it follows, by writing

$$R_n(x) = \frac{x^n}{(n-1)!} \int_0^1 (1-t)^{p-1} (1-t)^{n-p} f^{(n)}(tx) dt,$$

where p is any rational number such that $0 < p \leq n$, and then applying the First (Integral) Theorem of Mean Value, that

$$R_n(x) = \frac{x^n}{(n-1)!} (1-\theta)^{n-p} f^{(n)}(\theta x) \int_0^1 (1-t)^{p-1} dt, \quad 0 < \theta < 1,$$

$$\text{or} \quad R_n(x) = \frac{x^n (1-\theta)^{n-p}}{(n-1)! p} f^{(n)}(\theta x), \quad 0 < \theta < 1. \quad (7)$$

If $p = n$, this becomes *Lagrange's form of the remainder*

$$R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta x), \quad 0 < \theta < 1, \quad (8)$$

and, if $p = 1$, it becomes *Cauchy's form of the remainder*

$$R_n(x) = \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta x), \quad 0 < \theta < 1. \quad (9)$$

Thus, if (8) is used when $f(x) = \sin x$,

$$R_n(x) = \frac{x^n}{n!} \sin(\theta x + \frac{1}{2}n\pi),$$

so that
$$|R_n(x)| \leq \left| \frac{x^n}{n!} \right|,$$

which tends to zero when $n \rightarrow \infty$.

Example 1.—Show that, for all real values of a and x ,

(i) $\cos(a + x)$

$$= \cos a - \frac{x}{1!} \sin a - \frac{x^2}{2!} \cos a + \frac{x^3}{3!} \sin a + \dots,$$

(ii) $\sin(a + x)$

$$= \sin a + \frac{x}{1!} \cos a - \frac{x^2}{2!} \sin a - \frac{x^3}{3!} \cos a + \dots;$$

and deduce that

(iii) $\cos x = \cos a - \frac{x-a}{1!} \sin a - \frac{(x-a)^2}{2!} \cos a + \dots,$

(iv) $\sin x = \sin a + \frac{x-a}{1!} \cos a - \frac{(x-a)^2}{2!} \sin a - \dots$

Example 2.—Prove that

(i) $\cos^2 x = 1 - \frac{2x^2}{2!} + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \dots;$

(ii) $\sin^2 x = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \dots$

[Apply the identities

$$2 \cos^2 x = 1 + \cos 2x, \quad 2 \sin^2 x = 1 - \cos 2x.]$$

Example 3.—Calculate the percentage error in taking $1 - \frac{1}{2}\theta^2$ as an approximation to the value of $\cos \theta$ when $\theta = \frac{1}{2}\pi$. (Given that $\pi^2 = 9.8696$.)

Ans. -0.36 .

Alternative Method.—For all real values of x

$$1 - \cos x \geq 0.$$

Now assume that x is positive, and integrate from 0 to x . Since the integrand is positive or zero and x increases

throughout the range of integration the integral

$$\int_0^x (1 - \cos x) dx$$

is itself positive. Hence

$$x - \sin x > 0, \quad x > 0 \quad . \quad . \quad (10)$$

By repeated integration over the same range it can be shown that

$$\frac{x^2}{2!} - 1 + \cos x > 0, \quad x > 0, \quad . \quad . \quad (11)$$

$$\frac{x^3}{3!} - \frac{x}{1!} + \sin x > 0, \quad x > 0, \quad . \quad . \quad (12)$$

$$\frac{x^4}{4!} - \frac{x^2}{2!} + 1 - \cos x > 0, \quad x > 0, \quad . \quad (13)$$

and so on.

In this way a series of inequalities for $\cos x$ and $\sin x$ can be obtained. The general forms of these inequalities are

$$\cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots - \frac{x^{4n-2}}{(4n-2)!}, \quad x > 0, \quad (14)$$

$$\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{x^{4n}}{(4n)!}, \quad x > 0, \quad . \quad (15)$$

$$\sin x > x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots - \frac{x^{4n-1}}{(4n-1)!}, \quad x > 0, \quad (16)$$

$$\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{x^{4n+1}}{(4n+1)!}, \quad x > 0. \quad (17)$$

They can be established by induction. For instance, if it is assumed that (14) is true, then

$$\frac{x^{4n-2}}{(4n-2)!} - \frac{x^{4n-4}}{(4n-4)!} + \dots + \frac{x^2}{2!} - 1 + \cos x > 0, \quad x > 0.$$

On integrating this polynomial four times in succession over the range from 0 to x it is found that

$$\frac{x^{4n-1}}{(4n-1)!} - \frac{x^{4n-3}}{(4n-3)!} + \dots + \frac{x^3}{3!} - \frac{x}{1!} + \sin x > 0, \quad x > 0, \quad (18)$$

$$\frac{x^{4n}}{(4n)!} - \frac{x^{4n-2}}{(4n-2)!} + \dots + \frac{x^4}{4!} - \frac{x^2}{2!} + 1 - \cos x > 0, \quad x > 0, \quad (19)$$

$$\frac{x^{4n+1}}{(4n+1)!} - \frac{x^{4n-1}}{(4n-1)!} + \dots - \frac{x^3}{3!} + \frac{x}{1!} - \sin x > 0, \quad x > 0, \quad (20)$$

and

$$\frac{x^{4n+2}}{(4n+2)!} - \frac{x^{4n}}{(4n)!} + \dots - \frac{x^4}{4!} + \frac{x^2}{2!} - 1 + \cos x > 0, \quad x > 0,$$

so that

$$\cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots - \frac{x^{4n+2}}{(4n+2)!}, \quad x > 0. \quad (21)$$

Thus if (14) holds for one value of n it is true also for the value $(n + 1)$. But, from (11) it holds for $n = 1$; hence it holds for all values of n .

The other inequalities can be established in a similar manner, or deduced from (14) by means of (18), (19) and (20).

From (14) and (15) it is clear that, if $x > 0$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots - \frac{x^{4n-2}}{(4n-2)!} + R_{2n}(x),$$

where $R_{2n}(x) = \theta \frac{x^{4n}}{(4n)!}$, $0 < \theta < 1$,

and from (15) and (21) that, if $x > 0$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{x^{4n}}{(4n)!} - R_{2n+1}(x),$$

where $R_{2n+1}(x) = \theta \frac{x^{4n+2}}{(4n+2)!}$, $0 < \theta < 1$.

In each case the remainder tends to zero when $n \rightarrow \infty$
Hence

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,$$

the remainder on stopping at any term being numerically less than, and of the same sign as, the succeeding term.

In the same way it can be shown that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,$$

the same rule holding for the remainder after n terms.

If x is negative, let $x = -y$; then

$$\begin{aligned} \cos x = \cos y &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \end{aligned}$$

and

$$\begin{aligned} \sin x = -\sin y &= -\left\{y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right\} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \end{aligned}$$

so that (2) and (3) hold also for x negative. If x is zero the expansions are obviously true.

The rules stated above for the remainders hold also when x is negative.

Example 4.—If $x > 0$, show that $\cos x > 1 - x$.

Example 5.—Show that

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= 0. \\ \frac{1}{\sin x} - \frac{1}{x} &= \frac{x - \sin x}{x \sin x}, \\ &= \frac{x - (x - \theta x^3/3!)}{x \sin x}, \quad 0 < \theta < 1 \\ &= \theta \frac{x}{3!} \cdot \frac{x}{\sin x} \rightarrow 0 \text{ when } x \rightarrow 0. \end{aligned}$$

Theorem on Limits.—Let $f(x)$, $\phi(x)$ and their first $(n + 1)$ derivatives be continuous near $x = 0$. Then, if $f(x)$, $\phi(x)$

and their first $(n - 1)$ derivatives all vanish when x is zero, and if $\phi^{(n)}(0) \neq 0$,

$$\lim_{x \rightarrow 0} \frac{f(x)}{\phi(x)} = \frac{f^{(n)}(0)}{\phi^{(n)}(0)}. \quad \dots \quad (22)$$

This follows from formulæ (4) and (5).

Thus, in Example 5, if $f(x) = x - \sin x$, $\phi(x) = x \sin x$, $f(0)$, $f'(0)$, $f''(0)$, $\phi(0)$, $\phi'(0)$ all vanish, while $\phi''(0) = 2$. Therefore

$$\lim_{x \rightarrow 0} \frac{f(x)}{\phi(x)} = \frac{f''(0)}{\phi''(0)} = 0.$$

Example 6.—Show that

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{d}{dx} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \frac{1}{6}. \\ \frac{d}{dx} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \frac{\sin^2 x - x^2 \cos x}{x^2 \sin^2 x} \\ &= \frac{1 - \cos 2x - 2x^2 \cos x}{2x^2 \sin^2 x} \\ &= \left(1 - \left(1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \theta \frac{64x^6}{6!} \right) \right) / (2x^2 \sin^2 x), \quad \begin{cases} 0 < \theta < 1 \\ 0 < \theta' < 1 \end{cases} \\ &\quad - 2x^2 \left(1 - \frac{x^2}{2!} + \theta' \frac{x^4}{4!} \right) \\ &= \frac{1}{6} \frac{x^2}{\sin^2 x} + \lambda \frac{x^4}{\sin^2 x} \rightarrow \frac{1}{6} \text{ when } x \rightarrow 0, \end{aligned}$$

λ being bounded.*

Example 7.—Show that

$$\begin{aligned} \text{(i)} \quad \int_0^\infty \frac{\sin \theta}{\theta} d\theta &= \frac{\pi}{2}; \\ \text{(ii)} \quad \sum_{n=1}^\infty \frac{\sin n\theta}{n} &= \frac{\pi - \theta}{2}, \quad 0 < \theta < 2\pi. \end{aligned}$$

We have

$$\cos \theta + \cos 2\theta + \dots + \cos n\theta = \sin \left(n + \frac{1}{2} \right) \theta / (2 \sin \frac{1}{2} \theta) - \frac{1}{2}.$$

* The variable λ is said to be *bounded* if $A \leq \lambda \leq B$, where A and B are constants.

On integrating from 0 to θ , we get, if $0 < \theta < 2\pi$,

$$\sum_1^n \frac{\sin n\theta}{n} = \int_0^\theta \frac{\sin(n + \frac{1}{2})\theta}{\theta} d\theta + \int_0^\theta \sin(n + \frac{1}{2})\theta \left(\frac{1}{2 \sin \frac{1}{2}\theta} - \frac{1}{\theta} \right) d\theta - \frac{1}{2}\theta.$$

Hence, substituting ϕ for $(n + \frac{1}{2})\theta$ in the first integral, and integrating by parts in the second, we have

$$\sum_1^n \frac{\sin n\theta}{n} = \int_0^{(n+\frac{1}{2})\theta} \frac{\sin \phi}{\phi} d\phi - \frac{\cos(n + \frac{1}{2})\theta}{n + \frac{1}{2}} \left(\frac{1}{2 \sin \frac{1}{2}\theta} - \frac{1}{\theta} \right) + \frac{1}{n + \frac{1}{2}} \int_0^\theta \cos(n + \frac{1}{2})\theta \frac{d}{d\theta} \left(\frac{1}{2 \sin \frac{1}{2}\theta} - \frac{1}{\theta} \right) d\theta - \frac{1}{2}\theta. \quad (A)$$

Here put $\theta = \pi$, and let $n \rightarrow \infty$; thus

$$\lim_{n \rightarrow \infty} \int_0^{(n+\frac{1}{2})\pi} \frac{\sin \phi}{\phi} d\phi = \frac{\pi}{2},$$

from which (i) follows. To obtain (ii), let $n \rightarrow \infty$ in equation (A).

