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Craftsmanship in the Teaching of Elementary Mathematics

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Craftsmanship in the Teaching of Elementary Mathematics

BY

F. W. WESTAWAY

Formerly one of H. M. Inspectors of Secondary Schools
Author of "Scientific Method, its Philosophical Basis and its Modes of A
"Science Teaching: What it Was—What it Is—What it Might B
"The Endless Quest: 3000 Years of Science" &c.

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" Have some wine," the March Hare said in an encouraging tone.

Alice looked round the table, but there was nothing on it but tea. " I don't see any wine," she remarked.

" There isn't any," said the March Hare.

" Then it wasn't very civil of you to offer it," said Alice angrily.

* * * * *

" You are sad," the White Knight said. " Let me sing you a song to comfort you."

" Is it very long?" Alice asked, for she had heard a good deal of poetry that day.

" It's long," said the Knight, " but it's very, very *beautiful. Everybody that hears me sing it—either it brings the* tears *into their eyes, or else——"*

" Or else what?" said Alice, for the Knight had made a pause.

" Or else it doesn't, you know.—The song *is called* ' WAYS AND MEANS ', *but that's only what it's* called, *you know!"*

" Well, what is *the song, then?" said Alice.*

" I was coming to that," the Knight said. " The song really is ' A-SITTING ON A GATE ': *and the tune's my own invention."*

By F. W. WESTAWAY

SCIENCE TEACHING

What it Was — What it Is
What it Might Be

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"Get the book and read it; it is the best thing yet. It is packed with practical advice which will always be of value."
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PREFACE

When asked to write a book on the teaching of Elementary Mathematics, I felt doubtful as to the avenue by which the subject might be best approached. During the present century, the general " policy " and " attitude " to be adopted in mathematical teaching have been discussed by so many authorities that there seemed very little new to say. Finally I decided that class-room *craftsmanship* might be made a suitable basis of treatment. Thus the book is not intended for the experienced teacher who has already acquired skill in his art, but for the still struggling beginner. In the leading schools, mathematical craftsmanship probably leaves little to be desired, but the leaven has yet to work its way into the mass.

From the great variety of topics that come within the ambit of the various mathematical subjects, I have selected for treatment those which, in my experience, seem to give young teachers most difficulty. To treat all topics that come within the daily practice of mathematical teachers is impossible; it would mean writing a dozen books rather than one.

I have sometimes been asked if, as an Inspector pursuing the same daily round year after year, decade after decade, I am, when listening to lessons in mathematics, ever amused, ever really interested, ever inclined to be severely critical, ever bored.

Amused? Yes; for instance, when a young master tells his boys that mathematics is by far the most important subject they learn, inasmuch as it is the only one that leads them into the region of " pure thought ".

Really interested? Yes, every day of my life. In the craftsmanship of even a beginner there is almost always some element of interest; in the craftsmanship of a really skilful mathematical teacher there is to me always a veritable joy. I never enter a classroom without hoping to find *something* which will make an appeal, and I am not often disappointed. Sometimes disappointed, of course; unfortunately not all mathematical teachers have come down from heaven.

Severely critical? Yes, occasionally, more especially at the rather slavish adoption of certain doubtful forms of traditional procedure. For instance, a teacher may include in the work of the bottom " Set " of a Form the Italian method of division, well knowing that two-thirds of the boys will thenceforth always get their sums wrong. Another teacher may adopt " standard form ", not because he has examined it and found it to be good, but because " everybody does it nowadays ". Instead of saying, " I thought we *had* to do these things," why do not mathematical teachers hold fast to the faith which is really in them? If their faith, their *faith*, includes the Italian method, standard form, and the score of other doubtful expedients that spread like measles from school to school, I have nothing more to say.

Bored? Yes, though not often. The petrifying stuff often doled out to Sixth Form specialists, the everlasting Series and Progressions, the old dodges and devices and bookwork "proofs" *ad nauseam* in preparation for scholarship examinations, all this is virtually the same now as forty years ago. True, teachers

are not much to blame for this. Boys *have* to be prepared for the scholarship examinations, and according to prescription. But that does not soothe an Inspector who has to listen to the same thing year after year, and I admit that, with Sixth Form work, sometimes I am almost bored to tears.

If I had to pick out those topics which in the classroom make the strongest appeal to me, I should include (i) Arithmetic to six-, seven-, and eight-year-olds, when well taught; (ii) Beginners' geometry; (iii) Upper Fourth and Lower Fifth work when the rather more advanced topics in algebra, geometry, trigonometry, and mechanics are being taken for the first time (*not* the Upper Fifth and its revision work); (iv) Upper Sixth work when examinations are over and the chief mathematical master really has a chance to show himself as a master of his craft. Sixth and Upper Fifth Form work often savours too much of the examination room to be greatly interesting: everything is excluded that does not pay. But inasmuch as examination success is a question of bread and cheese to the boy, the teacher is really on the horns of a dilemma, and very naturally he prefers to transfix himself on that horn that brings him the less pain.

Why, of all the subjects taught, is mathematics the least popular in girls' schools? and why is it the one subject in which the man in the street feels no personal interest?

It is not because mathematics is difficult to teach. My own opinion is that it is probably the easiest of all subjects to teach. When it is taught by well-qualified mathematicians, and when those mathematicians are skilled in their teaching craft, success seems always to follow as a matter of course, in girls' schools equally with boys'. The failure to make any headway, even under the best conditions, on the part of a small

proportion of boys and a rather larger proportion of girls is probably due to a natural incapacity for the subject. Had I my own way, I would debar any teacher from teaching even elementary mathematics who had not taken a strong dose of the calculus, and covered a fairly extensive field of advanced work generally. It is idle to expect a mathematical teacher to handle even elementary mathematics properly unless he has been through the mathematical mill. And yet I have heard a Headmaster say, "He can take the Lower Form mathematics all right; he is one of my *useful* men: he took a Third in History."

As long as University Scholarships are what they are, so long will Sixth Form specialists' work proceed on present lines. But one purpose of the book is to plead for consideration of the many neglected byways in mathematics and for their inclusion in a course for *all* Sixth Form boys; suggestions to this end are made in some of the later chapters. We want a far greater number of ordinary pupils to become mathematically interested, interested in such a way that the interest will be permanent; and we want them to learn to *think* mathematically, if only in a very moderate degree. Why do ordinary pupils shrivel up when they find a mathematician in their midst? It is simply that they are afraid of his cold logic.

There is, in fact, a curious popular prejudice against mathematicians as a class. It probably arises from the fact that we are not a nation of clear thinkers, and we dislike the few amongst us who are. Foreigners—at least the French, the Germans, and the Italians—are mathematically much keener than we are. They seem to become immediately interested in a topic with mathematical associations, whereas we turn away from it, disinclined to take part in a discussion demanding

rigorous logical reasoning. Competent observers agree that this is in no small measure traceable to the fact that our school mathematics has not been of a type to leave on the minds of ordinary pupils impressions of permanent interest.

We have driven Euclid out of Britain, but we must all admit that he stood as a model of honest thinking, and we miss him sadly. Were he to come back, frankly admitting his failings and promising reform, not a few mathematicians would give him a warm welcome. It is only a very few years since I heard my last lesson in Euclid, and that, curiously enough, was at a Preparatory School. It was a pleasure to hear those 12-year-old boys promptly naming their authority (e.g. I, 32; III, 21; II, 11) for every statement they made, and it was exceedingly difficult to improvise the necessary frown of disapproval. Of course those small boys did not understand much of what they were doing, and of actual geometry they knew little. But in spite of this they were learning to think logically, and to produce good authority for every assertion they made. Our modern ways are doubtless better than the old ways, but when we emptied the bath, why did we throw out the baby?

The very last thing I desire to do is to impose on teachers my ideas of methods. Anything of the nature of a standardized method in English schools is unthinkable. The Board of Education, as I knew it, never issued decrees in matters affecting the faith and doctrines of our educational system; it confined itself to making suggestions. Admittedly, however, democracy has now come to stay, and its unfortunate though inevitable tendency to standardize everything it controls may ultimately prove disastrous to all originality in teaching methods, and reduce the past high average of initiative and of intellectual

independence in schools. Let every teacher strive to base his methods on a venturous originality. Let him resist to the death all attempts of all bureaucrats to loosen the bonds of obligation to his art, or to mar his craftsmanship.

But though I plead for originality I desire to utter a warning against a too ready acceptance of any new system or method that comes along, especially if it is astutely advertised. It is perhaps one of our national weaknesses to swallow a nostrum too readily, whether it be a new patent medicine or a new method of teaching. What good reason have we for thinking that a teacher of 1931 is a more effective teacher than one of 1881, or for that matter of 2000 years ago? What is there in method, or in personal intelligence, that can give us any claim to be better teachers, better *teachers*, than were our forefathers? When a new method is announced, especially if it be announced with trumpets and shawms, write to the nearest Professor of Education, and more likely than not he will be able to give you the exact position of the old tomb which has been recently ransacked.

A method which is outlined in a lecture or in a book is only the shadow of its real self. A method is not a piece of statuary, finished and unalterable, but is an ever-changing thing, varying with the genius of the particular teacher who handles it. It is doubtful wisdom to try to draw a sharp antithesis between good methods and bad: the relative values of abstractions are invariably difficult to assess. The true antithesis is between effective and ineffective teaching. The method itself counts for something, but what counts for very much more is the life that the craftsman when actually at work breathes into it.

The regular working of mathematical exercises is essential,

for the sake not only of the examination day that looms ahead but also for illuminating ideas and impressing these on the mind. Nevertheless, the working of exercises tends to dominate our work far too much, and to consume time that might far more profitably be devoted both to the tilling of now neglected ground of great interest and to the serious teaching of the most fundamental of all mathematical notions, namely, those of number, function, duality, continuity, homogeneity, periodicity, limits, and so forth. If boys leave school without a clear grasp of such fundamental notions, can we claim that their mathematical training has been more than a thing of shreds and patches? I plead for a more adequate treatment of these things.

The terms " Forms " and " Sets " I have used in accordance with their current meaning. The average age of each of the various sections of Form II, III, IV, and V is considered to be 12+, 13+, 14+, and 15+, respectively, the units figure of the age representing the Form: this is sufficiently accurate for all practical purposes. The age range of Form VI is taken to be 16+ to 18+. " Sets " represent the redistributed mathematical groups within any particular Form; for instance, 100 boys in the different sections of Form IV might be redistributed into 4 Sets, α, β, γ, and δ. Admittedly it is in the lower Sets where skilled craftsmanship is most necessary.

On reading through the manuscript I find that I have sometimes addressed the teacher, sometimes the boy, rather colloquially and without much discrimination. I crave the indulgence of my readers accordingly.

All teachers of mathematics should belong to the Mathematical Association. They will then be able to fraternize

periodically with the best-known and most successful of their fellow-workers. The *Mathematical Gazette* will provide them with a constant succession of lucidly written practical articles, of hints and tips, written by teachers known, by reputation if not personally, to everybody really interested in mathematical education; also with authoritative reviews of new mathematical books. Members may borrow books from the Association's Library, and the help and advice of specialists are always to be had for the asking.

In writing the book my own views on numerous points have been checked by constant reference to Professor Sir Percy Nunn's *Teaching of Algebra* and its two companion volumes of Exercises, books I have not hesitated to consult and to quote from, in several chapters. I am old enough to remember the great reputation Professor Nunn enjoyed as a gifted teacher of mathematics when he was an assistant master thirty years ago. The methods he advocates are methods which have been amply tested and found to be sound and practical. His book deals with algebra in the broadest sense and gives advice on the teaching of trigonometry, the calculus, and numerous other subjects.

F. W. W.

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CHAPTER I

Teachers and Methods

Mathematical Knowledge

"That man is one of the finest mathematicians I have ever had on my Staff, but as a teacher he is no good at all."

"Oh yes, he can teach all right. He can hold a class of 40 boys in perfect order for an hour. The trouble is that his knowledge of mathematics is so superficial."

These criticisms of Head Masters are not altogether uncommon. A Head Master naturally looks for two things in members of his Staff: sound knowledge of a subject, and skill in teaching that subject.

Suppose that a teacher has spent three or four years at the University, has obtained the coveted First in mathematics, and has then spent a year in the Training College Department of a University or of a University College. Can he then claim to be a competent mathematician and a skilful teacher? The answer is *no*.

The knowledge of mathematics obtained in a four years' University course forms an admirable foundation on which to build, but how much mathematics can be learnt in so short a period as four years? At the end of that time it is a common thing for University students hardly to have touched the serious mathematics of physics, of chemistry, of engineering, of machinery and structures, of aviation, of astronomy, of biology, of statistics, or to have mastered more than the barest elements of the philosophy of the subject. Assuming that it takes a year to acquire an elementary knowledge of each of the applied

subjects just named, the newly-fledged graduate has still in front of him a long spell of hard work before he can claim to be a mathematician in the broader sense of the term. As for the philosophy of mathematics, he might still be a babe in the subject after five years' hard reading. Mathematics touches life at so many points that the all-round training of a mathematician is a very long business. If by the age of 35 a mathematician has acquired a fair general knowledge of his subject, he has done well. Consider the mathematics of physics alone: what a vast field! The field is, of course, ploughed up and sown by the teachers of physics, with the result that there is always a rich harvest for the mathematical staff to reap. *Is that reaping always done?*

Skill in Teaching: Training

Then as to teaching. How can a mathematical Training College student claim to be an efficient teacher at the end of his training year? Skill in any art can be acquired only by much practice, and the art of teaching is a particularly difficult art. Although mathematics is one of the ~~easiest, perhaps~~ *the* easiest, of all subjects to teach, it is a rare thing for a mathematical teacher to be able to feel at all satisfied with his professional skill before the age of 30. He is lucky if other people adjudge him efficient before the age of 35. Every mathematical subject is full of teaching problems. Every one of these problems can be solved in a variety of ways. Every one of these ways is worth testing. And all this takes a long time. As for lucidity of presentation—a prime necessity in all mathematical teaching—that is in itself an art which half a life-time does not seem long enough to perfect.

I have often been asked by mathematical teachers who have not been formally trained what good they would have done by going to a Training College. They are inclined to argue that very few of the Training Colleges have on their Staffs a front-rank mathematician, and that therefore such Training Colleges are not in a position to deal with the subject effec-

tively; that even if the Training Colleges are able to impress into their service members of the mathematical staff of the local University, little real help is obtainable, for " although a University Professor can teach me mathematics, he cannot teach me how to teach boys mathematics ". The general contention is not without its points, but the strictures certainly do not apply to *all* Training Colleges, as trained mathematical teachers who have been through the hands of, say, Professor Sir Percy Nunn are the first to admit.

The further criticism that the time spent on Psychology at the Training Colleges is " absolutely wasted ", since " it has no practical value in the solution of actual classroom problems ", is, perhaps, rather more justified. It is possibly true that the almost useless introspective psychology of half a century ago still hangs about some of the Training Colleges; I do not know. But young teachers should make themselves acquainted with the valuable experimental work which is now being done by psychologists all over the world. These experiments are often based on masses of actual data derived from the classroom. It is true that the definite results obtained so far are rather patchy; a complete body of psychological doctrine has yet to be built up into something that may claim to be " science ". But no teacher can afford to ignore the work that has been done and is being done, if only because such a large part of it has a very close bearing on present-day school practice.

One broad distinction between the outlook of a teacher who has been through a Training College and that of one who has not is that the trained teacher has usually had knocked out of him the prejudice which he (very naturally) felt for his own special subject, whether mathematics, classics, or what not. He has learnt that in the Common Room he will become a member of a community representing all the subjects of the curriculum, and that " all these subjects, not his own subject alone, answer to deep-seated needs of the human spirit, all of them essential currents in the great stream of movement called civilization ". The Training College does

its best, of course, to turn out competent craftsmen, but it does much more. It leads its students to understand the real meaning of education and something of its significance in relation to the many-sided business of life. It shows them how much wider education is than mere teaching. Moreover, the students are day by day in contact with men who have reflected deeply upon both education in the broader sense and teaching in the technical sense.

In a good Training College, the embryo mathematical teacher is taught not only how to convey to his pupils a knowledge of arithmetic, algebra, geometry, and the rest, but also how to make himself, in the greatest measure possible, an active intellectual adventurer in the realms of number and space, how to follow up the labours of the great masters of mathematical thought, and how to catch something of their spirit and outlook. He is encouraged to question accepted mathematical values, and to inquire, in a critical spirit, what parts of the traditional curriculum are really vital and what parts have only a conventional value. He is made aware that many of the textbooks contain a considerable amount of useless lumber, and he is taught how to discriminate between methods that are sound and methods that are otherwise.

Then again, in a good Training College the student is able to obtain expert advice on every kind of difficulty he may meet with in his teaching practice. At what stage, for example, should " intuition " work in geometry give way to rigorous proof? How can the best approach be made in the teaching of ratio and proportion? How is the theory of parallels to be treated? The Training College may have to tell him frankly that in his present state of pupilage he is probably not yet fitted to deal with the theory of parallels except in an empirical way, since the inherent difficulties of the theory can only be grappled with after a prolonged and careful study of the researches of modern geometry; only then will he be in a position to disentangle logic from intuition, and so be able to devise a treatment suitable for Sixth Form boys. Briefly, the Training College will point out quicksands, as well as firm rock, for in

mathematical teaching quicksands abound. A teacher who is not trained will often not only get into the quicksands but take his boys with him.

It may be said that the great majority of Secondary School teachers, especially teachers in the Conference * schools, are not trained. That is true, and the result has often been that such teachers have bought their first few years' experience at the expense of their boys. But not always. If a young man straight down from the University goes as a Student Teacher, that is as an observer and learner, in a large school where mathematics is known to be well taught, and if the Head of the mathematical department has sufficient leisure to guide him in his reading and to act as his professional friend, the year's training may not be inferior to that at a Training College. Some authorities urge that it may be superior, inasmuch as the Student Teacher spends the year with real practitioners rather than with theoreticians. I do not attempt to decide this question.

Still another alternative—a very common one—is for the embryo teacher, just down from the University, to join the Staff of a big school, to admit freely that he is a neophyte, and to beg for all the help and criticism he can obtain from his mathematical colleagues. Let him invite his seniors to come to hear him teach, and to criticize him, and let him beg the privilege of being present occasionally at their lessons. Let him seek their advice as to a suitable course of reading. But let him not think that he is a teacher sent straight from heaven to rectify the views and methods of the old fossils of 25 and upwards he may find in the Common Room: that way lies trouble.

One thing that the untrained beginner should never do is to join the Staff of a small school where he has to undertake the responsibility of the whole of the mathematical teaching. In the first place it is not fair to the school; the boys are sent there to be *taught*. In the second place it is not fair to himself,

* The term "Public" school is now ambiguous, and is better not used. All grades of schools are either "public" or "private".

for how is he to learn his job? Why not open an office, and set up as a Consulting Civil Engineer instead? He is just as competent to do the one thing as the other.

In the old days, all would-be craftsmen—joiners, bricklayers, mechanics, and others—served a seven years' apprenticeship, and they really learned their business; they became skilled craftsmen. The system is dead, their work being done mostly by machinery. But we cannot *teach* by machinery—yet; and skilled craftsmanship in teaching can be acquired only by a great deal of practice.

Conventional Practice

How is the value of a lesson in mathematics to be assessed? I do not refer to the ordinary things in which every teacher with a year's experience ought to be reaching efficiency—class-management, discipline, use of the blackboard, expertness in questioning and in dealing with answers, and so forth—but to the lesson as a mathematical lesson. The commonest fault of the young mathematical teacher is that he talks too much; he lectures, and, if he is teaching the Sixth Form, he often uses his University notes. It takes some young teachers a long time to learn the great lesson that the thing that matters most is not what they give out but what the boys take in; that their work is teaching, not preaching.

Another common fault of young mathematical teachers, and not all experienced mathematical teachers are guiltless of it, is the adoption of a particular method because it is mathematically neat, the sort of method that appeals to a mathematician *as* a mathematician, not the method that is the most suitable for demonstrating a particular principle or teaching a particular rule so that the child can understand it. If a mathematical teacher thinks that a mathematician is listening to him, he is more often than not keener to reveal his knowledge of mathematics than to exhibit his teaching power. But the observer's object is usually to discover what the boys are *learning*, and to assess the value of the *teaching*; and the

very " neatness " of the method adopted is quite likely to be the cause of the boys learning next to nothing. To be effective, a method must be simple and be clear to the boys. Mathematical rigour may thus have to be sacrificed, though the rigour then sacrificed will come later.

Again, young mathematical teachers are apt to be hidebound by conventions. Generation after generation of boys are told, for example, that, in an expression like $m + n \times p$, the multiplication sign should take precedence over the addition sign. *Why should it?* Doubtless the original suggestion is hidden away in some old textbook, but it has been consistently adopted by modern writers as if it were something sacrosanct. Surely if the signs are not to be taken in their natural order from left to right, it is the business of the person who frames the question to insert the necessary brackets, and not leave the wretched little learner to do it.

Let the Head of a mathematical department in a big school remember that the place of honour for himself or for any exceptionally gifted member of his Staff is in the lower Forms. The beginners' geometry is, more than any other mathematical work, in need of skilful teaching. The hackneyed stuff usually done by the Sixth Form specialists can quite well be taken by a youngster just down from the University. He may not be able to teach, but he is mathematically fresh, and, if the specialists in the Sixth have been previously well trained, they can usually take in the new mathematics even if it is rather clumsily presented to them.

The textbooks mentioned in the course of this volume are intended, in the main, for those teachers to read who are technically untrained. The object is not to recommend this book or that book for adoption in schools: that is not part of our purpose at all. The object is to suggest a book, written by some skilful teacher, for the novice to read right through, critically. He should ask himself why the writer has approached and developed the subject in that particular way. He should then read a second book, then a third, and so on, noting the different ways of approach and of development, and the different

ways in which different teachers do things. Then he should settle down and evolve methods for himself. He should not, unless in exceptional circumstances, copy another teacher's method. Let his methods be part of himself, things of his own creation, things for which he has an affection because they are his own children.

Let him realize that methods of teaching mathematics, as of teaching other subjects, are largely conventional. What is a "best" method, and how is it to be determined? Is it a specially "neat" method, invented by some clever mathematician? If so, is it a simple method? Is it productive of accuracy? Here, psychology teaches us a little, though not yet very much, and to say that one method is "better" than another is, more often than not, merely to express a personal preference. The teacher should always ask himself, *which method works out best in practice?* Let every teacher make up his own mind, and not be led away either by the textbooks or by the critics, though the textbooks will always help, and the critics, if competent, are worth listening to. But, however good the books and however competent the critics, let him take their help and advice critically.

Psychology has helped us a great deal over certain points in the teaching of arithmetic. Experiments have been directed mainly to discovering which of possible alternative methods is productive of greatest accuracy amongst children. If such experiments are sufficiently numerous and varied, and if the results of the tests are fairly uniform, we may feel it advisable to consider a particular method favourably. But people who experiment in this way must set out with an entirely unprejudiced mind. Results that are not arrived at objectively carry no weight.

Whether psychology has yet succeeded in devising convincing tests of personal mathematical ability, I am uncertain. The validity of some of the criteria used has been seriously questioned by recognized authorities. The relations between mathematical ability and general "intelligence" have certainly not been clearly determined. We have probably

all met highly intelligent men with keen logical powers who were no good at all at mathematics, and have known brilliant mathematicians whose lack of general intelligence in non-mathematical affairs was amazing. We do not yet really know if mathematical ability can be trained, or whether it is, so to speak, a fixed quantity at birth. The deductions we can legitimately draw from mere examination successes are by no means certain; even poor mathematicians may become adepts in the use of crammers' dodges.

The "Dalton Plan"

The question is sometimes asked, what is the **Dalton** plan of teaching, and can it be made to apply to mathematics?

The plan originated in America in 1920 and has since been introduced into a certain number of English schools. "The aim is to provide for the differences encountered in individual pupils." Class teaching as such is abolished, and gives way to organized private study, in which the pupil, not the teacher, becomes the principal and responsible agent. Instead of a course of lessons prescribed according to a time-table, an "assignment" of work, to last for a month, is prepared by the teacher. The whole "plan" hinges on these assignments. The month's task is divided into four weekly allotments, which are further subdivided into daily units. Instead of working to a time-table, the pupil is free to work at whatever subject he pleases. The rooms are no longer "classrooms", but subject rooms, each being in charge of a specialist teacher and being provided with the necessary books and material. The pupils move freely about from room to room. The instructors are consulted at any time by any pupil; it is their duty to advise and help whenever asked to do so. Conferences and collective discussions are, however, arranged at specified hours.

There is a certain amount of acceptable opinion in favour of the plan as regards subjects like English and History, but as regards Mathematics, Science, and Modern Languages,

the balance of opinion is undoubtedly against it. For one thing, the majority of mathematical textbooks are unsuitable; they do not demonstrate and elucidate principles simply enough for average pupils to understand, with the consequence that, in some schools which are working on Dalton lines, formal lessons on new principles precede the work by assignments, which, for mathematics, are not much more than a few general directions, and exercises to be worked.

In short, the plan does not at present seem to be favoured very much by the majority of teachers. A teacher who adopts the plan, no matter what his subject, must be prepared for greatly increased personal labour; if his subject is mathematics, he must be prepared for some measure of disappointment too. On the other hand, the able mathematical boy, if given a free hand, with just occasional help when difficulties are serious, seems to run away quickly from all the others. The plan seems to pay with Sixth Form specialists, who have been well trained up to the Fifth. Such boys, if provided with good textbooks, can, with very little formal teaching or other help, make remarkably rapid progress.

The one general conclusion that seems to emerge from Dalton experiments is that pupils would do better if left to wrestle more for themselves, and that in the past we have all tended to teach too much. Although the plan *as* a plan is, in the estimation of not a few good judges, rather too revolutionary for general adoption, it must, on the other hand, be admitted that a clever teacher who loves teaching for its own sake may be something of a danger; he may do too much of the thinking, and leave the boys too little to do for themselves.

No boy can become a successful mathematician unless he fights hard battles on his own behalf.

Some General Principles

The last statement does not mean that mathematical teaching is not necessary. For all pupils save perhaps the very best, it is fundamentally necessary, and above all things the teaching must be clear.—Strive day by day to make the expression of your meaning ever clearer. Choose your words carefully and use them consistently. Never mind the correct formal definitions of difficult terms. Use a term over and over again always in exactly the same sense but associated in different ways with different examples, until its exact significance imposes itself on the pupil's mind. It is merely a question of the child continuing to learn new words much in the same way as he learnt the stock of common words which are already in his possession. His mother did not define for him as a baby such words as *milk*, *mamma*, *toes*, *pussie*, *sleep*, *naughty*, yet he learnt to understand their meaning almost before he could walk.

In short, do not worry beginners with formal definitions, or abstractions of any other sort. Of course, almost from the first, the boy makes crude use of all sorts of crudely acquired abstract terms, for in his enumeration work and in his early quantitative measurements, which he has always associated with concrete objects, intuition and guesses have played a large part. But the mathematical ideas and processes which he uses for solving different practical problems gradually become clearer, and he begins to see interrelations between principle and principle, and to distinguish those which are mutually connected from those which are independent. As the subject proceeds, it tends to become more abstract; experience grows; and the teacher has to choose his own time for stepping in and exacting greater and greater logical rigour. Below the Sixth Form, mathematics is essentially a *practical* instrument, not a subject for philosophic speculation. Never press forward formal abstract considerations until experience has paved the way

What is the use of discussing even with Fifth Form pupils the rival merits of Euclid's parallel postulate and Playfair's alternative version? For all lower and middle forms, some such statement as, " lines which intersect have different directions; lines which have the same direction do not meet but are parallel ", is good enough, and it need not be subjected to criticism until the Sixth. Then, criticism is desirable.

The organization of mathematical work in a large school is a simple matter; between the Junior Forms and the Sixth there may be 4 blocks of 3 or 4 Sets each. When the Set system prevails, gradation is easy. Let the work of the top Set of a block be much sterner and more exacting than in the bottom Set, and do not attempt to include in the work of the bottom Set all the subjects, or even all the topics of a particular subject, that are allotted to the better Sets. For instance, all top Sets will learn logarithms. But bottom Sets? Why should they? What difficult calculations will they have to engage in that logarithms will really help? None in school, and none after leaving school. Why then should such dull boys be made to waste their time by poring over the pages of a numerical lexicon and then getting their sums wrong instead of right? It is unutterably silly. It is sometimes done because teachers have not the courage to say what they really think.

The timid teacher may be inclined to argue, "but how are we to provide for the boy who during the year happens to be promoted a Set?" That is certainly a real problem of school organization and must be faced. But the needs of the occasional boy must not be catered for at the expense of a whole class. And, after all, there will be much the same *minimum* of work for the various Sets within a block, and inter-block promotions after the first year or two will be rare.

Again, suppose that somebody comes along and asks if you teach, say, Vectors. If you do not, you probably have a good reason for it, perhaps because Lord Kelvin himself poured scorn on them. In that case do not hesitate to say so. *Hold fast to your faith.* But re-examine the grounds of your

faith from time to time. It *may* be that you will find new arguments in favour of vectors, arguments which will induce you to revise your opinions. And so with scores of other things. Keep an eye on your defences, but remain captain of your own quarter-deck.

Mathematical Reasoning

All mathematical teachers should reflect carefully on the nature of mathematical reasoning, and should see that their pupils are made more and more conscious of what constitutes mathematical rigour. Mathematical reasoning is not, as commonly supposed, *deductive* reasoning; it is based upon an initial analysis of the given and, being analytical, is in essence *inductive*. The threads of the web once disentangled, synthesis begins, and the solution of the problem is set out in deductive dress. We arrange our arguments deductively in order that other people may easily follow up the chain to our final conclusion. If this mere setting out were the whole story, how simple it would be! Consider this syllogism, in form typically Euclidean and deductive:

Major premiss: All professional mathematicians are muddle-headed.

Minor premiss: The writer of this book is a professional mathematician.

Conclusion: Therefore the writer of this book is muddle-headed.

Now the conclusion is quite possibly true, and it is certainly the correct conclusion to be drawn from the two premisses. But both the major and the minor premisses are false (the writer of this book is not a professional mathematician: heaven forbid! he is only a teacher), and therefore the conclusion, even if materially true, is logically absurd. In fact the main source of fallacious reasoning almost always lies in false premisses. The truth of the conclusion cannot be more true than the truth of the premisses, and a scrutiny and a rigorous analysis of these is therefore always necessary.

At bottom, all reasoning is much of the same kind, and it usually turns on the truth or falsehood of the premisses. Clear thinking is thus indispensable: probabilities have to be weighed, irrelevant details discarded, the general rules according to which events occur have to be divined, hypotheses have to be tested; the general rules once established, the derivation of particular instances from them is a simple matter.

But in elementary mathematics for beginners, the provision of concrete particular instances comes first in importance. In the handling of his subject in the classroom, the mathematical teacher cannot be too concrete. As the boys advance from Form to Form, they will gradually begin to understand, and in the Sixth to realize fully, that the solving of every mathematical problem consists first of disentangling, then of setting out and classifying, then of tracing similarities and finding possible connecting links, then of linking up and generalizing; in other words, of analysis followed by synthesis. Although without generality there is no reasoning, without concreteness there is neither importance nor significance. But in schools logical rigour is a thing of exceedingly slow growth.—We shall return to the question of mathematical reasoning in a later chapter.

More often than not, present-day writers of standard textbooks in mathematics strain after both ultra-precision of statement and the utmost rigidity of proof. But any attempt in schools to be perfectly exact all at once, to include in every statement all the saving clauses and limitations that can be imagined, inevitably ends in failure. This is where the beginner, untrained, just down from the University, so often blunders. He is inclined to argue that, unless his classroom logic is as unassailable as that of his University Professor, his work will be open to serious criticism. The work of even Sixth Form specialists cannot be placed on an unimpeachable logical basis. The degree of rigour that can be exacted at any stage must necessarily depend on the degree of intellectual development of the pupil. A school can never become a place for mathematical asceticism.

But, as boys get older, they should be encouraged to read their own textbooks " up and down, backwards and forwards ". In their study, let us say, of the calculus, let them first obtain an insight into general elementary processes, and then proceed at once to simple applications. Ample practice in differentiation and integration is, of course, necessary, but the study of geometrical and dynamical applications must not be unduly delayed. It is these that will excite interest, and will help greatly to produce an appreciation of fundamental principles. But again and again go back to a more critical examination of those principles. The applications will have taught the boys a great deal of the inner meaning of the processes, and the more abstract discussions will then be made much easier by the fact that the learner has acquired a fair stock of more or less concrete ideas.

The Fostering of Mathematical Interest

The general standard of mathematical attainments in Sixth Forms is now reasonably satisfactory, and entrants at the newer universities are beginning work of much the same grade as entrants at Cambridge. But though Sixth Form specialists are doing solid work (of a very restricted type, it must be added), the amount of mathematical work being done by all the other pupils who have obtained the School Certificate is, as a rule, slight, too slight and much too academic for the fostering of a life-long interest in the subject. Let the younger race of teachers wake up to this important fact, and help to put things right. We shall refer to this point again.

Books to consult:

1. *Didaktik des mathematischen Unterrichts*, Alois Höfler.
2. *A Study of Mathematical Education*, Benchara Branford.

CHAPTER II

Which Method: This or That?

Old and New. Rational and Rule-of-thumb.

An intelligent woman, who is known to have done a fair amount of mathematics in the days of her youth, recently received a bill for 8*s*. 7¼*d*., representing the cost of 7 lb. 6 oz. of lamb. She was " sure " that the ounces and farthings had been included merely for the purpose of cheating her, and she telephoned to the butcher to know the price of the meat per lb. She was quite unable to calculate the amount for herself (1*s*. 2*d*.).

A well-known Inspector of the nineties dictated this sum to a class of 11-year-olds: " Take one million ten thousand and one from ten millions one thousand one hundred ". As might be expected, hardly any children had the sum right. The Inspector looking grieved, the Teacher gently asked him if he would himself work the sum on the blackboard. Very unwisely the Inspector tried to do so, and made a hopeless mess of it—to the delight of the boys.

The first story illustrates one of the commonest faults of school mathematics: teachers are apt to push on into more advanced work before foundations have been well and truly laid. The second story shows that a non-mathematician should not be allowed to criticize mathematical teachers. To the non-specialist, mathematics is full of pitfalls, and it may be hoped that the time will come when every teacher of the subject will be a trained mathematician, even if he has to teach nothing but elementary arithmetic.

Not the least important question for a teacher of elementary arithmetic to consider is the method of approach to a new rule. Should that rule be given to the child dogmatically, given as a *rule*, to be followed by the working of examples

until it is thoroughly assimilated? or should the rule be "explained", approached "intelligently", and be thoroughly "understood", before it is applied to examples? In other words, is it immoral or is it legitimate to provide a child with a working tool before the nature of the tool is explained?

To put it another way: suppose that we teach a rule "intelligently", and the children get 50 per cent of their sums right; or suppose that we teach by rule of thumb and the children get 80 per cent of their sums right. Which plan should we adopt?

Should we give credit merely for "getting sums right"? or should we forgive mere slips if the working shows some grasp of the process?

Again: suppose we find that some of the newer and popular methods, methods that have superseded those in common use forty or fifty years ago, are less productive of speed or accuracy or both, are we, or are we not, justified in feeling a little suspicious of the newer methods?

Some of these questions have been answered for us by the psychologists, who in recent years have adopted various devices for testing the comparative merits of the methods we use in teaching arithmetic. The old school of psychologists trusted too much to intuition, and their views were doctrinaire. Present-day psychologists, on the other hand, are devoting themselves to experiment, to the garnering of facts, to making careful deductions from those facts. For instance, some of them have arranged with schools for tens of thousands of simple sums, of varying types, to be worked by different methods. From such large numbers of results legitimate deductions may be drawn, especially when different psychologists arrive at similar conclusions from different sets of examples. It is on such evidence as this that different methods have been compared and some sort of priority determined. No thoughtful mathematical teacher would now pronounce dogmatically in favour of his own method of doing a thing, even if he has used it all his life. He would subject it, and other methods as well, to prolonged tests

selecting different groups or classes of children all "new" to the principle to be taught; and he would compare the results in different ways, for instance for intelligence, for accuracy, and for speed; and he would make sure that the general conditions of the tests, for instance the time of day when they were given, were equalized as far as possible.—It is in such matters that psychologists are helping us greatly.

Perhaps the first essential of all is accuracy, especially accuracy in all kinds of computation. What would be the use of a bank clerk who made mistakes in running up a column of figures? A tradesman inaccurate in his calculations might soon find himself a bankrupt. Indeed, accuracy ranks as a cardinal virtue; it is a main factor of morality. A boy who gets a sum wrong should be made to get it right. *Never* accept a wrong answer. This does not mean that credit should not be given for intelligence: anything but that. For instance, a boy may be given a stiff problem and get it wrong. But that problem may include half a dozen little independent sums, each of them to be thought out before it can be actually worked; five of them may be right and one wrong. In such a complex operation, a margin of error may be legitimately allowed for.

If we think of our own personal operations in arithmetic, those we are engaged in day by day, we must admit that most of our working is by rule of thumb; the actual rationale of a process does not enter our heads. We have become almost mathematical automata. Yet, if called upon to do so, we could, of course, explain the rationale readily enough. But the average boy, the *average* boy, however intelligently he may have been taught, not only works by rule of thumb but could not for the life of him give an adequate explanation of the process. This is admittedly brutal fact. Test any average class of 30 boys, twelve months after they have been taught a new rule, and it is highly improbable that more than 8 or 10 will explain the mathematical operation adequately and intelligently. The experienced teacher never expects it.

Nevertheless, no teacher worth his salt would ever dream

of teaching a new rule without approaching it rationally. He would do his best to justify every step of the process, illustrating and explaining as simply as possible. Perhaps 4 or 5 of the boys in a class will see the whole thing clearly, and their eyes may sparkle with satisfaction. A few more, perhaps 8 or 10, will follow the argument pretty closely, though if asked to repeat it they will probably bungle pretty badly. But the rest? No. *They* want the rule, simply and crisply put, a rule they can follow, a rule they can trust and hold fast to. And no teacher need break his heart that the majority can do no more. No inspector, if he is a mathematician, ever expects more; he is too familiar with the mathematical limitations that nature has imposed on the average boy of British origin.

As a boy goes up the school and his intelligence is developed, the fundamental processes of arithmetic may be made clearer to him. Any average boy of 13 or 14 may be made to understand the main facts of our decimal system of notation, whereas at 7 or 8 he may have failed to grasp the real significance of even a three-figure number. *Every teacher of mathematics should remember that he cannot clear the ground finally as he goes along*; he has to come back again and again.

Do not worry young children with such terms as abstract and concrete. Nothing is gained by telling a child to add 8 sheep to 9 sheep instead of 8 to 9. Actual arithmetical processes are all abstract, and the notion of casting every sum into problem form has become a silly fetish. Present-day children are suffering from a surfeit of oranges and apples. Of course when little children are beginning to count, to add, to subtract, &c., the use of real things is essential, and in this matter we may learn much from the efficient Kindergarten teacher. Some of the very best arithmetic teaching is to be seen in Kindergarten schools. It is a pleasure to watch children who are little more than toddlers getting a real insight into number and numeration. The *worst* teaching of arithmetic I have ever seen was in the lower forms of the old grammar schools of 40 years ago. In those days it was not

an uncommon thing for the lowest forms to be placed in charge of an unqualified hack. Those were dark days indeed.

"Practical" mathematics includes manipulative work of some kind, actual measuring as well as calculating, and the more of this in the Seconds, Thirds, and Fourths the better, especially if there is a mathematical laboratory available. It is *concrete* mathematics, but do not give it that label. In fact, put the label into the waste-paper basket. As for the label *abstract*, burn it.

Books to consult:

1. *The Approach to Teaching*, Ward and Roscoe.
2. *The New Teaching*, Adams.

(These are not books specially directed to mathematics, but to teaching generally. They are books to be read by every teacher, for they are full of good things. Mr. Ward was for many years chief Inspector of Training Colleges; Mr. Roscoe is Secretary to the Teachers' Registration Council, and was formerly Lecturer on Education at the University of Birmingham; Sir John Adams was formerly Professor of Education in the University of London.)

CHAPTER III

"Suggestions to Teachers"

The *Handbook of Suggestions to Teachers*, 1928, issued by the Board of Education, contains useful hints "for the consideration of teachers and others concerned in the work of Public Elementary Schools". The practical hand is revealed on every page, and there can be no doubt that the best teaching practice known in the country is embodied in it. The book deals specifically with the requirements of Elementary Schools as they are likely to be developed during the next few years—Infant Schools, Junior Schools, and Senior

Schools, including "Selective" Central Schools; but what is said about mathematics, especially arithmetic, is equally applicable to schools of all types.

The Board are of opinion that, by the age of 11, "a minimum course should at least include a thorough groundwork in notation, a knowledge of the first four rules applied to money, and the ordinary English measures of length, area, capacity, weight, and time; an elementary acquaintance with vulgar and decimal fractions, together with simple notions of geometrical form and some skill in practical measurements." By that age, "accuracy in simple operations should in great measure be automatic. It depends first on a ready knowledge of tables, and secondly upon concentration, but in the case of written work is greatly assisted by neatness of figuring and clear statement."

(The Board contemplate that, in future, the arithmetic of all Senior Elementary Schools (where the age will extend from 11 to 14 or 15 or even 16) will be associated with mensuration, scale drawing, geometry, graphs, and (for boys) algebra, trigonometry, and practical mechanics.) The course of mathematical work mapped out for such schools is particularly suggestive and should be read by mathematical teachers in all schools.

The Board seem also to contemplate for Senior Elementary Schools some form of mathematical laboratory where practical work can be done. This work is to be associated with the geometry, mensuration, surveying, mechanics, and manual instruction, and even for the lower classes useful hints are given for practical work in weights and measures. Much of the work which at one time constituted the preliminary course of practical physics might be included as well—the use of the vernier and the micrometer screw gauge, the volumes of irregular solids by displacement, densities and specific gravities, U-tube work, and experimental verifications of such principles as those involved in the lever and pulleys, in the pendulum, and in Hooke's law.

The time has gone by when arithmetic, even in Elemen-

tary Schools, should be looked upon as a self-contained subject. Although arithmetic is the subject dealing with numerical relations, it is geometry which deals with space relations, and the two should be taught together. Algebra is just a useful mathematical instrument, full of ingenious labour-saving devices for both arithmetic and geometry. Trigonometry is the surveyor's subject, a useful application of algebra and geometry together. A graph is a geometrical picture, showing arithmetical and algebraic relations of some sort. The various subjects fuse together as parts of a single puzzle, and quite young boys may be given a working insight into them all. Arithmetic alone is dry bread indeed, far too beggarly a mathematical fare even for a Junior School.

From the first, keep the mathematical work in close contact with the problems of practical life. Let matters reasoned about be matters with which the children are either already familiar or can be made to understand clearly. Do not take the children for excursions into the clouds; what is perfectly clear to you may be very foggy to them. There is no independent "faculty" of reasoning, independent of the particular facts and relations reasoned about, stored away ghost-like in the brain, to be called upon when wanted. Hence, always endeavour to ensure that the things which you call upon a boy to discuss are seen by him as in a polished mirror.

Be scrupulous over the exact use of mathematical terms. If you take care always to use such terms in their exact sense they need rarely be defined. Even very small children have to learn the terms *add*, *subtract*, *sum*, *difference*, *remainder*, *whole*, *part*, *less*, *equal*, *equals*, *total*; and older children must acquire an exact knowledge of such terms as *interest*, *discount*, *gross*, *net*, *balance*, *factor*, *prime*, *measure*, *multiple*, and dozens of others. If *you* use them consistently, the children will soon learn to appreciate their exact significance.

Let part of your stock-in-trade be price-lists of some of the big London stores, the Post Office Guide, Bradshaw, Whitaker's Almanack, and the like. Ask yourself what sort of mathematical knowledge the children are likely to require

in after life, and, as far as you can, provide accordingly. But it is not merely a question of giving them practical tips: train them to think mathematically. Train them to care for accuracy. Train them to appreciate some of the marvels of the universe—the very great and the very small.

Do not despise old-fashioned methods that have stood the test of time, and do not be too ready to adopt the new-fangled methods of some new prophet. Any new educational lubricant which is advertised to be a tremendous accelerator of the classroom machinery generally proves to be nasty clogging stuff, making life a burden for those who use it.

Book to consult:

Handbook of Suggestions for Teachers, H.M. Stationery Office.

CHAPTER IV

Arithmetic: The First Four Rules

Numeration and Notation. Addition

We have already mentioned that the laborious work of psychologists has taught us much about the pitfalls experienced by beginners when learning arithmetic. Few young teachers realize the number of separate difficulties felt by children in learning to do ordinary addition sums, even after the addition table to $9 + 9$ is known.

For instance, a child has to learn:

(1) To keep his place in the column;
(2) To keep in mind the result of each addition until the next number is added to it; and
(3) To add to a number in his mind a new number he can see;

(4) To ignore possible empty spaces in columns to the left;
(5) To ignore noughts in any columns;
(6) To write the figure signifying units rather than the total number of the column, specially learning to write 0 when the sum of the column is 20, 30, &c.;
(7) To carry.

A teacher should analyse in this way *every* general arithmetical operation, and provide an adequate teaching of every separate difficulty. Unless at least the slower pupils are thus taught, they may break down in quite unsuspected places. Another important thing is the *grading* of difficulties. For instance, we now know that the average beginner finds the addition sum

$$\begin{array}{r} 21 \\ 43 \\ 35 \\ \hline \end{array} \qquad \text{easier than} \qquad \begin{array}{r} 4 \\ 21 \\ 3 \\ 35 \\ \hline \end{array}$$

and the latter very much easier than $21 + 43 + 35$. He seems to have more confidence in the completed columns, and the vertical arrangement appeals more strongly to his eye than does the horizontal arrangement.

Here is a series of first subtraction sums, graded according to the difficulty experienced by beginners:

$$\begin{array}{rrr} 8 & 8 & 8 \\ \underline{3} & \underline{8} & \underline{0} \end{array}$$

Teach *one thing at a time*; see to it that this one thing does not conceal a number of separate difficulties; and let that one thing be taught thoroughly before the next is taken up.

The bare elements of numeration and notation will have been taught in the Infant School or Kindergarten School, and on entry to the Junior School or Department the children will clearly apprehend the inner nature of a 3-figure number, that, for instance,

$$357 = 300 + 50 + 7.$$

If that is thoroughly understood, but not otherwise, numeration and notation should give little further difficulty.

Numbers of more than 6 figures will seldom be required in the Junior School or Department, and children soon learn to write down 6-figure numbers correctly. Let beginners have two 3-column ruled spaces, thus:

Thousands.			H.	T.	U.
H.	T.	U.			
2	4	3	5	9	6

Tell them they have to fill up the spaces under " thousands " exactly as they fill up the old familiar 3-column space on the right. Dictate " 243 thousands ", and pause; the child writes 243 under " thousands ". Now go on: " listen to what comes after thousands; 596 ". The child soon learns to write down a dictated number of " thousands ", just as he would write down a dictated number of " books ". With a class of average children of 10 or 11 years of age, one lesson ought to be enough to enable them to write down even 9-figure numbers accurately, if these are properly dictated, and if the children are first made to understand that after " millions " there must always be two complete groups each of 3 figures, the first of these groups representing thousands.

Millions.			Thousands.			H.	T.	U.
H.	T.	U.	H.	T.	U.			

The teacher dictates: " Write down 101 million 10 thousand and one."

" How many millions?" " 101." " Write 101 under *millions*."

" How many thousands?" " 10." " Write 10 under *thousands*."

" What comes after thousands?" " 1." " Write 1 in the right-hand 3-column space."

	1	0	1		1	0		1

" Now fill up with noughts."

Numbers without noughts should come first. Introduce the noughts gradually. Remember that they provide constant pitfalls for beginners.

Dispense with the ruled columns as soon as possible, but let the successive triads always be separated by commas; 14,702,116 (14 millions, 702 thousands, 116).

Subtraction

On entry to the Junior School the child will already have been taught that the subtraction sum

$$\begin{array}{r} 867 \\ 142 \\ \hline 725 \end{array}$$

is a shortened form of

$$\begin{array}{r} 800 + 60 + 7 \\ 100 + 40 + 2 \\ \hline 700 + 20 + 5 \end{array}$$

bundles of sticks or bags of counters being used to make the process clear.

They will also have been taught to decompose the top line in such a sum as this, leaving the lower line alone:

$$\begin{array}{r} 534 \\ 386 \\ \hline \end{array}$$

Thus:

$$\begin{array}{r} 500 + 30 + 4 \\ 300 + 80 + 6 \\ \hline \end{array} = \begin{array}{r} 400 + 120 + 14 \\ 300 + 80 + 6 \\ \hline 100 + 40 + 8 \end{array} = 148$$

I have often seen excellent results in such an instance, the small children handling their bundles or bags, untying them and regrouping, in a most business-like way. They really did seem to have grasped the essentials of the process.

But can the method be regarded as the most suitable permanent possession for older children? Consider this sum:

$$\begin{array}{r} 80{,}003 \\ 47{,}167 \\ \hline \end{array}$$

The necessary decomposition is a complicated matter for young children. They have to take 10,000 from the 80,000; leave 9000 of the 10,000 in the thousands column and carry 1000 on; leave 900 of this 1000 in the hundreds column and carry 100 on; leave 90 of the 100 in the tens column and carry 10 on to the units column. And thus we have:

$$\begin{array}{l} 80{,}000 \quad +3 \\ 40{,}000+7000+100+60+7 \end{array} = \begin{array}{l} 70{,}000+9000+900+90+13 \\ 40{,}000+7000+100+60+\ 7 \\ 30{,}000+2000+800+30+\ 6 = 32{,}836 \end{array}$$

In practice this is what we see:

$$\begin{array}{r} 7\ 9\ 9\ 9 \\ \not{8}\ \not{0},\ \not{0}\ \not{0}^{13} \\ 4\ 7,\ 1\ 6\ 7 \\ \hline 3\ 2,\ 8\ 3\ 6 \end{array}$$

I confess that, *judged by the number of sums right*, the best results I have ever met with were in a school where this decomposition of the minuend was taught, although the teacher responsible was not only not a mathematician but was entirely ignorant of the principles underlying the plan she had adopted. She would give the children a sum like this:

$$\begin{array}{r} 70{,}024 \\ 30{,}578 \\ \hline \end{array}$$

Before the children began actually to subtract, they had to examine each vertical column of figures, beginning with the units column. If the upper figure was smaller than the

lower, they would borrow 1 from " next door ", prefix it to the unit figure in the top line, and show the borrowing by diminishing by 1 the figure they had borrowed from. Thus they wrote:

$$\begin{array}{r} {\scriptstyle 1} \\ 7\,0,0\,\not{4}^{14} \\ 3\,0,5\,7\,8 \\ \hline \end{array}$$

Then they would examine the tens column. If, as before, they found the upper figure smaller than the lower, they would borrow from next door again—if they could; if not, they would pass along to the place where borrowing was possible, in this case 7, change the 7 to 6, and prefix the borrowed 1 to the 1 in the tens column, always changing into 9's the 0's they had passed over. They always " borrowed from next door when anybody was at home, putting a 9 on the door of every house they found empty ". Thus the sum was made to look like this:

$$\begin{array}{r} 6\,{\scriptstyle 9,9\,11} \\ \not{7}\,\not{0}\,\not{0}\,\not{4}^{14} \\ 3\,0,5\,7\,8 \\ \hline \end{array}$$

Then the subtracting was begun, and, of course, it was all plain sailing: 8 from 14, 7 from 11, 5 from 9, 0 from 9, 3 from 6.

Over and over again I tested that class, and not a child had a sum wrong. But the children had no idea of the " why " of the process; neither had the teacher. The accuracy was the result of *a clear understanding of an exactly stated simple rule*. The children followed the rule blindly.

But this case does not typify my *general* experience, which is that the decomposition method is not productive of anything like the accuracy obtained by the alternative method of equal additions. Although, therefore, I am driven to favour the equal additions method, this method does not seem quite so susceptible of simple concrete explanation for very young beginners. Still, such explanation is possible.

First try to make the pupils understand that equal additions

to the minuend and subtrahend * *will not affect the difference.* The ages of two children provide as good an illustration as anything:

Jack is 7 and Jill is 10; their difference is 3. In 4 years' time, Jack will be 11 and Jill will be 14; their difference will still be 3. And so on. Their difference will always be 3.

A first lesson on brackets will serve to reinforce the idea:

$$10 - 7 = 3$$
$$(10 + 4) - (7 + 4) = 14 - 11 = 3$$
$$(10 + 6) - (7 + 6) = 16 - 13 = 3$$

Then

$$\begin{array}{r} 73 \\ 21 \\ \hline 52 \end{array} \quad \text{or,} \quad 73 + 5 = 78,\ 21 + 5 = 26,\ \begin{array}{r} 78 \\ 26 \\ \hline 52 \end{array} \quad \text{or,} \quad 73 + 10 = 83,\ 21 + 10 = 31,\ \begin{array}{r} 83 \\ 31 \\ \hline 52 \end{array}$$

In this way, get the child to grasp the cardinal fact that in any subtraction sum we may, before subtracting, add anything we like to the top line if we add the same thing to the bottom line; the answer will always be the same. Another example:

$$\begin{array}{r} 73 \\ \\ 48 \end{array} \quad = \quad \begin{array}{r} 70 + 3 \\ \\ 40 + 8 \end{array} \quad = \quad \begin{array}{r} 70 + 13 \\ \swarrow \\ 50 + 8 \end{array}$$

Here we have added 10 to the top line, turning 3 into 13, and we have added 10 to the bottom line, turning 40 into 50. (The double arrow usefully draws attention to the two additions.) Thus the answer to the altered sum will be the same as to the original sum. In this way it is easy to give a clear understanding of the so-called " borrowing " process.

But the small child is not quite so happy when working by this method in the concrete, as he is with the decomposition method. When he is given the two extra bundles of 10, he does not always believe that the sum can be the same.

* Do not use these terms with beginners; wait until the senior school. The *from* line and *take* line will do, or the *top* line and *bottom* line, or the *upper* line and *lower* line. Adopt simple terms of some kind, and adhere to them until the children are better prepared to adopt a stricter nomenclature.

However, *some* kindergarten teachers seem to have little trouble about it.

In a sum like the following, the teaching jargon * we should use for beginners would probably be something like this:

" Whenever we give 1 to the top line, we must always give 1 to the bottom line as well, but next door."

9023
3765

" 5 from 3 we cannot; give 1 to the top line and so turn 3 into 13; 5 from 13 is **8**. Now give **1** to the bottom line, next door; **1** and 6 is **7**."

" 7 from 2 we cannot; give 1 to the top line, and so turn 2 into 12; 7 from 12 is **5**. Now give 1 to the bottom line, next door; 1 and 7 is **8**."

" 8 from 0 we cannot; give 1 to the top line and so turn 0 into 10; 8 from 10 is **2**. Now give 1 to the bottom line, next door; 1 and 3 is **4**.

" 4 from 9 is **5**."

The words *borrow* and *pay back* tend to mislead the slower boys, since we borrow from one line and pay back to another. To them this seems unfair, especially when we say we borrow 10 and pay back only 1.

Personally I prefer to *give* 1 to the top line and never talk about paying back, but compensate by giving 1 to the bottom line. But the 2 parts of each double transaction must be worked in association *at once*; this satisfies the children's sense of justice.

In each of the first several lessons, ask what the giving of 1 really signifies. " When we turned 3 into 13, the 1 given was really 10; did we give the bottom line the same number?

* The term *jargon* is rather suggestive of slang, but, of course, what I really mean is the simple homely language which we mathematical teachers all invent for teaching small boys, language which rather tends to offend the ear of the English purist. But that does not matter. The important thing is to express ourselves in words which convey an exact meaning to the children's minds.

Yes, because the 6 which by adding 1 we turned into 7 is in the tens column.

" When we turned the 2 into 12, the 2 was really 20, and the 1 we gave to it was really 100; did we give the bottom line the same number? Yes, because the 7 which by adding 1 we turned into 8 is in the hundreds column."

And so on. A very considerable proportion of the children will not *at this stage* understand the process at all. But do not worry about that. Come back to it in a year's time.

It will be weeks, even months, before the slower child will have had enough practice to do subtraction quickly and accurately, and it is best to adhere all the time to precisely the same form of teaching " jargon ".

There remains the question, shall we teach the children (i) actually to *subtract*, or (ii), to *add* (complementary addition), or (iii), *first to subtract from* 10 and *then add the difference to the figure in the top line*? For instance,

$$\begin{array}{r} 13 \\ 5 \\ \hline 8 \end{array}$$

Shall we say 5 from 13 is **8**?
or, shall we say 5 and **8** is 13?
or, shall we say 5 from 10 is 5 and 3 is **8**?

The last must be ruled out of court; it is productive of great inaccuracy amongst beginners, though later on it is useful in money subtraction. The second plan is popular, but it has been proved to be less productive of accuracy than the first; and it is something of a sham, for the number to be added must be obtained by subtraction.* Hence the first method, honest subtraction, is strongly advocated, and that demands ample practice in both the subtraction and the addition tables. Thus the child learns:

both	5 and 1 is 6 5 and 2 is 7 5 and 3 is 8	and	5 from 6 is 1 5 from 7 is 2 5 from 8 is 3	&c.

* Cf. algebraic subtraction.

"5 from 8?" is as effective a form of question as (and is much more elegant than) "5 and what makes 8?".

The Tables

The child must learn the addition table to 9 + 9 perfectly. He must be able to say at once that, e.g., 9 and 8 is 17. He must also be able to say at once that 9 from 17 is 8, and that 8 from 17 is 9. In fact, the addition and subtraction tables should be learnt in close association. Very young children when learning to count, to add and subtract, will, of course, be shown how to find out that $8 + 3 = 11$ and that $11 - 3 = 8$, but the time must soon come when they can give those results *pat*, without calculation or thinking of any kind; and this means a great deal of sheer ding-dong work from which lower forms and classes can never escape. Never mind the charge of unintelligence; be assured that the people who make such a silly charge have never had to face the music themselves. Table accuracy is the one key to accurate arithmetic.

Each subtraction table is, of course, as already indicated, the mere complement of an addition table. For instance, the 4 times addition table begins 4 and 1 is 5 and ends 4 and 9 is 13; the corresponding subtraction table begins 4 from 5 is 1 and ends 4 from 13 is 9. Carry the addition tables to 9 + 9 and the subtraction tables to 18 − 9.

How many repetitions are necessary to ensure permanent knowledge? All experienced teachers know that this varies enormously. It may be that only 10 repetitions are required, but it *may* be 500, according to the individual. Test, test, test, day by day. Do not waste the time of a whole class because further drill is necessary with a few.

Helpful blackboard tests may be given in a variety of forms, e.g.,

$$9 + 6 = x \qquad 17 - 4 = x$$
$$5 + 8 = x \qquad 13 + 8 = x \text{ \&c.}$$

In examples of this kind we have the germ of equations,

as we had with the examples in brackets. Explain that x is a symbol for the number to be found. Call on a member of the class and point to the first x, then call on another member and point to the second x. But do not call on members in order. Keep every child in expectation. Call on Smith the shirker half a dozen times a minute. If the answers are not given at once, without any calculations, the tables are not known, and more drill is necessary.

Draw a circle of small numbers on the board and have them added together, as they are pointed to. The answers must be instantaneous—or the tables are not known.

Make the children count forwards and backwards, by 1's, then by 2's, then by 3's, &c.

1, 4, 7, 10, 13, 16, &c.
100, 96, 92, 88, 84, 80, &c.

This sort of practice helps the tables greatly.

But do not expect that, because a boy knows $7 + 6$ is 13, he will therefore know that $27 + 6 = 33$. Such extended examples require special practice, and the practice must be continued day by day until the boy knows *at once* that a 7 added to a 6 always produces a 3. Similarly with subtraction; a boy must be able to say *at once* that a 7 taken from a 6 always produces a 9.

Write on the blackboard, say, a 7. "Let us add 6's." Smith? 13; Brown? 19; Jones? 25; dodging about the class. The response must be *instant*. Similarly with subtraction.

Let your schemes for testing the tables be as varied as possible. Do not be satisfied as long as there is a single mistake. Do not forget that dull boys may require 10 times, perhaps 50 times, the practice that quick boys require. There must be no counting on fingers, no strokes, no calculations of any kind.

So with the multiplication and division tables. Beginners are taught, of course, that multiplication is just a shortened form of a succession of additions, and division a shortened form of a succession of subtractions. That fact grasped,

then come the tables, multiplication to 9×9 and division to $81 \div 9$.

Do not be intelligently silly and teach a boy " to find out for himself " the value of 9×8 by making him set out 9 rows of 8 sticks each and then count to discover 72. Make him *learn* that $9 \times 8 = 72$. When he *begins* multiplication and division, a few very easy concrete examples will be given him, to make the fundamental ideas clear. Then make him learn his tables, **learn** his tables.

As with the addition and subtraction tables, write the multiplication and division tables side by side. The sign for " equals " may well be substituted for " is ".

$1 \times 7 = 7$	7's into $7 = 1$
$2 \times 7 = 14$	7's into $14 = 2$
$3 \times 7 = 21$	7's into $21 = 3$
.....	.....
$9 \times 7 = 63$	7's into $63 = 9$

Mental work:

Seven threes?
Three sevens?
Sevens into twenty-one?
Threes into twenty-one?

} Let the 3, the 7, and the 21 hang together in all 4 ways. And so on.

Ask for the factors of such numbers as 42, 77, 28, &c.

Blackboard Work:

Write down a number consisting of 15 or 18 figures, and ask the class to give the products of successive pairs of figures, as rapidly as possible: e.g.

371498652498.

Answers:

21, 7, 4, 36, 72, &c.,

Again:

$7 \times 3 = x$; $3 \times 7 = x$; $3 \times x = 21$; $7 \times x = 21$; $x \times 3 = 21$; $x \times 7 = 21$.

$$\frac{21}{3} = x; \quad \frac{21}{7} = x.$$

And so on. Point to an x, and call on a particular pupil for the answer.

Mental work in preparation for multiplication and division sums:

$(3 \times 7) + 1 = x$.	$(3 \times 7) + 2 = x$	$(8 \times 7) + 5 = x$.
3's into $22 = x$.	3's into $23 = x$.	8's into $61 = x$.

Ample practice in this type of example is necessary. The examples are of course one step beyond the simple tables; there are two operations, one in multiplication or division, one in addition or subtraction. Hence instantaneous response is hardly to be expected from slower children. But it is surprising how quickly the answers come from children *who know their tables*, who *know* that $8 \times 7 = 56$ and that $56 + 5 = 61$, though it is well to remember that a mental effort is required to keep in mind the first answer while it is being further increased or diminished.

The 11 times table is hardly worth learning. The 12 times table may be postponed until money sums are taken up. The 15 times is easy to learn and is useful for angle division. So is the 20 times table. Mental work on simple multiples is easy to provide, e.g. 18×9 = twice 9×9; (16×7) = twice (8×7).

But when actually teaching the tables, it is a safe rule not to complicate matters by giving tips for exceptional cases. Do not, for example, tell a beginner that, when he is adding a column of figures, he should look ahead to see if two of them added together make 10. If he has to find the sum of 4, 8, 3, 7, 5, teach him to say, 4, 12, 15, 22, 27, not to look ahead and to discover that $3 + 7 = 10$, and then to say 4, 12, 22, 27. Such a plan with beginners makes for inaccuracy. Good honest straightforward table work must come first. Short cuts may come later, when they may be more readily assimilated.

Multiplication

It is easy to make any average child who is well grounded in numeration and notation understand that 4 times 273 means the sum of four 273's, i.e.

$$\begin{array}{r} 273 \\ 273 \\ 273 \\ 273 \\ \hline \end{array}$$

and that therefore the answer is

(4 times 200) + (4 times 70) + (4 times 3);

and he sees readily enough that the teacher's shortened form

$$\begin{array}{r} 273 \\ 4 \\ \hline 1092 \end{array} \quad = \quad \begin{array}{r} 200 \\ 4 \\ \hline 800 \end{array} \quad + \quad \begin{array}{r} 70 \\ 4 \\ \hline 280 \end{array} \quad + \quad \begin{array}{r} 3 \\ 4 \\ \hline 12 \end{array}$$

But the slower child will *not* understand, though he will learn the ordinary rule of multiplication fairly readily.

In teaching multiplication, the advisable succession of steps seems to be:

(*a*) Easy numbers by 2, 3, and 4; no carrying; no zeros in multiplicand.

(*b*) Easy numbers by 2, 3, and 4; no carrying; zeros in multiplicand.

(*c*) Easy numbers by 2, 3 . . . 9, with carrying; no zeros in multiplicand.

(*d*) Easy numbers by 2, 3 . . . 9, with carrying; zeros in multiplicand.

(*e*) The same with larger multiplicands.

(*f*) Multiplication by 10.

(*g*) Multiplication by 2-figure numbers not ending in a zero.

(*h*) Multiplication by 2-figure numbers ending in a zero.
(*i*) Multiplication by 3-figure numbers, zeros varied.

Be especially careful to show clearly the effect of multiplying by 10, viz. the shifting of every figure in the multiplicand one place to the left in the notational scheme, i.e. each figure is made to occupy the next-door position of greater importance. Then show the effect of multiplying by 100, by 1000, by 20, 200, 6000, &c. Bear in mind that the work has particular value, inasmuch as ultimately it will lead on to decimals.

From the outset, use the term *multiplier* and the term *product*, but let the difficult term *multiplicand* wait until the senior school stage. The term *top line* will do for juniors.

When we come to ordinary 2-figure and 3-figure multipliers, which of the following processes is preferable, the first or the second?

34261	34261
43	43
102783	1370440
137044	102783
1473223	1473223

The first is the old-fashioned method; the second is newer and at present is popular. The second is often advocated because (1) it leads on more naturally to the rational multiplication of decimals, (2) it is preferable to multiply by the more important figure first, if only because the first partial product is a rough approximation to the whole product.

The first reason does not appeal to me at all, for I am very doubtful about the allied method of multiplication of decimals. The second reason is undoubtedly a good one.

Numerous tests of the comparative merits of the two methods have shown that the old method leads to a much greater accuracy than the new, and to me that seems greatly to outweigh the advantage of the new method. Slower boys seem to have much more confidence in a method where they have to begin with both units figures, as they do in addition

and subtraction. In any case I deny that the newer method is " more intelligent " than the old.

Division

Begin by instructing the children to write down in standard division form such little division sums as they know from their tables. Teach them the terms *dividend*, *divisor*, and *quotient*: we can hardly do without them.

$2|\underline{6}$ $\quad$ $3|\underline{9}$ $\quad$ $3|\underline{7}$ $\quad$ $5|\underline{9}$

$\underline{3}$ $\quad$ $\underline{3}$ $\quad$ $\underline{2}$, and 1 over $\quad$ $\underline{1}$, and 4 over.

Now teach them the use of the term *remainder*, and to write the letter *R* for it.

$5|\underline{9}$ $\quad$ $4|\underline{6}$

$\underline{1}$, R 4 $\quad$ $\underline{1}$, R 2

Now 2-figure dividends, within the tables they know.

$4|\underline{36}$ $\quad$ $8|\underline{47}$ $\quad$ $9|\underline{79}$

$\cdot\underline{9}$ $\quad$ $\cdot\underline{5}$, R 7 $\quad$ $\cdot\underline{8}$, R 7

Now 2-figure dividends beyond the tables they know.

$4|\underline{93}$

" 4's into 93? the tables do not tell us. Then let us take our 4 times table further:

$$10 \times 4 = 40$$
$$11 \times 4 = 44$$
$$\cdots\cdots\cdots$$
$$23 \times 4 = 92$$
$$24 \times 4 = 96$$

" Evidently 4's into 93 are 23, and 1 R. Hence

$4|\underline{93}$

$\underline{23}$, R 1

" But we need not have written out that long table; we may work in this way:

" 4's into 9? **2** and 1 over; write down the **2**.
" By the side of the 1 over, write down the 3, to make **13**.
" 4's into 13? **3**, and 1 over. Write down the **3**.
" The last 1 over is our Remainder.

" But what does this mean? When we said 4's into 9 we really meant 4's into **90**, and when we wrote down the 2, the 2 really meant **20**. Here is a better way of showing it all, and we will write the figures of the answers *above* the dividend, instead of below it.

```
  23
4)93
  8   = 80 = 20 times 4
  13
  12      =  3 times 4
  1 R
```

" First we took from the 93, **20** times 4, and had 13 left.
" Then we took from the 13, **3** times 4, and had 1 left.
" Altogether we took from the **93, 23** times 4, and had 1 left."

A little work of this kind will suffice to justify the process to the brighter children; a few will grasp it fully. The dullards will not understand it all; they want the clear-cut rule, and explanations merely worry them.

Now consider an ordinary long division sum; say, divide 45329 by 87. Let the children write out the 87 times table, to 9 × 87.

1 × 87 = 87	4 × 87 = 348	7 × 87 = 609
2 × 87 = 174	5 × 87 = 435	8 × 87 = 696
3 × 87 = 261	6 × 87 = 522	9 × 87 = 783

(In making a table like this note that 3 times = 2 times

+ 1 time, 5 times = 3 times + 2 times, &c., and so save the labour of multiplying; only multiplication by 2 is necessary; all the rest is easy addition.)

```
  ··
87)45329
```

" 87's into 4? won't go: 4 is not big enough; put a dot over it.

" 87's into 45? won't go: 45 is not big enough; put a dot over the 5.

" 87's into 453? *will* go, because 453 is bigger than 87. How many times?"

Look at the table, and take the biggest number (435) that can be subtracted from 453. The 435 is **5** times 87. Place the 5 over the 3 in the dividend, write the 435 under the 453, and subtract; the difference is 18.

Bring down the 2 from the dividend, placing it to the right of the 18, making 182. Look at the table again, and take the biggest number (174) that can be taken from the 182, &c.

```
   ··521
87)45329
   435
   ·182
    174
   ··89
     87
     ·2 R
```

" Thus we know that **87** is contained **521** times in **45329**, and that there is 2 (the *Remainder*) to spare.

" What is the biggest R we could have? Could it be 87? Why not?"

Teach the children the usual verification check: multiply the divisor by the quotient, add R to the product, and so obtain the original dividend.

(Do not forget, when introducing formulæ later, to utilize the *D*, *d*, *Q*, and *R*. D = *d*Q or *d*Q + R.)

Now pour a little gentle scorn upon making out a special multiplication table for every division sum: " We must give up such baby tricks ". But that leads us to what some beginners in division find very difficult—how to tell the number of times a big divisor will go into one of the numbers derived from the dividend:

$$69\overline{)342}$$

" Instead of saying ' 69's into 342 ', we cut off the last figure of the 69 and of the 342 and say 6's into 34 instead. This seems to be 5, but the 5 *may* be too big, because of the carry figure; we find it *is* too big, so we try 4 instead."

Warn the children that if, after subtracting at any step, they have a difference greater than the divisor, the figure they have just put into the quotient is too small. Rub this well into the dullards.

Warn them, too, that, above every figure in the dividend, they *must* place either a dot or a figure for the quotient.

$$\begin{array}{r} \cdots 341,\ \mathrm{R} = 267. \\ 329\overline{)112456} \end{array}$$

Similarly in short division except that the dots and figures go below:

$$\begin{array}{r} 7\overline{)13259} \\ \cdot 1894,\ \mathrm{R} = 1. \end{array}$$

A dot is not a very suitable mark, owing to confusion with a decimal point; it is, however, in common use. If no mark is used, figures get misplaced and errors made. The marks may be dropped later.

The advantage of the method of placing the quotient *over* instead of to the right of the dividend, is that children are less likely to forget to write down noughts when these are required.

Let division by factors stand over until the senior school. The calculation of the remainder is puzzling to beginners. Divide 34725 by 168. Suitable factors of 168 : $4 \times 6 \times 7$.

4|34725 units.
6|·8681 fours, R = 1 unit.
7|·1446 twenty-fours, R = 5 fours.
··206 one hundred and sixty-eights, R = 4 twenty-fours.

$$\text{Total Remainder} = (24 \times 4) + (4 \times 5) + 1$$
$$= 117.$$
$$\textit{Quotient} = 206, \text{R} = 117.$$

Avoid the Italian method, except perhaps with A Sets. With average children the method is productive of great inaccuracy.

In fact, avoid all short cuts until main rules are thoroughly mastered. For instance if a boy has to multiply by 357, do not teach him to multiply by 7, and then multiply this first partial product by 50 to obtain his second partial product; it is simply asking for trouble.

Of course, practised mathematicians do these things, but we have to think of *beginners*. Teach a straightforward method, and stick to it. Hints as to " neat dodges " and about " short cuts " are for the few, not for all.

CHAPTER V

Arithmetic: Money

Money Tables

No part of arithmetic is more important than the various manipulative processes of money. It is with us every day of our lives, and accuracy is indispensable. The ordinary money tables must be *known*, and thus more ding-dong work is necessary. This is mainly a question of a knowledge of the 12 times table. Five minutes' brisk mental work twice

a day will *pay*, sometimes with and sometimes without the blackboard, and sometimes on paper.

Associate with the 12 times table:

$1 \times 12 = 12$	$12d. = 1s.$
$2 \times 12 = 24$	$24d. = 2s.$
$3 \times 12 = 36$	$36d. = 3s.$, &c. &c.

Day by day drill:

$80d. = ?$ $83d. = ?$ $84d. = ?$ $89d. = ?$ &c.
Pence in $7s.$? in $7s.\ 3d.$? in $9s.\ 9d.$? &c.

and so every day until accurate answers up to $144d.$ are instantaneous. If the boys are familiar with two definite landmarks in each " decade ", 20 and 24, 30 and 36, 40 and 48, &c., the necessary additions for the other numbers of each decade are simple.

Associate the farthings table with the 4 times table, and the shillings table with the 20 times table, which is easily learnt.

Let every mental arithmetic lesson at this stage include simple addition and subtraction of money, especially the addition of short columns of pence.

Elementary facts concerning the coinage should be associated with the money tables, and in this connexion do not forget *guineas* (which often figure in subscriptions and in professional fees) and Bank of England *notes*.

At a later stage the boys should be taught such commonplace facts about the coinage as every intelligent person ought to know, e.g. the nature of " standard " gold and silver, degrees of " fineness ", the nature of the present legalized alloy for " silver " coinage, the market prices of pure gold and silver, the nature of bronze; alloys; tokens; the Mint.

Reduction

Reduction is not likely to give serious trouble, if the tables are known. The commonest mistake is to multiply

instead of divide, or vice versa. Teach the boy to ask himself every time whether the answer is to be *smaller* or *larger*; if smaller, to divide; if larger, to multiply. But " guineas to pounds ", and the like, is a type of sum that baffles the slow boy and requires special handling.

Subtraction

There is something to be said for avoiding, at first writing farthings in the usual fractional form and for giving them a separate column:

£	*s.*	*d.*	*f.*
47	14	6	1
21	19	4	3
25	15	1	2

let the children use the fractional forms a little later, when they may be made a useful introduction to fractions. The alternative is to omit farthings altogether in the early stages.

Multiplication

How is this to be done? Whatever method is adopted, a percentage of inaccurate answers seems almost to be inevitable. We set out a sum by each of the 4 methods commonly used. Multiply £7, 15*s.* 10½*d.* by 562.

I.

£	*s.*	*d.*			
7	15	10½			
		10			
77	18	9	= £7	15	10½ × 10
		10			
779	7	6	= £7	15	10½ × 100
		5			
3896	17	6	= £7	15	10½ × 500
15	11	9	= £7	15	10½ × 2
467	12	6	= £7	15	10½ × 60
4380	1	9	= £7	15	10½ × 562

II.

562
15
20|8430*s.*
£421, 10*s.* 0*d.*

12|5620*d.*
20| 468*s.* 4*d.*
£23, 8*s.* 4*d.*

2|562*h.*
12|281*d.*
23*s.* 5*d.*

£7	0	0	× 562 =	£3934	0	0
£0	15	0	× 562 =	£ 421	10	0
£0	0	10	× 562 =	£ 23	8	4
£0	0	½	× 562 =	£ 1	3	5
£7	15	10½	× 562 =	£4380	1	9

III.

562 at £7, 15*s.* 10½*d.*

7

	3934	0	0	= £7	0	0	× 562
10/- = ½ of £1	281	0	0	= 0	10	0	× 562
5/- = ½ of 10/-	140	10	0	= 0	5	0	× 562
-/7½ = ⅛ of 5/-	17	11	3	= 0	0	7½	× 562
-/3 = 1/20 of 5/-	7	0	6	= 0	0	3	× 562
	£4380	1	9	= £7	15	10½	× 562

IV.

£7, 15*s.* 10½*d.* × 562
= £7·79375 × 562
= £4380·0875
= £4380, 1*s.* 9*d.*

My own experience, and this corresponds to the results of many inquiries, is that the second method produces the best results; then the third (" practice ") method, provided that pupils have been well drilled in aliquot parts (though some always seem to find division more difficult than multiplication); then the first method. The last is a good method for older pupils who have learnt to decimalise money readily, but not for younger pupils or for slower older pupils.

The ordinary method (the first method) is curiously productive of errors; in the course of a long experience I have *never* known a whole class, without exception, get a sum right by this method, even after they had had several months' practice. The second method generally leads to untidy and unsystematic marginal work. This marginal

work should be made an integral part of the working of the sum, and should not be looked upon as scrap work.

Division

Dr. Nunn's suggestion that the process of working may be set out in the following way may well be followed. All pounds are kept in one vertical column, shillings in another, and so on. It is very neat and compact. Allow plenty of space across the paper. Example: Divide £3541, 14*s.* 9½*d.* by 47.

£	*s.*	*d.*	*f.*
··75	7	1	1
47)3541	14	9	2
329	→320	→60	→88
·251	334	69	90
235	329	47	47
·16 × 20	··5 × 12	22 × 4	43 R

Answer: £75, 7*s.* 1¼*d.* and 43 farthings over.

Other methods have been devised, but this old method is probably best and most readily learnt.

To ensure a full understanding of the nature of the " remainder " a sum like the above should be followed up by two others:

1. Take 43 farthings from the dividend; then divide again.
2. Add 4 farthings to the dividend; then divide again.

Even slower boys can usually explain the (to them) rather surprising new answers.

CHAPTER VI

Weights and Measures

Units and Standards

Consider what weights and measures are used, to what extent, and how, *in practical life*. Teach these, and teach them well, and let all the rest go. A coal merchant concerns himself with tons, cwt., qr., never with lb. and oz.; a grocer with cwt., qr., lb., oz., never with drams and rarely with tons; a farmer with acres, quarter-acres, and perches; a builder with yards, feet, and inches; a surveyor with chains and links; and so generally. The completer tables of weights and measures are generally given in the textbooks as a matter of convenience, but, in practical life, the whole table is seldom wanted by any one person. A teacher who gives boys reduction, multiplication, or division sums, say, from tons to drams (or even to ounces), or from square miles to square inches, is simply proclaiming aloud his incompetence: perhaps he is the slave of some stupidly written textbook; certainly he is lacking in judgment. The main thing is to make the boys thoroughly familiar with the few weights and measures that are commonly used, and to give them a fair amount of practice in the simpler transformations of comparatively small quantities; and to let all the rest go.

Teach clear notions of *units* and *standards*. Show how unintelligent we British people have always been in our choice of units. We have, for instance, determined our *inch* by placing three grains of barley in line; we have selected our *foot*, because 12 of the inches roughly represent the length of a man's foot; we have determined our smallest weight (the *grain*), by adopting the weight of a dried grain of wheat. That such things vary enormously did not trouble our forefathers at all. Tell the boys that at one time the French people had similar unsatisfactory weights and measures

but that now they have changed to a system much more rational.

Let the various tables be *learnt* and learnt perfectly.

Weight

1. *Avoirdupois* (not used for the precious metals).—Let the table to be learnt include the *oz.*, *lb.*, *qr.*, *cwt.*, *ton.* Note that the standard weight is the *pound*, which consists of 7000 *grains*. (A dried grain of wheat, though roughly a grain in weight, is not, of course, a standard. A *grain* is 1/7000 part of a *pound*.)

Teach the *stone* as a separate item: normally 14 lb., but for dead meat, 8 lb.

Give easy sums for practice in:

(1) *tons*, *cwt.*, *qr.* (coal and heavy goods).
(2) *cwt.*, *qr.*, *lb.* (wholesale grocery).
(3) *cwt.*, *st.*, *lb.* (wholesale meat purchases).
(4) *lb.*, *oz.* (retail grocery and meat).

Note that an oz. of water or any other fluid is an avoirdupois ounce, like the ounce of any common solid, and contains 7000/16 or 437½ grains.

In making up arithmetical examples, utilize as far as possible the quantities (sacks, bags, chests, &c.) representing the unit purchases of tradesmen and others, though the problem given will often depend on the locality. For instance, problems on *crans* and *lasts* of herrings, or on *trusses* of hay or straw, would be quite inappropriate in big inland towns. The teacher should, for problem purposes, make a note of points like the following: weight of a chest of tea, ¾ cwt.; sack of coal or of potatoes, 1 cwt.; bag of flour, 1¼ cwt.; bag of rice, 1½ cwt.; truss of straw, 36 lb.; truss of new hay, 60 lb., of old hay 56 lb.; a brick, 7 lb.; 1000 bricks, 3⅛ tons; 100 lb. of wheat produces 70 lb. of flour which produces 91 lb. of bread; and so on. Everyday quantities of this kind

give a reality to problems in arithmetic that even the non-mathematical boy appreciates.

2. *Troy*, used by jewellers.—Let the table to be learnt include the *grain*, *dwt.*, *oz.*, *lb.* It is important to remember that the Troy *ounce* is heavier than the common (avoirdupois) ounce, since it contains 480 grains, as against $437\frac{1}{2}$. But the Troy *pound* is lighter than the common pound, since it contains only 12 ounces and therefore 5760 grains, as against 7000 in the avoirdupois lb.

3. *Apothecaries'*.—The old weights have gone out of use. Drugs are generally used in very small quantities, and the basic weight is the *grain* (the grain is a constant weight for all purposes). A quantity of drugs weighing more than a few grains is expressed as a fraction of an ounce avoirdupois.

N.B.—Ignore the avoirdupois *dram* ($\frac{1}{16}$ oz.) and the druggists' old *scruple* and *drachm* weights. The *dram* and *drachm* were not the same.

Length

Let the main table to be learnt include the *in.*, *ft.*, *yd.*, *pole*, *fur.*, *mile*, and let the *link*, *chain*, and *fur.* be included in a separate table. Remind the boys that the chain is the length of a cricket pitch.

Give easy sums for practice in:

(1) *yd.*, *ft.*, *in.* (builders, &c.).
(2) *poles*, *yd.*, *ft.* (farmers, &c.).
(3) *miles*, *chains*, *links* (surveyors).
(4) *miles*, *yd.* (road distances, &c.).

Measures that may be drawn from practical life for problem use are almost innumerable. The sizes of battens, deals, and planks will be learnt in the manual room; notes of the sizes of other materials used by builders—slates, glass, door-frames, &c., &c.—may be made from time to time; size of an ordinary brick, $8\frac{3}{4}'' \times 4\frac{1}{4}'' \times 2\frac{3}{4}''$ (note the $\frac{1}{4}''$ all

round for jointing), square tile, $9\frac{3}{4}'' \times 9\frac{3}{4}'' \times 1''$ or $6'' \times 6'' \times 1''$; machine-printed wall-paper, $11\frac{1}{2}$ yd. $\times$ 21″; hand-printed, 12 yd. $\times$ 21″; French, 9 yd. $\times$ 18″; sheets of paper, foolscap, $17'' \times 13\frac{1}{2}''$ (see *Whitaker* for other sizes); bound books, foolscap 8vo, $6\frac{3}{4}'' \times 4\frac{1}{4}''$ (see *Whitaker*); skein of yarn = 120 yd., hank = 7 skeins; railway gauge $4' 8\frac{1}{2}''$ (12′ of roadway for single track, 23′ for double); equator, 24,902 miles; polar diameter, 7926 miles; fathom, 6′; knot, 6080′ (40 knots = 46 miles). These are only a tithe of the everyday measurements that may be used for making up problems. Such problems are far more valuable than the hackneyed reduction sums given in the older textbooks.

N.B.—The *ell*, *league*, and such foreign lengths as the *verst*, may be ignored. The *cubit* is worth mentioning.

Area

Let the table to be learnt include the sq. *in.*, *ft.*,, *yd. pole*, the *rood*, the *acre*, sq. *mile*. It is useful to remember that an *acre* = 10 sq. chains, or a piece of ground 220 yd. $\times$ 22 yd., or a piece just about 70 yd. square.

Give easy sums for practice in:

(1) *sq. miles*, *acres* (areas of counties, &c.).
(2) *ac.*, *ro.*, *sq. poles* (farmers, &c.).
(3) *sq. yd.*, *sq. ft.*, *sq. in.* (builders, &c.).

Familiar areas for problem purposes: Lawn tennis court, 78′ $\times$ 36′ or 78′ $\times$ 27′; Association football ground, 120 yd. $\times$ 80 yd.; Rugby, 110 yd. $\times$ 75 yd.; croquet lawn, 105′ $\times$ 84′; Badminton court, 44′ $\times$ 20′; &c.

Volume

Table: c. *in.*, *ft.*, *yd.*

Let sums for practice be of the simplest, e.g. the number of cubic yards of earth excavated from a trench; the number of

cubic feet of brickwork in a wall; the cubic capacity of a room or of a building; the volume of the Earth in cubic miles.

Capacity

1. *Liquids.*—Table: *gill, pt., qt., gall.* Casks have a variety of names: *barrel* of ale = 36 gall.; *hogshead* of ale = 54 gall., of wine = 63 gall., &c. A wine bottle = $\frac{1}{6}$ gall.; Winchester quart = $\frac{1}{2}$ gall.

2. *Dry Goods* (corn, &c.).—Table: *peck, bushel, quarter.* The quarter-peck is called a " quartern "; the half-peck is the equivalent of a gallon. The gallon is a kind of link between the liquid and dry measures.

There is now a tendency to substitute *weight* for measure. Problems on capacity reduction are hardly worth doing, except small problems that may be done mentally. But problems involving transformations between capacity and weight are common, and ample practice is necessary. *N.B.*— 1 gall. of water weighs 10 lb. " A pint of pure water weighs a pound and a quarter."

Liquid medicine measure (mainly solutions in water).

Table: 60 minims = 1 fluid drachm.
8 fluid drachms = 1 fluid ounce.
20 fluid ounces = 1 pint.

The fluid ounce is the same as the common (avoirdupois) ounce, and therefore weighs 437$\frac{1}{2}$ grains. But it contains 480 minims, and therefore a minim weighs rather less than a grain. The *minim* may be thought of as a " drop ", though of course drops vary greatly in size.

Doctors' prescriptions may be discussed, rather than sums worked. If a solid drug is prescribed, the amount is expressed in *grains* or in fractions of an *ounce*; if liquid, then *minims, drachms,* or *ounces.*

Time

The *second*, *minute*, *hour*, *day*, *week*, give little trouble. The variable *month* requires careful explanation. Teach the doggerel "Thirty days hath September", &c., or furnish some alternative mnemonic. Explain "leap" year and its determination.

Few problems of reduction are necessary, and these should be easy. A few on the calendar are advisable, and a few dealing with speeds.

Useful Memoranda

Other useful memoranda for problem-making.—(The quantities are approximations only and should be memorized. They are useful when closer approximations have to be evaluated):

1 *cubic foot of water* = $6\frac{1}{4}$ gall. = $62\frac{1}{2}$ lb.
1 *cubic inch of water* = $252\frac{1}{2}$ grains.
A common cistern $4' \times 3' \times 2\frac{1}{2}' = 30$ c. ft. $= 187$ gall. $= \frac{5}{6}$ ton.
1 *ton of water* = 36 c. ft. = 224 gall.
1 *gallon of water* = $277\frac{1}{2}$ c. in. = 10 lb.
1 *ton of coal* occupies about 40 c. ft. (hence 25 tons need a space $10' \times 10' \times 10'$).

Wall Charts

A few permanent charts are useful on a wall of the classroom where weights and measures are taught: an outline plan of (1) the town or village showing the over-all dimensions, length, breadth, and area; (2) the school-site and buildings; (3) the school itself; (4) the actual classroom. (5) Diagram to scale to show that $5\frac{1}{2}$ yd. $\times$ $5\frac{1}{2}$ yd. = $30\frac{1}{4}$ sq. yd. (often used for a first lesson in fractions). (6) A chart giving the *weights* of a few familiar objects in and about the school, and the *capacities* of a few others. ***See that these charts are used and known.***

The Metric System

Some knowledge of this system is necessary, if only because of the work in the physical laboratory. The beginner may be shown a metre measure side by side with a yard measure, and simply be told that it is rather longer, and had its origin in France. As the boy goes up the school he will learn that its length is about 39·37 in., and is the measured fraction of a quadrant of the earth's surface. A little later still, he will be told how the French measured the actual length of an arc of one of their meridians, and how they determined the latitude of each place at the end of the arc. This easy astronomical problem is usually worked out in a Fifth Form geography lesson.

The cubic decimetre and the litre, the cubic centimetre and the gram, are, as derivations of the initial metre, always a source of interest to boys.

The boys should memorize the few usual approximate equivalents between the British and metric systems, e.g. 1 metre = 39·37 in.; 1 kilogram = 2·2 lb.; 1 litre = 1·76 pints; 1 gram = 15·43 grains; 1 are = $\frac{1}{40}$ acre. With these they can quickly estimate quantities in terms of metric units. For instance, a Winchester quart will hold 4/1·76 litres = 2·27 litres = 2270 c. c.; 1 hectare = $2\frac{1}{2}$ acres; and so on.

But do not forget to enter a defence in favour of our own system of weights and measures, if the metric system is advocated on purely *scientific* grounds. Sixth Form boys are always interested in this. In the first place, the metre was not measured accurately; in the second place, it is a local and not a universal unit; it depends upon the length of a particular meridian in a particular country. The meridian the French measured was an ellipse, not a circle, and not a true ellipse at that. Had they utilized the polar axis (a fixed length) instead of a meridian (a variable length), their unit would have been more scientific, for it would have been universal, and it could have been measured more accurately.

The length of the polar axis is *very nearly* 500,500,000 in., so that *the inch already bears a simpler relation to the polar axis than the metre does to its own meridian quadrant.* If we adopted a *new* inch, viz. 1/500,000,000 of the polar axis, it would make but a very slight change in our linear measurements, and then, curiously enough, a cubic foot of water would weigh almost exactly 1000 oz. (instead of 997). Our present ounce weight would have to be increased by only $\frac{1}{18}$ part of a grain! Moreover, the new cubic foot would contain exactly 100 half-pints. Such a new system would be incomparably more scientific than the metric system.

Thus the opponents to the adoption of the metric system have sound arguments to support their views. The metre has on its side the virtue of being the basic unit of a convenient and simple system; but scientifically it is a poor thing.

There is no need for the boys to learn the metric tables. But they should learn the three Latin prefixes *deci, centi, milli,* and know that these represent *fractions*; and the three Greek prefixes *deca, hecto, kilo* representing *multiples.* These learnt thoroughly, the tables as such are unnecessary. But with three or four exceptions the multiples and sub-multiples are hardly ever wanted.

The best exercises on the metric system are those based on laboratory operations.

CHAPTER VII

Factors and Multiples

The term " factor " and " multiple " should be used when the tables are being taught, though without formal definition. " $3 \times 7 = 21$; we call 3 and 7 *factors* of 21."

Give me a factor of 6? 3; another? 2; a factor of 30? 2;

another? 3; another? 5. A *multi*tude of people means *many* people, and a *multi*ple of a number means a bigger number containing it *many* times, though "many" *may* not be greater than 2. Now think of your 5 times table. Give me a multiple of 5? 15; another? 30; another? 35.—After a little of this work, the terms *factor* and *multiple* will become part of the boys' familiar vocabulary. "Common" factor and multiple will come later. *One idea at a time.*

Tests of Divisibility

Prime Factors.—Tests of divisibility for 2, 3, 5, and 10 may readily be given in the Junior School or Department; those for 4, 8, 9, 11, 12, 25, 125, a year or two later. At first, give the rules dogmatically.

"A number is divisible by 2 if it is an even number.
" " " 3 if the sum of its digits is divisible by 3.
" " " 5 if it ends in a 5
" " " 10 if it ends in a 0."

Justification, not "proofs", of such rules is commonly given in Form IV. The reasoning, which is quite simple, depends on the principle that a common factor of two numbers is a factor of their sum or their difference. Never mind the general proof; at this stage merely *justify* the principle by considering a few particular instances, and these readily emerge from the multiplication table; for instance:

5 fours = 20	9 fives = 45
7 fours = 28	7 fives = 35
12 fours = 48	2 fives = 10

We know that 5 fours added to 7 fours make 12 fours, i.e. 4 is a factor of 20 and of 28, and is also a factor of 20 + 28.

Again, we know that 7 fives from 9 fives is 2 fives, i.e.

5 is a common factor of 35 and 45, and is also a factor of $45 - 35$.

Divisibility by 2.—Consider any even number, say 754; $754 = 750 + 4$. Since 2 is a factor of 10 and therefore of the multiple 750, and is also a factor of 4, it is, by our rule, a factor of $750 + 4$ or 754.

Divisibility by 5.—Consider any number ending in 5, say 295; $295 = 290 + 5$. Since 5 is a factor of 10 and therefore of the multiple 290, and is also a factor of 5, it is, by our rule, a factor of $290 + 5$ or 295.

Divisibility by 3.—Consider any number, say 741.

$$\begin{aligned} 741 &= 700 + 40 + 1 \\ &= (100 \times 7) + (10 \times 4) + 1 \\ &= (99 \times 7) + 7 + (9 \times 4) + 4 + 1 \\ &= (99 \times 7) + (9 \times 4) + 7 + 4 + 1 \\ &= (99 \times 7) + (9 \times 4) + 12. \end{aligned}$$

Now 3 is a factor of 9 and therefore of all multiples of 9; it is also a factor of 12. Since 3 is a factor of 99×7 and of 9×4 and of 12, it is a factor of their sum, i.e. of 741. Hence, &c.

A formal proof of the principle used should be associated with the algebra later.

The justification of the rule for 4 and 25, 8 and 125, and 9 is equally readily understood, but that for 11 is a little more difficult.

Primes and Composite Numbers

Quite young boys quickly see the distinction between a prime and a composite number and are always interested in the sieve of Eratosthenes.

Third Form boys should be made to memorize the squares of all numbers up to 20; $13^2 = 169$; $17^2 = 289$; &c. (The squares of 13, 17, and 19 must really be learnt; 14^2, 16^2, and 18^2 can be mentally calculated in a second or two, if forgotten.) Then give a little mental practice in extracting

square roots: of 81? of 256? of 361? (Mention that the root sign ($\sqrt{}$) we use is merely a badly written form of the initial letter R.)

Make the class write down, in order, the successive pairs of factors of, say, 36:

$$\begin{array}{r} 2 \times 18 \\ 3 \times 12 \\ 4 \times 9 \\ 6 \times 6 \\ 9 \times 4 \\ 12 \times 3 \\ 18 \times 2 \end{array}$$

Then point out that the second column is the first column reversed, and that the 3 lower horizontal lines are the 3 upper horizontal lines reversed. Hence when we have to write dowu the factors of 36, we need not proceed beyond the fourth line, viz. 6×6, for then we already have all the factors; and the 6, the last trial number, is $\sqrt{36}$. The boys can now appreciate the common rule: When resolving a number into factors, *it is unnecessary to carry our trials beyond its square root*, unless the number is not a perfect square, and then it is advisable to consider the next square number beyond it.

For instance, write down all the factors of 120; 120 is not a square number, but the next square number is 121, the square root of which is 11. Hence we need not proceed with our trial numbers beyond 11, but as 11 does not happen to be a factor, we do not proceed beyond 10. Thus by trial we find that 2, 3, 4, 5, 6, 8, and 10 are factors; and, dividing 120 by each of these, we obtain other factors which pair off with them. The 14 factors of 120 are

2	3	4	5	6	8	10
60	40	30	24	20	15	12

The next step is to teach factor resolution by trials of prime numbers only. This causes no additional difficulty, but the boys should recognize *at once* all the prime numbers

up to, say, 41 (1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41).

" Write down the factors of 391. The next square number beyond is 400. Hence we need not carry our trials beyond $\sqrt{400}$ or 20. By trial we find that the lowest factor is 17. By division we obtain 23, which we recognize as another prime number. Thus 17 and 23 are the only factors."

I have known a C Set of a Fourth Form become expert in factorizing 3-figure numbers, after one lesson. It is work that most boys like.

" Express 360 as a product of factors which are all prime. —Divide by the lowest prime number, 2 if possible; and again and again if necessary; then by the next prime, 3, if possible; and again and again, if necessary. Then by 5, if possible; then 7; then 11; and so on.

$$\begin{aligned} 360 &= 2 \times 180 \\ &= 2 \times 2 \times 90 \\ &= 2 \times 2 \times 2 \times 45 \\ &= 2 \times 2 \times 2 \times 3 \times 15 \\ &= 2 \times 2 \times 2 \times 3 \times 3 \times 5. \end{aligned}$$

Obviously we now have *all* the factors, though our trial division did not proceed beyond 3.

" A neater way of writing down the prime factors of 350 is

$$2^3 \times 3^2 \times 5.$$

The little 3 at the top right-hand corner of the 2 shows the *number* of twos and is called an index.

" Express 18900 as the product of factors which are all prime.

$$\begin{aligned} 18900 &= 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 7 \\ &= 2^2 \times 3^3 \times 5^2 \times 7. \end{aligned}$$

We read, 2 squared into 3 cubed into 5 squared into 7. —An index serves as a useful means of shortening our written work."

At this stage two or three minutes' brisk mental work occasionally will help to impress upon the pupils' minds the

values of the lower powers of the smaller numbers: 2^3, 2^4, 2^5, 3^2, 3^3, 3^4, 4^2, 4^3, 4^4, 5^2, 5^3, 5^4, &c.

Common Factors

" Give me a common factor of 36 and 48: 2; another? 3; another? 4; another? 6; another? 12. Which is the *greatest* of these common factors? 12. We call 12 the *Greatest Common Factor* of 36 and 48. If we write down the prime factors of the different numbers, we can almost *see* the G.C.F. at once.

$$36 = \mathbf{2} \times \mathbf{2} \times \mathbf{3} \times 3.$$
$$48 = \mathbf{2} \times \mathbf{2} \times 2 \times 2 \times \mathbf{3}.$$

Evidently **2** is a common factor of both numbers, and another **2**, and a **3**. Hence, the G.C.F. $= 2 \times 2 \times 3 = \mathbf{12}$, i.e. 12 is the greatest number that will divide exactly into 36 and 48."

" It is neater to write down the factors in the index form. —What is the G.C.F. of 540, 1350, 2520?

$$540 = 2^2 \times 3^3 \times 5^1.$$
$$1350 = 2^1 \times 3^3 \times 5^2.$$
$$2520 = 2^3 \times 3^2 \times 5^1 \times 7$$

We see that 2, 3, and 5 are factors common to all three numbers; from the indices we see that *one* 2, *two* 3's, and *one* 5 are common. Hence the G.C.F. is $2^1 \times 3^2 \times 5^1 = 90$. Note that we write down each prime factor that is common and attach to it the *smallest* index from its own group."

However clear the teaching, I find that there is usually a small number of slow boys who are puzzled by the index grouping. Hence in lower Sets the extended non-indexed groups of factors are preferable. *Always sacrifice a neat method if it leads to puzzlement and inaccuracy.*

Common Multiples

'Give me a multiple of 5: 25; another? 35; another? 55. Give me a multiple of 3: 21; another? 15; another? 60. Give me a common multiple of 3 and 5: 60; another? 15; another? 30. Which is the least of all the common multiples of 3 and 5? 15; i.e. 15 is the smallest number into which 3 and 5 will divide exactly. We call it the *Least Common Multiple.*

"Find the L.C.M. of 18, 48, and 60."

Write down the numbers as products of their factors, expressed in primes.

$$18 = 2 \times 3 \times 3.$$
$$48 = 2 \times 2 \times 2 \times 2 \times 3.$$
$$60 = 2 \times 2 \times 3 \times 5.$$

The L.C.M., being a multiple of the three given numbers, must contain all the factors of the numbers, but it must not contain more, or it will not be the *least* common multiple.

"1. The L.C.M. has to contain all the factors of 18; write them down as part of the answer:

$$\text{L.C.M.} = 2 \times 3 \times 3 \times$$

"2. In order that the L.C.M. may contain all the factors of 48, it must include four 2's and one 3. We have already written down one 2; hence we must write down three more. As we have already written down two 3's, another is not necessary. Hence,

$$\text{L.C.M.} = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times$$

"3. In order that the L.C.M. may contain all the factors of 60, it must include two 2's, one 3, and one 5. We already have two 2's, and a 3, but no 5. Hence we must include a 5.

$$\text{L.C.M.} = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5$$
$$= 720.$$

" It is neater to write down the factors in the index form. —What is the L.C.M. of 54, 72, 240?"

$$54 = 2 \times 3^3.$$
$$72 = 2^3 \times 3^2.$$
$$240 = 2^4 \times 3^1 \times 5^1.$$

We may write down the L.C.M. at once, by writing down every one of the prime factors and attaching to each the *greatest* index of its group:

$$\text{L.C.M.} = 2^4 \times 3^3 \times 5$$
$$= 2160$$

It is well to provide pupils with some little mnemonic, to enable them to keep in mind that:

the *smallest* index concerns the *greatest* C.F.
and the *greatest* index concerns the *least* C.M.

There is much to be said for using the terms *greatest* and *smallest* (or *least*) in arithmetic, and the terms *highest* and *lowest* in algebra. The former terms are obviously correctly applicable to magnitudes. Beginners do not find it easy to appreciate the exact significance of *highest* and *lowest*. The term *measure* is best avoided. The distinction between it and *factor* is a little subtle for boys.

The old-fashioned division methods of G.C.F. and L.C.M. are cumbrous and unnecessary, and slower boys never understand the processes, the formal " proofs " of which are quite difficult enough for Fifth Forms. Numerical illustrations of the principle that a common factor of two numbers is a factor of their difference may be utilized to justify the ordinary G.C.F. procedure, *if* the procedure itself is considered necessary.

CHAPTER VIII

Signs, Symbols, Brackets. First Notions of Equations

Terminology and Symbolism

Mathematical terms should always be used with precision; then formal definitions in all early work will be unnecessary. *Sum*, *difference*, *product*, *quotient* are terms which should be quite familiar even to Juniors; they are the A, B, C of the whole subject. So should the signs $+$, $-$, $\times$, and $\div$. *Multiplicand* and *multiplier*, *dividend* and *divisor* should also become current coin at an early stage, though there is difference of opinion about the first term in this group. I am not quite sure about *subtrahend* and *minuend*, even in the senior school; they are commonly confused. If we bear in mind the English significance of the Latin **-nd-** (gerundive), the **-nd** terms can be explained in a group.

*Multiplica***nd** is a number that **has to be** multiplied.
*Divide***nd** is a number that **has to be** divided.
*Subtrahe***nd** is a number that **has to be** subtracted.

And of course *minue***nd** is a number that **has to be** minus-ed or reduced, but boys *will* confuse minuend and subtrahend. If the terms *are* used let subtrahend come first, and minuend a good deal later.

As for the division sign $\div$, hammer in the fact that the dots stand for numbers, that when we write, e.g., $\frac{4}{5}$ we mean 4 divided by 5, and that we might write out our division table,

either $\left|\begin{array}{l} 5 \div 1 = 5 \\ 10 \div 2 = 5 \\ 15 \div 3 = 5 \end{array}\right|$ or $\left|\begin{array}{l} \frac{5}{1} = 5 \\ \frac{10}{2} = 5 \\ \frac{15}{3} = 5 \end{array}\right|$

since both mean exactly the same thing.

Algebraic letter symbols may be introduced at a very

early stage. (Do not look upon arithmetic and algebra as distant cousins, but as twin brothers, children to be brought up *together*.) Begin with the simple consideration of lengths and areas. Establish by a few numerical examples that the area of a rectangle may be determined by multiplying length by breadth. Select rectangles whose sides are exact inch-multiples, ignoring all fractions until later. Then introduce the notion of a " formula "—a convenient shorthand means of keeping an important general arithmetical result in our mind. " We have found that however many inches long, and however many inches broad, a rectangle is, the area in square inches is equal to the product of the inches length and the inches breadth. It is easy to remember this by taking the first letter of the word length (l), of the word breadth (b), and of area (A), and writing the result so:

$$l \times b = \text{A}.$$

But we generally save time by writing $lb = \text{A}$, omitting the multiplication sign. Always remember that when in algebra two letters are written side by side, a multiplication sign is supposed to be between them. Instead of the letters l, b, and A, any other letters might be used."

Rub in well the principle taught, giving a few simple evaluations.

Now consider a square area, $l \times l$, or $m \times m$; ll or mm. (Distinguish between inches square and square inches.)

Follow this up with cases of rectangular solids, and establish such formulæ as $\text{V} = l \times b \times h = lbh$; then the cube, $\text{V} = aaa$.

" When we were working factors, we adopted a plan for shortening our work. Instead of writing $4 \times 4 \times 4$, we wrote 4^3, the little 3 at the top right-hand corner (which we called an *index*) showing the *number* of 4's to be multiplied together. So in algebra.

aa may be written a^2,
aaa may be written a^3.

Then what does a^5 mean? a^3b^2?" And so on.

Avoid all difficult examples at this stage. The main thing is to teach the new *principle*. Keep the main issue clear. Let hard examples wait.

Brackets

"A pair of brackets is a sort of little box containing something so important that it has to receive special attention. The brackets generally contain a little sum all by itself. If I write

$$9 + (7 + 3)$$
$$\text{or } 9 + (7 - 3),$$

I mean that the answer to the little sum inside the brackets has to be added to the 9. If I write

$$9 - (5 + 2)$$
$$\text{or } 9 - (5 - 2),$$

I mean that the answer to the little sum inside the brackets has to be *subtracted* from the 9.

"Now I will work out the four sums:

$$9 + (7 + 3) = 9 + 10 = 19.$$
$$9 + (7 - 3) = 9 + 4 = 13.$$
$$9 - (5 + 2) = 9 - 7 = 2.$$
$$9 - (5 - 2) = 9 - 3 = 6.$$

Are the brackets really of any use? Let us write the same sums down again, leaving the brackets out, and see if we get the same answer:

$$9 + 7 + 3 = 19$$
$$9 + 7 - 3 = 13$$
$$9 - 5 + 2 = 6$$
$$9 - 5 - 2 = 2.$$

The first two answers *are* the same, the last two are not. But look at the last two again. It looks as if they had been changed over. Thus

$9 - (5 + 2)$ *is the same as* $9 - 5 - 2$,
and $9 - (5 - 2)$ *is the same as* $9 - 5 + 2$."

With a few easy examples like this, we are in a position to justify the rule that a + sign before a bracket does not affect the + and − signs within, but that a − sign before a bracket has the effect of converting + and − signs within to − and + respectively. Thus you are able to give the rule and to justify it. That is enough at present. Give enough easy examples to ensure that the rule is known and can be applied with certainty. " Proof " should play no part at this early stage. The algebraic minus sign comes later.

Now show the effect of a multiplier.

$$4(6 + 3) = 4 \times 9 = 36.$$

" We *might* have multiplied the two numbers separately in this way,

$$4(6 + 3) = 24 + 12 = 36,*$$

and when the brackets contain both letters and numbers we ***must*** do it in that way:

$$5(N + 3) = 5N + 15$$

for we cannot add N to 3."

First Notions of Equations

Simple equated quantities.—For convenience at this early stage we may call the following an equation:

$$7 + 5 = 21 - 9.$$

Establish the **fundamental fact** about an equation that we may add to, subtract from, multiply, or divide each side of an equation by any number we like, provided that we use the same number for both sides. Give several examples, to illustrate each of the four operations. To enable the class to

* A repetition of the sign of equality in the same line should never be allowed in school practice; it is almost always ambiguous. We do it sometimes in this book merely to save space.

see the operations more clearly, put the original quantities in brackets. Thus:

$$\begin{aligned} (7+5)+4 &= (21-9)+4 \\ (7+5)-4 &= (21-9)-4 \\ 4(7+5) &= 4(21-9) \\ \frac{7+5}{4} &= \frac{21-9}{4} \end{aligned}$$

Do not talk about " proofs "; you are merely verifying particular instances, to enable the boys to see that your rules are not arbitrary but are based on reason. A little practice in such easy examples as the following may usefully follow.

5 times a certain number is 65. *What is the number?*

" We have to find a certain unknown number. Let us call it N. The sum tells us that

$$5N = 65.$$

Divide each side of the equation by 5; then, N = 13, the number we require."

The class is not quite ready for such an example as the following, but they can follow out their teacher's reasoning, and their appetite is whetted.

Divide 32 *into two parts, so that* 5 *times the smaller is* 3 *times the greater.*

" The two parts added together must make 32, so that one part taken from the 32 must give the other.

Let S stand for the smaller number.
Then 32 — S must represent the greater.
The sum tells us that

5 times the smaller = 3 times the greater.

So we may write

$$5S = 3(32 - S).$$

Removing the brackets, by multiplying by 3,

$$5S = 96 - 3S.$$

We cannot see the value of S from this, because we have S's on both sides of the equation. But, adding 3S to each side, we have:

$$\begin{aligned} 5S + 3S &= 96 - 3S + 3S \\ \therefore \quad 8S &= 96 \\ \therefore \quad S &= 12, \text{ the smaller number} \\ \text{and} \quad 32 - S &= 20, \text{ the greater number.} \end{aligned}$$

Now let us verify the results;" &c.

CHAPTER IX

Vulgar Fractions

First Notions of Fractions

Vulgar or Decimal Fractions first? The first notions of vulgar fractions will be given in the preparatory Forms, where the significance of at least halves and quarters will be understood and the manner of writing them down known. In the lower Forms of the senior school, it is probably wise first to give a few lessons on the nature and manipulation of vulgar fractions, then to proceed with decimals, and to return to the more difficult considerations of vulgar fractions later.

The first thing is to get clearly into the child's mind that a fraction is a *piece* of a thing, a piece " broken off " a thing. Take one of several similar things (sticks, apples), and break off or cut off a " fraction " of it. Cut one of the things into 2 equal parts, and introduce the term halves; into 3, and introduce the term thirds; into 4, and the term

quarters; and see that the terms halves, thirds, quarters, fifths, &c., are made thoroughly familiar. "I have cut this apple into 8 parts: give me 1 eighth; give me 5 eighths. I add 2 of the eighths and 5 of the eighths together: how many eighths have I?"

We have a special way of writing down fractions. We draw a line; under it we write the *name* of the parts we cut the apple into, over it we write a figure to show the *number* of the parts we take: thus

$$\frac{3}{\text{fifths}}, \quad \frac{4}{\text{sevenths}}.$$

Parts of the same name may be added together. Just as we say

$$2 \text{ apples} + 3 \text{ apples} = 5 \text{ apples},$$

so we may say,

$$2 \text{ sevenths} + 3 \text{ sevenths} = 5 \text{ sevenths},$$

and we write,

$$\frac{2}{\text{sevenths}} + \frac{3}{\text{sevenths}} = \frac{5}{\text{sevenths}}.$$

Let the child see clearly that the fraction shows

$$\frac{\textit{number} \text{ of parts}}{\textit{name} \text{ of parts}},$$

and, a little later on, introduce the terms numerator and denominator:

$$\frac{\text{numerator}}{\text{denominator}},$$

where *num* = number and *nom* = name. If the children learn Latin, give the Latin words.

Then come to fractions of *collections* of things: $\frac{1}{2}$ of the class of children, $\frac{1}{4}$ a basket of apples, $\frac{1}{10}$ of a lb. of cherries. The way is now paved to fractions of mere numbers: $\frac{1}{4}$ of 32; $\frac{1}{9}$ of 27; and so on. But at this stage avoid the terms abstract and concrete.

For illustrating fractional processes, every teacher will utilize concrete examples of some kind drawn from everyday life. As the number 60 contains numerous easy factors, fractions of a crown (60*d.*) and of an hour (60 minutes) make good examples for mental work.

Mental work may profitably be undertaken as soon as the nature of a fraction is fully grasped.

" Number of pence in $\frac{1}{3}$ of 1/-? in $\frac{2}{3}$? in $\frac{1}{6}$? in $\frac{4}{6}$? in $\frac{1}{12}$? in $\frac{8}{12}$?" Let the children thus discover that $\frac{2}{3} = \frac{4}{6} = \frac{8}{12}$, that different fractions may therefore have the same value. Thus we come to the notion of " cancelling " and its converse.

Again:

" How many minutes in $\frac{1}{3}$ of an hour? 20.
" " " $\frac{1}{4}$ " " ? 15.
" " " $\frac{1}{5}$ " " ? 12.

Thus $(\frac{1}{3} + \frac{1}{4} + \frac{1}{5})$ of an hour = 47 minutes.

How may we express 47 minutes as the fraction of an hour? $\frac{47}{60}$.

$$\therefore \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} = \tfrac{47}{60}.$$

Can you see that this is true? No.

" Well, we have seen that $\frac{1}{3}$ of an hour = 20 minutes, and since 1 minute = $\frac{1}{60}$ of an hour, 20 minutes = $\frac{20}{60}$ of an hour; &c. Hence we may write $\frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ in this way:

$$\tfrac{20}{60} + \tfrac{15}{60} + \tfrac{12}{60}$$

and *now* it is easy to see why the answer is $\frac{47}{60}$.

" Thus if we want to add fractions together, we must first see that they are fractions of the same **name**, i.e. that they have the same denominator.

" But how are we to change fractions of *different* denominators to fractions of the *same* denominator?"—And so we come to L.C.M.s, &c.

On the whole, however, I prefer to illustrate fractional

processes by means of diagrams, rectangles rather than lines. A rectangle is conveniently divided up into smaller rectangles by lines drawn in two directions, and thus the fraction of a fraction is easily exhibited. We append a few diagrams. (A squared blackboard or squared paper is always advisable.)

Diagrammatic Illustrations

Cancelling.— $\frac{8}{12} = \frac{4}{6} = \frac{2}{3}$.

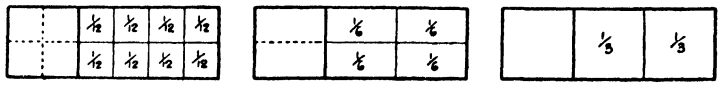


Fig. 1

Addition.— $\frac{1}{3} + \frac{1}{6} + \frac{1}{8} = \frac{8}{24} + \frac{4}{24} + \frac{3}{24} = \frac{15}{24}$.

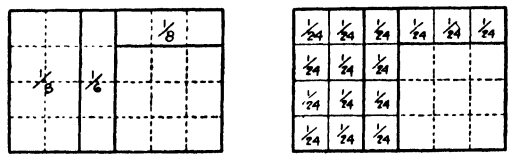


Fig. 2

Subtraction.— $\frac{5}{6} - \frac{7}{12} = \frac{3}{12}$.

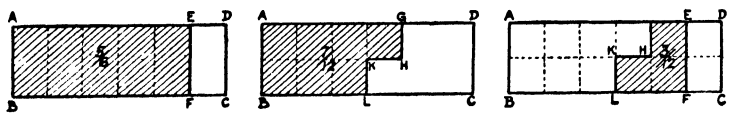


Fig. 3

These figures illustrating subtraction will puzzle the slower children, but squared paper and scissors will soon help to make things clear.

Multiplication

Multiplication by a fraction is always a little puzzling at first. A child naturally expects a multiplication sum to

produce an answer bigger than the multiplicand. It is best to begin with mixed numbers.

The child knows that 2/- × 3 is 6/-; by 4, is 8/-; and by $3\frac{1}{2}$, is 7/-. Hence multiplying by the $\frac{1}{2}$ seems to him *somehow* to have been a real multiplication, inasmuch as the 6/- has been increased to 7/-. The multiplication may be considered, as usual, as an addition, viz. of 3 florins and a half-florin. Multiplying a florin by $\frac{1}{2}$ is to take the half " of " a florin. Give other examples to show clearly the meaning of the word " of " when we speak of multiplying by a fraction.

Example: *Multiply* 2 *sq. in. by* $3\frac{1}{3}$.

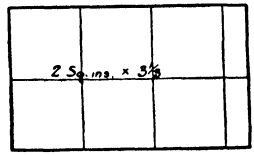


Fig. 4

The figure shows 2 sq. in. one above the other, then another 2 sq. in., then another, then $\frac{1}{3}$ part of 2 sq. in. The last piece shows multiplication by a real fraction, viz. 2 in. × $\frac{1}{3}$, i.e. the strip is $\frac{1}{3}$ **of** 2 sq. in. Thus the whole figure is $6\frac{2}{3}$ sq. in. Hence $2 \times 3\frac{1}{3} = 6\frac{2}{3}$.

Another example: *Draw a figure to show* $3\frac{1}{4}$ *sq. in.; then show this multiplied by* $2\frac{1}{3}$.

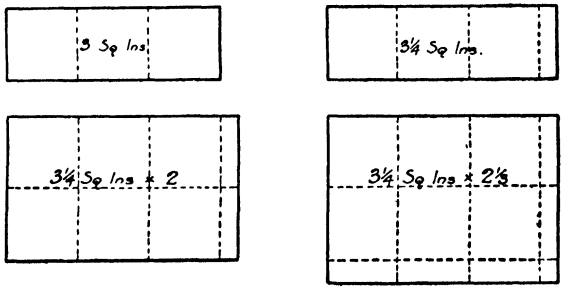


Fig. 5

" How from these figures may we obtain an answer to the sum $3\frac{1}{4} \times 2\frac{1}{3}$? Let us first think of money, say shillings, instead of inches.

$$3\tfrac{1}{4}s. = 3s.\ 3d. = 39d.$$

Twice $39d. = 78d.$; *one-third* of $39d. = 13d.$

$\therefore$ $2\frac{1}{3}$ times $39d. = (78 + 13)d. = 91d. = 7s.\ 7d. = 7\frac{7}{12}s.$

Apparently, then, $3\frac{1}{4} \times 2\frac{1}{3} = 7\frac{7}{12}$. Thus, if we divide the last figure up into twelfths, we ought to see whether this result is true. Fig. 6 shows that it *is* true; the squares need not be true squares; oblongs will show the fractions just as well.

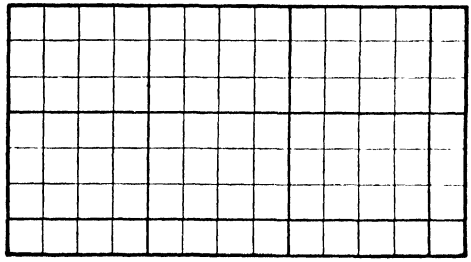


Fig. 6

Number of twelfths in the figure:

In the 6 big squares or oblongs, 12 each	=	72
In the 2 strips at the sides, 3 each	=	6
In the 3 strips at the bottom, 4 each	=	12
In the small strip at the corner	=	1
Total		91

Thus in the figure we have 91 twelfths $= \frac{91}{12} = 7\frac{7}{12}$, as expected."

Such a figure is satisfactory for illustrating the multiplication of mixed numbers, but for multiplication purposes mixed numbers should rarely be turned into improper fractions. A different example is therefore advisable.

Draw a figure to show $\frac{4}{7}$ *of* $\frac{2}{3}$. Lead up to the necessary figure by showing, first, $\frac{2}{3}$; then, by dividing the thirds into

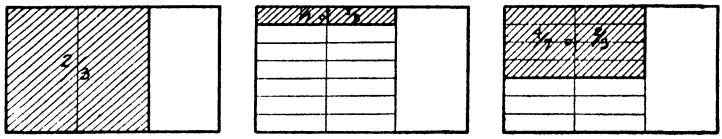


Fig. 7

7 parts, show $\frac{1}{7}$ of $\frac{2}{3}$; lastly, show $\frac{4}{7}$ of $\frac{2}{3}$. Obviously, now, $\frac{4}{7} \times \frac{2}{3} = \frac{8}{21}$.

" We have found out that:

(i), $2 \times 3\frac{1}{3}$, or $\frac{2}{1} \times \frac{10}{3}$, $= \frac{20}{3}$;
and (ii), $3\frac{1}{4} \times 2\frac{1}{3}$, or $\frac{13}{4} \times \frac{7}{3}$, $= \frac{91}{12}$;
and (iii), $\frac{4}{7} \times \frac{2}{3} = \frac{8}{21}$.

Now look: in every case the numerators multiplied together give the numerator in the answer, and the denominators multiplied together give the denominator in the answer."

Now the teacher is in a position to enunciate the rule. He has done nothing to *prove* the rule, but he has justified it, so far as it can be justified with beginners.

Division

Just as a child naturally expects a multiplication always to produce an increase, so he expects a division always to produce a decrease.

" When you have divided a number by another, the dividend is always made smaller. Do you agree?"—Yes. " Always smaller?"—Yes. " Quite certain?"—Yes.

" Let us divide 36 pence amongst some boys.

I give them	12*d.*	each:	how many boys?		3.
,, ,,	6*d.*	,,	,,	,,	6.
,, ,,	3*d.*	,,	,,	,,	12.
,, ,,	2*d.*	,,	,,	,,	18.
,, ,,	1*d.*	,,	,,	,,	36.

" Do you mean to say I cannot divide the 36 pence

amongst more than 36 boys?"—Yes, if you give them less than 1*d.*

"Then give them $\frac{1}{2}d.$ each. How many?" 72.

"Then 36 divided by $\frac{1}{2}$ is 72. Thus, although I have *divided* 36 I have a quotient *bigger* than 36. So you were wrong!" And so on.

A suitable scheme of diagrammatic division is easily devised, but it is best approached by the division of whole numbers.

When we divide 24 by 4, the quotient is 6, and the 4 sixes may be arranged in 4 lines thus:

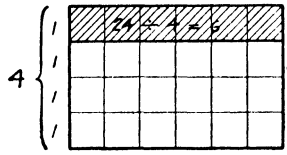


Fig. 8

The shaded section denotes the quotient (6 units); it is a row of units *in line with* 1 *of the units of the divisor*, 4. Any other row would have done equally well, for any other row would have been in line with 1 of the units of the divisor, 4.

So generally; a rectangle representing the dividend may always be divided up in such a way that each horizontal row of units represents the quotient; there are as many horizontal rows as there are units in the divisor. *Opposite any unit of the divisor* (we select the first) *is a horizontal row of units representing the quotient.*

We show 12 divided by 2, by 3, and by 4:

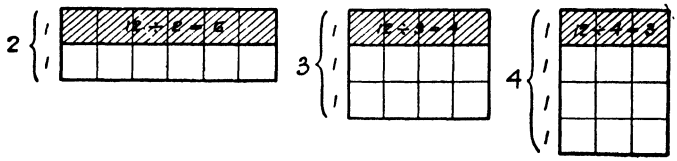


Fig. 9

Let the first fractional problem be to divide $4\frac{1}{2}$ by $1\frac{1}{3}$. We may ask how many times $1\frac{1}{3}s.$ is contained in $4\frac{1}{2}s.$, i.e. how many times $16d.$ is contained in $54d.$ We may show this division in the ordinary way, $\frac{54}{16}$ or $\frac{27}{8}$, which is equal to $3\frac{3}{8}$. Thus the *answer* to the sum is $3\frac{3}{8}$. How are we to show this in a diagram?

We will first divide 3, represented by a rectangle of 3 sq. in., by 4, by 3, by $2\frac{1}{2}$, by 2, by $1\frac{1}{2}$, by 1, by $\frac{4}{5}$.

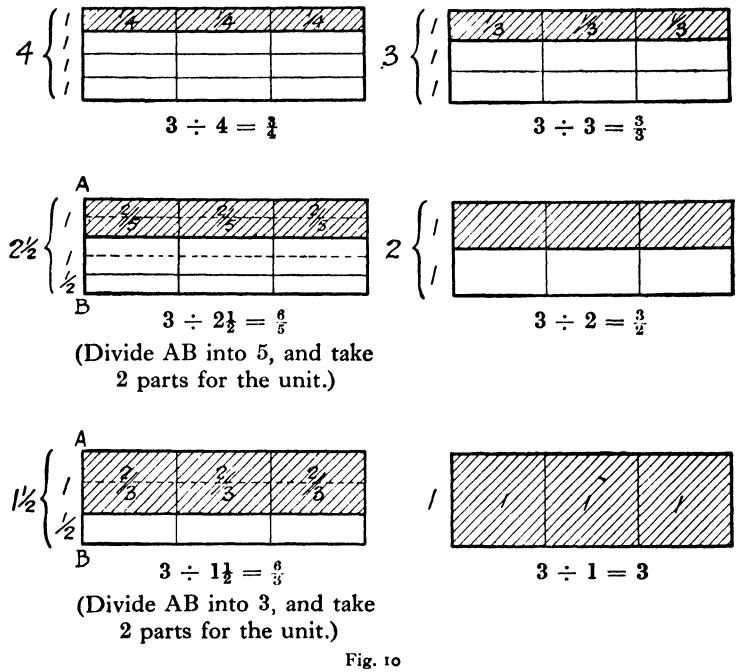
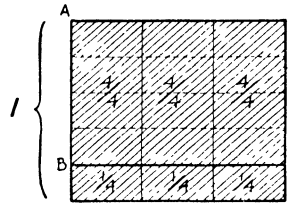


Fig. 10

Examine the six diagrams, and note how the quotient (the shaded part) increases as the divisor diminishes. If then we diminish the divisor further, the quotient (the shaded part) must be still bigger. As before, the shaded quotient must occupy a space opposite one complete unit of the divisor. But in this case AB is not long enough to show a

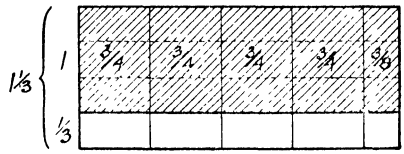
complete unit, only $\frac{4}{5}$ of one. So we must extend it to make $\frac{5}{5}$ $(=1)$.



$3 \div \frac{4}{5} = 3\frac{3}{4}$

Fig. 11

We may now return to our original example: *Divide* $4\frac{1}{2}$ *by* $1\frac{1}{3}$.



$4\frac{1}{2} \div 1\frac{1}{3} = (\frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{8}) = 3\frac{3}{8}$

Fig. 12

One more example: *Divide* $\frac{2}{3}$ *by* $\frac{3}{4}$.

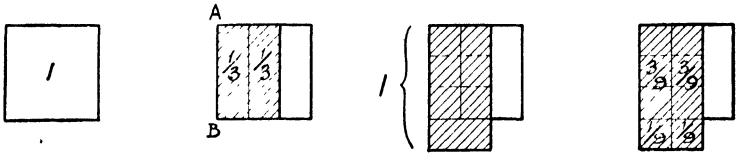


Fig. 13

The first figure shows 1 sq. in.

The second figure shows $\frac{2}{3}$ sq. in., the part to be divided by $\frac{3}{4}$.

The third figure shows AB extended to one complete unit, since AB itself represents only $\frac{3}{4}$ of one.

The last figure shows the result of the division, viz. $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$.

We may now collect up our results:

$$\begin{array}{l|l|l} \frac{3}{1} \div \frac{4}{1} = \frac{3}{4} & \frac{3}{1} \div \frac{2}{1} = \frac{3}{2} & \frac{3}{1} \div \frac{4}{5} = \frac{15}{4} \\ \frac{3}{1} \div \frac{3}{1} = \frac{3}{3} & \frac{3}{1} \div \frac{3}{2} = \frac{6}{3} & \frac{9}{2} \div \frac{4}{3} = \frac{27}{8} \\ \frac{3}{1} \div \frac{5}{2} = \frac{6}{5} & \frac{3}{1} \div \frac{1}{1} = \frac{3}{1} & \frac{2}{3} \div \frac{3}{4} = \frac{8}{9} \end{array}$$

An average class will soon discover that by inverting the divisor the quotient is then obtained by treating the sum as a multiplication sum, e.g.

$$\frac{9}{2} \div \frac{4}{3} = \frac{9}{2} \times \frac{3}{4} = \frac{27}{8}.$$

But let the teacher be under no delusion. Only a very small minority will, at the time, appreciate the purpose of the diagrams. Over and over again I have seen a majority completely baffled, even with very skilful teaching. No matter. Come back to the demonstration again, a year or two later. You have justified your rule as far as you can. Now state it in clear terms and—for the present—be satisfied that the boys are able to get their sums right. There is probably nothing more difficult in the whole range of arithmetic than the division of fractions, i.e. *for boys to understand the process when it is first taught.*

The following kind of argument is sometimes useful:

To divide a fraction, say $4\frac{1}{2}$, by 5 is the same thing as taking $\frac{1}{5}$ of $4\frac{1}{2}$. But to take $\frac{1}{5}$ "of" $4\frac{1}{2}$ is the same thing as multiplying $4\frac{1}{2}$ by $\frac{1}{5}$; i.e.

$$4\tfrac{1}{2} \div 5 = 4\tfrac{1}{2} \times \tfrac{1}{5}.$$

Now, if we divide $4\frac{1}{2}$ by $\frac{5}{7}$, we divide it by a number 7 times as *small* as when we divided by 5; therefore our answer must be 7 times as *large* as before; i.e.

$$\begin{array}{l} \text{Since } 4\frac{1}{2} \div 5 = 4\frac{1}{2} \times \frac{1}{5} \\ \therefore \quad 4\frac{1}{2} \div \frac{5}{7} = 4\frac{1}{2} \times \frac{7}{5}; \end{array}$$

i.e to divide by $\frac{5}{7}$ is the same thing as multiplying by $\frac{7}{5}$. Hence, once more, the rule of inverting the divisor.

But the argument is quite beyond the average beginner, as every experienced teacher knows.

Cancelling.—When a boy is told he may "cancel" thus:

$$\begin{matrix} 2 & 5 \\ \frac{\not{44}}{\not{45}} & \times \frac{\not{25}}{\not{22}} \\ 9 & 1 \end{matrix}$$

he is likely to ask, why? He will already have learnt:

(1) That, e.g., $7 \times 9 = 9 \times 7$;
(2) Reduction of fractions to their lowest terms;
(3) Multiplication of fractions.

Thus he will understand that

$$\frac{44}{45} \times \frac{25}{22} = \frac{44 \times 25}{45 \times 22},$$

and that this may be written:

$$\frac{44 \times 25}{22 \times 45} \text{ or } \frac{44}{22} \times \frac{25}{45}.$$

He now readily sees that he is justified in reducing each of these to its lowest terms, and that the final result is the same as when he cancelled terms in different fractions.

CHAPTER X

Decimal Fractions

A Natural Extension of Ordinary Notation

If care is taken to teach the inner nature of decimal notation thoroughly, decimals need present little difficulty.

"A decimal fraction is merely a particular kind of vulgar fraction, viz. one with a denominator 10 or power of 10,

e.g. $\frac{7}{10}$, $\frac{73}{100}$, $\frac{8192}{1000}$, $\frac{3}{10000}$. But we do not generally write them this way; we write them as follows:

$$\cdot 7,\ \cdot 73,\ 8\cdot 192,\ \cdot 0003$$

" Let us consider a number consisting entirely of *ones* (any other figure would do equally well).

The number $11 = 10 + 1$
The number $111 = 100 + 10 + 1$
The number $1111 = 1000 + 100 + 10 + 1$.

The 1 of least importance in each number is the 1 on the extreme right; each 1 to the left is 10 times as important as, and is 10 times the value of, its right-hand neighbour. The 1 on the extreme right represents just one unit.

" But we often break up a unit into parts, e.g. a sovereign, or a bag of nuts. These parts are fractions, and we might carry our *ones* to the right, to represent these fractions, devising some means of separating the fractions from the whole numbers: a straight line would do.

Thus, 1111|1 may stand for $1000 + 100 + 10 + 1 + \frac{1}{10}$,
and, 11|111 may stand for $10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$.

Any sort of separating mark will do. Generally we use the smallest possible mark, a dot, written half-way up the height of the figures.

" Thus $276\cdot347 = 200 + 70 + 6 + \frac{3}{10} + \frac{4}{100} + \frac{7}{1000}$. If we add the 3 fractions together, we get $\frac{347}{1000}$. Thus $276\cdot347 = 276\frac{347}{1000}$.

" Note the two ways of writing down the same thing:

$$\cdot 31 = \frac{3}{10} + \frac{1}{100} = \frac{31}{100}.$$
$$\cdot 4501 = \frac{4}{10} + \frac{5}{100} + \frac{0}{1000} + \frac{1}{10000} = \frac{4501}{10000}.$$

" Perhaps there are no tenths, and we begin with hundredths:

$$\cdot 042 = \frac{0}{10} + \frac{4}{100} + \frac{2}{1000} = \frac{42}{1000}.$$

We could not write this ·42 because $\cdot 42 = \frac{4}{10} + \frac{2}{100} = \frac{42}{100}$.

Thus we may have one or more noughts between the decimal point and the "significant" figures.

"But noughts on the extreme right hand of a decimal have no meaning:

$$\cdot 034 = \tfrac{0}{10} + \tfrac{3}{100} + \tfrac{4}{1000} = \tfrac{34}{1000}$$
$$\cdot 03400 = \tfrac{0}{10} + \tfrac{3}{100} + \tfrac{4}{1000} + \tfrac{0}{10000} + \tfrac{0}{100000} = \tfrac{34}{1000}$$

as before."

Give ample practice in conversion and reconversion, until the change can be written down mechanically:

$$37\cdot 063 = 37\tfrac{63}{1000}; \quad \tfrac{421}{10000} = \cdot 0421; \quad \&c., \&c.$$

Ready conversion of either decimal form into the other is essential. This is the key to all future work.

Let the rules for conversion be stated in the simplest possible words.

A few exercises of the following nature are useful:

"If $x = 10$, find the value of these expressions, writing down the answers in both decimal forms (do not cancel as you would vulgar fractions):

$$3 + \frac{4}{x} + \frac{7}{x^2} = 3 + \tfrac{4}{10} + \tfrac{7}{100} = 3\tfrac{47}{100}, \text{ or } 3\cdot 47.$$

$$5x^3 + 4x + 9 + \frac{3}{x^2} + \frac{6}{x^4} = 5049\tfrac{306}{10000}, \text{ or } 5049\cdot 0306.$$

Multiplication and Division by 10, by 100, &c.

"If we *multiply* 347 by 10 we obtain 3470, the 7 becoming 70, the 40 becoming 400, and the 300, 3000.

Th.	H.	T.	U.
	3	4	7
3	4	7	0

Every figure is moved one place to the *left*, and its value is increased 10 times.

" If we *divide* 2180 by 10 we obtain 218, the 2000 becoming 200, the 100 becoming 10, and the 80, 8.

Th.	H.	T.	U.
2	1	8	0
	2	1	8

Every figure is moved one place to the *right*, and its value is diminished 10 times.

" So it is with decimal fractions, or decimals as we often call them."

Multiply 3·164 *by* 10.

$$\begin{aligned} \mathbf{3{\cdot}164 \times 10} &= (3 + \tfrac{1}{10} + \tfrac{6}{100} + \tfrac{4}{1000}) \times \mathbf{10} \\ &= 30 + 1 + \tfrac{6}{10} + \tfrac{4}{100} \\ &= 31\tfrac{64}{100} = \mathbf{31{\cdot}64;} \end{aligned}$$

i.e. the decimal point has been moved one place to the *right*, and every figure occupies a place 10 times as important as before.

So $\quad 5{\cdot}623 \times 100 = 562{\cdot}3;$
$\quad\quad {\cdot}005623 \times 10000 = 56{\cdot}23.$

Divide 3·164 *by* 10.

$$\begin{aligned} \mathbf{3{\cdot}164 \times \tfrac{1}{10}} &= (3 + \tfrac{1}{10} + \tfrac{6}{100} + \tfrac{4}{1000}) \times \tfrac{1}{10} \\ &= \tfrac{3}{10} + \tfrac{1}{100} + \tfrac{6}{1000} + \tfrac{4}{10000} \\ &= \tfrac{3164}{10000} = \mathbf{{\cdot}3164;} \end{aligned}$$

i.e. the decimal point has been moved one place to the *left*, and every figure occupies a place reduced in importance 10 times.

Give a number of varied examples in both multiplication and division, and help the pupils to deduce the rules.

Give plenty of mental work of the following kind:

tens × tens = hundreds
hundreds × tens = thousands
hundreds × hundreds = ten thousands
tenths × tenths = hundredths
hundredths × tenths = thousandths.

Continue this kind of work until instant reponse is obtained as to the significance of moving the decimal point so many places to the right or so many to the left. *Let the notation be mastered*; then the rest will give little trouble.

Addition and Subtraction

Do not forget the common cause of inaccuracy, blanks in the fractional columns, especially if the numbers are arranged horizontally, e.g. 7·612 + 3·1 + 2·0151.

