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THE PRINCIPLES
OF
PROJECTIVE GEOMETRY

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THE PRINCIPLES
OF
PROJECTIVE GEOMETRY
APPLIED TO THE
STRAIGHT LINE AND CONIC

by

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PREFACE

THIS book has been written in the hope of placing in the hands of the pupil, who has mastered the portions of Euclid usually read, most, if not all, of the Pure Geometry which he requires in order to proceed to an Honours degree in Mathematics at any one of the Universities of Oxford, Cambridge, London and Manchester While the subject has been primarily considered from the projective point of view, considerable trouble has been taken to deduce the more important metrical properties of Conics from the projective theorems with which they are related In an addendum a certain number of elementary theorems of a non-projective nature have been collected for reference

The author trusts that this book may do something to encourage the student not to neglect the methods of pure geometry In every other branch of mathematics analysis now reigns supreme—even in geometry it is fast gaining a predominance Twenty years' experience as a Teacher of Projective Geometry and ten years' experience as an examiner of the University of London have led the author to regard this as a misfortune When the great landmarks of Projective Geometry—the theorems of Pascal and Brianchon, of Carnot and Desargues, together with their immediate consequences—are clearly placed before the student, the author has found that even in the younger student an enthusiasm is aroused which is wanting in his study of other branches of Mathematics In the examination room it has been found that students who have mastered and absorbed the principles of Pure Geometry have taken a superior place to those who depend on a facility for handling analytical expressions Such a facility with practice may undoubtedly be acquired by most pupils, but for all who are worthy to take a mathematical degree the study of Pure Geometry is a matter of primary importance

To the works of Salmon and Cremona the author—like all others who now attempt to deal with the subject—is deeply indebted He has freely consulted the examination papers of the Universities of Oxford, Cambridge and London, which set the standard of geometrical teaching in this country To the Savilian Professor of Geometry in the University of Oxford he has to express his indebtedness for much valuable assistance and advice He has freely consulted the notes of the lectures delivered by the late Professor H S Smith To Mr G S Le Beau, Mr S G Soal and other members of the mathematical staff of the East London College he offers his sincere thanks for their assistance more especially in connexion with the proof sheets of this book

The hopelessness of tracing the various theorems to their original discoverers has led the author to abstain from a task which is beyond his power to perform satisfactorily, and as a consequence names have only been associated with theorems when they are in ordinary use for their identification In most cases the enunciations of theorems have been given before the proofs and an effort has been made to present the proofs in a form suitable for reproduction In the case of most of the important theorems alternative proofs have been given The properties of the involution have been largely used as they render it possible to overcome some of the difficulties which otherwise arise through points and lines being real in certain cases and in certain cases imaginary The last chapter is to some extent an introduction to the future consideration of the "imaginary" in connexion with the conic This subject has been postponed for a subsequent volume in which the author hopes to deal with it from a geometrical point of view

J L S H.

East London College,
March 1913

CONTENTS

CHAPTER I

PAGE

Notation 1
Principle of duality 2
Definitions 4

CHAPTER II

Measurement of distances 6
Measurement of angles 9
Projective forms anharmonic 12

CHAPTER III

Projective forms harmonic 24

CHAPTER IV

Conical projection and plane perspective 31

CHAPTER V

Applications of projection 50
Applications of plane perspective 52
Triangles in perspective 54

CHAPTER VI

Anharmonic forms 58
Ranges and pencils in perspective 58
Ranges and pencils projective 61
Ranges and pencils superposed 70

CHAPTER VII

Harmonic forms 81
Harmonic property of the quadrangle and quadrilateral 82
Harmonic perspective 85

CHAPTER VIII

Involution 91

CHAPTER IX

PAGE

Coplanar figures 111
Problems of the first degree 120

CHAPTER X

Projective forms in relation to the circle 124
Anharmonic property 125
Pole and polar 130
Circles in self perspective 136

CHAPTER XI

Special properties of the circle 142
Inverse points 142
Coaxal circles 144
Circles in perspective with each other 148

CHAPTER XII

Projective theorems for the circle 160
Carnot's theorem 160
Pascal's theorem 162
Desargues' theorem 163

CHAPTER XIII

Projective forms in relation to the conic 165
Definition, construction and anharmonic property of the conic 165
General properties of a conic 172
Properties of the foci of a conic 177
Alternative definitions of a conic 179

CHAPTER XIV

Projective forms in relation to the conic 186
Carnot's theorem 186
Pascal's theorem 189
Desargues' theorem 194

CHAPTER XV

Deductions from the anharmonic property of the conic 200
from the pole and polar properties of the conic 202
from Carnot's theorem 207

CHAPTER XVI

Deductions from Pascal's theorem 212
from Desargues' theorem 217
from the complete inscribed quadrangle and circumscribed quadrilateral of a conic 229
Conics in self-perspective 230
The rectangular hyperbola 232

CHAPTER XVII

	PAGE
Construction of self-corresponding elements	235
Problems of the second degree	237
Construction of conics under various conditions	240

CHAPTER XVIII

Triangle loci and envelopes	247
Pole and polar loci and envelopes	249
Anharmonic and general loci and envelopes	254

CHAPTER XIX

Six points on a conic and six tangents to a conic	259

CHAPTER XX

Theorems concerning two conics	266
Common conjugate points	266
Conics in self-perspective	274
Conics in perspective with each other	276
Conics with double contact	283
Harmonic locus of two conics	293

CHAPTER XXI

The constant of an involution	302
Focal properties and confocal conics	307
Geometrical correspondence	312
Characteristics of systems of conics	323
Pascal hexagons for six points on a conic	328

CHAPTER XXII

Involution in connexion with the conic	335

ADDENDUM

Non projective proofs of properties of straight lines	349
of properties of circles	354

INDEX OF TERMS AND DEFINITIONS	359
INDEX OF THEOREMS	361

NOTE

The student, who has no previous knowledge of the subject, may with advantage read as a first course the large type portions of Chapters I to VIII, X, XII, XIII and XIV It is also advisable in most cases for the student to attempt the examples for himself before referring to the solutions given in the text The word "line" is used throughout the book to signify a straight line although out of respect to custom the term "straight line" has been frequently retained The author would be pleased to be informed of any mistakes or misprints

CHAPTER I

NOTATION PRINCIPLE OF DUALITY DEFINITIONS

1 Notation

Geometrical figures and the proofs of theorems are rendered more intelligible, when a uniform notation is employed In this book, the following notation is adopted

Points are denoted by the	capital letters	$A, B, C,$	X, Y, Z
Lines „ „	small letters	$a, b, c,$	x, y, z
Planes „ „	Greek letters	$\alpha, \beta, \gamma,$	χ, ψ, ω

The point where the line a meets the plane α may be denoted by $a\alpha$, if the lines a and b are situated in the same plane then point of intersection may be similarly denoted by ab, the line of intersection of the planes α and β may be denoted by $\alpha\beta$, the line joining the points A and B by AB, while Aa may be taken to represent the plane through the point A and the line a

Hence it is seen that

Points may be denoted by		$a\alpha, b\beta, c\gamma, \alpha\beta\gamma,$	
		or if $a\ b\ c$ are coplanar by	ab, bc, ca
Lines „ „	„ „	$\alpha\beta, \beta\gamma,$	$AB, BC,$
Planes „ „	„ „	$Aa, Bb,$	$ABC,$

This method of representation may be carried still further Thus a meaning may be attached to such expressions as $\alpha\ BC$, $A\ \beta\gamma$, $\alpha\ Bc$, $A\ \beta c$ The first represents the point of intersection of the line BC with the plane α, the second the plane through the point A and the line $\beta\gamma$, the third the line of intersection of the planes α and Bc, and the last the line joining the point A to the point βc

It should be noticed in the last case that if $Ac\ \beta$ had been written in place of $A\ \beta c$ the meaning would not have been the same $Ac\ \beta$ represents the line of intersection of the planes Ac and β which is not the same as the line joining the point A to the point βc

In cases where there is no fear of confusion angles are denoted by Greek letters

2 Principle of Duality

Every figure in a plane may be regarded as being constructed either of points or of lines If it is regarded as made up primarily of points, the lines which occur in it are looked upon as the connectors of pairs of points If it is considered to be constructed by means of lines, the points in it are determined as the intersections of pairs of lines In the former case, the curves, which occur in the figure, may be looked upon as consisting of series of points and the tangent at any point is the line joining two consecutive points on the curve The curve is in this case termed the *locus* of the points situated on it In the latter case, curves are determined by an infinite number of straight lines which touch them and the point of intersection of any two consecutive tangents gives a point on the curve The curve is then called the *envelope* of the lines which touch it Hence, if a figure be constructed or a theorem be proved by using points as the fundamental elements, another figure may be constructed or another theorem be proved in identically the same way by substituting lines for points, thus in geometry theorems occur in pairs This is called the *Principle of Duality*

It is usual to prove a theorem in the first instance looking upon points as the fundamental elements and to deduce the corresponding theorem—which is termed the *correlative*—by substituting lines for points The full importance of this principle will be gradually realised together with the method of its application As an illustration certain definitions and theorems—which will be proved hereafter—together with their correlatives, are given in the next article

3 Instances of the Principle of Duality

Two points A and B determine a straight line AB

The line AB is the locus of a series of points which are situated on the line

Any series of points, situated on the same straight line, is called *a range of points*

Any four points A, B, C, D—no three of which are collinear—determine a *complete quadrangle*

Two straight lines a and b determine a point ab

The point ab is the envelope of the group of lines which pass through the point

Any group of lines, which pass through the same point, is called *a pencil of rays*

Any four straight lines a, b, c, d—no three of which are concurrent—determine a *complete quadrilateral*

In the figure

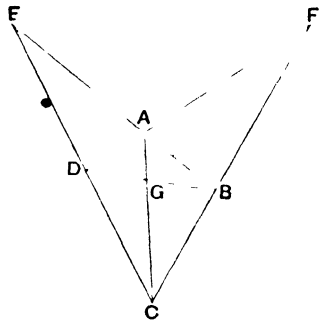
the lines AB and DC determine E,
„ „ BC and DA „ F,
„ „ BD and CA „ G

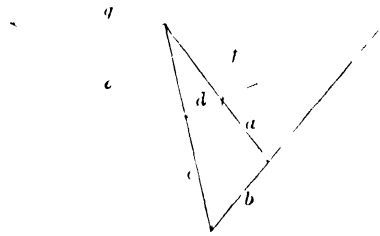
The points E, F, G are termed the diagonal points of the quadrangle and the triangle EFG the *diagonal points triangle* of the quadrangle

In the figure

the points ab and dc determine e,
„ „ bc and da „ f,
„ „ bd and ca „ g

The lines e f g are termed the diagonals of the quadrilateral and the triangle efg the *diagonal triangle* of the quadrilateral





The following are instances of theorems and their correlatives which will be proved in the following chapters

(*a*) If two triangles ABC and $A'B'C'$ are such that the straight lines AA', BB', CC' intersect at a point S, then the three points of intersection of corresponding sides, viz

$$AB \quad A'B',$$
$$BC \quad B'C',$$
$$CA \quad C'A'$$

lie on a straight line s (Art 13 *a*)

If two triangles abc and $a'b'c'$ are such that the points aa', bb', cc' lie on a straight line s, then the lines joining the corresponding vertices, viz

$$ab \quad a'b',$$
$$bc \quad b'c',$$
$$ca \quad c'a'$$

pass through a point S (Art 13 *a*)

(*b*) If two complete quadrangles $ABCD$, $A'B'C'D'$ are such that five pairs of sides, AB and $A'B'$, BC and $B'C'$, CA and $C'A'$, AD and $A'D'$, BD and $B'D'$ intersect in five points on a straight

If two complete quadrilaterals $abcd$, $a'b'c'd'$ are such that five pairs of vertices, ab and $a'b'$, bc and $b'c'$, ca and $c'a'$, ad and $a'd'$, bd and $b'd'$, lie on five straight lines through a point S, then the

line s, then the remaining pair CD and $C'D'$ also intersect on the line s (Art 13 b)

(c) If the three sides of a variable triangle pass through three collinear points A, A' and S, and two of the vertices move on fixed lines s and i, then the locus of the third vertex is a straight line which passes through the point si (Art 71 (1))

In the above (c) is a restatement of (a)

remaining pair cd and $c'd'$ also lie on a straight line through S (Art 13 b)

If the three vertices of a variable triangle are situated on three concurrent straight lines a, a' and s, and two of the sides pass through fixed points S and I, then the third side passes through a fixed point which lies on the line SI (Art 71 (1))

4 Definitions

The following definitions will be seen to be closely connected with the principle of duality as already explained

(1) A *Range* or *Row* of points A, B, C, ... is a figure composed of points lying on a straight line termed the *base*

(2) A *flat pencil*, or shortly, a *pencil* of lines a, b, c, ... is a figure composed of straight lines lying in the same plane and radiating from a point termed the *centre* or *vertex*

(3) A *plane figure* (plane of points or plane of lines) is a figure which consists of points and straight lines, such that the points of intersection of the lines and the lines joining the points lie in a *plane*

5 Extension of the Principle of Duality

The Principle of Duality may also be applied to figures in space In all there are three forms of the principle of duality including the case already considered

(1) *Figures in a plane* in which the point and straight line are the fundamental elements, the duality being between the point and line

(2) *Figures in the sheaf* (sheaf of lines, sheaf of planes) which are made up of lines and planes which all pass through a given point The duality in this case is between the line and the plane If a sphere be described with its centre at the given point, the lines and planes determine points and great circles, on the sphere Hence the consideration of such a sheaf leads to the geometry of figures on a sphere and to the results obtained by spherical trigonometry

(3) *A solid figure* (figure of planes, figure of points) which consists of planes and points In this figure straight lines are determined either as the intersection of planes or as the connectors of points and the duality is between planes and points

In the consideration of the preceding *axial pencils* arise, which consist of planes, which all pass through a given line termed the axis of the pencil Thus, by analogy, it is possible to speak of a *line of points*, *line of planes*, as a figure made up of points which lie on a given line and of the planes which pass through it

6. Connexion between the Principle of Duality and Polar Reciprocation

The student, who is acquainted with polar reciprocation, will notice that in this case, as in that of the principle of duality, one figure is deduced from the other by substituting line for point and point for line The methods of proof in the two cases are, however, different In the Principle of Duality the same proof is interpreted in two different ways to prove a theorem and its correlative In Polar Reciprocation a theorem is proved and another theorem is deduced therefrom by means of the properties of pole and polar It was, however, by means of Polar Reciprocation that the Principle of Duality was first arrived at and its importance realised

CHAPTER II

MEASUREMENT OF DISTANCES AND ANGLES PROJECTIVE FORMS ANHARMONIC

7 Measurement of Distances

The expression AB is used in geometry with two different meanings According to our definition (Art 1) it represents the straight line joining the points A and B, and carries with it no idea of length It is, however, also used to denote the distance measured along this line from A to B It is sometimes an advantage to denote this distance by the symbol AB This will be done in some cases, but in accordance with general custom AB will be written for AB when there is no fear of ambiguity

As a convention it is usual to assume that $BA = -AB$ so that

$$AB + BA = 0$$

Hence if A, B, C be on the same straight line

$$AB + BC + CA = 0,$$

whatever be their relative positions This formula is of importance because in a slightly altered form it enables all distances measured along a straight line to be expressed in terms of the distances of various points on the line from some given point, which given point may be looked upon either as an origin or as a variable point on the line The formula then becomes

$$AB = OB - OA = PB - PA,$$

O being considered as an origin and P is a variable point on the line

To prove that if A, B, C, D are any four collinear points

$$BC \cdot AD + CA \cdot BD + AB \cdot CD = 0$$

If all the lengths are expressed as distances from A this relation becomes

$$(AC - AB)\,AD + CA\,(AD - AB) + AB\,(AD - AC) = 0,$$

which is identically true

To prove that if A, B, C, O are any four collinear points

$$\bar{OA}^2\ \bar{BC}+\bar{OB}^2\ \bar{CA}+\bar{OC}^2\ \bar{AB}=-\bar{BC}\ \bar{CA}\ \bar{AB}$$

Express all lengths in terms of distances from O, then

$$\begin{aligned}-\bar{BC}\ \bar{CA}\ \bar{AB}&=-(OC-OB)(OA-OC)(OB-OA)\\&=OA^2(OC-OB)+OB^2(OA-OC)+OC^2(OB-OA)\\&\qquad+OA\ OB\ OC-OA\ OB\ OC\\&=OA^2(OC-OB)+OB^2(OA-OC)+OC^2(OB-OA)\\&=OA^2\ BC+OB^2\ CA+OC^2\ AB\end{aligned}$$

Examples (1) *If O, A, B are collinear prove that*

$$OA^2+OB^2=AB^2+2\ OA\ OB$$

(2) *If C be the middle point of AB prove that*

(*a*) $OC=\frac{1}{2}(OA+OB)$,

(*b*) $OA\ OB=OC^2-AC^2$,

(*c*) $OA^2+OB^2=CA^2+CB^2+2\ OC^2$,

(*d*) $OA^2-OB^2=2\ AB\ CO$

(3) *Show that the second of the relations proved in the text holds if O is not on the straight line ABC*

The relations connecting the distances of points situated on a straight line may be written in either of two ways according as they are regarded as relations, generally symmetrical, connecting the distances between the points, or as relations connecting the distances of certain variable points, P, Q, ... from certain fixed points, A, B, C, ...

The more important are as follows, viz

Three points (i) $AB+BC+CA=0$ or $AB=PB-PA$ (*a*)

Four points (ii) $AB\ CD+CA\ BD+BC\ AD=0$

or $PA\ BC+PB\ CA+PC\ AB=0$ (*a*)

Obtained from (i) (*a*) by substituting for BC, etc. in terms of distances from P

$$PA^2\ BC+PB^2\ CA+PC^2\ AB=BC\ CA\ AB \qquad (b)$$

Obtained from (i) (*a*) by substituting for BC, etc. in terms of distances from P

$$PA\ PB\ BA+PC\ PA\ AC+PB\ PC\ CB=BC\ CA\ AB \qquad (c)$$

Obtained from (ii) (*b*) by means of (ii) (*a*)

Five points (iii)

$$PA\ QA\ BC+PB\ QB\ CA+PC\ QC\ AB=BC\ CA\ AB$$

or $$PA\ DA\ BC+PB\ DB\ CA+PC\ DC\ AB=-BC\ CA\ AB \qquad (a)$$

Subtract from (ii) (*b*) and, after removing factor PQ, the relation is reduced to (ii) (*a*)

$$PA\ BC\ CD\ DB+PB\ CD\ DA\ AC+PC\ DA\ AB\ BD+PD\ AB\ BC\ CA=0 \qquad (b)$$

Substitute for products $AB\ BC\ CA$, etc from (II) (*b*)

$$PA^2\ BC\ CD\ DB+PB^2\ CD\ DA\ AC+PC^2\ DA\ AB\ BD+PD^2\ AB\ BC\ CA=0 \qquad (c)$$

Substitute for the products $AB\ BC\ CA$, etc from (II) (*b*)

Six points (IV)

$$PA\ QA\ BC\ CD\ DB+PB\ QB\ CD\ DA\ AC+PC\ QC\ DA\ AB\ BD \\ +PD\ QD\ AB\ BC\ CA=0 \qquad (a)$$

Subtract from (III) (*c*) Remove factor PQ and the relation reduces to (III) (*b*)

The preceding results may be easily proved analytically by means of determinants Let $a, b, c, d,$ be the distances of P from the points $A, B, C, D,$ Then

(III) (*a*) $PA\ DA\ BC+PB\ DB\ CA+PC\ DC\ AB$

$$=a(d-a)(b-c)+b(d-b)(c-a)+c(d-c)(a-b)$$

$$-d\Sigma a(b-c)-\Sigma a^2(b-c)=d\begin{vmatrix} a & b & c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$=+(a-b)(b-c)(c-a)=-AB\ BC\ CA$$

(III) (*b*) $-\Sigma PA\ BC\ CD\ DB=\Sigma a(b-c)(c-d)(d-b)=\begin{vmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \end{vmatrix}=0$

(III) (*c*) $-\Sigma PA^2\ BC\ CD\ DB=\Sigma a^2(b-c)(c-d)(d-b)=\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \end{vmatrix}=0$

(IV) (*a*) $\Sigma PA\ QA\ BC\ CD\ DB=\begin{vmatrix} a(q-a) & b(q-b) & c(q-c) & d(q-d) \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \end{vmatrix}=0$

Other results may be obtained by this method Thus

(1) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$

Therefore $\Sigma PA^3\ BC+AB\ BC\ CA\ (PA+PB+PC)=0$

(2) $\begin{vmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}=0$

Therefore $\Sigma PA\ (PB+PC+PD)\ BC\ CD\ DB=0$

Determination of Position by means of Ratios

If two points A and B are given, the position of any point P on the line AB is uniquely determined if the ratio $\frac{AP}{BP}$ is given If the ratio $\frac{AP}{BP}$ is negative the point P lies between A and B If it is positive, it lies on the other portions of the line AB For different positions of P the ratio $\frac{AP}{BP}$ can have any (real) value, positive or negative The expression $\frac{AP}{BP}$ is termed *the ratio* of P with respect to A and B

The point P, whose ratio is p, can be constructed as follows Draw any line AQ through A On AQ take a point R such that $\frac{AQ}{RQ} = p$, the point R being taken on the same side of Q as A or on the opposite side according as p is positive or negative Join RB The line through Q parallel to RB meets AB in the required point P

If $p = 1$ the point R coincides with A, and the line through Q, parallel to RB, is parallel to AB The point P is then said to be at an infinite distance In this case $AP = BP$ This may be interpreted to mean that the point P is at such a distance from A and from B that the finite distance AB may be neglected in comparison with these distances The point P in this case is termed *the point at infinity* on AB

8 Measurement of Angles

The angle between the line a and the line b, measured from a to b, may be denoted by $\widehat{ab}$ and the angle measured from b to a by $\widehat{ba}$ As a convention it is assumed that $\widehat{ab} = -\widehat{ba}$ so that

$$\widehat{ab} + \widehat{ba} = 0$$

If a, b, c are any three concurrent straight lines,

$$\widehat{ab} + \widehat{bc} + \widehat{ca} = 0$$

Hence whatever be the relative position of three concurrent lines a, b, o,

$$\widehat{ab} = \widehat{ob} - \widehat{oa}$$

This enables all the angles at a point to be expressed in terms of the angles made by lines through the point with a given line The

segment of the line a from which $\widehat{ab}$ is measured must however be the same in every case If a' is the extension of a through ab then

$$\widehat{ab} + \widehat{ba'} = \pi$$

If a circle be described with its centre at the point of intersection of given lines the angles at this point are measured (in circular measure) by the arcs cut off by these lines divided by the radius Any homogeneous relation connecting the angles at this point may therefore be interpreted as a relation connecting arcs of a circle whose centre is at the point of intersection of the lines

Thus if $\widehat{ab}$ be a length measured along the arc of a circle it is seen by the method of the last article that

$$\widehat{bc}\ \ \widehat{ad} + \widehat{ca}\ \ \widehat{bd} + \widehat{ab}\ \ \widehat{cd} = 0,$$

and likewise that

$$\widehat{oa}^2\ \ \widehat{bc} + \widehat{ob}^2\ \ \widehat{ca} + \widehat{oc}^2\ \ \widehat{ab} = -\ \widehat{bc}\ \ \widehat{ca}\ \ \widehat{ab}$$

The whole of the examples of the last article may be similarly interpreted

9 Projection in a Plane

If any number of points $A, B, C, D, E \ldots W$ situated on a straight line u be joined to any point S—not on u—by straight lines $a, b, c, d, e \ldots w$ and these lines meet a second line u in points $A', B', C', D', E' \ldots$ then the points $A', B', C', D', E' \ldots$ are said to be the projections of the points $A, B, C, D, E \ldots$ from the centre of projection S on the line u'

Similarly the points $A', B', C', D', E' \ldots$ may be projected from another centre of projection S' on another line u'' into points $A'', B'', C'', D'', E'' \ldots$

The ranges $ABCDE \ldots$, $A'B'C'D'E' \ldots$, $A''B''C''D''E'' \ldots$ are said to be projective as one is derived from the other by a series of projections and the points $AA'A'', BB'B'', CC'C'', \ldots$ are said to be corresponding points

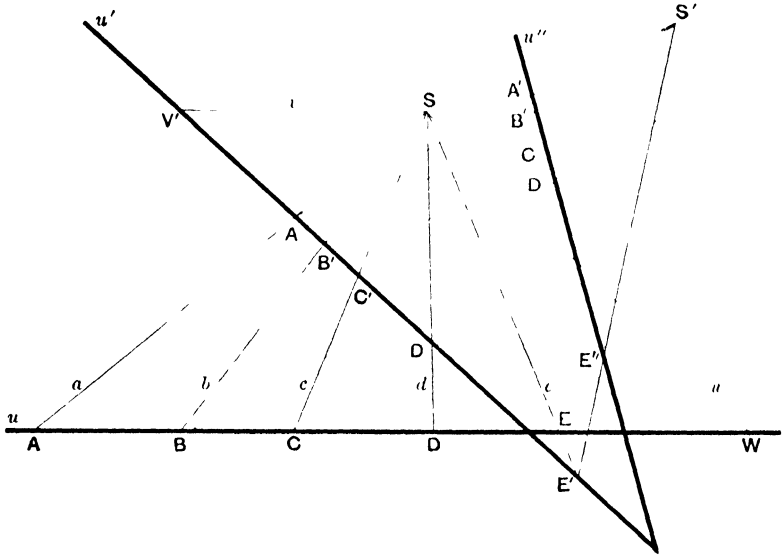
The ranges on u and u' which can be obtained by direct projection from each other are said to be in *Perspective* Similarly the ranges on u' and u'' are said to be in Perspective The ranges on u and u'' are projective but they are not in Perspective

If through the point S a straight line w be drawn parallel to u' it will meet u in a point W The projection of this point W on the line u' is the intersection of w and u' According to Euclid's definition these lines being parallel do not meet In projective geometry they are regarded as meeting at an infinite distance and their point of intersection is termed *the point at infinity* on either of the lines Thus when

the ranges on u and u' are considered the point W, which corresponds to the point at infinity on u', is obtained by drawing through S a line parallel to u' to meet u at W

Similarly the point V' of the range $A'B'C'D'$ which corresponds to the point on u, which is at an infinite distance, is obtained by drawing through S a line parallel to u to meet u' in V' The points W and V' are called the *vanishing points* of the projective ranges on u and u'

The point at infinity on any line is considered to be at such a distance from all points, which are at a finite distance, that its distance from all such points may be regarded as being the same Hereafter



(Art 22) it will be shown that all lines which are parallel may be regarded as meeting at one and the same point, which point is at an infinite distance

A relationship which holds for the distances between the points A, B, C, D, E is said to be a *projective relationship* if the same relationship necessarily holds for the corresponding distances between the corresponding points on any range derived from it by projection

Likewise if an expression involving the distances between the points A, B, C, D, E is such that the same expression, involving the corresponding distances on any range derived from the given one by a series of projections, has the same value, then that expression is said to be a *projective expression*

To one point of a given range corresponds one and only one point—the corresponding point—on any other range derived from it by a series of projections

The subject of projection is fully treated in Chapter IV

10 Projective Forms Simple Ratio

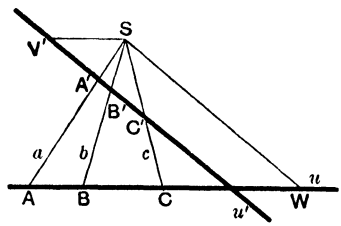
A simple ratio of the distances between three collinear points is not generally projective

There are, however, two special cases that require consideration

(1) If the cutting lines u and u' are parallel then

$$\frac{AC}{BC} = \frac{A'C'}{B'C'}$$

and the simple ratio is projective



(2) Let the lines joining ABC to S meet u' in $A'B'C'$.