Example 8.—If m is a positive integer, prove that

$$\begin{aligned} \text{(i)} \quad & \int_0^{\frac{\pi}{2}} \frac{\sin(2m+1)x}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2m-1)x}{\sin x} dx, \\ \text{(ii)} \quad & \int_0^{\frac{\pi}{2}} \frac{\sin(2m+1)x}{\sin x} dx = \frac{\pi}{2}, \\ \text{(iii)} \quad & \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}. \end{aligned}$$

Example 9.—**The Binomial Theorem.**—In (4) let

$$f(x) = (1+x)^m,$$

where $x > -1$, m is any rational number, and that value of $(1+x)^m$ is taken which is equal to 1 when $x = 0$. Then

$$(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1)\dots(m-n+2)}{(n-1)!}x^{n-1} + R_n,$$

where $R_n = u_n \int_0^1 (1-t)^{n-1} (1+tx)^{m-nd} dt,$

and $u_n = \frac{m(m-1)\dots(m-n+1)}{(n-1)!}x^n.$

Write $R_n = u_n \int_0^1 \left(\frac{1-t}{1+tx} \right)^{n-1} (1+tx)^{m-1} dt,$

and assume that $-1 < x < 1$. Then $0 \leq (1-t)/(1+tx) \leq 1$, so that

$$0 \leq \left(\frac{1-t}{1+tx} \right)^{n-1} \leq 1.$$

Also $1 - |x| \leq 1 + tx < 2,$
and therefore

$$\begin{aligned} (1 - |x|)^{m-1} &\leq (1 + tx)^{m-1} \leq 2^{m-1}, \quad m - 1 \geq 0, \\ (1 - |x|)^{m-1} &\geq (1 + tx)^{m-1} > 2^{m-1}, \quad m - 1 < 0. \end{aligned}$$

Let M be the greater of 2^{m-1} and $(1 - |x|)^{m-1}$. Then

$$|R_n| \leq |u_n| \int_0^1 M dt = |u_n| \cdot M.$$

But (Chap. XV, § 9, Example 8), when $n \rightarrow \infty, u_n \rightarrow 0,$ and therefore $R_n \rightarrow 0.$ Hence, if $|x| < 1,$

$$(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots$$

§ 6. Gregory's Series for the Inverse Tangent

By means of the formula

$$\tan^{-1} x = \int_0^x \frac{du}{1+u^2}, \quad -\frac{1}{2}\pi < \tan^{-1} x < \frac{1}{2}\pi \quad (23)$$

it is possible to obtain a series for $\tan^{-1} x$ in powers of $x.$ For

$$\begin{aligned} \frac{1}{1+u^2} &= 1 - u^2 + u^4 - \dots + (-1)^{n-1} u^{2n-2} \\ &\quad + (-1)^n \frac{u^{2n}}{1+u^2}, \end{aligned}$$

and consequently, on integrating from 0 to $x,$ we have

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + R_n(x),$$

where $R_n(x) = (-1)^n \int_0^x \frac{u^{2n}}{1+u^2} du.$

Now, assume that x is positive; then

$$|R_n(x)| = \int_0^x \frac{u^{2n}}{1+u^2} du < \int_0^x u^{2n} du = \frac{x^{2n+1}}{2n+1}.$$

Hence, if $0 \leq x \leq 1$, $R_n(x) \rightarrow 0$ when $n \rightarrow \infty$. It follows that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots; \quad (24)$$

and the formula holds for the range

$$-1 \leq x \leq 1, \quad -\frac{1}{4}\pi \leq \tan^{-1} x \leq \frac{1}{4}\pi.$$

For, if $-1 \leq x < 0$, let $x = -y$, so that $0 < y \leq 1$. Then

$$\begin{aligned} \tan^{-1} x &= -\tan^{-1} y = -\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \dots\right) \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \end{aligned}$$

Formula (24) is *Gregory's Series* for $\tan^{-1} x$.

In particular, when $x = 1$, $\tan^{-1} x = \frac{1}{4}\pi$, and (24) becomes

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad (25)$$

§ 7. Expansions of Circular Functions of Multiple Angles

In Chapter XV, § 5, it was shown that, if n is a positive integer,

$$\cos n\theta = A_0 c^n + A_2 c^{n-2} + A_4 c^{n-4} + \dots, \quad (26)$$

$$\text{and } \sin n\theta = s(B_1 c^{n-1} + B_3 c^{n-3} + B_5 c^{n-5} + \dots), \quad (27)$$

where $c = \cos \theta$, $s = \sin \theta$, $A_0 = B_1 = 2^{n-1}$.

The other coefficients may be determined by differentiating each series twice and comparing the series so obtained with the original series. Thus, on differentiating (26), it is found that

$$\begin{aligned} -n \sin n\theta &= -s\{nA_0 c^{n-1} + (n-2)A_2 c^{n-3} \\ &\quad + (n-4)A_4 c^{n-5} + \dots\}. \quad (28) \end{aligned}$$

If, now, the sign is changed and the resulting series differentiated, it is found that

$$\begin{aligned}
 n^2 \cos n\theta &= c\{nA_0c^{n-1} + (n-2)A_2c^{n-3} \\
 &\quad + (n-4)A_4c^{n-5} + \dots\} \\
 &+ (c^2-1)\{n(n-1)A_0c^{n-2} + (n-2)(n-3)A_2c^{n-4} \\
 &\quad + (n-4)(n-5)A_4c^{n-6} + \dots\}. \quad (29)
 \end{aligned}$$

Now the expressions for $n^2 \cos n\theta$ given by (26) and (29) are equal for all values of θ , and therefore for all values of c between -1 and 1 . Hence, by a well-known theorem in algebra, the equation

$$\begin{aligned}
 n^2\{A_0c^n + A_2c^{n-2} + A_4c^{n-4} + \dots\} \\
 = \{n^2A_0c^n + (n-2)^2A_2c^{n-2} + (n-4)^2A_4c^{n-4} + \dots\} \\
 - \{n(n-1)A_0c^{n-2} + (n-2)(n-3)A_2c^{n-4} \\
 + (n-4)(n-5)A_4c^{n-6} + \dots\}
 \end{aligned}$$

obtained by equating these expressions is an identity, and consequently the coefficients of the various powers of c on each side of the equation must be equal. The identity may be written

$$\begin{aligned}
 n(n-1)A_0c^{n-2} + (n-2)(n-3)A_2c^{n-4} \\
 + (n-4)(n-5)A_4c^{n-6} + \dots \\
 + 2(2n-2)A_2c^{n-2} + 4(2n-4)A_4c^{n-4} \\
 + 6(2n-6)A_6c^{n-6} + \dots = 0,
 \end{aligned}$$

and in this form the coefficients of the different powers of c must all vanish. Thus, since $A_0 = 2^{n-1}$,

$$\begin{aligned}
 A_2 &= -\frac{n}{1}2^{n-3}, \quad A_4 = \frac{n(n-3)}{2!}2^{n-5}, \\
 A_6 &= -\frac{n(n-4)(n-5)}{3!}2^{n-7}, \dots
 \end{aligned}$$

If it is assumed that

$$A_{2r} = (-1)^r \frac{n(n-r-1)(n-r-2) \dots (n-2r+1)}{r!} 2^{n-2r-1}, \quad (30)$$

$$\begin{aligned}
 \text{then } A_{2r+2} &= -\frac{(n-2r)(n-2r-1)}{2^2(r+1)(n-r-1)} A_{2r} \\
 &= (-1)^{r+1} \frac{n(n-r-2)(n-r-3) \dots (n-2r-1)}{(r+1)!} 2^{n-2r-3};
 \end{aligned}$$

so that, if the formula (30) holds for A_{2r} , it holds also for

A_{2r+2} . But it holds when $r = 2$ or 3 ; hence it holds for all values of r .

Accordingly, if n is a positive integer,

$$\begin{aligned} 2 \cos n\theta &= (2c)^n - \frac{n}{1!} (2c)^{n-2} + \frac{n(n-3)}{2!} (2c)^{n-4} \\ &\quad - \frac{n(n-4)(n-5)}{3!} (2c)^{n-6} + \dots \\ \dots + (-1)^r \frac{n(n-r-1)(n-r-2) \dots (n-2r+1)}{r!} (2c)^{n-2r} \\ &\quad + \dots \quad \dots \quad (31) \end{aligned}$$

By comparison of (27) and (28) it follows that, if n is a positive integer,

$$\begin{aligned} \frac{\sin n\theta}{\sin \theta} &= (2c)^{n-1} - \frac{n-2}{1!} (2c)^{n-3} \\ &\quad + \frac{(n-3)(n-4)}{2!} (2c)^{n-5} - \frac{(n-4)(n-5)(n-6)}{3!} (2c)^{n-7} \\ &\quad + \dots + (-1)^r \frac{(n-r-1)(n-r-2) \dots (n-2r)}{r!} (2c)^{n-2r-1} \\ &\quad + \dots \quad \dots \quad (32) \end{aligned}$$

Example.—Show that, if n is even,

$$\begin{aligned} \text{(i)} \quad (-1)^{\frac{1}{2}n} 2 \cos n\theta &= (2s)^n - \frac{n}{1!} (2s)^{n-2} + \frac{n(n-3)}{2!} (2s)^{n-4} \\ &\quad - \frac{n(n-4)(n-5)}{3!} (2s)^{n-6} + \dots; \\ \text{(ii)} \quad (-1)^{\frac{1}{2}n-1} \frac{\sin n\theta}{\cos \theta} &= (2s)^{n-1} - \frac{n-2}{1!} (2s)^{n-3} \\ &\quad + \frac{(n-3)(n-4)}{2!} (2s)^{n-5} - \dots; \end{aligned}$$

while, if n is odd,

$$\begin{aligned} \text{(iii)} \quad (-1)^{\frac{n-1}{2}} 2 \sin n\theta &= (2s)^n - \frac{n}{1!} (2s)^{n-2} \\ &\quad + \frac{n(n-3)}{2!} (2s)^{n-4} - \frac{n(n-4)(n-5)}{3!} (2s)^{n-6} + \dots; \\ \text{(iv)} \quad (-1)^{\frac{n-1}{2}} \frac{\cos n\theta}{\cos \theta} &= (2s)^{n-1} - \frac{n-2}{1!} (2s)^{n-3} \\ &\quad + \frac{(n-3)(n-4)}{2!} (2s)^{n-5} - \dots \end{aligned}$$

EXAMPLES XVI

1. Express $\sin^8 \theta$ in terms of cosines of multiples of θ , and deduce the value of

$$\int_0^{\frac{\pi}{6}} \sin^8 \theta \, d\theta.$$

Ans. $\frac{1}{2^7}(\cos 8\theta - 8 \cos 6\theta + 28 \cos 4\theta - 56 \cos 2\theta + 35),$
 $\frac{1}{2^8}\left(\frac{35}{3}\pi - \frac{169}{8}\sqrt{3}\right).$

2. Express $\cos^4 \theta \sin^6 \theta$ in terms of cosines of multiples of θ , and hence evaluate

$$\int_0^{\frac{\pi}{4}} \cos^4 \theta \sin^6 \theta \, d\theta.$$

Ans. $\frac{1}{2^9}(6 - 2 \cos 2\theta - 8 \cos 4\theta + 3 \cos 6\theta + 2 \cos 8\theta - \cos 10\theta),$
 $\left(\frac{3}{4}\pi - \frac{4}{5}\right)/256.$

3. Show that

$$\int_0^{\frac{\pi}{3}} \sin^4 \theta \cos^6 \theta \, d\theta = (\pi + \frac{63}{80}\sqrt{3})/256.$$

4. Express $\sin^6 \theta \cos^2 \theta$ as a sum of cosines of multiples of θ , and find the value of

$$\int_0^{\frac{\pi}{8}} \sin^6 \theta \cos^2 \theta \, d\theta.$$

Ans. $2^{-7}(5 - 4 \cos 2\theta - 4 \cos 4\theta + 4 \cos 6\theta - \cos 8\theta),$
 $2^{-7}\left(\frac{5}{8}\pi - \frac{2}{3}\sqrt{2} - 1\right).$

5. Express $x^{2n}/(x^{2n+1} + 1)$ as a sum of partial fractions.

Ans. $\frac{1}{2n+1} \left\{ \frac{1}{x+1} + 2 \sum_{r=0}^{n-1} \frac{x - \cos\left(\frac{2r+1}{2n+1}\pi\right)}{x^2 - 2x \cos\left(\frac{2r+1}{2n+1}\pi\right) + 1} \right\}.$

6. If m and n are positive integers, m being less than or equal to $2n$, prove that

$$\frac{x^{m-1}}{x^{2n} + 1} = \frac{1}{n} \sum_{r=0}^{n-1} \frac{\cos(m-1)\alpha_r - x \cos m\alpha_r}{x^2 - 2x \cos \alpha_r + 1},$$

where $\alpha_r = (2r+1)\pi/(2n).$

7. Obtain the real quadratic factors of

$$x^{2n} - 2a^n x^n \cos \theta + a^{2n},$$

and deduce that

$$\frac{nx^{n-1}(x^n - a^n \cos \theta)}{x^{2n} - 2a^n x^n \cos \theta + a^{2n}} = \sum_{r=0}^{n-1} \frac{x - a \cos \frac{\theta + 2r\pi}{n}}{x^2 - 2ax \cos \frac{\theta + 2r\pi}{n} + a^2}.$$

8. If n is a positive integer, show that

$$\frac{nx^{2n-1}}{x^{2n} - 1} = \frac{x}{x^2 - 1} + \sum_{r=1}^{n-1} \frac{x - \cos \frac{r\pi}{n}}{x^2 - 2x \cos \frac{r\pi}{n} + 1}.$$

9. Express $1/(x^{2n+1} - 1)$ in partial fractions with linear denominators, and deduce that

$$\mathcal{L}_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{2n+1}{x^{2n+1}-1} \right) = n.$$

$$\text{Ans. } (2n+1)/(x^{2n+1}-1) = \frac{1}{x-1}$$

$$+ \sum_{r=1}^n \left(\frac{\cos \frac{2r\pi}{2n+1} + i \sin \frac{2r\pi}{2n+1}}{x - \cos \frac{2r\pi}{2n+1} - i \sin \frac{2r\pi}{2n+1}} \right)$$

$$+ \sum_{r=1}^n \left(\frac{\cos \frac{2r\pi}{2n+1} - i \sin \frac{2r\pi}{2n+1}}{x - \cos \frac{2r\pi}{2n+1} + i \sin \frac{2r\pi}{2n+1}} \right)$$

10. Show that

$$\frac{2nx}{(1+x)^{2n} - (1-x)^{2n}} = \sum_{r=1}^{n-1} \frac{(-1)^{r-1} \sin^2 \frac{r\pi}{2n} \left(\cos \frac{r\pi}{2n} \right)^{2n-4}}{x^2 + \tan^2 \frac{r\pi}{2n}}.$$