Multiplication

$$
\begin{aligned}
&72{\cdot}314 \times {\cdot}32 \\
&= 72\tfrac{314}{1000} \times \tfrac{32}{100} \\
&= \tfrac{72314}{1000} \times \tfrac{32}{100} \\
&= \tfrac{2314048}{100000} \\
&= \mathbf{23{\cdot}14048.}
\end{aligned}
$$

The whole process resolves itself into (1) conversion, (2) multiplication of whole numbers, (3) reconversion.

Note that "conversion" does not mean conversion to *vulgar* fractions, but to the alternative form of *decimal* fractions, with denominators consisting of powers of 10.

The multiplication of the denominator is really nothing more than the mere addition of noughts, and it is obvious from this multiplication that the number of decimal places in the product is equal to the sum of the numbers of decimal places in the multiplicand and multiplier, and that from the very nature of the case this must always be so.

Hence the simple rules:

1. Ignore the decimal point and perform the multiplication as if the multiplicand and multiplier were whole numbers.

2. Add together the decimal place in the multiplicand and multiplier; this gives the number in the product. Fix the point by counting back that number of places from the right.

72·314
·32
144628
216942
23·14048 No. of dec. places = (3 + 2) = 5

Is the method *intelligent*? It is at least as intelligent as any other method, and it has this advantage—that the boy works his sums exactly as he works ordinary simple multiplication. And the procedure is easily and immediately justified, by conversion and reconversion.

Give other examples, using the same numbers but changing the position of the decimal points. The answers shall be given mentally and at once:

$$\begin{aligned} \cdot 72314 \times 3\cdot 2 &= 2\cdot 314048. \\ \cdot 0072314 \times 32 &= 0\cdot 2314048. \\ 72314 \times \cdot 00032 &= 23\cdot 14048. \\ 723\cdot 14 \times 320 &= 7231\cdot 4 \times 32 \\ &= 231404\cdot 8 \end{aligned}$$

If preferred, the boy might set out his working thus:

$$\begin{aligned} \mathbf{72\cdot 314 \times \cdot 32} &= \tfrac{72314}{1000} \times \tfrac{32}{100} \\ &= \tfrac{2314048}{100000} \\ &= \mathbf{23\cdot 14048} \end{aligned}$$

and show his actual multiplication neatly on the left.

Possible objections to the method.

1. "The most important digit in the multiplier is not used first."—Granted. But this disadvantage is outweighed by the advantage of greater accuracy.

2. "The decimal points are not kept in a vertical column."—This is of no material consequence, though it is quite easy to teach the boy, if it is thought worth while, to place the points in the successive products. For instance:

72·314
·32
1·44628
21·6942
23·14048

The boy multiplies through by 2 and then says, " When I multiplied 4 by 2, I multiplied thousandths by hundredths, and this gives me hundredths of thousandths, which occupy the 5th decimal place; therefore the point goes in front of the first 4." He argues similarly when he has multiplied by the 3, though he would soon learn that the position of the point in the first partial product gives the key to its position in all the other products. Thenceforth he would work mechanically. Does not the time come when we *all* work mechanically in *all* types of calculation? does not the *rationale* of procedure tend to fade away, until something turns up demanding revivification?

Is there a more intelligent plan than teaching the boy to complete the actual multiplying before considering the decimal point at all? I doubt it. And I am quite sure that no other plan is productive of a greater degree of accuracy. The boy has *confidence* in a method so closely akin to one with which he is already familiar.

Standard Form

It has been gravely said that " standard form " was the invention of the devil. In reality it was not quite so bad as that. It was invented* by an old personal friend of my own, the senior mathematical master of one of our great Public Schools, who decided that he " must adopt some new method to prevent his boys from getting so many sums right, in order to take the conceit out of them ".

Why are the apologists of the method always so faint-hearted?

* The method was suggested by " standard " form in logarithms, where of course it is very useful.

Division

As division is the reverse process of multiplication, the analogous method for fixing the decimal point may be adopted.

$$
\begin{aligned}
&23{\cdot}14048 \div {\cdot}32 \\
&= \tfrac{2314048}{100000} \times \tfrac{100}{32} \\
&= \tfrac{2314048}{32} \times \tfrac{100}{100000} \\
&= 72314 \times \tfrac{1}{1000} \\
&= \tfrac{72314}{1000} \\
&= \mathbf{72{\cdot}314.}
\end{aligned}
$$

The actual simple division by 32 may be neatly shown to the left.

The simple rules are:

1. Ignore the decimal points and divide as in simple division.

2. Subtract the number of decimal places in the divisor from the number in the dividend. This gives the number in the quotient. Fix the point by counting back this number from the right.

The whole process resolves itself into (1) conversion, (2) division by whole numbers, (3) reconversion.

Again: is the method *intelligent?* Again the answer is that it is as least as intelligent as any other method, and it certainly makes for accuracy. Here is an example with the working as commonly shown: Divide 2·0735 by 8·72.

```
          23
8·72)2·0735
     1 744
      3295      Decimal places = 4 − 2 = 2.
      2616      Thus the quotient is ·23, and a remainder.
       679
```

The division may now be continued to any number of places.

If, before dividing, we add 0's to the dividend and continue the dividing further, this does not affect the

decimal point in the quotient: e.g. divide 2·073500 by 8·72.

```
          2377
8·72)2·073500
     1 744
     -----
      3295
      2616
      ----
       6790      Decimal places = 6 − 2 = 4.
       6104      Thus the quotient is ·2377, and a remainder.
       ----
        6860
        6104
        ----
         756
```

Hence, if a given dividend contains a smaller number of decimal places than the divisor, add 0's to make the number equal (and more if necessary). Example: divide ·001 by 7·0564. Write:

$$7{\cdot}0564\overline{)\cdot001}$$

We cannot proceed with the division until we add at least 5 more 0's.

```
                 1
7·0564)·00100000
       70564      Decimal places = 8 − 4 = 4
       -----
       29436
```

Thus the answer is **·0001** . . . and a remainder. The quotient can be carried to as many places as may be required.

The value of the remainder.—It is desirable to make the abler boys see the real value of the quantities in the successive steps of the division. Example: divide **·07925 by 3·7.**

```
      214
3·7)·07925
    74
    --
     52
     37      Decimal places = 5 − 1 = 4.
     --
     155     Quotient = ·0214 and a remainder.
     148
     ---
       7
```

What is the value of the " 74 " in the first step? It is the product of 3·7 and ·02 (as we may see from the quotient), and must therefore contain (1 + 2 or) 3 decimal places; hence its value is **·074**. Similar arguments apply to each step. Hence, more correctly, the division may be set out in this way:

$$
\begin{array}{rl}
\cdot0214 & \\
\hline
3{\cdot}7)\cdot07925 & \\
\cdot074 & = 3{\cdot}7 \times \cdot02 \\
\hline
\cdot0052 & \\
\cdot0037 & = 3{\cdot}7 \times \cdot001 \\
\hline
\cdot00155 & \\
\cdot00148 & = 3{\cdot}7 \times \cdot0004 \\
\hline
\cdot00007 &
\end{array}
$$

Thus the quotient (to 4 figures) is **·0214**, and the remainder is **·00007**. The abler boys will soon learn to assign the correct value to the remainder, by merely glancing at the dividend vertically above it.

Verification should be encouraged:

$$
\begin{aligned}
\text{Dividend} &= (\text{quotient} \times \text{divisor}) + \text{remainder} \\
&= (\cdot0214 \times 3{\cdot}7) + \cdot00007 \\
&= \cdot07918 + \cdot00007 \\
&= \mathbf{\cdot07925}
\end{aligned}
$$

Practice in manipulation of the following kind is useful:

$$\frac{3{\cdot}204}{\cdot0701} = \frac{3204}{70{\cdot}1} = \frac{\cdot03204}{\cdot000701} = \frac{320{\cdot}4}{7{\cdot}01}.$$

The boy sees at once that the same quotient must result from all the division sums. The only real defence for the reduction of the divisor to a form approximating a small whole number is that it enables a boy to obtain a rough answer by easy calculation. For instance, in the last of the four forms above, $\frac{320{\cdot}4}{7{\cdot}01} = \frac{320}{7}$ approximately, and thus the

answer to this group of division sums is roughly $\frac{1}{7}$ of 320, i.e. a number between 40 and 50.

$$\text{Again: } \frac{\cdot 000983}{\cdot 047} = \frac{\cdot 0983}{4 \cdot 7} \text{ or } \frac{\cdot 983}{47}.$$

Thus the answer is roughly $\frac{1}{47}$ of 98 hundredths, i.e. about 2 hundredths, i.e. about ·02.

This is useful for final verification, but the decimal point is best fixed by the rule already given. If the simple multiplication and division are accurately performed, the correct fixing of the decimal point is a simple matter to even an unintelligent boy.

Recurring Decimals

These will probably rarely be used, except in a very simple form. Every boy ought, however, to know their significance, though as a subject of general exposition they are now generally ignored, perhaps unwisely. The younger race of mathematicians are losing familiarity with much that is interesting in the theory of numbers. Most people know of course that $\frac{1}{7} = \cdot 142857142857 \ldots = \cdot\dot{1}42857\dot{}$, and that if we multiply this group of 6 figures by 2, 3, 4, 5, and 6, respectively, we obtain products giving the same group of figures in the same order, each succeeding group beginning with the next higher figure of the group. It is, however, less commonly known nowadays that this remarkable property of numbers is not uniquely characteristic of the sevenths but applies to all prime numbers whatsoever, 7 and beyond, and that the grouping within the groups is sometimes of an extraordinarily interesting character. Teachers of arithmetic probably lose not a little of the potential effectiveness and interest in this subject if they do not familiarize their pupils with some of the properties of numbers, properties which to beginners seem almost uncanny. (See Chapter XLI.)

As to circulating decimals, a boy should be taught at least this much:

Show him that he may at any point bring to an end the quotient of a decimal he is dividing, showing the remainder as a vulgar fraction.

Thus $\frac{1}{7} = \cdot 14\frac{2}{7}$ or $\cdot 1428\frac{4}{7}$ or $\cdot 14285\frac{5}{7}$.

Reconvert, say, the first: $\cdot 14\frac{2}{7} = \frac{1}{10} + \frac{4\frac{2}{7}}{100} = \frac{14\frac{2}{7}}{100} = \frac{100}{700} = \frac{1}{7}$.

Then the boy sees that the scheme is justified.

Again: $\frac{1}{3} = \cdot 3333\ldots$ apparently without end,

so, $\frac{1}{9} = \cdot 1111\ldots$ apparently without end,

and, $\frac{71}{99} = \cdot 717171\ldots$ apparently without end.

But we can bring the division to an end anywhere, e.g.:

$$\frac{1}{3} = \cdot 3333\tfrac{1}{3}, \text{ or } \cdot 33\tfrac{1}{3} \text{ or } \cdot 3\tfrac{1}{3}.$$
$$\frac{1}{9} = \cdot 111\tfrac{1}{9} \text{ or } \cdot 1\tfrac{1}{9}.$$
$$\frac{71}{99} = \cdot 7171\tfrac{71}{99} \text{ or } \cdot 71\tfrac{71}{99}.$$

Reconvert these:

$$\cdot 3\tfrac{1}{3} = \frac{3\frac{1}{3}}{10} = \frac{10}{30} = \frac{30}{90} = \frac{3}{9}.$$

$$\cdot 1\tfrac{1}{9} = \frac{1\frac{1}{9}}{10} = \frac{10}{90} = \frac{1}{9}.$$

$$\cdot 71\tfrac{71}{99} = \frac{7}{10} + \frac{1\frac{71}{99}}{100} = \frac{7}{10} + \frac{170}{9900} = \frac{7100}{9900} = \frac{71}{99}.$$

Now show the repeated figures in a decimal division this way:

$$\cdot 3333\ldots = \cdot\dot{3}; \quad \cdot 1111\ldots = \cdot\dot{1}; \quad \cdot 717171\ldots = \cdot\dot{7}\dot{1}.$$

Thus we have learnt that

$$\cdot\dot{3} = \tfrac{3}{9}; \quad \cdot\dot{1} = \tfrac{1}{9}; \quad \cdot\dot{7}\dot{1} = \tfrac{71}{99}.$$

Hence to convert any repeating decimal into a vulgar fraction, we make a denominator of 9's, viz. as many as there are places in the decimal.

Thus $\cdot\dot{3}\dot{1} = \frac{31}{99}$; $\cdot\dot{0}\dot{9} = \frac{09}{99} = \frac{1}{11}$.

If there are non-repeating figures as well as repeating figures, e.g. $\cdot 5\dot{1}\dot{6}$, then

$$\cdot 5\dot{1}\dot{6} = \frac{5\cdot\dot{1}\dot{6}}{10} = \frac{5\frac{16}{99}}{10} = \frac{511}{990};$$

$$\cdot 01\dot{6} = \frac{1\cdot\dot{6}}{100} = \frac{1\frac{6}{9}}{100} = \frac{15}{900} = \frac{1}{60}.$$

The commoner forms should be known, especially the thirds, sixths, and twelfths:

$$\tfrac{1}{3} = \cdot\dot{3}; \quad \tfrac{2}{3} = \cdot\dot{6}.$$
$$\tfrac{1}{6} = \cdot 1\dot{6}; \quad \tfrac{5}{6} = \cdot 8\dot{3}.$$
$$\tfrac{1}{12} = \cdot 08\dot{3}; \quad \tfrac{5}{12} = \cdot 41\dot{6}; \quad \tfrac{7}{12} = \cdot 58\dot{3}; \quad \tfrac{11}{12} = \cdot 91\dot{6}.$$

The boys should know that when the denominator of a vulgar fraction contains any prime numbers except 2 and 5, the conversion of the fraction to a decimal is bound to give a *circulating* decimal.

Simplification of Vulgar and Decimal Fractions

There are certain conventional rules about signs; for instance:

1. Multiplication and division must be given precedence over addition and subtraction:

$3 \times 18 + 15 \div 3 - 2$ means $(3 \times 18) + (15 \div 3) - 2$.

2. Multiplication and division alone must be worked in the natural sequence from left to right:

$36 \div 9 \times 2$ means $(36 \div 9) \times 2$.

But the conventions are not whole-heartedly accepted; they are without reason, and they are traps for the unwary. It is unjust for examiners to assume that they will be followed. The above examples should have been written with the brackets. If brackets are inserted mistakes need not arise.

Here is a complex fraction to be simplified, taken from one of the best textbooks in use. Doubtless the question

was taken from an examination paper. If so, the examiner should have been put in the stocks.

$$\frac{1{\cdot}463}{7{\cdot}315} + \frac{\frac{2}{5} \text{ of } 141{\cdot}75 - \frac{4}{5} \times 38{\cdot}125}{5 \times 18{\cdot}9 + 25 + 1{\cdot}22} \times 3{\cdot}39525.$$

If given at all, it should have been written

$$\frac{1{\cdot}463}{7{\cdot}315} + \left\{\frac{(\frac{2}{5} \text{ of } 141{\cdot}75) - (\frac{4}{5} \times 38{\cdot}125)}{(5 \times 18{\cdot}9) + 25 + 1{\cdot}22} \times 3{\cdot}39525\right\}.$$

It is often an advantage to work in decimals instead of vulgar fractions. Example: What fraction of £21, 5*s.* 6*d.* is $\frac{\cdot 04255 \times \cdot 32}{\cdot 00016}$ of £1, 11*s.* 3*d.*?

$$\text{Fraction} = \frac{\frac{\cdot 04255 \times \cdot 32}{\cdot 00016} \text{ of } 31{\cdot}25 \text{ shill.}}{425{\cdot}5 \text{ shill.}}$$

$$= \frac{\cdot 04255 \times \cdot 32 \times 31{\cdot}25}{\cdot 00016 \times 425{\cdot}5}$$

$$= \frac{\overset{1}{\cancel{4255}} \times \overset{2}{\cancel{32}} \times 3125}{\underset{1}{\cancel{16}} \times \underset{1000}{\cancel{4255000}}}$$

$$= 6{\cdot}25$$

$$= 6\tfrac{1}{4}.$$

At the third step, both numerator and denominator were multiplied by 10^9, to get rid of the decimals. Boys feel more confidence when cancelling whole numbers. But in A Sets such conversion should not be necessary.

Decimalization of Money

The common method of performing arithmetical operations on money, weights, and measures reduced to their lowest denomination has the advantage of simplicity but the disadvantage of tediousness and cumbrousness. It is certainly an advantage to work in the highest denomination when possible, decimalizing all the lower denominations. For

instance, if we have to multiply £432, 17*s.* 4½*d.* by 562, it is obviously an advantage to multiply £432·86875 instead, *provided* we can convert into the decimal form at once.

But the rules for conversion to more than 3 places are a little too difficult for slower boys, and it must be remembered that if multiplication is in question (and this is often the case), *exact* decimalization is necessary, or the multiplied error may be too serious to be negligible. The *mil* invariably causes trouble. On the whole, decimalization methods are advisable in A Sets, not in others.

But all boys should be taught to give in pounds the approximately equivalent decimals of sums of money, i.e. to call every 2/-, ·1; every odd 1/-, ·05; every -/6, ·025; every farthing, ·001.

Thus: £3, 17*s.* 10½*d.*

$$\begin{aligned} &= £3 + 16/- + 1/- + -/6 + 18f. \\ &= £(3 + \cdot 8 + \cdot 05 + \cdot 025 + \cdot 018) \\ &= £3{\cdot}893. \end{aligned}$$

Greater certainty and greater accuracy is obtained by the ordinary method:

$$\begin{aligned} ½d. &= \cdot 5d. \\ 10½d. &= 10{\cdot}5d. \\ &= \cdot 875s. \\ 17s.\ 10½d. &= 17{\cdot}875s. \\ &= £{\cdot}89375 \\ £3,\ 17s.\ 10½d. &= £3{\cdot}89375, \end{aligned}$$

the boys dividing by 12 and 20 without putting down the factors.

The converse operation, the conversion of decimally expressed money into pounds, shillings, and pence, is most safely and quickly performed by the old-fashioned multiplication method (by 20 and by 12).

Consider £307·89275.

$$\begin{aligned} £{\cdot}89275 &= 17{\cdot}855s. \\ \cdot 855s. &= 10{\cdot}26d. \\ \therefore £307{\cdot}89275 &= £307,\ 17s.\ 10{\cdot}26d. \\ &= £307,\ 17s.\ 10¼d.\ (+\cdot 01d.). \end{aligned}$$

Numerous tests in recent years have shown conclusively that the usual decimalization rules are productive of much inaccuracy amongst slower boys. But decimalization and reconversion by the ordinary methods of division and multiplication are easy to effect and are often advantageous in practice.

A quick boy who wanted to multiply 15*s.* 9*d.* by 2420 would probably use the practice method (15*s.* = $\frac{3}{4}$ of £1, 9*d.* = $\frac{1}{20}$ of 15*s.*, and he would see at once that the product is £1815 + £90·75 or £1905, 15*s.*), and he would not decimalize. But to a slow boy a choice of methods is only an embarrassment. He wants *one* method, and that method without frills of any kind.

Contracted Methods

Contracted methods of multiplying decimals are productive of so much inaccuracy that their use with average boys is not advised. In A Sets, of course; in B Sets, perhaps; in C and D Sets, no; though in A and B Sets logarithms will usually be used instead, unless the sum to be worked is so simple that ordinary methods are quicker. To slower boys logarithms are puzzling, and their use in lower Sets is not recommended. But no boy ought to be allowed, in Forms above the Fourth, to show in his working the figures to the right of and below the heavy line in a sum like the following.

Divide 5·286143 by 37·29 (to 4 places).

```
          ·1417
      ------------
37·29)5·286143
      3 729 |
      ------|
       1557 | 1
       1491 | 6
      ------|--
         65 | 54
         37 | 29
      ------|---
         82 | 253
      ------|----
         26 | 103
      ------|----
          2 | 150
```

I have found that even slower boys soon gain confidence in cutting out figure after figure in the divisor, instead of bringing down figures from the dividend. The doubtful "carry" figures worry him a little at first, but not for long, and he soon learns to understand what to do to ensure accuracy to a given number of decimal places.

Never encourage average boys to adopt the expert mathematician's plan of multiplying and subtracting at the same time (Italian method). Boys hate it, rarely become expert at it, and make mistakes galore. It should of course be used by boys having any sort of real mathematical bent.

CHAPTER XI

Powers and Roots. The A, B, C of Logarithms

Powers and Roots

I have seen four- and five-figure logarithms deftly used in Preparatory Schools, but it is probably not wise to expect much facility before the age of 14. The A, B, C of logarithms, as a simple extension of work on powers and roots, may, however, readily be taught a little sooner.

At first, powers, indices, and roots should always be treated arithmetically, not algebraically. The later generalizations are then much more likely to be understood.

Some typical preliminary exercises:

(i) $5^4 = 5 \times 5 \times 5 \times 5$; $5^3 = 5 \times 5 \times 5$;

$\therefore 5^4 \times 5^3 = (5 \times 5 \times 5 \times 5) \times (5 \times 5 \times 5)$

$= (5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5)$

$= 5^7$,

$\therefore 5^4 \times 5^3 = 5^7$.

Thus lead up to the rule, and then state it clearly, that in multiplication of this kind the indices are **added.** But impress on the boys that the operation concerns powers of the *same number* (in this case, 5), though any number may be similarly treated.

(ii) $7^9 = 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7;$

$7^4 = 7 \times 7 \times 7 \times 7;$

$\therefore 7^9 \div 7^4 = \dfrac{\cancel{7} \times \cancel{7} \times \cancel{7} \times \cancel{7} \times 7 \times 7 \times 7 \times 7 \times 7}{\cancel{7} \times \cancel{7} \times \cancel{7} \times \cancel{7} \times \cancel{7}}$

$= 7^5;$

$\therefore \dfrac{7^9}{7^4} = 7^{9-4} = 7^5.$

Now lead up to the associated rule of *subtraction* of indices, in division of this kind.

(iii) $7^4 = 7 \times 7 \times 7 \times 7;$

$\therefore \dfrac{7^4}{7^4} = \dfrac{7 \times 7 \times 7 \times 7}{7 \times 7 \times 7 \times 7} = 1;$

$\therefore 7^{4-4} = 1$ or $7^0 = 1.$ So $3^0 = 1;\ 10^0 = 1.$

(iv) $(7^2)^3 = 7^2 \times 7^2 \times 7^2$

$= (7 \times 7 \times 7 \times 7 \times 7 \times 7) = 7^6.$

Thus lead up to the rule as to *multiplication* of indices. State categorically that such results always hold good, and that a convenient way of remembering them is this:

$$a^m \times a^n = a^{m+n};\ a^m \div a^n = a^{m-n};\ (a^m)^n = a^{mn};\ a^0 = 1.$$

But at this stage do not talk about " general laws ". Let the above expressions be looked on merely as a kind of shorthand for collecting up several results actually worked out arithmetically.

The following is a summary of a particularly effective first lesson I once heard, given to a class of boys of 13, on fractional and negative indices.

" The **square root** of a number is that number which when squared produces the original number; e.g. the square root of 16 is 4; of 81 is 9. We write R16 = 4; R81 = 9.

(The mathematician writes his R like this: '$\surd$', and calls it 'root'. Thus '$\surd 36 = 6$' reads 'root 36 is 6'.) Note carefully that $\surd 6 \times \surd 6 = \surd 36 = \surd 6^2 = 6$; that $\surd 11 \times \surd 11 = 11$; and so on.

"The **cube root** of a number is that number which when cubed produces the original number. We show the operation by writing a little 3 inside the $\surd$. Thus $\sqrt[3]{125} = \sqrt[3]{5 \times 5 \times 5} = \sqrt[3]{5^3} = 5$; $\sqrt[3]{1000} = \sqrt[3]{10 \times 10 \times 10} = \sqrt[3]{10^3} = 10$; $\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5} = \sqrt[3]{125} = 5$.

"And so on. $\sqrt[4]{81} = \sqrt[4]{3 \times 3 \times 3 \times 3} = \sqrt[4]{3^4} = 3$.

"Now suppose the index is a fraction, and not a whole number. What does $5^{\frac{1}{2}}$ mean? Well, we have learnt that $5^2 \times 5^2 = 5^{2+2} = 5^4$, so apparently we may assume that $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$.

"But $\surd 5 \times \surd 5 = 5$; therefore $5^{\frac{1}{2}} = \surd 5$.

In other words, $5^{\frac{1}{2}}$ is merely another way of writing down $\surd 5$. Similarly, $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = 5^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 5^1 = 5$; therefore $5^{\frac{1}{3}} = \sqrt[3]{5}$, i.e. $5^{\frac{1}{3}}$ is another way of writing down $\sqrt[3]{5}$.

"Similarly, $\sqrt[4]{7} = 7^{\frac{1}{4}}$; $\sqrt[6]{2} = 2^{\frac{1}{6}}$.

"What does $8^{\frac{2}{3}}$ mean? We know that $(3^4)^3 = 3^{12}$, so apparently we may assume that $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}}$, i.e. that $8^{\frac{2}{3}}$ means the cube root of 8^2, or $\sqrt[3]{8^2}$, or $\sqrt[3]{64}$, or 4.

"Similarly $5^{\frac{3}{4}} = \sqrt[4]{5^3} = \sqrt[4]{125}$.

"Thus we have learnt that the *numerator* of a fractional index indicates a *power*, and that the *denominator* indicates a *root*.

"Again, what does 6^{-2} mean?

"Since $6^5 \times 6^2 = 6^{5+2} = 6^7$, apparently we may assume that

$$6^5 \times 6^{-2} = 6^{5-2} = 6^3.$$

But

$$6^5 \div 6^2 = \frac{6^5}{6^2} = 6^3;$$

$$\therefore \quad 6^5 \times 6^{-2} = \frac{6^5}{6^2}$$

or

$$6^{-2} = \frac{1}{6^2}.$$

Similarly, $5^{-3} = \frac{1}{5^3}$; also $\frac{1}{7^4} = 7^{-4}$.

Thus we may conveniently remember that a^{-n} and $\frac{1}{a^n}$ are two ways of writing down the same thing.

" Examples: $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$.

$3^{-\frac{1}{3}} = \frac{1}{3^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{3}}$.

$7^{-\frac{2}{3}} = \frac{1}{7^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{7^2}} = \frac{1}{\sqrt[3]{49}}$.

" A root form in a denominator is often troublesome, since it leads to difficult arithmetic; and we may often get rid of it in this way:

$$\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5} = \tfrac{1}{5}\sqrt{5}."$$

The lesson was followed up by a few very easy exercises. There was no algebra, save the " shorthand " expressions utilized as mere mnemonics. The teacher's purpose was to make the boys familiar with the basic facts of indices (integral and fractional, + and −), and with the alternative forms of writing down the same thing. Naturally many more examples were given than the few above cited, and by the end of the lesson the boys were remarkably accurate in their answers to " mental " test exercises that were made somewhat severely searching.

Boys should know their squares up to 20^2. Extraction of square roots may be taught when $(a + b)^2$ is known in algebra, though boys should be made to break up numbers into factors whenever possible, and then to obtain square roots by inspection. Encourage boys to leave certain types of answers in surd form, but, generally, to rationalize their denominators; thus the answer $\frac{5}{\sqrt{7}}$ would not be acceptable,

but $\frac{1}{7}\ 5\sqrt{7}$ would. All boys should know the values of $\sqrt{2}$ $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, to 2 places of decimals.

The Beginnings of Logarithms

" Mathematicians were long ago clever enough to see how they could use indices for working long sums in multiplication and division. Suppose they wanted to multiply together two large numbers, each of which was a power of 3. They would turn to a book of ' tables ' showing the powers of 3. In fact, we may easily make up a little table for ourselves: e.g. $3^1 = 3$; $3^2 = 9$; $3^3 = 27$; $3^4 = 81$; &c. Here is a table from 3^1 to 3^{16}. In the first column we write the *index*, in the second the corresponding *number*.

Index.	Number.	Index.	Number.
1	1	9	19,683
2	9	10	59,049
3	27	11	177,147
4	81	12	531,441
5	243	13	1,594,323
6	729	14	4,782,969
7	2187	15	14,348,907
8	6561	16	43,046,721

" Now for some exercises.

1. Multiply 19,683 by 729.
   Answer: $19{,}683 \times 729 = 3^9 \times 3^6 = 3^{15} = \mathbf{14{,}348{,}907.}$
2. Divide 43,046,721 by 531,441.
   Answer: $\dfrac{43{,}046{,}721}{531{,}441} = \dfrac{3^{16}}{3^{12}} = 3^{16-12} = 3^4 = \mathbf{81.}$
3. What is the square of 2187?
   Answer: $(2187)^2 = (3^7)^2 = 3^{14} = \mathbf{4{,}782{,}969.}$

How easy! Instead of working hard sums, we simply refer to our table, and add, subtract, or multiply little numbers like 9, 6, &c.

" But the mathematician would set out the first sum something like this:

$$\begin{aligned}\text{Index of the answer} &= \text{index of } (19{,}683 \times 729).\\ &= \text{index of } 19{,}683 + \text{index of } 729.\\ &= 9 + 6.\\ &= 15.\end{aligned}$$

" But in the table the number corresponding to the index 15 is 14,348,907.

$$\therefore \textit{Answer} = \mathbf{14{,}348{,}907}.$$

Rather a roundabout way, isn't it? And he has a rather grand word which he prefers to the word index, it is ' logarithm '. The second part of the word, *-arithm*, you already know; the first part, *log* means ' rule ' or ' plan '. Although logarithms are only indices, the word itself suggests a clever " arithmetical plan " for shortening our work, and you must try to master it.

" Our little table contains only a few numbers, and there are big gaps between them; e.g. there is no number between 27 and 81. Now $27 = 3^3$ and $81 = 3^4$. Would it be possible to obtain a number between 27 and 81 by finding the value of $3^{3\frac{1}{2}}$? Certainly we should *suspect* that the value of $3^{3\frac{1}{2}}$ is somewhere between 27 and 81.

" We know that $3^{3\frac{1}{2}} = 3^{\frac{7}{2}} = \sqrt{3^7} = \sqrt{2187} = 46{\cdot}8$ (by calculation). Hence $3^{3\frac{1}{2}}$ *does* lie between 3^3 (=27) and 3^4 (=81). Obviously it is possible to put into our table as many fractional indices as we like, and so make the table more complete.

" The **3** which we have made the base of our calculations the mathematician calls a **base**. Any other number might be used instead, and in point of fact **10** is generally used."

The boy is now in a position to understand that $(\textit{base})^{\textit{log}} = \textit{natural number}$, and he may be introduced to a short table of three-figure logarithms, a table that may be included in a single printed page. Give a variety of very easy examples, and avoid great masses of figures. It is enough at this stage to drive home the main principle. There is much to be said

at first for avoiding the word logarithm altogether, and for letting the boy work from the relation, *number* $= 10^{index}$. But we are anticipating Form IV work. (See Chap. XVII.)

CHAPTER XII

Ratio and Proportion

Simple Equations Again

If 4 *chairs cost* £20, *what is the cost of* 15 *chairs?*

4 chairs cost £20.

$\therefore$ 1 chair costs $\frac{£20}{4}$.

$\therefore$ 15 chairs cost $\frac{£20 \times 15}{4} = £75$.

This method, " the method of unity ", is a good childish way of working such a sum, and it is the method suitable for boys up to the age of 11. At about this age the notion of ratio should be introduced and it should gradually supersede the unitary method.

First, revise the work on very simple equations.

We may begin a sum by saying, " What number of chairs . . . ?" The number we do not know; we have to discover it. It is customary to let the symbol x represent a number not yet discovered, and to argue about the x just as we argue about any ordinary number.

How many chairs can I buy for £45, *if* 1 *costs* £5?

Let the number of chairs be x.

Then £5 $\times$ x = £45,

$\therefore$ $5x = 45$.

$\therefore$ $x = \frac{45}{5} = 9$.

Similarly, if $17x = 51$, $x = \frac{51}{17} = 3$.

" We have already learnt that we may multiply or divide the two sides of an equation by any number we please, provided that we treat the two sides alike; e.g.

$$\text{If } x = 12, \text{ then } 3x = 36, \text{ or } \frac{x}{10} = \frac{12}{10} \text{ and so on.}$$

If we have an equation involving fractions, it is an advantage to get rid of them as soon as we can, and we may always do this by multiplying both sides of the equation by the L.C.M. of the denominators: e.g. let the equation be $\frac{x}{9} = \frac{20}{12}$. The L.C.M. of 9 and 12 is 36. Multiplying both sides by 36, we have

$$4x = 60.$$
$$\therefore \quad x = 15.$$

Instead of using the L.C.M. for our multiplier, any other C.M. will do, though this will mean rather harder arithmetic. We might, for instance, use the *product* of the denominators, viz. 108.

$$\frac{x}{9} = \frac{20}{12}.$$
$$\therefore \quad 12x = 9 \times 20.$$
$$\therefore \quad x = 15 \text{ (as before).}$$

In this form we see in the simplified second line all four terms of the original equation (x, 9, 20, 12), and this simplified second line might have been obtained *at once* from the original equation by **cross multiplying**, i.e. by multiplying each numerator by the opposite denominator. This cross-multiplying is often very useful, in algebra and geometry as well as in arithmetic.

" From cross-multiplying it follows that if we have two equated fractions, a numerator and the opposite denominator may be interchanged; e.g. $\frac{3}{7} = \frac{12}{28}$, $\frac{28}{7} = \frac{12}{3}$, $\frac{3}{12} = \frac{7}{28}$.

" Now we come to *Ratio* and *Proportion*."

Ratio and Proportion

If 1 sheep cost £3, then,

2	sheep	cost	£6	7	sheep	cost	£21
3	,,	,,	£9	10	,,	,,	£30
4	,,	,,	£12	13	,,	,,	£39
5	,,	,,	£15	21	,,	,,	£63

As we increase the number of sheep we increase **in the same proportion** the number of pounds.

Take any pair of numbers (sheep) from the first column, and the corresponding pair of numbers (pounds) from the second, say the last but two and the last in each case, and convert them into fractions, thus:

$$\frac{10}{21}, \quad \frac{30}{63}.$$

We see that these fractions are **equal**. That we should expect, for 10 bears **the same relation** to 21 as 30 bears to 63. A better way of saying it is that the **ratio** of 10 to 21 is **equal** to the **ratio** of 30 to 63.

" We know that the sign of division is $\div$, and that if in the place of the two dots we write numbers, e.g. $\frac{5}{6}$, we have a fraction, and that the fraction means 5 *divided by* 6. Thus a fraction represents a quotient. Similarly a ratio represents a quotient. A **ratio** merely shows the **relation** between two quantities, viz. how many times one is contained in the other. When two ratios are equal, as in the case of the sheep and pounds, we **write** them thus:

$$\frac{10}{21} = \frac{30}{63},$$

and we **read**,

10 bears the same ratio to 21 as 30 bears to 63.

Such a statement is **a statement in proportion.** Sometimes we read ' 10 is to 21 as 30 is to 63 ', and sometimes ' 10 over 21 equals 30 over 63 '.

" Remember, then: **a statement in proportion is a statement of the equality of two ratios.**

How many pounds of tea can I buy for 40*s. if* 6 *lb. cost* 15*s.?* Call the unknown number of pounds, x. We have 4 terms viz. 2 lots of pounds, 2 lots of shillings. Write:

Lb.	*Shillings.*
x cost	40.
6 ,,	15.

Convert each pair of terms into a ratio or fraction, equate, and solve:

$$\frac{x}{6} = \frac{40}{15}.$$

$$\therefore 5x = 80.$$

$$\therefore x = \mathbf{16}.$$

16 *lb. of tea cost* 40*s.*; *how many pounds can I buy for* 15*s.?*

Lb.	*Shillings.*
16 cost	40.
x ,,	15.

Equating, $$\frac{16}{x} = \frac{40}{15}.$$

$$\therefore 40x = 240.$$

$$\therefore x = \mathbf{6}.$$

Find the cost of 6 *lb. of tea if* 16 *lb. cost* 40*s.*

Lb.	*Shillings.*
6 cost	x.
16 ,,	40.

Equating, $$\frac{6}{16} = \frac{x}{40}.$$

$$\therefore 2x = 30.$$

$$\therefore x = \mathbf{15}.$$

6 *lb. of tea cost* 15*s.*; *what is the cost of* 16 *lb.?*

Lb.	*Shillings.*
6 cost	15.
16 ,,	x.

Equating, $$\frac{6}{16} = \frac{15}{x}.$$

$$\therefore 6x = 240.$$

$$\therefore x = \mathbf{40}.$$

The simple scheme applies to all cases of *direct* proportion write down the 4 terms in pairs; equate; solve for x.

I sell a horse for £47, 10s., thereby losing 5 per cent. What should I have sold him for if I had gained 5 per cent?

Prices in pounds	*Representative percentages.*
$47\frac{1}{2}$	95
x	105

Equating, $\frac{47\frac{1}{2}}{x} = \frac{95}{105}$.

$$\therefore x = \frac{105 \times 47\frac{1}{2}}{95} = \frac{105}{2} = £52,\ 10s.\ 0d.$$

The prices of the horse are in direct proportion to the representative percentage numbers.

Inverse Proportion

But, of course, *inverse* proportion is another story. In practice it is relatively rare, and is thus sometimes overlooked.

To cover a floor with carpet 72 in. wide I require 40 yd. from the roll; if the carpet is only half the width, I require twice the number of yards from the roll; if only one-third of the width, then 3 times the number of yards. We may tabulate thus:

Running yards.	*Width in inches.*
40	72
80	36
160	18
320	9

Clearly we cannot select a pair of terms from one column and equate them to the corresponding pair from the other. *One* pair has to be inverted, e.g.

$$\frac{80}{160} = \frac{18}{36}.$$

Thus when one quantity varies *inversely* as another, the in-

version of one ratio (it matters not which) is necessary before equating.

Teach the boys to distinguish between direct and inverse proportion by asking themselves whether when one quantity increases the other increases or decreases, and to distinguish them on paper by pointing arrows in the same direction to indicate direct proportion, in opposite directions to indicate inverse proportion.

3 *lb. of tea cost* 8*s.*; *how many lb. will cost* 24*s.*?

Lb.	*Shillings.*
$\downarrow$ 3	8 $\downarrow$
x	24

6 *men can do a piece of work in* 20 *days*; *how many could do it in* 15 *days?*

Men.	*Days.*
$\downarrow$ 6	$\uparrow$ 20
x	15

Boyle's Law is the best known example of inverse proportion in science, but in practical life examples of inverse proportion are much less common than those of direct, and the consequence is that very artificial examples are often invented to illustrate it. " Men and work " sums are often silly. " If it takes 20 men to build a house in 20 days ", more than one maker of an arithmetic book will ask us to believe that, as a logical consequence, 1000 men could build the house in $\frac{2}{5}$ of one day.

Never use the old form of proportional statement, : :: : . A common (and meaningless) form of statement sometimes found in a boy's exercise book is

$$36 : 40 :: 24.$$

Always let the equated ratios consist of two fractions, and make the boy realize that the particular position of the x (first, second, third, or fourth place) is entirely without significance.

Examples Acceptable and Unacceptable

We give two more examples.

1. *A clock which was* $1\frac{4}{9}$ *minutes fast at* 10.45 *p.m. on* 2*nd December was* 8 *minutes slow at* 9 *a.m. on* 7*th December. When was it exactly right?*

This problem, like most other problems, requires a preliminary discussion. By judicious questioning, help the boys to cast it in a simpler form:

A slow-going clock loses $9\frac{4}{9}$ *of its false minutes in* $106\frac{1}{4}$ *true hours. In how many hours will it lose* $1\frac{4}{9}$ *of its false minutes?*

"Minutes" lost by slow clock	during	*Hours of true clock*
$\downarrow 9\frac{4}{9}$		$106\frac{1}{4} \downarrow$
$1\frac{4}{9}$		x

$$\frac{9\frac{4}{9}}{1\frac{4}{9}} = \frac{106\frac{1}{4}}{x}.$$

$$\therefore \frac{85}{13} = \frac{425}{4x}.$$

$$\therefore \frac{1}{13} = \frac{5}{4x}.$$

$$\therefore x = 16\tfrac{1}{4},$$

i.e. the slow clock was right $16\frac{1}{4}$ hours after 10.45 p.m. on 2nd December, i.e. at 3 p.m. on 3rd December.

2. *It takes* 8 *men* 6 *days to mow a field of grass. How long would it take* 20 *men to do it?*

Days	*Men*
$\downarrow 6$	8
x	$20 \uparrow$

$$\frac{6}{x} = \frac{20}{8}.$$

$$\therefore x = 2\tfrac{2}{5}.$$

But although $2\frac{2}{5}$ days is the orthodox answer, the time would really be rather less. Men mowing a field for 6 days would

find, in the growing season, the work much harder on for example, the sixth day than on the first, so that the amount of grass cut would not be equally distributed over the 6 days. The answer as calculated is but a rough approximation. Writers of textbooks, and some examiners, are so often out of touch with practical life that it may be useful to append a few absurd questions of the type supposed to be examples of Ratio and Proportion:

1. It takes 3 minutes to boil 5 eggs. How long would it take to boil 6 eggs?
2. A man rides a bicycle at the rate of 20 miles an hour. How far could he travel in $92\frac{1}{2}$ hours?
3. My salary is £500 a year and I save £50 a year. How long shall I take to save £10,000?
4. My brother weighed 24 lb. when he was 3 years old. How much will he weigh when he is 45 years old?
5. A rope stretches $\frac{1}{2}$ in. when loaded with 1 cwt. How much will it stretch when loaded with 10 tons?
6. It cost £1 to dig and line a well 2 ft. deep. How much will it cost to dig and line a well 100 ft. deep?
7. A stone dropped down an empty well 16 ft. deep reaches the bottom in 1 second. What is the depth of another well, if a stone takes 5 seconds to reach the bottom?

Another point: if the answer to problems concerning men and work comes out to, say $4\frac{1}{4}$ men; instruct the boy to say **5** men, with an explanatory note.

Until a boy is thoroughly well grounded in Ratio and Proportion, the formal statement of the ratios is desirable. But at least the abler boys in the top Forms may be allowed to do as mathematicians themselves do—write down the odd term and multiply at once by the fraction determined by the ratio of the other two terms.

$3\frac{2}{3}$ lb. of tea cost $11\frac{3}{8}$ shillings; find the cost of $7\frac{1}{5}$ lb.

$$\text{Cost} = 11\tfrac{3}{8}s. \times \frac{7\frac{1}{5}}{3\frac{2}{3}}.$$

"Compound" Proportion

"Double" or "Compound" "Rule of Three". For the most part the typical sums given by the textbooks to illustrate this "rule" have little relation to practical life. Occasionally they are legitimate enough, and then they may be regarded as just a simple extension of the simpler two-ratio examples already considered. The terms may be arranged in their natural pairs, converted into ratios, these marked (with arrows) direct or inverse, then multiplied out.

If 16 *cwt. are carried* 63 *miles for* £6, 6*s.*, *what weight can be carried* 112 *miles for* £2, 16*s.*?

cwt.	*miles.*	*cost in shillings*
$\downarrow \begin{matrix} x \\ 16 \end{matrix}$	$\uparrow \begin{matrix} 112 \\ 63 \end{matrix}$	$\downarrow \begin{matrix} 56 \\ 126 \end{matrix}$

$$x = 16 \times \frac{63}{112} \times \frac{56}{126}$$
$$= 4.$$

Here is another, one of the commoner types, taken from one of the best of the textbooks: *If* 36 *men working* 8 *hours a day for* 16 *days can dig a trench* 72 *yd. long*, 18 *ft. wide*, 12 *ft. deep*, *in how many days can* 32 *men working* 12 *hours a day dig a trench* 64 *yd. long*, 27 *ft. wide*, *and* 18 *ft. deep?*

The example is not practicable. Men working 12 hours a day can *not* do $1\frac{1}{2}$ times as much work as men working 8 hours a day. The cost of digging a trench 18 ft. deep is *more* than $1\frac{1}{2}$ times the cost of digging one 12 ft. deep. The deeper the trench the more expensive it is to get out the excavated earth. The cost does not necessarily vary as the *width* of the trench; if timbering the sides is necessary (a serious additional item of expenditure), a little extra width would not add appreciably to the cost. But more than this: for excavation work, steam navvies have largely replaced manual labour.

So it is with a large number of the textbook exercises:

they have no relation to practical life. Here is one more, from a really excellent textbook.—*If* 10 *cannon which fire* 3 *rounds in* 5 *minutes kill* 270 *men in* $1\frac{1}{2}$ *hours, how many cannon which fire* 5 *rounds in* 6 *minutes will kill* 500 *men in* 1 *hour?* Did the man who made up this problem claim to be a mathematician, or a soldier, or a humorist? It is a shocking thing that school boys are made to waste their time over the pretence of " solving " problems of this kind.

CHAPTER XIII

Commercial Arithmetic

No branch of arithmetic is more important, and yet it need not take up a very great deal of time. For the most part, the work consists of the application of principles, already learnt, to business relations in practical life. Once the boy grasps the inner nature of the business relation, the arithmetic should give him little trouble. But " hard " sums, especially sums involving great masses of figures, are rarely if ever necessary. Give ample practice in working easy exercises and so make the boy thoroughly familiar with the A B C of commercial life.

Percentages

Teach the meaning of " per cent " thoroughly. We require a numerical standard of reference of some kind, and the number 100 has been accepted as the most convenient, though any other would do instead. It is a disadvantage that 100 is not divisible by 3.

5 per cent means 5 per 100 or $\frac{5}{100}$. Drive this cardinal fact well home: everything hangs upon it. 5 per cent of

£1 = $\frac{5}{100}$ of 20*s.* = 1*s.*; 2½ per cent of £1 = 6*d.* Let these two results be the pegs of plenty of mental arithmetic; e.g. 7½ per cent of £1 = 3 times 6*d.* = 1*s.* 6*d.*, and 7½ per cent of £20 = 1*s.* 6*d.* × 20 = 30*s.*; and so on.

Representative percentage numbers (as they are usefully called) is the next thing to drive home. When we buy a thing, it may be assumed that we buy it at the standard price which is represented by 100. If we sell the thing at 10 per cent profit, we sell it at a price represented by 110; if we sell it at a loss of 15 per cent, we sell it at a price represented by the number 85. This notion is of fundamental importance. The majority of exercises grouped under the term "percentages" or "profit and loss" are cases of simple proportion, the two terms of one ratio consisting of money and the two terms of the other ratio consisting of representative percentage numbers.

How much is 12¾ *per cent of* £566, 13*s.* 4*d.*?

$$\frac{12\frac{3}{4}}{100} \text{ of } £566\tfrac{2}{3} = \frac{51}{400} \text{ of } £\frac{1700}{3} = £72,\ 5s.\ 0d.$$

Direct proportion example (common): *If a debt after a deduction of* 3 *per cent becomes* £210, 3*s.* 4*d.*, *what would it have become if a deduction of* 4 *per cent had been made?*

Representative % Nos.	*Reduced debts.*
↓ 97	↓ $£210\frac{1}{6}$
96	x

$$x = £\frac{1261}{6} \times \frac{96}{97}$$

$$= £208.$$

Inverse proportion example (comparatively rare): *A fruiterer buys shilling baskets of cherries,* 30 *in a basket. He also sells them at a shilling a basket, but* 24 *in a basket. What profit per cent does he make?*

The *smaller* the number of cherries he sells in a basket, the *larger* his profit. Hence the proportion is inverse.

Cherries per Basket.	*Representative % Nos.*
↓ 30	↑ 100
24	x

$$x = 100 \times \frac{30}{24}$$
$$= 125.$$

This representative percentage number shows that the profit is **25** per cent. (Strictly, the answer is not right, as no allowance is made for the necessary purchase of new baskets.)

Another inverse proportion example: *If eggs are bought at* 21 *for* 1*s., how many must be sold for a guinea, to give a profit of* $12\frac{1}{2}$ *per cent?*

The selling price is represented by $112\frac{1}{2}$, a number *greater* than 100; the number of eggs sold for a guinea must be *smaller* than the number bought for a guinea. Hence the proportion is *inverse*.

No. for 21/-	*Representative % Nos.*
↑ 441	↓ 100
x	$112\frac{1}{2}$

$$x = 441 \times \frac{100}{112\frac{1}{2}}$$
$$= 441 \times \frac{8}{9}$$
$$= 392.$$

Simple Interest

The kind of examples really necessary should cause little trouble. Even a slow boy readily understands the main principles. As soon as he has learnt what 5 per cent per annum means, he can follow this reasoning:

Interest on £100 at 5 per cent per annum for 1 year
$= £100 \times \frac{5}{100}$.
Interest on seven times £100, i.e. on £700 for 1 year
$= £700 \times \frac{5}{100}$.
Interest on £700 for 3 years $= £700 \times \frac{5}{100} \times 3$.

There is now an excellent opportunity for establishing a simple algebraic formula:

Let I = Interest.
„ P = Principal.
„ R = Rate per cent.
„ T = Time in years.

Then $$I = \frac{P \times T \times R}{100} = \frac{PTR}{100}.$$

The technical term " amount " should also be explained: $A = P + I$.

As interest is usually paid half-yearly, " 5 per cent per annum " (as in the case of Government Stock) generally represents rather more than its nominal value. This should be explained.

The use of the formula is quite legitimate, *provided* the boy has learnt to establish it from first principles; and equally he may be allowed to deduce the subsidiary formulæ, arguing in this way:

Since from first principles

$$I = \frac{PTR}{100},$$

$$\therefore I \times 100 = PTR.$$

$$\therefore P = \frac{I \times 100}{TR}; \quad T = \frac{I \times 100}{PR}; \quad R = \frac{I \times 100}{PT}.$$

But in practical life these subsidiary problems (to find P or T or R) are very rarely wanted, and it is not worth while to let boys waste time over working a large number. An occasional example, mainly to give facility in the use of the formula, is enough.

Compound Interest

It is enough to tell a boy to find out what will be due to him if he places in the Bank £100 on deposit and allows it to remain there for 2 or 3 years, the interest, say at " 4 per cent ", being undrawn. Two minutes of explanation will show him how to work the sum, each half-year's interest being added to the Principal as it becomes due. A little later on, instruction will be necessary as to shorter procedure in calculation, but to give up time to the working of numerous examples is inadvisable. Bankers never work compound interest sums: they merely refer to ready-made tables, prepared by mathematical hacks for all the world to use. Do not let the boys waste time over such useless work, especially as the time is so badly wanted for other things. On the other hand, see that they really do understand main principles, and can readily apply them to simple cases.

The subject may, of course, be resumed in the Fifth or Sixth Form, should the general mathematical theory of interest and annuities be taken up.

Present Worth and Discount

Here again the *principles* are important and are very easily mastered. Their use may be amply illustrated by reference to a few easy examples. Do not forget to give a clear explanation of Bills of Exchange and Promissory Notes.

The boy already knows from his interest sums that

Amount = Principal + Interest.

In Discount sums, three new terms are used, and really they are identical with the three just mentioned:

Sum Due = Present Worth + Discount.

Here are two exactly analogous examples in direct proportion of the normal type.

What is the **Principal** *that will produce an* **Amount** *of* £840 *in* 3 *years at* 4 *per cent?*

When we have found the *Principal*, we can subtract it from the *Amount*, and so obtain the *Interest*.

Principal.	*Amount.*
x	£840

$$\frac{x}{100} = \frac{840}{112}.$$

$$\therefore £x = 100 \times \frac{\overset{15}{\cancel{840}}}{\underset{2}{\cancel{112}}}$$

$$= £750 \text{ (Principal)}.$$

Interest (if required)

$$= £840 - £750 = £90.$$

What is the **Present Value** *of* £840, *the* **Sum Due** *at the end of* 3 *years, the interest being* 4 *per cent?*

When we have found the *Present Value*, we can subtract it from the *Sum Due*, and so obtain the *Discount*.

Present Value.	*Sum Due.*
x	£840

$$\frac{x}{100} = \frac{840}{112}.$$

$$\therefore £x = 100 \times \frac{\overset{15}{\cancel{840}}}{\underset{2}{\cancel{112}}}$$

$$= £750 \text{ (Present Value)}.$$

Discount (if required)

$$= £840 - £750 = £90.$$

But there is only one term for each of the Ratios. Where are the others? We have to invent them.

We *do not know* the value of x, the Principal that will amount to £840 in 3 years at 4 per cent.

We *do not know* the value of x, the Present Value of the sum £840 due in 3 years at 4 per cent.

But we may take *any sum we please* and invest it for 3 years at 4 per cent. £100 is as good a sum as any.

£100 invested for 3 years at 4 per cent yields £12 interest.

Thus £100 is the *Principal* that *Amounts* to £112 in 3 years at 4 per cent, and £100 is the *Present Value* of £112, the sum due in 3 years at 4 per cent. Now we may complete our Ratios.

Principals.	*Amounts.*
$\downarrow x$	$\downarrow$ £840
100	112

Present Values.	*Sums Due.*
$\downarrow x$	$\downarrow$ £840
100	112

Remind the boy that this *True* discount is never heard of in practice. The Bill Broker's discount, which he deducts, is really the interest on the whole Sum Due. It is exactly the same as calculating *Interest* on the *Amount* instead of on the *Principal*, a thing the banker would (naturally) never dream of doing. Let the boy compare the two things, and see for himself that when the Banker deducts interest on the *Sum Due* instead of on the *Present Value*, the customer receives, as Present Value, a sum rather less than by arithmetic he is entitled to.

Exercises in Present Value and Discount are hardly worth doing, unless they are very simple and can be done quickly.

Stocks and Shares

Nothing is more important in arithmetic than a working knowledge of stocks and shares and of financial operations. Whatever views political extremists may take of a roseate financial future, we have to deal with the hard facts of the present day, when it behoves every member of society to save, and to invest his savings.

But do not make boys waste their time by working through the useless examples on stocks and shares given in many of the older textbooks.

The first stile for the boy to get over is the distinction between stock and money, and there is no better way than to turn the whole class into an imaginary Limited Liability Company with its own Directors. To play a game of this kind is worth while. Let the Directors draw up a simple Prospectus and invite subscriptions at par. A week later let the Directors report some disaster—perhaps the destruction of property by fire—and an inevitable fall in the expected interest. Some shareholders will become anxious and will be willing to sell at 90 or even lower. And so on. A little reality of this kind is worth ten times the value of a long sermon on the subject. If a boy pays £100 (any sort of paper

token will do) and receives a Certificate for 100 £1 shares, and then has to part with his shares at, say, 18*s*. each, a sense of reality is brought home to him. Perplexity about stocks and shares is almost always due to a hazy understanding of the reality which underlies it all. As always, the trouble is with the slower boys. The quicker boys pick up the threads readily enough.

There are numerous facts for the boys to understand and remember, as well as sums to work. Explain the nature of debentures, preference shares, the different kinds of ordinary shares, their relative value and relative safety. Warn the boys never to invest without taking advice, and never in any circumstances to invest in a new flotation. Explain " gilt-edged " securities, and point out the relative safety of Government stock, though even this may fall seriously in value (compare the present price of Consols with the price fifty years ago). Insist that a large interest connotes a big risk, that financial greed spells disaster. Impress upon the boys that the financial world is full of sharks.

The old days of a brokerage of $\frac{1}{8}$ per cent have passed away, and thus many of the sums in the older text-books are out of date. Stockbrokers' charges now include Government Stamp Duty, Company's Registration fee, and Contract Stamp. Give the class a short table of charges to be entered in their notebooks, for permanent reference, e.g.

Purchases £50+ to £75, total charges 18*s*. 3*d*.
Purchases £75+ to £100, total charges £1, 3*s*. 3*d*.,
and so on.

All ordinary " examples " in stocks and shares are instances of simple proportion (nearly always direct: there is little point in puzzling young boys with the rule " the amount of stock held is inversely proportional to the price "), and they call for no comment.

Examples on the purchase and on the sale of small amounts of stock and small numbers of shares are the only exercises

that need be given. Let the exercises be typical of those that in practical life the average man engages in.

Other Commercial Work

Rates and Taxes.—A simple explanation of and a variety of exercises in these are of great importance. Explain the increase in both rates and taxes during the last few years. Distinguish carefully between expenditure by the Government and expenditure by Local Authorities, and show why both kinds of expenditure are inevitable. Explain how taxes are imposed and how rates are levied. Let exercises be easy, but devise them to illustrate principles and to give an inner meaning to things. " Rateable values " is another thing to be explained.

And there are numerous other things, of which it behoves every intelligent person to have at least an elementary knowledge, things which only a mathematical teacher can handle effectively. We mention a few: Income Tax and its assessment, its schedules, its forms and the correct method of filling them up; rent, house purchase, mortgages; the raising of loans by public bodies and by private persons; insurance of all kinds, especially life insurance, Health and Unemployment insurance; policies (especially " all-in " policies) and premiums; pensions, annuities, the keeping of personal accounts, thrift, household economics; banks and saving; the Post Office bank and National Saving Certificates; co-operative stores and their financial basis; building societies; insurance tables and how to read them (a Sixth Form ought to have some knowledge of their actuarial basis). There are numerous tables of very useful kinds in *Whitaker* that every boy ought to be made to understand, and by means of them an arithmetic teacher may devise exercises of a very valuable kind.

A particularly useful syllabus on the arithmetic of citizenship is given in the appendix of the 1928 Report of

the Girls' Schools Sub-committee of the Mathematical Association.

Books on arithmetic to consult:

1. *The Psychology of Arithmetic*, Thorndike.
2. *Leçons d'Arithmétique*, Tannery (Armand Colin).
3. *The Teaching of the Essentials of Arithmetic*, Ballard.
4. *The Tutorial Arithmetic*, Workman.
5. *The Groundwork of Arithmetic*, Punnett.
6. *The Small Investor*, Parkinson.

CHAPTER XIV

Mensuration

Simple Formulæ

Easy problems involving actual measurements will be embodied in the mathematical course for children below the age of 11, by which time a boy ought to be familiar with the mensuration of rectangular areas and rectangular solids and to be able to work easy conversion (reduction) sums in linear, square, and cubic measures. He ought also to have learnt to measure up the area of the classroom floor and walls, and to express his results in formula fashion, e.g. area of floor $= l \times b$; area of the four walls $= 2(l + b)h$

He should now be taught, if he has not been taught before, to make paper models of cubes and cuboids, and from a consideration of the " developed " surfaces of these, laid out in the form of " nets ", to devise formulæ for calculating the areas; e.g. of a cube, $6l^2$; of a square prism, $4al + 2a^2$ or $2a(2l + a)$; of a brick, $2(lb + lt + bt)$. The memorizing of these formulæ is not worth while, but they are worth working out as generalizations from particular examples; and when, once more, numerical values are assigned to them, it makes early algebra very real.

The Papering of Rooms

Some attention must be given to the stock problems on the papering of rooms, but it is not worth while to take time over measuring up doors, windows, and fire-places; assume that the walls are unbroken planes, and the room rectangular. Nor is it worth while to divide the perimeter of the room by 21 in. to find the necessary number of strips of paper. There is bound to be a good deal of paper wasted, especially if the pattern is elaborate. Hence it is enough to take the total wall area $2(l + b)h$, divide this by the area of one roll of paper, 36 ft. $\times$ $1\frac{3}{4}$ ft. or 63 sq. ft. or 9 sq. yd., and so obtain the necessary number of rolls. If the answer comes out to $13\frac{1}{2}$ rolls, evidently 14 are wanted, perhaps 15 because of waste; perhaps 13 or even 12 would do, because of windows, doors, &c. A paper-hanger never measures up a room with any degree of accuracy; his estimate is very rough and always done by rule of thumb. There is really no point in giving boys such problems to work, especially when it is remembered what a large number of problems, depending on accurate measurements, may be culled from the boys' physics course.

The Carpeting of Floors

The carpeting of floors is generally considered to give an easier type of problem than the papering of walls, but the problem *in practice* is a little tricky. If from an ordinary 27-in. wide roll a carpet has to be made up to fit the usual rectangular room, it is unlikely that the width of the room is an exact multiple of 27 in., in which case the last of the strips cut off the roll will be too wide, and there will be waste; and yet the whole of that strip will have to be purchased, as the pattern cannot be " matched ". If the carpet is plain, and the purchaser does not object to patching, then the exact amount required may be cut from the roll,

though the vendor might not agree to cut to the small fraction of a yard.

Consider a floor 18′ × 12′, and carpet 2′ 3″ wide.

1. Let the carpet be *plain* (patternless). Area of floor = 24 sq. yd. Required number of running yards from the roll = $24 \div \frac{3}{4} = \mathbf{32}$. This will give 5 strips, each 6 yd. long, and 2 running yards (a piece 6′ × 2′ 3″) over. This strip of 6′ × 2′ 3″ will have to be cut up to cover a space 18′ by 9″, so that it will be cut into 3 pieces each 6′ long, placed end to end, the width of these being 9″.

2. Let the carpet show a *design*, the width being the same as before. Evidently *at least* 6 strips, each 6 yd. long, or 36 yd. in all, must be cut from the roll. It is highly improbable that, if the strips are cut to exact length, they would match when laid side by side. There would be a good deal of waste, depending on the size of the design. The problem cannot be brought within the scope of classroom arithmetic: all the factors are not available.

3. A more practical problem for the classroom is to estimate the amount of plain carpet required to cover a room of given size with a minimum number of complete strips, allowing the surplus width to determine an equal all-round border (to be stained or covered with linoleum). For instance, the 5 strips above mentioned would leave a surplus width of 9″. If the strips are placed together centrally, there will be a width of $4\frac{1}{2}$″ to spare at each end of the room. Hence we must arrange for a complete border of $4\frac{1}{2}$″ all round the room. Thus the 5 strips will not now be 18′ long, but 17′ 3″ long, and the amount to be cut from the roll will be 17′ 3″ by 5 ($28\frac{3}{4}$ yd.).

Thus the area of the room = 18′ × 12′; of the carpet, 17′ 3″ × 11′ 3″; and the required number of running yards from the 2′ 3″ wide roll = $\mathbf{28\frac{3}{4}}$.

If the *whole floor* had been covered, and patching was allowable, the number of running yards required = **32**; if patching was not allowable, the number of running yards = **36** (leaving a waste piece 6 yd. long and 18″ wide, with an area of 3 sq. yd., equal to 4 running yards).

Does it not all come round to this—that these mensuration problems concerning wall-paper and carpet are rather futile, especially when whole chapters in arithmetic books are devoted to them? Children are much better employed in mensuration problems that really do enter into the practical business of life.

Border Areas

Make these a matter of *subtraction*, whenever possible as, for instance, in estimating the area of a garden path 4′ wide between a rectangular lawn and the rectangular garden wall the garden being 108′ × 72′.

$$\text{Area} = \{(108 \times 72) - (100 \times 64)\} \text{ sq. ft.}$$

Do not allow boys to find the area of the path piecemeal.

Rectangular Solids

For the mensuration of these, boys can, with very little help, establish the necessary simple formulæ and interpret them in some brief form of words easily remembered. Problems on the excavation of trenches, the cubical content of cisterns, the air space of school dormitories, and the like, will readily occur to the teacher. The cubical content of a solid " shell " (e.g. of iron in a cistern, of stone in a rectangular trough) should, whenever possible, be made a problem of subtraction. Example: *Find the weight of a stone trough* 6″ *thick, external dimensions* 10′ × 3′ × 2′ 6″, *the weight of stone being* $1\frac{1}{2}$ *cwt. to* 1 *cubic foot.*

$$\begin{aligned}\text{No. of c. ft. of stone} &= (10 \times 3 \times 2\tfrac{1}{2}) - (9 \times 2 \times 2) \\ &= 75 - 36 \\ &= 39.\end{aligned}$$

$$\text{Weight} = 1\tfrac{1}{2} \text{ cwt.} \times 39 = 2 \text{ tons, } 18\tfrac{1}{2} \text{ cwt.}$$

A boy should never attempt to cube up the stone piecemeal.

If a gasholder (" gasometer ") at an ordinary gas-works is made the subject of a mensuration problem, remember that

(1) a gasholder has no bottom, (2) its top is not flat. Not all writers of arithmetic books seem to realize this.

Mensuration beyond the very elementary stage is best associated, primarily, with the geometry rather than with the arithmetic.

CHAPTER XV

The Beginnings of Algebra

Informal Beginnings

Regarded as simple generalized arithmetic, algebra will have been begun at the age of about 9 or 10. Quite young boys will have measured up rectangular areas and will have learnt to express intelligently the meaning of the formula $A = l \times b$. In their lessons on physical measurements, rather older boys will probably have evaluated π, $2\pi R$, πR^2; in their arithmetic lessons they will have established the formula $I = \frac{PTR}{100}$; in their first lessons on Ratio and Proportion, they will have learnt the significance of $\frac{x}{6} = \frac{7}{14}$ and will have obtained the first notions of an equation. Formally, algebra will not have been begun; informally, foundations will have been laid.

Never begin the teaching of the subject according to the sequence of the older textbooks. The difficult examples in mechanical work so often given on the first four rules, on H.C.F.s and L.C.M.s, on fractions, &c., are not only calculated to make boys hate the subject but are wholly unprofitable either at an early stage or later.

Suppose you are asking questions in mental arithmetic to a class of boys 10 or 11 years of age, and you suddenly spring upon them the sum, " add together all the numbers

from 1 to 100."—" We cannot do it, sir."—" Well, let us try. Let us first take an easier sum of the same kind: add together all the numbers from 1 to 12. We will do it in this way:

" Add together the first number and the last, 1 and 12? 13.

" Add the 2nd from the beginning and the 2nd from the end, 2 and 11? 13.

" Add the 3rd from the beginning and the 3rd from the end, 3 and 10? 13.

" 4 and 9? 13; 5 and 8? 13; 6 and 7? 13.

Now we have included them all. How many 13's? 6. What are six 13's? 78. This 78 must be the answer to the question."[1] Smiles of agreement.

" Now let us make up a little formula that we can use for similar sums: How did we obtain the first 13? We added together the *first* number and the *last*.

" What is the first letter of the word *first*? f.

" What is the first letter of the word *last*? l.

" How can we show the *sum* of f and l? $f + l$.

" How far along the line from 1 and 12 was our multiplier, 6? Half-way.

" What is the first letter of the word *half*? h.

" Now I will show you how to write down $f + l$ multiplied by h." Then follows a brief explanation, and $h(f + l)$.

" Now let us work the harder sum, 1 to 100.

" $f = ?$" 1; " $l = ?$" 100; " $h = ?$" 50.

" $\therefore h(f + l) = 50(1 + 100) = (50 \times 101) =$ **5050.**

Now add together all the numbers from 1 to 1000."—And so on. " I have been giving you an algebra lesson which I sometimes give to boys 2 or 3 years older. An interesting subject, isn't it?" Yes. " And useful?" Yes.

Or we might begin straight away with problems producing equations. First notions of an equation will already have

been given in arithmetic. By means of a few easy exercises, revise the principle that the two sides of an equation may be added to, diminished, multiplied, or divided, by any number we please, provided that the two sides are treated exactly alike. The rule of cross multiplication should also be revised. But naturally at this stage no equation should be given with a binomial in a denominator.

Here are two examples, in a teacher's own phraseology (summarized, except that, to save space, his many admirably framed questions are omitted), once taken with a class of beginners of 11.

The sum of £50 is to be divided among 2 men, 3 women, and 4 boys, so that each man shall have twice as much as each woman and each woman 3 times as much as each boy. Required the share of each.

" In sums of this kind it is always well to consider first the person who is to have the *least*, in this case a boy.

Let x be the number of £'s in each boy's share.
Then $3x$ is the number of £'s in each woman's share.
And $6x$ is the number of £'s in each man's share.
Hence we have, in £'s,

$$\begin{aligned} \text{The share of the 4 boys} &= 4x. \\ \text{The share of the 3 women} &= 9x. \\ \text{The share of the 2 men} &= 12x. \end{aligned}$$

But the sum of all these shares amounts to £50.

$$\begin{aligned} \therefore\ 4x + 9x + 12x &= £50, \\ \therefore\ 25x &= £50, \\ \therefore\ x &= £2. \qquad \therefore, \text{ \&c.}" \end{aligned}$$

I want to divide some nuts among a certain number of boys. If I give 4 nuts to each boy, I shall have 2 nuts to spare; if I give 3 to each boy, I shall have 8 to spare. How many boys are there?

" Let x be the number of boys.

" There are two parts to this problem, both beginning with the word *if*; each part enables us to write down *the*

total number of nuts, though not *exactly* as in arithmetic, because we have to use x.

" 1. If I give 4 nuts to each of x boys, I give away $4x$ nuts. But the total number of nuts is 2 more than that.

$$\therefore \textit{ the total number of nuts} = 4x + 2.$$

" 2. If I give 3 nuts to each of x boys, I give away $3x$ nuts. But the total number of nuts is 8 more than that.

$$\begin{aligned} \therefore \textit{ the total number of nuts} &= 3x + 8. \\ \therefore\ 4x + 2 &= 3x + 8, \\ \therefore\ x + 2 &= 8, \\ \therefore\ x &= 6." \end{aligned}$$

This class of boys subsequently spent the next three or four lessons in working through a chapter of problems (some of them pretty difficult) producing equations. They gained confidence quickly, and from the first looked upon their new subject as interesting and useful.

Formal Beginnings. Signs as "Direction Posts"

Of course the time comes when algebra must be treated formally. There are certain fundamental difficulties that have to be faced, and, of these, algebraic subtraction is to beginners a difficulty of a serious kind.

This does not mean that boys need be taken through the elaborate subtraction sums of the older textbooks. It means that they have to be taught the inner meaning of, say, " take $-3a$ from $-2a$ ". To the boy, what does that *mean*? At first it can mean nothing but juggling with *arithmetical* values, juggling of which he is naturally suspicious.

Consider, first, what the boy has already done (or ought to have done) in his arithmetic. He is familiar with this kind of sum.

$$\begin{aligned} & 17 - 6 - 5 + 3 - 9 + 14 \\ &= 17 + 3 + 14 - 6 - 5 - 9 \\ &= (17 + 3 + 14) - (6 + 5 + 9) \\ &= 34 - 20 \\ &= 14. \end{aligned}$$

He has been taught to collect up his numbers in this way, and to realize that the plan of adding together all the minus numbers and taking them away "in a lump" is a much better plan than taking them away separately. Thus he sees that

$$34 - 6 - 5 - 9$$

must be the same as

$$34 - (6 + 5 + 9),$$

and he therefore gets a first notion of the effect of a minus sign before a pair of brackets. Still, the work so far is wholly numerical, and nothing more. If the sum had been

$$20 - 34$$

he would probably have been taught to prefix a + sign to the 20:

$$+20 - 34,$$

to take the difference between the 20 and the 34, and to prefix the sign of the larger number (34); thus, -14. Naturally the boy would call it a "subtraction" sum, and would say that the -14 is "less than nothing"! The teacher might, however, allow the use of this illogical expression provisionally, comparing it with *debts* as against *assets*.

But the boy must soon come to grips with the fundamental algebraic notion of *direction*, as well as of numerical value.

In arithmetic always, in algebra commonly, the + sign before an initial term is omitted. But in the early stages of algebra it is advisable that it be consistently written.

In algebra it is necessary to have some means of distinguishing direction to the *right* from the opposite direction to the *left*, and direction *upwards* from the opposite direction *downwards*. The opposed signs + and − are used for this purpose. These signs are the algebraic *signposts* or *direction posts*; the two signs **direct** the numbers to which they are attached. It has been agreed that direction upwards and to the right shall be called a + (positive) direction, and direction

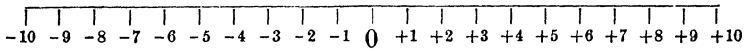
downwards and to the left a — (negative) direction. The converse would have done equally well, but the decision has been universally accepted. It is just a convention. If the boy asks *why*? tell him there is no answer.

Consider the centigrade thermometer, with the freezing-point marked 0°. If the temperature is −5° and *rises* 20°, every boy knows that it rises to +15; i.e. −5 + 20 = +15; also that if the temperature is, say, +10°, and *falls* 25°, it falls to −15°, i.e. +10 − 25 = −15. The results may be obtained by actual counting, upwards or downwards, on the scale. Upward counting means adding + numbers; downward counting means adding — numbers. The thermometer provides an excellent means of giving a first lesson on algebraic *direction*.

Addition and Subtraction

Now consider a more general case. We will choose a horizontal scale, with + numbers and — numbers to the right and left, respectively, of a zero.

Adding + quantities means counting to the right.
Adding — quantities means counting to the left.



Four **addition** *sums:*

	Addition Sums.	Starting Point on Scale.	Count or Add on Scale.	Direction, R or L.	New Point on Scale and ∴ *Ans.*
i	(+5) + (+3)	+3	+5	R	+8
ii	(+5) + (−3)	−3	+5	R	+2
iii	(−5) + (+3)	+3	−5	L	−2
iv	(−5) + (−3)	−3	−5	L	−8

Four **subtraction** *sums.*—Where we have to work a subtraction sum, say 12 − 7, we may work it by asking what

we must **add** to 7 to make 12. Thus in the four subtraction sums below we may say,

(i) What must we **add** to (+3) to make (+5)?
(ii) „ „ „ (−3) „ (+5)?
(iii) „ „ „ (+3) „ (−5)?
(iv) „ „ „ (−3) „ (−5)?

	Subtraction Sums.	Starting Point on Scale.	Scale Point to count to.	Direction, R or L.	Number of Points counted = *Answer*.
i	(+5) − (+3)	+3	+5	R	+2
ii	(+5) − (−3)	−3	+5	R	+8
iii	(−5) − (+3)	+3	−5	L	−8
iv	(−5) − (−3)	−3	−5	L	−2

Let the boys now examine the two groups of answers and note from them that:

$$(+5) + (+3) = (+5) - (-3)$$
$$(+5) + (-3) = (+5) - (+3)$$
$$(-5) + (+3) = (-5) - (-3)$$
$$(-5) + (-3) = (-5) - (+3)$$

They thus learn that in every case we can turn a subtraction sum into an addition sum merely by *changing the sign* of the subtrahend.

They ought now to understand that in *arithmetical* addition the total is increased by each term added; that in *algebraic* addition the numbers indicate movements or distances backwards or forwards along a line from a zero point or " origin ".

They ought also to see that in algebraic *addition* we may drop the sign which separates the components, and deal with the components in accordance with their own signs, e.g.

$$(+5) + (+3) = +5 + 3$$
$$(+5) + (-3) = +5 - 3$$
$$(-5) + (+3) = -5 + 3$$
$$(-5) + (-3) = -5 - 3$$

For algebraic *subtraction*, let them substitute algebraic

addition, at the same time always reversing the sign of the second term (subtrahend). Since the sum is now an addition sum, we may drop the connecting + sign as before:

$$(+5) - (+3) = (+5) + (-3) = +5 - 3$$
$$(+5) - (-3) = (+5) + (+3) = +5 + 3$$
$$(-5) - (+3) = (-5) + (-3) = -5 - 3$$
$$(-5) - (-3) = (-5) + (+3) = -5 + 3$$

Examples: Add $17x$ and $-19x$:

$$+17x - 19x = \mathbf{-2x}.$$

From $17x$ take $-19x$:

$$+17x - (-19x) = 17x + 19x = \mathbf{+36x}.$$

For the slow boys, indeed for all boys, the whole process crystallizes into three simple little rules:

1. *Addition sums.*
   (i) *Like signs*: add, and prefix the same sign.
   (ii) *Unlike signs*: find the difference between the two numbers and prefix the sign of the larger.
2. *Subtraction sums.* Reverse the sign of the second term (subtrahend) and treat the sum as an addition sum.

Teachers are not always quite happy about this question of directed numbers, and often ask if it is not unwise even to make the attempt to deal with it, and if a statement of just the rules ought not to suffice. Of the answer I have no doubt. Boys who do not grasp the significance of directed numbers can never get to the bottom of their algebra; their work all through will inevitably be mechanical. Admittedly, however, the non-mathematical boy fails to understand, and for him the rules, as rules, must suffice. A Sets can and must master the difficulty, and I think B Sets too. But with C Sets, and especially with D Sets, be content to state the rules and to give the boys plenty of practice in them. Such boys will never make mathematicians, and nobody expects that they will. It is best to admit that the application of signs to component and resultant scale distances is a little too subtle for the non-mathematical boy.

Multiplication

Here again the rule of signs can be understood only by a clear grasp of the effect of direction. The usual train illustration is as good as any.*

Graph the route of a train travelling *northwards* through O (say Oxford) at the rate of 40 miles an hour, and thus show the position of the train at all points on its journey.

Let horizontal lengths to the *right* of MOM′ measure *times after* train reaches O, and let the times be indicated by + numbers; and let those to the *left* of MOM′ measure *times before* train reaches O, and let these times be indicated by — numbers.

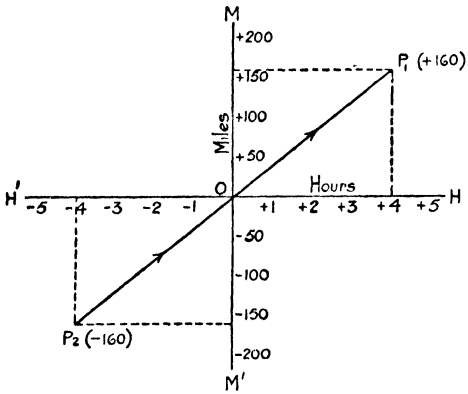


Fig. 14

Let lengths *above* H′OH measure *distances north* of O, and let these be indicated by + numbers; and let those *below* H′OH measure *distances south* of O, and let these be indicated by — numbers.

We will mark the positions of the train 4 hours before reaching O and 4 hours after passing O. (The scales used are 5 mm. to 50 miles and 5 mm. to 1 hour.) At 40 miles an hour, the train must, at these times, be 160 miles short of O and 160 beyond O, respectively. Plot points P_2 and P_1 to show this. P_2 must be directly below -4 on H′H, and to the left of -160 on M′M; P_1 must be directly above $+4$ on H′H and to the right of $+160$ on MM′. The line P_2P_1 evidently passes through O, and represents the train route.

* See Nunn, *Teaching of Algebra*, Chap. XVIII.

Now how can we determine the two positions by calculation?

We may utilize the formula $d = vt$ (" distance = speed × time "), and by making the three symbols stand for directed numbers, the formula will give us information about the *direction* as well as the magnitude. Hence we must use the term *velocity*. Let velocity northwards (40 miles an hour) be considered +.

1. Position of train at P_2:

$$d = vt = (+40) \times (-4)$$
$$= -160 \text{ (as graphed)} = 160 \text{ miles S.}$$

2. Position of train at P_1:

$$d = vt = (+40) \times (+4)$$
$$= +160 \text{ (as graphed)} = 160 \text{ miles N.}$$

Now consider the train travelling southwards. Let velocity southwards (40 miles an hour) be considered negative (—).

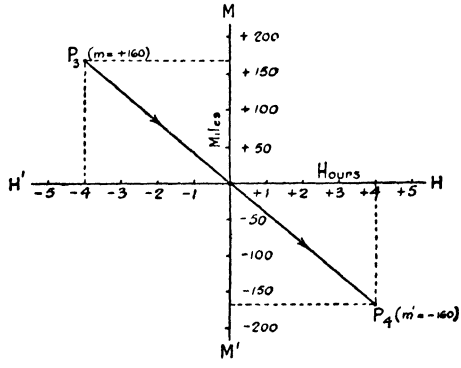


Fig. 15

3. Position of train at P_3:

$$d = vt = (-40) \times (-4)$$
$$= +160 \text{ (as graphed)} = 160 \text{ miles N.}$$

4. Position of train at P_4:

$$d = vt = (-40) \times (+4)$$
$$= -160 \text{ (as graphed)} = 160 \text{ miles S.}$$

Comparing the 4 results we have:

$$(+40) \times (+4) = +160$$
$$(+40) \times (-4) = -160$$
$$(-40) \times (-4) = +160$$
$$(-40) \times (+4) = -160$$

This is enough, *at this stage*, to justify the sign rule for multiplication. A more rigorous generalization may, if necessary, come later.

(The boys should be made to see that the sloping lines in the above graphs do not graphically show the actual railway track, which is supposed to run due north-south.)

It may be urged that the whole thing seems to be a little artificial. So it is. But the rule of signs is a universally accepted convention. The convention is perfectly self-consistent, and is easily justified, but by its nature it admits of no " proof ".

Book to consult: *The Teaching of Algebra*, Nunn.

CHAPTER XVI

Algebra: Early Links with Arithmetic and Geometry

Algebra and Arithmetic in Parallel

Get the boys to see that an algebraic fraction is only a shorthand description of actual arithmetical fractions, and that there is really no difference in the treatment. The working processes are practically identical.

The arithmetical fraction $\frac{8}{11}$ may be written $\frac{8}{8+3}$, which shows clearly that the denominator is greater by 3 than the numerator. So does the fraction $\frac{a}{a+3}$, and that is *all* it means. Thus in the fraction $\frac{1}{a+7}$, $a+7$ represents

a single number; as in arithmetic, it must be moved *as a whole* from one place in the expression to another. In algebra, beginners sometimes forget this, and treat the parts of a binomial denominator separately. So with 2 or more binomial denominators: for instance in $\frac{1}{x-5}+\frac{1}{x+11}$, $x-5$ and $x+11$ express *single numbers*.

Show a few corresponding arithmetical algebraic processes side by side. It helps the slower boys much.

1. Let $a=4$, $b=7$.

$$\frac{1}{4}+\frac{1}{7} \qquad \frac{1}{a}+\frac{1}{b}$$
$$=\frac{7}{28}+\frac{4}{28} \qquad =\frac{b}{ab}+\frac{a}{ab}$$
$$=\frac{11}{28}. \qquad =\frac{b+a}{ab}.$$

2. Let $a=3$, $b=4$, $c=5$.

$$\frac{3}{20}+\frac{4}{15}+\frac{5}{12} \qquad \frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$$
$$=\frac{3^2}{60}+\frac{4^2}{60}+\frac{5^2}{60} \qquad =\frac{a^2}{abc}+\frac{b^2}{abc}+\frac{c^2}{abc}$$
$$=\frac{3^2+4^2+5^2}{60}. \qquad =\frac{a^2+b^2+c^2}{abc}.$$

3. Let $a=7$, $b=4$.

$$\frac{1}{7-4}+\frac{1}{7+4}-\frac{14}{49-16} \qquad \frac{1}{a-b}+\frac{1}{a+b}-\frac{2a}{a^2-b^2}$$
$$=\frac{1}{3}+\frac{1}{11}-\frac{14}{33}$$
$$=\frac{11}{33}+\frac{3}{33}-\frac{14}{33} \qquad =\frac{a+b}{a^2-b^2}+\frac{a-b}{a^2-b^2}-\frac{2a}{a^2-b^2}$$
$$=\frac{11+3-14}{33} \qquad =\frac{a+b+a-b-2a}{a^2-b^2}$$
$$=\frac{0}{33} \qquad =\frac{0}{a^2-b^2}$$
$$=0. \qquad =0.$$

There is little or no need to take fractions beyond quite simple binomial denominators. Denominators of a higher order are seldom required in practice. Hence all H.C.F.s and L.C.M.s should be evaluated by factorizing, exactly as in arithmetic. The *principle* of the cumbrous divisional processes for finding factors should be familiar to boys in A Sets, who, however, may be referred to their textbooks. Do not waste time over such things in class.

Geometrical Illustrations

Factors, multiplication, division, simple expansions, &c., should all, in the early stages, be illustrated geometrically, and thus be given a reality. When this reality is appreciated, but not before, the illustrations may be given up. Second power expressions should be consistently associated with areas. We append a few illustrative examples.

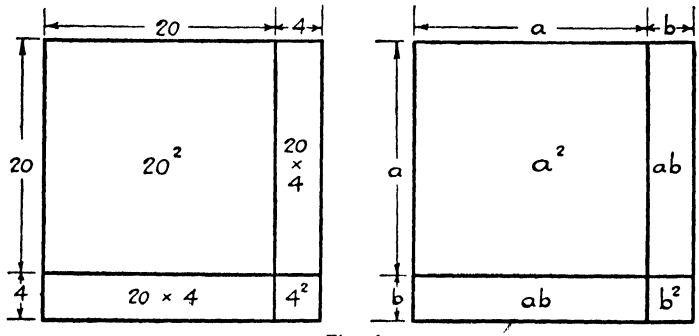


Fig. 16

1. Compare the square of 24 (i.e. 20 + 4) with the square of $a + b$.

$$24^2 = (20 + 4)^2 = 20^2 + 2.20.4 + 4^2.$$

$$(a + b)^2 = a^2 + 2ab + b^2.$$

2. Compare 24×27, i.e. $(20+4)(20+7)$, with

$$(a+4)(a+7).$$

$$(20 + 4)(20 + 7) = 20^2 + 20.7 + 20.4 + 4.7 = 20^2 + 20(7 + 4) + 4.7.$$

$$(a + 4)(a + 7) = a^2 + 7a + 4a + 4.7 = a^2 + a(7 + 4) + 4.7.$$

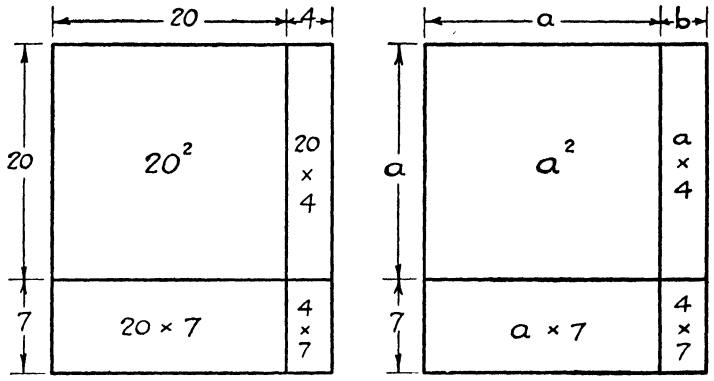


Fig. 17

3. Show graphically that

$$(2a + 5b)(a + 3b) = 2a^2 + 11ab + 15b^2.$$

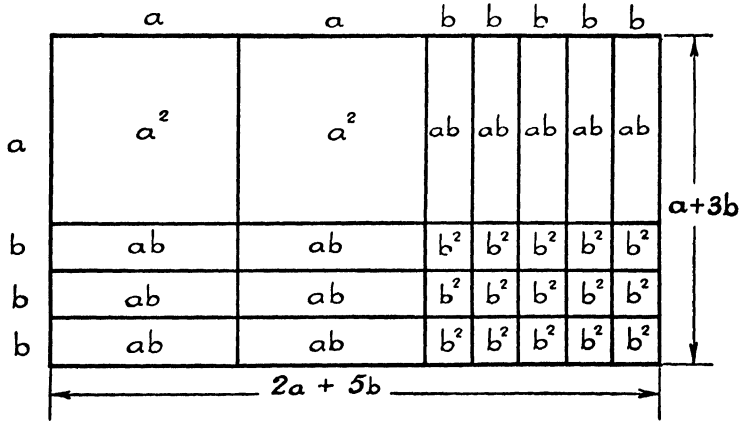


Fig. 18

The result is seen at a glance.

4. Show graphically that $(a - 2)(a - 3) = a^2 - 5a + 6$.

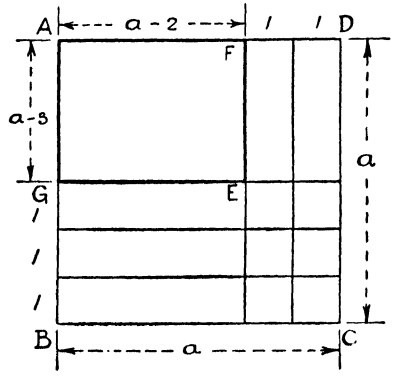


Fig. 19

$AE = (a - 2)(a - 3)$, | $FC = 2a$,
$AC = a^2$, | $GC + FC - EC = 5a - 6$,
$GC = 3a$, | $AE = AC - (GF + FC - EC)$

i.e. $(a - 2)(a - 3) = a^2 - 5a + 6$.

5. Show graphically that $(a + 2)(a - 3) = a^2 - a - 6$.

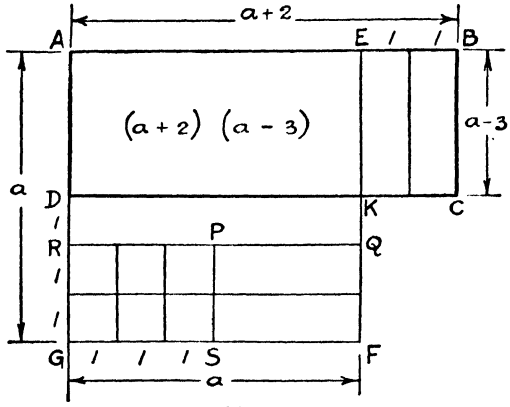


Fig. 20

$(a + 2)(a - 3) = AC$ | $= AK + PF$
$= AK + EC$ | $= AF - DQ - RS$;

i.e. $(a + 2)(a - 3) = a^2 - a - 6$.

6. Show graphically that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

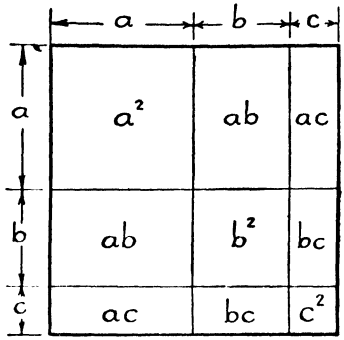


Fig. 21

The result is seen at a glance.

CHAPTER XVII

Graphs

From the Column to the Locus

Begin with *column* graphs, that is with mere verticals with the tops unconnected; say, the amount of gas consumed each week for a quarter, or the height of a barometer each morning for a week. Now join the tops of the columns, first by straight lines, then by a curved line. Do the straight lines teach anything? Which is likely to be the more correct, the straight lines or the curved line? Can intermediate columns be inserted, and, if so, what would they signify? Are there any cases where intermediate columns would be absurd?

Now discuss the locus graph, as distinct from the column

graph. The barograph is a useful example especially for the contrast of a *gentle* slope and a *steep* slope, and hence as an introduction to a *gradient* and what it signifies. What does a chart of closely packed isobars signify? of open isobars? Or graph the vertical section of a piece of hilly country, by taking heights from an ordnance survey map. Here the gentle slope and the steep slope appeal at once, the closely packed isobars and the closely packed contours being akin.

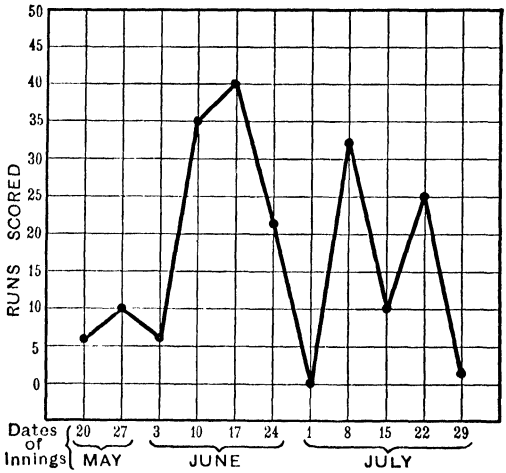


Fig. 22

The significance of a gentle gradient and of a steep gradient is fundamental. It is really the key to all subsequent work. Let the boys graph their cricket scores for the previous summer term, and discuss the resulting gradients. Familiar and personal data of this kind often provoke animated discussion of a useful character. In the first lesson or two, much of the work can be done on the blackboard, exact numerical values playing only a minor part. Give the beginners a *general* notion of the graph and its significance. A few instances may be culled from chemistry and physics, say solution curves (common salt, with its

very slowly rising straight line; nitre, with its steep curve); the experimental results, with their subsequent pictorial illustrations, are always impressive. Other useful graphs from practical work are a straight-line graph from a loaded spiral spring, or from a F.-C. scale comparison; an inverse proportion graph, say a time-speed curve during a journey. Thus prepare the way for formal work.

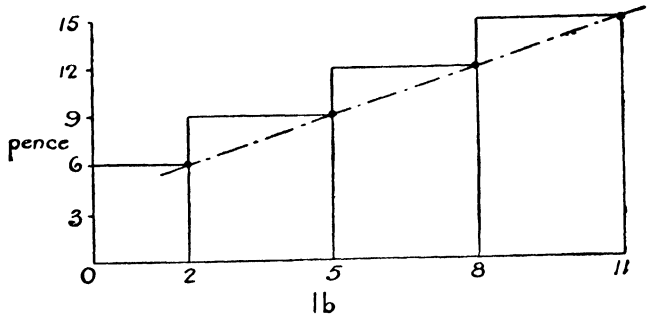


Fig. 23

A parcel post graph is easy of interpretation and, by its gradient of equal steps, leads on naturally from a column graph to a direct proportion graph. It may be called a "stepped" graph. There is a minimum charge of 6*d.* for any weight of parcel up to 2 lb.; the charge is 9*d.* for any weight over 2 lb. and up to 5 lb.; a shilling for any weight over 5 lb. and up to 8 lb.; and so on. A straight line can be drawn through the corners of the figure, but this straight line does not pass through the origin.

The mere *plotting* of a graph nowadays gives little trouble. Most modern books give instructions both simple and satisfactory. But a clear understanding of what has been done and a satisfactory interpretation of the completed graph often leave much to be desired. It is the interpretation that is the all-important thing. A graph is essentially a kind of picture, a picture to be understood. The pictorial element admits of a *general* interpretation simple enough for be-

ginners to understand, but as time goes on this interpretation must be made more and more exacting.

The study of $y = mx + c$. Direct Proportion

Experience convinces me that the study of the form $y = mx$ should precede that of the form $xy = c$. But proportionality of one kind or another underlies the whole thing, and the straight line and rectangular hyperbola should occupy a first place.

Do not attempt to define for beginners the term *function.* The term should, however, be used from the first. " Here is an expression involving x, that is, a function of x." In time, drop the words " expression involving " and simply say " function of ". Let the word be used constantly; it will gradually sink in and become part of the boys' own mathematical vocabulary.

Begin with *a straight-line graph passing through the origin.*

(i) $y = x$. What does this mean? That y is always equal to x, i.e. that the ordinate is always equal to the abscissa,

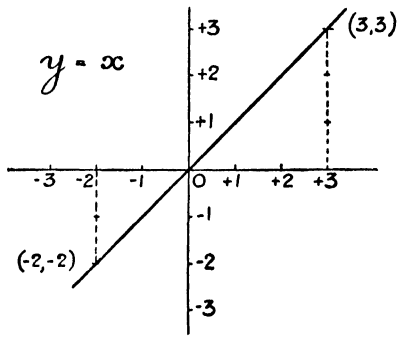


Fig. 24

no matter what point on the line is taken, whether in the first or third quadrant. Thus in the figure we have the point (3, 3) in the first quadrant and the point $(-2, -2)$ in the third.

(ii) $y = -x$. This is practically the same as before. The length of y is equal to the length of x, i.e. the length of the ordinate is equal to the length of the abscissa, *but now the signs are different*, whether a point is taken in the second

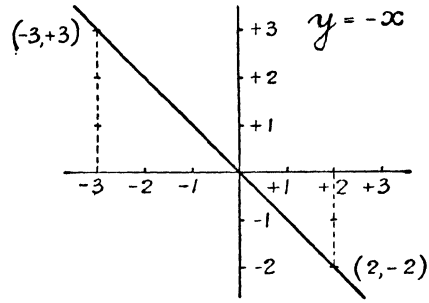


Fig. 25

quadrant (as -3, 3) or in the fourth (as 2, -2). The graph runs *from the left downwards*, from the second to the fourth quadrant.

(iii) $3y = 2x$. This means that three times the length of the ordinate is equal to twice the length of the abscissa. We may write, more simply, $y = \frac{2}{3}x$, and then we see that the

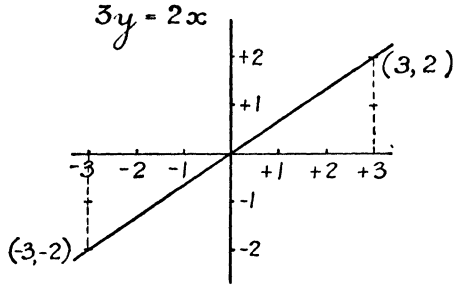


Fig. 26

ordinate is always $\frac{2}{3}$ of the abscissa. This is easily seen from any pair of values (save 0, 0) in a table:

$x =$	-3	-2	0	$+3$	$+4$	$4\frac{1}{2}$
$y =$	-2	$-1\frac{1}{3}$	0	$+2$	$+2\frac{2}{3}$	3

No matter what point in the line is chosen, the ratio of (1) the $\perp^r$ to the x axis to (2) the intercept on the x axis, i.e. the ratio $\frac{y}{x}$, is always $\frac{2}{3}$. This ratio is constant; the triangles formed by drawing perpendiculars are all similar. *The slope of the line is always the same*, i.e. *the gradient of the graph is constant.*

(iv) $2y = -3x$ or $y = -\frac{3}{2}x$. Here the length of the ordinate is always $1\frac{1}{2}$ times the length of the abscissa, but

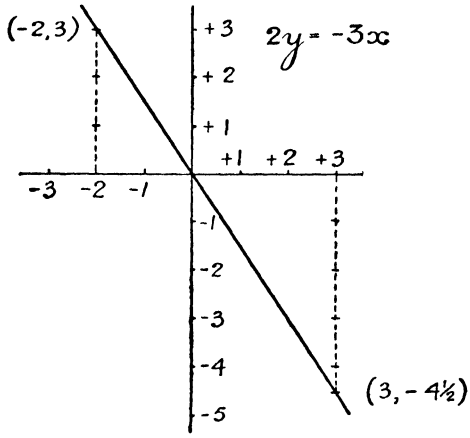


Fig. 27

the two are of opposite signs, as may be seen from any pair of values (save 0, 0).

$x =$	-2	-1	0	2	3	4
$y =$	3	$1\frac{1}{2}$	0	-3	$-4\frac{1}{2}$	-6

The graph runs *from the left downwards*, from the second to the fourth quadrant.

Before proceeding further, give the class plenty of mental work from the squared blackboard, using a metre scale or a rod to represent the graph, holding it in various positions but always passing through some selected named point and through the origin, and asking the class to name the equations.

I have known a class of thirty boys give almost instant response, one after the other, when tested in this way.

See that the boys become thoroughly familiar with the difference between $y = mx$ (same signs, slope from left upwards) and $y = -mx$ (opposite signs, slope from left downwards). Also see that they are not caught by the alternative forms to these, viz. $y - mx = 0$, $y + mx = 0$.

The next step is to see that the boys *understand the significance of the* m in the equation $y = mx$. They already know that when the coefficient of y is unity, the coefficient of x is a ratio representing $\frac{y}{x}$, i.e. the " steepness ", the " slope ", or the " gradient " of the graph, and they are thus prepared for the *general* method of writing this ratio, viz. by the letter m. Do not *begin* with the general form m, and say that it represents the slope of the line, and then illustrate it with numerical examples. Begin with the numerical examples, in order that the boys may really understand the principle; then introduce the m as a sort of shorthand registration of facts which they already know.

The next step is **to move the graph about** *parallel to itself*, and to study the effect upon the written function; and so lead the boys to see that a graph which does not pass through the origin necessarily cuts off pieces (intercepts) from both axes (we neglect the case of a graph parallel to an axis). We may begin by graphing a few particular cases of the function $y = mx \pm c$, say $y = \frac{2}{3}x \pm c$:

$$y = \tfrac{2}{3}x + 2$$
$$y = \tfrac{2}{3}x + 1$$
$$y = \tfrac{2}{3}x$$
$$y = \tfrac{2}{3}x - 1$$
$$y = \tfrac{2}{3}x - 2.$$

Show the pupils how to tabulate two or three pairs of values of each case, and how then to draw the graphs. They may then compare their results.

They will readily discover that the $+2$, $+1$, 0, -1, -2, represent merely the number of units the graph has been raised or lowered (the third case, $y = \frac{2}{3}x$, being an old friend). The function proper, $y = \frac{2}{3}x$, is *the same in all cases*; the slope is constant; the five lines are parallel. A perpendicular (ordinate) dropped from any point on the graph to the x axis

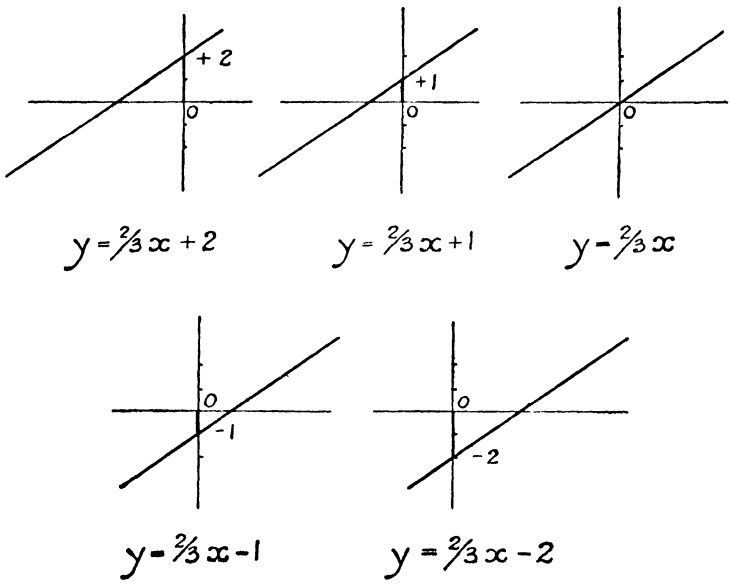


Fig. 28

shows a right-angled triangle similar to all other similarly drawn triangles. In every case, the ratio of the sides round the right angle is given by the m, the coefficient of x. The number (the c) added or subtracted represents merely the bit of the y axis intercepted between the graph and the x axis. For this reason we call such bits of the y axis, *intercepts*.

But when we raise or lower the graph above or below the origin, the graph really intercepts *both* axes. If the graph is raised *above* the origin, a portion of the y axis above the origin is intercepted, and a portion of the x axis to the *left*

of the origin, as well. If the graph is lowered *below* the origin, a portion of the y axis below the origin is intercepted, and a portion of the x axis to the *right* of the origin, as well. How in each case are the two intercepts related?

Consider the first of the above five expressions, viz. $y = \frac{2}{3}x + 2$. Instead of expressing y in terms of x, we may express x in terms of y, thus:

$$\because \quad y = \tfrac{2}{3}x + 2$$
$$\therefore \quad 3y = 2x + 6$$
$$\therefore \quad x = \tfrac{3}{2}y - 3.$$

Here the x intercept is -3, where we have precisely the same graph as before when the y intercept was $+2$. The

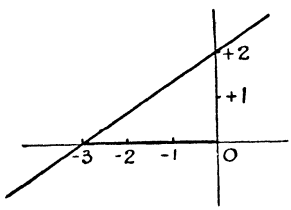


Fig. 29

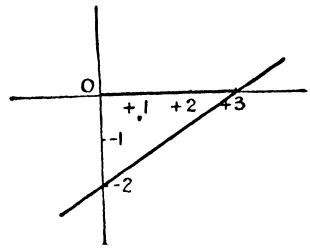


Fig. 30

function is unaltered. So with the last of the five expressions, viz. $y = \frac{2}{3}x - 2$. If we express x in terms of y, we have $x = \frac{3}{2}y + 3$. The x intercept is $+3$, and, as before, the y intercept is -2, the graph being identically the same. The **function** is unaltered, we have merely expressed it differently.

Generally, however, we express y in terms of x, and the added or subtracted quantity (the c) represents a y intercept.

The analogous results from the function $y = -\frac{2}{3}x \pm c$ may now be rapidly dealt with in the same way.

Let the pupils occasionally check a graph by means of other pairs of tabulated values. For instance, from the function $y = \frac{2}{3}x + 2$ we have:

x	-3	0	$+3$	$+6$	$+8$
y	0	$+2$	$+4$	$+6$	$+7\frac{1}{3}$

Consider the last point (8, $7\frac{1}{3}$), where OS = 8 and PS = $7\frac{1}{3}$. The slope of the graph is determined by the sides round the right angle of *any* right-angled triangle determined in the manner aforementioned. In the main figure we see *two* such triangles (shown also as separate figures

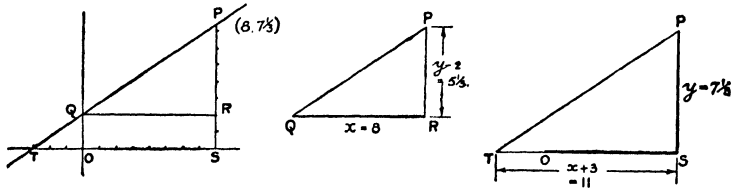


Fig. 31

with the ordinates, x and y, in dark lines). The slope is determined

either by $$\frac{PR}{RQ} = \frac{y-2}{x} = \frac{7\frac{1}{3}-2}{8} = \frac{2}{3},$$

or by $$\frac{PS}{TS} = \frac{y}{3+x} = \frac{7\frac{1}{3}}{3+8} = \frac{2}{3}.$$

Hence we may write *either* $\frac{y-2}{x} = \frac{2}{3}$; *or*, $\frac{y}{3+x} = \frac{2}{3}$. The two are identical.

Beginners are apt to confuse the value of m with the co-ordinates of some arbitrarily chosen point; e.g. to take the value (8, $7\frac{1}{3}$) of the above point P, to convert it into the fraction $\frac{7\frac{1}{3}}{8}$, and to call it m. It is a thing that wants watching.

The boys ought now to realize that, in $y = mx + c$, the c is of little consequence compared with the all-important m; and that it may sometimes be convenient to ignore the c and to plot the graph in its fundamental form $y = mx$. Since it then passes through the origin, the function is more easily recognizable.

The linear function should thus provide the boy with a

preliminary training to enable him to see clearly how the relation between variables may be represented not only in equation form but pictorially. He should be able to discover the relation between the variables, that is, to discover the equation or law connecting them, to discover what function y is of x, to discover m.

The beginner is often perplexed when told that $Ax + By + C = 0$ is the general form of a linear equation. Why those capital letters, he wonders. But if he first sees that his now familiar friend $y = -\frac{3}{4}x + \frac{18}{4}$ may be written $3x + 4y = 18$, he will understand that the new form provides a neater way of writing down the function, though the all-important m no longer reveals itself so readily. "When we write this new and neater form $Ax + By + C = 0$, the only reason for using capital letters is that it enables us to identify it readily. Other forms and their specific uses you will learn all in good time. Why should we not have different ways of writing down the same function? May we not weigh up in the laboratory a piece of brass in ounces or in grams? Convenience dictates a choice of method."

It is a good general plan to lead up to a general form through a few particular examples. To spring suddenly upon a class such a general form $Ax + By + C = 0$, before they have been suitably prepared, is not the sort of thing that an experienced teacher ever does.

Independent and *dependent* variables are terms to be introduced gradually. *Make quite clear* that the x axis is always used for the quantity which is *under our control* and is quite "independent" of the other quantity, and that for this reason it is given the name *independent variable*; and that the y axis is used for values calculated from the formula, or for values observed in experiment, i.e. values which "depend" on the selected and controlled x values, and it is therefore called the *dependent variable*. Each time we change the value of our selected x quantity, calculation or observation gives us a related y quantity; and the graph we draw is a picture to show not only how these pairs of quantities are

related but to show that this relation is the same for every pair.

Another way of expressing the connexion between the two variables is to say that the dependent variable is a *function* of the independent variable, the latter being often called the *argument* of the function, since we make it the basis of our argument. The graph of an equation shows how the *function* varies as the *argument* varies and is called the *graph of the function*; the abscissa is selected for the argument, and the ordinate thus represents the function.

The Circle

There is little to gain in spending much time over the circle, as it will rarely be used except to illustrate the solution of such simultaneous equations as $x^2 + y^2 = 52$, $xy = 24$. But it does serve to illustrate simply how a formula is affected where the graph is " pushed about ". We give the same circle in four different positions.

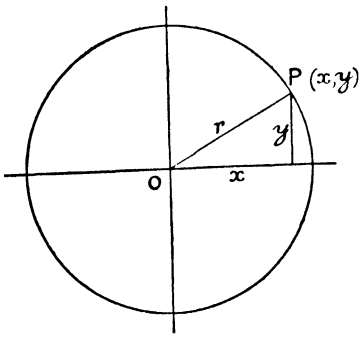


Fig. 32

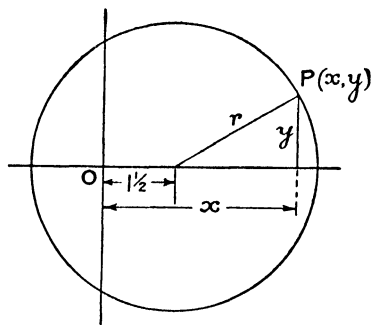


Fig. 33

Centre of circle at origin. Equation: $x^2 + y^2 = r^2$.

The centre is pushed $1\frac{1}{2}$ units to the right; its co-ordinates are $(1\frac{1}{2}, 0)$. The horizontal of the right-angled triangle is no longer x, but x diminished by $1\frac{1}{2}$.

Equation: $(x - 1\frac{1}{2})^2 + y^2 = r^2$.

The centre is pushed $2\frac{1}{4}$ units up; its co-ordinates are $(0, 2\frac{1}{4})$. The vertical of the right-angled triangle is no longer y, but y diminished by $2\frac{1}{4}$.

Equation: $x^2 + (y - 2\frac{1}{4})^2 = r^2$.

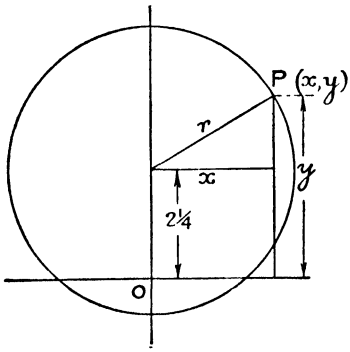


Fig. 34

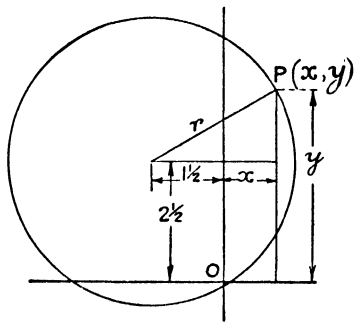


Fig. 35

The centre is pushed $1\frac{1}{2}$ units to the left and $2\frac{1}{2}$ units up. The horizontal of the right-angled triangle is $x + 1\frac{1}{2}$, and the vertical is $y - 2\frac{1}{2}$.

Equation: $(x + 1\frac{1}{2})^2 + (y - 2\frac{1}{2})^2 = r^2$.

The Study of $xy = k$. Inverse Proportion

The *direct* proportion graph we found to be a straight line. The *inverse* proportion graph (the rectangular hyperbola) is naturally the next for investigation.

Let the learner himself plot some simple case: " 32 men take 1 day to mow the grass in the fields of a farm. How many days would it take 16, 8, 4, and 2 men, and 1 man to do it?" (An absurd example, really, but for our present purpose the weather conditions and the growth of the grass may be ignored.)

With half the number of men, twice the number of days would be required.

With one-third the number of men, three times the number of days would be required.

And so on. Hence, for graphing, we may write down these pairs of values.

men	32	16	8	4	2	1
days	1	2	4	8	16	32

The graph is evidently a smooth curve. Lead the class to discover that the product of each pair of values is constant, that xy is 32 in all cases.

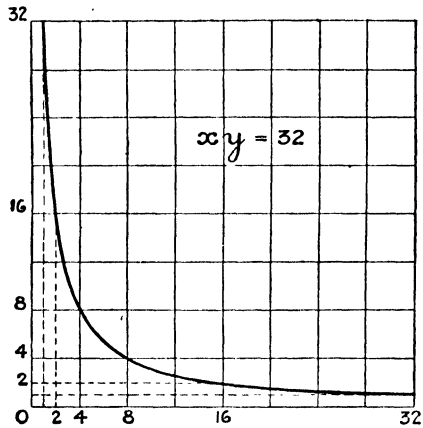


Fig. 36

Now plot $xy = k$ for several values of k, e.g. $k = 25$, 49, 64, 100, 225, 400, and examine the curves as a family. How are they related?

1. A line bisecting the right angle at O divides all the curves symmetrically.

2. The point where that line cuts the curve is the point nearest the origin; it is the " head " or *vertex* of the curve.

3. At a vertex V, $x = y$. Hence, $\therefore xy = k$, $x = y = \sqrt{k}$; $\therefore$ in $xy = 25$, the co-ordinates of the vertex V are (5, 5).

4. Each curve approaches constantly nearer the axes, but never reaches them. However great the length of x, $y = \frac{k}{x}$ and y can therefore never be zero. Neither can x ever be zero. Either may be indefinitely small because the other may be indefinitely large, but neither can be absolutely zero. Hence we say that the axes are the *asymptotes* of the curve.

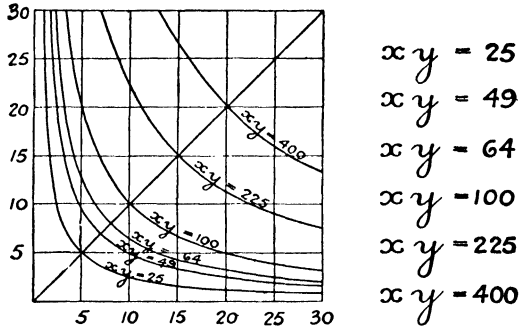


Fig. 37

This term means that the line and the curve approach each other more and more closely but never actually meet (*asymptote* = " not falling together ").

5. The successive curves are really similar, although at first they do not appear so. But draw any two straight lines through the origin to cut the curves and examine the intercepted pieces of the curves (it is best to cover the parts of the figure outside these lines), and each outer bit of curve will be seen to be a photographic enlargement of the next inner bit.

Boyle's Law is the commonest example of inverse proportion in physics. But the data (p and v) obtained from school experiments are usually too few to produce more than a small bit of curve, much too small for ready inter-

pretation. But inasmuch as the law $pv = k$ seems to be *suggested* by the data, this may be verified in two ways: (1) find the product of p and v for each pair of related values and see if the product is constant; (2) convert the apparently inverse proportion into a case of direct proportion by plotting not v against p but $\frac{1}{v}$ against p. The points thus obtained *ought* to lie on a straight line, and the line may be tested by means of a ruler, or a piece of stretched cotton. Does the line pass through the origin? Why?

There is probably little advantage in teaching boys to " push about " into new positions the rectangular hyperbola, though for purposes of illustration one or two examples may usefully be given. If the graph $xy = 120$, or $y = \frac{120}{x}$, is raised, say, 3 units, the function becomes $y = \frac{120}{x} + 3$ or $y - 3 = \frac{120}{x}$. If it is lowered 3 units, $y + 3 = \frac{120}{x}$. If it is raised 3 units and then moved 4 units to the right, the function reads $y - 3 = \frac{120}{x-4}$ or $y = \frac{120}{x-4} + 3$. But the beginner is apt to find this a little confusing. It is best to let him keep the curve in a symmetrical position, and to continue to use the asymptotes for his co-ordinate axis.

Negative values.—Instruct the class to graph $xy = 100$ for both positive and negative values. Then proceed in this way.

" When we plotted pairs of quantities from a linear function, we passed from negative values through the origin to positive values (or vice versa), and the graph was *continuous*—an unbroken straight line. Apparently, then, the rectangular hyperbola, though consisting of two separated parts, ought to be regarded as a single continuous curve. Is this possible?

" The curve in the third quadrant is certainly an exact reproduction of that in the first.

" Suppose the x axis indefinitely extended both ways, and a point Z far out to the right to travel along it towards O the origin. At any position it may be regarded as the foot of the ordinate of a corresponding point P on the curve.

As (for instance) Z_1 moves to Z_2, P_1 moves round the curve to P_2, and as ZO diminishes in length (Z_1O to Z_2O), the ordinate PZ increases (P_1Z_1 to P_2Z_2). But however long PZ may be, it gets still longer as Z gets still nearer O. In fact, it seems to become endlessly long, and yet we cannot say that the curve ever really meets the y axis, for it is absurd

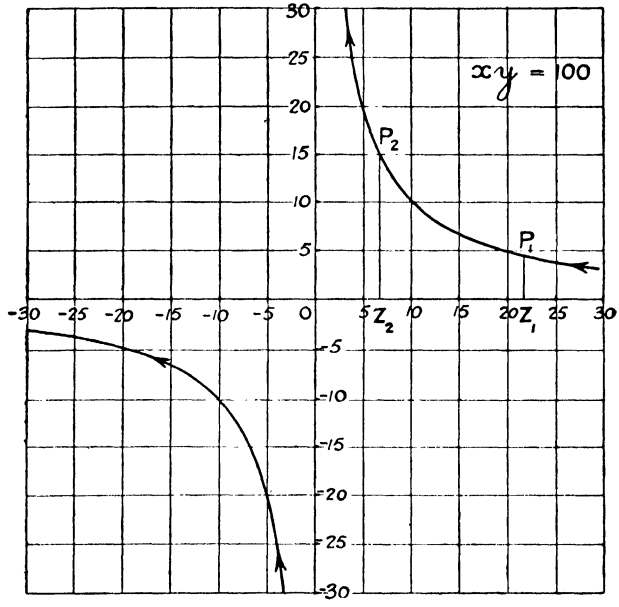


Fig. 38

to speak of the quotient $\frac{100}{0}$. But if Z continues its march, it must eventually pass to the other side of O. And yet no interval can be specified to the left and right of O so short that there are no corresponding positions of P *still nearer* to the y axis—on the right at an endless height and on the left at an endless depth. As Z proceeds along Ox, P simply repeats in reverse order along the curve in the third quadrant its previous adventure along the curve in the first. The crossing of Z over the y axis at O seems to have taken P

instantaneously from an endless northern position to an endless southern position. We feel bound to regard the two curves as two branches of the same graph, for both are given by the function $xy = k = 100$.

" If you plot $xy = -k$, the branches appear in the second and fourth quadrants."

The above argument is always appreciated by A Sets, though naturally its implications are too difficult for them to understand until later. With lower Sets, it is futile to discuss the subject at all.

With A Sets, too, the use of the term " hyperbolic function " is quite legitimate. We called $ax + b$ a *linear function* of x because the graph of $y = ax + b$ is a straight *line.* Similarly we may call any function that may be thrown into the form $\frac{k}{x+a} + b$ a *hyperbolic function* of x, because the graph $y = \frac{k}{x+a} + b$ is a (rectangular) *hyperbola.*

The Study of $y = x^2$. Parabolic Functions

The pupil should master two or three new principles before he proceeds to the quadratic function.

1. The first is the nature of a "*root*" *of a simple equation.* A very simple case will suffice to make the notion clear. The boy knows already that the root of the equation $x - 3 = 0$ is 3. Now let him graph the function $y = x - 3$.

Since $y = x - 3$ we have:

$x =$	0	1	2	3
$y = x - 3 =$	-3	-2	-1	0

The line *crosses the* x *axis* at $+3$, that is when $y = 0$, $x = 3$, and we say therefore that 3 is the " root " of the equation $x - 3 = 0$. Of course we should never let a boy waste his time by actually solving an equation in this manner, but it serves to teach him that when the value of a function

equals 0, then *the intercept on the x axis gives the root of the equation* represented by the function. (Fig. 39.)

The roots of *related* equations are easily derived. For instance, solve the equation $x - 3 = 1$. (Fig. 40.)

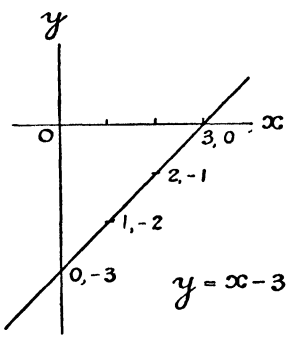


Fig. 39

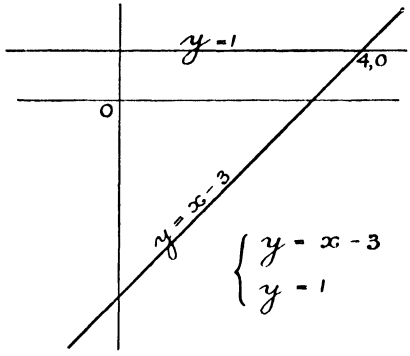


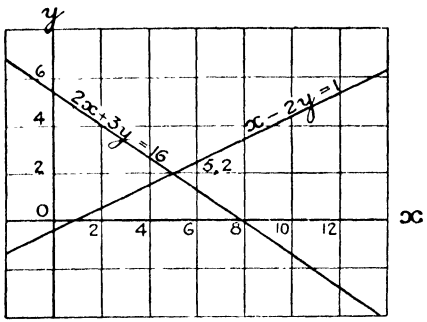
Fig. 40

Write $y = x - 3 = 1$; i.e. $y = x - 3$, and $y = 1$. The graph of $y = x - 3$ is the same as before; the graph of $y = 1$ is a line parallel to the x axis, 1 unit above. The value of x in the equation $x - 3 = 1$ is given by the intercept that $y = x - 3$ makes with $y = 1$, i.e. 4. In other words the root of the equation is the x value of the point of intersection of the two lines.

Evidently we have the clue for solving graphically two "simultaneous" equations, say, $x - 2y = 1$, and $2x + 3y = 16$. The lines cross each other at the point P (5, 2). This pair of values satisfies both equations (let class verify). A line drawn through this point parallel to the x axis is $y = 2$. Hence the value of x for both lines where they cross $y = 2$ is 5. The 5 represents the *intercept* on the line $y = 2$, made by each of the given lines. (Fig. 41.)

2. A second preliminary principle to be mastered concerns the method of making out tables of values for graphing. Having decided what values of x are to be used (this is a question of experience), write them down in a *row*, then

evaluate the successive parts of the function, *one complete row at a time*. The mental work proceeds much more easily this way than when columns are completed one at a time. For the sake of comparison, we will set out selected values



$x - 2y = 1$
$2x + 3y = 16$

Fig. 41

of the function $4x^2 - 4x - 15$, in two ways, one by addition, one by multiplication. Show the learner why the results are necessarily identical.

$x =$	-3	-2	$-1\frac{1}{2}$	-1	0	$\frac{1}{2}$	$+1$	$+2$	$+2\frac{1}{2}$	$+3$	$+4$
$4x^2 =$	36	16	9	4	0	1	4	16	25	36	64
$-4x =$	12	8	6	4	0	-2	-4	-8	-10	-12	-16
$y = 4x^2 - 4x - 15 =$	33	9	0	-7	-15	-16	-15	-7	0	9	33

Since the function factorizes into $(2x + 3)(2x - 5)$, we may set out the values of the factors and multiply, instead of adding as before:

$x =$	-3	-2	$-1\frac{1}{2}$	-1	0	$\frac{1}{2}$	$+1$	$+$	$+2\frac{1}{2}$	$+3$	$+4$
$(2x+3) =$	-3	-1	0	$+1$	3	4	5	7	8	9	11
$(2x-5) =$	-11	-9	-8	-7	-5	-4	-3	-1	0	1	3
$y = 4x^2 - 4x - 15 =$	33	9	0	-7	-15	-16	-15	-7	0	9	33

3. A third preliminary principle concerns *scales*. Different scales for the two axes are often desirable, though in the early stages of graphing different scales are not advisable. The learner should recognize the normal slope of the straight line and the normal shape of the curve. Only in this way can he recognize and analyse the purely geometrical properties of the graph. But with the study of the parabolic function, if not before, the " spread " of the numbers should be taken into account. Moreover, a good " spread " to the parabola is an advantage, in order to obtain accurate readings of the x intercepts.

We now come to the actual graphing of the function.

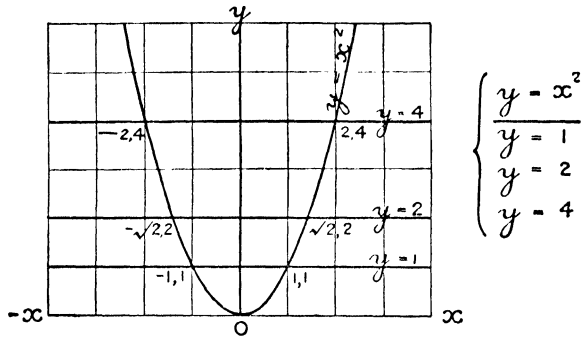


Fig. 42

Let the boy be first made familiar with the graph of the normal function $y = x^2$, the parabola being head down and the co-ordinates of its head (vertex) being (0, 0). Let him see that the curve cuts any parallel to the x axis in two points, e.g. 1 and -1, $\sqrt{2}$ and $-\sqrt{2}$, &c. The curve is symmetrical with respect to the y axis. Note that, with the same scale for both axes, there is not much spread to the curve.

Now we will graph the function $y = 4x^2 - 4x - 15$, taking the sets of values for x and y from either of the tables on the previous page. To obtain a greater " spread ", we adopt a larger scale for the x axis. The curve cuts the x axis (when $\therefore y = 0$) in two points, viz. $-1\frac{1}{2}$ and $2\frac{1}{2}$ (these values

are also seen in the tables), and these are therefore the roots of the equation $4x^2 - 4x - 15 = 0$.

From the same graph we may obtain the roots of the equations $4x^2 - 4x - 15 = 9$, or $4x^2 - 4x - 15 = -7$, or $4x^2 - 4x - 15 = z$, where z = any number whatsoever. It is simply a question of drawing across the curve a parallel

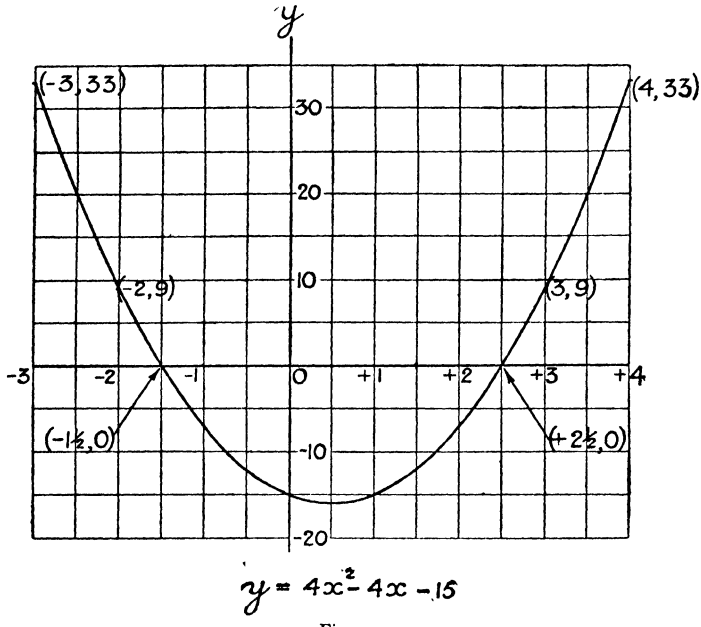


Fig. 43

to the x axis, and of reading the values of x from the points of intersection. For instance, if $4x^2 - 4x - 15 = 9$, the parallel to be drawn is $x = 9$, and this cuts the curve in $x = -2$ and 3, which are therefore the roots of the equation. These values of x may, of course, be seen from our tables where $y = 4x^2 - 4x - 15 = 9$, but they are easily estimated from the graph itself, if this is reasonably accurate.

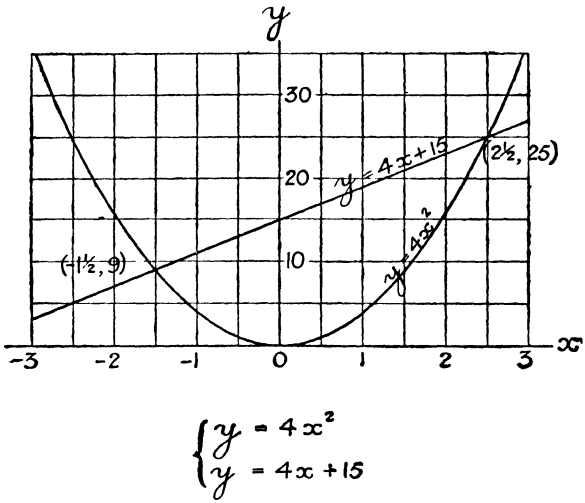
A function may sometimes be conveniently divided into two parts, and each part treated as a separate function and

graphed. The intersection of the two graphs will then give the roots of the equation. Really we have two simultaneous equations; e.g.

if $4x^2 - 4x - 15 = 0$,
then $4x^2 = 4x + 15$.

Hence we may write $y = 4x^2$
and $y = 4x + 15$

The line cuts the curve at the points $x = -1\frac{1}{2}$ and $2\frac{1}{2}$, and these are the roots of the equation $4x^2 - 4x - 15 = 0$, as



$$\begin{cases} y = 4x^2 \\ y = 4x + 15 \end{cases}$$

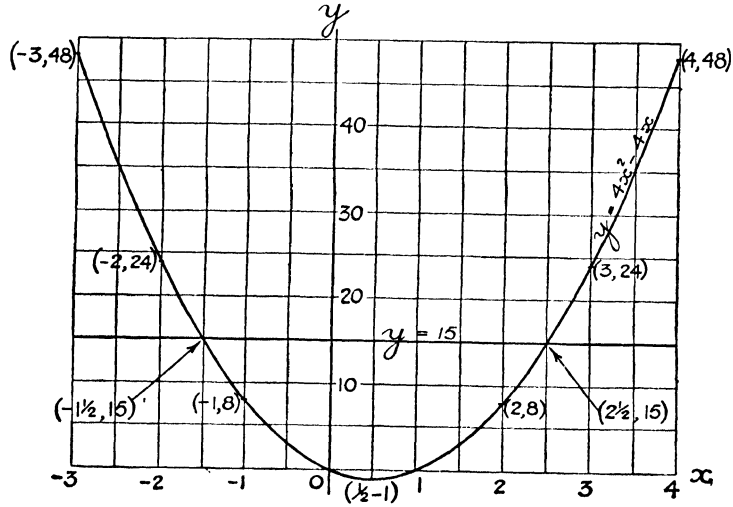
Fig. 44

before. It should be noticed that this last figure does not represent the graph of the function $y = 4x^2 - 4x - 15$, though this graph is now easily drawn by superposing the $4x^2$ graph on the $4x + 15$ graph. If $Y_1 = 4x^2$, $Y_2 = 4x + 15$, and $Y = 4x^2 - 4x - 15$, then $Y = Y_1 - Y_2$. Hence any ordinate of Y may be obtained by taking the algebraic difference of the corresponding ordinates of Y_1 and Y_2. Let the pupils draw the Y graph from their Y_1 and Y_2 graphs, and verify.

The function might have been broken up in another way

If $4x^2 - 4x - 15 = 0$,
then $4x^2 - 4x = 15$.

Hence we may write $y = 4x^2 - 4x$
and $y = 15$.



$$\begin{cases} y = 4x^2 - 4x \\ y = 15 \end{cases}$$

Fig. 45

Here are the graphs of these two functions. The latter cuts the former at $x = -1\frac{1}{2}$ and $2\frac{1}{2}$, the same roots as before.

The easiest way to discover where the parabola $4x^2 - 4x - 15$ crosses the x axis is to express the quadratic function as a product of two linear functions, viz. $(2x + 3)(2x - 5) = 0$. Hence either $2x + 3 = 0$ or $2x - 5 = 0$, i.e. $x = -\frac{3}{2}$ or $+\frac{5}{2}$. Thus from the two linear functions we form two simple equations, the roots of which are the roots of the quadratic equation.

We will plot these two linear functions (see the second table, p. 156). (The lines happen to be parallel. Why?) The graph of the quadratic function is readily obtained by multiplying together corresponding y values (again refer to second table, p. 156). For instance, at -2 the y value of $2x + 3$ is -1 and the y value of $2x - 5$ is -9. The product of -1 and -9 is $+9$. Hence at -2 the y value of the quadratic function is $+9$, i.e. the point $(-2, +9)$ is a point

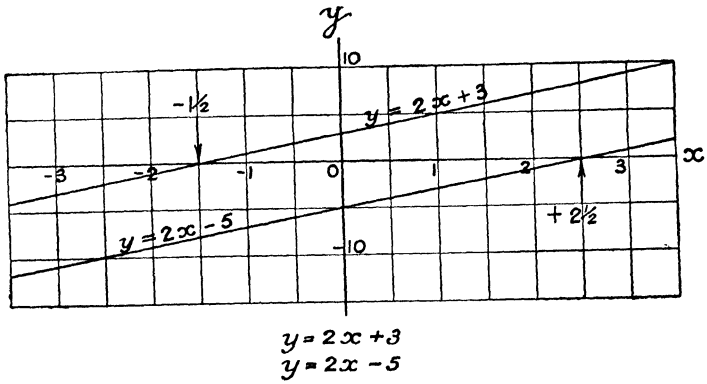


Fig. 46

on the curve. By pursuing this plan we may obtain fig. 43 over again.

The boy ought now to realize that he may graph his function in a variety of ways. But do not encourage him to think that the normal process of solving a quadratic equation is to graph the function. Not at all. The important thing for the boy to understand is that every algebraic function can be thrown into a picture and that this picture tells a story. What the algebra means to the geometry and what the geometry means to the algebra are the things that matter. We are dealing with the *same thing*, though in two different ways, and the closeness of the relationship should be seen clearly. As with the linear function, so with the parabolic

function: the boy must see the result of " pushing the graph about ".

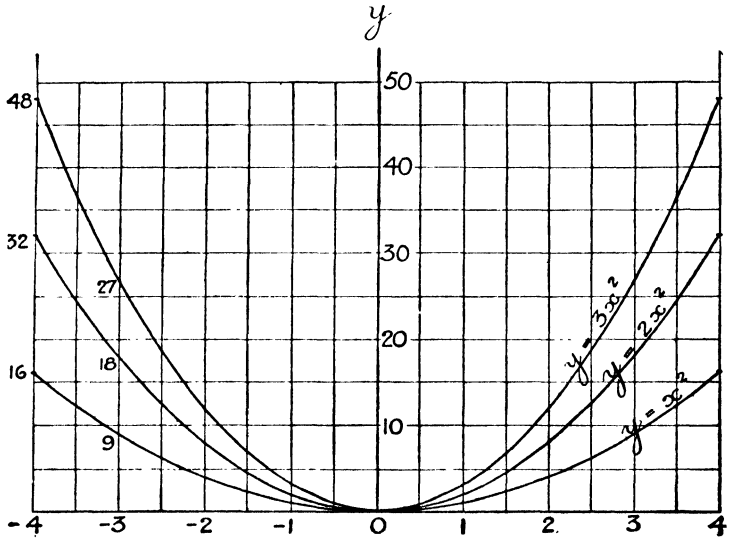
If $4x^2 - 4x - 15 = y,$
then $4x^2 - 4x + 1 = y + 16,$
or $(2x - 1)^2 = y + 16.$

If we compare this with the normal form $x^2 = y$, we see that:

$2x - 1$ has taken the place of x
and $y + 16$ has taken the place of y,

i.e. instead of $x = 0,\ 2x - 1 = 0,$ or $x = \frac{1}{2},$
and instead of $y = 0,\ y + 16 = 0,$ or $y = -16,$

i.e. the head of the parabola is not (0, 0) but $(\frac{1}{2}, -16)$ as in fig. 43. Clearly the graph of $4x^2 - 4x - 15$ is *identical*



$$\begin{cases} y = x^2 \\ y = 2x^2 \\ y = 3x^2 \end{cases}$$

Fig. 47

with the graph of $4x^2$, except that it has been pushed $\frac{1}{2}$ unit to the right, and 16 units down. (The scale difference must, of course, be borne in mind.)

This identification of similar functions is of great importance throughout the whole range of algebra. One of the greatest difficulties of beginners is to see how the form of a normal function may be obscured by mere intercept values.

Family of parabolas.—Let the boy graph a few related parabolas like the following: $y = x^2$; $y = 2x^2$; $y = 3x^2$; &c. For $2x^2$, the ordinates of x^2 are doubled; for $3x^2$, tripled; and so on. Grouping of this kind helps to impress on the learner's mind the relationship of the curves.

A metal rod bent into the shape of a parabola, with an inconspicuous cross-piece for maintaining its shape and for moving it about the blackboard, is useful for oral work in class.

Contrast the parabola $y = ax^2 + bx + c$ when a is negative with that when a is positive. With a negative, the curve is " head up "; e.g. $7 + 3x - 4x^2$ gives such a parabola. Fig. 49 shows another. Give the boys a little practice in drawing parabolas in this position. They should also draw one or two of the type $x = y^2$ and $x = -y^2$, and carefully note the positions with respect to the axis.