Then from the triangle ASC,

$$\frac{AC}{AS} = \frac{\sin \widehat{ac}}{\sin SCA}$$

Also from the triangle BSC, $\dfrac{BC}{BS} = \dfrac{\sin \widehat{bc}}{\sin SCA}$

Therefore $\dfrac{AC}{BC} \div \dfrac{AS}{BS} = \dfrac{\sin \widehat{ac}}{\sin \widehat{bc}}$ and similarly $\dfrac{A'C'}{B'C'} \div \dfrac{A'S}{B'S} = \dfrac{\sin \widehat{ac}}{\sin \widehat{bc}}$

The expression $\dfrac{AC}{BC} \div \dfrac{AS}{BS}$ may be conveniently written $(ABC\ S)$ and, from the above, it is seen that it is projective, when S is the centre of projection, for $(ABC\ S) = (A'B'C'\ S)$

Elements at Infinity If A be the point at infinity on u, which may be denoted by ∞, then A' becomes the vanishing point V' on u'. Similarly, if B' be the point at infinity on u', denoted by ∞', B becomes the vanishing point W on u.

The relation $(ABC\ S) = (A'B'C'\ S)$ then becomes

$$(\infty\ WC\ S) = (V'\infty'C'.S)$$

Hence
$$\frac{\infty C}{WC} \div \frac{\infty S}{WS} = \frac{V'C'}{\infty'C'} \div \frac{V'S}{\infty'S}$$

Therefore, since $\infty C = \infty S$ and $\infty' C = \infty' S$,

$$\frac{WS}{WC} = \frac{V'C'}{V'S},$$

or
$$WC\ V'C' = WS\ V'S = \text{a constant}$$

Hence *the product of the distances of two corresponding points of two projective ranges from the vanishing points of these ranges is constant*

If A_1, A_2, A_3 B, C are collinear points it can be shown that a relation of the form

$$l\frac{BA_1}{CA_1}+m\frac{BA_2}{CA_2}+n\frac{BA_3}{CA_3}+\quad=0$$

is projective

Divide each term by $\frac{BS}{CS}$ Then the above relation becomes

$$l(BCA_1\ S)+m(BCA_2\ S)+n(BCA_3\ S)+\quad=0,$$

which is projective since by (2) each term is unaltered by projection from S

11 Ratio of a Ratio Anharmonic Ratio

Given four collinear points the *ratio of the ratios* of their distances is subject to certain conditions projective This is by far the most important deduction from (2), Art 10

Consider in Figure, Art 9, the four collinear points A, B, C, D Let the centre of projection be S

Then $(ABC\ S)$ and $(ABD\ S)$ are projective

Therefore
$$\frac{(ABC\ S)}{(ABD\ S)}=\frac{\dfrac{AC}{BC}\ \dfrac{AS}{BS}}{\dfrac{AD}{BD}\ \dfrac{AS}{BS}}=\frac{AC}{BC}\ \frac{AD}{BD},$$

and the expression $\frac{AC}{BC}\ \frac{AD}{BD}$ is projective

When A, B, C, D are collinear $\frac{AC}{BC}\ \frac{AD}{BD}$ is written $(ABCD)$ and is called *the anharmonic ratio* (or one of the anharmonic ratios) of the four points A, B, C, D

From Art 10 (2) it is seen that

$$\frac{AC}{BC}\ \frac{AS}{BS}=\frac{\sin\widehat{ac}}{\sin\widehat{bc}}\quad\text{and}\quad\frac{AD}{BD}\ \frac{AS}{BS}=\frac{\sin\widehat{ad}}{\sin\widehat{bd}},$$

$$(ABCD)=\frac{AC}{BC}\ \frac{AD}{BD}=\frac{\sin\widehat{ac}}{\sin\widehat{bc}}\ \frac{\sin\widehat{ad}}{\sin\widehat{bd}}$$

The latter expression is sometimes written $(abcd)$ and is called the anharmonic ratio of the pencil $abcd$ The anharmonic ratio of the pencil formed by joining any four points A, B, C, D to any point P is

denoted by $(P\ ABCD)$ and every line cuts such a pencil in four points which have the same anharmonic ratio as the anharmonic ratio of the pencil. In some books $\frac{AC}{BC}\ \frac{AD}{BD}$ is written as $(ACBD)$, *i.e.* the middle letters are interchanged.

Examples. If A, B, C, D, C', D' are collinear points,

(1) $\frac{(ABCD)}{(ABC'D)} = (ABCC')$,

(2) $\frac{(ABCD)}{(ABCD')} = (ABDD')$,

(3) $(ABCD)(ABC'D') = (ABCD')(ABC'D)$,

(4) $(ABCD)(ABDD') = (ABCD')$.

12 Determination of the changes in the value of the Anharmonic Ratio arising from a change in the order in which the elements are taken

Let A, B, C, D be collinear points, then

$$(ABCD) = \frac{AC}{BC}\ \frac{AD}{BD} = \lambda \text{ (suppose)}$$

(1) $$(ABDC) = \frac{AD}{BD}\ \frac{AC}{BC} = \frac{1}{\lambda},$$

$$(BACD) = \frac{BC}{AC}\ \frac{BD}{AD} = \frac{1}{\lambda}$$

Therefore *if the first and second or the third and fourth elements of an anharmonic ratio are interchanged the value of the anharmonic ratio is changed from* λ *to* $\frac{1}{\lambda}$.

(2) $BC\ AD + CA\ BD + AB\ CD = 0$ (Art. 7.)

$$1 - \frac{AC\ BD}{BC\ AD} - \frac{AB\ CD}{CB\ AD} = 0,$$

$$1 - (ABCD) - (ACBD) = 0,$$

$$(ACBD) = 1 - \lambda$$

Similarly $$(DBCA) = 1 - \lambda$$

Therefore *if the second and third or the first and fourth elements of an anharmonic ratio are interchanged the value of the anharmonic ratio is changed from* λ *to* $1 - \lambda$.

(3) $(CBAD) = 1 - (CABD)$ by (2),

$= 1 - \dfrac{1}{(ACBD)}$ by (1),

$= 1 - \dfrac{1}{1-(ABCD)}$ by (2),

$= 1 - \dfrac{1}{1-\lambda} = \dfrac{\lambda}{\lambda - 1}$

Similarly $(ADCB) = \dfrac{\lambda}{\lambda - 1}$

Therefore *if the first and third or the second and fourth elements of an anharmonic ratio are interchanged the value of the anharmonic ratio is changed from* λ *to* $\dfrac{\lambda}{\lambda - 1}$

If in (1), (2) or (3) after one pair of elements have been interchanged, the remaining pair are interchanged, the value of the anharmonic ratio is again λ. Hence the following important result is obtained

If the four elements, which make up an anharmonic ratio, are divided in any way into two pairs, and the elements in both these pairs are interchanged, the value of the anharmonic ratio is unaltered

It follows as an immediate consequence that if λ is the anharmonic ratio $(ABCD)$ the values of the anharmonic ratios of the points A, B, C, D taken in any different order must be either

$$\lambda, \quad \frac{1}{\lambda}, \quad 1-\lambda, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda}{\lambda - 1} \quad \text{or} \quad \frac{\lambda - 1}{\lambda}$$

Hence only six different values of the anharmonic ratios of four given points can be obtained by varying the order in which these points are taken

Given the anharmonic ratio of four points of a range and the position of three of the points—no two of which coincide—the fourth point is uniquely determined

Suppose that the three given points are A, B, C and that the position of a fourth point D has to be determined

Then $(ABCD) = \dfrac{AC}{BC} \dfrac{AD}{BD} = \lambda$ (suppose),

therefore $\dfrac{AD}{BD} = \dfrac{1}{\lambda} \dfrac{AC}{BC} =$ a known quantity

Therefore D is determined as the point which divides the distance between A and B in a given ratio, externally or internally, according as $\frac{1}{\lambda}\frac{AC}{BC}$ is positive or negative. Such a point by Art 7 is uniquely determined.

From the preceding it may be deduced at once that *given the anharmonic ratio of a pencil of four rays and the position of three of the rays—no two of which coincide—the position of the fourth ray is uniquely determined.*

Hence *if the anharmonic ratios of two ranges or pencils are equal, and three pairs of corresponding elements coincide, the fourth pair must likewise coincide.*

Coincidence of Points forming an Anharmonic Range

Given the position of three points A, B, C, no two of which coincide, to find the values of the anharmonic ratio $(ABCD)$ for which the point D coincides with one of the three given points.

If $(ABCD)=1$ then $\frac{AC}{BC}=\frac{AD}{BD}$. Therefore C and D coincide.

If $(ABCD)=0$ then $(ACBD)=1$. Therefore B and D coincide.

If $(ABCD)=\infty$ then $(BCAD)=1$. Therefore A and D coincide.

Hence *if the anharmonic ratio $(ABCD)$ equals ∞, 0 or 1, the point D coincides with A, B, or C.* For other values the point D is distinct from A, B, and C.

If two out of four points A, B, C, D coincide, to find the value of the anharmonic ratio $(ABCD)$ for all positions of the other two points.

(1) Let A and B coincide, then

$$(AACD)=\frac{AC}{AC}\frac{AD}{AD}=1 \text{ for all positions of } C \text{ and } D,$$

unless C or D coincide with $A\,(\equiv B)$ in which case the anharmonic ratio is indeterminate.

(2) Let A and C coincide, then

$$(AB1D)=\frac{AA}{B1}\frac{1D}{BD}=0 \text{ for all positions of } B \text{ and } D,$$

unless B or D coincide with $A\,(\equiv C)$ in which case the anharmonic ratio is indeterminate.

(3) Let 1 and D coincide, then

$$(ABC1)=\frac{AC}{BC}\frac{A1}{B1}=\infty \text{ for all positions of } B \text{ and } C,$$

unless B or C coincide with $A\,(\equiv D)$ in which case the anharmonic ratio is indeterminate.

Hence *when two and only two points of a range coincide the value of the anharmonic ratio is either 1, 0 or ∞ and the other two points may occupy any positions.*

Anharmonic ratio of four points three of which coincide

The anharmonic ratio in this case is indeterminate, but a meaning may be assigned to it in certain cases by considering the limiting process by which the three points come to coincide

Let the points, which become coincident, be A, B, and C, and let the distances between A and C and between C and B be ld and md, where d in the limit is infinitely small



Then $$(ABCD) = \frac{AC}{BC} \frac{AD}{BD}$$

$$= \frac{ld}{-md} \frac{ld+CD}{CD-md} = -\frac{l}{m} \text{ when } d \text{ is infinitely small}$$

Therefore, wherever the point D is situated, the anharmonic ratio is $-\frac{l}{m}$ An important instance of this will arise hereafter

Sign of an Anharmonic Ratio

When the points A, B, C, D occur in this order on a straight line, the anharmonic ratio $(ABCD) = \frac{AC}{BC} \frac{AD}{BD}$ is positive

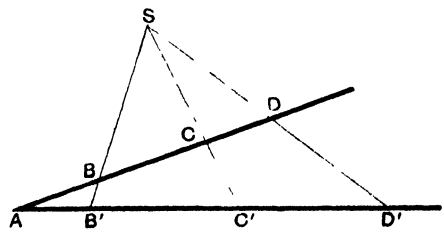
If any one of the points moves along the line till it coincides with one of the other points the value of the anharmonic ratio becomes either zero, infinity or unity In the two former cases the anharmonic ratio, when the points cross, changes its sign but not in the last case

Thus if the four points be divided into two groups, viz (1) A and B and (2) C and D, a change in the relative position of the points in either group does not affect the sign of the anharmonic ratio, but any alteration of the position of the points in one group relative to those in the other causes the sign of the anharmonic ratio to change

Hence *if the points A, B, C, D are situated on a straight line in the order in which the letters occur, the anharmonic ratio $(ABCD)$ is positive and every change of position of A or B with respect to C or D changes the sign of the anharmonic ratio*

Anharmonic condition that three lines should be concurrent

If through a point A two lines $ABCD$ and $AB'C'D'$ be drawn and the anharmonic ratio $(ABCD)$ equals the anharmonic ratio $(AB'C'D')$ then the three lines BB', CC', DD' are concurrent



Let BB' and CC' meet at S Join SD meeting $AB'C'$ in D''

Then

$$(AB'C'D') = (ABCD)$$
$$= (AB'C'D''),$$

$$\frac{AC'}{B'C'}\frac{AD'}{B'D'} = \frac{AC'}{B'C'}\frac{AD''}{B'D''},$$

$$\frac{AD'}{B'D'} = \frac{AD''}{B'D''}, \qquad D'' \text{ coincides with } D' \text{ (Art 7)}$$

Anharmonic condition that three points should be collinear

If through two points on a line a, three sets of lines bcd and b'c'd' be drawn and the anharmonic ratio of the pencil (abcd) be equal to the anharmonic ratio of the pencil (ab'c'd') then the points bb', cc', dd' are collinear

Let s the line joining bb' and cc' meet d in D and let the line joining D to $b'c'$ be d''

Then $$(ab'c'd') = (abcd) = (ab'c'd'')$$

Hence the lines d' and d'' coincide

Projection of a Simple Ratio

The statement at the beginning of Art 10 to the effect that "a simple ratio of the distances between three collinear points is not generally projective" is subject to an important qualification Consider three collinear points A, B, C On the line ABC there is a point at infinity, which may be regarded as equally distant from every point of the figure which is at a finite distance If this point be denoted by ∞

$$\frac{AC}{BC}\frac{A\infty}{B\infty} = (ABC\infty)$$

If A, B, C, ∞ are projected from any centre into $A'B'C'V'$ then

$$\frac{AC}{BC}\frac{A\infty}{B\infty} = (ABC\infty) = (A'B'C'V')$$

$$= \frac{A'C'}{B'C'}\frac{A'V'}{B'V'}$$

By means of this fact theorems may often be generalised

Similar ranges Two ranges are said to be *similar* when their points at infinity are corresponding points In this case

$$(ABC\infty) = (A'B'C'\infty),$$

$$\frac{AC}{A'C'} = \frac{BC}{B'C'}$$

Hence corresponding segments are proportional If two pairs of corresponding segments are proportional then every pair will be proportional, and the ranges are similar

Equal ranges Two ranges are said to be equal when the corresponding segments are equal When the corresponding segments have the same sign the ranges are said to be *directly equal*, when they have opposite signs, the ranges are said to be *oppositely equal* If two pairs of corresponding segments are equal, all corresponding segments are equal and the ranges are equal

Equal pencils Two pencils are said to be equal when the angles between pairs of corresponding rays are equal When the angles are equal and have the same sign the pencils are said to be *directly equal* When the signs are different the pencils are said to be *oppositely equal* If two pairs of angles between corresponding rays are equal, the pencils are equal

The student may be helped in recognising projective ranges by the fact that two ranges are projective

(1) When the sum, the difference, the ratio or the product of the distances of corresponding points from two fixed points on the bases is constant,

or (2) When pairs of corresponding points subtend equal angles at some fixed point or points

13 Application of Anharmonic Ratios to Important Theorems of Projective Geometry and Plane Perspectives

(*a*) *If the lines joining the three vertices of two triangles in pairs are concurrent at a point* (*S*) *then the corresponding sides intersect in pairs on a straight line* (*s*)

Let AA', BB', CC', the connectors of the vertices of the triangles meet in S Let AC, $A'C'$ intersect in Q and AB and $A'B'$ in R Let QR be s, and let AA', BB', CC' meet s in K, L and M Then

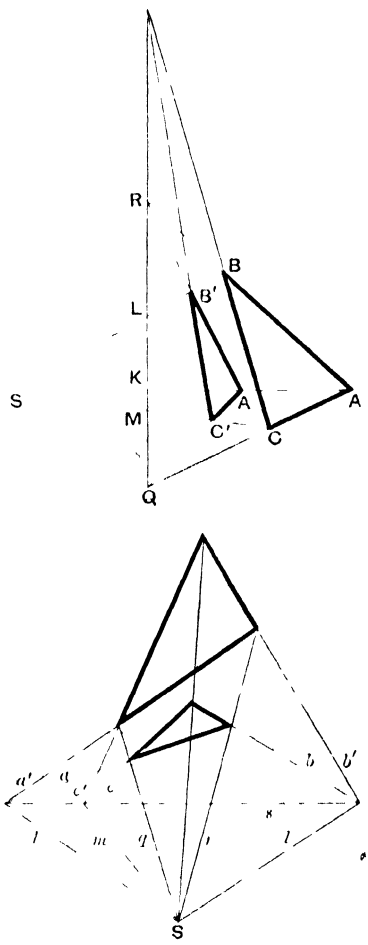
$$(SMC'C) = (SKA'A) = (SLB'B)$$

Hence the lines ML, $B'C'$ BC are concurrent, and therefore, $B'C'$ and BC intersect in a point on s

Conversely and correlatively

If the sides of two triangles intersect in pairs on a straight line (*s*) *then the lines joining the corresponding vertices are concurrent at a point* (*S*)

Let aa', bb', cc', the pairs of corresponding sides of the triangles, intersect on s Let the line joining



ac and $a'c'$ be q and that joining ab and $a'b'$ be r. Let q and r intersect in S. Join aa', bb', cc' to S by k, l and m.

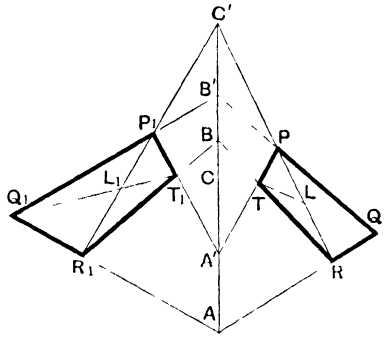
Then $$(smc'c) = (skc'a) = (slb'b)$$

Hence the points ml, $b'c'$, bc are collinear, and therefore the line joining $b'c'$ to bc passes through lm or S.

The triangles in the above case are said to be in plane Perspective. S is termed the centre and s the axis of Perspective.

(*b*) *If five pairs of sides of two quadrangles intersect on a straight line, then the sixth pair intersect on the same straight line.*

Let the quadrangles be $PQRT$ and $P_1Q_1R_1T_1$ and let PR, PQ, RT, QT and PT intersect P_1R_1, P_1Q_1, R_1T_1, Q_1T_1 and P_1T_1 respectively in $C'B'BCA'$ on the line s.



It is necessary to prove that A, the point of intersection of QR and Q_1R_1, lies on s.

Since the sides of the triangles QTP and $Q_1T_1P_1$ intersect in pairs on s, these triangles are in perspective, and PP_1, QQ_1, TT_1 are concurrent at some point S. Similarly the triangles PRT and $P_1R_1T_1$ are in perspective and the lines PP_1, RR_1, TT_1 are concurrent at the same point S. Therefore, since RR_1, QQ_1, TT_1 are concurrent at S, the triangles QTR and $Q_1T_1R_1$ are in perspective. Therefore the points B, C and A are collinear, and A lies on (s).

This theorem may be proved directly as follows. Let QR and Q_1R_1 meet s in A_1 and A_2, then in the figure

$$\begin{aligned}(T\ A\ CBC') &= (T_1\ A'CBC'),\\ (PLRC') &= (P_1L_1R_1C'),\\ (Q\ PLRC') &= (Q_1\ P_1L_1R_1C'),\\ (B'C'A_1C') &= (B'C'A_2C')\end{aligned}$$

Hence, since $B'CC'$ are collinear, A_1 and A_2 must coincide in a point A on s.

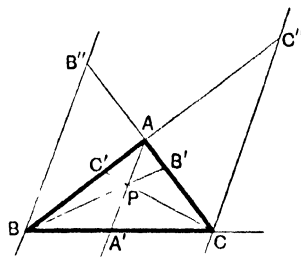
The correlative theorem, which may be proved by similar methods, is as follows.

If five pairs of vertices of two quadrilaterals are collinear with a given point, then the sixth pair are also collinear with this point.

From this it follows that if two quadrangles $ABCD$ and $A'B'C'D'$ are such that five of the sides of one are parallel to the five corresponding sides of the other, then the sixth pair of sides are also parallel. The axis of Perspective is the line at infinity.

(*c*) **Ceva's Theorem** *If the lines joining any point P to the vertices ABC of a triangle meet the opposite sides in $A'B'C'$ respectively, then*

$$\frac{BA'}{CA'}\,\frac{CB'}{AB'}\,\frac{AC'}{BC'} = -1$$



Draw through B and C lines parallel to AA' to meet CA and BA respectively in B'' and C'''. In this case ∞ the point at infinity on AA', is also the point at infinity on BB'' and CC'''. Then

$$(B\ AA'P\infty) = (C\ AA'P\infty)$$

$$(ACB'B'') = (ABC'C''')$$

by taking intercepts on AC' and AB

$$\frac{AB'}{CB'}\,\frac{AB''}{CB''} = \frac{AC''}{BC''}\,\frac{AC'''}{BC'''},$$

$$\frac{AB'}{CB'}\,\frac{BC'}{AC'} = \frac{AB''}{CB''}\,\frac{AC'''}{BC'''} = \frac{AB}{CB}\,\frac{AC}{BC} = -\frac{BA'}{CA'},$$

$$\frac{AB'}{CB'}\,\frac{BC'}{AC'}\,\frac{CA'}{BA'} = -1$$

Conversely If points $A'B'C'$ are taken on the sides of a triangle ABC such that

$$\frac{BA'}{CA'}\,\frac{CB'}{AB'}\,\frac{AC'}{BC'} = -1,$$

then the lines AA', BB', CC' are concurrent.

Let the lines BB', CC' intersect in P, and let AP meet BC in A. Then

$$\frac{BA'}{CA'}\,\frac{CB'}{AB'}\,\frac{AC}{BC} = -1,$$

$$\frac{BA}{CA}\,\frac{BA}{CA} \text{ (from data)},$$

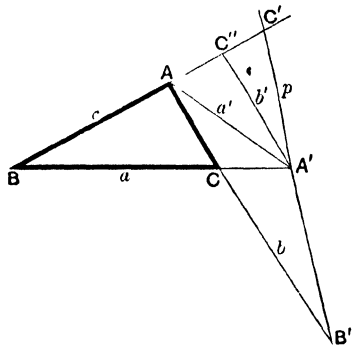
A coincides with A'.

(*d*) **Menelaus' Theorem** *If any straight line meets the sides of a triangle ABC in points $A'B'C'$, then*

$$\frac{BA'}{CA'}\,\frac{CB'}{AB'}\,\frac{AC''}{BC''} = 1$$

Let the sides of the triangle be a, b, c and let the straight line be p Through A' draw b', parallel to b, to meet c in C''

Let AA' be a' and let the point at infinity on b' be ∞



Then $(b\ aa'pb') = (c\ aa'pb')$,

$$(CAB'\infty) = (BAC'C'')$$

by taking intercepts on b and c

$$\frac{CB'}{AB'} = \frac{BC'}{AC'}\,\frac{BC''}{AC''},$$

$$\frac{CB'}{AB'}\,\frac{AC'}{BC'} = \frac{AC''}{BC''} = \frac{CA'}{BA'},$$

$$\frac{CB'}{AB'}\,\frac{AC'}{BC'}\,\frac{BA'}{CA'} = 1$$

Conversely If points $A'B'C'$ are taken on the sides of a triangle ABC such that

$$\frac{CB'}{AB'}\,\frac{AC'}{BC'}\,\frac{BA'}{CA'} = 1,$$

then the points $A'B'C'$ are collinear

Let the line joining $B'C'$ meet a in A'' Then

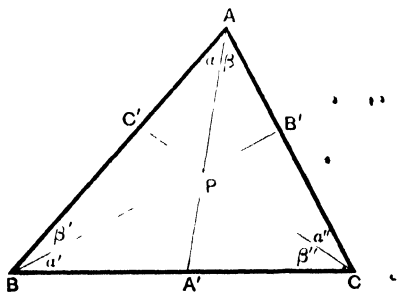
$$\frac{BA''}{CA''}\,\frac{CB'}{AB'}\,\frac{AC''}{BC''} = 1,$$

$$\frac{BA'}{CA'} = \frac{BA''}{CA''} \text{ (from data)},$$

A'' coincides with A'

Secondary form of Ceva's Theorem

Let the sides of the triangle be denoted by a, b, c, and the lines AP, BP, CP by a_1, b_1, c_1



Then $$\frac{BA'}{CA'} = \frac{c \sin ca_1}{b \sin ba_1}$$

with similar expressions for

$$\frac{CB'}{AB'} \text{ and } \frac{AC'}{BC'}$$

Therefore by substituting in Ceva's relation

$$\frac{\sin ca_1}{\sin ba_1}\,\frac{\sin ab_1}{\sin cb_1}\,\frac{\sin bc_1}{\sin ac_1} = -1$$

or, denoting the angles as in the figure and disregarding the signs,

$$\frac{\sin\alpha}{\sin\beta}\,\frac{\sin\alpha'}{\sin\beta'}\,\frac{\sin\alpha''}{\sin\beta''} = +1$$

14. Analytical Expressions for Anharmonic Ratios

If two points B and C are given, the position of any other point A on the line joining them is determined by the ratio $\frac{BA}{CA}$ It is sometimes a convenience to speak of this as the ratio of the point A and to denote it by some expression such as x

If B and C are given points and x_1, x_2, x_3, x_4 are the ratios of four other points A_1, A_2, A_3, A_4 on the line BC, then

(1) $(BCA_1A_2)=\frac{x_1}{x_2}$,

(2) $(A_1A_2A_3A_4)=\frac{(x_3-x_1)(x_4-x_2)}{(x_3-x_2)(x_4-x_1)}$

This result follows at once from the fact that

$$A_1A_2=a\frac{x_1-x_2}{(x_1-1)(x_2-1)} \text{ where } a=BC$$

(3) If x_1, x_2 are the roots of the equation $a'x^2+2h'x+b'=0$ and x_3, x_4 the roots of the equation $a''x^2+2h''x+b''=0$, then

$$[2h'h''-a'b''-a''b']^2=4\left[\frac{1+\lambda}{1-\lambda}\right]^2(h'^2-a'b')(h''^2-a''b'')$$

where λ is a value of the anharmonic ratio $(A_1A_2A_3A_4)$

(4) If x_1, x_2, x_3, x_4 are the roots of the equation $ax^4+4bx^3+6cx^2+4dx+e=0$ and λ is any one of the anharmonic ratios of $A_1A_2A_3A_4$, then

$$\frac{1}{3^3}\frac{(\lambda+1)^2(\lambda-\frac{1}{2})^2(\lambda-2)^2}{(\lambda^2-\lambda+1)^3}=\frac{\{c^3-ace+ad^2-2bcd+b^2e\}^2}{\{3c^2+ae-4bd\}^3}$$

EXAMPLES

(1) If $(ABCD)=\lambda$, $\frac{1-\lambda}{AB}=\frac{1}{AD}-\frac{\lambda}{AC}$

(2) If $(ABPD)=(ABCQ)$ show that $(PQAC)=(PQDB)$

(3) If E and F are two fixed points and the anharmonic ratio

$$(AA'EF)=(BB'EF)=(CC'EF)=(DD'EF)$$

show that $$(ABCD)=(A'B'C'D')$$

(4) If four collinear points A, B, P, Q are such that $(ABPQ)$ is constant and A and B are fixed, then the ranges described by P and Q are projective

(5) If four concurrent lines a, b, p, q are such that $(abpq)$ is constant and a and b are fixed, then the pencils described by p and q are projective

CHAPTER III

PROJECTIVE FORMS HARMONIC

15. Harmonic Ranges and Pencils.

In Chapter II, Art 12, the case of four points whose anharmonic ratio has the value ∞, 0 or 1 was considered and it was proved that two out of the four points must coincide In this chapter another important case of particular values of the anharmonic ratio of four points will be considered

The range formed by the four points A, B, C, D is said to be *Harmonic* when

$$(ABCD) = (ABDC)$$

By Art 12, $$(ABDC) = \frac{1}{(ABCD)}$$

Therefore in this case $$(ABCD) = \frac{1}{(ABCD)}$$

Hence $$(ABCD)^2 - 1 = 0,$$

and therefore $$(ABCD) = 1 \text{ or } -1$$

If $(ABCD) = 1$ it is seen, by Art 12, that two of the four points coincide. Therefore if the points are distinct

$$(ABCD) = -1.$$

Also (Art 12) in this case

$$(ACBD) = 2 \text{ and } (CABD) = \tfrac{1}{2}.$$

Hence, if the anharmonic ratio of four points is either -1, 2 or $\frac{1}{2}$, the four points form a Harmonic range

The points C and D, which may be interchanged without altering the values of the above anharmonic ratios, are said to be Harmonic Conjugates of A and B But since

$$(ABCD) = (CDAB) \text{ and } (ABDC) = (CDBA)$$

therefore $$(CDAB) = (CDBA)$$

Hence A and B may likewise be interchanged without altering the value of the anharmonic ratio, and are therefore also called harmonic conjugates of C and D.