11. Show that

$$\mathcal{L}_{x \rightarrow a} \frac{a \cos \frac{x}{a} - x \cos \frac{a}{x}}{\frac{a}{x} \sin \frac{x}{a} - \frac{x}{a} \sin \frac{a}{x}} = a \frac{\frac{1}{2} + \tan 1}{-1 + \tan 1}.$$

12. Prove that

$$\begin{aligned} \frac{\tan 2n\theta}{\tan \theta} &= 2n \prod_{r=1}^n \frac{1 - \tan^2 \theta \cot^2 \frac{r\pi}{2n}}{1 - \tan^2 \theta \cot^2 \left(\frac{2r-1}{4n} \pi \right)} \\ &= \frac{1}{n} \sum_{r=1}^n \frac{\operatorname{cosec}^2 \left(\frac{2r-1}{4n} \pi \right)}{1 - \tan^2 \theta \cot^2 \left(\frac{2r-1}{4n} \pi \right)}. \end{aligned}$$

[From Chapter XV, (7), it is clear that

$$\tan 2n\theta = \sum_{r=1}^n \frac{A_r \tan \theta}{1 - \tan^2 \theta \cot^2 \left(\frac{2r-1}{4n} \pi \right)}.$$

Hence by formula (K), p. 285,

$$\begin{aligned} A_r &= \lim_{\theta \rightarrow \frac{2r-1}{4n}\pi} \left\{ \frac{1 - \tan^2 \theta \cot^2 \left(\frac{2r-1}{4n} \pi \right)}{\cos 2n\theta} \frac{\sin 2n\theta}{\tan \theta} \right\} \\ &= \frac{1}{n} \sec^2 \theta \cot^2 \frac{2r-1}{4n} \pi, \text{ where } \theta = \frac{2r-1}{4n} \pi \\ &= \frac{1}{n} \operatorname{cosec}^2 \left(\frac{2r-1}{4n} \pi \right). \end{aligned}$$

13. Show that

$$n \tan n\theta = \sum_{r=0}^{n-1} \frac{\sin \theta}{\cos \theta - \cos \alpha_r},$$

where $\alpha_r = (2r+1)\pi/(2n)$, $r = 0, 1, 2, \dots, n-1$.

14. Show that

$$n \cot n\theta = \cot \theta - \sum_{r=1}^{n-1} \frac{\sin \theta}{\cos \theta - \cos (r\pi/n)}.$$

15. Show that

$$\begin{aligned} \frac{2n+1}{\cos (2n+1)\theta} &= \sum_{r=0}^{n-1} \frac{(-1)^r 2 \sin \alpha_r \cos \theta}{\cos^2 \theta - \cos^2 \alpha_r} + \frac{(-1)^n}{\cos \theta} \\ &= \sum_{r=0}^{n-1} (-1)^{r-1} \{ \operatorname{cosec} (\theta - \alpha_r) - \operatorname{cosec} (\theta + \alpha_r) \} + (-1)^n \sec \theta, \end{aligned}$$

where $\alpha_r = (r + \frac{1}{2})\pi/(2n+1)$, $r = 0, 1, 2, \dots, n-1$.

16. Show that

$$n \tan n\theta = \begin{cases} \sum_{r=1}^N \frac{\sin 2\theta}{\sin^2 \alpha_r - \sin^2 \theta}, & n \text{ even,} \\ \tan \theta + \sum_{r=1}^N \frac{\sin 2\theta}{\sin^2 \alpha_r - \sin^2 \theta}, & n \text{ odd,} \end{cases}$$

where $\alpha_r = (2r - 1)\pi/2n$ and $N = \frac{1}{2}n$ or $\frac{1}{2}(n - 1)$ according as n is even or odd.

17. Prove that

$$n \cot n\theta = \begin{cases} \cot \theta - \tan \theta + \sum_{r=1}^N \frac{\sin 2\theta}{\sin^2 \theta - \sin^2 (r\pi/n)}, & n \text{ even,} \\ \cot \theta + \sum_{r=1}^N \frac{\sin 2\theta}{\sin^2 \theta - \sin^2 (r\pi/n)}, & n \text{ odd,} \end{cases}$$

where $N = \frac{1}{2}(n - 2)$ or $\frac{1}{2}(n - 1)$ according as n is even or odd.

18. Show that

$$(i) \sin (2n + 1)\theta \cdot \operatorname{cosec} \theta = (-1)^n \prod_{r=1}^n \left(1 - \cos^2 \theta \sec^2 \frac{r\pi}{2n+1} \right);$$

$$(ii) \frac{(2n + 1) \sin \theta}{\sin (2n + 1)\theta} = \sum_{r=1}^n \frac{(-1)^r \sin \frac{r\pi}{2n+1} \sin \frac{2r\pi}{2n+1}}{\cos^2 \frac{r\pi}{2n+1} - \cos^2 \theta}.$$

19. Show that

$$(i) \frac{n \sin n\phi}{\cos n\theta - \cos n\phi} = \sum_{r=0}^{n-1} \frac{\sin (\phi + 2r\pi/n)}{\cos \theta - \cos (\phi + 2r\pi/n)};$$

$$(ii) \frac{n(\sin n\theta - \sin n\phi)}{\cos n\theta - \cos n\phi} = \sum_{r=0}^{n-1} \frac{\sin \theta - \sin (\phi + 2r\pi/n)}{\cos \theta - \cos (\phi + 2r\pi/n)}.$$

20. If
$$y = \tan^{-1} \left(\frac{x \sin \alpha}{1 + x \cos \alpha} \right),$$

show that

$$\frac{d^n y}{dx^n} = (-1)^{n-1} (n - 1)! \sin n\phi \left(\frac{\sin \phi}{\sin \alpha} \right)$$

where

$$\tan \phi = \sin \alpha / (x + \cos \alpha).$$

21. If $y = x^2/(x^3 - 1)$, prove that

$$(-1)^n \frac{3}{n!} D_x^n y = \frac{1}{(x-1)^{n+1}} + \frac{2 \cos(n+1)\phi}{(x^2+x+1)^{\frac{1}{2}(n+1)}}$$

where

$$\tan \phi = \frac{\sqrt{3}}{2x+1}.$$

22. If $P = \prod_{r=1}^n \sin(x - a_r)$,

and $P_r = \prod_{s=1}^n \sin(a_r - a_s)$, $r = 1, 2, \dots, n$,

the dash denoting that the factor corresponding to $s = r$ is omitted, prove that

$$(i) \frac{\sin^{n-1} x}{P} = \sum_{r=1}^n \frac{\sin^{n-1} a_r}{P_r} \frac{1}{\sin(x - a_r)};$$

$$(ii) \frac{\sin^{n-2} x}{P} = \sum_{r=1}^n \frac{\sin^{n-2} a_r \cos(x - a_r)}{P_r \sin(x - a_r)}.$$

[For (i) divide the numerator by $\cos^{n-1} x$ and the denominator by $\cos^n x \cos a_1 \cos a_2 \dots \cos a_n$, and then find partial fractions for the resulting expression regarded as a function of $\tan x$. For (ii) apply the same procedure to $\sin^{n-2} x \cos x/P$, then multiply the result by $\cos x$ and add it to (i) multiplied by $\sin x$.]

23. If $f(x, y)$ is a homogeneous polynomial in x and y of degree $n - 1$, show that, with the notation of Example 22,

$$\frac{f(\sin x, \cos x)}{P} = \sum_{r=1}^n \frac{f(\sin a_r, \cos a_r)}{P_r \sin(x - a_r)}.$$

24. Sum to n terms, and to infinity, the series

$$\frac{1}{a(a+2)} + \frac{1}{(a+2)(a+4)} + \frac{1}{(a+4)(a+6)} + \dots$$

where a is not a negative integer.

$$\text{Ans. } S_n = \frac{1}{2} \left(\frac{1}{a} - \frac{1}{a+2n} \right), \quad S = \frac{1}{2a}.$$

25. Sum to n terms, and to infinity, the series

$$(i) \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \dots;$$

$$(ii) \frac{1}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \frac{3}{4 \cdot 5 \cdot 6} + \dots$$

$$\text{Ans. (i) } S_n = \frac{1}{4} \left\{ \frac{1}{1 \cdot 3} - \frac{1}{(2n+1)(2n+3)} \right\}, S = \frac{1}{4};$$

$$(ii) S_n = \frac{1}{4} - \frac{2n+3}{2(n+2)(n+3)}, S = \frac{1}{4}.$$

$$\left[\text{For (ii) } u_n = -\frac{1}{2} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \frac{1}{2} \left(\frac{1}{n+2} - \frac{1}{n+3} \right) \right]$$

26. Sum to n terms, and to infinity, the series

$$(i) \frac{3}{1 \cdot 2 \cdot 3} + \frac{5}{2 \cdot 3 \cdot 4} + \frac{7}{3 \cdot 4 \cdot 5} + \dots;$$

$$(ii) \frac{2}{3 \cdot 4 \cdot 5} + \frac{3}{4 \cdot 5 \cdot 6} + \frac{4}{5 \cdot 6 \cdot 7} + \dots$$

$$\text{Ans. (i) } S_n = \frac{5}{4} - \frac{1}{2n+2} - \frac{3}{2n+4}, S = \frac{5}{4};$$

$$(ii) S_n = \frac{5}{24} + \frac{1}{2n+6} - \frac{3}{2n+8}, S = \frac{5}{24}.$$

27. Show that the sum of n terms of the series

$$\frac{3}{2} - \frac{5}{8} + \frac{7}{12} - \frac{9}{20} + \dots$$

is equal to $1 - \frac{1}{n+1}$ or $1 + \frac{1}{n+1}$ according as n is even or odd, and state what kind of series it is.

Ans. Convergent.

$$\left[u_n = (-1)^{n-1} \frac{2n+1}{n(n+1)} = (-1)^{n-1} \left(\frac{1}{n} + \frac{1}{n+1} \right) \right]$$

28. Show that

$$\sum_{n=1}^{\infty} \left(\sin \frac{x}{n} - \sin \frac{x}{n+1} \right) = \sin x.$$

29. Prove that

$$(i) 4 \cos x \sin^2 x = \frac{x^2}{2!}(3^2-1) - \frac{x^4}{4!}(3^4-1) + \frac{x^6}{6!}(3^6-1) - \dots;$$

$$(ii) \left(\frac{\sin x}{x} \right)^2 = 2 \left\{ \frac{1}{2!} - \frac{(2x)^2}{4!} + \frac{(2x)^4}{6!} - \dots \right\}.$$

30. Show that, for all values of x ,

$$\frac{x^3}{3!} - \frac{2x^5}{5!} + \frac{3x^7}{7!} - \frac{4x^9}{9!} + \dots = \frac{x}{2} \left(\frac{\sin x}{x} - \cos x \right).$$

31. Prove that, if powers of x higher than x^6 be neglected, and if $|x| < 2\sqrt{5}$,

$$\sin x = \frac{x(60 - 7x^2)}{60 + 3x^2}.$$

32. Establish the following limits :

$$(i) \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2} = \frac{1}{2}(b^2 - a^2);$$

$$(ii) \lim_{x \rightarrow 0} \frac{x - \sin x}{\sin x - x \cos x} = \frac{1}{2}; \quad (iii) \lim_{x \rightarrow 0} \frac{2 \sin ax - \sin 2ax}{\sin^3 x} = a^3;$$

$$(iv) \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}; \quad (v) \lim_{x \rightarrow 0} \frac{1 - \cos mx}{1 - \cos nx} = \frac{m^2}{n^2};$$

$$(vi) \lim_{x \rightarrow 0} \frac{1 - x \tan 2x - \cos 2x}{x^2 \sin^2 x} = -\frac{10}{3}.$$

33. Show that, if $|x| < 1$,

$$\tan^{-1} \left(\frac{1-x}{1+x} \right) = \frac{\pi}{4} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \dots$$

34. Prove that, if $\frac{1}{2}\pi < \theta < \frac{3}{2}\pi$,

$$\theta = \frac{1}{2}\pi - \cot \theta + \frac{1}{3} \cot^3 \theta - \frac{1}{5} \cot^5 \theta + \dots$$

35. Show that

$$(i) \cos^3 x = \sum_{n=0}^{\infty} (-1)^n \frac{3 + 3^{2n}}{4} \frac{x^{2n}}{(2n)!};$$

$$(ii) \sin^3 x = \sum_{n=0}^{\infty} (-1)^n \frac{3 - 3^{2n+1}}{4} \frac{x^{2n+1}}{(2n+1)!}.$$

36. Expand $2 \sin \left(\frac{1}{3}\pi + \frac{1}{2}x \right) \cos \left(\frac{1}{3}\pi + \frac{1}{4}x \right)$ in ascending powers of x .

$$\text{Ans. } \frac{3}{2} + \sum_{n=1}^{\infty} (-1)^n x^{2n} / (2n)!.$$

37. B is the mid-point of an arc ABC of a circle whose centre is O, and the radius OB meets the chord AC at D. If

$$BD/DC = x < 1,$$

show that

$$\frac{\text{arc ABC}}{\text{chord AC}} = 2 \left\{ \frac{1}{2} + \frac{x^2}{3} - \frac{x^4}{3 \cdot 5} + \dots + \frac{(-1)^{n+1} x^{2n}}{(2n-1)(2n+1)} + \dots \right\}.$$