Turning-Points. Maximum and Minimum Values

The pupil has learnt that in $y = 4x^2 - 4x - 15$, the head of the parabola is $(\frac{1}{2}, -16)$. He sees that the equation $4x^2 - 4x - 15 = 0$ has two roots whenever y is greater than -16. For example if $y = 9$, $x = -2$ and 3; if $y = -7$, the roots are -1 and 2; if $y = -15$, the roots are 0 and $+1$. But if $y = -16$ the two roots are equal, each being $\cdot 5$. If a line parallel to the x axis is down below $y = 16$, it does not cut the curve at all, so that if y is less than -16, x has no values, or, as is generally said, " the equation has no roots ". For instance, if we give y the value -17, and work out the equation $4x^2 - 4x - 15 = -17$ in the ordinary way, we

find that $x = \frac{\pm\sqrt{-1} + 1}{2}$. But these values of x have no reality because we cannot have the square root of a negative number. The graph tells the true story. Instead of saying that the equation has two unreal or "imaginary" roots, we may more correctly say that, when the value of y is less than -16, x has no value at all, simply because the y line

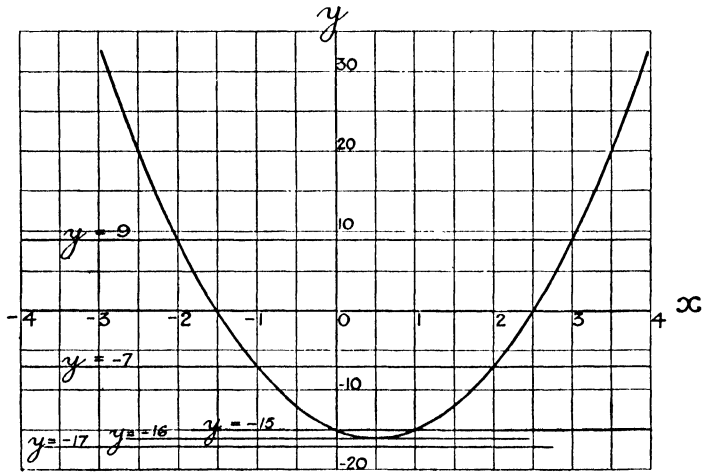


Fig. 48

does not now cut the curve at all. The y line is "out of the picture".

As a point moves along the curve from the left downwards, the ordinate of the point decreases until it reaches the value -16, then a turn upwards is made, and the ordinate begins to increase as it ascends to the right. The point $(+\cdot5, -16)$ is the *turning-point* of the graph, and the value -16 of the ordinate is called the *turning value* of the ordinate (or of the function). That value of the ordinate is its *minimum value*. If the graph was one with its head upwards, the turning-point would be at the top and would be a *maximum* value (see fig. 49).

Thus the pupil must understand clearly that, in the case of any parabolic function, the head (vertex) represents a kind of limiting value of y. Each value of y corresponds to two different values of x, though the head of the curve seems to be an exception. Strictly speaking, the head corresponds to only one value of x, but it is convenient to adopt the convention that x has in this case two identical values. Beyond the head, outside the curve, x can have *no* values. Some quadratic equations have two roots, some one (two identical), some none. Do not talk of "imaginary" roots: that is nonsensical. We shall refer to this point again, in the chapter on complex numbers (see Chap. XXVII).

The pupil should note how slowly the length of the ordinate changes near the turning-point of a parabola. In fact this characteristic of *slow change* near a turning-point is characteristic of turning-points in all ordinary graphs. Let the pupil plot on a fairly large scale $y = x^2$ for small values of x.

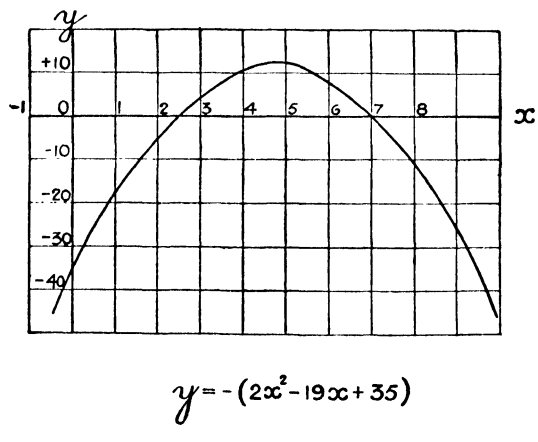


Fig. 49

Show the pupil how the graph tells him at a glance where the values of y (the function) are positive, say for $y = 19x - 2x^2 - 35$. The part of the curve above the x axis

corresponds to values of x between $2\frac{1}{2}$ and 7. But the values of y ($= 19x - 2x^2 - 35$) above the x axis are positive. Hence the expression $19x - 2x^2 - 35$ is positive between the values $2\frac{1}{2}$ and 7. If any values outside these are tested algebraically, the expression is seen to be negative. (Fig. 49.)

It may be emphasized again that quadratic equations should be looked upon as merely one interesting and useful feature in the general elementary theory of parabolic functions. Do not forget practical applications of the parabolic function; e.g. falling bodies in mechanics.

Simultaneous Equations

Practice in solving various types of simultaneous equations should be given less with the idea of finding the actual roots of the equations than for the purpose of studying the relative positions and the intersections of the graphs. We will refer briefly to two typical examples.

1. Consider the equations:

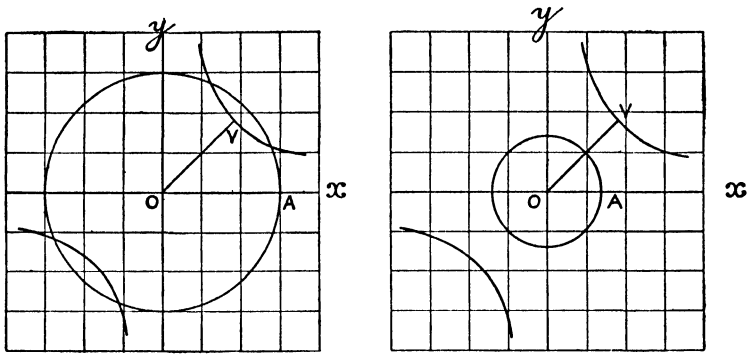
$$\left.\begin{aligned} x^2 + y^2 &= 97 \\ xy &= 36 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x^2 + y^2 &= 20 \\ xy &= 36 \end{aligned}\right\}$$

$x^2 + y^2 = 97$ is a circle with its centre at the origin and radius $\sqrt{97}$; and $xy = 36$ is a rectangular hyperbola symmetrically placed in the first and third quadrants, with its vertices at a distance of $\sqrt{2 \times 36}$ from the origin. As $\sqrt{2 \times 36}$ is less than $\sqrt{97}$, the circle cuts the hyperbola in four points, symmetrically placed. In the second case, since $\sqrt{20}$ is less than $\sqrt{2 \times 36}$, the circle does not cut the hyperbola, and there are no roots. (Fig. 50.)

2. Consider the equations:

$$\left.\begin{aligned} y &= (x-2)^2 \\ x &= y^2 - 1 \end{aligned}\right\}$$

Here we have two parabolas, one with its apex downwards, touching the axis of x, two units to the right of the origin,

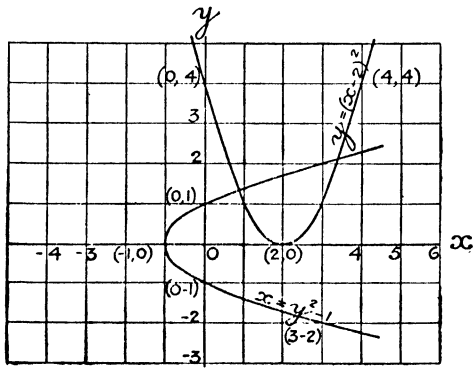


$x^2 + y^2 = 97$
$xy = 36$
$OV = \sqrt{2 \times 36}$; $OA = \sqrt{97}$

$x^2 + y^2 = 20$
$xy = 36$
$OA = \sqrt{20}$; $OV = \sqrt{2 \times 36}$

Fig. 50

the other symmetrically astride the x axis, with its apex at -1 to the left. The roots are readily obtained approximately by measurement of the co-ordinates of the intersections. (Fig. 51.)



$y = (x-2)^2$
$x = y^2 - 1$

Fig. 51

Higher Equations

The pupils should study a few cubics graphically, if only that they may gain confidence in a method of general application.

The normal form of the cubic ($y = x^3$) is easily graphed and remembered.

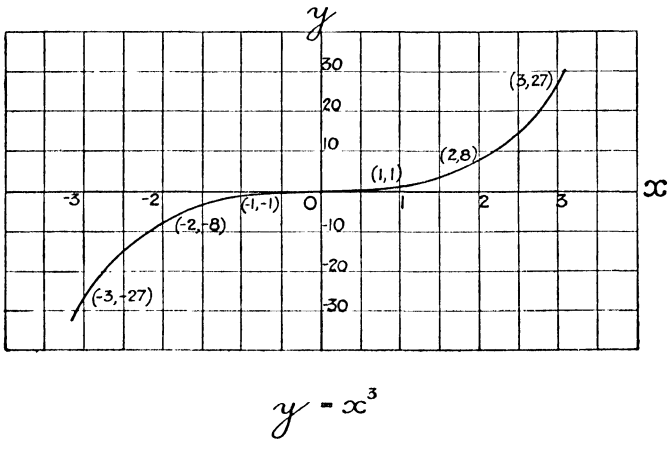


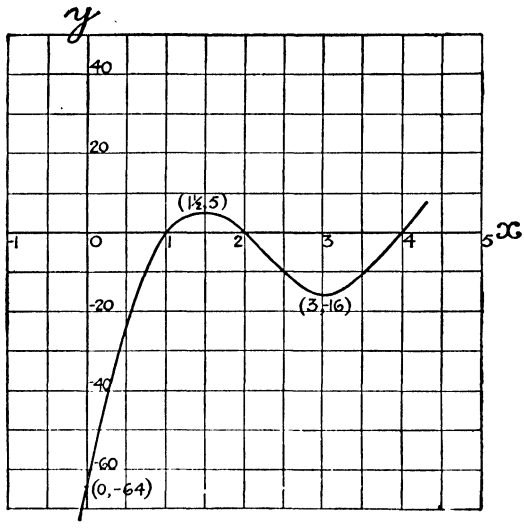
Fig. 52

Consider the equation $8(x - 1)(x - 2)(x - 4) = 0$.

Let $8(x - 1)(x - 2)(x - 4) = y$.

$x =$	0	1	$1\frac{1}{2}$	2	3	4	5
$(x - 1) =$	-1	0	$\frac{1}{2}$	1	2	3	4
$(x - 2) =$	-2	-1	$-\frac{1}{2}$	0	1	2	3
$(x - 4) =$	-4	-3	$-2\frac{1}{2}$	-2	-1	0	1
$y = 8(x - 1)(x - 2)(x - 4) =$	-64	0	5	0	-16	0	96

The curve cuts the x axis at points 1, 2, and 4, which are therefore the roots of the equation (as, of course, we know at once from the factors). (Fig. 53.)



$y = 8(x-1)(x-2)(x-4)$

Fig. 53

Now consider the equation $x^3 - 7x + 4 = 0$.

Since $x^3 - 7x + 4 = 0$; $\therefore\ x^3 = 7x - 4$.

Let $x^3 = Y_1$; $7x - 4 = Y_2$; $Y_3 = Y_1 - Y_2 = x^3 - 7x + 4$.

We will tabulate values for Y_1, Y_2, and Y_3.

		−3	−2	−1	0	+1	+2	+3
Y_1 =	x^3 =	−27	−8	−1	0	+1	+8	+27
Y_2 =	$7x - 4$ =	−25	−18	−11	−4	+3	+10	+17
Y_3 =	$x^3 - 7x + 4$ =	−2	+10	+10	+4	−2	−2	+10

We will now plot Y_1 ($= x^3$), a normal cubic, and Y_2 ($= 7x - 4$) a straight line, and so solve the equation. The latter cuts the former in three points, viz. where $x = -2{\cdot}90$, $-{\cdot}60$, $2{\cdot}29$, which are therefore the three roots. But the

figure (fig. 54) does not show the graph of $y = x^3 - 7x + 4$, the original function. To draw this graph, we may **either** use the values of Y_3 in the table, **or** superimpose the above two graphs, Y_1 and Y_2, on each other, remembering that

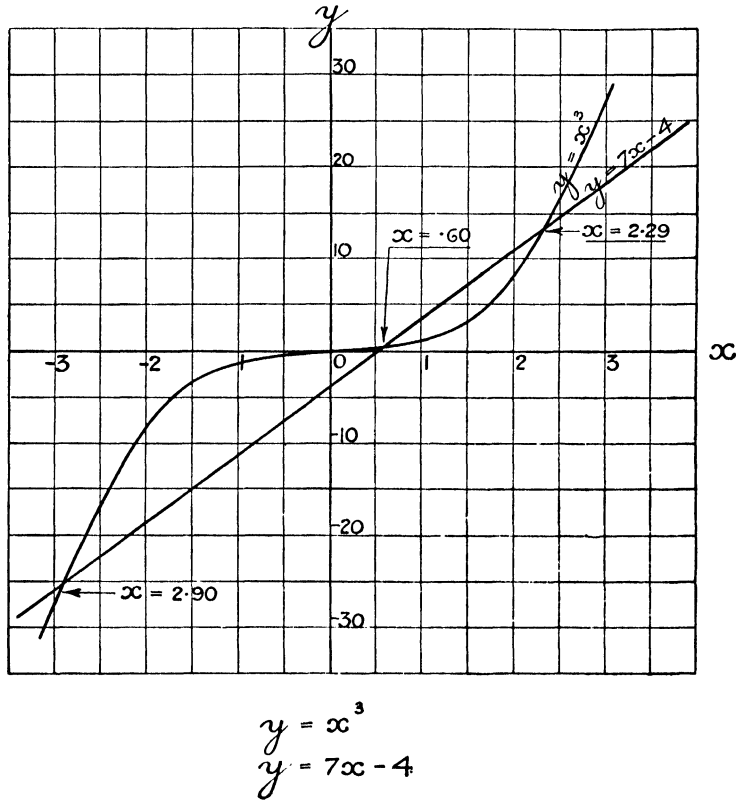
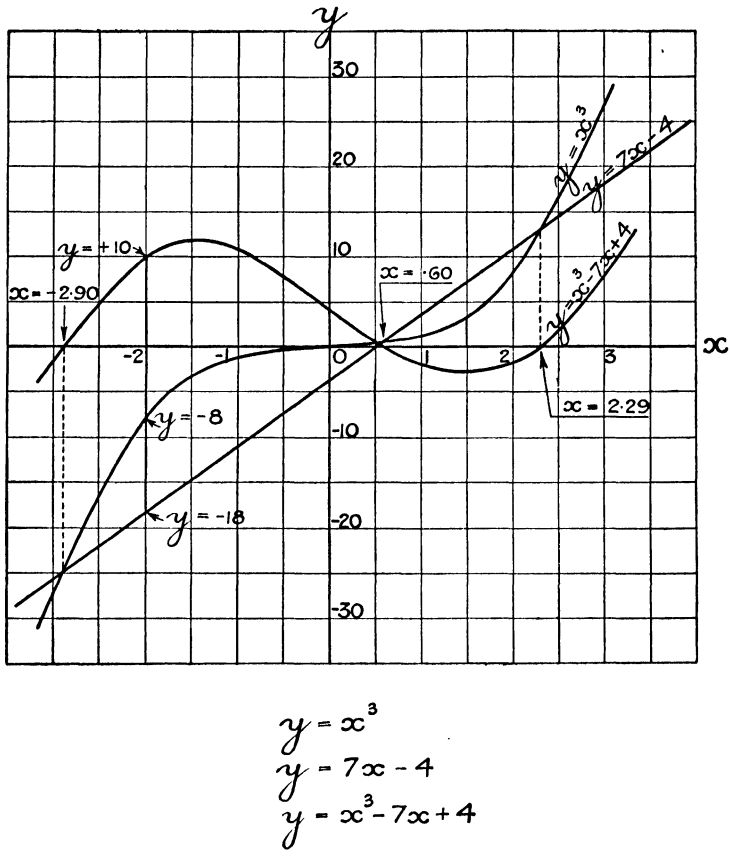


Fig. 54

$Y_3 = Y_1 - Y_2$, and that therefore we may obtain any ordinate for Y_3 by taking the *difference* of the corresponding ordinates for Y_1 and Y_2. For instance, the ordinate at $x = 2$ is -8 for $Y_1 = x^3$, and -18 for $Y_2 = 7x - 4$, and for $Y_3 (= Y_1 - Y_2)$ is therefore $-8 + 18$, or $+ \mathbf{10}$. And so generally. This



$$y = x^3$$
$$y = 7x - 4$$
$$y = x^3 - 7x + 4$$

Fig. 55

time the roots of the equations are given by the intersection of the curve with the x axis, the values ($-2{\cdot}90$, $\cdot 60$, $2{\cdot}29$) being, of course, the same as before. (Fig. 55.)

The Logarithmic Curve

We dealt with the A B C of Logarithms in Chapter XI, and we now come to the logarithmic curve, the use of which is, of course, not as a substitute for the tables but as a justification of the extension of the laws of indices from positive integers to fractional and negative values. The boy has to learn, too, that the curve is really a *picture* of a small set of tables. He should therefore be taught to plot a curve from first principles, and to use it as far as he can.

Let him first become familiar with the general form of the curve. For instance he might plot $y = 2^x$, 3^x, 5^x.

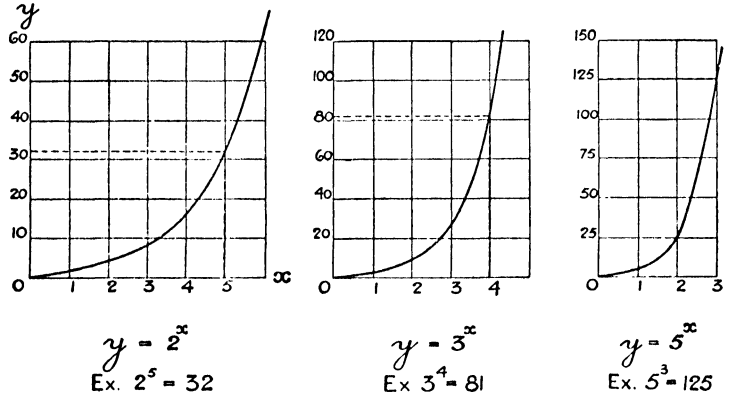


Fig. 56

Show the advantage of changes of scales. Draw two or three extended logarithmic curves on the blackboard, and spend a few minutes in oral work, e.g. 2^8? 3^6? 5^4? (approximate answers are of course, all that can be expected).

The next step is to deal with the evaluation of fractional indices in $y = 10^x$. Let the class graph $y = 10^x$ up to $x = 3$, on a fairly large scale, drawing the graph from the integral values $x = 1, 2, 3$. " If the index law holds good, we *ought* to be able to obtain by readings from the graph such values as $10^{1\frac{1}{2}}$ and $10^{2\frac{1}{2}}$. But our graph is necessarily very rough;

we had such a few points with which to plot it. We must try to construct a better curve.

" Let us use our arithmetic for constructing the curve, say a curve representing values from 10^0 to 10^1. The more values we find, the more points we shall have for plotting our curve. How many? Say 7 between 10^0 and 10^1, viz.

$$10^{\frac{1}{8}},\ 10^{\frac{2}{8}},\ 10^{\frac{3}{8}},\ 10^{\frac{4}{8}},\ 10^{\frac{5}{8}},\ 10^{\frac{6}{8}},\ 10^{\frac{7}{8}}."$$

Begin with $\mathbf{10^{\frac{4}{8}} = 10^{\frac{1}{2}} = \sqrt{10} = 3{\cdot}162.}$

Then $\mathbf{10^{\frac{2}{8}} = 10^{\frac{1}{4}} = \sqrt{10^{\frac{1}{2}}} = 1{\cdot}779.}$

Then $\mathbf{10^{\frac{1}{8}} = \sqrt{10^{\frac{1}{4}}} = 1{\cdot}333.}$

We have 4 more to find, viz.

$$10^{\frac{3}{8}},\ 10^{\frac{5}{8}},\ 10^{\frac{6}{8}},\ 10^{\frac{7}{8}}.$$

$\mathbf{10^{\frac{3}{8}}} = (10^{\frac{1}{8}})^3 = (1{\cdot}333)^3 = \mathbf{2{\cdot}371.}$

$\mathbf{10^{\frac{5}{8}}} = (10^{\frac{1}{8}})^5 = (1{\cdot}333)^5 = (1{\cdot}333)^2 \times (1{\cdot}333)^3$
$= 1{\cdot}779 \times 2{\cdot}371 = \mathbf{4{\cdot}217.}$

$\mathbf{10^{\frac{6}{8}}} = 10^{\frac{3}{4}} = (10^{\frac{1}{4}})^3 = (1{\cdot}779)^3 = \mathbf{5{\cdot}623.}$

$\mathbf{10^{\frac{7}{8}}} = (10^{\frac{1}{8}})^7 = (1{\cdot}333)^7 = (1{\cdot}333)^4 \times (1{\cdot}333)^3$
$= 3{\cdot}162 \times 2{\cdot}371 = \mathbf{7{\cdot}497.}$

If the arithmetic is distributed amongst the class, it is quickly done; very little explanation is necessary, provided previous elementary work in powers and roots was understood.

Now the boys can make up their table of values, changing the vulgar fractions into decimal fractions; then plot their points, and draw the curve.

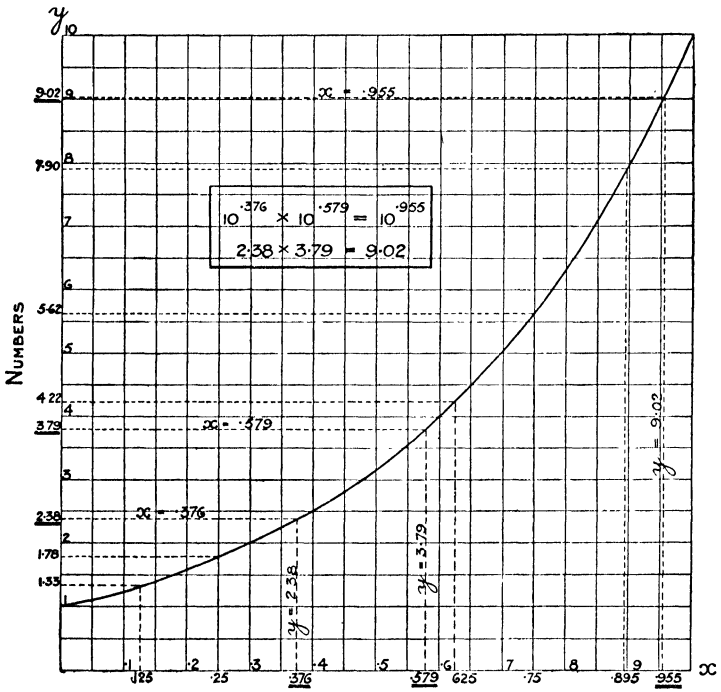
$x =$	0	·125	·25	·375	·5	·625	·75	·875	1
$y = 10^x =$	1	1·333	1·779	2·371	3·162	4·217	5·623	7·497	10

The class may now be given a few multiplication and division sums to work, for the purpose of checking their curve. (Of course they cannot read to more than 2 places of decimals.)

1. Multiply $3{\cdot}79 \times 2{\cdot}38$.

From the graph, $3{\cdot}79 = 10^{\cdot 579}$ and $2{\cdot}38 = 10^{\cdot 376}$.
$\therefore$ $\mathbf{3{\cdot}79 \times 2{\cdot}38 = 10^{\cdot 579} \times 10^{\cdot 376} = 10^{\cdot 955} = 9{\cdot}02}$ (from the graph).

Now verify by actual multiplication.



POWERS TO BASE 10.

Fig. 57

2. Divide 9·02 by 2·38.

$$9{\cdot}02 \div 2{\cdot}38$$
$$= 10^{\cdot 955} \div 10^{\cdot 376} = 10^{\cdot 955 - \cdot 376} = 10^{\cdot 579} = \mathbf{3{\cdot}79}.$$

Now verify by actual division

Now let the class write into their graph, by interpolation, the index values of the integral numbers 1 to 10. (Some

teachers make the boys learn off these values to 3 places of decimals.) The boys' interpolations resulting from their own measurements will necessarily be very rough and at this stage a prepared graph of the following kind might be given them.

$x =$	0	·301	·477	·602	·699	·778	·845	·903	·954	1·00
$y = 10^x =$	1	2	3	4	5	6	7	8	9	10

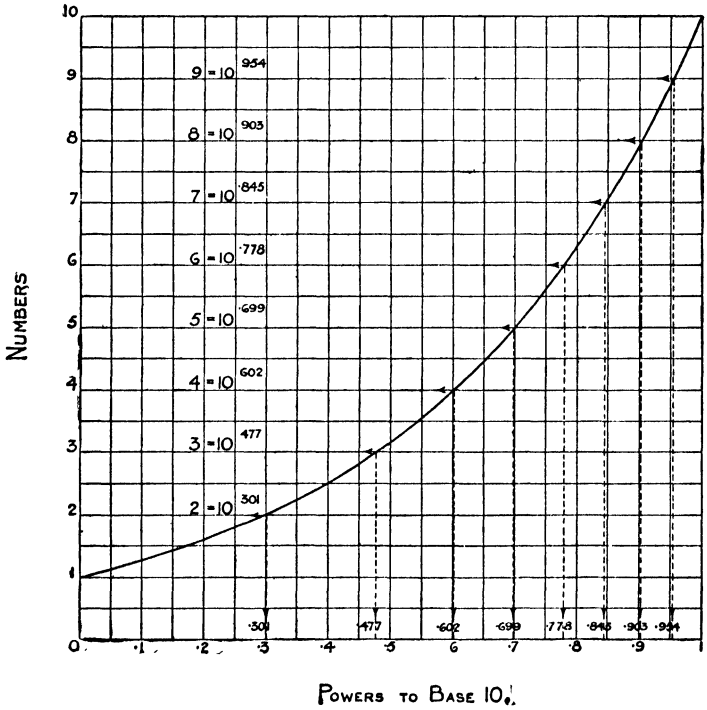


Fig. 58

The term "logarithm" may now be introduced. "It is just another name for index." Set out a multiplication

sum in parallel, showing the related methods. Emphasize the fact that the two things are the same, except in appearance.

Multiply 4·73 by 1·84.

1.	2.
Let $4{\cdot}73 \times 1{\cdot}84 = x$.	Let $4{\cdot}73 \times 1{\cdot}84 = x$.
$x = (4{\cdot}73 \times 1{\cdot}84)$	$\log x = \log (4{\cdot}73 \times 1{\cdot}84)$
$= 10^{\cdot 675} \times 10^{\cdot 265}$ (graph)	$= \log 4{\cdot}73 + \log 1{\cdot}84$
$= 10^{\cdot 675 + \cdot 265}$	$= {\cdot}675 + {\cdot}265$ (graph)
$= 10^{\cdot 940}$;	$= {\cdot}940$;
$\therefore x = 8{\cdot}70$ (graph).	$\therefore x = 8{\cdot}70$ (graph).

Now give the boys just one page of 4-figure logarithms, make them work out a few examples in both ways, and see they understand that the two ways represent exactly the same thing.

It ought now to be possible for the boys to proceed with logarithms in the usual way, and really to understand what they are doing.

Graphs and the "Method of Differences"

The nature of a graph may easily be investigated by means of the method of differences. A series of equidistant ordinates is drawn, beginning at any point on the graph. The heights of the ordinates are measured, and a table is made of the first, second, third, . . . differences. If the graph is a straight line, the first difference will be constant; if a parabola, the second difference; if a function of the third degree, the third difference; and so on. Hence by examining the differences of the ordinates, we can determine the degree of the function which corresponds to the graph. This is a useful principle for the boys to know.

Books to consult:

1. *Graph Book*, Durell and Siddons.
2. *Graphs*, Gibson.

CHAPTER XVIII

Algebraic Manipulation

Common-form Factors

During the last 30 years there has been amongst the older boys of schools a serious falling off in their power of algebraic manipulation. Nowadays, there is often a sad lack of easy familiarity with even the simpler transformations in algebra and trigonometry. Although a great deal of bookwork is done and mastered, the valuable old transformation exercises receive too little attention, with the result that there is often a good deal of uncertainty about everyday working algebraic procedure.

Readiness in manipulation is the key to algebraic success. Pupils *must* acquire facility in the manipulation of common algebraic expressions.

The factors to be mastered in the first year of algebra are few, but they are of fundamental importance and must be taught thoroughly. In the early stages they should be associated with arithmetic and geometry, if only in order that the pupils may be convinced of their usefulness.

The early forms are,

$$ab \pm ac = a(b \pm c),$$

and $$a^2 - b^2 = (a + b)(a - b);$$

and the expansions $$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

and perhaps $$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$

Let factors be first looked upon as a device for simplifying formulæ, and for putting these into shape for arithmetical substitution. It is a good plan to begin with obvious geometrical relations and base upon these an algebraic identity. But do not talk of " proving " the truth of the geometrical proposition. The illustrations in Chapter XVI, pp. 134–7 typify the kind of thing to be done.

The elementary standard forms $(a + b)^2$, $(a - b)^2$, $a^2 - b^2$, being well known, verified by a few numerical examples, and illustrated geometrically, a first element of complexity may be introduced into them.

The a and the b may be regarded, respectively, as, say, a square and a circular box, into each of which we may put any algebraic expression we please. Thus we may write:

$$\square^2 - \bigcirc^2 = (\square + \bigcirc)(\square - \bigcirc),$$

and then fill up, say with p^2 and q^2 respectively, in this way:

$$\boxed{p^2}^2 - \textcircled{q^2}^2 = \left(\boxed{p^2} + \textcircled{q^2}\right)\left(\boxed{p^2} - \textcircled{q^2}\right)$$

$$\text{i.e.} \quad p^4 - q^4 = (p^2 + q^2)(p^2 - q^2)$$
$$= (p^2 + q^2)(p + q)(p - q).$$

Such a device is very useful, but do not carry such an extension very far *at first*. Wait a year, and then with harder examples push the principle home.

The expansions $(a \pm b)^3$ are probably best postponed until the second year, though when they are taken up they should be associated with a geometrical model. A 6-in. or 8-in. cubical block, sawn through by cuts parallel to each pair of parallel faces, makes a suitable model, and may be prepared in the manual instruction room. Or a cube cut from a bar of soap may be used, if a very thin-bladed knife is available for cutting the sections. We deal first with the identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. If each edge of the cube is cut into two parts a and b, the original edge being $a + b$, and a being $> b$, the cut-up cube evidently consists of eight portions, viz. a larger cube a^3, three square slabs of area a^2 and thickness b, three square prisms of length a and square section b^2, and a smaller cube b^3.

Then the class sees at once that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

But they should discover this identity from the model for themselves, and not be told.

The same model may be used for the identity

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$

but the manipulation is a little more troublesome. The whole composite cube must now be called a^3, and the thickness of the other seven parts (slabs, prisms, and small cube) should be called b. The larger of the two cubes within the whole composite block is evidently $(a - b)^3$. When actually handling the model it is easy to see that this cube $(a - b)^3$ with the three slabs $(3a^2b)$ and the little cube (b^3) are together equal to the whole composite block (a^3) plus the three prisms $(3ab^2)$, i.e.

$$(a - b)^3 + 3a^2b + b^3 = a^3 + 3ab^2,$$

or $$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

The three slabs " overlap ", a fact which tends to perplex most pupils.

It is really better to cut up *two* cubes and have two models, one to be kept in its eight separate pieces, the other

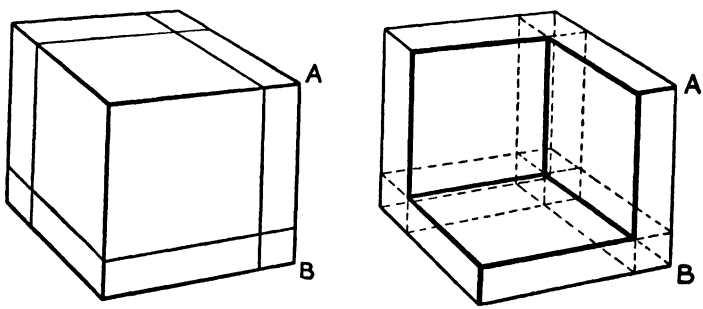


Fig. 59

to be glued up again without the cube $(a - b)^3$ and looking something like three of the six sides of a cubical box. Unless the teacher is pretty deft in manipulating such a model, it had better not be used, or the class will get more amusement

than instruction from his efforts. It is obvious that since $a = AB$, the second model (the three-sided shell) is *less* than the three slabs $3a^2b$ by *the three prisms* $3ab^2$ *diminished by the little cube* b^3.

$$\begin{aligned} \text{I.e.} \qquad \text{shell} &= 3a^2b - (3ab^2 - b^3) \\ &= 3a^2b - 3ab^2 + b^3; \end{aligned}$$

add the removed cube $(a - b)^3$ to each side:

$$\begin{aligned} \text{shell} + (a - b)^3 &= (a - b)^3 + 3a^2b - 3ab^2 + b^3, \\ \text{i.e.} \qquad a^3 &= (a - b)^3 + 3a^2b - 3ab^2 + b^3, \\ \text{i.e.} \qquad (a - b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \text{ (as before).} \end{aligned}$$

This on paper looks complicated. With the model in the hand it may be made clear at once. The case seems complicated because what we have called a slab a^2b consists of four pieces of wood, each of the thickness b, viz. a slab $(a - b)^2$ in area, two square prisms each $(a - b)$ long, and a cube b^3.

The boys always look upon it as a pretty little puzzle. Let them build up the cube a^3 themselves, beginning with the cube $(a - b)^3$, and adding and subtracting the other pieces one by one. The whole difficulty comes about from calling the edge of the whole cube a as compared with the previous example when a referred to *part* of the edge.

A further identity for the boys to discover from their model is:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

The whole cube may be called a^3 and the removable cube b^3. Lay out the seven pieces, all of thickness $(a - b)$, on the table. The united area obviously is:

$$\begin{aligned} & 3ab + (a - b)^2 \\ &= a^2 + ab + b^2; \\ \therefore \quad \text{volume} &= (a - b)(a^2 + ab + b^2), \\ \text{i.e.} \qquad a^3 - b^3 &= (a - b)(a^2 + ab + b^2). \end{aligned}$$

Verify all these identities by a variety of numerical calculations, and so emphasize the utility of the alternative forms.

It is a curious fact that Form IV boys are prone to forget

the factors of $a^4 + a^2b^2 + b^4$. It is a good thing to ask them occasionally for the factors of $(a^6 - b^6)$. They will give them readily enough:

$$= (a^3 + b^3)(a^3 - b^3)$$
$$= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2).$$

Now ask them to multiply the four factors together again, in pairs:

$(a + b)(a - b) = a^2 - b^2$ (readily given),
$(a^2 - ab + b^2)(a^2 + ab + b^2)$ (generally forgotten).

If the product is not forthcoming, ask for the factors of $a^4 + a^2b^2 + b^4$ and give them the hint of adding and subtracting a^2b^2, thus:

$$a^4 + a^2b^2 + b^4$$
$$= (a^4 + 2a^2b^2 + b^4) - a^2b^2$$
$$= (a^2 + b^2)^2 - (ab)^2$$
$$= (a^2 + ab + b^2)(a^2 - ab + b^2).$$

Come back to this twice a term, until it is *known*.

Algebraic Phraseology

Each successive school year will demand its quota of further manipulative work until in the Upper Fifth, especially the top Set, the boys become expert. The four or five years' course of instruction must be organized in such a way that the difficulties of manipulation are carefully graded. Impress on the boys that ready manipulation is the key to success in the greater part of algebra and therefore to the greater part of trigonometry, conics, and the calculus.

Let your phraseology be accurate, and use it consistently, exercise after exercise, lesson after lesson, and see that the boys gradually acquire the use of phraseology of the same degree of accuracy.

"Jones, what is the first thing to do?"—"Rearrange the terms."

" How?"—" Write down all the plus terms first, and then all the minus terms."

" Then?"—" Put the plus sign . . ."

" No. That is not the way we decided to say things."—" Add up all the plus terms and write down the sum, prefixed by a plus sign; then add up all the minus terms and write down the sum, prefixed by a minus sign."

" Smith: lastly?"—" Take the difference between the two sums, and prefix the sign of the larger."

Remember the slow boys and the amount of practice they need until the soaking in is complete. Then all is well.

There are certain common algebraic terms which, though of fundamental importance, are often loosely used. Formal definitions to be learnt by rote are unnecessary, but consistently accurate usage should be adopted from the outset. Introduce the terms one at a time and make each new one part of the everyday jargon of each lesson for a few weeks. We refer to such terms as *mononomial*, *binomial*, *degree* and *dimensions*, *homogeneity* and *symmetry*, and so forth.

" In algebra, a letter, or a product of two or more letters, or of letters and numbers, in which there is no addition or subtraction, is called a *term*, or a *mononomial*, e.g. x, x^2, x^2y, $3bx^2$.

" If the same letter occurs more than once in a term we write the letter down once, and at the top right-hand corner we write a figure to show the number of times it occurs, e.g. *xxx* is written x^3, *aaaa* is written a^4.

" A term may be *integral*, as ab^2; or *fractional*, as $\frac{ab^v}{x}$.

" The *degree* or the *dimension* of a term is the sum of the indices of the named letters; e.g. the term x^2y^3 is a term of the fifth degree, or a term of five dimensions.

" A *binomial* consists of two terms connected by the sign + or −; a *trinomial* of three terms; a *polynomial* of more than three."

All this is just the stock phraseology of the classroom. But let it be carefully thought out and consistently used, in order

that the boy may soon get to know the precise significance of the new vocabulary.

We have already referred to the term *function*. Use it consistently and use it often.

Such a term as the *law of commutation* is hardly worth mentioning at all unless it be in Form VI, where algebraic theory is being minutely discussed. The boys will know from their arithmetic that the mere order in which terms are arranged for addition purposes is immaterial. So with multiplication: the notion of commutation is imbibed with the multiplication table; 5 sevens gives the same product as 7 fives. Thus, any elaborate formal explanation that $d + c + a + b = a + b + c + d$, or that b^2ac is the same as ab^2c, is unnecessary. It is, as a rule, enough to point out the close analogy with arithmetic, though in a first-year course of algebra attention must repeatedly be called to the fact that abc is not in form a faithful copy of 345, and that 345 means $300 + 40 + 5$. In the main, let early algebraic processes grow out of corresponding arithmetical processes.

Typical Expressions for Factor Resolution

1. $ac + bc + ad + bd = (a + b)(c + d)$.
2. $x^2 + (a + b)x + ab = (x + a)(x + b)$.
3. $acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$.

These depend on a redistribution of terms, and too much care cannot be paid to the teaching of the principle involved.

We know that

$$\begin{aligned}(a + b)(c + d) &= a(c + d) + b(c + d)\\ &= ac + ad + bc + bd,\end{aligned}$$

and therefore, conversely,

$$\begin{aligned}ac + ad + bc + bd &= a(c + d) + b(c + d)\\ &= (a + b)(c + d).\end{aligned}$$

If then we are given the expression $ac + bd + ad + bc$,

and we rearrange it so that both the a terms come first, we have a suitable distribution for finding the factors:

$$\begin{aligned} & ac + bd + ad + bc \\ &= ac + ad + bc + bd \\ &= a(c + d) + b(c + d) \\ &= (a + b)(c + d). \end{aligned}$$

Boys are often puzzled about the derivation of the last line from the last line but one, but their difficulty is cleared up when it is pointed out to them that if they had to multiply $(a + b)$ by $(c + d)$, they would begin by writing down $a(c + d) + b(c + d)$.

Emphasis must be laid on this intermediate step of a partial redistribution and on how we proceed forwards and backwards from it.

$$\begin{aligned} & (a + b + c)(d + e) \\ &= a(d + e) + b(d + e) + c(d + e) \\ &= ad + ae + bd + be + cd + ce, \\ \text{which } \therefore \quad &= a(d + e) + b(d + e) + c(d + e) \\ &= (a + b + c)(d + e), \text{ with which we began.} \end{aligned}$$

We append two rather harder examples. It is always a question of arranging according to the powers of some selected letter, though which letter only experience can tell.

(i) $$\begin{aligned} & x^2 + (a + b + c)x + ab + ac \\ &= x^2 + ax + bx + cx + ab + ac. \end{aligned}$$

Arranging in powers of a, we have

$$\begin{aligned} & ax + ab + ac + x^2 + bx + cx \\ &= a(x + b + c) + x(x + b + c) \\ &= (a + x)(x + b + c). \end{aligned}$$

(ii) $$a^2 + 2ab - 2ac - 3b^2 + 2bc.$$

We note the letter c in two terms. Try grouping them together.

Then $$\begin{aligned} & a^2 + 2ab - 3b^2 - 2ac + 2bc \\ &= (a^2 + 2ab - 3b^2) - 2c(a - b) \\ &= (a + 3b)(a - b) - 2c(a - b) \\ &= (a + 3b - 2c)(a - b). \end{aligned}$$

If boys feel a difficulty about accepting the last line as another form of the last line but one, give them an example of the reverse kind:

$$\begin{aligned} & (a+b)(c+d+e) \\ \textit{either} \quad & = (a+b)c + (a+b)d + (a+b)e \\ \textit{or} \quad & = (a+b)(c+d) + (a+b)e. \end{aligned}$$

Both redistributions yield exactly the same result.

Illustrate with a numerical example:

$$\begin{aligned} & 47 \times 365 \\ \textit{either} \quad & = (47 \times 300) + (47 \times 60) + (47 \times 5) \\ \textit{or} \quad & = (47 \times 360) + (47 \times 5), \end{aligned}$$

i.e. we can perform our multiplication in little bits or in bigger bits, just as we please.

The type $x^2 + (a+b)x + ab = (x+a)(x+b)$ seldom gives much trouble. Examples:

$$x^2 + 8x + 15.$$
$$x^2 - 8x + 15.$$

The two rules (1) for signs, (2) for determining the coefficients of x, should be kept separate. Both admit of very simple statement.

For the first example we begin by writing $(x+\ \ )(x+\ \ )$, and for the second example we begin by writing $(x-\ \ )(x-\ \ )$. For both examples we ask the question, What two numbers multiplied together give us 15 and when added together give us 8? Answer, 5 and 3. Hence the factors $(x+5)(x+3)$ and $(x-5)(x-3)$.

Other examples:

$$x^2 + 2x - 15$$
$$x^2 - 2x - 15.$$

As the last term is a minus term, the second term of the two factors will be of opposite signs. Hence we may begin by writing down for each case $(x+\ \ )(x-\ \ )$. "Find two numbers whose product is 15 and whose difference is 2."

Answer, 5 and 3. " Give the larger number the sign before the middle term." Hence we have:

$$x^2 + 2x - 15 = (x + 5)(x - 3)$$
$$x^2 - 2x - 15 = (x - 5)(x + 3).$$

Of course these are mere rules, to be remembered; but they should be first worked out from an examination of the different products, three or four sets being taken for confirmation purposes.

$$(x + 3)(x + 5) = x^2 + 8x + 15$$
$$(x - 3)(x - 5) = x^2 - 8x + 15$$
$$(x + 5)(x - 3) = x^2 + 2x - 15$$
$$(x - 5)(x + 3) = x^2 - 2x - 15.$$

Help the boys to examine the products and to discover:

(1) That if the last term of the trinomial is +, the signs of both factors are the same, the same as the middle term.

(2) That if the last term of the trinomial is —, the signs of the two factors are different, the factor with the larger number taking the sign of the middle term.

(3) That the *last* term of the trinomial is always the algebraic *product* of the second terms of the two factors (hence the signs).

(4) That the *middle* term of the trinomial is always the algebraic *sum* of the second terms of the two factors (hence the signs).

The mere *rules* must be mastered by all Sets, but experience shows that the *justification* of the rules, by an analysis of a series of products, is beyond lower Sets, though upper Sets always appreciate them. Do not talk of " proving " the rules.

The type, $acx^2 + (ad + bc)x + bd$

This common type of expression boys generally find rather troublesome to factorize. I remember seeing a Fourth

Form trying to factorize $35x^2 - 59x - 48$. There had been a preliminary discussion on the necessarily long succession of " trial " factors, and the 33 boys were actually working out with the patience of 33 Jobs the possible combinations, the first factors being $35x \pm 1$, $7x \pm 1$, $5x \pm 1$, $x \pm 1$, $35x \pm 2$, $7x \pm 2$, $5x \pm 2$, $x \pm 2$, and so on with ± 3, ± 4, ± 6, ± 8, ± 12, ± 16, ± 24, and ± 48, 80 possible first factors in all! Naturally the lesson was not long enough for this single set of trials to be completed. In any circumstances the particular example would be very difficult for class practice. But the " trial " method is unnecessary. All ordinary cases can be dealt with by a method which is much simpler.

Consider the example $(6x^2 + 17x + 12) = (3x + 4)(2x + 3)$. Let us multiply the factors together in the ordinary way.

$$\begin{array}{r} 3x + 4 \\ 2x + 3 \\ \hline 6x^2 + 8x \\ \quad + 9x \quad + 12 \\ \hline 6x^2 + 17x + 12 \end{array}$$

We might have multiplied out, thus:

$$\begin{aligned} & (3x + 4)(2x + 3) \\ = \; & 2x(3x + 4) + 3(3x + 4) \\ = \; & 6x^2 + 8x + 9x + 12 \\ = \; & 6x^2 + 17x + 12. \end{aligned}$$

To find the factors, why not reverse this process?

$$\begin{aligned} & 6x^2 + 17x + 12 \\ = \; & 6x^2 + 8x + 9x + 12 \\ = \; & 2x(3x + 4) + 3(3x + 4) \\ = \; & (3x + 4)(2x + 3). \end{aligned}$$

Yes, why not? But how could we tell that the $17x$ in the first line should be divided into $8x$ and $9x$, instead of, say, into $3x$ and $14x$, or into $5x$ and $12x$? *That* is the trouble, that the only difficulty. How are we to find the two correct numbers?

Let us suppose these unknown; call them m and n.

Now $m + n = 17$ (that we know).
And $mn = 72$.

[How do we know that? Because 72 is the product of the 6 and 12 which we obtained (in the multiplication sum) by multiplying 3 by 2 and by multiplying 4 by 3; and from these same 4 numbers, 3, 2, 4, 3 we obtained the 9 and the 8 also in the multiplication sum. Thus the 72 is the product of the 6 and 12 in the first and third terms of the trinomial.]

Hence all we have to do is to find two numbers which when added together come to 17 and which when multiplied together come to 72. The numbers are easily seen to be 8 and 9, and therefore we now know that the $17x$ must be divided into $8x$ and $9x$.

Another example: $14x^2 - 25x + 6$.

Here $m + n = -25$ and $mn = 14 \times 6 = 84$. By trial, the two required numbers are -21 and -4.

$$\begin{aligned} \therefore \quad & 14x^2 - 25x + 6 \\ &= 14x^2 - 21x - 4x + 6 \\ &= 7x(2x - 3) - 2(2x - 3) \\ &= (7x - 2)(2x - 3). \end{aligned}$$

Another example: $6x^2 - 11x - 10$.

This time we have to find two numbers whose product is -60 and whose sum is -11. The numbers are evidently -15 and $+4$.

$$\begin{aligned} \therefore \quad & 6x^2 - 11x - 10 \\ &= 6x^2 - 15x + 4x - 10 \\ &= 3x(2x - 5) + 2(2x - 5) \\ &= (3x + 2)(2x - 5). \end{aligned}$$

Thus we have this simple rule. *Redistribute the terms of the expression, splitting the coefficient of the middle term into two parts, m and n, so that m and n is the product of the coefficients in the first and last terms.* Then factorize the redistributed product in the usual way.

For top Sets the rule can be stated more formally from

the expression $acx^2 + (ad + bc)x + bd$, where the relations stated in the rule are obvious.

At least top Sets should be made to see how, as regards both coefficients and signs, all the different cases may be brought under a single rule. Let the general expression be $ax^2 + bx + c$. Then: *divide* b *into two parts* $m + n$ *so that* $mn = ac$. Now let the class apply the rule to all possible different cases, say:

$$x^2 \pm 8x + 15,$$
$$x^2 \pm 2x - 15,$$
$$6x^2 \pm 19x + 15,$$
$$6x^2 \pm x - 15.$$

Difficult cases where m and n cannot be obtained readily from mn and $m + n$ at once by mental arithmetic may be solved quadratically. Examples:

$x^2 + 2x - 360.$
Write, $x^2 + 2x - 360 = 0.$
$\therefore x^2 + 2x + 1 = 361,$
$\therefore x + 1 = \pm 19,$
$\therefore x = 18$ or $-20,$
$\therefore$ factors $= (x - 18)(x + 20).$

$x^2 + 12x - 405.$
Write, $x^2 + 12x - 405 = 0.$
$\therefore x^2 + 12x + 36 = 441,$
$\therefore x + 6 = \pm 21,$
$\therefore x = -27$ or $+15,$
$\therefore$ factors $= (x - 15)(x + 27).$

Do not let the pupils look upon these as quadratic equations but simply as a plan for finding the factors. Quadratic equations will come a little later. The quadratic principle may be applied to any case, but more often than not it is merely a clumsy substitute for the method first mentioned. For instance, consider $6x^2 - 11x - 10$.

$$6x^2 - 11x - 10 = 6(x^2 - \tfrac{11}{6}x - \tfrac{5}{3})$$

Solving the quadratic $x^2 - \frac{11}{6}x - \frac{5}{3} = 0$, we have,

$$x^2 - \tfrac{11}{6}x + (\tfrac{11}{12})^2 = \tfrac{5}{3} + (\tfrac{11}{12})^2.$$
$$\therefore x - \tfrac{11}{12} = \pm \tfrac{19}{12}.$$
$$\therefore x = \tfrac{5}{2} \text{ or } -\tfrac{2}{3}.$$
$$\therefore x^2 - \tfrac{11}{6}x - \tfrac{5}{3} = (x - \tfrac{5}{2})(x + \tfrac{2}{3}).$$
$$\therefore 6x^2 - 11x - 10 = 6(x - \tfrac{5}{2})(x + \tfrac{2}{3})$$
$$= (2x - 5)(3x + 2).$$

Complex derivatives from type forms are a prolific source of errors with all but the ablest pupils. Much care is necessary in substituting. Example: factorize $8a^3 - (a + 2)^3$.

Type: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.
Thus $x = 2a;\ y = a + 2$.

$$\begin{aligned}\therefore\quad & (2a)^3 - (a + 2)^3 \\ &= \{2a - (a + 2)\}\{(2a)^2 + 2a(a + 2) + (a + 2)^2\} \\ &= (a - 2)(4a^2 + 2a^2 + 4a + a^2 + 4a + 4) \\ &= (a - 2)(7a^2 + 8a + 4).\end{aligned}$$

Product Distribution Generally

There comes a time, probably towards the end of the Upper Fourth year or the beginning of the Lower Fifth, when a boy's accumulated facts concerning products must be summarized and analysed, and reduced to laws of some kind. We will run rapidly over the necessary ground.

$(a + b)(c + d)$. Here we have two factors, each of two terms. We have to multiply each term of the first factor by c and then by d and so we have four terms in all, viz.

$$ac + bc + ad + bd.$$

Compare this with the ordinary arithmetical multiplication.

$$\begin{aligned}& 37 \times 24 \\ &= (30 + 7)(20 + 4) \\ &= (30 \times 20) + (30 \times 4) + (7 \times 20) + (7 \times 4),\end{aligned}$$

and show the close analogy. We have and *must* have four products both in the algebra and in the arithmetic.

For similar reasons:

(i) $(a + b + c)(d + e)$ will give 6 products.
(ii) $(a + b + c)(d + e + f)$ will give 9 products.
(iii) $(a + b)(c + d)(e + f)$ will give 8 products.
(iv) $(a + b + c)(d + e + f)(g + h + k)$ will give 27 products.

The last will be quite clear if it be observed that each of the 9 products in (ii) has to be multiplied by g, then by h, then by k. Clearly, then, if the factors consist of p, q, and r terms, the number of products will be $p \times q \times r$; *and this will be quite general.* Hence we can tell how many products to expect in an algebraic multiplication.

But in the above cases, all the terms are different. There is neither condensation owing to like terms occurring more than once, nor reduction owing to terms destroying each other. Either or both of these things may happen.

Consider the product $(a + b)(a + b)$. By the general rule the distribution will give 4 terms. But only 2 different letters, a and b, occur in the product, and with these only 3 really distinct products of 2 factors can be formed with them, viz. a^2, ab, b^2. Hence, among the 4 terms, at least 1 must occur more than once, and, in fact, $a \times b$ occurs twice. The result of the distribution therefore is $a^2 + 2ab + b^2$.

Thus we may write:

$$(a + b)^2 = a^2 + 2ab + b^2.$$
$$\text{Similarly} \quad (a - b)^2 = a^2 - 2ab + b^2.$$

In the case of $(a + b)(a - b)$, the term ab occurs twice, but as the two terms are of opposite signs they destroy each other. Nevertheless the main rule still holds good: the product really consists of 2×2 or 4 terms.

What are all the possible products of 3 factors that can be made with the 2 letters a and b? Evidently

$$aaa, \quad aab, \quad abb, \quad bbb;$$
$$\text{or,} \quad a^3, \quad a^2b, \quad ab^2, \quad b^3; \quad \textbf{4 in all.}$$

Hence in the distribution of $(a + b)^3$, i.e. of $(a + b)(a + b)(a + b)$, which by the general rule will give 8 terms, only 4 really distinct terms can appear. What terms recur and how often?

a^3 and b^3 evidently appear each only once, because to get 3 a's or 3 b's we must take one from each bracket, and this can be done in only one way.

a^2b may be obtained:

(i) by taking b from the *first* bracket, and a from each of the others;

(ii) by taking b from the *second* bracket, and a from each of the others;

(iii) by taking b from the *third* bracket, and a from each of the others.

ab^2: the same holds as for a^2b.

Thus the result is,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Similarly,
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$
$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

If we remember that the possible *binary* products of 3 letters, a, b, c, are a^2, b^2, c^2, ab, ac, bc (6 in all), then

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

The *ternary* products of 3 letters, a, b, c, are easily enumerated if we first deal with the letter a, writing down the terms in which it occurs thrice, then those in which it occurs twice, then those in which it occurs once; then deal similarly with b, for such forms as are not already written down; then with c. Thus we have (10 in all):

$$a^3,\ a^2b,\ a^2c,\ ab^2,\ ac^2,\ abc,$$
$$b^3,\ b^2c,\ bc^2,$$
$$c^3.$$

Hence, following the rule, we have:

$$\begin{aligned}(a+b+c)^3 &= (a+b+c)(a+b+c)(a+b+c)\\ &= a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2\\ &\quad + 3b^2c + 3bc^2 + 6abc.\end{aligned}$$

The result may be verified by successive distribution:

$$\begin{aligned}(a+b+c)^3 &= (a+b+c)^2(a+b+c)\\ &= (a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)(a+b+c)\\ &= \&c.\end{aligned}$$

Another example: $(b + c)(c + a)(a + b)$.

Here not all the 10 permissible ternary products can occur. for a^3, b^3, c^3 are excluded by the nature of the case, a appearing in only 2 of the brackets, b in only 2, and c in only 2.

$$(b + c)(c + a)(a + b) = bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + 2abc.$$

But although we do not get the 10 *ternary* products, we do get 8 ($= 2 \times 2 \times 2$) *products*, according to the general rule.

In the product $(b - c)(c - a)(a - b)$ the term abc occurs twice but with opposite signs, and there is then a further reduction:

$$(b - c)(c - a)(a - b) = bc^2 - b^2c + ca^2 - c^2a + ab^2 - a^2b.$$

Σ Notation

Upper Sets in the Fifths should be taught to use this notation. It is easy to understand, though average boys are a little shy of it. " Σ " stands for, *the sum of all the terms of the same type as*, though its exact meaning in any particular case depends on the number of variables that are in question.

If 2 variables, a and b, Σab means simply ab.

If 4 variables, a, b, c, d, Σab means $ab + ac + ad + bc + bd + cd$.

If 2 variables, a, b, Σa^2b means $a^2b + ab^2$.

If 3 variables, a, b, c, Σa^2b means $a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2$.

The context usually makes clear how many variables are to be understood.

" Choose any one of the terms and place Σ before it."

The use of the sign certainly saves labour: thus

$$(a + b)^3 = \Sigma a^3 + 3\Sigma a^2b.$$
$$(a + b + c)^3 = \Sigma a^3 + 3\Sigma a^2b + 6abc.$$
$$(b + c)(c + a)(a + b) = \Sigma a^2b + 2abc.$$

More Complex Forms

Quick boys soon pick up the method of manipulating more complex forms based on those already familiar to them, but the slower boys require much practice and should not be worried by forms so complicated as to be puzzling. The slower boys should always first be given forms with + signs only. The added difficulty of negative signs should come a little later.

(i) Type forms, with the addition of coefficients; e.g.

(α) $(3a + 2b)^3 = (3a)^3 + 3(3a)^2(2b) + 3(3a)(2b)^2 + (2b)^3$.

(β) $(a + 2b + 3c)^2 =$ &c.

(ii) Type forms with a mononomial replaced by a binomial; e.g.

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Replacing b throughout by $(b + c)$, we have

$$(a + b + c)^3 = a^3 + 3a^2(b + c) + 3a(b + c)^2 + (b+c)^3$$
$$= \text{\&c.}$$

(iii) Association of parts of factors of more than 3 terms; e.g.

(α) $(a + b + c - d)(a - b + c + d)$
$= \{(a + c) + (b - d)\}\{(a + c) - (b - d)\}$
$= (a + c)^2 - (b - d)^2$, &c.

(β) $(a + b + c)(b + c - a)(c + a - b)(a + b - c)$
$= \{(b + c) + a\}\{(b + c) - a\}\{a - (b - c)\}\{a + (b - c)\}$
$= \{(b + c)^2 - a^2\}\{a^2 - (b - c)^2\}$
$= (b^2 + 2bc + c^2 - a^2)(a^2 - b^2 + 2bc - c^2)$
$= \{2bc + (b^2 + c^2 - a^2)\}\{2bc - (b^2 + c^2 - a^2)\}$
$= (2bc)^2 - (b^2 + c^2 - a^2)^2$
$= 4b^2c^2 - (b^4 + c^4 + a^4 + 2b^2c^2 - 2c^2a^2 - 2a^2b^2)$
$= 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$.

The type forms are few and are easily remembered, and all boys should have them at their finger-ends. The possible applications and developments are, of course, very diverse,

but do not perplex boys with expressions that are beyond their manipulative skill at any particular stage.

Detached Coefficients. First Notions of Manipulation

Here is a useful general theorem, easy for upper Sets to remember.—If all the terms of all the factors of a product be simple letters unaccompanied by numerical coefficients and all $+$, **the sum of the coefficients** in the distributed value of the product will be $l \times m \times n \ldots$, where l, m, n, are the numbers of the terms of the respective factors.

Thus in the evaluated products of the following, we have:

	Coefficients.	Sum of Coefficients.
$(a + b)^2$	$1 + 2 + 1$	$4 = 2^2$
$(a + b)^3$	$1 + 3 + 3 + 1$	$8 = 2^3$
$(a + b)^4$	$1 + 4 + 6 + 4 + 1$	$16 = 2^4$
$(a + b)^5$	$1 + 5 + 10 + 10 + 5 + 1$	$32 = 2^5$
$(b + c)(c + a)(a + b)$	$1 + 1 + 1 + 1 + 1 + 1 + 2$	$8 = 2^3$

The theorem is useful in connexion with expansions.

Simple Expansions and First Generalizations therefrom.—Let an upper Set in a Fourth Form obtain the following results by continued multiplication, the second being obtained by multiplying the first by $(x + a)$, the third by multiplying the second by $(x + a)$, and so on.

$(x + a)^2 = x^2 + 2xa + a^2.$
$(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3.$
$(x + a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4.$
$(x + a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5.$
$(x + a)^6 = x^6 + 6x^5a + 15x^4a^2 + 20x^3a^3 + 15x^2a^4 + 6xa^5 + a^6.$

Now help the boys to generalize, and to establish the usual rules. Afterwards, they may work out a few higher expansions and see that these follow the same rules.

1. The power to which we have carried $(x + a)$ gives

the index of the highest terms of the expansion and is therefore the degree of the function.

2. The function has one term more than that index. Thus the expansion $(x + a)^4$ has 5 terms.

3. The powers of x run in descending order from the first term to the last term but one; the powers of a run in ascending order from the second term to the last. (There is no objection to writing the first term $x^n a^0$ and the last $x^0 a^n$, if the class understand that $x^0 = a^0 = 1$. Then both x and a appear in *every* term.)

4. The dimensions of all the terms are the same and are always equal to the power to which $(x + a)$ is carried.

5. The coefficients follow a regular law.—We may detach them from their terms (detached coefficients may often be usefully considered alone), and place them in order, thus:

$(x + a)^1$	1	1					
$(x + a)^2$	1	2	1				
$(x + a)^3$	1	3	3	1			
$(x + a)^4$	1	4	6	4	1		
$(x + a)^5$	1	5	10	10	5	1	
$(x + a)^6$	1	6	15	20	15	6	1

The sum of *any* 2 *successive coefficients in any line* gives the coefficient standing in the next line immediately below the second of these. Thus, in the third line the 6 is the sum of 3 and 3 in the second line; in the last line, the 15 is the sum of 10 and 5 in the fifth line. Show that this is the simple result of continued multiplication. For instance, if we multiply $(x + a)^3$ by $(x + a)$ we have:

$$
\begin{array}{l}
1 + 3 + 3 + 1 \\
1 + 1 \\
\hline
1 + 3 + 3 + 1 \\
\quad 1 + 3 + 3 + 1 \\
\hline
1 + 4 + 6 + 4 + 1
\end{array}
$$

The second partial product is arranged *one place to the right* under the first partial product. Thus any coefficient for any

expansion may be found by taking the coefficient of the corresponding term in the previous expansion and adding to it its predecessor. Let the boys continue the table: they like the work. They soon see that when they have written the expansion of, say, $(x + a)^{10}$, they can *immediately* write down that for $(x + a)^{11}$; it is merely a question of carrying on the game already begun.

" There is something still more interesting to learn about the coefficients. Consider the expansion of $(x + a)^5$. The coefficient of the second term is 5; we may write it $\frac{5}{1}$. The coefficient of the next term is 10, which we may write $\frac{5 \times 4}{1 \times 2}$. That of the next term is again 10, which we may write $\frac{5 \times 4 \times 3}{1 \times 2 \times 3}$. And so generally. Examine the other expansions and see if a similar rule is followed; for instance, $(x + a)^6$

$$(x + a)^6 = x^6 + \frac{6}{1}x^5a + \frac{6.5}{1.2}x^4a^2 + \frac{6.5.4}{1.2.3}x^3a^3 + \frac{6.5.4.3}{1.2.3.4}x^2a^4 + \frac{6.5.4.3.2}{1.2.3.4.5}xa^5 + \frac{6.5.4.3.2.1}{1.2.3.4.5.6}a^6.$$

With one or two leading questions, the boys will see that the coefficient of a^6 is the same as that for x^6, that for xa^5 the same as for x^5a, that the coefficient $\frac{6.5.4.3}{1.2.3.4}$ is the same as $\frac{6.5.4}{1.2.3}$, and so generally. Let them formulate the obvious rule for themselves.

Let them write down the first few terms of such an expansion as $(x + a)^{20}$.

First they write the terms without coefficients:

$$x^{20} + x^{19}a + x^{18}a^2 + \&c.$$

Then they work out their coefficients and insert them:

$$1 + \frac{20}{1} + \frac{20.19}{1.2} + \&c.$$

The object of all this is not to teach the Binomial Theorem: that will come later. It is to impress boys with the wonderful simplicity and regularity that underlies algebraic mani-

pulation. Never mind the generalized form $(x + a)^n$, yet. Never mind the general term. Never mind $n\mathrm{C}r$. When these things are taken up later, the way will have been paved, and the work will give little trouble.

The Remainder Theorem

This theorem must be known in order that the Factor Theorem may be clearly understood. It may be approached in this way.—We know that $(x - 5)$ is a factor of $(x^2 + x - 30)$, and in order to find out the other factor we may conveniently set out the process of division, exactly as in arithmetic.

$$\begin{array}{l} x - 5)x^2 + x - 30(x + 6 \\ \quad\; x^2 - 5x \\ \quad\quad\quad\; 6x - 30 \\ \quad\quad\quad\; 6x - 30 \end{array}$$

Of course there is no " remainder " (R), but if $(x - 5)$ was not a factor there would be a R. Divide $(3x^2 - 2x + 4)$ by $(x - 5)$.

$$\begin{array}{l} x - 5)3x^2 - 2x + 4(3x + 13 \\ \quad\; 3x^2 - 15x \\ \quad\quad\quad\; 13x + 4 \\ \quad\quad\quad\; 13x - 65 \\ \quad\quad\quad\quad\quad\; 69 = \mathrm{R}. \end{array}$$

The remainder is 69, and by analogy with arithmetic we know that

$$\text{Dividend} = (\text{Quotient} \times \text{Divisor}) + \text{R}.$$

Suppose we had to divide $(x^2 + px + q)$ by $(x - a)$.

$$\begin{array}{l} x - a)x^2 - px + q(x + (a - p) \\ \quad\; x^2 - ax \\ \quad\quad\quad\; x(a - p) + q \\ \quad\quad\quad\; x(a - p) - a(a - p) \\ \quad\quad\quad\quad\quad\; a(a - p) + q = \mathrm{R}. \end{array}$$

Note that $a(a - p) + q$ really is the remainder, for it does

not involve x, and we cannot proceed with the division any farther.

Now let us set out in this way the previous example, treating the figures as if they were letters, and not actually multiplying and subtracting as we did before.

$$
\begin{array}{l}
x - 5)3x^2 - 2x + 4(3x + (3.5 - 2) \\
\quad 3x^2 - 3.5x \\
\quad \overline{\qquad\qquad x(3.5 - 2) + 4} \\
\quad\qquad\quad x(3.5 - 2) - 5(3.5 - 2) \\
\quad\qquad\quad \overline{\qquad\qquad\qquad 5(3.5 - 2) + 4 = \mathrm{R}.}
\end{array}
$$

As might be expected, the R is the same, viz. 69.

Now compare the Remainders and the Dividends in both the last examples.

Dividend $= x^2 - px + q.$	Dividend $= 3x^2 - 2x + 4.$
Remainder $= a^2 - pa + q.$	Remainder $= 3.5^2 - 2.5 + 4.$

Clearly, then, if the remainder was the only thing we wanted, we could have written it down at once, for it is exactly of the same *form* as the dividend. We merely have to substitute for the x in the dividend the second term of the divisor (a in the first case, 5 in the second), treating these, however, as if they were positive.

Give the pupils several examples, and convince them of the truth of the rule.

The Remainder " Theorem ", as it is called, provides us with a simple means of calculating the remainder of a particular kind of division sum in algebra, without actually performing the division.

The particular kind of division sum is that in which the divisor and the dividend are functions of the same letter (say x), and in which the divisor is a linear expression such as $(x - 5)$ with unity as the coefficient of x.

Example: if we divide

$(x^3 - 7x^2 + 5x - 1)$ by $(x - 9)$, the R is
$(9^3 - 7.9^2 + 5.9 - 1) = 206.$

We have merely substituted 9 for x in the dividend.

Hence:

(1) *The Theorem.*—When a function of x is divided by $(x - a)$, the R is obtained by substituting a for x in the function.

(2) The *why* of it. We know that,

$$\text{Dividend} = (\text{Quotient} \times \text{Divisor}) + \text{R}.$$

If we make a equal to x, the divisor $(x - a) = 0$.

$$\therefore \text{Dividend} = (\text{Quotient} \times 0) + \text{R},$$
$$= \text{R},$$

i.e. by substituting a for x in the Dividend, we have the R.

The Factor Theorem

What is the remainder when we divide $(x^2 - 7x + 10)$ by $(x - 5)$?

Substituting 5 for x in $x^2 - 7x + 10$,
we have $5^2 - 7.5 + 10$
$= 0.$

Since $\text{R} = 0$, $(x - 5)$ divides exactly into $(x^2 - 7x + 10)$ and is therefore a factor of it.

Thus we have a *method* of finding out whether an expression of the type $(x - a)$ is a factor of a given expression

Example: Is $(3x^3 - 2x^2 - 7x - 2)$ divisible by $(x - 2)(x + 1)$?

Writing 2 for x in the first expression, we have

$24 - 8 - 14 - 2 = 0$. Hence $(x - 2)$ is a factor.

Again, writing -1 for x we have

$-3 - 2 + 7 - 2 = 0$. Hence $(x + 1)$ is a factor.

Homogeneous Expressions

A homogeneous expression is one in which all the terms have the same dimensions; e.g.

$$x^2 + xy + y^2, \quad \text{or} \quad a^3 + b^3 + c^3 + 3abc.$$

It may sometimes be necessary to state what letters are regarded as giving dimensions; e.g. $x^3 + ax^2y + 2xy^2 + 3y^3$ is homogeneous in x and y but is not homogeneous if a is regarded as having dimensions.

Obviously the product of two homogeneous expressions is itself homogeneous.

The only homogeneous integral functions of x and y of the first and second degrees are,

$$Ax + By,$$
$$Ax^2 + Bxy + Cy^2.$$

For 3 variables the corresponding functions are,

$$Ax + By + Cz,$$
$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy.$$

The class may usefully write down functions of the third degree. Upper Form boys should be thoroughly familiar with all such general expressions.

Symmetry

A function which is unaltered by the interchange of any two of its variables is said to be *symmetrical with regard to these variables*; e.g. $x^2 - xy + y^2$ is symmetrical with regard to x and y; $(y + z)(z + x)(x + y)$ is symmetrical with regard to x, y, and z. But $x^2y + y^2z + z^2x$ is *not* a symmetrical function of x, y, and z, for the 3 interchanges x with y, y with z, z with x, give, respectively,

$$y^2x + x^2z + z^2y,$$
$$x^2z + z^2y + y^2x,$$
$$z^2y + y^2x + x^2z,$$

and although all these are equal to each other, none of them is equal to the original expression.

But the term " symmetry " is not used in quite the same sense by all writers in algebra. " Cyclic symmetry " expresses a much clearer connotation.

Cyclic Expressions

An expression is said to be " cyclic " with regard to the letters a, b, c, d, . . . k, arranged in this order, when it is unaltered by changing a into b, b into c, . . . k into a. Thus the expression $a^2b + b^2c + c^2d + d^2a$ is cyclic with regard to the letters a, b, c, d, arranged in this order, for by interchanging a into b, b into c, c into d, d into a, we get $b^2c + c^2d + d^2a + a^2b$, the same as before. Note that the first term is changed to the second, the second to the third, and so on. It is merely a question of beginning at a different point on the circle, but always going round in the same direction.

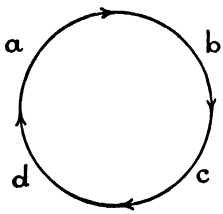


Fig. 60

If in a cyclic expression a term of some particular type occurs, the terms which can be derived from this by cyclic interchange must also occur, and the coefficients of these terms must be equal. Thus, if x, y, and z are the variables, and the term x^2y occurs, then all the terms x^2y, x^2z, y^2z, y^2x, z^2x, z^2y must occur. The cycle is easily seen if the six terms are thus collected up:

$$x^2(y + z) + y^2(z + x) + z^2(x + y).$$

Expressions which are unaltered by a cyclical change of the letters involved in them are called **cyclically symmetrical.** Thus $(b - c)(c - a)(a - b)$ is cyclically symmetrical, since it is unaltered by changing a into b, b into c, and c into a, that is " by starting at a different point in the circle ".

Legitimate Arguments from Cyclical Symmetry

Find the factors of

$$a^2(b - c) + b^2(c - a) + c^2(a - b). \quad . \quad . \quad \text{(i)}$$

Here is the solution from one of the best textbooks we have.

" If we put $b = c$ in the expression, the result is zero, and it therefore follows from the Remainder theorem that $(b - c)$ is a factor.

" In a similar manner we can prove that $(c - a)$ and $(a - b)$ are factors.

" Now the given expression is of the third degree; it can therefore have only 3 factors.

" Hence the expression is equal to

$$N(b - c)(c - a)(a - b), \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

where N is some number which is always the same for all values of a, b, c.

" We can find N by giving values to a, b, c, Thus, let $a = 0$, $b = 1$, $c = 2$; then (i) $= -2$, and (ii) $= +2N$. Hence $N = -1$.

" Therefore the factors are $-(b - c)(c - a)(a - b)$."

This argument is open to criticism. It is wholly unnecessary to say, " in a similar manner we can prove that $(c - a)$ and $(a - b)$ are factors". Once we know that $(b - c)$ is a factor, it *follows at once* that $(c - a)$ and $(a - b)$ are factors. What applies to $(b - c)$ *must* apply to $(c - a)$ and $(a - b)$. This is the very essence of cyclical symmetry. Nay, it is the very essence of all algebraic manipulation. That $(c - a)$ and $(a - b)$ are also factors requires no argument of any sort or kind, except, " it follows from cyclical symmetry "; and no further argument should be tolerated.

Another example. Find the factors of $a^3(b - c) + b^3(c - a) + c^3(a - b)$. As in the last example $(b - c)$, $(c - a)$, $(a - b)$ are all factors. Now the given expression is of the fourth dimension; hence there must be a *fourth* factor, and that of the first dimension. Since this factor must be *symmetrical* with respect to a, b, c, it is necessarily $(a + b + c)$. Thus the required factors are

$$N(b - c)(c - a)(a - b)(a + b + c),$$

N being found in the usual way. Any sort of more elaborate process or argument should be sharply criticized.

Another example. Find the product of $(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$.

Each of the two factors is symmetrical in a, b, c, and therefore the product will be symmetrical in a, b, c.

Obviously the term a^3 occurs with the coefficient unity; hence the same must be true of b^3 and c^3.

Obviously, too, the term b^2c has the coefficient 0; hence by symmetry the five other terms b^2c, c^2a, ca^2, ab^2, a^2b belonging to the same group must have the coefficient 0.

Lastly, the term $-abc$ is obtained (i) by taking a from the first bracket and $-bc$ from the second; hence it is also obtained (ii) by taking b, and (iii) by taking c, from the first bracket. Thus the term abc must have the coefficient -3. Hence the product

$$= a^3 + b^3 + c^3 - 3abc.$$

Boys should gain complete confidence in arguments from symmetry. In at least the A Sets of the Fifth Form, cumbrous processes should be prohibited whenever arguments from symmetry are possible.

Identities to be Learnt

The following identities should be at the finger ends of all Fifth Form boys.

1. $(b - c) + (c - a) + (a - b) = 0.$
2. $a(b - c) + b(c - a) + c(a - b) = 0.$
3. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$
4. $(a + b + c)^3 = a^3 + b^3 + c^3 + 3b^2c + 3bc^2 + 3c^2a + 3ca^2 + 3a^2b + 3ab^2 + 6abc.$
5. $(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab) = a^3 + b^3 + c^3 - 3abc.$
6. $(b - c)(c - a)(a - b) = -a^2(b - c) - b^2(c - a) - c^2(a - b)$
   $= -bc(b - c) - ca(c - a) - ab(a - b).$
7. $(b + c)(c + a)(a + b) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc.$
   $= bc(b + c) + ca(c + a) + ab(a + b) + 2abc.$

and perhaps,

8. $(a + b + c)(a^2 + b^2 + c^2)$
$= bc(b + c) + ca(c + a) + ab(a + b) + a^3 + b^3 + c^3.$
9. $(a + b + c)(bc + ca + ab)$
$= a^2(b + c) + b^2(c + a) + c^2(a + b) + 3abc.$
10. $(a + b + c)(b + c - a)(c + a - b)(a + b - c)$
$= 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4.$

Books to consult:

1. *Textbook of Algebra* (2 vols.), Chrystal (still the leading work in the subject).
2. *A New Algebra*, Barnard and Child.

CHAPTER XIX

Algebraic Equations

Equations of Different Degrees

"Either . . . or?"

More than once I have heard a small boy round on his teacher for this kind of argument:

Solve the equation $x^2 - 7x + 12 = 0$. "Factorizing, we have $(x - 4)(x - 3) = 0$. Hence *either* $(x - 4)$ *or* $(x - 3)$ must be zero, i.e. x must be either 4 or 3, and therefore *both* 4 *and* 3 must be roots of the equation."

Says the boy: "You said *either* $(x - 4)$ *or* $(x - 3)$ must be zero; how then can it follow that x is *both* 4 *and* 3?"

The criticism is just, for the reasoning is faulty.

A formal approach to equations may be successfully made by such general arguments as follows.

It is advisable in the first place to distinguish between an *equation* and an *identity*, and consistently to use the same form of words when referring to them. For instance: "When two expressions are equal for *all the values* of the quantities

involved, the statement of their equality is called an *identity*;" e.g. that $m - (n - p) = m - n + p$ is true for all values of the letters m, n, and p.

" But when two expressions are equal for *only particular values* of the quantities involved, the statement of their equality is called an *equation*." Thus $x + 7 = 10$ is an equation; it is true *only* where $x = 3$.

If in an equation we bring all the terms from the right-hand side to the left-hand side, and equate the whole to 0, e.g.

$$x + 7 - 10 = 0,$$

then by giving x its own particular value, the expression " vanishes ", e.g.

$$3 + 7 - 10 = 0,$$

i.e. $3 + 7 - 10$ is seen really to be 0, and has therefore " vanished ".

The value of the unknown quantity that makes the two sides of an equation equal is said to *satisfy* the equation. The process of finding that value, the *root*, is called *solving* the equation.

Consider the equation $3(x - 2) = 2(x - 1)$.

For what value of x is $3(x - 2)$ equal to $(2x - 1)$?

Try a few values, say the numbers 1 to 10. The only one of the ten that makes the expressions equal is 4, i.e.

$$3(4 - 2) = 2(4 - 1),$$

and so we say that 4 is the root of the equation.

If we simplify the original equation, we have

$$\begin{aligned} 3x - 6 &= 2x - 2, \\ \therefore\ 3x - 2x &= 6 - 2, \\ \therefore\ x &= 4, \text{ as expected.} \end{aligned}$$

We may write $x = 4$ as $x - 4 = 0$, and when in the expression $x - 4$ we write 4 for x, the expression vanishes, for $4 - 4 = 0$.

Again, for what value does $x^2 - x = 6$?

Try a few numbers as before. We find that in this case

there are *two* values which satisfy the equation, viz. 3 and -2, and that there are no others. Substituting, we have

$$3^2 - 3 = 6,$$
and $$(-2)^2 - (-2) = 6.$$

If we write the equation in the form $x^2 - x - 6 = 0$, the expression on the left-hand side vanishes when we write either $x = 3$ or $x = -2$. Thus

$$3^2 - 3 - 6 = 0,$$
and $$(-2)^2 - (-2) - 6 = 0,$$

and it does not vanish for any other value.

With equations of the second degree, we may always find *two* values of x that will satisfy the equation.

Since $x^2 - x - 6 = 0,$
and since $x^2 - x - 6 = (x - 3)(x + 2),$
$\therefore (x - 3)(x + 2) = 0.$

Now a product *cannot* be equal to zero *unless* one of the factors is equal to zero, and hence $(x - 3)(x + 2)$ can be equal to zero only (1) when $x - 3 = 0$, and (2) when $x + 2 = 0$, and never otherwise. Thus when we have $(x - 3)(x + 2) = 0$, we may equate the two factors equal to 0 separately, solve the two simple equations, and obtain the two roots:

Since $x - 3 = 0, \quad \therefore x = 3,$
and since $x + 2 = 0, \quad \therefore x = -2;$ as before.

Thus we have a *method* of solving a quadratic equation. Bring all the terms to the left-hand side and equate to 0; break up the expression into factors, and equate each of these to 0; solve the two resulting simple equations.

Note that a **quadratic** equation has two roots.

Here is a quotation from a well-known textbook: "If the product of two quantities is nothing, one of the quantities is nothing." One objection to this phraseology is that "nothing" is not a mathematical term.

Suppose we have an equation of the third degree, a

cubic equation as it is called, say, $x^3 - 6x^2 - 11x = 6$. Bring all the terms to the left-hand side and equate to 0. By a series of trials we may discover that there are 3 and *only* 3 values of x which will make the expression on the left-hand side vanish, viz. 1, 2, and 3 Thus

$$x^3 - 6x^2 + 11x - 6 = 0$$
$$1^3 - 6(1)^2 + 11 - 6 = 0$$
$$2^3 - 6(2)^2 + 22 - 6 = 0$$
$$3^3 - 6(3)^2 + 33 - 6 = 0.$$

Hence the roots of the equation are 1, 2, 3. But the trials would have been tedious. Let us factorize as before.

$$\text{Since} \quad x^3 - 6x^2 + 11x - 6 = 0,$$
$$\therefore \quad (x - 1)(x - 2)(x - 3) = 0.$$

A product of factors can be equal to 0 only when one of its terms is equal to 0. Obviously, in the equation, this may happen in three different ways, when $(x - 1) = 0$, when $(x - 2) = 0$, when $(x - 3) = 0$, and in no other way. If then we solve these three simple equations, we get $x = 1$ or 2 or 3, as before. A **cubic** equation has three roots.

Suppose we have an equation of the fourth degree, a " biquadratic " equation as it is called. It is quite easy to solve if we can factorize the expression made by bringing all the terms to the left-hand side. Usually this is a difficult job, but here is an easy one.

$$x^4 + 9x^2 + 38x = 8x^3 + 40,$$
$$\therefore x^4 - 8x^3 + 9x^2 + 38x - 40 = 0,$$
$$\therefore (x - 1)(x + 2)(x - 4)(x - 5) = 0.$$

This product can be zero only if one of its factors is zero. This can happen in 4 ways, and only in 4, viz. when $x - 1 = 0$, $x + 2 = 0$, $x - 4 = 0$, $x - 5 = 0$. Thus the orginal equation is equal to these 4 separate simple equations, and the roots are $1, -2, 4, 5$.

Thus a **biquadratic** equation has 4 roots. And so we might go on.

The general rule for solving an equation of a degree

beyond the first is, then, to bring all the terms to the left-hand side, to reduce the resulting expression to a series of linear factors, to equate each of these to 0, and then to solve them as simple equations.

It is therefore clear that the roots of an equation of any degree may be written down at once, provided we can resolve into linear factors the expression which results from bringing all the terms of the equation to the left-hand side. Generally speaking, *the* trouble is to find the factors, and it is often necessary to resort to indirect methods.

The Need for Verifying Roots

When solving equations, we frequently adopt the device of multiplying or dividing both sides by some quantity, and sometimes we square, or take the square root of, each of the two sides. Is this always allowable?

Consider an equation of the simplest form, one having only one solution, say $x - 3 = 2$. Since $x - 3 = 2$, $x - 5 = 0$, and $\therefore x = 5$. Let us multiply both sides of the original equation by, say, $(x - 6)$. Thus

$$\begin{aligned} (x - 3)(x - 6) &= 2(x - 6), \\ \therefore\ x^2 - 9x + 18 &= 2x - 12, \\ \therefore\ x^2 - 11x + 30 &= 0, \\ \therefore\ (x - 5)(x - 6) &= 0, \\ \therefore\ &\text{the values of } x \text{ are } 5 \text{ and } 6. \end{aligned}$$

Thus by introducing the factor $(x - 6)$ we have transformed the equation into another completely different. The new root 6 does *not* satisfy the original equation.—Evidently when we have solved an equation we must see if the roots really *satisfy* the equation.

Another example: consider the very simple equation $x = 3$. Square both sides,

$$\begin{aligned} x^2 &= 9, \\ \therefore\ x^2 - 9 &= 0, \\ \therefore\ (x + 3)(x - 3) &= 0. \end{aligned}$$

Hence there are 2 roots, $+3$ and -3, as compared with

only one root (+3) in the original equation. Thus the squaring has introduced an extraneous root.

Another example:

$$3x - \sqrt{x^2 - 24} = 16.$$
$$\therefore\ 3x - 16 = \sqrt{x^2 - 24}$$
$$\therefore\ 9x^2 - 96x + 256 = x^2 - 24$$
$$\therefore\ 8x^2 - 96x + 280 = 0,$$
$$\therefore\ x^2 - 12x + 35 = 0,$$
$$\therefore\ (x - 7)(x - 5) = 0,$$
$$\therefore\ x = 7 \text{ or } 5.$$

But on examination we find that only 7 satisfies the original equation; 5 does not. Hence there is only one root. We seem to have solved the equation in the usual way: have we done anything wrong? Let us see if by working our way *backwards* we can discover any sort of mistake.

$$(x - 5)(x - 7) = 0,$$
$$\therefore\ x^2 - 12x + 35 = 0.$$
Multiplying by 8, $8x^2 - 96x + 280 = 0.$
Adding $x^2 - 24$ to each side, $9x^2 - 96x + 256 = x^2 - 24.$
Extracting the square root of each side, $3x - 16 = \pm\sqrt{x^2 - 24}.$
$$\therefore\ 3x \mp \sqrt{x^2 - 24} = 16.$$

The steps are exactly the same until we come to the last but one; then we had to prefix the double sign. Hence at the second step in our forward process, we really introduced a new and extraneous root, since the square of $+\sqrt{x^2 - 24}$ is also the square of $-\sqrt{x^2 - 24}$. Thus from that step onwards, the equations ceased to represent the original equation.

When we multiply or divide by ordinary arithmetical numbers, no difficulty will arise. When we multiply or divide by an algebraic expression, we sometimes run a risk. What is wrong with the following, for instance?

Suppose $x = y.$
Then $x^2 = xy,$
$$\therefore\ x^2 - y^2 = xy - y^2,$$
$$\therefore\ (x + y)(x - y) = x(x - y),$$
$$\therefore\ x + y = x,$$
$$\therefore\ x + x = x,$$
$$\therefore\ 2 = 1, \text{ which is absurd.}$$

We have divided both sides in the fourth line by $(x - y)$, i.e. by $(x - x)$, i.e. **0**. This is quite illegitimate, and it inevitably leads to an absurdity. Can you see now why our *first* example went wrong? We had, really, $(x - 5) = 0$; and then $(x - 5)(x - 6) = 0(x - 6)$, though we did not show it this way.

Another example:

$$\frac{x^2 - 3x}{x^2 - 1} + 2 + \frac{1}{x - 1} = 0.$$

Multiply by $x^2 - 1$, the L.C.M.,

$$x^2 - 3x + 2(x^2 - 1) + x + 1 = 0,$$
$$\therefore 3x^2 - 2x - 1 = 0,$$
$$\therefore (3x + 1)(x - 1) = 0,$$
$$\therefore x = -\tfrac{1}{3} \text{ and } 1.$$

But by testing we find that 1 is *not* a value of the original equation and must therefore be rejected. Multiplying by $(x^2 - 1)$ led to this trouble. Here is a more correct way of solving:

$$\frac{x^2 - 3x}{x^2 - 1} + 2 + \frac{1}{x - 1} = 0,$$
$$\left(\frac{x^2 - 3x}{x^2 - 1} + \frac{1}{x - 1}\right) + 2 = 0,$$
$$\therefore \left(\frac{x^2 - 3x + x + 1}{x^2 - 1}\right) + 2 = 0,$$
$$\therefore \frac{(x - 1)^2}{x^2 - 1} + 2 = 0,$$
$$\therefore \frac{x - 1}{x + 1} + 2 = 0,$$
$$\therefore x = -\tfrac{1}{3}, \text{ the only root.}$$

The former method is quite acceptable, provided the roots found are checked, and that one is rejected if found unacceptable.

Strictly speaking, *either . . . or* are "disjunctive" they therefore suggest *alternatives*. But sometimes they

are equivalent to *both . . . and*; or, *alike . . . and*; and it is this exceptional use which more correctly represents the algebraic argument. But the use of *either . . . or* in connexion with equations is best avoided. When a boy says "*either* 6 *or* 5" he naturally thinks that if one is accepted the other is necessarily rejected.

The Theory of Quadratics

The work on equations should be closely associated with the work on graphs. The graph helps to elucidate all sorts of difficulties. See, for instance, fig. 46, p. 161, in connexion with the "either . . . or" argument.

The elementary theory of quadratics, as far as it is necessary for a Fifth Form, seldom gives trouble. The more elementary facts should be known thoroughly and should be consistently used for checking and other purposes. But do not forget that the quadratic *function* is of far greater importance than the quadratic *equation* (see the chapter on Graphs).

The formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ may be used as other formulæ are used, but it should not be used as the stock method of solving quadratics; boys are apt to forget its significance if used in that way. They should clearly realize that the formula represents the roots of the equation $ax^2 + bx + c = 0$, and that these roots are real and different, real and equal, or unreal and different, according as the discriminant $b^2 - 4ac$ is $+$, 0, or $-$.

The pupils should frequently make use of the further facts that if x_1 and x_2 are the roots of the equation $ax^2 + bx + c = 0$, then $x_1 + x_2 = \frac{-b}{a}$, and $x_1 x_2 = \frac{c}{a}$. The method of evaluating these from $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ should be remembered.

Equations Solved like Quadratics

Group the common types together. They are usually easy, though attention must be paid to all the roots involved; the boys are apt to overlook some of them. We append examples of the main type, and add the sort of hint that ought to suffice to enable the boys to set to work.

1. $x^4 - 10x^2 + 9 = 0.$

   Write $x^2 = y$, and solve for y; then from the 2 values of y obtain the 4 for x.

2. $(x^2 + 2)^2 - 29(x^2 + 2) + 198 = 0.$

   Write $x^2 + 2 = y$ and solve for y.

3. $2x^2 - 4x + 3\sqrt{x^2 - 2x + 6} - 15 = 0.$

   $\therefore\ 2(x^2 - 2x + 6) + 3\sqrt{x^2 - 2x + 6} - 27 = 0.$

   Write $\sqrt{x^2 - 2x + 6} = y$ and solve for y.

4. $(x - 1)(x - 3)(x - 5)(x - 7) = 9,$

   $\therefore\ (x - 1)(x - 7)(x - 3)(x - 5) - 9 = 0,$

   $\therefore\ (x^2 - 8x + 7)(x^2 - 8x + 15) - 9 = 0.$

   Write $x^2 - 8x = y$, and solve for y.

5. $\dfrac{x + 4}{x - 4} - \dfrac{x - 4}{x + 4} = \dfrac{9 + x}{9 - x} - \dfrac{9 - x}{9 + x},$

   $\therefore\ \dfrac{16x}{x^2 - 16} = \dfrac{36x}{81 - x^2}$ (obviously $x = 0$ is a root),

   $\therefore\ \dfrac{4}{x^2 - 16} = \dfrac{9}{81 - x^2}$ (obviously 2 more roots).

6. $x^2 + \dfrac{18}{x^2} - 11 = 0,$

   $\therefore\ x^4 - 11x^2 + 18 = 0.$

7. $x^3 + 1 = 0.$ (Factorize.)

8. $x^6 - 7x^3 - 8 = 0.$ (Factorize.)

9. $7x^3 - 13x^2 + 3x + 3 = 0.$ (Factorize: $(x - 1)$ evidently a factor.)

Boys soon see through all these types and solve examples fairly readily.

Simultaneous Equations

Do not spend much time over these, unless you are unlucky enough to have to prepare for an unintelligent examination in which far-fetched examples are given. Let a result of each main type be graphed (see the Chapter on Graphs) and all the roots be pictorially explained.

Teach the pupils to pair off the roots correctly.

We append an example of the commoner types.

1. $x + y = 1$
   $xy = 14$

2. $x^2 + y^2 = 13$
   $xy = 6$

Show that once we know the value of $x + y$ and $x - y$, we may obtain the separate values of x and y by mere addition and subtraction; and that $x + y$ can always be obtained from $x^2 + 2xy + y^2$, and $x - y$ from $x^2 - 2xy + y^2$.

3. $x - y = 2$
   $x^2 + y^2 = 100$

4. $x^3 + y^3 = 152$
   $x + y = 8$

5. $x^3 - y^3 = 98$
   $x^2 + xy + y^2 = 49$

By division we obtain quotients which enable us to proceed as in examples 1 to 3.

6. $x^4 + x^2y^2 + y^4 = 133$
   $x^2 + xy + y^2 = 19$

By division, we obtain $x^2 - xy + y^2$, which, subtracted from $x^2 + xy + y^2$, gives us xy. Then as in examples 1 to 5.

7. $3x^2 + 4xy + 5y^2 = 31$
   $x + 2y = 5$

From the second express y in terms of x, and substitute in the first.

8. $x^2 + 3xy - y^2 = 9$
   $2x^2 - 2xy + 3y^2 = 7$

Expressions are homogeneous. Convert into fractions, simplify, and factorize. Thus we have:

$$11x^2 - 39xy + 34y^2 = 0,$$
$$\therefore (11x - 17y)(x - 2y) = 0,$$
$$\therefore x = \tfrac{17}{11}y \text{ and } 2y; \text{ \&c.}$$

All these types are easily taught and remembered. It is waste of time for boys to be given the far-fetched and exceptional types worked out (often elegantly it is true) in the textbooks. School life is not long enough.

Problems producing Equations

These have been given a place greatly beyond their value, and important mathematical principles are often treated rather superficially in order that more time may be devoted to " problems ". Unfortunately, however, problems have become entrenched in all mathematical examinations, and there is nothing for it but to teach boys how to solve them. And, after all, problems do test boys' knowledge of certain principles, and a correct solution is always a source of satisfaction.

The veriest tyro of a teacher can write out on the blackboard the solution of a problem which the boys themselves have been unable to solve. But what do the boys gain from that? The mere setting out of a solution deductively, after the manner of a proposition in Euclid, gives the boys no inner light at all. The boys want to be initiated into a plan of effective attack, to be taught how to analyse and how to utilize the data of a problem, to be told exactly how the teacher himself *discovered* the solution.

Be it remembered that a solution most suitable for a class of boys is by no means necessarily the " neat " solution so dear to the heart of a mathematician.

The main difficulty felt by boys in solving most algebra problems is the translation of the words of the problem into suitable equating formulæ. Much practice is necessary if facility in this translation is to be gained. Once expressed in algebraic form, the equation is generally easy of solution.

The boy knows that an equation consists of two parts connected together by the sign $=$. The first thing to search for in a given problem is therefore the word " equal ", or some words which imply " equal ", or such words as " greater than " or " less than ". If the problem concerns money matters, the boy may be able to dig out of his own knowledge some relation of equality, e.g.

Cost price + profit = selling price;

or, if he is dealing with racing problems, he may be able to utilize the already familiar relation:

$$\text{distance} = \text{speed} \times \text{time};$$

or, if he is dealing with a clock sum, he may be able to split up a component angle in two different ways, and so equate

$$\alpha + \beta = \gamma + \delta \text{ (or some modification of this).}$$

We append a few problems, with teaching hints.

1. *If* 4 *be added to a certain number, and the sum be multiplied by* 5, *the product will be* **equal** *to the number added to* 32. *Find the number.*

The question tells us,

$$\text{a product} = \text{the number} + 32. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Let us try to arrange our equation accordingly.

What have we to find? A *number*. Then let x represent the number. The " product " is 5 times the sum of x and 4; how shall we write this down? $5(x + 4)$. (i) tells us that this product is *equal* to $x + 32$;

$$\therefore 5(x + 4) = x + 32. \qquad \therefore x = 3.$$

2. *Find a number such that if it be multiplied by* 5, *and* 2 *be taken from the product, one-half the remainder shall exceed the number by* 5.

The question says

$$\text{half a remainder } \textbf{exceeds} \text{ the number by } 5,$$
$$\text{i.e. half a remainder} = \text{the number} + 5. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

How shall we represent the number? by x. 5 times the number? $5x$. What is the remainder when 2 is taken from this product? $5x - 2$. What is half this remainder? $\frac{1}{2}(5x - 2)$. Then how from (i) can we make up our equation?

$$\tfrac{1}{2}(5x - 2) = x + 5. \qquad \therefore x = 4.$$

3. *A man spent £10 of his money, and afterwards one-quarter of the remainder. He had £30 left. How much had he at first?*

The word *left* suggests the relation:

(money at first) − (expenditure) = £30. . . . (i)

Let us try to arrange our equation accordingly.

Let x represent the number of pounds he had at first. Then

x − expenditure = £30. . . . . (ii)

What is the expenditure?

First expenditure = £10; ∴ $x - 10$ = remainder.
Second expenditure = ¼ of remainder = ¼$(x - 10)$,
∴ Total expenditure = $10 + \frac{1}{4}(x - 10)$.

Now we may substitute this in (ii):

$$\therefore\ x - \{10 + \tfrac{1}{4}(x - 10)\} = 30. \quad \therefore\ \boldsymbol{x = £50.}$$

4. *A man buys a flock of sheep at £3 a head, and turns them into a field to graze for 3 months, for which he is charged 45s. a score. He then sells them at £3, 10s. a head, and so makes a clear profit of £77, 10s. How many sheep were there in the flock?*

Clearly:

(Money laid out) + (profit) = (Proceeds of sale). . (i)

What have we to find? The *number* of sheep bought.

Let x represent no. of sheep bought; then $\frac{x}{20}$ = no. of scores of sheep bought.

1. *Money laid out*:

(*a*) Cost of x sheep at £3 each = $3x$ pounds

(*b*) Cost of grazing $\frac{x}{20}$ scores of sheep at £2¼ a score = $\left(\frac{x}{20} \times 2\tfrac{1}{4}\right)$

∴ total money laid out = $3x + \left(\frac{x}{20} \times 2\tfrac{1}{4}\right)$. . . (ii)

2. *Proceeds of sale*:

Sale of x sheep at £3½ each = $3\frac{1}{2}x$ pounds. . . . (iii)

According to (i), (ii) + £77½ = (iii),

$$\text{i.e.}\quad 3x + \left(\frac{x}{20} \times 2\tfrac{1}{4}\right) + 77\tfrac{1}{2} = 3\tfrac{1}{2}x. \qquad \therefore\ x = \mathbf{200}$$

5. *A boy was born in March. On the* 18*th of April he was* 5 *times as many days old as the month of March was on the day before his birth. Find his date of birth.**

This examination absurdity is simple enough, once the wording is unravelled. Note that if a boy is born, say, on 4th May, he is 20 days old on 24th of May. It is a case of simple subtraction.

In the problem we have to deal with two ages, expressed in days:

(i) *the age of the boy* on 18th of April,
(ii) *the age of March* on the day before the boy was born

The former = 5 times the latter. Hence we can make up our equation . . . . . (i)

(i) *The age of the boy on* 18th *of April*:

Let the boy be born on the xth of March.
By the end of March he is $(31 - x)$ days old.
By April 18th he is $(31 - x + 18)$ days old. . . . . (ii)

(ii) *The age of March on the day before the boy was born.*

The boy was born on the xth day of March.
Hence March was then x days old.
The day before that, March was $(x - 1)$ days old. . . (iii)

From (i) we know that (ii) is 5 times (iii).

$$\text{i.e.}\quad (31 - x) + 18 = 5(x - 1). \qquad \therefore\ x = \mathbf{9}.$$

6. *The* 3 *hands of a watch are all pivoted together centrally. When first after* 12.0 *will the seconds hand, produced backwards, bisect the angle between the other* 2 *hands?*

We have to remember that the seconds hand moves 60

* The problem is not well worded. For instance, March is not, strictly, nine days old until midnight on March 9th. We have assumed that the boy was born at midnight, and we have reckoned ages from midnight.

times as fast as the minute hand, and the minute hand 12 times as fast as the hour hand. Thus the relative speeds are 720 : 12 : 1.

At noon (N) all the hands are together. The watch circumference is divided into 60 equal arcs, and we may measure the angles in terms of these arcs. Let the seconds hand move round to its position S* in x seconds; i.e. arc NS $= x$. Since the minute hand also moves round to its position M in x seconds, the arc NM measures $\frac{x}{60}$.

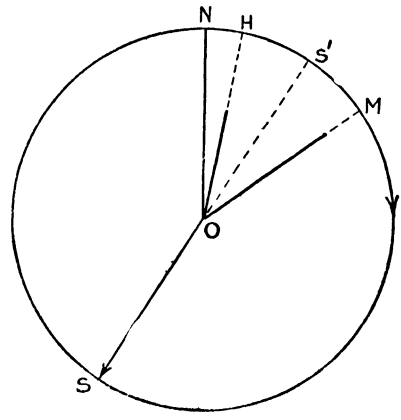


Fig. 61

And since the hour hand also moves round to its position H, in x seconds, the arc NH measures $\frac{x}{720}$.

Now the seconds hand OS produced backwards, making OS′, bisects the angle HOM; i.e. the arc HS′ = the arc S′M.

We ought therefore to be able to make up an equation by means of the pieces of arc between N and M, e.g.

$$\text{NM} = \text{NS}' + \text{S}'\text{M}. \quad \ldots\ldots\ldots \quad \text{(i)}$$

We know that $\text{NM} = \frac{x}{60}$,

that $\text{NS}' = x - 30$,

and that $\text{S}'\text{M} = \frac{1}{2}\text{HM} = \frac{1}{2}(\text{NM} - \text{NH}) = \frac{1}{2}\left(\frac{x}{60} - \frac{x}{720}\right)$,

$\therefore$ from (i) we have $\frac{x}{60} = (x - 30) + \frac{1}{2}\left(\frac{x}{60} - \frac{x}{720}\right)$,

$$\therefore x = 30\tfrac{390}{1427} \text{ (secs. after 12.0).}$$

* The angles in the figure are necessarily much exaggerated.

7. *54 minutes ago, it was 5 times as many minutes past 5 as it is now minutes to 7. What is the time now?*

Most watch and clock problems can be solved on the basis of the principle illustrated in the last example, and one careful analysis, to exemplify the method, is usually enough to enable the boys to attack successfully most of the problems given in a textbook. But this problem, another absurdity from an examination paper, does not fit into any general scheme. Though easy, its translation into an equation may at first puzzle most average pupils.

The basis for equalizing quantities is pretty obvious at the outset:

(54 min. ago, no. of min. past 5) = 5 (no. of min. to 7 now). . (i)

The question to be answered is, what is the time *now*?

The problem mentions 5.0 and 7.0, and refers to the time now as a *number of minutes to seven.*

Hence, let the time now be x *minutes to* 7. . . . (ii)

We also require to know what the time was 54 *minutes ago*; this must have been $(x + 54)$ minutes to 7.

But we have to express this time in terms of minutes past 5.0. Now 5.0 is 120 minutes before 7.0.

Hence, 54 minutes ago the number of minutes past 5.0 was $120 - (x + 54)$. . . . . (iii)

From (i), (iii) is 5 times (ii),

$$\text{i.e. } 120 - (x + 54) = 5x. \qquad \therefore \ x = \mathbf{11}.$$

All answers to equations should be checked; checking in a case like this is particularly necessary.

Since $x = 11$, the time now is 11 minutes to 7, or **6.49**. The time 54 minutes ago was **5.55**, or the number of minutes (55) then past 5.0 is 5 times the number of minutes (11) now to 7.0.

8. *Three friends going on a railway journey take with them luggage amounting in all to* 6 *cwt. Each has more than can be carried free, and the excess charged them is* 2s. 6d., 7s., *and* 10s., *respectively. Had the whole belonged to one person, he would have had to pay* 34s. 6d. *excess. How much luggage is each passenger allowed to carry free, what is the excess charge per lb., and what is the weight carried by each of the three friends?*

Consider first a simple case. If I am allowed to take with me, say, 100 lb. free, and have to pay, say, $\frac{1}{2}d.$ on every lb. exceeding 100, then if I take with me a total of, say, 150 lb. the excess I have to pay is $\frac{1}{2}d. \times (150 - 100)$.

Thus a possible form of equation seems to be:

(excess charge per lb.) $\times$ (no. of lb. excess) = (total charge for excess),

and as there are two separate though similar statements concerning excess, we ought to be able to formulate *two* equations, say in x and y.

What have I to find out? (1) lb. per person carried free, and (2) excess charge per lb. Hence:

Let each passenger

(1) Carry x lb. free.
(2) Pay y pence on each lb. excess.

Assume that one of the friends takes 3 tickets and shows them to the porter, who on weighing the luggage and finding it to be 672 lb., deducts $3x$ from the 672, and charges y pence per lb. on the difference, viz. $(672 - 3x)$. Since the sum actually paid for excess $= 2s.\ 6d. + 7s. + 10s. = 19s.\ 6d. = 234d.$,

$$\therefore (672 - 3x)y = 234. \quad \ldots \text{(i)}$$

But if all 672 lb. had belonged to one person, he would have taken only 1 ticket, and the porter would have charged y pence on each of $(672 - x)$ lb. Since the sum actually paid in this case for excess $= 34s.\ 6d. = 414d.$,

$$\therefore (672 - x)y = 414. \quad \ldots\ldots \text{(ii)}$$

Dividing (i) by (ii) we have

$$\frac{672 - 3x}{672 - x} = \frac{13}{23}, \qquad \therefore\ x = 120$$

= no. of lb. allowed free.

Hence from (i) or (ii),

$y = \frac{3}{4}d.$ = charge per lb. excess.

Weight in lbs. carried by each person

$$= 120 + \frac{30}{\frac{3}{4}},\ 120 + \frac{84}{\frac{3}{4}},\ 120 + \frac{120}{\frac{3}{4}},\ \text{respectively.}$$

This problem is worth giving a class a second time, say three months after the first. Boys seem to find a first analysis a little difficult.

9. *An express train and an ordinary train travel from London to Carlisle, a distance of* 300 *miles. The ordinary train loses as much time in stoppages as it takes to travel* 25 *miles without stopping. The express train loses only three-tenths as much time in stoppages as the ordinary train, and it also travels* 20 *miles an hour quicker. The total times of the two trains on the journey are in the ratio* 26 : 15. *Find the rate of each train.*

We have to think of *distance*, *speed*, and *time*, and their relation $d = st$.

The **ratio** of the *total times taken* by the two trains is given, viz.

$$26 : 15. \qquad \ldots\ldots\ldots \text{(i)}$$

Hence if we can express these total times in some other way we can formulate our equation.

Distance: 300 miles.

Speeds: Let ordinary train travel x miles an hour.
Then express train travels $x + 20$ miles an hour.

Times: Time taken by ordinary train $= \frac{300}{x}$ + stoppages (in hours). $\ldots\ldots\ldots\ldots\ldots$ (ii)

Time taken by express train $= \frac{300}{x + 20}$ + stoppages (in hours). $\ldots\ldots\ldots\ldots\ldots$ (iii)

We now require to know the amount of time lost over stoppages.

(1) *Ordinary train*: time lost over stoppages is equal to that taken in travelling 25 miles:

Train travels x miles in 1 hour,

$$= 1 \text{ mile in } \frac{1}{x} \text{ hour}$$

$$= 25 \text{ miles in } \frac{25}{x} \text{ hours}$$

$$\therefore \frac{25}{x} \text{ hours} = \text{time lost over stoppages.}$$

(2) *Express train*: time lost over stoppages $= \frac{3}{10}$ that of ordinary train,

$$= \frac{3}{10} \text{ of } \frac{25}{x} \text{ hours} = \frac{15}{2x} \text{ hours.}$$

We can now express (ii) and (iii) in the following forms:

$$\text{Time taken by ordinary train} = \left(\frac{300}{x} + \frac{25}{x}\right) \text{ hours.} \quad \text{(iv)}$$

$$\text{Time taken by express train} = \left(\frac{300}{x + 20} + \frac{15}{2x}\right) \text{ hours.} \quad \text{(v)}$$

Hence we have from (i), (iv), (v),

$$\frac{\frac{300}{x} + \frac{25}{x}}{\frac{300}{x + 20} + \frac{15}{2x}} = \frac{26}{15}.$$

$$\therefore x = 30 \text{ (miles an hour).}$$

If in " racing " and analogous problems the relation $d = st$ is kept in view, the necessary analysis is usually quite simple.

10. *What is the price of sheep per* 100 *when* 10 *more in* £100 *worth lowers the price by* £50 *per* 100?

We must avoid confusion between (1) the number of sheep for £100 and (2) the cost of 100 sheep.

We can find the number of sheep costing £100 if we know the price of 1, and we can find the price of 1 if we know the price of 100.

A possible equation seems to be:

(First no. of sheep for £100)
= (second no. of sheep for £100) − 10. . . . (i)

What have we to find? *The price of* 100 *sheep.*

(1) First price of 100 sheep. Call this £x.

$\therefore$ 1 sheep costs $\frac{£x}{100}$,

$\therefore$ *number obtainable for* £100 $= \frac{£100}{\frac{£x}{100}} = \frac{10{,}000}{x}$. . . . (ii)

(2) Second price of 100 sheep. This = £$(x - 50)$.

$\therefore$ 1 sheep costs $\frac{£(x - 50)}{100}$,

$\therefore$ *number obtainable for* £100 $= \frac{£100}{\frac{£(x - 50)}{100}} = \frac{10{,}000}{x - 50}$. . . . (iii)

(i) shows us how (ii) and (iii) are related, and then we may make up our equation.

$$\frac{10{,}000}{x} = \frac{10{,}000}{x - 50} - 10, \qquad \therefore x = 250.$$

i.e. the price of 100 sheep is £250, or £2, 10*s.* each, or £100 worth = 40 sheep.

(If 50 for £100, each costs £2, or cost of 100 = £200, i.e. £50 less than before.)

I have found that even Sixth Form boys are sometimes baffled by the analysis of this little problem.

Problems which are at all unusual in form are always worth repeating after an interval.

Books to consult (on the general technique of teaching algebra):

1. *The Teaching of Algebra*, Nunn.
2. *Elements of Algebra*, 2 vols., Carson and Smith.
3. *A New Algebra*, Barnard and Child.
4. *Algebra*, Godfrey and Siddons.
5. *A General Textbook of Elementary Algebra*, Chapman.
6. *Elements of Algebra*, De Morgan. (A valuable old book. So are De Morgan's other books, especially his Arithmetic.)

CHAPTER XX

Elementary Geometry

Early Work

Some of the younger generation of teachers have never read Euclid, and seem to be totally unacquainted with the rigorous logic of the old type of geometry lesson. Not a few of the old generation regret the disappearance of Euclid, urging that the advantages of the newer work are outbalanced by the loss of the advantages of the older.

The real distinction between the older and the newer work is, however, sometimes forgotten. Essentially, Euclid wrote a book on *logic*, using elementary geometry as his raw material. The amount of actual geometry, qua geometry, which he taught was, relatively speaking, trifling. Boys in existing technical schools do ten times as much geometry as is found in Euclid. But as an exposition of deductive reasoning from an accepted set of first principles, Euclid has never been equalled.

Until forty years ago, Euclid was universally taught in secondary schools, but the collective opinion of experts had gradually hardened against it, partly because the average boy found it difficult, partly because some of its propositions were too subtle for schoolboys, partly because its foundations were far from being unassailable, and partly because the actual geometry it expounded was too slight to be of much practical service.

But the geometry that was substituted for Euclid—the geometry now exemplified in all the ordinary school text-books—is still *Euclidean* geometry, i.e. it is a geometry based, in the main, on the same foundations as Euclid. These foundations consist of a number of quite arbitrarily chosen axioms. Other sets of axioms might be substituted for them, and then we should get an entirely new system of geometry

of a non-Euclidean character. Reference to such geometry will be made in a future chapter.

In practice, the difference between Euclid and the geometry now taught is in the choice of working tools. In Euclid, the proof of every proposition was ultimately traceable to the axioms, and every schoolboy had to substantiate every statement he made by referring it to something already proved, and this in its turn to something that had gone before, and so back to the axioms. In those days the axioms were really the working tools. But those axioms were so subtle that the boys' confidence in them was entirely misplaced. Nowadays, the working tools consist of a small number of fundamental propositions. By means of carefully selected forms of practical work, the truth of these propositions is shown to beginners to be highly probable, but the formal proofs of such propositions are not considered until the boys reach the Sixth Form. Examination authorities no longer call for the formal proofs at the School Certificate stage.

These working tools once thoroughly mastered, beginners plunge into the heart of the subject and make rapid headway. In the old Euclidean days a year or more was spent on these propositions and a few others, and at the end of that time the average boy had but very vague notions about them, though the mathematically-minded boy certainly did seem to appreciate the rigour of the reasoning presented to him.

I find it a little difficult to describe the methods of the pre-eminently successful teacher of geometry. The methods are not the reflection of any particular book but of the man himself. By the gifted teacher who happens to be a sound mathematician, a new principle is often illuminated by so many side-lights that even the dullard can hardly fail to see and understand. Successful teachers of geometry seem to be those who have given special attention to the foundations of the subject, who possess exceptional ingenuity in making things clear, and who at an early stage make use of symmetry and of proportion and similarity.

Those teachers who are not successful are often those

who confine their work to the limits of the ordinary textbook written for the use of boys; who fail to survey the whole geometrical field; who are still unacquainted with, or at all events do not teach, the great unifying principles of geometry —duality, continuity, symmetry, and so forth. The little textbooks are all right for the boys, but the teaching of geometry connotes something outside and beyond such textbooks, especially the principles underlying the grouping and regrouping of the thousand and one facts that the beginner necessarily learns as facts more or less isolated. The accumulated facts can be given many different settings, each setting forming a perfect picture, all the pictures different yet closely related.

Work up to 13 or 14

As I have said in another place,* the following is an expression of authoritative opinion as to the nature of the work which it is most advisable to do with young boys:

1. The main thing should be to give the boys an intelligent knowledge of the elementary *facts* of geometry.

2. No attempt should be made to develop the subject on rigorously deductive lines, from first principles, though, right from the first, precise reasons for statements made should be demanded.

3. Young boys are never happy and are often suspicious if they feel they are being asked to prove the obvious, but they can follow a fairly long chain of reasoning if the facts are clear.

4. All subtleties should be avoided, and, therefore, proofs of propositions concerning angles at a point, parallels, and congruent triangles should not be attempted, such proofs being a matter for later treatment in the upper Forms.

5. These main working tools of geometry, angles at a point, parallels, and congruent triangles, should be presented

* *Lower and Middle Form Geometry*, Preface.

in such a way as to enable the boys to understand them clearly and to use and apply them readily.

6. Young boys can easily understand Pythagoras, elementary facts about areas, and the main properties of the circle and of polygons; and these facts should be taught.

7. The simple commensurable treatment of (i) the proportional division of lines, and (ii) similar triangles, should be included in the work to be done at the age of 12 to 14; young boys soon become expert in the useful practice of writing down equated ratios from similar triangles.

8. By about the age of 13, a boy ought to be able to write out a simple straightforward proof formally and to attack easy riders.

9. Throughout the course, all possible use should be made of the boys' intuitions and of their knowledge of space-relations in practical life, relations in three dimensions as well as in two.

10. Responsible teachers should always express themselves in exact geometrical language, and should make precision and accuracy of statement an essential part of the boys' training.

11. The boys should be taught how to formulate their own definitions, and, under the guidance of the teacher, to polish up these definitions as accurately as their knowledge at that stage permits; and these definitions should be learnt. Definitions should never be provided ready-made.

12. The boys should be taught to realize exactly what properties are implied by each definition, and all other properties must be regarded as derivative properties requiring proof.

13. The boys may usefully be given an elementary training in the principle that a *general* figure necessarily retains its basic properties even when it becomes more and more particularized, but that, as the figure becomes less and less general, it acquires more and more properties; and vice versa.

14. In short, clear notions of the all-important principle

of continuity should, by the time a boy is about 13, " be in his very bones ".

15. A young boy's natural fondness for puzzles of all kinds may often usefully be employed for furthering his interest in geometry.

16. In one respect we have drifted too far away from Euclid: boys' knowledge of geometry is too often vague, too seldom exact.

Teachers differ in opinion about the degree of accuracy to be demanded in beginners' geometrical drawing. *Some* training in the careful use of instruments is certainly desirable, but time should not be wasted over elaborate drawings when freehand sketches can be made to serve adequately. In Technical schools, accurate drawing with instruments is an essential part of much of the pupils' work. Even in Secondary schools, where the geometry is necessarily given an academic bias, a preliminary training in the careful use of instruments serves a useful purpose, but there is no point in making Secon dary school boys spend time over elaborate pattern drawings and designs. Exercises in accurate work of a more telling type may be found in the theorems of Brianchon, Desargues, Pascal, and others. As *theorems*, these are, of course, work for the Upper Forms; as geometrical constructions, they are useful in the lower Middle Forms, where they may be learnt as useful and interesting geometrical facts.

All pupils should be taught the wisdom of drawing good figures for rider solving purposes.

Boys in Technical schools often have a better all-round knowledge of geometry than those in Secondary schools because they do more work in three dimensions. Solid geometry of a simple kind may with great advantage be included in the early stages of any geometry course.

A short course on simple projection at about the age of 13 helps later geometry enormously. Boys soon pick up the main principles, and the work helps greatly to develop their geometrical imagination. So does simple work with the polyhedra, work which always appeals to boys.

Let logic of the strictly formal kind wait until foundations are well and truly laid. The increasing difficulty felt by beginners in geometry is largely an affair of increasing difficulty of logic, and thus we now recognize that parts of the third, fourth, and sixth books of Euclid are easier than parts of the first book.

Push ahead. Do not paddle about year after year in the little geometrical pond where examiners fish for their questions. Even for the examination day such paddling most certainly does not pay.

It is convenient, though not defensible, to preserve the old distinction between axioms, postulates, and definitions. But if any teacher still believes that axioms should be considered in a beginner's course of geometry, let him consult Mr. Bertrand Russell, and he will soon be disabused.

Early Lessons

Here are a few early lessons in geometry, selected at random from the book already cited. The sections have been renumbered, seriatim, for convenience of reference, but actually the lessons are drawn from all parts of the book.

Lesson I

Planes and Perpendiculars

1. Carpenters, bricklayers, blacksmiths, and plumbers, all have to know something about geometry. Architects, builders, surveyors, and engineers have to know a great deal about it. All of them have to know how to measure things, and how to make things perfectly level, perfectly upright, perfectly square, perfectly "true"; and much more besides.

You have already learnt how to use a **ruler** or **scale**, marked with inches and parts of an inch on the one edge, and with centimetres and parts of a centimetre on the other. Note that there are very nearly, but not exactly, $2\frac{1}{2}$ centimetres to an inch.

You know already that, when two lines meet at a point, they form an **angle**. Here are three angles.



Fig. 62

The middle one is the angle you know best. It is the angle you see at the corner of an ordinary picture-frame, or of a door, or of a window-frame, or of a table-top. Such an angle is called a square angle, or **right angle.**

2. You have probably seen a carpenter **planing** a piece of wood, perhaps for a shelf. He begins by planing one of the " faces " of the wood, and, as soon as he thinks that the face is a true **plane,** he tests it. To do this he uses a **try-square,** which consists of a steel *blade* with parallel edges perfectly straight, fixed at right angles into a wooden *stock.* (A *try*-square is a *test*-square.) He holds the stock in his hand, and to the planed face of the wood he applies the outside edge of the blade, " trying " it in many places and in different directions, along and across. If he can see daylight anywhere between the blade and the wood, he knows that the planed face is not yet **a true plane,** and that he must continue his planing. When it is true, he marks it *face-side.*

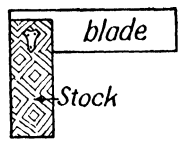


Fig. 63

3. Now he turns up the wood so that an edge rests on the bench, and he planes the edge at the top. Not only has he to make this *face-edge* (as it is called) a true plane like the *face-side,* but he has to make the two planes at right angles to each other, or, as we usually say, **perpendicular** to each other. The carpenter is not satisfied until the inside right angle of his try-square fits exactly, at the same time, the face-side and

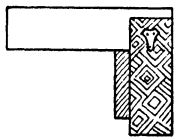


Fig. 64

the face-edge, the blade fitting against the one, the stock against the other, the test being made at several places along the wood.

You might use a big try-square to see if a flag-staff or a telegraph-post is perpendicular to the ground. If it did not fit exactly in the angle between the post and the ground, no matter where tested round the post, you would know that either the ground is not level or the post is not upright. **Two planes, or two lines, or a line and a plane, are perpendicular to each other if and only if they are at right angles to each other.** A perfectly upright post in a sloping bank is not perpendicular to the bank *because* it does not make right angles with the bank.

4. The maker of a try-square guarantees the accuracy of the *inside* angle, but not of the outside right angle. Thus you may use it for testing the right angles of a table-top, of a door, of the *outside* of a box. It is not advisable to use it for testing the *inside* right angles of a box, or of a drawer, or of a door-frame. (Strictly speaking, it ought not to have been used for testing the right angle round the flag-staff or telegraph-post.) For testing inside right angles, we use an architect's **set-square,** a flat triangular piece of wood with a true right angle. You will be given two of these to work with, a little later on.

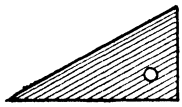


Fig. 65

5. You have learnt that, when you want to find out if a surface is a true plane, you must test it with an accurately made straight-edge of some kind. If, on the other hand, you are doubtful about the accuracy of a straight-edge (an ordinary ruler, for instance), you can test it by applying it to a plane known to be true. Thus, a true straight-edge may be used for testing a plane, and a true plane may be used for testing a straight-edge. *One* must be true, and then it may be used for testing the other.

(*When numbered statements in dark type are followed by the letter "L", the statements are to be* ***learnt****, perfectly.*)

6. A PLANE SURFACE (or a PLANE) is a surface in which a true straight-edge will everywhere fit exactly. (L.)

LESSON II

Horizontal, Vertical, and Oblique Lines and Planes

7. Borrow a **spirit-level** from the Geography Master, or from the school carpenter, and see if the floor of your room, the top of the table, the window ledge, and the mantelpiece are **horizontal** (perfectly level). (Your master will explain how the spirit-level is made and used.) If a plane surface is everywhere horizontal, the surface is called a **horizontal plane,** and straight lines drawn on that surface are **horizontal lines.** The surface of still water (in a basin, for instance) is a horizontal plane, and floating lead-pencils may be regarded as representing horizontal lines. The edge of a book-shelf, the edge of a table-top, the joints of floorboards, the line where the floor meets a wall, are other examples of horizontal lines.

8. You have probably seen a bricklayer use a **plumb-line**—a cord stretched straight by a hanging leaden weight. He uses it to see if the walls he is building are **vertical** (perfectly upright). Make a plumb-line for yourself, and see if your school walls are vertical. If they are vertical and if they are plane, their surfaces are **vertical planes.** Cover the plumb-line with chalk, hold it close to the wall and let it come to rest, then pull it out towards you a little way and let it go suddenly. It springs back and leaves a straight chalk-line on the wall. This straight line is a **vertical line.** The balusters on a stair-case, hanging chains, hanging ropes, telegraph poles, the lines where any two walls of a room meet each other, may all be regarded as representing vertical lines. Rain-drops fall in vertical lines, unless there is a wind. A telegraph post fixed in horizontal ground is both vertical and perpendicular; if fixed in a sloping bank, it is vertical but not perpendicular. Why?

Vertical lines always point downwards, towards the centre of the earth.

Butterflies alight with their wings in vertical planes, moths with their wings in horizontal planes.

9. Planes and straight lines which are neither vertical nor horizontal are called **oblique.** Oblique means slanting or sloping.

10. On the vertical surface of a wall, it is easy enough to draw both horizontal and vertical and oblique lines, but vertical and oblique lines cannot be drawn on a horizontal sheet of paper lying on the table. All lines on a horizontal plane are horizontal. Yet it would be inconvenient to have to draw lines on a sheet of paper which is pinned to the wall, though sometimes your master certainly does draw lines on a vertical blackboard. It has been decided, just as a matter of convenience when drawing, to represent the three different kinds of straight lines all on a horizontal plane, and in this way: horizontal lines, parallel to the top and bottom edges of your paper; vertical lines, parallel to the left- and right-hand edges of your paper; oblique lines, lines in any other direction.



Horizontal lines



Vertical lines



Oblique lines

But remember that, as long as your paper is lying on the horizontal table, it is not strictly true to say that the lines you draw on it are anything but horizontal. We do not obtain a true picture unless we hold the paper in a vertical plane (against the wall, for instance). Then the vertical lines may be made to appear *really* vertical.

11. HORIZONTAL PLANES are planes which are perfectly level. (L.)

12. HORIZONTAL LINES are straight lines in a horizontal plane. (L.)

13. VERTICAL PLANES are planes which are perfectly upright. (L.)

14. VERTICAL LINES are straight lines in a vertical plane that point downwards towards the centre of the earth. (L.)

15. Planes and straight lines which are neither horizontal nor vertical are called OBLIQUE. (L.)

LESSON III

Solids and Surfaces

16. Here is a brick. Measure it. It is 9″ long, 4½″ broad, 3″ thick. It is a **solid** body, but, in geometry, we call it a solid not because it is made throughout of a particular kind of hard stuff but because it occupies a certain amount of **space.** If the brick were hollow and made of paper, we should still call it, in our geometry lessons, a solid.

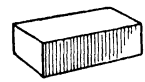
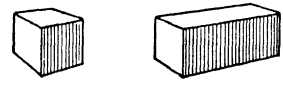


Fig. 66

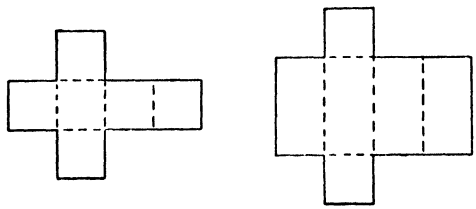
A room of a house is a solid; so is an empty box. Both have **length, breadth,** and **thickness.** But we do not usually speak of the *thickness* of a house or of a box. We say that a house has length, breadth, and **height,** and a box length, breadth, and **depth.** But all have three **dimensions;** that is, we can *measure* them from front to back, from side to side, and from top to bottom. (Both the word *dimension* and the word *mensuration* are derived from the same Latin word, *mensura*, a measure.)

17. Here are a **cube** (fig. 67, i) and a **square prism** (ii). You have probably seen them before, and know their names. If they were made of paper, we could run a knife along some of their edges and lay them out flat like this:



i ii

Fig. 67



i ii

Fig. 68

Plans of this kind are called the **nets** of the solids. Later on in the geometry book, you will find instructions how to cut out nets from stiff paper and how to fold and bind them up into the solids they represent.

18. The surface of both the cube and the prism consists of six **faces**. All six faces of the cube are **squares**. Only two faces of the prism are squares, the other four being **oblongs**. Sometimes we speak of the two square faces of the prism as **ends** or **bases**, and the four oblong faces as the **sides**. In each case, all the faces are, of course, planes. Any two adjoining planes of the cube or of the prism meet in an **edge**, or, as we sometimes say in geometry, the **two planes intersect in a straight line.**

19. Here are four more solids which you have probably seen before; a **square pyramid**, a **cylinder**, a **cone**, and a **sphere.**

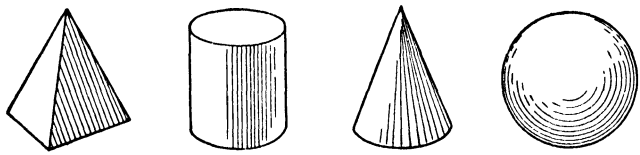


Fig. 69

20. In shape, the **square pyramid** reminds you of the famous Egyptian pyramid. Its surface consists of five plane faces, namely, one square base, and four triangular faces meeting in a point called the **vertex**. The vertex is exactly over the centre of the base * (fig. 69, i).

21. The **cylinder** reminds you of a garden roller, of a jam-jar, or of part of a pipe or tube. You can imagine it spinning on an axis. When rolled on the ground it runs in a straight line. The complete surface of the cylinder consists of two circular plane surfaces separated by a curved surface (fig. 69, ii).

22. The **cone** reminds you of the old-fashioned candle-

* It is convenient to be able to refer to the "centre" of a square, but it is not strictly correct. A circle has a true centre, so has a sphere.

extinguisher, or of the sugar loaf. It has a circular base, and it is so far like a pyramid that it has a vertex over the centre of the base. You can imagine it spinning on an axis. When rolled on the ground it runs round in a circle. The complete surface of the cone consists of one circular plane surface and a curved surface (fig. 69, iii).

23. The **sphere** reminds you of a ball of some kind, and it is a ball which is perfect in this way—the point called the centre is exactly the same distance from every point on the surface. You can imagine it spinning on an axis, like the earth. When rolled on the ground, it will run in any direction. The surface of a sphere is everywhere curved (fig. 69, iv).

24. When we speak of a " cylindrical surface " or of a " conical surface ", we usually refer to only the *curved* surface of the cylinder or cone. It is important to notice that this curved surface of the cylinder and the cone is very different from the curved surface of a sphere. If you place a sphere upon a plane (say a table), it touches the plane in a **point.** If you allow a cylinder or a cone to lie with its curved surface on a plane, it touches the plane in a **line.**

25. We can make nets of a cylinder and a cone, but not of the sphere. Here are nets of a square pyramid, a cylinder,

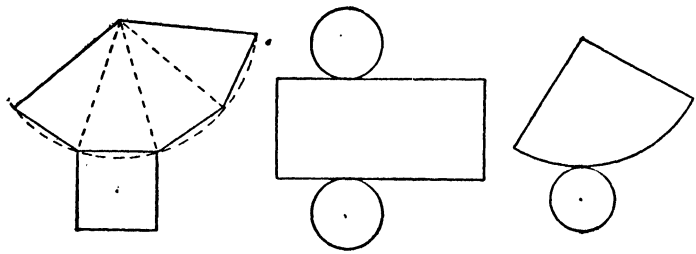


Fig. 70

and a cone, but to make models from the nets of the last two is a little difficult.

26. You have learnt to recognize a **square,** an **oblong,** a **triangle,** and a **circle.** All these are called **plane figures,**

because each encloses, within a boundary line, part of a plane surface. **All plane figures have closed boundary lines.** The letter O and the letter D are geometrical figures, but not the letter C or the letter W. **Straight-lined** figures like squares and triangles are called **rectilineal figures.** A circle is a **curved** figure.

27. A PLANE FIGURE is part of a Plane, and it is separated from the rest of the Plane by a boundary line. (L.)

28. A PLANE RECTILINEAL FIGURE is a straight-lined figure on a Plane. (L.)

29. RECTI-LINEAL means straight-lined. (L.)

LESSON IV

Angles

30. If I stand facing the east and the drill sergeant says "left turn", I turn and face the north, and I have then turned through a right angle. If he repeats the order, I turn to the west, and I have then turned through another right angle. If he repeats the order twice more, I turn and face south and then turn and face east, by which time I shall have turned through four right angles. I have made one complete **rotation** (Lat. *rota* = a wheel). Note the little arrow showing my first quarter-rotation or right angle.

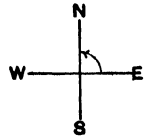


Fig. 71

Evidently an angle may be smaller or greater than a right angle. Whenever you look at a clock, the two hands are making an angle with each other. In fact, they are making new angles with each other all day long. Even when they are exactly together they have just completed a new angle and are just beginning to make others.

31. On paper an angle is represented by two lines meeting in a point. The two lines are called the **arms** of the angle, and the point where the two arms meet is called the **vertex**

of the angle. The same angle may have long arms or short arms. If a big clock and a little watch are both keeping correct time, the angles between their hands are always exactly the same. An angle always represents an amount of **movement**, namely, the movement of **rotation**. One arm shows where the rotation began, and the other where it finished, and you must always think of an angle in this way.

32. You can measure angles of different sizes fairly well by opening and closing your dividers, but the joint prevents you from making an angle of a whole rotation. A more convenient form of **angle-measurer** is necessary, and you may make one in this way. Take two nicely planed strips of wood, say about 12″ long, $\frac{1}{2}$″ wide, $\frac{1}{8}$″ thick, and pivot them together, something like your dividers, by means of a tiny brass bolt with rounded head and nut, generally obtainable for a penny or two from the ironmonger's. If you cannot obtain these things, two strips of cardboard will do, pivoted on a long drawing-pin, head downwards, with a protecting bit of cork over the point.

Place your angle-measurer on the table before you, the vertex O to the left, the two arms OA, OB together as if

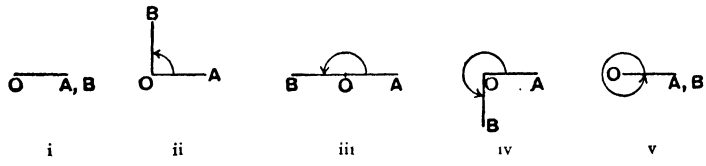


Fig. 72

they were both pointing to III on a clock-face (fig. 72, i). Keep the under arm OA fixed, and rotate the upper arm OB. Rotate it in an anti-clockwise direction (this is the custom in geometry), and make angles equal to one, two, three, and four right angles. Draw the four angles, and in each case show the amount of rotation by means of little curved arrows (fig. 72, ii, iii, iv, v).

33. In measuring different quantities, weights and measures for instance, big units like tons and miles are some-

times inconvenient. We do not weigh our tea in tons or measure our pencils in miles; we use smaller units like ounces and inches. So with angles. A right angle is a rather big unit, and sometimes we use a smaller unit called a **degree.** If we make a right angle as before, but move OB into position gradually, in ninety equal steps, each of these steps is **an angle of one degree.** It is a very small angle, too small to be shown clearly on paper unless we make the arms very long. The figure shows that even an angle of 5 degrees is very small.

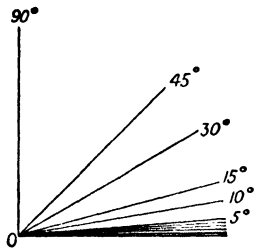


Fig. 73

The sign for " degree " is a little circle placed at the top right-hand corner of the given number. Thus for " 35 degrees " we write, " 35° ".

34. We may make up a little table:

90 degrees make a right angle,
2 right angles make a straight angle,
2 straight angles make a perigon.

A perigon is an angle of one complete rotation (*peri* = round; *gon* = angle). It is equal to four right angles, or 360°. A **straight angle** contains 180° (§ 32, fig. 72, iii).

We choose the number 360 for the perigon simply because it is the number which contains many useful factors (2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180). Any other number would do, but it would be less useful. The French use the number 400; they prefer to divide up the right angle into 100 degrees (they call them *grades*) instead of 90.

35. Note the number of degrees in the angles of fig. 74. The dotted lines show the right angles, and help the eye to estimate the numbers of degrees.

Practise drawing angles of different sizes, and estimating the number of degrees. The most important angles of all are 30°, 45°, 60°, 90°, 180°. The easiest to make is, of course

an angle of 90°. Divide it into two equal parts as accurately as you can, and so obtain 45°. It is pretty easy to divide,

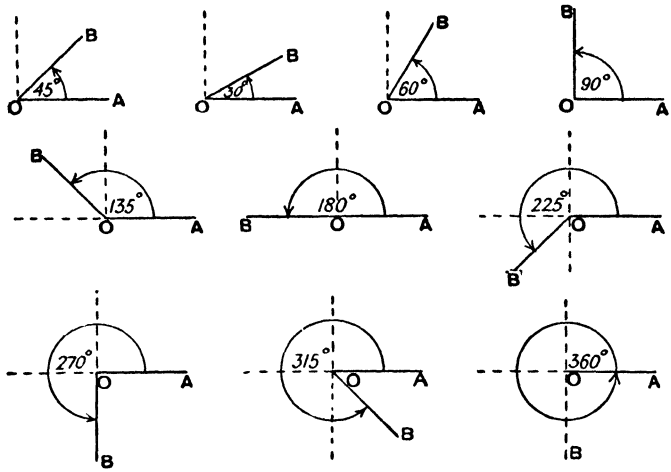


Fig. 74

with fair accuracy, 45° into three equal parts, in order to obtain 15° and 30°. And so on.

But guesswork will certainly not always do. It is often necessary to draw given angles accurately, and for this purpose you must use a **protractor.** A protractor is a semicircular * piece of brass or celluloid, with numbers from 0° to 180° round the circumference, in both directions, and by means of it you can make an angle of any size.

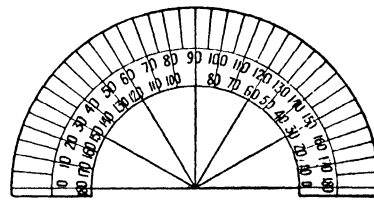


Fig. 75

Suppose you have to draw a line, at say, 55° with a given line. Place the straight diameter of your protractor against the given line, in such a way that **the marked midpoint**

*A surveyor's protractor is circular, and is numbered up to 360°. But in geometry we do not very often require angles greater than 180°.

of the diameter is against that end of the line that is to be the vertex of the angle. At the number 55 on the circumference, mark a point on the paper. Remove the protractor, and through that point draw the second arm of the angle.

But, you will say, there are *two* 55's on the circumference. How are we to choose between them? That is easy, for you know that 55° is less than a right angle, and you choose the 55 which will give you such an angle. The other 55 would be used if the vertex of the angle had to be at the other end of the line.

36. When two straight lines stretch out from one point, like two spokes from the hub of a wheel, they form an angle. (L.)

37. The two lines are called the arms of the angle, and the point where they meet is called the vertex of the angle. (L.)

38. An angle always shows a certain amount of ROTATION round the vertex, one arm showing where the rotation began, the other arm showing where it ended. (L.)

39. A PERIGON is an angle of one complete rotation. (L.)

40. A STRAIGHT ANGLE is an angle of a half rotation. (L.)

41. A RIGHT ANGLE is an angle of a quarter rotation. (L.)

42. AN ANGLE OF ONE DEGREE is an angle of $\frac{1}{360}$ part of a rotation.

LESSON V

Surveyors and their Work

43. A surveyor's work is to measure up land, and to draw plans and maps. For measuring lengths, he uses a long chain of 100 links. For measuring angles, he uses an

angle-measurer which is like yours in this respect—that it consists of two pivoted arms; but it is much more elaborate than yours, for he has to measure angles very accurately. He also uses a levelling-staff, to help him measure differences of level. A levelling-staff is merely a pole, graduated to show heights above the ground.