Thus a Harmonic Range consists of two pairs of points These pairs of points are interchangeable, as also are the points in each pair The points in each pair of harmonic conjugates divide the distance between the other pair, internally and externally, in the same ratio, for if

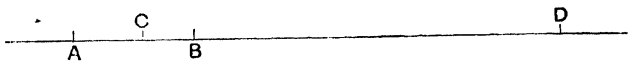
$$(ABCD) = -1, \text{ then } \frac{AC}{BC} = -\frac{AD}{BD}.$$

If C and D are harmonic conjugates of A and B and D is at infinity, C bisects the distance between A and B, and conversely, if C bisects the distance between A and B, the range made up of A, B, C and the point at infinity on the line is a harmonic range. Consequently the projection of this range on any line is also a harmonic range

If four points A, B, C, D forming a harmonic range be joined to any point S by lines a, b, c, d, it follows from Art 11 that $(abcd) = -1$ and the pencil a, b, c, d is then said to be a *harmonic pencil* It is obvious that any harmonic pencil is cut by any transversal in a harmonic range.

Case of Coincident Elements of a Harmonic Range.

In Art 12 it was proved that if two and only two of the elements of any anharmonic range coincide the value of the anharmonic range must be either ∞, 0 or 1 Therefore, if two elements of a harmonic range coincide, a third point must likewise coincide in the same point Hence a pair of conjugate points must coincide with one of the other pair of conjugates



It has likewise been shown that if

$$AC = l\ d \text{ and } CB = m\ d$$

$$(ABCD) = -\frac{l}{m}$$

in the limit when d is infinitely small, and that D in this case may be any point at a finite distance from C If $l = m$ the range is harmonic

Hence *if two points A and B equally distant from a point C are supposed to move uniformly towards the point C and in the limit coincide with it, the range formed by ABC and any other point on the line forms in the limit a harmonic range of which A and B are a pair of conjugate points*

16. Properties of Harmonic Ranges and Pencils.

If O be the middle point between two conjugates A and B of a harmonic range $ABCD$ and o be a line bisecting the angle $\widehat{ab}$ between a pair of conjugate lines of a harmonic pencil $abcd$, the chief properties of Harmonic Ranges and Pencils may be stated as follows

(a)	$\frac{2}{AB}=\frac{1}{AC}+\frac{1}{AD}$,	(a')	$2\cot\widehat{ab}=\cot\widehat{ac}+\cot\widehat{ad}$,
(b)	$CA\ CB=CO\ CD$,	(b')	$2\sin\widehat{oa}\ \sin\widehat{cb}=\sin 2\widehat{co}\ \tan\widehat{cd}$,
(c)	$OC\ OD=OA^2=OB^2$,	(c')	$\cot\widehat{oc}\ \cot\widehat{od}=\cot^2\widehat{oa}=\cot^2\widehat{ob}$,
(d)	$\frac{OC}{OD}=\left(\frac{AC}{AD}\right)^2=\left(\frac{BC}{BD}\right)^2$,	(d')	$\frac{\sin 2\widehat{oc}}{\sin 2\widehat{od}}=\left(\frac{\sin\widehat{ca}}{\sin\widehat{da}}\right)^2=\left(\frac{\sin\widehat{cb}}{\sin\widehat{db}}\right)^2$

The results on the left-hand side may be deduced from the fact that

$$\frac{AC}{BC}\ \frac{AD}{BD}=-1,$$

and those on the right-hand either from

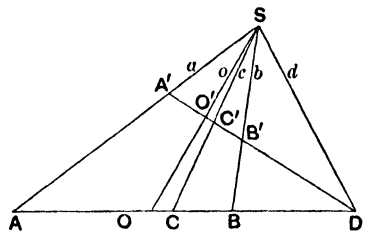
$$\frac{\sin\widehat{ac}}{\sin\widehat{bc}}\ \frac{\sin\widehat{ad}}{\sin\widehat{bd}}=-1,$$

or from the [illegible] relation on the left-hand

(a) Express all the distances in $(ABCD)=-1$ in terms of distances from A Then after simplification it is seen that

$$\frac{2}{AB}=\frac{1}{AC}+\frac{1}{AD}$$

(a') Let S be the vertex of the pencil Drop a perpendicular $DB'C'A'$ on a, meeting c and b in C' and B' (not shown in figure) Then $A'B'C'D$ is harmonic



Hence $$2\frac{1}{A'B'}=\frac{1}{A'C'}+\frac{1}{A'D},$$

$$2\frac{SA'}{A'B'}=\frac{SA'}{A'C'}+\frac{SA'}{A'D},$$

$$2\cot\widehat{ab}=\cot\widehat{ac}+\cot\widehat{ad}$$

(b) From (a) since $2\ CO=CA+CB$,

$$\frac{2}{CD}=\frac{1}{CA}+\frac{1}{CB}=\frac{CA+CB}{CA\ CB}=\frac{2\ CO}{CA\ CB},$$

$$CO\ CD=CA\ CB$$

(b') Similarly, since $2\widehat{co}=\widehat{ca}+\widehat{cb}$,

$$2\cot\widehat{cd}=\cot\widehat{ca}+\cot\widehat{cb}=\frac{\sin 2\widehat{co}}{\sin\widehat{ca}\ \sin\widehat{cb}},$$

$$2\sin\widehat{ca}\sin\widehat{cb}=\sin 2\widehat{co}\tan\widehat{cd}$$

(*c*) Since
$$\frac{AC}{CB} = \frac{AD}{BD},$$
$$\frac{AC-CB}{AC+CB} = \frac{AD-BD}{AD+BD},$$
$$\frac{2\ OC}{2\ OB} = \frac{2\ OB}{2\ OD},$$
$$OC\ OD = OB^2 = OA^2$$

(*c'*) Drop a perpendicular from D on o meeting a, b, c and o in $A'B'C''O'$ Then O' is the middle point of $A'B'$, therefore, since $A'B'C'D$ is harmonic,
$$O'C'\ \ O'D = O'1'^2,$$
$$\frac{SO'}{O'C'}\ \frac{SO'}{O'D} = \left(\frac{SO'}{O'1'}\right)^2,$$
$$\cot \widehat{oc} \cot \widehat{od} = \cot \widehat{oa} = \cot^2 \widehat{ob}$$

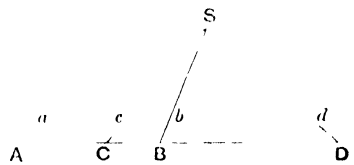
(*d*) From (*c*)
$$\frac{OA}{OC'} = \frac{OD}{O1'} = \frac{1D}{1C'}$$
by subtracting numerators and denominators
$$\frac{OD}{OC'} = \left(\frac{1D}{1C'}\right)^2$$
Similarly
$$\frac{OD}{OC'} = \left(\frac{BD}{BC'}\right)$$

(*d'*) From (*c'*)
$$\frac{\cot \widehat{oc}}{\cot \widehat{oa}} = \frac{\cot \widehat{oc}}{\cot \widehat{od}} = \frac{\sin \widehat{oc}}{-\sin \widehat{da}}\ \frac{\sin \widehat{oc}}{\sin \widehat{od}},$$
$$\frac{\cot \widehat{oc}}{\cot \widehat{od}} = \left(\frac{\sin \widehat{oc}}{\sin \widehat{da}}\right)^2 \left(\frac{\sin oc}{\sin od}\right)^2,$$
$$\frac{\sin 2\ \widehat{oc}}{\sin 2\ \widehat{od}} = \left(\frac{\sin \widehat{oc}}{\sin da}\right)^2$$
Similarly
$$\frac{\sin 2\ \widehat{oc}}{\sin 2\ od} = \left(\frac{\sin \widehat{cb}}{\sin db}\right)^2$$

The internal and external bisectors of any angle form with the lines that include the angle a harmonic pencil, and conversely,

If two conjugate rays of a harmonic pencil are at right angles they are the internal and external bisectors of the angle formed by the other pair of conjugate lines

Let any line s meet the lines a and b in A and B, and c and d the bisectors of the angle $\widehat{ab}$ in C and D



Then from the triangle ASB,
$$\frac{AC}{BC} = \frac{AS}{BS} = -\frac{AD}{BD} \quad \text{(Euclid VI 3 and A)}$$

Hence the range $ABCD$ is harmonic

Conversely, if the pencil $abcd$ is harmonic and the rays c and d are at right angles draw o to bisect the angle ASB

Then from (b') $\quad 2 \sin \widehat{ca} \sin \widehat{cb} = \sin 2\widehat{co} \tan \widehat{cd}$

But, since $\widehat{cd} = \frac{\pi}{2}$, $\tan \widehat{cd}$ is infinitely great while $\sin \widehat{ca} \sin \widehat{cb}$ is finite

Therefore $\quad \sin 2\widehat{co} = 0$ and consequently $\widehat{co} = 0$

Hence the line c coincides with o and is a bisector of the angle $\widehat{ab}$ Since d and c are at right angles d is the other bisector

17 Analytical Expressions in connexion with Harmonic Ranges

Adopting the notation of Art 14 and denoting the ratios of the points A_1, A_2, A_3, A_4 with reference to the points B, C by the letters r_1, r_2, r_3, r_4 respectively, the following results are obtained

(1) If $\frac{r_1}{r_2} = -1$ the range BCA_1A_2 is harmonic, therefore the points r_1 and $-r_1$ are harmonic conjugates of B and C

(2) If r_1 and r_2 are the roots of the equation $ar^2+b=0$ then the range BCA_1A_2 is harmonic

(3) The condition that the four points whose ratios are r_1, r_2, r_3, r_4 should form a harmonic range is

$$(r_1+r_2)(r_3+r_4) = 2r_1r_2 + 2r_3r_4$$

(4) The condition that A_3, A_4 should be harmonic conjugates of B and A_2 is

$$r_2(r_3+r_4) = 2r_3r_4$$

(5) If r_1, r_2 are given as roots of the equation $a'r^2+2h'r+b'=0$ and r_3, r_4 as roots of the equation $a''r^2+2h''r+b''=0$ then the condition that A_1, A_2 should be harmonic conjugates of A_3, A_4 is

$$a''b' + a'b'' - 2h'h'' = 0$$

(6) If r_1, r_2, r_3, r_4 are the roots of the equation $ax^4+4bx^3+6cx^2+4dx+e=0$ the condition that the range $A_1A_2A_3A_4$ should be harmonic is

$$c^3 - ace + ad^2 - 2bcd + b^2e = 0$$

(7) The common harmonic conjugates of B, C and A_1, A_2 are the points

$$+\sqrt{r_1r_2} \text{ and } -\sqrt{r_1r_2}$$

EXAMPLES

(1) If C, D be harmonic conjugates of A, B and P be any point on the line $ABCD$, then

$$2\ PA\ PB + 2\ PC\ PD = (PA+PB)(PC+PD)$$

and, conversely, if this relation holds $ABCD$ is harmonic

Express $(ABCD) = -1$ in terms of distances from any point P on the line and the required result is obtained

*(2) If Q and Q' be the harmonic conjugates of P with respect to AB and CD respectively, all the points being collinear, show that $ABCD$ is harmonic if

$$\frac{1}{PA}\frac{1}{PB}+\frac{1}{PC}\frac{1}{PD}=\frac{2}{PQ}\frac{2}{PQ'}$$

Since Q is a harmonic conjugate of P with respect to AB, $\frac{2}{PQ}=\frac{1}{PA}+\frac{1}{PB}$, similarly $\frac{2}{PQ'}=\frac{1}{PC}+\frac{1}{PD}$ Therefore

$$\frac{2}{PA}\frac{2}{PB}+\frac{2}{PC}\frac{2}{PD}=\left(\frac{1}{PA}+\frac{1}{PB}\right)\left(\frac{1}{PC}+\frac{1}{PD}\right)$$

or $2\ PC\ PD+2\ PA\ PB=(PB+PA)(PD+PC)$ Hence from (1) the result follows

(3) If C_1, C_2, C_3 be the middle points of A_1B_1, A_2B_2, A_3B_3, which are Harmonic Conjugates of AB, and P be any point on the line AB, then

$$C_2C_3\ PA_1\ PB_1+C_3C_1\ PA_2\ PB_2+C_1C_2\ PA_3\ PB_3=0$$

If C be the middle point of AB, then by (1) $PA_1\ PB_1+PA\ PB=2PC_1\ PC$

Substituting in the given relation it becomes $\Sigma C_2C_3(2\ PC_1\ PC-PA\ PB)=0$, or $2PC\Sigma C_2C_3\ PC_1-PA\ PB\ \Sigma C_2C_3=0$

Hence for the given relation to hold $\Sigma C_2C_3\ PC_1$ or $\Sigma(PC_3-PC_2)PC_1$ must be zero, but this is the case

(4) If E, F be the middle points of AB, CB respectively, two segments of a straight line, which cut each other harmonically, then

$$EF^2=EB^2+FD^2$$

Describe circles on AB and CD as diameters cutting at P Then

$$EP^2=EB^2=EC\ ED$$

Hence the circles cut at right angles and therefore

$$EF^2=EP^2+PF^2=EB^2+FD^2$$

Note From this a solution is obtained of the problem "to place two given segments so that their ends may form a harmonic range"

(5) If AB be harmonic conjugates of XY and of PQ, then

$$\frac{PX}{QX}\frac{PY}{QY}=\left(\frac{PA}{QA}\right)^2$$

(6) If $ABCD$ be a harmonic range and P any point on the line

$$2\frac{PB}{AB}=\frac{PC}{AC}+\frac{PD}{AD}$$

(7) If AB be harmonic conjugates of XY and of PQ, then

$$XP\ QA\ AY=-YQ\ PA\ AX$$

* See also Examples (3) and (6), Chapter VIII

(8) If AA' be harmonic conjugates of BB' and P any point on the line prove that

$$(AA'BP) = -(AA'B'P)$$

(9) A, B, C, D are four collinear points, determine P and Q which are harmonic conjugates of AB and CD and prove that

$$AC \cdot AD \cdot BC \cdot BD = AP^2 \cdot BP^2$$

(10) If $ABCD$ be a harmonic range A and B being conjugate and C and D, then

(i) $AB \cdot CD = 2AD \cdot CB$,

(ii) $2OR \cdot OR' = OP \cdot OP' + OQ \cdot OQ'$,

where O is any point on the line and PP', QQ', RR' are the mid points of BC, AD, CA, BD, AB and CD

(11) If A', B', C', D' be harmonic conjugates of A, B, C, D with respect to EF, prove that $(A'B'C'D') = (ABCD)$

(12) A, B, A', B' are four points on a straight line, P, P' the harmonic conjugates of any point in the line with regard to AB, $A'B'$, prove that P, P' describe homographic ranges and determine the self-corresponding points and the vanishing points of the ranges

(13) A and A' are any pair of harmonic conjugates of E and F and P and Q any pair of fixed points on the line EF Prove that

$$\frac{PA}{EA} \cdot \frac{QA}{FA} + \frac{PA'}{EA'} \cdot \frac{QA'}{FA'} \text{ is constant}$$

Express all lengths as distances from O the middle point of EF Reduce to one fraction remembering that $OA \cdot OA' = OE^2 - OF^2$, then the factor $2OE^2 - OA^2 - OA'^2$ can be removed from numerator and denominator, and the given expression becomes

$$\frac{OE^2 - OP \cdot OQ}{OE^2}$$

(14) If $(ABKD) = (ABCL)$, where A, B, C, D, K, L are collinear points, then

$$\frac{KA}{LA} \cdot \frac{KB}{LB} = \frac{KC}{LC} \cdot \frac{KD}{LD}$$

CHAPTER IV

CONICAL PROJECTION AND PLANE PERSPECTIVE

18 There are four kinds of Projections which are in general use, viz,

(1) Orthogonal Projection,
(2) Projection in a Plane,
(3) Conical Projection
(4) Homology, or Plane Perspective

In addition to the above, it is also possible to project from any given axis (1) and (2) are particular cases of (3), and (4) may also be derived from (3)

The fundamental theorems of Conical Projection and of Homology or Plane Perspective are similar, although in some cases they have to be proved in different ways In Conical Projection the proofs are generally simpler and bring home more directly to the learner the reason why the theorems are true In Homology, on the other hand, there is a simplification in the fact that the figures lie in one plane, and the learner is not confronted with the difficulty of comprehending figures in space The fundamental theorems of Conical Projection and of Homology will be given side by side together with their proofs Afterwards, it will be shown how the latter may be deduced from the former

Conical Projection

General method of obtaining one plane figure from another plane figure by Conical Projection

Let the given figure consisting of points A, B, C, and lines a, b, c, be situated in the plane σ

Take any point S outside the plane σ

Construct the lines SA, SB, SC, ... and the planes Sa, Sb, Sc, ...

Consider any plane σ' (not coincident with σ). The lines SA, SB, SC, ... will intersect this plane in points, say, A', B', C', ... The planes Sa, Sb, Sc, ... will intersect this plane in lines, say, a', b', c', ...

The figure made up of the points A', B', C', ... and the lines a', b', c', ... in the plane σ' is said to be a projection upon the plane σ' of the original figure made up of the points A, B, C, ... and the lines a, b, c, ...

The points A', B', C', ... in one figure are said to correspond to the points A, B, C, ... in the other, and the lines a', b', c', ... in one figure are said to correspond to the lines a, b, c, ... in the other.

It will be seen that every pair of corresponding points are collinear with S, and, since any two corresponding lines a and a' are the intersections of a plane through S with the planes σ and σ', the lines a and a' meet the line $\sigma\sigma'$ in the same point.

Homology, or Plane Perspective

General method of obtaining one plane figure from another plane figure by Homology or Plane Perspective

Let the given figure consisting of points A, B, C, ... and lines a, b, c, ... be situated in the plane σ. Take any point S outside the plane σ.

On any plane σ', not coincident with σ, construct with centre S a projection of the given figure consisting of points A', B', C', ... and of lines a', b', c', ...

Take any other centre of projection S'. From S' project the figure consisting of points A', B', C', ... and lines a', b', c', ... upon the plane σ, so as to form a figure in the plane σ consisting of the points A_1, B_1, C_1, ... and lines a_1, b_1, c_1, ...

The figure consisting of the points A_1, B_1, C_1, ... and the lines a_1, b_1, c_1, ... is said to be in plane perspective with the figure consisting of the points A, B, C, ... and the lines a, b, c, ... The points A_1, B_1, C_1, ... in one figure are said to correspond to the points A, B, C, ... in the other, and the lines a_1, b_1, c_1 in one figure are said to correspond to the lines a, b, c in the other.

Every pair of corresponding points A, A_1 are collinear with the point where SS' meets σ, for these points are situated on the line in which the plane $SS'A'$ meets σ.

Any two corresponding lines a and a_1 meet $\sigma\sigma'$ in the point where it is met by a'. Therefore all pairs of corresponding lines intersect on the fixed line $\sigma\sigma'$.

Hence two figures in plane perspective are such that every pair of corresponding points are collinear with a given point, termed the *centre of Perspective*, and every pair of corresponding lines intersect on a given line, termed the *axis of Perspective*

In practice, these theorems are used to construct figures in plane perspective and the consideration of figures in space is thus excluded

19 From the preceding it follows that both in Conical Projection and in Plane Perspective

(1) To points correspond points and to straight lines correspond straight lines

(2) To collinear points correspond collinear points and to concurrent lines correspond concurrent lines

(3) Corresponding curves are the loci of corresponding points and corresponding envelopes are the envelopes of corresponding lines

(4) To points or lines in one figure which coincide in the limit correspond in the other figure points or lines which likewise coincide in the limit

(5) To a point of intersection of a line and a curve corresponds a point of intersection of the corresponding line and curve

(6) A line meets a curve in the same number of points as the corresponding line meets the corresponding curve Hence a curve and its corresponding curve are said to be of the same *degree*

(7) To a tangent to a curve at a point corresponds the tangent at the corresponding point to the corresponding curve

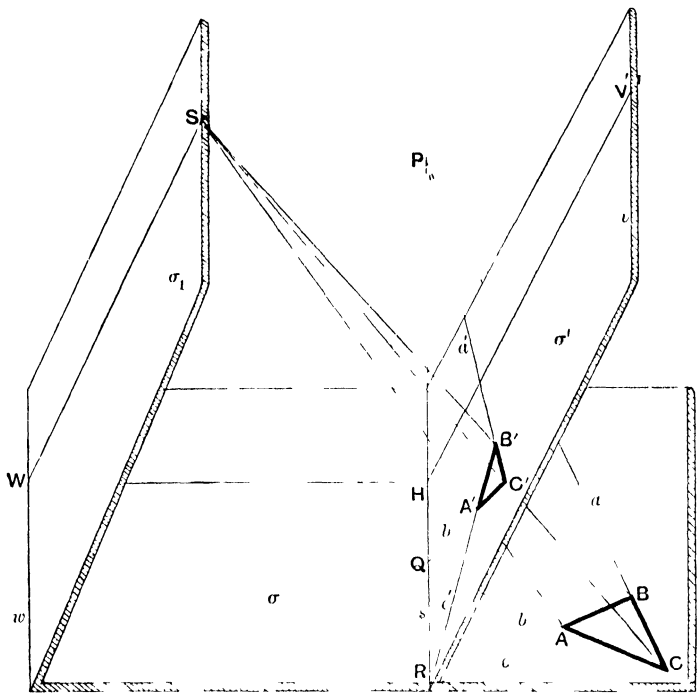
(8) To the tangents from any point to a curve correspond the tangents from the corresponding point to the corresponding curve Hence a curve and its corresponding curve are said to be of the same *class*

(9) The anharmonic ratio of any four collinear points in one figure is equal to the anharmonic ratio of the corresponding points of the other figure For the two ranges are in plane perspective (see Art 9) with S for centre of perspective

(10) A harmonic range is projected into a harmonic range

(11) Either figure may be deduced from the other by the same process

These points are more fully illustrated in the following article



20 Construction of figures by Conical Projection

Let the given figure of which A, B, C are three points be in the plane σ (Figure page 34)

Take S any point outside the plane σ

Join S to A, B and C and on SA, SB, SC take any points A', B', C' These points lie in some plane σ'

Since AA' and BB' meet in S, AB and $A'B'$ lie in a plane and meet in some point R

Similarly, BC and $B'C'$ meet at a point P and AC and $A'C'$ meet at a point Q

Construction of figures in Plane Perspective

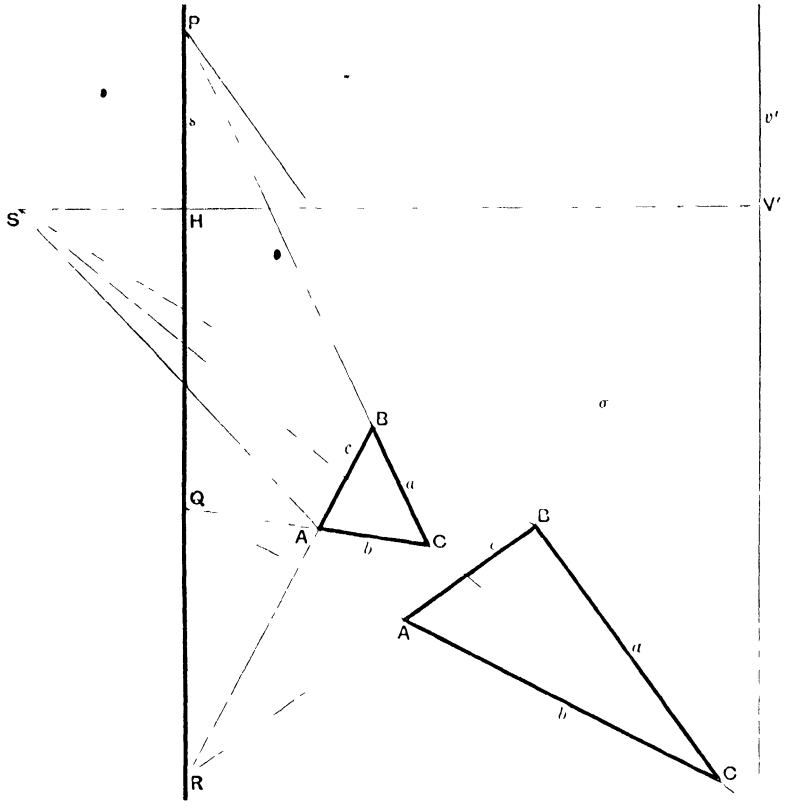
Let the given figure of which A, B, C are three points be in the plane σ (Figure page 35)

Take S any point in the plane σ

Join S to A, B and C, and on SA, SB, SC take any points A', B', C' These points lie in the plane σ

Since AA' and BB' are in the plane σ, AB and $A'B'$ meet in some point R

Similarly, BC and $B'C'$ meet at a point P and AC and $A'C'$ meet at a point Q



But P, Q and R must lie on the line of intersection of σ and σ' and are therefore collinear

The line PQR or $\sigma\sigma'$ is usually denoted by the letter s

The figure obtained by projection on the plane σ' is now completely determined, for the point corresponding to any given point D in the plane σ can be found

But the triangles ABC and $A'B'C'$ are such that the lines joining their vertices are concurrent, and therefore P, Q and R (Art 13 (a)) are collinear

The line PQR is usually denoted by the letter s and is called the axis of homology or perspective

The figure in perspective with the given one is now completely determined, for the point corresponding to any given point D can be found, since (Art 18) corresponding points are collinear with S and corresponding lines intersect on s

For join D to S

Then the point corresponding to D is that in which SD meets the plane σ'

For join D to S Let DA meet s in T

Then the point D' corresponding to D is that in which SD meets TA' For DA and $D'A'$, being corresponding lines, must intersect on s

21 Determining elements of a Conical Projection

Given the figure to be projected, the determining elements of the projection may be

(1) The point S and any three points A', B', C' on SA, SB, SC respectively which are to correspond to A, B, C

(2) The point S and the plane σ', or, what is equivalent thereto,

S and s together with A', the point corresponding to A, which must be collinear with S and A

or correlatively,

s and S together with a', the line corresponding to a, which must be concurrent with s and a

(3) Any two points A' and B' which correspond to A and B, such that AA' and BB' meet at some point S, and any point C'' on SC which corresponds to C'

Note Even if C' is not given, a point R on s is determined and the plane σ' for different positions of C' will rotate round the line $RA'B'$

or correlatively,

any two lines a' and b', which correspond to a and b, such that aa' and bb' lie on a line s, and any line c' which corresponds to c and intersects c on s