38. Show that, if n is odd,

$$\begin{aligned} \text{(i) } \sin n\theta &= ns + n(1^2 - n^2) \frac{s^3}{3!} \\ &+ n(1^2 - n^2)(3^2 - n^2) \frac{s^5}{5!} + \dots + (-1)^{\frac{n-1}{2}} 2^{n-1} s^n; \end{aligned}$$

$$\begin{aligned} \text{(ii) } \cos n\theta / \cos \theta &= 1 + (1^2 - n^2) \frac{s^2}{2!} \\ &+ (1^2 - n^2)(3^2 - n^2) \frac{s^4}{4!} + \dots + (-1)^{\frac{n-1}{2}} 2^{n-1} s^{n-1}; \end{aligned}$$

while, if n is even,

$$\begin{aligned} \text{(iii) } \sin n\theta / \cos \theta &= ns + n(2^2 - n^2) \frac{s^3}{3!} \\ &+ n(2^2 - n^2)(4^2 - n^2) \frac{s^5}{5!} + \dots + (-1)^{\frac{n-2}{2}} (2s)^{n-1}; \end{aligned}$$

$$\begin{aligned} \text{(iv) } \cos n\theta &= 1 - n^2 \frac{s^2}{2!} + n^2(n^2 - 2^2) \frac{s^4}{4!} \\ &- n^2(n^2 - 2^2)(n^2 - 4^2) \frac{s^6}{6!} + \dots + (-1)^{\frac{n}{2}} 2^{n-1} s^n. \end{aligned}$$

39. Prove that, if n is even

$$\begin{aligned} \text{(i) } (-1)^{\frac{n}{2}} \cos n\theta &= 1 - n^2 \frac{c^2}{2!} \\ &+ n^2(n^2 - 2^2) \frac{c^4}{4!} - \dots + (-1)^{\frac{n}{2}} 2^{n-1} c^n; \end{aligned}$$

$$\begin{aligned} \text{(ii) } (-1)^{\frac{n-2}{2}} \sin n\theta / \sin \theta &= nc + n(2^2 - n^2) \frac{c^3}{3!} \\ &+ n(2^2 - n^2)(4^2 - n^2) \frac{c^5}{5!} + \dots + (-1)^{\frac{n-2}{2}} (2c)^{n-1}; \end{aligned}$$

while, if n is odd,

$$\begin{aligned} \text{(iii) } (-1)^{\frac{n-1}{2}} \cos n\theta &= nc - n(n^2 - 1^2) \frac{c^3}{3!} \\ &+ n(n^2 - 1^2)(n^2 - 3^2) \frac{c^5}{5!} - \dots + (-1)^{\frac{n-1}{2}} 2^{n-1} c^n; \end{aligned}$$

$$(iv) (-1)^{\frac{n-1}{2}} \sin n\theta / \sin \theta = 1 - (n^2 - 1^2) \frac{c^2}{2!} \\ + (n^2 - 1^2)(n^2 - 3^2) \frac{c^4}{4!} - \dots + (-1)^{\frac{n-1}{2}} (2c)^{n-1}.$$

40. Show that

$$\cos^2 x + \cos^2 2x + \cos^2 3x + \dots + \cos^2 nx \\ = \frac{2n-1}{4} + \frac{\sin(2n+1)x}{4 \sin x},$$

and deduce that

$$\sin 2x + 2 \sin 4x + 3 \sin 6x + \dots + n \sin 2nx \\ = \frac{\sin(2n+1)x}{4 \sin x} \cot x - \frac{(2n+1) \cos(2n+1)x}{4 \sin x}.$$

41. Prove that

$$\cos x + \cos 3x + \cos 5x + \dots \text{ to } n \text{ terms} = \frac{\sin 2nx}{2 \sin x},$$

and deduce that

$$\sin x + 3 \sin 3x + 5 \sin 5x + \dots \text{ to } n \text{ terms} \\ = \frac{\sin 2nx \cos x - 2n \cos 2nx \sin x}{2 \sin^2 x}.$$

42. Show that the sum of the first $2n + 1$ terms of the series

$$1 - 2 \cos 2x + 2 \cos 4x - 2 \cos 6x + 2 \cos 8x - \dots$$

is $\sec x \cos(4n + 1)x$, and deduce that

$$\int_0^{\pi/2} \frac{\cos(4n+1)x}{\cos x} dx = \frac{1}{2} \pi.$$

43. Show that the fraction $\sin mx \sin nx / \sin x$, where n is a positive integer, can be expressed as a series of sines of n angles in arithmetical progression, and find the value of

$$\int_0^{\pi/4} \frac{\sin 4x \sin 3x}{\sin x} dx.$$

Ans. $\sum_{r=0}^{n-1} \sin(m - n + 1 + 2r)x, \quad 1\frac{1}{8}.$

44. If

$$S = \frac{\alpha}{n} \left(\cos \frac{\alpha}{n} + \cos \frac{2\alpha}{n} + \cos \frac{3\alpha}{n} + \dots + \cos \frac{n\alpha}{n} \right),$$

find the limit to which S tends when n tends to infinity, and interpret your result as a theorem in integration.

Ans. $\sin \alpha.$

45. The sum of the first n terms of the series

$$\sin (a+h) \cos h + \sin (a+2h) \cos 2h + \sin (a+3h) \cos 3h + \dots$$

is denoted by S_n , and the angles are measured in radians. If the product nh is a constant c , prove that, when n tends to infinity, h tends to zero, and that

$$\lim_{n \rightarrow \infty} \frac{cS_n}{n} = \frac{1}{2} \{ \sin (a+c) \sin c + c \sin a \}.$$

Show that the expression cS_n/n gives an approximation to the area under a certain curve, and test the result by integration.

46. ABC is a variable triangle inscribed in a fixed circle; show that

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

CHAPTER XVII

THE LOGARITHMIC AND THE EXPONENTIAL
FUNCTIONS : HYPERBOLIC FUNCTIONS

§ 1. The Logarithmic Function

IN this chapter the logarithmic function will be introduced in the form of a definite integral, the exponential function being then defined as the function inverse to the logarithmic function. This definition of the logarithm is practically the same as that given by Napier of Merchiston. The connection between the logarithmic function thus defined and the function as defined in books on elementary algebra will be made clear later on. As the definition of a definite integral may be based on geometrical conceptions, this procedure makes it possible to avoid some of the difficulties of defining the logarithmic and exponential functions for irrational values of their arguments. This same difficulty was avoided in connection with the circular functions, which were defined as geometrical ratios. Thus it was found possible to develop the theory of trigonometry without the necessity of basing the discussion on strictly arithmetical considerations.

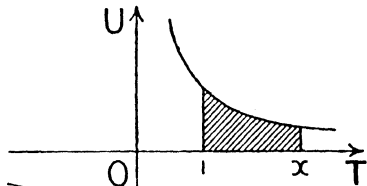


FIG. 1.

Definition.—If x is positive, the function $\log x$ is defined by the equation

$$\log x = \int_1^x \frac{dt}{t} \dots \dots \dots (1)$$

The value of $\log x$ is represented by the shaded area (Fig. 1) between the curve $u = 1/t$, the t -axis and the ordinates $t = 1$ and $t = x$.

At a later stage it will be shown that this function is identical with the logarithmic function as defined in textbooks on algebra, the base being a certain number which is denoted by e and is defined below.

The restriction $x > 0$ is necessary, as the integral has no meaning when the range of integration includes the point $t = 0$. The lower limit 1 is chosen in order that the function may vanish when $x = 1$, thus agreeing with the logarithm to any base as usually defined. Hence

$$\log 1 = 0. \quad . \quad . \quad . \quad . \quad (2)$$

If $0 < x < 1$, $\log x$ is negative.

§ 2. Properties of $\log x$

A number of important properties of the function will now be established. From the definition it is clear that $\log x$ is continuous, and increases with x . Its derivative is

$$D_x \log x = \frac{1}{x}. \quad . \quad . \quad . \quad . \quad (3)$$

Example 1.—Show that, if $ax + b > 0$,

$$D_x \log(ax + b) = \frac{a}{ax + b}.$$

Example 2.—If $f(x)$ is positive, show that

$$D_x \log \{f(x)\} = f'(x)/f(x).$$

Example 3.—If $f(x)$ is negative, show that

$$D_x \log \{-f(x)\} = f'(x)/f(x).$$

If x_1 and x_2 are any positive numbers

$$\log(x_1 x_2) = \log x_1 + \log x_2. \quad . \quad . \quad . \quad (4)$$

For

$$\begin{aligned} \log(x_1 x_2) &= \int_1^{x_1 x_2} \frac{dt}{t} = \int_1^{x_1} \frac{dt}{t} + \int_{x_1}^{x_1 x_2} \frac{dt}{t} \\ &= \log x_1 + \int_1^{x_2} \frac{dv}{v}, \quad \text{where } t = x_1 v, \\ &= \log x_1 + \log x_2. \end{aligned}$$

Again, if x is any positive number and n any rational number,

$$\log x^n = n \log x, \quad . \quad . \quad . \quad (5)$$

it being understood that, if n is fractional, the principal value of x^n is to be taken.

For

$$\begin{aligned} \log x^n &= \int_1^{x^n} \frac{dt}{t} \\ &= n \int_1^x \frac{dv}{v}, \quad \text{where } t = v^n, \\ &= n \log x. \end{aligned}$$

In particular, when $n = -1$,

$$\log \frac{1}{x} = -\log x. \quad . \quad . \quad . \quad (6)$$

From (4) and (6) it follows that, if x_1 and x_2 are positive,

$$\log \frac{x_1}{x_2} = \log x_1 - \log x_2. \quad . \quad . \quad (7)$$

When x tends to infinity the function $\log x$ also tends to infinity; that is

$$\mathcal{L}_{x \rightarrow \infty} \log x = \infty. \quad . \quad . \quad . \quad (8)$$

Take x greater than 2^n , where n is a positive integer. Then, when $n \rightarrow \infty$, x also $\rightarrow \infty$. Thus

$$\log x > \log 2^n = n \log 2.$$

But $\log 2$ is positive. Hence, when $n \rightarrow \infty$, $n \log 2 \rightarrow \infty$, and therefore x and $\log x$ tend to infinity together.

When $x \rightarrow 0$, $\log x \rightarrow -\infty$; that is

$$\mathcal{L}_{x \rightarrow 0} \log x = -\infty. \quad . \quad . \quad . \quad (9)$$

For, if $x = 1/u$, $u \rightarrow \infty$ when $x \rightarrow 0$. Hence, from (6),

$$\mathcal{L}_{x \rightarrow 0} \log x = -\mathcal{L}_{u \rightarrow \infty} \log u = -\infty.$$

Graph of log x.—The form of the graph is shown in Fig. 2. As x increases from 0 to 1, $\log x$ increases from $-\infty$ to 0,

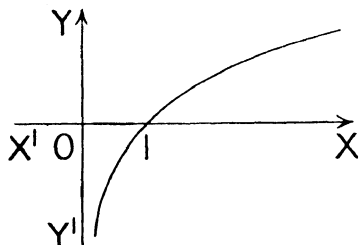


FIG. 2.

while the gradient $1/x$ decreases from $+\infty$ to 1; as x increases further from 1 to $+\infty$, $\log x$ increases from 0 to $+\infty$, while the gradient decreases from 1 to 0.

If $y = \log x$, to each value of y between $-\infty$ and $+\infty$ there corresponds one and only one value of x (between 0 and $+\infty$). Let the value of x which corresponds to $y = 1$ be denoted by e . Then

$$\log e = 1, \quad \dots \dots (10)$$

or

$$\int_1^e \frac{dt}{t} = 1.$$

From (5) it follows that, if y is any rational number,

$$\log e^y = y \log e = y.$$

Thus, if $\log x = y$, $x = e^y$, provided that y is a rational number, and that e^y takes its positive real value.

Logarithmic Differentiation.—Let

$$w = u_1 u_2 \dots u_n,$$

where the u 's are functions of x . Then

$$|w| = |u_1| \cdot |u_2| \dots |u_n|,$$

or $(\pm w) = (\pm u_1) \cdot (\pm u_2) \dots (\pm u_n),$

where in each bracket that sign is taken which makes the expression positive. Hence, on taking logarithms, we have $\log(\pm w) = \log(\pm u_1) + \log(\pm u_2) + \dots$

$$\dots + \log(\pm u_n), \quad (A)$$

the arguments of the logarithms being all positive.

Now, if u_r is positive,

$$D \log u_r = \frac{u_r'}{u_r},$$

while, if u_r is negative,

$$D \log (-u_r) = \frac{-u_r'}{-u_r} = \frac{u_r'}{u_r}.$$

Therefore, on differentiating both sides of (A), we have

$$\frac{w'}{w} = \sum_{r=1}^n \frac{u_r'}{u_r},$$

a formula which was established otherwise in Chapter XVI, § 1.