44. Here is a problem in which the necessary angle measurements may be correctly and easily made with one of your set-squares. To solve it you must make a drawing to scale.

A *and* B *are two towns* 20 *miles apart. Another town* C *is* 60° *east of north from* A *and* 30° *west of north from* B. *Draw a plan to show the position of* C, *and give its distances from* A *and* B.

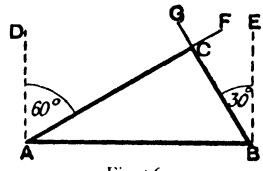


Fig. 76

The line AB is 20 miles long, and we have to draw it to a suitable scale. A scale of $\frac{1}{8}''$ to the mile would do. Thus we make AB 20 eighth-inches, or $2\frac{1}{2}''$, long.

If C were exactly north of A, it would be somewhere in the line AD. But it is 60° *east* of this line, and we therefore make the angle DAF equal to 60°. Again, if C were exactly north of B, it would be somewhere in the line BE. But it is 30° *west* of this line, and we therefore make the angle EBG equal to 30°. We know now that the town C lies on both AF and BG. But the only place where it can lie on both is where they meet. Hence, mark this point, C. We have thus found the *position* of C.

To find the *distances* CA and CB, we measure them to scale. CA is nearly $17\frac{1}{3}$ eighth-inches long, and CB is 10 eighth-inches long. Thus C is $17\frac{1}{3}$ miles from A and 10 miles from B.

But this problem was a problem on paper. No part of the work was done with measuring instruments in the field. Let us come back to the surveyor.

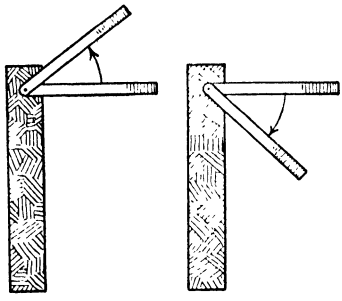
Sometimes a surveyor works on level ground, and has to measure angles in a horizontal plane. Sometimes he works

on hilly ground and has to measure angles in a vertical plane.

45. Measuring an angle in a horizontal plane.—Suppose you are standing at a place P in a field, and you imagine a line drawn from your eye to each of two distant trees, T_1 and T_2. What is the angle between the lines? Set up a table at P, with a piece of drawing-paper pinned on it. (A camera tripod stand with a drawing-board fixed on it horizontally about the height of your top waistcoat button would do nicely.) Place an angle-measurer on the table, swing one arm round to point to T_1, and the other round to point to T_2. Hold the arms firmly and draw the two angle lines (against the *inside* of the arms), remove the measurer, and with your protractor find the number of degrees in the angle.

(Ask the geography master to show you his *plane-table* and to explain how he measures angles made by distant objects. With his angle-measurer pivoted to the centre of a circular protractor on the table, he is able to read at once any angle made by the two arms.)

46. Measuring an angle in a vertical plane.—Pivot your angle-measurer to the side of a short post, or to the side of a stout stick thrust vertically into the ground, in order that the arms may swing in a vertical plane. An ordinary drawing-pin makes a poor pivot, for it is then difficult to make the arms remain in a particular position. An angle-measurer made of wood, with a fairly tight wooden or metal pivot, is much more satisfactory than the pivoted cardboard strips. **An angle measured in a vertical plane is always an angle with a horizontal arm;** the other arm points upwards or downwards as may be necessary.



Angle of elevation Angle of depression

Fig. 77

If you are on low ground and want to measure the angle made by, say, a cottage at the top of the hill, point the one arm of your angle-measurer upwards to the cottage, and measure **the angle of elevation.** If you are on high ground, say the top of a cliff, and want to measure the angle made by a boat in the water below, point the one arm of your angle-measurer downwards to the boat, and measure the **angle of depression.** Since, for measuring different angles, the arms may have to swing round in different vertical planes, it is an advantage to be able to turn the post round in the ground, and it should therefore have a rounded point, something like the point of a cricket-stump, prolonged.

(Ask the geography master to show you his *clinometer*, and to explain how he reads, from the cardboard protractor, angles of elevation and depression. Try to understand the use of the little plumb-line, and observe the pivot on which the protractor turns.)

47. How can I find the width of a river which I cannot cross?—To solve this problem we have to measure angles in a horizontal plane.

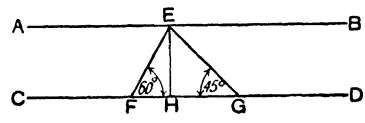


Fig. 78

Let AB and CD represent the two banks. I note some object E on the opposite bank, and I measure any length FG, say 100 yards, on the near bank. Then I measure the horizontal angles at F and G, in each case pointing one arm of my angle-measurer along CD and the other arm to the object E. I note that angle EFG = 60°, and angle EGF = 45°. Now I am ready to make a drawing to scale. A scale of 1″ to 50 yards seems convenient, so that FG = 2″. The width of the river is shown by a perpendicular EH drawn from E to FG. Measuring EH to scale, I find it is very nearly **63½ yards.**

48. How can I find the height of the flagstaff in the school field?—To solve this problem I have to measure an angle in a vertical plane, and as one arm of my angle-measurer

will have to point upwards, the angle will be an angle of elevation.

Let AB be the flagstaff, and let CD be the post to which my angle-measurer is attached: a convenient height of this attachment is 4′ above the ground. The post may be fixed **at any measured distance** from the flagstaff, say 20′. Thus DB = 20′. The horizontal arm of the angle-measurer points to E in the flagstaff; E is therefore 4′ above the ground. The other arm points to the top of the flagstaff. I now measure the angle ACE, and find it is 60°. Now I am ready to make a drawing to scale, say 1″ to 10′, so that CE (= DB) = 2″, angle ACE = 60°, angle AEC = 90°. The length of AE, measured to scale, is 34·6′. Hence AB = 34·6′ + 4′ = **38·6′.**

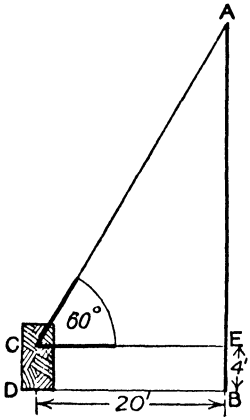


Fig. 79

(The length BE is exaggerated in the printed figure.)

49. The next time you see a surveyor at work, ask him to show you the instruments he uses for measuring horizontal and vertical angles, and to explain how he is able to measure even very small fractions of a degree. Also ask him to tell you something about his levelling-staff and his chain.

LESSON VI

Symmetry

50. Stand in front of a looking-glass, with a book (or some other object) in your right hand. In the glass you see an **image** of yourself, but the image holds the book in his left hand. Close your left eye; the image closes his right eye.

Hold open your right hand in front of the glass, and look at the image of the palm. Compare this image with the

palm of your real left hand. They are exactly alike. For instance, the two thumbs point in the same direction.

Thus **the image of your right hand is a left hand.** In short, your two hands are not in all respects alike; each is the " image " of the other.

51. Place a pair of gloves side by side on the table, backs upwards, thumbs touching. Each is the image of the other. Turn the left glove inside out; it has become a right-hand glove. You now have two right-hand gloves, no longer images of each other but **like** each other.

52. Fold a sheet of white paper, like a sheet of note-paper, and smooth down the crease. Open again, and let a drop of ink fall in the crease. Now fold, and press the folded paper fairly hard, to make the ink run and form a pattern. Open; the two half-patterns are right- and left-handed; each is the image of the other. When the paper is folded on its crease and held up to the light, the two half-patterns are seen to fit over each other exactly.

53. Right- and left-handed patterns that can be folded exactly together in this way, and are thus images of each other, are said to be **symmetrical.** The dividing line represented by the crease is called the **axis of symmetry.** We say that the doubled pattern is **symmetrical with respect to the axis.**

54. Take another sheet of paper, and fold as before. Let a drop of ink fall inside, but at some distance from the crease. Press down the doubled paper, and so form ink figures. The figures are images of each other and fold together exactly as before, but this time they do not touch the crease (the axis). That does not matter. They are still **symmetrical with respect to the axis.**

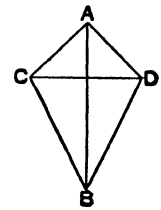


Fig. 80

55. In the accompanying figure (a kite), the line AB is evidently an axis of symmetry, for the half ACB can be folded over on AB and be made to fit exactly on the other half. The one half is the image of the other. Hence **corresponding lines on**

the left and right must be equal in length. If we hold the doubled paper up to the light, we can *see* that the lines are equal. **Corresponding angles must also be equal.**

56. ABC and DEF are two figures symmetrical with respect to the axis MN. Hence, if we fold on MN, the figures will fit together exactly. On ABC, mark the two points G and H, fold over, and prick through G and H, on DEF. On opening out we shall find the two points K and L in **positions corresponding exactly** to G and H, and KL is evidently equal to GH.

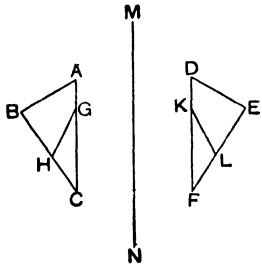


Fig. 81

57. Every point, every line, every angle, in one of two symmetrical figures has an image in the other. The image always corresponds exactly to the original. The two have exactly corresponding positions.

58. Thus a point and its image are always equidistant from the axis. In the last figure, for instance, A and D are equidistant from MN, for they come together when the figures are folded about the axis. Hence if we join AD, the axis must bisect AD.

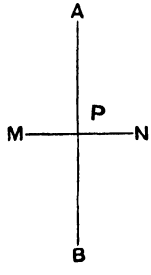


Fig. 82

59. Fold a piece of paper and mark the crease as an axis AB. On one side of the axis, make a point M. Fold, and prick through M to obtain its image N. Join MN. PM = PN (by § 58). Angle APM = angle APN (by § 57). But angle MPN is a straight angle, and thus the equal angles APM and APN are both right angles. Also, the angles vertically opposite these are equal. Hence all four angles at P are right angles. We see now that the axis not only bisects MN but is perpendicular to it, that is, the axis is the **perpendicular bisector** of MN. Observe that whenever you fold a sheet of paper a second time, as when you put it into an envelope, you make two axes of symmetry

perpendicular to each other, the four perfect right angles fitting exactly together in the envelope. When the paper is opened out, you see the complete perigon they form.

60. Fold a piece of paper, and then fold a second time, thus obtaining two axes of symmetry, MN and PQ, and four right angles. The four divisions are sometimes called **quadrants.** Prick through all four thicknesses of the folded paper, in four or five points not in the same straight line. Open out, and join up the points, in the same manner, in the four quadrants, thus making four figures. Convince yourself that both MN and PQ are really axes, by first folding on MN, holding up to the light and seeing that the left and right halves of the whole fit, then folding on PQ and seeing that the upper and lower halves fit. Join any point, say A, to its image on the other side of each axis, namely to B on the other side of MN, and to C on the other side of PQ. Observe that the axes are the perpendicular bisectors of the respective joining lines, MN of AB, and PQ of AC. So it is generally.

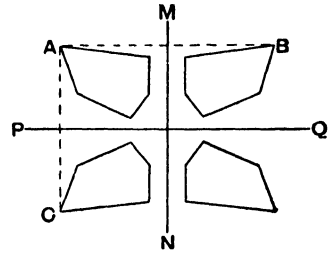


Fig. 83

61. When two figures are symmetrical with respect to an axis, they are right- and left-handed, and when they are folded about the axis, they fit together exactly. (L.)

62. An axis of symmetry is the perpendicular bisector of the line joining any point on one side of the axis to its image on the other side.

LESSON VII

Congruent, Symmetrical, Similar

63. We often require a word to describe two figures which are alike in all respects—corresponding lines the same length, corresponding angles the same, areas the same, appearances the same.

When two figures are exactly alike in all respects, and can be made to fit exactly together, they are said to be **congruent.** (*Congruent* means *exactly agreeing.*) Here are three pairs of congruent figures.

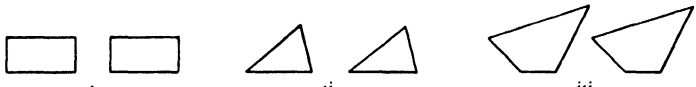
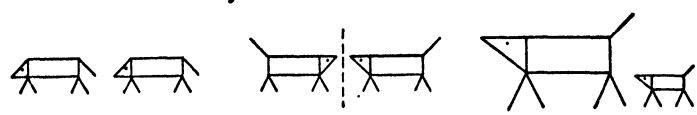


Fig. 84

64. Symmetrical figures are exactly alike in all respects save one: they are right- and left-handed. To make two symmetrical figures fit exactly together, we have to turn one of them over through a straight angle (180°), round the axis of symmetry. It is like picking one up, turning it upside down, and putting it down again. Then the two will fit exactly.

Strictly speaking, we ought not to call symmetrical figures congruent, because they are not alike in **all** respects; they are right- and left-handed. But it has become customary to call even symmetrical figures congruent, because they *can* be made to fit exactly if one is turned over.



i. Congruent (but not symmetrical)

ii. Symmetrical (and congruent)

iii. Similar (neither congruent nor symmetrical)

Fig. 85

65. But **similar** is another term altogether. Similar

figures are figures of **the same appearance, irrespective of their size.** (See fig. 85.)

66. Congruent figures which are not symmetrical may be made to fit together exactly by *sliding* one over the other. But symmetrical figures cannot be made to fit by sliding; one has first to be turned over.

You might say that since congruent figures are alike in appearance, we might call them similar. That is true, but in geometry we do not usually apply the term similar to congruent figures unless they are of different sizes.

67. Notice **two important things about similar figures:** (1) all the **angles** in the one are equal to the corresponding angles in the other; (2) the **proportions** in the one are equal to the proportions in the other. (If, for instance, the big pig's tail is one-third the length of his back, the little pig's tail is one-third the length of *his* back.) You will learn more about " proportions " later on.

68. CONGRUENT figures are figures exactly alike in all respects. One can be made to slide over the other and fit. (L.)

69. SYMMETRICAL figures are right- and left-handed congruent figures. To make them fit, one has to be turned over through 180°. (L.)

70. SIMILAR figures are figures of different sizes, but they have the same appearance, the same proportions, and the same angles. (L.)

LESSON VIII

Classifying and Defining

71. When we arrange a number of things in separate classes, we are said to **classify** them.

We may, for instance, arrange all school exercise-books in two quite distinct classes, namely, *ruled* and *unruled.* Such a classification is good. But suppose we say that all the people in London are either *males*, or *females*, or

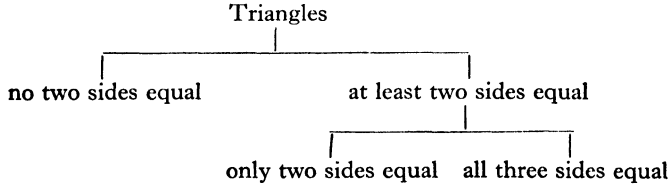
Australians. The classification is bad, for the Australians have been included twice over; they are all males or females.

Here is another example of a good classification. In a certain school, the 100 boys in Form IV are grouped in four divisions, according to the languages they learn in addition to English and French.

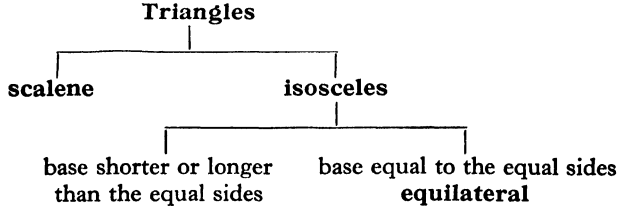
Form IVa learn both Latin and Greek, but not German.
Form IVb learn both Latin and German, but not Greek.
Form IVc learn Latin, but not Greek or German.
Form IVd learn German, but not Latin or Greek.

There are four distinct divisions. Every boy is included once, and only once.

72. Now we will classify triangles. All triangles are either *isosceles* or *scalene*. But isosceles triangles are of two kinds, those with two sides equal, those with all three sides equal. Thus we may arrange the classes in this way:



or we may arrange in this way:



73. Defining.—A **thing** (it may be a dog or it may be a triangle) has a **name**, and that name is a **word.** In order to say what that word **means**, we have to make a short statement which will show how the **thing** is dis-

tinguished from all other things. That short statement is a **definition.** We **define** a **word,** and the definition must include **the leading property of the thing.**

We begin by thinking of the class to which the thing belongs. Suppose, for instance, we have to define a **chair.** To what class of things does a chair belong? Evidently to the class **articles of furniture.** Thus we may begin by saying,

A chair is an article of furniture . . .

Now we have to pick out the particular property which distinguishes a chair from all other articles of furniture. What is a chair specially used for? For sitting on. Thus we may now say,

A chair is an article of furniture for sitting on.

But **benches, sofas,** and **stools** are also used for sitting on. How are we to distinguish a chair from these? Benches and sofas are made for more than one person to sit on. So we may say,

A chair is an article of furniture for one person to sit on.

But this might apply to a **stool.** How are we to distinguish? A chair has a back, a stool has not. We therefore say,

A chair is an article of furniture for one person to sit on and to lean back against.

Again, define **a pair of compasses.** To what class does it belong? Mathematical instruments. What is its special use? For drawing circles. Thus we make up the definition:

A pair of compasses is a mathematical instrument for drawing circles.

74. Define a **triangle.**

To what class does it belong? Plane rectilineal figures.

What property distinguishes it from all other plane rectilineal figures? It has three sides. Therefore we say,

A triangle is a plane rectilineal figure with three sides.

Define an **isosceles triangle.**

To what class does it belong? Triangles.

What distinguishes isosceles triangles from the other great class of triangles (scalene)? Equality of the two sides from the vertex to the base.

Therefore we say,

An isosceles triangle is a triangle in which the two sides from the vertex to the base are equal.

Define an **equilateral triangle.**

To what class does it belong? Isosceles triangles.

What distinguishes it from other isosceles triangles? The base is equal to each of the other two sides.

Therefore we say,

An equilateral triangle is an isosceles triangle in which the base is equal to each of the other two sides.

Another definition of an equilateral triangle is sometimes given: *an equilateral triangle is a triangle with three equal sides.* But this definition is not so good as the other.

75. We might, if we liked, classify triangles according to their angles, and ignore their sides. The sum of the three angles of a triangle is 180°. Hence, if a triangle has an obtuse angle, the other two angles must be acute; or if it has a right angle, the other two angles must be acute; if it has neither an obtuse angle nor a right angle, all three angles must be acute. Thus we have a new classification: **All triangles are either obtuse-angled triangles, or right-angled triangles, or acute-angled triangles.**

But do not mix up the two classifications of triangles. That would take us back to the Australians!

Such lessons are easily within the range of very young boys.

Some teachers are, however, curiously afraid of the principle of symmetry, urging that it does not lend itself to

strictly deductive proof. Personally I would use it very much more for teaching even advanced geometry; I always did in my teaching days. For elementary work at all events, it is a singularly useful weapon. Though proof by means of it is difficult for beginners to set out, it produces *conviction* in the beginner, a great gain.

It will be observed that, for framing definitions, we have used the old device *per genus et differentiam*. This is probably the only safe method for beginners. From schoolboys we must be satisfied with something much less than perfection in their definitions. In particular, do not worry about " redundant " definitions. In the early stages they are inevitable; they are then almost to be encouraged. It is much better to let a young boy say that " a rectangle is a right-angled parallelogram " than " a rectangle is a parallelogram with a right angle ". A beginner naturally regards the latter with suspicion. It is doubtful wisdom ever to ask a boy to define a *straight line* or an *angle*. He has clear notions of these things already, and these notions he cannot express in language that is entirely satisfactory. If a boy says that " a straight line is the shortest distance between two points ", strictly the definition is unacceptable, because of the vague term " distance ". If he adds *as tested by a stretched string*, we should feel that the idea in his mind was clear and distinct; and what more can we want from *him?* As for an *angle*, I have often asked boys for a definition, not because I expected a satisfactory one, but in order to show them that, whatever definition they put forward, it was open to criticism. Who has ever defined either a *straight line* or an *angle* satisfactorily? Again: ordinarily we distinguish between a circle and its circumference, and a useful distinction it is. And yet we all talk about drawing a *circle* to pass through three points. However, matters of this kind are not for beginners but for the Sixth Form, which is the proper place for a final polishing up of all such things.

Working Tools for Future Deductive Treatment

These consist of the familiar propositions concerning:

1. Angles at a point.
2. Parallels.
3. Congruency.
4. Pythagoras.,
5. Circles; such properties as can be established from considerations of symmetry.

The formal proof of Pythagoras is easily mastered in the Fourth Form, but proofs of the other theorems may wait until the Sixth. Meanwhile all the propositions must be thoroughly known as geometrical *facts*, facts which can readily be used and referred to in all subsequent work. Although formal proofs are beyond beginners, the probable truth of the propositions must be substantiated in some way. *Justification* is always possible at this stage, though rigorous proof is not. Most of the more recent textbooks provide " practical " proofs of a kind which to the beginner really do seem to justify the claims made by the theorems. Here, little need be said about such proofs.

First considerations of *angles at a point* naturally arise when the nature of an angle itself is being discussed. Acute, obtuse, adjacent, reflex, complementary, supplementary, and vertically opposite angles may all be brought into an early lesson, provided that the rotational idea of the angle is clearly demonstrated. Angles up to 360° should be considered from the first.

Here is a first lesson on **parallels** and transversals.

Parallel Lines and Transversals

76. You have already learnt that the blue lines on the pages of your exercise books are **parallel,** that is, **they run in the same direction and are always the same distance apart.** When we speak of " distance apart " we mean the

shortest distance, and that distance is represented by a perpendicular from one line to the other. But can we be quite sure that a line which is perpendicular to one of the parallel lines is also perpendicular to the other?

77. A line that is drawn across two or more other lines is called a **transversal** (*trans* means *across*). Draw a transversal PQ across the parallel lines AB and CD, cutting AB in M and CD in N.

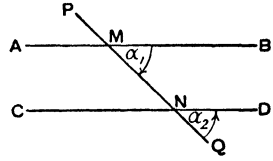


Fig. 86

Imagine a man to walk along AB and, on reaching M, to turn to the right and walk along MN. He has turned through the angle α_1, for he was first walking towards B, and is now walking towards Q. On reaching N, let him turn to the left, and walk along ND. He has now turned through the angle α_2, for he was walking towards Q and is now walking towards D. But now that he is walking along ND he is walking **in the same direction** as when he was walking along MB. Hence the angle he turned through on reaching N is equal to the angle he turned through on reaching M, that is, the angle α_2 is equal to the angle α_1.

We might have expected this, for the two angles α_1 and α_2 **look** alike. They are called **corresponding angles.**

78. When a transversal is drawn across two parallel lines, the corresponding angles are equal. (L.) Hence,

79. A transversal which is perpendicular to one of two parallel lines is also perpendicular to the other. (L.) Conversely,

80. If two lines are both perpendicular to a transversal, they are parallel to each other. (L.)

81. Just as we showed that the corresponding angles α_1 and α_2 are equal, so we may show that the corresponding angles β_1 and β_2 are

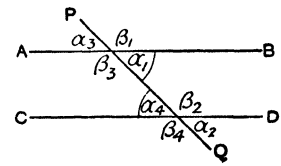


Fig. 87

equal; also α_3 and α_4; also β_3 and β_4. (Fig. 87.) But α_2 and α_4 are also equal, because they are vertically opposite angles.

Since	$\alpha_1 = \alpha_2$,	(§ 78)
and since	$\alpha_2 = \alpha_4$,	
therefore	$\alpha_1 = \alpha_4$.	

The angles α_1 and α_4 are on **opposite** sides of the transversal, and are called **alternate** angles.

Similarly it can be shown that the alternate angles β_3 and β_2 are equal.

82. When a transversal is drawn across two parallel lines, the alternate angles are equal. (L.)

83. Observe that, in the eight marked angles of the last figure, there are four pairs of opposite angles, four pairs of corresponding angles, two pairs of alternate angles, every pair being equal. The four α's are equal, and the four β's are equal.

The four angles **between** the parallel lines are called **interior** angles.

The four angles **outside** the parallel lines are called **exterior** angles.

The following is very important:

	$\alpha_1 + \beta_3$ = a straight angle,	
	= two right angles.	
But	$\beta_3 = \beta_2$,	(§ 82)
therefore	$\alpha_1 + \beta_2$ = two right angles.	

Similarly we may show that $\alpha_4 + \beta_3 =$ two right angles.

84. When a transversal is drawn across two parallel lines, the two interior angles on the same side of it are together equal to two right angles, that is, they are supplementary. (L.)

Considerations of *congruency* are best led up to by actual practical work on the construction of triangles from given data. One lesson is enough for the boys to discover that a triangle can be described if

(1) the 3 sides are given,
(2) 2 sides and the included angle are given,
(3) 1 side and 2 angles are given;

and that therefore two triangles are congruent if there is correspondence and equality of

(1) 3 sides,
(2) 2 sides and the included angle,
(3) 1 side and 2 angles.

Further than this with beginners it is unnecessary to go.

As for *Pythagoras*, it is enough to give beginners one or two of the many well-known dissection figures.

Here $T = W + V$.

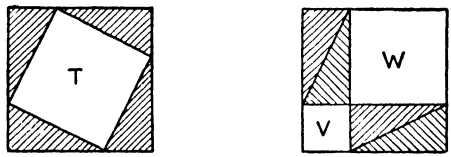


Fig. 88

Here the big square is cut up to form the two little squares,

$P = Q = R = S = P' = Q' = R' = S'$,
$M = M'$.

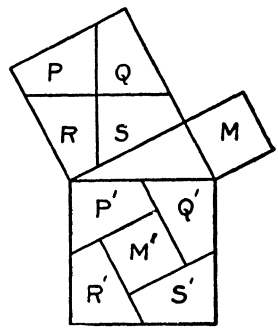


Fig. 89

The fact must be emphasized that in these early stages any attempt at formal *proof* is out of place. Nevertheless adequate *reasons* may be found, and should be provided, in support of all statements made concerning these fundamental propositions.

Here is a lesson on the **centre of the circle as a centre of symmetry.**

85. The centre of the circle as a centre of symmetry. —Fold a circle on a diameter AB as an axis of symmetry, prick through the two halves at C, open out and call the corresponding points C_1 and C_2, join C_1 and C_2 and join each to the centre O, thus forming the isos. $\Delta\ C_1OC_2$. The arc of the sector AC_1OC_2 and the arc of the segment AC_1C_2 are the same.

Take a piece of celluloid (a piece of tracing-paper will do, if you use it carefully), pin it down over the circle (fig. 90, i) by means of a pin thrust through it and through the

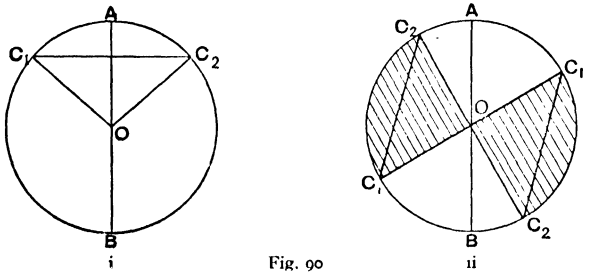


Fig. 90

centre O, and trace on the celluloid the radii OC_1, OC_2, the chord C_1C_2, and the arc C_1AC_2; now rotate the celluloid round the pin (fig. 90, ii).

Since the traced sector and segment on the celluloid preserve their shape and size while rotating, all lengths and angles remain unchanged, and the arc C_1AC_2 is seen always to fit exactly on the circumference below it. We say that the rotating sector and segment are **symmetrical with respect to the centre of the circle,** because of this exact fitting during the whole of a rotation. Thus the length of the arc, the length of the chords, the angle between the radii, all remain constant. We see all this plainly in fig. ii, where the rotating sector and segment are shown in two positions. Hence:

86. Equal chords in a circle are equidistant from the centre. (L.)

87. Equal chords in a circle are subtended by equal arcs. (L.)

88. Equal angles at the centre of a circle are subtended by equal chords and by equal arcs. (L.)

89. If two chords of a circle are equal, they cut off equal segments. (L.)

90. All these things (§§ 86–89) which apply to one circle also apply to equal circles, since equal circles will fit together exactly.

Early Deductive Treatment

Do not expect any rigorous logic from beginners. We suggest a lesson easily within the comprehension of young boys at the end of their First Year. Note: (1) the proofs though of the simplest kind are enough to convince young boys; (2) there is a logical grouping of the different kinds of parallelograms; (3) the gradual extension of the properties, as the variety within the species becomes more particularized, is brought out. This gradual extension of properties should always be borne in mind in the teaching of geometry.

Quadrilaterals as Parallelograms

91. A quadrilateral is a plane rectilineal figure with four sides. (L.) There are different kinds of quadrilaterals. We will begin with the parallelogram.

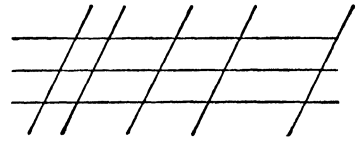


Fig. 91

92. Draw a few parallel transversals across the parallel lines of your notebook. You see a number of four-sided figures with their opposite sides parallel. These are **parallelograms.** (*Gram* means *line.*)

How can we define a parallelogram? First put it into its class. (§ 73.)

A parallelogram is a quadrilateral . . .

What special property distinguishes a parallelogram? Its opposite sides are parallel. Hence the definition:

A parallelogram is a quadrilateral with its opposite sides parallel. (L.) Now let us discover the other properties of a parallelogram.

93. Any side of a parallelogram may be regarded as a transversal across two parallel lines. Let ABCD be a parallelogram, with the four angles marked as shown.

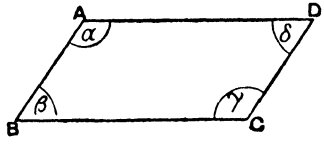


Fig. 92

$\alpha + \beta = 2$ rt. $\angle$s, (§ 84)
$\beta + \gamma = 2$ rt. $\angle$s; (§ 84)
$\therefore \alpha + \beta = \beta + \gamma.$
$\therefore \alpha = \gamma.$
Similarly, $\beta = \delta.$

Thus, **the opposite angles of a parallelogram are equal.** (L.)

94. Join two opposite vertices by a line. Such a line is called a **diagonal.** This diagonal (AC) is a new transversal to both pairs of parallel lines. Mark the two pairs of equal alternate angles, a_1, a_2, β_1, β_2, and the pair of equal opposite angles, γ_1, γ_2. In the two $\angle$s ABC, ADC, the diagonal forms a side belonging to both, and the three angles of the one are equal, respectively, to the three angles of the other. Hence the two Δs are congruent.

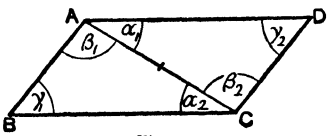


Fig. 93

$\therefore$ AB = CD and AD = BC.

Thus, **the opposite sides of a parallelogram are equal.** (L.)

95. Draw the *two* diagonals, intersecting at E; and consider the two triangles AEB, CED. In these Δs, the three $\angle$s of the one are equal to the three $\angle$s of the other (§ 82), and AB = CD (§ 94). Hence the Δs are congruent.

$\therefore$ AE = CE and BE = DE.

Thus, **the diagonals of a parallelogram bisect each other.** (L.) (Of course we might have used the other Δs AED and BEC.)

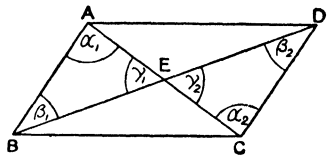


Fig. 94

96. We may now collect up the various properties of a parallelogram:

In any Parallelogram,

(1) **The opposite sides are parallel.** (*known from the definition*)
(2) **The opposite angles are equal.** (*proved*)
(3) **The opposite sides are equal.** (*proved*)
(4) **The diagonals bisect each other.** (*proved*)

97. Imagine a $\square$m to be jointed at the four angles, and let it move from position A to position C. It remains a

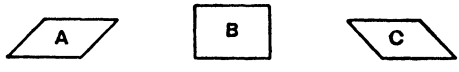


Fig. 95

$\square$m all the time, and therefore keeps all its properties. But on its journey to C it passes through B, where the angles are right angles. A parallelogram with right angles is called a rectangle. **A RECTANGLE is a right-angled parallelogram.** (L.) Since a rectangle is a $\square$m, it has all the properties of a $\square$m (§ 96); and it has certain additional properties:

98. (i) **The angles of a rectangle are right angles.** (This follows from the definition.)

99. (ii) Draw the two diagonals of the rectangle ABCD, and examine the two Δs ABC and DCB (which, it will be seen, partly overlap and have a common base). The two sides AB, BC are equal to the

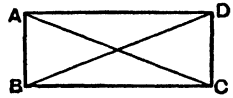


Fig. 96

two sides DC, CB (§ 96, 3); and the included angles ABC and DCB are equal, both being right angles. Hence the Δs are congruent. Therefore AC is equal to BD. Thus, **the diagonals of a rectangle are equal.** (L.)

100. Suppose a rectangle gets shorter and shorter until its length and breadth are equal. It remains a rectangle



Fig. 97

all the time and therefore keeps all its properties. When the length exceeds the breadth, as in A and B, the rectangle is called an **oblong**; when its length and breadth are equal, it is called a **square.**

101. An OBLONG is a rectangle with its length exceeding its breadth. (L.)

102. A SQUARE is a rectangle with all four sides equal. (L.) Since a square is a rectangle, it has all the properties of a rectangle (§§ 97–99); and it has certain additional properties:

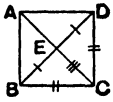


Fig. 98

103. (i) **All four sides of a square are equal.** (This follows from the definition.)

104. (ii) Draw the two diagonals of the square ABCD, intersecting in E, and examine the two Δs BEC and DEC.

BE = DE, (§§ 95, 97)
BC = DC, (§ 103)
EC is common to both Δs;
$\therefore$ ΔBEC $\equiv$ ΔDEC;
$\therefore$ $\angle$BEC = $\angle$DEC;
$\therefore$ each $\angle$ = $\frac{1}{2}$ st. $\angle$ or 1 rt. $\angle$.

Similarly we may show that each of the other two $\angle$s at E are rt. $\angle$s. Hence, **the diagonals of a square bisect each other at rt. $\angle$s.** (L.)

105. We still require names for the two non-rectangular parallelograms. The non-rectangular parallelogram with all four sides equal is called a **rhombus** (fig. 99, i). The non-

rectangular parallelogram with only its opposite sides equal is called a **rhomboid** (ii). The rhomboid is the parallelogram we began with (§§ 93–96). It is the **most general** form of parallelogram. If we made all its sides equal, or if we made all its angles right angles, we should make it a **particular** kind of parallelogram.

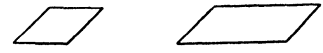


Fig. 99

106. Just as an **oblong** may be reduced in length and made a **square,** so a rhomboid may be reduced in length and made a **rhombus.**

Just as a square has all the properties of an oblong and certain additional properties, so a rhombus has all the properties of a rhomboid and certain additional properties.

107. All four sides of a rhombus are equal. (This follows from the definition.)

108. The diagonals of a rhombus are the perpendicular bisectors of each other. (L.) This property may be discovered in this way. The word **rhombus** really means a *spinning-top.* If we stand it on an angle, it looks something like a spinning-top. We see at once that either diagonal is an axis of symmetry, B and D being corresponding points about the axis AC, and A and C being corresponding points about the axis BD. Thus each diagonal is the perpendicular bisector of the other.

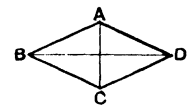


Fig. 100

But this also applies to a square. What is the *difference* between a rhombus and a square?

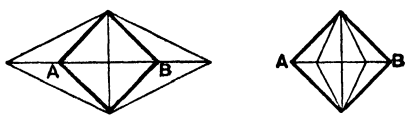


Fig. 101

109. If we lengthen (or shorten) equally the two halves of one of the diagonals, AB, of a square, we stretch out (or

contract) the square into a rhombus. The diagonals of a square are equal; those of a rhombus are unequal.

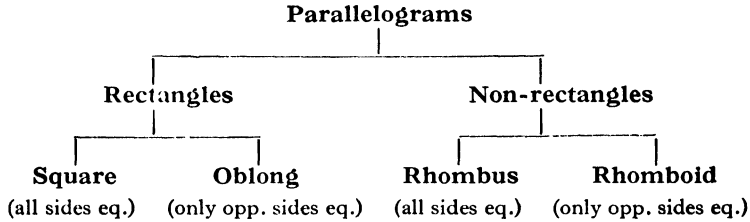
A rhombus **differs** from a square in these ways:

1. Its angles are not right angles.
2. Its diagonals are not equal.

A rhombus **resembles** a square in these:

1. The four sides are equal.
2. Each diagonal is the perpendicular bisector of the other.
3. Each diagonal is an axis of symmetry.

110. We may classify the four kinds of parallelograms in this way:



How easy it is to make up definitions from this scheme:

111. A SQUARE is a rectangular parallelogram with all four sides equal (§ 102).

112. An OBLONG is a rectangular parallelogram with only its opposite sides equal (§ 101).

113. A RHOMBUS is a non-rectangular parallelogram with all four sides equal. (L.)

114. A RHOMBOID is a non-rectangular parallelogram with only its opposite sides equal. (L.)

115. We might classify parallelograms according to their axes of symmetry. A **square** has four axes of symmetry, viz. two diagonals and two medians (a median is the line joining the middle points of opposite sides); an **oblong** has two,

viz. the two medians; a **rhombus** has two, viz. the two diagonals; a **rhomboid** has none. But in definitions we do not usually refer to symmetry; symmetry is useful mainly for discovering properties.

Remember that the two halves of a figure folded on an axis of symmetry will fit together exactly. Remember, too, that a figure can always be imagined to **spin** on an axis of symmetry.

116. More about definitions. We have defined a **square** as a **rectangle with all four sides equal.** Therefore,

(i) Since it is a rectangle, it has **right angles** and is a **parallelogram** (§ 97).

(ii) As it is a parallelogram, **its opposite sides are parallel** (§ 92). Thus, **our definition of a square tells us three things:**

1. **The four sides are equal,**
2. **The four angles are right angles,**
3. **The opposite sides are parallel.**

But the definition tells us nothing at all about the diagonals. Properties of the diagonals must be discovered either by congruence (§ 104) or by symmetry.

117. We *might* define a square as a **quadrilateral** with four equal sides and four right angles. But all that *this* definition tells us is that:

1. The four sides are equal,
2. The four angles are right angles.

It is quite a good definition, but it does not tell us that the square is a parallelogram, and therefore it does not tell us that the opposite sides are parallel. Hence this property is one we should have to find out (perhaps by congruence) if we used the new definition. Let us decide not to use it.

118. We can now classify the properties of parallelograms:

	Rhomboid.	Oblong.	Square.	Rhombus.
1. Opp. sides ||.	×	×	×	×
2. Opp. sides eq.	×	×	×	×
3. Opp. ∠s. eq.	×	×	×	×
4. Diags. bisect each other.	×	×	×	×
5. All four ∠s. rt. ∠s.		×	×	
6. Diags. eq.		×	×	
7. All four sides. eq.			×	×
8. Diags. at rt. ∠s.			×	×

Observe that the rhomboid is the most general of the parallelograms, and has fewest properties; and that the square is the most special of the parallelograms, and has most properties. No other parallelogram has all the properties of the square.

This lesson may usefully be followed up by the consideration of quadrilaterals that are not parallelograms.

Proportion and Similarity

A knowledge of proportion and similarity is so fruitful throughout the whole range of the study of geometry that the subject should be introduced at an early stage, though naturally incommensurables are then ignored entirely. We append a lesson suitable for the second year of the geometry course. Note the little device for constructing a triangle with sides simply commensurable. The proofs given are rigorous enough *at this early stage*. The important thing is to provide learners with a serviceable weapon—rough and unpolished, for the moment, it is true; but that is of no consequence.

(The nature of a ratio, of cross-multiplication, &c., has already been referred to in the chapters on arithmetic and algebra, but the three subjects should be brought into line when a principle common to them all is under consideration.)

119. Take a piece of paper ruled in $\frac{1}{4}''$ squares, and on it draw this triangle: the base AB of the Δ is to be on one of the ruled horizontal lines 4″ or 5″ down the paper, and the vertex in a parallel line 3″ above, i.e. in the twelfth parallel line above. Fix the point A towards the left-hand end of the line selected for the base, and with a radius of 3·6″ draw a circle to cut the top line in C. With C as centre, and with a radius of 4·5″, draw a circle to cut the base line in B. Join AB (its length does not matter) and so complete the ΔABC.

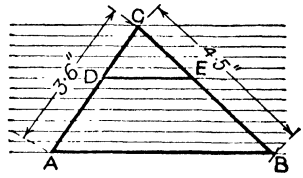


Fig. 102

Since AC = 3·6″, it can be divided into twelve equal parts of ·3″ each, and each division will fall on one of the ruled horizontal lines. Since CB = $4\frac{1}{2}''$, it can be divided into twelve parts of $\frac{3}{8}''$ each, and again each division will fall on one of the ruled horizontal lines. But the ruled lines are all **parallel** to each other. We therefore seem to have the following result:

120. If the two sides of a triangle are divided into the same number of equal parts, and the corresponding points of division in the two sides are joined, all the joining lines are parallel to the base. (L.)

It has been found that this result is always true, no matter how it is tested. But the real proof is too difficult for you to understand at present. The following particular case is often useful:

121. If two sides of a triangle are bisected, the line joining the points of division is parallel to the base. (L.)

122. We may now learn that if a line is drawn parallel to one side of a Δ, it cuts the other side proportionally. Consider, for instance, the fifth parallel DE, from the top in the figure to § 119. CD is $\frac{5}{12}$ of

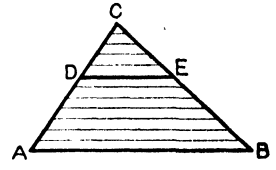


Fig. 103

CA, and DA is $\frac{7}{12}$ of CA; CE is $\frac{5}{12}$ of CB, and EB is $\frac{7}{12}$ of CB.

$$\therefore \frac{CD}{DA} = \frac{5}{7} \text{ and } \frac{CE}{EB} = \frac{5}{7},$$

$$\therefore \frac{CD}{DA} = \frac{CE}{EB}.$$

So with any other parallel. Or a part of a side may be compared with the whole. For instance,

$$\frac{CD}{CA} = \frac{CE}{CB},$$

for each is equal to the fraction $\frac{5}{12}$.

We can imagine the $\triangle CDE$ to be a small $\triangle$ fitting over the top of the larger $\triangle CAB$ (fig. 104, i), and CD being made to slide down CA so that the small $\triangle CDE$ occupies the

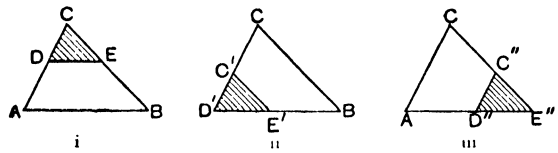


Fig. 104

position C′D′E′, D taking the place of A (fig. 104, ii). Just as the three $\angle$s of the small $\triangle$ are respectively equal to the three $\angle$s of the large $\triangle$ in fig. 104, i (see § 78), so they must be in the second, since corresponding angles are equal. Hence C′E′ is $\parallel$ CB, and therefore C′E′ cuts the two sides D′C and D′B proportionally; and just as C′D′ is $\frac{5}{12}$ of CD′, so D′E′ must be $\frac{5}{12}$ of D′B,

$$\text{or } \frac{D'C'}{C'C} = \frac{D'E'}{E'B}, \text{ or } \frac{D'C'}{D'C} = \frac{D'E'}{D'B}.$$

Similarly by making the little $\triangle$ slide down to the other corner (fig. 104, i to iii), so that E takes the place of B, C″D″ is $\parallel$ to CA, and therefore

$$\frac{E''C''}{C''C} = \frac{E''D''}{D''A}, \text{ or } \frac{E''C''}{E''C} = \frac{E''D''}{E''A}.$$

Hence:

123. If in a triangle a line is drawn parallel to any side, it cuts the other sides proportionally. (L.) (This is always true, but the real proof is too difficult for you to understand at present.)

124. We might detach the small $\triangle CDE$ from the large one CAB, and place the two side by side. They look alike. They *are* alike. They are **similar.** Although the sides of the two $\triangle$s differ so much in length, the three $\angle$s of the one are respectively equal to the three $\angle$s of the other. (Why?) In other words, the two similar $\triangle$s are **equiangular.** And we know already that the corresponding sides are **proportional.** This we should expect in similar figures of any kind. In a photograph of yourself, for instance, you would expect the " proportions " of your body to be accurately preserved. If the ratio of the lengths of your outstretched forearm and upper arm is $\frac{5}{3}$, you would expect that ratio to be preserved in the photograph (or the photographer would probably hear about it!).

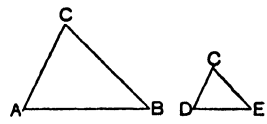


Fig. 105

125. If, then, ABC and DEF are two similar $\triangle$s, the corresponding sides are proportional. But note that we may express the ratios in two different ways: (1) **two sides of one $\triangle$ as a ratio** equal to the ratio of the corr. two sides of the other $\triangle$; (2) **one side of one $\triangle$ and the corr. side of the other $\triangle$ as a ratio** equal to the ratio of any second side of the first $\triangle$ and the corr. side of the second $\triangle$. Consider, for instance, the two sides AB, BC in the $\triangle ABC$, and the two corr. sides DE, EF in the $\triangle DEF$. We may say,

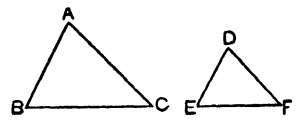


Fig. 106

$$\textit{either}\quad \frac{AB}{BC}=\frac{DE}{EF},\quad \textit{or}\quad \frac{AB}{DE}=\frac{BC}{EF}.$$

The two proportional statements are really the same thing

since we obtain the same product from the cross-multiplication of either:

$$AB \,.\, EF = BC \,.\, DE. \quad (\textit{The full stop is used instead of } \times.)$$

It is often an advantage to interchange one form for the other; really we interchange the second term of the first ratio and the first term of the second.

We have learnt that:

126. SIMILAR TRIANGLES are equiangular, and their corresponding sides are proportional. (L.)

127. When expressing ratios between two sides of each of two similar Δs, be careful to select **corresponding**

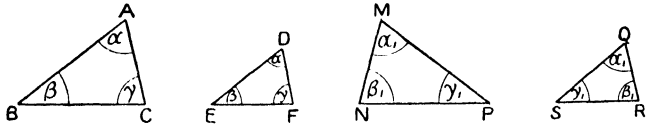


Fig. 107

sides, i.e. sides taken in the same order round corresponding angles. In these two pairs of Δs, the corresponding $\angle$s are marked with the same Greek letters. From the first pair we may equate ratios thus, six equations in all:

$$\frac{AB}{BC} = \frac{DE}{EF}; \frac{AB}{AC} = \frac{DE}{DF}; \frac{AC}{CB} = \frac{DF}{FE}; \frac{AB}{DE} = \frac{BC}{EF}; \frac{AB}{DE} = \frac{AC}{DF}; \frac{AC}{DF} = \frac{BC}{EF}.$$

From the second pair, we may do exactly the same thing:

$$\frac{MN}{NP} = \frac{QR}{RS}; \frac{MN}{MP} = \frac{QR}{QS}; \frac{MP}{PN} = \frac{QS}{SR}; \frac{MN}{QR} = \frac{NP}{RS}; \frac{MN}{QR} = \frac{MP}{QS}; \frac{MP}{QS} = \frac{PN}{SR}.$$

Yet there **appears** to be a difference. That is because in the second pair the Δs are right- and left-handed. If you have any doubt, turn one of the pair over, through 180°, as you would turn over a page of a book. Then the pair will look alike. But if you mark the corresponding angles correctly, you ought to have no difficulty.

The sides of the triangles may usefully be named by means of single small letters; then the writing of the ratios is simplified; e.g. $\frac{c}{a}$ instead of $\frac{AB}{BC}$. The small letter selected for a side is always the same as the capital letter naming the opposite angle.

128. Sometimes each of a pair of similar Δs is similarly divided by a perpendicular from a vertex to the opposite side. The resulting pair of Δs in the one case are evidently similar, respectively, to the resulting pair in the other case, for they are equiangular, i.e. Δs ABG and DEH are similar, and Δs ACG and DFH are similar (check, by sum of angles). We may reason in this way:

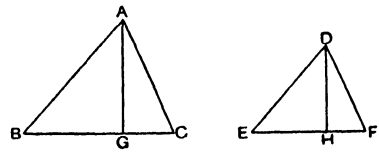


Fig. 108

In the Δs ABG, DEH, since $\frac{AG}{DH} = \frac{AB}{DE}$

and in the Δs ABC, DEF, since $\frac{AB}{DE} = \frac{BC}{EF}$

$$\therefore \frac{AG}{DH} = \frac{BC}{EF}.$$

An exchange of ratios may often usefully be made in this way. From this particular exchange we learn that the altitudes AG, DH are proportional to the bases BC, EF. Hence:

129. When similar triangles are divided by perpendiculars drawn from corresponding vertices to opposite sides, an exchange of ratios may often be usefully made. (L.)

130. In similar triangles, the altitudes are proportional to the bases. (L.)

131. We know that when two ratios are equated to form a proportion, they may be cleared of fractions by cross-

multiplication. For instance, in the two similar △s ABC, DEF, we know that

$$\frac{AB}{BC} = \frac{DE}{EF}; \qquad \textit{(an equation consisting of 2 ratios)}$$

$$\therefore AB \,.\, EF = DE \,.\, BC. \qquad \textit{(an equation consisting of 2 products)}$$

What does this mean? AB, BC, DE, EF, all represent **lines** of a particular length; a length multiplied by a length gives an **area.** Thus each of the two products AB . EF and DE . BC represents a **rectangle.**

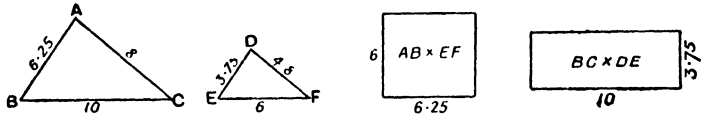


Fig. 109

Note that we begin with ratios, i.e. with **quotients,** representing a length **divided** by a length. After cross-multiplying, we have **products** representing **areas,** or a length **multiplied** by a length. (The measured lengths are shown to scale. Check the numerical ratios and the products. For instance, are $\frac{6{\cdot}25}{10}$ and $\frac{3{\cdot}75}{6}$ equal? and are $10 \times 3{\cdot}75$ and $6 \times 6{\cdot}25$ equal?)

132. By cross-multiplication, two equated ratios of lengths give two equated rectangular areas. ("Equated" means expressed as an equation.)

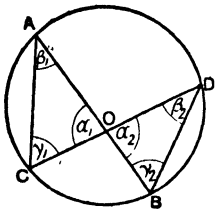


Fig. 110

133. The last result is useful in all sorts of ways. For instance, AB and CD are two chords of a circle, intersecting at O. Join AC and DB, and we have two similar △s, the △s being equiangular (angles in the same segment; see § 126). Taking ratios (see § 127) we have,

$$\frac{OA}{OC} = \frac{OD}{OB};$$

$$\therefore \text{ rect. } OA \,.\, OB = \text{rect. } OC \,.\, OD.$$

Hence, **if two chords intersect in a circle, the rectangle contained by the two segments of the one is equal to the rectangle contained by the two segments of the other.** (L.) (The term " segment " here applies to the *parts* of the chords.)

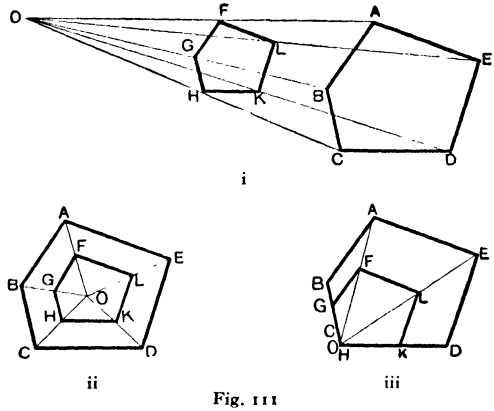


Fig. 111

The proportional division of lines and the construction of similar figures should follow at a slightly later stage. Be sure that the boys master the principle exemplified in these three figures: for the construction of the similar pentagon the position of the point O is quite immaterial.

The centre of similarity problems are readily followed by those on centre of similitude. Insist on the point that any two circles may be regarded as similar figures, since, like rectilineal similar figures, they may be looked upon as the same figure drawn to different scales.

Circles and Polygons

The ordinary properties of the circle give little trouble—angles in a segment, the cyclic quadrilateral, tangents, alternate segment property, circles in contact, and intersecting circles. Do not forget to group properties around a

common principle; e.g. (i) the tangent to a circle, (ii) the external common tangent to two circles, (iii) the transverse common tangent to two circles, should be taken in that order, and be made to follow on the key proposition that the angle in a semicircle is a right angle. All these propositions on the circle being quite simple, formal proofs should now be consistently exacted.

Regular polygons, too, need give little trouble. Their angle properties are interesting, easy to understand, and always appeal to a boy. The pentagon excepted (see the next section), they are not much wanted. The hexagon and octagon involve the simplest geometry, easy work for beginners. The decagon is easily constructed from the pentagon, and the dodecagon from the hexagon. The heptagon and nonagon are hardly ever used; the latter is easily constructed from its angle properties; the former is not, inasmuch as its angle properties involve fractions of a degree and hence some sort of approximation method is required for its construction. The best is probably the following, especially as it is common to all polygons.

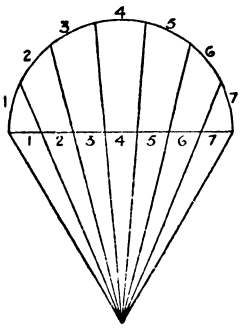


Fig. 112

On one side of a straight line draw a semicircle and on the other side an equilateral triangle. If the line be divided into x equal parts, and lines be drawn from the apex of the triangle through the points of division, to meet the semicircle, the semicircle is divided into the same number of parts as the line. This is not susceptible of proof, simply because it is not mathematically true, but the approximation is so near that the most careful measurement usually fails to detect an error. Architects generally use it. Evidently by drawing radii from the points of division of the semicircle, we divide 180° into x equal parts.

The conversion of polygons (regular and irregular) into

triangles, triangles into rectangles, and rectangles into squares, which is often wanted, is simple straightforward work, though some little practice in manipulating the figures is necessary. To the beginner, a polygon with one or more re-entrant angles is puzzling.

Golden Section and the Pentagon

We append the following lesson as an example of linking up different Euclidean propositions (II, 11; IV, 10, 11) and of utilizing algebra in solving geometrical problems.

134. To divide a line into two parts so that the rectangle contained by the whole and one part is equal to the square on the other part.

This is sometimes stated:

To divide a line in medial section.
or, **To divide a line in extreme and mean ratio.**
or, **To divide a line in golden section.**

The problem is very easy to do and to understand if we can solve easy quadratic equations. It is the *kind* of problem in which algebra can help us much.

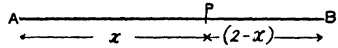


Fig. 113

Let AB be the line to be divided, and let it be, say, 2″ long. Suppose the point of division is P.

Let AP be x inches long; then PB $= (2 - x)$ inches long.

The line has to be divided so that AB . BP $= AP^2$,

$$\text{i.e. } 2(2 - x) = x^2.$$

We must now solve this equation, and find the value of x

$$x^2 = 2(2 - x);$$
$$\therefore\ x^2 + 2x = 4.$$
$$\therefore\ x^2 + 2x + 1 = 5.$$
$$\therefore\ x + 1 = \pm\sqrt{5}. \quad \text{(We may neglect the minus sign.)}$$
$$\therefore\ x = \sqrt{5} - 1,$$

i.e. AP = $(\sqrt{5} - 1)$ inches. Can we measure off this length and so find P? Yes, by the theorem of Pythagoras. We do it in this way:

Erect a ⊥ BC at B, 1″ long, and join CA.

$$AC^2 = (AB^2 + BC^2) = (2^2 + 1^2) = 5;$$
$$\therefore AC = \sqrt{5},$$

i.e. AC is $\sqrt{5}$ inches long (fig. 113*a*, i).

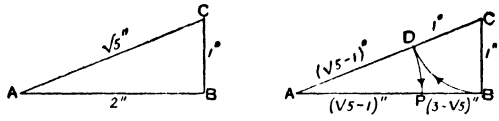


Fig. 113*a*

But we require a line $(\sqrt{5} - 1)$ inches long.

Since CB = 1″, with centre C and radius CB, cut CA in D (fig. 113*a*, ii); CD = 1″.

Thus AD = $(\sqrt{5} - 1)$ inches.

But we require a part of AB equal to $(\sqrt{5} - 1)$ inches.

Hence, with centre A and radius AD, cut AB in P; AP = $(\sqrt{5} - 1)$ inches.

Thus P is the point required.

The length of PB is evidently $2 - (\sqrt{5} - 1)$ in., i.e. $(3 - \sqrt{5})$ inches.

If we have done "surds" in algebra, we can show that the result is correct: AB . BP has to be equal to AP^2. Now AB . BP = $2(3 - \sqrt{5}) = 6 - 2\sqrt{5}$; and $AP^2 = (\sqrt{5} - 1)^2 = 6 - 2\sqrt{5}$, as before.

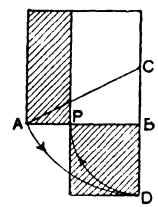


Fig. 114

Here is Euclid's figure.—He does not cut off a piece from CA; he makes CD equal to CA, so that CD = $\sqrt{5}$, and BD = $\sqrt{5} - 1$. Then he makes BP equal to BD = $\sqrt{5} - 1$, so that P is found as before, except that PB is now the longer instead of the shorter section. The shaded parts of the figure show the rectangle AB . AP $2(3 - \sqrt{5})$; and the square on PB, $(\sqrt{5} - 1)^2$.

Now examine a regular pentagon and its 5 contained diagonals. Give the boys a few hints (such as the following) and then leave them to construct the pentagon themselves.

(1) *Angles.*—$a_1 = a_2 = a_3$; hence it is clear that the 15 angles at the 5 vertices of the pentagon are all equal, and that each = 36°.

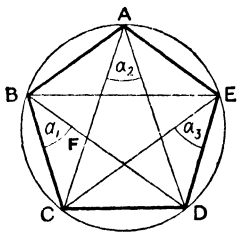


Fig. 115

(2) *Lines.*—Each diagonal is divided by 2 others into 3 parts. Is there any relation between the parts? e.g. does CF bear any relation to FA or to the side CD?

Draw the triangle ACD and the line FD separately, and write in all the angles. Evidently AFD and DFC are isosceles triangles. ∴ AF = FD, CD = FD, ∴ AF = CD. Hence if we put a circle round the triangle AFD, CD is a tangent (relation

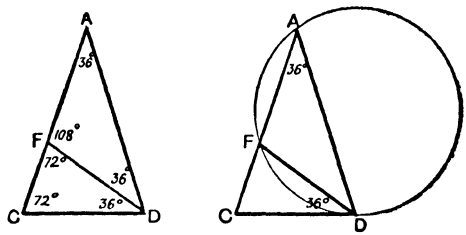


Fig. 116

= ?); also CA is a secant; ∴ CF . CA = CD^2; ∴ CF . CA = FA^2, i.e. CA is divided at F in golden section.

To construct a pentagon, therefore, we begin by drawing any line AC, and dividing it in golden section in F. With A as centre and AC as radius, we draw a circle (not shown) and draw in it the chord CD equal to FA, and then join AD. This gives us the triangle ACD, round which we circumscribe a circle and so obtain part of fig. 115; to obtain the points B and E we bisect the angles ADC, ACD.

Teach the boys one or two special ways of drawing the pentagon; e.g. let them tie into a simple knot a strip of paper

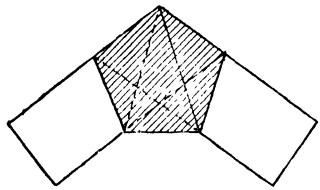


Fig. 117

of uniform width. It is a useful exercise to make them prove that the figure produced really is a pentagon.

The Principle of Continuity

This is an ambiguous term, for in each of several branches of knowledge it is given a special significance. Even in the single subject mathematics, it is used in different senses. One standard textbook of geometry states: " The principle of continuity, the vital principle of modern geometry, asserts that if from the nature of a particular problem we should expect a certain number of solutions, and if in any particular case we find this number of solutions, then there will be the same number of solutions in all cases, although some of the solutions may be imaginary. For instance, a straight line can be drawn to cut a circle in two points; hence we state that every straight line will cut a circle in two points, although these may be imaginary or may coincide. Similarly we may say that two tangents may be drawn from any point to a circle, but they may be imaginary or coincident." *

But in geometry the term " continuity " has come to be used more loosely than that. It is used to indicate *generality*, a generalizing of some fundamental principle, or the grouping of a number of allied instances around some central principle. We give a few instances of different kinds, from which the

* Lachlan, *Modern Pure Geometry.*

reader will see more clearly what is meant. As regards the teaching of geometry, the principle is one of the very greatest importance.

1. *The particularizing of a general figure and the extension of properties.* We have already given an instance of this in the lesson on parallelograms.

2. *Varying the figure to include different cases.* These three figures tell their own story. If the parts of such figures

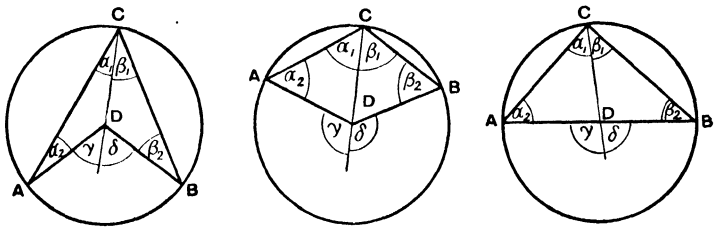


Fig. 118

are similarly named, as a rule exactly the same words apply in all cases to the proof. What difference there is is generally a difference of mere sign.

3. *Generalizing a term to include its natural extensions,*

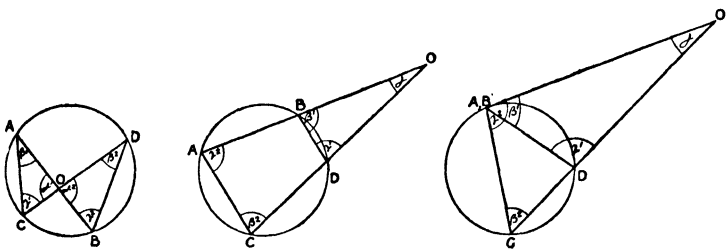


Fig. 119

e.g. a chord as a secant, and a secant as a tangent. From the case of intersection O *inside* the circle, we pass to the case of intersection *outside* the circle, and then from the two secants to a secant and a tangent. The three cases may first be separately

taken and then generalized. If the lettering is consistent, the arguments are identical, though for the tangent-secant case we should generally argue rather differently. In all three cases we have two similar triangles, OAC and OBD, and OA/OC = OD/OB.

Another general chord-secant-tangent property is seen in the following four figures, showing the measure of an angle

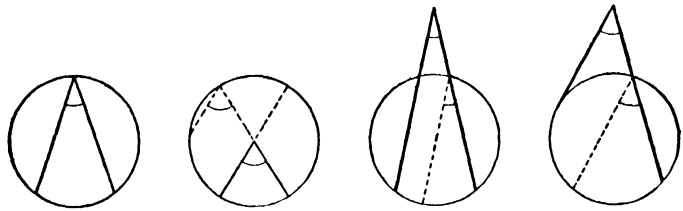
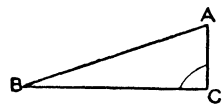


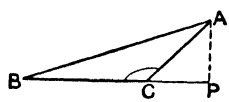
Fig. 120

inscribed in a circle by reference to the intercepted arcs; again the argument may be made perfectly general.

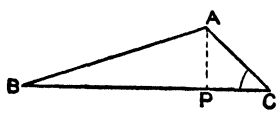
4. *The extensions of Pythagoras form another series.* It is the general custom nowadays to give the boys Pythagoras towards the end of their first year, to serve as a useful working tool; to give a formal proof during the second year, and to take the extensions (Euclid II, 12, 13) a few months later still; most boys are then familiar with the results in the following form.



$AB^2 = AC^2 + CB^2$ **exactly.**



$AB^2 = AC^2 + CB^2 + 2BC.CP$



$AB^2 = AC^2 + CB^2 - 2BC.CP$

where CP is the projection of CA on BC. (L.)

But figures to illustrate the extensions are less often provided. Here is a suggestion:

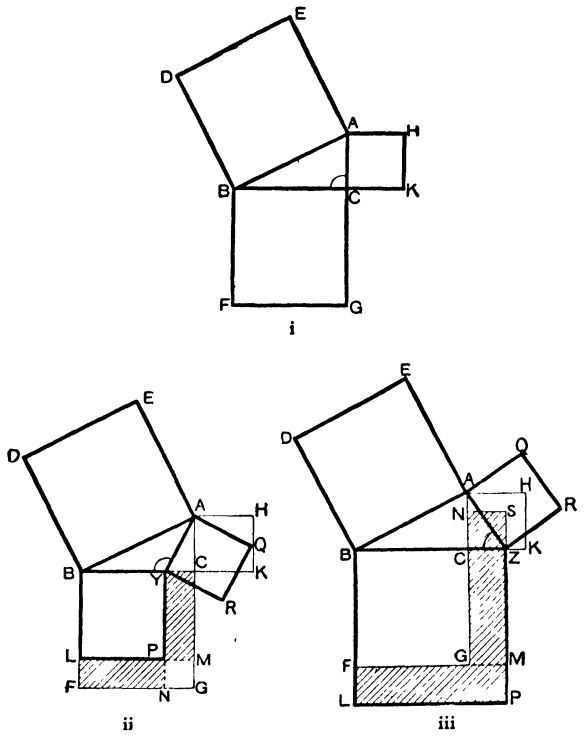


Fig. 121

Fig. (i) illustrates Pythagoras. In (ii) compare the squares on the new sides BY, YA with the squares on the old sides BC, CA. In (iii) compare the squares on the new sides BZ, ZA with the squares on the old sides BC, CA. The dissections are interesting, though they tend to puzzle slower boys.

5. *Summing the exterior angles of a polygon: "walking the polygon"*. First consider an ordinary convex polygon. Mark in the angles systematically: "always turn to the

left"; the angle to be worked is that between the old direction and the new.

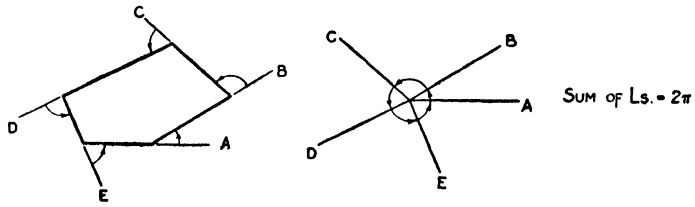


Fig. 122*a*

Secondly, a polygon with one re-entrant angle:

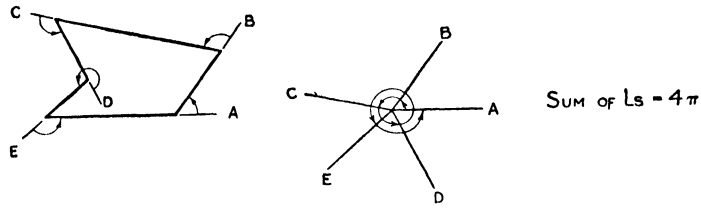


Fig. 122*b*

Thirdly, a cross polygon; also with one re-entrant angle:

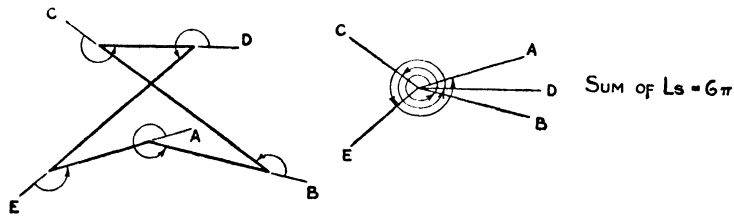


Fig. 122*c*

The point in this example is to see how exactly the same principle is followed out: always turn to the *left*, always measure the exterior angle between the old direction (produced) and the new. The result must always be a multiple of 2π.

6. *Euclid, Book II.* Given an algebraic basis, suitable

figures, and a rational grouping, props. 4 and 7, 5 and 6, 9 and 10, can be taught in a single lesson. *Never* make the boys go through the Euclidean jargon; life is not long enough.

7. *The Sections of a Cone* (for more advanced pupils). Let a plane perpendicular to the plane of the paper rotate round the point P, first cutting the cone ABC parallel to the base; then obliquely to cut the slant surface; then more obliquely, parallel to AC and cutting the base; then perpendicularly to the base and cutting the base. Since the motion may be regarded as continuous, we should expect no sudden changes in the properties of the curves made by the rotating plane as it cuts the cone. Why should there be? The boys' knowledge of geometry ought by this time to make them revolt against the idea of any *fundamental* differences in the properties of the curves. The curves may all be described as conics possessing certain common properties. In particular positions the curves have certain additional and special properties, but the common properties will remain. Let the boys understand that for convenience we study the curves separately first, and collectively later. But make them see at the outset that the circle is just a particular case of an ellipse, just as the ellipse is a case of the more general conic. The elliptic orbit of the earth, for example, is so very nearly a circle that a correct figure drawn on paper is virtually indistinguishable from a circle. Astronomical figures are often purposely exaggerated.

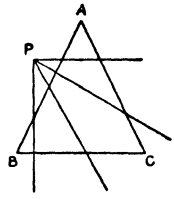


Fig. 123

8. *The Polyhedra* (see Chapter XXXVIII). These form an even better illustration of the principle of continuity than those already cited.

The principle applies, in fact, to the whole range of geometry. To deal with a proposition as an unrelated unit is, generally speaking, to offend almost every canon of geometrical teaching.

The Principle of Duality

This is best exemplified by a few well-known pairs of theorems:

1. If the *sides* of a triangle are equal, the opposite *angles* are equal.

2. If two triangles have two *sides* and the included *angle* respectively equal, the triangles are congruent.

3. If a quadrilateral be *inscribed* in a circle, the sum of one pair of opposite *angles* is equal to the sum of the other pair.

4. If a hexagon be *inscribed* in a circle, the three *points* of intersection of pairs of opposite *sides* are *collinear*.

1. If two *angles* of a triangle are equal, the opposite sides are equal.

2. If two triangles have two *angles* and the included *side* respectively equal, the triangles are congruent.

3. If a quadrilateral be *circumscribed* about a circle, the sum of one pair of opposite *sides* is equal to the sum of the other pair.

4. If a hexagon be *circumscribed* about a circle, the three diagonal *lines* connecting opposite *angles* are concurrent.

Such pairs of propositions are said be *dual* or *reciprocal*.

There is, in short, a remarkable analogy between descriptive propositions concerning figures regarded as assemblages of points and those concerning corresponding figures regarded as assemblages of straight lines. Any two figures of which the *points* of one correspond to the *lines* of the other are said to be reciprocal figures. When a proposition has been proved for any figure, a corresponding proposition for the reciprocal figure may be enunciated by merely interchanging the terms *point* and *line*; *locus* and *envelope*; *point of intersection of two lines* and *line of intersection through two points*; &c. The truth of the reciprocal or dual proposition may usually be inferred from what is called " the principle of duality ".

The teacher should always be on the look-out for examples of this principle which gives boys so much insight into geometry. Numerous examples of concurrency and collinearity will occur to him at once. The principle is especially useful

in the treatment of more advanced work, for instance in the theory of perspective and in the theory of the complete quadrilateral (tetrastigms and tetragrams).

CHAPTER XXI

Solid Geometry

Preliminary Work

First notions of solid geometry will have been given in the Preparatory School. Even in the Kindergarten School the children are made acquainted with the shapes of common geometrical figures and solids. Lower Form arithmetic is closely linked up with practical mensuration, and quite young boys are made familiar with the methods of measuring up rectangular surfaces and solids. The practical mensuration associated with early measurements in physics forms another introduction to solid geometry. First notions of projection are given in early geography lessons; very young boys soon acquire facility in building up vertical cross-sections from contoured ordnance maps, and when projection is first formally taken up in the mathematical lessons, say in the Pythagoras extensions or in early trigonometry, the main idea is already familiar. All the way up the school, three-dimensional geometry in some form should be made to serve as a handmaid to the plane geometry. Indeed, first notions of the geometry of the sphere are required at a very early stage in the teaching of geography, and if these notions are to be properly implanted the mathematical Staff should make themselves responsible, for not all geography teachers are mathematicians.

Only a minority of boys acquire readiness in reading geometrical figures of three dimensions. With the majority,

the training of the geometrical imagination is a slow business. For the clear visualization of the correct spatial relations in an elaborate three-dimensional figure, or for that matter even in a simple one, models of some kind are, in the earlier stages, essential.

Supplies of useful little wooden models of the geometrical solids are often found in the physics laboratory, though why physics teachers so frequently relieve their mathematical colleagues of this particular work I have never been able to discover. If models in wood are not available, models may be readily cut from good yellow bar soap; the material is cleaner to handle than raw potato or clay or plasticine. By means of a roughly-cut model, the correct shape of a transverse section of a geometrical solid can be realized at once. Personally I prefer models made from " nets " of cartridge paper or thin cardboard; these are easy to make and are permanent, but the making consumes a good deal of time.

Useful skeleton models are readily made from pieces of long knitting needles, sharpened at each end and thrust into small connecting corks. Two slabs of cork to represent the Horizontal and Vertical Planes, tacked to a pair of hinged boards, and a few pointed knitting needles, make excellent provision for the first lessons on orthographic projection.

The natural sections of an orange, or the cut sections of a well-shaped apple, are useful when teaching the geometry of the sphere.

The small varnished wooden models of the cylinder, sphere, and cone, of the same diameter and height, are useful for showing, by displacement of water in a measuring jar, that the volumes are 3 : 2 : 1.

A slated sphere, mounted, should be part of the equipment of all mathematical teachers.

Even such a simple device as two intersecting sheets of paper, each sheet being slit half-way across, to show the intersection of two planes at any angle, is often useful.

But of course all these props should gradually be withdrawn, and the eye made to depend on two-dimensional

drawings. Still, it is always an advantage, even for the trained mathematician, to put a few shading lines into such drawings. They help the eye greatly.

Stereographic photographs, or even hand-made stereograms, are also a great aid in teaching solid geometry. These are easily provided, and stereoscopes are cheap. Mr. E. M. Langley used them with great effect as far back as the nineties.

Do not forget that even for plane geometry models may be useful. The pantograph is particularly useful when teaching similarity (see Carson and Smith's *Geometry*). When teaching loci, encourage the boys to make wooden or cardboard "linkages" to represent engineering motions and astronomical movements. The loci are then given a reality.

The boys should also be encouraged to make "nets" of the commoner geometrical solids, in cartridge paper or cardboard. Boys of 11 or 12 learn to make these readily, and at that age time can be spared. I have known boys of 10 make almost perfect paper models of the five regular polyhedra.

In naming triangular pyramids, name the vertex first, then the three corners of the base, thus, A.BCD. Note that *any* corner of such a pyramid may be regarded as a vertex, the other three being the corners of the base (just as any corner of a Δ may be regarded as a vertex, and the other two corners as the ends of the base).

A problem like the following is better understood if a prism is actually cut up, perhaps a wooden one made in the carpenter's shop; or one may be cut neatly from a bar of soap. It is easier to cut up the latter with a thin knife than to cut up the former with a saw.

A suitable model for showing that a prism is equal to three times the volume of a pyramid on the same base and of the same height is a little troublesome to make. Inasmuch as it is particularly useful in the demonstration of that important principle, we give a few hints for constructing it.

Fig. 124, i, represents the complete triangular prism, with bases ABC, DEF. From it, cut the pyramid E.ABC

by holding the knife (or saw) at E, and cutting through to AC. iii shows the pyramid cut off.

Now we have to cut the remaining piece (ii) into two other pyramids. Cut from it the pyramid C.DEF. To do

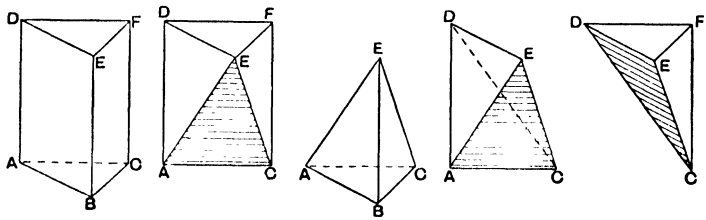


Fig. 124

this, hold the knife again at E, and cut down to DC. v shows the pyramid cut off, its new face being shaded. iv shows the part left. It is a curious-looking wedge-shaped pyramid. We will name it E.DAC.

We may show (since pyramids on equal bases and of the same vertical height have the same volume) that the three pyramids (iii, iv, v) are equal in volume. The bases of iii and v, E.ABC and C.DEF, are equal, since they are the bases of the prism; and the heights are equal, for FC = EB, and these are two of the long edges of the prism. Again, if we name iv and v E.ACD and E.FDC, we see that the bases are equal, for they are the halves of DACF, one of the faces of the prism; and their vertical heights are equal, since the two have a common vertex E. Hence the volumes of all three pyramids are equal.

It is a simple matter to make the "nets" (fig. 125) of the three pyramids, and fold them up to make models. The models may then be placed together to form the prism. Each net will, of course, consist of four triangles, the sides of all of which will be edges, or diagonals of the faces, of the prism.

The formal mensuration of geometry of the pyramid, then of the cone, then of the cylinder, is interesting and valuable,

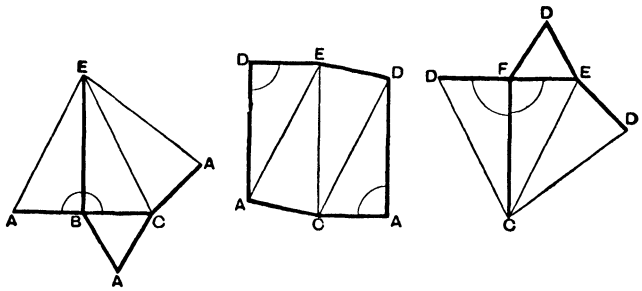


Fig. 125

and I do not find that it gives teachers much trouble, always provided that the necessary preliminary work from Euclid XI on lines and planes has been done well.

The calculation of the areas of the surfaces of solids is also simple, including even the surface of the sphere, provided that suitable figures are drawn.

Euclid XI

All the essential propositions from Euclid, Book XI, are now included in the leading schoolbooks on geometry. Most boys find the reasoning easy enough, but many have great difficulty in understanding the figures, unless models are available to help visualization.

It will suffice to touch upon Euclid XI, 4 and 6.

XI, 4. *If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane containing them.*

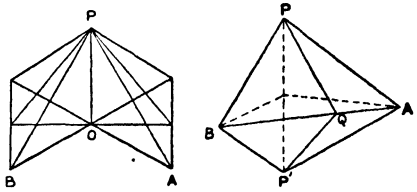


Fig. 126

The first of the following figures is Euclid's own, and to most boys it is incomprehensible. The second is that found in many modern textbooks. This is a case where a model is certainly desirable. Failing that, *two* figures should be drawn, from which the

different planes may easily be picked out. The following figures are suitable: in the first, the horizontal plane and the vertical planes are easily seen; in the second, the two oblique planes. If such figures are steadily gazed at, with one eye,

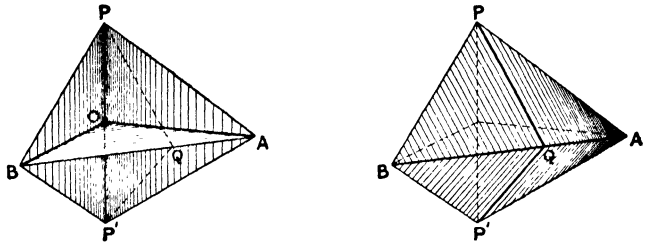


Fig. 127

through a very small hole in, say, a piece of cardboard, they quickly assume an appearance of three dimensions.

XI, 6. *If two straight lines are perpendicular to the same plane, they are parallel.*

The first figure is Euclid's (again a poor thing); the second is that commonly found in school textbooks. In

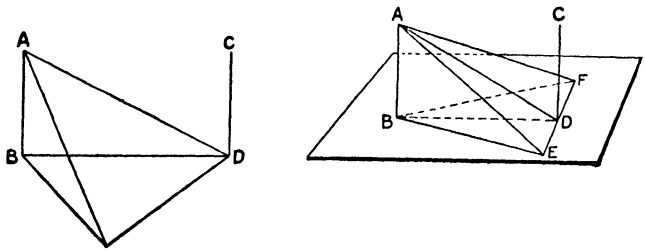


Fig. 128

this case, again, the planes want sorting out, to help visualization. Figure 129 is more suitable, with the horizontal plane shaded. The two perpendiculars AB, CD are shown by rather thicker lines. The two congruent triangles FDA, EDA in the oblique plane AFE are easily picked out; so are the two BDE, BDF in the horizontal plane. But it is so difficult to draw a figure that will show, to a beginner's eye,

the two congruent triangles ABE, ABF in their separate vertical planes, that either a wire model or a pair of stereograms are certainly desirable.

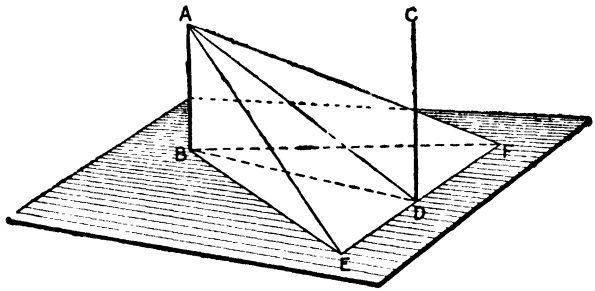


Fig. 129

It is unfortunate that so many boys experience difficulty in visualizing three-dimensional figures. But the fact has to be allowed for, and provision made accordingly.

Do not press too far the argument that such aids as models should be withdrawn in order that the boys' imagination may be given opportunity to develop. The boys' developed imagination will be a poor thing if it has to be nurtured on the teacher's badly-drawn figures.

CHAPTER XXII

Orthographic Projection

Elementary Work

Below we reproduce subject-matter suitable for two or three preliminary lessons on orthographic projection to the Middle Forms. Time can seldom be found for much ruler and compass work, but freehand drawings, rapidly executed

in association with the teacher's own blackboard demonstrations, may be made to serve a useful purpose in laying the foundations of the subject. Higher up the school, if time permits, more advanced work should be taken. It helps the ordinary geometry, plane and solid, greatly.

In preparing drawings for builders, architects make **plans** and **elevations** of buildings to be erected. A plan of a thing is an outline on a horizontal plane; an elevation is an outline on a vertical plane.

Push the table up against the wall. On the table place a rectangular block with two faces parallel to the wall. Chalk on the table an outline of the base of the block, and thus make a *plan* of the block. Now push the block against the wall, and chalk an outline on the wall. This is an elevation of the block; the elevation *a'b'c'd'* in the figure is a projection of the face *abcd* of the block.

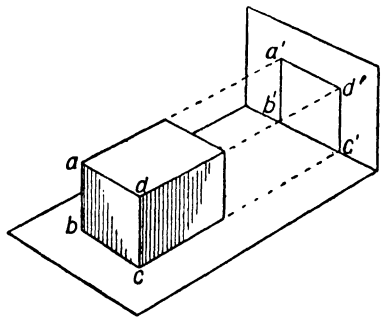


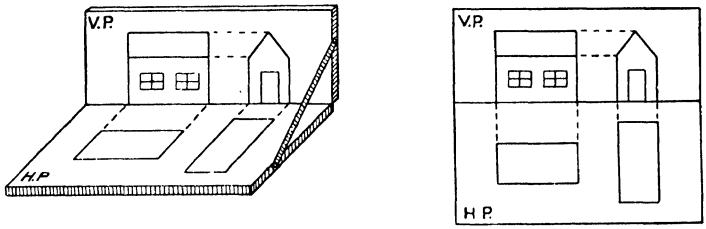
Fig. 130

It is a very simple kind of projection, because most of the work to be done depends on the drawing of **perpendiculars** and **parallels.** The **projectors** and other working lines are nearly all perpendiculars and parallels. A word implying perpendiculars and parallels is " orthographic ", and the projection is sometimes called **orthographic projection.**

As it is not very convenient to draw on the wall, we sometimes use two boards hinged at right angles. The next figure shows such a pair, first of all hinged in position, then unhinged and the vertical plane turned back into the horizontal. The figure shows two plans and elevations of the hut in fig. 136. The first plan shows the long sides of the hut ***parallel*** to the vertical plane and the elevation a **side eleva-**

tion. The second plan shows the long sides *perpendicular* to the vertical plane, and the elevation an **end elevation.** The term **front elevation** is also sometimes used. An elevation is often spoken of as a **view.**

An architect would not draw two plans of one building, but he would always draw two or more elevations, in order



i ii

Fig. 131

to make the builder understand exactly what the building was to be like.

Note that all the projectors (shown as broken lines) are perpendiculars and parallels.

The two boards are shown hinged merely to help you to understand how the plan and elevation are related, but plans and elevations are commonly drawn as in fig. ii. An architect would not trouble to draw the outlines of the two boards. He just rules a line across the paper (marking it XY sometimes), draws the plan below, and the elevation above. More frequently than not, he uses separate pieces of paper, but he always remembers how the separate drawings are related. The XY line is sometimes called the **ground line:** it is the line of contact of the vertical plane with the ground. It is usual to keep the plan a little distance away from this line, but to let all elevations stand on the line.

Although your plans and elevations will always be drawn on the flat, you will sometimes find it useful to fold your paper at right angles on the XY line, and to place the object, if small enough, in position on the horizontal plane. You can then see more plainly what the elevation on the vertical

plane will be. For instance, in the figure at the beginning of this section, suppose a pencil is placed in contact with the edge *ad* of the block, and the block removed. You can see at once that the plan of the pencil is *bc*, and that the elevation is *a'd'*.

Remember that plans and elevations of any object are obtained **by drawing perpendicular projectors** to the Horizontal Plane (H.P.) and Vertical Plane (V.P.). The feet of these projectors are then joined in such a way that the lines correspond to the edges of the object itself.

Here are some examples of plans and elevations. Copy them full size. Then fold your paper on the ground line and turn the V.P. into position.

1. Plans and elevations of a line 3″ long. Hold a piece of wire or a short pencil in position, so that, looked at from

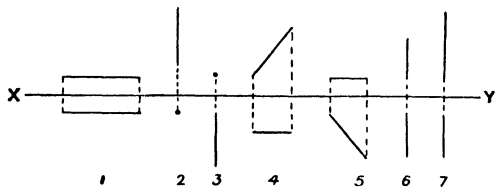


Fig. 132

above, it covers the plan, and looked at from the front, it covers the elevation. Then, in each case, try to describe the position of the pencil with reference to the two planes, checking your descriptions by the correct descriptions below.

	1.	2	3.	4.	5.	6.	7.
To the H.P. the line is	∥	⊥	∥	60°	∥	45°	60°
To the V.P. the line is	∥	∥	⊥	∥	45°	45°	30°

2. Plans and elevations of a rectangular sheet of paper, 3″ × 2″:

	The Plane of the Paper is	The Long Edges are		The Plane of the Paper is	The Long Edges are
1.	‖ to V.P.	‖ to H.P.	5.	‖ to V.P.	30° to H.P.
2.	‖ ,, V.P.	⊥ ,, H.P.	6.	‖ ,, H.P.	60° ,, V.P.
3.	‖ ,, H.P.	‖ ,, V.P.	7.	45° ,, both planes	‖ ,, both planes
4.	‖ ,, H.P.	⊥ ,, V.P.	8.	⊥ ,, both planes	45° ,, both planes

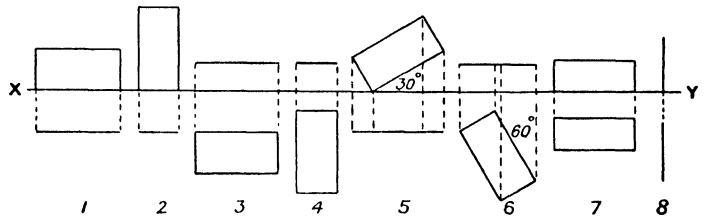


Fig. 133

3. Plans and elevations of a square prism:

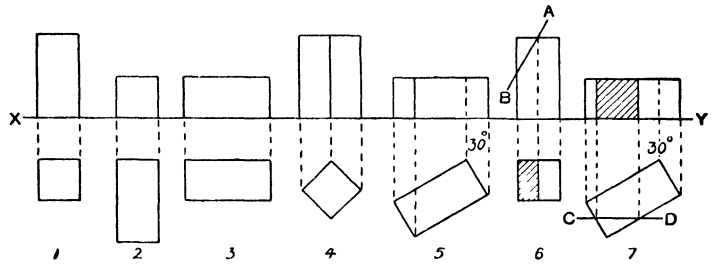


Fig. 134

Positions as follows:

(1) Standing on base, two sides ‖ to V.P.
(2) Lying on a side, all sides ⊥ to V.P.
(3) Lying on a side, bases ⊥ to both planes.
(4) Standing on a base, one diag. of base ⊥ to V.P.
(5) Lying on a side, two sides 30° with V.P.
(6) Same as No. 1, with a section AB ⊥ to V.P.
(7) Same as No. 5, with a section CD ⊥ to H.P.

It is sometimes necessary to know the shapes of **sections** of solids. Two are shown above. The cut surfaces are indicated by cross hatching. When in doubt about such a shape, make a rough model and cut through it.

The positions of objects with respect to the two planes may be described in more than one way. In the first of the last series, for instance, we might have said two sides ⊥ to V.P. In the third, we might have said two sides ‖ to the V.P. and two ‖ to H.P. But the description must always be sufficient to fix the object in a particular position.

4. Plans and elevations of other solids:

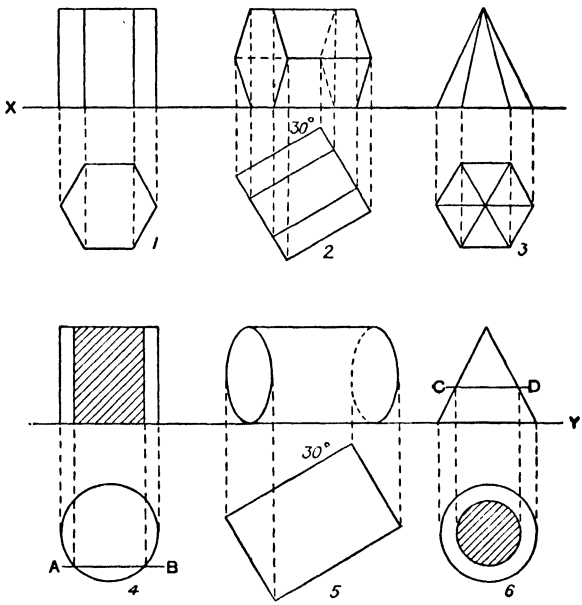


Fig. 135

Position of solids:

(1) Hexagonal prism standing on base, two sides ‖ to V.P.
(2) Hexagonal prism lying on a side, long edges 30° to V.P.

(3) Hexagonal pyramid standing on base, two edges of base $\parallel$ to V.P.

(4) Cylinder standing on base, with section AB $\perp$ to H.P.

(5) Cylinder lying down, axis 30° to V.P.

(6) Cone standing on base, with section CD $\perp$ to V.P.

Sections which are cut obliquely to one of the two projection planes may give a little trouble, especially if their shapes cannot be first imagined. The shape of an oblique section through a cylinder may be shown by half filling a round bottle with water and holding the bottle obliquely; the water-surface gives the shape of the section—an ellipse. So with a square bottle, or a conical flask. Or you may push the solid obliquely into the ground, down to the level of the section line. The shape of the mouth of the hole is the shape of the section.

More Advanced Work

Here are types of problems suitable for more advanced pupils:

1. Determine the projection of three spheres of different radii, resting on the ground in mutual contact.

2. Determine the projections of the curve of intersection of a cone penetrating a cylinder, the axes of the two solids intersecting at a given angle.

3. Determine the shadow cast by the hexagonal head of a bolt with a cylindrical shaft, the bolt standing vertically on its screw end, from given parallel rays.

4. Determine the shadow cast by a cone standing on the ground, the direction of the light being so arranged as to throw part of the shadow on the vertical plane.

For shadow-casting problems, it is a good plan to place the object in strong sunlight, so that the shadow can actually be cast on the horizontal plane (and vertical plane, too, if necessary), and examined. The problems then become very

simple. Shadows cast by artificial lights are less serviceable, since the light rays are necessarily not parallel.

As a rule there is no time for, and there is very little point in, making projections of *groups* of objects, but cases of simple interpenetrations make such good problems that one or two are worth doing.

Speaking generally, the ground covered in orthographic projection should be enough to enable Sixth Form boys to solve, readily and intelligently, such stock theorems of projection as these:

The projection on a plane of an area in another plane; and particular cases, e.g.:

(α) Projection of an ellipse into a circle, and the ratio of their areas.

(β) The projective correspondence between the perpendicular diameters of a circle and conjugate diameters in an ellipse.

(γ) Extension of the properties of polars from the circle to the ellipse.

CHAPTER XXIII

Radial Projection

First Notions

For Sixth Form boys learning mathematics seriously, a knowledge of radial projection is at least as important as a knowledge of orthographic projection. Here is the substance of a lesson for beginners: it is the sort of lesson one might expect to hear an intelligent art teacher give.

Stand about 18″ or 24″ from a window, keeping your head perfectly steady, and, with a piece of wet chalk, trace accurately on the glass an outline of a distant building.

When you have finished, it is easy to imagine straight threads passing from all the principal points in the building, through the corresponding points in your sketch on the glass, to your eye. Every line in the sketch exactly covers the corresponding line in the building. The drawing is another kind of projection. But the **projectors** are no longer perpendiculars; they all **radiate** from your eye, and they all pass through the vertical plane on which you have made the sketch, to the building. The vertical plane on which you have made the picture is called the **picture plane.** This kind of projection is called **radial projection** or **perspective projection.** Perspective drawings are the kind of drawings made by artists. Pictures are painted in accordance with the rules of perspective. The camera also follows these rules. Pictures and photographs represent things as they **appear** to the eye.

Here is the perspective projection of a hut:

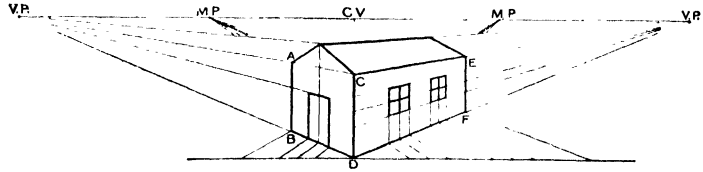


Fig. 136

In the hut itself, the three vertical lines, AB, CD, EF, are all equal. In the drawing, CD, the one nearest the observer, is the longest, and those farther away are shorter. So with the verticals in the doors and in the windows. All parallels which recede from the observer seem to approach each other, and at last to meet at a point on a line level with the eye. Equal parts of a horizontal in the object are unequal in the drawing (compare the horizontal window-bars in the two windows). The farther away a thing is taken, the shorter it becomes in the drawing. If you have made an accurate chalk-drawing on the window, you can teach yourself a good deal about perspective.

If, however, your chalk-drawing is not satisfactory, do this instead. Take a rectangular wooden frame of some sort (an old picture-frame will do), say about 15″ × 10″. Drive in tacks two-thirds of their length, at equal distances apart, say 1″, all round the edge. Stretch cotton across the frame and round the tacks in such a way as to divide up the frame into squares. Now divide up a piece of drawing paper into the same number of squares. Place the frame in a vertical position between your eye and a suitable object or view that may be sketched. If you sketch a house (a very suitable object) get back far enough to see the whole house easily within the frame. Now observe what part of the object appears within a particular square of the frame, and sketch that part in the corresponding square on your paper. And so on. With care you may make a fairly accurate drawing, and can then learn a good deal about perspective, more particularly about converging lines and diminishing lengths. You may also learn much from a large photograph of a building, especially if you can compare the photograph with the building itself.

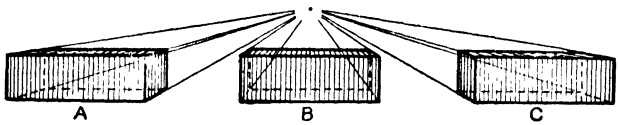


Fig. 137

Here is a perspective sketch of three bricks in a row. It is as they would appear in a photograph. The middle brick

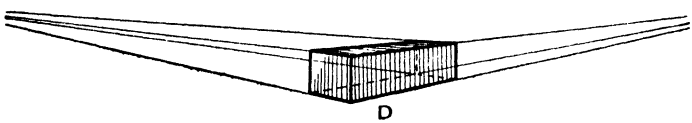


Fig. 138

is in the middle of the picture, and the photographer points his camera towards it. If he were photographing, say, brick C alone, he would turn his camera round and point towards

that brick. His picture would then be like D. Neither A nor C is the correct drawing of a brick unless the brick is to the left or right of a *group* of things, as in fig. 137.

The Picture Plane. Use of Models

The ordinary perspective text-book prepared for Art teachers is, generally, just a book of rules, rules with only the faintest tinge of mathematics in them. I have known boys make faultless and most elaborate perspective drawings of groups of objects in different positions, and yet they have had the most hazy ideas of the inner nature of the rules they have been applying. And yet, at bottom, the whole thing is a study, and a simple study, too, of similar triangles.

This figure shows the Picture-plane 12 ft. from the observer, with his eye 12 ft. away and 5 ft. above the ground at S. P. (his Station Point). The Picture-plane meets the ground in the *ground-line*. The point on the Picture-plane immediately opposite the eye is the *Centre of Vision*. The horizontal through the Centre of Vision is called the *Horizontal Line*. Radial projectors run from

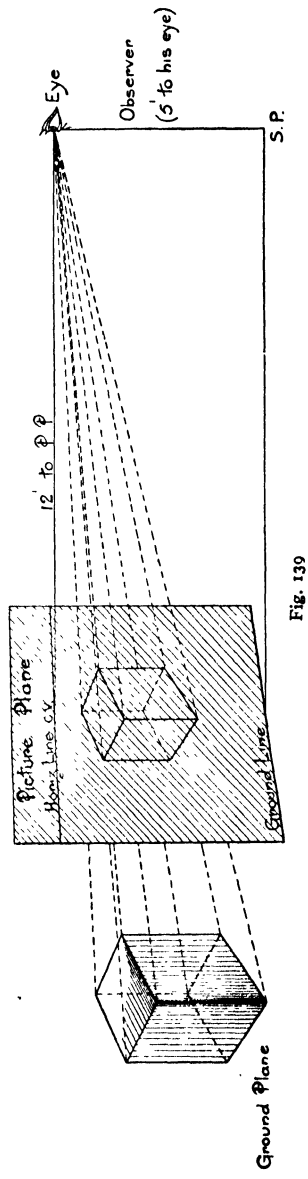


Fig. 139

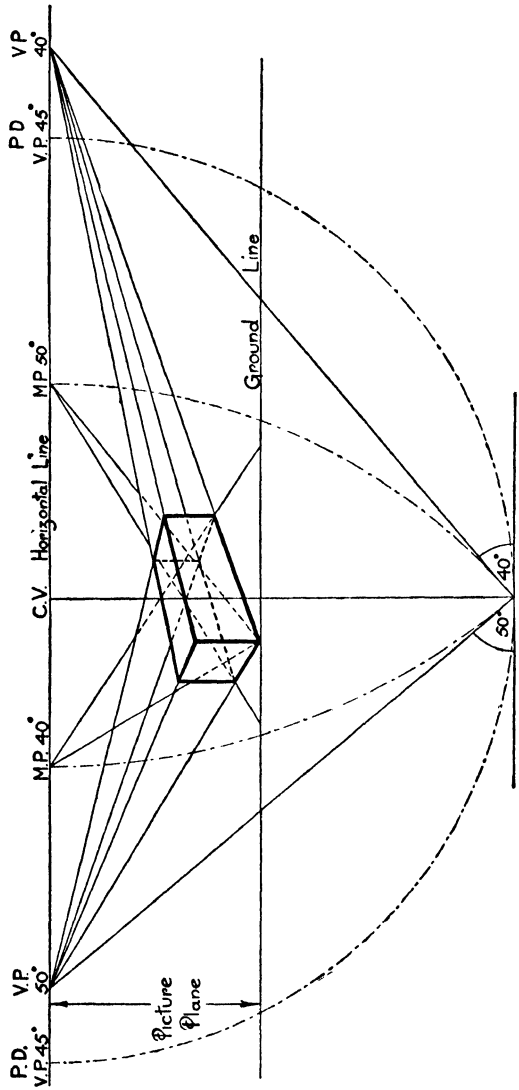


Fig. 140

the eye to each corner of a block fixed behind the Picture-plane and cut the Picture-plane in points which, when

joined up, give on the Picture-plane a perspective picture of the block.

Fig. 140 shows the sort of perspective drawing that appears in the textbooks. The pupil *must* see the relation between figs. 139 and 140. In fig. 139 the line from the eye to the C. of V. is represented at right angles to the P.P. That line must be supposed to be hinged at C. of V. and to turn on the hinge through 90° until it comes into the same plane as the P.P., as in fig. 140, which represents *a drawing in one plane*, the plane of the paper. It is imperative that the boys see fig. 139 as a model. Only then will they be able to understand fig. 140 completely. *Then* the points of distance, vanishing points, and measuring points are all a matter of very easy geometry.

In practice, it is an advantage to substitute for the glass P.P. a sheet of perforated zinc, or a square of stretched black filet net of $\frac{1}{10}''$ mesh, and to run threads (fastened with drawing pins to the corner of the rectangular block or other object being sketched) through the appropriate holes in the zinc or net to the ring representing the eye, where they may be secured. A drawing may then be represented in threads of another colour, run from hole to hole in the zinc or net, instead of in chalk as when glass is used.

Main Principles

The main principles of perspective, mathematically considered, are all reducible to a small handful of three-dimensional problems. One will suffice to illustrate the degree of difficulty.

Given any point on the ground-plane, to determine its position on the picture-plane.

Since solids are determined by planes, planes by lines, and lines by points, it will suffice to determine the position in the picture-plane of just one point on the ground-plane. This really solves the general problem, inasmuch as any other point may be similarly determined.

Let M be the point on the ground-plane. Drop a perpendicular MN on the picture-plane, and another EC from the eye E to the C. of V. on the P.P. Since EC is parallel to MN, both EC and MN, and also CN and ME, are in the shaded oblique plane. NC is the complete projection of the line NM extended to an unlimited length behind the

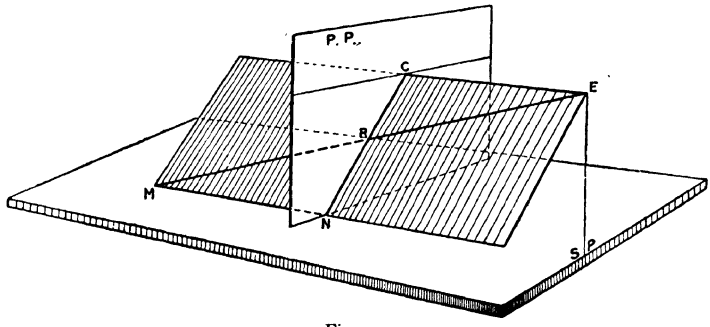


Fig. 141

P.P.; and the point R, where ME intersects CN, is the projection of M. Hence R is the point to be determined.

Now in the oblique plane we have the two similar triangles RCE, RMN. Hence $\frac{NR}{RC} = \frac{MN}{CE}$, i.e. CN is divided at R in the ratio:

$$\frac{\text{distance of point M from P.P.}}{\text{distance of observer from P.P.}}$$

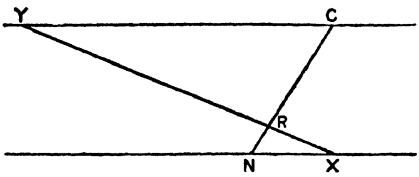


Fig. 142

Thus CN being drawn, R can be determined at once.

Suppose M is 3′ behind the P.P. and the observer is 10′ in front. It is required to divide CN in the ratio 3 : 10. Through C and N draw any pair of parallels. Measure off NX equal to 3 units and CY equal to 10 units. We have

two similar triangles. Hence $\frac{NR}{RC} = \frac{NX}{CY} = \frac{3}{10}$, i.e. the position of R in CN is determined.

Hence if the space between the two parallels represents the P.P., if C is the C. of V., and if CN is the complete projection of a line perpendicular to the picture-plane and meeting it in N, the projection of any point in this perpendicular line may be found by the above method.

In practical perspective we use as a pair of parallels the horizontal eye-line and the ground line. This is a mere matter of convenience; any other pair of parallels drawn on the P.P. would do equally well.

Measuring points, vanishing points, and the rest are all determined by the consideration of virtually the same principle. In fact the complete art of perspective projection lies in that principle. With the model, the whole thing becomes simplicity itself. The perforated zinc (or net) P.P. with strings passing through to the eye, and the projection of the figure threaded in with threads of a different colour, make the main principles so clear that there is little need for formal demonstration. The similar triangles then *in situ* tell the whole story.

Sixth Form Work

When these main principles underlying the practice of perspective have been mastered, the subject should be followed up in the Sixth by a few of the stiffer propositions associated with the general theory of perspective, treated formally and deductively, more especially those concerned with triangles in perspective, so far as these are necessary for the understanding of the chief properties of the hexastigm in a circle; at least Pascal's theorem should be known, though as a mere *fact* in practical geometry this theorem should be known lower down the school; its later theoretical consideration is always a delight to the keen mathematical boy.

But to attack such theorems of perspective before some

understanding of the practice of perspective has been acquired is to attack theorems that are lifeless.

CHAPTER XXIV

More Advanced Geometry

A Possible Outline Course

What is sometimes called "Modern" Geometry or "Pure" Geometry usually occupies a subordinate position in Sixth Form work. This is to be regretted.

It may be readily admitted that analysis is a powerful instrument of research, and doubtless for this reason alone mathematicians have given it a very important place in recent years. Accordingly, Sixth Form boys tend to devote much time to preparation for the work of that kind which is demanded of them at the University. But it cannot be denied that an intimate acquaintance with geometry is only to be obtained by means of "pure" geometrical reasoning. In the classroom no branch of mathematics is so productive of sound reasoning as is pure geometry. The ordinary geometrical theorem admits of a simple, rigorous, and completely satisfactory proof, a proof that is convincing and not open to question. An elementary knowledge of the properties of lines and circles, of inversion, of conic sections, treated geometrically, of reciprocation, and of harmonic section, ought to be expected from all Second Year Sixth Form boys. Many boys now leave school without any conception of some of the remarkable properties of the triangle and circle; and this ought not to be.

There is much to be said for beginning with rectilineal figures, including a fairly complete study of the tetragram and tetrastigm, the more elementary properties of the polygram

and polystigm, and then for proceeding with harmonic section. The remaining topics follow simply. We outline for teaching purposes one or two of the different subjects.

The Polygram and Polystigm

These may be regarded either as systems of lines intersecting in points, or as systems of points connected by straight lines. The simplest figure is that determined by 3 lines or 3 points. If we have any 3 lines which are not concurrent, or if we have any 3 points which are not collinear and which may therefore be connected by 3 straight lines, we have two systems which are virtually the same, and we may give the name *triangle* to either.

But with more than 3 lines or points, the resulting figures though closely related are not identical.

Rectilineal figures regarded as systems of *lines* are called *polygrams*; as systems of points, *polystigms*.

A *tetragram* in its most general form is a complete rectilineal figure of *four lines*, no 3 of which are concurrent, and no 2 parallel. Each line is therefore intersected by the other 3. If the lines be named *a*, *b*, *c*, *d*, their points of intersection may be named by combining the 2 letters which denote the intersecting lines. Since there are 3 points of intersection in each of the 4 lines, we seem to have 12 points of intersection in all, but these are reduced to 6, since each is named twice. The 6 points of intersection are called **vertices.**

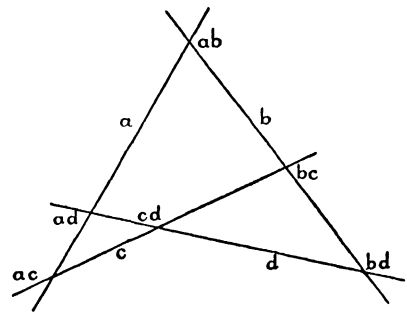


Fig. 143

A *tetrastigm* in its most general form consists of *four primary points*, no 3 of which are collinear and which do

not fall in pairs in parallel lines. If the points be named A, B, C, D, their connectors may be named in the usual way, AB, BC, &c. Since there are 3 connecting lines terminating in each of 4 primary points, we seem to have 12 connecting lines in all, but these reduce to 6, since each is named twice. The 6 lines are called **connectors.**

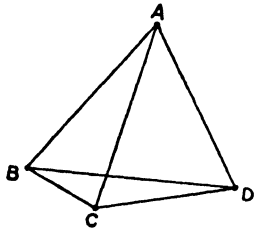
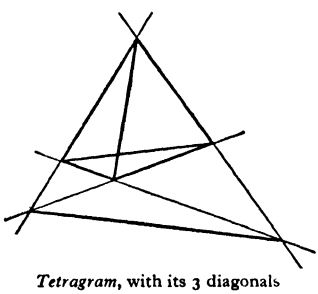


Fig. 144

From suitable figures, the number of vertices and connectors in the *pentagram* and *pentastigm* is seen to be 10, and in the *hexagram* and *hexastigm*, 15.

We infer that in a **polygram** of n lines, and in a **polystigm** of n points, the number of *vertices* and *connectors* are, respectively, $\frac{n(n-1)}{2}$; for the tetragram and tetrastigm give us $\frac{4\times 3}{2}$, the pentagram and pentastigm $\frac{5\times 4}{2}$, and the hexagram and hexastigm $\frac{6\times 5}{2}$.

In a *tetragram* a **diagonal** may be drawn from each of the vertices to another vertex; the 6 diagonals reduce to 3.


Tetragram, with its 3 diagonals

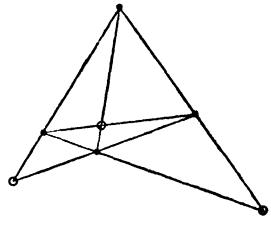

Tetrastigm, with its (4 primary and) 3 secondary points

Fig. 145

In a *tetrastigm*, each of the 6 connectors can intersect another connector at a point other than the 4 primary points; the 6 reduce to 3. These 3 new points are called the **secondary points** of the tetrastigm.

From suitable figures the number of diagonals and

secondary points in the pentagram and pentastigm is seen to be 15, and in the hexagram and hexastigm, 45.

Since

$$3 = \frac{4 \times 3 \times 2 \times 1}{8},\quad 15 = \frac{5 \times 4 \times 3 \times 2}{8},\quad 45 = \frac{6 \times 5 \times 4 \times 3}{8},$$

we infer that in a **polygram** of n lines, and in a **polystigm** of n points, the number of *diagonals* and of *secondary points* respectively is $\frac{n(n-1)(n-2)(n-3)}{8}$.

The pupil should check for the pentagram and pentastigm. The figures for the hexagram and hexastigm are complicated and their analysis is hardly worth while. The pupil should note that a polygram and polystigm of the same order are *reciprocal* figures; they give us an excellent example of the principle of duality.

Derived Polygons

The pupil may be encouraged to establish, from an examination of a few particular cases, the principle that the number of derived polygons from a polygram or polystigm is $\frac{(n-1)!}{2}$.

For instance, the number of derived tetragons from a tetragram or tetrastigm is $\frac{3 \times 2 \times 1}{2} = 3$; of derived pentagons from a pentagram or pentastigm is $\frac{4 \times 3 \times 2 \times 1}{2} = 12$; of hexagons, 60; and so on.

Here is a *tetragram* and its 3 derived tetragons:

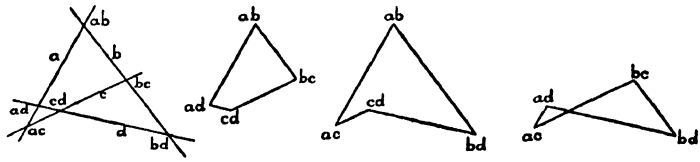


Fig. 146

A tetragram consists of 4 lines with 6 consequent vertices, and 3 vertices lie on each of the 4 lines. But in a tetragon

there are only 2 vertices in a line, viz. those at the extremities of the line; there are thus 4 vertices in all. Hence, for a derived tetragon, we have to select 4 vertices out of the 6, in such a way that 2, and 2 only, may lie on and determine the extremities of each of the 4 lines. Such a selection is known as *a complete set of vertices* for a derived tetragon. Note that, whatever vertex is chosen as a starting-point, that vertex must be the point where the figure is completed.

Here is a *tetrastigm* and its 3 derived tetragons.

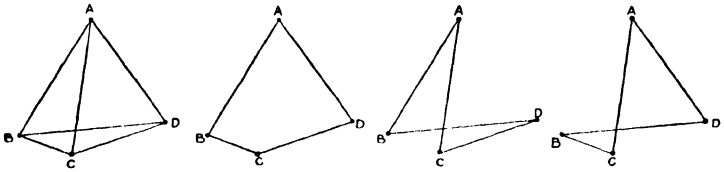


Fig. 147

Analogous reasoning applies. We have to select 4 connectors out of 6, in such a way that 2, and only 2, may terminate in each of the 4 vertices. The selection is known as *a complete set of connectors* for a derived tetragon.

The boys may be given the task of drawing the 12 pentagons from a pentagram and the 12 from the pentastigm. But they must set to work systematically or there will be confusion. Consider the pentastigm with its 5 primary points, A, B, C, D, E. Select AB as the initial connector. Associated with it as a second connector we may have BC, BD, or BE; we then have the first two connectors formed in 3 different ways, viz. AB, BC; AB, BD; AB, BE. The first two connectors being fixed, the remaining 3 can be selected in 2 different ways, and thus we have 6 different pentagons formed with AB as a first connector. Now do exactly the same thing with the other 3 connectors terminating in A. And so on.

Do not forget that the polystigm is the key to many of the mediæval tree-planting problems. *Given n trees, what is the greatest number of straight rows in which it is possible*

to plant them, each row to consist of m trees? For instance, given 16 trees, plant them in 15 rows of 4.

Construct a regular pentagram with such of its diagonals as are necessary to form an inner second pentagram. The introduction of these diagonals gives 6 new points, which, with the 10 vertices of the pentagram, make 16 points.

The *general* problem has never been completely solved.

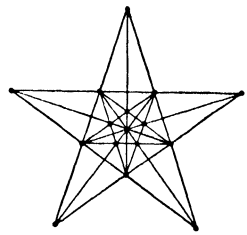


Fig. 148

Harmonic Division *

The Pythagoras relation, golden section, and harmonic division, are the 3 keys of pure geometry, yet harmonic division frequently receives but very scant attention. The principle itself once fully grasped, the actual proofs of theorems involving it are generally of the simplest.

The approach to the subject and its problems may be effectively made in this way:

(1) *Divide a line internally and externally in the same ratio,*

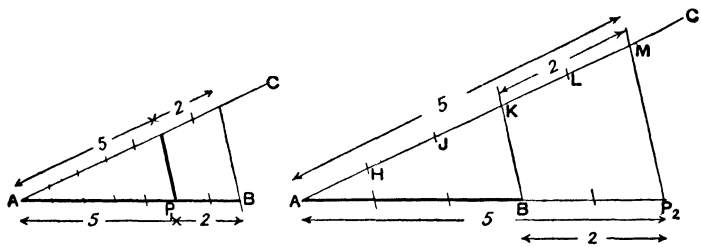


Fig. 149

say 5 : 2. Note that the correct reading of the ratios is *from* the extremities of the line *to* the point of division; thus for

* In speaking of cross-ratios, avoid the term "anharmonic", since it implies "not harmonic", whereas a cross-ratio may be harmonic, for it may be the cross-ratio of an harmonic range.

internal division we have $\frac{AP_1}{BP_1}$; for external division $\frac{AP_2}{BP_2}$. Also note the *sign* as well as the magnitude; e.g. for internal division, AP_1 and BP_1 are measured in *opposite* directions and the ratio is therefore negative; for external division AP_2 and BP_2 are measured in the *same* direction, and the ratio is therefore positive. Give important examples of this, for instance the theorems of Ceva and Menelaus.

(2) *Algebraic Harmonic Progression.*—Definition: If a, b, and c are in H.P., then $\frac{a-b}{b-c} = \frac{a}{c}$; or, $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$; or $b = \frac{2ac}{a+c}$.

(3) *Compare the geometry and the algebra.*—A line AQ is said to be " harmonically divided " at P and B when, if $AQ = a$, $AB = b$, $AP = c$, a, b, and c are in H.P.

Since $\frac{a-b}{b-c} = \frac{a}{c}$ (by definition), $\therefore \frac{BQ}{BP} = \frac{AQ}{AP}$,

or $AQ \times BP = AP \times BQ$; (Cf. fig. 150.)

i.e. product of whole line and middle segment equals product of external segments. Hence if AB is divided harmonically

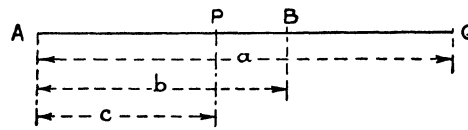


Fig. 150

at P and Q, PQ is divided harmonically at A and B. In other words, AB is divided internally and externally at P and Q in the same ratio; *and* PQ is divided internally and externally at B and A in the same ratio.

(4) *Harmonic Ranges.*—If a line AB is divided harmonically at P and Q, the range of points {AB, PQ} is called a harmonic " range ". The pair of points A and B are said to be *conjugate* to each other; so with the points P and Q. We may

conveniently name a harmonic range thus {AB, PQ}, the comma being inserted to distinguish the pairs of conjugate points.

(5) *Harmonic Pencils.*—Define " ray " and " pencil ". Every section of a harmonic pencil is a harmonic range, e.g. (AB, PQ), (A'B', P'Q'). A pencil O.APBQ is harmonic if

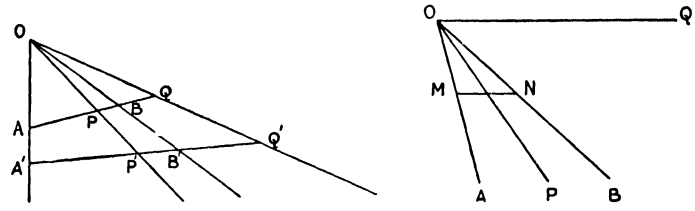


Fig. 151

a transversal MN parallel to one ray OQ is bisected by the conjugate OP.

A range may be read {AB, PQ} *or* {APBQ}, and a pencil may be read O(AB, PQ) *or* O.APBQ. Adopt *one* plan and adhere to it, or the boys may be confused. It is a good plan to use coloured chalks for every harmonically divided line on the blackboard, and always of the same colour. Harmonic division is so useful that its immediate recognition is desirable.

The Complete Quadrilateral

The commonest theorems involving harmonic section concern (1) the complete quadrilateral, (2) pole and polar.

Fig. 152 (i) shows a tetragram with its 3 diagonals (2 produced to meet) which are indicated by heavy lines. *Each of the* 3 *diagonals is harmonically divided* by the other two. Fig. (ii) shows a tetrastigm with its 6 connectors also indicated by heavy lines, and with lines (faint) joining the secondary points in pairs. *Each of the* 6 *connectors is harmonically divided* by (1) the secondary point through which it passes, and (2) the line joining the other two secondary points. If we superpose one figure on the other, we get a remarkable series of harmonic

pencils, of mutual harmonic intersections, and of collinearities Actually, of course, they can be picked out in fig. 152 (ii).

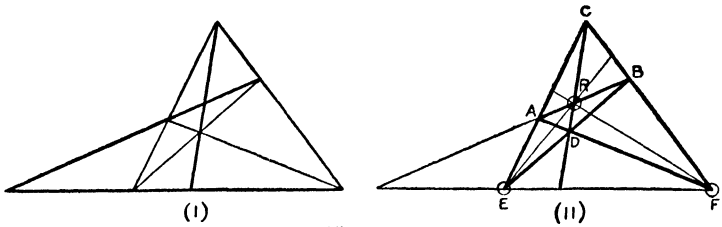


Fig. 152

If the complete quadrilateral is approached in this way, the boy's interest and curiosity is aroused. He is greatly surprised to discover that a simple thing like a quadrilateral has so many remarkable properties.

But the important thing is for the boy to realize that this general quadrilateral *generalizes* the theorems of all particular quadrilaterals. For instance, suppose that the line EF in the last figure is removed to an " infinite " distance from C, the 4 points C, A, D, B, become the vertices of a parallelogram; and since R is the harmonic conjugate of the point in which CD intersects EF with respect to the points C and D, R becomes the middle point of CD. Thus the theorem of the complete quadrilateral is a generalization of the theorem that the diagonals of a parallelogram bisect each other.

Impress on the boys the importance of the use of a *general* figure in their geometrical work, and the fact that from its properties the special properties of a particular figure may often be inferred.

Pole and Polar

The dark lines in the figures represent harmonically divided lines: (i), (ii), every chord which passes through the pole P is cut harmonically by the polar; (iii), if the polar of P_1 passes through P_2, the polar of P_2 passes through P_1. In all such cases, let the pole and the corresponding point

on the polar be consistently marked, say, by P's. Also let the circumference be cut in Q's. A consistent system of naming all harmonically divided lines is a great advantage.

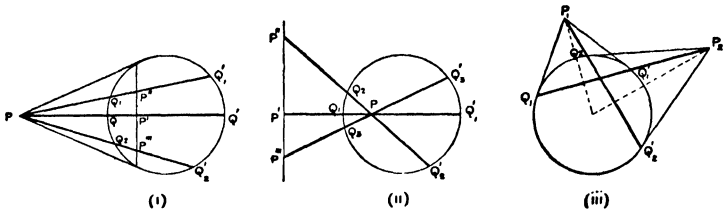


Fig. 153

For a good general problem on harmonic section, see *Scientific Method*, pp. 387–9.

Concurrency and Collinearity

Many theorems involving these principles form excellent practical problems for careful work in junior classes. Encourage young boys to bisect the angles of a triangle, to bisect the sides, to draw perpendiculars from the mid-points, &c., and to make discoveries for themselves. Let them thus obtain the *facts*. A little later, the simpler theorems and their proofs may be given; e.g. concurrent lines through the vertices of a triangle, the medians, the perpendiculars to the opposite sides, two exterior bisectors and the internal bisector of the three angles. A little later still: given the Menelaus relation, prove that the points are collinear; given the Ceva relation, prove that the points are concurrent. Pascal's and Brianchon's theorems will, of course, always be included. Another good type of theorem is this: four points on a circle and the tangents at those points form, respectively, two quadrilaterals whose internal diagonals are concurrent and form a harmonic pencil, and whose external diagonal points are collinear and form a harmonic range. The *principles* of concurrency and collinearity are so important that they cannot be too strongly

emphasized, but with most boys facility comes only after much practice with varied types of problems.

Pascal's theorem suggests the study of the hexastigm, of which that theorem is the simplest property. The theorem is usually quoted, " The opposite sides of any hexagon inscribed in a circle intersect in 3 collinear points ", but a more precise statement is, " The 3 pairs of opposite connectors of a hexastigm inscribed in a circle intersect in 3 collinear points ". Fully expressed, this comes to, " The 15 connectors of a hexastigm inscribed in a circle intersect in 45 points which lie 3 by 3 on 60 lines ".—I have seen *one* passable figure prepared by a boy; he was looked upon as the fool of his Form, though he was extraordinarily successful in the use of ruler and compasses. Elaborate drawing of this kind is largely a waste of time, and, after all, the hexastigm still remains to be investigated fully.

The Further Study of the Triangle and Circle

There are numerous theorems on the triangle, many of them simple, many useful, many beautiful. For instance, those concerning triangles in perspective, pole and polar with respect to a triangle, symmedian points of a triangle, Brocard points of a triangle.

So with the circle: the nine-point circle, escribed circles, the cosine circle, the Lemoine circle, the Brocard circle.

It is important to leave on the boy's mind a vivid impression of the remarkable properties that even now are frequently being discovered concerning the triangle and circle. Boys who, when they leave school, know no more pure geometry than that contained within the limits of School Certificate requirements are certainly not likely to devote leisure moments to a systematic playing about with circles and triangles, in the hope of hitting on some new and perhaps remarkable property yet undiscovered.

Conic Sections

The pure geometry and the algebraic geometry of the cone should be studied side by side. If either has to be sacrificed, let it be the latter. Algebraic manipulation is all very well, but the cone is a thing which occupies *space*, and when its spatial relations are reduced to symbols, these symbols may assume, in the pupils' minds, an importance which is not justified, and the geometry proper may be overshadowed.

CHAPTER XXV

Geometrical Riders and their Analysis

In those schools where riders are, as a rule, solved readily, schools where boys take a real delight in attacking new ones, the secret of success seems to be that right from the first every new theorem and every new problem has been presented, not as a thing to be straight-away learnt, but as a thing to be investigated and its secret discovered. The boys do not learn a new theorem or problem until they have been taught how to analyse it, and to discover how it hangs on to what has gone before.

General instructions should include advice as to the necessity of drawing a *general* figure, of drawing that figure accurately, and of setting out definitely what is " given " and what is to be proved. We append a few instances of problems and theorems actually solved in the classroom, with a brief summary of the sort of arguments used.

1. *O is the mid-point of a straight line PQ, and X is a point such that $XP = XQ$. Prove that the $\angle XOP$ is a right angle.*

We argue in this way:

(1) What facts are **given**?

$$OP = OQ,$$
$$XP = XQ.$$

(2) What have I to **prove**?

That $\angle XOP$ is a right angle.

(3) Since I have to prove that $\angle XOP$ is a rt. $\angle$, I must join XO.

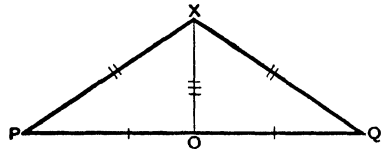


Fig. 154

(4) How have I been able to prove before, that an $\angle$ is a rt. $\angle$?

(i) Sometimes by finding it to be one of the two eq. adj. $\angle$s making a str. $\angle$.

(ii) Sometimes by finding it to be an angle in a semicircle.

(iii) Sometimes by finding it to be at the intersection of the diag. of a sq. or a rhombus.

(5) The first of these looks possible here. Are the adj. $\angle$s at O equal? Yes, if the two $\triangle$s XOP and XOQ are congruent.

(6) *Are* these $\triangle$s congruent? Yes, three sides in the one are equal to three sides in the other, as marked.

Now I know how to write out the proof, in the ordinary way: I therefore begin again, and make up a new figure as I proceed.

Proof. Join XO. In the Δs XOP, XOQ,

$OP = OQ$, (*given*)
$XP = XQ$, (*given*)
XO is common,
$\therefore \Delta XOP \equiv \Delta XOQ$; (*three sides*)
i.e. the two Δs are equal in all respects.
$\therefore \angle XOP = \angle XOQ$,
$\therefore \angle XOP = 90°$. (*half the str.* $\angle POQ$)
(which was to be proved)

2. *ABCD is a parallelogram; E is the mid-point of BC, and AE and DC produced intersect at F. Prove that AE = EF.*

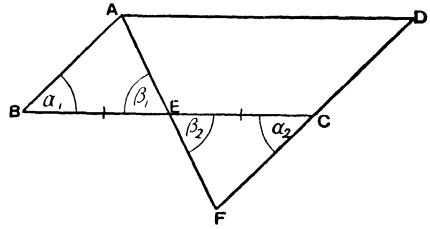


Fig. 155

Argument.

(1) What facts are **given**?

(i) ABCD is a $\square$m; $\therefore$ its opp. sides are $\parallel$.
(ii) BE = EC (by constr.).

(2) What have I to **prove**?

That AE = EF.

(3) How have I been able to prove before, that two lines are equal?

(i) Sometimes by finding them in two congruent Δs.
(ii) Sometimes by finding them in a Δ with two angles equal.
(iii) Sometimes by finding them to be the opp. sides of a $\square$m.

(4) Does either of these plans seem possible, to prove AE eq. to EF?
(5) Yes, the first, for the △s ABE and FCE *look* congruent.
(6) *Are* they congruent?
(7) Yes. Two ∠s and a side, as marked.

Now I know how to write out the proof in the ordinary way

Proof. In the △s ABE, FCE,

BE = EC,	(*constr.*)
∠AEB = ∠FEC,	(*vert. opp. ∠s*)
∠ABE = ∠FCE,	(*alt. ∠s; BC across ‖s AB, DF*)
∴ △ABE ≡ △FCE,	(2 *∠s and a side*)
∴ AE = EF.	(which was to be proved)

3. *Draw a circle of* $\frac{1}{2}''$ *radius to touch the given line AB and the given circle CDE.* (The given line must not be more than 1″ from the given circle.)

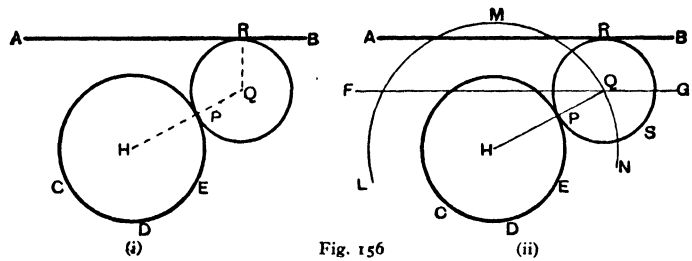


Fig. 156

This is a **problem,** and we have to discover the method of **construction.** I assume the problem done, and I make a sketch of the required circle in position, as accurately as possible (fig. 156, i). I examine the figure, and I observe that the line QR to the pt. of contact R = $\frac{1}{2}''$ and is ⊥ AB; that HQ = HP + $\frac{1}{2}''$ and passes through the pt. of contact P.

Argument.

(1) *Since the required ⊙ has to touch the line AB,* **its**

centre must lie somewhere on a line FG || AB, and $\frac{1}{2}''$ from AB (fig. 156, ii).

(2) *Since the required* ⊙ *has to touch the* ⊙ *CDE*, its centre must lie somewhere on the circle LMN having the same centre as CDE and having a radius HQ equal to radius HP + $\frac{1}{2}''$ (fig. 156, ii).

(3) Since the centre of the required ⊙ lies both on the line FG and on the circle LMN, it must be at a point of intersection of FG and LMN.

Now I know how to construct the circle.

Construction.

(1) Draw a line FG || AB, $\frac{1}{2}''$ away from it.

(2) From centre H, with radius equal to HP + $\frac{1}{2}''$, draw ⊙ LMN.

(3) From one of the pts. of intersection of this line and circle, say Q, as centre, draw a circle RSP of $\frac{1}{2}''$ radius. This is the required ⊙.

Proof.

(1) The circle RSP has a radius of $\frac{1}{2}''$. (*constr.*)

(2) The circle touches AB (in R), for any circle of $\frac{1}{2}''$ radius having its centre on the line FG must touch AB, a ⊥ from the centre Q passing through the pt. of contact. (*constr.*)

(3) The circle touches the given circle CDE (in P), for any circle of $\frac{1}{2}''$ radius having its centre on the circle LMN must touch CDE, the line joining the centres passing through the point of contact. (*constr.*)

(4) Therefore the circle is constructed in accordance with the given conditions. (which was to be done).

4. *From the right angle of a right-angled triangle, one straight line is drawn to bisect the hypotenuse, and a second is drawn perpendicular to it. Prove that they contain an angle*

equal to the difference between the two acute angles of the triangle.

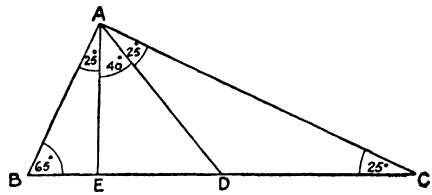


Fig. 157

Given.

(1) a rt. ∠ at A.
(2) rt. ∠s at E.
(3) DB = DC.
(4) DA = DB = DC. (Since a ⊙ will go round the △ABC on BC as diameter.)

Further, an examination of the figure shows that in the 2 rt. ∠d △s BAC, BEA, ∠ABE is common; ∴ the two △s are equiangular. This fact may prove useful.

Required to prove: ∠EAD = (∠ABC − ∠ACB).

Argument.—This is a type of problem in which we may first usefully test a particular case by assigning to some angle a number of degrees, and then calculating the number in some or all of the other angles. For instance, let ∠ABC = 65° (not 30°, or any other factor of 360°, lest a fallacy creep into our argument).

If ∠ABC = 65°, ∠ACB = 25° (the complement).
If ∠ACD = 25°, ∠CAD = 25° (for AD = DC).
Also ∠BAE = 25° (equiangular △s, as above).

Again, if ∠ABC = 65°, ∠BAD = 65° (for DB = DA),
and ∠EAD = ∠BAD − ∠BAE = 65° − 25° = 40°.

But ∠ABC − ∠ACB = 65° − 25° = 40°,
∴ ∠ABC − ∠ACB = ∠EAD.

Thus the theorem is true in this particular case. We are therefore now in a position to generalize the result *and to set out the proof in the ordinary way.*

Proof.

(1) DB = DA. (*given*)
$\therefore$ $\angle$DBA = $\angle$DAB.

(2) BAC and BEA are equiangular rt. $\angle$d $\triangle$s. (*given*)
$\therefore$ $\angle$ACB = $\angle$EAB.

(3) $\therefore$ $\angle$DBA $-$ $\angle$ACB = $\angle$DAB $-$ $\angle$EAB (from 1 and 2),
= $\angle$DAE.
(which was to be proved)

5. *The figure shows an equilateral triangle ABC within a rhombus ADEF, a side of the former being equal to a side of the latter. Determine the magnitude of the angles of the rhombus.*

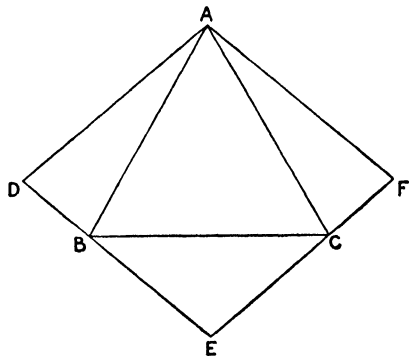


Fig. 158

Argument.

From an examination of the figure we know the following facts:

(1) *Rhombus*: 4 equal sides; opp. sides ||; opp. $\angle$s equal.
(2) *Equil.* $\triangle$: sides equal; $\angle$s equal.
(3) $\triangle$s ADB, AFC isosceles.
(4) $\triangle$EBC isosceles (by symmetry).

All that we *know* about ∠ magnitudes from the figure are:

(1) Angle-sum of any $\Delta = 180°$.
(2) Each ∠ of $\Delta ABC = 60°$.

We have therefore to try to express the ∠s of the rhombus in terms of these values.

From the rhombus, $\angle FAD + \angle ADB = 2$ rt. ∠s.
Also, $\angle ABD + \angle ABE = 2$ rt. ∠s.
But $\angle ADB = \angle ABD$, (*isos.* Δ)
$\therefore \angle FAD = \angle ABE$.

Obviously, therefore, $\angle FAD = \angle FED = \angle ABE = \angle ACE$

Now the sum of the last 3 of these ∠s
= (sum of ∠s of ΔEBC) + $\angle ABC + \angle ACB$
$= 180° + 120°$
$= 300°$.

$$\therefore \angle FAD = \frac{300°}{3} = \mathbf{100°};$$

$$\therefore \angle ADE = (180° - 100°) = \mathbf{80°}.$$

The estimate may be set out formally in almost the same sequence.

6. *Show that the* 4 *straight lines bisecting the angles of any quadrilateral form a cyclic quadrilateral.*

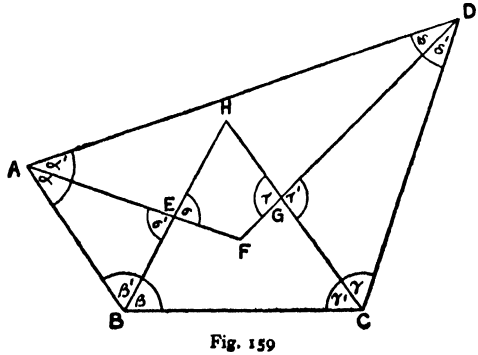


Fig. 159

Let ABCD be the given quadrilateral, and let the bisectors of the ∠s form the quadrilateral EFGH.

Let the ∠s be marked as shown.

Given: $\alpha = \alpha'$; $\beta = \beta'$; $\gamma = \gamma'$; $\delta = \delta'$.

Argument.

If a circle will go round EFGH, the sum of any 2 opp. ∠s of EFGH = 2 rt. ∠s; thus $\sigma + \tau$ = 2 rt. ∠s.

If $\sigma + \tau$ = 2 rt. ∠s, $\alpha + \beta' + \gamma + \delta'$ = 2 rt. ∠s since the sum of all the ∠s of the 2 △s ABE and CDG = 4 rt. ∠s.