Note Even if c' is not given, the line of intersection of the planes aa' and bb' passes through S, and the point S will move along this line of intersection, for different positions of c'

Determining elements of figures in Homology or Perspective

Given the figure whose perspective is required, the determining elements of the perspective may be

(1) The point S and any three points A', B', C' on SA, SB, SC respectively which are to correspond to A, B, C

(2)

S and s together with A', the point corresponding to A, which must be collinear with S and A

or correlatively,

s and S together with a', the line corresponding to a, which must be concurrent with s and a

(3) Any two points A' and B' which correspond to A and B, such that AA' and BB' determine some point S, and any point C'' on SC which corresponds to C'

Note Even if C' is not given, a point R on s is determined and the line s for different positions of C' will rotate round the point R

or correlatively,

any two lines a' and b', which correspond to a and b, such that aa' and bb' determine a line s, and any line c' which corresponds to c and intersects c on s

Note Even if c' is not given, the line joining ab to $a'b'$ must pass through S, and the point S will move along this line, for different positions of c'

(4) Any two points A' and B' corresponding to A and B, and such that AA' and BB' intersect, and the direction of s in the plane σ

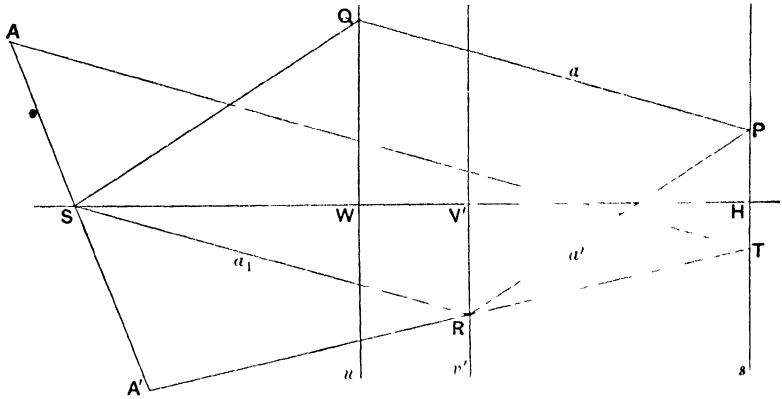
(4) Any two points A' and B' corresponding to A and B, and the direction of s

Note For another method of construction see Art 28

In each of the above cases the point S, and the plane σ' or the axis of perspective s, can be found

22 Vanishing Lines

The loci of the points in one figure which correspond to the points at infinity in the other are straight lines termed the vanishing lines There is one vanishing line in each figure



To find the point (see Fig page 34) corresponding in the second figure to the point at infinity on a line a, draw through S a line a_1 parallel to a This line will meet σ' in the point $a_1\sigma'$ Similarly the points corresponding to the points at infinity on $b, c, d,$ are obtained as the points of intersection of the plane σ' with the lines $b_1, c_1, d_1,$ drawn through S parallel to b, c, d These lines lie in a plane through S parallel to σ and the line of intersection of this plane with σ' gives the locus of the points corresponding to the points at infinity on σ This line is parallel to s, and is termed the *vanishing line*

To find the point (see Fig page 37) corresponding in the second figure to the point at infinity on a line a, draw through S a line a_1 parallel to a If A' corresponds to A and a parallel to a be drawn through A meeting s in T, then the required point is given as the intersection of a_1 and TA' By considering the points corresponding to the points at infinity on any other lines $b, c, d,$ any number of points on the locus may be obtained This locus by Addendum I is a straight line parallel to s, and is termed the *vanishing line*

This vanishing line may be conveniently denoted by v'

Similarly the points at infinity in the figure σ' correspond to the points lying on a vanishing line w, which is obtained by drawing a plane through s parallel to σ'

As the points at infinity in either figure correspond to points on a vanishing line, it is convenient to regard all the points at infinity in either figure as being on a straight line termed *the line at infinity* Thus in the first figure the points at infinity are regarded as being on a line v which corresponds to v' and in the second on a line w' which corresponds to w

The *vanishing point* of a line is the point where the line meets the vanishing line in its figure This point corresponds to the points at infinity on the corresponding line

A system of parallel lines in one figure corresponds to a system of lines in the other which intersect in a point on a vanishing line

For the above construction gives the same point on the vanishing line as corresponding to the point at infinity on each line of the system

Determination of Position of Vanishing Lines

Through S draw a plane perpendicular to s meeting

s in H,
w in W,
v' in V'

This vanishing line may be conveniently denoted by v'

Similarly the points at infinity in the second figure correspond to the points lying on a vanishing line w, which is also parallel to the axis of Perspective s

As the points at infinity in either figure correspond to points on a vanishing line, it is convenient to regard all the points at infinity in either figure as being on a straight line termed *the line at infinity* Thus in the first figure the points at infinity are regarded as being on a line v which corresponds to v' and in the second on a line w' which corresponds to w

The *vanishing point* of a line is the point where the line meets the vanishing line in its figure This point corresponds to the points at infinity on the corresponding line

A system of parallel lines in one figure corresponds to a system of lines in the other which intersect in a point on a vanishing line

For the above construction gives the same point on the vanishing line as corresponding to the point at infinity on each line of the system

Determination of Position of Vanishing Lines

Through S draw a line perpendicular to s meeting

s in H,
w in W,
v' in V'

Then $SWHV'$ is a parallelogram (Figure page 34)

Therefore

$$SW = V'H \text{ and } SV' = WH$$

Therefore *the distance of either vanishing line from the centre of Projection* (S) *is equal to that of the other vanishing line from the line of intersection of the planes* (s)

Hence, given S and any two of the lines s, w and v' the remaining line is completely determined

23 The connector of S with the vanishing point of any line a, viz (S wa) in the figure σ, is parallel to the corresponding line a' For (S wa) passes through $w'a'$ the point at infinity on a'

Hence a line in one figure is parallel to the line joining S to the vanishing point of the corresponding line in the other figure

As an immediate consequence the following important result is obtained

The angle subtended by the vanishing points of two lines at S is equal to the angle between the corresponding lines

Let a and a' be two corresponding lines which therefore meet in a point P on s

Let a meet w in Q and a' meet v' in R (Figure page 37)

Then aw (or Q) corresponds to the point at infinity on a' Therefore SQ is parallel to a' Similarly SR is parallel to a Therefore $SQPR$ is a parallelogram, and

$$SW = V'H \text{ and } SV' = WH$$

Therefore *the distance of either vanishing line from the centre of Perspective* (S) *is equal to that of the other from the axis of Perspective* (s)

Hence, given S and any two of the lines s, w and v', the remaining line is completely determined

The connector of S with the vanishing point of any line a, viz (S wa) in one figure, is parallel to the corresponding line a For (S wa) passes through $w'a'$ the point at infinity on a'

Hence a line in one figure is parallel to the line joining S to the vanishing point of the corresponding line in the other figure

As an immediate consequence the following important result is obtained

The angle subtended by the vanishing points of two lines at S is equal to the angle between the corresponding lines

24 Nature of the figure obtained by Conical Projection

If the centre of Projection S, the plane σ, and the vanishing line w are given, the plane σ' may move parallel to itself, and the lines s and v' will then move parallel to w and at a constant distance apart, equal to the distance of S from w

In this case the Projections will be similar For sections of a cone by parallel planes are similar

The linear dimensions of figures obtained by projection are proportional to the perpendicular from the vertex on the plane of section or to the intercepts by these planes on any line through S drawn in a given direction The intercept SV' is such a length But

$$SV' = WH$$

Nature of the figure obtained by Plane Perspective

If the centre of Perspective S, the first figure, and the vanishing line w are given, the lines s and v' may move parallel to w and will be at a constant distance apart, equal to the distance of S from w

In this case the Perspective figures are similar For if a and b, any two lines of the given figure, meet w in aw and bw, then the corresponding lines in every figure in perspective are parallel to the lines joining these points to S Hence in all the perspective figures the angles between pairs of lines which correspond to a given pair of lines in the original figure are equal Therefore the figures are similar

Let v' and s be the vanishing line and axis of perspective of one figure, and v_1' and s_1 be the vanishing line and axis of perspective of another figure

Let two lines of the given figure be a and b Through S draw lines parallel to a and b meeting v' and v_1' in P' and P_1' and in Q' and Q_1' These points correspond to the points at infinity on a and b Hence, if the central line meets v', s, v_1', s_1 in V', H, V_1' and H_1,

$$\frac{P'Q'}{P_1'Q_1'} = \frac{SV'}{S'V_1'} = \frac{WH}{WH_1}$$

Therefore the linear dimensions of the similar figures obtained by projection are proportional to WH, the distance of the vanishing line from the intersection of the planes σ and σ'

Hence the nature of a projection depends on the position of the plane $(S\ w)$ and the position of S in this plane

By taking any line in σ as w (the vanishing line) that line may be projected into the line at infinity

Therefore the linear dimensions of the similar figures obtained by plane perspective are proportional to WH, the distance of the vanishing line from the axis of perspective s

Hence the nature of the perspective figure depends on the position of w in the plane and on the position of S relative to it

By taking any line in the plane as w (the vanishing line) that line may be made to correspond to the line at infinity in the perspective figure

25 To form a figure by Projection such that the angles between the two pairs of lines, which correspond to two given pairs of lines, shall have given values, and the Projection of a given line shall be the line at infinity

Let a and b, c and d be the lines the projections of which are to contain given angles α and β Take as vanishing line of the figure w, the line which is to be projected to infinity Draw any plane σ_1 through w Construct the vanishing points aw, bw, cw, dw of the lines a, b, c, d These will be on the line of intersection w of the planes σ_1 and σ

In the plane σ_1 describe on aw, bw a segment of a circle to contain an angle α and on cw, dw describe a segment of a circle to contain an angle β

To form a figure by Plane Perspective such that the angles between the two pairs of lines, which correspond to two given pairs of lines, shall have given values, and the line corresponding to a given line shall be the line at infinity

Let a and b, c and d be the lines whose corresponding lines are to contain given angles α and β Take as vanishing line of the figure w, the line which is to correspond to the line at infinity Construct the vanishing points aw, bw, cw, dw of the lines a, b, c, d

On aw, bw describe a segment of a circle to contain an angle α and on cw, dw describe a segment of a circle to contain an angle β

Let these segments intersect in S*

If S be taken as centre of projection and any plane σ' parallel to σ_1 as the plane of section, then the lines a', b', c', d' corresponding to a, b, c, d will be parallel to the lines joining S to aw, bw, cw and dw

Therefore the angles between a' and b', c' and d' will be of the given magnitudes α and β Also the given line w will be projected to infinity Hence the figure obtained by this Projection meets the given conditions

Let these segments intersect in S*

If S be taken as centre of perspective and any line s parallel to w as axis of perspective, a', b', c', d' the lines corresponding to a, b, c, d will be parallel to the lines joining S to aw, bw, cw and dw

Therefore the angles between a' and b', c' and d' will be of the given magnitudes α and β Also the given line w will be projected to infinity Hence the figure obtained by this Plane Perspective meets the given conditions

26 Connexion between Conical Projection and Plane Perspective

In the preceding, the fundamental theorems of conical projection and of plane perspective have been proved independently The latter, however, may be deduced from the former

Suppose that the figures in the planes σ and σ' are given and likewise the correspondence between them Let the plane σ' rotate about the line of intersection of the planes σ and σ' (i e s) Let a, b, c and a', b', c' be any three pairs of corresponding lines which intersect in pairs of corresponding points A, B, C and A', B', C' (see Fig page 34) When σ' is rotated round $\sigma\sigma'$, the lines aa', bb', cc' will continue to intersect on $\sigma\sigma'$ Hence the lines AA', BB', CC' will continue to intersect in some point S The pairs of points AA', BB' may be regarded as fixed and determining the position of S and CC' as any pair of corresponding points Hence the figures, when their planes are rotated, will continue to have a centre of projection S

Consider the parallelogram $SWHV'$ Because the figure in σ' is given and likewise the correspondence, HV' is constant Therefore WS

* If the segments aw, bw and cw, dw overlap, the circles described thereon will intersect in a pair of real points and S is real If they do not overlap, the circles, in certain cases, will not intersect and in this case S will be an imaginary point The projection is then said to be imaginary The criterion as to whether or not the projection or perspective is

is constant and since W is fixed, S describes a circle round W in the plane perpendicular to the lines w, v', s

If the plane σ' be rotated round s till σ and σ' coincide, the two figures will be superposed in plane perspective in the plane σ with S as centre of perspective at a distance HV' from w, and s for axis of perspective The rotation may take place in either of two ways, so that S may lie on either side of w, and v' on either side of H

27 Any two plane quadrangles may be derived the one from the other by a series of projections

Let $ABCD$ and $A'B'C'D'$ be the quadrangles situated in the same or different planes

Through D draw a plane distinct from the plane $ABCD$ and project the quadrangle $A'B'C'D'$ into a quadrangle in this plane so that the point corresponding to D' coincides with D The problem then reduces to the following

"Given two planes σ and σ', three points A, B, C in the former, three points A', B', C' in the latter and D a point on their line of intersection, to derive $A'B'C'D$ from $ABCD$ by projection"

There are two ways of proceeding

(*a*) Through D draw a line s in the plane σ to meet BC, CA and AB in K, L, and M, and a line s' in the plane σ' to meet $B'C'$, $C'A'$ and $A'B'$ in K', L' and M'

Since s and s' intersect at D they lie in a plane σ_1, in which KK', LL', MM' form a triangle $A_1B_1C_1$

The triangle ABC is projective with the triangle $A_1B_1C_1$ since the sides intersect in K, L, M on the line $\sigma\sigma_1$, and, since D is on $\sigma\sigma_1$, the point D corresponds to itself

Similarly the triangle $A'B'C'$ is projective with the triangle $A_1B_1C_1$ since the sides intersect in K' L', M' on the line $\sigma'\sigma_1$ and, since D is on $\sigma'\sigma$ the point D corresponds to itself

Therefore the points A', B', C', D can be derived from the points A, B, C, D by two projections

(*b*) Let AB meet DC in E and $A'B'$ meet DC' in E'

Since DCE and $DC'E'$ are in one plane, CC' and EE' will meet in a point S_1 Draw a plane σ_1 through DCE and project $A'B'C'D$ from the point S_1 on the plane σ_1 The points D, C, E' will thus be projected into the points D, C, E, and A' and B' will be projected into points A'', B'' in the plane σ_1, which are collinear with E

Since the lines ABE and $A''B'E$ are concurrent, AA'' and BB'' are

concurrent at some point S, and from this point the quadrangle $A''B''CD$ can be projected into the quadrangle $ABCD$

This result should be compared with that obtained in Art 65, where it is shown that any two quadrangles can be placed in a plane so as to be in plane perspective

If three triangles situated in different planes are two by two in perspective and have the same axis of perspective, their three centres of perspective are collinear

Let the three triangles be (1) ABC, (2) $A'B'C'$, (3) $A''B''C''$ Then since the triangles are two by two in perspective and have the same axis of perspective, any three pairs of corresponding sides, say, AB, $A'B'$, $A''B''$, meet at a point on s the common axis of perspective

Let S be the centre of perspective of (1) and (2),

S' „ „ „ (2) and (3),

S'' „ „ „ (3) and (1)

Then the three points S, S' and S'' are situated in the plane $AA'A''$, and also in the plane $BB'B''$ Hence they are all on the line of intersection of the planes $AA'A''$ and $BB'B''$ and are collinear It is seen also that $CC'C''$ passes through this line

The Correlative theorem is

If three triangles situated in different planes are two by two in perspective and have the same centre of perspective, their three axes of perspective are concurrent

28 Anharmonic ratios in connexion with figures in Plane Perspective

If A and A' be any two corresponding points of two figures in plane perspective, and AA' meets s in P, then if S be the centre of perspective the anharmonic ratio $(SPAA')$ is constant

Take any other pair of corresponding points B and B' Let BB' meet s in Q Then the lines AB and $A'B'$ intersect in some point R on s Hence $(SPAA') = (SQBB') =$ a constant, for every pair of corresponding points From this it follows that

(1) If S, s and the anharmonic ratio $(SPAA')$ be given, the Perspective is completely determined $(SPAA')$ may be termed the anharmonic ratio of the Perspective

(2) If A be taken at infinity, $(SPAA') = 1$ $\dfrac{SA'}{PA'} = \dfrac{PA'}{SA'}$ In this case, A' is any point on the vanishing line, which may be denoted by V', and its locus is clearly a line parallel to s such that

$$\frac{IV'}{SV'} = (SPBB') = \text{the anharmonic ratio of the perspective}$$

(3) Similarly $\dfrac{HW}{SW} = (SPB'B) = \dfrac{1}{(SPB'B)} = \dfrac{SV'}{HV'}$,

$$\frac{HS + SW}{SW} = \frac{SH + HV'}{HV'}$$

$$SW = V'H$$

29 Particular cases of Conical Projection and Plane Perspective

Conical Projection

(1) Let s be at infinity In this case, the cutting planes are parallel and the figures are similar The vanishing lines coincide with the line at infinity and the ratio of any pair of corresponding segments is constant

(2) Let S be at infinity In this case, the cone formed by joining points in the figures to S is replaced by a cylinder or prism The line at infinity corresponds to itself Orthogonal projection is a particular instance of this case

Plane Perspective

(1) Let s be at infinity Then the figures are said to be similar and similarly placed The vanishing lines coincide with the line at infinity and S is termed the centre of similitude

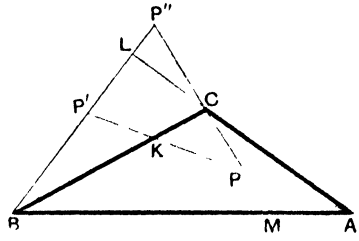
(2) Let S be at infinity Then the line joining any pair of corresponding points is parallel to a fixed direction The line at infinity corresponds to itself Such figures are termed "homological by affinity"

(3) Let the anharmonic ratio of the plane perspective be -1 The Perspective is then said to be harmonic In this case $(SV\,I\,I') = (SV\,I\,A)$, so that pairs of corresponding points mutually correspond Important cases of this will arise hereafter

If ABC be a triangle and σ' be the perspective of a figure σ, I and BC being the centre and axis of perspective, and σ'' be the perspective of σ', B and AC being the centre and axis of perspective, and σ''' be the perspective of σ', C and AB being the centre and axis of perspective, then, provided the anharmonic ratios of the perspective be the same in each case, the figures σ and σ'' coincide

Let P be any point of σ and let the anharmonic ratio of the perspective be in each case λ Then, if AP meet BC in K, the point P' of σ' corresponding to P is determined by $(IKPP') = \lambda$, so P'' the point of σ'' corresponding to P' is given by $(BLP'P'') = \lambda$

$$(AKPP') = (BLP'P'') = (LBP''P)$$



Therefore P and P'' are collinear with C

Also $(CMP''P) = (B\ CMP''P)$

$$= (KAP'P) = (AKPP') = \lambda$$

Hence P corresponds to P'', when C and BA are the centre and axis of Perspective

An important particular case of the preceding arises when the three perspectives are harmonic (See Art 107)

30 The important theorem of Art 25—if for the present the use of imaginary points be excluded—requires the two circles to intersect in a pair of real points From the following theorem it may be ascertained when this will be the case

A, A', B, B' are four collinear points and circles are described on AA' and BB' to contain angles α and β respectively

(1) *If the points occur in the order ABA'B', the circles will always intersect in real points*

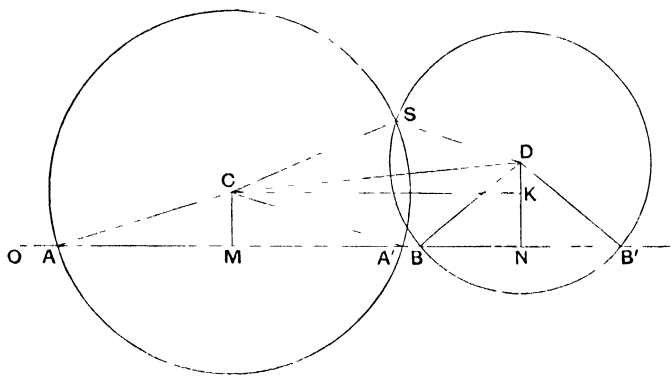
(2) *If the points occur in the order AA'BB', the circles will intersect in real points, if*

$$(BB'AA') > \frac{\cos^2 \frac{\alpha - \beta}{2}}{\cos^2 \frac{\alpha + \beta}{2}}$$

(3) *If the points occur in the order ABB'A', the circles will intersect in real points, if*

$$(BB'AA') < \frac{\sin^2 \frac{\alpha - \beta}{2}}{\sin^2 \frac{\alpha + \beta}{2}}$$

(1) In this case the theorem is obvious



(2) Let C and D be the centres of the circles, M and N the feet of the perpendiculars for these points on $AA'BB'$, and S a point of intersection of the circles Then the condition that S should be real is that

$$(CS + SD)^2 > CD^2 \qquad (1)$$

Then angle ACM = angle $MCA' = \alpha$,

and angle BDN = angle $NDB' = \beta$

Let O be any point on the line $AA'BB'$ and CK the perpendicular from C on DN. Then (i) becomes

$$(CS+SD)^2>DK^2+CK^2 \qquad \text{(ii)}$$

But $$CS=\frac{OA'-OA}{2\sin\alpha}, \qquad SD=\frac{OB'-OB}{2\sin\beta},$$

$$CK=\frac{OB'+OB}{2}-\frac{O1'+O1}{2}, \qquad DK=\frac{OB'-OB}{2\tan\beta}-\frac{O1'-O1}{2\tan\alpha}$$

Hence (ii) becomes

$$\left(\frac{OA'-OA}{2\sin\alpha}+\frac{OB'-OB}{2\sin\beta}\right)^2>\left(\frac{OB'-OB}{2\tan\beta}-\frac{O1'-O1}{2\tan\alpha}\right)^2+\left(\frac{OB+OB}{2}-\frac{O1+OA}{2}\right)^2$$

or $$(OA'-OA)(OB'-OB)\frac{1+\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\sin\beta}>2(OB-O1)(OB-OA'),$$

$$\frac{1+\cos(\alpha-\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}>\frac{1B}{11'}\frac{1B'}{BB'} \text{ or } >(B1'AB)$$

Hence the required condition is

$$(BB'1A')>\frac{\cos^2\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha+\beta}{2}}$$

(3) As in (2) it may be shown that the condition in this case is that

$$(BA'AB')>\frac{\cos(\alpha-\beta)-1}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}>\frac{\sin^2\frac{\alpha-\beta}{2}}{\sin^2\frac{\alpha-\beta}{2}-\sin^2\frac{\alpha+\beta}{2}}$$

or $$(BBA1')<\frac{\sin^2\frac{\alpha-\beta}{2}}{\sin^2\frac{\alpha+\beta}{2}}$$

EXAMPLES

(1) Given in a plane four lines a, b, a', b', and two points 1 and 1, what (if any) condition must hold if a, b, 1 correspond to a', b', A' in plane perspective? Construct the perspective.

A' must be situated on the line joining $a'b'$ to the point, where s the axis of perspective meets $A\ ab$.

(2) Two pairs of straight lines ab and de are given and c is the line joining ab to de. On c any point C is taken and through C any two lines are drawn to cut a and e in A and E, and b and d in B and D.

Show that the points ae, bd and $AB\ DE$ are collinear.

Consider the perspective in which

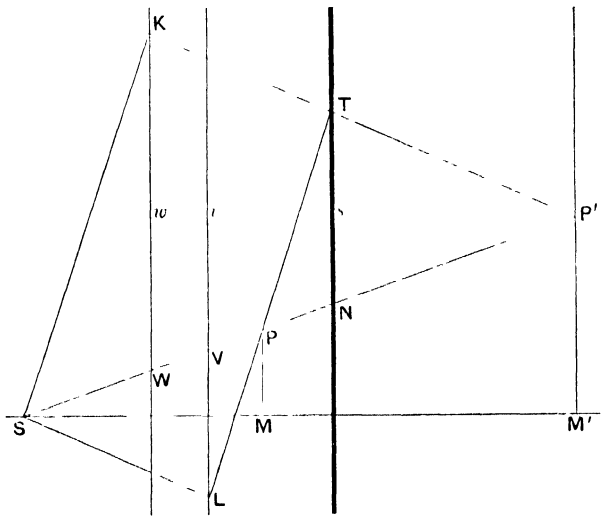
a corresponds to e
b „ „ d

and C is the centre of perspective

A corresponds to E
B „ „ D
AB „ „ DE

Therefore ae, bd, and AB DE are collinear on s the axis of perspective

(3) In two figures σ and σ' in plane perspective prove that the ratio of the distances of two corresponding points P and P' from S is equal (1) to the ratio of the distance of the point P from the vanishing line in its figure, to the distance of the other vanishing line from S, and (2) to the ratio of the distances of the two points from the central line



Let S and s be the centre and axis of perspective Let P and P' be a pair of corresponding points which must be collinear with S Let SPP' meet s in N Join P and P' to T any point on s Let a line parallel to PT through S meet TP' in K and a line through S parallel to TP' meet TP in L Then L and K are respectively on the vanishing lines v and w', which are parallel to s Let these vanishing lines meet SPP' in V and W' and drop perpendiculars PM and $P'M'$ from P and P' on the central line, *i e* on the perpendicular from S on s

Then $\frac{SP}{SP'} = \frac{PM}{P'M'}$ by similar triangles

Also $$(SPNP') = (S\infty NW),$$

$$\frac{SW'}{PA} = \frac{SP}{PP'} \qquad \frac{SW'}{SW' - PN} = \frac{SP}{SP - PP'}$$

$$\frac{SW'}{VP} = \frac{SP'}{SP}, \text{ since } SW' = VN$$

(4) Prove that if the lines joining the corresponding angular points of two triangles in the same plane are concurrent, then the triangles are, in an infinite number of ways, the projections from two points in space of the same triangle lying outside their plane

(5) Prove that the locus of the vertex from which a system of four fixed points in a plane can be conically projected into a square is a circle in a plane at right angles to the third diagonal of the quadrilateral formed by the four points

CHAPTER V

APPLICATIONS OF PROJECTION AND OF PLANE PERSPECTIVE TRIANGLES IN PERSPECTIVE

31 *If the three sets of points* BCA_1A_2, CAB_1B_2, ABC_1C_2 *are collinear, the expressions*

$$\begin{matrix} BA_1 & CB_1 & AC_1 \\ CA_2 & AB_2 & BC_2 \end{matrix} \quad \textit{and} \quad \begin{matrix} BA_1 & BA_2 & CB_1 & CB_2 & AC_1 & AC_2 \\ CA_1 & CA_2 & AB_1 & AB_2 & BC_1 & BC_2 \end{matrix}$$

are projective

The first expression may be written

$$\begin{pmatrix} BA_1 & BS \\ CA_1 & CS \end{pmatrix} \begin{pmatrix} CB_1 & CS \\ AB_1 & AS \end{pmatrix} \begin{pmatrix} AC_1 & AS \\ BC_1 & BS \end{pmatrix}$$

or, with notation of Art 10,

$$(BCA_1\ S)(CAB_1\ S)(ABC_1\ S)$$

Each of these expressions is projective by Art 10, and therefore their product is also projective

The second expression is projective because it is the product of two expressions of the form of the first expression

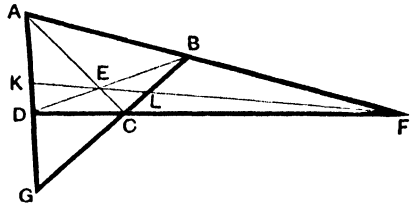
Projection

By means of Conical Projection one figure may be deduced from another as explained in the last chapter and the properties of one figure may be inferred from those of the other From a general theorem a particular case can be deduced or from a particular case a general theorem may be obtained The theorems of Art 25 and Art 27 are those most usually employed These are as follows, viz