Example 4.—If

$$u_n = (1+x)(1+x^2)(1+x^4) \dots (1+x^{2^{n-1}}),$$

where $-1 < x < 1$, prove that the sequence (u_n) converges to the limit $1/(1-x)$.

Find the sum of the series

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \dots$$

$[u_n = (1-x^{2^n})/(1-x)$. If this equation is differentiated logarithmically it is found that

$$S_n = -\frac{2^n x^{2^n-1}}{1-x^{2^n}} + \frac{1}{1-x},$$

where S_n is the sum of the first n terms of the series. The result, $1/(1-x)$, is then obtained by making n tend to ∞ .]

Example 5.—Show that

$$(i) \sin x = 2^n \cos \frac{x}{2} \cos \frac{x}{2^2} \dots \cos \frac{x}{2^n} \sin \frac{x}{2^n};$$

$$(ii) \cot x = -\frac{1}{2} \tan \frac{x}{2} - \frac{1}{2^2} \tan \frac{x}{2^2} - \dots$$

$$\dots - \frac{1}{2^n} \tan \frac{x}{2^n} + \frac{1}{2^n} \cot \frac{x}{2^n}$$

$$(iii) \sum_{r=1}^{\infty} \frac{1}{2^r} \tan \frac{x}{2^r} = \frac{1}{x} - \cot x.$$

Example 6.—Show that

$$(i) \sin 3x = \sin x(1 + 2 \cos 2x);$$

$$(ii) \sin (3^n x) = \sin x \prod_{r=0}^{n-1} \left\{ 1 + 2 \cos (2 \cdot 3^r x) \right\};$$

$$(iii) \sum_{r=0}^{n-1} \frac{4 \cdot 3^r \sin(2 \cdot 3^r x)}{1 + 2 \cos(2 \cdot 3^r x)} = \cot x - 3^n \cot(3^n x).$$

§ 3. The Exponential Function

This function is the inverse of the logarithmic function ; that is, if $y = \log x$, $x = \exp(y)$ is the exponential function of y , x being that value of the argument of $\log x$ which corresponds to the value y of $\log x$. From Fig. 2 it is clear that, as y increases from $-\infty$ to $+\infty$, $\exp(y)$ increases from 0 to $+\infty$. When y is rational

$$\exp(y) = e^y.$$

In elementary text-books on algebra a^x is defined only for rational values of x . Wider definitions will be given later, but at this point e^x may be defined for all real values of x by the equation

$$e^x = \exp(x). \quad . \quad . \quad . \quad (11)$$

Thus if $x = e^y$, $y = \log x$. It follows that the logarithmic function as defined in this chapter is identical with the logarithmic function to the base e , as defined in text-books on algebra. Hence

$$\log_e x = \int_1^x \frac{dt}{t}.$$

In higher mathematical work it is always assumed that $\log x$ means $\log_e x$, unless the contrary is explicitly stated.

Note.—The function e^x as defined by (11) is a single-valued function, and is real and positive if x is real. For instance, if $x = 1/n$, where n is a positive integer, e^x is the real positive n th root of e . In particular, $e^0 = 1$.

The graph of the equation $y = e^x$ (Fig. 3) is obtained from that of $y = \log x$ (Fig. 2) by interchanging x and y .

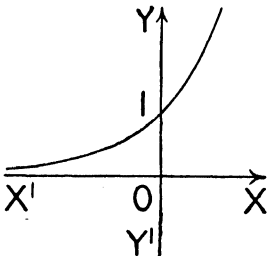


FIG. 3.

When $y = \log x$, $x = e^y$, and therefore

$$x = e^{\log x}, \quad . \quad . \quad . \quad (12)$$

while

$$\log e^y = y. \quad . \quad . \quad . \quad (13)$$

§ 4. Properties of the Exponential Function

If $y = e^x$, $x = \log y$, and therefore

$$\frac{dy}{dx} = 1 \left/ \frac{dx}{dy} = 1 \left/ \left(\frac{1}{y} \right) = y = e^x. \right.$$

Thus

$$D_x e^x = e^x. \quad . \quad . \quad . \quad (14)$$

As e^x is always positive, the gradient of the graph of e^x is always positive, and increases from 0 to 1 as x increases from $-\infty$ to 0, and from 1 to $+\infty$ as x increases from 0 to $+\infty$.

Since e^x has a definite value for any value of x , it follows from the definition of a derivative that e^x is continuous.

Example 1.—Show that $D_x e^{px} = pe^{px}$.

Again, let $y_1 = e^{x_1}$, $y_2 = e^{x_2}$, where x_1 and x_2 are any real numbers. Then $x_1 = \log y_1$, $x_2 = \log y_2$, and therefore, from (4),

$$\log (y_1 y_2) = x_1 + x_2.$$

Hence

$$y_1 y_2 = e^{x_1 + x_2},$$

or

$$e^{x_1} e^{x_2} = e^{x_1 + x_2}. \quad . \quad . \quad . \quad (15)$$

Thus the first index law holds for all real values of x , rational or irrational. It will be shown in § 5 that the law

$$(e^x)^y = e^{xy}$$

also holds for all real values of x and y .

Expansion in Series.—If $f(x) = e^x$ it follows from (14) that $f^{(n)}(x) = e^x$, so that $f^{(n)}(0) = 1$. Hence, by Maclaurin's Theorem,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!} + R_n(x),$$

where

$$R_n(x) = \frac{x^n}{n!} \int_0^1 n(1-t)^{n-1} e^{xt} dt.$$

Now $e^{xt} \leq e^{|x|}$, so that

$$\begin{aligned} |R_n(x)| &\leq \left| \frac{x^n}{n!} \right| e^{|x|} \int_0^1 n(1-t)^{n-1} dt \\ &\leq \left| \frac{x^n}{n!} \right| e^{|x|}. \end{aligned}$$

But, when $n \rightarrow \infty$, $x^n/n! \rightarrow 0$: hence $R_n(x) \rightarrow 0$, and therefore, for all real values of x ,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (16)$$

This is the *Exponential Series*.

In particular, when $x = 1$,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

Example 2.—Show that $e = 2.71828 \dots$

§ 5. The General Power

The function a^x as defined in works on elementary algebra has only one value if x is an integer, positive or negative. If x is rational but not integral, a^x has a number of values, which can be found by the method explained in Chapter XIV, § 7. If x is irrational the function has not, so far, been defined, except for the value e of a . In the following definition it is assumed that a is positive, and that the value of a^x given by the definition is itself positive.

When x is a rational fraction it may be put in the form p/q , where p and q are integers with no common factor and q is positive: then a^x has q different values, and, when a is positive, one and only one of these is positive. Denote this value by y ; then

$$y = a^{\frac{p}{q}}$$

and therefore

$$\log y = \frac{p}{q} \log a = x \log a.$$

Hence

$$y = e^{x \log a},$$

or

$$a^x = e^{x \log a}. \quad (17)$$

Now $e^{x \log a}$ has been defined as a single-valued continuous function for all real values of x , a being positive. Equation (17) is therefore employed to define the real value of a^x when x is irrational (a positive). This value of a^x is its *principal value*.

A number of properties of this function will now be given. If a is positive

$$\log a^x = x \log a \quad . \quad . \quad . \quad (18)$$

for all real values of x . This follows from (17); it is a generalisation of formulæ (5) and (13).

The Index Laws.—The index laws hold for all real values of the indices. For, if a is positive,

$$a^x \times a^y = e^{x \log a} \times e^{y \log a} = e^{(x+y) \log a},$$

by (15), and therefore

$$a^x \times a^y = a^{x+y} \quad . \quad . \quad . \quad (19)$$

Also $(a^x)^y = e^{y \log a^x} = e^{xy \log a},$

by (17) and (18), and therefore

$$(a^x)^y = a^{xy} \quad . \quad . \quad . \quad (20)$$

In particular

$$(e^x)^y = e^{xy} \quad . \quad . \quad . \quad (21)$$

Derivatives.—If a is positive

$$D_x a^x = a^x \log a \quad . \quad . \quad . \quad (22)$$

For $D_x a^x = D_x e^{x \log a} = e^{x \log a} \log a = a^x \log a.$

Again, if x is positive, the formula

$$D_x x^n = nx^{n-1} \quad . \quad . \quad . \quad (23)$$

holds for irrational as well as for rational values of n .

For $D_x x^n = D_x e^{n \log x} = \frac{n}{x} e^{n \log x} = \frac{n}{x} x^n = nx^{n-1}.$

Example 1.—Show that $Dx^x = x^x(1 + \log x).$

Example 2.—Show that

$$\lim_{x \rightarrow a} \frac{x^a - a^x}{x^x - a^a} = \frac{1 - \log a}{1 + \log a}.$$

Expansion in Series.—Since $a^x = e^{x \log a}$, it follows from (16) that

$$a^x = 1 + \frac{x \log a}{1!} + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \dots \quad (24)$$

Example 3.—Show that

$$\lim_{x \rightarrow 0} \frac{a^x - x \log(a+x) - 1}{1 - \cos x} = (\log a)^2 - \frac{2}{a}.$$

Example 4.—Show that the Binomial Expansion (Chap. XVI, § 5, Example 9) holds for all real values of the index m .

Example 5.—Prove that

$$\lim_{x \rightarrow \infty} x^n e^{-x} = 0.$$

From (16)

$$x^n e^{-x} = \frac{x^n}{e^x} = \frac{x^n}{1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots}.$$

Now let v be a positive integer greater than n ; then, if $x > 0$,

$$x^n e^{-x} < \frac{x^n}{x^v} = v! \frac{1}{x^{v-n}},$$

and this tends to zero when $x \rightarrow \infty$.

Example 6.—Prove that, if $n > 0$,

$$\lim_{x \rightarrow 0} (x^n \log x) = 0.$$

Let $x = e^{-y}$, so that, when $x \rightarrow 0$, $y \rightarrow \infty$. Then

$$x^n \log x = e^{-ny} \log e^{-y} = -y e^{-ny}.$$

But this tends to zero when $y \rightarrow \infty$; hence $x^n \log x \rightarrow 0$ when $x \rightarrow 0$.

§ 6. Some important Limits

From formula (K), p. 285, it follows that

$$\lim_{h \rightarrow 0} \frac{1}{h} \log(1+h) = 1. \quad . \quad . \quad (25)$$

Now let μ be a fixed number, and let h and n be variable quantities such that, when $h \rightarrow 0$, $nh \rightarrow \mu$ and $n \rightarrow +\infty$ or $-\infty$. Then

$$\lim_{h \rightarrow 0} n \log(1+h) = \mu. \quad (26)$$

For $n \log(1+h) = nh \cdot \frac{1}{h} \log(1+h)$, and this tends to $\mu \cdot 1$ or μ .

Again if, when $h \rightarrow 0$, $nh \rightarrow \mu$ and $n \rightarrow \pm\infty$,

$$\lim_{h \rightarrow 0} (1+h)^n = e^\mu. \quad (27)$$

For, since e^x is a continuous function of x , when $x \rightarrow x_1$, $e^x \rightarrow e^{x_1}$. Thus, when $n \log(1+h) \rightarrow \mu$, $e^{n \log(1+h)} \rightarrow e^\mu$. But by (18),

$$e^{n \log(1+h)} = e^{\log(1+h)^n} = (1+h)^n.$$

Hence the result follows.

COROLLARY.—

$$\lim_{n \rightarrow \pm\infty} \left(1 + \frac{x}{n}\right)^n = e^x. \quad (28)$$

In (27) put $h = x/n$, so that $nh = x$.

In particular

$$\lim_{n \rightarrow \pm\infty} \left(1 + \frac{1}{n}\right)^n = e. \quad (29)$$

Example 1.—Show that $\left(\cos \frac{\theta}{n}\right)^n \rightarrow 1$ when $n \rightarrow \pm\infty$.

$$\left[\left(\cos \frac{\theta}{n}\right)^n = \left(1 - 2 \sin^2 \frac{\theta}{2n}\right)^n, \text{ and, when } n \rightarrow \pm\infty, \right. \\ \left. n \times -2 \sin^2 \frac{\theta}{2n} \rightarrow 0.\right]$$

Example 2.—Show that, when $n \rightarrow \pm\infty$,

$$\left(\cos \frac{\theta}{n}\right)^{n^2} \rightarrow e^{-\frac{1}{2}\theta^2}.$$

Example 3.—Show that

- (i) $\lim_{n \rightarrow \infty} \left\{ \cos \sqrt{\left(\frac{\theta}{n}\right)} \right\}^n = e^{-\frac{1}{2}\theta}, (\theta \geq 0);$
- (ii) $\lim_{x \rightarrow 0} \{\log(e+x)\}^{\frac{1}{x}} = e^{\frac{1}{e}};$
- (iii) $\lim_{x \rightarrow 0} \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} = \sqrt{ab}, (a \text{ and } b \text{ positive}).$

§ 7. The Logarithmic Series

From the definition of the logarithmic function it is clear that, if $x > -1$,

$$\log(1+x) = \int_1^{1+x} \frac{dt}{t} = \int_0^x \frac{du}{1+u},$$

where $t = 1 + u$. Now, if $v \neq 1$ and n is a positive integer,

$$\frac{1-v^n}{1-v} = 1 + v + v^2 + \dots + v^{n-1}.$$

In this identity put $v = -u$; then, if $u \neq -1$,

$$\frac{1}{1+u} = 1 - u + u^2 - \dots + (-1)^{n-1} u^{n-1} + (-1)^n \frac{u^n}{1+u}.$$

Hence, on integrating, it is found that, if $x > -1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + R_n(x),$$

where
$$R_n(x) = (-1)^n \int_0^x \frac{u^n}{1+u} du.$$

If $x > 0$,

$$|R_n(x)| < \int_0^x u^n du = \frac{x^{n+1}}{n+1}.$$

Hence, if $0 < x \leq 1$, $R_n(x) \rightarrow 0$ when $n \rightarrow \infty$.