But $\alpha + \beta' + \gamma + \delta'$ we *know* are equal to 2 rt. ∠s, for $2\alpha + 2\beta' + 2\gamma + 2\delta'$ = 4 rt. ∠s (the 4 ∠s of the quadl.)

Thus we have found the key.

Proof.

The sum of the 4 ∠s of the quadl. ABCD = 4 rt. ∠s.

∴ the sum of the halves, $\alpha + \beta' + \gamma + \delta'$ = 2 rt. ∠s.

But the sum of all the ∠s of △s ABE and CDE = 4 rt. ∠s.

∴ $\sigma' + \tau'$ = 2 rt. ∠s,

∴ $\sigma + \tau$ = 2 rt. ∠s,

∴ the points E, F, G, H are concyclic.

(which was to be proved)

7. *Three points D, E, and F in the sides of a triangle ABC are joined to form a second triangle, so that any two sides of the latter make equal angles with that side of the former at which they meet. Show that AD, BE, and CF are at right angles to BC, CA, and AB, respectively.* (You may not assume properties of the pedal triangle.)

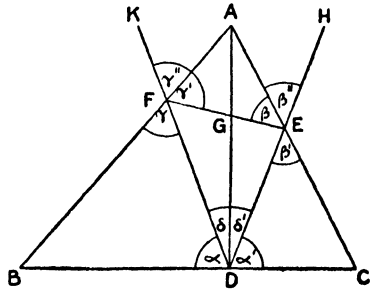


Fig. 160

Given: $\alpha = \alpha'$; $\beta = \beta'$; $\gamma = \gamma'$.

Required to prove: AD is $\perp$ to BC, &c.

Argument.—Assume that AD *is* $\perp$ to BC.

$$\text{Then } \because \alpha = \alpha',\ \delta = \delta'.$$

$$\therefore \text{ in } \triangle DEF,\ \frac{DF}{DE} = \frac{FG}{GE},$$

$$\therefore \frac{DF}{FG} = \frac{DE}{GE}.$$

But this is *known*, since each ratio $= \frac{DA}{AG}$.

[This is easily shown: Produce DE to H; EA is the bisector of the external $\angle GEH$ of the $\triangle GDE$; $\therefore \frac{DE}{EG} = \frac{DA}{AG}$. Similarly, by producing DF to K, it is seen that $\frac{DF}{FG} = \frac{DA}{AG}$.]

Thus we can make these known ratios our starting-points, and set out the proof in the usual way.

Proof.

Produce DE to H and DF to K. Then EA and FA are the bisectors, respectively, of ext. $\angle GEH$ of $\triangle GDE$, and of ext. $\angle GFK$ of $\triangle GDF$.

$$\therefore \frac{DE}{EG} = \frac{DF}{FG}, \text{ for each} = \frac{DA}{AG},$$

$$\therefore \frac{DE}{DF} = \frac{EG}{FG},$$

$\therefore$ DG bisects $\angle EDF$,

$\therefore \delta = \delta'$,

$\therefore \delta + \alpha = \delta' + \alpha'$,

$\therefore$ **AD is $\perp$ to BC.**

Similarly, we may show that BE ⊥ AC, and CF ⊥ AB.
(which was to be proved)

8. *Show that the perpendicular drawn from the vertex of a regular tetrahedron to the opposite face is* 3 *times that drawn from its own foot to any of the other faces.*

Let ABCD be the tetrahedron, and let AE be the ⊥ from the vertex A to the opp. face BCD. Then E is the centroid of the △BCD.

Let a ⊥ EF be drawn to the face ACD; F will meet the median AG.

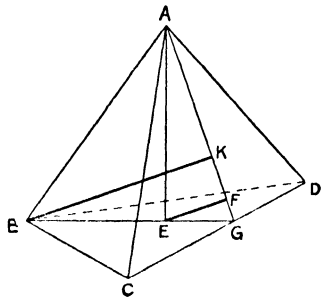
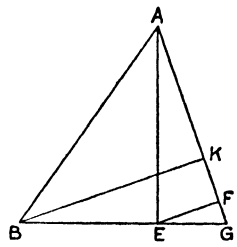


Fig. 161



SECTION THROUGH TETRAHEDRON

Fig. 162

To prove: AE = 3EF.

Argument: Consider the vertical section through ABG.

We know that, since E is the centroid of BCD, EG = ⅓ BG, Is there an analogous relation between EF and AE?

If we draw BK ⊥ AG in the face ACD, BK must be equal to AE.

$$\frac{EF}{BK} = \frac{EG}{BG} = \frac{1}{3},$$

$$\therefore\ EF = \tfrac{1}{3}BK = \tfrac{1}{3}AE$$

For the other faces similar results follow from symmetry.

Proof.

Let a ⊥ BK from B meet the median AG in K; BK = AE.

$$EG = \tfrac{1}{3}BG. \qquad (E \text{ is the centroid of } BCD)$$

$$\frac{EF}{BK} = \frac{EG}{BG} = \tfrac{1}{3},$$

$$\therefore EF = \tfrac{1}{3}BK$$

$$= \tfrac{1}{3}AE. \qquad \text{(which was to be proved)}$$

If a rider is in any way of an unusual character, pupils sometimes have difficulty in writing out a proof concisely. We give an example of an acceptable proof for such a rider.

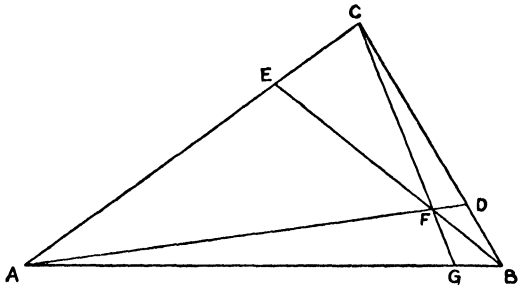


Fig. 163

In a given triangle ABC, BD is taken equal to one-fourth of BC, and CE equal to one-fourth of CA. Show that the straight line drawn from C through the intersection F of BE and AD will divide the base into two parts at G which are in the ratio 9 *to* 1.

$$\triangle BEA = 3\ \triangle BEC,$$

$$\triangle FEA = 3\ \triangle FEC,$$

$$\therefore \triangle BFA = 3\ \triangle BFC,$$

$$= 12\ \triangle BFD,$$

$$\therefore AF = 12\,FD,$$

$$\therefore \triangle AFC = 12\ \triangle DFC$$

$$= 36\ \triangle BFD$$

$$= 9\ \triangle BFC.$$

Now the Δs AFC, BFC are on the same base FC. Hence the vertical height of ΔAFC above this base = 9 times the vertical height of ΔBFC above this base.

$$\therefore \Delta AGC = 9\,\Delta BGC, \quad \text{(on the extended base, GC)}$$
$$\therefore \quad AG = 9\,GB.$$

Any reasonable examiner would accept a proof given in this form and would be glad to be saved from the trouble of reading defensive explanatory matter.

Books on geometry to consult:

1. *Plane Geometry*, 2 vols., Carson and Smith.
2. *Geometry*, Godfrey and Siddons.
3. *Geometry*, Barnard and Child.
4. *Elementary Concepts of Algebra and Geometry*, Young.
5. *Elementary Geometry*, Fletcher.
6. *Cours de Géométrie*, d'Ocagne (Gauthier Villars).
7. *A Course of Pure Geometry*, Askwith.
8. *Modern Pure Geometry*, Lachlan.
9. *Sequel to Elementary Geometry*, Russell.
10. *Geometry of Projection*, Harrison and Baxandall.
11. *Projective Geometry*, Matthews.
12. *An Elementary Treatise on Cross-Ratio Geometry*, Milne.
13. *Foundations of Geometry*, Hilbert.
14. *The Elements of Non-Euclidean Geometry*, Sommerville.
15. *Space and Geometry*, Mach.
16. *Analytical Conics*, Sommerville.
17. *Curve Tracing*, Frost (new edition). An old and faithful friend.
18. *Euclid*, 3 vols., Heath. *The* work on the subject.

CHAPTER XXVI

Plane Trigonometry

Preliminary Work

The pupils' first acquaintance with the tangent, sine, and cosine should be made during their elementary lessons in geometry. Boys soon learn that the symbols for the trigonometrical ratios may enter into formulæ which can be manipulated algebraically; and since, in the algebra course, the study of x^n and a^x is included, it is difficult to exclude from it the study of $\sin x$ and $\tan x$. Each represents a typical kind of function. To each corresponds a specific form of curve—its own particular picture, the graphic picture of the function. Algebra and trigonometry should be much more closely linked together, and much of the purely formal side of trigonometry might with advantage be sacrificed, and greater stress be laid on the practical and functional aspects of the subject. The needs of co-ordinate geometry and the calculus, of mechanics and physics, should always be borne in mind; in fact, much of the work done in trigonometry might be directed towards these subjects.

The notion of an angle as a rotating line should be given at the very outset of geometry, so that, when in trigonometry angles greater than 180° are discovered, the notion will already be familiar. The angle of " one complete rotation ", and its subdivisions, straight angle, right angle, and degree, will, of course, be known, and pupils should be able to draw freehand, at once, to a fairly close approximation, an angle of any given size, the 30°, 45°, and 60° angles being quite familiar from the half equilateral triangle and the half square.

Co-ordinate axes and the four quadrants will also be familiar from previous work on graphs; so will directed algebraic numbers. Angles of elevation and depression will already have been measured in connexion with practical

problems in geometry and mensuration. Pythagoras should be at the pupils' finger-ends; so should the fundamental idea of projection. Similar triangles should also be known, and ratios of pairs of sides should be equated with readiness. Unless all these things are known, really *known*, the earlier work in trigonometry is much hampered by time-consuming preliminaries.

Do not scare the class in the first lesson by hurling at their heads all six trigonometrical ratios. Only the tangent, sine, and cosine need be studied at first, and these *one at a time*, each as a natural derivative of practical problems of some kind.

The Tangent

The tangent should come first. Revise a few simple geometry problems in heights and distances, and let the new trigonometrical term gradually replace the geometrical ratio which the boys already know.

We might begin in this way.

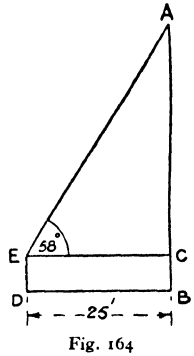


Fig. 164

Measure the height of the school flag-staff AB.

Set up the 4′ high theodolite at D, at a distance of, say, 25′ from B, and measure the angle AEC (= 58°). Make a scale drawing. By scale, AC = 40′. Hence AB = AC + CB = 40′ + 4′ = **44′**.

Thus the **ratio** $\frac{AC}{CE} = \frac{40}{25} = \mathbf{1{\cdot}6}$.

In other words, when the angle E is 58°, AC = 1·6 EC. Now look at a *series* of right-angled triangles with the base angle 58°. In every case the ratio AC/CE is the *same*, since the triangles are similar. Thus in each case AC = 1·6 EC. Hence, whatever the length of EC, we can find the length of AC by multiplying EC by 1·6. (Fig. 165.)

Thus the number 1·6 is evidently associated with the particular angle 58°. How? It measures the **ratio** AC/CE, i.e. the $\frac{\text{perpendicular}}{\text{base}}$ of the right-angled triangle AEC. If, then, we make a note of this value 1·6, as belonging to the particular angle 58°, we are likely to find it very valuable when dealing with right-angled triangles having an angle

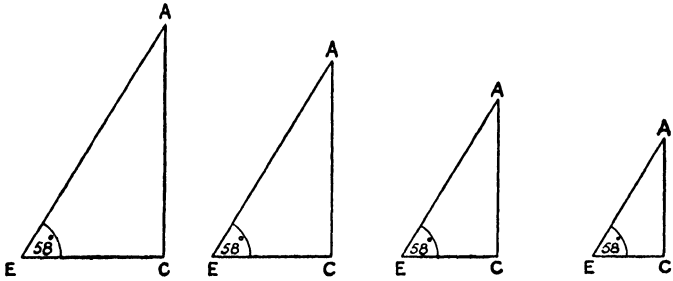


Fig. 165

of 58°; if we know the base we have merely to multiply it by 1·6 to obtain the perpendicular.*

Obviously *every* angle, not merely 58°, must have a special value of this kind. We may take a series of right-angled triangles, with different base angles, say 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, measure their perpendiculars and bases to scale, calculate their ratios, and make up a little table for future use.

If we liked, we could draw these triangles independently, though that would make the arithmetic rather tedious. An easier way is to draw a base of exactly 1″ in every case; then our arithmetic is easy (fig. 166). (Any number instead of 1 would do, but that would mean a little more arithmetic.)

$$\text{For } 40°,\ \frac{AC}{BC} = \frac{\cdot 84}{1} = \cdot 84; \quad \text{for } 30°,\ \frac{DC}{BC} = \frac{\cdot 58}{1} = \cdot 58;$$

$$\text{for } 20°,\ \frac{EC}{BC} = \frac{\cdot 36}{1} = \cdot 36; \quad \text{for } 10°,\ \frac{FC}{BC} = \frac{\cdot 18}{1} = \cdot 18; \text{ &c.}$$

* Do not mention the term hypotenuse at all. Let that wait until the sine is dealt with.

Mathematicians sometimes make the perpendicular a *tangent* to the circle, fig. 167 (they always remember that an angle is concerned with rotation): and for convenience they call the ratio $\frac{\text{perpendicular}}{\text{base}}$ the **tangent** of the angle. Thus they say, tangent 10° = ·18; tangent 20° = ·36; and so on. They generally write **tan** for tangent.

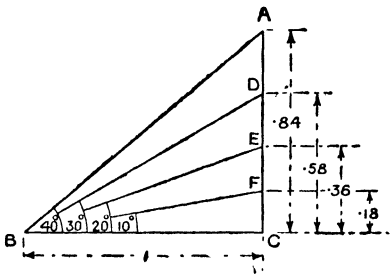


Fig. 166

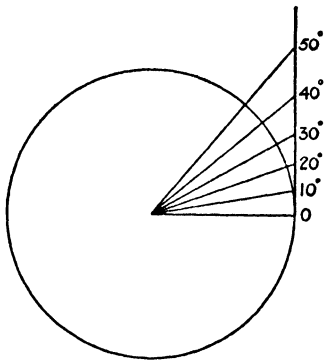


Fig. 167

But remember that *the tangent of an angle* is just a **number** which shows **how many times the perpendicular is as long as the base**; in other words, it is the ratio $\frac{\text{perp.}}{\text{base}}$.

Since $\frac{\text{perp.}}{\text{base}} = \tan$, $\therefore$ *perp.* = *base* × *tan*; hence in the triangle ABC, AC = BC × tan 35°, i.e. the tan of an angle is the **multiplier** for converting the base into the perpendicular. (Fig. 168.)

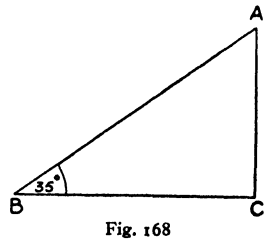


Fig. 168

There are better ways of finding these values than by merely drawing to scale; in fact, values to 7 places of decimals have been found, the work to be done with them (by surveyors, for instance) having often to be very accurate. Here is a little

table giving the values of the tangents of 10 angles, to 4 places of decimals.

tan 10° = ·1763	tan 50° = 1·1918
„ 20° = ·3640	„ 60° = 1·7321
„ 30° = ·5774	„ 70° = 2·7475
„ 40° = ·8391	„ 80° = 5·671
„ 45° = 1·0000	„ 89° = 57·29

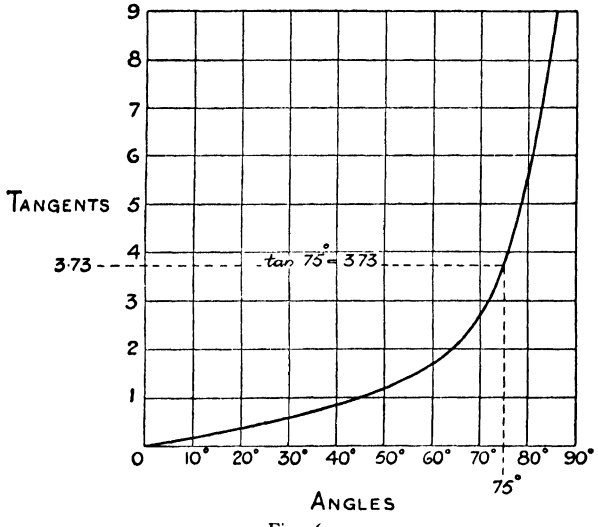


Fig. 169

There is no tangent for 90°. Can you see why? Can you see why the tan of 89° is so large? look at fig. 167. Can you see why the tan of 89° 59′ 59″ must be enormously large?

You will remember how, when we had graphed a function of x, we were able to obtain other values by interpolation. We may do the same with the tan graph; in fig. 169, plotted from the above table, you may see that the tan of 75° is *about* 3·73. To get anything like *accurate* values, we should have to have a very large graph.

We give one or two easy practical exercises.

A ladder leaning against a house makes an angle of 20° *with the wall. Its foot is* 10′ *away. How high up the house does it reach?*

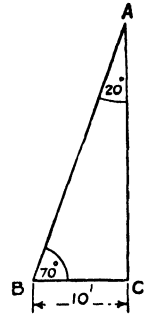


Fig. 170

We have to obtain the height AC, and we therefore require to know the tan of the angle B. Since A = 20°, B = 70°.

$$\frac{AC}{BC} = \tan B = \tan 70°$$
$$= 2{\cdot}7475 \text{ (see table or graph).}$$
$$\therefore AC = BC \times 2{\cdot}7475$$
$$= 10' \times 2{\cdot}7475$$
$$= \mathbf{27{\cdot}475'}.$$

Two boys are on opposite sides of a flag-staff 50′ *high. Their angles of elevation of the top of the staff are* 20° *and* 30°, *respectively. How far are they apart?*

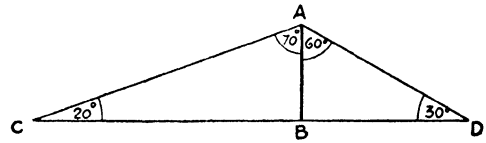


Fig. 171

Given, length of AB; Required, length of BC and BD. Since the angles at C and D are given, we may mark in the angles at A.

$$\text{Distance of boys apart} = CD$$
$$= CB + BD$$
$$= AB \tan 70° + AB \tan 60°$$
$$= 50(2{\cdot}7475 + 1{\cdot}7321)$$
$$= \mathbf{223{\cdot}98} \text{ (feet).}$$

Give ample practice in easy examples of this kind until the boys are thoroughly familiar with the fact that the tan is just a *multiplier*, sometimes less than 1, sometimes greater,

for calculating the length of the base from the perpendicular. Vary the exercises, so that the base is not always a horizontal.

The Sine

To beginners, navigation problems for introducing the sine seem to be a little difficult, and may best be taken a little later. Here is a suitable first problem. *A straight level road AB, 20 miles long, makes an angle of 37° with the west-east direction AC. How much farther north is B than A?*

In the figure we have to find the length BC. It is easy to find this length from a scale drawing: BC = 12 miles, i.e. B is 12 miles north of C.

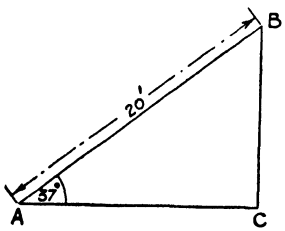


Fig. 172

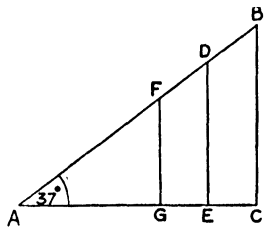


Fig. 173

Now examine the ratio $\frac{BC}{BA}$. As long as the angle A in a right-angled triangle remains 37°, the ratio must always be the same, no matter what the length of the sides, e.g. $\frac{BC}{BA} = \frac{DE}{DA} = \frac{FG}{FA}$. If then we know the value of this ratio for one triangle, we know it for all similar triangles; its value is $\frac{12}{20}$ or ·6. Thus, if AD = 14, DE = ·6 of 14 = ·84; and so on.

This new ratio is $\frac{\text{perpendicular}}{\text{hypotenuse}}$ and is called **sine.** It is a mere *number*, and represents how many times the perpendicular is as long as the hypotenuse. We ought really

to say, represents what *fraction* the perpendicular is of the hypotenuse, since the value is always less than 1. Thus sine 37° = ·6 (we generally write sine, *sin*, though we pronounce " sin " as " sine ").

Just as with the tangents, so with the sines: we might draw a series of right-angled triangles with base angles successively equal to, say, 10°, 20°, 30°, &c., and so construct a table. When we constructed fig. 166 for the tangents, we made a triangle with a *base* of 1 unit, because we wanted to

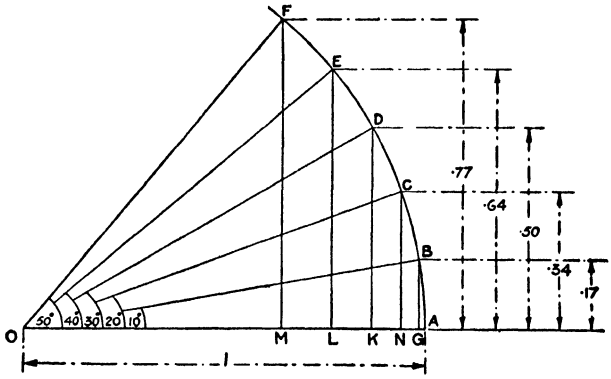


Fig. 174

make the arithmetic easy, and then the base was the denominator of the ratio. In the case of the sine, we will also make the denominator of the ratio unity, i.e. we must now make the hypotenuse unity. Here is a plan for doing this.— With O as centre, and unit radius, draw a circle. With the protractor, mark in the angles 10°, 20°, 30°, &c.; each radius OB, OC, &c., is equal to unity. From the ends B, C, D, &c., of these radii, drop perpendiculars to the base, BG, CN, DK, &c., and measure them. Since OA = OB = OC (&c.) = 1″, the perpendiculars will be fractions of 1″. Now we may obtain the sines: $\sin 10° = \frac{BG}{BO} = \frac{·17}{1} = ·17$; $\sin 20° = \frac{CN}{CO} = \frac{·34}{1} = ·34$; sin 30° = ·50, &c. By careful measurement, we

may obtain sines to 2 decimal places. Here is a little table to 4 places.

sin 10° = ·1736	sin 50° = ·7660
„ 20° = ·3420	„ 60° = ·8660
„ 30° = ·5000	„ 70° = ·9397
„ 40° = ·6428	„ 80° = ·9848
„ 45° = ·7071	„ 90° = 1·0000

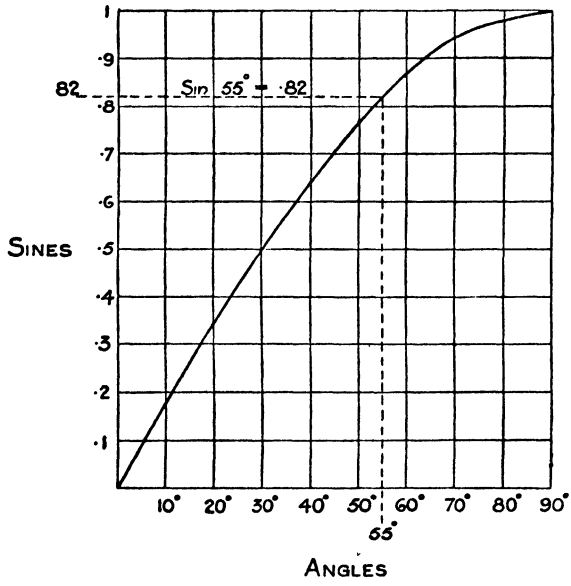


Fig. 175

By drawing the sine graph, we may obtain the sine of any other angle up to 90°, by interpolation; e.g. sin 55° is *about* ·82.

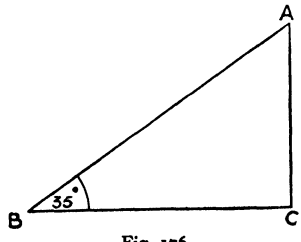


Fig. 176

Remember that the *sine of an angle* is just a **number**. Since $\frac{\text{perpendicular}}{\text{hypotenuse}} = \text{sine}$, $\therefore$ *perpendicular* = *hypotenuse* × *sine*. Hence in the triangle ABC, AC = AB sin 35°,

i.e. the sine of an angle is the **multiplier** for converting the hypotenuse into the perpendicular. In this case the multiplier happens to be always a fraction.

Here are one or two easy typical problems:

A ladder 30′ *long stands against a vertical wall. It makes an angle of* 70° *with the ground. What is the height above the ground of the top of the ladder?* (Fig. 177.)

Given, AB = 30′; ∠ABC = 70°. Required AC.

$$\frac{AC}{AB} = \sin 70° = \cdot 94 \text{ (from table or graph)},$$

$$\therefore AC = AB \times \cdot 94 = 30' \times \cdot 94 = \mathbf{28 \cdot 2'}.$$

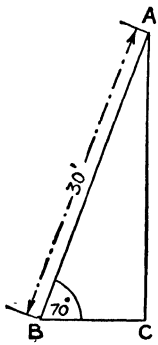


Fig. 177

A railway slopes at an angle of 10° *for a distance of* 1000 *yards. What is the difference in level of its two ends?* (Fig. 178.)

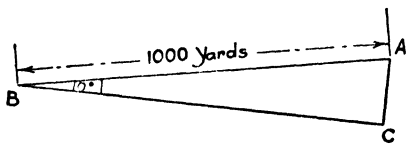


Fig. 178

Given, AB = 1000 yards; ∠ABC = 10°. Required AC.

$$\frac{AC}{AB} = \sin 10° = \cdot 1736.$$

$$\therefore AC = AB \times \cdot 1736 = 1000 \text{ yd.} \times \cdot 1736 = \mathbf{173 \cdot 6 \text{ yd.}}$$

The Cosine

Projection problems form a suitable beginning.—*AB represents a sloping road* 500 *yd. long. A surveyor finds that it makes an angle of* 30° *with the horizontal. What is the*

projected length on a horizontal line, such as would be shown on an ordnance map?

The *projection* of a line AB on another line MN is the distance between two perpendiculars drawn to MN from the

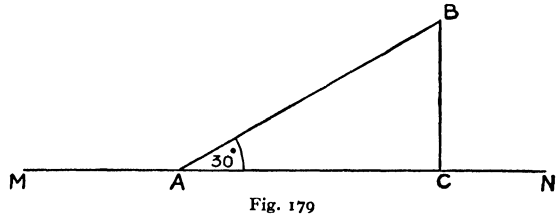


Fig. 179

ends of AB. If MN passes through A, one end of the road, only one perpendicular (BC) is necessary. The projection is then AC.

Given, AB = 500 yd.; $\angle BAC = 30°$. To find AC.

From a scale drawing we find that AC = 433 yd.

Now examine the ratio $\frac{AC}{AB}$. As long as the angle A in a right-angled triangle remains 30°, the ratio must always be

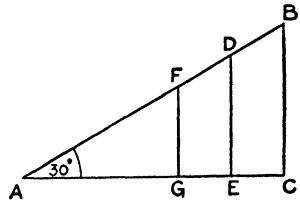


Fig. 180

the same, no matter what the length of the sides, e.g. $\frac{AC}{AB} = \frac{AE}{AD} = \frac{AG}{AF}$, for the triangles ABC, ADE, AFG are all similar. If then we know the value of this ratio for one triangle, we know it for all similar triangles. Its value is $\frac{433}{500}$ or **·866.** This new ratio, $\frac{\text{base}}{\text{hypotenuse}}$, is called the **cosine** (generally written

cos). It is a mere *number*. Since $\frac{\text{base}}{\text{hyp.}} = \cos$, *base* = *hypotenuse* × *cos*. Hence, in the triangle ABC, AC = AB cos 30°, i.e. the cosine of an angle is the **multiplier** for converting the hypotenuse into the base. In this case, again, the multiplier always happens to be a fraction.

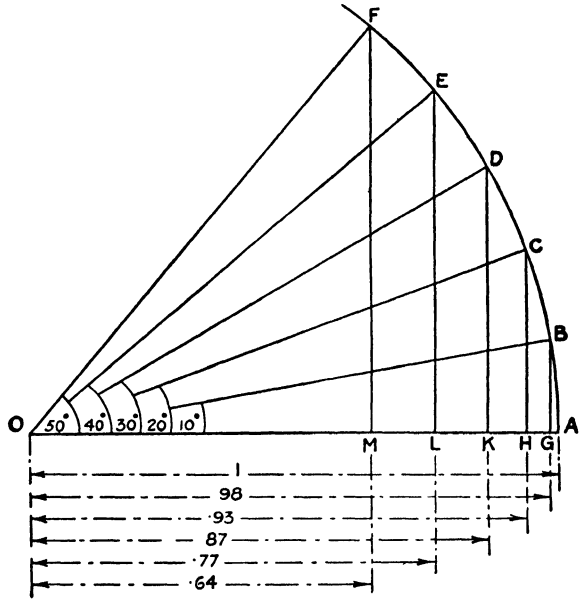


Fig. 181

Just as with the tangent and sine, so with the cosine: we may draw a series of right-angled triangles with base angles successively equal to say 10°, 20°, 30°, &c., measure them up, and so construct a table. And as in the case of the sine, we will so construct our triangles that the length of the hypotenuse is always unity.

$$\cos 10° = \frac{OG}{OB} = \frac{·98}{1} = ·98; \quad \cos 20° = \frac{OH}{OC} = \frac{·93}{1} = ·93;$$

$$\cos 30° = \frac{OK}{OD} = \frac{·87}{1} = ·87; \text{ &c.}$$

Here is a little table of cosines, to 4 places of decimals:

cos 10° = ·9848	cos 50° = ·6428
,, 20° = ·9397	,, 60° = ·5000
,, 30° = ·8660	,, 70° = ·3420
,, 40° = ·7660	,, 80° = ·1736
,, 45° = ·7071	,, 90° = 0

By drawing a cosine graph from the above values, we can, by interpolation, obtain the value of any other angle up to 90°, e.g. cos 35° = ·82 (approx.).

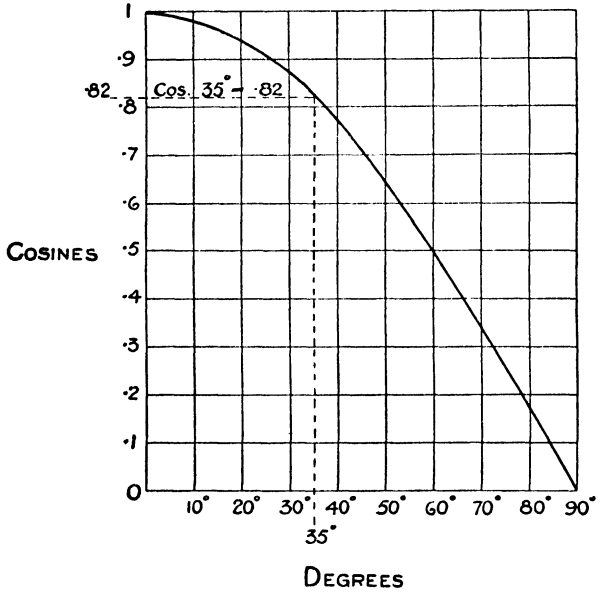


Fig. 182

Compare the sine and cosine graphs. Each is an exact looking-glass reflection of the other. Now look at the two tables of sines and cosines. Each is the other turned upside down. Evidently there is a curious connexion between sines and cosines.

It is easy to draw both sine and cosine curves by means

of intersecting points made by (1) parallels from an angle-divided quadrant, and (2) perpendiculars from the correspondingly divided abscissa. Note how the two curves together form a symmetrical figure, and how they cut in one point. What do you infer about this point common

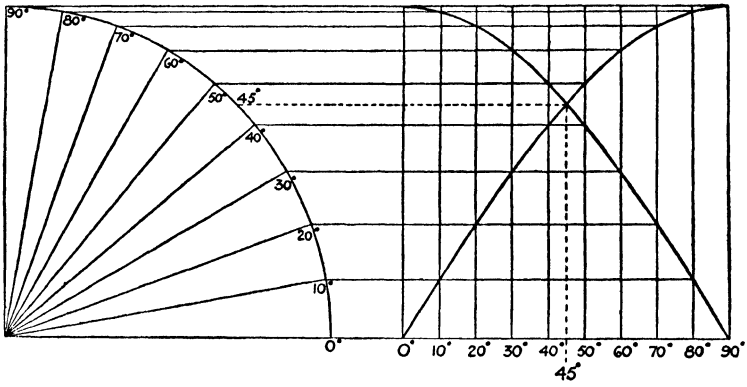


Fig. 183

to the two curves? There is evidently *some* angle the sine and cosine of which have the same value. Look at the two tables.

Easy cosine problems.—(1) *The legs of a pair of compasses are* 5″ *long. Find the distance between the points when the legs are opened to an angle of* 80°.

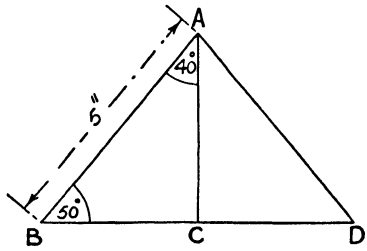


Fig. 184

Given: AB = AD = 5″; ∠BAD = 80°. If AC is the bisector of ∠BAD, ∠BAC = 40°; hence ∠ABC = 50°.

Required: length of BD (= 2BC).

$$\frac{BC}{AB} = \cos 50°; \quad \therefore BC = AB \cos 50°$$
$$= 5'' \times \cdot 64,$$
$$\therefore BD = 10'' \times \cdot 64 = \mathbf{6 \cdot 4''}.$$

(2) *C is any point in the line XY. CA and CB are drawn on the same side of XY so that* $CA = 4''$, $CB = 5''$, $\angle XCA = 40°$, $\angle YCB = 60°$. *Find the projection of ACB on XY.*

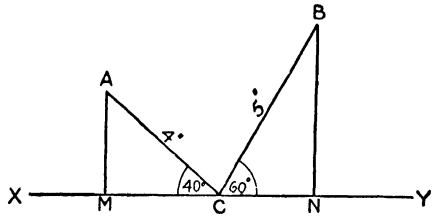


Fig. 185

Drop perpendiculars AM, BN, on XY. Then the projection of ACB on XY is MN. Required: the length of MN.

$$MN = CM + CN$$
$$= AC \cos 40° + BC \cos 60°$$
$$= (4'' \times \cdot 77) + (5'' \times \cdot 50)$$
$$= \mathbf{5 \cdot 58''}.$$

Now give the boys the same two problems again, making them use the sine instead of the cosine. Hence give them the first notion that the sine and cosine are so closely related that one may sometimes be used instead of the other. Make them remember this:

If the hypotenuse is given,

(1) use the sine to find the perpendicular;
(2) use the cosine to find the base.

The sin, cos, and tan: Simple Inter-relations

Introduce the notation a, b, and c to represent the number of units of length in the sides opposite the angles correspondingly named. Also show that since $\angle A + \angle B = 90°$ $A = 90° - B$, and $B = 90° - A$. Now tabulate:

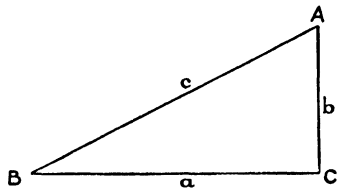


Fig. 186

$$\frac{b}{a} = \tan B, \text{ or } b = a \tan B; \quad \frac{a}{b} = \tan A, \text{ or } a = b \tan A.$$

$$\frac{b}{c} = \sin B, \text{ or } b = c \sin B; \quad \frac{a}{c} = \sin A, \text{ or } a = c \sin A.$$

$$\frac{a}{c} = \cos B, \text{ or } a = c \cos B; \quad \frac{b}{c} = \cos A, \text{ or } b = c \cos A.$$

Hence,

(1) Since $\frac{a}{b} = \tan A$, and $\frac{b}{a} = \tan B$, $\therefore$ $\mathbf{\tan A = \frac{1}{\tan B}}$.

(2) Since $\frac{a}{c} = \sin A = \cos B$, $\therefore$ $\mathbf{\sin A = \cos B}$.

(3) Since $\frac{b}{c} = \cos A = \sin B$, $\therefore$ $\mathbf{\cos A = \sin B}$.

(4) $\frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b}$, $\therefore$ $\mathbf{\frac{\sin A}{\cos A} = \tan A}$.

(5) Similarly, $\mathbf{\frac{\sin B}{\cos B} = \tan B}$.

(6) Since $B = 90° - A$, and $\sin A = \cos B$,
$$\therefore \sin A = \cos(90° - A).$$

(7) Since $B = 90° - A$, and $\cos A = \sin B$,
$$\therefore \cos A = \sin(90° - A).$$

All these relations must be carefully committed to memory.* Note that the last two may be summed up in this way: *the sine of an angle is the cosine of its complement.* Explain the significance of *co-* in cosine.

Some teachers prefer the words *opposite* and *adjacent* instead of *perpendicular* and *base*, but experience suggests that for beginners the latter terms are preferable. The main thing is to adopt *one* form of words and stick to it.

The *secant*, *cosecant*, and *cotangent*. — These should be remembered as the **reciprocals** of the cos, sin, and tan, respectively. Give easy examples to show the appropriateness of the forms beginning with *co*.

The ratios of common angles.—The sin, cos, and tan of the common angles 30°, 45°, and 60° should be memorized

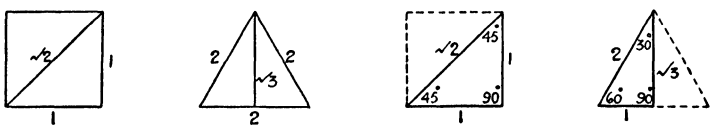


Fig. 187

as soon as the nature of the three functions is understood. Teach the boys to visualize the half square and the half equilateral triangle—the obvious aids to memory.

*Do not despise some simple form of mnemonics when, with beginners, confusion is almost inevitable, as in the case of the three trigonometrical functions; e.g. remember

(1) **Tan** $= \frac{\text{Perp}}{\text{Base}}$ by the words **Tanned Post Boy**,

(2) **Sin** $= \frac{\text{Perp}}{\text{Hyp}}$ by the words **Sign, Please, Henry**,

(3) **Cos** $= \frac{\text{Base}}{\text{Hyp}}$ by the words **Costly Black Hat**,

or some other form of catchy words.

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

A little later, the table should be extended to 0° and to 90°, and eventually to 180°. When discussing the 0° and 90° values, draw a series of right-angled triangles, beginning with a very small acute angle A and *very nearly* 0°, and ending with an angle A *very nearly* 90°. A discussion of just one general figure, without reference to the actual values of particular cases, is, with beginners, almost profitless. Do not say that the tan of 90° is " infinity ", a term which is beyond the comprehension of beginners. Adopt some such non-committal form of words as " immeasurably great ".

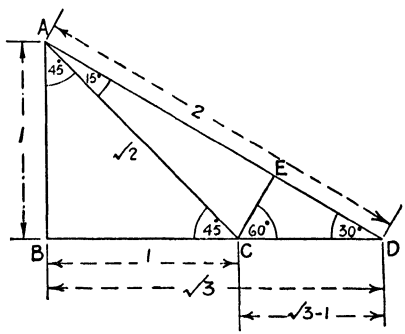


Fig. 188

The ratio for 15° is easily obtained from this figure, derived from fig. 187. ABC is an isosceles rt. ∠d Δ, sides 1, 1, 2, angles 45°, 45°, 90°. ABD is a half equil. Δ, sides 1, 2, $\sqrt{3}$, angles 30°, 60°, 90°. Thus ∠CAD = 15°, and

$CD = (\sqrt{3} - 1)$. From C, drop a perpendicular on AD. Since CED is a half equil. $\triangle$, $CE = \frac{1}{2}CD = \frac{\sqrt{3} - 1}{2}$. Hence $\sin 15° = \frac{CE}{CA} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$. From this the other ratios of 15° are easily found, and then those of 75°.

For 18°, fig. 113*a* is the key. The small angles of a regular pentagram are 36°, and hence the sine of half the angle is $\frac{\sqrt{5} - 1}{4}$. Let the boys work this out for themselves; it is a good exercise; the other ratios may be derived arithmetically, but the first (the sine) *must* be established geometrically. The derivation for multiples of 18° (36°, 54°, 72°) is suitable work a year later.

The following identities may readily be established geometrically.

1. **$\sin^2 A + \cos^2 A = 1$.**—This is seen from a figure to be a direct application of Pythagoras. Let the derivatives also be noted: $\sin A = \sqrt{1 - \cos^2 A}$, $\cos A = \sqrt{1 - \sin^2 A}$.

2. **$1 + \tan^2 A = \sec^2 A$.**—Here a hint is necessary to the boys to work " backwards ". We have to prove:

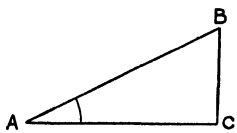


Fig. 189

$$1 + \frac{BC^2}{AC^2} = \frac{AB^2}{AC^2},$$

$$\text{i.e. } \frac{AC^2 + BC^2}{AC^2} = \frac{AB^2}{AC^2}.$$

The boys now observe that the numerators form the simple Pythagoras relation. Hence they write out:

$$AC^2 + BC^2 = AB^2, \qquad (Pythag.)$$

$$\therefore \frac{AC^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AB^2}{AC^2}, \qquad (Div.\ by\ AC^2)$$

$$\therefore \quad 1 + \tan^2 A = \sec^2 A. \qquad \text{Q.E.D.}$$

The obvious derivatives should follow.—Give several easy

examples to verify the rule that if any one trigonometrical ratio of an angle be given, the other ratios may all be calculated without reference to tables. But all fundamental relations *must* be established geometrically. Geometry must take precedence over algebra.

Heights and Distances

It is surprising what a great variety of problems, in three as well as in two dimensions, may be solved by means of the small amount of trigonometry already touched upon. Give plenty of such problems until the sin, cos, and tan are as familiar as the multiplication table, are, indeed, a *part* of the multiplication table. Insist all along that every problem on heights and distances is really a geometry problem with an arithmetical tail, but that the arithmetic is made easy for us because all the necessary multiplication sums have been worked out and the answers put into a book of tables, the multipliers having been given the rather fanciful names of sin, cos, tan, &c. In every problem we are concerned with a triangle; the length of one side is always given, and the multipliers in the book of tables enable us to find the other sides; to *find* the multipliers, we have to know the *angles* of the triangle.—Four-figure tables of natural sines, cosines, and tangents, for whole degrees only, are enough for beginners. Let logs wait. Let the problems be easy and varied. Three-dimensional problems may be included quite soon, though at least a little solid geometry should have been done previously.

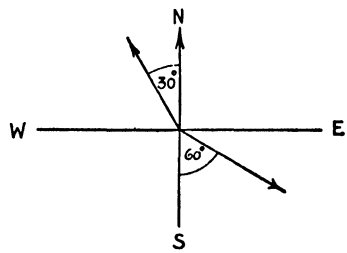


Fig. 190

When setting problems involving " bearings ", avoid, as a rule, the old terms " north-west ", " south-east ", &c., and adopt the surveyor's plan, always placing N. or S. first,

then so many degrees W. or E., thus N. 30° W., S. 60° E. the angle always being measured from the N.—S. line.

The drawing of figures for heights and distances. If a figure lies wholly in a horizontal plane, there is seldom much difficulty, especially if drawing to scale has been properly taught in the Junior Forms. Figures in a vertical plane are also readily drawn, though the angles of elevation and depression are sometimes confused by boys whose early practical geometry has not been properly taught.

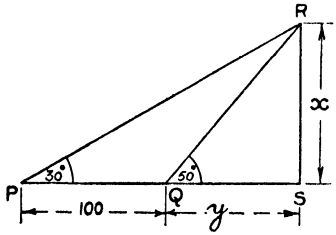


Fig. 191

Consider this old problem: *From a point P in a horizontal plane, an observer notes that a distant inaccessible tower subtends an angle of* 30°. *He walks to Q, a distance of* 100 *ft., towards the tower, and finds that the tower then subtends* 50°. *Find the height of the tower and the man's distance from it.*

Explain how easy it is to work with tangents, as the figure readily shows.

(1) RS = PS $\tan 30°$, i.e. $x = (y + 100) \tan 30°$.
(2) RS = QS $\tan 50°$, i.e. $x = y \tan 50°$,
$\therefore (y + 100) \tan 30° = y \tan 50°$.

Hence y can be found, then x by substitution. The long succession of statements in some of the textbooks is unnecessary and merely serves to bewilder the boys.

The problem is, of course, easy enough. It is only when the measured distance PQ is not in the same plane as PRS, i.e. is not directly towards the tower, that the boys are baffled, because of the difficulty of drawing a suitable figure in 3 dimensions.

We will deal with the three-dimensional figure difficulty in one or two problems:

A wall 12 *ft. high runs east and west. The sun bears*

S. 60° *W. at an elevation of* 32°. *Calculate the breadth of the shadow of the wall on the ground.*

This is taken from one of the best of our textbooks. It is, of course, very simple, yet I have given it to several lots of boys, and the necessary figures have nearly always puzzled them. Had the sun been directly south, a stick placed vertically at O would have had its shadow cast on the ground in the direction ON (fig. 192). But as the sun was S. 60° W., the stick at O would have had its shadow cast on the ground in the *direction* OQ_1, so that the shadow makes an angle of 30° with the vertical plane (wall) in EW (fig. 193).

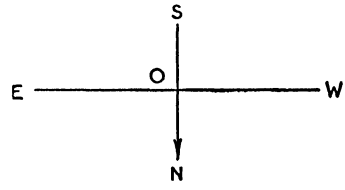


Fig. 192

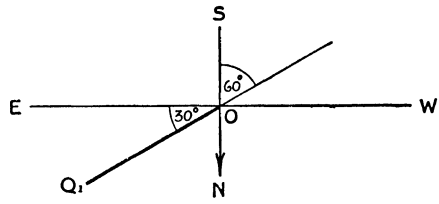


Fig. 193

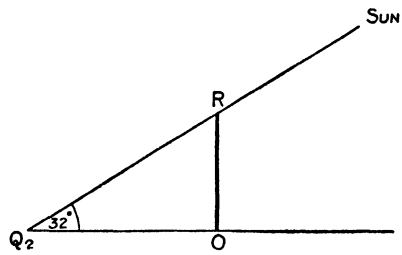


Fig. 194

But as the sun has an elevation of 32°, the *length* of the shadow of the stick RO would be Q_2O, Q_2 being the far end of the shadow on the ground (fig. 194). (During the day

this shadow would occupy a succession of positions, just as if it were pivoted on the stick, following the sun round.)

We have to consider these two things, the *direction* and the *length* of the shadow in a three-dimensional figure. Let ABCD be the wall running east-west; it may be looked upon as a close set of palings, with one paling RO taking the place of the stick. Of course the shadow of the whole wall will be cast, but we will first consider the shadow of RO only. As the sun is at 32°, the shadow must be cast

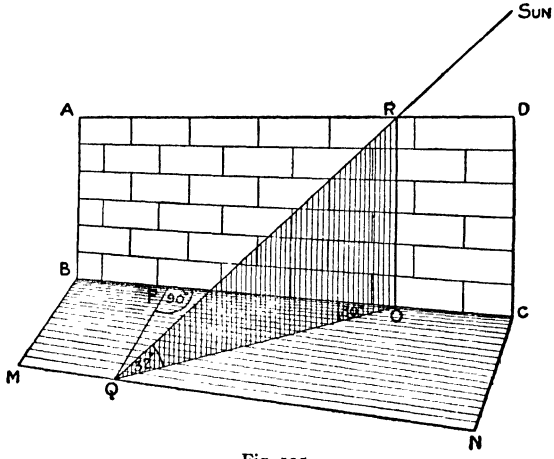


Fig. 195

somewhere on the ground as a length OQ. That " somewhere " is given us by the sun's position (irrespective of its height) at a particular time in the day, viz. S. 60° W., i.e. OQ will make 30° with the east-west wall, or $\angle BOQ = 30°$.

Now all the palings will cast shadows parallel to OQ, and thus we shall have a belt of shadow, on the ground BMNC, of a *breadth* equal to the perpendicular QP to the wall. Thus we have to find the **length of PQ.**

To find the length of PQ we may solve the $\triangle PQO$ in the H.P. In the $\triangle RQO$ (vertical plane), $RO = 12'$; $QO = RO \cot 32° = RO \tan 58° = 12' \times 1{\cdot}6 = \mathbf{19{\cdot}2'}$. In the $\triangle PQO$, $\mathbf{PQ = QO \sin 30° = 19{\cdot}2' \times {\cdot}5 = 9{\cdot}6'}$.

For beginners a model is far better than a sketch; then the angles do not mislead. Even a book held upright on the desk to represent a vertical plane, and then a pencil placed in position to represent a line in an oblique plane, will help the eye greatly. But some long hat-pins stuck vertically into a board, with pieces of cotton tied round under the heads (a snick made with the laboratory file will help to secure the cotton), stretched and held fast by a twist under the head of a drawing-pin, will enable the boys to make in a minute or two a model of almost any figure that may be required.

Here is another problem and the provided figure from the same excellent textbook. The figure has puzzled several lots of boys.

"*A hillside is a plane sloping at* 27° *to the horizontal. A straight track runs up the hill at an angle of* 34° *with a line of greatest slope. What angle does the track make with the horizontal?*"

"AB is the line of intersection of the hillside and a horizontal plane ABC. AF, BE are lines of greatest slope

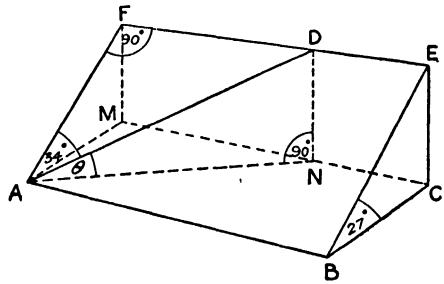


Fig. 196

meeting a horizontal at F, E. Let the track AD cut EF at D. Draw DN, EC perpendicular to the H.P., ABC. Then AN is the projection of AD on ABC. It is required to find $\angle DAN = \theta$, say." Then follows the solution, simple enough, of course, from considerations of the 3 rt. ∠d △s ECB, AND, AFD, the first two in V.P.s, the last in an oblique

plane. A model with 3 hat-pins at FM, DN, and EC, and drawing-pins at A, B, C, M, N, and connecting threads, would make the whole thing clear at once. Otherwise a few shading lines might be added, as in fig. 197; the 3 planes are then shown clearly.

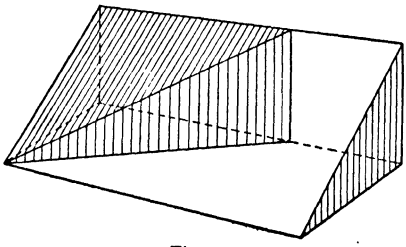


Fig. 197

Here is a simple problem from another book, the figure for which has often given beginners trouble.—*The extremity of the shadow of a flag-staff FG, 6′ high, standing on the top of a square pyramid 34′ high, just reaches the side of the base and is distant 56′ and 8′ respectively from the extremities of that side. Find the sun's altitude.*

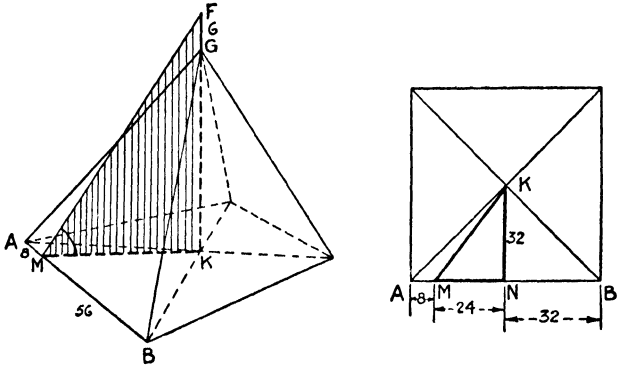


Fig. 198

FK = FG + GK = 6′ + 34′ = 40′. We have to find the ∠FMK in the rt. ∠d △ FMK. In this △ we know FK; and we can find MK by Pythagoras from △ KMN in the plan (second figure):

$$KM = \sqrt{32^2 + 24^2} = 8 \times 5 = 40.$$

$$\text{Tan FMK} = \tfrac{40}{40} = 1; \quad \therefore \angle FMK = \mathbf{45°}.$$

In practical problems, boys are constantly blundering over compass bearings. Impress on the class that the *difference between the bearings* of two distant objects is the angle made by the two lines, *drawn in the H.P.*, from the observer to the objects. If the objects are *above* the H.P., the difference between the bearings is still an angle on the H.P., viz. the angle between the two vertical planes drawn through the observer and each of the objects.—An observer is at S in a H.P., his south-north line being SN. PQ and RT are two vertical poles. He takes the bearings of the two poles and

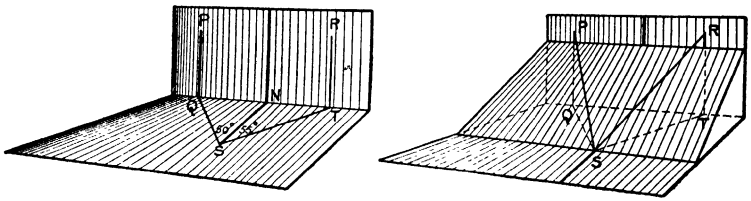


Fig. 199

finds that their horizontal angles are respectively N. 50° W. and N. 55° E., i.e. the difference between their bearings is **105°**.

Now suppose he could not see the bottom of the poles, because of an intervening hill. The observer would have to point his telescope at the pole-tops P and R, and he could then, if he wished, take the angles of elevation. But his purpose now is to take *the difference between the bearings*, and he would therefore observe where each vertical plane containing the tilted telescope cut the horizontal plane. The angle to be measured ($\angle$QST) would be exactly the same as before. Impress on the boys that the observer could not measure the angle PSR in the oblique plane; his theodolite does not permit of that. And even if he could, the angle would not represent the angle between the bearings.

Here is an illustrative problem. *Find the distance between the tops of the spires of two distant inaccessible churches.* (It would be taken rather later in the course.)

Measure off a base line AB in a suitable position, and from each end take the bearings of both spires, P and Q, draw

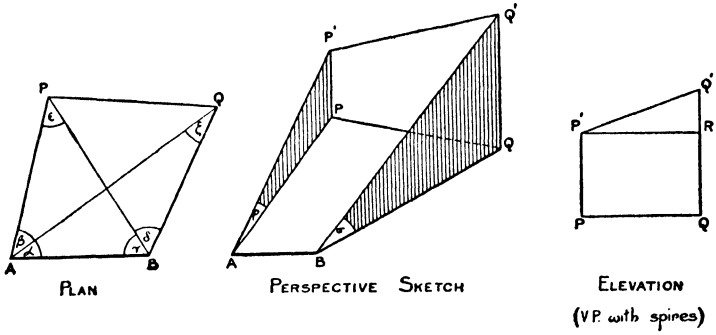


Fig. 200

a ground plan, and mark in the ∠s α, β, γ, δ. The ∠s ϵ and ζ may be calculated if wanted.

In the △PAB, AB and the ∠s are known;
∴ **PA** and **PB** can be calculated.
In the △QAB, AB and the ∠s are known;
∴ **BQ** can be calculated.
In the △PBQ, PB and BQ are calculated, and ∠δ known,
∴ **PQ** can be calculated.

Now examine the perspective sketch, with the 2 spires P'P and Q'Q in position. We have to find P'Q'. We know AP, BQ, PQ. Measure the ∠s of elevation ρ and σ; $P'P = AP \tan\rho$, $Q'Q = BQ \tan\sigma$; hence **P'P** and **Q'Q** are known. Hence in the elevation, everything is known except P'Q', and this is easily calculated by Pythagoras.

Make the boys do this practically. Any two distant tall objects will do.

The Obtuse Angle

Angles up to 180° should be considered at an early stage, but, before angles greater than 180° are considered, substantial progress on the practical side of the subject is desirable.

Remind the boys that the rotating arm of the angle, regarded as the hypotenuse of a rt. ∠d Δ, may be carried round from 90° to 180°, the pivot being the point of intersection of the co-ordinate axes. Refer to the work on graphs, and the rule of signs for the second quadrant; all x values measured to the left of the origin are regarded as negative. If, for instance, we consider the triangle BOC in the second quadrant, the hypotenuse and perpendicular are positive as in the first quadrant, but the base OC is negative. Proof? There is none. It is merely an accepted convention.

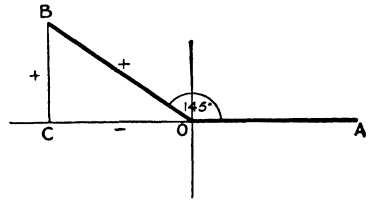


Fig. 201

Suppose, then, we have an angle greater than 90°, say 145°. How am I to find the value of its *sine*, *cosine*, and *tangent*?

Exactly as before. From any point on the rotating arm OB, drop a perpendicular to the fixed arm OA (produced backwards, because necessary), and take the ratios in the same way as we did for acute angles. But *remember the signs*. For this angle 145°, the hypotenuse is OB, the base OC, the perpendicular BC.

Hence: $\sin 145° = \frac{BC}{OB}$; $\cos 145° = \frac{-OC}{OB}$, $\tan 145° = \frac{BC}{-OC}$.

It may not *look* as if the perpendicular BC concerned the angle AOB (145°), but how else could a perpendicular for 145° be drawn?

Now consider an *acute* angle equal to the angle BOC in fig. 201. Evidently it is (180° − 145°) or 35°. Fig. 202

shows OB′ equal to OB in fig. 201. Hence the triangle B′OC′ has sides of exactly the same length as the triangle BOC. $\text{Sin}\,35° = \frac{B'C'}{OB'}$; $\cos 35° = \frac{OC'}{OB'}$; $\tan 35° = \frac{B'C'}{OC'}$.

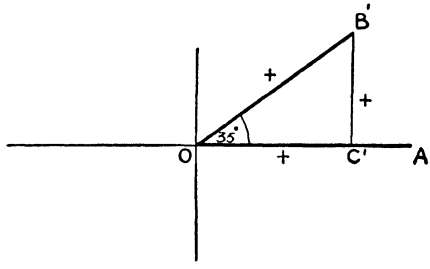


Fig. 202

Comparing the ratios for 145° and 35°, we see that:

$$\sin 35° = \sin 145°,$$
$$\cos 35° = -\cos 145°,$$
$$\tan 35° = -\tan 145°.$$

The same thing must apply to *any* pair of angles whose sum is 180°. Thus we may say,

$$\sin A = \sin(180° - A),$$
$$\cos A = -\cos(180° - A),$$
$$\tan A = -\tan(180° - A).$$

If the above demonstration is attempted from a single figure (as it might well be from fig. 201), slower boys will inevitably be confused.

It is an excellent plan to make boys in Upper Sets express their ratios in terms of co-ordinates, i.e. to call the rotating arm r, and its extremity P (x, y).

Give plenty of oral practice in the obtuse angle relations, e.g. $\tan 100° = -\tan(180° - 100°) = -\tan 80° = -5{\cdot}67$.

The General Triangle and its Subsidiary Problems

Before proceeding to the solution of triangles, revise carefully the geometry of congruent triangles, and note what various sets of data are necessary and sufficient for copying a triangle. A triangle is determined uniquely if we are given (1) the 3 sides, (2) 2 sides and the included angle, (3) 1 side and 2 angles. If we are given 2 *sides and the angle opposite one of them*, there may be 2 solutions, or 1 solution, or no solution.

A triangle cannot be determined unless the data include at least one side.

Thus the necessary data include 3 elements, at least one of them being a length.

All the formulæ in this section *must* be established geometrically. As geometrical exercises they are all first-rate.

1. *In any triangle ABC*, $a = b\cos C + c\cos B$.—Show that this relation holds good for both acute-angled and obtuse-angled triangles. It is simply a question of dropping a perpendicular and considering separately the two resulting right-angled triangles.

Do not forget the sister expressions in this and subsequent formulæ. The one thing to keep in mind is the cyclic order of the letters A, B, C, and a, b, c. For instance, the above identity may be written,

$$b = c\cos A + a\cos C,$$

$$\text{or,}\quad c = a\cos B + b\cos A.$$

2. *The sine formula.*—In any triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. As before, show that the relation holds good for obtuse-angled as well as for acute-angled triangles.

The "ambiguous case" should receive special attention. Link up the work with the closely analogous case in geometry. In fact the problems are the same. Readily understood as they generally are, they are often half forgotten. They must be regarded as sufficiently tedious and troublesome as to merit special and repeated attention.

3. *The cosine formula.*—In any triangle ABC, $c^2 = a^2 + b^2 - 2ab \cos C$. Again be careful to consider both acute-angled and obtuse-angled triangles. Link up carefully with Pythagoras and its extensions (Euclid, I, 47; II, 12, 13). The solution is straightforward and seldom gives trouble.

When solving triangles, use sine or cosine formulæ?

If given (1) 3 *sides*, or (2) 2 *sides and included angle*, use *cosine* formula for first operation; then continue with the quicker *sine* formula, using it to find the smaller of the two remaining angles.

If given (3) 2 *angles and* 1 *side*, or (4) 2 *sides and a not-included angle*, use *sine* formula.

N.B. (1) If given 3 sides, *find the smallest angle first.*

(2) If the given triangle is isosceles, use neither formula, but drop a perpendicular to the base.

4. *The tangent formula.*—In any triangle ABC (where $b > c$),

$$\frac{\tan \frac{1}{2}(B - C)}{\tan \frac{1}{2}(B + C)} = \frac{b - c}{b + c}.$$

This is a useful alternative, more suitable for log calculations, when 2 sides and the included angle are given. The *cos* formula is often cumbrous in application, not being suitable for log calculation.

The boys *must* learn to establish the formula geometrically, from first principles, and not derive it from other trigonometrical formulæ. But for beginners it is generally puzzling. Begin by giving them a particular case to which they may apply the formula. Let the sides of a triangle be, say, 11, 13, 16, and let the boys work out the angles from their cos

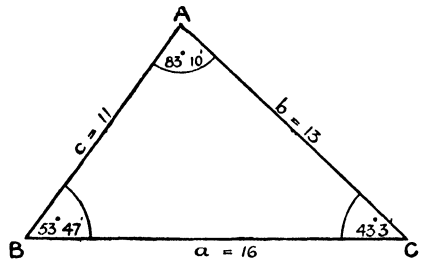


Fig. 203

and sine rules, using four-figure tables. The angles shown in the figure are, to the nearest minute,

$$\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)} = \frac{\tan\frac{1}{2}(53°\,47' - 43°\,3')}{\tan\frac{1}{2}(53°\,47' + 43°\,3')}$$

$$= \frac{\tan 5°\,22'}{\tan 48°\,25'} = \frac{\cdot 094}{1\cdot 127} = \frac{1}{12}.$$

Again: $$\frac{b-c}{b+c} = \frac{13-11}{13+11} = \frac{1}{12}.$$

Thus they see that, at least in this particular case, the theorem holds good. Working out a particular case in this way, they grasp the fact that $\tan\frac{1}{2}(B-C)$ is, after all, just the tan of a simple angle. So with $\frac{1}{2}(B+C)$.

The problem now is to devise a figure which shall actually show these angles $\frac{1}{2}(B-C)$ and $\frac{1}{2}(B+C)$; also the sum $(b+c)$ and the difference $(b-c)$ of the sides.

There are two subsidiary points to note first.

(1) In any triangle,

since $A + B + C = 180°$,

$$\therefore \quad \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = 90°,$$

$$\therefore \quad \frac{B}{2} + \frac{C}{2} = 90° - \frac{A}{2};$$

or

$$\frac{B + C}{2} = 90° - \frac{A}{2}.$$

Again:

$$\frac{B+C}{2} - C = \frac{B+C}{2} - \frac{2C}{2},$$

or

$$\frac{B+C}{2} - C = \frac{B-C}{2}.$$

Similarly:

$$\frac{C + A}{2} - A = \frac{C - A}{2}; \text{ \&c.}$$

Give plenty of oral work on these points, with blackboard figures to illustrate.

(2) How have we been able in geometry to show the sum and difference of two sides of a triangle?

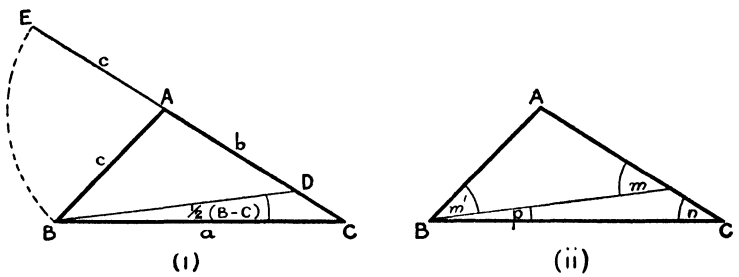


Fig. 204

The *sum* of *b* and *c* may be shown by swinging round AB on A to AE, so that AE = AB; hence CE = (*b* + *c*); the *difference* may be shown by cutting from AC a part AD equal

to AB; thus DC = AC − AB = $(b - c)$. The same figure shows $\frac{1}{2}$(B − C). For (fig. ii), since $m = n + p$, $\therefore m' = n + p$, $\therefore m' + p = n + 2p$, or B = C + 2$p$; $\therefore p = \frac{1}{2}$**(B − C).**

(i) Now we may draw the required figure.

With centre A and radius AB, describe the circle EBD, and produce CA to E. Evidently EC = $(b + c)$, DC = $(b - c)$. Join EB. ∠E (at circf.) = $\frac{1}{2}$A (at centre). We know that ∠DBC = $\frac{1}{2}$(B − C), but there seems to be no obviously

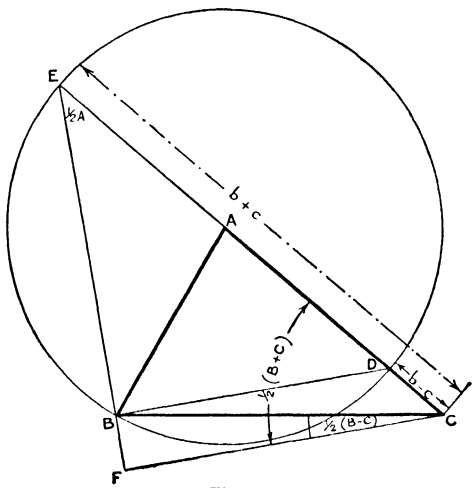


Fig. 205

simple way of using it. But if we draw CF parallel to BD, to meet EB produced in F, ∠**BCF** = ∠DBC = $\frac{1}{2}$**(B − C).**

Again, in the right-angled triangle EFC, since ∠E = $\frac{1}{2}$A, ∠**ECF** = $\frac{1}{2}$**(B + C).**

Thus we have the two angles and the two lengths for the tan formula:

$$\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)} = \frac{\tan BCF}{\tan ECF} = \frac{\frac{BF}{FC}}{\frac{EF}{FC}} = \frac{BF}{EF} = \frac{DC}{EC} = \frac{b-c}{b+c}.$$ Q.E.D.

(ii) A boy might very well ask if we could use the figures

made by drawing the circle with radius AC instead of AB.

Exterior $\angle A$ at centre $= B + C$; $\therefore$ $\angle D$ at circumference $= \frac{1}{2}(B + C)$.

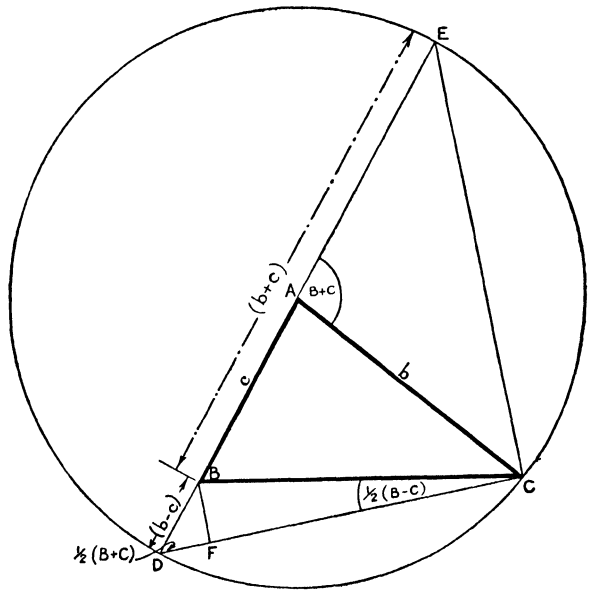


Fig. 206

Also $\angle BCD = \angle B - \angle D$ (ext. $\angle$ property) $= \angle B - \frac{1}{2}(B + C) = \frac{1}{2}(B - C)$.

We may take the tan of the last angle by dropping the $\perp$ BF.

$$\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)} = \frac{\tan BCF}{\tan BDF} = \frac{\frac{BF}{FC}}{\frac{BF}{FD}} = \frac{FD}{FC} = \frac{BD}{BE} = \frac{b-c}{b+c}.$$ Q.E.D.

There is no essential difference between the two proofs.

(iii) Or a boy might ask if we could not derive the angle $\frac{1}{2}(B + C)$ from the $\frac{1}{2}A$ obtained by actually bisecting A.

Let AD be the bisector, and let CD meet it at right angles. Draw BF perpendicular to CD produced, and BE perpendicular to AD.

Evidently $\angle BCF = \frac{1}{2}(B - C)$, and $\angle ABE = \angle ACD = \frac{1}{2}(B + C)$.

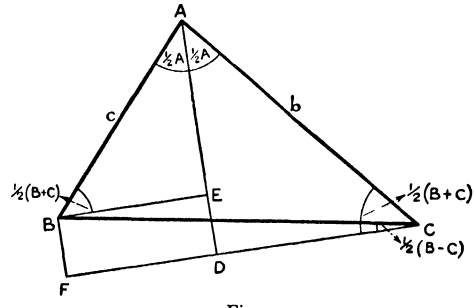


Fig. 207

The figure does not give us a length AC + AB ($= b + c$), or a length AC − AB ($= b - c$). But we can *project* AB and AC on to FC; FD ($=$ BE) is the projection of AB, and DC is the projection of AC, and so we may obtain what we want in this way:

(1) $AD = b \sin\frac{1}{2}(B + C)$; $AE = c \sin\frac{1}{2}(B + C)$;

$$\therefore \mathbf{BF} = AD - AE = \mathbf{(b - c) \sin\tfrac{1}{2}(B + C).}$$

(2) $DC = b \cos\frac{1}{2}(B + C)$; $FD = c \cos\frac{1}{2}(B + C)$:

$$\therefore \mathbf{FC} = DC + FD = \mathbf{(b + c) \cos\tfrac{1}{2}(B + C).}$$

(3) $$\therefore \frac{BF}{FC} = \frac{(b - c) \sin\frac{1}{2}(B + C)}{(b + c) \cos\frac{1}{2}(B + C)} = \frac{b - c}{b + c} \tan\tfrac{1}{2}(B + C).$$

But $$\frac{BF}{FC} = \tan\tfrac{1}{2}(B - C);$$

$$\therefore \frac{b - c}{b + c} \tan\tfrac{1}{2}(B + C) = \tan\tfrac{1}{2}(B - C),$$

or $$\frac{b - c}{b + c} = \frac{\tan\frac{1}{2}(B - C)}{\tan\frac{1}{2}(B + C)}.$$

This last method is not quite so simple as the first, but it appeals to A Sets to whom alone (perhaps) it should be given.

5. Other formulæ that should be worked out geometrically:

(i) $\text{Cos}\,2a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$.

(ii) $\text{Sin}\,2a = 2\sin a \cos a$.

(iii) $\text{Cos}\,C = \dfrac{a^2 + b^2 - c^2}{2ab}$ (and thence, algebraically, the half-angle formulæ, $\sin \tfrac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, &c.).

(iv) Area $= \tfrac{1}{2}bc \sin A$; &c.

(v) Circles of a triangle: circumscribed, inscribed, escribed.

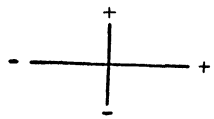
(vi) Medians, angle bisectors, pedal triangle, orthocentre, &c.

The *geometry* of these basic formulæ is the important thing. The derivatives may be obtained algebraically.

Angles up to 360°. The Four Quadrants

It is best to begin by showing the boys how surveyors in their work often find it an advantage to consider angles up to 360°. We have therefore to decide how the ratios of angles between 180° and 360° can be expressed. Thus we have to consider the 3rd and 4th quadrants.

Remind the boys that there are no *proofs* of our conventions concerning the signs in the four quadrants. The conventions are just a matter of convenience, arrived at by general consent, and consistent with one another. It is this consistency which is the important thing. The boys must be drilled in the quadrant signs until the last shred of doubt disappears.



" Plus: right and above,"
" Minus: left and below."

Fig. 208

Other important memos.

1. The *fixed* arm of the angle is always in the 3 o'clock position.

2. The *rotating* arm of the angle always moves counter-clockwise.

3. *Never take a short cut* by moving clockwise.

From any point in the rotating arm we may drop a perpendi cular PM on the abscissa, form a right-angled triangle,

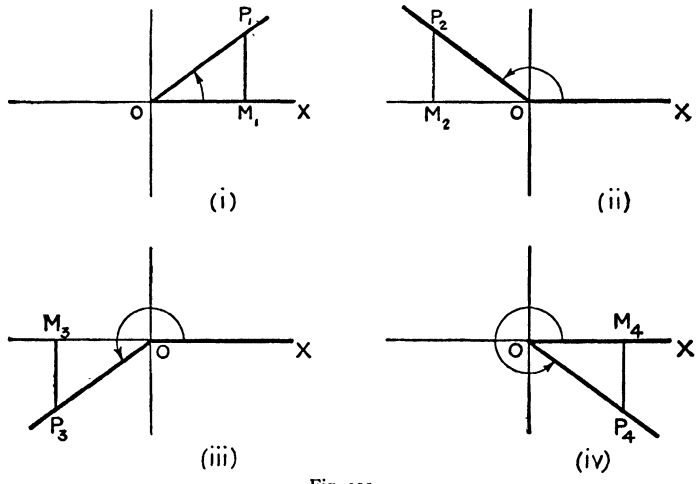


Fig. 209

and so take any ratio of any angle in any quadrant. Taking e.g., the tangent, we have:

$$\tan XOP_1 = \frac{+P_1M_1}{+OM_1} = +\frac{P_1M_1}{OM_1},$$

$$\tan XOP_2 = \frac{+P_2M_2}{-OM_2} = -\frac{P_2M_2}{OM_2},$$

$$\tan XOP_3 = \frac{-P_3M_3}{-OM_3} = +\frac{P_3M_3}{OM_3},$$

$$\tan XOP_4 = \frac{-P_4M_4}{+OM_4} = -\frac{P_4M_4}{OM_4}.$$

Beginners are often puzzled about the re-entrant angles in the 3rd and 4th quadrants. Make them understand that if they take the smaller angles in these quadrants, they have taken a clockwise rotation of the moving arm, and this is not allowable. (Postpone the consideration of negative rotations until the main principle is grasped thoroughly.)

As already suggested, an alternative plan is to call the length of the rotating arm r, and to call the point P which we fix in it (x, y), x and y being the co-ordinates of the point. But if the boys are at first well drilled in the use of the terms *hypotenuse*, *base*, and *perpendicular*, these terms will probably continue to be used, at least mentally. In A Sets, the co-ordinate notation is preferable: its advantages are obvious.

Make the boys memorize the following scheme: it merely amplifies what was said on a previous page.

	1.	2.	3.	4.
Sin ..	+	+	−	−
Cos ..	+	−	−	+
Tan ..	+	−	+	−

" *Sin*, *cos*, and *tan* are + in the 1st quadrant, and each is + in one other, viz. *sin* in 2nd, *tan* in 3rd, *cos* in 4th."*

Give plenty of oral work on the ratios of angles in all four quadrants. Boys should recognize the landmarks 90°, 180°, 270°, 360°, and know at once in which quadrant a given angle occurs.

Beginners are often caught: they take the complement of the angle instead of the angle itself. Point out again and again that whatever angle we may have in the first quadrant there must be angles with *exactly the same numerical ratio* in the other three quadrants. The four resulting triangles

* One or two schools use this mnemonic:

s ← |– all
↓ |
t –|→ c

" positively all silver tea cups ".

formed by dropping a perpendicular from the same point P on the rotating arm must be congruent.

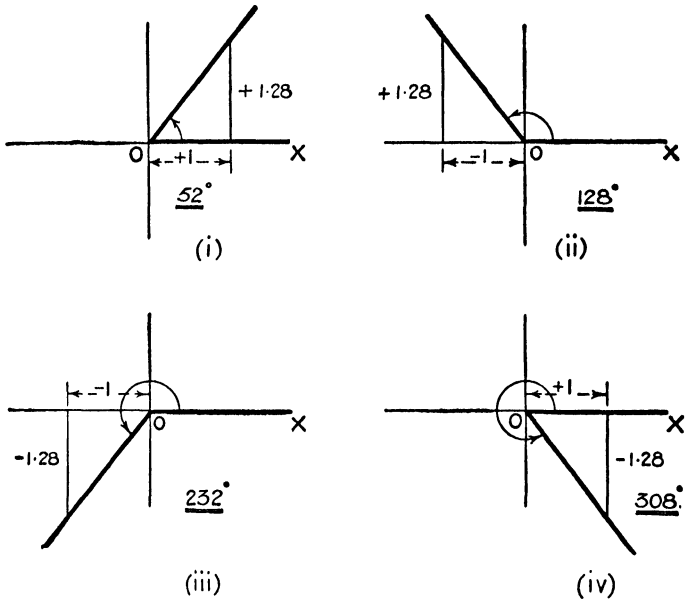


Fig. 210

Note the 4 angles: **52°**; 180° − 52° = **128°**; 180° + 52° = **232°**; 360° − 52° = **308°**.

Note also the 4 tangents: * $\tan 52° = +\frac{1{\cdot}28}{1}$; $\tan 128° = -\frac{1{\cdot}28}{1}$; $\tan 232° = +\frac{1{\cdot}28}{1}$; $\tan 308° = -\frac{1{\cdot}28}{1}$.

The angle in the second quadrant is obtained by **subtracting** 52° from 180°.

The angle in the third quadrant is obtained by **adding** 52° to 180°.

The angle in the fourth quadrant is obtained by **subtracting** 52° from 360°.

* Quite by chance the angle in the second quadrant (128°) has the appearance of being 100 times the value of the tangent (1·28).

The four angles do *not* form an arithmetical progression, and they cannot do so unless the angle in the first quadrant is 45°.

Any such group of 4 angles forms a symmetrical figure:

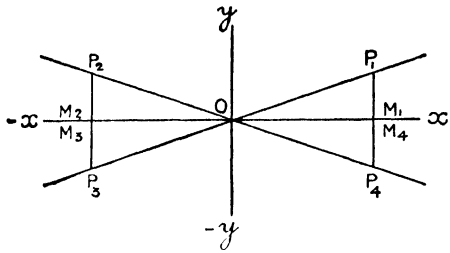


Fig. 211

Whenever we take a trigonometrical ratio of an angle from the tables, the angle is one belonging to the first quadrant. But there are three other angles having the same numerical value. If α is the angle in the first quadrant, the other three are $180° - \alpha$, $180° + \alpha$, $360° - \alpha$. But each ratio in each quadrant has its own signs as we have already seen.

To evaluate the ratios of angles greater than 90°, we may remember the formula, $n180° \pm \alpha$, though this is really intended to include angles greater than 360°. Let the boys make up this general formula from an examination of a number of particular cases.

First Notions of Periodicity

The boys are already familiar with the notion that the rotating arm of the angle may proceed beyond one revolution; the movement of the pedal of an ordinary bicycle serves to convey the notion of angles of $n360°$ or $n360° + \alpha$. Show clearly that the ratios of any angle α are exactly the same as those of any angle that differs from α by any number of

complete revolutions. Thus, $\sin(n360° + \alpha) = \sin\alpha$, where n is any integer; so with all the ratios. For example,

$$\cos 700° = \cos(720° - 20°) = \cos(2 \cdot 360° - 20°)$$
$$= \cos - 20° = \cos 20°.$$

Give ample oral practice to emphasize the fact that the addition or subtraction of any multiple of 360° does not alter the value of any ratio of an angle. The general rule may be expressed: "If α is an angle, any ratio of $2n\pi \pm \alpha$ is *numerically equal* to the same ratio of α". The sign to be attached depends on the quadrant. (The radian notation should be familiar by this time.)

For purposes of illustrating continuous functions, graphs may be obtained, with sufficient accuracy, by 30° and 60°

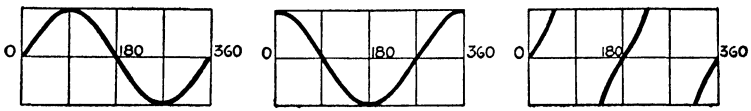


Fig. 212

parallels and perpendiculars as in fig. 183. The two intermediate points in each quadrant are enough to determine the curve fairly readily. The boys should be able to sketch the curves rapidly and should become thoroughly familiar with them. They should note that if the graph of $\cos\alpha$ be moved 90 units along the x axis, it coincides with that of $\sin\alpha$, and that this is equivalent to saying that $\sin(\alpha + 90°) = \cos\alpha$.* Superpose the cos graph on the sine graph and discuss the intersecting points and the ratios of the angles indicated by those points. Draw a continuous sine graph up to 5π or 6π. Select some first quadrant angle, say 40°, raise a perpendicular to cut the graph, and through the point of intersection run a parallel to the x axis and another the same distance below the axis. Discuss and compare the

* Slower boys *will* confuse $90° + A$ with $180° - A$. Show that they are necessarily different unless $A = 45°$.

sines of all the angles indicated by the successive points of intersection. Show clearly that there is a *period* of 2π, and that $\sin x$ may therefore suitably be called a *periodic function* of x. So with $\cos x$. Tan x is likewise a periodic

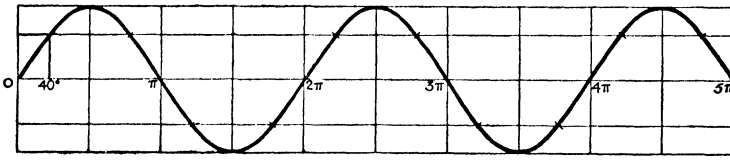


Fig. 213

function, but with a period of π (not 2π); show how this may be inferred from the parallel tan curves.

Compound Angles

1. $\text{Sin}(A + B) = \sin A \cos B + \cos A \sin B$.
2. $\text{Cos}(A + B) = \cos A \cos B - \sin A \sin B$.
3. $\text{Sin}(A - B) = \sin A \cos B - \cos A \sin B$.
4. $\text{Cos}(A - B) = \cos A \cos B + \sin A \sin B$.

Beginners naturally think that $\sin 50° = \sin 20° + \sin 30°$, that $\cos 70° = \cos 80° - \cos 10°$. Give a few examples, with free reference to the four-figure tables, to show that this is *not* so.

Of the four identities named above, at least one should be proved geometrically and mastered thoroughly. The neatest method is the projection method, and with A Sets the general case can readily be proved by this method. With B Sets and certainly with C Sets the problem is best considered merely from the point of view of positive acute angles. All the books give the solution, but the boys should be taught to analyse the conditions of the problem, not merely to follow out a book solution.

The following sequence of arguments is suitable for teaching purposes.

Let OX rotate through $\angle A$ to OC, then through $\angle B$ to OD; in its complete journey to OD it has rotated through the complete $\angle (A + B)$. We have to prove that $\sin(A + B) = \sin A \cos B + \cos A \sin B$, and in connexion with the three angles this means the consideration of five ratios, viz.

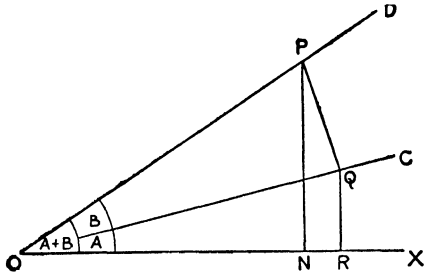


Fig. 214

the sines of A, B, and A + B, and the cosines of A and B. We will try to show all these in one figure.

Evidently we require three perpendiculars, since there are three angles.

(1) From any point P in OD, drop the $\perp$ PN to OX. The sine of $\angle(A + B)$ may be considered from the rt. $\angle$d Δ PON.

(2) From P, drop a $\perp$ PQ on OC. The sine and the cos of $\angle B$ may be considered from the rt. $\angle$d Δ POQ.

(3) From Q, drop a $\perp$ QR on OX. The sine and cos of $\angle A$ may be considered from the rt. $\angle$d Δ QOR.

When we have to prove that a simple expression is equal to a more complex expression, it is a good general rule to begin with the latter, try to simplify it, and get back to the former. Thus we may begin:

$$\sin A \cos B + \cos A \sin B$$
$$= \frac{QR}{QO} \cdot \frac{QO}{OP} + \frac{OR}{QO} \cdot \frac{PQ}{OP}.$$

But how are we to proceed now? True the OQ's seem to

cancel out in the left-hand term, but we do not seem to be able to simplify any further.

Since a circle will go round ONQP (on OP as diameter), $\angle MPQ = \angle A$. If then we draw QM $\perp$ PN, we have a

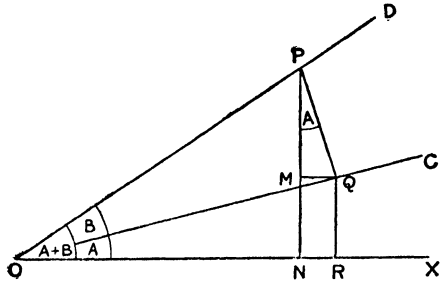


Fig. 215

ΔPMQ similar to ΔQRO. Thus, as far as *ratios* are concerned we may consider ΔPMQ instead of ΔQRO. Now let us try simplification again:

$$\sin A \cos B + \cos A \sin B$$

$$= \frac{QR}{\cancel{QO}} \cdot \frac{\cancel{QO}}{OP} + \frac{PM}{\cancel{PQ}} \cdot \frac{\cancel{PQ}}{OP}$$

$$= \frac{QR}{OP} + \frac{PM}{OP} = \frac{PN}{OP} = \sin(A + B).$$

We may now set our proof as an examiner would expect to see it.

$$\sin(A + B) = \frac{PN}{OP} = \frac{QR}{OP} + \frac{PM}{OP}$$

$$= \frac{QR}{OP} \cdot \frac{OQ}{OQ} + \frac{PM}{OP} \cdot \frac{PQ}{PQ}$$

(*each term multiplied by* 1)

$$= \frac{QR}{OQ} \cdot \frac{OQ}{OP} + \frac{PM}{PQ} \cdot \frac{PQ}{OP}$$

$$= \sin A \sin B + \cos A \sin B. \quad \text{Q.E.D.}$$

The three analogues now follow on simply. All four identities should be verified by a few particular cases (4-figure logs will do), e.g.

$$\sin 80° = \begin{cases} \sin 55° \cos 25° + \cos 55° \sin 25°, \\ \sin 10° \cos 70° + \cos 70° \sin 10°. \end{cases}$$

Some teachers prefer this proof instead:

Let the acute ∠s A and B be the ∠s of a △ABC. Draw a circle round the △, and the radii OA, OB, OC. Evidently ∠AOC = 2B, ∠BOC = 2A. If ⊥s from the centre be drawn, they bisect the sides of the △. Hence, AB = d sin (A + B), AC = d sin B, CB = d sin A; also AB = AC cos A + CB cos B. By equating the first and last of these, and substituting from the 2nd and 3rd:

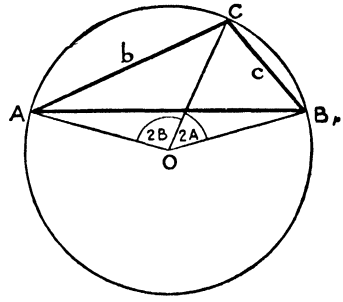


Fig. 216

$$\begin{aligned} d \sin(A + B) &= AC \cos A + CB \cos B \\ &= d \sin B \cos A + d \sin A \cos B, \\ \text{or} \quad \sin(A + B) &= \sin A \cos B + \cos A \sin B. \end{aligned}$$

This proof does not seem to appeal to boys so readily as the former does.

We now come to the *general* case. B and C Sets find it difficult, and as a rule it should be given to A Sets only. Of the various methods of proof the two following are the simplest for teaching purposes.

1. *The* **Projection** *Method.*—This method is productive of mistakes *unless* the boys have mastered the elementary principles of projection.

Give the class one or two preliminary exercises of the following kind:

The angle A of the regular pentagon ABCDE touches the X axis, with which AB makes an angle of 12°. *Find* (1) *the horizontal distance of the vertex D from A, and* (2) *the height of D above the X axis.*

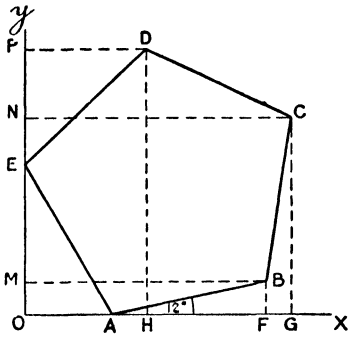


Fig. 217

(1) *Horizontal distance of D from A* = projection of AB + projection of BC + projection of CD. Remember that projection means *projection with proper sign attached*, and we must take the ∠s which AB, BC, CD make with the + direction of OX. Take AB as unity.

$$\therefore \textit{Distance } AH = \text{AB}\cos 12° + \text{BC}\cos 84° + \text{CD}\cos 156°$$
$$= \cdot 9781'' + \cdot 1045'' - \cdot 9135''$$
$$= \mathbf{\cdot 1691''}.$$

(2) *Height of D above OX.*

$$\text{Height HD} = \text{OM} + \text{MN} + \text{NP}$$
$$= \text{AB}\sin 12° + \text{BC}\sin 84° + \text{CD}\sin 156°$$
$$= \cdot 2079'' + \cdot 9945'' + \cdot 4067''$$
$$= \mathbf{1 \cdot 6091''}.$$

Now we come to the identity $\sin(\text{A} + \text{B}) = \sin\text{A}\cos\text{B} + \cos\text{A}\sin\text{B}$. Let the ∠s be the same as in fig. 214. From any

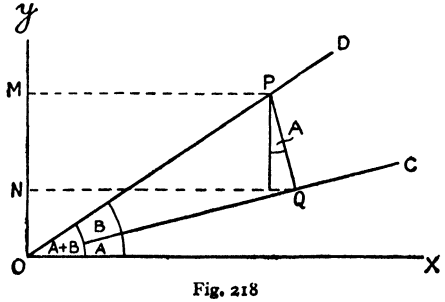


Fig. 218

point P in OD, draw PQ ⊥ OC. Project the three sides of the △POQ on the Y axis.

$$OM = ON + NM,$$

$\therefore$ projection of OP = sum of projections of OQ and QP.

$$\therefore OP \sin XOP = OQ \sin XOC + QP \cos XOC.$$

$$\therefore OP \sin(A + B) = OP \cos B \sin A + OP \sin B \cos A,$$

$$(OQ = OP \cos B,\ QP = OP \sin B)$$

$$\text{i.e. } \mathbf{\sin(A + B) = \sin A \cos B + \sin B \cos A.}$$

In a similar manner, by projecting the three sides on the X axis, we may prove that

$$\mathbf{\cos(A + B) = \cos A \cos B - \sin A \sin B.}$$

Note that the method is perfectly general, being applicable to *any* angles.

2. *The* **Cosine Rule** *Method.*—This is based on the rules (1) that $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ and (2) that if P and Q are the two co-ordinate points (x_1, y_1), (x_2, y_2), then $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

The identity usually considered is $\cos(A - B) = \cos A \cos B + \sin A \sin B$, the others being treated as derivatives.

Whatever two angles are given, the initial line for each is the positive direction of the X axis. Note that we are taking the *difference* between two angles, not their sum. No matter what two angles are taken, $\cos POQ = \cos(A - B)$.

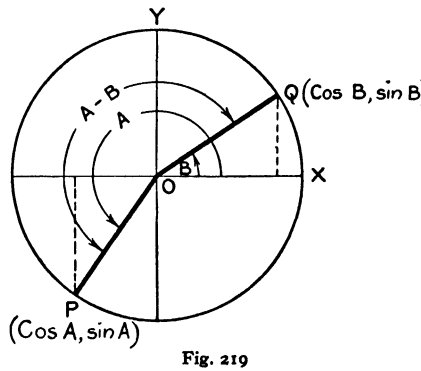


Fig. 219

The simplest way is to take a circle of unit radius, and

to let the co-ordinates of P be (cos A, sin A) and of Q, (cos B, sin B). Note that OP = OQ = 1; hence the denominators of the cosine ratios need not be written.

$$\begin{aligned}
&\textbf{Cos}(A - B)\\
&= \cos POQ\\
&= \frac{OP^2 + OQ^2 - PQ^2}{2OP \cdot OQ}\\
&= \frac{2 - PQ^2}{2}\\
&= \frac{2 - \{(\cos A - \cos B)^2 + (\sin A - \sin B)^2\}}{2}\\
&= \frac{2 - \{\cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B - 2\sin A \sin B\}}{2}\\
&= \frac{2 - \{2 - (2\cos A \cos B + 2\sin A \sin B)\}}{2}\\
&= \textbf{cos A cos B + sin A sin B.}
\end{aligned}$$

There does not seem to be much to choose between this method and the projective method. To able boys both methods appeal. To boys of poor mathematical ability, both methods are equally hateful.

One or other of the four identities, preferably the one proved by the projective method, should be regarded as basic, and the others should be treated as derivatives.

Other necessary derivatives are:

(1) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$, and its three analogues.

(2) $\sin C + \sin D = 2\sin\frac{C + D}{2}\cos\frac{C - D}{2}$, and its three analogues.

(3) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

(4) $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$.

(5) $\sin 3A = 3\sin A - 4\sin^3 A$.

(6) $\cos 3A = -3\cos A + 4\cos^3 A$.

All the formulæ should be learnt off. If mnemonics can be devised, they will help the lame ducks much.

Let the boys verify all the formulæ established, by means of a few simple exercises. Use four-figure logs for this purpose, and so cover a good deal of ground in a short time. For instance, show that

$$\begin{aligned} 2\sin 50^\circ \cos 24^\circ &= \sin(50^\circ + 24^\circ) + \sin(50^\circ - 24^\circ) \\ &= \sin 74^\circ + \sin 26^\circ, \\ (2 \times \cdot 7660 \times \cdot 9135) &= (\cdot 9613 + \cdot 4384), \text{ &c.} \end{aligned}$$

Books to consult:

1. *Trigonometry*, Siddons and Hughes.
2. *Elementary Trigonometry*, Durell and Wright.
3. *Advanced Trigonometry*, Durell and Robson.
4. *The Teaching of Algebra*, Nunn.
5. *Elementary Trigonometry*, Heath.
6. *Trigonometry*, Lachlan and Fletcher.
7. *A Treatise on Plane Trigonometry*, Hobson.

CHAPTER XXVII

Spherical Trigonometry

Spherical trigonometry enters into the work of the map-maker, the navigator, and the astronomer; also into the work of the surveyor if that work extends over larger areas, as in the case of the Ordnance Survey. But for an understanding of the *essentials* of surveying, map-making, navigation, and astronomy, little more than the A, B, C of spherical trigonometry is required, and all this can be included in a very few lessons. The elementary *geometry* of the sphere should already have been done.

The following are the chief points for inclusion in the necessary elementary course. (Many of the difficulties can

be elucidated by the use of simple illustrations. The orange, with its natural sections, is very useful. Well-shaped apples lend themselves to the making of useful sections. A slated sphere, mounted, should always be available).

1. Great and small circles.

2. Shortest distance that can be traced between two points on the surface of a sphere—the arc of the great circle passing through them. (A simple experimental verification is good enough for beginners.)

A suitable argument: If a string be stretched between two points on the surface of a sphere, it will evidently be the shortest distance that can be traced on the surface between the points, since, by pulling the ends of the string, its length between the points will be shortened as much as the surface will permit. Any part of the stretched string, being acted on by two terminal tensions, and by the reaction of the surface which is everywhere normal to it, must lie in a plane containing the normal to the surface. Hence the plane of the string contains the normals to the surface at all points of its length, i.e. the string lies in a great circle. (Sixth Form boys ought to appreciate such an argument.)

3. Axes; pole and polar.

4. Primary and secondary circles.

5. The angle between two great circles is measured by:

(i) the angle between their planes,
(ii) the arc intercepted by them on the great circle to which they are secondaries,
(iii) the angular distances between their poles.

6. The spherical triangle—that portion of the surface of a sphere bounded by the arcs of three great circles. Parts: 3 sides and 3 angles.

7. Since 3 great circles intersect one another to form 8 triangles, that particular triangle is selected which has 2, or if possible 3, sides each less than a quadrant.

Cut an orange or an apple into two equal parts;

hold the two parts together, and cut again into two equal parts, this time by a plane oblique to the first; hold the four parts together, and cut still again into two equal parts, by a plane oblique to both of the other planes.

8. The analogy between theorems in *plane* and *spherical* trigonometry, e.g. any two sides of a triangle are together greater than the third.

9. Polar triangles, i.e. triangles so related that the vertices of the one are the poles of the sides of the other.

10. Angular limits of the sides and angles of a spherical triangle.

11. Fundamental formulæ:

(i) Any spherical triangle:

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad (\textit{and analogues}).$$

(ii) Right-angled triangles:

$$\sin A = \sin a / \sin c;$$
$$\cos A = \tan b / \tan c;$$
$$\tan A = \tan a / \sin b.$$

(iii) Sine rule:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a \sin b \sin c}.$$

All the proofs are simple. The only trouble is in the drawing of suitable figures.

12. The *Latitude* problem. This is perhaps the most important of the elementary problems of the sphere.

The navigator's " dead reckoning " depends on his knowledge of two things: (1) his *course* (direction), (2) the *distance run* (determined by log). He has to resolve his distance-course into separate mileage components of northing and southing, easting and westing. Then he has to convert his

northing and southing mileage into degrees and minutes of latitude, his easting and westing into degrees and minutes of longitude.

There is no difficulty with the former. The meridians of longitude are all great circles. When we know the length of the circumference of these circles, a simple calculation will give the change of latitude produced by a given northing and southing. (Polar circumference = 24,856 miles; therefore length of degree of latitude = 69 miles; $\frac{1}{60}$ of 69 miles = nautical or sea mile = 6080 ft. Thus 60 sea miles = 1 degree of latitude, and 1 sea mile = 1 minute of latitude. " Knots " = sea miles per hour.)

But parallels of latitude are small circles decreasing from the equator to the poles. Only along the equator itself does 1 sea mile imply 1 degree of longitude.—We have to discover a law which the length of a degree of longitude follows.

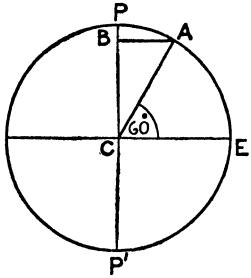


Fig. 220

This law does *not* show a length proportional to the distance from the pole. The greatest distance between two meridians is not halved at 45°, but at 60°. *Why* has the parallel of 60° half the circumference of the equator?

CE = radius; A = point in lat. 60°. Let figure rotate on PP'. The circle will trace out the surface of the globe, E will trace out the equator, and A the parallel of 60° of which AB is the radius.

$$\begin{aligned} & CE = CA = R \text{ (say).} \\ \text{Then } & AB = AC \sin ACB, \\ & = R \cos 60°, \\ & = \tfrac{1}{2}R. \end{aligned}$$

Since AB = $\frac{1}{2}$R, circf. of the 60° parallel = $\frac{1}{2}$ length of equator, ∴ the length of a degree in 60° lat. is half the length of a degree along the equator.

Thus a voyage of a given number of sea-miles along the

60th parallel implies a change of longitude twice as great as the same distance along the equator.—With the help of the slated globe, show the class how short the degrees of longitude necessarily are in the neighbourhood of the Pole.

We give a suitable figure* for showing the general case. If R be the radius of the equator, and r the radius of the

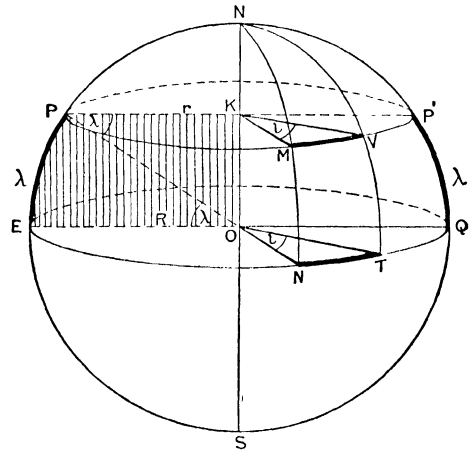


Fig. 221

parallel of latitude λ, passing through a given point, then $r = R \cos\lambda$.

Q = Lat. 0°, long. 0°.
P′ = Lat. λ, Long. 0°.
V = Lat. λ, Long. P′V west.
T = Lat 0°, Long. QT west.

Difference of longitude of M and V = arc MV = $\angle MKV = \angle l$.

Give the boys the little problem to prove that the length of 1 minute of longitude measured along a parallel of latitude λ is, 1 *nautical mile* $\times$ $\cos\lambda$.

* The figure is designed to show merely the main geometrical facts. When the boys are familiar with these facts, the correct notation of polar co-ordinates, and the accepted astronomical sign convention, should be introduced.

It requires very little skill in soldering to make a *wire model*, and then the demonstration is exceedingly simple.

In spherical geometry and trigonometry, good figures are essential, or very few boys will understand the problems considered. Here is an example of a problem from one of our very best books on the subject. We reproduce the original figure.

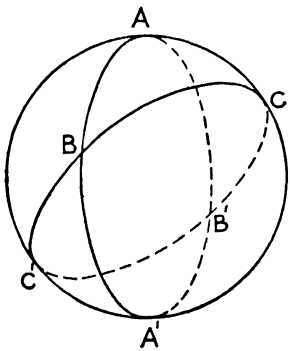


Fig. 222

The excess of the sum of the three angles of a spherical triangle over two right angles is a measure of its area.

Let ABC be a spherical triangle; then, since the sum of the three spherical segments (lunes) ABA′C, A′BC′B′, ACBC′, exceeds the hemisphere ACA′ by the two triangles ABC, A′B′C′; and since,

(i) the measures of the three spherical segments are, respectively, the angles A, B, C, of the spherical triangle,

(ii) the measure of the hemisphere is 2 right angles,

$\therefore$ *the sum of the three angles exceeds* 2 *right angles.* . (i)

If A is the number of degrees in the angle A, S the surface of the hemisphere, the area of the spherical segment $= \frac{A}{180^\circ} . S$;

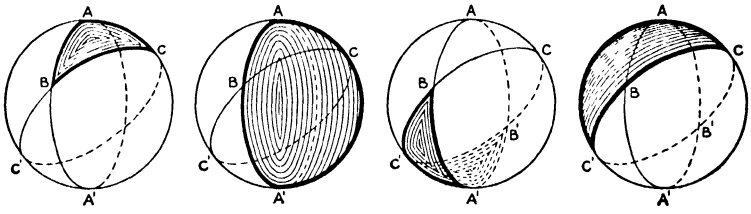
$\therefore$ since ABC is equal to its symmetric triangle A′B′C′, the result of (i) is that if Σ is the area of the spherical triangle,

$$\left(\frac{A + B + C}{180} - 1\right) S = 2\Sigma,$$

$$\text{or} \quad \Sigma = \frac{A + B + C - 180}{360} . S,$$

i.e. the area Σ is proportional to the excess of A + B + C over 2 rt. ∠s.

I have given this theorem to boys on several occasions, but they have almost invariably failed to visualize the figure properly. They failed to pick out the spherical segments. We append four new figures. The first shows the spherical triangle plainly; the next three show the three lunes, separately shaded. The real trouble is that half the second lune (iii), viz. the part A′B′C′ (= the symmetric triangle of ABC) is not visible. When the shaded lunes of ii, iii, iv are added



(i) (ii) Fig. 223 (iii) (iv)

together, it is seen that the ΔABC is included twice and the hidden A′B′C′ once. Hence the sum of the three shaded areas exceeds the hemisphere by the two triangles ABC and A′B′C′.

Books to consult:

1. *Spherical Trigonometry*, Murray.
2. *Practical Surveying and Elementary Geodesy*, Adams.

CHAPTER XXVIII

Towards De Moivre. Imaginaries

Interpretation of $\sqrt{-1}$

" Please sir, what is the *good* of De Moivre's theorem? What is it really all about? What is the use of talking about *imaginary* roots to equations?"

Thoughtful boys often ask such questions. It is our business to see that our answers satisfy them.

The symbol $\sqrt{-1}$, if interpreted as a *number*, has no meaning. But algebraic transformations which involve the use of complex quantities of the form $a + bi$ (where a and b are numbers, and $i = \sqrt{-1}$) yield propositions which do relate purely to numbers, and those propositions are now known to be rational and acceptable

Boys should understand that algebra does not depend on arithmetic for the validity of its laws of transformation. If there were such a dependence, it is obvious that as soon as algebraic expressions are arithmetically unintelligible, all laws respecting them lose their validity. But the laws of algebra, though suggested by arithmetic, do not depend on it. The laws regulating the manipulation of algebraic symbols are identical with those of arithmetic, and it therefore follows that no algebraic theorem can ever contradict any result which could be arrived at by arithmetic, for the reasoning in both cases merely applies the same general laws to different classes of things. If an algebraic theorem is interpretable in arithmetic, the corresponding arithmetical theorem is therefore true. Sixth Form boys seem to gain confidence when once they realize that algebra may be conceived as an independent science dealing with the relations of certain marks conditioned by the observance of certain conventional laws.

It is true that the present-day use of imaginary quantities, in accordance with the authoritative interpretation now given them, does not involve any sort of contradiction and is therefore presumably valid, for absence of logical contradiction is certainly a good test of valid reasoning. But Mr. Bertrand Russell is perhaps going a little far when he says (*Prin. of Maths.*, Vol. I, p. 376) that the theory of imaginaries has now lost its philosophical importance by ceasing to be controversial. There is still a hesitancy in the treatment of the subject in Sixth Forms, which suggests that in

the minds of at least some teachers there is a lingering doubt about the accepted interpretation.

Let the early treatment of the subject be frankly dogmatic. Let discussions as to validity stand over for a while.

Define the symbol $\sqrt{-1}$ merely as an expression, (1) the square of which $= -1$, and (2) which follows the ordinary laws of algebra. And deduce the inference that since the squares of all numbers, whether positive or negative, are always positive, it follows that $\sqrt{-1}$ can*not* represent any *numerical* quantity.

Deduce the further inference that, since $\sqrt{-a^2} = \sqrt{-1 \times a^2} = \sqrt{-1} \times a$, $\sqrt{-a^2}$ cannot represent any numerical quantity. Thus $\sqrt{-1} \times a$ may be called an "imaginary" expression. It therefore follows that such a statement as $A + B\sqrt{-1} = a + b\sqrt{-1}$ can only be true when $A = a$ and $B = b$.

Numbers like $a + b\sqrt{-1}$, where a and b are real numbers, which consist of a real number and an imaginary number added together, are called **complex** numbers.

At this stage it is advisable to revert to the significance of ordinary negative quantities. If $+a$ indicates a certain *number* of linear units in some chosen direction, $-a$ indicates the same number of linear units in the same line but in the *opposite* direction. Hence when working out, with algebraic symbols, a problem concerning distance, we interpret the *minus* symbol to mean a complete reversal of direction.

It is desirable to take some little trouble to convince the pupils that, on the face of things, there is nothing in the expression $a + b\sqrt{-1}$ to make it more "absurd" than in an expression like $-x$. The result symbolized by $b - a$ where b is less than a is certainly "imaginary", unless we add to the conception of magnitude, which necessarily belongs to it as a number, the further conception of direction.

Quantities which contain $\sqrt{-1}$ as a factor are obviously in some ways very different from quantities which do not contain it.

What interpretation, then, can be given to the result of multiplying a distance by $\sqrt{-1}$? Argand put forward an ingenious hypothesis, which has now received general acceptance.

As we have seen, the effect of multiplying a distance by -1 is to *turn* the distance *through two right angles.*

Hence, whatever interpretation we give to $\sqrt{-1}$, it must be such that the multiplication of a distance by

$$\sqrt{-1} \times \sqrt{-1},$$

i.e. by -1, must have the effect of *turning* a distance *through two right angles.*

Thus it seems worth while to consider how far we may interpret the effect of multiplying a distance by $\sqrt{-1}$, by supposing that it *turns* the distance *through* **one** *right angle.* Evidently we have to devise *some* scheme by which a reversal of direction will be effected in two identical operations.

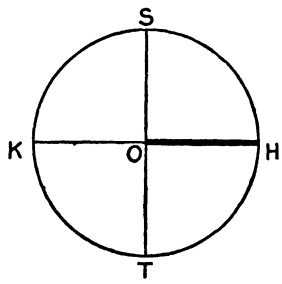


Fig. 224

One possible plan is to revolve OH through a right angle either in the direction of S or in the direction of T, for each of these operations, if repeated, would bring H into coincidence with K. Further double applications of the same operation would successively bring the point to H$(+1)$, to K(-1), to H again, and so on indefinitely.

Clearly the two *algebraic* operations which, by definition, must produce, when applied in this way, the sequence $+1$, -1, $+1$, -1, . . ., represent a repeated multiplication, either by $+\sqrt{-1}$ or by $-\sqrt{-1}$.

Thus *for exactly the same reason* that we identify -1 with a unit step taken along a line in a reverse direction to the unit represented by $+1$, we may identify $+\sqrt{-1}$ with the revolution of a line through a right angle in one sense,

and $-\sqrt{-1}$ with an equal revolution in the opposite sense.

This is the accepted interpretation.

OS is regarded as the i (or $\sqrt{-1}$) direction, and OT as the $-i$ (or $-\sqrt{-1}$) direction.

Complex Numbers

Revise the early work on the significance of co-ordinates. —Given a fixed line OA, and a fixed origin as at O, there are two convenient ways of fixing the position of a point P.

1. *Rectangular co-ordinates*: $Op = 5$, $Pp = 2$.
2. *Polar co-ordinates*: $\angle AOP = 22°$, $OP = 5{\cdot}4$.

Evidently we may regard the *rectangular* co-ordinates as specifying not merely measurements which define the position of P, but also movements by which P could be reached from O. The two movements would be, one of $+5$ along OA and one of $+2$ at right angles to OA. The *polar* co-ordinates specify much the same thing, though in a different way. If to begin with we are at O and facing A, then the polar co-ordinates may be taken as instructions, first to turn through an angle of 22°, and then advance along OP a distance of 5·4.

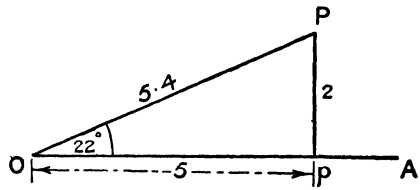


Fig. 225

If *along a straight line* a point takes two successive movements OA, AB, the *length* and *direction* of OB is the *algebraic* sum of the two movements.

If OA and OB are straight lines or *vectors* which represent two movements *not* in the same straight line, the directed line OB which closes the triangle OAB may again be called the "sum" of OA and OB, since it

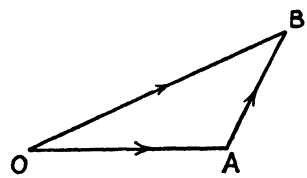


Fig. 226

represents the single movement equivalent to the combination of the two movements. Thus in fig. 225, OP may be called the sum (more fully, the *vector sum*) of the movements $+5$ along OA and $+2$ at right angles to OA. But if the movement OP be represented by the symbol R, we cannot in this case write $R = (+5) + (+2)$, for this would represent a movement of $+7$ from O along the line OA. But we may still represent R as a sum, provided we do something to indicate *that the component movements are at right angles.* For this purpose the letter i is prefixed to that directed number *which represents the component at right angles* to the initial line. This is in accordance with our interpretation of $\sqrt{-1}$. Thus the movement of OP would be represented by the notation $(+5) + i(+2)$. (Fig. 227.)

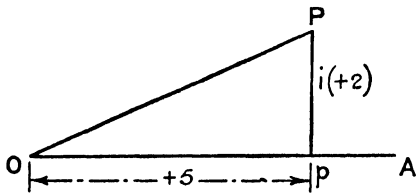


Fig. 227

Of course, if P is confined to the line OA, a single directed number will suffice to define its position after a series of movements. But if P is *forced to move about over the whole plane* of the paper, its position may be fixed just as definitely by such an expression as, say, $(-13) + i(+21)$.

Thus we may regard an expression of the form $a + ib$ as a **complex number** which serves *to fix the position of a point in a plane*, just as the simple number a or b fixes its position in a straight line.

But bear in mind that the term " complex number " is only a convenient label, suggested by analogy; $a + ib$ is not really *one* number but a combination of *two* numbers, together with a symbol i which stands for no number at all. **The symbol i is merely a direction indicator**—to show that the movement or measurement represented by the second number of the complex number is at right angles to that represented by the first.

Let a, b be the rectangular co-ordinates, and r, α the

polar co-ordinates of a point P. Then, since $a = r\cos\alpha$ and $b = r\sin\alpha$,

$$a + ib = r\cos\alpha + i(r\sin\alpha).$$

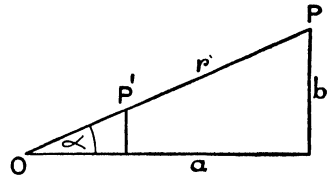


Fig. 228

Again, let P′ be the point on OP at unit distance from O. Then the movement OP′ may be represented by the complex number, $\cos\alpha + i\sin\alpha$. But since r steps, each of length OP′, would carry a point from O to P, we may write:

$$a + ib = (\cos\alpha + i\sin\alpha) \times r,$$

or, more conveniently,

$$a + ib = r(\cos\alpha + i\sin\alpha).$$

It follows that we may write:

$$(r\cos\alpha) + i(r\sin\alpha) = r(\cos\alpha + i\sin\alpha).$$

The conclusion is important, for it shows that we may, at least in this connexion, proceed *just as if i stood for a number*. Otherwise we could not legitimately assume that the two expressions are equivalent.

Note that the non-directed number r is called the *modulus* of the complex number $a + ib$, and the angle α its amplitude.

The operation which carries OA from its original position to OB, then to OC, then to OP, in equal jumps, may be looked upon as the repetition of a *constant factor*, viz. a factor of the form

$$\cos\alpha + \sqrt{-1}\sin\alpha, \quad \text{i.e.} \quad \cos\alpha + i\sin\alpha,$$

where α is the constant angle between the rays from O. Since two rays divide the ∠AOP into three equal parts, we may infer that $m - 1$ rays would divide it into m equal parts.

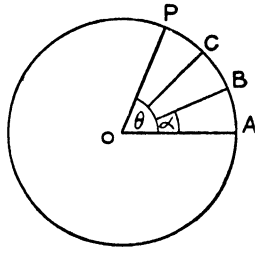


Fig. 229

Hence, if $\angle AOP = \theta$, $\theta = m\alpha$. Since $OA = r$, the line **OP** may be represented by the expression:

$$r(\cos\theta + i\sin\theta)$$

(or, by $a + ib$, where $a = r\cos\theta$ and $b = r\sin\theta$).

Again, since the factor $\cos\alpha + i\sin\alpha$ represents the turning of the line from its original position OA through the angle α, the factor

$$(\cos\alpha + i\sin\alpha) \times (\cos\beta + i\sin\beta)$$

must, presumably, represent a turning through the angle $(\alpha + \beta)$, and therefore be equivalent to the factor

$$\cos(\alpha + \beta) + i\sin(\alpha + \beta).$$

Obviously, then, the identity

$$\cos m\alpha + i\sin m\alpha = (\cos\alpha + i\sin\alpha)^m$$

is foreshadowed.—The usual sequel is obvious and simple.

Practice in the addition and subtraction of complex numbers is desirable; it is quite easy. Devise examples to enforce the notion that i is just a direction indicator, providing us with a simple means of fixing a point P anywhere in a plane containing an initial line; that it serves to show that the second element of a complex number is at right angles to the first. Practice in multiplication and division should follow; this is also quite easy, once the boys see that $\cos\alpha + i\sin\alpha$ is merely a "direction coefficient", i.e. a complex number which, when it multiplies another number, produces a result which corresponds to the turning of a line through the angle α. De Moivre easily follows.

The term "imaginary number" is not a happy one; $\sqrt{-1}$ is just a symbol which can be treated in certain cases *as if* it were a number. In the complex number $a + ib$, a is often called the **real**, and ib the **imaginary** part.

The fruitful suggestion was made by Gauss that instead of calling $+1$, -1, and $\sqrt{-1}$, positive, negative, and imaginary units, we should call them direct, inverse, and

lateral units. To Gauss the radical difference between a complex number and a rational number was that while the latter denotes the position of points *along a line*, the former denotes the position of points *in a plane*.

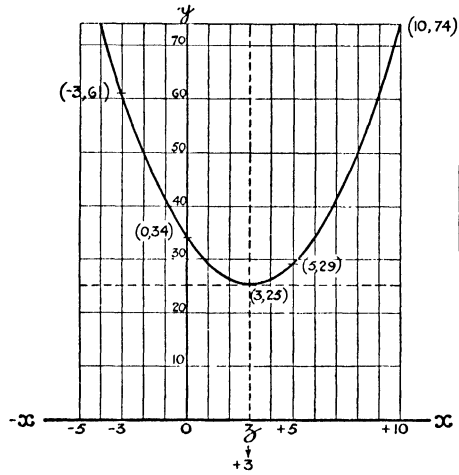
$a + b\sqrt{-1}$ must be regarded as the typical number of algebra, " real " numbers being merely special cases in which $b = 0$. If we are confined to real values of the variables in $y = f(x)$, we must admit that in the case of most functions there are either values of x to which no values of y correspond, or values of y which are not produced by any value of x. But if the variables are complex numbers, these exceptions never occur. To a value of x of the form $a + b\sqrt{-1}$, there corresponds, in the case of every possible function, a value of y of the form $A + B\sqrt{-1}$, a, b, A, B being themselves real numbers.

The principle is so important that it must be understood thoroughly by all pupils. Emphasize strongly the fact that *real* numbers correspond to points in a straight *line*, *complex* numbers to points in a *plane*.—If we represent the values of x by points in one line, and those of y by points in another, we cannot say that any function $y = f(x)$ establishes a one-to-one correspondence between all the points on the two lines; in most cases, whole stretches of points will remain outside the correspondence. But if we take **two planes**, and represent the values of x by the points of one of them, and the values of y by points of the other, we then obtain, in every function, a one-to-one correspondence between *all* the points in the two planes. This is the key to the secret of quadratic equations with " imaginary " roots.

Quadratic Equations and (so-called) Imaginary Roots

Complex numbers can be used to explain certain difficulties met with in the study of quadratic equations. Consider the example $x^2 - 6x + 34 = 0$; the roots of which are

sometimes said to be $3 \pm \sqrt{-25}$. But $x^2 - 6x + 34$, i.e. $(x - 3)^2 + 5^2$, cannot be factorized; hence (we usually argue) there is no value of x for which y (in $y = x^2 - 6x + 34$) is zero; in other words, the equation has no real roots. Another way of stating this is that the parabola $y = x^2 - 6x + 34$ has no points below $y = 25$ and therefore does not cross the axis of x. Here is a graph of the function:



$y = x^2 - 6x + 34$

x =	-3	0	+3	+5	+10
x^2 =	9	0	9	25	100
$-6x$ =	18	0	-18	-30	-60
34 =	34	34	34	34	34
y =	61	34	25	29	74

Fig. 230

But if i be treated as a number whose square is -1, we may write,

$$(x - 3)^2 + 5^2 = (x - 3)^2 - i^2 \cdot 5^2$$
$$= (x - 3 + 5i)(x - 3 - 5i).$$

Apparently, then, $y = 0$ if $x = +3 \pm 5i$.

It is usual to say that these values are " imaginary roots " of the equation, or that they describe imaginary points where the parabola may be supposed to cross the axis of x.

But from what we have already said about the nature of i, there is clearly an alternative way of regarding this, a way much more rational. The values $+3 \pm 5i$ describe points not *on* the axis of x, but elsewhere in some plane containing

that line. It is obvious that it cannot be the plane of the paper, and we must therefore look for points in the plane which is at right angles to the plane of the paper.

The necessary figure (231) consists of two parabolas, each $y = x^2 - 6x + 34$, head to head, with a common axis but in two planes at right angles to each other. A suitable sketch is a little difficult to make, but it may be done in this way.—Let ABCD, EFGH be a rectangular block with square ends. Bisect the block by the mid-perpendicular planes JKLM, NPQR, STUV. The first and second intersect in the line *ab*, of which V is the mid-point. In the horizontal plane *ab*LM, draw the parabola $y = x^2 - 6x + 34$, with vertex at V. In the vertical plane *cd*TS, draw the same parabola, also with its vertex at V. The line *m*V*n* is the common axis of both parabolas. The heavy lines in the plane JKLM (xOx and Oy) are the co-ordinate axes of the primary parabola in the horizontal plane. The axis of the parabola intersects the x axis in z. As in the previous figure, Oz = 3, zV = 25.

If, instead of $y = (x - 3)^2 + 5^2$ the parabola was $y = (x - 3)^2$, the parabola would touch the axis of x at z (= +3), but when the parabola moves into the position $y = (x - 3)^2 + 5^2$, its vertex is at V, $(5)^2$ units from the x axis. Hence, the points given by the complex values of x answering to $y = 0$ are at *a distance* 5 *above and below the plane* of the primary parabola, and on a similar parabola to the first, viz. the parabola in the vertical plane. Evidently the points are on a line through z, m', and n', each 5 units from z.

Thus, when we take into account complex values of x, the complete graph corresponding to real values of the function $y = (x - 3)^2 + 5^2$ is not one parabola but two, lying in two planes. The parabola in the perpendicular plane contains all points answering to complex values of x which satisfy the given relation. Figs. 230 and 231 should be compared.

Note that the line $y = 25$ lies in the plane NPQR, which is tangential to both parabolas.

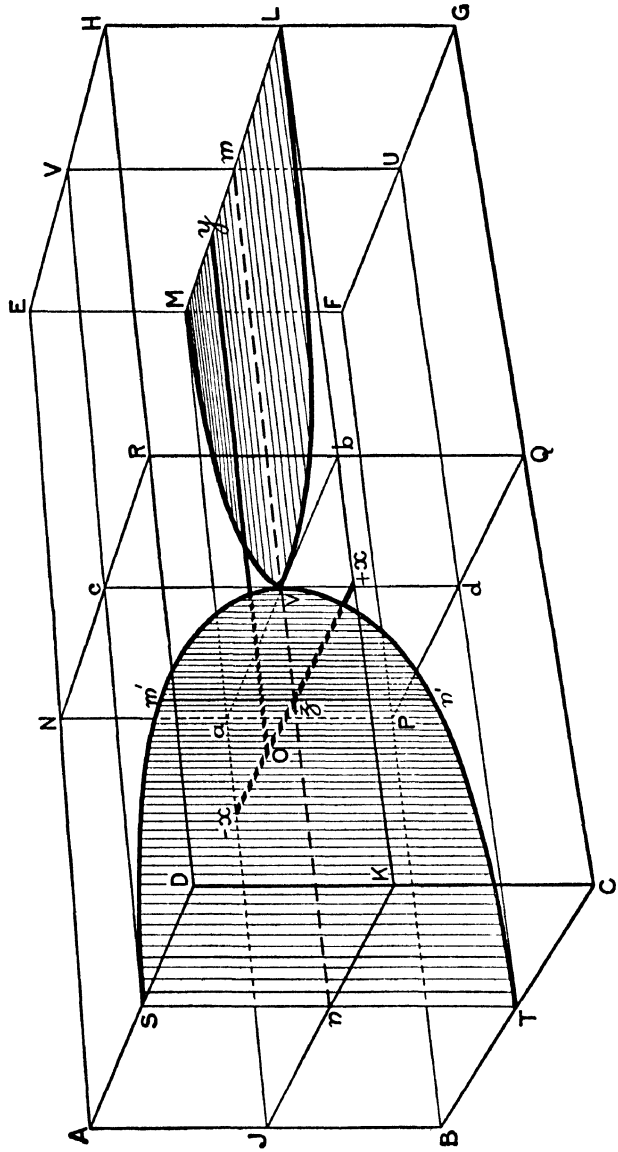
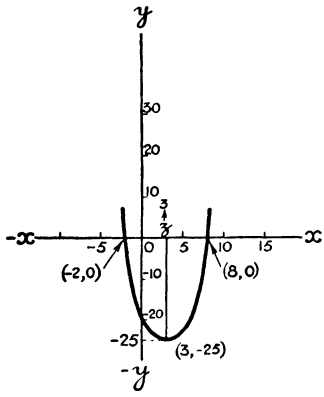


Fig. 43

Note also the difference between these two equations:

$$x^2 - 6x - 16 = 0.$$
$$\therefore\ x^2 - 6x + 9 = 25,$$
$$\therefore\ (x-3)^2 = \pm 5,$$
$$\therefore\ x = 3 \pm 5.$$

$$x^2 - 6x + 34 = 0.$$
$$\therefore\ x^2 - 6x + 9 = -25,$$
$$\therefore\ (x-3)^2 = \pm\sqrt{-25} = \pm i5,$$
$$\therefore\ x = 3 \pm i5.$$



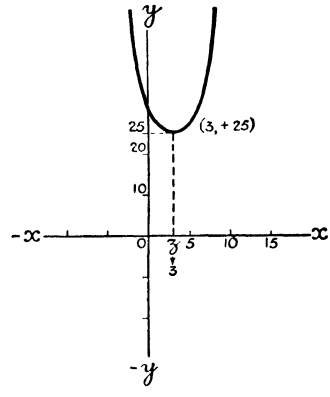


Fig. 232

To obtain the points on the curve we proceed from the origin to z, $+3$ units away, in the x axis, and then, *also in the x axis*, we proceed from z, $+5$ and -5 units, and so reach the points $+8$ and -2. The vertex is 25 units *below* the x axis (see fig. above).

The journey is a journey in *one line*, the x axis. The two 5's are measured from z.

To obtain the points on the curve, we proceed from the origin to z, $+3$ units away, in the x axis, and then we proceed $+i5$ and $-i5$ units from z, i.e. $+5$ and -5 units in a plane perpendicular to the plane of the parabola, where we reach points on a similar parabola in this new plane.

The vertex of the primary parabola is 25 units *above* the x axis (see fig. above and fig. 231).

The journey is a journey in *two lines* perpendicular to each other. The two 5's are measured from z as before, but in a perpendicular plane.

Unless provided with a wire model (fig. 231), or with a really good perspective sketch, boys are apt to be puzzled by this problem. A model is much to be preferred; then the effect of increasing and decreasing the distance of the primary parabola from the x axis is easily observed.

Warn the boys not to be led away by the remarkable and perfectly logical consistency of the hypothesis concerning $\sqrt{-1}$. It is *only* an hypothesis after all. Still, it is not advisable for learners to talk about "imaginary" roots of equations but rather to explain such roots in the light of the hypothesis in question.

We have touched upon *vector* algebra. The subject receives considerable attention in Technical Schools but very little in Secondary. This is a pity, for it is a cunningly wrought instrument and is as useful as it is illuminating. Quite the best introduction to it is Part I (*Kinematic*) of Clifford's *Elements of Dynamic*. The first two parts of the book, *Steps*, and *Rotation*, should be read by all teachers of mathematics, and the third part, *Strains*, by all teachers of mechanics. Maxwell's *Matter and Motion* is a little book dealing admirably with the same subject. Henrici and Turner's *Vectors and Rotors* is also useful.

CHAPTER XXIX

Towards the Calculus

Co-ordinate Geometry

Teachers differ in opinion whether the calculus should be preceded by a course of co-ordinate geometry. Certainly anything like a complete course of co-ordinate geometry is not a necessary preliminary. On the other hand, some little knowledge of its fundamentals is advisable, and this is easily developed from the previous knowledge of graphs. The notion of the differential coefficient is nearly always made to emerge from considerations of the tangent to the parabola, but, more frequently than not, the common properties of the parabola have not previously been taught. This partly explains the haze which often enshrouds the notions underlying the new subject.

A minimum of preliminary work in co-ordinate geometry may be outlined.

The boys already know that $y = mx + c$ represents a line making an angle whose tangent is m with the axis of x; that, in short, m represents the *slope* of the line; and that, in whatever other form the equation may be written, it may be re-cast into the $y = mx$ form, and its slope be determined at once. For instance, the intercept form $\frac{x}{\sqrt{3}} + \frac{y}{3} = 1$ may be written $y = -\sqrt{3}x + 3$, and the slope is seen to be $-\sqrt{3}$.

The boys must be able to determine the equation of a line satisfying necessary conditions. They already know that if they are told a straight line must pass through a given point, this condition alone is not enough to determine the line, since any number of lines may pass through the point. But if they are given some second condition, e.g. the *direction* of the line, or the position of a second point through which

it passes, then the two data completely fix the line. This fits in with the fact that the equation of the line must contain two constants.—Make the boys thoroughly familiar with the ordinary rules for finding the equation of a line satisfying two given conditions.

Types of suitable exercises for blackboard oral work:

1. Find the equation of a straight line cutting off an intercept of 2 units' length on the axis of y and passing through the point (3, 5).

2. Find the equation of a straight line drawn through the point (3, 5), making an angle of 60° with the axis of x.

3. Find the equation of a straight line passing through the points (2, 3), (−4, 1).

The last exercise is a type with which the boys should be thoroughly familiar. It will be required often in future work. The equation should therefore be familiar in its general form, and should be illustrated geometrically.

Find the equation of a line passing through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$. We may proceed in this way:

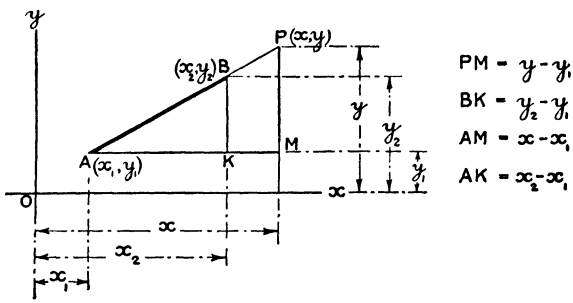


Fig. 233

Suppose $y = mx + c$ represents the equation, where m and c are unknown.

The particular point (x_1, y_1) is on the line, $\therefore y_1 = mx_1 + c$. (i)
The particular point (x_2, y_2) is on the line, $\therefore y_2 = mx_2 + c$. (ii)
The point (x, y), *any* point, is on the line, $\therefore y = mx + c$. (iii)

Subtracting (i) from (iii),

$$y - y_1 = m(x - x_1).$$

Subtracting (i) from (ii),

$$y_2 - y_1 = m(x_2 - x_1).$$

$\therefore$ by division

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1},$$

$$\text{or} \qquad (y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

which is the required equation, $\frac{y_2 - y_1}{x_2 - x_1}$ representing m (the **slope**) in $y = mx$.

Beginners rarely see this clearly, unless the algebra is clearly illustrated geometrically. From the last figure,

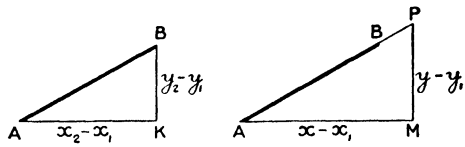


Fig. 234

take out the two similar triangles, and show the lengths of their perpendiculars and bases in terms of co-ordinates. The *slope of the line AB* is given from the first triangle; $\frac{BK}{AK} = \frac{y_2 - y_1}{x_2 - x_1}$, A and B being the two specified points on the line.

The *slope of the line AB* is also given from the second triangle, $\frac{PM}{AM} = \frac{y - y_1}{x - x_1}$, A being a specified point, and P being *any* point on the line.

But the triangles are similar, and the slopes are therefore identical.

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

$$\text{or} \quad (y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1), \text{ as before.}$$

Or again: take the slope found in the last case, say $\frac{y_2 - y_1}{x_2 - x_1}$, and substitute for m in $y = mx + c$. We have

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + c. \quad \ldots\ldots\ldots \text{(i)}$$

and since the line passes through (x_1, y_1), [or (x_2, y_2) might be chosen if preferred],

$$y_1 = \frac{y_2 - y_1}{x_2 - x_1}x_1 + c. \quad \ldots\ldots\ldots \text{(ii)}$$

By subtraction, we have from (i) and (ii)

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1), \text{ as before.}$$

Every step *must* be substantiated geometrically. It is fatal to allow the boys to look upon the problem as mere algebra. The boys should be able to write down *instantly* the equation of a line passing through two points and understand its full significance. But in evaluations of this kind, do not be satisfied with just the typical textbook formal solutions. *Vary* the work.

The Parabola and its Properties

Throughout the teaching of co-ordinate geometry, let all principles be established first by *pure* geometry. Let the picture come first, and teach its new lessons. Then let symbolism follow. Geometry treated as pure algebra tends to lose all semblance of its essential space relations. At

least the parabola, if not the ellipse, will already have received some attention. It will have been touched upon in connexion with graphs and quadratic equations, and the boys may have learnt something about the paths of falling bodies; they may also know that in certain circumstances the chains of a suspension bridge, and vertical sections of the surface of a rotating liquid, form parabolas.

It is useful to give the boys a mechanical means of readily drawing a parabola. It saves much time, and encourages them to use good figures. Here is one way.

An ordinary T-square slides along AB, the left-hand edge of a drawing board, in the usual way, the edge AB answering

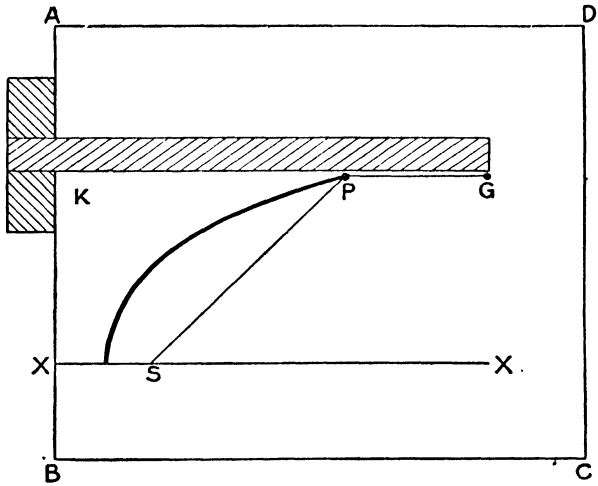


Fig. 235

as a directrix. A string equal in length to KG is fastened at G, and at a fixed point S in a line XX perpendicular to AB. A pencil point P keeps the string stretched and remains in contact with the edge KG of the T-square. As the T-square slides up and down the edge of the board, the pencil traces out a parabola with focus S.

A parabola is the locus of a point whose distance from a

fixed point is equal to its distance from a fixed straight line.—Help the boys to see that from this definition certain properties follow at once:

(1) ⊥ PM (diam.) = PS.
(2) ES (semi-latus rectum) = EM′ = SX.
(3) AS = AX.
(4) SX = 2SA.
(5) ES = 2SA.
(6) EF (latus rectum) = 4SA.

The main property to be mastered is *the slope of the tangent*, and to this end the following summary of preliminary work is suggested. All principles should be established first geometrically, then analytically, and the boys *must* be made to see that the results are identical.

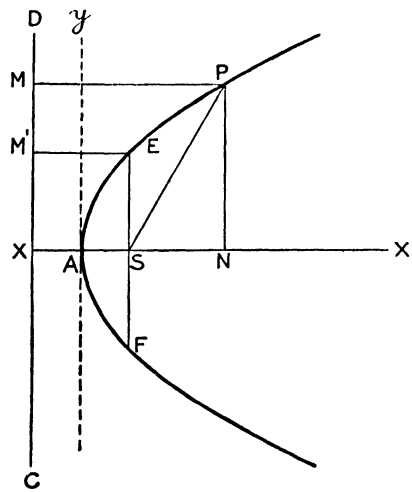


Fig. 236

1. *The principal ordinate of any point P on a parabola is a mean proportional to its abscissa and the latus rectum.*

i.e. $PN^2 = 4AS.AN$.

Analytically: call the point P, (x, y); SA = AX = a (say). Then $\mathbf{y^2 = 4ax}$. (Fig. 236.)

If the directrix MX is the y axis, the equation becomes $y^2 = 4a(x - a)$.

2. *If a chord PQ intersects the directrix in R, SR bisects the external angle QSP′ of the triangle PSQ.*

Drop ⊥s PM and QV on directrix. △s PMR and QVR are similar.

$$\therefore \frac{PR}{QR} = \frac{PM}{QV} = \frac{PS}{QS}.$$

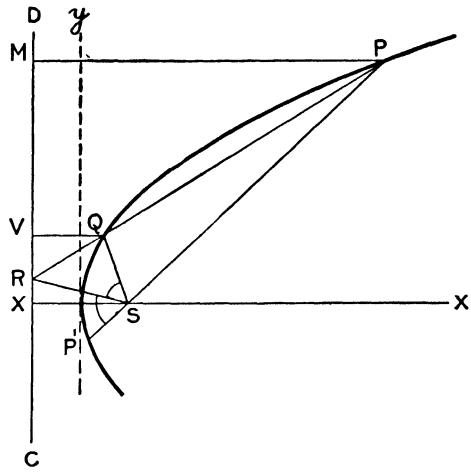


Fig. 237

3. *If the tangent at P meets the directrix in R, the angle PSR is a right angle.* —Deduce this from the preceding proposition.—When Q coalesces with P, each of the marked equal angles becomes a right angle. (See next figure.)