(i) From any figure σ another figure σ' may be obtained by projection, such that (a) the angles between any two pairs of lines in σ', corresponding to any two given pairs of lines in σ, may have given values, and (b) the line corresponding to any given line in σ may be the line at infinity in σ'

(ii) From any figure σ another figure σ' may be obtained by projection, such that any four given points in σ are the projections of any four given points in σ

(1) *A* *nto a square*



This is a case of (11) It may be deduced from (1) as follows Let *ABCD* be the quadrangle and *E*, *F*, *G* its diagonal points Project *GF* into the line at infinity and the angles *ABC* and *BEC* into right angles Then the resulting figure must be a square

(2) *The line joining any pair of diagonal points (E and F) of a quadrangle (ABCD) is cut harmonically by a pair of sides (at K and L)*

Project the quadrangle into a square, *EG* being projected into the line at infinity The theorem then reduces to proving that if a line be drawn through the centre of a square parallel to a side, the intercept made on it by the other pair of sides is bisected at the centre

From this it follows that any side of a quadrangle is divided harmonically at the diagonal point on the side and at the point where the side is met by the opposite side of the diagonal points triangle

As a particular case it is seen that if through a diagonal point of a quadrangle a straight line be drawn parallel to the opposite side of the diagonal points triangle, the intercept on this line by a pair of opposite sides is bisected at the diagonal point

(3) *If the corresponding sides of two triangles ABC and A'B'C' intersect in three collinear points on a straight line s, then the three lines joining pairs of corresponding vertices are concurrent at a point S* Art 13 (*a*)

Project the line *s* into the line at infinity and the theorem reduces to 2 (*a*) of the addendum

(4) *If the lines joining corresponding vertices of two triangles ABC and A'B'C' are concurrent at a point S, then the three pairs of points of intersection of corresponding sides are collinear on a straight line s* Art 13 (*a*)

Project the point *S* to infinity, and the theorem reduces to 2 (*b*) of the addendum

(5) *If a transversal meet the sides of a triangle ABC in A'B'C', then*

$$\frac{AC'}{BC'}\cdot\frac{BA'}{CA'}\cdot\frac{CB'}{AB'}=1 \quad \text{(Menelaus' Theorem)}$$

The left-hand side of this equation is unaltered by projection Project the transversal into the line at infinity In this case *A'*, *B'*, *C'* are at infinity, and each of the three ratios which compose the expression is equal to unity Therefore their product is in this, and in all cases, equal to unity

(6) *If the lines joining the vertices of a triangle ABC to any point P meet the opposite sides in A', B', C', then*

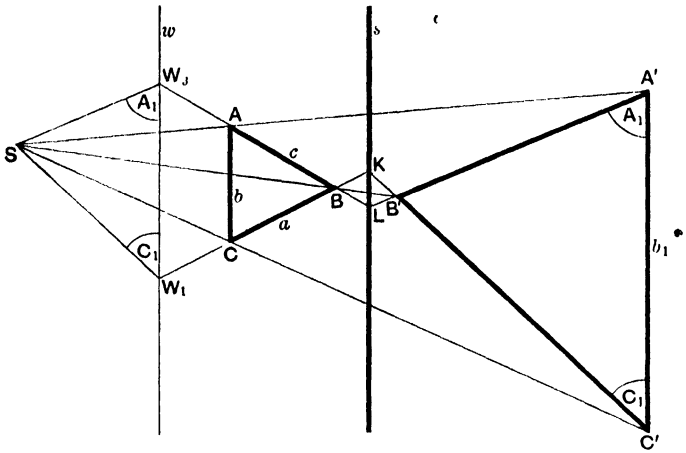
$$\frac{AC'}{BC'}\cdot\frac{BA'}{CA'}\cdot\frac{CB'}{AB'}=-1 \quad \text{(Ceva's Theorem)}$$

Take any triangle and its centroid Project the triangle *ABC* into this triangle and the point *P* into the centroid (Art 27) Then each of the ratios which compose the expression on the left hand side becomes equal to -1 Therefore the product is -1

32. Plane Perspective

Plane Perspective like Conical Projection may be employed to obtain one figure from another or to deduce one theorem from a related theorem The following are examples of the construction of figures by Plane Perspective

(*a*) *To construct the perspective of a given triangle ABC, so that it may be equal in dimensions to a given triangle* $A_1B_1C_1$



Denote the opposite sides of the triangles by the corresponding small letters Take any line w parallel to b as vanishing line Let a and c meet w in W_1 and W_3 Through W_1 and W_3 draw lines making angles C_1 and A_1 with w, to meet at S Take S as centre of perspective On SC take C' such that $\frac{SC'}{SC} = \frac{b_1}{b}$ Through C' draw $C'A'$ parallel to w to meet SA in A' Through A' and C' draw lines parallel to SW_3 and SW_1 to meet AB and BC in L and K and to intersect in B'

Then, in the perspective with S as centre, w as vanishing line and C and C' as corresponding points, B and B' correspond Hence ABC and $A'B'C'$ are in perspective and consequently KL (s) is parallel to w The triangles $A_1B_1C_1$ and $A'B'C'$ are similar by construction Also since

$$\frac{AC}{A'C'} = \frac{SC}{SC'} = \frac{b}{b_1},$$

the sides $A'C'$ and A_1C_1, and therefore the triangles $A_1B_1C_1$ and $A'B'C'$, are equal

(*b*) *To construct the perspective of a given quadrangle ABCD so that it may be*

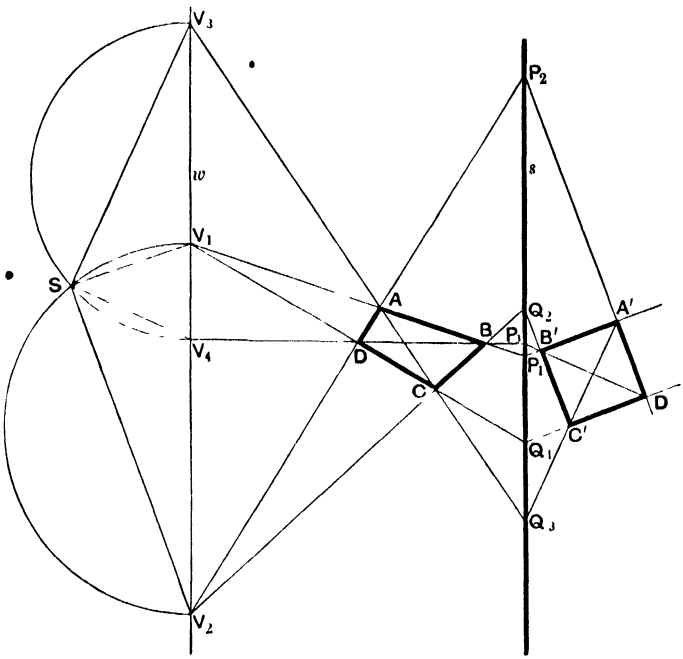
(1) *a square*

Let V_1 and V_2 be a pair of diagonal points of the quadrangle Take V_1V_2 as vanishing line and let the third pair of sides of the quadrangle meet V_1V_2 in V_3 and V_4 On V_1V_2 and on V_3V_4, as diameters, describe two circles which will intersect at some point S Take S as centre of perspective and some line s parallel to w as axis The perspective of the quadrangle will then be a square

For since V_1 and V_2 are on the vanishing line, the resulting figure is a parallelogram

Since SV_1 and SV_2 are at right angles and are parallel to the sides, it is a rectangle

Since SV_3 and SV_4 are at right angles and are parallel to the diagonals, it is a square



(ii) *a square of given dimensions*

The dimensions of the square in (i) vary directly as the distance between w and s (Art 24) Let d be the length of the side of the given square On V_2Q_2 take a point Q_2' such that

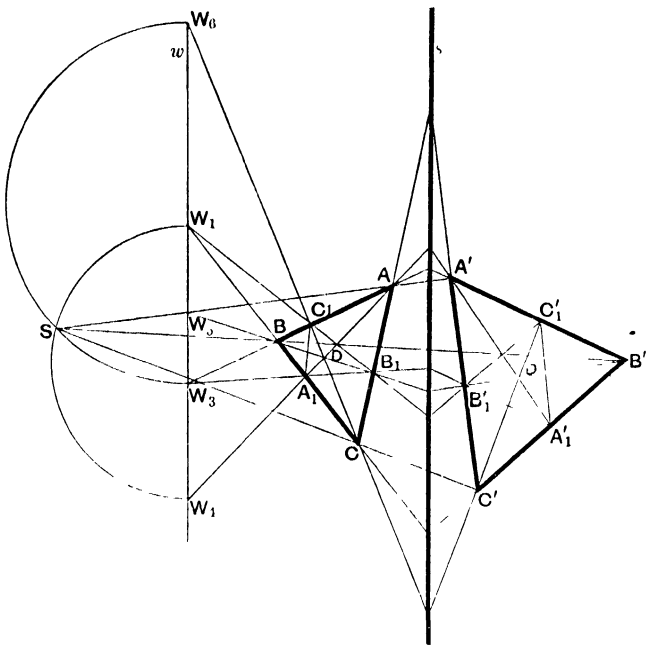
$$\frac{V_2Q_2'}{V_2Q_2} = \frac{d}{B'C'}$$

If a line parallel to w through Q_2' be taken as axis of perspective, the resulting figure will be a square of side d

(iii) *an equilateral triangle and its orthocentre*

Let $ABCD$ be the quadrangle and let AD, BD, CD meet BC, CA, AB respectively in $A_1B_1C_1$ Let BC meet B_1C_1 in W_1 and AB meet A_1B_1 in W_3 Then the ranges ABC_1W_3 and CBA_1W_1 are harmonic (Art 31 (2)) Take W_1W_3 as vanishing line (w) Let AD and CD meet w in W_4 and W_0 respectively On W_1W_4 and on W_3W_0 as diameters describe circles intersecting in S Take S as centre of perspective and any line parallel to w as axis of perspective

In the perspective figure the points corresponding to $A_1B_1C_1$ will bisect the sides of the triangle corresponding to ABC and the lines corresponding to AD, BD, CD will be perpendicular to those corresponding to BC, CA, AB Hence the figure meets the requirements of the case Its dimensions depend on the position of s



From the above it is seen that if two of the three segments determined on a line by any four points subtend a right angle at a given point, the third segment also subtends a right angle at that point

(iv) *a quadrangle of given dimensions* See Art 65, Chapter IX

33 Particular cases of Triangles in Perspective

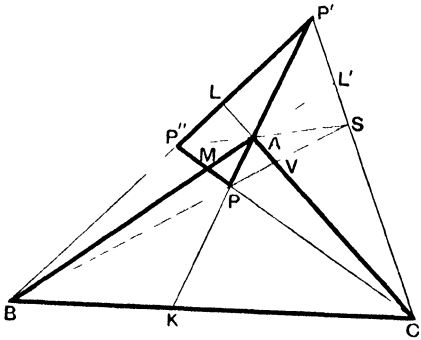
(1) *If three coplanar triangles are two by two in perspective, and have the same axis of perspective, their three centres of perspective are collinear*

Let the triangles be (1) ABC, (2) $A'B'C'$ and (3) $A''B''C''$, and let their opposite sides be denoted by the corresponding small letters Since the triangles are two by two in perspective with the same axis of perspective, the sides a, a', a'' are concurrent Therefore the triangles $BB'B''$ and $CC'C''$ are in perspective Therefore the points BB' CC', $B'B''$ $C'C''$, $B''B$ $C''C$ are collinear But these points are the centres of perspective of the triangles (1), (2) and (3) taken two by two

(II) *If three coplanar triangles are two by two in perspective and have the same centre of perspective, then three axes of perspective are concurrent*

Let the triangles be (1) abc, (2) $a'b'c'$, and (3) $a''b''c''$, and let their opposite vertices be denoted by the corresponding large letters. Since the triangles are two by two in perspective with the same centre of perspective, the vertices A, A', A'' are collinear. Therefore the triangles $bb'b''$ and $cc'c''$ are in perspective. Therefore the lines bb' cc', $b'b''$ $c'c''$, $b''b$ $c''c$ are concurrent. But these lines are the axes of perspective of the triangles (1), (2) and (3) taken in pairs.

(III) *If a triangle be circumscribed to another triangle and is in perspective with it, the three vertices of the former triangle may be deduced from each other as harmonic perspectives with respect to the vertices and opposite sides of the latter triangle.*



Let the triangle $PP'P''$ be circumscribed to the triangle ABC and in perspective with it, as in the figure, S being the centre of perspective. Let AP, BP' and CP'' meet the opposite sides of ABC in K, L and M. Then from the quadrangle $PP''P'S$,

$$(BLPS) = -1$$

But $$(ALPP') = (BLPS) = -1$$

Similarly the ranges $(BLP'P'') = -1$ and $(CMP''P) = -1$.

Conversely. If P, P', P'' are harmonic perspectives of each other with respect to the vertices A, B, C, and the opposite sides of a triangle ABC, the triangles $PP'P''$ and ABC are in perspective.

Take the perspective of P with centre B and axis AC, this will be a point S such that $(BLPS) = -1$. Then (Art. 29) S is the perspective of P' with regard to C and AB. To show that $P''A$ passes through S it is necessary to prove that LB, LC and $P''S$ are concurrent.

This is the case since $(P'P''BL) = -1 = (P'SLC)$.

The correlative theorems are

If a triangle be inscribed in another triangle and in perspective with it, the three sides of the former triangle may be deduced from each other as harmonic perspectives with respect to the vertices and opposite sides of the latter triangle.

Conversely. If p, p', p'' are harmonic perspectives with respect to the vertices A, B, C, and the opposite sides of a triangle abc, the triangles $pp'p''$ and abc are in perspective.

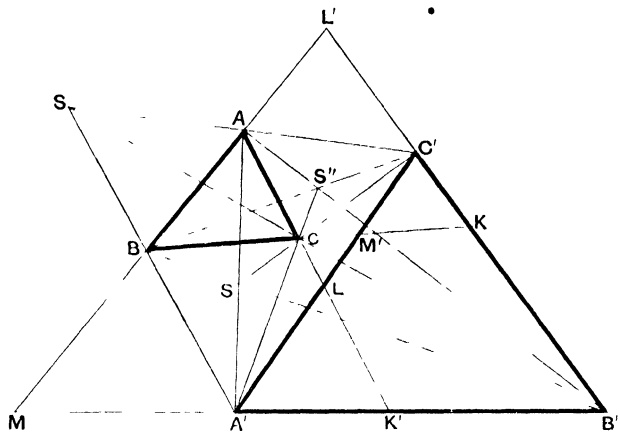
(IV) *If two coplanar triangles are copolar in two different ways, such that each pair of corresponding vertices in the one case are non-corresponding vertices in the other, they are also copolar in a third way and the three poles are the vertices of a triangle copolar in three ways to either of the former triangles.*

Let ABC and $A'B'C'$ be the triangles and denote the opposite sides by the corresponding small letters. In the figure, the triangles are in perspective with

the centres of perspective at S and S' It is required to prove that $A'C$, AB' and BC' are concurrent at some point S''

Because the triangles are in perspective with S as centre of perspective, $K(a\ a')$, $L(b\ b')$, $M(c\ c')$ are collinear Also in the perspective centre S'

$K(a\ a')$ corresponds to $K'(c'\ b)$,
$L(b\ b')$ „ „ $L'(a'\ c)$,
$M(c\ c')$ „ „ $M'(b'\ a)$



Hence, since K, L, M are collinear, K', L', M' are collinear Therefore the triangles are in perspective with c', a', b' corresponding to b, c, a respectively and the lines BC', CA', and AB' are concurrent at some point S''

The triangle ABC is obviously in perspective with the triangle $SS'S''$ from A', B' and C''

EXAMPLES

(1) Given five points, no three of which are collinear, project them into five points such that one may be the centroid and the other the orthocentre of the remaining three

(2) Through the points of intersection of the diagonals of a quadrilateral straight lines are drawn parallel to the four sides to meet the sides which are respectively opposite to those to which they are drawn parallel Prove that the four points of intersection are collinear

(3) Show that if two coplanar triangles are in perspective and a vertex of one lies on a non corresponding side of the other (*e.g.* A' on AB), then provided the centre of perspective is not at B, a pair of corresponding sides must coincide examine the position of the centre and axis of perspective

(4) If two quadrangles are in perspective in three ways with the vertices and opposite sides of a triangle as centres and axes of perspective, the vertices are collinear in pairs with a fourth point

(5) If $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles in perspective, the triangle $B_1C_2\ B_2C_1$, $C_1A_2\ C_2A_1$, $A_1B_2\ A_2B_1$ is in perspective with $A_1B_1C_1$ and $A_2B_2C_2$ and the three centres of perspective lie on a straight line.

(6) If on each of three concurrent straight lines a, b, c three points are taken, these form the vertices of 36 sets of triangles in perspective in pairs, and the axes of perspective pass three by three through 36 points which lie four by four on 27 straight lines.

(7) If four triangles (1) $A_1B_1C_1$, (2) $A_2B_2C_2$, (3) $A_3B_3C_3$, (4) $A_4B_4C_4$ are such that (1) and (2), (2) and (3), (3) and (4), (4) and (1) are in perspective with $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ for corresponding vertices, and $S_1S_2S_3S_4$ for centres of perspective, and $s_1s_2s_3s_4$ for axes, prove that, if $S_1S_2S_3S_4$ are collinear, $s_1s_2s_3s_4$ are concurrent.

CHAPTER VI

ANHARMONIC FORMS —RANGES AND PENCILS IN PERSPECTIVE RANGES AND PENCILS PROJECTIVE RANGES AND PENCILS SUPERPOSED

34 Self-corresponding elements

If two projective ranges are constructed on the same base, they are termed *superposed projective ranges* Any point upon the base may be looked upon as an element of either range and to a given point, considered as an element of one range, there generally corresponds some other point on the base regarded as an element of the other range

In certain cases a pair of corresponding elements coincide in one point and such a point is termed *a self-corresponding point* of the given projective ranges

A pair of projective ranges on different bases may also have a self-corresponding point at the point of intersection of the bases

If two projective pencils are constructed with the same vertex, they are termed *superposed projective pencils* Any line through the vertex may be looked upon as an element of either pencil and to a given line, considered as an element of one pencil, there generally corresponds some other line through the vertex regarded as an element of the other pencil

In certain cases a pair of corresponding elements coincide in one line and such a line is termed *a self-corresponding ray* of the given projective pencils

A pair of projective pencils with different vertices may also have a self corresponding ray in the line joining their two vertices

Ranges and pencils in perspective

Two ranges are said to be in perspective when the lines joining pairs of corresponding points pass through a fixed point

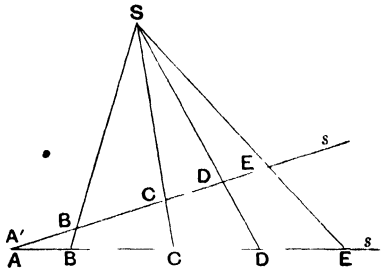
Two pencils are said to be in perspective when the points of intersection of pairs of corresponding rays lie on a fixed line

If in two projective ranges

(1) *the connectors of three pairs of corresponding points are concurrent,*

or (2) *the point of intersection of the bases is a self-corresponding point,*

the two ranges are in perspective



Let the projective ranges $A, B, C, D, E,$ and $A', B', C', D', E',$ be situated on bases s and s'

(1) Let BB', CC', DD' be concurrent at the point S

Join S to E any point on s to meet s' at E_1

Then $(BCDE) = (B'C'D'E_1)$,

but $(BCDE) = (B'C'D'E')$

since the ranges are projective Therefore E_1 and E' coincide and every pair of corresponding elements are collinear with S

(2) Let ss' be a self-corresponding point, viz A or A', and let BB', CC' meet at S

Join S to E any point on s to meet s at E_1

Then $(ABCE) = (A'B'C'E_1)$,

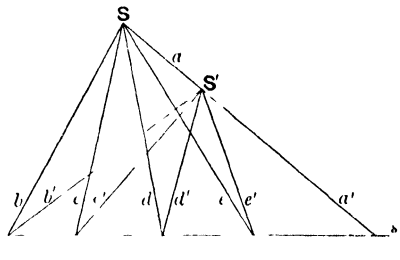
but $(ABCE) = (A'B'C'E')$

If in two projective pencils

(1) *the points of intersection of three pairs of corresponding rays are collinear,*

or (2) *the connector of the vertices is a self-corresponding ray,*

the two pencils are in perspective



Let the projective pencils $a, b, c, d, e,$ and $a', b', c', d', e',$ have for their vertices S and S'

(1) Let bb', cc', dd' be collinear on the line s

Join the intersection of s with e, any line through S, to S' by a line e_1

Then $(bcde) = (b'c'd'e_1)$,

but $(bcde) = (b'c'd'e')$

since the pencils are projective Therefore e_1 and e' coincide and every pair of corresponding elements intersect on s

(2) Let SS' be a self-corresponding ray, viz a or a', and let bb', cc' be the lines

Join the point of intersection of s with e, any line through S, to S' by a line e_1

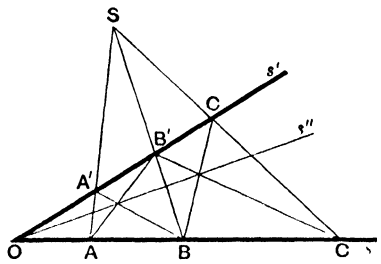
Then $(abce) = (a'b'c'e_1)$,

but $(abce) = (a'b'c'e')$

since the ranges are projective Therefore as before E_1 and E coincide

From the above it follows that two pairs of corresponding elements completely determine two ranges in perspective

If two ranges are in perspective the transverse connectors of any two pairs of corresponding points intersect on a fixed straight line



Let $A, B, C,$ and $A', B', C',$ be the ranges, in which $AA', BB', CC',$ are collinear with a fixed point S Consider the triangles $A'BC'$ and $AB'C$ They are in perspective with S for centre of perspective Therefore AB' BA' and BC' CB' lie on a line s'' which passes through O the point of intersection of the bases

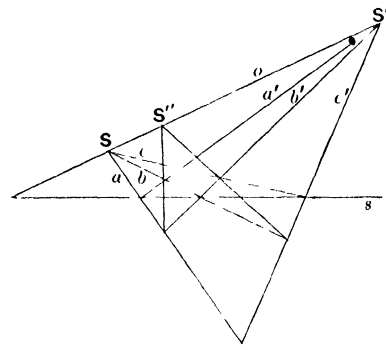
If AA', BB' are regarded as two pairs of fixed corresponding points, which determine the ranges, the line s'' is fixed and the variable point BC' CB' lies on this line

This result also follows from the properties of Harmonic Perspective when S is the centre and s'' the axis of Perspective

since the ranges are projective Therefore as before e_1 and e' coincide

From the above it follows that two pairs of corresponding rays completely determine two pencils in perspective

If two pencils are in perspective the points of intersection of pairs of the non-corresponding rays are collinear with a fixed point



Let $a, b, c,$ and $a', b', c',$ be the rays, in which $aa', bb', cc',$ lie on a fixed line s Consider the triangles $a'bc'$ and $ab'c$ They are in perspective with s for axis of perspective Therefore ab' ba', bc' cb' intersect in a point S'' which lies on o the connector of the vertices

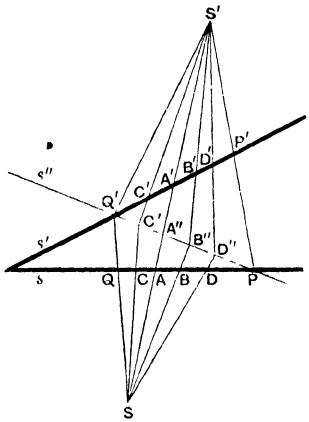
If aa', bb' are regarded as two pairs of fixed corresponding rays, which determine the pencils, the point S'' is fixed and the variable line bc' cb' passes through this point

This result also follows from the properties of Harmonic Perspective when s is the axis and S'' the centre of Perspective

35 Projective ranges and pencils

*Construction of corresponding elements of projective forms**

Given three pairs of corresponding points A and A', B and B', C and C' on two bases s and s', to construct the projective ranges determined by these corresponding points



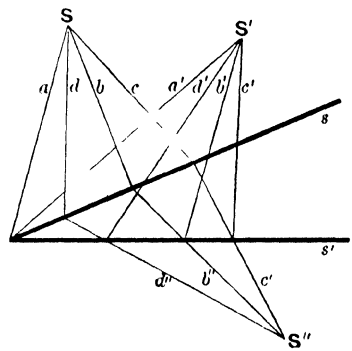
Take S and S' any two points on AA' Construct s'' the line joining SC $S'C'$ to SB $S'B'$ Then the pencils S ABC

and S' $A'B'C'$

have the self-corresponding ray SS', and are therefore in perspective, with s'' as axis of perspective (Art 34)

To construct the point corresponding to any point D on s, join SD to meet s'' in D'' Join $D''S'$ to meet s' in D' Then D' is the point corresponding to D,

Given three pairs of corresponding rays a and a', b and b', c and c' passing through two vertices S and S', to construct the projective pencils determined by these corresponding rays



Take s and s' any two lines through aa' Construct S'' the point of intersection of sc $s'c'$ and sb $s'b'$ Then the ranges s abc

and s' $a'b'c'$

have the self-corresponding point ss', and are therefore in perspective, with S'' as centre of perspective (Art 34)

To construct the ray corresponding to any ray d through S, join sd to S'' by d'' Join $d''s'$ to S' by d' Then d' is the ray corresponding to d, for the same

* For construction of self corresponding elements of two superposed ranges see Art 109

for the same projections which determine $A'B'C'$ from ABC also determine D' from D

If s'' meet s and s' in P and Q', P' and Q, the points corresponding to P and Q', are found as the intersects of $S'P$ and SQ' with s' and s respectively

Any line may be taken as the line s'' For, if s'' meet s and s' in P and Q', and P' and Q be the corresponding points, then S and S' are found as the points of intersection of QQ' and PP' with the line joining any pair of corresponding points A and A'

projections which determine $a'b'c'$ from abc also determine d' from d

If the lines joining S'' to S and S' be p and q', p' and q, the rays corresponding to p and q', are found as the connectors of $s'p$ and sq' to S' and S respectively

Any point may be taken as the point S'' For, if the lines joining S'' to S and S' be p and q', and p' and q be the corresponding rays, then s and s' are found as the lines joining the points qq' and pp' to the point of intersection of any pair of corresponding rays a and a'

From the preceding it follows that

(1) *A range can always be constructed with which each of two given projective ranges is in perspective*

A pencil can always be constructed with which each of two given projective pencils is in perspective

(2) *Three pairs of corresponding points determine two projective ranges*

Three pairs of corresponding rays determine two projective pencils

(3) *Two ranges, such that the anharmonic ratio of any four points of the one is equal to the anharmonic ratio of the four corresponding points of the other, are projective*

Two pencils, such that the anharmonic ratio of any four rays of the one is equal to the anharmonic ratio of the four corresponding rays of the other, are projective

Theorem (3) of the above may be proved by taking A, B, C and A', B', C' to determine the ranges, and D and D' for corresponding points The point D_1 determined by the above construction as corresponding to D, is such that

$$(A'B'C'D_1) = (ABCD)$$

and therefore coincides with D'

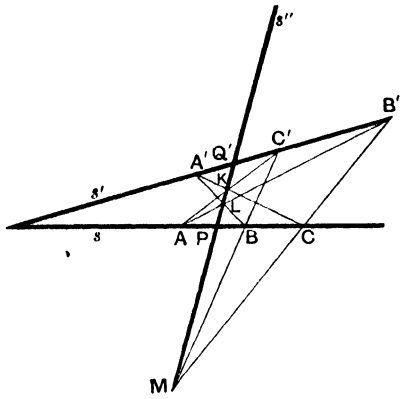
Theorem (3) of the above may be proved by taking a, b, c and a', b', c' to determine the pencils, and d and d' for corresponding rays The ray d_1 determined by the above construction as corresponding to d, is such that

$$(a'b'c'd_1) = (abcd)$$

and therefore coincides with d'