Again, if $-1 < x < 0$, put $x = -y$, so that $0 < y < 1$; then

$$R_n(x) = (-1)^n \int_0^{-y} \frac{u^n}{1+u} du = - \int_0^y \frac{v^n}{1-v} dv,$$

where $v = -u$.

Now, in this range of integration,

$$1 - v \geq 1 - y,$$

and therefore

$$|R_n(x)| < \frac{1}{1-y} \int_0^y v^n dv = \frac{1}{1-y} \frac{y^{n+1}}{n+1} = \frac{1}{1+x} \frac{|x|^{n+1}}{n+1}.$$

Thus $R_n(x) \rightarrow 0$ when $n \rightarrow \infty$ for $-1 < x < 0$. Hence

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \quad (30)$$

where $-1 < x \leq 1$. (For $x = 0$ both sides of the equation have the value zero.)

When $x = 1$ the formula becomes

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad (31)$$

Example 1.—Show that, if $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$,

$$\log \sec \theta = \frac{1}{2} \tan^2 \theta - \frac{1}{4} \tan^4 \theta + \frac{1}{6} \tan^6 \theta - \dots$$

Example 2.—Show that, if $0 \leq x \leq 1$ and $m > 0, n > 0$,

$$(i) \int_0^x \frac{x^{m-1} dx}{1+x^n} = \frac{x^m}{m} - \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n} - \dots;$$

$$(ii) \int_0^{\frac{\pi}{4}} (\tan \theta)^{\frac{2m-n}{n}} d\theta = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \dots$$

Example 3.—Show that

$$\sin \theta + \sin 2\theta + \dots + \sin n\theta = \frac{1}{2} \cot \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta / (2 \sin \frac{1}{2}\theta),$$

and, by integrating from θ to π , deduce that, if $0 < \theta < 2\pi$,

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \dots + (-1)^{n-1} \frac{1}{n} + \frac{\cos \theta}{1} + \frac{\cos 2\theta}{2} + \dots + \frac{\cos n\theta}{n} \\ = -\log \sin \frac{1}{2}\theta - (-1)^n / (2n + 1) \\ + \sin(n + \frac{1}{2})\theta / \{(2n + 1) \sin \frac{1}{2}\theta\} \\ - \frac{1}{2n + 1} \int_{\theta}^{\pi} \frac{\sin(n + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta} \cot \frac{1}{2}\theta d\theta. \end{aligned}$$

Hence, show that

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\log(2 \sin \frac{1}{2}\theta), \quad 0 < \theta < 2\pi.$$

§ 8. The Function $\log_a x$

If

$$x = a^y = e^{y \log a}, \quad a > 0,$$

then

$$y \log a = \log x,$$

so that

$$y = \frac{\log x}{\log a}.$$

This function is defined for all positive values of x and a , and, when y is rational, it is identical with the function $\log_a x$ —the logarithm of x to the base a —as defined in text-books on algebra. The function $\log_a x$ can therefore be defined for all positive values of x and a by the equation

$$\log_a x = \frac{\log x}{\log a}, \quad x > 0, a > 0. \quad . \quad . \quad (32)$$

Thus, if $y = \log_a x$, $x = a^y$.

From (32) it follows that

$$\log_x a = \frac{\log a}{\log x}.$$

Hence

$$\log_a x = \frac{1}{\log_x a}. \quad . \quad . \quad (33)$$

Again

$$\log_b a \log_c b = \frac{\log a}{\log b} \cdot \frac{\log b}{\log c} = \frac{\log a}{\log c},$$

and therefore

$$\log_c a = \log_b a \log_c b. \quad . \quad . \quad (34)$$

Since

$$\log_a x = \frac{\log x}{\log a} = \log x \log_a e, \quad . \quad . \quad (35)$$

it follows that

$$D_x \log_a x = \frac{1}{x \log a} = \frac{1}{x} \log_a e. \quad . \quad . \quad (36)$$

Example.—If $x > 0$, $y > 0$, $a > 0$, show that

$$\log_a (x^m y^n) = m \log_a x + n \log_a y,$$

where m and n are any real numbers.

Common Logarithms.—The logarithms to the base 10, which are employed for numerical work, are connected with the logarithms to the base e by the equations

$$\log_{10} x = \frac{\log_e x}{\log_e 10} = \log_e x \log_{10} e, \quad . \quad . \quad (37)$$

where $\log_e 10 = 2.30258 \dots$ and $\log_{10} e = 0.43429 \dots$

§ 9. The Hyperbolic Functions

It is now possible to define the functions known as the hyperbolic functions in terms of the exponential function.

The definitions are as follows :

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad . \quad . \quad . \quad (38)$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad . \quad . \quad . \quad (39)$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad . \quad . \quad (40)$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad . \quad . \quad (41)$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad . \quad . \quad (42)$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \quad . \quad . \quad (43)$$

Example 1.—Derive the following formulæ from the definitions :

- (i) $\cosh 0 = 1$;
- (ii) $\sinh 0 = 0$;
- (iii) $\cosh (-x) = \cosh x$;
- (iv) $\sinh (-x) = -\sinh x$;
- (v) $\tanh (-x) = -\tanh x$;
- (vi) $\cosh^2 x - \sinh^2 x = 1$;
- (vii) $\operatorname{sech}^2 x = 1 - \tanh^2 x$;
- (viii) $\operatorname{cosech}^2 x = \coth^2 x - 1$;
- (ix) $\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y$;
- (x) $\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y$;
- (xi) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$;
- (xii) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$;
- (xiii) $D_x \cosh x = \sinh x$;
- (xiv) $D_x \sinh x = \cosh x$;
- (xv) $D_x \tanh x = \operatorname{sech}^2 x$.

[See also Examples XVII, 78 to 81.]

The graphs of $\sinh x$, $\cosh x$ and $\tanh x$ are given in Fig. 4.

Example 2.—Draw the graphs of $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\coth x$.

The hyperbolic functions are connected with the hyperbola and the rectangular hyperbola in much the same way as the circular functions are connected with the ellipse and the circle. For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

parametric equations are

$$x = a \cos \theta, \quad y = b \sin \theta;$$

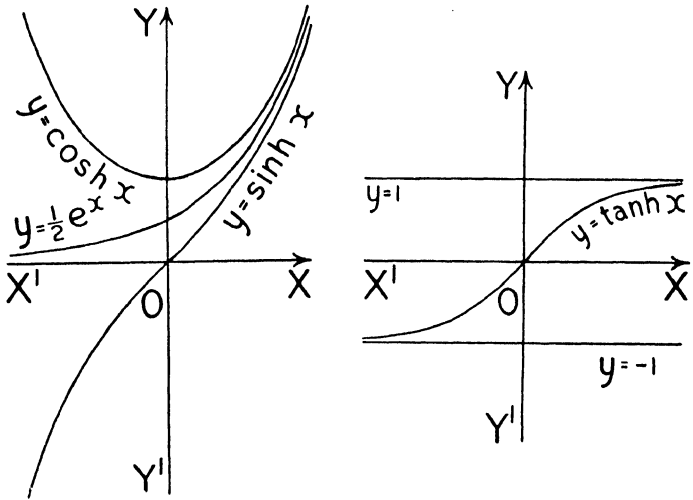


FIG. 4.

and, in particular, for the circle

$$x^2 + y^2 = a^2$$

the co-ordinates of any point can be put in the forms

$$x = a \cos \theta, \quad y = a \sin \theta.$$

Similarly, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

parametric equations are

$$x = \pm a \cosh u, \quad y = b \sinh u;$$

and, in particular, for the rectangular hyperbola

$$x^2 - y^2 = a^2$$

the co-ordinates of any point can be put in the forms

$$x = \pm a \cosh u, \quad y = a \sinh u.$$

Example 3.—If O is the origin, A and P the points $(a, 0)$ and $(a \cosh u, a \sinh u)$ respectively on the right-hand branch of the rectangular hyperbola $x^2 - y^2 = a^2$, show that the area of the sector OAP, bounded by the radii OA and OP and the arc AP, is $\frac{1}{2}a^2u$.

EXAMPLES XVII

1. Find the derivatives of

(i) $(x^2 + a^2) \log \sqrt{(x^2 + a^2) - \frac{1}{2}x^2}$;

(ii) $x^{-1} \sqrt{(x^2 - a^2)} - \log \{x + \sqrt{(x^2 - a^2)}\}$;

(iii) $\log \left\{ \frac{\sqrt{(1+x^2)} + x}{\sqrt{(1+x^2)} - x} \right\}$;

(iv) $\log \sqrt{\left(\frac{1 + \sin x}{1 - \sin x} \right)}$;

(v) $\sin^{-1}\left(\frac{x}{a}\right) - \log \left\{ \frac{a + \sqrt{(a^2 - x^2)}}{x} \right\}$;

(vi) $\log \{x + \sqrt{(x^2 - a^2)}\} - \sin^{-1}\left(\frac{a}{x}\right)$;

(vii) $\log \sqrt{(\sin x)} + \log \sqrt{(\cos x)}$.

Ans. (i) $2x \log \sqrt{(x^2 + a^2)}$; (ii) $-x^{-2} \sqrt{(x^2 - a^2)}$;

(iii) $2/\sqrt{(1+x^2)}$; (iv) $\sec x$;

(v) $x^{-1} \sqrt{\{(a+x)/(a-x)\}}$; (vi) $x^{-1} \sqrt{\{x+a\}/\{x-a\}}$;

(vii) $\cot 2x$.

2. If $y = x\{A + B \log x + \frac{1}{2}(\log x)^2\}$, where A and B are constants, prove that

$$x^2 D^2 y + y = x(1 + Dy).$$

3. If $y = A \cos(\log x) + B \sin(\log x)$, prove that

$$x^2 D^2 y + x Dy + y = 0.$$

4. If $x = y \log(xy)$, prove that

$$\frac{dy}{dx} = \frac{xy - y^2}{xy + x^2}.$$

5. Integrate with regard to x the following :

- (i) $x^2 \log x$; (ii) $x^3 (\log x)^2$; (iii) $\log(1 - x^2)$;
 (iv) $1/\sqrt{x^2 + k}$; (v) $\sqrt{x^2 + k}$.

Ans. (i) $\frac{1}{3}x^3(\log x - \frac{1}{3})$; (ii) $\frac{1}{4}x^4\{(\log x)^2 - \frac{1}{2}\log x + \frac{1}{8}\}$;
 (iii) $(1+x)\log(1+x) - (1-x)\log(1-x) - 2x$;
 (iv) $\log\{x + \sqrt{x^2 + k}\}$;
 (v) $\frac{1}{2}x\sqrt{x^2 + k} + \frac{1}{2}k \log\{x + \sqrt{x^2 + k}\}$.

6. Determine the constants A, B in the identity

$$16 \cos \theta + 11 \sin \theta \equiv A(3 \cos \theta - 2 \sin \theta) - B(3 \sin \theta + 2 \cos \theta),$$

and hence evaluate the integral

$$\int \frac{16 \cos \theta + 11 \sin \theta}{3 \cos \theta - 2 \sin \theta} d\theta.$$

Ans. $A = 2, B = -5, 2\theta - 5 \log(3 \cos \theta - 2 \sin \theta)$.

7. If $f(m, n) = \int x^m (\log x)^n dx$, show that

$$f(m, n) = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} f(m, n-1).$$

8. Evaluate the following integrals :

$$(i) \int_1^\infty \frac{dx}{x\sqrt{x^2+1}}; \quad (ii) \int_{-3}^{-2} \frac{dx}{x+1}; \quad (iii) \int_1^2 (x^2-1)^{3/2} dx.$$

Ans. (i) $\log(\sqrt{2}+1)$; (ii) $-\log 2$;
 (iii) $\frac{3}{4}\sqrt{3} + \frac{3}{8}\log(2+\sqrt{3})$.

9. Show that

$$(i) \int_0^{\frac{\pi}{2}} \log \sin x \, dx = -\frac{1}{2}\pi \log 2;$$

$$(ii) \int_0^{\pi} x \log \sin x \, dx = -\frac{1}{2}\pi^2 \log 2.$$

[For (i) apply the theorem

$$\int_0^a F(x) dx = \int_0^a F(a-x) dx;$$

then taking the mean of the two integrals we find that the given integral is equal to

$$\frac{1}{2} \int_0^{\frac{\pi}{2}} \log(\frac{1}{2} \sin 2x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{1}{2} \pi \log 2,$$

which gives the required result. For (ii) apply the same method.]