4. *The tangents at the extremities of a focal chord intersect*

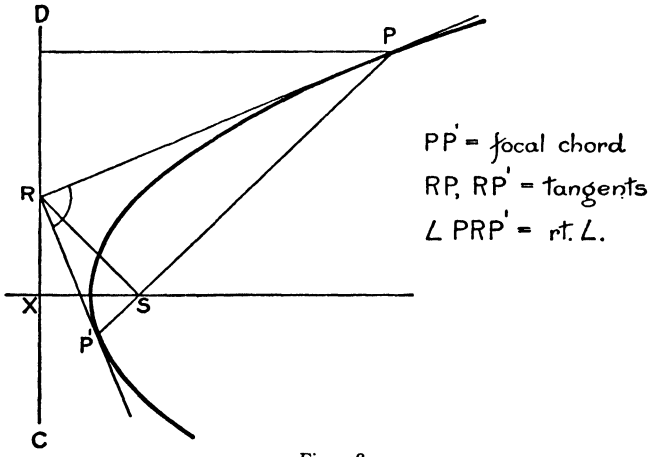


Fig. 238

at right angles on the directrix, i.e. the tangents at P and P', the extremities of the focal chord PSP', make a right angle

at R, where they meet on the directrix. Observe that RS meets the focal chord at right angles (cf. No. 3).

5. *The subtangent is equal to twice the abscissa*, i.e. TA=AN or, TN = 2AN.

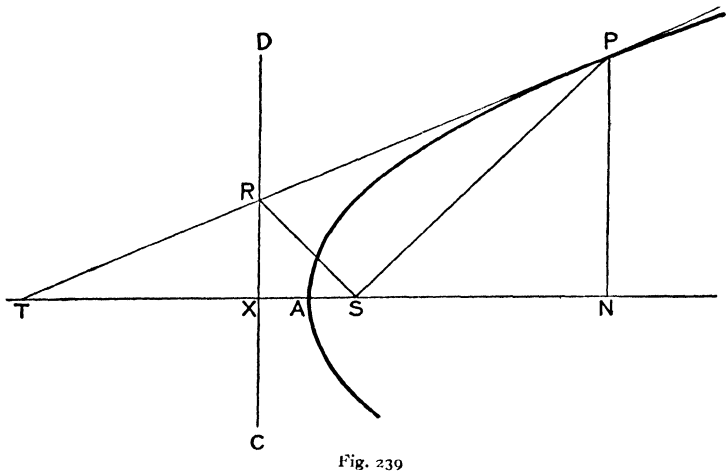


Fig. 239

6. *The foot of the focal perpendicular of any tangent lies on the tangent at the vertex*, i.e. Y, the foot of the ⊥ SY to the tangent PT, lies on the tangent at A.

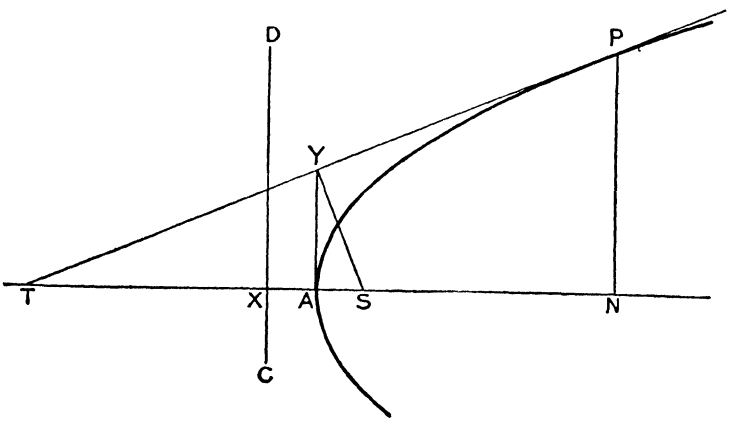


Fig. 240

7. *The slope of any tangent* $= \dfrac{\frac{1}{2} \text{ latus rectum}}{\text{ordinate}}$. For in the last figure the triangles YAS and TNP are similar. Hence slope of tangent PT

$$= \frac{PN}{NT} = \frac{SA}{AY} = \frac{2AS}{PN} = \frac{\frac{1}{2} \text{ latus rectum}}{\text{ordinate}},$$

i.e. the slope of the tangent to the *axis* of the parabola. If the figure is turned round, so that the slope of PT is to the tangent AY at the vertex, then slope $= \dfrac{\text{ordinate}}{\frac{1}{2} \text{ latus rectum}}$.

How may this slope be expressed in rectangular co-ordinates? The equation of a secant cutting the curve in $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$y(y_1 + y_2) = 4ax + y_1y_2.$$

If a figure be drawn accurately (this is not easy), actual measurement will show that the slope of the secant is the ratio

$$\frac{\frac{1}{2} \text{ latus rectum}}{\text{mean of ordinates}}.$$

To obtain the equation of the tangent at (x_1, y_1) we take Q indefinitely close to P, so that ultimately $y_2 = y_1$. The equation to the secant then becomes:

$$\begin{aligned} 2yy_1 &= 4ax + y_1^2 \\ &= 4ax + 4ax_1. \\ \therefore\ yy_1 &= 2a(x + x_1) \\ \text{or}\quad y &= \frac{2a}{y_1}(x + x_1), \end{aligned}$$

which is the equation to the tangent, and the *slope* of the tangent is thus $\dfrac{2a}{y_1}$, i.e. $\dfrac{\frac{1}{2} \text{ latus rectum}}{\text{ordinate}}$, as before.

The Tangent to the Parabola

If future work is to be understood, the tangent to the parabola, and its various implications, must receive close attention. The necessary further elucidation may thus be summarized.

1. *To find the condition that the straight line* $y = mx + c$ *may touch the parabola* $y^2 = 4ax$.

$$\begin{aligned} &\text{Since } y = mx + c, \\ &\therefore\ y^2 = (mx + c)^2; \\ &\text{and since } y^2 = 4ax, \\ &\therefore\ (mx + c)^2 = 4ax. \end{aligned}$$

By solving this equation we obtain the abscissæ of the two points in which the straight line *cuts* the curve. The line will *touch* the curve if the two points coincide, and the condition for this is that the roots must be equal,

i.e. in
$$\begin{aligned} m^2x^2 + 2x(mc - 2a) + c^2 &= 0, \\ 4(mc - 2a)^2 &= 4m^2c^2, \\ \text{i.e. } a = mc, \quad \text{or} \quad c &= \frac{a}{m}. \end{aligned}$$

Hence the line $y = mx + c$ touches the curve $y^2 = 4ax$ if $c = \frac{a}{m}$ (where m is the slope which the tangent makes with the axis).

2. *To find the point where the tangent* $y = mx + \frac{a}{m}$ *touches the parabola* $y = 4ax$.

As before,

$$\begin{aligned} (mx + c)^2 &= 4ax, \\ \therefore\ \left(mx + \frac{a}{m}\right)^2 &= 4ax, \\ \text{or} \quad \left(mx - \frac{a}{m}\right)^2 &= 0, \\ \therefore\ x &= \frac{a}{m^2}, \\ \text{and since } y^2 = 4ax, \quad y &= \frac{2a}{m}. \end{aligned}$$

Hence the point required is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

3. *Compare the two forms of the equation of the tangent, viz.* $yy_1 = 2a(x + x_1)$, *and* $y = mx + \frac{a}{m}$.

The first may be written,

$$y = \frac{2a}{y_1}x + \frac{2ax_1}{y_1}.$$

Hence we may write the two forms in parallel thus:

$$\left.\begin{aligned} y &= \frac{2a}{y_1}x + \frac{2ax_1}{y_1} \\ y &= mx \quad + \frac{a}{m} \end{aligned}\right\}$$

They represent the same line; $\therefore \frac{2a}{y_1} = m$, and $\frac{2ax_1}{y_1} = \frac{a}{m}$. That these two last equations are consistent may easily be shown by evolving the second from the first, the connecting link being $y_1^2 = 4ax_1$.

Evidently, then, the equation $y = mx + \frac{a}{m}$ is the tangent at the point (x_1, y_1), i.e. $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

4. *Verify geometrically that the tangent* $y = mx + \frac{a}{m}$ *touches the parabola* $y^2 = 4ax$ *at the point* $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$. (This verification is of great importance.)

(The abler boys ought to do this without any further help).

AY = tangt. at vertex = y axis.
AX = x axis.
S = focus.
TA = AN (*subtangent* = 2 *abscissa*)
SZ meets the tangent PT at rt. ∠s at Z, since AZ is the tangent at vertex.

The rt. angled Δs TAZ, ZAS, TZS, TNP are all similar.
Since TA $= \frac{1}{2}$TN, $\therefore$ ZA $= \frac{1}{2}$PN.
AS $= a$ = dist. of focus to vertex.

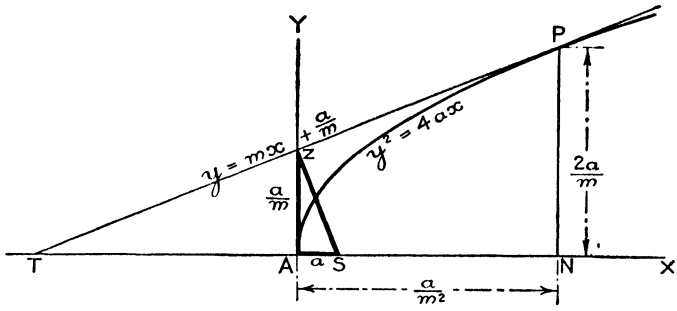


Fig. 241

$$\text{AZ} = \frac{a}{m} = \text{intercept on } y \text{ axis.}$$

$$\frac{\text{PN}}{\text{NT}} = m = \text{slope of tangent.}$$

$$\frac{\text{TA}}{\text{AZ}} = \frac{\text{AZ}}{\text{AS}}, \text{ or } \text{TA} = \frac{\text{AZ}^2}{\text{AS}},$$

$$\therefore \text{AN} = \frac{\text{AZ}^2}{\text{AS}} = \frac{\left(\frac{a}{m}\right)^2}{a} = \frac{a}{m^2} = \text{abscissa of P,}$$

and $$\text{PN} = 2\text{ZA} = \frac{2a}{m} = \text{ordinate of P.}$$ Q.E.D.

Observe, again, that the slope of the tangent $= \dfrac{\frac{1}{2}\text{ latus rectum}}{\text{ordinate}} = \dfrac{2a}{\frac{2a}{m}} = m.$

This pictorial parallelism between the geometry and algebra is essential whenever it is possible. Let the boys see that co-ordinate geometry is **geometry** and not mere algebra. But of course the geometrical figure is also a graph, to be interpreted algebraically.

We have taken the subject of co-ordinate geometry

thus far, less for its own sake than as an introduction to the next chapter. Co-ordinate geometry is an easy subject to teach, and boys like it, *provided* the geometry itself is made clear. As a subject of mere algebraic manipulation, unassociated with pure geometry, its value is slight, and time should not be spent over it.

Methods of Approximation

The calculus is such a valuable mathematical weapon, and the fundamental ideas underlying it are so simple, that the subject should find a place in every Secondary school. It might be begun in the Fifth Form, if not in the Fourth, though naturally the first presentation must be of a simple character. This simple presentation is easily possible. The more technical side of the subject, as elaborated in the standard textbooks, is wholly unnecessary in schools.

It was, I believe, Professor Nunn who pointed out that the history of the subject suggests the best route for teachers to follow. Although Newton and Leibniz are rightly given the credit of being the creators of the calculus as a finished weapon, the preliminary work of certain of their predecessors, especially Wallis, from which the main idea of the calculus was derived, must always be borne in mind. Wallis's work is merely a special kind of algebra and may readily be understood by a well-taught Fourth Form.

If we are thus to begin with approximation work, there is much to be said, as pointed out by Professor Nunn, for beginning with integration rather than with differentiation. The necessary arguments are so simple and the results so valuable that the rather radical departure from normal sequence is justified. For all practical purposes, Wallis was the actual inventor of the integral calculus, and Wallis's own work and methods serve to give young pupils a clear insight into the new ideas.

This early work, in differentiation as well as in integration, should be taught as a calculus of approximations. The

pupils should learn that such investigations give results which may be regarded as true to any degree of approximation, though not absolutely true. When later the calculus itself is formally taken up, and the pupils are able to grasp the modern theory of " limits ", they should be able to see that the new arguments, if properly stated, do as a matter of fact give results which are unequivocally exact. They must not be allowed to assume, at that later stage, that the arguments of the calculus prove merely approximately true results, and yet that these may be treated *as if* they were exact truths. This illegitimate jump from possible truth to certain truth is often made, it is true; but the deduction commonly involves the fallacious use of such terms as " infinitely small ", " infinitely great ", and the like.

" Methods of approximation are inferior methods and do not yield exact results." Granted. But these methods are best for beginners, if only because they form a good introduction to the exact methods of the calculus, and they are, after all, based upon a kind of reasoning which is rigorous enough for practical purposes. But the important thing is to make the pupils feel that they must never be finally satisfied until they have mastered a method which yields results that admit of no doubt at all.

The beginner has already learnt, or should have learnt, from his graph work the main idea of the real business to be taken in hand, and that is the nature of a *function*: that the value of one variable can be calculated from the value of another by the uniform application of a definite rule expressed algebraically.

We append a few suggestions for work in suitable approximations.

1. *Rough approximations.*

(i) Revise certain exercises in mensuration, e.g. find the area of a circle and of some irregular figures by the squared paper method.

(ii) Surveyor's Field-Book exercises; measure up some

irregular field, or other area, but insist that all such results are only rough approximations.

2. *Closer approximations*, and the methods involved.

(i) Revise the work on expansion (in physics). For instance, the coefficient of linear expansion of iron is ·00001. Justify the rule of accepting ·00002 instead of $(\cdot 00001)^2$ for area expansion, and ·00003 instead of $(\cdot 00001)^3$ for cubical expansion. Show the utter insignificance of the rejected decimal places. Refer to the geometrical illustrations of $(a + b)^2$ and $(a + b)^3$.

(ii) Estimate the area of a triangle as the sum of a number of parallelograms. The more numerous the parallels and the

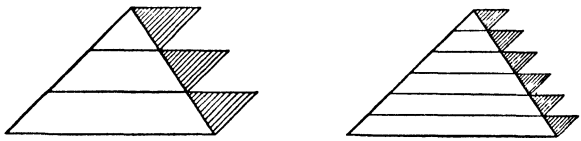


Fig. 242

more numerous the parallelograms, the more negligible do the projecting little triangles become. Observe that if the number of parallelograms is doubled, each shaded triangle is reduced to one-fourth; and so on.

(iii) Estimate the volume of a pyramid as the sum of the volumes of a number of flat prisms, gradually diminished in thickness.

(iv) Estimate the volume of a sphere regarded as the sum of a number of pyramids formed by joining the centre of the sphere to the angular points of a polyhedron, the number of whose faces is increased indefinitely. The pyramids formed have as their bases the faces of the polyhedron; the volume of each pyramid = (face × height)/3, hence the volume of the sum of the pyramids = (sum of faces) × height/3. If the number of faces be increased, the sum of the faces becomes more nearly equal to the area of the spherical surface, and then the height of the pyramid is more nearly equal to the

radius r of the sphere. But the sum of the faces can *never* be *quite* equal to the surface of the sphere, though we can so increase the number of faces of the polyhedron that the approximation may be closer than any degree we like to name. The *spherical* surface is necessarily greater than the sum of the *flat* faces of the polyhedron and can never be reached: it is an unreachable limit. If the sum of the faces *could* become equal to the surface of the sphere the sum would be $4\pi r^2$ and then the height of each pyramid would be equal to r. Hence the volume of the sphere would be $4\pi r^2 \times \frac{r}{3} = \frac{4}{3}\pi r^3$. Now this result agrees with the result arrived at by other methods, and it is correct. Still, to obtain the result, we had to jump from flat-faced pyramids (though these may have been made inconceivably small) to corresponding bits of spherical surface which were not flat.—We have still to discover whether such a method is allowable.

(v) The value of π. The pupils may be allowed to assume —from a figure they will readily guess—that if 2 regular polygons with the same number of sides be, respectively, inscribed within and circumscribed without a circle, the length of the circumference of the circle will be less than the perimeter of the circumscribed polygon but greater than that of the inscribed polygon. Show the pupils that the determination of π is thus merely a question of arithmetic, though of very laborious arithmetic, inasmuch as we have to determine the perimeters of polygons of a very large number of sides; the greater the number of sides, the greater the degree of approximation of the value of π.—Give a short history of the evaluation of π, from the time of Archimedes onwards. Point out that the irrationality of π has now been definitely demonstrated, so that it is useless for computors to waste any more time over it. Make the pupils see that the method of evaluating π is only a method of approximation, and that in this case no better method is ever to be hoped for; that we can obtain values more and more approximating

to the ratio of the circumference to the diameter, but there can be no *final* value, as the decimal can never terminate.

Area under a Parabola

We cut off a parabola by a line P'P perpendicular to its axis OA, and enclose it in the rectangle P'M'MP. We will calculate the shaded area OPM, i.e. the area " outside " or " under " the half parabola AOP. Let the parabola be $y=kx^2$

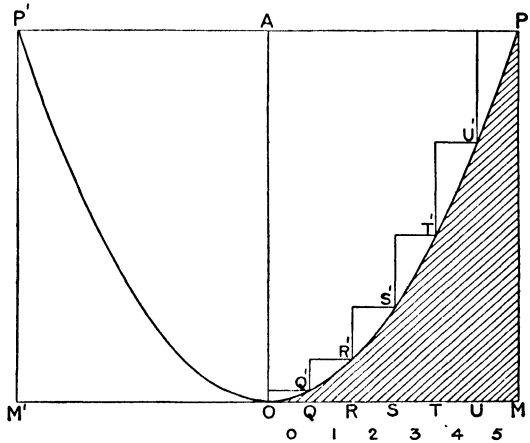


Fig. 243

We may divide OM into any number of equal parts, and on these parts construct a number of rectangles of equal breadth, set side by side as shown in the figure. The added areas of the rectangles are evidently in excess of the area OPA, but by increasing the number of rectangles indefinitely, the excess is indefinitely diminished.

We will begin by dividing OM into a small number of parts, and then increase the number gradually. As the first division OQ is gradually to be diminished, we will consider the rectangle on it to be of zero area. Hence OR $= (2 - 1)$ $= 1$ unit; $\therefore$ RR' $= 1^2$ units, OS $= (3 - 1) = 2$ units;

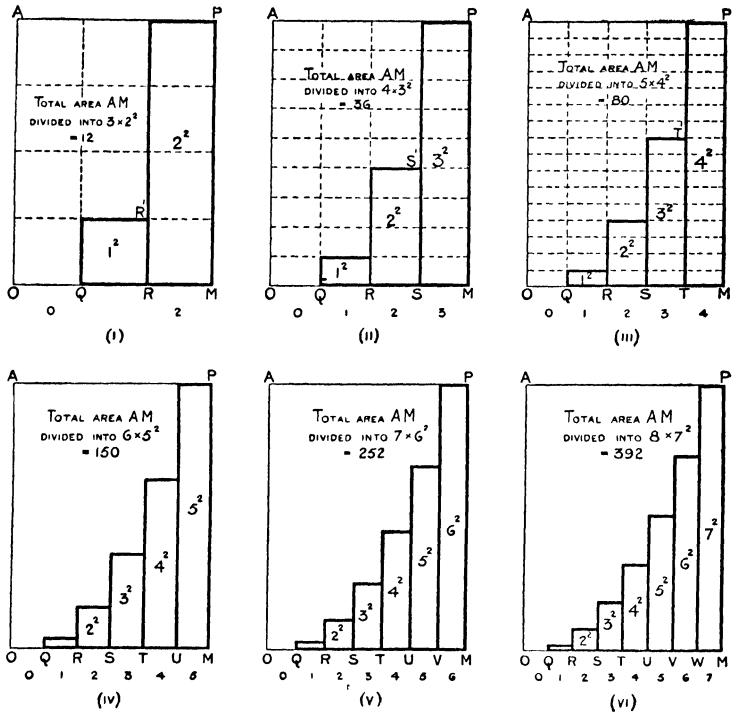


Fig. 244

$\therefore$ SS′ $= 2^2$ units; OT $= (4 - 1) = 3$ units, $\therefore$ TT′ $= 3^2$ units; &c.

We may tabulate the results thus:

Linear Units in		Square Units of Area in		Ratio of (*b*) to (*a*).	
OM.	PM.	(*a*) Rect. AM.	(*b*) Sum of contained Rectangles.	Lowest Terms.	Re-written.
3	2^2	3×2^2	$1^2 + 2^2$	$\frac{5}{12}$	$\frac{5}{12}$
4	3^2	4×3^2	$1^2 + 2^2 + 3^2$	$\frac{7}{18}$	$\frac{7}{18}$
5	4^2	5×4^2	$1^2 + 2^2 + 3^2 + 4^2$	$\frac{3}{8}$	$\frac{9}{24}$
6	5^2	6×5^2	$1^2 + 2^2 + 3^2 + 4^2 + 5^2$	$\frac{11}{30}$	$\frac{11}{30}$
7	6^2	7×6^2	$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$	$\frac{13}{36}$	$\frac{13}{36}$
8	7^2	8×7^2	$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$	$\frac{5}{14}$	$\frac{15}{42}$

The rewritten ratios show the numerators in *A.P.*, and the denominators as multiples of 6. Obviously if m be the number of units in QM, the ratio may be written $\frac{2m+1}{6m}$ or $\frac{1}{3}+\frac{1}{6m}$.

Thus the area of the added rectangles is equal to $\frac{1}{3}$ the area of the rectangle AM + a fraction depending on the value of m. It is easy to prove that the law holds good for all values of m.

The ratio $\frac{2m+1}{6m}$ enables us to write down as many terms as we please. For instance if $m = 1000$, the ratio $= \frac{2001}{6000}$ or $\frac{1}{3}+\frac{1}{6000}$. Hence if we built up a figure with 1000 rectangles, the total area of the rectangles would be equal to $\frac{1}{3}$ of AM + a small area equal to $\frac{1}{6000}$ of AM.

Evidently by taking m large enough, the fraction $\frac{1}{6m}$ becomes so small as to be insignificant, and thus the combined area of the rectangles can be made to differ as little as we please from $\frac{1}{3}$ the area of AM. And as the rectangles are made narrower and narrower, the area they cover will eventually become indistinguishable from the area under the curve OP (fig. 243); e.g. if $m = 1000$ the sum of the top left-hand corners of the rectangles projecting outside the curve is only $\frac{1}{6000}$ of AM. Finally the tops of the rectangles will be indistinguishable from the curve itself. We conclude, therefore, that this area under the curve is, at least to a very great degree of accuracy, $\frac{1}{3}$ of the rectangle AM. Since OM $= x$, and PM $= kx^2$ (fig. 243), we express the conclusion by the formula

$$\mathrm{A} = \tfrac{1}{3}kx^3.$$

It follows that the area AOP is $\frac{2}{3}$ the area AM. Hence the whole area of the parabola up to P′P is $\frac{2}{3}$ OA × P′P.

A point for emphasis: "Having *proved* that the area under the curve is, apparently to an unlimited degree of closeness, $\frac{1}{3}$ of the rectangle AM, we are *almost forced to*

believe that the former is exactly $\frac{1}{3}$ of the latter." Still, the fact remains that what we have proved is only an approximation. Do not disguise the theoretical imperfection of the conclusion. Do not slur over the fact that we have merely an approximation formula, though it is quite proper to emphasize the other fact that no limit can be set to the closeness of the approximation which it represents.

The particular approximation result arrived at is easily extended.—Let an ordinate start from the origin and move to the right. If it has a constant height, $y = k$, it will, in moving through a distance x, trace out an area, $A = kx$. If its height is at first zero, but increases in accordance with one of the laws $y = kx$, $y = kx^2$, $y = kx^3$, the area traced out will be given by the corresponding law, $A = \frac{1}{2}kx^2$, $A = \frac{1}{3}kx^3$, $A = \frac{1}{4}kx^4$. These results we might establish by proceeding exactly as before. Calling the function which gives the height of the ordinate, the *ordinate function*, and the function which gives the area traced as the *area function*, the results may be summarized simply

Ordinate functions	kx^0	kx^1	kx^2	kx^3	kx^{n-1}
Corresponding area functions	$\frac{1}{1}kx^1$	$\frac{1}{2}kx^2$	$\frac{1}{3}kx^3$	$\frac{1}{4}kx^4$	$\frac{1}{n}kx^n$

This summary exhibits Wallis's Law.

Books to consult:

1. *Teaching of Algebra*, Nunn.
2. *Cartesian Plane Geometry*, Scott.
3. *Modern Geometry*, Durell.

CHAPTER XXX

The Calculus: some Fundamentals

First Notions of Limits

To boys the two terms "infinity" and "limits" are always bothersome, and it is doubtful if the first ought to be used in Forms below the Sixth. A misapprehension as to the significance of both terms is responsible for much faulty work, much fallacious reasoning.

What is a *point*? "A point is that which marks position but has no magnitude." But how can a thing with *no* magnitude indicate position? If it has magnitude, is it of atomic dimensions, say 10^{-24} cm.? Or is it 10^{-10} of this? Obviously if it has magnitude *at all*, a certain definite number side by side would make a centimetre. But this is entirely contrary to the mathematician's idea of a point.

If a line is composed of points, the number of points certainly cannot be finite; otherwise, if the number happened to be odd, the line could not be bisected. Again, if the side and the diagonal of a square each contained a finite number of points, they would bear a definite numerical ratio to each other, and this we know they do not. The existence of incommensurables proves, in fact, that every *finite* line must, if it consists of points, contain an infinite number. In other words, if we were to take away the points one by one, we should never take away all the points, however long we continued the process. The number of points therefore cannot be counted. This is the most characteristic property of the infinite collection—that *it cannot be counted.*

Consider two concentric circles. From any number of points on the circumference of the outer circle, draw radii to the common centre. Each radius cuts the circumference of the inner circle, so that there is a one-to-one correspondence between all the points on the outer circle and all the points

on the inner. Imagine the outer circle to be so large as to extend to the stars, and the inner one to be so small as to be only just visible to the naked eye. Further, imagine an indefinitely large number of points packed closely round the circumference of the big circle, and all the radii drawn; the number of corresponding points on the inner circumference must be the same as the number on the outer. Clearly, in any line however short, there are more points than any assignable number. However large a number of points we imagine in a line, no one of them can be said to have a definite successor, for between any two points, however close, there must always be others.

Again, consider the class of positive integers. They may be put into one-to-one correspondence with the class of all even positive integers, by writing the classes as follows:

1	2	3	4	5	6	. . . . .
2	4	6	8	10	12	. . . . .

To any integer a of the first class there corresponds an integer $2a$ of the second. Hence the number of all finite numbers is not greater than the number of all even finite numbers. Evidently we have a case of the whole being not greater than its part.

Thus we have another characteristic of classes called infinite: *a class is said to be infinite if it contains a part which can be put into a one-to-one correspondence with the whole of itself.*

It is possible to imagine any number of sequences whose numbers have a one-to-one correspondence with all the integers, for instance all the multiples of 3, or of 7, or of 97. The characteristics of all such sequences are: (1) there is a definite first number; (2) there is no last number; (3) every number has a definite successor. Hence they must all be supposed to have the same infinite number of members.

It is important to notice that, given any *infinite* collection of things, any finite number of things can be added or taken away without increasing or decreasing the number in the collection.

It will be agreed that the nature of an infinite number is beyond the conception of an ordinary boy, and the boy should not be allowed to use the term. Even the ordinary teacher may ponder over the paradox of Tristram Shandy. —A man undertakes to write a history of the world, and it takes him a year to write up the events of a day. Obviously if he lives but a finite number of years, the older he gets the further away he will be from finishing his task. If, however, he lives for ever, no part of the history will remain unwritten. For the series of days and years has no last term; the events of the *n*th day are written in the *n*th year. Since any assigned day is the *n*th, any assigned day may be written about, and therefore no part of the history will remain unwritten.

Neither Tristram Shandy nor Zeno is meat for babes, but there are certain elementary considerations of number sequences with which boys should be familiar.

" Number " in the more general sense means simply the ordinary integers and fractions of arithmetic. All numbers in mathematics are based on the primitive series of integers. A fraction is, strictly speaking, a *pair* of integers, associated in accordance with a definite law. This law enables us to substitute for each single integer a pair of integers which are to be taken as equivalent to it. Thus $\frac{5}{1}$ is equivalent to 5. In this way we get an infinite number of numerical rationals of the same form.

Between any two numbers of the sequence of integers, there is an infinite number of rationals. For instance, between 8 and 13 there is an infinite series of rationals, or between 8 and 9. Obviously, then, the rationals between, say, 8 and 9 form a sequence that is endless both ways. Between 8 and 9 we have, for instance, 8·5; between 8 and 8·5 we have 8·25; between 8·25 and 8·5 we have 8·375; and so on indefinitely. *Consecutive* fractions, that is, fractions between which, for example, a mean cannot be inserted, are inconceivable, just as are consecutive points in space. The integers 8 and 9 are the

first numbers met with *beyond* the sequence. We call these numbers 8 and 9 the upper and lower **limits** of the sequence. There is no last rational less than 9 and no first rational greater than 8. It is erroneous to say that the terms of a sequence ultimately coincide with the limit. The limit is always *outside* the sequence of which it is the limit.

Consider the sequence $2 - \frac{1}{1}, 2 - \frac{1}{2}, 2 - \frac{1}{3}, \ldots 2 - \frac{1}{n}$, as n increases endlessly. Here successively higher integral values form a sequence of rationals which constantly rise in value but have *no last term*. There is, however, a rational number, 2, which comes *next after* all possible terms of the sequence. That is to say, if any rational number be named less than 2, there will always be some term less than $2 - \frac{1}{n}$ between it and the number 2. This is what is meant by calling 2 the **limit** of the sequence.

A and B are a given distance apart, say 2″. We attempt to reach B from A by taking a series of steps, the first step being half the whole distance; the second, half the remaining distance; the third, half the still remaining distance; and so on. When would B be reached? Obviously the answer is *never*. For any step taken is only half the distance still remaining. Thus the successive distances, in inches, are, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, and so on. However many of these distances are added together, the sum would never be equal to the whole distance, 2″. It is thus absurd to talk about summing a series to infinity. The **limit**, 2, is not a member of the series; it is unreachable and stands, a challenger, *right outside the series*, as a limit always does.

The example is well worth pursuing.

$$\text{The sum of the series } 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{12}} = 2 - \frac{1}{2^{12}}.$$

$$\text{The sum of the series } 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} = 2 - \frac{1}{2^n}.$$

We can, of course, take a number of terms of the series that will be great enough to make the sum fall short of 2 by less than any fraction that can be named, say less than 1/100,000. We have to calculate the value of n so that $\frac{1}{2^n}$ may be equal to or less than 1/100,000. By trial we find $\frac{1}{2^{16}} = \frac{1}{65536}$ and $\frac{1}{2^{17}} = \frac{1}{131072}$. The latter is less than 1/100,000. Thus if we take 17 terms of the series, the sum differs from 2 by less than 1/100,000.

It is impossible to take enough terms to make the sum equal to 2. There is always a gap $1/2^n$. However great n may be, $1/2^n$ is always something; it is never zero. We may say that the sum of n terms becomes nearer and nearer 2 as n becomes greater and greater, or that it tends towards 2 as n becomes indefinitely great. Do not use the term infinity.

Consider another example, the decimal ·11111....

$$\cdot 11111\ldots = \tfrac{1}{10} + \tfrac{1}{100} + \tfrac{1}{1000} + \tfrac{1}{10000} + \&c.$$

$$\text{The sum of the first 2 terms} = \frac{11}{100} = \frac{1}{9\frac{1}{11}}.$$

$$\text{The sum of the first 3 terms} = \frac{111}{1000} = \frac{1}{9\frac{1}{111}}.$$

$$\text{The sum of the first 4 terms} = \frac{1111}{10000} = \frac{1}{9\frac{1}{1111}}.$$

By increasing the number of terms, the sum can be made to differ by less and less from $\frac{1}{9}$, and this difference can be made smaller than any quantity that can be named. Hence $\frac{1}{9}$ is the *limit* to which the sum tends, though *this limit can never be actually reached.*

So in cases like the *area* under a curve. Where we say that PM is the *limit* of the ordinate *pm*, we mean that by taking *m*M constantly smaller, *pm* may be brought constantly nearer PM, and that it never occupies a position so near that

it could not be still nearer. It always remains the opposite side of the rectangle, and never actually coincides with PM, for PM stands outside all possible positions of *pm*. But PM is the *first* ordinate that stands outside the series, that is, there is no other ordinate between PM and the series of all possible positions of *pm*.

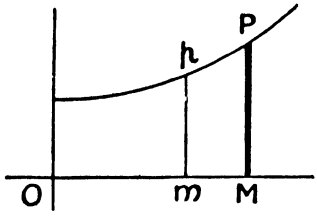


Fig. 245

The teacher should devise other illustrations of the nature of a limit. The notion is fundamental, and the pupils *must* understand it.

Rate

The old geometers were concerned more with drawing tangents to curves, and with finding the areas enclosed by curves, than with rate of change in natural phenomena; but the latter idea as well as the former one was certainly in Newton's mind, and was embodied in the language of the calculus, as we now call it, which he and Leibniz invented. The two ideas, tangency and rate, are virtually just two facets of the same idea, and in teaching the calculus the two should be kept side by side.

Pupils will have learnt something already about tangency. And if they have begun dynamics, as they ought to have done, they will have some idea of the nature of " Rate ". Rate is one of those rather subtle terms which are much better consistently used than formally defined. Even in the lower Forms, boys should be given little sums in which the term is correctly used: " at what rate was the car running?" and so forth. But before the calculus is begun, the notion of rate must be clarified. This means presenting the notion, in some way, in the concrete. Practical work is essential, even if the experiments are only of a rough and ready character. Suitable experiments are described in any modern book on dynamics.

Consider a train in motion. How can we determine its

velocity at some instant, say at noon? We might take an interval of 5 minutes which includes noon, and measure how far the train has gone in that period. Suppose we find the distance to be 5 miles; we may then conclude that the train was running at 60 miles an hour. But 5 miles is a long distance, and we cannot be sure that *exactly* at noon the train was running at that speed. At noon it may have been running 70 miles an hour, or perhaps 50 miles, going downhill or uphill at that time. It will be safer to work with a smaller interval, say 1 minute, which includes noon (perhaps half a minute before to half a minute after Big Ben begins to strike), and to measure the distance traversed during that period. But even greater accuracy may be required: one minute is a rather long time. In practice, however, the inevitable inaccuracy of our measurements makes it useless to take too small a period, though in theory the smaller the period the better, and we are tempted to say that for ideal accuracy an " infinitely small " period is required. The older mathematicians, Leibniz in particular, yielded to this temptation, and so gave wrong explanations of the working of the new mathematical instrument (the calculus) which they invented.

Revise rapidly some of the easier graph work and show how change of rate is indicated by change of steepness in the slope.

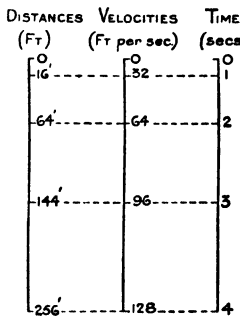


Fig. 246*a*

The careful study of a falling body will go far to make clear the notion of rate. Refer to Galileo's experiments on falling bodies. Generally speaking, it will not be possible to repeat such experiments, and so obtain first-hand data; the necessary data must therefore be provided otherwise.—Let fig. 246*a* represent the path of a body falling from a tower or down a well. The three lines allow the three sets of values (distances, velocities, times) to be shown in parallel, the distances and velocities being

shown at the end of 1, 2, 3, and 4 seconds, respectively. Use the data to verify (perhaps in some degree to establish) the formulæ $v = ft$, $v = u + ft$, $s = \frac{1}{2}(u + v)t$, $s = \frac{1}{2}ft^2$.

Boys are often puzzled about the 32 (the acceleration constant). In the first place, it is a power of 2, and they confuse it with t^2. (It is really best to use the nearer value 32·2, even though the arithmetic is a little more difficult.) In the second place the boys are apt to forget that this acceleration number is merely the value attached to a particular interval of time, viz. 1 second. They should be given a little practice with smaller intervals, say $\frac{1}{4}$ seconds. The second figure, a modification of the previous figure, is therefore useful. It represents the happenings in the first second, at quarter-second intervals. Since the same amount of extra velocity is added on *per second*, we have to take one-quarter of this for each quarter of a second. Observe that although this figure really represents the happenings in the first second of the previous figure, the two figures have an identical appearance so far as the line-divisions are concerned. The one second is divided up exactly as the four seconds were divided up. It impresses boys greatly that this sort of magnification or photographic enlargement might go on " for ever ". If, for instance, we take the first quarter-second of the last figure (246*b*), and magnify the distance line 16 times (as we did in the case of fig. 246*a*), we get still another replica, this time with the quarter-second divided up to show the happenings during each sixteenth-second. However short the distance, there is acceleration, and the acceleration has a constant value. The acceleration is " uniform ".

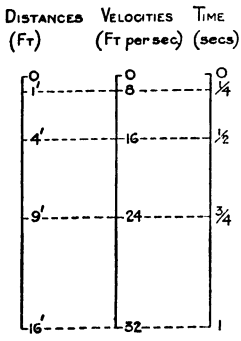


Fig. 246*b*

" *Uniform acceleration* is measured by the amount by which the velocity increases in unit time."—Many boys have difficulty in understanding what " uniform " accelera-

tion, such as acceleration due to gravity, really implies. "If only you would accelerate by adding on velocity in definite chunks at equal intervals, we could understand it."

Let the boy have his definite chunks, at first, and utilize these for approaching the main idea. Go back to the graph.

Suppose a train to move for 1 minute at a uniform velocity of 5 miles an hour; then to be suddenly accelerated to 10 miles an hour and to travel for 1 minute at that velocity; then to be accelerated to 15 miles an hour for a third minute; to 20 miles an hour for a fourth minute; to 25 for a fifth; and to 30 for a sixth. How far would it have travelled altogether?—A velocity-time graph shows this at once. The number of units of area under the graph is $1 + 2 + 3 + 4 + 5 + 6 = 21$, and this gives us the number of miles travelled.

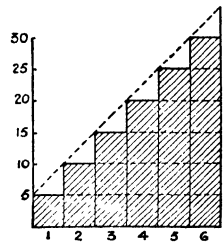


Fig. 247

The dotted line passing through the top left-hand corners of the rectangles can easily be proved to be straight, and this evidently indicates *some* sort of uniformity in the motion. But the whole of the area under this line is not enclosed by the rectangles; there are 6 little triangles unaccounted for. How are these triangles to be explained? By the fact that really we have imagined an impossible thing, viz. that at certain times the train's speed was instantaneously increased 5 miles an hour.

Now although in practice we know that even in the very best trains acceleration is really brought about by sudden jerks, these jerks are virtually imperceptible, and it is therefore not impossible to imagine an acceleration free from such sudden increases. It may be easily illustrated by running water: the following ingenious illustration we owe to Professor Nunn.

Attached to my bath is a tap so beautifully made that by means of the graduated screw-head I can regulate the amount of water running in up to 8 gallons a minute.

I turn on the tap for one minute, the water running at the rate of 1 gallon a minute; in that time 1 gallon has been delivered. Then I turn the tap on further, to deliver water at the rate of 2 gallons a minute, and allow it to run for one minute; during this minute, 2 gallons have been delivered. Thus I continue for 8 minutes, 8 gallons running in during the eighth minute. The graph (fig. 248, i) shows the water run in during the successive minutes; the shaded rectangle, for instance, represents the amount of water (5 gallons)

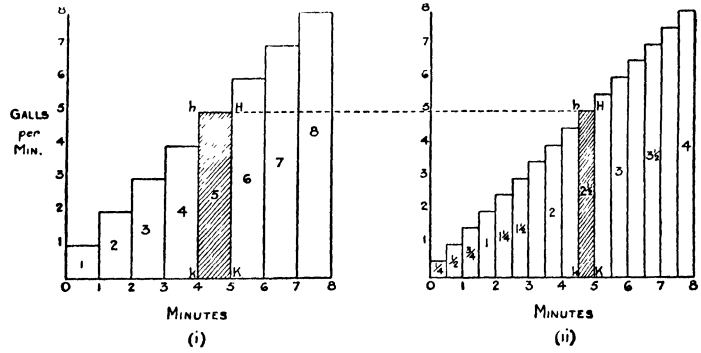


Fig. 248

run in during the fifth minute. Total number of gallons delivered = **36.**

I now repeat the operation, but this time I turn the tap on every *half* minute, beginning by running in ½ gall. a minute, and increasing by ½ gall. each half minute. The first delivery will be ¼ gall., the next ½ gall., and so on, the last being 4 gall. But note (fig. 248, ii) that during the last half of the fifth minute, when 2½ gall. were delivered, the *rate* of delivery was 5 gall. a minute; this column has the same height as the corresponding column in (i), but, of course, only half its area. The *rate* of flow during the half minute was the same, though only half the 5 gall. was actually delivered. The rate of flow during the last half of the eighth minute was 8 gall. a minute, though only 4 gall. were delivered. Total number of gallons delivered = **34.**

I repeat again, this time allowing the water to be increased every $\frac{1}{4}$ minute, beginning by running in $\frac{1}{4}$ gall. a minute, and increasing by $\frac{1}{4}$ gall. each $\frac{1}{4}$ minute. The first delivery is thus $\frac{1}{16}$ gall., the next $\frac{1}{8}$ gall., the last $\frac{3}{16}\frac{2}{6}$ or 2 gall. Note (fig. iii) that during the last $\frac{1}{4}$ of the fifth minute, when $1\frac{1}{4}$ gall. were delivered, the *rate* of delivery was still 5 gall. a minute; the column has the same height as the corresponding columns in the first two figures, but of course only $\frac{1}{4}$ of the area of the column in the first figure. The

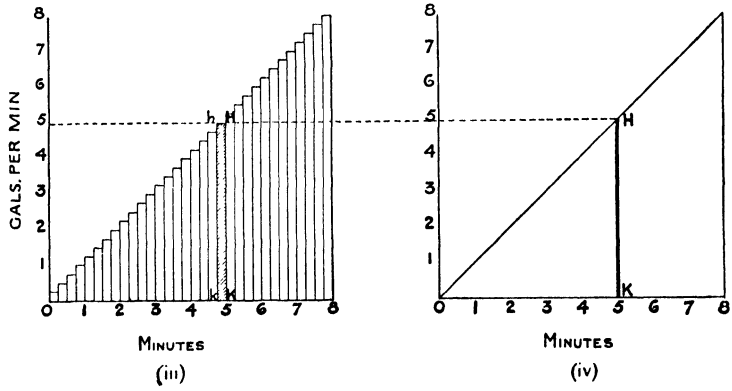


Fig. 249

rate of flow in the column preceding HK was the same in all 3 cases. Total number of gallons delivered (fig. 249, iii) = **33.**

I repeat the operation once more, this time turning on the tap *gradually* and continuously, in such a way that at the end of the first minute the water is running at the *rate* of 1 gall. a minute, though only momentarily; and so on. At the end of the eighth minute I turn off, i.e. at the very moment when the rate of flow has reached 8 gallons a minute. The graph (fig. 249, iv) is now a straight line, and its area is $\frac{1}{2}(8 \times 8)$ or **32** units, the number of gallons delivered.

Observe that, in all 4 figures, the rate of delivery at the end of any particular minute is the same, for instance at the end of the fifth minute, represented by HK; though in

the last case, when the tap is *gradually* turned on, the rate at any particular time is only momentary, since the rate is continuously changing.

In the last figure, HK no longer bounds a rectangle, as it did in the previous three figures; all the columns have become indefinitely narrow. The column which HK bounded has shrunk to a mere line which therefore cannot represent any *quantity* of water delivered. Still, as it has the same height as the series of gradually narrowing columns which it bounded, we say that it represents a *rate* of flow of 5 gall. a minute, just as the columns did. But this rate of flow is clearly not a rate of flow during any interval of time, however small. Hence we say it is the rate of flow *at* the end of the fifth minute.

Boys ought now to understand clearly that the *velocity* of a body at any instant is measured by the *rate* per unit time in which distance is being traversed by the body *when in the immediate neighbourhood* of that instant.

A body cannot move over *any* distance in *no* time, so that we could not find its velocity by observing its position *at* one single instant. To find its rate of motion, we must observe the distance traversed during some interval of time near the given instant, this interval of time being the shortest possible. Hence the term velocity *at* any instant must be regarded as an abbreviation for *average velocity during a very small interval of time, including the given instant.* But we have no means of finding such a velocity by actual experiment. We have to adopt other means.

It is sometimes said that *acceleration* at a given instant of time is measured by the rate *per* unit time at which the velocity is increasing *in the immediate neighbourhood* of the given instant, or the *average acceleration in a small interval of time including the given instant.*

The question, what is meant by the statement that at a certain moment a thing is moving *at the rate* of so many feet, a second ought now to be answered by all average

pupils. Sixth Form boys should grasp the full significance of the following formal statement: "if the magnitude possessed by any increasing or decreasing quantity be represented by an area-function, the *rate* of increase or decrease of the quantity at any specified point is given by the corresponding ordinate function."

Thus if any given function is regarded as an area-function, the corresponding ordinate function may be called the *rate-function* of the former.

Calculation of Rate-functions

We may consider again the rate-function corresponding to the area-function ax^3. According to the results at the end of the last chapter, this should be $3ax^2$.

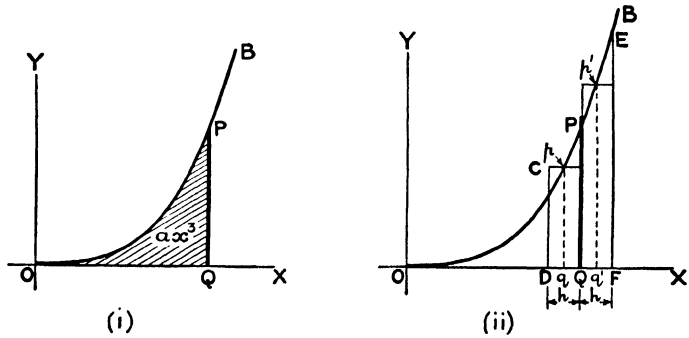


Fig. 250

Let the curve in fig. 250 have the property that the area under it from the y axis up to any ordinate PQ is ax^3. How may we determine the exact height of PQ?

Take two other ordinates (fig. ii) CD, EF, each at distance h from PQ. Draw upon DQ, QF rectangles whose areas are respectively equal to those of the strips under the curve between CD and PQ, and PQ and EF. Let the curve cut the upper ends of these rectangles in p and p'. Draw the ordinates pq and $p'q'$.

Although we cannot calculate PQ directly, it is easy to calculate pq and $p'q'$. We have:

$$pq \times h = \text{area CQ}.$$
$$\therefore pq = \frac{\text{area CQ}}{h}$$
$$= \frac{ax^3 - a(x-h)^3}{h}$$
$$= (3x^2 - 3xh + h^2)a,$$
$$\text{i.e. } pq = \{3x^2 - h(3x-h)\}a.$$

$$p'q' \times h = \text{area EQ}.$$
$$\therefore p'q' = \frac{\text{area EQ}}{h}$$
$$= \frac{a(x+h)^3 - ax^3}{h}$$
$$= (3x^2 + 3xh + h^2)a,$$
$$\text{i.e. } p'q' = \{3x^2 + h(3x+h)\}a.$$

Whatever value x may have, h may be taken smaller; hence h must be smaller than $3x$. Thus $h(3x - h)$ and $h(3x + h)$ must both be positive, and pq will necessarily be less than PQ, and $p'q'$ greater than PQ. By making h small enough, we can make pq and $p'q'$ differ from $3ax^2$ as little as we please. In other words, PQ must lie between all possible positions of pq and $p'q'$, and thus the value $3ax^2$ is the only value left for it to possess.

The Rate as a Slope.—Here is another way of considering a rate-function. Let OQ′ be the curve $y = ax^3$. Let the abscissa of any point P be x, and the abscissæ of two neighbouring points Q and Q′, $x - h$ and $x + h$, respectively. While x increases from $x - h$ to x, and from x to $x + h$, y increases by Qq and $q'Q'$, respectively. (Fig. 251.)

Hence the *average* rate of the latter increases must be $\frac{Qq}{h}$ and $\frac{q'Q'}{h}$, i.e. tan PtX and tan Pt'X, respectively.

$$\tan \text{P}t\text{X} = \frac{Qq}{h}$$
$$= \frac{\text{PM} - \text{QN}}{h}$$
$$= \frac{ax^3 - a(x-h)^3}{h}$$
$$\text{i.e. } \tan \text{P}t\text{X} = \{3x^2 - h(3x-h)\}a.$$

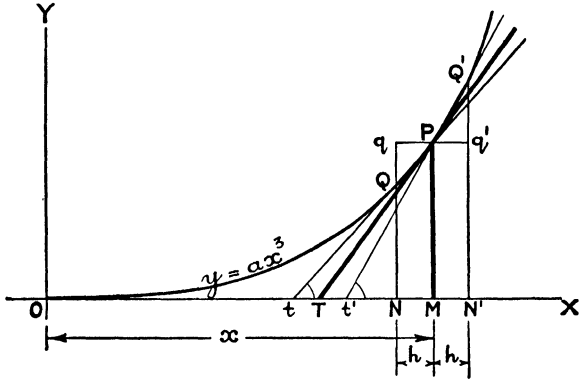
$$\tan \text{P}t'\text{X} = \frac{q'Q'}{h}$$
$$= \frac{\text{Q}'\text{N}' - \text{PM}}{h}$$
$$= \frac{a(x+h)^3 - ax^3}{h},$$
$$\text{i.e. } \tan \text{P}t'\text{X} = \{3x^2 + h(3x+h)\}a,$$

(both as in the last example)

As h gets smaller, Q and Q′ approach P, tan PtX being always

less, and $\tan Pt'X$ always greater, than $3ax^2$, though by taking h small enough, they may be made to differ as little as we please from $3ax^2$.

If PT be drawn, so that $\tan PTX = 3ax^2$ exactly, then PT is evidently the tangent at P. For a line through P ever so little divergent from PT would make with the x axis an angle greater or less than PTX, and so would cut the curve



h is purposely exaggerated to make the figure clear

Fig. 251

in one of the possible positions of Q or of Q'. Hence PT is the only line which meets the curve at P but does not cut it.

Thus PT holds among secants such as PQ or PQ' the same unique position that HK holds among the rectangles (figs. 248, 249), or that PQ holds amongst the other ordinates (fig. 250).

In fig. 251, the slopes of PQ and PQ' measure the *average rate of change of the function* during the changes of x represented by NM and N'M. The slope of PT does not measure the change *during* any intervals, but evidently measures what has been defined as the rate of change of the function **at** the moment (or for the value of x) represented by OM.

Meaning of "Limit"

The common element in the three cases considered, HK (figs. 248, 249), PQ (fig. 250), PT (fig. 251), is described by saying that all three are examples of a *limit*. In all three cases, members of a series have been brought nearer and nearer the limit, but they have never been so near that they could not have been brought nearer. They have always remained "in the neighbourhood" of the limit, but in every case the limit has been unreachable. In all three cases, the limit is the first number outside the series.

A *rate-function* is sometimes given this general definition: Take the given function of x, and find how much its value changes when x is raised or lowered by any positive number h. Divide this change by h, and so obtain the *average rate of change* for a change of the variable from $x - h$ to h, or from x to $x + h$. The *rate-function* is the *limit of the quotient* and is indicated more and more closely as h gets smaller and smaller.

The Two Main Uses of Limits

1. *To define the velocity of a given point at a given moment.* —If we define velocity as the quotient of a distance travelled, by the time in which it is traversed, then "the velocity at a given moment" is not a velocity at all.

On the other hand, if we consider the distance travelled by the point during a series of constantly decreasing intervals of time, and divide each distance by the length of the corresponding interval, we shall again fail, as a rule, to obtain anything that can be called *the* velocity of the point, for all the results will be different, except in the special case of uniform motion. But if the sequence of average velocities thus calculated follows some definite law of succession as the interval is taken smaller, then it will generally have a definite limit as the interval approaches zero. Thus the limit is a perfectly definite number, associated in a perfectly

unambiguous way, both with the given moment and with the endless sequence of different average velocities. Moreover, for small intervals of time, the average velocities are sensibly equal to the limit, the differences being of theoretical rather than of practical importance. It follows that although the " velocity at the given moment " is not really a velocity at all, it is quite the most useful number to quote in order to describe the behaviour of the moving point while it is *in the neighbourhood* of the place which it occupies at the given moment.

2. *To determine a magnitude which cannot be evaluated directly.*—Consider again fig. 250. We had to determine the height of the ordinate PQ. We found (i) that it lies between, and is the limit of, a lower sequence consisting of ordinates pq and an upper sequence consisting of ordinates $p'q'$; (ii) that it lies similarly between, and is the limit of, the sequences of numbers represented by $\{3x^2 - h(3x - h)\}a$ and $\{3x^2 + h(3x + h)\}a$; and (iii), that the latter sequence corresponds to the former, term by term. From these premises it seems to be an inescapable conclusion that the height of PQ is exactly $3ax^2$, for PQ is the *only* line between the two sequences of (i), and $3ax^2$ is the *only* number between the two sequences of (ii).

For blackboard revision work occasionally, devise questions to emphasize these principles (the term gradient might now be used generally):

(1) The gradient of a chord is the average gradient of the arc.

(2) A tangent is the limiting position of a secant.

(3) The gradient of a tangent at P is the gradient of the curve at P.

(4) The gradient of the tangent is the rate at which the function is changing.

(5) The limiting value of the slope of a secant is the slope of the tangent.

" *In the neighbourhood of.*"—We have spoken of the

members of a series being " in the neighbourhood of " a limit. What is a *neighbour*? That is a question of degree. In Western Canada, a man's nearest neighbours might be 40 or 50 miles away; in an English country district, perhaps a single mile; in a town, only a few yards; round one's own table, only a few inches. So with number sequences: it is just a question of degree. For instance, we know that π comes within the interval 3·1 and 3·2, and therefore 3·1 and 3·2 are neighbours of π. But π also comes within the smaller interval 3·13 and 3·15, which are therefore closer neighbours of π. Again, π comes within the interval 3·1414 and 3·1416, which are therefore still closer neighbours of π. And so we might go on. However close our selected neighbours of π, we can always find still closer neighbours. Thus π *always* has neighbours no matter how small the interval in which he is enclosed. It is all a question of standard of approximation. The important thing, when dealing with limits, is that we must never think of the interval shrinking to nothing. Think of the interval as always large enough for standing room both for π itself and some neighbours. The neighbours *cannot* be thought of as disappearing altogether.

Secant to Tangent Again

The gradient of a straight-line graph AB is determined easily enough: it is the ratio, ordinate y/abscissa x. The ratio may be determined from any selected bit of the line. Or, if we like, we may increase the line, say to BC, and take the ratio CD (= *increment* of y) to BD (= *increment* of x).

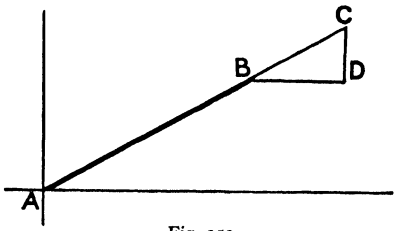


Fig. 252

But if the graph is a curve, the gradient at any specified point on the curve is determined by the tangent at that point. A ruler held against the edge of an ordinary

dish is, practically, a tangent at a point on the ellipse. If, then, we want to determine the gradient of a curve, why not just draw the tangent and measure the angle it makes with the x axis?

With a circle this would be easy enough: we should draw a radius to the point and then a line at right angles; and there are simple rules for certain other curves. But merely to hold a ruler against a curve, and to draw a line, is not to draw a tangent that we can accept.—Circulate amongst the class copies of a mechanically drawn parabola, tell the boys to draw a tangent at the point P, and then to measure the angle that the tangent makes with the x axis. The angles will probably be all different. Clearly the method will not do, for the angles *ought* to be the same in all cases.

If we draw a secant instead of a tangent, and find the gradient of the secant, we shall evidently have the *average* gradient of the curve between the two points P and Q where the secant intercepts the curve. Would that help?

Yes, but if the points are far apart, as P and Q_1, the slope of the secant, and therefore the average gradient of the curve between the two points, differs much from the gradient of the tangent PT. If we bring the points closer together, say P and Q_2, the gradient of the secant is nearer the gradient of the tangent. If we bring Q down to Q_3 the gradient of the secant PQ_1 is still nearer the gradient of the tangent. It is this gradient of the tangent that we have to find somehow.

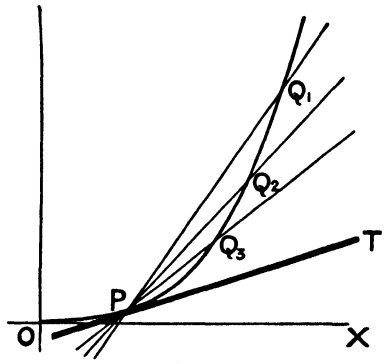


Fig. 253

[Some teachers prefer that numerical considerations like those that follow should precede the more general work

concerning the graph, as outlined in the earlier part of this chapter. I have seen equally satisfactory final results obtained from both sequences.]

Let us actually calculate the gradients of successive secants, and see if we can learn anything from the results. On a parabola we will select a point P where $x = 1$ (and $\therefore\ y = 1^2 = 1$), and keep this fixed. We will also place a point Q on the curve, at first where $x = 1{\cdot}5$ (and $\therefore\ y = (1{\cdot}5)^2 = 2{\cdot}25$). Thus, since P is (1, 1) and Q is (1·5, 2·25), the increment of x is ·5, and the increment of y is 1·25. (The piece of line PQ may be looked upon as an "increase" of the line AP; hence the term "increment" may usefully be applied to the corresponding increases of x and y.)

The gradient of the secant $= \dfrac{QV}{PV} = \dfrac{1{\cdot}25}{{\cdot}5} = \mathbf{2{\cdot}5}$. Now

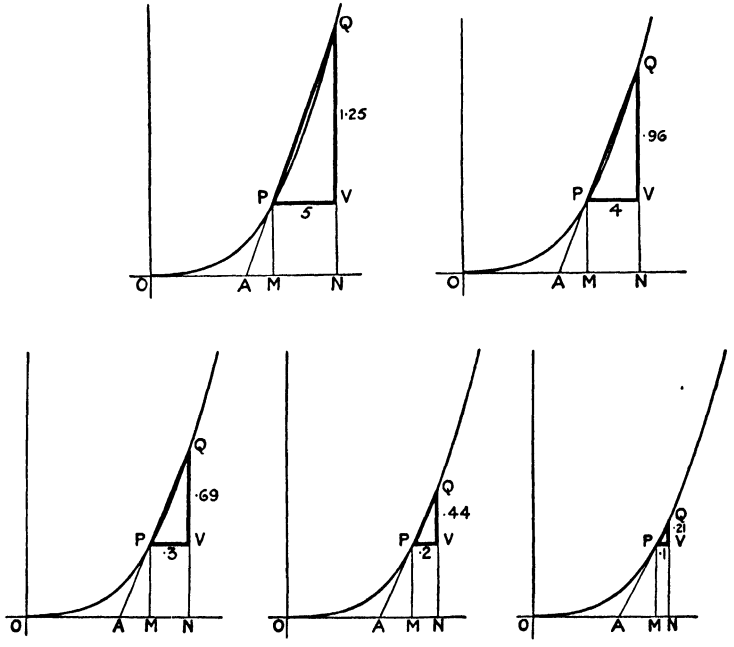


Fig. 254

bring Q gradually closer to P. Let the next four x values be 1·4, 1·3, 1·2, 1·1; then the corresponding y values are $(1{\cdot}4)^2$, $(1{\cdot}3)^2$, $(1{\cdot}2)^2$, $(1{\cdot}1)^2$. The gradient calculations may be summarized thus (the x and y increments are often indicated by h and k, respectively):

$x = \text{ON} =$	1·5	1·4	1·3	1·2	1·1
$y = x^2 = \text{ON}^2 = \text{NQ} =$	2·25	1·96	1·69	1·44	1·21
$k = \text{QV} =$	1·25	·96	·69	·44	·21
$h = \text{PV} = \text{MN} =$	·5	·4	·3	·2	·1
Gradient $= \frac{k}{h} = \frac{\text{QV}}{\text{MN}} =$	2·5	2·4	2·3	2·2	2·1

Note how the value of the gradient has diminished from 2·5 to 2·1. We cannot write $h = 0$, or the denominator of our ratio would equal 0, and the ratio would have no meaning. But we may continue to diminish the values of the x increment, and calculate the gradient as before. We may make the increments as small as we please. Let us calculate the gradient when the successive values of x for Q are 1·01, 1·001, 1·0001, 1·00001, so that the x increments are ·01, ·001, ·0001, ·00001. We cannot draw the figure, for the increments are much too small to be shown.

$x = \text{ON}$	1·01	1·001	1·0001	1·00001
$y = x^2 = \text{ON}^2 = \text{NQ} =$	1·0201	1·002001	1·00020001	1·0000200001
$k = \text{QV} =$	·0201	·002001	·00020001	·0000200001
$h = \text{PV} = \text{MN} =$	·01	·001	·0001	·00001
Gradient $= \frac{k}{h} = \frac{\text{QV}}{\text{MN}} =$	2·01	2·001	2·0001	2·00001

We observe (1) that however small we make h (the increment of x), the value of the gradient always exceeds 2; (2) that the smaller we make the increment, the smaller is the excess of the gradient over 2. Evidently we can approach to within any degree of approximation we like to name; it is only a question of making h small enough to start with. We observe also that the more nearly the value of the gradient approaches 2, the more nearly does the secant approach the position of the tangent. As long as the secant remains a secant, it can never be a tangent, and it must always have a gradient in excess of 2. But the successive gradients seem to compel us to *infer* that the gradient of the tangent itself, and therefore of the curve, is 2. Thus we regard 2 as the *limiting value* of the gradient of all possible secants. It is a value that is never *quite* reached by any secant, for the tangent stands alone, outside them all, four-square and defiant!

Thus we have found that, for the function $y = x^2$, the gradient of the point P, where $x = 1$, is 2.

We may arrive at the same result by arguing more generally, merely calling the increments, h and k.

The co-ordinates of P are (1, 1).

The co-ordinates of Q are $(1 + h, 1 + k)$.

Since $y = x^2$,

$$(1 + k) = (1 + h)^2$$
$$= 1 + 2h + h^2,$$
$$\therefore\ k = 2h + h^2,$$
$$\therefore \text{ gradient} = \frac{\text{QV}}{\text{PV}} = \frac{k}{h} = \frac{2h + h^2}{h}.$$

From this point on, argument nowadays commonly proceeds thus:

As Q approaches P, so h tends towards 0. We have to find the limit to which $\frac{k}{h}$ or $\frac{2h + h^2}{h}$ tends as h tends towards 0

As long as h is $\neq 0$, $\frac{2h + h^2}{h} = 2 + h$, and as $h \to 0$

$2 + h \to 2$. If we decide that $2 + h$ must differ from 2 by less than 1/1000000, there is no difficulty; we merely give to h a value less than that, e.g. 1/1000001.

In the limit, as $h \to 0$, $\frac{k}{h} \to 2$.

Hence the gradient of the curve at P = 2.

We may now find the gradient at *any* point P (x, y). The co-ordinates of Q are $(x + h, y + k)$.

$$\therefore (y + k) = (x + h)^2$$
$$= x^2 + 2xh + h^2,$$
$$\therefore k = 2xh + h^2.$$

Hence the gradient of PQ $= \frac{k}{h} = 2x + h$ if $h \neq 0$.

Now as Q approaches P, $h \to 0$.

$\therefore$ the gradient of the curve at P = limit of $(2x+h)$ as $h \to 0$,
$= 2x$.

I am not quite happy about the language of this argument, though it is now in common use and has been designed to get over the old difficulty of infinitesimals and of the absurdity of dividing by 0. But even able boys in the Sixth sometimes admit that the reasoning is not clear to them, saying that they feel they take a leap over the final gap to the limit. The teacher must insist that the gap is really never crossed, that the interval still remains, that the limit is always there with a crowd of neighbours who vainly strive to reach him; that every neighbour has a value a little greater than $2x$ (or, in some of our earlier illustrations, a little less), and that the value $2x$ is a solitary value, which therefore we feel bound to assume is the value which belongs to the Limit, and to the Limit alone.

Revise: *The function* $y = x^2$.—To calculate the *ordinate* for any value of x, work out the value of x^2. To calculate the *gradient* for any value of x, work out the value of $2x$.

Thus x^2 may be described as the formula for the ordinate, and $2x$ as the formula for the gradient. In other words, the function x^2 gives the ordinate, and the function $2x$ the gradient, for any value of x.

x^2 is the original function which defines the curve; $2x$ is called the *derived function* of x^2. The process of finding the derived function of a given function is called *differentiation.*

Since the gradient of the tangent to $y = x^2$ at any point P is $2x$, the gradient where $x = 1$, is 2; where $x = 2$, is 4; where $x = 3$, is 6; &c. Does this square with the work we have done in pure geometry? We found (p. 409),

$$\text{gradient of tangent to axis of parabola} = \frac{\frac{1}{2}\text{ latus rectum}}{\text{ordinate}},$$

$$\textit{or}\ \text{gradient of tangent to tangent at vertex} = \frac{\text{ordinate}}{\frac{1}{2}\text{ latus rectum}}.$$

Let the tangent at the vertex be the x axis, and let the axis

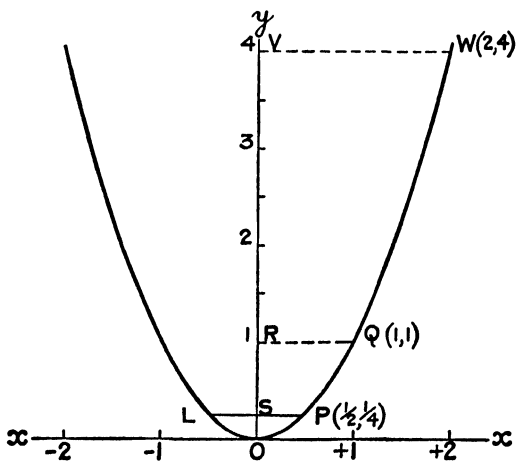


Fig. 255

of the parabola be the y axis. Let S be the focus, and let the latus rectum LSP be unit length.

Half the latus rectum = SP = $\frac{1}{2}$. Since PN = half the distance of P from the directrix (not shown), PN = $\frac{1}{2}$PS = $\frac{1}{4}$. Hence the co-ordinates of P are ($\frac{1}{2}$, $\frac{1}{4}$).

At the point Q (**1**, 1), gradient of tangent to OX

$$= \frac{\text{ordinate}}{\text{half latus rectum}} = \frac{1}{\frac{1}{2}} = \mathbf{2}.$$

At the point W (**2**, 4), gradient of tangent to OX

$$= \frac{\text{ordinate}}{\text{half latus rectum}} = \frac{2}{\frac{1}{2}} = \mathbf{4}.$$

At a point Z (**3**, 9), gradient of tangent to OX

$$= \frac{\text{ordinate}}{\text{half latus rectum}} = \frac{3}{\frac{1}{2}} = \mathbf{6}.$$

Clearly then, the new method of finding the slope of the tangent does produce a result absolutely accurate, not merely approximately accurate. Evidently the " limit " argument is sound, though we must always remember that the limit is *outside* the sequence under consideration, never reached by any member of the sequence.

The Calculus Notation

We have used the letters h and k to denote the increases (" increments ") in the values of x and y. But the increments always actually considered are very small, and the symbol generally used to denote them is the Greek letter *delta* (Δ or δ) prefixed to the value of x or y from which the increment begins. Pronounce Δx as " delta x "; the symbol Δx must be taken as a whole; Δ is not a multiplier and has no meaning apart from the x and y to which it refers. Remember, then, to write Δx instead of h, and Δy instead of k.

Δx means "the increment of x"; Δy means "the increment of y".

$\frac{\Delta y}{\Delta x}$ means the ratio $\frac{\text{increment of } y}{\text{increment of } x}$. The Δ's cannot be cancelled.

Treat $\frac{\Delta y}{\Delta x}$ exactly as if written $\frac{k}{h}$; it measures the *average gradient of the graph* over **the interval** between x and $x + \Delta x$.

The **limit** of $\frac{\Delta y}{\Delta x}$ is the gradient of the graph at the point given by x. It is sometimes written $\mathrm{D}(y)$, sometimes $\frac{dy}{dx}$. But the curious thing is that, although $\frac{dy}{dx}$ looks like a ratio or a fraction, it is *not* a ratio or fraction. The symbols dy, dx, written separately, have no meaning. The limit of Δx is not dx; the limit of Δy is not dy. $\frac{dy}{dx}$ is just a single symbol. $\frac{\Delta y}{\Delta x}$ is always a ratio of real value; $\frac{dy}{dx}$ is not a ratio at all and is therefore very misleading to the eye.

The process of finding $\mathrm{D}(y)$ or $\frac{dy}{dx}$ is called *differentiation.* $\mathrm{D}(y)$ or $\frac{dy}{dx}$ has received various names:

(1) The derivative of y or $f(x)$ with respect to x.
(2) The differential coefficient of y with respect to x.
(3) The derived function of y with respect to x.

We will differentiate x^4. Let $y = x^4$. When x is increased to $x + \Delta x$, let y be increased to $y + \Delta y$. Then:

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^4 \\ &= x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4; \\ \therefore \Delta y &= 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4, \\ \therefore \frac{\Delta y}{\Delta x} &= 4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3. \end{aligned}$$

Hence as $\Delta x \to 0$, $\frac{\Delta y}{\Delta x} \to 4x^3$.

$$\therefore \frac{dy}{dx} = 4x^3.$$

Let the class discuss the result (or one like it) critically. In particular, discuss the significance of the arrows. Forms and language that might pass muster in an examination room should be subjected to the closest scrutiny in class. It is a fact that those boys who have acquired facility in working out the ordinary stock exercises in the calculus are often nonplussed when cross-examined in the underlying fundamental notions.

There can be no doubt that the idea of derived functions is best introduced as a generalization of the familiar ideas of connexions between area functions and ordinate functions, ordinate functions and gradient functions, &c. The notation $\frac{dy}{dx}$ should not be introduced too soon. $D(y)$ is much preferable. The symbol $\frac{dy}{dx}$ originated with Leibniz (not with Newton), and it expresses a view of the nature of a differential coefficient that is out of harmony with modern ideas and conflicts with the doctrine of limits. Originally the view was that any finite value of the variables y and x is really the sum of a vast number of "infinitesimal" values which, though immeasurably small, have yet a definite magnitude. Thus the differential coefficient $\frac{dy}{dx}$ was looked upon as simply the ratio of the "infinitesimals" of two variables, the ratio being finite and measurable (much as the weights of atoms are measurable), in spite of the smallness of the terms. This view is no longer held. The expression $\frac{dy}{dx}$ is not a ratio at all but only the **limit** which the ratio of the increases of the variables approaches as the increment of x approaches zero. Naturally the learner is greatly puzzled

if he is told to write the derivative in the form of a fraction and is then forbidden to think of it as a fraction. Thus it is much the best plan to withhold the Leibnizian notation at first. Use the symbol D(y) instead; this symbol reminds the pupil that he is seeking a *function* which he is to *derive* from the given *function* y by means of a definite rule of procedure. This relationship between functions is the essence of the whole matter.

Integration.—Like current ideas about the nature of a differential coefficient, those about the nature of an integral also show traces of the erroneous mathematical philosophy of earlier days. The problems first systematically studied by Wallis came to be regarded as having for their aim the summation of an " infinite " number of " infinitesimals " dy, of the form $y.dx$. This view is still represented not only by the usual notation $I = \int y.dx$, which (like dy/dx) was introduced by Leibniz, but also by the common statement that an integral is the sum of an infinite number of infinitely small magnitudes. With the rejection of the notion of an infinitesimal as a definite atomic magnitude, this statement, and the notation which expresses it, have become inadmissible. If dx has any numerical significance at all, it stands for the increment h when h is zero. Hence the product $y.dx$ is also zero for all values of y, and therefore the summation represented by $\int y \,.\, dx$ is the summation of a series of zeros! I is *not* the sum of an infinite number of products; it is simply the **limit** of the sum of a *finite* number of products.

There is neither need nor warrant for introducing the term " infinite " at any point of the discussion. If we substitute the useful Δx for the absurd dx, we may still usefully retain the Leibnizian mode of expression $I = \int y \,.\, \Delta x$, but the symbol " $\int$ " must now be read, " limit of the sum as Δx approaches zero ".

Interpretation of $\frac{dy}{dx}$.

1. $\frac{\Delta y}{\Delta x} = \frac{\text{distance}}{\text{time}}$ = average speed during Δx;

   $\frac{dy}{dx}$ = **limit** of average speed = " instantaneous " speed.

2. $\frac{\Delta y}{\Delta x}$ = average slope of curve during interval Δx;

   $\frac{dy}{dx}$ = **limit** of average slope = limit at point P.

The two problems (1) to determine the rate of increase of a function and (2) to draw a tangent to a curve, are really identical; if we have a general method of determining the rate of increase of a function $f(x)$ of a variable x, we are able to determine the slope of the tangent at any point (x, y) on the curve.

Points for emphasis.—We will once more stress the points to be kept in the forefront of the teaching.

The pupils should be told plainly that the old idea of infinitely small quantities has been definitely abandoned. The real explanation of the whole thing was first put forward by a German mathematician, Weierstrass, about the middle of the nineteenth century.—The subject had been sound enough; so, virtually, had been the mathematical procedure, but the explanation had been wrong.

The problem was to retain an interval of length h, over which to calculate the average increase, and at the same time to treat h as if it were zero. As Professor Whitehead says, " As long as we look upon ' h tending to a ' as a fundamental idea, we are in the clutches of the infinitely small, for we imply the notion of h being infinitely near to a. This is what we want to get rid of." " The limit of $f(h)$ at a is a property of the neighbourhood of a." " In finding the limit

of $\frac{h(2x+h)}{h}$ at the value 0 of the argument h, the *value* (if any) of the function at $h = 0$ is *excluded*. But for all values of h except $h = 0$ we can divide through by h." " In the neighbourhood of the value 0 for h, $2x + h$ approximates to $2x$ within every standard of approximation, and therefore $2x$ is the limit of $2x + h$ at $h = 0$. Hence, at the value 0 for h, $2x$ is the limit of $\frac{(x+h)^2 - x^2}{h}$."

The difficulty of former mathematicians was that on the one hand they had to use an interval h over which to calculate the average increase, and on the other hand they wanted to put $h = 0$. " Thus they seemed to land themselves with the notion of an existent interval of zero size." Present-day mathematicians avoid that difficulty by using the notion that, corresponding to any and every possible standard of approximation, there is still some *interval*.

My own experience is that when Sixth Form boys are puzzled over this question, their puzzlement is almost always due to the fact that they have got hold of the term infinity, and do not understand what the term signifies.

Books to consult:

1. *The Teaching of Algebra*, Nunn.
2. *An Elementary Treatise on the Calculus*, Gibson.
3. *Course of Pure Mathematics*, Hardy.
4. *Applied Calculus*, Bisacre. (An outstanding book.)

CHAPTER XXXI

Wave Motion: Harmonic Analysis: Towards Fourier

Sine and Cosine Curves. Composition

The pupils will, of course, be thoroughly familiar by this time with the radian notation, and will understand that the reason for measuring angles in radians is that theoretical arguments are simplified. They will know that π radians $= 180°$; that as an angle of θ radians is subtended by an arc of θr, the length of an arc of a circle $= r\theta$; that the symbols θ and ϕ are commonly used in circular measure, and the symbols α, β, γ for measurements in degrees. They ought also to know that, when an angle is small, its circular measure may, in approximation calculations, be substituted for its sine (or tangent); and that, when it is not small, the values of the sine and cosine may still be expressed approximately in circular measure by means of the simple formulæ $\sin\theta = \theta - \frac{\theta^3}{6}$, $\cos\theta = 1 - \frac{\theta^2}{2}$. The proofs of these may be given at an appropriate stage, but a simple graphic method is easily devised to *suggest* that the formulæ are approximately true.

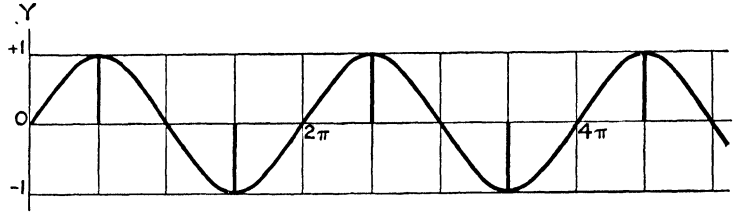
When boys are first introduced to angles greater than 360°, they are inclined to doubt if they are dealing with real things, and to be a little sceptical about the practical value of the work in hand. Light, however, begins to dawn when they are introduced to Simple Harmonic Motion, to waves, and to spirals.

They must be made to understand clearly that the values of the ratios connected with an angle are repeated endlessly in cycles as the angle rises.

They will, of course, be thoroughly familiar with the sine and cosine curves. With very little practice they can make

a supply of these curves for themselves by running them off from Fletcher's trolley arranged for uniform motion; with care, these curves may be obtained to a surprising degree of accuracy. Draw tangents to the succession of crests, then the axis midway between them. The chief ordinates are the perpendiculars at those points of the axis midway between the nodes.

Point out that all sine curves have the same general shape and properties; that $\sin x$ gives a wave curve of period



2π with successive maximum and minimum values at $+1$ and -1 respectively; that $\sin px$ gives a wave curve of period $\frac{2\pi}{p}$; that $a\sin(px + e)$ gives a wave curve of period $\frac{2\pi}{p}$, with successive maximum and minimum values of $+a$ and $-a$, respectively, the effect of e being merely to displace the curve along the axis.—These fundamentals must be mastered. The p, the a, and the e are veritable traps for the unwary beginner; the inner significance of the three symbols should be expounded and emphasized again and again.

From his earlier knowledge of graphs, a boy may, without further instruction, graph one or two easy cases of compound periodic functions. We give two examples adapted from examples in Siddons and Hughes' *Trigonometry*, the first, $2\sin x + 3\cos x$, consisting of two periodic functions of the same period (fig. 256), and the second $\sin 3x + 2\sin x$, periodic functions of different periods (fig. 257). In each case the two functions are first plotted separately (the curves are shown by lighter lines), then the required composite

curve is obtained by means of points determined by taking the algebraic sum of the ordinates of the constituent curves.

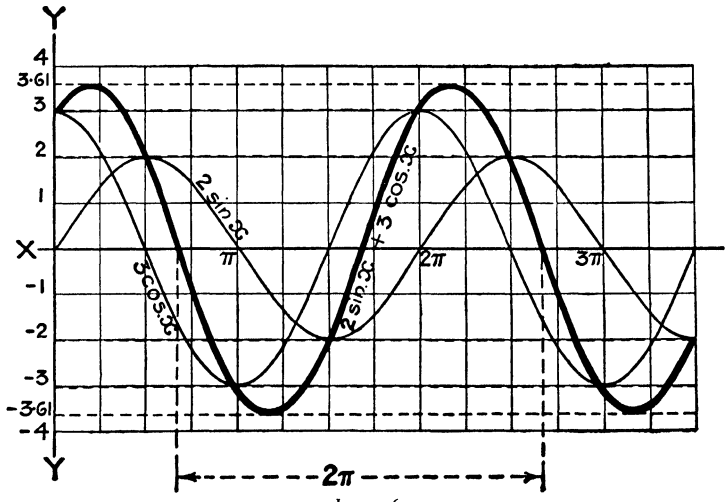


Fig. 256

For instance, in the second case $pm = pn + pq$. Note in the first case where we are compounding functions of the

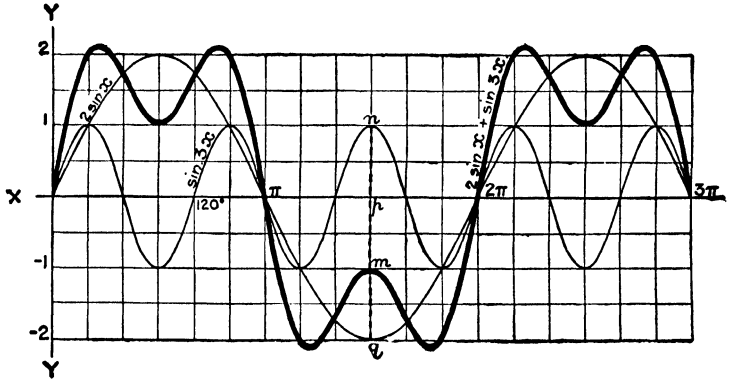


Fig. 257

same period, the result is a *sine* curve; in the second case, where the functions to be compounded are not of the same

period, the result, though a periodic curve, is *not* a sine curve. If in the second case we slide the half curve π to 2π along the x axis up to the y axis, the upper and lower halves will easily be seen to be symmetrical. Observe that in the case of $\sin 3x$ the period is $\frac{360^\circ}{3} = 120^\circ$.

In all such cases the shape of the composite curve can be obtained only by plotting a wide range of ordinate values, though this is always simply done by algebraic addition, and a pair of dividers will soon give the necessary number of points. The curves are a little tricky to draw, because of the minus quantities to be added.

This kind of exercise need take but little time. I have known boys work through half a dozen in an hour. The general shapes of the sine and cosine curves are already familiar, and as the axis can be divided up and the principal ordinate put in at once, the constituent curves can be sketched in in less than a minute. It is assumed, of course, that the significance of p, a, and e has been fully grasped. But the negative quantities to be considered when building up the composite curve nearly always give trouble. *No calculations are, however, necessary.* Let the dividers do the work, unless, in some very exceptional instances, rigorous accuracy is wanted. The *general form* and what it teaches is the main thing.

Waves and their Production

Since, in most instances, waves are periodic phenomena, they afford excellent concrete examples of periodic functions. No one can appreciate the most striking triumphs of physical science who has not given some attention to the mathematics of wave motion. In fact, wave motion now forms the very basis of the study of the greater part of physics, and, after all, the necessary mathematics of the subject is, in all its main factors, quite simple. It need hardly be said that that new and rather formidable subject, Wave Mechanics, is outside the scope of school practice.

The teaching of such characteristics of waves as resistance, persistence, and over-shooting the mark, is part of the business of the physics master. The mathematical master is concerned mainly with considerations of the *form* of the wave and its analysis. Let beginners first read through Fleming's *Waves and Ripples*, and so supplement the work they have already done in the physics laboratory; the mathematics will then give them little trouble. But, if they begin the mathematics of wave motion before they have acquired in the laboratory a considerable amount of practical knowledge of the subject, they will never be quite sure of their ground.

The boys will probably be familiar with the device of producing a train of waves by means of a length of narrow stair carpet, or a sand-filled length of rubber tubing, or a length of heavy rope: these things are part of the stock in trade for teaching wave motion in the physics laboratory. An instructive experiment is the following: take a common blind-roller about 5′ long, with a pulley runner fixed at each end. Into the roller drive 37 4″ nails, at $1\frac{1}{2}$″ intervals, in the form of a uniform spiral of 3 complete turns. The nails should be separated from one another by a uniform interval of 30°, so that the 1st, 13th, 25th, and 37th are in the same straight line; the 2nd, 14th, and 26th in another straight line; and so on. Support the roller in a horizontal position in front of a white screen, and turn it by means of an improvised crank. Let a distant light throw on the screen a shadow of the rotating roller. Observe how the shadows of the nail-heads exhibit progressive wave motion. Observe the movement of any one particular shadow; it is an example of simple harmonic motion (see Chapter XXXVII). The travelling shadow-wave, constituted by equal simple harmonic motions of the shadows of the nail-heads, is a progressive harmonic wave. The shadow of any head differs in phase from that of its neighbours by a constant amount of 30°. Note that each nail remains in its own vertical plane; the progressive horizontal movement is one of *form* only. The boys *must* distinguish between (1) the actual to and fro

movements of elements in a wave-medium, and (2) the movement of the wave itself. The second is merely an appearance, resulting from the successive movements of the first. The first has the effect of making successive sections of the medium (as we may conveniently call it) assume one after another the same shape. The shape therefore *appears to be something moving along.*

The waves on the surface of the sea, away from the shore, are good examples of progressive waves. If their outline were exactly a sine curve, as theoretically it should be, we should have an example of a *harmonic progressive wave.*

Common Wave Formulæ

In figure 258,

the *wave-length* $= L_1L_2 = L_2L_3 = \lambda$,
the *amplitude* $= PQ = a$.

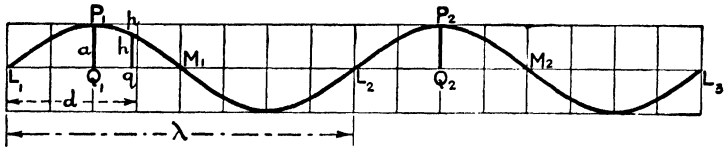


Fig. 258

If T is the periodic time of the wave (i.e. time to complete a vibration), it follows from first principles that $v = \lambda/T = n\lambda$, where v = velocity and n = frequency.

Let L_1q be d; let pq be h. Since the curve is a sine curve, $\lambda = 360° = 2\pi$. Hence the number of degrees in $d = \frac{2\pi}{\lambda} \times d$. For $\frac{2\pi}{\lambda}$ write p. Then, wherever q is taken along L_1L_2, it may be found, by actual measurement, that $h = a \sin pd$. This is a fundamental formula.

Let $y = a \sin px$ describe the wave outline in a given position, x being measured from L_1. If the curve move to the

right with a velocity v, its form after t seconds is given by the formula,

$$y = a \sin p(x - vt). \quad \ldots\ldots\ldots \quad \text{(i)}$$

If to the left with the same velocity,

$$y = a \sin p(x + vt). \quad \ldots\ldots\ldots \quad \text{(ii)}$$

These formulæ are simply applications of the general principle that if a graph is moved a distance d parallel to the x axis, $(x - d)$ must be substituted for x in the formula.

Since $p = \frac{2\pi}{\lambda}$, (i) and (ii) may be expressed thus:

$$y = a \sin \frac{2\pi}{\lambda}(x \pm vt).$$

The actual significance, in the graph, of each symbol in this formula *must* be understood.

Compound Harmonic Waves

Let two boys near each other on the edge of a pond, or other suitable sheet of still water, each produce a series of waves by striking the water rhythmically with a stick. Let the frequency of the blows be 2 to 3 (say 2 in 2 seconds and 3 in 2 seconds, easily done after a little practice with watch in hand), and suppose the waves to travel with the velocity v. A pattern will result from the five waves which are produced every two seconds, and it will be regularly repeated, though gradually fading away into ripples. But this pattern will no longer represent simple harmonic waves, for the shape which appears to move along the water beyond the ends of the line joining the centres of disturbance, is no longer a simple sine wave; the length, the frequency, and the amplitude of the resultant waves will be different from the length, the frequency, and the amplitude of the component waves. At points reached simultaneously by crests and troughs belonging to the component wave-trains,

the elevation or depression of the surface is exaggerated.—All this should be confirmed by observation.

To calculate the resultant disturbance due to the two component waves (assumed to be simple harmonic waves), we adopt the principle, which accords with observation, that the actual displacement at any point is equal to the algebraic sum of the displacements due to the waves separately.

If the first wave-train existed alone, the displacements produced would be represented by moving the curve $y = a_1 \sin\frac{2\pi}{\lambda_1}x$ with velocity v towards the right. If the second wave-train existed alone, the displacement produced would be represented by moving the curve $y = a_2 \sin\frac{2\pi}{\lambda_2}(x-c)$ with velocity v towards the right. Here, c is the x co-ordinate (at $t = 0$) of the nearest point of the wave from the centre of disturbance, comparable with L in fig. 258.

Thus the actual character of the resultant composite wave is represented by the graph

$$y = a_1 \sin\frac{2\pi}{\lambda_1}x + a_2 \sin\frac{2\pi}{\lambda_2}(x - c),$$

moving to the right with a velocity v.

This evaluation from first principles is really very simple, but, unless it is associated with at least a little experimental work, it may prove difficult for average boys. The formula is a key formula and should be mastered.

Had there been *three* boys at the pond side, each producing waves by striking the water rhythmically, all the waves being of different length, the composite waves would have been more complex, and the necessary formula for the graph would have consisted of three terms. So generally.

Comparison of Periodic and non-Periodic Functions

Functions of the form $\sin px$ and $\cos px$ $\left(\text{where } p = \frac{2\pi}{\lambda}\right)$ have much the same relation to periodic curves as x has to non-periodic curves. The simplest non-periodic curve is the straight line $y = Ax$ (we write it in different forms, according to circumstances; e.g. $y = mx + c$); and the simplest periodic curve is $y = A \sin px$ (also written in different forms according to circumstances).

With our former non-periodic work we soon learnt that the curve $y = A_1x + A_2x^2$ was more complex than $y = Ax$, and that $y = A_1x + A_2x^2 + A_3x^3$ was more complex still; and so on; a quadratic function was more complex than a straight-line function, a cubic more complex than a quadratic. Still, however complex the function, it was always a question of the addition of a number of terms; the actual graphing was simple enough though tedious if the terms were many.

So it is with periodic functions, where the curve, whether simple or complex, recurs endlessly, and makes a continuous *wave*. The effect of adding to $y = A \sin px$ the term $A^2 \sin 2px$ may be compared with that of adding to $y = Ax$ the term A_2x^2; in each case we obtain a form of greater complexity. By adding further terms we get, in each case, still further complexity, save that in the former case the successive curves all have the period $\lambda = \frac{2\pi}{p}$ or a submultiple of this.

The standard form of a periodic function may be written:

$$y = A_1 \sin px + A_2 \sin 2px + A_3 \sin 3px + \ldots + A_r \sin rpx.$$

It was the French mathematician Fourier who first observed that a periodic function of unlimited complexity may be described by a formula of this type. The process of determining the components of which a given periodic function

is the resultant is known as *harmonic analysis.* Fourier's statement is known as Fourier's theorem.

Let the boys consider an illustration of this kind. Let them inagine a water wave sent out with a velocity v, of length λ, and frequency 1 per second; the wave would form a simple sine curve, such as we see on any disturbed water surface. Now let them imagine a second wave, sent out from the same point, independently but at the same moment, at a frequency of 2 per second, with the same velocity v and therefore of a wave-length $\lambda/2$. This second wave would not have the appearance of its independent self but would be imposed on the other, and what we should see travelling along the water surface would be a composite wave. Now let them imagine a third wave to be sent out from the same point, independently but at the same moment as the other two, at a frequency of 3 per second, with the same velocity v, and therefore of a wave-length $\lambda/3$. (Remember that $v = n\lambda$, always.) This third wave will not, any more than the second, show itself independently; it will simply make the previous composite wave still more complex. And so we might go on. The waves sent out independently might be shown thus:

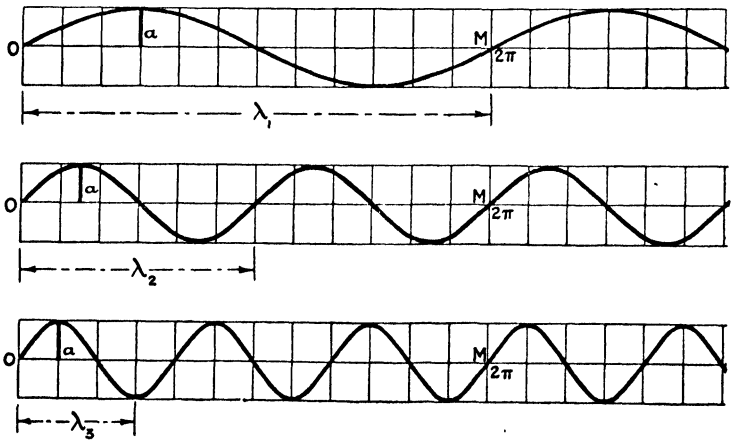


Fig. 259

All 3 waves start from O, and since they travel with the same velocity they reach M at the same moment, but by that time the first will have completed 1 of its periods, the second 2, the third 3. Imagine a whole series of waves sent out in this way, each of them with a wave-length which is a submultiple of λ, though not necessarily *all* the members of the sequence λ, $\lambda/2$, $\lambda/3$, $\lambda/4$, &c.: some of the series may be missing. Now imagine the water to be suddenly frozen, so that the wave would be set in ice, and its section readily drawn. We might have a composite wave like fig. 260, OM composing a unit which would be repeated endlessly until the wave died away in a ripple. The problem is, how

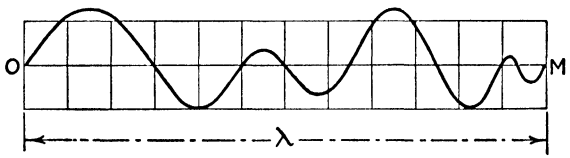


Fig. 260

are we to *analyse* this curve, in order to discover all the simple curves of which it is compounded.

The waves need not all have been sent out with the same amplitude (a), as shown in fig. 259. Neither need they have been sent out from exactly the same point; one might have been started 40° or 50° farther along the axis than the others. And remember that a cosine curve is produced from a sine curve merely by pushing it forwards or backwards 90° along the axis. We may therefore easily find cosines as well as sines in our formulæ; it is all a question of convenience, depending on the particular curves under consideration.

Briefly, Fourier's statement was this: Any repeated complex wave pattern of length λ may be produced by adding to a certain fundamental sine or cosine curve of length λ, sine or cosine curves of the proper amplitudes whose lengths are $\lambda/2$, $\lambda/3$, $\lambda/4$, &c. Conversely, the complex pattern may be revolved into its original component sine and cosine

curves, since any of the unknown amplitudes may be determined at will.

Observe that the *main* difficulty in analysing the complex curve arises from the fact that the component curve may be of different *amplitudes*. The general expression for the complex curve may be written in different ways, though they all mean the same thing.

$$\text{(i) } y = a_0 + (a_1 \sin px + b_1 \cos px) + (a_2 \sin 2px + b_2 \cos 2px) + \ldots$$

$$\text{(ii) } y = a_0 + a_1 \sin(x + \alpha_1) + a_2 \sin(2x + \alpha_2) + a_3 \sin(3x + \alpha_3) + \ldots$$

$$\text{(iii) } y = a_0 + a \sin(\theta + \alpha) + b \sin(2\theta + \beta) + c \sin(3\theta + \gamma) + \ldots$$

Remember that $p = 2\pi/\lambda$.

The constant a_0 meets the case in which the x axis is not identical with the common axis of the various harmonic curves.

Observe that *each term* in the above expressions represents a simple harmonic function. Those harmonics in which the coefficient of x is an odd number are called *odd* harmonics; those in which the coefficient of x is even are called *even* harmonics. Observe, too, that the *second* term gives a curve with *twice* as many complete waves, the *third* term a curve with *three times* as many complete waves (and so on), as the first or fundamental term. This is exemplified in fig. 259, where the " period " of the second term is $\frac{1}{2}$ the period of the first, the period of the third is $\frac{1}{3}$ the period of the first; and so on. The *frequencies* are therefore twice, three times, &c., the frequency of the first.

Curve Composition

Let us first consider curve *composition*. It is very simple. Plot to the same axis the successive components, $a_1 \sin(x+\alpha_1)$, $a_2 \sin(2x+\alpha_2)$, &c., and then add the corresponding ordinates to obtain the respective ordinates of the composite curve

(cf. figs. 256, 257). Since the first or fundamental term is represented by the period 0 to 2π, the wave will consist of repetitions of the first portion between $x = 0°$ and $x = 360°$.

We give the graph of

$$100 \sin x + 50 \sin(3x - 40°),$$

from 0° to 360°. The first or fundamental term, $100 \sin x$, represents the *first* harmonic with an amplitude 100 ($=a_1$ in

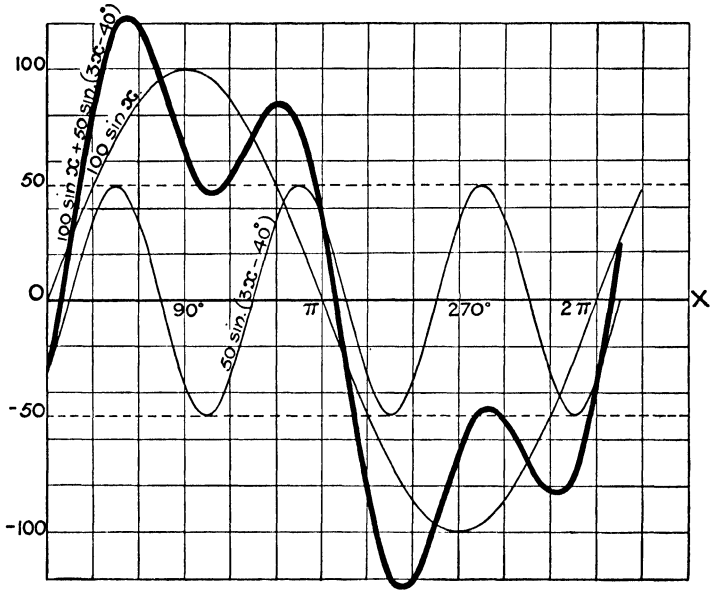


Fig. 261

general formula). The second term, $50 \sin(3x - 40°)$, is the *third* harmonic with an amplitude of 50 ($=a_2$), and consists of three complete waves within the period of the first.

The function consists of only *odd* harmonics, and in virtue of this fact the graph possesses a special kind of symmetry characteristic of all curves containing only odd harmonics. If the portion of the graph from π to 2π be made

to slide to the left, to the position 0 to π, it will be the reflected image of the half above the axis. (Cf. fig. 257.) Note that the composite curve is *not* a sine curve.

Had the function contained the absolute term a_0, say

$$y = 70 + 100 \sin x + 50 \sin(3x - 40°),$$

the graph would be the same as before but raised vertically 70 units. The line of symmetry referred to above would then no longer be the x axis.

We give a second example, this time consisting of the first and second harmonics:

$$y = 10 \sin(\theta + 30°) + 5 \sin(2\theta + 45°).$$

We will plot the graph from a tabulated series of values, though this is really unnecessary.

If $10 \sin(\theta + 30°) = y_1$, and $5 \sin(2\theta + 45°) = y_2$, then when $\theta = 0°$, $y_1 = 10 \sin 30° = 5$, and $y_2 = 5 \sin 45° = \frac{5\sqrt{2}}{2}$; also when $\theta = 30°$, $y_1 = 10 \sin 60° = 8{\cdot}66$ and $y_2 = 5 \sin 105° = 5 \sin 75° = 4{\cdot}8$. Similarly other values may be calculated.

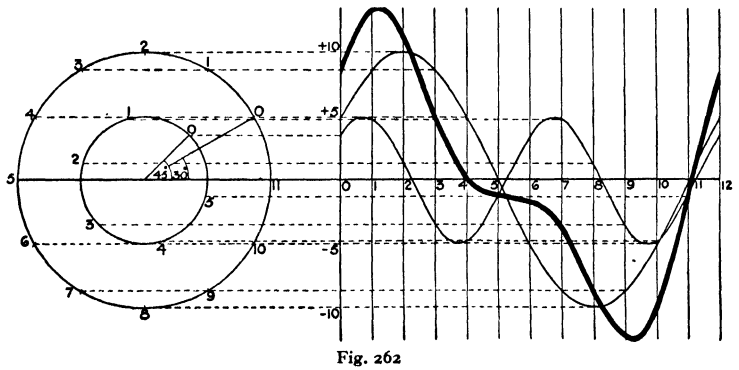


Fig. 262

Note the device of running off horizontals from a graduated circle. Since the " period " in the x axis is divided into 12 equal parts, we divide the circumference of the circle also into 12 equal parts.

Values of θ →	0°.	30°.	60°.	90°.	120°.
$y_1 =$	5	8·65	10	8·65	5
$y_2 =$	3·5	4·8	1·3	−3·5	−4·8
$y_1 + y_2 =$	8·5	13·45	11·3	5·15	0·2

Observe that as the point on the smaller circle rotates at twice the rate of a point on the larger, it is only necessary to divide the smaller circle into half as many parts as the larger. Set up, say, 12 ordinates for the whole line 0° to 360°, then divide the circumference of the larger circle also into 12 parts, and run off parallels to cut the ordinates. Each second harmonic will embrace only 6 of the 12 ordinates, and hence only 6 parallels from the smaller circle are required. The radii of the circles are, of course, equal to the amplitudes of the respective harmonics. Observe the plan for fixing the first point of each harmonic.

Functions with more terms than two are treated in exactly the same way, but naturally the composition is a tedious operation.

Curve Analysis

Secondly, we come to the decomposition or analysis of a composite curve. This is much less simple than the reverse operation.

The composite curve may be the resultant of two or more, perhaps a large number, of harmonics. But it does not at all follow that, because a particular harmonic, say the ninth, has been included in the building up, therefore all the earlier ones (in this case the first 8) of the series are included too. How are we to discover *which* harmonics are included, and how are we to draw them?

Whatever scheme we adopt, it is advisable, when we have discovered the component harmonics, to draw them all carefully, to compound them again, and to see if the result corresponds to the original curve.

Let integration wait until a later stage. Let the boys first learn what the new thing is really about. Let them consider the few simple cases which may easily be solved graphically, and after all, these are the cases of greatest practical importance (for instance those that are concerned with the theory of alternating currents). Such cases may, with sufficient approximation, be represented by the sum of two or three harmonic terms.

We select as an example one of Mr. Frank Castle's engineering problems. The curve in the figure is drawn

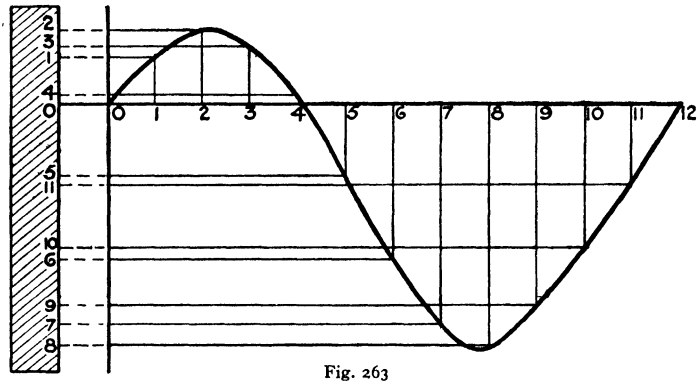


Fig. 263

through 12 successive positions of a slide valve, corresponding to intervals of 30° of the crank, beginning at the inner dead point. It is required to analyse the motion so as to express, in the form of a series of harmonics, the displacement of the valve from its mean position. (*Practical Mathematics*, p. 459.)

Were the curve divided more symmetrically by the x axis, we should suspect comparatively little deviation from the first harmonic, i.e. an ordinary sine curve. But, fairly obviously, it is compounded with other harmonics as well.

Run off the lengths of the ordinates to the edge of a paper strip as shown in fig. 263. Use the strip for plotting the points in fig. 264, but first *reverse* it, so that point 8 is at the top and point 2 at the bottom.

For the first harmonic.—Let 0 of the strip coincide with O in fig. 264, and mark off these distances: 0 to 6 on the ordinate through O, 1 to 7 on the ordinate through 1, 2 to 8 on the ordinate through 2, 3 to 9 on 3, 4 to 10 on 4, 5 to 11 on 5 (six measurements in all). Observe that these distances on the

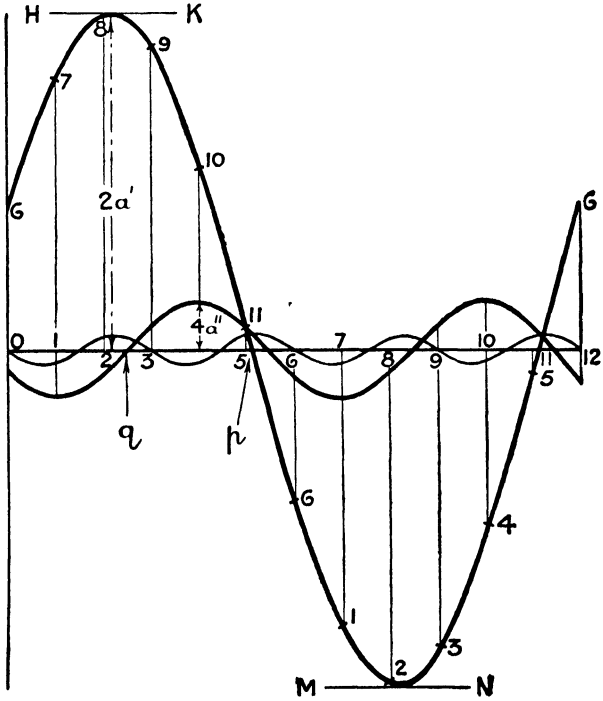


Fig. 264

strip give, successively, $y_0 - y_6$, $y_1 - y_7$, $y_2 - y_8$, $y_3 - y_9$, $y_4 - y_{10}$, $y_5 - y_{11}$. Draw a curve through the points, then the second half of the curve, below, through points obtained from the same measurements, reversed. Draw a tangent to this curve at a maximum or minimum point, HK, MN. The amplitude a' is **half** the distance from the tangent to the axis; it is $7/2 = 3{\cdot}5$.

The magnitude of the angle α can be obtained by measuring the length Op. The distance O to 6 = 180°; hence Op = 151·8°. Thus $\alpha = 180° - 151{\cdot}8° = \mathbf{28{\cdot}2°}$.

For the second harmonic.—The successive distances for the ordinates to be taken from the strip are (0 to 3) + (6 to 9), (1 to 4) + (7 to 10), (2 to 5) + (8 to 11). Observe that these distances are $(y_0 - y_3) + (y_6 - y_9)$, $(y_1 - y_4) + (y_7 - y_{10})$, $(y_2 - y_5) + (y_8 - y_{11})$. Draw a curve through the points, and repeat below; and do the same thing again for the second period of the wave.

To obtain the *amplitude* a'', draw a tangent, and take $\frac{1}{4}$ of the distance to the x axis, = ·**25**. To obtain the angle β, measure Oq; Oq = $\frac{5}{6}$ of O3 = 150°, hence $\beta = \mathbf{210°}$.

For the third harmonic.—The successive distances for the ordinates to be taken from the strip are, first (0 to 2) + (4 to 6) +(8 to 10), then (1 to 3) + (5 to 7) + (9 to 11). The curve almost coincides with the x axis, and as the distance to the crests has to be divided by 6 to obtain the amplitude a''',

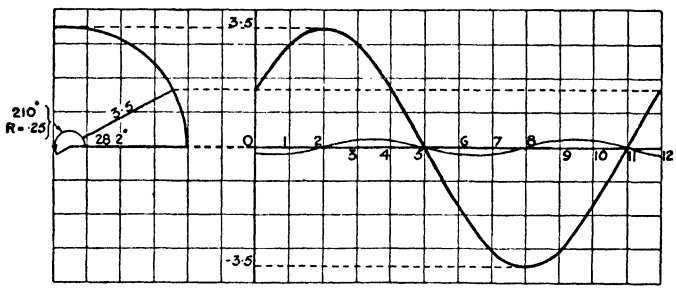


Fig. 265

it is evident that this third term of the harmonic series is negligible. Hence the equation may be written:

$$y = 3{\cdot}5 \sin(x + 28{\cdot}2°) + {\cdot}25 \sin(2x + 210°).$$

Now draw the two harmonics to scale (fig. 265), recompose, compare the result with the original graph, and thus check the work.

Boys are always keen to know what is behind such unusual procedure. The explanation is really very simple.

Let (i), fig. 266, be the first harmonic (the fundamental), (ii) the second harmonic, (iii) the third, and (iv) the fourth. We can cut the whole wave in (ii) into two equal and similar parts, and slide the right-hand half along the axis and superpose it on the left-hand half. We may cut (iii) into three equal and

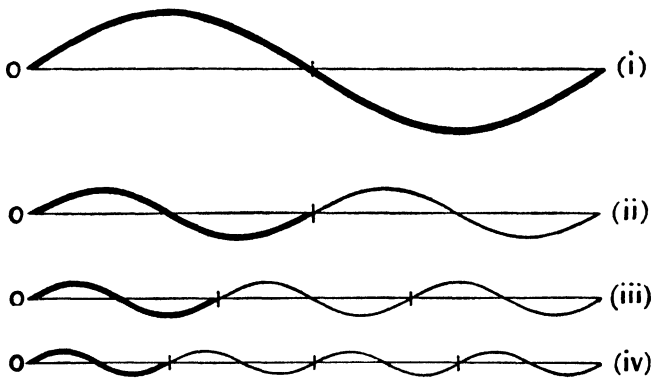


Fig. 266

similar parts, slide the second and third parts along and superpose them on the first. We may cut (iv) into four equal and similar parts, and again superpose.

Now suppose we have a compound curve of unknown composition. If it consisted of the *first* harmonic only, it would be just a simple sine curve, like fig. (i).

If the *second* harmonic is present, fig. (ii) represents that component. To test for its presence, cut the composite curve 0 to 2π into *two*, slide along and superpose, add the corresponding ordinates of the two parts thus superposed (y_0+y_6, y_1+y_7, &c., algebraically, of course), take the average of each of these sums by dividing by 2, and plot the curve. That curve is the *second harmonic* together with any of its multiples, if any of these are components; but the curve

does not contain any *other* harmonic than these multiples; i.e. the curve so obtained is,

$$y = a_2 \sin(2x + \alpha_2) + a_4 \sin(4x + \alpha_4) + \&c.$$

If the *third* harmonic is present, fig. (iii) represents the component. To test for its presence, cut the composite curve 0 to 2π into *three*, slide along and superpose, add the corresponding ordinates of the 3 parts thus superposed, take the average of each sum by dividing by 3, and plot the curve. The curve is the *third harmonic* together with any of its multiples, if any of these are components, but the curve does not contain any *other* harmonic than those multiples; i.e. the curve obtained is,

$$y = a_3 \sin(3x + \alpha_3) + a_6 \sin(6x + \alpha_6) + \&c.$$

So with harmonics beyond the third. But these are rarely required; they affect the result too slightly.—The proofs of these rules are very simple, and should be given.

Inasmuch as there is no advantage in giving for analysis any composite curves containing harmonics beyond the third, this graphic work need not be carried further. But the boys ought now to return to the example represented by figs. 263 and 264, and penetrate the mystery of the paper strip: the additions from the strip are really the additions of superposed ordinates resulting from cutting up the composite curve, sliding to the left, and superposing. The *reversal* of the strip is readily seen to be a simple device for converting subtraction into addition.

Teachers who think well of this method of Professor Runge may refer to *Zeitschrift für Mathematik und Physik*, Vol. 48, 443–56.

Professor Nunn's Plan: the Principle

A much more important curve-decomposition method may be briefly considered. The fundamental principle

underlying it is the obvious fact that the total area of either a complete sine curve or of a complete cosine curve is *zero*, since it is equally divided by the x axis. Professor Nunn's exposition (*Algebra*, pp. 521–3) is particularly illuminating, though I have sometimes found Sixth Form boys, who had



Fig. 267

not had a good training in solid geometry, puzzled over the geometrical figures. I append an outline of the exposition, together with a few new " solid " figures.

On one side of a line AB of length l, draw the semi-sine curve $y = a \sin\frac{\pi}{l}x$, choosing any value for the amplitude HK (= a). On the other side draw similarly the curve $y = \sin\frac{\pi}{l}x$, with amplitude KL (= unity). Cut the figure out and fold it about AB until the planes of the two curves are at right angles. Now mould a solid, in clay, plasticine, soap, or any similar soft material, to fill up the space between the curves.

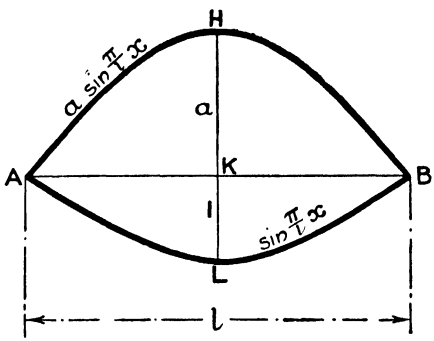


Fig. 268

In practice, the best way to do this is first to mould a rectangular prism l units long with cross-section KH × KL. Then draw the curve $a \sin \frac{\pi}{l}x$ on the face of the prism FCDE (i), and the curve $\sin\frac{\pi}{l}x$ on the top of the prism MFEG (ii). Pare off horizontally round the curve as in (i),

and vertically round the curve as in (ii). The result is (iii), the solid we require, ALBH; the plan of the solid is the figure bounded by AB and the curve $\sin\frac{\pi}{l}x$, and the elevation in the figure bounded by AB and the curve $a \sin\frac{\pi}{l}x$.

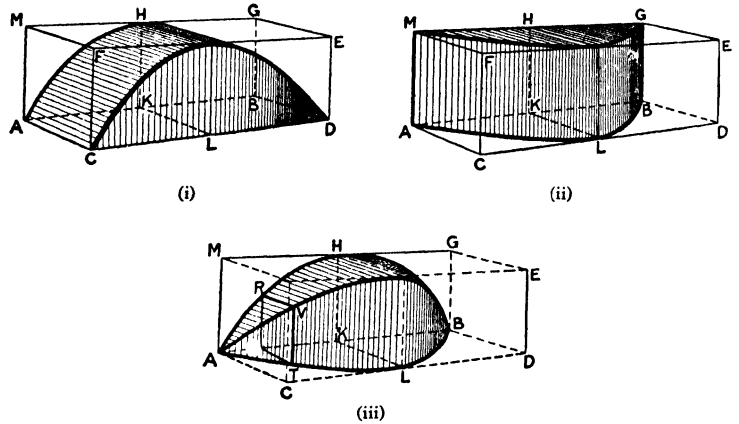


Fig. 269

It is important to note that *any section* of the solid by a plane at right angles to AB is a rectangle (e.g. RSTV) whose adjacent sides are $a \sin\frac{\pi}{l}x$ and $\sin\frac{\pi}{l}x$, x being the distance of the section from A. Note that the two lengths may be measured either on the flat surfaces behind and below or on the curved surfaces in front and above. Unless the solid is actually constructed, many boys will have difficulty in seeing this.

$$\begin{aligned}\text{The area of the section} &= a \sin \pi x/l \times \sin \pi x/l \\ &= a \sin^2 \pi x/l \\ &= \frac{a}{2}(1 - \cos 2\pi x/l) \\ &= \frac{a}{2} - \frac{a}{2}\cos\frac{2\pi x}{l};\end{aligned}$$

that is, the area of any section of the solid is equal to the algebraic difference between a constant area $\frac{a}{2}$ and a variable area $\frac{a}{2}\cos 2\pi x/l$.

For convenience, each of these areas may be looked upon as rectangles, each of height $a/2$. Thus the base of the former would be unity, and that of the latter $\cos 2\pi x/l$.

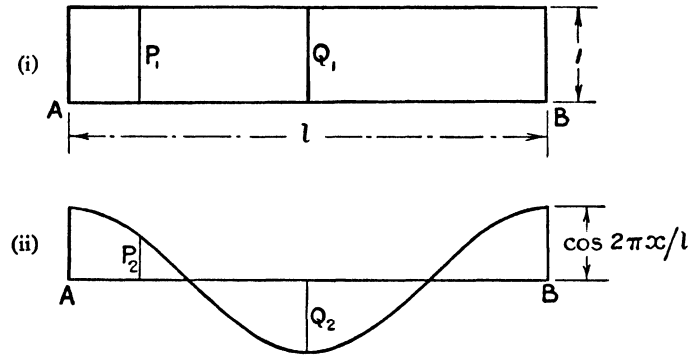


Fig. 270

The two rectangles may be regarded as cross-sections of two new solids of length AB ($= l$) and of uniform height $a/2$. Above are their plans. Note the neat, though obvious,

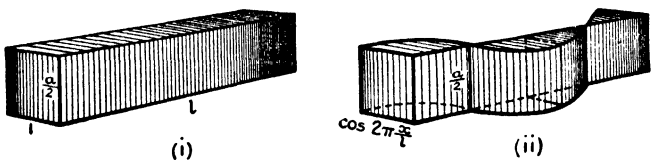


Fig. 271

device for showing the width of the second. Fig. 271 shows perspective sketches (for the sake of clearness, figs. 270 and 271 are drawn very considerably out of proportion, compared with figs. 268 and 269).

Let a plane at right angles to AB cut the solids at P_1P_2, corresponding to RSTV in fig. 269 (iii). Then the section RSTV is equal to the *difference* between the sections P_1 and P_2; and so with any other vertical section. At KL in fig. 269 (iii), the difference is between Q_1 and Q_2, but since Q_2 is negative, the difference is the arithmetical *sum*. This is as might be expected, for the section on HL is the full section of the original rectangle. In the case of section P_2, the width $\cos 2\pi x/l$ is positive; in the case of Q_2, it is negative. Thus the area of the section P_2 must be reckoned positive and that of Q_2 negative.

It follows that the part of the solid above AB in fig. 270 (ii) must be reckoned positive, and that below AB negative. Hence we must regard the total volume of the solid in fig. 271 (ii) as 0. But the volume of the solid in fig. 269 (iii) is equal to the *difference* of the volumes of the two solids in fig. 271 (i) and (ii). *Hence the volume of the solid in fig.* 269 (*iii*) *is equal to the volume of the simple prism in fig.* 271 (*i*). *The volume of the solid in fig.* 269 (*iii*) *is therefore* $al/2$.—This result is always a surprise to the boys, and they are much inclined to question it. They should be made to think about it carefully and to search for the fallacy they suspect. It will pay to make the boys work out one or two particular cases. Let them bear in mind that the volume of the rectangular blocks in fig. 269 (see fig. 268) is $l \times a \times 1 = al$.

On one or two occasions I have known Sixth Form boys cut out their models so carefully that, when checked by weighing, the results have been surprisingly accurate. To cut fig. 269 (iii) out of soap, and to weigh the model against the parings, may afford a very convincing check.

The Principle Applied

Consider the following figure, one complete element (O to 2π) of a composite wave. The problem is to determine the amplitudes of the various component harmonics; that done, the harmonics are easily drawn. Since the right-hand half of the curve is the " image " of the left-hand half, it is sufficient to consider the left-hand half alone; call its length l. We will assume that there are *two* components, viz. $y = a_1 \sin \pi x/l$ and $y = a_2 \sin 2\pi x/l$, in other words that the given curve is made up of the first and second harmonics. (We know from the *kind* of symmetry that the third harmonic is not a component (see p. 463).)

On the line M'N' (= MN = l), draw the curve $y = \sin \pi x/l$ inverted, i.e. a sine curve with amplitude unity; and

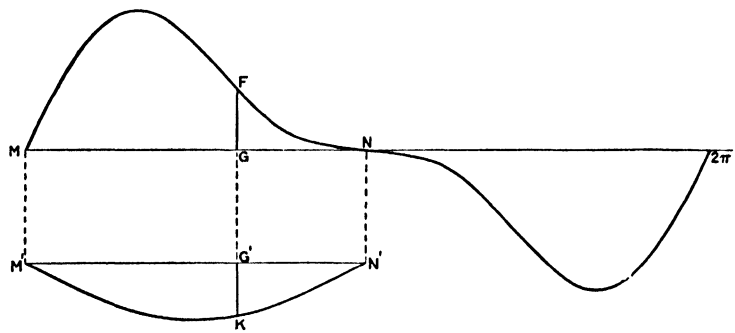


Fig. 272

make a model of the solid determined by the two curves when the lines M'N' and MN are made to coincide and the planes of the figures are at right angles. Note that any section FGK at right angles to MN is rectangular, as in the solid of fig. 269. The solid is not an easy one to model accurately (fig. 273).

The *volume* of the composite solid is equal to the

sum of the two solids determined by the curves,

$$\text{(i) } y = \sin \pi x/l \text{ and } y = a_1 \sin \pi x/l,$$
$$\text{and (ii) } y = \sin \pi x/l \text{ and } y = a_2 \sin 2\pi x/l.$$

But the latter of these volumes is easily proved equal to 0 (cf. fig. 271 (ii)), and the volume of the former is $a_1 l/2$ (cf. fig. 271 (i)). Hence

$$\textit{total} \text{ volume of solid} = a_1 l/2. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

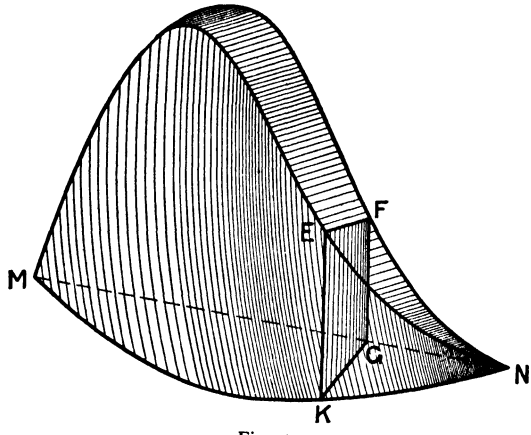


Fig. 273

But the volume may also be determined *directly*, by calculating the mean value of its cross-section. Consider, for instance, the vertical section at G on MN where MG $= 2l/3 = x$, so that $\pi x/l = 2\pi/3$ radians or 120°. The section in a rectangle whose sides FG, GK are closely analogous to the sides RS, ST in fig. 269 (iii). Of these two sides, FG may easily be determined by actual measurement from the curve, while GK $= \sin 120° = \sqrt{3}/2$. The product gives the area of the vertical section through FGK.

In this way we may find the area of any number of such vertical sections. For convenience, divide MN into 12 equal parts. Calculate the areas of the respective sections through

the dividing points, and then by Simpson's rule* the volume of the solid. Deduce from this the average cross-section A_1 by dividing by l.

$$\begin{aligned} \text{Thus} \quad & \text{vol.} = A_1 l. \\ \text{But} \quad & \text{vol.} = a_1 l/2 \qquad \text{(by (i)).} \\ \therefore \quad & \frac{a_1 l}{2} = A_1 l, \\ \text{or} \quad & a_1 = 2A_1. \end{aligned}$$

In a similar manner, by supposing a second solid to be formed by combining the given half curve with the curve $y = \sin 2\pi x/l$, the value of a_2 may be determined. If the given curve contained any other harmonic components, their amplitudes might be determined in the same way.

The principle of the method is that any sine curve $y = \sin r\pi x/l$ when combined with half the given composite curve determines a solid whose volume $(a_r l/2)$ depends on the amplitude a_r of the component $y = a_r \sin r\pi x/l$, and *not at all on the amplitude of any other component.* In this way, the successive sine components can be dealt with one by one, and their amplitudes determined. The determination of the amplitudes is, of course, the very essence of the problem.

The work of computing the average cross-sections can be divided up amongst the members of the class. Instruct them to carry out the following operations, and to tabulate the results:

(1) *To determine the amplitude of the first harmonic.*

(α) Divide up MN into twelve 15°-phase differences; erect the ordinates and measure their lengths in millimetres. In accordance with Simpson's rule, only half the height of the first and last ordinates is required in the calculations, but as, in this instance, these happen to be zero, the halving makes no difference.

(β) Calculate the successive values of $\sin n15°$, n being the number of the ordinates.

* "Add half the first and last areas and the whole of the intermediate areas, and multiply the sum by the common interval."

(γ) Multiply (α) by (β) and so obtain the areas of the successive sections of the solid (fig. 273).

i.	ii.	1st Harmonic.			2nd Harmonic.		
Number of Ordinate.	Length of Ordinate in mm.	iii. Angle.	iv. Sine.	v. Area ii or iv.	vi. Angle.	vii. Sine.	viii. Area ii or iv.
0	0	0°	0	0	0°	0	+0
1	13·0	15°	·26	3·38	30°	·5	+6·50
2	23·0	30°	·5	11·50	60°	·87	+19·93
3	29·5	45°	·71	20·90	90°	1·0	+29·50
4	31·5	60°	·87	27·40	120°	·87	+27·40
5	29·5	75°	·97	28·60	150°	·5	+14·75
6	24·0	90°	1·0	24·00	180°	0	0
7	17·5	105°	·97	16·90	210°	−·5	−8·75
8	10·3	120°	·87	8·92	240°	−·87	−8·92
9	5·0	135°	·71	3·54	270°	−1·0	−5·00
10	2·0	150°	·5	1·00	300°	−·87	−1·74
11	1·0	165°	·26	·26	330°	−·5	−·50
12	0	180°	·0	0	360°	0	0
			Total =	146·40		Total =	73·16
			Average area = A_1 =	12·20		Av'ge = A_2 =	6·097

Amplitude of the 1st harmonic = $a_1 = 2A_1$ = (12·2 mm. × 2)
= 24·4 mm.
= **2·44** cm.

(2) *To determine the amplitude of the second harmonic.*— Corresponding to the half curve MN will be a *complete* sine curve of the second harmonic. Hence the angles will now be $n30°$, and the sines from 180° to 360° will be negative The ordinate lengths will be the same as before.

Amplitude = $a_2 = 2A_2$ = (6·097 mm. × 2)
= **1·22** cm.

Hence the original curve is,

$$y = 2{\cdot}44 \sin \pi x/l + 1{\cdot}22 \sin 2\pi x/l.$$

The periods being known, and the amplitudes having been found, the angles follow at once.

Let the boys realize fully that the essence of the problem is the discovery of the *amplitudes* of the component harmonics.

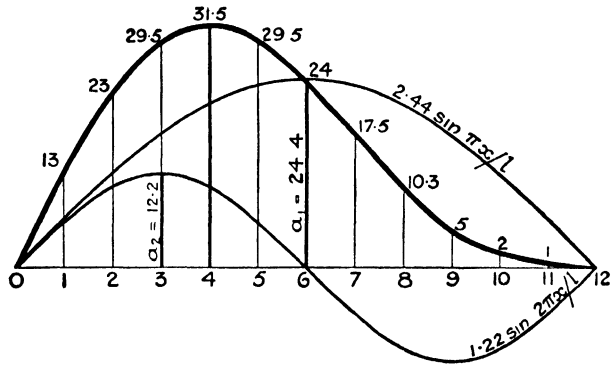


Fig. 274

By improvising the solids and devising two different schemes for determining their volumes, we obtain two different formulæ each involving a in terms of A. It is true that A appears as an area, but, by taking one of the dimensions of the area as unity, A becomes a linear value, and of course we begin by giving the solid a base consisting of a sine curve of unit amplitude.

The subject can be followed up by integration. The boys are now ready for it, for they have learnt what the subject is really about.

Books to consult:

1. *The Teaching of Algebra*, Nunn.
2. *Manual of Practical Mathematics*, Castle.
3. Any modern standard work on Sound.

CHAPTER XXXII

Mechanics

The Teacher of Mechanics

The most successful teachers of mechanics whom I have known are those who have had a serious training in a mechanical laboratory; who know something of engineering, and are familiar with modern mechanism; who are competent mathematicians; and who have mastered Mach's *Mechanics*, especially Chapters I and II.* Mach's book is universally recognized as *the* book for all teachers of mechanics. It deals with the development of the fundamental principles of the subject, traces them to their origin, and deals with them historically and critically. The treatment is masterly. The book might with advantage be supplemented by Stallo's *Concepts of Modern Physics* (now out of date from some points of view), Karl Pearson's *Grammar of Science*, and Clifford's *Common Sense of the Exact Sciences* and *Lectures and Essays* (still first-rate, though written 50 years ago).

It is of great advantage to a teacher of mechanics to be familiar with the subject historically. The main ideas of the subject have almost always emerged from the investigation of very simple mechanical processes, and an analysis of the history of the discussions concerning these is the most effective method of getting down to bedrock.

Who were the great investigators? The scientific treatment of statics was initiated by Archimedes (287–212 B.C.), who is truly the father of that branch of mechanics. The work he did was amazing, but there was then a halt for 1700 or 1800 years, when we come to Leonardo, Galileo, Stevinus, and Huygens; to Torricelli and Pascal; and to Guericke and Boyle. For dynamics, we go first to its founder Galileo

* Hertz also wrote a *Mechanics* of the same masterly kind, but there is no English translation, so far as I know.

(falling bodies, and motion of projectiles), then to Huygens (the pendulum, centripetal acceleration, magnitude of acceleration due to gravity), and then to Newton (gravitation, laws of motion). The great principles established by Newton have been universally accepted almost down to the present time, and, so far as ordinary school work is concerned, will continue to be used—at least during the present generation.

A boy is always impressed by Newton's argument that since the attraction of gravity is observed to prevail not only on the surface of the earth but also on high mountains and in deep mines, the question naturally arises whether it must not also operate at greater heights and depths, whether even the moon must not be subject to it. And the boy is still more impressed by the story of the success of Newton's subsequent investigation.

Newton's four rules for the conduct of scientific investigation (*regulæ philosophandi*) are the key to the whole of his work, and should be borne in mind by his readers.

The First Stage in the Teaching of Mechanics

How do successful teachers begin mechanics with boys of about 12 or 13? They usually begin by drawing upon the boys' stock of knowledge of mechanism.* Most boys know something of mechanism, some will have had enough curiosity to discover a great deal, and a few will probably have had experience of taking to pieces machines of some sort and of putting them together again. This stock of knowledge may be sorted out, and the topics classified and made the subjects of a series of lessons. By means of an informal lesson on some piece of mechanism, an important principle may often be worked out, at least in a rough way.

I have known a teacher give his first lesson on mechanics in the school workshop, utilizing the power-driven lathe and the drilling-machine; another first lesson in the school playground, an ordinary bicycle being taken to pieces. I have

* See Chapter VIII, *Science Teaching*.

seen a model steam-engine used for the same purpose, and I have known beginners taken to a local farm to watch agricultural machinery at work. In all these instances the boys learnt that their new subject seemed to have a very close relation with practical life. They were not made to look upon it as another branch of mathematics, and a rather difficult branch at that.

Let the early lessons be lessons to establish very simple principles. Never mind refinements and very accurate measurements. Do not bother about small details, and avoid all complications. Let the boy get the *idea*, and get it clearly. Very simple arithmetical verifications are quite enough at this stage. The boy's curiosity is at first qualitative; let that be whetted first, and then turned into a quantitative direction gradually. Encourage the boy to find out things for himself, and do not tell him more than is really necessary. Encourage him to ask questions, but as often as possible answer these by asking other questions which will put him on a new line of inquiry. Let him accumulate knowledge of machines and machine processes. Give him some scales and weights, and a steelyard, and tell him just enough to enable him to discover the principle of moments, but do not talk at first about either " principle " or " moments ". It is good enough if at this stage he suggests that

$$\text{long arm} \times \text{little weight} = \text{short arm} \times \text{big weight}.$$

He has the *idea*, and the idea is expressed in such a form that it *sticks*. Give him a model wheel and axle, give him a hint that it is really the lever and the lever-law over again, and make him show this clearly. Give him some pulleys and let him discover, with the help of one or two leading questions, how a small weight may be made to pull up a big weight, and let him work out the same law once more, but now in the form that what is gained in power is lost in speed. Give him a triangular block and an endless chain, let him repeat Stevinus' experiment, and so discover the secret of the inclined plane. Let him use a jack to raise your motor-car (and inci-

dentally learn something about " work "); now tell him something about the pitch of the screw, something about Whitworth's device for measuring very small increases in length, something about the manufacture of a Rowlands grating. Encourage him to give explanations of mechanical happenings in everyday life, and use his suggestions as pegs on which to hang something new.

A term of this kind of work pays. The boy is accumulating knowledge of the right sort, and when the subject is taken up more formally and with a more logical sequence, rapid progress may be made. Once he has been taught to read elementary mechanism, it is easy enough to teach him its grammar. Surely this is the right sequence. Mechanism must come before mechanics. The mathematics of the subject is a superstructure, to be built upon a foundation of clear ideas.

Of course, if the preliminary work of the preparatory school or department has been properly done, the way is paved for an earlier treatment of a more formal kind.

The Second Stage

The second stage should consist of work of a more systematic character, but still work essentially practical, though arranged on a logical string. Ideas will now be classified, and mathematical relations gradually introduced. But the physical thing and the physical action must still remain in the front of the boy's mind. The mathematics will take care of itself.

Let the teaching be inductive as far as possible. Obtain all necessary facts from experiments, and do not use experiments merely for verifying a principle enunciated dogmatically.

The basic principles to be taught are really very few, and a boy who knows these thoroughly well can work most ordinary problems on them. Mechanics is, after all, largely a matter of common sense. The laws of equilibrium, together with the ratio of stress to strain, covers almost the whole range of statical problems, including those of hydrostatics; while Newton's Laws of Motion covers practically everything else. But

of course these are basic principles. If they are known, *known*, derived principles are learnt easily enough; if they are only vaguely known, derived principles are never really mastered.

Statics or dynamics * first? Teachers do not agree. There is much to be said for beginning with dynamics, first using the ballistic balance for studying colliding bodies, and the momentum lost by one and gained by another; it is then an easy step to pass on to the idea of force. But a boy who is led to think of a force as something analogous to muscular effort will always be in trouble, and in any case he is likely to form a very vague idea of acceleration. Of course, uniform acceleration is anything but common in practical life: we nearly always refer either to falling bodies or to a train moving out from a station. And it is this difficulty that makes many teachers take up statics first. Although, at the outset, a boy's working idea of force is necessarily crude, a spring balance, for simple quantitative experiments, helps to put the boy on the right track, and there is much to be said for allowing him to assume, to begin with, that *weight* is the fundamental thing to be associated with force. At an early stage he may verify, to his own satisfaction, the principles of the parallelogram and triangle of forces, but he must be warned that he has not yet " proved " these principles and cannot yet do so. But since the parallelogram of forces is such a useful working principle, it would be foolish not to allow the boy to use it before he can prove it formally. At this stage formal proofs are difficult, and it is simply dishonest to encourage a boy to reproduce a page of bookwork giving a proof of something quite beyond his comprehension, though this was common enough thirty or forty years ago.

Do not employ graphic statics at too early a stage, or the real point at issue may be obscured.

Now as to dynamics. What is the best approach? We have already referred to the ballistic balance. Should Atwood's machine be used? It may be used, perhaps, for

* The terms kinetics and kinematics are falling into disuse.

illustrating the laws of motion, but not as a practical method of finding g.

Atwood's machine has been superseded by Mr. Fletcher's trolley,* by means of which practically the whole of the principles of dynamics may be satisfactorily demonstrated. It lends itself to many experiments, all of which provide a space-time curve ready made, and, from that, speed-time and acceleration-time curves may be plotted. In a paper read at the York meeting of the British Association, Mr. C. E. Ashford gave details of a large number of trolley experiments as performed at Dartmouth, a school where the teaching of mechanics is well known to be of a high order. Reference should be made to Mr. Fletcher's own article in the *School World* for May, 1904. In it he shows how boys may be given sound ideas of the physical meaning of the terms, moment of inertia, angular momentum, moment of momentum, and therefore of moment of rate of change of momentum and moment of force. Useful teaching hints may also be found in Mr. S. H. Wells's *Practical Mechanics* and Mr. W. D. Eggar's *Mechanics*.

Once the foundations of mechanics have been well and truly laid the superstructure may be erected according to traditional methods. To leave the subject just as developed in the laboratory would be to leave it unfinished. But the superstructure may now be built properly. When necessary formulæ have been evolved from experiment, the physical things behind the formulæ have to the boy a reality of meaning which the older "methods of applied mathematics" teaching could not possibly give him.

If principles are not understood, proofs have no meaning.

Throughout the whole of a mechanics course every opportunity should be taken to excite the boys' interest in new mechanical inventions. It helps the more academic work

* The friction of the trolley may be eliminated either by tilting the plane to the necessary angle, or by attaching a weight that will just maintain uniform motion. The friction of the pulley over which the thread passes cannot be compensated, and it is therefore necessary to use a good pulley.

enormously, and makes the boys feel that the subject is really worth taking trouble over. Examples occur on every side—variable speed gears, transmission gears, taximeters, boat-lowering gear, automatic railway signalling, automatic telephones, the self-starter in a motor-car, the kick-starter in a motor-cycle, and so on. Some mechanical devices depend, in their turn, on electricity, and their place of introduction into a teaching course would be determined accordingly. Complex mechanisms like the air-plane, the submarine, the paravane, should not be wholly forgotten. Boys can read up such things for themselves, and perhaps prepare and read papers on them to the school science society.

Hydrostatics

The mechanics of fluids is an exceedingly difficult subject to teach effectively. Even a Sixth Form boy is sometimes held up by questions on the barometer or on Dulong and Petit's equilibrating columns. The work of Archimedes and Pascal for liquids and of Boyle for gases cannot be too well done. Above all, the U-tube must receive careful attention, and especially the surface level above which pressures are compared. Do not buy Hare's apparatus from an instrument-maker's. The standard pattern is always made with two straight tubes, of the same bore, fixed vertically. Let the boys make a variety of forms of this apparatus for themselves, and work out the vertical height law from data as varied as possible. Approach the whole subject of hydrostatics from the point of view of familiar phenomena, e.g. measure the water pressure from a tap in the basement and again from a tap in the top story of the school, and see if there is any sort of relation between the difference of these pressures and the height of the school. *Do not try to establish a principle formally until the phenomenon under investigation is clearly understood as a physical happening.* Let boys know really *what* they are going to measure before they begin to measure.*

* The preceding paragraphs are taken from *Science Teaching* (pp. 121–8).