36 Particular case of the preceding

Take S at any point A' of the range on s', and S'' at the corresponding point A on s



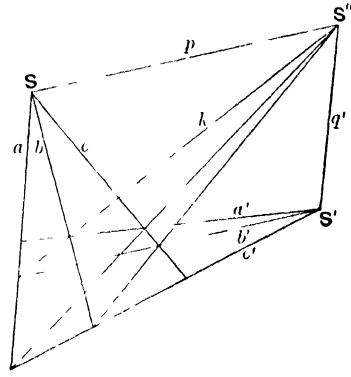
Let s'' meet s and s', as before, in P and Q'

Then, by construction, P corresponds to the point ss' regarded as a point of the range on s', and Q' corresponds to the point $s's$ regarded as a point of the range on s

Hence Q' and P are fixed points and s'' is therefore a fixed line, as long as S and S'' are taken at pairs of corresponding points of the ranges But A, A' and B, B' are any two pairs of corresponding points, and AB and $A'B$ intersect in s''

It follows therefore that the lines joining transversely any two pairs of corresponding points of two projective ranges intersect on a fixed line

Take s as the ray a' of the pencil vertex S', and s' as the corresponding ray a of the pencil vertex S



Let the lines joining S and S'' to S'' be as before p and q'

Then, by construction, p corresponds to line SS' regarded as ray of pencil S, and q' corresponds to line $S'S$ regarded as ray of pencil S

Hence q' and p are fixed lines and S'' is therefore a fixed point, as long as s and s' are a pair of corresponding rays of the pencils But a, a' and b, b' are any two pairs of corresponding rays and ab' and $a'b$ are collinear with S''

It follows therefore that the points of intersection of the non-corresponding rays of any two pairs of corresponding rays of two projective pencils are collinear with a fixed point

If any three pairs of points AA', BB', CC' be taken, these determine two projective ranges and the points

AB' BA', BC' CB', CA' AC'

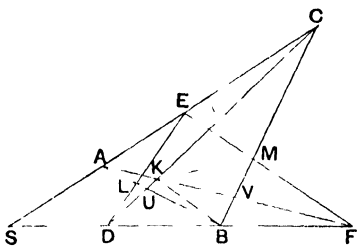
are collinear

From this there follows

Pascal's theorem for a Pair of Lines, viz

If the alternate vertices of a hexagon lie three by three on two straight lines, then the three points of intersection of the pairs of opposite sides are collinear

This theorem may also be proved as follows



Let $ABCDEF$ be a hexagon, such that A, E, C and D, B, F lie on two straight lines which meet at S Let K, L, M be the points of intersection of pairs of opposite sides, and let U and V be the points AB DC and AF CB Join KL and KM The condition that K, L, M should be collinear is

$$(K\ LAUB)=(K\ MVCB),$$

or $$(LAUB)=(MVCB),$$

or $$(D\ EACS)=(F\ EACS),$$

which is true

If any three pairs of rays aa', bb', cc' be taken, these determine two projective ranges and the lines

ab' ba', bc' cb', ca' ac'

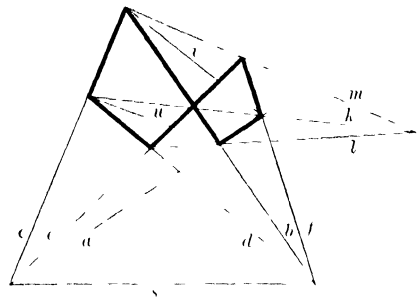
are concurrent

From this there follows

Brianchon's theorem for a Pair of Points, viz

If the alternate sides of a hexagon pass three by three through two fixed points, then the three lines joining the pairs of opposite vertices are concurrent

This theorem may also be proved as follows



Let $abcdef$ be a hexagon, such that a, e, c and d, b, f meet at two points the connector of which is s Let k, l, m be the connectors of pairs of opposite vertices, and let u and v be the lines ab de and af cb The condition for k, l, m to be concurrent is

$$(k\ laub)=(k\ mvcb),$$

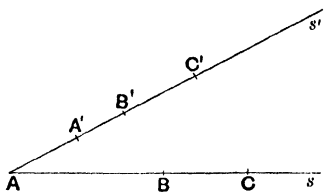
or $$(laub)=(mvcb),$$

or $$(d\ eacs)=(f\ eacs),$$

which is true

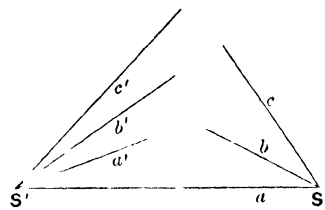
37 Connexion between Projective Pencils and Ranges and those in Plane Perspective

Two projective ranges may be placed in plane perspective by moving either range along its base



Let A' be the point of one range which corresponds to the point of intersection of the bases looked upon as a point of the other If the points A', B', C' are moved through a distance AA' along s' the ranges will have a self-corresponding point at A and be in perspective with the point BB' CC' (in the new position) for centre of perspective

Two projective pencils may be placed in plane perspective by rotating either pencil round its vertex



Let a' be the ray of one pencil which corresponds to the line joining the vertices of the pencils looked upon as a ray of the other If the rays a', b', c' are rotated round S' through an angle $\widehat{aa'}$ the pencils will have a self-corresponding ray in a and be in perspective with the line bb' cc' (in the new position) for axis of perspective

In the preceding (right-hand side) the rotation may be performed in either of two ways

(1) the ray a' may be rotated through the acute angle $a'S'a$, or

(2) through the angle $\pi - a'S'a$

The axis of perspective is the same in both cases

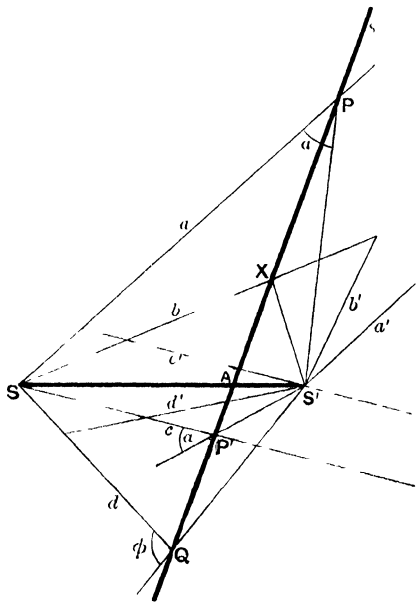
Conversely, *any two projective pencils may be obtained by the rotation round its vertex of either of some two pencils in plane perspective*

38 *Two projective pencils may have two pairs, one pair or no pair of parallel corresponding rays*

Rotate one of the pencils round its vertex S' till the pencils are in plane perspective and let s be the axis of perspective in the displaced position

(1) Let the axis of perspective s meet SS' between S and S' at A

Pairs of corresponding rays of the pencils in the displaced position are obtained by joining a variable point X on s to S and S' When the point X is very distant from A in the upper part of the figure the angle θ between corresponding rays is zero As X moves towards A this angle increases till at A it is π After passing A the external

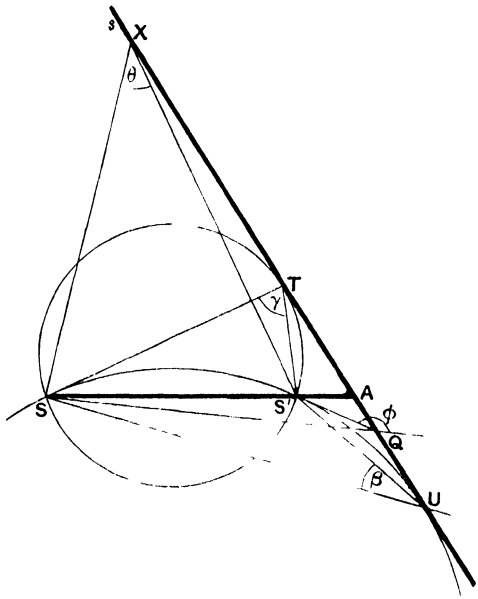


angle, marked ϕ at Q, increases from 0 to π, which value it reaches when X is taken at an infinite distance below A Hence the internal angle between the rays above SS' and the external angle below SS' each have any given value α $(< \pi)$ Therefore, when the pencil vertex S' is rotated back to its original position round S', through an angle α, there will be two pairs of parallel corresponding rays These correspond to the two pairs of rays passing through P and P' which originally contained an angle α

(2) Let the axis of perspective s of the pencils meet SS' at A, where A is not situated between S and S'

In this case when X is taken at infinity at the top of the figure the angle θ between the lines joining X to S and S' is zero This angle θ

increases from zero till P is at T in which case STS' is a maximum (γ) The point T is determined as the point of contact of a circle through S and S' which touches the axis of perspective The angle θ then decreases till at A, where SS' meets the axis of perspective, it is again zero



Hence for a positive rotation of S' through an angle between o and γ there are two pairs of parallel corresponding rays

Below A the angle to be considered is the external angle ϕ between SQ' and SQ (produced)

This angle ϕ is initially π and decreases till at U it is $\pi - \beta$, where U is again the point of contact of a circle through S and S' which touches the axis of perspective From U to the lower point at infinity on the axis ϕ increases from $\pi - \beta$ to π

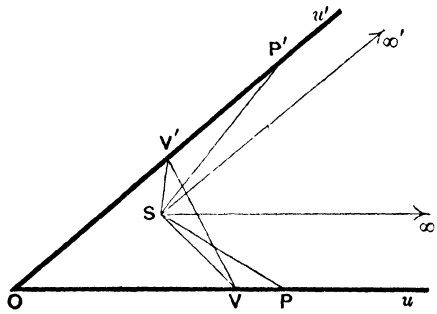
Hence, for a positive rotation between $\pi - \beta$ and π of the pencil round S' there are again two pairs of parallel corresponding rays

For a rotation of the pencil S' in a positive direction of γ or $\pi - \beta$ there is one pair of parallel corresponding rays

For a rotation of the pencil S' in a positive direction between γ and $\pi - \beta$ there is no pair of parallel corresponding rays

39 (*a*) **There are two points at which pairs of corresponding points of two projective ranges on different bases subtend equal angles**

Let the ranges be situated on the bases u and u' which intersect at O, and let them be determined (Art 35) by three pairs of corresponding points V, P, ∞ and ∞', P', V'



Every pair of corresponding points will subtend equal angles at S, if S is such that

$$V'SP' = \infty SP$$

and $$P'S\infty' = PSV$$

In this case the angle $OVS = \infty SV = V'S\infty' = OV'S$ Hence the lines joining V and V' to S, make equal angles with u and u' respectively

If the second pencil is rotated round S through such an angle that SP and SP' lie on each other, the pencils will coincide, and the two ranges will be the intercepts made on two different transversals u and u', by the same pencil

But it was shown (Art 10) that in this case

$$VP \cdot V'P' = \text{constant} = VS \cdot V'S$$

Hence the product $VS \cdot V'S$ is a known quantity

Hence S is found as the vertex of a triangle described on VV' as base, having the difference of the base angles equal to the difference of the angles $OV'V$ and OVV', and having the product of its sides equal to a given quantity

Such a triangle can be constructed (Addendum 4) on each side of VV'

That any two pairs of corresponding points subtend equal angles at S, determined as above, may be proved as follows

Consider the triangles VPS and $V'SP'$

$$V'P' \cdot VP = V'S \cdot VS,$$

$$\frac{V'P'}{V'S} = \frac{VS}{VP},$$

also the angles $SV'P'$ and SVP are equal

Therefore the triangles $V'SP'$ and VPS are similar and the angle $V'SP'$ = the angle VPS

Take two other corresponding points Q and Q'

Then $\quad V'SQ' = VQS,$

and $\quad V'SQ' = VQS - VPS = PSQ$

(*b*) *There are generally two straight lines on which pairs of corresponding rays of two projective pencils with different vertices determine equal segments*

There are generally two pairs of parallel corresponding rays of two projective pencils (Art 38) Draw any transversal parallel to the direction of one of these pairs of parallel corresponding rays to meet the pencils in A, B, C, ∞ and A', B', C', ∞ respectively, where ∞ is the point at infinity on the transversal Then

$$(ABC\infty) = (A'B'C'\infty) \quad \text{or} \quad \frac{AC}{BC} = \frac{A'C'}{B'C'}.$$

Hence, if AC and $A'C'$ be equal, BC and $B'C'$ will also be equal But (Addendum 3) a straight line may be drawn in a given direction so that the intercepts on it made by two given pairs of lines are equal The line so drawn is one on which three pairs of corresponding rays of the pencils determine equal segments, and on which therefore all pairs of corresponding rays determine equal segments A second line, parallel to the other pair of parallel corresponding rays of the pencils, may be obtained in a similar manner

From (*a*) it follows that *any pair of projective ranges may be obtained by the section of the same pencil by a pair of transversals, one transversal being afterwards rotated round the vertex of the pencil*

From (*b*) it follows that *generally any pair of projective pencils may be obtained by projecting the same range from two different vertices, one pencil being afterwards moved parallel to the base on which the range is situated*

40 Summary of the Properties of two Projective Ranges

In the case of two projective ranges in which A, B, C correspond to A', B', C'

(*a*) There is a vanishing point (W or V) on each base such that

$WA \cdot V'A' = WB \cdot V'B' =$ constant (Arts 10 and 39)

(*b*) Corresponding points may be obtained by projecting on the bases any point on the line joining SA $S'A'$ to SC $S'C'$ where S and S', the centres of projection, are on the connector of a pair of corresponding points (Art 35)

(*c*) Corresponding points may also be obtained by rotating a given angle round either of two fixed points (Art 39)

(*d*) The ranges are in plane perspective, if the point of intersection of the bases corresponds to itself (Art 34)

(*e*) The ranges may be placed in plane perspective by moving one of them, so that the point of intersection of the bases is a self-corresponding point (Art 37)

41 Superposed Projective Ranges and Pencils

Two superposed projective ranges or pencils may have one or two, but not more than two, self-corresponding points *

Let the projective pencils abc and $a'b'c'$ have a common vertex S Through any point S' in their plane draw lines $a_1b_1c_1$ parallel to $a'b'c'$ Then the pencils abc and $a_1b_1c_1$ are perspective Hence (Art 38) they have two pairs, one pair or no pair of parallel corresponding rays If they have two pairs of parallel corresponding rays e, e_1 and f, f_1, the rays e' and f' of the pencil $a'b'c'$ which correspond to e_1 and f_1 must coincide with the rays e and f of the pencil abc Hence the two given pencils have two self-corresponding rays

Similarly, if the pencils abc and $a_1b_1c_1$ have one pair or no pair of parallel corresponding rays, the pencils abc and $a'b'c'$ have one or no [illegible] ray It is obvious that two pencils cannot have more than two self-corresponding rays without entirely coinciding

If the points of two superposed projective ranges are joined to any vertex two superposed pencils are obtained, and, corresponding to the self-corresponding rays of the pencils, there may be two self-corresponding points, one self-corresponding point, or no self-corresponding point of the ranges

42 Maximum and Minimum Segments of Superposed Projective Ranges

Let V and V' be the vanishing points of two superposed projective ranges Then if P and P' are a pair of corresponding points $VP \cdot V'P' = k$ constant (Arts 10 and 39)



Suppose this constant to be $-K^2$ so that $VP \cdot V'P' = -K^2$ Then

$$VP(V'P + PP') = -K^2,$$

$$PP' = -\frac{K^2}{VP} - V'P = -\frac{K^2}{VP} - (V'V + VP),$$

$$PP' = \{+2K - V'V\} - \frac{\{K + VP\}^2}{VP} \qquad (\text{I}),$$

$$PP' = \{-2K - V'V\} - \frac{\{K - VP\}^2}{VP} \qquad (\text{II})$$

also For the construction of self-corresponding elements of two superposed projective Th see Art 109

angle)

The least value of the numerator of the second expression is in each case zero

Consider (i)

If $2K - V'_1 V$ is positive PP' is a minimum when

$$VP = -K$$

If $2K - V'V$ is negative PP' is (numerically) a maximum when

$$VP = -K$$

Consider (ii)

If $-2K - V'V$ is positive PP' is a maximum when

$$VP = +K$$

If $-2K - V'V$ is negative PP' is (numerically) a minimum when

$$VP = +K$$

Hence for the values $VP = \pm K$ the segments PP' are one a maximum and the other a minimum These are termed the maximum and minimum segments They will be denoted by EE' and FF'

If $VP \cdot V'P' = +K^2$ the maximum and minimum segments are imaginary

The self-corresponding points (K and L) are given the relation

$$VP \cdot V'P = -K^2$$

They are therefore situated between V and V' and are equally distant from these points

It will be seen from the preceding that if O be the middle point of VV' then

$$EE' = -2(K + OV), \qquad FF' = +2(K - OV),$$
$$OE = K + OV, \qquad OE' = -(K + OV),$$
$$OF = -K + OV, \qquad OF' = (K - OV),$$
$$OK^2 = OL^2 = OV^2 = K^2$$

The square of the distance between the self corresponding points equals the product of the maximum and minimum segments

The maximum and minimum segments are

$$2(K - OV) \text{ and } -2(K + OV)$$

But
$$KL^2 = +(OV^2 - K^2)$$
$$= -(K + OV)(K - OV)$$
$$= EE' \cdot FF'$$

If the self corresponding points are real, EE' and FF' have the same sign, one is a maximum (numerically) and one a minimum (numerically) If the self corresponding points are imaginary EE' and FF' have opposite signs, and are then numerically both minima

The distance between the vanishing points is equal to the semi-sum of the maximum and minimum segments

The maximum and minimum segments are

$$2K - V'V \text{ and } -2K - V'V$$

Therefore their semi sum is $-V'V$, the distance between the vanishing points

The product of the distances of a pair of corresponding points from their respective vanishing points equals $-\frac{1}{4^2}$ *the square of the difference of the maximum and minimum segments*

The difference of the maximum and minimum segments is

$$(2K - V'V) - (2K - V'V) = 4K,$$

$$K^2 = \frac{1}{4^2} \text{ (difference of segments)}^2$$

Hence the maximum and minimum segments are such that

(1) They are bisected at the mean point, *i e* the mid-point of VV'

(2) Their product equals the square of the distance between the self-conjugate points

(3) Their semi sum equals the distance between the vanishing points

(4) The square of their difference is (-16) times the product of the distances of a pair of corresponding points from their vanishing points

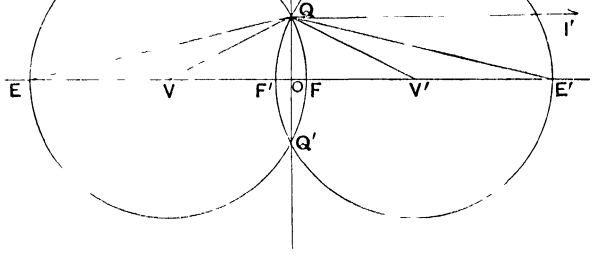
43 *If EE' and FF' are the maximum and minimum segments and circles be described on EF and $E'F'$ as diameters, the self corresponding elements are the limiting points of the system of coaxal circles* determined by these circles, and if the circles intersect at Q and Q', corresponding segments of the ranges subtend equal angles at Q and Q'*

The centres of the circles described on EF and $E'F''$ as diameters are at V and V' the vanishing points and the two circles are equal If they do not intersect the real points

$$OE \cdot OF = OV^2 - K^2 = OK^2 = OL^2 \quad \text{(Art 42)}$$

Therefore K and L are the common harmonic conjugates of EF and $E'F''$ Hence they are the limiting points of the coaxal system when real The circle on LM cuts the circles on EF and $E'F''$ orthogonally

If the circles on EF and $E'F''$ intersect as in the figure in Q and Q' then the angle $E'QE = \pi - F'QF$ Let I be the point at infinity on a parallel to the base through Q



* See Art 83

Then $I'QE' = QE'E = QEE' = EQV, \qquad I'QV = E'QE$

Hence EE', FF', $I'V$ all subtend equal angles or angles whose sum is π at Q and also at Q'

If $VP \cdot V'P' = K^2$ the self-corresponding points are real and the maximum and minimum segments do not exist Hence

(1) If the maximum and minimum segments EE' and FF' are imaginary the self-conjugate points are real

(2) If the maximum and minimum segments are real and the circles on EF' and $E'F'$ do not intersect the self corresponding points are real and are then limiting points

(3) If the maximum and minimum segments are real and the circles on EF' and $E'F'$ intersect in Q and Q' then the self corresponding points are imaginary, and corresponding points subtend equal angles at Q and Q'

It therefore follows that in the case of two superposed projective ranges there are either

(1) A pair of self corresponding elements S and S', in which case, if A and A' be a pair of corresponding points, the point P' corresponding to any point P of either range may be found from the fact that

$$(SS'AP) = (SS'A'P') \quad \text{(Example 5, page 76)}$$

or

(2) There are two points Q and Q' at which pairs of corresponding elements subtend equal angles

In case (2) it is obvious that the points Q and Q' must be symmetrically situated in regard to the base, and that the maximum and minimum segments are obtained when the arms of the rotated angle are equally inclined to the line QQ'

Summary of the Properties of two Superposed Projective Ranges

In the case of two projective ranges on the same base in which $A, B, C \ldots$ correspond to $A', B', C' \ldots$

(a) There is a vanishing point W or V' of each range such that

$$WA \cdot V'A' = WB \cdot V'B = \text{constant}$$

(b) There is a point O half-way between the vanishing points which may be called the mean point

(c) There are two self-corresponding points S and S' which may be real or imaginary

(d) If S and S' are imaginary—and sometimes when they are real—there are maximum and minimum segments KK', LL'

(e) If S and S' are imaginary, circles described on KL, $K'L'$ intersect in two points Q and Q' at which pairs of corresponding points subtend equal angles

(*f*) If S and S' are real, any pair of corresponding points may be constructed by means of the fact that $(SS'AP) = (SS'A'P')$, where P and P' correspond If S and S' are imaginary, corresponding points may be obtained by the rotation of a constant angle round Q or Q'

(*g*) The limiting points of the circles described on KL and $K'L'$ as diameters are the self-corresponding points of the ranges

44 Connexion between Ranges determined in different ways

The different relationships between ranges of points (or pencils of rays) may be defined as follows, viz

If two ranges (or pencils) are such that one may be deduced from the other by a series of projections, the ranges (or pencils) are said to be *projective*

If two ranges (or pencils) are such that the anharmonic ratio of any four elements of the one is equal to the anharmonic ratio of the four corresponding elements of the other the ranges (or pencils) are said to be *equianharmonic*

If two ranges (or pencils) are such that pairs of corresponding elements are determined by means of an algebraic relation, and to any element of the one corresponds one and only one element of the other, then the ranges (or pencils) are said to be *uniquely determined* and to have one to one correspondence

It has already been shown (Arts 11 and 35) that the terms projective and equianharmonic, as defined above, when applied to ranges or pencils, are equivalent terms It will now be shown that the terms equianharmonic and uniquely determined are, in this case, also equivalent Equianharmonic ranges are (Art 12) uniquely determined and thus it is only necessary to prove that uniquely determined ranges are equianharmonic

Let x be the distance of any point of the first range from some origin on its base and x' the distance of the corresponding point of the other from an origin on its base Then if the points are uniquely determined each from the other, by an algebraic relation, the relation between x and x' must be of the form

$$Axx' + Bx + Cx' + D = 0$$

Therefore

$$x = -\frac{Cx' + D}{Ax' + B} \qquad (1)$$

Suppose that x_1', x_2', x_3', x_4' are the coordinates of four points of the second range corresponding to the points of the first range whose

coordinates are x_1, x_2, x_3, x_4 The anharmonic ratio of these points is

$$\frac{x_1 - x_3}{x_2 - x_3} \frac{x_1 - x_4}{x_2 - x_4} \qquad (11)$$

But from (1) $x_1 - x_3 = (x_1' - x_3') \dfrac{AD - CB}{(Ax_1' + B)(Ax_3' + B)}$

Substituting in (11) this and the corresponding values for

$$x_2 - x_3, \quad x_1 - x_4, \quad x_2 - x_4$$

it is seen that

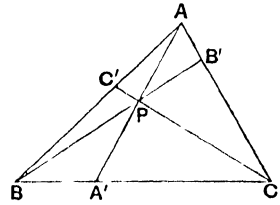
$$\frac{x_1 - x_3}{x_2 - x_3} \frac{x_1 - x_4}{x_2 - x_4} = \frac{x_1' - x_3'}{x_2' - x_3'} \frac{x_1' - x_4'}{x_2' - x_4'}$$

which proves the theorem

The corresponding theorems for pencils may be deduced without difficulty by considering the ranges formed by the pencils on any transversal

Determination of Position by Ratio

If B and C are two given points on a line the position of any point A' on the line is known if the ratio $\dfrac{BA'}{CA'}$ be given (See Art 7)



In a triangle ABC the position of points situated on the three sides may be given by their ratios with respect to the vertices Ceva's and Menelaus' Theorems may then be stated as follows

When three points are situated on the sides of a triangle, if the product of their ratios with respect to the vertices is -1, *the lines joining the points to the vertices are concurrent, and, if the product of their ratios is* $+1$, *the points are collinear*

EXAMPLES

(1) Two sets of four collinear points are such that the points of one set are at distances $2, \frac{1}{2}, -1, +1$ from a fixed point on the line, and those of the other set at distances $-3, \frac{3}{2}, -\frac{1}{3}, -\frac{3}{2}$ from the point Prove that two projective four point ranges can be formed out of the two sets respectively

(2) $ABCD$ and $A'B'C'D'$ are two ranges of collinear points such that

$$\frac{\lambda}{AD} + \frac{\mu}{A'D'} = \frac{\lambda}{BD} + \frac{\mu}{B'D'} = \frac{\lambda}{CD} + \frac{\mu}{C'D'}.$$

Prove that they have the same anharmonic ratios

From the given relations $\lambda \left\{ \dfrac{1}{AD} - \dfrac{1}{BD} \right\} = -\mu \left\{ \dfrac{1}{A'D'} - \dfrac{1}{B'D'} \right\}$,

$$\frac{1}{AD} \frac{BA}{BD} = -\frac{\mu}{\lambda} \frac{1}{A'D'} \frac{B'A'}{B'D'},$$

so
$$\frac{1}{AD}\frac{CA}{CD} = -\frac{\mu}{\lambda}\frac{1}{A'D'}\frac{C'A'}{C'D'},$$
$$\frac{BA}{BD}\frac{C1}{CD} = \frac{B'A'}{B'D'}\frac{C'A'}{C'D'},$$
$$(ADBC) = (A'D'B'C')$$

(3) If P and P' correspond in the two homographic ranges $ABC\ldots$ and $A'B'C'\ldots$ show that
$$\frac{AB}{A'B'}\frac{CP}{C'P'} + \frac{AC}{A'C'}\frac{PB}{P'B'} + \frac{AP}{A'P'}\frac{BC}{B'C'} = 0$$