10. Prove that

$$(i) \int_0^{\frac{\pi}{2}} \log \left(\frac{1 + \sin x}{1 + \cos x} \right) dx = 0;$$

$$(ii) \int_0^{\frac{\pi}{4}} \log (1 + \tan x) dx = \frac{1}{8} \pi \log 2.$$

11. Find the length of that part of the curve

$$y = \log (\sin x + \cos x)$$

which lies between the points where $x = 0$ and $x = \frac{1}{2}\pi$. Indicate the shape of the curve between these points.

$$\text{Ans. } \log (3 + 2\sqrt{2}).$$

12. For the curve

$$x = c \sec \phi, \quad y = c \log (\sec \phi + \tan \phi)$$

prove that

$$\frac{ds}{d\phi} = c \sec^2 \phi,$$

and show that the length of the curve between the points for which $\phi = 0$ and $\phi = \frac{1}{4}\pi$ is c . Sketch this portion of the curve and show that when it rotates about the x -axis it generates surface area of amount

$$2\pi c^2 \{ \log (\sqrt{2} + 1) - \sqrt{2} + 1 \}.$$

13. Prove that

$$\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \dots \sin \frac{(n-1)\pi}{n} = n 2^{1-n},$$

and deduce that

$$\int_0^{\pi} \log \sin x \, dx = -\pi \log 2.$$

14. P is the point (α, β) , where α and β are positive, on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, A is the vertex of the hyperbola nearest to P, and $\frac{1}{2}u$ is the area of the sector OAP; show that

$$u = ab \log \left(\frac{\alpha}{a} + \frac{\beta}{b} \right).$$

15. If $y = x^2 \log (1/x)$, show that the maximum value of y is $1/(2e)$.

$$16. \text{ If } f(x) = \log (1+x) - \frac{2x}{x+2},$$

$$\text{and } \phi(x) = \log (1+x) - \frac{2x}{x+2} - \frac{x^3}{6(x+1)(x+2)},$$

show that, when x is positive, $f'(x)$ is positive and $\phi'(x)$ is negative. Deduce that, if n is positive

$$(i) \frac{2}{2n+1} < \log \left(1 + \frac{1}{n}\right) < \frac{2}{2n+1} + \frac{1}{6n(n+1)(2n+1)};$$

$$(ii) e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1 + \frac{1}{12n(n+1)}}.$$

17. If $|x| < 1$, show that

$$\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \dots = \frac{1+2x}{1+x+x^2}.$$

18. Show that

$$(i) D^n(e^x \cos x) = 2^{\frac{1}{2}n} e^x \cos \left(x + \frac{1}{4}n\pi\right);$$

$$(ii) D^n(e^x \sin x) = 2^{\frac{1}{2}n} e^x \sin \left(x + \frac{1}{4}n\pi\right);$$

and deduce that, for all values of x ,

$$(iii) e^x \cos x = \sum_{n=0}^{\infty} \frac{(x\sqrt{2})^n}{n!} \cos \left(\frac{1}{4}n\pi\right);$$

$$(iv) e^x \sin x = \sum_{n=0}^{\infty} \frac{(x\sqrt{2})^n}{n!} \sin \left(\frac{1}{4}n\pi\right).$$

19. Prove that

$$(i) D^n\{e^{x \cos \alpha} \cos(x \sin \alpha)\} = e^{x \cos \alpha} \cos(x \sin \alpha + n\alpha);$$

$$(ii) D^n\{e^{x \cos \alpha} \sin(x \sin \alpha)\} = e^{x \cos \alpha} \sin(x \sin \alpha + n\alpha);$$

and deduce that

$$(iii) e^{x \cos \alpha} \cos(x \sin \alpha) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cos n\alpha;$$

$$(iv) e^{x \cos \alpha} \sin(x \sin \alpha) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sin n\alpha.$$

20. Prove that

$$\frac{d^n}{dx^n} \left(\frac{x e^{-x}}{x^2 + a^2} \right) = (-1)^n e^{-x} \sum_{r=0}^n \frac{n! \cos(r+1)\theta (\sin \theta)^{r+1}}{(n-r)! a^{r+1}},$$

where $\sin \theta = a/\sqrt{(x^2 + a^2)}$.

21. Show that points in the Argand Diagram representing the numbers $z, z^2/a, z^3/a^2, \dots, z^n/a^{n-1}$, where

$$z = c(\cos \gamma + i \sin \gamma),$$

and a is a positive constant, all lie on the spiral curve

$$r = ae^{\theta \cot \alpha},$$

where $\gamma \cot \alpha = \log(c/a)$.

22. If $y = x^2 e^{ax}$, find the value of $D^n y$.

$$\text{Ans. } a^{n-2} e^{ax} \{a^2 x^2 + 2nax + n(n-1)\}.$$

23. If $y = e^{ax^2}$ and $D^n y = e^{ax^2} u_n(x)$, prove that

$$\begin{aligned} \text{(i) } u_{n+1} &= \frac{du_n}{dx} + 2axu_n; \\ \text{(ii) } u_{n+1} &= 2axu_n + 2nau_{n-1}; \\ \text{(iii) } \frac{d^2 u_n}{dx^2} + 2ax \frac{du_n}{dx} &= 2nau_n. \end{aligned}$$

24. If $e^{-y/x} = a + bx^{-1}$, express y as a function of x ; and prove that

$$x^3 D^2 y = (y - xDy)^2.$$

25. Find the maximum and the minimum values of

$$(x^2 - x + 1)e^x.$$

$$\text{Ans. } 3e^{-1}, 1.$$

26. Solve the equation

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = e^{3x} + x,$$

given that $y = \frac{dy}{dx} = -\frac{1}{4}$ when $x = 0$.

$$\text{Ans. } y = -\frac{1}{2}(x+3)e^{2x} + e^{3x} + \frac{1}{4}(x+1).$$

27. Solve the equation

$$\frac{d^2 y}{dx^2} - 4y = x^2 + e^{2x},$$

subject to the condition that $y = \frac{dy}{dx} = 0$ when $x = 0$.

$$\text{Ans. } y = \frac{1}{8}e^{-2x} + \frac{1}{4}xe^{2x} - \frac{1}{8}(2x^2 + 1).$$

28. If $x = e^t \cos t$, $y = e^t \sin t$, find the co-ordinates of the centre of curvature at the point whose parameter is t , and show that the evolute can be obtained by revolving the curve about the origin through a right angle.

$$\text{Ans. } (-e^t \sin t, e^t \cos t).$$

29. Prove that, if n is a positive integer,

$$D^n(x^{n-1} e^x) = (-1)^n x^{n-1} e^x.$$

30. Show that

$$\lim_{x \rightarrow a} \frac{ae^{\frac{x}{a}} - xe^{\frac{a}{x}}}{\frac{a}{x}e^{\frac{x}{a}} - \frac{x}{a}e^{\frac{a}{x}}} = -\frac{1}{4}ae^2.$$

31. Integrate

- (i) x^2e^{-2x} ; (ii) $e^{-x} \cos^2 x$; (iii) $e^{\sin x} \sin 2x$;
 (iv) $(x+1)(e^x + e^{-x})$.

- Ans. (i) $-\frac{1}{2}(x^2 + x + \frac{1}{2})e^{-2x}$;
 (ii) $\int_0^1 e^{-x} (2 \sin 2x - \cos 2x - 5)$;
 (iii) $2e^{\sin x} (\sin x - 1)$; (iv) $xe^x - (x+2)e^{-x}$.

32. Prove that

$$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{3}} e^{-x} \sin \left(x - \frac{2\pi}{3} \right) dx = \frac{1}{2} (e^{-2\pi/3} + e^{-5\pi/3}).$$

33. If $I_n = \int_0^1 e^{ax} x^n dx$, $n > 0$,

prove that $I_n = \frac{1}{a}(e^a - nI_{n-1})$,

and evaluate $\int_0^1 e^{\frac{1}{2}x} x^3 dx$.

$$\text{Ans. } 96 - 58e^{\frac{1}{2}}.$$

34. Show that

$$\int_0^{\infty} \frac{dx}{e^x + 1} = \log 2.$$

35. If $x = e^{-t} (\cos t - \sin t)$, $y = e^{-t} (\cos t + \sin t)$, show that the length of the arc measured from the point where $t = 0$ is $2(1 - e^{-t})$.

36. If $r = ae^{\theta \cot \alpha}$, calculate (i) the area bounded by the curve and the radii for which $\theta = 0$ and $\theta = \beta$; (ii) the length of the corresponding arc of the curve.

Ans. (i) $\frac{1}{4}a^2 \tan \alpha (e^{2\beta \cot \alpha} - 1)$; (ii) $a \sec \alpha (e^{\beta \cot \alpha} - 1)$.

37. Prove that, for all values of x ,

$$(1+x)e^{-x} - (1-x)e^x = \sum_{n=1}^{\infty} \frac{4n}{(2n+1)!} x^{2n+1}.$$

38. Prove that

$$\sum_{n=1}^{\infty} \frac{n^3 x^n}{n!} = x(x^2 + 3x + 1)e^x.$$

39. Show that

$$(i) 1 + \frac{2x}{1!} + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \dots = e^x(x+1);$$

$$(ii) 1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots = 2e.$$

40. Show that

$$\lim_{x \rightarrow 0} \left\{ \frac{\log(e^x - 1)}{\log x} \right\} = 1.$$

41. Differentiate

$$e^{ax} \left\{ 1 - ax + \frac{a^2 x^2}{2!} - \dots + (-1)^n \frac{a^n x^n}{n!} \right\}.$$

$$\text{Ans. } (-1)^n e^{ax} a^{n+1} x^n / n!$$

42. If $f(x) = e^a \sin^{-1} x$, show that

$$(i) (1 - x^2)f''(x) - xf'(x) - a^2 f(x) = 0;$$

$$(ii) (1 - x^2)f^{(n+2)}(x) - (2n + 1)xf^{(n+1)}(x) - (n^2 + a^2)f^{(n)}(x) = 0.$$

43. If $y = Ae^{-kx} \cos(ax + b)$, show that

$$\frac{d^{n+2}y}{dx^{n+2}} + 2k \frac{d^{n+1}y}{dx^{n+1}} + (a^2 + k^2) \frac{d^n y}{dx^n} = 0.$$

44. If $f(x) = e^x(x^2 - 6x + 12) - (x^2 + 6x + 12)$, calculate the first three derivatives of $f(x)$, and show that these derivatives are all positive when x is positive. Deduce that, when x is positive,

$$\frac{1}{1 - e^{-x}} - \frac{1}{x} - \frac{1}{2} < \frac{x}{12}.$$

To what does the expression on the left of this inequality tend when x tends to zero?

Ans. Zero.

45. Show that

$$(i) \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1; \quad (ii) \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{\sin bx} = \frac{a}{b};$$

$$(iii) \lim_{x \rightarrow c} \frac{x - c}{\log x - \log c} = c;$$

$$(iv) \lim_{x \rightarrow a} \left\{ \log \left(2 - \frac{x}{a} \right) \cdot \cot \left(\frac{\pi x}{a} \right) \right\} = -\frac{1}{\pi};$$

$$(v) \lim_{x \rightarrow \frac{1}{2}\pi} \{ \cos x \log(x \cos x) \} = 0.$$

46. If $x > 1$, show that

$$\frac{1+x}{x} < e^{\frac{1}{x}} < \frac{x}{x-1},$$

and deduce that

$$(i) \quad \mathcal{L}_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n} \right) = \log 3;$$

$$(ii) \quad \mathcal{L}_{n \rightarrow \infty} \left\{ \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) \frac{1}{\log n} \right\} = 1.$$

47. If a and b are real, show that

$$(i) \quad \text{De}^{ax} (\cos bx + i \sin bx) = (a + ib)e^{ax} (\cos bx + i \sin bx);$$

$$(ii) \quad \int e^{ax} (\cos bx + i \sin bx) dx = e^{ax} (\cos bx + i \sin bx) / (a + ib).$$

48. Show that

$$(i) \quad \mathcal{L}_{n \rightarrow \infty} \left(\frac{n^2 + n + 1}{n^2} \right)^n = e;$$

$$(ii) \quad \mathcal{L}_{n \rightarrow \infty} \frac{n^{2n}}{(1+n^2)^n} = 1;$$

$$(iii) \quad \mathcal{L}_{n \rightarrow \infty} \left\{ n^2 \log \cos \frac{a}{n} \right\} = -\frac{1}{2}a^2.$$

49. Show that

$$\mathcal{L}_{n \rightarrow \infty} \frac{1}{n} \left\{ 1 + \left(\frac{2}{1} \right)^2 + \left(\frac{3}{2} \right)^3 + \left(\frac{4}{3} \right)^4 + \dots + \left(\frac{n}{n-1} \right)^n \right\} = e.$$

[Cf. Examples XV, 90.]

50. Prove that

$$\mathcal{L}_{x \rightarrow 0} (\sec x)^{\cot^2 x} = e^{\frac{1}{2}}.$$

51. If $y = (\sin x)^{\tan x}$, ($x > 0$), show that, when $x \rightarrow 0$, $y \rightarrow 1$, and that, when $x \rightarrow \frac{1}{2}\pi$, $y \rightarrow 1$.