The Johannesburg British Association Meeting

At the Johannesburg meeting of the British Association, an animated discussion took place on the general question of the teaching of mechanics. It followed on a paper read by Professor Perry. We append a few suggestive extracts.

Professor Perry.—" The very mathematical man often does not know anything of mechanics; it is the subject of applied mathematics that he has studied and that he cares for.

" The two elementary principles of statics, (1) if forces are in equilibrium, their vector sum is zero, and (2) the sum of their moments about any axis whatsoever is zero, ought to be so clear to a pupil that it is practically impossible for him to forget them. They ought to be as much a part of his mental machinery as the power to walk is part of his physical function.

" I lay no stress upon mere abstract proofs of propositions in mechanics. When understanding is affected there is no difficulty about the proofs. It is quite usual to find men who can prove everything, without having any comprehension of what they have proved."

Mr. W. H. Macaulay.—" I agree with the taking of statics before dynamics. I also agree that graphical statics is a subject full of dodges, though very good to learn if you want to use them every day."

Professor Boys.—" I absolutely agree as to the desirability of dealing with fundamental principles, and of not worrying about innumerable details. . . . A friend of mine heard Lord Kelvin say in one of his lectures, ' And now we come to the principle of the lever. You will understand that levers are divided into three orders, levers of the first order, of the second, and of the third—but which of them is which I cannot for the life of me tell you.' Textbooks were at one time filled up with futile and unnecessary kinds of discrimination which had nothing whatever to do with the subject."

Professor Bryan.—" The idea of mechanics which appeals most readily to a young boy is that it has something to do with machines, and that machines have something to do with turning out useful work. There is no better way of stimulating interest in the subject than showing the beginner that when you have got your machine for changing one kind of work into another, you are no better off than when you started."

Professor Hicks.—" My own experience is in approaching mechanics from a kinetic point of view. First let the boys find out by experiment that momentum remains constant. Of course the first thing depends on what mass is; then we must proceed to show that when two bodies collide with equal velocities they come to rest. By making experiments of velocities of colliding bodies, boys get to realize that momentum remains unalterable. Given two colliding bodies in a straight line, the momentum lost by one is gained by the other. By getting a large number of experiments, pupils come to a realized knowledge of that."

Sir David Gill.—" I remember Clerk Maxwell illustrating the misuse of definitions by a funny story. He said he went into his room one day, and there was a white cat which jumped out of the window. He and his friends ran to the window to see what had become of the cat, and the animal had disappeared, no one being able to solve the mystery. At last he solved the problem. He said it must be this. The white cat jumped out of the window, fell a certain distance with a certain velocity, and collided with an ascending black cat. There were therefore two equal and opposite cats meeting with equal and opposite velocities, the result being no cat.—Without a proper understanding of definitions of these things, one might arrive at such an absurdity as this story illustrates."

Professor Forsyth.—" The first stage in teaching mechanics is not the stage in which pupils have to prove, or attempt to prove, or can be expected to prove, anything. That belongs to a later stage. The first thing to do is accustom the pupils to the ordinary relations of bodies and of their properties."

Mr. W. D. Eggar.—" I should like to see a penny-in-the-slot automatic weighing machine in every passenger lift, so that the fundamental experiment of showing a connexion between force and acceleration could be within the reach of everybody."

Professor Minchin.—" I hope to see the term ' centrifugal force ' utterly banished."

Mr. C. Godfrey.—" Statics is a fairly easy matter if one begins with experiment. Nor need experiment cease after the first stage; any school should be able to get hold of some bit of machinery with plenty of friction in it, say a screw-jack, and investigate efficiency. Plotting ' load ' against ' effort ' leads to very striking results.

" There is the question of mass and weight. In vain one resorts to the centre of the earth; it is all too hypothetical. I remember as a boy being puzzled to understand how the weight of a train (acting vertically) could have anything to do with its acceleration under a pull (horizontal) from the engine.

" We might give a touch of reality to the kinetics course by brake horse-power determinations. It should be possible to rig up for a few shillings a brake-drum on a motor (electric or water); even a motor-cycle on a stand or a foot-lathe might serve the purpose.

" Engineers talk in a very confusing way about centrifugal force. When a particle moves in a circle uniformly, the force on the particle is centripetal and the force on the constraints is centrifugal. But the popular use of language and the popular belief is that there is an outward force on the particle."

" Applied " Mathematics

The old school of " pure " mathematicians very cleverly picked out from the whole subject of mechanics and engineering such problems as lent themselves to algebraic and geometrical treatment, and left the residue, rather disdainfully labelled " applied mechanics ", to be dealt with by

teachers of lower degree. Note the term " applied ". The *real* mechanics was the mechanics that could be done from an easy chair, and was a mathematicians' job. The building of the Assouan dam and of the Forth Bridge were trivial things which any " ordinary engineer " could take in hand, trivial things that had no relation whatever to " pure " thought. This temper survived even until the present century. When the two Wrights were risking their lives by experimenting with the first air-plane, a well-known mathematician wrote to the press protesting against such folly, inasmuch as mathematicians had not yet worked out the mathematical principles of flight!

The mathematician's proper share of such work is to begin where the inventor or the engineer leaves off; it is not his business to invent paper air-planes, but to learn from the real thing the principles of flight and to see if these rest on secure mathematical foundations; if they do not, he may be able to offer fruitful suggestions. Of course if the mathematician happens to have been trained as an engineer, that is a different matter. Unless the teacher of mechanics knows something of actual engineering, his mechanics is likely to have but a remote connexion with actual mechanism. There are still teachers of mechanics who have had neither workshop nor laboratory experience, and naturally they tend to shirk those parts of the subject that do not come within the four corners of algebra and geometry. It is not an uncommon thing for a course of lessons on elementary statics to include not a single word about, for instance, the equilibrium and stability of walls, the effect of buttresses, the thrust along rafters, or about roof-trusses or cranes. Friction may be the subject of a lesson with no mention whatever of lubricants. Energy may be the subject of others, and yet no reference be made to energy storage in, for example, accumulators and fly-wheels. The transmission of motion and power is rarely touched upon seriously in a course of mechanics lessons. And yet all such things as are thus ignored are just those things that have already been included, in

some measure, within the four corners of the boys' daily experience. Subjects like tension and compression, shearing and torsion, beams, girders, and frameworks, are passed over hurriedly as of little importance. Why is elementary hydrostatics so often given such short shrift? Why is it not followed up by the subject which really matters, viz. elementary hydraulics—the flow of water through orifices and pipes, the pressure in a water-main, water-wheels, turbines, the propulsion of ships and air-planes, and hydraulic machines? As for capillarity and surface tension, which lend themselves to all sorts of delightful experiments, they are too often an affair of just blackboard and chalk. Do not put off that interesting section of physics, " properties of matter " (the twin-sister of mechanics), until the Sixth Form. The mathematics of it in the Sixth, yes; but the necessary laboratory course can be taken in the Fourth and Fifth.

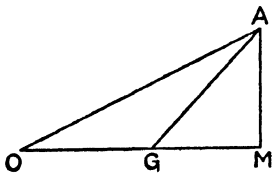
In short, the mathematics of mechanics is very serious Sixth Form work. The practical work that *must be done* before the mathematical work can profitably be attempted may be done earlier.

We will give an extract from an elementary textbook on *Mechanics for Beginners*, with a very well known name on the title-page. It is an introduction to *Moment of Inertia*.

" Let the mass of every particle of a body be multiplied into the square of its distance from an assigned straight line; the sum of these products is called the *moment of inertia* of the body about that straight line. The straight line is often called an *axis*.

" *The moment of inertia of any body about an assigned axis is equal to the moment of inertia of the body about a parallel axis through the centre of gravity of the body, increased by the product of the mass of the body into the square of the distance between the axes.*—Let m be the mass of one particle of the body; let this particle be at A. Suppose a plane through A, at right angles to the assigned axis, to meet the axis at

O, and to meet the parallel axis through the centre of gravity at G. From A draw a straight line AM, perpendicular to OG or to OG produced. Let GM = x, where x is a positive or negative quantity according as M is to the right or left of G. By Euclid II, 12, 13, we have $OA^2 = OG^2 + Ga^2 + 2OG.x$; therefore,



$$m.OA^2 = m.OG^2 + m.GA^2 + 2OG.m.x.$$

A similar result holds good with respect to every particle of the body. Hence we see that the moment of inertia with respect to the assigned axis is composed of three parts, namely, first the sum of such terms as $m.OG^2$, and this will be equal to the product of the mass of the body into OG^2; secondly, the sum of such terms as $m.Ga^2$, and this will be the moment of inertia of the body about the axis through G; and thirdly the sum of such terms as $2OG.m.x$, which is zero. Hence the moment of inertia about the assigned axis has the value stated in the proposition."

Be it remembered that this book is a book for "beginners". I remember a Fourth Form once being given ten minutes to read up the subject-matter just quoted. Then came questions. Said one boy, "I thought inertia meant laziness."—"So it does, a *sort* of laziness."—"Then does 'moment of inertia' mean a moment of laziness?" Said another boy, "How are we to find the mass of one particle? Do we crush the thing up in a mortar, and weigh one of the particles? or do we weigh the thing first, then crush it up, count up the particles, and divide the weight by the number?" The teacher replied, "Don't be silly; moment of inertia is not real; it is only theory"!

Who could blame the boys for asking such questions? How could they have obtained the faintest insight into the nature of the subject under discussion?

Forty or fifty years ago, Todhunter's *Analytical Statics* was a standard work, used by mathematical students at the

University. There are cases on record of men who obtained Firsts in mathematics but who in the subject mentioned had read no other book at all, had never handled a piece of apparatus in their lives. Fortunately that age has passed away.

Mr. Fletcher's trolley, which is now in general use for teaching dynamics, is not always made so serviceable as it might be. (Readers should refer again to Mr. Fletcher's own comprehensive article in the *School World* for 1904.) In Perry's *Teaching of Elementary Mechanics*, already referred to, Mr. Ashford, formerly Head of the Royal Naval College, Dartmouth, gives some exceedingly useful hints on the further use of the trolley.

The early teaching of mechanics *must* be given an experimental basis. Mathematicians unacquainted with the mechanical laboratory should let the subject alone. It is better not taught at all than to be taught as mere algebra and geometry. Only if basic principles are established experimentally can the subsequent mathematical work be given a reality and a rigour that command respect.

" The Teaching of Mechanics in Schools "

A report on " The Teaching of Mechanics in Schools ", specially prepared for the Mathematical Association, was issued in 1930. The responsible sub-committee was appointed in 1927 by the General Teaching Committee of the Association. The sub-committee included such well-known teachers as Mr. C. O. Tuckey, Mr. W. C. Fletcher, Mr. W. J. Dobbs, Mr. C. J. A. Trimble, and Mr. A. Robson, and the Report will therefore carry great weight amongst all teachers of mathematics. Every page reveals the hand of the practical teacher. No teacher of mathematics should fail to give it his serious attention. We quote a few short paragraphs in order that the reader may gather some notion of the general tenor of the Report.

" There is perhaps no branch of mathematical instruction for which a pupil comes prepared with a larger body of

intuitional knowledge than he does for mechanics. The suggestions made in this report are based on the view that this body of knowledge should form the foundation of the teaching, and that the aim of the teaching should be largely concerned with a development of a taste for such accurate thought and consideration of mechanical facts as will make them more intelligible, increasing the interest which attaches to the mechanical behaviour of things, and leading to that insight which brings this behaviour more completely under control."

"Just as geometry has its roots in familiar phenomena of daily life, so has mechanics. The basic principles of both sciences can be gathered, at least crudely, from ordinary observation—this is the process we knew as abstraction."

"When we have carried the process some little way it becomes necessary, or at least economical, to arrange things so as to provide a more exact answer to a definite question than can be obtained from observation of unarranged or uncontrolled phenomena. So we get two processes, fading into one another no doubt in marginal cases, but in general easily distinguishable, viz. reflection on ordinary experience, and deliberately arranged experiment. In the former it may be noted—and it is perhaps an essential part of the distinction—experience comes before thought; we may or may not observe and reflect upon it and we may or may not make scientific use of it. In the latter, viz. experiment, as it has to be deliberately arranged, thought comes first—we must frame a question before we can arrange the experiment which is to give the answer."

"While there is room for difference of opinion and practice as to the place of experiment in the school treatment of mechanics, there is no question that observation and reflection on ordinary experience are essential for any proper grasp of the subject. The widespread neglect of this obvious truth is responsible for much lack of success in the teaching of the subject."

"In mechanics the crude facts lie open to direct observation, and the rôle of experiment is limited to rendering more

precise an answer which, in the rough, can be given without experiment."

" The function of experience is to provide a basis of reality for the abstract science of the textbook and the schoolmaster, and the paramount duty of the latter is to make his pupils conscious of their own experience, to get them to reflect upon it, to co-ordinate their existing store and to open their eyes to observe more closely and to see the significance and interest of much that the unobservant mind ignores. Training of this sort is essential if the subject is to have its real value. . . . In each fresh section of the work, the first thing to do is to collect and clear up existing experience bearing on the matter in hand."

The " Contents " of the Report are as follows:

1. Position in the Curriculum.
2. General Aims.
3. Experience and Experiment.
4. Order of Treatment.
5. The beginning of Statics.
6. The beginning of Dynamics.
7 Miscellaneous Topics:
   (i) Earlier teaching of Mechanics; (ii) Experiments; (iii) Initial difficulty of Statics; (iv) Kinematics; (v) Units and Dimensions; (vi) Horse Power; (vii) Formation of the Equations of Motion; (viii) Jointed Frames; (ix) Friction; (x) Torque, Couples; (xi) Geometrical and Algebraic Methods; (xii) Impact and the Law of Momentum and Energy; (xiii) Rotatory Motion; (xiv) Limitations of School Dynamics; (xv) Miscellaneous.
8. To examiners.
9. Appendices:
   (i) Wheeled Vehicles; (ii) Momentum Diagram.

Newtonian Mechanics superseded

It is commonly said that Einstein has dethroned Newton, and this in a sense is true, inasmuch as Newton's laws have been superseded; but Einstein has always regarded Newton as his master. Improved instruments have led to

the discovery of facts unknown to Newton, and Newton's laws have had to be amended in order that the new facts may be included, and this has been really Einstein's work.

At the end of last century, physical science recognized three indisputable universal laws: (1) conservation of matter; (2) conservation of mass; (3) conservation of energy; and on the strength of these laws physical science became almost aggressively dogmatic. They should, of course, have been regarded merely as working hypotheses. Since 1905, it has been recognized that energy of every conceivable kind has mass of its own. Mass is the aggregate of rest-mass and energy-mass. Mass is seen to be conserved only because matter and energy are conserved separately.

Then, again, as to the question of fixed axes. The trouble that some of us had when learning mechanics in the days of our youth arose (as we *now* see) from the assumption that axes were fixed in space. It is impossible not to feel that such able men as Kelvin, Tait, and Routh were not suspicious that the theory was in some way incomplete, but they seem to have acquiesced in giving to Newton's laws of motion a universality and finality which we now know the laws did not really possess.

Listen to Clerk Maxwell (as a mathematician probably second only to Newton), in his lighter moments:

" RIGID BODY (*Sings*)

" Gin a body meet a body
Flyin' through the air,
Gin a body hit a body
Will it fly? and where?
Ilka impact has its measure,
Ne'er a ane hae I,
Yet a' the lads they measure me
Or, *at least, they try.*

" Gin a body meet a body
Altogether free,
How they travel afterwards
We do not always see.

Ilka problem has its method
By analytics high;
For me, I ken na ane o' them,
But what the waur am I?"

How are the tremendously far-reaching twentieth century changes to affect our teaching? Probably not at all except in the Sixth Form, for another twenty years to come. Of course the changes are very slight, too slight to affect appreciably the actual practice of mechanics. But the *theory* of mechanics is another story altogether.

Books to consult:

1. *Mechanics*, J. Cox.
2. *Introduction to the Principles of Mechanics*, J. F. S. Ross.
3. *Theoretical Mechanics*, J. H. Jeans.
4. *Mechanics of Fluids*, E. H. Barton.
5. *Treatise on Hydrostatics*, G. W. Minchin.

Routh, and Lamb, should still be on every teacher's shelf. Elementary books like Ashford, Eggar, and Fawdry, are full of useful teaching hints. *The* book for every teacher to master is *Science of Mechanics* (Mach).

CHAPTER XXXIII

Astronomy*

Mathematics or Physics?

If astronomy is included in the school physics course, the necessary mathematical work will be mainly supplementary. If the subject has to be included wholly in the mathematical course, it is not likely to have any great value. Mathematical astronomy which is not based upon personal observations

*This chapter should be read in conjunction with Chapter XXVI of *Science Teaching*.

of any kind, with the telescope at least, if not with the spectroscope, is not likely to have much reality.

Elementary Work

A certain amount of introductory astronomy will necessarily be included in a school geography course. For instance:

1. The earth as a globe travelling round the sun and spinning all the time on its own axis inclined $66\frac{1}{2}°$ to the plane of the ecliptic, i.e. the plane of its path round the sun.
2. The consequences of these movements: day and night, the seasons.
3. The moon as a globe spinning on its own axis once a month, and travelling round the earth once a month, in a plane slightly inclined to the plane of the ecliptic. Phases of the moon.
4. Eclipses: comparative rarity of the phenomenon the result of the inclination of the orbits of the earth and moon.
5. Fixing positions on the earth's surface. Latitude and longitude. Elementary notions of map projection.

Older pupils who have done a fair amount of geometry, especially geometry of the sphere, have no difficulty in understanding these things from descriptions and diagrams. But younger pupils require more help, otherwise they cannot visualize the phenomena, they remain puzzled, and their written answers to questions are seldom satisfactory.

If an orrery is available, there is little difficulty, but more often than not the teacher has to manage with improvised models, perhaps a mounted globe to represent the earth, and painted wooden balls to represent the sun and moon. Personally I prefer to use a large porcelain globe (the kind used with the old-fashioned paraffin lamps) to represent the sun, the globe being fixed in position a foot or so above the centre of the table, and illuminated from the inside by the most powerful electric light available, the room being otherwise in

darkness. This makes an admirable sun, and gives a sharply defined shadow. The earth may be represented by a small wooden ball painted white, with a knitting-needle thrust through its centre to represent the axis, and with black circles to represent the equator and the $23\frac{1}{2}°$ and $66\frac{1}{2}°$ parallels, the ball being mounted so that its centre is the same height above the table as is the centre of the sun, and the axis being inclined at $66\frac{1}{2}°$. About one-half the " earth " is now brilliantly illuminated, and the other half is in shade. If the earth is moved round in its orbit, the successive positions of its axis maintaining a constant parallelism, the meaning of (i) day and night and their varying length in different parts of the world, and (ii) the seasons, may be made clear in a few sentences. If more serious work is to be done later, it is particularly necessary that the plane of the ecliptic should be clearly visualized, and this is easily done if the sun and the earth are supposed to be half immersed in water, the surface of the water representing the plane of the ecliptic. Make the pupils see clearly that half the earth's equator is always above, and the other half always below, this plane.

The phases of the moon are best taught by ignoring the model of the earth for the time being and considering models of the sun and moon alone. Let the laboratory sun illuminate a painted ball, to represent the moon; let the pupils move round this ball, from a position where they see the non-illuminated half to the position where they see the fully-illuminated half. One " phase " after another comes into view, and further teaching is unnecessary. Now put the "earth" in position, and show how the earth may get between the sun and the moon, and prevent the sun from shining on the moon; and how the moon may get between the earth and the sun, and prevent our seeing the sun. And thus we come to eclipses.

The first essential in teaching eclipses is to make pupils realize that a cone of shadow is a thing of three dimensions. Let the school sun cast the shadow of the much smaller school earth. The whole classroom remains brilliantly lighted

save for a cone of darkness on the far side of the earth (we ignore all other objects in the room), and the shape and size of this cone is easily demonstrated by holding a screen at varying distances behind the earth. With a second ball to represent the moon, correct notions of total, annular, and partial eclipses may be readily given. It is quite easy to show why eclipses are comparatively rare phenomena by making the moon move round in an orbit inclined to the earth's orbit.

More Advanced Work

A Sixth Form ought to carry the subject very much farther than the elementary aspects of it commonly included in a geography course, but the business of the mathematical teacher is not to give astronomy lectures in the wider sense but to teach boys to solve those problems which are suggested by the results of actual observation; for instance, the problem of fixing the positions of the stars by means of their co-ordinates, the related question of the diurnal revolution of the heavens, the daily movements of the sun and moon, the calculation of times of rising and setting, nautical problems of determining latitude and longitude, dialling problems.

Facts must not be confused with hypotheses. Thus the earth's daily rotation on its axis and its annual revolution round the sun are mere *hypotheses*, invented to account for facts of observation. The mathematical teacher is concerned with the face value of the facts observed. According to that face value, the stars move round the sky daily, and the sun and moon move amongst them. Any attempt to provide a theory of stellar movements must be preceded by an exact determination of the facts as they appear.

Quite low down the school the boys ought to have been made familiar with the globe (a blackboard surface is very useful) and a cardboard horizon fitting over it. And in the very early stages of geometry they will have been introduced to the theodolite, and will have been taught to measure altitudes and azimuths (though perhaps the term azimuth

has not been used). The theodolite may have been made in the school workshop, and a mere cardboard tube used instead of a telescope. But higher up the school an instrument designed for fairly accurate measurements should be used, and nowadays a good one may be purchased for a few pounds. Even Fourth Form boys can be taught to measure the azimuth and altitude of a given star as it appears to an observer at a given moment. It is easy and interesting work and they like it, though some of them seem to need repeated help with the setting up and initial adjustment of the instrument. I have known boys of 9 or 10 readily pick out the better known constellations, and such stars as the Pole Star, Vega, Capella, Sirius, and the Pleiades. This kind of observation work ought to be included in every Nature Study course. It creates an early interest that becomes permanent, and such basic facts are very useful for future mathematical work.

A school lucky enough to have a small observatory of its own will have an altazimuth (a theodolite is virtually a portable altazimuth), so that azimuths and altitudes (or zenith distances) may readily be found. An equatorial may also be available. If not, the altazimuth should be of such a kind that its telescope can be mounted equatorially when required. Then the boys can take Declinations and Right Ascensions, and become familiar with the celestial equator as well as with the celestial pole, and they will then soon look upon the rotating northern celestial hemisphere as an old familiar friend. Once they feel this familiarity, the making of reasonably accurate observations is child's play and the mathematics involved is not difficult. The sidereal clock and sidereal time are also easily mastered.

The solution of the common problem of determining the altitude and azimuth of a star when the hour-angle and declination are given (or vice versa) is an easy case of the solution of a spherical triangle, and should be familiar.

The sun-dial cannot profitably be taken up until the Sixth, and not even then unless the boys have been well grounded in the geometry of the sphere and its circles. The

geometrical method of graduating the dial (to be fixed either horizontally or on a south wall) is simple enough *if* the elementary geometry of the sphere has been mastered. The boys *must* be able to see that the key to the whole thing lies in the fact that the edge of the gnomon is parallel to the earth's axis and is therefore pointing in the direction of the celestial pole. If about this they are vague, the whole thing is vague.

There is no better way of introducing the young observer to the knowledge of the law of the sun's rotation than by leading him to see that, if a dial be so placed that the style (the edge of the gnomon) is parallel to the axis of the rotating celestial hemisphere, the shadow of the style will at all seasons of the year move uniformly over the receiving surface at the rate of 15° an hour.

The graduation of a sun-dial to be placed on a vertical wall is not difficult, but it is a good little puzzle for testing a boy's knowledge of the sphere and his powers of visualizing the true geometrical relations of the parts of a rather complicated figure.

Mathematical problems in astronomy are, of course, unlimited, but in school there is no time to touch upon more than the bare fundamentals.

Whitaker's Almanack is a mine of useful data for problem purposes.

Stellar Astronomy

The main interest of astronomers, and indeed that of the general public, is now concerned with the stars and nebulæ rather than with the solar system. With the main facts of the solar system every boy should be made familiar; but stellar astronomy is more difficult, the greater part of the available evidence being merely of an inferential character. In a very large measure we have to deal with probabilities, not certainties.

The astronomer's principal instruments are the telescope (mounted in different ways according to the work to be done), the spectroscope, the camera, and the interferometer. The

last-named is outside possible school practice, so is the camera. But the spectroscope is now in common use in schools, and as it ranks next to the telescope in the work of an observatory, its uses should be taught thoroughly.

A course of instruction may be expected to include the following:

1. Spectrum analysis. Displacement of lines: the causes; difficulty of interpretation; distance and speed effects considered separately.
2. The galactic system of stars.
3. The extra-galactic system: stars and nebulæ.
4. Stellar spectra. Interpretation of photographs.
5. Stellar magnitudes, movements, velocities, distances, temperatures; how determined.
6. Theories of stellar structure: for instance, (i) Eddington's, (ii) Jeans'.
7. Solar radiation. Energy and temperature of sun. Poincaré's theorem.
8. Stellar radiation and cosmic radiation generally. Hoffmann's determination of the sun's contribution to the total cosmic ultra-radiation; inferences therefrom. Hess's views.
9. Relativity. General outline. Einstein's proposed tests. Confirmation of the tests and final acceptance of the theory.
10. Modern cosmologies: (i) Einstein's, (ii) De Sitter's. Do they clash? Lemaitre's views—how an Einstein universe may expand to a De Sitter universe.
11. Rival theories as to the future of the universe. British physicists' views of a universe slowly running down to a state of thermodynamic equilibrium. Millikan's views of a universe being continually rebuilt. Evidence pro and con.

How much of this work will be done by the mathematical teacher? His task will probably be concerned mainly with two things: (i) some easy but extremely interesting arithmetic; (ii) the very difficult subject of Relativity.

Mathematical teachers differ in opinion as to the wisdom (or folly) of introducing relativity in a Sixth Form course.

But in view of the far-reaching, indeed fundamental, changes that the subject is bringing about in the whole domain of physics, it seems desirable that an attempt should be made to give Sixth Form specialists at least an outline of the subject. After all, the " special " theory of relativity is easily taught, and, this done, the much more difficult " general " theory may be so far touched upon that the final results of the theory may be fairly well understood by the abler boys. Professor Rice's and Mr. Durell's little books may be followed up by Einstein's own elementary book, and his by Nunn's *Relativity and Gravitation*, which is by far the best book on the subject from the teacher's point of view.*

The *arithmetic* of stellar astronomy deals with numbers so vast that it is likely to deceive all but the trained mathematician. How, for instance, may we bring home to a boy the real significance of the following:

1. The sun is losing weight by radiation at the rate of $1{\cdot}31 \cdot 10^{14}$ tons a year, yet $2 \cdot 10^{9}$ years ago it was only 1·00013 times its present weight.
2. Weight of sun $= 2 \cdot 10^{33}$ grammes.
3. Temperature of interior of sun $= 4 \cdot 10^{8}$ degrees.
4. Number of stars in galactic system $= 4 \cdot 10^{11}$.
5. The 2,000,000 extra-galactic nebulæ each contain enough matter to make $2 \cdot 10^{9}$ stars, that is $4 \cdot 10^{15}$ stars in all.
6. The extra-galactic nebulæ are at an average distance away of 140 million light-years (1 light-year $= 6 \cdot 10^{12}$ miles) and their average distance apart from each other is of the order of 2 million light-years.
7. Radius of universe is perhaps 2000 million light-years

$$= 2 \cdot 10^{9} \times 6 \cdot 10^{12} \text{ miles}$$
$$= 1{\cdot}2 \cdot 10^{22} \text{ miles.}$$

We shall refer to this subject again in a later Chapter.

Be consistent when using the terms " world ", " universe ", " cosmos ", " space ", " ether ", " space-time ". It

* For detailed suggestions see Chapter XXXII of *Science Teaching*.

is probably sufficient to tell a boy that the matter-containing universe, no matter how large, is itself within a limitless void. Do not let him think that the mathematician's convenient and necessary fiction " space-time " is any sort of glorified Christmas pudding mixture. The mathematical partnership is purely formal. Distinguish between an infinite void and a limited wave-carrying matter-containing universe.

In his address to the Mathematical Association, January, 1931, Sir Arthur Eddington said: "About every 1,500,000,000 years the universe will double its radius and its size will go on expanding in this way in Geometrical Progression for ever."—A rude boy might ask some very awkward questions on this point, and carry his teacher backwards as well as forwards in limitless time. It is of no use merely to go back to an assumed initial state of equilibrium. The boy is certain to say, *and before that?*

Books to consult:

In selecting books on Astronomy, don't forget some of the older writers, e.g. Herschel, Proctor, Lockyer, Ball. Eddington's, Jeans', and Turner's books should be known to all teachers of mathematics. Barlow and Bryan's *Elementary Mathematical Astronomy* is very useful. From the teacher's point of view, Sir Richard Gregory's books take quite the first place. Consult also Dingle's *Astrophysics*.

Readers who are specially interested in Relativity should read Dr. John Dougall's searchingly critical article in Vol. X of the *Philosophical Magazine*, pp. 81–100.

CHAPTER XXXIV

Geometrical Optics

Present Methods of Teaching often Criticized

We include this subject because it quite properly belongs to mathematics as well as to physics.

Probably no part of the teaching of mathematics or of physics is so severely criticized as the teaching of optics, no matter whether the subject is taught by the mathematics teacher or by the physics teacher. That there is an urgent need for some reform will be readily admitted from the discussion on " The Teaching of Geometrical Optics " that took place on April 26, 1929, reported fully on pp. 258–340 in No. 229 of the *Proceedings of the Physical Society*. Papers were read by a number of persons interested in optics, including several Public School and University teachers and representatives of the optical industry. A few of the teachers tried to defend the present system, though not very successfully. The conflict of opinion centred largely (1), round the place to be given and the purpose to be assigned, in a teaching course of optics, to the reciprocal equation $(1/u + 1/v = 1/f)$; and (2), round the question of " rays or waves ". My own quite definite conclusion from the discussion was that the best way of teaching the subject is to begin with elementary physical optics in Forms IV and V, and to defer geometrical optics until Form VI.

Several of the critics found fault with the present system because it fails to supply a sufficient practical knowledge of optical instruments and their performance; because pupils by the end of their course in optics have done little more than devote their time to elementary algebra and geometrical diagrams which have but a very slender relation to the subject under consideration; because, in short, the utility of the subject is extremely meagre.

My own main criticism takes another direction—that the mathematics and the theory of the subject at present tend to take too early a place in the teaching course, inasmuch as the physical phenomena underlying the mathematics and the theoretical arguments have not been studied, the arguments, therefore, having no real significance.

Rays or Waves?

Hitherto the " ray " method of teaching has been almost universal in our schools, but the mathematics has been too much divorced from experiment and its real significance has been ill understood. In the discussion already referred to, the method was defended mainly because of its simplicity, not because of its practical utility. The protagonist of the wave or curvature method was Dr. Drysdale, for many years head of the optical department at the Northampton Institute, London. He advocated the method on the grounds (amongst others) that (1) it simplifies the teaching; (2) it harmonizes the teaching of science with optical practice; and (3) it leads naturally to higher physical optics. The real advantage of the method seems to be that it places the whole of optical teaching on a physical basis, and leads naturally to the study of interference, diffraction, and polarization. Two well-known elementary books developing the subject on a wave basis are those of Mr. W. E. Cross and Mr. C. G. Vernon.

Whichever method is used, the teacher should be quite frank in stating that energy can be radiated in two forms, corpuscles and waves. Both forms are easily illustrated experimentally. For example, replicas of diffraction gratings (if gratings themselves are too expensive to buy) are suitable for illustrating the *periodic* character of light. In fact, the periodic character of light *must* be experimentally demonstrated in some way before the curvature method can logically be introduced, and this means a preliminary study of the velocity of light.

It is a good thing to teach both methods, and to teach them more or less in parallel. A ray may, for instance, be looked upon as a line representing an element of the wave-front, or as a normal to the wave-surface; or the wave-front may be traced as a series of arcs after the rays have been drawn graphically.

The best defence of the wave method is that the whole of physics is, fundamentally, a study of wave systems, and it is therefore difficult to justify the picking out of one branch and treating it on an entirely different basis. But the objection to the ray method largely disappears if the ray be thought of as an *element* of a wave, and to the lens designer the ray is the all-important thing.

Theories of Light

The whole question turns largely on an acceptable theory of light. But whose theory? Newton's? Fresnel's? Young's? Maxwell's? Planck's? de Broglie's?

Newton's corpuscular theory failed to account for certain observed facts. The wave theory which superseded it was also found to be defective, and to eliminate these defects the " quantum " theory has been devised. The new theory has shown that Newton was not wholly wrong in regarding light as corpuscular, for that theory is based on the experimental fact that a beam of light may be considered to be broken up into discrete units called " light-quanta " or " photons ", " with almost the definiteness with which a shower of rain may be broken up into drops of water, or a gas into separate molecules ". At the same time, the light preserves its undulatory character. Each photon has associated with it a perfectly definite quantity of the nature of a wave-length.

There seems to be no doubt at all that radiation of all kinds can appear now as waves, now as particles. But the fundamental units of matter, electrons and protons, can also appear now as waves, now as particles. In many circumstances the behaviour of an electron or proton is found to be too

complex to permit of explanation as the motion of a mere particle, and accordingly physicists have tried to interpret it as the behaviour of a group of waves, and in so doing have founded the branch of mathematical physics known as " wave-mechanics ".

In fact it may be fairly said that no single satisfactory theory of light exists to-day. The electromagnetic theory carries us a long way, but in its classical form it is quite inadequate to carry us the whole way. The powerful methods devised by Hamilton in geometrical * mechanics and geometrical optics are being used to found a wave-mechanics bearing to geometrical mechanics a relation similar to that which wave-optics bears to geometrical optics. The *quasi* light-particles emerge from this mechanics more or less naturally, so that we are practically back to Newton and working on Newton's lines. The two views are blended; neither is destroyed.

Geometrical optics is as worthy of serious study as geometrical mechanics. Each is the limiting form when $\lambda \to 0$, and for many purposes this limiting mathematical form is not only entirely sufficient but it is vastly simpler, mathematically, than the general wave-form, whether in optics or in mechanics. What is not worthy of study (at all events as physics) is the type of question often set, the solution of which depends wholly on some mathematical trick. Large numbers of these are found in such favourite old books as Tait and Steele's *Dynamics*, or Parkinson's *Optics*, or Heath's *Geometrical Optics*. Such problems are possibly good as training material in mathematics, but for the display of mathematical talent there is an abundance of excellent material that is, in itself, valuable in physics also.

* I.e. Newtonian.

The Teacher of Optics

Should optics be taught by the mathematics teacher or by the physics teacher? Admittedly a mathematics teacher who has had no training in physics is not likely to be able to appreciate the natural powers and limitations of optical instruments, or to grasp the significance of certain matters in optical theory. Admittedly, too, a physics teacher with no special knowledge of mathematics will be out of his depth in the Sixth Form where, in optics, mathematical considerations count for almost everything, though he will be easily able to cope with the first considerations of the reciprocal equation, which, after all, is essentially a natural development of fives-court and billiard-table geometry. There is thus very little doubt about the answer to our question. The physics teacher should be responsible for the physical optics in Forms IV and V, and the mathematics teacher for the geometrical optics to be done in VI. By geometrical optics is here meant the really serious mathematical work that should *follow* the physical work, work that is partly revisionary but mainly supplementary. The higher physical work in VI will, however, still have to be taken by the physics teacher.

Suggested Elementary Course: Mainly Physics

This elementary course is intended to be mainly experimental and to be done in the laboratory, all consideration of the theory of aberration being excluded. Let all serious mathematics and theoretical developments be postponed to VI.

If the wave method is adopted, wave motion and its significance will naturally be taught first. Of the many wave-producing machines in the market, select one, and see that the boys really understand what it teaches. The propagation of transverse waves may be shown by a ripple tank, illuminated stroboscopically, so that the apparent rate of propagation may

be slowed down. Carry out practical work with real beams of light, not by pin and parallax methods. The sunbeam offers a concrete starting-point.

Devote a lesson or two to showing how fallible the eye is as a measuring instrument, and why, therefore, instrumental aids are necessary. Devise experiments to show the limited power of the eye in unaided vision, and show the capacity of the eye for distinguishing detail under different conditions of illumination and size of aperture.

Make beginners familiar with the construction and use of optical instruments—the telescope, the microscope, and photographic lens. When a boy handles optical instruments, and learns to adjust, to test, and to use them, he acquires knowledge of their potentialities and limitations; and he also becomes acquainted with the language of the subject. Throughout the course keep in mind elementary notions both of physiological optics and of the psychology of vision; also that the eye as an optical instrument is very imperfect, deceptive, and inconstant. Teach beginners when using optical instruments the importance of correct illumination; and the uselessness of increasing magnification beyond the value suitable for the aperture actually effective in the experiment. Show that the apparent brightness of an extended object cannot be increased by optical means; the moon looks no brighter through a telescope.

The key to refraction is, of course, the mere retardation of velocity in a denser medium, and the boys must understand clearly that a refractive index is simply a velocity ratio. The slewing round of the wave-front must be understood to be just a natural and inevitable consequence of *any* such retardation and to be applicable universally and not merely in connexion with light. The trundling of a garden roller across a smooth lawn to a rough gravel drive affords a serviceable illustration. If the direction of motion across the lawn is normal to the line of separation between grass and gravel, there is merely retarded velocity; if oblique, there is a slewing round as well.

Suggested topics:

1. Nature and propagation of light.
2. Waves: motion, length, amplitude, frequency, velocity.
3. Illumination. Photometry, especially the measurement of illumination by daylight photometer.*
4. Experiments in brightness, colour, persistence of vision, fatigue, glare.
5. Reflection and refraction. Concept of the ray as a line representing the direction of movement of an element of the wave-front. The use of rays in optical diagrams. Huygens' principle.
6. Function of lenses; imprinting of curvature.
7. Interference, diffraction, polarization.
8. The spectrum; the spectroscope.
9. The spectrometer: first considerations.
10. The beginnings of mathematics; the reciprocal equation as a convenient memorandum for elementary work at the optical bench.
11. Inverse square law; the unit standard source of light, the unit of luminous flux, the unit of illumination, and their interrelations.

Suggested Advanced Course: Largely Mathematics

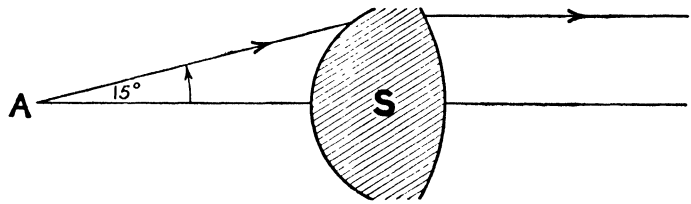
Whatever books on geometrical optics teachers use, especially if they are old favourites like Parkinson and Heath, it is a good plan either to compare these with modern standard works on the technical side of the subject, or to discuss them with a friend acquainted with the optical industry. The important thing is to find out if the principles laid down in a book will *work*.

* An examiner reports that at a recent university examination he set a simple question on the measurement of daylight illumination. Hardly any of the 240 candidates gave a complete answer. A common plan was to balance sunlight against an electric lamp, using, say, a grease spot, assume the sun to be 93,000,000 miles away, assume the inverse square law, and to calculate the candle-power of the sun!

There is no excuse whatever for teaching the subject by methods that are out of harmony with applied optics. Young computers who are taken on at optical works often find to their disgust that their school and textbook knowledge is valueless, and they have to be taught anew by technical experts.

The commonest mistake in optical teaching is due to the misuse or to the misunderstanding of the sign convention and notation. This is unaccountable, as the convention is the result of international agreement. In the optical discussion already referred to, an examiner said that a year or two ago he marked 250 scripts in the Higher Certificate examination, the candidates having been taught in schools in different parts of the country. In the Light paper was a simple question on a lens, and 247 of the candidates attempted it, but only 7 of the 247 obtained the correct result. Such a record of muddleheadedness is utterly inexcusable.

Remember that the basis of all lens work calculation should be the deviation in a ray *at each surface*. Suppose that a ray



which diverges from the point A at, say, 15° is to emerge from the lens system, S, parallel to the axis. Since the whole deviation is to be 15°, and if there are, say, two surfaces, are the two partial deviations to be $7\frac{1}{2}°$ each, or in some other proportion? What are the criteria for what is best? What are the aberrations? And so on.

After a few simple calculations on a *simple* lens for actual wide-angle cases, the boys will soon find that the rule given by Parkinson for the relative radii of the surfaces is by no

means always right; it is only right when $\alpha \leqq 10°$ (about), while in many lenses α is very much greater.

Wave optics must not, of course, be forgotten. For instance, the wave equation and its simple solution should be included.

Suggested topics:

1. Geometrical optics: the reciprocal equation more fully considered. " Wave " proofs and " ray " proofs compared.

2. The dioptre, sagitta (sag), focal power. Show that the curvature of a wave-front or surface is measured by the reciprocal of the radius; the surface with a radius of 1 m. is chosen as a standard. $R_{\text{dioptres}} = 1/r_{\text{metres}}$. Exhibit a curve of 1 m. radius so that the curvature may be visualized. Point out that for a chord of 8·95 cm. the curvature in dioptres is represented by the sag in mm. The application of Euclid, III, 35, to the sag. The dioptre spherometer.

3. The ideal lens contrasted with the actual lens. (The solution of problems arising out of actual lenses will in general be too difficult.)

4. Lenses; spectacles. How the optician is concerned with the *forms* of lenses as well as with their power.

5. Combination of lenses with prisms to correct defects of convergence in the eyes.

6. Thin lenses in contact.

7. Lens combinations.

8. Axial displacement.

9. Chromatic aberration.

10. Spherical aberration: the disc of confusion.

11. Astigmatism, coma, distortion.

12. Photometry further considered. How the distance of star clusters and spiral nebulæ have been determined by the measurement of the apparent brightness of Cepheid variables contained therein.

13. Modern instruments; the telephoto lens, range-finders, prism binoculars, kinema projectors.

14. The more elementary considerations of such subjects as defects of images, collineation between object space and image space, the optical sine theory, design of instruments.

The correction of aberrations by calculation will, in general, be too difficult; so will the higher order of aberrations considered by ray-tracing, though some notion of ray-tracing should certainly be given. The general theory of lenses will also be too difficult. In short, a good deal of this work is more suitable for the university than for the school. Much will depend upon the close collaboration of the mathematics and physics staff. The two aspects of the subjects *must* be considered together.

The real value of mathematical work in optics lies in the discovery of the general principles underlying the *actual behaviour* of real optical systems, as contrasted with the imagined behaviour of ideally perfect systems.

Technical Optics

Few teachers are familiar with technical optics. Very few have seen even the ordinary operation of grinding a lens. As for the designing of lenses for special purposes, or the art of producing optical glass, it is known to very few persons indeed. Few teachers realize that for ordinary industrial purposes the index of a glass is not considered known unless its value is obtained to the fourth decimal place, and for dispersion to the fifth.

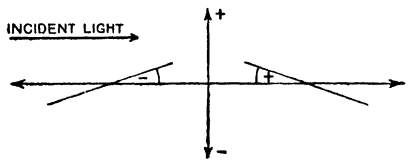
Formerly when an optical system had been designed, the material prepared by the designer was handed over to a number of computers expert in the use of logarithmic tables. But calculating machines are now used, to the operators of which the computation of the elements, individually and in combination, of the new optical system is entrusted. These operators need have no special mathematical equipment, other than that of a common knowledge of simple trigonometrical expressions. Particular rays are traced step by step through surface after surface for the purpose of determining at various stages the longitudinal and transverse aberrations. These values are assessed by the skilled computer, who decides at what particular part of the system a

modification can best be effected. His special skill is practical, the outcome of active practice in the industry itself. It involves, above all, good judgment in the balancing of one type of aberration against another, for no optical system can be free from all kinds of aberration.—Consider the amount of work involved in the computation of the optical system of a typical submarine periscope. Altogether the number of separate operations is something like 40,000, the mere recording of which would fill a book of some 250 pages.

Of course all this sort of work is entirely outside anything that can be done in school, but if a teacher himself is entirely ignorant of it, how can he help making his subject unreal, and talking about it in a foreign tongue?

The Sign Convention

Many of the difficulties underlying the teaching of elementary optics in the past have arisen because teachers have adopted different practices in the use of signs. The following diagram shows the sign convention that has been



agreed upon by the principal optical authorities in the country.

Books to consult:

1. *Optics*, W. E. Cross.
2. *Light*, C. G. Vernon.
3. *The Theory of Light* (new ed.), T. Preston.
4. *Introduction to the Theory of Optics*, A. Schuster.
5. *Experimental Optics*, C. F. C. Searle.
6. *Practical Optics*, B. K. Johnson.
7. *Theory of Optics*, P. Drude (trans. by Mann and Millikan).
8. *Optics*, Müller-Pouillet, 3rd ed.

9. *Principles and Methods of Geometrical Optics*, J. P. C. Southall.
10. *Optical Measuring Instruments*, Prof. L. C. Martin.
11. *Optical Designing and Computing*, Prof. Conrady.
12. *Proceedings of the Physical Society*, No. 229; the papers by Mr. T. Smith, Dr. Searle, Dr. Drysdale, Mr. C. G. Vernon, Captain T. Y. Baker, are all very instructive.

The reader may usefully refer to the memorandum prepared, in January, 1931, by the Council of the British Optical Instrument Manufacturers' Association. The facts adduced definitely establish the pre-eminence of the British position in the optical industry. The tests effected in the National Physical Laboratory are alone sufficient to make that clear.

CHAPTER XXXV

Map Projection

Developable and non-Developable Surfaces

It is the geography teacher's business to show how maps can be outlined on the particular graticule system prepared for him by the mathematician. This graticule system—a gridiron or lattice-work system of parallels and meridians—is in its very essence mathematical and should be included in every school mathematical course.

Fundamental principles of projection will already have been taught in the lessons on geometry. The principles of orthographic projection, including so-called "plans and elevations", should have been taught thoroughly. It is just an affair of parallels and perpendiculars, and thence to the idea of parallel rays of light from an indefinitely distant source is but a step.

The geometry of the sphere should also be known thoroughly; for instance, that the area of a circle is πR^2; of a sphere, πD^2 and therefore 4 times one of its great

circles; of a hemisphere, twice that of its great circle; and that the volume of a sphere is $\pi D^3/6$.

It should be realized that when we look at a sphere we cannot see the whole of a half of it. The portion of the visible surface is that encircled by a tangent cone with its apex at the eye (we neglect binocular vision). A photograph of a geographical globe would necessarily give a picture of rather less than a hemisphere.

Developable surfaces is another subject that should have been taught. A paper model of a cube, prism, or pyramid can be slit open along some of its edges and laid out on the flat, in other words, " developed ". A cylinder or cone can be similarly treated. On a cylinder or cone straight lines can be drawn in certain directions; if the cylinder or the cone is lying on the table, the line of contact with the table is one such straight line. But a spherical surface is altogether of a different type; no straight line can be drawn upon it; it cannot be developed. A sphere touches a plane at a point. We cannot cover a sphere with a sheet of paper as we can a cylinder or cone.

Now the earth is approximately spherical, and any correctly drawn map is part of that spherical surface. An atlas of true maps would consist of spherical segments, not flat sheets. Such an atlas has been made in metal, but it is clumsy to use and is expensive. For convenience we draw our maps on the flat, and thus they are all wrong. A map of England drawn to scale on the surface of an orange would be very small but large enough for a needle to be thrust through the orange along a chord from Bournemouth to Berwick. A perfectly straight tunnel driven between these towns would pass under Birmingham 4 *miles below the surface.* If a map of Europe be sketched to scale on a hollow india-rubber ball, and that portion of the ball be cut out, the portion has to be stretched a great deal to lie flat, and thus parts of the map are greatly distorted.

Evidently no map can be drawn on a flat surface accurately. How do map-makers set to work?

If we examine an ordinary geographical globe, we see the equator, the north and south poles, meridians of longitude running from pole to pole, and diminishing circles of latitude running " parallel " to the equator. And on this network of lines we see a true map of the world.

To draw a map, we first draw a network of lines corresponding as nearly as possible to those on the surface of the globe, though they are bound to differ very considerably from the originals. The network once drawn, we put into each little compartment, as accurately as we can, the corresponding bit of map on the globe. The real trouble is to draw the network.

An examination of an atlas shows that the various networks differ much in appearance. Sometimes one or both sets of lines are straight, sometimes curved, and the curvature seems to vary in all sorts of ways. Why? This we must try to find out.

In an ordinary *plan* drawn to scale, say of a house or of a town, we have the simplest form of projection, called the " orthographic ". To every point in the original corresponds a definite point in the drawing, and the spatial relations between the points are faithfully reproduced; only the scale is changed.

But in a *map*, the relations may all be changed. There will, however, still be a systematic one-to-one correspondence of points. Some sort of general resemblance to the original may always be easily detected, though there is certain to be distortion of form, or inequality in area, or both.

The map-maker is bound to sacrifice *something*. If he is making a map for teaching geography, he tries to represent correctly the relative *sizes* of land and sea areas and thus provides an *equal-area* projection. If he is making a map for a navigator, he tries to show correct *directions*, and does not trouble much about size. Or he may be concerned mainly with correct *shapes*, and not much with sizes and directions. Hence he has contrived projections for different purposes. He has to be content to represent a portion of

the earth's surface accurately in certain respects and to let other considerations go.

The plan adopted is to project the curved lines from the globe on to (1) a *plane* surface, or (2) a *developable* surface (cylinder or cone).

Some projections are readily effected geometrically; they are easy to draw and to understand. Other projections are not strictly geometrical: they are compromises, effected for some particular purpose, and are often called transformations. In these cases point-to-point correspondence is determined merely by formulæ which express the position of each point on the plane of the projection in terms of the position of the point on the spherical surface to which it corresponds.

Projection Shadows

It is possible to obtain geometrical projections by casting *shadows*. A light is placed in a suitable position, and a pencil outline of the shadow of the globe is traced on a conveniently placed plane. This done, it is easy to see how a better projection may be made with ruler and compasses.

Of course if we use a solid globe the shadow will be merely a black circle. We require a hollow translucent globe, with the meridians and parallels painted black on the surface, and a strong light inside. If the globe is fixed near a sheet of white paper on the wall or on the table, the shadows of some of the meridians and parallels will be cast on the paper, and those fairly near the globe will be clear enough to be pencilled over. A large white porcelain globe used for gas and electric lighting answers the purpose well.

When teaching 40 years ago, I found that a better plan was to use a spherical wire cage instead of a globe, made something after the pattern of the old-fashioned wire protectors of naked gas-flames in factories. Such a cage 2′ or 2′ 6″ in diameter is easily made in the school workshop. It is merely a question of bending wire and soldering a number of joints. For the equator, a rather stouter wire should be

used than for the other circles. The meridians are best not made of complete circles but of rather less than half circles, fastened into a ring 4″ or 5″ in diameter, after the manner of the ribs at the top of an umbrella. It is true that the actual north and south poles will be missing but this cannot be helped, the crossing of 12 wire circles at a common point not being practicable. The meridians and parallels may be placed at 15° intervals. Two half-cages are also desirable, one with a pole at its centre, one with a point on

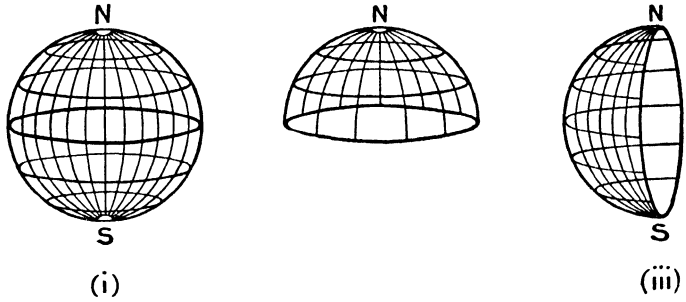


Fig. 275

the equator at its centre. The three should be mounted on suitable stands, in order that, in use, they may easily be kept in a fixed position.

The main difficulty is the provision of a suitable light. Theoretically we require the light to be concentrated at a point. As this is impossible, we use a small electric bulb, porcelain or similar material, with the most powerful light obtainable. A darkened room is, of course, necessary.

Main Types of Projection

The principal projections may be grouped under six main heads: (1) zenithal or azimuthal; (2) globular; (3) conical; (4) cylindrical; (5) sinusoidal; (6) elliptical. Of most of these there are various modifications.

(1) Zenithal or Azimuthal Projection

This projection derives its two names from the facts (1) the map is symmetrical about its central point, just as the stellar vault is symmetrical about the *zenith* of the observer; (2) the projection preserves the *azimuths* of distances measured from the map's centre.

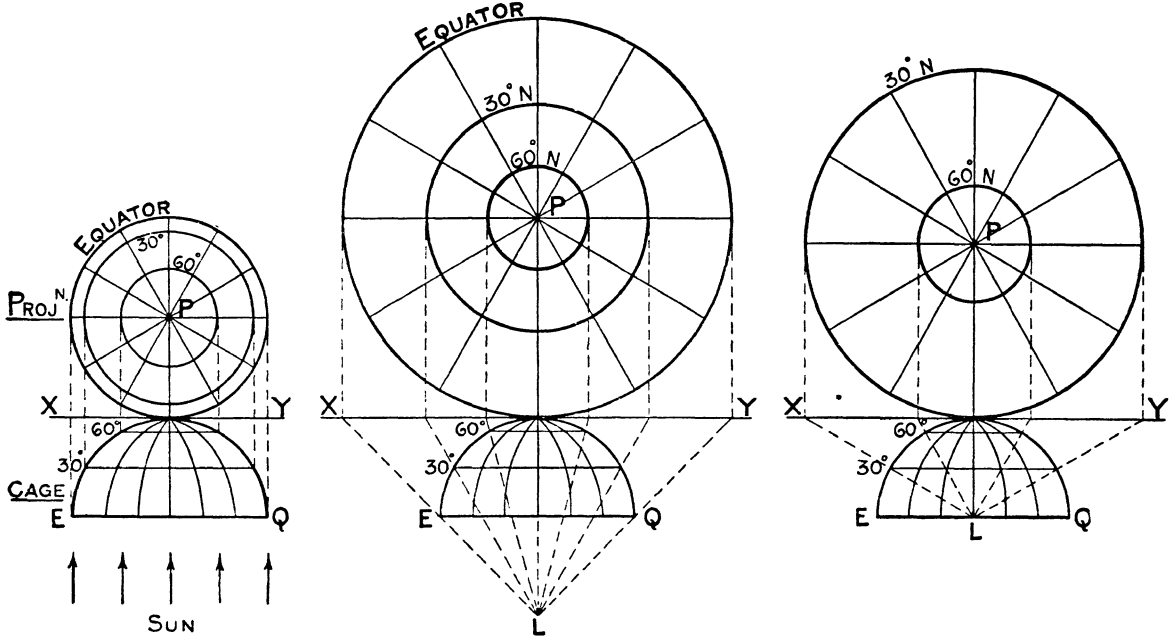
There are three distinct types of this projection: (1) *orthographic*; (2) *stereographic*; (3) *gnomonic*. (See fig. 276.)

1. *Orthographic.*—This is simply an affair of perpendiculars and parallels. As we cannot obtain parallel rays by artificial light, we must use the sunlight at some convenient hour. Let the paper prepared to receive the shadow be placed at right angles to the direction of the sun's rays. Fix the skeleton wire hemisphere (ii) so that the equator is parallel to the paper; the parallels of latitude will be projected as circles of true size, the meridians as radii of these circles. Geometrically, we draw the projection exactly as in geometry. Note that the scale along the circles is always true, but that the radial lines are foreshortened more and more as the distance from the centre increases.

2. *Stereographic.*—Set up the same wire hemisphere, with its equatorial plane vertical and parallel to the projection plane, and place the lamp at the further extremity of the diameter corresponding to the earth's axis, that is at the " south pole ". The parallels of latitude are again projected as circles, but they are enlarged, the equator being twice the size of the original. The meridians are projected as radii, as before.

This projection has a general similarity to the orthographic, and its geometrical construction is a useful exercise. The scale is increased equally along the meridians and parallels, and some good Sixth Form problems may be based on the projection. In particular, the projection provides a ready means of studying the sum of the angles of a spherical triangle.

3. *Gnomonic.*—For this projection, the light is placed at the *centre* of the sphere. Although the parallels of latitude are



ORTHOGRAPHIC STEREOGRAPHIC GNOMONIC (Pole to 30°N)

Fig. 276

still projected as true circles, they are still more enlarged, and the equator itself, having the light in its own plane, cannot be projected at all. The general appearance of the projection is similar to that of the other two, but obviously the areas very far from the pole are greatly distorted in the projection. The figure shows the polar region to within 30° of the equator. It makes a fairly good map for areas within 30° of the pole.

The three projections may be usefully compared in this way:

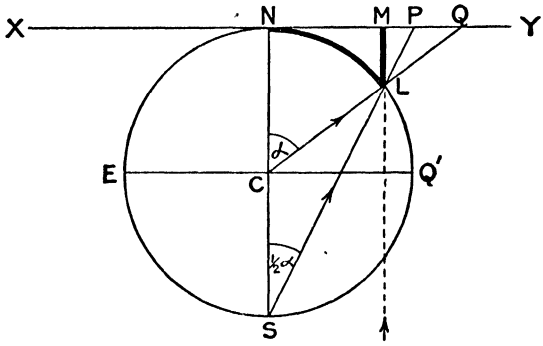


Fig. 277

Let NESQ′ be a meridian of the earth, and XY its projection plane. Take a point L at polar distance a. Then angle LSN $= \frac{1}{2}a$. Let radius be R.

Orthographic projection of arc NL $=$ NM $=$ R $\sin a$.
Stereographic „ „ $=$ NP $=$ 2R $\tan \frac{1}{2}a$.
Gnomonic „ „ $=$ NQ $=$ R $\tan a$.

Observe that in the orthographic projection the outer circles are crowded together, in the stereographic the outer are farther apart than the inner, and in the gnomonic the outer circles get so far apart as to be useless. It is sometimes convenient to arrange these circles at equal dis-

tances apart, and then we have the *zenithal equidistant* projection. It is also possible for the distances of the parallels of latitude so to be regulated that the area enclosed by any parallel is equal to the area of the globe cut off by the same parallel, and then we have the *zenithal equal-area projection.* Strictly, these are not true projections, but the associated geometry is interesting and instructive.

(2) Globular Projection

All three zenithal projections are sometimes called " perspective " projections, since they can be cast as shadows. But the globular projection cannot be cast as a shadow, and is therefore non-perspective.

The geometry is a useful exercise for beginners. The projection is commonly used for maps of the world in two hemispheres. The figure represents one hemisphere. Divide the equatorial diameter into an equal number of parts, say parts representing 30°. Divide the circumference similarly. The curves are all arcs of circles, each to be drawn through three points. The mathematics of the projection is of the simplest.

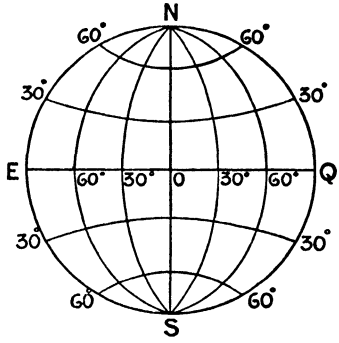


Fig. 278

(3) Conical Projection

For this we require the wire skeleton of the complete globe, with the light fixed at the centre. The shadow will be cast, not on a plane, but on the inner surface of a white paper cone.

Fold up, in the usual way, a common filter paper, and

fit it into a funnel. It makes a cone with a 60° apex. A half circle of paper would make the same cone, the two halves of the diameter being brought together. A sector having an apex of less than 180° folds up into a more pointed cone; one with an apex of more than 180° folds up into a flatter cone. A sector of 360° (a complete circle) necessarily remains a plane.

Make a white paper cone (of about 130° apex in the flat), slip it over the polar region of the skeleton globe so that the apex is in a line with the axis of the globe. The cone touches the sphere tangentially, viz. in a circle, and this circle is a parallel of latitude. If this corresponds with one of the wire

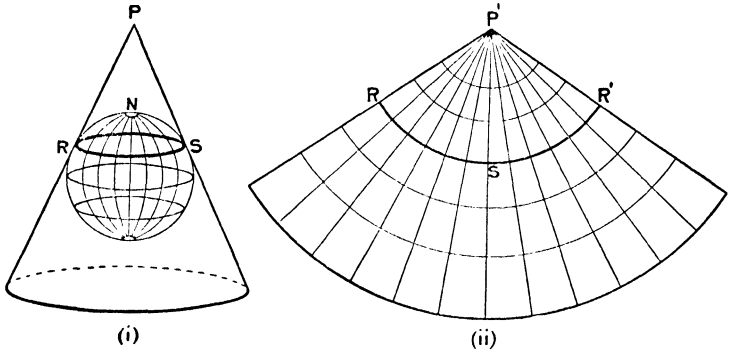


Fig. 279

circles so much the better. Now gently mark in the outlines of the cast shadow. This is pretty easy in the neighbourhood of the line just mentioned, but the cast shadow gets very faint as we get farther away from the line. Now open out the cone on the flat (fig. 279, ii), and we have an ordinary conical projection. The arc represented by a heavy line RSR′ is the circle of contact RS in (i), the " standard parallel ", and it is divided exactly as the circle it touches on the sphere is divided.

The solid angle N of the sphere = 4 right angles. The angle of the cone when developed is angle RP′R′. The ratio of the latter angle to the former is called the *constant*

of the cone. It is a simple Fifth Form problem to prove that this constant is the sine of the latitude of the standard parallel.

The geometrical construction is simple. Observe that the parallels are arcs of circles, and that the meridians are straight lines. Since meridians are great circles and their planes pass through the centre of the globe, these planes must bisect the cone and therefore cut its surface in straight lines. The projection is commonly used for countries in middle latitudes if the latitude is not of too great an extent, e.g. for England. The conical projection with *two* standard parallels (fig. 280) is a common projection for the larger European countries. Its principle is equally simple.

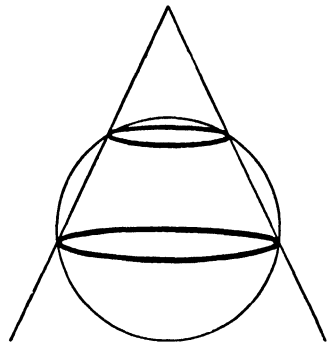


Fig. 280

(4) Cylindrical Projection

Take a large sheet of white paper and convert it into a cylinder of the same diameter as the skeleton wire sphere. Its length should be 3 or 4 times the diameter. Slip it over the sphere so that the equator is in about mid-position, and place the light at the centre of the sphere. A shadow of a part of the wire sphere is cast on the cylinder. Obviously the shadows of the two poles cannot be cast on the cylinder at all, and high latitudes are cast at great distances, with consequently great distortion. The small circle of latitude AB will appear as A′B′; in fact *all* circles of latitude will be projected as circles on the cylinder and will all be of the same size as the equator. All meridians, being great circles, will be cast as straight lines. Open out the cylinder on the flat (ii), and the projection is seen to consist of a *net of rectangles.* E′Q′ = πEQ and may be subdivided in the usual way

The projection is not of much practical value. Except in the immediate neighbourhood of the equator there is far too much distortion.

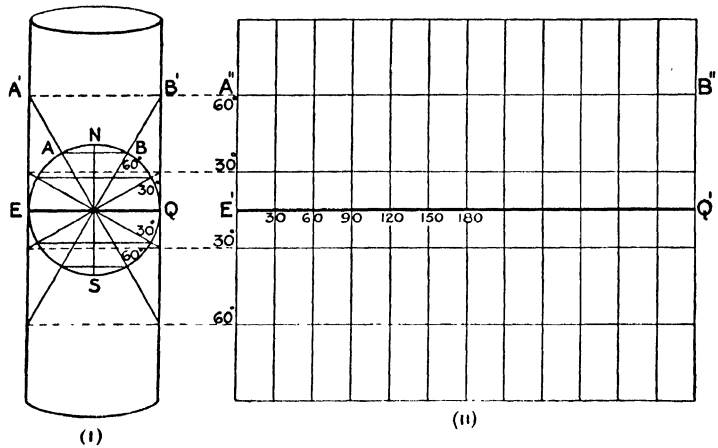


Fig. 281

But various modifications of this primary cylindrical projection have been adopted, two of them being noteworthy: (1) *Lambert's equal-area* projection, and (2) *Mercator's* projection.

1. *Lambert's projection.*—Construction: divide the quadrant

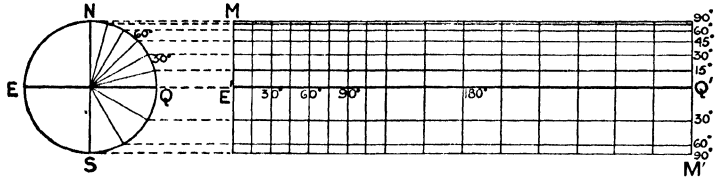


Fig. 282

NQ (fig. 282) into an equal number of parts, say 6 of 15° each, and draw parallels to EQ, and so obtain parallels of latitude. For meridians, make $E'Q' = \pi EQ$, and divide up into intervals of, say, 30°. Note that the parallels of lati-

tude are horizontal lines at a distance of $r \sin \lambda$ from the equator (λ = lat.).

It is a well-known theorem in geometry that the area between any two parallels on the enveloping cylinder is equal to that of the corresponding zone on the globe. Hence the *area* of the rectangle MM′ is *equal* to the area of the globe. The proof of the theorem should be given.

2. *Mercator's orthomorphic projection.*—This is the best-known of all projections; it is used for navigation purposes, and for maps of the world. But it is responsible for many geographical misconceptions, for instance the misleading appearance of the polar areas, which are greatly exaggerated. Greenland is made to appear larger than South America, though only one-tenth its size.

As with all cylindrical projections, the meridians are equidistant parallel lines; the parallels of latitude, on the other hand, increase in distance from one another the farther they are from the equator. This spacing of the parallels of latitudes is so arranged that at any point of intersection of parallels and meridians (in practice, any small area), *the scale in all directions is the same.* Hence the projection is *orthomorphic.* Literally the term means " preserving the *correct shape* ".

The essential characteristic of the projection, then, is this —that at any point the scale along meridian and parallel is the same. We give Dr. W. Garnett's ingenious illustration of the method of effecting this.

Dr. Garnett takes a very narrow gore, i.e. a strip between two meridians on the globe (cf. the surface of a natural division of an orange, selected for its narrowness), and spreads it out as flat as possible; if very narrow there is no great difficulty in spreading it out very nearly flat, without much distortion; then it is *very nearly* an *equal-area* strip, i.e. its area on the flat is *very nearly* the same as when it was part of the curved surface of the sphere. The *length* of the spread-out gore is, of course, half the circumference of the orange.

Let NAB represent the *half* gore, AB representing 10° at the equator; and let NM be the central meridian. Divide NM into 9 equal parts, and through the points of division draw the parallels shown in the figure; these represent 10° intervals of latitude from the equator to the pole. Suppose the gore to be made of malleable metal.

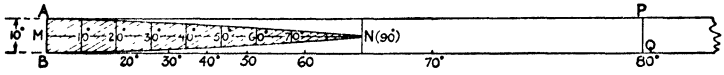


Fig. 283 *

Hammer it out in such a way as to cause it to spread to the uniform width AB. Clearly we cannot do this in the immediate neighbourhood of N: there would not be enough metal. Hence cut the gore off at about 85°. But the gore cannot be hammered out without expanding in length as well as in breadth. At, say, 40° little hammering will be required, and the additional length there will be slight; but at, say, 70° much hammering will be required to produce

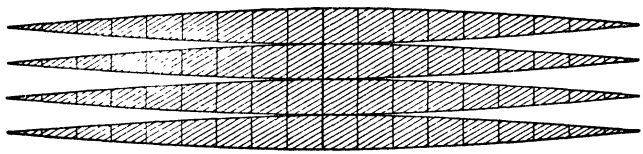


Fig. 284 *

the necessary additional width, and therefore there will be much additional length produced. At 45° the ratio of the increased width to the original width is $\sqrt{2} : 1$, and therefore the length of the strip at 45° is increased $\sqrt{2}$ times, and hence the area is increased there $\sqrt{2} \times \sqrt{2}$ times, that is, twice, and the thickness is therefore halved. At 60° there will be a doubling of width and therefore a doubling of length, i.e.

* Figs. 283 and 284 are made to lie down, to save space. Normally, the gores would be given an upright position.

the area will be multiplied by 4 and the thickness reduced to $\frac{1}{4}$.

It is easy to imagine the whole series of 36 gores (fig. 284 shows 4) placed side by side, and rolled out until the edges meet and 36 rectangles are formed.

Generally, every little strip parallel to the equator is increased both in length and breadth in proportion as the radius of the sphere is to the radius of the circle of latitude where the strip is situated. At 80° the area is increased about 33 times, and at 85° about 132 times. The figure (fig. 283) shows roughly how the gore between 0° and 80° is hammered out into the rectangle ABPQ.

If the 36 gores were extended to lat. 80° N. and S., and placed side by side, we should have a rectangle 36 times AB in length and twice AP in height, and we should have the framework for a Mercator map of the world between the parallels 80° N. and 80° S.

The point about the whole projection is the retention of true shape, though this applies to only very small areas. At the equator, areas are unchanged; at 80° they are increased 33 times.

The *shapes of small areas* are magnified, not distorted. *Strictly* the orthomorphism is applicable only to points and is therefore only theoretical.

Construction of a Mercator map.—The radius of a parallel of latitude on a sphere of radius r is $r\cos\theta$. Hence if a degree of longitude in latitude θ is to be made equal to a degree at the equator, its length must be divided by $\cos\theta$. If the scale of the map is to be increased in all directions in the same ratio, then the length of the degree of latitude measured along the meridian must also be increased in the same ratio. If y be the distance of the parallel of latitude θ from the equator in the Mercator map of a sphere of radius r,

$$y = r \log \cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right),$$

a formula which may be evaluated by Sixth Form boys.

The distances of the parallels from the equator are, in terms of the radius, approximately, for

10°	·176 R	50°	1·011 R
20°	·356 R	60°	1·317 R
30°	·55 R	70°	1·736 R
40°	·763 R	80°	2·436 R

These values should be checked from a Mercator in a good atlas: equator $= 2\pi . R$.

Mercator, and Great Circle Sailing.—The special merit of Mercator's projection lies in the fact that any given uniform compass course is represented by a straight line. All meridians are exactly north and south, and all parallels exactly east and west. Hence a navigator has only to draw a straight line between his two ports, and the angle this line makes with the meridian on the map gives his true course for the whole voyage.

Any straight line drawn in any direction on a Mercator is called a *rhumb line*; it crosses all parallels at a constant angle, and all meridians similarly. A sailor who is told to sail on a constant bearing simply sets his compass according to the rhumb line.

But this course *may* not be the shortest; it cannot be, unless it is along the equator or along a meridian, i.e. along a great circle. A rhumb course in any other direction is *not* along a great circle, and we know that the shortest distance between two points in a sphere is along the great circle passing through them. Economy makes the navigator take the shortest course if he can. How is he to find it?

A rough and ready way would be to take a wire hoop that would exactly fit round the equator or round one of the meridians (and therefore round a great circle: we neglect the ellipticity of the earth), hold it over the globe so that it passed through the two ports at the ends of the course under consideration, chalk in the curve, and then transfer the curve to the Mercator, freehand, as accurately as the corresponding graticules would allow.

A navigator always follows a great circle if he can, not the rhumb line, and for his special use great circle courses are calculated and laid down on a Mercator's chart.

If ARB is the rhumb line between two places A and B (the figure is a fragment of a Mercator chart), and AGCB is the great circle (and therefore *shorter* than the rhumb line), a navigator might sail along a series of chords AG, GC, CB, altering his course at G and C. He would not quite follow

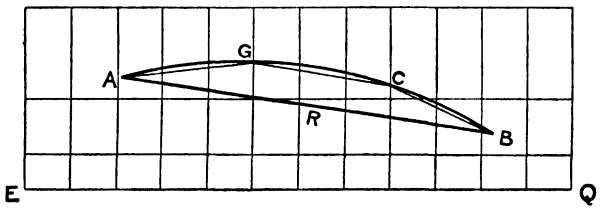


Fig. 285

the great circle, but he would follow a much shorter route than the rhumb line course.

Give the boys examples of the course between, say, Japan and Cape Horn, Plymouth and New Orleans, Cape Town and Adelaide. Let them mark in roughly both the rhumb line and the great circle courses on a Mercator chart. Remind them of the deceptive geometry, as in fig. 285, where the chord represents a *longer distance* than the arc it subtends.

To trace the course of a great circle on a Mercator chart.— Any great circle must cut the equator at two places and at a given angle. Hence it will cut (i) a given meridian at a point whose latitude can be determined, and (ii) a given parallel of latitude at a point whose longitude can be determined.

Assume that we are given:

(i) α, the inclination of the great circle to the plane of the equator;

(ii) λ, the longitude, measured from one of the points of section, of a meridian in latitude L.

Then the following equation may be established:

$$\tan L = \tan\alpha \,.\, \sin\lambda,$$
$$\text{or,} \quad \sin\lambda = \cot\alpha \,.\, \tan L.$$

From this equation, either the latitude can be determined at which the great circle cuts any meridian, or the longitude at which it cuts any parallel. The equation may therefore be used to trace the course of a great circle on a Mercator chart.—Sixth Form boys should work through a few of the exercises in *Nunn*, Exercises, Vol. II.

Aviators are naturally much interested in great circle sailing. Let the boys determine an aviator's route between two given places, say 5000 miles apart, by stretching a string over a geographical globe. Then ask them how an aviator would set his compass. Let them lay down the course on a Mercator chart (graphically and approximately will do), and see how it differs from the rhumb line, and how compass directions might be determined by a succession of chords.

(5) Sinusoidal Equal-area Projection

This is sometimes called the Sanson Flamsteed projection; it is used mainly for world maps. An equal-area or " homolographic " projection is a projection where shape is sacrificed to equality of area.

It differs widely from the geometrical and (mainly) shadow projections already considered.

The equator ($= 2\pi R$) is true to scale. The central meridian ($= \pi R$) is also true to scale. Parallels of latitude are equidistant horizontal lines. All the meridians are of the form of sine curves. Each parallel is equally divided by the meridians, which are nearer and nearer together towards the poles.

Fig. 286 shows a quarter of the complete projection of the world map; EZ = 2NZ. Divide NZ into, say, 6 equal parts (of 15° each), and EZ into 6 parts (of 30° each). Each horizontal straight line is equal in length to the corresponding

circle of latitude. Through the extremities of these lines draw the curve EN which represents the boundary of the quarter map. Divide every parallel into 6 equal parts, similar to EZ, and draw curves through the corresponding points

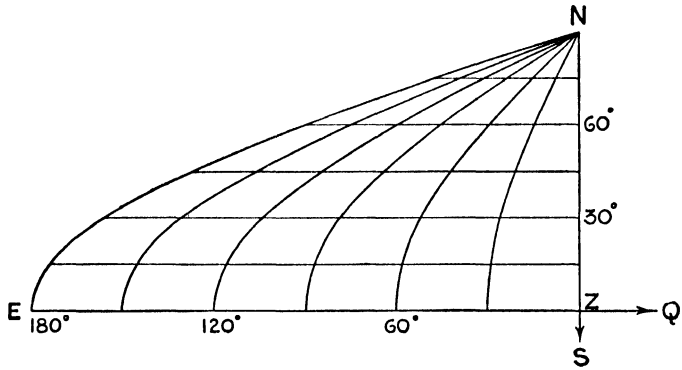


Fig. 286

of division; these curves are meridians (in the figure, half-meridians).

The disadvantage of the projection is that towards the edge the meridians are very oblique and thus the shape is much distorted. The graticules along the equator and central meridian, on the other hand, practically retain their original shape.

In any projection graticule, the horizontal lengths are exactly the same as in the graticule on the globe; and the vertical height of the projection graticule is equal to the length of the corresponding piece of meridian on the globe. Hence the *area* of any projection graticule is equal to the area of the corresponding graticule on the globe, or the whole area of the map is equal to the whole area of the globe.

Each curved meridian is a sine curve: why? Might the sine curves be drawn before the parallels?

The projection is very good for maps of Africa and South America. Why?

(6) Mollweide's Elliptical Projection

This projection is also used for world maps. Again the parallels are horizontal lines. The meridians are ellipses (there are two special cases: the central meridian is a straight line and the 90° meridian is a circle).

Again the area of the map is equal to the area of the surface of the globe.

Since the area of the surface of the sphere is equal to 4 times the area of its great circle, the area of the hemisphere is equal to twice the area of its circular base.

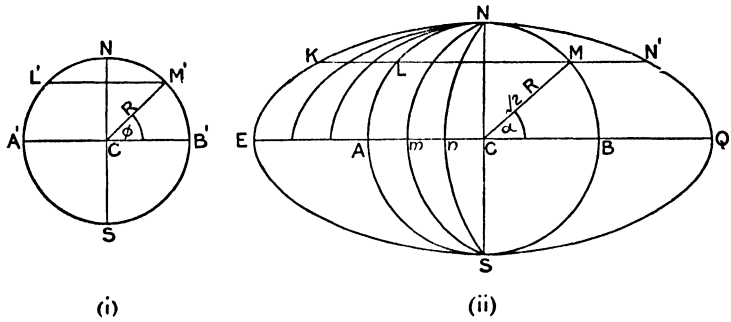


Fig. 287

Let the radius of the globe, fig. 287 (i), be R. Draw a circle (ii) of radius $\sqrt{2}$.R (= CB). Area = $2\pi R^2$ sq. in., which is the area of the half globe. Let C be the centre of the circle; draw a horizontal diameter ACB and a vertical diameter NCS. Produce AB so that CE = 2CA and CQ = 2CB. Draw an ellipse having EQ and NS for axes. It is one of the properties of the ellipse that if any line KLMN′ be drawn parallel to EQ cutting the ellipse and the circle, KN′ = 2LM; and as this is true for any such line, it follows that the *area of the ellipse* = twice the area of the circle = the *area of the globe.*

Divide EQ into equal parts and through the points of division draw ellipses with NS as a common axis; these are

the meridians. Evidently all gores (e.g. N*n*SC, N*m*S*n*) are equal in area. For an equal-area projection, it remains to divide these gores by parallels of latitude into the same areas as the corresponding gores between the meridians on the globe are divided. This is the only difficult part of the problem.

We have to draw KN′ so that it will correspond to some particular degree of latitude ϕ on the globe. Fig. 287 (i) represents a section of the globe through the great circle NA′SB′. In fig. (ii), the circle represents the *area* of the hemisphere and the ellipse the area of the whole sphere.

The area of the zone L′A′B′M′ on the *spherical surface* (radius = R) in fig. (i) is equal to twice the area of the zone LABM on the *plane surface* (radius = $\sqrt{2}$.R) in fig. (ii).

We have to find the angle MCB. Let it equal α. Then

$$2\alpha + \sin 2\alpha = \pi \sin\varphi.$$

It is not easy from this equation to obtain α in terms of ϕ, but it is quite easy to determine ϕ in terms of α. Hence if any parallel be drawn in the ellipse, and the angle α is measured, the latitude ϕ to which it corresponds is found at once.—The formula should be established by the Sixth Form.

Choice of Projection

Let the boys examine a good modern atlas in which the projections used are named; and get the boys to discover why a particular projection is used in each case. This may give rise to an interesting discussion.

Books to consult:

1. *A Little Book on Map Projection*, Garnett.
2. *The Study of Map Projection*, Steers.
3. *Map Projections*, Hinks.

CHAPTER XXXVI

Statistics

The Importance of the Subject

It is highly desirable that an elementary study of this subject shall be included in any Sixth Form course. Statistics enter largely into modern science and administrative practice, and the underlying principles have now been so well worked out and have become so definite, that no large office, government or local, can afford to be without at least one well-trained statistician. The newer developments of psychology depend almost entirely upon a rational interpretation of statistics. There are some teachers who are still ignorant of the principles underlying the correct handling of the statistics of everyday school practice; and thus they are necessarily unable to make the most effective use of, for instance, an ordinary sheet of tabulated examination results.

I have seen the subject taken up seriously in only two or three schools, and have therefore had little experience of the methods of teaching it. The teaching suggestions in Professor Nunn's *Algebra* are recognized as the most practicable yet made, and the topics he selects for inclusion in a school course seem to be just about right. The technical side of the subject is, of course, rather difficult for boys, but the fundamentals are easy to grasp, and it is possible to map out an excellent preliminary course that will give a good general insight into the subject and into its methods.

The main problems to be considered may be grouped under the three usual heads: (1) frequency distribution of a series of measurements or other statistics; (2) frequency calculation: probability; (3) correlation.

Frequency Distribution

Frequency distribution is concerned with the best ways of recording statistics and of expressing most simply and effectively the information which they contain. Suppose, for instance, a Local Education Authority has examined 20,000 children between the ages of 10 years 6 months and 11 years 6 months for scholarships to be held in the local Secondary Schools. What would be the best way of recording the results, so that not only might their significance and its implications be readily apprehended but also that the record might form a simple means of comparison with similar records elsewhere?

Suppose that all the examination papers were arranged in 20 piles, according to the percentage of marks awarded to each paper, 0% to 5%, 5% to 10%, and so on up to 95% to 100%. The height of the piles would exhibit to the eye the *frequency distribution* of the marks, the number of papers in a pile giving the *frequency* of the particular mark in that pile. A logically set-out record of the whole of the results might be called a *frequency table.* A *column graph* showing the number of papers in each pile would afford a useful alternative means of exhibiting the frequency distribution; such a *frequency diagram* is commonly called a *histogram.*

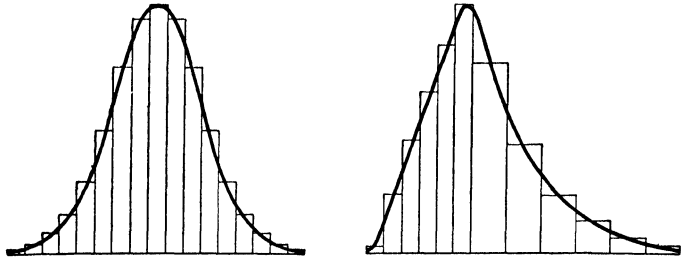
The first thing to do is to familiarize the pupils with the main forms of frequency distribution in tables and diagrams. The records in Government " Blue Books " are often useful in this connexion.

A frequency diagram differs from the ordinary graph of algebra and physics. The latter represents the relation between two variables, or the values of a function which correspond to different values of a single variable. But a frequency diagram serves simply to show how often each value of the variable is met with in some record.

Examples of the forms of frequency distribution may be drawn from very different sources—from anthropometric measurements, from economics, from meteorology, from

medical records, and from the records of the workings of what (in our ignorance) we call chance. The various forms display resemblances that are often most pronounced when the diversity of origin would seem to be greatest. The resemblances are brought out very clearly by the *frequency curves*.

The pupils should learn that the smooth curve really represents the interpretation of the ideal distribution to which actual samples might be expected to approach if they contained a sufficient number of cases drawn from a field sufficiently wide to be really representative. When the curve is drawn with this idea in view, it is always of the same general pattern,



(i) Fig. 288 (ii)

more or less bell-shaped. The curves are, however, easily sorted out into seven distinct types, two of them symmetrical and five of them asymmetrical or skew.

Fig. 288 (i) is an example of an almost perfect symmetrical curve, imposed on its histogram, which, however, is itself unlikely to be quite symmetrical; it is the " normal " type of curve. Fig. 288 (ii) is an example of a skew curve also imposed on its histogram. If a curve evaluated from an ordinary examination mark-sheet was very skew, i.e. varied considerably from the normal, it would suggest that an inquiry was necessary.

The normal curve was formerly spoken of as the graphic representation of the " law of error ", it being thought, perhaps naturally, that the mean of the distribution was the number (physical measurements in human beings, for instance) which represented nature's intention, deviations there-

from being " errors ". But it is now recognized that ordinary distributions are not, even ideally, normal, and that skewness in them is almost inevitable, though probably in most cases the skewness or asymmetry is moderate.

Enough should be done to teach the boys the fundamental fact that although when the data are few the columns which represent the frequencies of occurrence may exhibit no orderly arrangement, yet order invariably appears as the data become sufficiently numerous.

There will seldom be time for pupils to deal with the formulæ representing statistical graphs, though such work is certainly interesting and valuable. But the pupils should be made thoroughly familiar with the significance of the ordinary statistical phraseology.—The *mean* is the ordinary arithmetical mean and is easily found when all the individual measurements are given. The *median* is the *middle* measurement; it is the measurement corresponding to the ordinate which bisects the area of the curve; it may be found by calculation based upon the known properties of the curve, or (roughly) by using squared paper and counting up the squares under the curve. If we examine 5 boys and their percentage marks are, respectively, 85, 80, 60, 55, and 30, the *mean* (average) mark is 62, and the *median* (middle) mark is 60. The *lower* and *upper quartiles* are the two measurements one-fourth from the beginning and one-fourth from the end of the whole series. The *interquartile range* includes the middle half of the whole series. The *mode* is the measurement corresponding to the highest ordinate of the frequency curve drawn over the histogram.

Deviation.—A boy may obtain high marks or low marks; a man may be tall or short; in both cases there is deviation from a standard. In fact, in any series there is bound to be " dispersion ". To measure the deviation, what standard should be taken?

One recognized way of indicating the dispersion of a set of measurements, when something more concise than a frequency table is required, is to state the interquartile range,

or, more usually, the semi-interquartile range, of the statistics; only the middle half of the measured cases is considered. This semi-interquartile range is often called the *quartile deviation.*

Or we may strike an average of the arithmetical differences between the various measurements and some selected standard measurement, e.g. the *median*, the *mean*, or the *mode.* Since the sum of the differences of the measurements from the *median* is less than from any other standard ordinate, the "*mean deviation*" of a set of measurements is, as a rule, calculated with reference to their median.

But the most useful measure of dispersion is that which takes account not of the deviations themselves but of their *squares.* Squaring the deviations gives greater weight to the larger ones, and has the mathematical advantage of making the numbers positive. Take the deviations, square them, find the mean of the squares, and then take the root of the mean: this gives the useful measure of dispersion called the *standard deviation.*

Whether the subject is taught or not, the mathematical staff would find it a great advantage to graph their periodical examination results statistically. What does the graph teach? What does its skewness teach? What can be learnt from the interquartile range? What can be learnt from a comparison of the quartile, mean, and standard deviations? Which is the more useful mark in a mark-sheet, the mean or the median? Why? And so on.

The fact should be impressed upon learners that the graphic method of presenting statistical data has advantages over the tabular statement of the same data, though it is necessary to remember that the facts cannot be more *accurately* represented by a diagram than by the data from which the diagram is constructed. Indeed, the diagram may, from imperfect draftsmanship, fall short in accuracy of the statistical tables it represents. The graphical presentation has, however, the important advantage that it presents lengthy series of data in a form in which the majority of users

of them find it easier to grasp their sequence and their relations than when presented in the form of tables.

Frequency determined by Calculation

The previous section dealt with the analysis of frequencies actually *given*. It is now necessary to refer to the possibility of *predicting* them among events that have never been observed. It is in connexion with this problem that the topics, (1) combinations and permutations, and (2) probability, are best treated.

The calculation of probabilities is nothing more than the calculation of frequencies. Probability is not an attribute of any particular event happening on any particular occasion. It can only be predicted of an event happening, or conceived as happening, on a very large number of " occasions ", or of an event " on the average ", or " in the long run ". Unless an event can happen, or be conceived to happen, a great many times, there is no sense in speaking of its probability. Frequency would be a better word than probability in the study of the subject generally, but the latter word has become definitely established.

Make sure that the boys understand the notion of " independent events ". The fall of a tossed coin is an independent event. Whether it will fall " head " or " tail " the next time it is thrown depends not at all on how it fell last time, or the last thousand times. If, for example, there had been a run of a hundred heads, the " chance " that the next throw would also be a head is just as great as before.

Another idea the pupils must grasp is that frequency predictions are possible only in so far as the events predicted can be regarded as compounded of *independent* elementary events whose characteristic behaviour is already known. Thus, knowing that the spin of a coin is an independent event which will, in the long run, turn out heads and tails with equal frequency, we can predict with confidence what will happen (again in the long run) in the case of an event which consists in the tossing of (say) 10 coins.

1. *Combinations and Permutations.* — Apart from the "tricky" problems that occur in some of the textbooks (they are of no importance and may be ignored) this subject seems to be taught well. Most mathematical teachers seem to have neat little devices for working out *nPr*, *nCr*, &c., from first principles, it may be by ringing the changes systematically, and neatly classifying the results, for a group of a few letters on the blackboard. Even boys of average ability soon get to like the little stock-problems about people sitting round a table, or about the selection of elevens. Do not spend much time on the subject; it is not worth while. But give plenty of oral practice in such exercises as finding the value of ${}^{10}C_3$, ${}^{10}C_7$, &c., and do not forget the evaluation of coefficients in expansions.

2. *Probability.*—This is a more serious topic, though its more elementary considerations are easily within the range of school work.

Justification for teaching the subject is hardly necessary. It is by far the best application of the theory of permutations and combinations, but much more than that, it enters into the regulation of some of the most practical concerns of modern life, for instance in the use of mortality tables, insurance and annuity problems, and so forth. The following arguments and examples * may serve as a suitable introduction to the subject.

When we say that the probability that an event will happen in a certain way is $1/n$, what we mean is that the relative amounts of knowledge and ignorance we possess as to the conditions of the event justify the amount of expectation. The event itself will happen in some one definite way, exactly determined by causation; the probability does not determine that, but only our subjective expectation of it. It is from this combination of knowledge and ignorance that the calculation of probability starts.

Fundamentally, the theory of probability consists in

* *Scientific Method*, pp. 260 *seq.*

putting similar cases on an equality, and distributing equally among them whatever knowledge we possess. Throw a penny into the air, and consider what we know in regard to its way of falling. We know that it will certainly fall upon a side, so that either head or tail will be uppermost; but as to whether it will be head or tail, our knowledge is equally divided. Whatever we know concerning head, we know also concerning tail, so that we have no reason for expecting one more than the other. The least predominance of belief to either side would be irrational; it would consist in treating unequally things of which our knowledge is equal. *We must treat equals equally.*

The theory does not require that we should first ascertain by experiment the equal facility of the events we are considering. The more completely we could ascertain and measure the causes in operation, the more would the events be removed from the sphere of probability. The theory comes into play where ignorance begins, and the knowledge we possess requires to be distributed over many cases. Nor does the theory show that the coin will fall as often on the one side as the other. It is almost impossible that this should happen, because some inequality in the form of the coin, or some uniform manner in throwing it up, is almost sure to occasion a slight preponderance in one direction. But as we do not previously know in which way a preponderance will exist, we have no reason for expecting head more than tail.

Suppose that, of certain events, we know that some one will certainly happen, and that nothing in the constitution of things determines one rather than another; in that case, each will recur, in the long run, with a frequency in the proportion of one to the whole. Every second throw of a coin, for example, will, in the long run, give heads. Every sixth throw of a die will, in the long run, give ace.

The method which we employ in the theory consists in calculating the number of all the cases or events concerning which our knowledge is equal.

Let us suppose that an event may happen in three ways and fail in two ways, and that all these ways are equally likely to occur. Clearly, in the long run, the event must happen three times and fail two times out of every five cases. The probability of its happening is therefore $\frac{3}{5}$, and of its failing, $\frac{2}{5}$. Thus the probability of an event is the ratio of the number of times in which the event occurs, in the long run, to the sum of the number of times in which the events of that description occur and in which they fail to occur.

An event must either happen or fail. Hence the sum of the probabilities of its happening or failing is certainty. We therefore represent certainty by unity.

The usual algebraic definition of probability is as follows. If an event may happen in a ways and fail in b ways, and all these ways are equally likely to occur, the probability of its happening is $\frac{a}{a+b}$, and the probability of its failing is $\frac{b}{a+b}$. (In mathematical works, the word "chance" is often used as synonymous with probability.)

It should be noticed that $\frac{a}{a+b} + \frac{b}{a+b} = 1$; also that $1 - \frac{a}{a+b} = \frac{b}{a+b}$. Thus, if p be the probability of the happening of an event, the probability of its not happening is $1 - p$.

When the probability of the happening of an event is to the probability of its failure as a is to b, the *odds* are said to be a to b *for* the event, or b to a *against* it, according as a is greater or less than b.

Suppose that 2 white, 3 black, and 4 red balls are thrown promiscuously into a bag, and a person draws out one of them, the probability that this will be a white ball is $\frac{2}{9}$, a black ball, $\frac{3}{9}$, and a red ball, $\frac{4}{9}$.

A few simple problems will help to illustrate the principles involved.

1. What is the probability of throwing 2 with an ordinary

die?—Any one face is as likely to be exposed as any other face; there are therefore one favourable and five unfavourable cases, all equally likely. The required probability is therefore $\frac{1}{6}$.

2. What is the probability of throwing a number greater than two with an ordinary die?—Obviously there are 4 possible favourable cases out of a total of 6. The probability is therefore $\frac{4}{6}$ or $\frac{2}{3}$.

3. A bag contains 5 white, 7 black, and 4 red balls. What is the probability that 3 balls drawn at random are all white?—We have 16 balls altogether. The total number of ways in which 3 balls can be drawn is therefore ${}^{16}C_3$, and the total number of ways in which 3 white balls can be drawn is ${}^{5}C_3$. Therefore, by definition, the probability is ${}^{5}C_3/{}^{16}C_3$, that is, $\frac{1}{56}$.

By a *compound event*, we mean an event which may be decomposed into two or more simpler events. Thus, the firing of a gun may be decomposed into pulling the trigger, the fall of the hammer, the explosion of the cartridge, &c. In this example, the simple events are *not independent*, because, if the trigger is pulled, the other events will, under proper conditions, necessarily follow, and their probabilities are therefore the same as that of the first event. Events are *independent* when the happening of the one does not render the other either more or less probable than before. Thus the death of a person is neither more nor less probable because the planet Mars happens to be visible. When the component events are independent, a simple rule can be given for calculating the probability of the compound event, thus: *Multiply together the fractions expressing the probabilities of the independent component events.*

If, for instance, A occur once in 6 times, its probability is $\frac{1}{6}$, or 1 for and 5 against; if B occur once in 10 times, its probability is $\frac{1}{10}$, or 1 for and 9 against. The probability, or relative frequency in the long run, of the concurrence of the two is $\frac{1}{60}$, that is, 1 for and 59 against.

The justification of the rule may be shown thus.—If

two dice are thrown, the side which the one shows uppermost has nothing to do with the side which the other shows uppermost; but each die has 6 sides, each of which may fall uppermost, and each of these may with equal possibility coincide with any one of the 6 sides of the other; there are thus 36 possible cases, and the probability of each single one of them is $\frac{1}{36}$ $(= \frac{1}{6} \times \frac{1}{6})$.

We may add one or two more problems.

1. What is the probability of throwing an ace in the first only of two successive throws of a single die?—Here we require a compound even to happen, namely, at the first throw the ace is to appear, at the second throw the ace is not to appear. The probability of the first simple event is $\frac{1}{6}$, and of the second $\frac{5}{6}$. Hence the required probability is $\frac{5}{36}$ $(= \frac{1}{6} \times \frac{5}{6})$.

2. A party of 23 persons take their seats at a round table. Show that it is 10 to 1 against two specified individuals sitting next to each other.—The probability that a given person A is on one side of a given person B is $\frac{1}{22}$; the probability that A is on the other side of B is also $\frac{1}{22}$; hence, the probability of A being next to B is $\frac{2}{22} = \frac{1}{11}$. Thus the odds are 10 to 1 against A and B sitting together.

3. Find the probability of throwing 8 with two dice.—With two dice, 8 can be made up of 2 and 6, 3 and 5, 4 and 4, 5 and 3, and 6 and 2, that is 5 ways. The total number of ways is 36. The probability is therefore $\frac{5}{36}$, and the odds 31 to 5 against.

4. A pack of 52 cards consisting of 4 suits is shuffled and dealt out to 4 players. What is the chance that the whole of a particular suit falls to a particular player?

$$\text{Chance} = \frac{(3n!)\ (n!)}{4n!} = \frac{1}{7 \cdot 10^{11}} \text{ approximately,}$$

i.e. 1 in something less than a billion

The Laws of Probability rest upon the fundamental principles of reasoning, and cannot be really negatived by any possible experience It might happen that a person should

always throw a coin head uppermost, and appear incapable of getting tail by chance. The theory would not be falsified because it contemplates the possibility of the most extreme runs of luck. But the probability of the occurrence of extreme runs of luck is excessively slight. Whenever we make any extensive series of trials, as in throwing a die or coin, the probability is great that the results will agree pretty nearly with the predictions yielded by theory. Precise agreement must not, of course, be expected, for that, as the theory shows, is highly improbable. Buffon caused a child to throw a coin many times in succession, and he obtained 1992 tails and 2048 heads. The same experiment performed by a pupil of De Morgan's resulted in 2044 tails to 2048 heads. In both cases the coincidence with theory is as close as could be expected. Jevons himself made an extensive series of experiments. He took 10 coins, and made 2048 throws in two sets of 1024 throws each. Obviously, the probability of obtaining 10, 9, 8, 7, &c., heads is proportional to the number of combinations of 10, 9, 8, 7, &c., things chosen from 10 things. The results may therefore be thus conveniently tabulated:

Character of Throw.	Theoretical Numbers.	First Series.	Second Series.	Average.	Divergence.
10 Heads, 0 Tails	$^{10}C_0 = 1$	3	1	2	+ 1
9 „ 1 „	$^{10}C_1 = 10$	12	23	$17\frac{1}{2}$	+ $7\frac{1}{2}$
8 „ 2 „	$^{10}C_2 = 45$	57	73	65	+20
7 „ 3 „	$^{10}C_3 = 120$	129	123	126	+ 6
6 „ 4 „	$^{10}C_4 = 210$	181	190	$185\frac{1}{2}$	−$24\frac{1}{2}$
5 „ 5 „	$^{10}C_5 = 252$	257	232	$244\frac{1}{2}$	− $7\frac{1}{2}$
4 „ 6 „	$^{10}C_6 = 210$	201	197	199	−11
3 „ 7 „	$^{10}C_7 = 120$	111	119	115	− 5
2 „ 8 „	$^{10}C_8 = 45$	52	50	51	+ 6
1 „ 9 „	$^{10}C_9 = 10$	21	15	18	+ 8
0 „ 10 „	$^{10}C_{10} = 1$	0	1	$\frac{1}{2}$	− $\frac{1}{2}$
	1024	1024	1024	1024	0

The present writer repeated the same series of experiments, with the following results:

Character of Throw.	Theoretical Numbers.	First Series.	Second Series.	Average.	Divergence.
10 Heads, 0 Tails	$^{10}C_0 = 1$	4	0	2	+ 1
9 ,, 1 ,,	$^{10}C_1 = 10$	20	6	13	+ 3
8 ,, 2 ,,	$^{10}C_2 = 45$	40	40	40	− 5
7 ,, 3 ,,	$^{10}C_3 = 120$	83	150	$116\frac{1}{2}$	$-3\frac{1}{2}$
6 ,, 4 ,,	$^{10}C_4 = 210$	224	222	223	+13
5 ,, 5 ,,	$^{10}C_5 = 252$	250	209	$229\frac{1}{2}$	$-22\frac{1}{2}$
4 ,, 6 ,,	$^{10}C_6 = 210$	242	222	232	+22
3 ,, 7 ,,	$^{10}C_7 = 120$	115	107	111	− 9
2 ,, 8 ,,	$^{10}C_8 = 45$	28	60	44	− 1
1 ,, 9 ,,	$^{10}C_9 = 10$	14	6	10	0
0 ,, 10 ,,	$^{10}C_{10} = 1$	4	2	3	+ 2
	1024	1024	1024	1024	0

The whole number of single throws of coins amounted to 2048 × 10, or 20,480 in all, one half of which, or 10,240, should theoretically give heads. The total number of heads obtained by Jevons was 10,352 (5130 in the first series, and 5222 in the second). The number obtained by the present writer was 10,234 (5098 in the first series, and 5136 in the second). The coincidence with theory is in each case fairly close.

Boys should be encouraged to repeat on a small scale a few experiments of this kind. Their interest is kindled when they find that a practical result closely approximates a theoretical estimate.

Correlation

Suppose that a group of measurements give us data about two variables, say (1) the weight, (2) the stature, of a number of men. Then we may not only ask questions with regard to the variation of weight, and questions with regard to the

variation of stature, but we may also raise the further question of *the connexion between the two*. A boy who is taller than another is not necessarily heavier, and yet there is undoubtedly *some* connexion between height and weight. This question of *correlation* in statistical theory is becoming one of rapidly increasing importance.

The existence of the connexion itself may, of course, be in question. Is a boy who is good at sports likely, in the long run, to be a duffer in the classroom? Some very mathematical pupils are, and some are not, musical. Some very musical pupils are, and some are not, mathematical. Is it possible to discover a definite measurement of the degree of connexion between two things whenever the things themselves are capable of trustworthy estimation?

One of the simplest methods of measuring correlation is Professor C. Spearman's foot-rule method; it is easily mastered in five minutes. Whatever work of this kind may be attempted with schoolboys, not only the Spearman coefficient, but the Bravais-Pearson coefficient, should be familiar to all teachers.

From the teaching point of view, Spearman's method possesses the advantage that original material for illustrating its use is always available in schools. Investigations of the correlation between the performance of a class in different subjects, in the same subject in different terms, in different examinations in the same subject, in school performances which are not all academic subjects, all these would give valuable information to the teacher. The use of the correlation coefficient as the measure of the " reliability " of an examination test is of special importance.

The teacher should consult the works of Professor Spearman, Professor Thorndike, Professor Karl Pearson, Dr. W. Brown, Udny Yule, and A. L. Bowley.

Statistics has become such a big subject that teachers may decide against its introduction into schools. But at the very least boys should be warned of the seriously faulty

inferences drawn from statistics by the imperfectly-trained student of economics. University degrees in Economics may now be obtained by students with only a superficial knowledge of mathematics. Need we therefore feel surprised at the absurd economic opinions now often expressed by some of our younger politicians? One of the commonest political fallacies is to impose a correlation on two utterly unrelated graphs, perhaps those concerning (i) foreign trade and (ii) the marriage rate, on the sole ground that the graphs show somewhat similar variations.

Mathematics teachers should warn their pupils that opinions based on statistics cannot be more than *probably* true; the degree of probability may be very great, but there can be no absolute certainty.

Statistics is beginning to occupy an important place in theoretical physics. Dirac says: " When an observation is made on any atomic system . . . in a given state, the result will not in general be determinate, i.e. if the experiment is repeated several times under identical conditions, several different results may be obtained. If the experiment is repeated a large number of times, it will be found that each particular result will be obtained a definite fraction of the total number of times, so that we can say there is a *definite probability* of its being obtained any time the experiment is performed. This probability the theory enables us to calculate. In special cases, the probability may be unity, and the result of the experiment is then quite determinate."—Instead of the accuracy and precision which until a short time ago we have always ascribed to nature, we seem to have nothing but uncertainty and randomness. Nature seems to know nothing whatever of *simple* mathematics. Virtually the present-day physicist seems to be immersed in the study of the statistics of electron " jumps ". We can foretell what will happen *in the long run* when we throw up coins, and apparently we can quite definitely forecast what will happen *in the long run* when we experiment with vast crowds of atoms and electrons. The laws of averages and of probability are entering more and more

into the physics of small-scale things. The 2000-year-old question of causation (determinism) presents itself anew.

CHAPTER XXXVII

Sixth Form Work

The Normal Programme for Specialists

The work done by Sixth Form specialists is almost always work in preparation either for University Scholarships or for the Higher Certificate. It has become stereotyped in scope, and much of it has been described as " deadly dull ". Inasmuch, however, as the University Authorities seem to require sent up to them boys who have been "well grounded", boys who are proficient in the use of those mathematical weapons which will make attack on the University Course immediate and effective, mathematical teachers appear to have no option but to make their boys face the necessary drudgery. If boys are actually going on to the University, perhaps that does not much matter. But if they are not, it is pretty safe to say that their mathematical interest will, as a rule, cease as soon as they leave school.

In 1904, a Committee of the Mathematical Association, consisting of 34 of the leading mathematical masters in the country, reported on " Advanced School Mathematics ". The committee took into account the different classes of boys who study advanced mathematics in schools, e.g. candidates for army examinations, science students, engineering students, and boys who intend to read mathematics at the University, and they framed a course of instruction which, they hoped, would prove suitable for all. The following is a summary:

1. *Algebra*: partial fractions, elementary manipulation with complex numbers and geometrical applications thereof, the theory of equations so far as it treats of the numerical solution of equations, the notation and easy properties of determinants, the simpler tests of convergency, and the binomial, exponential, and logarithmic series; but *excluding* the theory of numbers, probability, continued fractions, and advanced theorems on inequalities, on indeterminate equations, and on summation of series.

2. *Differential and Integral Calculus*: introduction, and a free use of the calculus in subsequent work.

3. *Trigonometry*: graphical illustrations of De Moivre's theorem, simple work in trigonometrical series and factors.

4. *Conic Sections*: a treatment of the elementary parts of the subject, in which either the geometrical or the analytical method is used, that method being used in each particular case which is most suitable for the problem under discussion.

5. *Solid Geometry*: the elementary geometry of the plane, cone, cylinder, sphere, and regular solids, including practical solid geometry.

6. *Dynamics*: an introduction to the dynamics of rotation (in two dimensions), viz. the motion of a rigid body round a fixed axis with uniformly accelerated angular velocity, together with other simple cases of the motions of rigid bodies.

The Committee urged the importance of " a more intimate union between the teaching of mathematics and science, whereby theoretical and practical work may be brought into relation with one another ".

In 1907, the committee outlined a special schedule of work suitable for boys preparing for Oxford and Cambridge scholarships.

1. *Pure Geometry.*—Geometry of straight lines, circles, and conics; inversion, cross-ratios, involution, homographic ranges, projection, reciprocation and principles of duality,

elementary solid geometry including plans and elevations of simple solids.

2. *Analytical Geometry.*—Straight lines and curves of the second degree; tangential co-ordinates. *Excluding* homogeneous co-ordinates, invariants, and analytical solid geometry.

3. *Algebra*, including elementary theory of equations. *Excluding* recurring continued fractions, harder tests of convergence, theory of numbers, and probability.

4. *Geometrical Trigonometry.—Excluding* spherical trigonometry.

5. *Analytical Trigonometry.*—Properties of circular, hyperbolic, exponential and logarithmic functions. *Excluding* the *proofs* of the infinite products for sine and cosine, and of the series of partial fractions for the other trigonometrical ratios.

6. *Calculus.*—Total and partial differentiation; Taylor's and Maclaurin's theorems; elementary integration; simple applications to plane curves (especially to such as are of intrinsic importance), to maxima and minima, to areas and volumes, and to dynamics; curve-tracing, not as a rule to scale. *Excluding* the theory (but not the use) of differential equations.

7. *Dynamics.*—Elementary statics, including simple graphical statics; elementary kinematics and kinetics, including motion of a rigid body about a fixed axis, and motion of cylinders and spheres in cases where the centre of gravity describes a straight line. *Excluding* hydrostatics and hydrodynamics.

In a suggestive article which appeared in the *Mathematical Gazette* for January, 1923, Mr. F. G. Hall asks for a much greater unification of subjects in Sixth Form work. He points out that the separate consideration of the different subjects involves a great waste of time. He says that, for instance, ratio and proportion are treated algebraically, geometrically, and trigonometrically; logarithms occur in every type of textbook; variation is considered from the

point of view of formal algebra, algebraic graphs, trigonometrical graphs, and the calculus. He then outlines a revised scheme, under seven headings:

1. Ratio in algebra, geometry, and trigonometry.
2. The inter-relation of trigonometry and geometry.
3. Variation; the general study of functionality.
4. The elements of the differential calculus and its application to algebra, geometry, and trigonometry.
5. Logarithms and their use in arithmetic, algebra, and trigonometry.
6. The elements of the integral calculus and its application to mensuration.
7. Elementary Analysis: (α) the important expansions of algebra —the binomial, exponential, and logarithmic theorems; (β) further trigonometry, with additional work on De Moivre's theorem and the expansions to which it leads; (γ) easy treatment of the following: Rolle's theorem and the First Mean-Value Theorem; the different substitutions employed to effect the important types of integration; successive differentiation; elementary differential equations.—The underlying ideas of this section are (i) the analytical study of " form " in pure mathematics, and (ii) the development of manipulative power to enable the study of Higher Mathematics to be undertaken when the pupil proceeds to a University course.

The whole article is worth reading; it makes fruitful suggestions for economy of effort and for the linking up of different subjects.

Professor Nunn, discussing studies of the kind usually found in textbooks on " higher algebra ", urges that such studies do not offer the most suitable course of instruction for the general body of Sixth Form pupils. " For the student who is to be a teacher or engineer, or to engage in higher industrial or administrative work, as well as for the student who is continuing his mathematical studies as part of a general education, the best course would seem to be one which sets in clear relief the central aims and most vital notions of the main branches of mathematics, supplements exposition with sufficient practical exercises to give the

student a real training and the sense of mastery that comes with training, and, in particular, illustrates vividly the essential part which mathematics plays in so many departments of modern life and activity."

Excellent advice. Would that it were more generally followed!

Simpler Fare for the Non-Specialists

It will be observed in the last paragraph that Professor Nunn was speaking of the " general body " of Sixth Form pupils, whereas both the Committee of the Mathematical Association and Mr. F. G. Hall had in view the small section who intend to specialize in mathematics. Now it is a fact, and a very regrettable one, that Sixth Form boys who are not specializing in mathematics and science, or at least the great majority of them, do no mathematics at all. It is for these boys that I wish to enter a plea. I do not ask for a supersession of the present type of specialist Sixth Form work; in practice such supersession is not possible. The Universities know what they want and schools have no option but to prepare the boys accordingly. But I do ask for the provision, for all sections alike of the Sixth Form, say for two periods a week (assuredly a modest allowance), for a course of mathematical instruction that shall be at once less bookish, less academic, more informal, more interesting, in short, mathematics of the by-ways rather than the sterner stuff so much beloved by the successful mathematician. I want all boys to leave school with a cultivated *interest* in mathematics. Some people are of opinion that this may be effected by making the proposed general Sixth Form course wholly recreational, but I would make the work more exacting than that. I believe it might continue to be as exacting as the work already done in the Fifth. Nevertheless, I would willingly sacrifice much of the formality of the subject in order to ensure a permanence of interest. The formal training I would willingly subordinate to interest and

knowledge, especially knowledge of unsuspected points of contact between mathematics and nature.

With this purpose in view I throw out a few hints in seven short chapters:

1. Harmonic Motion.
2. The Polyhedra.
3. Mathematics in Biology.
4. Proportion and Symmetry in Art.
5. Numbers; their unexpected Significance.
6. Time and the Calendar.
7. Mathematical Recreations.

CHAPTER XXXVIII

Harmonic Motion

The Ordinary Bookwork of S.H.M.

The work commonly done in connexion with Simple Harmonic Motion (which has already been touched upon in Chapter XXXI) usually begins in this way:

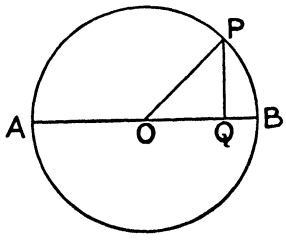


Fig. 289

Let P travel with uniform speed round the circumference of a circle. If Q is the foot of the perpendicular on the diameter AB, the motion of Q is an oscillation between A and B. Q is the projection of P on AB, and the motion of Q is *Simple Harmonic Motion.*

Thus we may define S.H.M. as the projection of a uniform circular motion on to a diameter of the circle.

Then usually follows a series of blackboard demonstrations:

Let the amplitude of the oscillation, i.e. the radius of the circle, be a.

Let μ be the square of the angular velocity of the point P about the centre O.

(i) Since the angular velocity of OP $= \sqrt{\mu}$,

$\therefore$ the linear velocity of P $= a\sqrt{\mu}$.

The *acceleration* of Q $= \mu.\mathrm{OQ}$ towards O;

$$\text{or,} \qquad \mu = \frac{\text{acceleration of Q}}{\mathrm{OQ}}$$

$$= \frac{\text{acceleration of Q}}{\text{displacement}}$$

(ii) The *period* of oscillation $= \dfrac{2\pi}{\sqrt{\mu}}$.

(iii) The *velocity* of the point Q (in any position) $= \sqrt{\mu(a^2 - \mathrm{OQ}^2)}$.

(iv) The period of a small oscillation of a simple pendulum $= 2\pi\sqrt{\dfrac{l}{g}}$.

(v) The length of a seconds pendulum $= l = \dfrac{g}{\pi^2}$.

This is all sound enough, but to many boys it is just an affair of algebra, with only the vaguest relation to practical life. The boys probably forget their practical exercises on loci in geometry when from models of linkages they learnt how a to-and-fro motion may be converted into a circular motion, or, if they remember, they do not associate the new work with the old. They are probably not made to realize that S.H.M. forms the basis of the investigation of most oscillatory movements that occur in nature, such as the small oscillation of a simple pendulum, the vibration of strings and other bodies emitting musical notes, the light vibrations in the æther, the vibrations producing Fletcher's trolley-wave; and so on. They have been studying S.H.M. in their mechanics and physics without knowing it, and now they are studying S.H.M. as something having only the vaguest relation to their mechanics and physics.

What is the use of playing about with the mere algebraic formula of simple harmonic motion before the motion has been studied and its full significance grasped in the related practical work in mechanics and physics? The laboratory, not the classroom, is the place for teaching S.H.M.

S.H.M. Experiments

Elaborate experiments are not necessary. Here are a few simple ones.

1. Let the boy stand facing a white wall, or a lantern screen, and at the back of the room let a strong light be placed at about the level of his head. His shadow will be cast on the wall or screen. Now let him whirl, in a horizontal plane, round his head a small heavy body attached to the end of a string about a yard long. After a little practice he can maintain his hand in almost the same position, and keep on whirling at a fairly constant velocity. As the small heavy body moves round his head, its shadow moves *to and fro* on the screen. There is a true projection of the circular motion, the shadow showing a true S.H.M. (A conical pendulum may also be used. Let the heavy body first hang vertically, then cause it to swing in such a way that the pendulum describes a conical surface. Note that the effective length of this pendulum is the vertical depth of the weight below the point of support.)

2. Let the boy continue to whirl the string-pendulum around his head; then set up an ordinary pendulum between him and the wall, and let it vibrate. The latter may be allowed to swing with rather more than " small " vibrations; these will not, it is true, be in strict S.H.M., but for a rough experiment that will not much matter. Now adjust the length of the string in the first experiment so that the to-and-fro movement of its shadow is about equal in length to the swing of the suspended pendulum, and so adjust the time of whirling that there is one complete revolution to a complete to-and-fro movement of the suspended pendulum. With

a little practice (it *needs* practice), the shadow of the heavy weight may be made to follow exactly the movement of the pendulum bob (except of course that the latter does not move quite in a horizontal plane). The experiment convinces the observer that the movement of the shadow is synchronous at all points with the movements of the pendulum bob.

3. Mount on a board a good sine curve as produced by a Fletcher's trolley running with uniform speed. Cut a slit in a piece of cardboard and arrange the cardboard so that

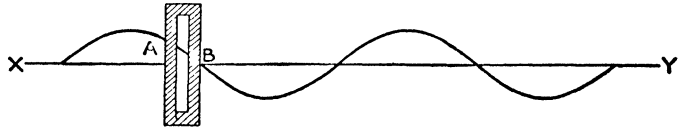


Fig. 290

its mid-line AB always coincides with the axis XY of the curve. Move the cardboard to and fro with uniform speed x. The bit of curve showing through the slit seems to move up and down with S.H.M.—A permanent model is worth making in the school workshop. A groove may be run along the board, coinciding with the XY axis, and at the back of the cardboard (a thin piece of wood is better) two bits of wood are fixed at AB, one on each side of the slit, to slide along the groove.—A better model may be devised by mounting the curve on a large wooden cylinder which is made to revolve, the cardboard with the groove being fixed medially in front of it. The resemblance of the bit of line moving up and down the groove to a pendulum movement is very striking.

4. Revise the laboratory experiments on the pendulum, and verify that the time of vibration of a pendulum varies as the square root of the length.

Lead the boys to see clearly that Fletcher's vibrating lath is really a pendulum, and the trolley simply a device for recording its vibrations pictorially; and that the model

in (3) above is, in its turn, another device for showing the lath vibrations *more slowly*, so that we can examine the way in which the lath really did move.

The term " harmonic " is justified because when the vibrations of the lath follow one another with sufficient rapidity, a definite musical note is heard which rises higher in pitch with increasing rapidity of the vibrations. So with vibrating things generally.

Compound Harmonic Motion

This subject is important in connexion with the study of sound. Lissajous curves are produced by compounding harmonic motions. The " compound pendulum " produces many of them in a simple and sufficiently effective manner. Here the mathematical master will be able to get help from the physics master.

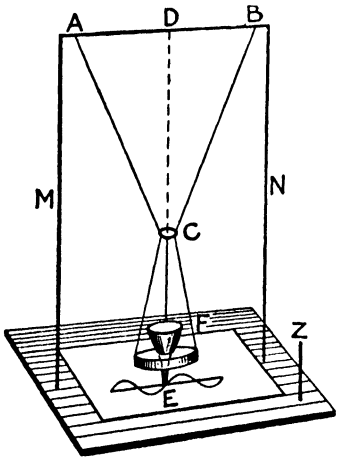


Fig. 291

A pendulum dropping sand affords the simplest means of illustrating the combined motions. A funnel F containing sand is suspended by two strings, passing through two small holes in a cork C, from two hooks A and B in the beam of the frame MN, and swings in S.H.M. at right angles to the frame; the sand dropped from the funnel makes a line on the paper below, also at right angles to the frame. If the paper is now steadily drawn across the baseboard with correct time adjustment, in a direction parallel to the frame, the sand will trace out a sine curve. Now draw the funnel aside so that it swings in the plane of the frame and the sand makes a line in that plane; the swing is again in S.H.M.;

slide the paper *across* the board, perpendicularly to the plane of the frame, and another sine curve results, but at right angles to the former. Finally, draw the funnel aside in a direction neither parallel nor perpendicular to AB (it might have been fastened back by a piece of cotton attached to the upright Z, fixed in a 45° position, and suddenly released by cutting or firing the cotton). The new motion combines the characteristics of the two former S.H.M.s, and the motion is said to be *compound* H.M. The precise form of the movement will depend (1) on the ratio of the two S.H.M.s, that is, the ratio of the lengths of the two really separate pendulums DE and CE, D and C being the respective points of support; and (2) the direction in which and the distance to which the funnel is drawn back. We may modify the first of these factors by sliding the cork C up and down the string; the length of the shorter pendulum will always be equal to CE. Of course the paper will not now be moved at all.

The pupils must understand clearly that if the pendulum swings at right angles to the frame MABN, its effective length is DE; that if it swings parallel to the frame its effective length is CE; that therefore by altering the position of C we may make the ratio of the two lengths any value we please.

Let the ratio be 1 : 4, e.g. let CE be 9″ and DE 36″. Set in motion by releasing from Z. The bob (the funnel) of the two pendulums cannot move in two directions at the same time and it therefore makes a compromise and follows a path compounded of the two directions. It traces over and over again fig. 292. Since the times of vibration vary as the square roots of the lengths of the pendulum, these times are as $\sqrt{1} : \sqrt{4}$, i.e. 1 : 2. Thus the short pendulum CE swings twice while the larger one DE swings once.

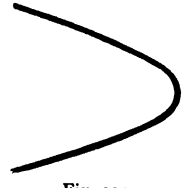


Fig. 292

If we wish the times of vibrations to be 2 : 3, that is if

we wish the larger pendulum to swing twice while the shorter swings three times), the ratio of the pendulum lengths must be $2^2 : 3^2$ or $4 : 9$, i.e. the shorter pendulum must be $4/9$ of the larger one. If the larger one DE is 36″, the shorter one CE must be 16″, and the cork C must be adjusted accordingly. The sand curve now traced, over and over again, is shown in fig. 293.

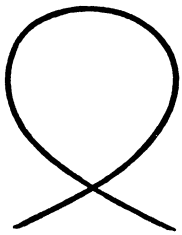


Fig. 293

Let the pupils try other simple ratios. Here is a short table of musical intervals formed by two notes which are produced by numbers of vibrations bearing to each other the same ratios as those given in the first column.

Frequency Ratio.	Pendulum Length Ratio.	Corresponding Musical Interval.
1 : 2	4 : 1	Octave (fig. 292)
2 : 3	9 : 4	Fifth (fig. 293)
3 : 4	16 : 9	Fourth
4 : 5	25 : 16	Major Third
5 : 6	36 : 25	Minor Third
3 : 5	25 : 9	Sixth

The new and more elaborate figures will interest the pupils, who must, however, realize that in all cases the movement actually executed is the resultant of two S.H.M.s perpendicular to each other.

In practice it is virtually impossible to set the two pendulums at the exact ratios given. The simple curves are therefore not maintained, but they open out and close again in a curious but regular movement. Here are examples of the changes seen in figs. 292 and 293, representing the octave and the fifth. But within a short time the curves are lost in the gradually spreading sand. The well-known successive phases of Lissajous figures, due to the tuning-forks not being in exact unison, are identical with the figures here shown, and, for all practical purposes, are produced in the

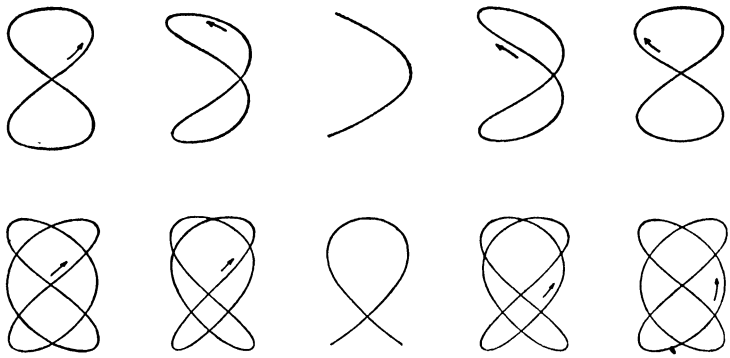


Fig. 294

same way. The boys should see them produced in the physical laboratory.

Obviously a better device than the swinging sand funnel is wanted for producing the figures. Pupils often suggest the substitution of a pencil or a pen for the sand, but of course this is not possible, inasmuch as the funnel does not move in a horizontal plane.

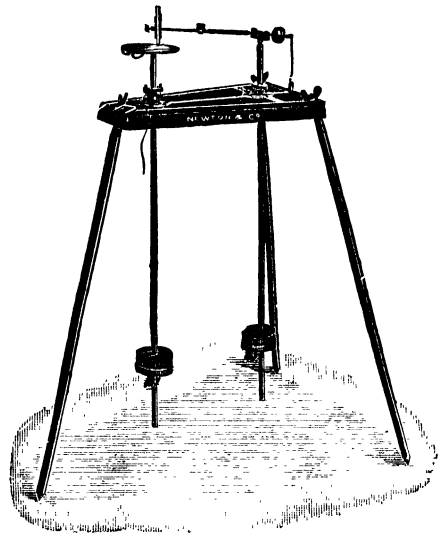


Fig. 295

This difficulty has been overcome by devising an entirely new type of compound pendulum.—Two pendulums are hung from a small wooden table supported on three legs, the pendulum rods passing through large holes in the table-top so that they can swing without touching it. To the top of one, a small table with paper is fixed so that it moves to

and fro while the pendulum swings. The top of the other carries a long rod which in turn carries a glass pen. Each pendulum can swing in one direction only, like a clock pendulum. Each pendulum swinging alone would simply describe a straight line of ink, again and again, until the motion died away. If the two pendulums are made to swing, not together, but in succession, there will be two straight lines at right angles to each other. But the two may be made to swing together, each to record whatever impulse may first be given to it, and so we get figures of the same type as the sand figures.

As the pen moves in a vertical plane, the surface of the receiving table top of the other pendulum is given the necessary slight cylindrical curvature, in order that the pen may always be just in contact with it.

We may vary as we please the impulses given to the pendulums. We may start them either at the same time, or one rather later than the other, say when the first has covered some definite fraction ($\frac{1}{2}$, $\frac{1}{4}$, or $\frac{3}{4}$) of its path. We may vary the length of either pendulum by raising or lowering the weights (bobs); the effective length of a pendulum is from the point of suspension to the centre of the bob.

Thus the figures produced may be varied greatly.

If the two pendulums are exactly equal, and if we start them swinging at exactly the same instant, we shall obtain one of these figures:

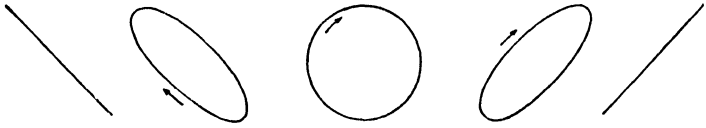


Fig. 296

These are really only different phases of the same figure, and they form the simplest group of Lissajous figures: all are ellipses, the straight line and the circle being merely particular cases. If the two are started at the same pace and at the same moment, the figure is a straight line; if the one

begins its movement when the other has already completed half its path, the figure is a circle; if the difference is greater or less than half a path, then an ellipse.

But although the pendulum continues to move, the pen does not continue to mark out *exactly* the same line, circle, or ellipse. We have to take friction into account. When the pen comes round to the point where, say, the circle began, it just misses it, and the second circle begins inside the first: really we get a spiral, ending at the centre of the paper when the pendulum comes to rest. By varying the friction, varying the ratio of the pendulums, and varying the impulses, we may obtain figures of extraordinary beauty. Many of these, exquisitely reproduced in colour, may be seen in Newton's *Harmonic Vibrations and Vibration Figures*. The more elaborate pendulums which have been designed for producing such figures (Benham's triple pendulum, for instance) are worth a careful examination for their mechanical ingenuity alone.

The mechanical difficulties associated with the compound pendulum (the harmonograph) are many, and only the exceptionally patient teacher is advised to purchase one. The feeding of the glass pen with a suitable ink, and the adjusting of the pen, are particularly teasing operations.

In two or three cases I have known the beautiful figures make a strong appeal to pupils who had previously professed their dislike of mathematics.

Various other instruments have been designed for producing curved designs mechanically. Fig. 297 shows two designs produced by the Epicycloidal Cutting Frame, but, generally speaking, the mathe-

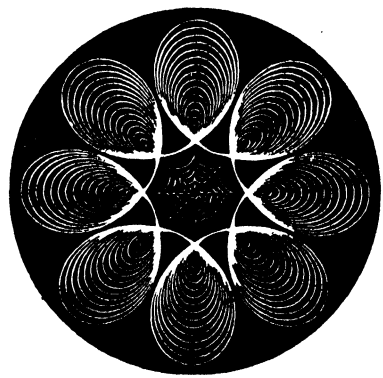


Fig. 297 (i)

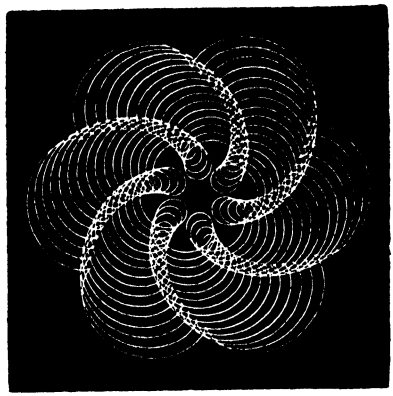


Fig. 297 (ii)

matics of such designs, though not advanced, is too tedious and elaborate to render its introduction into schools worth while.

Geometrical Construction of the Compound Curves

Let the pupils draw a few of the simpler figures geometrically.

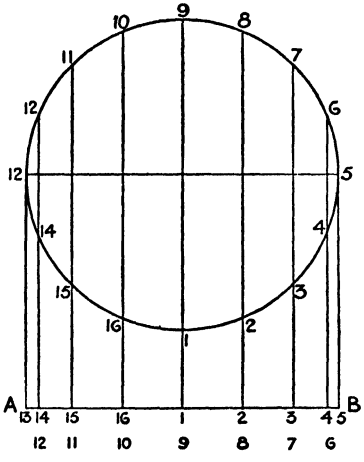


Fig. 298

They may refer again to the pendulum whirling round the head. Let a revolution take two seconds, so that the pendulum rotates through $\frac{1}{16}$ of the circle in $\frac{1}{8}$ of a second. To an observer watching the shadow on the screen, the bob appears to travel from 1 to 2 in the straight line, while it really travels from 1 to 2 in the circle. It appears to travel with greatest velocity at 1 and 9 in the straight line, and to be momentarily

at rest at 13 and 5; also to travel from A to B in 1 second, though in that time it has really travelled half-way round the circle. A comparison of the *unequal distances* covered in *equal times*, in AB, serves to show the varying velocity of a simple pendulum.

Remember that the *phase* of any point in S.H.M. is the fraction of a period that has elapsed since the point last passed through the position of maximum displacement.

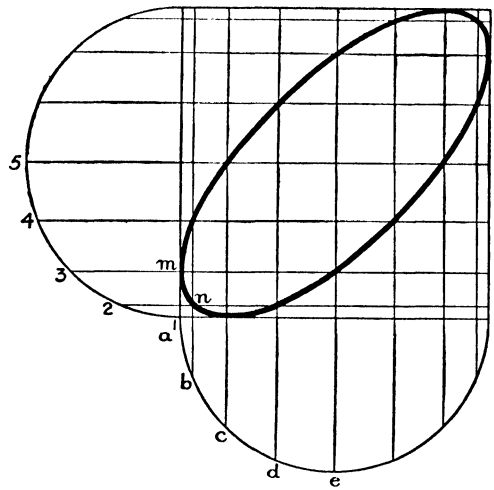


Fig. 299

Here is a figure (fig. 299: an ellipse) produced by two S.H.M.s of the same period but differing in phase by $\frac{1}{8}$ of a period. Semicircles suffice for obtaining the necessary projectors. The points for the ellipse are determined by the intersection of the two sets of projectors.

Projectors from *a* and 3 ($\frac{1}{8}$ of a phase apart) produce the point *m*.

Projectors from *b* and 2 ($\frac{1}{8}$ of a phase apart, $\frac{1}{16}$ in each circle) produce the point *n*. And so on. The curve follows the route marked out by the diagonals of successive parallelograms.

Boys who are poor draughtsmen sometimes make a hash of more complex figures, generally, however, because they have not grasped the main principle which is the same for all cases.

Here is another figure (fig. 300) showing the compounded motions of two pendulums whose lengths are as 16 : 25 and whose vibration ratio is therefore 4 : 5. Divide the semicircle BEC into 8 equal parts and the semicircle ADB into 10.

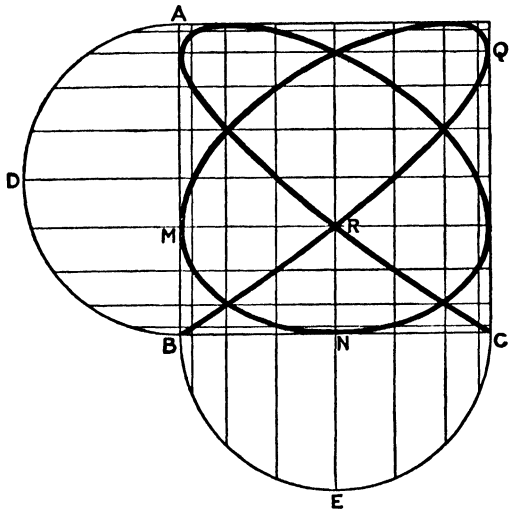


Fig. 300

Project on to the respective diameters, and continue across the square so that the projectors intersect as shown. The phases are intended to be the same, so that the two pendulums may be supposed to start together at B. The pendulum swinging towards A would reach M when the pendulum swinging towards C reached N—i.e. if the two were swinging independently; but inasmuch as they are moving together their motions are compounded; neither is able to take its own path, and the path actually followed is BR.

Observe that all the separate bits of line, horizontal and vertical, in the square, represent equal intervals of time.

Hence every point of intersection through which the curve passes is determined by compounding two distances representing equal intervals of time. But after the point Q, we have, of course, to begin to allow for a reversal of direction. The curve simply follows the route marked out by the diagonals of successive parallelograms.

Mathematically, the subject is hardly worth following very far, unless the pupils are seriously studying the subject of sound in physics. But there are a few exercises in Nunn's *Algebra* that all Sixth Form pupils might profitably work through.

Spirals

This subject of spirals, though outside the scope of the chapter, may be conveniently mentioned here.

A little work on spirals is worth doing, if only to emphasize the fact that there is a practical side to the notion that angles may have any magnitude. For instance, in *Archimedes'* spiral, $r = a\theta$; in the *Logarithmic* spiral, $r = a^{\theta}$; in the *Hyperbolic* spiral, $r\theta = a$; in the *Lituus*, $r^2\theta = a$. A little work on roulettes should also be done—cycloids, epicycloids, hypocycloids, but not trochoids. These are all best treated, for the most part, geometrically. Such peculiarities exhibited by curves as asymptotes, nodes, cusps, points of inflexion, &c., should be explained.

All these curves are full of interest; dwell on that side of them.

Spirals afford a good start for the study of vectors. In the Archimedean spiral, equal amounts of increase in the vectorial angle and radius vector accompany one another, i.e. if one is in A.P., so is the other. In the logarithmic spiral if the vectorial angles form an A.P., the corresponding radii form a G.P.

Books to consult:

1. Any standard work on *Sound*
2. *Harmonic Vibrations*, Martin.

CHAPTER XXXIX

The Polyhedra

Euclid, Book XIII

It is to be regretted that the displacement of Euclid has led to the abandonment of the greater part of the substance of his 13th book. It is true that the five regular solids are referred to in most modern textbooks of geometry, sketches of them given, "nets" for their development suggested, and perhaps Cauchy's proof of Euler's relation $E+2 = F+V$ worked out. But assuredly rather more than this ought to be done. The intimate relations of the five solids to each other, and the almost equally close relations between them and their many cousins, are so remarkable, and at the same time are so simple, that all boys ought to know something about them. The work forms a fitting completion to the course of solid geometry.

A regular polyhedron is one all of whose faces are equal and regular polygons, and all of whose vertices are exactly alike and lie on the circumscribed sphere. There are only 5, and the proof that there cannot be more is easy and should be provided. They are:

Name.	Number and Nature of Faces.	Sum of Plane ∠s at Vertex.	Remarks.
Tetrahedron	4 equilateral Δs	$60° \times 3 = 180°$	Found as natural crystals
Cube	6 squares	$90° \times 3 = 270°$	
Octahedron	8 equilateral Δs	$60° \times 4 = 240°$	
Dodecahedron	12 pentagons	$108° \times 3 = 324°$	Artificial
Icosahedron	20 equilateral Δs	$60° \times 5 = 300°$	

Closely related to the 3 natural polyhedra are 2 other natural solids, the *rhombic dodecahedron* (faces = 12 rhombuses), and the *trapezohedron* (faces = 24 kites). And closely related

to the 2 artificial polyhedra are 2 other artificial solids, the *triacontahedron* (faces = 30 rhombuses), and the *hexacontahedron* (faces = 60 kites). In all 9 cases, E + 2 = F + V.

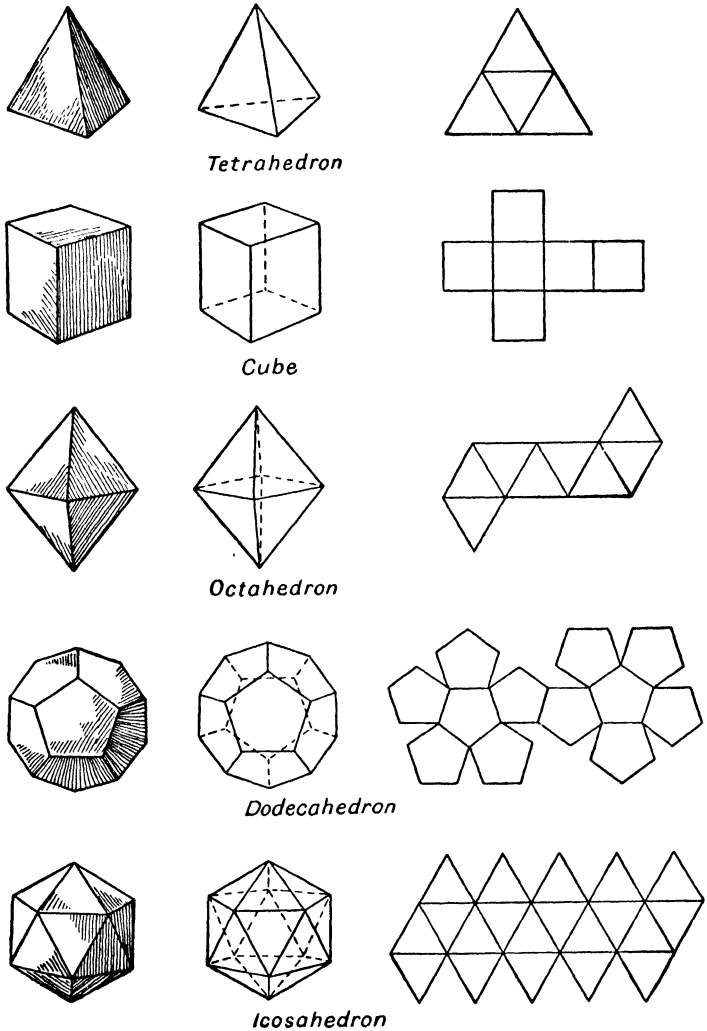


Fig. 301

On the previous page are sketches of the 5 polyhedra and nets for their construction. Third Form boys make them up readily enough (for practical hints, see *Lower and Middle Form Geometry*, pp. 202–3).