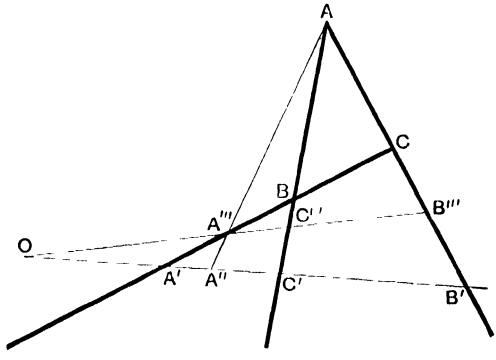
Since the ranges are projective, $(ACBP) = (A'C'B'P')$,
$$\frac{AB}{CB}\frac{CP}{AP} = \frac{A'B'}{C'B'}\frac{C'P'}{A'P'}$$
$$= \frac{\frac{1}{A'C'} - \frac{1}{A'P'}}{\frac{1}{A'C'} - \frac{1}{A'B'}},$$
$$\frac{1}{A'C'}\{AB\ CP - CB\ AP\} - \frac{1}{A'B'}AB\ CP + \frac{1}{A'P'}CB\ AP = 0,$$
$$\frac{AB}{A'B'}\frac{CP}{C'P'} + \frac{AC}{A'C'}\frac{PB}{P'B'} + \frac{AP}{A'P'}\frac{BC}{B'C'} = 0$$

(4) If Ω, Ω' be the double points of two projective ranges and AA' and BB' pairs of corresponding points then $(\Omega A\Omega' A') = (\Omega B\Omega' B')$

(5) If a straight line meet the sides of BA and BC of a triangle ABC in C_1 and A_1 respectively, and the lines joining P, any point on the line, to C and A meet the opposite sides in C'' and A', prove that
$$\frac{AC_1}{BC_1}\frac{AC''}{BC''} + \frac{CA_1}{BA_1}\frac{CA'}{BA'} = 1$$

By projection from P, $(ABC_1C') = (A'BA_1C) = (CA_1BA') = 1 - (CBA_1A')$,
$$(ABC_1C') + (CBA_1A') = 1$$

(6) Draw a line through a given point to meet the sides of a given triangle in three points which with the given point form a range of given anharmonic ratio



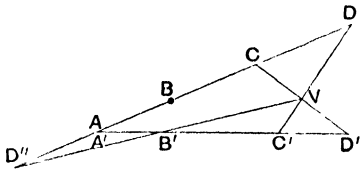
Let O be the given point Draw any transversal to meet the sides in $A'C''B'$ Take a point A'' on this line such that $(OA''C'B')$ has the given anharmonic ratio Join A'' to A to meet AB in A''' Let OA''' meet AB and AC in C'''' and B'''' Then $OA'''B''''C''''$ is the required transversal

(7) Prove the following test for collinearity of the three points O, M, N, $OABC$ and $OA'B'C'$ are any two straight lines through O, also AMB', BMA', $B'NC$, BNC' are straight lines Then the required condition for collinearity is that AA', BB', CC' are concurrent

In Art 35 s'' will not pass through O unless the ranges are in perspective, in which case AA', BB', CC' are concurrent

(8) Two ranges $ABCD$ and $A'B'C''D'$ on different lines have a common point A, CD' and $C'D$ meet at V Show that

$$(AB,\ CD)\times(AB',\ C'D')=(B'B,\ CD)$$



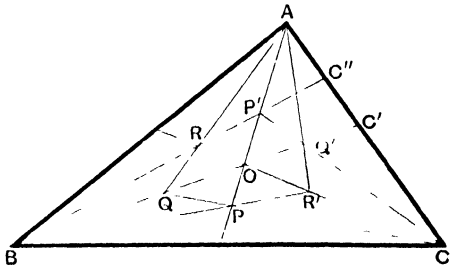
In the figure

$$(AB'C'D')=(V\ AB'C'D')=(AD''DC)$$

Therefore

$$(ABCD)(AB'C''D')=(ABCD)(AD''DC)$$
$$=(DCBA)(DCAD'')=(DCBD'')=(BD''DC)$$
$$=(V\ B'BC'D)$$

(9) AO, BO, CO connect the vertices of a triangle to O, P is taken on AO, PC meets BO in Q, QA meets CO in R, RB meets AO in P', $P'C$ meets BO in Q', and $Q'A$ meets CO in R' Prove that $R'BP$ are collinear



In the figure

$$(A\ BPR'C)=(A\ BOQ'C)=(C\ BRP'C'')=(A\ BQOC'')$$
$$=(C\ BQOC')=(C\ BPR'C')$$

The first and last of these pencils are in perspective and therefore B, P, R' are collinear

(10) If $ab'c$, $a'bc'$ be such that the three sides a, c, b are concurrent and likewise the sides b', a', c' show that the lines joining the opposite vertices, viz, ac' $a'c$, ab' $a'b$, cb' bc' are concurrent

Examples on Ceva's and Menelaus' Theorems

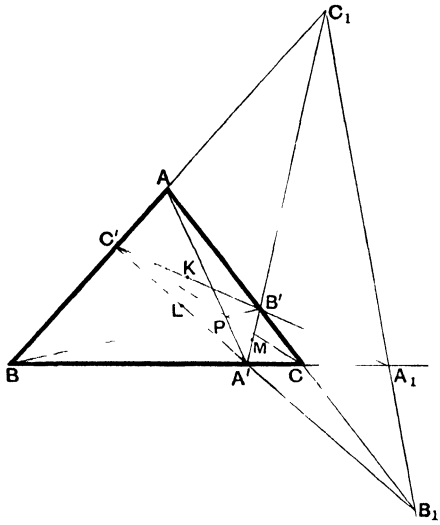
(11) If the lines joining the vertices of a triangle ABC to any point P meet the opposite sides respectively in $A'B'C'$ and K, L, M are the middle points of $C'B'$, $A'A'$, $A'B'$ respectively, and these lines meet the opposite sides in A_1, B_1, C_1 respectively, prove that

(1) A_1, B_1, C_1 are collinear,

and (2) AK, BL, CM are concurrent

(1) Since A', B', C_1 are collinear

$$\frac{AC_1}{BC_1} = \frac{1}{\frac{BA'}{CA'}\,\frac{CB'}{AB'}} \quad \text{(Menelaus' Theorem)}$$



By multiplying this by the two similar expressions, it is seen that

$$\frac{AC_1}{BC_1}\,\frac{BA_1}{CA_1}\,\frac{CB_1}{AB_1} = \frac{1}{\left(\frac{BA'}{CA'}\,\frac{CB'}{AB'}\,\frac{AC'}{BC'}\right)^2} = 1$$

Hence A_1, B_1, C_1 are collinear

(2) Let AK, BL, CM meet the opposite sides in K', L', M'

Then $$(C'B'KA_1) = (BCK'A_1)$$

$$\frac{C'K}{B'K}\,\frac{C'A_1}{B'A_1} = \frac{BK'}{CK'}\,\frac{BA_1}{CA_1},$$

or

$$-1\,\frac{C'A_1}{B'A_1} = \frac{BK}{CK'}\,\frac{BA_1}{CA_1}$$

By multiplying together this and the two similar relations, it is seen that

$$(-1)\ \Pi\frac{C'A_1}{B'A_1}=\Pi\frac{BK'}{CK'}\ \ \Pi\frac{BA_1}{CA_1},$$

or
$$(-1)\ 1=\Pi\frac{BK'}{CK'}\ \ 1 \text{ since } A_1,\ B_1,\ C_1 \text{ are collinear}$$

$$\Pi\frac{BK'}{CK'}=-1$$

Therefore by Ceva's Theorem AK', BL', CM' are collinear

(12) From the angular points of a triangle ABC lines AD, BE, CF are drawn cutting the opposite sides in D, E, F The lines AD, BE, CF form a triangle $A'B'C'$

Prove that
$$\frac{AF}{BF}\ \frac{CE}{AE}\ \frac{BD}{CD}=\frac{A'C}{A'B}\ \frac{B'A}{B'C}\ \frac{C'B}{C'A}$$

Consider the triangle ABC' By Menelaus' Theorem

$$\frac{AF}{BF}=\frac{1}{\frac{BA'}{C'A'}\ \frac{C'B'}{AB'}}$$

By multiplying together this and the two similar relations,

$$\frac{AF}{BF}\ \frac{CE}{AE}\ \frac{BD}{CD}=\frac{AB'}{B'A'}\ \frac{BC'}{CB'}\ \frac{CA'}{AC'}$$

(13) The middle points of the sides of a triangle ABC are A', B', C' and the feet of the perpendicular from the vertices are D, E, F Prove that if

$$(AEB'C')=(AFC'B)=(BDA'C)$$

the triangle must be equilateral

The given relationships may be written

$$(ACEB')=(ABFC')=(BCDA')=K \text{ (suppose)}$$

$$\frac{(ABFC')\,(BCDA')}{(ACEB')}=K,$$

$$\frac{AF}{BF}\ \frac{BD}{CD}\ \frac{CE}{AE}=K, \qquad K=1$$
$$\frac{}{AC'}\ \frac{}{BA'}\ \frac{}{CB'}$$
$$BC'\ \ CA'\ \ AB'$$

Therefore since $(ACEB')=1$ E and B' coincide Similarly, F and C' coincide, and D and A' Hence the triangle is equilateral

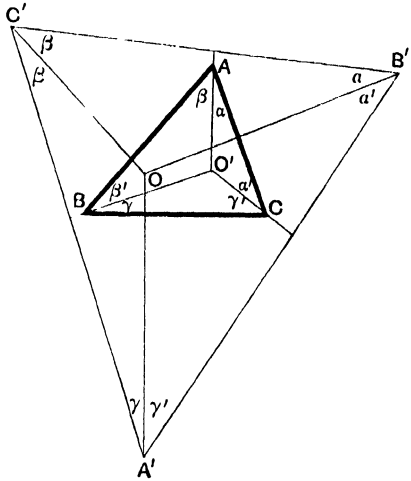
(14) ABC, $A'B'C'$, are two triangles such that the perpendiculars from A on $B'C'$, from B on $C'A'$, and from C on $A'B'$ are concurrent Show that the perpendiculars from A' on BC, from B' on CA, and from C' on AB are also concurrent

In the figure, the angles marked α, α', β, β', γ, γ' are equal

By the Sine form of Ceva's Theorem, the condition that the perpendiculars through ABC should be concurrent is

$$\frac{\sin\alpha \sin\beta \sin\gamma}{\sin\alpha' \sin\beta' \sin\gamma'} = 1$$

This is also the condition that the perpendiculars through $A'B'C'$ should be concurrent



(15) The triangle PQR is inscribed in the triangle ABC and is in perspective with it and the triangle XYZ is circumscribed about the triangle ABC and is in perspective with it Show that the triangles PQR, XYZ are in perspective

Use Sine form of Ceva's Theorem

(16) ABC is a triangle and O a point within it Through ABC and within the triangle lines are drawn making with AB, BC, CA angles equal respectively to OAC, OBA, OCB, prove that these lines meet at a point whose distances from the sides of the triangle are inversely proportional to the corresponding distances of O

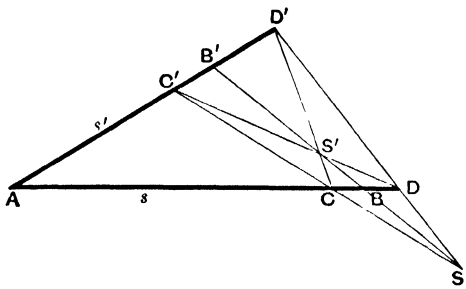
Use Sine form of Ceva's Theorem

CHAPTER VII

HARMONIC FORMS —HARMONIC PROPERTY OF THE QUADRANGLE AND QUADRILATERAL HARMONIC PERSPECTIVE

45 Harmonic Ranges and Pencils

If two harmonic ranges of four points have a self corresponding point at the point of intersection of the bases, then the ranges have two centres of perspective



Let $ABCD$ and $AB'C'D'$ be the ranges Let CD' $C'D$ be S' and DD' CC' be S

Since

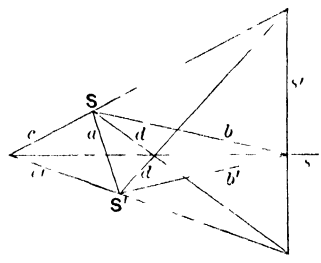
$$(ABCD) = -1 = (AB'C'D')$$

the ranges $ABCD$ and $AB'C'D'$ are projective Since A is a self-corresponding point they are in perspective

Therefore BB' passes through S

Similarly BB' passes through S' Hence the theorem is true

If two harmonic pencils of four rays have a self corresponding ray in the line joining their vertices, then the pencils have two axes of perspective



Let $abcd$ and $ab'c'd'$ be the pencils Let cd' $c'd$ be s' and dd' cc' be s

Since

$$(abcd) = -1 = (ab'c'd')$$

the pencils $abcd$ and $ab'c'd'$ are projective Since a is a self-corresponding ray they are in perspective

Therefore bb' is on s

Similarly bb' is on s' Hence the theorem is true

Conversely

If two ranges of four points, which have a self-corresponding point at the point of intersection of their bases, have also two centres of perspective, then the ranges are harmonic

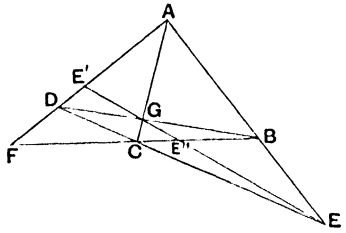
In the figure if S and S' be the centres of perspective,

$$(ABCD) = (AB'C'D') = (ABDC)$$

46 Harmonic Property of the quadrangle and construction of harmonic conjugates

The distance between any pair of vertices of a quadrangle which are taken with a given vertex of its diagonal points triangle, is divided harmonically by that vertex and the opposite side of the diagonal points triangle

This is practically the theorem proved in the last article, but on account of its importance the proof is restated



Let $ABCD$ be the quadrangle, and E, F, G its diagonal points triangle Let GE meet DA and CB in E' and E''

Conversely

If two pencils of four rays, which have a self-corresponding ray in the line joining their vertices, have also two axes of perspective, then the pencils are harmonic

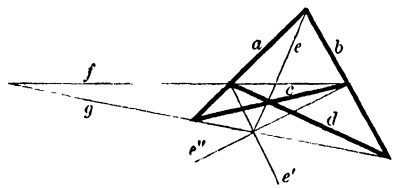
In the figure if s and s' be the axes of perspective,

$$(abcd) = (ab'c'd') = (abdc)$$

Harmonic Property of the quadrilateral and construction of harmonic conjugates

The angle between any pair of sides of a quadrilateral, which intersect on a given side of its diagonal triangle, is divided harmonically by that side and the line joining the opposite vertex of the diagonal triangle to the point of intersection of the pair of sides

This is practically the theorem proved in the last article, but on account of its importance the proof is restated



Let $abcd$ be the quadrilateral, and e, f, g its diagonal triangle Let the lines joining ge to da and cb be e' and e''

Then projecting from E,

$$(FE''CB) = (FE'DA)$$

Projecting from G,

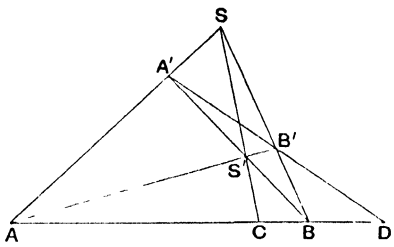
$$(FE''CB) = (FE'AD)$$

Therefore

$$(FE'DA) = (FE'AD)$$

Therefore A and D are harmonic conjugates of F and E'

Given a range of three points to construct graphically the harmonic conjugate of any one with respect to the other two



Let A, B, C be the given points It is required to construct the harmonic conjugate of C with respect to A and B

Take any two points S and S' collinear with C and join S and S' to A and B Let AS' SB and $S'B$ SA be B' and A' Then from the harmonic property of a complete quadrangle the line $A'B'$ meets the base in the required point D

Then taking sections on e,

$$(fe''cb) = (fe'da)$$

Taking sections on g,

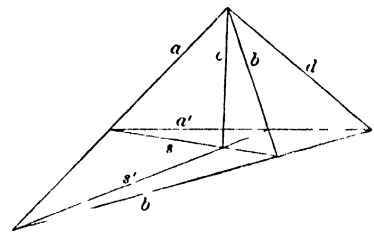
$$(fe''cb) = (fe'ad)$$

Therefore

$$(fe'da) = (fe'ad)$$

Therefore a and d are harmonic conjugates of f and e'

Given a pencil of three rays to construct graphically the harmonic conjugate of any one with respect to the other two



Let a, b, c be the given lines It is required to construct the harmonic conjugate of c with respect to a and b

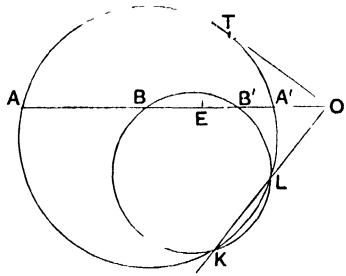
Take any two lines s and s' intersecting on c and construct the intersections of s and s' with a and b Let as' sb and $s'b$ sa be b' and a' Then from the harmonic property of the complete quadrilateral the line joining $a'b'$ to the vertex is the required ray d

47 *To find the common pair of harmonic conjugates of two given pairs of collinear points A, A' and B, B'*

Take any point K not on the straight line $ABB'A'$

Describe circles through AKA' and BKB' These circles will

intersect in a second point L. Let KL meet $ABB'A'$ in O. Draw a tangent OT to one of the circles and measure off distances OE and OF on AA' equal to OT.



Then

$$OE^2 = OF^2 = OT^2$$
$$= OA \cdot OA' = OB \cdot OB'$$

Hence E and F are harmonic conjugates of AA' and of BB' and are therefore the required points.

It should be noticed that

(1) E and F are uniquely determined,

(2) E and F are real if O is outside the circles and, for this to be the case, the segments AA' and BB' must not overlap.

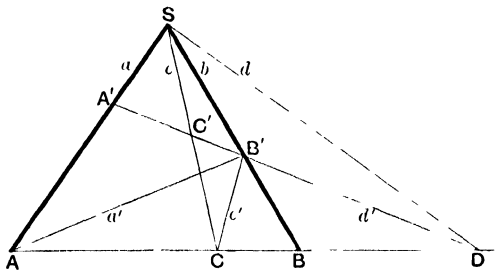
48 Conjugate Points—Pole and Polar

Conjugate points with respect to two given straight lines are such that their connector is cut harmonically by the pair of lines.

Thus, in the figure if $(ABCD) = -1$, C and D are conjugates with respect to a and b.

Conjugate lines with respect to two given points are such that they are harmonic conjugates of the lines joining their point of intersection to the given points.

Thus, in the figure if $(abcd) = -1$, a and b are conjugates with respect to C and D.



Hence, it follows that if through any fixed point D a variable transversal be drawn to cut two given lines in A' and B' and a point C' be taken on the line such that $A'B'C'D$ is harmonic (C' being a conjugate of D), then the locus of C' is a straight line c passing through the point of intersection of the given lines.

Hence, it follows that if on any fixed line b a variable point B' is taken and lines c' and d' are drawn to the fixed points C and D and a line a' is taken such that the pencil $(a'b'c'd')$ is harmonic (a' being the conjugate of b), then a' always passes through a fixed point A, which is on the line joining

The line c is called the *polar* of D with respect to the given lines

This line is the locus of the conjugates of D

The point D is called the *pole* of c with respect to the pair of lines

It should be noticed that

Every side of the diagonal points triangle of a quadrangle is the polar of the opposite vertex with respect to two pairs of connectors of the four points

the given points The point A is called the *pole* of b with respect to the pair of given points

This point is the envelope of the conjugates of b

The line b is called the *polar* of A with respect to the pair of points

Every vertex of the diagonal triangle of a quadrilateral is the pole of the opposite side with respect to two pairs of intersections of the four lines

49 Harmonic Perspective

Any pair of straight lines may be considered to be in Harmonic Perspective with each other, any point being the centre of perspective and the polar of this point with regard to the two lines being the axis

In the figure of Article 48, let SA and SB be the given lines Take D as centre of perspective and c the polar of D with regard to a and b as axis of perspective Then $(DC'B'A')$ is a harmonic range Therefore A' corresponds to B' and B' to A, in such a harmonic perspective

Hence, the lines $A'B$ and AB', being corresponding lines, intersect on SC the axis of perspective

Any quadrilateral (or quadrangle) may be regarded as in Harmonic Perspective with itself, any vertex of the diagonal (or diagonal points) triangle and the opposite side being the centre and axis of perspective

If the points ac and bd are situated on the side e of the diagonal triangle efg, then the poles of e with respect to a and c are on ac fg, and the poles of e with respect to d and b are on bd fg Therefore fg is a common pole of e with respect to a and c and to d and b Hence, with gf as centre and e as axis of harmonic perspective, a and c mutually correspond and so do also d and b Taking different vertices as centres of perspective the lines a, b, c, d will correspond in different pairs

The following are immediate consequences of the preceding

If any straight line u meets the sides a, b, c, d of a quadrilateral in K, L, M, N, and S be one of the vertices of its diagonal triangle (viz the intersection of ab cd with ad bc) and SK, SL, SM, SN meet c, d, a, b in K', L', M', N', then K', L', M', N' are collinear

Correlative

If any point U be joined to the vertices A, B, C, D of a quadrangle by lines k, l, m, n, and s be the third diagonal of the quadrangle, and sk, sl, sm, sn be joined to the opposite vertices of the quadrangle, viz C, D, A, B, by lines k', l', m', n', then k', l', m', n' are concurrent

A particular case of the preceding is the following

If through the point of intersection of the diagonals of a quadrilateral straight lines are drawn parallel to the four sides to meet the sides, which are respectively opposite to those to which they are drawn parallel, then the four points of intersection are collinear

Or correlatively

If through the vertices A, B, C, D of a quadrangle any system of parallel lines be drawn to meet the third diagonal in K, L, M, N, and K, L, M, N be joined to the opposite vertices of the quadrangle, then these four lines are concurrent

50. Generalisation of the theorems of Art 16

If AB, CD are two pairs of harmonic conjugates and O is the middle point of AB, then ∞, the point at infinity on the base, is the harmonic conjugate of O with respect to AB. Hence, if AB, CD, $O\infty$ are projected into $A'B'$, $C'D'$, $J'I'$, respectively, A', B' are the common harmonic conjugates of $C'D'$ and $J'I'$. Hence the theorem of Art 16 may be generalised as follows

(*b*) $\dfrac{CA}{\overline{CD}} = \dfrac{CO}{\overline{CB}}$, $\qquad \dfrac{CA}{\overline{CD}}\dfrac{\infty A}{\infty D} = \dfrac{CO}{\overline{CB}}\dfrac{\infty O}{\infty B}$,

$(ADC\infty) = (OBC\infty)$, $\qquad (A'D'C'I') = (J'B'C'I')$,

$(A'C'D'I') = (J'C'B'I')$,

$\dfrac{A'D'}{\overline{A'I'}}\dfrac{B'C'}{\overline{B'J'}} = \dfrac{D'C'}{\overline{I'J'}}$

(*c*) $OC \cdot OD = OA^2$, $\qquad \dfrac{OC}{OA} = \dfrac{OA}{\overline{OD}}$, $\qquad (CAO\infty) = (ADO\infty)$,

$(C'A'J'I') = (A'D'J'I')$,

$\dfrac{I'C'}{(J'A')^2}\dfrac{J'D'}{} = \dfrac{I'C'}{(I'A')^2}\dfrac{I'D'}{}$

(*d*) $\dfrac{OC}{\overline{OD}} = \left(\dfrac{AC}{\overline{AD}}\right)^2$, $\qquad (CDO\infty) = (CD\,I\infty)^2$,

$(C'D'J'I') = (C'D'A'I')^2$,

$\dfrac{C'I'}{D'J'}\dfrac{C'I'}{D'I'} = \left(\dfrac{C'A'}{\overline{D'A'}}\right)^2$

Extensions of the above theorems may also be obtained by introducing O' the middle point of DC. In this case O and O' are the harmonic conjugates of the point at infinity ∞ on the line with respect to AB and CD respectively

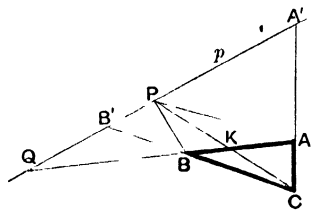
EXAMPLES

(1) Given a line p and three fixed points A, B, C, to find a point P on p such that p and PC are harmonic conjugates of PA and PB.

Let AB meet p in Q. Take K the harmonic conjugate of Q with respect to AB. Let KC meet p in P.

Then $(P\ QKAB)$ is harmonic and therefore P is the required point.

P may also be obtained as the harmonic conjugate of Q with respect to A', B', the points where CA and CB meet p.