52. Prove that

$$(i) \quad 2 \log x - \log(x+1) - \log(x-1) \\ = \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} + \dots, \quad x > 1;$$

$$(ii) \quad x + (1-x) \log(1-x) = \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + \dots, \\ -1 \leq x < 1.$$

53. Prove that, if $-1 < x < 1$,
 $\log(1+x)^{1+x} + \log(1-x)^{1-x}$

$$= x^2 + \frac{x^4}{2 \cdot 3} + \frac{x^6}{3 \cdot 5} + \frac{x^8}{4 \cdot 7} + \dots$$

54. Prove that, if $n > 1$,

$$\log(n+1) - \log(n-1) = 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots\right),$$

and hence show that

$$\begin{aligned} \log 2 &= \frac{2}{3} + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{5}\left(\frac{2}{3}\right)^5 + \dots \\ &= 2 \cdot \left(\frac{1}{3}\right) + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \frac{2}{5}\left(\frac{1}{3}\right)^5 + \dots \end{aligned}$$

55. Show that, if $x > 0$,

$$\begin{aligned} \frac{x-1}{x+1} + \frac{x^2-1}{2(x+1)^2} + \frac{x^3-1}{3(x+1)^3} + \dots \\ = 2\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \dots\right\}. \end{aligned}$$

56. Calculate $\log_e 10$ to 4 decimal places by means of the formula

$$\log 10 = 23 \log \frac{10}{9} - 6 \log \frac{25}{24} + 10 \log \frac{81}{80}.$$

Ans. 2.3026. [The value to 5 decimal places is 2.30258.]

57. If $|x| < 1$, show that

$$2x + \frac{1}{3}x^3 + \frac{6}{5}x^5 + \frac{8}{7}x^7 + \dots = \frac{x}{1-x^2} + \log \sqrt{\left(\frac{1+x}{1-x}\right)}.$$

58. If m and n are positive, show that

$$\log \frac{m}{n} = 2\left[\left(\frac{m-n}{m+n}\right) + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \dots\right].$$

59. Find an expansion in powers of x for

$$\log(1-x+x^2),$$

where $-1 < x \leq 1$.

$$\text{Ans. } \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n} - x^n}{n}.$$

60. If $x > 0$, show that

$$\begin{aligned} \log(1+x) = \log x \\ + 2\left\{\frac{1}{1+2x} + \frac{1}{3}\left(\frac{1}{1+2x}\right)^3 + \frac{1}{5}\left(\frac{1}{1+2x}\right)^5 + \dots\right\}. \end{aligned}$$

61. Show that, if $|x| < 1$,

$$\log(1-x+x^2-x^3) = \sum_{n=1}^{\infty} \frac{(-x)^n - x^{4n}}{n}.$$

62. Prove that, if $x > 2$,

$$\log \left\{ \frac{(x-1)^2(x+2)}{(x+1)^2(x-2)} \right\} = 2 \left\{ \frac{2}{x^3-3x} + \frac{1}{3} \left(\frac{2}{x^3-3x} \right)^3 + \frac{1}{5} \left(\frac{2}{x^3-3x} \right)^5 + \dots \right\}.$$

63. If $u_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$,

prove by induction that

$$u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n},$$

and deduce that the sequence (u_n) converges to $\log 2$.

64. Show that

$$(i) \frac{4}{1 \cdot 2} - \frac{7}{2 \cdot 3} + \frac{10}{3 \cdot 4} - \frac{13}{4 \cdot 5} + \dots = 2 - \log 2;$$

$$(ii) \frac{5}{1 \cdot 2 \cdot 3} + \frac{7}{3 \cdot 4 \cdot 5} + \frac{9}{5 \cdot 6 \cdot 7} + \dots = 3 \log 2 - 1.$$

65. If $0 < x < 1$, show that

$$\log(1+x) < x < -\log(1-x),$$

and deduce that, if $m > 1$,

$$(i) \log \frac{m+1}{m} < \frac{1}{m} < \log \frac{m}{m-1};$$

$$(ii) \log \frac{m+n+1}{m} < \frac{1}{m} + \frac{1}{m+1} + \dots + \frac{1}{m+n} < \log \frac{m+n}{m-1}.$$

66. Prove that

$$\log(2n+1) > 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

[Note that

$$\log \frac{2n+3}{2n+1} = \log \left(1 + \frac{1}{2n+2} \right) - \log \left(1 - \frac{1}{2n+2} \right) > \frac{1}{n+1},$$

and apply the induction method.]

67. Show that

$$(i) \int_{x \rightarrow \infty} x^{1/x} = 1; \quad (ii) \int_{x \rightarrow +0} x^x = 1.$$

[Note.—The symbol $x \rightarrow +0$ denotes that x tends to zero through positive values of x .]

68. Prove that

$$(i) \log_a(ab) + \log_b(ab) = \log_a(ab) \log_b(ab);$$

$$(ii) \log_a x \cdot \log_b y = \log_b x \cdot \log_a y.$$

69. If $x = \log_a(bc)$, $y = \log_b(ca)$, $z = \log_c(ab)$, prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1.$$

70. Show that

$$\log_{ab} x = \frac{\log_a x \cdot \log_b x}{\log_a x + \log_b x}.$$

71. Show that

$$\lim_{x \rightarrow 1} \frac{\log_a x - \log_b x}{x^a - x^b} = \frac{\log(b/a)}{(a-b) \log a \log b}.$$

72. Show that

$$D^{4n} (\cosh x \cos x) = (-4)^n \cosh x \cos x.$$

73. If $\frac{dy}{du} = \tanh u$ and $u = \log \{x + \sqrt{(x^2 - 1)}\}$, find $\frac{dy}{dx}$ in terms of x , and deduce an expression for y in terms of x .

$$\text{Ans. } 1/x; \log x + C.$$

74. P is the point (x, y) on the curve $y = a \cosh(x/a)$, M is the projection of P on the x -axis, and Q is the projection of M on the tangent at P. Prove that the length of MQ is constant and equal to a .

If s is the length of the arc AP of the curve, where A is the point where the curve crosses the y -axis, show that

$$s = a \sinh(x/a).$$

If, further, V is the volume generated by the revolution about the x -axis of the area bounded by this arc, the ordinates at its extremities and the x -axis, show that

$$V = \frac{1}{2} \pi a(ax + sy).$$

75. If $y = \cosh x$, show that $x = \log \{y \pm \sqrt{(y^2 - 1)}\}$, according as x is positive or negative.

76. If $y = \sinh x$, show that $x = \log \{\sqrt{(y^2 + 1)} + y\}$.

77. If $y = \tanh x$, show that $x = \frac{1}{2} \log \{(1 + y)/(1 - y)\}$.

78. Show that

$$\tanh(u + v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}.$$

79. Prove that

- (i) $\sinh(u + v) + \sinh(u - v) = 2 \sinh u \cosh v$;
- (ii) $\sinh(u + v) - \sinh(u - v) = 2 \cosh u \sinh v$;
- (iii) $\cosh(u + v) + \cosh(u - v) = 2 \cosh u \cosh v$;
- (iv) $\cosh(u + v) - \cosh(u - v) = 2 \sinh u \sinh v$.

80. Prove that

- (i) $\sinh u + \sinh v = 2 \sinh \frac{1}{2}(u + v) \cosh \frac{1}{2}(u - v)$;
- (ii) $\sinh u - \sinh v = 2 \cosh \frac{1}{2}(u + v) \sinh \frac{1}{2}(u - v)$;
- (iii) $\cosh u + \cosh v = 2 \cosh \frac{1}{2}(u + v) \cosh \frac{1}{2}(u - v)$;
- (iv) $\cosh u - \cosh v = 2 \sinh \frac{1}{2}(u + v) \sinh \frac{1}{2}(u - v)$.

81. Prove that

- (i) $\sinh 2u = 2 \sinh u \cosh u$;
 (ii) $\cosh 2u = \cosh^2 u + \sinh^2 u = 2 \cosh^2 u - 1$
 $\qquad\qquad\qquad = 1 + 2 \sinh^2 u$;
 (iii) $\tanh 2u = \frac{2 \tanh u}{1 + \tanh^2 u}$.

82. Show that, if n is a positive integer,

- (i) $\cosh nx = (\cosh x)^n \{1 + {}^nC_2 \tanh^2 x + {}^nC_4 \tanh^4 x + \dots\}$;
 (ii) $\sinh nx = (\cosh x)^n \{ {}^nC_1 \tanh x + {}^nC_3 \tanh^3 x + \dots \}$;
 (iii) $\tanh nx = \frac{{}^nC_1 \tanh x + {}^nC_3 \tanh^3 x + \dots}{1 + {}^nC_2 \tanh^2 x + {}^nC_4 \tanh^4 x + \dots}$.

[For (i) and (ii) use the identities :

$$\begin{aligned} 2 \cosh nx &= (\cosh x + \sinh x)^n + (\cosh x - \sinh x)^n, \\ 2 \sinh nx &= (\cosh x + \sinh x)^n - (\cosh x - \sinh x)^n. \end{aligned}$$

83. Prove that

$$\sum_{n=0}^{\infty} \frac{\sinh x}{\cosh nx \cosh (n+1)x} = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

$$\left[\frac{\sinh x}{\cosh nx \cosh (n+1)x} = \tanh (n+1)x - \tanh nx, \text{ so that} \right.$$

$$\left. S_{n+1} = \tanh (n+1)x. \right]$$

84. Prove that

$$(i) \cosh u + \cosh (u+v) + \cosh (u+2v) + \dots + \cosh \{u + (n-1)v\}$$

$$= \cosh \left(u + \frac{n-1}{2}v \right) \frac{\sinh \frac{nv}{2}}{\sinh \frac{v}{2}} ;$$

$$(ii) \sinh u + \sinh (u+v) + \sinh (u+2v) + \dots + \sinh \{u + (n-1)v\}$$

$$= \sinh \left(u + \frac{n-1}{2}v \right) \frac{\sinh \frac{nv}{2}}{\sinh \frac{v}{2}}.$$

85. If n is an even positive integer, show that

$$(i) \cosh nu = 2^{n-1} \prod_{r=0}^{\frac{1}{2}n-1} \left\{ \sin^2 \left(\frac{2r+1}{2n} \pi \right) + \sinh^2 u \right\} ;$$

$$(ii) \sum_{r=1}^{\frac{1}{2}n} \operatorname{cosec}^4 \left(\frac{2r-1}{2n} \pi \right) = \frac{n^2(n^2+2)}{6}.$$

86. Show that

- (i) $\tanh u \operatorname{sech} 2u + \tanh 2u \operatorname{sech} 4u + \dots$
 $\quad + \tanh (2^{n-1}u) \operatorname{sech} (2^n u) = \tanh (2^n u) - \tanh u;$
- (ii) $\operatorname{sech} u \operatorname{sech} 2u + \operatorname{sech} 2u \operatorname{sech} 3u + \dots$
 $\quad + \operatorname{sech} nu \operatorname{sech} (n+1)u = \operatorname{cosech} u \{ \tanh (n+1)u - \tanh u \};$
- (iii) $\operatorname{cosech} u + \operatorname{cosech} 2u + \dots + \operatorname{cosech} (2^{n-1}u)$
 $\quad = \operatorname{coth} (\frac{1}{2}u) - \operatorname{coth} (2^{n-1}u).$

87. Given that $y_0 = 0, y_1 = 1,$

and $2y_n = y_{n-1} + y_{n-2}, \quad n = 2, 3, 4, \dots,$

show that $y_n = \frac{2}{3} \{ 1 - (-\frac{1}{2})^n \}.$

Deduce that, if $x_0 = 1, x_1 = 10,$

and $x_n = \sqrt{(x_{n-1} x_{n-2})}, \quad n = 2, 3, 4, \dots,$

$$\lim_{n \rightarrow \infty} x_n = \sqrt[3]{100}.$$

88. If all the elements of the sequence (a_n) are positive, and if $a_n/a_{n-1} \rightarrow l$ when $n \rightarrow \infty$, show that $\sqrt[n]{a_n} \rightarrow l$ when $n \rightarrow \infty$.

$$\begin{aligned} & [\log \sqrt[n]{a_n} \\ &= \frac{\log a_1 + \log (a_2/a_1) + \log (a_3/a_2) + \dots + \log (a_n/a_{n-1})}{n}. \end{aligned}$$

Hence (Examples XV, 90) $\log \sqrt[n]{a_n} \rightarrow \log l$ when $n \rightarrow \infty$.

Therefore $\sqrt[n]{a_n} = e^{\log \sqrt[n]{a_n}} \rightarrow e^{\log l} = l$, when $n \rightarrow \infty$.]

89. Show that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1,$

and deduce that

$$\lim_{n \rightarrow \infty} \frac{1}{n} (1 + \sqrt{2} + \sqrt[3]{3} + \dots + \sqrt[n]{n}) = 1.$$

If $0 < \theta < \pi$, show that

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left\{ (\sin \theta) \left(2 \sin \frac{\theta}{2} \right) \left(3 \sin \frac{\theta}{3} \right) \dots \left(n \sin \frac{\theta}{n} \right) \right\}} = \theta.$$

90. Show that

$$\lim_{n \rightarrow \infty} \frac{1+n}{\sqrt[n]{(n!)}} = e.$$

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