In my own teaching days, we taught the interrelations of all 9 solids, but in these days of high pressure, one has to be content with a few main facts about the 5 regular solids alone. For those who wish to construct models of the other four, the following dimensions of the respective rhombuses and kites may be useful (fig. 302):

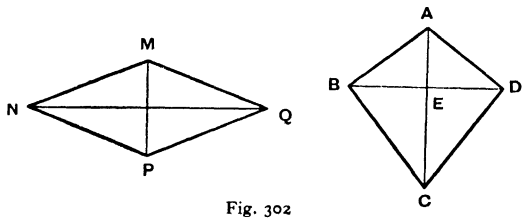


Fig. 302

Rhombic dodecahedron, $\frac{MP}{NQ} = \frac{1}{\sqrt{2}}$.

Triacontahedron, $\frac{MP}{NQ} = \frac{2}{\sqrt{5}+1}$.

Trapezohedron, $\frac{AC}{BD} = \frac{3\sqrt{3}}{4\sqrt{2}}$; $\frac{AE}{EC} = \frac{1}{2}$.

Hexacontahedron, $\frac{AC}{BD} = \frac{\sqrt{25+5\sqrt{5}}}{4\sqrt{2}}$; $\frac{AE}{EC} = \frac{2}{3+\sqrt{5}}$.

The key propositions to Euclid XIII are 13–18. The first five of these show how to construct the polyhedra, how to inscribe each in a sphere, and how to express the length of an edge in terms of the radius of the sphere. If E be the edge, and R be the radius of the sphere,

Proposition 13, tetrahedron, $E = \frac{2}{3}\sqrt{6}\,.\,R.$

„ 14, octahedron, $E = \sqrt{2}\,.\,R.$

„ 15, cube, $E = \frac{2}{3}\sqrt{3}\,.\,R.$

„ 16, icosahedron, $E = \frac{1}{5}\sqrt{10\sqrt{5}(\sqrt{5}-1)}\,.\,R.$

„ 17, dodecahedron, $E = \frac{\sqrt{3}}{3}(\sqrt{5}-1)R.$

Encourage the pupils to seek for interrelations amongst these. For instance, compare Propositions 15 and 17; in the latter, $E = \frac{1}{2}(\sqrt{5} - 1)$ of E in the former. Hence, if the edge of the inscribed cube be cut in golden section, the length of the greater segment is the length of the edge of the inscribed dodecahedron.

Euclid XIII, 18, is also useful.—To set out the edges of the five regular solids and to compare them with one another and with the radius of the circumscribing sphere.

Cutting One Polyhedron from Another

We shall refer to polyhedra "contained" within other polyhedra. The following volume relations should therefore be noted:

Containing Solid.	Contained Solid.		
	Octahedron.	Cube.	Tetrahedron.
Octahedron	—	9 : 2	27 : 2
Cube	6 : 1	—	3 : 1
Tetrahedron	2 : 1	9 : 1	—

The mutual relations of the two artificial solids are less simple. It may be noted that the ratio of the edges of an icosahedron and its contained dodecahedron is $6 : 1 + \sqrt{5}$.

A group of interesting problems are those of calculating the angles between edges, between faces, and between edges and faces, in the polyhedra. They are *good* problems, but some are a little tedious. In this connexion, the so-called 15th book of Euclid may be usefully consulted.

Practical demonstrations of polyhedral interrelations always appeal to boys. Encourage the boys to make their own models and to discover how easy it is to cut all sorts of regular sections through them. For instance:

1. A cube may be cut into two equal parts by means

of a section of the form of a regular hexagon; the section passes through the mid-points of 6 of the 12 edges of the cube.

2. A hexagonal section of an octahedron may be made similarly.

3. A hexagonal section of a dodecahedron, dividing this solid also into 2 similar halves, may be made by joining any 2 opposite angles of any pentagonal face (i.e. by drawing any one of the 5 diagonals of any face), and passing a plane through this join and the centre of the solid. The 6 edges of the hexagon are formed by 6 similar joins.

If such sections as the last are taken in all possible ways through a polyhedron, the intersection of the new planes will give clues to all sorts of interesting variations.

If time can be spared (e.g. after examinations, at the end of a Term), good preliminary work can be done in a Third or a Fourth Form in preparation for the more formal work in the Sixth. We may quote a few sections from *Lower and Middle Form Geometry*:

Transformations of Polyhedra

481. Any regular polyhedron may be cut from any other polyhedron, and the conversion is always a very simple matter. Suppose you cut a vertex (corner) from a cube, cutting the three edges back to the same extent; the section is an equil. Δ. Suppose you cut a vertex from an icosahedron, cutting all five edges back to the same extent; the section is a regular pentagon. Suppose you cut an edge from a cube, cutting back the faces symmetrically; the section is a rectangle. This is the sort of thing to be done in the following experiments. In one case, all the vertices will be cut off; in another, all the edges; always symmetrically and equally.

The best material to use is best yellow bar soap, from which you can easily cut cubes. It is better and cleaner than clay or plasticine. (If you are clever with carpenters' tools, a fine-grained wood is better still.) For cutting soap

a very thin-bladed knife is desirable. Do not attempt to cut off " chunks "; cut off shavings and do the cutting gradually.

482. **To cut a tetrahedron from a cube.**—A cube has eight vertices, a tetrahedron has four. The conversion is made by cutting away four of the eight vertices; the four new planes will be the four faces of the tetrahedron.

Draw the diag. AC of the top of the cube, and the opposite diag. FH in the base of the cube. These are two edges of the tetrahedron. Mark them by pushing into the soap three or four small pins in each line. Cut off the vertex D, and keep paring away until you reach the plane ACH, one of the faces of the tetrahedron. Scratch the letters A, C, and H on this new plane, or the model may become confusing. Now cut off the vertex G until you reach the plane CFH, a second face of the tetrahedron; now the vertex B, until you reach the plane AFC, a third face; lastly, cut off the vertex, E behind, until you reach the plane AFH, the fourth face. Now the cutting is completed. Note that the four vertices cut off are the four not concerned with the two diagonals first drawn. The two vertices at the end of each of these diagonals remain, and are the four vertices of the new tetrahedron.

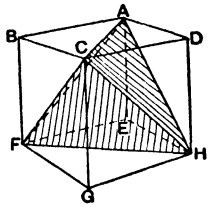


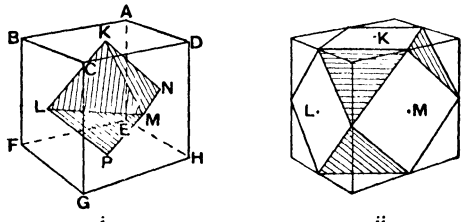
Fig. 303

The figure is a little difficult to follow, but the soap model itself will keep telling its own story, especially if the vertices are all lettered as in the figure.

483. **To cut an octahedron from a cube.**—This is done by cutting off the eight vertices of the cube, the eight new planes forming the eight faces of the octahedron.

Begin by marking in the six vertices of the octahedron. They are at " centres " (intersections of the diagonals) of the six faces of the cube. Show them by small pins thrust into the soap, K, L, M, N, P (and Q in face ABFE, not shown). Begin by cutting off the corner C symmetrically,

preserving the equil. Δ all the time, and pare down until you reach the plane made by the three pin-heads K, L, and M. KLM is one of the eight faces of the octahedron. Cut off the other seven corners of the cube similarly, and so obtain the other seven faces. Fig. 304 (ii) shows what the model would look like if the eight corners were cut down only to the mid-points of the edges. To make the octahedron, the cutting has to be continued to the centres



i ii

Fig. 304

of the faces, by which time the faces of the cube will have disappeared.

484. **To cut a dodecahedron from a cube.**—To construct a pentagon, Euclid used golden section. The faces of a dodecahedron consist of pentagons, and we shall use golden section for the necessary construction in this problem.

Golden section cuts a 2″ line into two parts, $\sqrt{5} - 1$ and $3 - \sqrt{5}$ inches long, i.e. 1·24 and ·76 inches, very nearly. A line 1″ long would be cut into parts of ·62 and ·38 inches, and a $3\frac{1}{2}$″ line into 2·17 and 1·33 inches. We will cut the dodecahedron from a $3\frac{1}{2}$″ cube. It is a convenient size for handling. If you cannot obtain a piece of soap or other material big enough, you must cut a smaller cube, but in the same proportion.

The twelve faces of the dodecahedron are the twelve new planes formed by cutting off the twelve edges of the cube. Hold a dodecahedron by two opposite edges between finger and thumb. These two edges may be regarded as medially placed in the top and bottom faces of the cube.

They are the middle parts of the medians of the faces of the cube. The four other edges that occupy medial positions in the other four faces of the cube can be easily traced as the model is thus held in the fingers.

The ratio of the length of the edge of the dodecahedron to the length of the edge of the containing cube is the ratio of the *shorter* section to the whole line in golden section.

In a $3\frac{1}{2}''$ line, the shorter section is 1·33 in. Call this

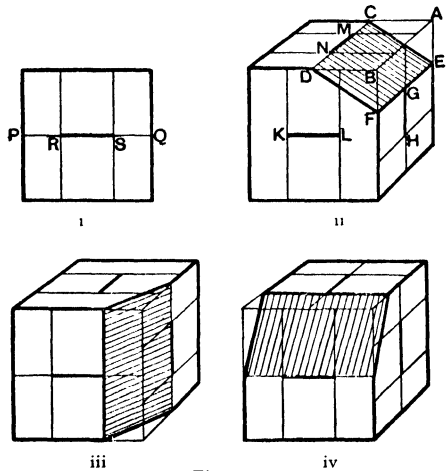


Fig. 305

$1\frac{1}{3}$ in. Place a line of this length centrally, as shown, in each of the six faces of the cube. (The three not shown in the figure—they are at the back and underneath—correspond in position to their opposite neighbours.) These six lines we call " medial " lines.

Fig. 305 (i) shows the construction for each square face. Since PQ $= 3\frac{1}{2}''$ and RS $= 1\frac{1}{3}''$, PR $=$ SQ $= 1\frac{1}{1\frac{1}{2}}''$.

The six medial lines of $1\frac{1}{3}''$ are six of the thirty edges of the dodecahedron. The other twenty-four edges will appear when the twelve new planes have been cut.

Mark the six medial lines by rows of three or four pins in each, one pin being placed at each end (G and H for

instance). These can be withdrawn as soon as the cuts nearly reach them, in order that the cuts may leave sharp edges.

Begin by cutting off the edge AB, cutting down to the median CD containing MN in the top face, and to the point G of the medial line GH in the side face. The new plane is shown, shaded. Obtain the other eleven planes in similar fashion, every plane extending from one medial to the next, line to point, as CD to G, the parallels CD and EF guiding the knife. Fig. 305 (iii) shows, separately, another edge cut off, and iv shows a third. As one new plane cuts through another, a part of the latter will disappear. Confusion will arise unless the points in the model are lettered to correspond with the letters in the figure.

It is best to cut off four parallel edges of the cube first (these four sections will be rectangles), then a second set of four parallel edges, then the third. The gradual formation of the dodecahedron will then be more clearly understood. The pentagons will appear as the last four planes are being cut.

485. **To cut an icosahedron from a cube.**—We might proceed as in the last case, first putting into medial position, in the six faces of the cube, the **longer** portion of a line divided in golden section, i.e. 2·17 in. in a cube of $3\frac{1}{2}$ in. edge. If an icosahedron be held by two opposite edges between finger and thumb, it is easy to see from the symmetry of the solid that these two and four other edges occupy medial positions in the six faces of the cube, exactly as in the last case. To obtain the twenty faces (equil. Δs) of the icosahedron, we have to cut off not only the twelve edges but also the eight vertices of the cube, the new planes by their intersections giving the faces. And all the cuts are determined by the medial lines in the six faces of the cube. But unless the edges, old and new, are carefully marked, the cutting is a complicated business. A simpler method is first to cut a dodecahedron from the cube (§ 484). To cut an icosahedron from a dodecahedron is then easy.

It is done by cutting off the twenty vertices of the

dodecahedron, continuing the cutting until the twenty new planes meet to form twenty equil. △s. Since each vertex of the dodecahedron is made up of three equal plane ∠s, an equil. △ is produced by cutting off a vertex symmetrically. Each cut must be continued until the new plane passes through the **centres * of the pentagons.** Hence, with twelve pins, mark the centres of the twelve pentagons; they show the positions of the twelve vertices of the icosahedron to be formed. If you begin by first carrying all the cuts to the mid-points of the edges, a very pretty solid is formed, consisting of equil. △s and small pentagons. But the cuts must be carried deeper, until they pass through the centres of the original pentagons.

486. **To cut a dodecahedron from an icosahedron.** —This case is almost identical with the last. Cut off a vertex of the icosahedron, symmetrically, and a regular pentagon results. Cut off all twelve vertices, and twelve pentagons result. Continue the cutting until each pentagon passes through the **centres * of the △s.** Hence, with twenty pins, begin by marking the centres of the twenty △s; they show the positions of the twenty vertices of the dodecahedron. If you begin by first carrying all the cuts to the mid-point of the edges, a pretty solid is formed, consisting of pentagons and small equil. △s. But the cuts must be carried deeper, until they pass through the centres of the original equil. △s.

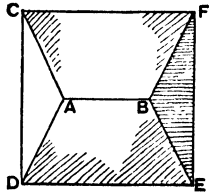


Fig. 306

487. **To cut a cube from a dodecahedron.**—Let AB be an edge of a dodecahedron, AC, AD two edges, radiating from A; BE, BF two radiating from B. If, in the solid, CD, DE, EF, FC are joined, a square is formed, and this is one of the faces of the contained cube. It is easy to mark off the other five faces, in a similar way, and then to cut out the cube.

Reductions of this kind lend themselves to useful geo-

* I.e. the intersection of the ⊥ bisectors of any two sides. In the case of the △, it coincides with the centroid.

metrical investigations by Sixth Form boys. Here, for instance, is an isometrical projection of a cube, showing all the construction lines for reducing the cube to an icosahedron. If the construction lines are based on the 3 initial lines ST, VW, XY, the angle at H (138·2°) should be checked by measurement; and vice versa. The advantage of the isometric projection is that all lengths parallel to the edges of the cube are true lengths. It is an excellent

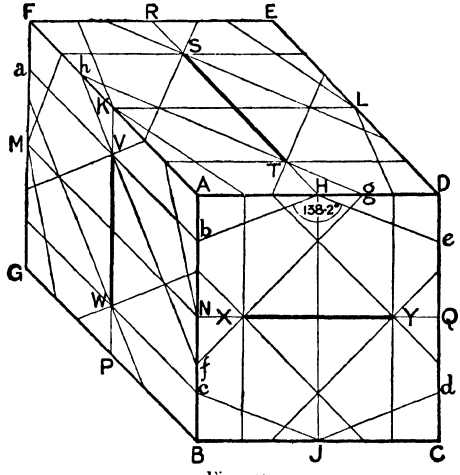


Fig. 307

exercise for boys to make cardboard models of the successive transition stages of reduction, e.g. the edge AF is first removed down to the plane R*ab*H; then the corresponding edges through B, C, and D. The face of the model with the 4 edges removed is then H*bcJde*. The other 2 groups of 4 edges each are treated similarly. Or, if it be decided to remove the 8 vertices first, symmetry will decide at once that, e.g., the plane cutting off the vertex A must pass through X, T, and V, and that $Af = Ag = Ah$. The real trouble comes when some of the edges or vertices have been cut away; the whole thing then seems to be " lost ". The important

thing is to keep the 3 initial lines ST, VW, XY in view all the time. When a model in soap is made, these initial lines may be marked at the outset by thrusting small pins into the soap, so that the heads act as guides all through the cutting process.

The following data are useful when effecting polyhedral transformations.

Points and Lines in these **containing** Polyhedra		become in the **contained** Polyhedra the following Points and Lines:
Tetrahedron	The 4 Cd's of the 4 F's	the 4 alt. V's of the **cube.**
	The 4 F's	4 of the 8 F's of the **octahedron.**
	The mid-pts. of the 6 E's	the 6 V's of the **octahedron.**
Cube	The 4 alt. V's	the 4 V's of the **tetrahedron.**
	The 6 D's, 1 in each F	the 6 E's of the **tetrahedron.**
	The 6 C's of the 6 F's	the 6 V's of the **octahedron.**
Octahedron.	The Cd's of the 8 F's	the 8 V's of the **cube.**
	The Cd's of the 4 alt. F's	the 4 V's of the **tetrahedron.**
Icosahedron.	The Cd's of the 20 F's	the 20 V's of the **dodecahedron.**
Dodecahedron.	The C. of G. of the 12 F's	the 12 V's of the **icosahedron.**

E = edge; F = face; V = vertex; C = centre; Cd = centroid; D = diagonal

A boy who can use the soldering iron may make some skeleton models in stout brass wire, fairly deep notches being cut with a file symmetrically at each vertex and elsewhere as required. Contained models may be exhibited in position by means of stout threads run continuously from vertex to vertex, from mid-point to mid-point, and so on. When the contained "solid" is completed, the thread is brought back to the first vertex and tied. The different contained models may be constructed in threads of different colours, and each is then easily distinguished from its neighbours.—We showed above how a cube might be cut from a dodecahedron: evidently 5 such cubes may be so cut, according to the edge we select to begin with. If all 5 cubes be threaded in a wire model of the dodecahedron, the resultant triacontahedron, with its 30 rhombuses, is effectively

shown. With care a 3rd polyhedron may be shown within a 2nd, and a 4th inside the 3rd, but girls are generally more expert than boys in handling and tying the threads.

The " centres " and centroids of the faces of the initial model may be shown by soldering into position diagonals or medians of very thin wire.

A useful present to a keen mathematical boy is Brückner's *Vielecke und Vielflache, Theorie und Geschichte* (Teubner). The many plates contain scores of photographs of beautiful models based on the polyhedra. The models are easily made in cardboard or stiff paper (I have known excellent specimens made by boys during the holidays), and the accompanying explanations and theoretical matter are well within the range of a Sixth Form boy. Here are reproductions of four of the photographs.

CHAPTER XL

Mathematics in Biology

General Ignorance of the Subject

There is a remarkable ignorance on the part of the average person in regard to the numerous matters of mathematical interest in botany and zoology. And not all mathematicians have interested themselves in this department of their subject, even though they may be perfectly familiar with the other branches of what they sometimes call " applied " mathematics.

The following topics are suggested for inclusion in the general mathematical course for Sixth Form non-specialists. For such boys only an elementary treatment will be possible, though the specialists, if they could spare the time, might carry the work much further. Each topic, animal locomotion,

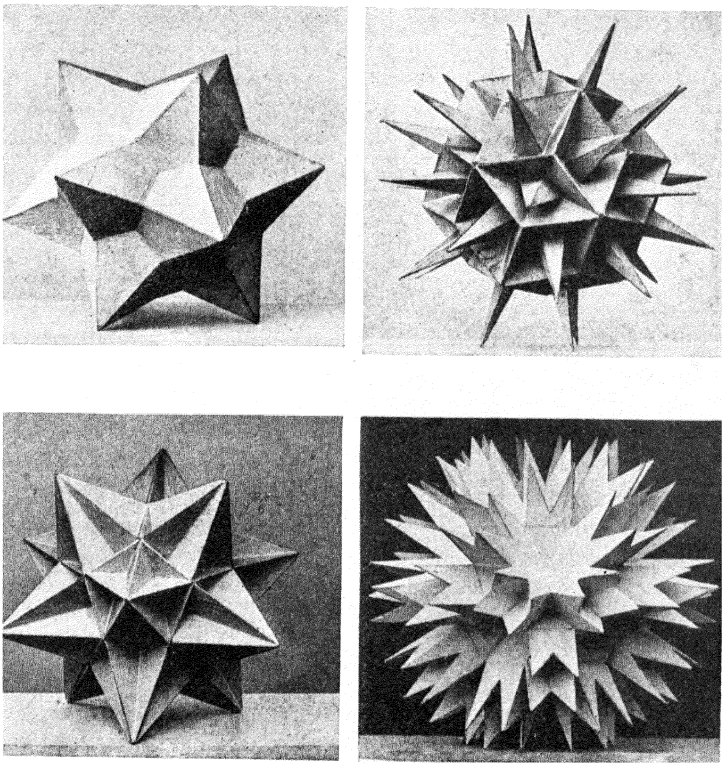


Fig. 308.—Polyhedral Models

[*Facing p.* 584

for instance, is really a very big subject, much too big for an exhaustive treatment. But for a boy to leave school entirely ignorant of the mathematical significance of the facts enumerated in the following paragraphs is a sad reflection on the narrowness of the course of mathematical work that schools commonly provide.

Biological Topics for Consideration

The principles of similitude in biological forms and structures.—Why there is necessarily a limit to the size of all plants and animals, as well as to all artificial structures. Stable equilibrium. How Eiffel secured an even distribution of strength in his tower by adopting the form of the logarithmic curve.

Mechanical efficiency.—The stream lines of a fish and the lesson to the naval architect; the stream lines of birds and the lesson to the student of aeronautics. The human skeleton from the engineer's point of view—compression and tension lines in the construction; ties (ligaments, tendons, muscles, &c.) and struts. The structure of a few of the principal bones of the body regarded as engineering units, and the mechanical distribution of compression and tensile stress. Compare the skeletal framework of a quadruped with a bridge supported by two piers: the quadruped is really an admirably jointed and flexible bridge. Strength and flexibility in aquatic animals. The remarkable strength of insects in proportion to their size.

Animal locomotion.—Locomotion on land, in air, in water. The wing regarded as a helix. The relation between the work which a bird does in moving itself forward and the linear dimensions of the wings. The minimum necessary speed in flight.

Rate of growth in the organic world.—Rate at different periods of "life"; its variability and its periodic retardation. The weight-length coefficient. (Growth graphs should receive special attention.)

Internal forms of organic cells.—Fields of force; their form; polarity. Effects of surface tension on cell division. Liquid films: minimal surfaces and figures of equilibrium. Plateau's experiment. Spiders' webs. Forms of globules, hanging drops, splashes. Unduloids in the infusoria (e.g. vorticella); fluted and pleated cells.

Cell aggregates.—Surfaces in contact; cell partitions; why partitions between cells of equal size are plane, and between cells of unequal size are curved. Tetrahedral symmetry; hexagonal symmetry. The geometry of the bee's cell and of bee-cell architecture. Minimal areas in nature's partitioning of space.

Spicules and spicular skeletons.—Concentric striation in nature. The fish's age as estimated by the concentric lamellation of its scales; compare with the concentric rings in the trunk of a tree. The skeletons of sponges. The radiolarian skeleton.

Geodesics.—The helicoid geodesic on cylindrical structures and its purpose; how " stretching tight " and constricting are effected by fibres arranged in geodesic fashion. The spiral coil in the tracheal tubes of an insect; the tracheides of a woody stem.

The logarithmic spiral.—Difference between spirals and helices. The curves of the horns of ruminants, of molluscan shells, of animals' tails, of the elephant's trunk. The properties of the logarithmic spiral in its dynamic aspect. Explain clearly why the molluscan shell, like the creature inside it, grows in size but does not change in shape: this constant similarity of form is *the* characteristic of the logarithmic spiral. The study of shells generally, morphologically, and mathematically. The spiral shells of the foraminifera. Torsion in the horns of sheep and goats. The deer's antlers. The curvature of beaks and claws.

Phyllotaxis.—Spirals. Symmetry.

Shapes of eggs.—An egg, just prior to the formation of its shell, is a fluid body, tending to a spherical shape, enclosed in a membrane. The problem of the shape of the

egg: given a practically incompressible fluid, contained in a deformable capsule which is either entirely inextensible or only very slightly extensible, and which is placed in a long elastic tube the walls of which are radially contractile: to determine the shape of the egg under pressure. At all points the shape is determined by the law of distribution of radial pressure within the oviduct; the egg will be compressed in the middle, and will tend more or less to the form of a cylinder with spherical ends. From the nature and direction of the peristaltic wave of the oviduct, the pressure will be greatest somewhere behind the middle of the egg; in other words, the tube will be converted for the time being into a more conical form, and the simple result follows that the anterior end of the egg becomes broader and the posterior end the narrower. The mathematical statement of the case is simple.

Comparison of related biological forms.—This is a very large subject, and applies to the whole region of biological morphology. Basically, it consists of the transformation of a system of co-ordinates and a comparative study of the original and transformed figures (wing, leg, bone, skull, or what not) in the co-ordinate system. The new figure in the transformed system shows the old figure under strain. The new figure is a function of the new co-ordinates in precisely the same way as the old figure is of the original co-ordinates.

The reader should examine figs. 404, 405, and 406 in *Growth and Form* (see below), where (i) the outlines of a human skull are enclosed in a co-ordinate system of squares; (ii) the outlines of a chimpanzee's skull are enclosed in another system determined by points exactly corresponding to the intersecting points in the first system. The new co-ordinate system though consisting of curved lines is of a strikingly regular type, and obviously bears a very simple mathematical relation to the first system. It is for the biologist to trace the transformed co-ordinates, for the mathematician to step in and show the relations between the new and the old co-ordinate systems, and then for the biologist to come in again and explain the relations—if he

can. That the relations are simple, and that they are continuous, are obvious. The logarithmic curve seems to make its appearance even once more. (See D'Arcy W. Thompson Chapter XVII.)

The best modern work on the whole subject is Professor D'Arcy Thompson's *Growth and Form.* Professor J. Bell Pettigrew's *Design in Nature* should also be consulted; it is a remarkable three-volume work with a multitude of useful facts and many hundreds of illustrations, but some of its opinions do not meet with general acceptance.

CHAPTER XLI

Proportion and Symmetry in Art

Proportion, Harmony, and Symmetry

Art is another subject which mathematics gathers into its ambit, though the mere suggestion is enough to stir the average artist to anger. But what about perspective? The subject is generally taught by art teachers as if it consisted of a number of incomprehensible stereotyped rules. And what about the geometry underlying design?

But, after all, these things are comparative trifles. Underlying a great deal of what counts for art is a mathematical foundation quite unsuspected even by many mathematicians.

The subject is much too far-reaching for more than a few references to it to be made here.

Those qualities in the general disposition of the parts of a building that are calculated to give pleasure to the observer are *proportion*, *harmony*, and *symmetry*. In the dimensions of a building, proportion itself depends essentially upon the employment of very simple mathematical ratios. Proportions such as those of an exact cube, or two cubes

placed side by side, or dimensions increasing by one-half (e.g. a room 20 ft. high, 30 ft. wide, 45 ft. long), please the eye far more than do dimensions taken at random. The great Gothic architects appear to have been guided in their designs by proportions based on the equilateral triangle.

By harmony is meant the general balancing of the several parts of the design. It is proportion applied to the mutual relations of the details. By symmetry is meant general uniformity in plan.

Accurate measurements have been made of the Parthenon and of several of the great cathedrals, and the unvarying simplicity of the mathematical ratios determining the various proportions is a very impressive fact. The same thing applies to natural objects. In particular it applies to the human figure. An artist does not, of course, measure up a model before making a selection, but his eye tells him at once if the proportions are satisfactory. If the human body approaches anything like perfection, from the crown of the head to the thigh joint is one-half the whole height; from the thigh-joint to the knee-joint, from the knee-joint to the heel, and from the elbow-joint to the end of the longest finger, are each one-fourth of the whole height; from the elbow-joint to the shoulder is one-fifth; from the crown of the head to the point of the chin is one-eighth. The proportions of a perfect face are even more remarkable; the ratios of the distances between the various facial organs, and of the lengths and widths of the organs, are singularly simple throughout.

In great architecture, even more remarkable than the linear measurements is the simplicity of the forms of the various rectilinear and curvilinear spaces. If we analyse a drawing of, say, the east front of Lincoln Cathedral, we can discover a series of striking relations amongst the parts. First enclose it in a rectangle, and then draw the bisecting vertical line; from the upper end of that vertical, draw 7 pairs of oblique lines, right and left, to the base and sides of the rectangle, as follows:

1. Lines (to the angles) which determine both the width of the design, the tops of the aisle windows, and the bases of the pediments on the inner buttresses.
2. Lines which determine the outer buttress.
3. Lines which determine the width of the great centre window.
4. Lines which determine the form of the pediment of the centre.
5. Lines which determine the form of the pediments of the smaller gables.
6. Lines which determine the height of the outer buttress.
7. Lines which determine the height of the inner buttresses

It will be found that (α) these lines determine the heights and widths of nearly all the main features of the design, and (β) the angles which the obliques make with the horizontal are all simple fractions of a right angle. Were the architect to depart from these simple ratios very appreciably, the eye would be offended; the trained eye would resent even a very small departure.

Ratio Simplicity

The key to the harmony of beauty in its more general sense seems to be the simplicity of the angle relations which determine or which underline the form of the thing considered beautiful.

It is sometimes said that there are three primary "orders" of symmetry, viz. those based on the numbers 2, 3, and 5, respectively. That of the *first* order is represented by the half square cut off by a diagonal, that is, the right-angled isosceles triangle, with angles 45°, 45°, 90° (angles 1 : 1 : 2); that of the *second* order is represented by the half equilateral triangle cut off by a median, that is, a triangle with angles 30°, 60°, 90° (angles 1 : 2 : 3); that of the *third* order is

represented by the half triangle from a pentagram cut off by a median from the vertex, that is a triangle of 18°, 72°, 90° (angles 1 : 4 : 5). Thus we may easily obtain angle ratios 1/1, 1/2; 1/2, 1/3, 2/3; 1/4, 1/5, 4/5; and these ratios, alone or compounded in a simple way, are architecturally fundamental.

An ellipse, a figure which enters very largely into architectural composition, may be constructed with its principal diameters of *any* ratio, but the ellipses which are acceptable for purposes of symmetrical beauty are those based upon the simplicity of the ratio of the angles made by the diagonal with the sides of the rectangle which encloses the ellipse. This ratio is usually one of those mentioned in the preceding paragraph.

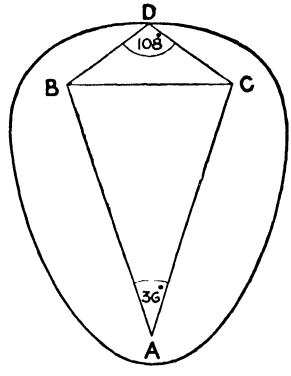


Fig. 309

But a much more subtle curve is the " oval ", better called a " composite ellipse ", since its axis is sometimes so short that the oval ceases to resemble an egg. This curve has 3 foci (A, B, C in figure), forming an isosceles triangle. If from the ends of the base of this triangle lines be drawn to that end of the axis, D, at the " flatter " end of the curve, they make an angle which (for the purposes under consideration) must have a very simple relation to the angle at the apex of the isosceles triangle (in the figure, 3 : 1). If this simplicity of ratio is departed from, the curve is not acceptable as an element of harmonious proportions.

It is a remarkable thing that a form is considered beautiful when the space which it encloses can be analysed in such a way that the resulting angles bear proportions to each other analogous to those which subsist among musical notes. The basis of musical harmony is that, when two sounds mingle agreeably, the numbers of vibrations of which they

are respectively composed bear a very simple ratio to each other. All the harmonies are represented by quite simple fractions, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c.

Some things to look for in objects considered beautiful

Boys should be encouraged to take an interest in the proportions, harmony, and symmetry of beautiful buildings and other objects. I have known many pupils who claimed to be beauty-blind, really awakened to a new life once they knew *what to look for* when examining a thing considered beautiful—a building, a piece of statuary, a picture, a vase, a piece of Gothic ornament.

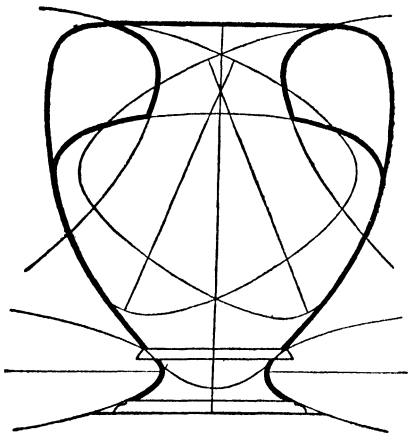


Fig. 310

Search for the beauty of form of a Greek or Etruscan vase: it does not take much finding. Stand in front, or behind, or at the side, of the Venus of Milo in the Louvre; the particular view matters little. The extraordinary beauty of the curves of the figure, despite their complexity, imposes itself upon the mathematician whether he will or not. To the artist the figure is beautiful for reasons which, though adequate, he finds a little difficult to explain, or at least difficult to analyse. The mathematician's approach is an entirely different approach, but the approach is intensely interesting; he desires to discover the secret of the artist's construction, and he sets to work to analyse. Once the artist's secret stands revealed, his first feeling is one of admiration for such subtle craftsmanship. The beauty of the thing gradually

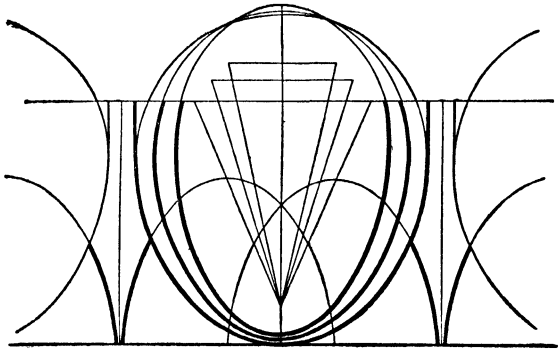


Fig. 311

grows upon him. The feeling is as much intellectual as it is emotional, and for that reason it is sometimes of a higher order than a feeling for beauty that may be emotional only. The boy who is a failure in the school studio can nevertheless be taught what to try to search for in a thing that the world calls beautiful.

We give figures of (1) an Etruscan vase showing its component curves and their tangential relations; (2) the construction of the echinus moulding in Greek architecture (note the 3 composite ellipses); (3) the outline of the human figure, showing nature's subtle construction of the sides of the head, neck, trunk, and outer surfaces of the legs. Note the tangency throughout. In an analysis of a perfect human form, these tangential relations seem to persist to the smallest detail. (Figs. 310, 311, 312.)

If one of the ends of life is the pursuit of beauty, then mathematics, properly understood, is one of the avenues we should follow. Consider even Einstein's work; what is its main value? that, underlying the diverse

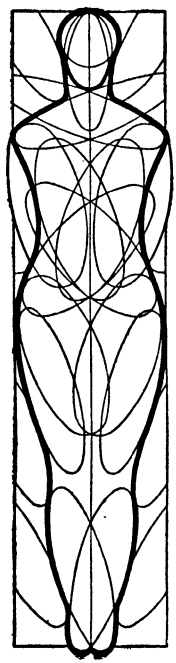


Fig. 312

phenomena of the natural world, there has been discovered a harmony more all-embracing than any ever before dreamed of.

I do not know if it may be adequately maintained that harmony is the most essential factor in beauty, but assuredly it is the desire for harmony that animates the modern searcher after the secret of the ordered relations of the universe, just as it animated the Greeks in their star-gazing and their geometry.

I find it very hard to distinguish the passion for truth from the quest of beauty. Certainly we need never despair of the beauty-blind boy if he is taught mathematics as it might be taught.

CHAPTER XLII

Numbers: Their Unexpected Relations

The Theory of Numbers

One noteworthy subject which is lacking in the equipment of many of our younger mathematical teachers is the theory of numbers. Forty or fifty years ago the subject was included as a matter of course in Sixth Form mathematics. In those happy days the mathematical work was not narrowed down to the requirements of the few boys who were going to read mathematics at the University. The ground then covered was more extensive, and in many ways was more interesting. Some schools devoted considerable attention to the theory of numbers (as the subject is called, though not very happily), such topics being included as the theory of perfect, amicable, and polygonal numbers; properties of prime numbers; possible and impossible forms of square numbers, of cubes, and of higher powers; the quadratic

forms of prime numbers; scales of notation; indeterminate equations; diophantine problems. The magic square and magic cube were also included. Altogether, the boys were given an interest in arithmetic and algebra that remained a permanent possession in after days, and was rarely forgotten.

A final blow was given to this branch of work when the fiat went forth that circulating decimals being useless and unpractical, their use must be abandoned. The futility of circulating decimals in the solution of practical problems may be granted. But if we ignore them altogether, we cut off from the learner some of the most striking properties of numbers; in fact we deny him most of the inner significance of numerical relations.

I plead for a revival of some of this work, and therefore indicate a few topics that may be included.

Within the last year or two, a highly competent young mathematical mistress told me that she had made what she considered to be a rather striking arithmetical discovery. Had I come across it before?

This is what she showed me, and then she pointed out a

$$\begin{aligned}
\tfrac{1}{7} &= \cdot\dot{1}42857\dot{} \\
\tfrac{2}{7} &= \cdot\dot{2}85714\dot{} \\
\tfrac{3}{7} &= \cdot\dot{4}28571\dot{} \\
\tfrac{4}{7} &= \cdot\dot{5}71428\dot{} \\
\tfrac{5}{7} &= \cdot\dot{7}14285\dot{} \\
\tfrac{6}{7} &= \cdot\dot{8}57142\dot{}
\end{aligned}$$

few of the well-known properties of this particular grouping: the same 6 figures in the same order in all 6 cases; the 1st group a factor of all the others; the 2nd group a factor of the 4th and 6th; the 5th group the sum of the 2nd and 3rd; and so forth. She said quite seriously that although she had, as a girl, won an open scholarship, she had *never* during her school days seen the completed decimal for any one of these half-dozen simple fractions.

I pointed out that she had not made up the group to the best advantage, and I modified it thus:

	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	$\frac{4}{7}$	$\frac{5}{7}$
$\frac{1}{7}$	1	4	2	8	5	7
$\frac{3}{7}$	4	2	8	5	7	1
$\frac{2}{7}$	2	8	5	7	1	4
$\frac{6}{7}$	8	5	7	1	4	2
$\frac{4}{7}$	5	7	1	4	2	8
$\frac{5}{7}$	7	1	4	2	8	5

The values of the 6 vulgar fractions may now be read off either from left to right *or* from top to bottom. Moreover, the diagonal lines running upwards from left to right consist each of the same figure.

Even then she could scarcely credit my statement that the same principle applied to all circulating decimals whatsoever. I suggested she should evaluate the 40 fractions $\frac{1}{41}, \frac{2}{41}, \frac{3}{41}, \ldots \frac{40}{41}$, which she did, and made the discovery that the 40 circulating decimals fell into 8 groups of 5 figures, all presenting the self-same symmetry.

$\frac{1}{41} = \cdot\dot{0}2439\dot{0}$	$\frac{2}{41} = \cdot\dot{0}4878\dot{0}$	$\frac{3}{41} = \cdot\dot{0}7317\dot{0}$	$\frac{4}{41} = \cdot\dot{0}9756\dot{0}$
$\frac{10}{41} = \cdot 24390$	$\frac{20}{41} = \cdot 48780$	$\frac{7}{41} = \cdot\dot{1}7073$	$\frac{23}{41} = \cdot\dot{5}6097$
$\frac{16}{41} = \cdot 39024$	$\frac{32}{41} = \cdot 78048$	$\frac{13}{41} = \cdot 31707$	$\frac{25}{41} = \cdot 60975$
$\frac{18}{41} = \cdot 43902$	$\frac{33}{41} = \cdot 80487$	$\frac{29}{41} = \cdot 70731$	$\frac{31}{41} = \cdot 75609$
$\frac{37}{41} = \cdot 90243$	$\frac{36}{41} = \cdot 87804$	$\frac{30}{41} = \cdot 73170$	$\frac{40}{41} = \cdot\dot{9}7560$
$\frac{5}{41} = \cdot\dot{1}2195\dot{0}$	$\frac{6}{41} = \cdot\dot{1}4634\dot{0}$	$\frac{11}{41} = \cdot\dot{2}6829\dot{0}$	$\frac{15}{41} = \cdot\dot{3}6585\dot{0}$
$\frac{8}{41} = \cdot 19512$	$\frac{14}{41} = \cdot 34146$	$\frac{12}{41} = \cdot 29268$	$\frac{22}{41} = \cdot\dot{5}3658$
$\frac{9}{41} = \cdot 21951$	$\frac{17}{41} = \cdot 41463$	$\frac{28}{41} = \cdot 68292$	$\frac{24}{41} = \cdot\dot{5}8536$
$\frac{21}{41} = \cdot\dot{5}1219$	$\frac{19}{41} = \cdot 46341$	$\frac{34}{41} = \cdot 82926$	$\frac{27}{41} = \cdot 65853$
$\frac{39}{41} = \cdot 95121$	$\frac{26}{41} = \cdot 63414$	$\frac{38}{41} = \cdot 92682$	$\frac{35}{41} = \cdot\dot{8}5365$

Make the boys evaluate these or other similar groups, and encourage them to search for the curious (though obviously necessary) relations between the members of each group and between group and group. For instance, the first decimals of the above 6 groups are 1 : 2 : 3 : 4 : 5 : 6; the sum of the 1st decimals of the 5th and 6th groups is equal to the first decimal of the 7th group; and so on almost indefinitely. An

examination of the numerators of the vulgar fractions gives the clue to an almost endless number of relations amongst the 40 decimal groups. Clearly all 40 decimals can be evaluated in 5 minutes; *only one* actual division is necessary, viz. that for $\frac{1}{41}$.—This sort of thing, which applies universally, was the A B C of Upper Form arithmetic half a century ago.

Suggested Topics

1. *Primes and composite numbers*; measures and multiples; tests of divisibility (with algebraic proofs). Familiarity with the factors of such common numbers as 1001 ($= 7 \times 11 \times 13$), of 999 ($= 27 \times 37$), in order to write down at once the factors of such numbers as 702,702 and 555,888. Eratosthenes' sieve. Fermat's theorem. The number of factors in a composite number; the number of ways in which a composite number may be resolved into factors.

2. *Perfect numbers.*—(A perfect number is one which is equal to the sum of all its divisors, unity included). $N = 2^{n-1}(2^n - 1)$, the bracketed factor being prime. Examples: 28, 496, 8128.

3. *Amicable numbers.*—(Amicable numbers are pairs of numbers, each member of a pair being equal to the sum of all the divisors of the other number.) Examples: 220 and 284; 18,416 and 17,296. The formulæ are rather long though easy to manipulate

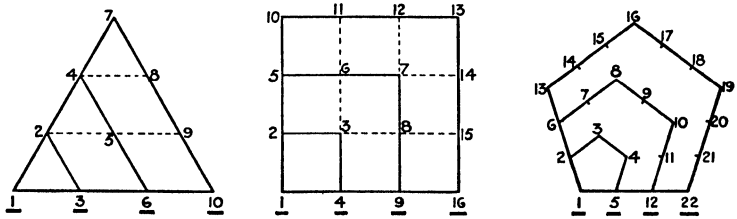


Fig. 313

4. *Polygonal numbers.*—Teach the pupils (1) to sum triangular, square, pentagonal, and hexagonal numbers, and

(2) to find the nth term of each series. All are easy and are full of interest. The necessary figures may be drawn readily. Pascal's arithmetical triangle.

5. *Scales of notation.*—Illustrate by some of the mediæval problems on age-telling cards; weighing with a minimum number of weights, e.g. binary scale weights 1, 2, 2^2, 2^3, &c., for one pan; ternary scale weights 1, 3, 3^2, 3^3, &c., for either or both pans; &c.

6. *Congruences.*—Use Gauss's notation $a \equiv b \pmod{m}$; e.g. $15 \equiv 8 \pmod{7}$, $36 \equiv 0 \pmod{12}$, $37 \equiv 19 \pmod{6}$. Emphasize the fact that a modulus is a *divisor*. The numbers 15 and 8 are *congruent*, or they *agree*, for the modulus 7, because they agree as regards the divisor 7; they "agree" in giving *the same remainder*, 1. The theory of congruences is necessary for a proper understanding of parts of the theory of numbers. Very little practice in a few of the more elementary theorems of congruences is required in order to give necessary facility in subsequent work; e.g.

(1) $72 \equiv 37 \equiv 30 \equiv 9 \equiv 2 \equiv -5 \equiv -12 \equiv -33 \pmod{7}$.

(2) $100 \equiv 15 \pmod{17}$.

i.e. $10^2 \equiv 15 \pmod{17}$,

$\therefore\ 10^4 \equiv 15^2 \pmod{17}$,

$\therefore\ 10^4 \equiv 225 \pmod{17}$,

$\therefore\ 10^4 \equiv 4 \pmod{17}$.

From the 225 we have subtracted 13 times the modulus and have thus brought it down to a number smaller than the modulus. The pupils should be familiarized with this principle.

7. *Circulating decimals.*—How many of our younger readers know how easy it is to write down, almost at top speed, the complete circulating decimal equivalent to a vulgar fraction in its lowest terms, if the denominator is prime, no matter how many figures the period may consist of? The following section will suffice for a general introduction to this interesting subject.

Circulating Decimals and Congruences

We will set out below the complete evaluation of the decimal corresponding to such a fraction, say $\frac{1}{29}$. This gives a recurring period of 28 places, and we shall therefore write down 1 followed by 28 ciphers and divide by 29 in the usual way. We will choose short division in order to show the successive quotient figures and their corresponding remainders clearly. Over the ciphers we will write the successive numbers 1 to 28, in order to be able to refer at once to any particular quotient figure or to any particular remainder. We will call the quotient figures, Q's, and the remainder, R's. Examine the 28 R's carefully: they consist of all the numbers 1 to 28, the last of them being 1; this typifies *all* evaluations of circulating decimals

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
29) 1 ·	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Q's →	·	0	3	4	4	8	2	7	5	8	6	2	0	6	8	9	6	5	5	1	7	2	4	1	3	7	9	3	1
R's →	10	13	14	24	8	22	17	25	18	6	2	20	26	28	19	16	15	5	21	7	12	4	11	23	27	9	3	1

(It may be observed that the unit figure of each R is identical with the corresponding Q; but this is not universal; it occurs only when the main divisor has 9 for its unit figure.)

The pupils' interest may readily be excited in this way:

Twenty-nine is a rather hard number to divide by. Can we substitute a smaller and easier number? Yes, at any rate after we have found the first few figures (Q's) of the answer by dividing by the 29, say as far as Q_{11}, i.e. when we have ·03448275862. Now begin again at the beginning and divide *this part-answer* by **5** (ignore the decimal point for the present).

$$5)3448275862$$
$$0689655, \text{ \&c.}$$

But these figures 0689655, &c., are the figures beginning at Q_{12} in the answer, and we may continue to divide by 5

until we reach the end. But note that we no longer bring down 0's, but the Q's we have previously written down.

But we need not have divided by 29 so far as Q_{11}. Suppose we had gone as far as Q_9. We write down as before the figures thus obtained (omitting the decimal point and the 0 which follows it), *prefix a* 4, and then divide by 7.

$$7)^{4}34482758$$
$$62068, \text{ \&c.}$$

But these figures 62068, &c., are the figures beginning at Q_{10} in the answer, and we may continue to divide by 7 until we reach the end. In this case we have shortened still more the original division by 29.

But we can shorten the division by 29 still further. Suppose we had proceeded as far as Q_3, and had obtained ·034. As before we write down the figures thus obtained (omitting the decimal point and the cipher), *prefix a* 4, and then divide by 9.

$$9)^{4}34$$
$$48, \text{ \&c.}$$

But these figures 48, &c., are the figures beginning at Q_4 in the answer, and we may continue to divide by 9 until we reach the end. Thus we may begin to divide by 9 after obtaining only 3 figures by dividing by 29.

Note carefully how we proceed with the division in the last case. We had, to begin with, ·034. To proceed with the division we prefix a 4 to the 3, put each new Q in its proper place, and remember to "bring it down" (not bring down a 0) when its turn comes.

9's into 43 = 4 and 7 over.	Hence	·0344.
9's ,, 74 = 8 ,, 2 ,,	,,	·03448.
9's ,, 24 = 2 ,, 6 ,,	,,	·034482.
9's ,, 68 = 7 ,, 5 ,,	,,	·0344827.

Really, however, it is unnecessary to divide by 29 more than 2 places, i.e. when we have obtained Q_1 and Q_2 (= ·03). This time we prefix to the 3 a 1, and thus obtain 13 for our

initial bit of new dividend. This time we can use the easy divisor 3.

3's	into	13	=	4	and	1	over.	Hence ·034.
3's	,,	14	=	4	,,	2	,,	,, ·0344.
3's	,,	24	=	8	,,	0	,,	,, ·03448.
3's	,,	08	=	2	,,	2	,,	,, ·034482.
3's	,,	22	=	7	,,	1	,,	,, ·0344827.

Observe carefully in this case that, at each step, the new quotient figure and the over figure give, *reversed*, the number to be divided at the next step; e.g. 4 and 1 over in the first line give 14, the number to be divided in the second line. And so generally.

To divide by 3 is so easy that we may evaluate the whole 28 figures of the period in half a minute.

But we need not divide at all. We may begin at the other end, and *multiply* instead.

Suppose we know the last 5 figures, . . . 37931 (Q_{24} to Q_{28}). We may multiply the end figure 1 by 11 and obtain Q_{23}, and then proceed in this way:

$1 \times 11 = 11$; 1 down (= Q_{23}) and 1 to carry.
Hence . . . . . 137931.

$(3 \times 11) + 1 = 34$; 4 down (= Q_{22}) and 3 to carry,
Hence . . . . 4137931.

$(9 \times 11) + 3 = 102$; 2 down (= Q_{21}) and 10 to carry,
Hence . . 24137931.

But other multipliers might be used. Suppose, for instance, we know only the very last figure, 1 (= Q_{28}). This is quite enough, if we use the multiplier 3, and we may finish the whole thing in a few seconds.

$1 \times 3 = 3$	. . . . . 31.
$3 \times 3 = 9$	. . . . . 931.
$9 \times 3 = 27$	. . . . . 7931.
$(7 \times 3) + 2 = 23$	. . . . 37931.
$(3 \times 3) + 2 = 11$	. . . 137931.

A similar scheme applies universally. Whence the secret?

The secret lies in the R's and in the Q's, and in the use of congruences.

Give the pupils a ten minutes' lesson on congruences. The Sixth Form specialists are almost certain to hit upon the solution. The following hints ought in any case to suffice.

Consider another example:

$$17)\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0$$

Q's → 0 5 8 8 2 3 5 2 9 4 1 1 7 6 4 7

R's → 10 15 14 4 6 9 5 16 7 2 3 13 11 8 12 1

The divisor 17 may be regarded as the modulus of a congruence.

(i) $10^1 \equiv 10$,
$\therefore (10^1)^2 \equiv 10^2$, i.e. $10^2 \equiv 100 \equiv \mathbf{15} = R_2$.

(ii) $10^1 \equiv 10$, and $10^2 \equiv 15$;
$\therefore 10^1.10^2 \equiv 10.15$, i.e. $10^3 \equiv 150 \equiv \mathbf{14} = R_3$.

(iii) $10^2 \equiv 15$, and $10^3 \equiv 14$;
$\therefore 10^2.10^3 \equiv 15.14$, i.e. $10^5 \equiv 210 \equiv \mathbf{6} = R_5$.

(iv) $10^3 \equiv 14$, and $10^5 \equiv 6$;
$\therefore 10^3.10^5 \equiv 14.6$, i.e. $10^8 \equiv 84 \equiv \mathbf{16} = R_8$.

All these results agree with the actual division. Clearly, then, when the first R has been found by actual division, any other remainder whatsoever may be found by applying the principles of congruences to the powers of 10.

It thus follows that once we have detected a multiple relation between a pair of R's, the multiple may be used as a general *divisor* for obtaining Q's, and actual dividing in the original division need be carried only a very little distance

Consider another example:

$$59)\,1\cdot0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0,\ \&c.$$

Q's → 1 6 9 4 9 1 5 2 5 4 2, &c.

R's → 10 41 56 20 54 9 31 15 32 26 14 32, &c.

An examination of the R's shows that $R_5 = 6R_8$. This gives us the clue to the relation between every pair of succes-

sive R's, and therefore to every pair of successive Q's. For instance,

$$\begin{aligned} R_5 &= 6R_6 && (54 = 6 \,.\, 9) \\ R_6 + 3M &= 6R_7 && (9 + 177 = 6 \,.\, 31) \\ R_7 + M &= 6R_8 && (31 + 59 = 6 \,.\, 15) \\ R_8 + 3M &= 6R_9 && (15 + 177 = 6 \,.\, 32), \end{aligned}$$

where the multiple relation suggests the divisor 6.

Consider the first of these, $R_5 = 6R_6$. If we have proceeded with the division as far as Q_6, we divide Q_6 by 6, and so obtain Q_7 (= 1) and 3 over; 6's into 31 = **5** and 1 over; &c.

Or, consider the last of the 4 relations: $R_8 + 3M = 6R_9$. If we have proceeded with the original division as far as Q_9, we prefix a 3 (representing 3M) to Q_9 (= 2), making 32, divide by 6, and so obtain Q_{10}.

Or, consider the third of the 4 relations: $R_7 + M = 6R_8$. If we have proceeded with the original division as far as Q_8, we prefix a 1 (representing 1M) to Q_8 (= 5), making 15, divide by 6, and so obtain Q_9.

We may begin to divide by 6 at any point after Q_1. It is merely a matter of prefixing a figure indicated by x in xM.

In our first example, $\frac{1}{29}$, we first divided by **5**, because we noted the relation $R_1 = 5R_{11}$; then we divided by 7, because we noted the relation $R_3 = 7R_{11}$; then we divided by 9, because we noted the relation $R_9 = 9R_{11}$. An easy divisor can always be obtained by examining the R's in this way. We finally divided by 3 because we noted that $R_{10} = 3R_{11}$.

But with a little practice we can dispense with the R's altogether, and detect a multiple relation amongst the early Q's, these being obtained, of course, by actual division. For instance

$$\tfrac{1}{23} = \cdot\dot{0}4347826086956521739\dot{1}3.$$

After obtaining about 5 Q's, we might notice that by prefixing 2 to Q_2, making 24, we might divide by 7 and obtain Q_3, and so on continuously.

Here is an example for the boys to complete. The recurring period consists of 646 places. A good Sixth Form boy ought to write down 1 figure per second, and so do the whole thing in 10 or 11 minutes.

$$\tfrac{1}{647} = \cdot\dot{0}01545595 \ldots 057187017\dot{}.$$

A suitable divisor or multiplier is 11. The division by 11 may be begun after the first 3 places are obtained (·001) by prefixing 6 to Q_1 and so obtaining 6001. (A smaller divisor may soon be found.)

11's into 60 = **5** (= Q_4) and 5 over;
11's ,, 50 = **4** (= Q_5) ,, 6 ,, ;
11's ,, 61 = **5** (= Q_6) ,, 6 ,, ;
11's ,, 65 = **5** (= Q_7) ,, 10 ,, ; &c.

In the multiplication,

$$7\ (= Q_{646}) \times 11 = 77;\ 7\ (= Q_{643}) \text{ and carry } 7;$$
$$\{1\ (= Q_{645}) \times 11\} + 7 = 18;\ 8\ (= Q_{642}) \text{ and carry } 1, \text{ &c.}$$

The result is easily checked by selecting other divisors.

If the numerator of the vulgar fraction is other than unity, the equivalent decimals will consist of the same figures as when the numerator is unity, and they will be in the same order, but the period will begin in a different place, easily discovered. But the subject, which is a source of delight to most boys, cannot be carried further here.

Magic Squares and Magic Cubes

Can time spent on this subject be justified? If as a mathematical topic for purposes of formal teaching, no. If as a subject for creating a lasting mathematical interest in the less mathematically inclined boys, yes.

Magic squares have interested the greatest mathematicians. The wonderful harmony and symmetry of the numbers so grouped have always tended to attract their attention. Boys are always impressed by the mysterious regularity that

emerges in so many ways when they study magic squares. Three or four lessons on the subject are well worth giving, though in so short a time the secrets of the construction of some of the remarkable squares that have been constructed by mathematicians could not be given.

The simplest and best-known construction (for a square with an odd number of sides) is the following: the method

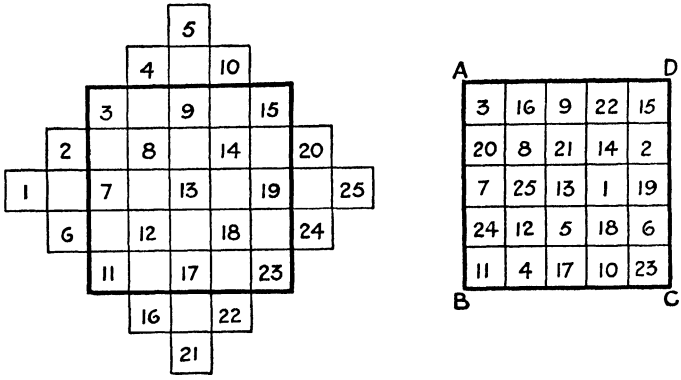


Fig. 314

underlying the symmetrical transfer of the numbers in the temporary outer cells is obvious. Horizontally, vertically, and diagonally, the sum of the numbers is 65.

Paste such a magic square round a roller, the circumference of which is equal to AB or BC. Two squares should be prepared, one to be rolled round from side to side, so that AB coincides with DC, and one from top to bottom so that AD coincides with BC. The consecutive numbers in the various diagonals of the rolled up squares give the learner the real secret of the construction of the simpler squares.

The famous Benjamin Franklin was the inventor of magic squares with properties that always fill the learner with astonishment. The construction of his 8×8 and 16×16 squares is quite simple, and this very simplicity goes far to create the astonishment. They are to be found in all the textbooks on the subject, and every boy should know them.

The problem of finding the *number of different ways* in which the numbers, say, 1 to 25, may be arranged in a square is worth looking into, though no general solution has yet been discovered. Boys find out at once, of course, that variations are easy to make and are numerous.

Not all the textbooks point out the device for making magic squares so that the *products* of the columns and rows

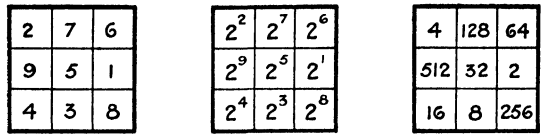


Fig. 315

are constant. It is simply a question of using the numbers in the ordinary squares as *indices* of some selected number for the new square. We append the usual 3×3 square (common sum $= 15$) and one of its cousins (common product $= 2^{15} = 32768$).

" Magic cubes " may be touched upon, say $3 \times 3 \times 3$ (1 to 27). The sum of the numbers in each row is 42, not

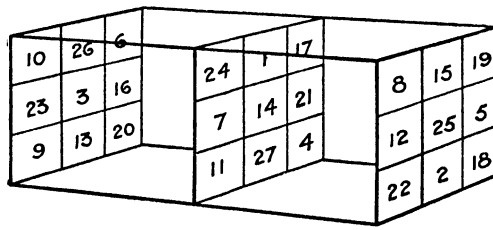


Fig. 316

only in each face shown, but *through* the faces, front to back; also the diagonals of the *cube* as well as the diagonals of some of the squares.

Magic circles, pentagrams, &c., are hardly worth spending time over.

The best books on the subject are (1) *Magic Squares and*

Magic Cubes, John Willis: (2) *Magic Squares and Cubes*, W. S. Andrews; (3) *Les Espaces Arithmétiques Hypermagiques*, Gabriel Arnoux; (4) *Les Carrés Magiques*, M. Frolow; (5) *Le Problème d'Euler et les Carrés Magiques*, *Atlas*, M. Frolow.

Magnitudes. Great and Small

When attempting to help a boy to form a clearer conception of the significance of very large numbers, say those concerning stellar distances or atomic magnitudes, it is essential for the teacher to eliminate from the problem every kind of avoidable complexity. To form a conception of a great number is quite difficult enough in itself, and to a boy the difficulty may prove insuperable. On one occasion I heard a teacher attacking our old friend the " light-year ", in favour of its new rival the " parsec ", simply on the ground that the latter made astronomers' computations easier. Now the light-year is a perfectly well understood thing. In mechanics we often define distance as the product of velocity and time ($s = vt$), as every child knows; and we apply this self-same principle to the distance known as a light-year, the new unit being determined by the product of the velocity of light (miles per second) and the number of seconds in a year. But the parsec is the distance corresponding to the parallax of 1″, and a simple calculation shows that it is 3·26 times as long as the light-year; and this trigonometrical method of determining star distances compels the learner to think in terms of the semi-major axis of the earth's orbit. The complexity is entirely unnecessary in school work; it tends to obscure the main thing the boy is supposed to be thinking about.

If the learner already knows that the velocity of light is 186,000 miles a second, simple arithmetic tells him that the length of the light-year is, approximately,

$$(186{,}000 \times 60 \times 60 \times 24 \times 365) \text{ miles, i.e. } 6 \times 10^{12} \text{ miles,}$$

or 6 billion miles. Thus, when the learner is told that α *Cen-*

tauri is 4 light-years distant, he knows that this means 24 billion miles; and that the 1,000,000 light-years representing the probable distance of the remoter nebulæ is a distance of 6×10^{18} (six trillion) miles. Or, he may be told that the mass of the H atom is $1{\cdot}66 \times 10^{-24}$ grams, when he sees at once that 10^{24} H atoms together must weigh $1\frac{2}{3}$ grams.

But are these vast numbers anything more than mere words to the boy? What does a quadrillion signify to him? or even a trillion or a billion? or even a million? Is it of any use to try to make the boy realize the significance of such numbers? or just to leave them as mere words? or not to mention them at all and merely to give some such illustration as Kelvin's earth-sized sphere full of cricket-balls?

I have tried the experiment of giving to boys such illustrations as these: (1) the number of molecules in 1 c. c. of gas is about 200 trillions (2×10^{20}), a number equal to the number of grains of fine sand, 70,000 to the cubic inch, in a layer 1 foot deep, covering the whole surface of England and Wales; (2) the number of molecules in a single drop of water is about 1700 trillions ($1{\cdot}7 \times 10^{21}$), a number just about equal to the number of drops of water in a layer $7\frac{1}{2}$ inches deep completely covering a sphere the size of the earth. —But I have always found that such illustrations merely give rise to vague wonderment. The pupil himself *makes no personal effort* to realize the magnitude of the numbers; and this is fatal.

Such an effort is indispensable. The best plan, perhaps, is to make the pupil first consider carefully the magnitude of a million, then of a billion, a trillion, a quadrillion, successively (10^6, 10^{12}, 10^{18}, 10^{24}). For instance, an ordinary watch ticks 5 times a second or 1000 times in about 3 minutes, or a million times in about 2 days and 2 nights. Let this fact be assimilated as a basic fact, first. Now let the boy think about a billion. Evidently, a watch would take (for the present purpose, all the underlying assumptions may be accepted) about 6000 years to tick a billion times (2 days $\times 10^6$), so

that if a watch had started to tick at the time King Solomon was building the Jewish temple, it would not yet have ticked *half* a billion times. Now proceed to a trillion, and then to a quadrillion. Evidently the watch would take 6000 million years to tick a trillion times, and 6000 billion years to tick a quadrillion times.—An approach of this kind to the subject does not take long, and a boy fond of arithmetic may be encouraged to invent illustrations of his own. It is worth while. It is worth one's own while, if the attempt has never been made before. It is, indeed, hard to realize the significance of the statement that light-waves tap the retina of the eye billions of times a second. Yet how are we to escape accepting this frequency if we accept the measured velocity of light and the measured length of light-waves? Impress upon the boys the fact that the inference is inescapable.

The description of the manufacture of such a thing as a diffraction grating with lines ruled 20,000 to the inch, or of Dr. J. W. Beams' mechanical production of light flashes of only 10^{-7} second duration, serves to impress pupils with the sense of reality of small things.

Boys must understand that both stellar magnitudes and atomic magnitudes are, for the most part, *calculated* values and not directly measured values, and that the calculations are, in the main, based on inferential evidence, the inferences being drawn partly from known facts, partly from hypotheses. But converging evidence of different kinds justifies a feeling of confidence in the probability of the truth of the estimate. So much so is this the case, that the natural repugnance of the mind to accept statements which seem to be so contradictory of everyday experience, and therefore to " common sense ", is overcome. Still, the *nature* of the evidence available must be borne in mind. So must the amazing nature of the results.

The same subject. Sir James Jeans' Methods

Sir James Jeans, in his two recent books, *The Universe Around Us* and *The Mysterious Universe*, has adopted various devices for helping the mathematically uninitiated to realize the significance of large numbers. The first book is " written in simple language " and is intended to be " intelligible to readers with no special scientific attainments ". The second book " may be read as a sequel ".

In order to find out how the books might appeal to some of their readers, I induced seven non-mathematical friends (two specialists in Classics, two in History, one in Modern Languages, two in Science) to read the books through and then submit to be questioned on the meaning of the following (and a few other) extracts:

1. " Less than a thousand thousand millionth part."
2. " 15 million million years."
3. " 2000 million light-years."
4. " The nearest star, Proxima Centauri, is 25,000,000 million miles away."
5. " An average star contains about 10^{56} molecules."

The chemist was on the spot at once. The biologist had to do a good deal of thinking, but he got there at last. But the other five? They failed utterly to understand what the numbers signified, though all when at school had taken mathematics in the School Certificate (or its equivalent), and one in the Higher Certificate. " What is the difference between ' 15 million million years ' and ' 15 million years '?" —" Oh, the former means just a few more millions than the latter, I suppose." To the Higher Certificate man I said, " Compare the increase of 10^4 to 10^7 with the increase 10^{53} to 10^{56}." He replied: " It is exactly the same thing, for in each case you have increased the index by 3. If you take 10^4 from 10^7 you get 9,990,000, so that if you take 10^{53} from 10^{56} the difference must be just the same "! And so generally. The five examinees had but the vaguest notions of what the

numbers meant. Finally they all admitted that, elementary as the mathematics of the books appeared to be, they simply did not understand it. The books had left in their mind feelings of intense wonderment, but the real facts they had not grasped at all.—It would be interesting if other teachers would test some of their own friends similarly.

I am not sure that we gain anything by writing " 15 million million " instead of " 15,000,000,000,000 ", or " 15 billion ", or " 15.10^{12} ". Are the words " million million " significant?

Some of Sir James Jeans' illustrations are well worth examining. Here is one:

1. Age of telescopic astronomy,	300	years.
2. Age of astronomical science,	3000	„
3. Age of man on earth,	300,000 *	„
4. Age of life on earth,	200,000,000	„

A teacher can make much of such a comparative device, especially if it is illustrated by a time-line.

Here is a second:

The earth's orbit is 600,000,000 miles. Represent this by a pinhead $\frac{1}{16}''$ in diameter. Then,

1. *Sun* is 1/3400″ in diameter.
2. *Earth* is 1/340,000″ in diameter (invisible under the most powerful microscope).
3. *Nearest star* is 225 miles away.
4. *Nearest nebula* is 30,000 miles away.
5. *Remotest nebulæ* are 4,000,000 miles away.

A third:

The average temperature of the sun's interior is 50,000,000 degrees (it is probably very much higher). Think of a pinhead of matter of this temperature. It would require the energy of an engine of 3000 billion horse-power to maintain it. The

* Possibly much longer than this.

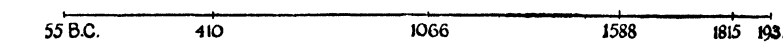
pinhead would emit enough heat to kill anybody 1000 miles away.

This illustration is less successful than the others. There is no gradation. A radius of 1000 miles represents an area nearly as large as Europe, but to say that a pinhead of matter would be so hot as to kill off the whole population of Europe does not leave on the mind a sufficiently definite mathematical impression. A succession of preparatory stages is desirable. To me, however, the temperature in question is utterly unimaginable; I cannot get much beyond the mere arithmetic, though I am fairly familiar with the highest temperatures that have been produced artificially.

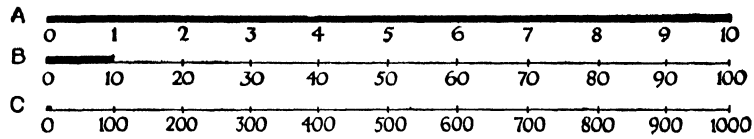
A fourth:

For $\frac{499}{500}$ of its journey, the light by which we see the remotest nebulæ travelled towards an earth not yet inhabited by man; yet the radius of the universe is *14 times as great* as the distance of the remotest nebulæ.—This is obviously a better way of bringing home the fact than by giving the length of the radius of the universe in miles (2000 million light-years = 12,000 trillion miles), though if this number were given it would add to the boy's interest to tell him that this number is roughly comparable to the number of molecules in a single drop of water.

In dealing with large numbers, it is a sound teaching principle (1) to illustrate them by diagrams of some sort, (2) to approach them by stages; and it is a simple matter to show the boys how these things may best be done. There is probably no better plan than that of a succession of distance or time lines drawn to gradually diminishing scales. The boys will have gleaned the main idea from their history lessons. Here is a skeleton history time-line, 55 B.C. to A.D. 1931:



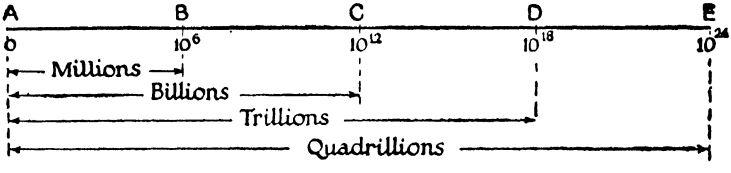
Let the line A represent 10 units; B, 100; C, 1000.



Point out how A has shrunk to $\frac{1}{10}$ of itself in B and to $\frac{1}{100}$ of itself in C. The first line might represent a ten-year-old boy's 10 birthdays; the second line would then show the boy's life history in $\frac{1}{10}$ the length of the first, and C in $\frac{1}{100}$. Develop this general idea carefully.

Now deliberately set a trap that will catch 90 per cent of the class:

"Let us devise a number line the successive parts of which shall represent the comparative sizes of really big numbers. You know that 10^6, 10^{12}, 10^{18}, 10^{24} represent, respectively, millions, billions, trillions, quadrillions.



"Is this all right?"—"Yes."—"**Caught.** Surely if AC represents a billion it must be *a million times as long as AB*, and AD a million times as long as that. If AB is 1 inch, AC must be made 16 miles long; AD, 16 million miles; and AE 16 billion miles." And so on.

Books to consult:

1. *Théorie des Nombres*, Desmarest (Hachette).
2. *Theory of Numbers*, Peter Barlow (an old book, but still very suggestive).

CHAPTER XLIII

Time and the Calendar

This subject is essentially mathematical, and it should be the business of the mathematical staff to see that the following topics are included in their scheme of instruction.

1. Greenwich mean time. How the length of the solar day is affected, (α) by the variable movement of the earth in its orbit; (β) by the fact that the axis of the earth is not perpendicular to the plane of the orbit.

2. The modern clock. The astronomical time-keeper is a " free " pendulum swinging in a vacuum chamber. How its swing is maintained, and how the pendulum of the " slave " clock (which does the work of moving the hands round the dial) is made to swing synchronously with it.

3. Sidereal time; its significance and use.

4. Summer time; opposition to its adoption. The legal definition.

5. Zone standard times for different countries. How the zone of other countries differs from Greenwich time by an integral number of hours. Why 5 standard times in U.S.A. and Canada and why 3 in Brazil.

6. The Date or Calendar line. Let the boys examine a good Mercator map and discover for themselves how the line differs from the 180th meridian. Boys are often puzzled about the reason for different days, say Monday and Tuesday, on the two sides of this line. Let them think the thing out for themselves. The practical difficulty of running the line *through* a group of islands instead of round them.

7. Calendar problems. Successive reforms of the calendar. Opposition to further reform religious and social, not scientific. Why not 13 months of 28 days each, and one non-calendar

(2 in leap year) day during the year? or some other scheme of a more even division than at present? Why even a fixed Easter is opposed. How Easter is determined for each year. The League of Nations and the Reform of the Calendar.

8. The history of time-measuring. Clepsydras, sand clocks, graduated candles, sundials, clocks.

CHAPTER XLIV

Mathematical Recreations

The multitude of problems usually classified under this heading may be made a very serious factor of the mathematical course. The average boy will face a good deal of drudgery if it is a question of solving a puzzle, or seeing his way through a trick, or liberating himself from a trap. The majority of the so-called mathematical recreations may be grouped around definite mathematical principles; if they are thus grouped, if the underlying principle of a group is thoroughly mastered, and if the members of the group are treated as applications of the principle, the work becomes as serious as it is interesting. There is no better means of giving a boy a permanent interest in mathematics than to help him to achieve a mastery of the commoner forms of mathematical puzzles and fallacies. A few principal topics may be suggested:

1. Arithmetical puzzles, especially those from mediæval sources.

2. Geometrical problems and paradoxes. Shunting and ferry-boat problems. Paradromic rings.

3. Chess-board problems.

4. Unicursal problems. Mazes.

5. Playing-card tricks.
6. Cats' cradles.
7. Ciphers and cryptographs.

} Only partly mathematical, but the necessary analysis is instructive.

8. Algebraical and geometrical fallacies.

An isolated problem like Kirkman's school-girls problem is also well worth doing, if only for the patient analysis that a solution of the problem demands.

Nearly all the necessary material may be found in the late Mr. Rouse Ball's *Mathematical Recreations*, but the literature of the subject is extensive. Every mathematical teacher should have on his shelves the works of Édouard Lucas; they include everything of interest. The late Henry Dudeney's books are also useful.

I append a manageable figure (it is new) to illustrate the principle of Captain Turton's geometrical fallacy, one of

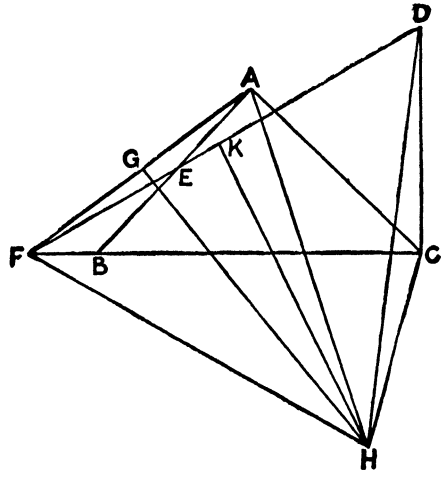


Fig. 317

the best I know. Be it remembered that in all cases like this the figure is the one thing that matters, if the fallacy is to be well concealed.

ABC is an isosceles Δ of 45°, 45°, 90°.
Draw CD equal to CA and $\perp$ BC.
Bisect AB in E, join DE and produce to F in CB produced.
Bisect AF in G and DF in K. Draw GH $\perp$ FA, and KH $\perp$ FD, meeting in H. Join FH, AH, DH, CH.

$$\Delta FGH \equiv \Delta AGH, \quad \therefore FH = AH;$$
$$\Delta FKH \equiv \Delta DKH, \quad \therefore FH = DH;$$
$$\text{Hence} \quad AH = DH.$$

In the Δs ACH, DCH,

$$CA = CD \quad \text{(constr.)},$$
$$AH = DH \quad \text{(proved)},$$
$$CH \text{ is common},$$
$$\therefore \text{ the } \Delta\text{s are congruent};$$
$$\therefore \angle ACH = \angle DCH.$$

Take away the common angle FCH;

$$\therefore \angle ACF = \angle DCF,$$
$$\text{i.e.} \quad 45° = 90°, \text{ or } 1 = 2.$$

CHAPTER XLV

Non-Euclidean Geometry

What does "non-Euclidean" Mean?

My experience of the teaching of non-Euclidean geometry has been slight (not more than 3 or 4 lessons in all) and not very encouraging, though in all cases the teachers were certainly competent and the boys (Sixth Form specialists) able and well-grounded. Nevertheless, the opinion of many prominent mathematical teachers is that boys ought to

know *something* about the subject. Personally I think it is too difficult, and is best taken at the University later.

It is essential for a teacher who decides to include the geometry in his Sixth Form course, first to familiarize himself both with the whole subject and with its implications. Here is a possible first course of reading:

1. The controversies of the last hundred years concerning Euclid's parallel postulate.
2. Some such book as Hilbert's *Foundations of Geometry*.
3. Part II of Poincaré's *Science and Hypothesis*.
4. Mr. Fletcher's article in No. 163 of the *Mathematical Gazette*, viz. " A method of studying non-Euclidean geometry ".

Captain Elliott writes a suggestive short article on " Practical non-Euclidean Geometry " in No. 177 of the *Gazette*.

At the beginning of the nineteenth century, almost simultaneously, Lobatscheffski, a Russian, and Bolyai, a Hungarian, showed irrefutably that a proof of the parallel postulate is impossible. It will be remembered that Euclid himself seemed to recognize a difference in the degree of conviction carried to the mind by his statement concerning parallels, compared with that of his other fundamental assumptions; and he called the statement a *postulate* rather than an *axiom*. It is incorrect to include the statement as his 12th axiom, as is commonly done.

Lobatscheffski assumed that through a point an *infinite* number of parallels may be drawn to a given straight line, but he retained all the other basic assumptions of Euclid. On these foundations he built up a series of theorems which are perfectly self-consistent and non-contradictory; the geometry is as impeccable in its logic as Euclid's. The theorems are, however, at first sight disconcerting; for instance, (1) the sum of the angles of a triangle is always less than two right angles, and the difference between that sum and two right angles is proportional to the area of the triangle; (2) it is impossible to construct a figure similar to a given

figure but of different dimensions. Lobatscheffski's propositions have little or no relation to those of Euclid, but they are none the less logically interconnected.—Let the reader try to reconstruct, say, the first 32 propositions of Euclid, Book I, on the assumption that the parallel postulate (the " 12th axiom ") is untenable; he will probably be more than a little surprised.

Riemann, a German mathematician, likewise rejected Euclid's parallel postulate. He also rejected the axiom that only one line can pass through two points. Otherwise he accepted Euclid's assumptions. The system of geometry which he then built up does not differ essentially from spherical geometry. On a sphere, through two given points, we can *in general* draw only one great circle, the arc of which between the two points therefore represents the shortest distance, and hence the straightest line between them. But there is one exception; if the two given points are at the ends of a diameter, an infinite number of great circles can be drawn through them. In the same way, in Riemann's geometry, through two points only one straight line can in general be drawn, but there are exceptional cases in which through two points an infinite number of straight lines can be drawn. Riemann's " space " is finite though unbounded, just as the surface of a sphere is finite though unbounded.

Thus Lobatscheffski's and Riemann's geometries, though both non-Euclidean, were, in a measure, opposed to each other.

In Euclid's geometry, the angle-sum of a triangle is two right angles, in Lobatscheffski's less than two right angles, in Riemann's greater than two right angles.

In Euclid's geometry, the number of parallel lines that can be drawn through a given point to a given line is one; in Lobatscheffski's, an infinite number; in Riemann's, none.

Euclidean geometry (which includes all school geometries) retains the parallel postulate; non-Euclidean geometries are those which reject the postulate.

Which Geometry is True?

Mr. Fletcher asks the question, which geometry is true? and answers it by saying that they are all true, though the whole question turns on the nature of a straight line. The straight line, being elementary and fundamental, cannot, however, be " defined ", for there is nothing simpler in terms of which it can be expressed. Thus we are driven to an indirect definition by axioms. But inasmuch as the axioms of the three geometries differ, it is obvious that they define different things. But by " things " we mean not material objects but ideas suggested by them. All three geometries are true but only as applying to the " things " known or unknown to which they refer.

But when it comes to the application of geometry to the facts of the external world, the question is, as Mr. Fletcher points out, different. " Now we have to ask: which of these absolute pure sciences applies the most conveniently or the most exactly to the facts with which we are dealing? In large scale work, where alone the differences in the results of the geometries are large enough to be apparent, we are dealing chiefly with the form of a ray of light, or the line of action of gravitation. It is easy to see that these forms may differ according to circumstances; that while, as seems now to be probable, Euclid's geometry may be applicable with all necessary exactness to those rays ' at an infinite distance ' from gravitating matter, Lobatscheffski's, or more probably Riemann's, may afford a better tool for dealing with them in the neighbourhood of such matter."

Since in the material world in which we live, Euclid's parallel postulate seems to be satisfied, Euclid's geometry will continue to be the geometry of practical life and hence of our schools.

" The essential requisite for clear thinking on the subject," Mr. Fletcher says, " is the maintenance of the distinction between the pure science and the applied. The ' things ' with which the former deals are ideas, abstractions; it can

only proceed from axioms, but on that basis its results are absolute. The latter deals with ' facts ', with ' external things '; its basis is experimental, and its results approximate."

Impress upon Sixth Form specialists that in our ordinary geometry we always argue as if we were living on a *plane*, whereas really we are living on a *sphere*. The surface of the very table we write on is, strictly, part of the surface of a sphere of about 4000 miles radius. *Practically*, it is a plane, of course, but if we build up a theoretical system (as Euclid did) on the assumption that we are dealing with actual planes instead of with parts of a spherical surface, how can our system be free from possible fallacy? And if we apply that system to the measurement of stellar distances, how can we logically assume that it is strictly applicable?

Relativity

It is not a difficult matter to give boys a clear understanding of the *special* theory * of Relativity, but the *general* theory is much too difficult for them. Even so, it is possible to give them one or two comparatively elementary lessons on the general theory, to enable them to see that the final acceleration-difference in Newton's and Einstein's gravitation formulæ, unlike as these formulæ are in appearance, is almost insignificant. Do not confuse " space-time " with hyperspace, a totally different thing. Space-time is merely a mathematical abstraction devised to meet the indispensable need of considering time and three-dimensional space together. The greater part of Professor Nunn's admirably written book on Relativity can be understood by Sixth Form specialists, as I know from experience.

Hyperspace

When we come to the question of hyperspace, we are in a region of difficulty too great for all but the very exceptional

* The Relativity of Simultaneity is apt to be a little puzzling. For suggestions see *Science Teaching*, pp. 357–72.

boy. Even some professional mathematicians still have deeply-rooted prejudices against N-dimensional space. The crudest form of prejudice is what may be called the " common-sense " opinion that as space cannot have more than three dimensions, any consideration of hyper-space is obviously nonsense. When Einstein announced his general theory, a distinguished Oxford philosopher wrote an indignant letter to *The Times*, pointing out (amongst other things) that inasmuch as Aristotle himself had pronounced space to be just long and broad and deep, in other words three-dimensional, there was nothing more to be said: Einstein's irreverence was almost unpardonable. Evidently the writer of that letter was under a complete misapprehension as to the nature of Relativity. Einstein's space is not four-dimensional but three-dimensional, though cosmically it is probably not quite homaloidal but slightly curved. Einstein's fourth dimension is *time*, not space. The same writer also probably misunderstood the nature of geometry as a science. Geometry certainly did start as a form of " earth-measuring ", but even in the time of the ancient Greeks it had developed into a semi-abstract science, to be deduced from a limited number of axioms and definitions. For more than 2000 years after Euclid, it was supposed that axioms were self-evident truths about the real world. Only one axiom, that concerning parallels, fell short of the high standard of the others: it was not self-evident. Attempts to prove it all failed, and at last it was realized that a logical system of geometry could be constructed by starting with the denial of the axiom, or postulate as it ought to be called.

Students of Relativity need not concern themselves much with hyperspace, but it must not be thought that hyper-geometry can have no application to the geometry of the real world or to physics. Beginners in wave mechanics naturally assume that the three dimensions required in Schrödinger's theory of the motion of a single particle are the three dimensions of ordinary space, but as soon as we come to two particles *six* dimensions are required. Many problems in

thermodynamics require a number of dimensions exceeding three, though perhaps " degrees of freedom " rather than " dimensions " is a term more acceptable to some people.

Books to consult:

1. Einstein's Nottingham University Lecture of June 6, 1930. (See *Nature* for June 14.)

2. Professor Forsyth's address, *Dimensions in Geometry*, to the Mathematical Association. (See *Gazette*, 212.) This address is most illuminating.

3. Professor Sommerville's *Geometry of N Dimensions.*

4. Professor Baker's *Principles of Geometry*, Vol. IV.

(The last two books should be read by teachers of mathematics, but, of course, for advancing their own knowledge, not for actual teaching purposes.)

CHAPTER XLVI

The Philosophy of Mathematics

Mathematical teachers will be well advised to admit that the philosophical foundations of mathematics is a subject which is outside the limits of Sixth Form work, save in the case of very exceptional boys. There are, however, a few points of a sufficiently simple character that can be included, if only in order that boys may, before leaving school, lose some of the " cocksureness " that early mathematical success so often excites, and learn that mathematical truth is, after all, something that is still far from being absolute, something that is still relative.

Mathematical and other Reasoning

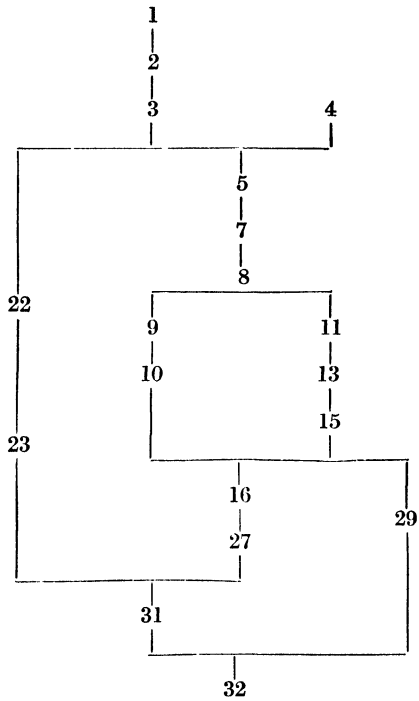
One or two formal lessons on the general nature of reasoning, deductive and inductive, are advisable. This does not mean that time should be spent on formal logic, except in so far as is necessary for a clear understanding of the

syllogism, and that *is* necessary. The mere setting-out of mathematical truth, as distinguished from the search and discovery of it, is essentially syllogistic, synthetic, and deductive in character. But mathematical *reasoning* is not deductive; it is above all things analytical, inductive.

Let the boys understand clearly that the elementary scheme of syllogistic reasoning that at one time passed for " logic ", the formal deductive logic of the last 2000 years, does not really represent our ordinary modes of reasoning, but is rather a scheme by which we try to show other people how our conclusions follow from our premisses. The arranging of a string of syllogisms, *as* syllogisms, presents no serious difficulty; from accepted premisses, a logical conclusion follows, with almost mechanical precision. Professor Jevons actually invented a logical machine, almost as simple as a penny-in-the-slot-machine, which made clear that syllogistic reasoning was at bottom mechanical. In reasoning the real difficulty is concerned with the premisses that compose the syllogism, not with the syllogism itself. Can we certify that the premisses are true, and do we all agree about the exact significance of the terms we use? *This* is the trouble, and the only real trouble, involved in ordinary reasoning. To follow out the sequence of syllogisms in a Euclidean proposition is child's play.

Opposite is a scheme showing Euclid's chain of reasoning for proving I, 32. The proposition and all the propositions on which it depends are easy enough to follow up, and may (with certain exceptions) be accepted. But when we come to the axioms which form the ultimate premisses of the propositions, and examine them carefully, we begin to feel doubt and difficulty. In short, we have reached the point where the work of serious reasoning begins.

This is the first thing for boys to bear in mind: that although mathematical demonstrations have every appearance of being mere chains of syllogisms, we may *not* infer that mathematical reasoning is deductive. The very contrary is really the case. Mathematical reasoning is above all things



analytical, inductive; it cannot be reduced to the rules of deductive logic. We dress the results up syllogistically merely for exhibition purposes.

Impress upon the boys that in *all* reasoning beginners tend to believe that the premisses are true if acceptable consequences seem to follow from them, and the longer the chain of apparently sound intermediate links, the less suspicious they become of any weakness in the first link; but that conclusions are quite worthless unless the premisses on which they first depend are unassailable.

It was John Stuart Mill who in the middle of the last century first shook our faith in Aristotelian logic. Mill's own position has since been shaken, but it was he who first gave to the subject its proper outlook.

The " new " logic, the logic especially of the last thirty years, insists upon this: that verbal explanations of meaning, so long as they remain merely verbal, are futile. Merely to " infer " one proposition from another, and to go on doing this for ever, gives us nothing but unexplained " propositions " at every step of the process. The old logic never even became aware of the fatal confusion between assertion and sentence that is covered by the word " proposition ". According to it, if we get one proposition from another or others, we have arrived at the end of a process of inference and have obtained a "conclusion". If "all men are feathered animals" happened to be one of the premisses, the old logic did not question that premiss at all, but proceeded to draw " conclusions "; the truth or falsehood of the premisses was none of its business. Its conclusions were thus often meaningless and misleading; sham, not real. The new logic, on the other hand, pays first attention to the premisses, knowing that the subsequent process of inference is a process relatively elementary.

We must not allow our admiration of the Greeks to blind us to their limitations and their failures. Greek mathematics was great; modern mathematics is greater. In certain regions the Greeks failed where we feel they ought to have succeeded. We feel, for example, that they ought to have anticipated Descartes; and we feel that, with their power of generalization and their love of philosophy, they ought to have given some coherent account of the foundations of geometry. The early pages of Euclid are definitely unsatisfactory.

Apparently the Greeks never realized that the foundations of geometry were necessarily abstract. Even Euclid himself could only look at space vaguely, and give some sort of popular description of what he thought he saw there. And to the Greeks generally, geometry was always the "science of space" of the physical space of experience in which we live. It is quite impossible to base a coherent geometry on such a foundation; the superstructure may be magnificent, but it is always likely to overturn because of the instability of its

foundations. That the theorems of geometry are not affected by earthquakes, that the Greeks could understand, but they could not, or at least did not, understand that geometry has nothing to do with physical space; that its " space " is its own creation; and that it is simply the statement of the logical relations between objects, defined by these relations alone.

Non-mathematicians still commonly suppose that the early pages of Euclid, his " axioms " and " postulates ", are profound and never to be questioned. There is, in point of fact, not one of these axioms or postulates, the parallel postulate alone excepted, which has anything but an historical interest, or which embodies any permanent contribution to science. Here the Greeks failed altogether.

Here is Mr. Bertrand Russell's opinion of Euclid.— " When Euclid is attacked for his verbosity or his obscurity or his pedantry, it has been customary to defend him on the ground that his logical excellence is transcendent, and affords an invaluable training to the youthful powers of reasoning. But at a close inspection, this claim vanishes. His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious.

" The first proposition assumes that the circles used in the construction intersect—an assumption not noticed by Euclid because of his dangerous habit of using a figure. The fourth proposition is a tissue of nonsense. Superposition is a logically worthless device; for if our triangles are spatial, there is a logical contradiction in the notion of using them; if they are material, they cannot be perfectly rigid, and when superposed they are certain to be slightly deformed from the shape they had before. The sixth proposition requires an axiom for proving that if D be in AB (the side of the isosceles triangle ABC), and BD is $<$ BC, the triangle DBC is $<$ the triangle ABC. The seventh proposition is so thoroughly fallacious that Euclid would have done better not to attempt a proof. I, 8 involves the same fallacy as I, 4. In I, 9 we require the equality of all right angles, which is not

a true axiom since it is demonstrable. I, 12 involves the assumption that a circle meets a line in two points or in none, which has not in any way been demonstrated. I, 26 involves the same fallacy as I, 4 and I, 8. Many more criticisms might be passed on Euclid's methods, and on his conception of geometry; but the above definite fallacies seem sufficient to show that the value of his work as a masterpiece of logic has been very grossly exaggerated."

Still, the Greeks did conceive the universe as a cosmos subject to rule; they did recognize that the universe is, at bottom, a mathematical affair. It was not until 2000 years later that the foundations of their science were carefully examined and found to be lacking in any sort of solidity or permanence.

Axioms, Postulates, Definitions

Every conclusion rests on premisses. These premisses are either self-evident and require no demonstration, or they can be established only by demonstration from other proofs. Since we cannot thus proceed in this latter fashion *ad infinitum*, geometry must be founded on a certain number of undemonstrable propositions.

To these undemonstrable propositions, some of the greatest mathematicians of the last fifty years have devoted very serious attention. That there is still divergence of opinion shows how difficult the subject really is. The several books which are the separate or joint productions of Mr. Russell and Professor Whitehead will be familiar to most readers, and the books of equally eminent French and German mathematicians will also be familiar to some. Such a mass of authoritative literature will convince any non-expert that the subject is a very thorny one.

The standard of logical rigour in mathematics is now greater than it has ever been. It is, however, still quite permissible, in teaching, to make use of small boys' intuitions—to use them, in some measure, as a reinforcing

basis when establishing elementary principles from data derived from the boys' experience or from special experiment. For later serious work, geometrical intuitions are not sufficiently trustworthy. One well-known authority describes intuitions as " a mere mass of unanalysed prejudice ". Certainly boys' intuitions are necessarily often crude, often meaningless, just shots in the dark.—We shall return to the question of intuition in the next chapter.

What are geometrical axioms? Kant said they were synthetic *a priori* intuitions. But in that case they would be imposed upon us with such a force that we could not conceive their contraries, and this we now know we can do.

Are they, then, experimental truths?—We do not make experiments on ideal lines or ideal circles; we can make them only as material objects. On what, therefore, would experiment serving as a foundation for geometry be based? The answer is simple. Since we constantly reason as if geometrical figures behaved like solids, geometry can borrow from experiments only the properties of those bodies. But this involves an insurmountable difficulty, for if geometry were an experimental science, it would not be an *exact* science; it would be subjected to continual revision. Indeed no rigorously invariable solid exists.

Thus the axioms are neither synthetic *a priori* intuitions nor experimental facts; they are merely *conventions*.

Our choice amongst all possible conventions is *guided* by experimental fact, but it remains free, and it is limited only by the necessity of avoiding every contradiction. Thus different geometries are possible according to our initial choice of conventions. One of these geometries is not more " true " than another (cf. the last chapter); it can only be more convenient. Euclidean geometry is the most convenient because (1) it is the simplest; (2) it sufficiently agrees with the properties of natural solids, those bodies which we can compare and measure by means of our senses.

Henrici (no mean geometer) called Euclid's definitions " axioms in disguise ". Poincaré (a much abler mathematician,

perhaps the greatest of the last fifty years) called Euclid's axioms " definitions in disguise ". As to Euclid's postulates, some writers have seen in them merely statements which limit the use of instruments in geometrical construction to ruler and compasses, but this is a mere side issue, and has nothing whatever to do with the nature of geometrical reasoning.

Euclid himself grouped the 10th, 11th, and 12th axioms together with the three postulates, in one class, under the name *αἰτήματα*; the arrangement found in the modern editions of his *Elements* is the work of his successors, the ground of the alteration being, " the distinction between postulates and axioms, which has become the familiar one, is that they are the indemonstrable principles of construction and demonstration, respectively ". This distinction is not accepted by the modern school of geometers and should be ignored. Views on the question are, however, discordant.

We referred in an earlier chapter to the desirability of teaching boys carefully how to formulate their own definitions. The great purpose of a definition is the *precise discrimination*, yes or no, of actual cases. The definition must leave no doubt at all as to the identity of the thing defined. With beginners there is bound to be superfluity of statement, but refinement will come as the years go on. *Never provide pupils with definitions ready made.*

I find that very few boys are able to grasp the significance of the argument, *pro* and *con*, as to whether axioms (if, as they seem to be, they are definitions in disguise) are *a priori* intuitions, or experimental truths, or geometrical abstractions, or whether they define conventions. The arguments are for maturer minds. But it is easy to bring a boy to see that we are all liable on occasions to admit as self-evident propositions which we subsequently recognize not to be so.

Ask a class (a Fourth Form will do) to consider the axiom, " two straight lines cannot enclose a space ". Accept their usual definition of a straight line, viz. " the shortest distance between two points ", as determined by a stretched string.

Then obtain from them the admission that the straightest —the most direct—line (the geodesic) that can be drawn between two points on the surface of a sphere may be determined in the same way; then show that this line is necessarily part of a great circle. Next, show that any two such lines on the surface of a sphere, being parts of great circles, intersect in two points; hence, at all events on the surface of a sphere, any two " straight " lines *must* enclose a space.

But we live on the surface of the sphere, and any surface we call plane, no matter how small, must be part of a spherical surface, and therefore any two " straight " lines we draw, being parts of great circles, must meet, at two points 180° apart. Even if the two lines we draw on the paper are what we call " parallel ", the same thing applies.

Point out that this is not mere theory, it is prosaic fact, for (1) the earth is known to be a sphere and therefore all our so-called planes are parts of a spherical surface; and (2) canal engineers actually have to allow 8″ in the mile for sphericity; if they made the bottom of the canal absolutely "level", they would be cutting a channel which would eventually emerge at the spherical surface. If 3 poles of equal length be set up at equal intervals on a perfectly " level " stretch of land, say along the straight six miles of the Bedford level between Witney Bridge and Welsh's Dam, and a telescopic sight be taken from the top of the first to the top of the third, the line of sight will be seen to pass 5 or 6 feet *below* the top of the second. If from one end of the Corinth canal we look through the clear Greek air along the canal surface, the earth's curvature is readily *seen*; and the amount of curvature is easily measured by taking from one end of the canal a tangential telescopic sight to meet some object, say the hull of a ship, at the other end.*

In a Sixth Form the argument can be carried farther,

* There are no locks on the canal, and two of its four miles of length run through a cutting with an average depth of 200 feet. Tested at any point of its length, the canal surface is found to be perfectly "level", yet the water half-way along is demonstrably several feet " higher " than at either end.

and conviction carried home. At all events, enough can be done (with other axioms as well as this) to make boys critical, in the future, of principles that claim to be fundamental. An elementary lesson on " infinity " may serve to lead up to an analysis of such an axiom as " the whole is greater than its part ". On two or three occasions I have known useful Sixth Form discussions take place after the boys had read the more elementary parts of Mr. B. Russell's chapters on *Infinity*: see, for instance, pp. 179–82 of his *Knowledge of the External World*. *Some* notion of the mathematical significance of the term infinity should be given to Sixth Form boys; but the difficulties are real. Mr. Russell is our ablest expositor of these difficulties. His critical views are drastically destructive; his constructive views are not accepted by all mathematicians.

Mathematical Proof

We have all heard a small boy, when engaged in a lively argument with another small boy, say suddenly, " You can't prove it." What does he really want when he thus calls for a certificate of proof? It is very hard to say. Verification by demonstration? Authority?

Strictly, the *proof* of a proposition is its directly logical syllogistic derivation from other propositions which we know to be certain and necessary, and ultimately derivative, therefore, from definitions and axioms. *To that extent* every deduction from definitions and axioms is also the proof of the conclusion reached by it.

When a distinction is made between proof and deduction, the *proof* is regarded as the problem of deciding as to the truth of an hypothesis, of confirming it or refuting it. The proof of an affirmative is the refutation of the negative; and vice versa.

The mathematical forms of statement that have been devised to record the facts of a proof include the explicit mention of all the considerations needed to justify it against

any attack. But this record is made *after the proof has been achieved*, and in setting out the proofs we keep out of the record all our unsuccessful attempts, all our " scrap-paper " work, and include just those few links in the main chain that are well and truly forged.

Ultimately we go back to our definitions and axioms, and it is here where we are really so susceptible to attack. We never seem to be able to make ourselves armour-proof. Mr. Bertrand Russell, or somebody like him, will come along and inevitably find a joint where he can inflict a nasty wound. " Proof " is purely a question of degree. It is hopeless to attempt to find a means of dissolving mathematical error completely, and of exhibiting Truth in a white light, unassailable. The most we can do with boys is to train them both to be always on the look-out for mathematical shams and to hunt these down relentlessly.

It is fatuous to make Third or Fourth Form boys write out the general " proof " of, say, the division process of finding the algebraic H.C.F. What does the " proof " signify to them? So it is with " proofs " all the way up the school. When a Fifth Form boy has " proved " the binomial theorem ask him what it is all about; closely cross-examine him and show him how the term " prove " has been improperly used. Let him *establish* the binomial theorem; let him *show* that for a fractional index the theorem takes the same form as for an integral index, but do not let him pretend to " prove " either. Beware of using the term proof in connexion with mathematical induction. Even the sometimes substituted phrase " *proof* by repetition " is open to criticism.

" Pure " Mathematics

If a strictly logical treatment of mathematics implies, as some present-day mathematicians and others contend, a strictly abstract treatment, the objects with which mathematics deals are just symbols, devoid of content except such as is implied in the assumptions concerning them. This abstract

symbolism constitutes what is sometimes called " pure " mathematics, everything else being, strictly, " applied " mathematics, since it deals with concrete applications of an abstract science. Thus all the ideas of pure mathematics can (so it has been seriously contended) be defined in terms which are not strictly mathematical at all, but are involved in complicated thought of *any* description. If this be true, all the propositions of mathematics might be deduced from propositions of formal logic.

The distinction is not the same as the distinction of thirty or forty years ago. Then, " pure " mathematics included all ordinary work in algebra, geometry, and the calculus; applied mathematics included such subjects as mechanics, surveying, and astronomy. Pure mathematics was the mathematics of the blackboard; applied mathematics was supposed to be the mathematics of the laboratory, but too often experiment played no part at all. The " pure " mathematician was a very exclusive sort of person, rather despising those who did the weighing and measuring—necessary hacks they admitted, but hacks all the same.

Nowadays " pure " mathematics tends to shrink into a smaller compass. Formerly, the laboratory experiment with, e.g., Fletcher's trolley came (as it still comes) within the ambit of applied work, but all subsequent considerations of the curve produced belonged to algebra and trigonometry, and therefore constituted " pure " work. But not so now. The curve itself is now recognized as a thing of ink or chalk, and is therefore material; it is not really a *geometrical* curve, it is a black thing or a white thing that we make, and is only a crude representation of the true curve which, if we are " pure " mathematicians, it behoves us to consider.

Thus " pure " mathematics tends to become a new subject, a subject in the border region between the mathematics that ordinary people learn, and philosophy. The subject is a very serious one, a subject within the realm of " pure " or abstract thought; but it is not a subject for schools. It was a prominent Church paper that, a few years ago, be-

lauded the " pure " mathematician because he was a " good " man; the " applied " mathematician was necessarily led by his " impure " work to free thinking and infidelity! The small circle of eminent philosophical mathematicians whom we recognize as authorities on abstract mathematics must be proud of their testimonial. But the great majority—teachers and all others who are engaged in ordinary workaday mathematics, merely " applying " the basic principles laid down by the few—should ponder over the fate that is said to await all infidels!

Is it not a little—just a little—absurd to pretend that we teach " pure " mathematics in schools. The work we do is all applied work, the different forms of which are all a question of degree. As we go up the school, the concrete work receives a gradually deepening tinge of abstractness, but even in the Sixth the work is never more than partially abstract. To claim that the Upper Form geometry we teach is more " refined ", is " purer ", or is " more intellectual " than mechanics or astronomy is merely to provoke ribaldry.

It is sometimes said that Newton was a " pure " mathematician. But was he? He spent his life in rounding off the work of the astronomers from the time of the ancient Greeks and Egyptians to the time of his predecessor Galileo. Even the new mathematical weapon (the calculus) which he forged was forged for the purpose of pushing ahead with his investigations among real things. But he was certainly also a philosopher if by this term we mean a thinker who looks to his foundations. His researches did not, however, take him very far into purely abstract mathematics. He was too busy with such problems as that of showing that a falling stone and a falling moon are subject to the same law.

Though men could not deny the tremendous success of Newton's system of mechanics, though Laplace acclaimed it as final, yet there remained questionings. One distinguished critic after another felt doubts about his absolute space and time, and Einstein, setting out to satisfy them, not only did so, but in the doing he evolved a theory that included not

only all that Newton had done but those other points which Newton's theory could not be made to include.

Mathematics, like all other subjects, has now to take its turn under the microscope and reveal to the world any weaknesses there may be in its foundations. But this is not work for schoolboys. To boys the main objects of mathematical study must continue to be real things, even if those real things are only figures produced by ruler and compasses. Mr. Russell criticized Euclid for drawing figures, because those very figures were partly responsible for preventing Euclid from building up the flawless system he aimed at. Schoolboys, working on a lower plane, would, without figures, be helpless.

Relative Values

In one respect at least, mathematics seems to be a subject quite unlike other subjects. Its discoveries are permanent. The theorem of Pythagoras for instance is as valid to-day as 2000 years ago. The majority of the mathematical truths we now possess we owe to the intellectual toil of many centuries, and a student who desires to become a mathematician must go over most of the old ground before he can hope to embark on serious research. To the uninitiated it is impossible to make mathematical truths clear. The great theorems and the great results of mathematics cannot be served up as a popular dish, and this inaccessibility of the subject tends to make it rather odious to those whose early grounding was of little account.

Although mathematics does not lead to results which are absolutely certain, the results are incomparably more trustworthy than those of any other branch of science. Still, if mathematics stands aloof, if it is not turned to practical account in other branches of science, it remains a useless accumulation of capital, almost an accumulation of lumber.

Can it be maintained that, as an intellect-developing instrument, mathematics ranks first amongst the different subjects of instruction, or even "first among equals"?

Mathematical reasoning is in some respects simpler than scientific reasoning; the data are clearer. It is simpler than linguistic reasoning; there are no probabilities to weigh. It is simpler than historical reasoning; there are no difficult human factors to consider. But in one respect mathematical reasoning is the most difficult of all, and this is because of the inherent difficulties of mathematical analysis, whether the analysis is the Third Form analysis of the data of a simple geometrical rider, or whether it is the more serious analysis in Sixth Form work. Relatively speaking, there is no other difficulty. The analysis once effected, the rest is plain sailing.

I attach very great value to mathematical instruction, but I deny that the virtue of the instruction lies in anything of the nature of super-certitude.

Picture-making by Physicists: the Dangers

It has been said that, from the broad philosophical standpoint, the outstanding advances in the physics of the last 20 years have been the theory of Relativity, the theory of Quanta, the theory of Wave-mechanics, and the dissection of the atom. But it would be more correct to say that it is the surrender of our rather aggressive certitude about the nature of things, and our recognition that we are still ignorant of the nature of ultimate reality. We now know that our pictures were all wrong. How faithful we were to the æther as a quivering jelly of inconceivable density! How we loved to tie knots in ætherial vortex-rings!

Certain seaside resorts are on a cliff, with an upper esplanade brilliantly illuminated at night, and, 10 or 15 feet below, an unlighted walk protected by a low wall. The shadows of passing people and vehicles on the upper esplanade are cast upon this low wall, and may be watched by a person seated on the lower level in the dark. Imagine such a person to have been entirely cut off from human kind since his early childhood, to be fastened to his seat permanently in

the dark, and to see nothing but the shadows in front of him. We may consider him endowed with powers of reasoning and with some amount of mathematical power.

We can imagine him observing the shapes, sizes, movements, and velocities of the shadows, gradually sorting out resemblances and differences, and eventually establishing a number of equations embodying the whole of his sense data. These equations would be strictly representative of reality *as he knew it.* But suppose he now began to speculate, and to attempt to infer from his equations the nature and properties of the original things, animate and inanimate, that had cast the shadows on the wall. Would not the rapidly moving motor-car be given pride of place, and would not the slow-moving human being be looked upon as of secondary importance? Would not all his conjured-up mental pictures unfailingly be a mere travesty of reality? In his allegory of the cave, Plato warned us, more than 2000 years ago, of fallacies of this kind.

Physicists are learning that the greater part of their observations are not observations of reality but of the shadows of reality. When they invent an atomic gymnasium and a system of electronic gymnastics, they know well that they are just speculating wildly. On the other hand they know that the mathematical formula they have established is, though uninterpretable, in *some* way representative of reality.

Einstein's formula for gravitation is universally accepted. His cosmology is *not* accepted. No cosmology can be, for it is necessarily hypothetical, speculative, fanciful.

Teach the boy that the physicist as a research worker and mathematician is a man to be respected, but that while we may admire his pretty pictures, we are quite certain that none of these will ever make Old Masters.

Books to consult:

1. Professor Whitehead's books.
2. Mr. Bertrand Russell's books.
3. *Mathematical Education*, Carson.

4. *Mysticism in Modern Mathematics*, Hastings Berkeley.
5. *Science and Hypothesis*, Poincaré.
6. *Les étapes de la philosophie mathématique*, Brunschvicg
7. *De la Certitude logique*, Milhaud.
8 *The Human Worth of Rigorous Thinking*, Keyser.

CHAPTER XLVII

Native Genius and Trained Capacity

Russell *versus* Poincaré

Mathematical philosophers, like the philosophers of other schools, naturally have greater faith in their own systems than in the systems of their rivals. Over one point in particular they are hopelessly at variance, namely, as to the respective rôles that logic and intuition play in the origin and development of mathematical ideas.

It was Aristotle who worked out the principles of deductive logic, and his scheme was universally accepted almost down to the close of the Victorian era. In the middle of the last century, George Boole, a distinguished mathematician, pointed out how deductive logic might be completely symbolized in algebraic fashion. Given any propositions involving any number of terms, Boole showed how, by a purely symbolic treatment of the premisses, logical conclusions might infallibly be drawn.

At the beginning of the present century (in 1901), Mr. Bertrand Russell said,* " Pure mathematics was discovered by Boole. His work was concerned with formal logic, and this is the same thing as mathematics "; and again:† " The fact that all mathematics is symbolic logic is one of the greatest discoveries of our age, and the remainder of the

* In the *International Monthly*. † *Principles of Mathematics*.

principles of mathematics consists in the analysis of symbolic logic itself." In their *Principia Mathematica*, the aim of Mr. Russell and Professor Whitehead is to deduce the whole of mathematics from the undefined logical constants set forth in the beginning. And in Signor Peano's *Formulario*, the different branches of mathematics are " reduced to their foundations and subsequent logical order ". Moreover, in his work *Les Principes des Mathématiques*, M. Couturat expresses the opinion that the works of Russell and Peano have definitely shown not only that there is no such thing as an *a priori* synthetic judgment (i.e. a judgment that cannot be demonstrated analytically or established experimentally), but also that mathematics is entirely reducible to logic, and that intuition plays no part in it whatever.

But Henri Poincaré, who was described by Mr. Russell himself as " the most scientific man of his generation ", flouted the logistic contention. He denied that logistic (mathematical logic) gave any sort of proof of infallibility, or that it is even mathematically fruitful. It did certainly force us to say all that we commonly assume, and it forced us to advance step by step. But its labels are labels of consistency and do not in any way refer to objective truth. " The old logistic is dead." " True mathematics will continue to develop according to its own principles." " Fundamentally its development depends on intuition."

Mr. Russell, in reply, said that mathematical logic was not " opposed to those quick flashes of insight in mathematical discovery " which Poincaré " so admirably described ". Nevertheless, the main outlooks of the two men seem to be radically opposed.

Mr. Russell has said elsewhere, " Mathematics is the science in which we do not know whether the things we talk about exist, nor whether the conclusions are true ". Apparently, then, Mr. Russell admits at least that logistic is not capable of discovering the mathematician's ultimate premisses, and is therefore not capable of establishing the truth of its final conclusions. It does, however, determine

the consistency of our conclusions with the premisses, and this is its undeniable merit.

The Origin of New Mathematical Truths

If ultimate mathematical truth is not discoverable by logistic, whence is its origin? Has it already an existence (as some contend) independent of us personally, something supra-sensible, already complete in itself, existing from the beginning of time, waiting to be discovered? is it of *a priori* origin? or is it actually created by mathematicians?

The term *a priori* is ambiguous. Literally it signifies that the knowledge to which it applies is derived from something *prior* to it, i.e. is derivative, inferred, mediate. The metaphor involved in " prior " suggests an infinite series of premisses. But the term *a priori* is also often used to indicate that certain general truths come to the mind, to begin with, as heaven-born conceptions of universal validity, and are thus " prior " to all experience. Strictly, however, all *a priori* truths are derived truths. But derived from what?

The mind seems to have a natural capacity for dictating the forms in which its particular experimental data may be combined. We may therefore correctly speak of the mind's creative *powers*, though not of its innate *ideas*.

The mind's undoubted power of detecting identity and difference, co-existence and succession, seems to be original and inborn. Still, the power is exercised only on a contemplation of actual things, from without or from within, and all such primitive judgments are individual. The mind compares two things and proclaims them to agree or disagree. The judgment is immediate, and it is felt to be necessary; it is irresistible and does not admit of doubt; it seems to be independent and to hang upon nothing else, and seems therefore to be primitive. But although the power is innate, this does not mean that the judgments themselves are innate.

As primitive judgments are immediate, they are sometimes described as *intuitive*.

Intuition and Reasoning

An intuition seems to be a general judgment immediately pronounced concerning facts perceived. But an intuitive judgment is as liable to error as is a reasoned judgment.

There is a natural tendency to ascribe to intuition a peculiar authority, for it seems to confront us with an irresistible force foreign to the products of voluntary and reflective experience. But knowledge derived from intuition is as much experiential knowledge as directly conscious knowledge, and it is just as fallible.

If we put on one side our purely primitive judgments, it seems very probable that, fundamentally, intuition and reasoning are identical, the former being instantaneous, the latter involving the notion of succession or progress. The difference then would be merely difference of time, every judgment of the mind being preceded by a process of reasoning, whether the individual is able to recollect it or not.

There are times when a great new truth suddenly comes to the mind of a mathematician. The combination of factors contributing to it seem to be a garnered knowledge derived from accumulated experience, a complete analysis of the given, a conscious connected reasoning, a systematic method of working, a natural capacity, and, finally, a flash of intuition. At some particular moment, the new truth flashes upon the vision as if light from all the other contributing factors was suddenly focused on the same point.

The Limitations of the Teacher's Work

Does not something of the same kind happen on a small scale when an intelligent schoolboy is solving a difficult problem? All ordinary methods of systematic attack may have failed him, yet light suddenly comes. Whence? Who shall say? Something from the rules of logistic, doubtless; something from the boy's store of mathematical knowledge;

something from the boy's power of analysis of data; but most of all from the boy's own native capacity. It has been suggested that the truth suddenly emerges from a chance combination of the boy's data. We may put it that way if we like, but the " chance " seems to be very much more than an affair of mere randomness.

If the boy's own native capacity is small, will the light appear? Can skilful teaching make up for native deficiency? In a considerable measure, yes; in a large measure, no. I do not think that great mathematical skill can ever be acquired by a boy with little natural mathematical endowment. We may meet with a considerable measure of success when we teach mathematics to average boys, but we shall never succeed in making such boys mathematicians. The boy whose average mark is 60 per cent is probably greatly indebted to his teacher. The boy whose average mark is 90 per cent probably owes very much more to nature.

Who can draw the line between the work of the boy's mind and what is skilfully presented to his mind to work upon? As teachers we have to teach methods, we have to teach analysis, we have to teach logic, we have to provide different types of mathematical knowledge. But can we do more? Can we increase the boy's own native mathematical capacity? Competent opinion answers that the old proverb of the silk purse still holds good.

CHAPTER XLVIII

The Great Mathematicians of History

Few boys know anything about the great mathematicians except by name. " No time," says the mathematical master, a statement that cannot be denied. Yet, in order to include

at least a little of the history of mathematics, I would sacrifice something else. Let a few of the really great mathematicians live once more, and let them be presented to the boys not merely as mathematicians but as human personalities. If it is to be but a few, who are the few to be? Personally I would select, from the ancients, *Archimedes* and *Pythagoras*; from Englishmen, *Newton* and *Cayley*; from Scotsmen, *Napier* and *Maclaurin*; from Irishmen, *Hamilton*; from Frenchmen, *Descartes* and *Pascal*; from Germans, *Gauss* and *Leibniz*. Others will plead for the inclusion of Sylvester, Clerk Maxwell, Poincaré, Einstein, and many more. Very well: the more the better. Let those selected come back and live their lives over again, and tell us exactly how they came to make their great discoveries. Let the boys know something of each great mathematical discovery, of each mathematical leap forward, and of the men responsible.

One of the most readable books on the subject is Sir Thomas Heath's *A History of Greek Mathematics*, but of course it deals only with one period. It may be supplemented by Mr. Rouse Ball's *A Short Account of the History of Mathematics*, D. E. Smith's *History of Mathematics*, Professor Turnbull's *The Great Mathematicians*, and F. Cajori's *A History of Mathematics*.

CHAPTER XLIX

Mathematics for Girls

Opinions still differ about the relative mathematical ability of boys and girls. My own conclusions, derived from observations extending over a long period, are these, though I express them with some reserve:

1. That a very small minority of girls in an average

Form are as able, mathematically, as the ablest boys in the corresponding Forms in boys' schools, perhaps 3 per cent of girls against 10 per cent of boys.

2. That the *average* mathematical ability of a Form of girls is rather lower than that of the corresponding Form of boys.

3. That the interest of girls in mathematics is decidedly less than that of boys.

4. That all girls should be compelled to take mathematics for 2 years (say, 11 to 13), but at the end of this time girls making no useful progress should give up the subject, arithmetic excepted, though the arithmetic should contain a certain amount of quite informal algebra and geometry, mainly by way of mensuration and the free use of formulæ. Thus, in a large school with three or four parallel Forms at each stage from 13+ to 16+, the bottom Form would, as a rule, be a non-mathematical Form.

However, the question is a woman's question, I would even say a question for women teachers generally rather than for mathematical mistresses only. Men are not likely to be able to consider adequately all the factors involved, or, indeed, to know them; and I am doubtful if a committee consisting only of mathematical mistresses would find it easy to free themselves entirely from the prejudice which naturally attaches to one's own subject. A committee for discussing the question should include a proportion of non-mathematical mistresses, but only those who could discuss the question objectively.

The Girls' School Committee of the Mathematical Association issued a special Report on Mathematics in Girls' Schools in 1916. The Report was revised in 1928, and reissued in 1929. The Report is full of valuable suggestions, and should be read by all who are engaged in teaching mathematics to girls. There are, however, a few very able mathematical mistresses whose attitude towards the Report is a little critical.

CHAPTER L

The School Mathematical Library and Equipment

The Library

In the 134th number of the *Mathematical Gazette* Dr. W. P. Milne writes a particularly interesting and suggestive article on this subject. A few paragraphs may be quoted:

"There is nothing more extraordinary in the educational world at the present day than the change of attitude amongst teachers towards the subject they have to teach. Teachers of bygone ages were to a large extent content to regard their pupils as so many buckets into which they were content to pour a prescribed amount of intellectual material. In most cases the teacher himself did not know the sources whence came his stock-in-trade. Compare the attitude of teachers of the present day, which is consciously or unconsciously, explicitly or implicitly, utterly different. For the teachers of to-day, their subject is not a finished structure; it is an organic growth, ever growing, ever changing, ever being added to, ever having new methods devised and old cast aside.

"I do not think we quite realize what we owe to the Technical Colleges in getting rid of the old view. For the students of Technical Colleges, mathematics is essentially an instrument. They are willing to take much upon trust. They are always on the look-out to use what they have learned, as a weapon. The result is inevitable. Mathematics acquired under these conditions may be imperfect, may be rough-and-ready, but the subject is at all events alive; it is confined within no rigid barriers, and is eternally moving and growing and changing. In the Technical College classroom, the propositions of geometry do not pass before the

audience in stiff and stately procession as the actors on the Greek stage; they are rather as the people of the market-place, hot and throbbing with life. Another great and incalculable influence in helping to overthrow the old rigid and detailed view of a subject lies in the fact that men trained on research lines are slowly percolating through the staffs of the schools, and their attitude as teachers reacts on the taught. The combined results of these movements and tendencies is that both master and pupil recognize that it is possible to know only a very little about this ever-growing ever-changing subject, but what knowledge they do possess fills them with an eagerness to ask for more. There is in fact a widespread feeling among teachers of mathematics that every school ought to possess an up-to-date mathematical library, so that both teachers and taught may keep abreast of the times in their school work, and also catch a glimpse of the great regions of mathematical thought that lies ahead.

"The school library should consist of two parts, one for scholars and one for masters, though these two should not be mutually exclusive. In general tone and constitution the masters' portion should be altogether heavier and more serious than the scholars', because the teachers are older, more experienced, and have greater width of knowledge and intellectual power than the scholars. On the other hand, the senior division of the library is not intended for highly specialized experts in various branches of mathematics; the books should present rather 'First Courses', as they are popularly called, in the various subjects in which they deal. What schoolmaster does not want to know something more about that rapidly extending subject of Nomography? What modern teacher of mathematics does not want to know something about the theory of gunnery, submarines, stability of aeroplanes? On the other hand, the scholars' library should be more suited to their age and capacity; mathematical recreations and puzzles; Rouse Ball's *History of Mathematics*; elementary history and theory of astronomy; the recent

book on Astronomy by Professor R. A. Gregory; *Pioneers in Science*, by Sir Oliver Lodge; and so forth.

" One cannot and ought not to lay down hard and fast rules about the composition of the school library. A convenient method, however, would be to collect and catalogue the books under the headings of : (1) Biography, (2) History, (3) Philosophy, (4) Mathematical Analysis, (5) Geometry, (6) Applied Mathematics, (7) General. Such a composition gives plenty of scope for variety and elasticity."

The whole article should be read by all mathematical teachers.

In 1926, the Mathematical Association issued a *List of Books* " likely to prove useful for reference to teachers and their more advanced scholars ". There are about 100 in all, mostly selected standard works, tried and proved, and recommended for their real worth. No school can do better than begin with these, and add others from time to time. It should not be forgotten that the French and the Germans are much keener mathematicians than we are, and that some of their standard works in mathematics rank as classics.

Those unfamiliar with German may be interested to know that enough about the language may easily be mastered in a month for any technical German work to be read with ease. (This does not of course apply to the speaking or writing of German!)

At the end of the Report on *Elementary Mathematics in Girls' Schools* (Mathematical Association) are some useful appendices:

Appendix C.—Lists of Books for the Libraries of Girls' Schools (an admirable selection).

Appendix D.—List of some of the Articles from the *Gazette* that are of general interest to teachers. (Many others may be found by referring to back numbers of the *Gazette*.)

Appendix E.—List of Reports published by the Mathematical Association.

A bound copy of the *Gazette* should be added every year to the teachers' section of the school library. The articles

are almost always helpful to the practical teacher, and every copy of the *Gazette* is full of useful hints.

The teachers' section of the mathematical library should also contain the Board of Education's *Special Reports on Educational Subjects*, numbers 12, 13, 15, 16, 18–34, all dealing with mathematics in some form or other, all written by well-known experts.

Mathematical teachers may be reminded that, whilst their school libraries are in the making (or at any other time), they may, if they are members of the Mathematical Association, borrow books from the Association's own library, which is under the care of Professor Neville at Reading. A list of books in this library is published by the Association.

Mathematical teachers should keep an eye on the columns of the *Mathematical Gazette*, *Nature*, *The Educational Outlook and Educational Times*, *The Times Educational Supplement*, and *The Journal of Education*, for reviews of new books.

Equipment

Schools provided with a separate Mechanical Laboratory are usually well stocked with apparatus for teaching mechanics and practical mathematics, including such things as screw gauges, calipers, verniers, spherometers, opisometers, slide rules, a planimeter, cycloidal curve tracers, an ellipsograph, link-motion apparatus, a binomial cube, a variety of wooden and wire geometrical models, and models to illustrate projection. When there is a special mathematical laboratory, the room is usually given up to the Lower and Middle Forms, and few instruments of precision are to be found in it. Indeed these are generally too expensive to be purchased, but they can always be seen and examined at the Science Museum at South Kensington.

Not all mathematical teachers are aware of the fine collection of instruments in the mathematical section of the Science Museum. I have often seen classes of boys under the guidance of a teacher, examining the instruments, apparatus,

and working models in the physics, mechanical, and engineering sections of the Museum, but the mathematics and the geodesy and surveying sections are usually deserted. Amongst the things that all boys should see in these sections are the following:

1. Instruments for drawing curves: trammels, ellipsographs, parabolographs, &c.
2. Ellipsoids, hyperboloids, paraboloids: various models of curved surfaces formed by (i) intersecting layers of stiff paper, (ii) series of stretched strings. (Some of these are singularly attractive and wonderfully instructive. I remember as a youth spending my very scanty supplies of pocket-money for several months in purchasing materials and tools for reproducing some of these.) Pairs of intersecting cylinders and of cones. Roof structures. Skew bridges.
3. Instruments for preparing perspective drawings from plans and elevations.
4. Sundials and clocks
5. Calculating machines: Pascal's, Morland's, Stanhope's, the Brunsviga.
6. Difference and analytical engines: Babbage's, Scheutz's.
7. Slide rules: straight, circular, spiral, cylindrical, and gridiron types.
8. Instruments for solving equations.
9. Linear integrators, planimeters, integrometers, integraphs. Professor Boys' curve-drawing integrator, and his model of a polar planimeter.
10. Harmonic analysers and integrators.
11. Jevons' logical machine.
12. Rangefinders and tacheometers.
13. Clinometers, azimuth compasses, prismatic compasses, early theodolites, altazimuth theodolites, geodetic theodolites, zenith sectors and telescopes.
14. Mine surveying and marine surveying instruments.
15. Air survey maps, geodetic triangulation maps, &c., &c.

The services of a well-informed museum guide may be

obtained for the asking; he will accompany visitors on their rounds, and explain the instruments under examination.

Many of the machinery models in the Science Museum are in motion, being driven by compressed air. Visiting boys should always be taken to see them.

Visits may also profitably be made to such places as Cussons' Technical Works, Manchester; to Sir Howard Grubb, Parsons, & Co.'s optical works at Newcastle; to Hilger's; and to some of the better-known instrument makers.

Teachers interested in practical measuring might read Mr. F. H. Rolt's *Gauges and Fine Measurements*, the standard work on measuring machines, instruments, and methods.

I am doubtful if any part of the mathematical equipment, even instruments of historical interest, should be kept in the school museum. The school museum is apt to be a holy place, to be visited seldom, and then silently lest the dust should be disturbed.

APPENDIX I

A QUESTIONNAIRE FOR YOUNG MATHEMATICAL TEACHERS

1. In teaching arithmetic, to what extent is it advisable to give a logical justification of the rules you teach at the time you teach them? Illustrate your answer by reference to division of vulgar fractions, and to multiplication of decimals.

2. What is your general plan for teaching the tables to young children? Doubtless you are convinced of the necessity for plenty of ding-dong work to make the tables perfect. What is the best means of dealing with a visitor who suggests that such work is unintelligent?

3. Draft a few notes suitable for a Student Teacher, instructing him how to proceed when teaching subtraction to beginners.

4. What is the proper function of so-called " mental " arithmetic? Discuss the merits of (*a*) oral work, and (*b*) practice on paper, in mathematical teaching, and assign to each type its proper place in the different Forms of the school.

5. Where in the school would you begin algebra, and how? Outline a suitable course of work for the first two years, and show how you would link it up with the mensuration, arithmetic, and geometry.

6. A slow-witted boy of Form IV tells you that he does " not quite see " how the product of $\sqrt{3}$ and $\sqrt{5}$ can be $\sqrt{15}$. On questioning him you find that he has failed to grasp the essentials of the lesson you have just given. Sketch out a new and very simple lesson to meet the case.

7. Would you include the theory of annuities in a school mathematical course? If so, on educational grounds or on utilitarian grounds?

8. What is your experience of teaching logarithms to bottom Sets? Is the practice desirable? If so, on what grounds?

9. The pedagogical treatment of parallel lines is admittedly very difficult, and no teacher would suggest that a rigorous treatment at the outset is possible. How would you make the subject more and more logically exacting in the successive Forms from II to VI?

10. Defend the principle that formal definitions in geometry should never be provided ready-made by the teacher.

11. Criticize the value of the work on graphs commonly done in schools. How would you modify such work?

12. In what part of the school mathematical course would you first introduce the idea of incommensurables? Should it be included at all in a school with a leaving age of 16? If so, justify your opinion.

13. Sketch out a course of numerical trigonometry for D Sets (Forms III to V), introducing as much field work as possible and reducing academic work to a minimum.

14. Whose business is it to teach mechanics, the teacher of mathematics or the teacher of physics? If the former, how is he to proceed if he has had no training as an engineer? If the latter, how is the mathematical side of the subject to be dealt with effectively?

15. In a Sixth Form course of mechanics, how far is it (*a*) possible, (*b*) desirable, to replace some of the elegant but rather academic and useless problems in ordinary dynamics by an elementary course in the dynamics of astronomy and geology?

16. Kepler's Third Law is found to hold good for the earth as well as for the other planets. How would you demonstrate to a Sixth Form that this fact alone affords strong evidence that the earth itself is a planet?

17. Do you find that school mathematics has improved since the calculus has been taken up in Forms below the Sixth? Defend (*a*) its inclusion in, (*b*) its exclusion from, such Forms. What do you consider to be the educational gain of a course of analysis in school work for boys not going on to the University?

18. Mathematicians have investigated many geometrical curves that have little relation to practical life. On the other hand, they have almost neglected to investigate the curvature of such common things as eggs. The reason sometimes put forward is that no two exactly similar eggs have ever been discovered. Assuming this to be a fact, does the fact justify the neglect? If it does, what have you to say about the mathematics of biology generally?

19. The physiological exchange which is inseparable from active life is conducted through limiting surfaces, external and internal. Provided the form remains unchanged, the bulk of a growing cell, tissue, or organ increases as the cube of the linear dimensions, the surface only as the square. Accordingly, as growth proceeds, the proportion of surface to bulk decreases, until a point of physiological inefficiency is reached.—Your biological colleague, who is not a mathematician, appeals to you for help over this difficult question of inter-relation between growth and form, whether applied to external surfaces or to internal conducting tracts in plants and animals. Sketch out a course of lessons dealing with the mathematics of this biological "size-factor".

20. Sixth Form work on (*a*) Capillarity and (*b*) Viscosity is almost

always treated mathematically in the main, experiment playing a very small part. Assuming that this treatment is correct, it is clearly the business of the mathematical master to be responsible for it. Give an outline of three or four lessons in each subject, and say exactly what help, and where, you would expect from your Physics colleague.

21. A Sixth Form boy tells you his Physics teacher has said that Van der Waals' equation is not sufficiently representative of the facts, and he asks you if you will explain " the mathematical fallacy in the equation ". Devise a suitable reply.

22. Write out brief notes of a lesson on Lissajous figures, including a demonstration of the method of establishing the general equations $\frac{x}{a} = \sin m\theta$, $\frac{y}{b} = \sin(n\theta + \beta)$, where m and n are proportional to the frequencies of the horizontal and vertical vibrations. Should the mathematics or the physics master demonstrate the fact that the gradual changes from one figure to another depend on the gradual change of β? Why?

23. Have you found School Certificate examination requirements clash with your own thought-out plans for mathematical teaching? If so, how? Are you quite sure that the clashing is real? If so, take a recent School Certificate paper, modify it in accordance with your own views, and see if your colleagues agree.

24. Here are two well-known old problems, much too difficult for the average boy:

(i) A spider at one corner of a semicircular pane of glass gives chase to a fly moving along the curve before him, the fly being 30° ahead when the chase begins. Each moves with its own uniform speed. The spider catches the fly at the opposite corner. (1) Trace the spider's path; (2) determine the ratio of the speeds of the two insects.

(ii) A horse is tethered to a stake in the hedge of a circular field, the rope being just long enough to enable him to graze over half the area of the field. Determine the length of the rope in terms of the diameter of the field.

Is it worth while spending time over such examples of the type of (i) the first, (ii) the second? Discuss examination conundrums generally and their injurious effect on rational school practice.

25. Young mathematical teachers generally try to adopt methods which, in exposition, shall be mathematically flawless.—Have you ever known " intelligent " teaching to be productive of unintelligent reception? If so, how do you explain it? Who is likely to be the more to blame, the teacher or the boy?

26. What do you understand by a *proof*? Criticize the term as commonly used in mathematical teaching. At what stage do you advise the introduction of simple proofs? When would you introduce

formal proofs of the fundamental theorems of geometry (congruent triangles, angles at a point, parallels, &c.)? What is your opinion of the value, to boys, of the usual book proofs of H.C.F. (algebra) and the Binomial Theorem?

27. What do you consider to be the essential distinction between " pure " and " applied " mathematics? A chalk or a plumbago line drawn with the compasses is as concrete a thing as the iron rim to a flywheel: are you therefore doing " purer " work in the classroom than in the laboratory? How do you defend the use of the term " pure " in any of your mathematical teaching except perhaps in the work of those Sixth Form specialists who are capable of grasping the A B C of mathematical philosophy?

28. How do you account for so few persons being interested in mathematics, although the great majority of educated people must have done a fair amount of mathematics at school? How would you modify the present course of school mathematics in order to ensure a greater permanent interest in the subject on the part of learners?

29. Plan out a course of work for the Sixth Form, specialists and non-specialists alike, calculated to develop and maintain a life-long and intelligent interest in mathematics, the course not to be so recreational that rigour is seriously sacrificed.

30. What commonly taught topics would you advise should be deleted from a school course? Why? What others would you substitute?

31. A boy asks the question, " Exactly when did the twentieth century begin?" How would you answer it? How would you deal with his meridian difficulty, especially if he lived on the 180° meridian? How would you make this difficulty a good jumping-off place for introducing first notions of Relativity?

32. Devise courses of instruction for:

(i) Junior Elementary Schools, 8+ to 11+.
(ii) Senior Elementary Schools, 11+ to 14+.
(iii) Central Elementary Schools, 11+ to 15+.
(iv) Preparatory Schools, 8+ to 13+.
(v) Secondary Schools, 11+ to 18+.

In the last case assume that the five Forms Upper III, Lower and Upper IV, Lower and Upper V, each consists of four graded Sets, A, B, C, and D. Show how the work of the Sets should differ in (i) content, (ii) treatment.

33. Your school is of sufficient importance to receive each year from a leading Training College a few students who have taken good degrees in mathematics, and you are entrusted with the purely practical side of their professional training. What measures would you adopt to make such training effective?

34. It is sometimes said that a classical training is " necessarily " productive of a much greater appreciation of the beautiful than is a mathematical training; that a real appreciation of a beautiful thing

is always accompanied by a glow, whereas a mathematician's appreciation is always reasoned and therefore cold. The contention is, of course, silly, but how would you formulate a defence that would carry conviction to an opponent's mind?

35. You are appointed chief mathematical master of a new school where a room 36 ft. by 25 ft. has been set aside for a mathematical laboratory. Give full details of the fittings and equipment you would provide for elementary and for advanced work, the expenditure to be limited to £500.

36. What is the proper function of school textbooks in mathematics, apart from the exercises they provide? Distinguish between textbooks for Lower Forms and those for Higher.

APPENDIX II

NOTE ON AXES NOTATION

In drawing graphs I have followed the usual notation. But the following figures indicate a useful reform:

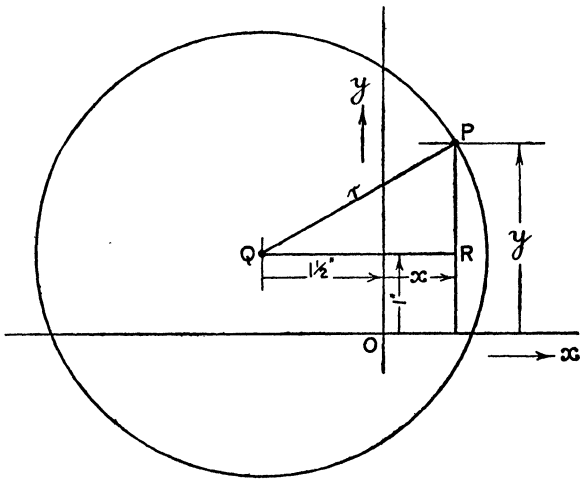


Fig. 1

1. Mark the x and y axes merely by arrows pointing in the positive direction. (Figs. 1 and 2.)

2. Indicate dimensions by means of single-headed lines instead of the draughtsman's usual double-headed lines, the direction of the arrow indicating the actual direction in which the line is to be measured. For instance, the length QR is $+(x + 1\frac{1}{2})$, and the length RP is $+(y - 1)$; the latter is the difference in the dimensions measured in the positive direction. (Fig. 1.)

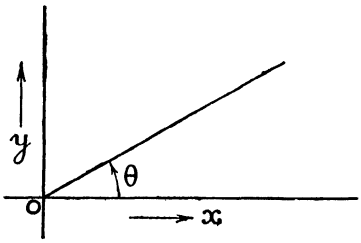


Fig. 2

3. Do not use a dimensional line for length r, since by convention a radius vector is always taken with the positive sign. (Fig. 1.)

4. For angles, use a single-headed (not a double-headed) arrow, and let the arrowhead indicate the direction of rotation. (Fig. 2.)

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