(2) Given three points A, B, C and any other point P to draw through P a line which shall be the harmonic conjugate of PC with respect to PA and PB

Let PC meet AB in K On AB take Q the harmonic conjugate of K with respect to AB Then PQ is the required line

(3) Given four lines l_1, l_2, l_3, l_4, and four points P_1, P_2, P_3, P_4 on them such that the connectors of P_1, P_2, P_3, P_4 to C are harmonic conjugates of l_1, l_2, l_3, l_4 with respect to the connectors of P_1, P_2, P_3, P_4 to A and B, prove that the pencil $(C\ P_1P_2P_3P_4)$ is projective with the range formed on AB by the lines l_1, l_2, l_3, l_4

(4) The lines joining any point O to four collinear points A, B, C, D cut any transversal through D in α, β, γ, D, respectively, and $B\gamma$ meets OA in O', $O'\beta$ meets AB in B' Prove that if $(AB, CD) = -1$ then $AB'\ CB = 2\ AC\ BB'$

It is necessary to prove that $\dfrac{AB'}{BB'}\dfrac{AC}{BC} = -2$ or that $(ABB'C) = -2$

But
$$(ABCD) = (\alpha\beta\gamma D) = (AB'BD) = -1,$$
$$(ABB'D) = 2 \text{ and } (ABCD) = -1,$$
$$(ABB'C) = \frac{(ABB'D)}{(ABCD)} = -2$$

(5) $KLMN$ is a simple quadrangle and A, B are two points such that $(K\ LNAB)$, $(L\ KMAB)$, $(M\ LNAB)$ are harmonic

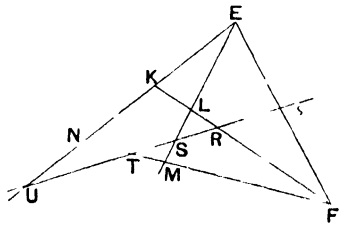
Prove that

(i) $(N\ MKAB)$ is harmonic, and

(ii) A, B must be situated on the third diagonal of the quadrangle

If the transversal AB meets the sides as in the figure in $RSTU$,

$$(ABUR) = -1,$$
$$(ABRS) = -1,$$
$$(ABST) = -1$$
$$\frac{AU}{BU} = -\frac{AR}{BR} = \frac{AS}{BS} = -\frac{AT}{BT}$$



Therefore the pencil $(N\ KMAB)$ is harmonic Since $\dfrac{AU}{BU} = \dfrac{AS}{BS}$, U and S coincide or AB passes through E Similarly AB passes through F Hence A, B are any pair of harmonic conjugates of E and F

(6) If $ABCD$ be a quadrangle and $AC\ BD$ bisects DB, then the connector of $AB\ CD$ to $AD\ CB$ is parallel to BD, or $ABCD$ is a parallelogram

(7) If a transversal cut the sides BC, CA, AB of a triangle in P, Q, R respectively, then

(i) if P', Q', R' be the harmonic conjugates of P, Q, R with respect to BC, CA, AB respectively, AP', BQ', CR' meet in a point, and

(ii) if X, Y, Z be the middle points of PP', QQ', RR' respectively, X, Y, Z lie on a straight line

Since
$$(BCP'P) = -1,$$
$$\frac{P'B}{P'C} = -\frac{PB}{PC}$$

Similarly

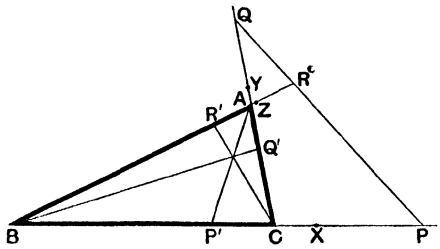
$$\frac{QC}{Q'A} = -\frac{QC}{QA},$$

and

$$\frac{RA}{R'B} = -\frac{RA}{RB}$$

$$\frac{P'B}{P'C}\frac{Q'C}{Q'A}\frac{R'A}{R'B} = (-1)^3\frac{PB}{PC}\frac{QC}{QA}\frac{RA}{RB}$$

$$= -1 \text{ by Menelaus' Theorem}$$



Therefore by Ceva's Theorem AP', BQ' CR' are concurrent

Since $\frac{XB}{XC} = \left(\frac{PB}{PC}\right)^2$ it follows that

$$\frac{XB}{XC}\frac{YC}{YA}\frac{ZA}{ZB} = (-1)^6\left(\frac{PB}{PC}\frac{QC}{QA}\frac{RA}{RB}\right)^2 = +1$$

Therefore by Menelaus' Theorem X, Y, Z are collinear

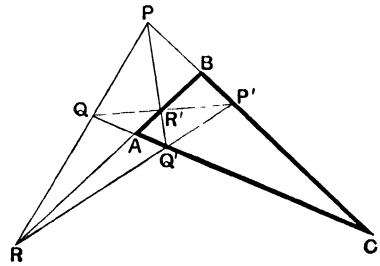
Given that AP', BQ', CR' are concurrent, the converse theorem can be proved

The point of concurrency of AP', BQ', CR' is termed *the pole of the line PQR with respect to the triangle*

This result should be compared with Example 11, Chapter VI

(8) The middle points of the diagonals of a quadrilateral are collinear

In the figure the sides of the triangle ABC are cut by a transversal PQR and from the quadrangle $QQ'P'P$, PP', QQ', RR' divide the sides of ABC harmonically Hence the middle points of PP', QQ', RR' are collinear But PP', QQ', RR' are the diagonals of the quadrilateral $QPQ'P'$ Hence the theorem is true



(9) The three straight lines joining the vertices A, B, C of a triangle to a point cut the opposite sides in points P, Q, R respectively, and R' is the point on BA which divides it externally in the same ratio as R divides it internally Prove that the straight line joining the middle points of AP, BQ passes through the middle point in CR'

Let S be the point to which the vertices are joined The line PQ will pass through R' Then BQ, AP, $R'C$ are the diagonals of the quadrilateral, BA, AQ, QP, PB Hence the middle points are collinear

(10) Show that the three poles of a straight line with respect to the three pairs of points of intersection of four given straight lines (*i e* the ends of diagonals of the quadrilateral) lie upon another straight line conjugate to the first straight line with respect to each of the three pairs of points

Take the diagonal triangle ABC as triangle of reference and let its sides meet the line in A', B', C' Let the ratios of R, T, S three collinear ends of diagonals be x_1, y_1, z_1, then those of the other ends of the diagonals R', T', S' will be

$$-x_1, -y_1, -z_1$$

Let the ratios of A', B', C' be a', b', c', then if a_2, b_2, c_2 be those of the harmonic conjugates of these points with respect to RR', SS', TT',

$$a_2 = \frac{x_1^2}{a'}, \quad b_2 = \frac{y_1^2}{b'}, \quad c_2 = \frac{z_1^2}{c'},$$

$$a_2 b_2 c_2 = \frac{x_1^2 y_1^2 z_1^2}{a'b'c'} = \frac{1^2}{1} = 1$$

these points are collineal

This theorem is an extension of Example 8

(11) If through any point P two lines be drawn to meet the sides of a triangle ABC in C', B' and C'', B'' and points A', A'' be taken on BC such that the triangles $A'B'C'$ and $A''B''C''$ are in perspective with ABC Then if Q be the point of intersection of $A'B'$ and $A''B''$, P and Q are collineal with B and the pencil A $PQBC$ is harmonic

Consider the pencils $(B'$ $ABC'A')$ and $(B''$ $ABC''A'')$ which have a self corresponding ray $AB'B''$ and whose other rays intersect in B, P, Q Let BB' CC' be S' and BB'' CC'' be S'' The rays of $(B'$ $ABC'A')$ intersect AB in A, B, C' and a point C_1 But from the quadrangle $B'S'A'C$ this range is harmonic,

$$(B'\ ABC'A') = -1, \text{ so } (B''\ ABC'A'') = -1$$

Therefore P, B, Q are collineal and $(A$ $PQBC)$ is harmonic

(12) Through a point O in the plane of a triangle ABC three straight lines are drawn meeting the sides BC, CA, AB in $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$, respectively, so that the ranges $(OD_1E_1F_1)$, $(OE_2F_2D_2)$, $(OF_3D_3E_3)$ are all harmonic Show that D_1, E_2, F_3 are collineal and also $D_1E_3F_2$, $D_2E_1F_3$, $D_3E_2F_1$

(13) If through one vertex C of a triangle ABC a chord be drawn to meet the opposite side in K and two points C' and K' be taken on CK such that C, C are harmonic conjugates of KK', then the pencils AC' and AK' are projective

For $(CC'KK') = -1$, $\quad (CKC'K') = 2$, $\quad \dfrac{CC'}{KC'} = 2\dfrac{CK'}{KK'}$

Hence the ranges K' and C' are projective

(14) If a transversal meets the sides BC, CA, AB of a triangle, ABC, in points K, L and M, and points K', L', M' are taken on the sides such that

$$(BCKK') = (CALL') = (ABMM'),$$

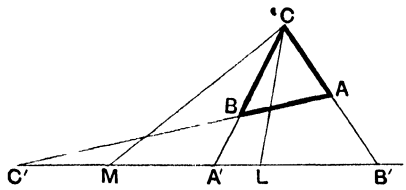
prove that the poles of KLM with respect to the triangle ABC and all triangles $K'L'M'$ coincide

This theorem may be proved directly or the straight line KLM may be projected into the line at infinity and the triangle into an equilateral triangle in which case the theorem becomes the following, which can be easily proved to be true

If on the sides of an equilateral triangle ABC points K', L', M' are taken such that they divide the sides in the same ratios, then the centroids of the triangles ABC and $K'L'M'$ coincide

(15) If ABC be a triangle and L and M are two points such that $(C\ LMAB)$ is harmonic, then $(A\ LMCB)=-(B\ LMCA)$

Let AB, BC, CA meet LM in C', A', and B' Take L, M as points of reference and let a, b, c be the ratios of A', B', C'



Then

$$(C\ LMAB)=(LMB'A')=\frac{b}{a},$$

$$b=-a,$$

$$(A\ LMCB)=(LMB'C')=\frac{b}{c},$$

$$(B\ LMCA)=(LMA'C')=\frac{a}{c}=-\frac{b}{c},$$

$$(A\ LMCB)=-(B\ LMCA)$$

(16) If ABC be a triangle and L and M are a pair of points such that

$$(C\ LMAB)=\lambda, \text{ then } (A\ LMCB)=\lambda(B\ LMCA)$$

CHAPTER VIII

INVOLUTION

51 There are two ways of defining an involution

First definition of an involution

If pairs of points AA', BB', CC', ... are taken on a straight line, termed the base, such that if O be a given point on the base

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \text{a constant quantity},$$

then the points AA', BB', CC', ... are said to form an involution of which A and A', B and B', C and C', ... are termed conjugate points

The properties of an involution differ according as the constant is positive or negative When the constant is positive the involution may be said to be of the *first kind* and when negative of the *second kind*

Involution of the first kind

If $OA \cdot OA'$ is a positive quantity, A and A' are situated on the same side of O, and OA and OA' must be both positive or both negative If a length OK be taken such that $OA \cdot OA' = OK^2$, two points L and M are obtained at equal distances OK from O in the positive and in the negative direction respectively such that each is its own conjugate These points are termed the *double points* of the involution Since O is the middle point of LM and $OM^2 = OL^2 = OA \cdot OA'$, it follows that L and M are harmonic conjugates of A and A' Similarly L and M are harmonic conjugates of B and B' and of every other pair of conjugate points Hence pairs of conjugate points of the involution may in this case be constructed as pairs of harmonic conjugates of L and M the double points of the involution



Involution of the second kind

If $OA \cdot OA'$ is a negative quantity, A and A' are situated on different sides of O, and OA and OA' must have different signs If

double points are sought for, as in the previous case, the equation $OA\ OA' = -OK^2$ is obtained That is to say, L and M are at distances $+\sqrt{-1}\ OK$ and $-\sqrt{-1}\ OK$ from O Hence L and M, the double points, are what may be termed imaginary points Assuming such points to exist on the base they are equally distant from O, and consequently, since $OL^2 = OM^2 = OA\ OA'$, A and A' may be looked upon as harmonic conjugates of these points There are, in this case, a pair of points equally distant on either side from O

In both kinds of involution the point O corresponds to the point at infinity on the base This point is called the centre of the involution

The anharmonic ratio of any four points of an involution is equal to that of their four conjugates

To prove this it must be shown that

$$(ABCD) = (A'B'C'D')$$

But $OA = K\dfrac{1}{OA'}$, $OC = K\dfrac{1}{OC'}$, where K is a constant,

$$AC = -K\frac{A'C'}{OA'\ OC'}$$

Similar relations hold for the distances between other pairs of points

$$\frac{AC}{BC}\frac{AD}{BD} = \frac{A'C'}{B'C'}\frac{OB'}{OA'}\frac{A'D'}{B'D'}\frac{OB'}{OA'}$$

$$= \frac{A'C'}{B'C'}\frac{A'D'}{B'D'},$$

$$(ABCD) = (A'B'C'D')$$

Second definition of an involution

If two superposed projective ranges have a pair of elements, which mutually correspond to each other, then every other pair of elements mutually correspond and the ranges are said to form an involution

If A and A' are the pair of mutually corresponding elements and B' of the second corresponds to B of the first range, then P the point of the first which corresponds to B of the second is given by

$$(AA'BP) = (A'AB'B)$$

But $$(A'AB'B) = (AA'BB'),$$

$$(AA'BP) = (AA'BB'),$$

therefore P coincides with B' This proves the theorem involved in the definition

This result may also be obtained for Arts 10 and 37 For if V and V' be the vanishing points of the ranges and P and P' mutually correspond, $VP\ V'P' = \text{constant} = VP'\ V'P$ Therefore V and V' coincide in a point O Hence the equation $VP\ V'P' = \text{constant}$ becomes $OP\ OP' = \text{constant}$ and each pair of corresponding points mutually correspond

The relation on which the first definition of an involution depends may be deduced at once as follows

If O be the point corresponding to the point at infinity (∞) on the involution, then by definition

$$(AB'\infty\ O) = (A'BO\infty),$$

$$\frac{A\infty}{B'\infty}\ \frac{AO}{B'O} = \frac{A'O}{BO}\ \frac{A'\infty}{B\infty},$$

therefore $$AO\ A'O = BO\ B'O = \text{a constant}$$

If the constant be positive, two points L and M may be taken on either side of O such that $OL^2 = OM^2 = OA\ OA'$ and L and M are then harmonic conjugates of AA' and of every other pair of conjugate points

52 Involution Pencil

Since an involution is a particular case of two superposed projective ranges, an involution is projected from any centre upon any line—in the plane of the centre of projection and the base—into another involution The pencil formed by joining the points of an involution to any point is termed an involution pencil and is cut by every line in its plane in another involution It should be noticed that the centre of an involution is not projected into the centre of the corresponding involution, but the double points are projected into double points The new centre, when the double points are real, may be determined as their mid-point The centre is always the conjugate of the point at infinity on the base

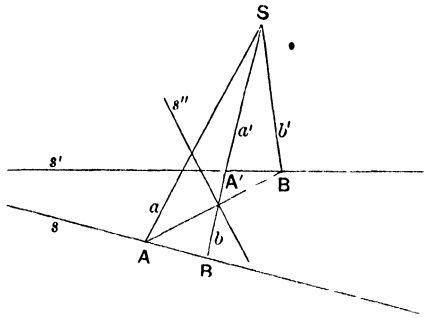
On reference to Art 48 it will be seen that a system of collinear conjugate points with regard to two straight lines form an involution

The case in which two projective pencils become an involution may be graphically considered by reference to Art 35

Generally in two projective pencils with the same vertex S any ray a' or b corresponds to a different ray a or b' according as it is considered to belong to the one or the other of the pencils Let the pencil ab determine a range AB on s and the pencil $a'b'$ a range $A'B'$ on s', where b and a' are the same line

Then AB' BA' will intersect on some fixed line s'' (Art 35)

Now if s'' does not pass through S the lines SA and SB' cannot coincide If s'' passes through S, AB must pass through S because it passes through the point where s'' meets $A'B$ In this case the lines SB' and SA do coincide Hence $SA'B$ has the same corresponding ray $SB'A$ to whichever pencil it is supposed to belong Similarly, every ray has the same corresponding ray to whichever pencil it is supposed to belong In this case, the two projective pencils become an involution pencil



53 An involution is completely determined by

(1) Two pairs of conjugate points,
(2) The two double points,
(3) The centre and a double point,
(4) A pair of conjugate points and the centre,
(5) A pair of conjugate points and a double point

If two pairs of conjugate points AA', BB' be given, the conjugate of any point C is uniquely determined by the relation

$$(AA'BC') = (A'AB'C)$$

A graphic construction for C' is given (Art 59)

From the fact that $(ABA'C') = (A'BAC)$,

$$\frac{AA'}{BA'}\frac{AC'}{BC'} = \frac{A'A}{B'A}\frac{A'C}{B'C'}$$

$$AB'\ BC'\ CA' + A'B\ B'C\ C'A = 0$$

This is a relation which must be satisfied by three pairs of conjugate points, AA', BB', CC' of an involution

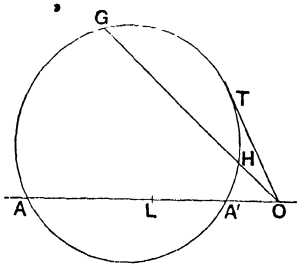
Also since $(AA'BC') = (A'AB'C)$,

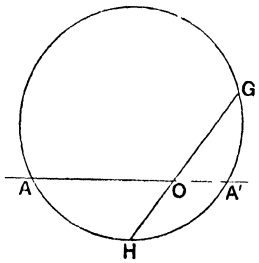
$$\frac{AB}{A'B}\frac{AC'}{A'C'} = \frac{A'B'}{AB'}\frac{A'C}{AC}$$

Hence $$\frac{AB}{A'B}\frac{AB'}{A'B'} = \frac{AC'}{A'C'}\frac{AC}{A'C} = \text{constant}$$

Hence if pairs of points of an involution are determined by ratios bb', cc', , A and A', any pair of conjugate points, being the points of reference, $bb' = cc' = a$ constant

To construct on a given base an involution having given the centre and the constant of the involution



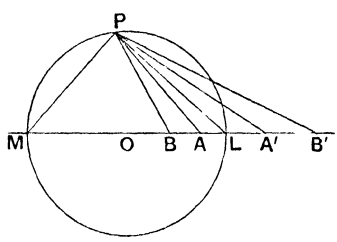


Take any point G not on the base, and on OG, where O is the centre, take a point H such that $OG \cdot OH$ equals the constant of the involution, H and G being on the same or on opposite sides of O according as the constant is positive or negative. If A be any point on the base, the circle through AGH will meet the base in A' the point conjugate to A, for $OA \cdot OA' = OH \cdot OG$ = the constant of the involution.

When the constant of the involution is positive a tangent OT can be drawn to the circle and the double points L and M are at distances OT from O.

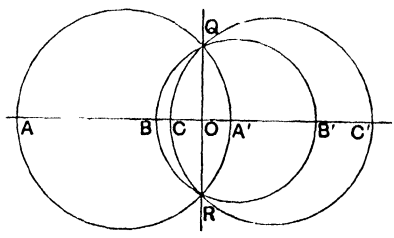
When the constant of the involution is negative, OG and OH can be taken equally distant from O on the perpendicular through O to the base and in this case every pair of conjugate points A and A' subtend a right angle at G and H.

The following illustrates the difference between an involution with real double points and one with imaginary double points.



Let the double points M and L be real.

Describe a circle on ML as diameter.



Let the double points be imaginary.

Take Q and R on the perpendicular through O such that $OQ = -OR$ and $OQ \cdot OR = OA \cdot OA'$.

Join P, any point on this circle, to ML, and to pairs of conjugate points AA', BB', etc

Since PM and PL are at right angles and the pencils $(P\ MLAA')$ and $(P\ MLBB')$ are harmonic, PL bisects the angles APA', BPB', etc (Art 16)

Therefore the angles APB and $A'PB'$ are equal

Therefore corresponding segments of an involution with real double points subtend equal angles at any point of the circle described on the line joining the double points as diameter

Describe circles through QR They will have their centres on the base and determine on the base pairs of conjugate points AA', BB', etc

Since AQA' and BQB' are right angles, therefore the angles AQB and $A'QB'$ are equal

Therefore corresponding segments of an involution with imaginary double points subtend equal angles at two points, and pairs of corresponding points of the involution subtend right angles at these points

When the double points are real, if P be any point on the circle described on the line joining them as diameter,

$$\frac{AP}{A'P} = \frac{AL}{A'L} = \frac{AM}{A'M}$$

54 It has been shown that an involution may be looked upon as a particular case of two superposed projective ranges On reference to Art 43 it will be seen that in the case of an involution the points W, V' and O coincide and that the properties there proved for two superposed projective ranges become as follows

(*a*) $OA \cdot OA' = OB \cdot OB' = \text{constant}$

(*b*) The self-corresponding points S and S' may be real or imaginary

(*c*) If S and S' are real, corresponding points are harmonic conjugates of S and S'

If S and S' are imaginary, corresponding points may be constructed by the rotation of a constant angle, which in this case is a right angle, round Q or Q' (Art 53), two points situated on the perpendicular through O

(*d*) If S and S' are imaginary, there are two coincident minimum segments, KK', and a circle described on KK' as diameter passes through Q and Q'

There are two special forms of an involution

(1) If one of the double points is at infinity, the other double point bisects the distance between every pair of conjugate points

This follows from the fact that every pair of conjugate points are harmonic conjugates of the double points

(2) If the two double points coincide, every point on the line is a conjugate of the point at which the double points coincide This is the particular case of a harmonic range dealt with in Art 15

These special cases are of importance in connexion with the Conic Section

55 *To find the common pair of conjugate elements of two superposed involutions*

Let O and O' be the centres of the two involutions

(1) Let the double points EF and $E'F'$ of both involutions be real

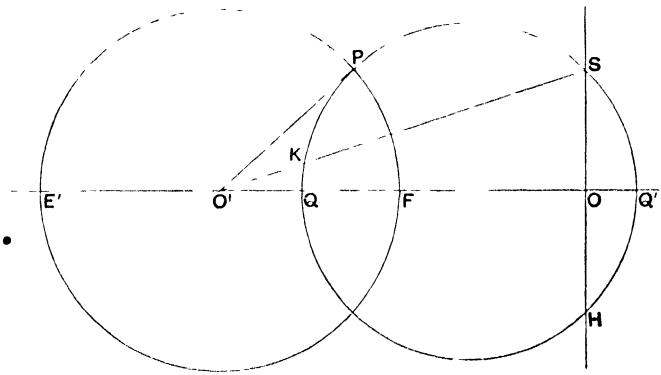
Then the common pair of conjugates of the two involutions are the common harmonic conjugates of EF and $E'F'$ (Art 51) These are real unless EF and $E'F'$ overlap (Art 47)

(2) Let the double points of both involutions be imaginary

Construct points S, H equally distant vertically above and below O such that any circle through S and H determines conjugate points of the involution centre O Similarly construct S', H' for the other involution Then a circle can be described through $SHS'H'$ and this circle determines the common pair of conjugates of the involutions

(3) Let the double points of the involution centre O be imaginary and those of the involution centre O' be real

Construct S and H as above for the involution centre O and let E' and F' be the double points of the involution centre O'



Join $O'S$ and on it take a point K such that $O'F'^2 = O'K \cdot O'S$ Describe a circle through K, S, and H to meet OO' in Q, Q' Q and Q'

are the required points, for they are of the involution centre O, since the circle which determines them passes through S and H and they are conjugates of the involution centre O' since

$$O'Q \cdot O'Q' = O'K \cdot O'S = O'F'^2$$

The only case in which the common [illegible] can be imaginary is when both involutions have real double elements, which overlap

The common conjugates of two involution pencils can be obtained from the common conjugates of the involutions which they determine on any line

The construction of the double points of an involution is given in Chapter XVII, Art 110

Given two pairs of lines a, a' and b, b' to find the pair of common harmonic conjugates of a, a' and b, b' which are parallel

Through aa' draw a pair of rays b_1, b_1' parallel to b, b' Construct e and f the common harmonic conjugates of aa' and b_1b_1' (Art 47) The rays through aa' and bb' parallel to e and f are the parallel common harmonic conjugates of a, a' and b, b'

56 Involution property of the quadrangle and quadrilateral

The three pairs of lines, which connect [illegible] are cut by any transversal in three pairs of conjugate points of an involution

Let S, T, Q, R be the four given points and AA', BB', CC' the pairs of points in which their connectors are met by any transversal s Let $SQ \cdot TR$ be the point P

Since the ranges obtained by projecting $RPTA$ from S and from Q on to s are projective,

$$(C'A'BA) = (B'A'CA) = (CAB'A)$$

Therefore AA', BB', CC' are pairs of conjugate points of an involution

If the transversal be drawn as

The three pairs of points, in which four straight lines intersect, determine by their connectors with any point three pairs of conjugate rays of an involution pencil

Let s, t, q, r be the four given lines and aa', bb', cc' the pairs of lines joining their intersections to any point S Let $sq \cdot tr$ be the line p

Since the pencils obtained by joining S to the points of intersection of s and q with $rpta$ are projective,

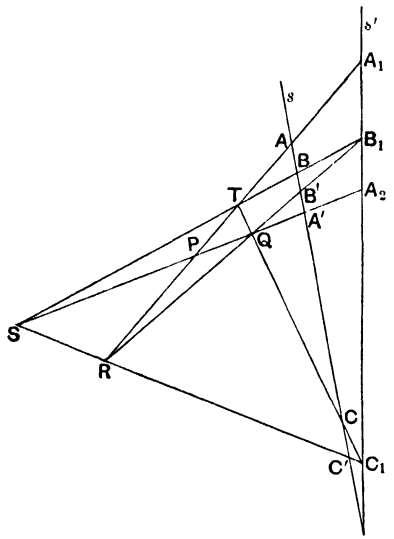
$$(c'a'ba) = (b'a'ca) = (cab'a')$$

Therefore aa', bb', cc are pairs of conjugate rays of an involution pencil

If the point S be situated at S',

s' to pass through TS QR and TQ RS as in the figure,

$$(C_1A_2B_1A_1) = (C_1A_1B_1A_2)$$



Hence A_1 and A_2 are harmonic conjugates of B_1 and C_1

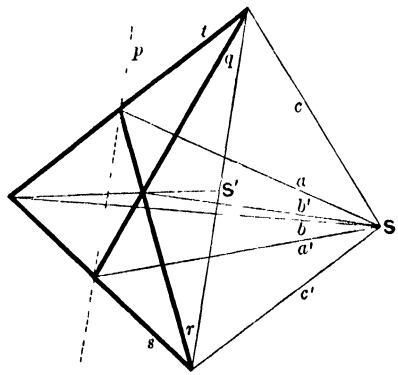
Cor

If AA', BB' and C are given, C' is fixed, since the points AA', BB', CC' form an involution

Hence, *if five of the sides of a quadrangle pass through five fixed collinear points, the sixth side must pass through a fixed point collinear with the other five points* (See Art 13 (*b*))

the point of intersection of ts qr and tq rs, b' coincides with b and c' with c

$$(ca'ba) = (cabc')$$



Hence a and a' are harmonic conjugates of c and b

Cor

If aa', bb' and c are given, c' is fixed, since the lines aa', bb', cc' form an involution

Hence, *if five of the vertices of a quadrilateral lie on five fixed concurrent straight lines, the sixth vertex must lie on a fixed line concurrent with the other five lines* (See Art 13 (*b*))

Graphic construction of conjugate elements of an involution

If two pairs of conjugate points A and A' and B and B' are given, the theorem of the last article enables us to construct C' the conjugate of any element C

If two pairs of conjugate rays a and a' and b and b' are given, the theorem of the last article enables us to construct c' the conjugate of any element c

Through C draw an arbitrary line on which take any two points Q and T

Join AT, $B'Q$ to intersect in R, and BT, $A'Q$ to intersect in S

Then SR meets the base in the required point C'

On c take any arbitrary point through which draw any two lines q and t

Join at and $b'q$ by a line r, and bt and $a'q$ by a line s

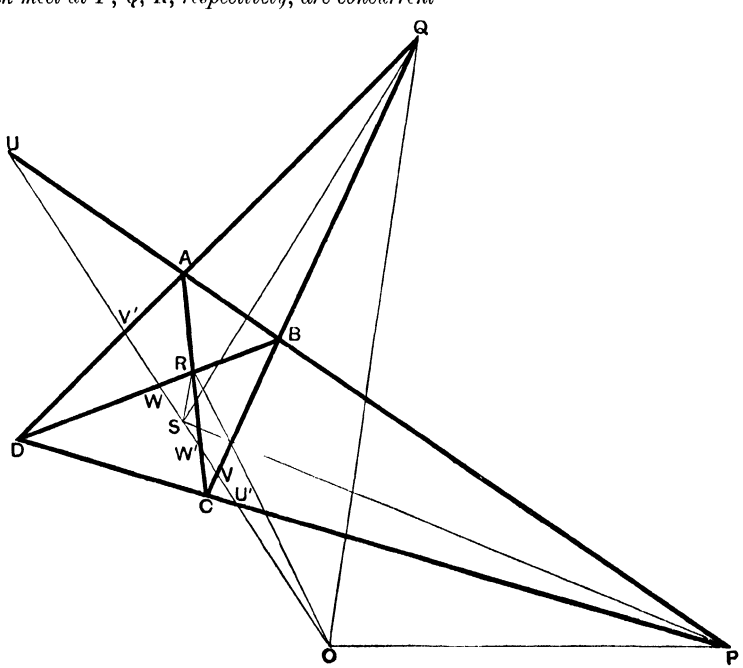
Then the line joining rs to the vertex of the pencil is the required ray c'

57 *Involutions determined by pairs of lines*

In Art 48 conjugate points with respect to pairs of lines were defined Collinear pairs of conjugate points form an involution and a pair of straight lines determines an involution on every line in their plane This involution is formed by pairs of harmonic conjugates of the two points of intersection of the line with the given pair of lines

There are two lines on which any two given pairs of lines determine the same involution, namely, the third pair of connectors of their four points of intersection

If $ABCD$ be a quadrangle, P, Q, R its diagonal points and O any point in its plane, then the harmonic conjugates of OP, OQ, OR, with respect to the sides of the quadrangle which meet at P, Q, R, respectively, are concurrent



Let the harmonic conjugates of OP and OQ with regard to the sides which meet at P and Q intersect in S

Let OS meet the pairs of opposite sides in UU', VV', WW' These points form an involution

But $(OSUU')$ and $(OSVV')$ are harmonic, and therefore O and S are the double points of this involution Therefore $(OSWW')$ is harmonic Therefore RS is the harmonic conjugate of RO with respect to RA, RB

O and S' are conjugate points with respect to the three pairs of opposite sides of the quadrangle

The correlative theorem is as follows

If abcd be a quadrilateral, p, q, r its diagonals and o any line in its plane, then the harmonic conjugates of op, oq, or, with respect to the vertices of the quadrilateral which lie on p, q, r respectively, are collinear

This may be proved by the correlative method Example 10, Chapter VII, is a restatement of the theorem and Example 8, Chapter VII, is a particular case

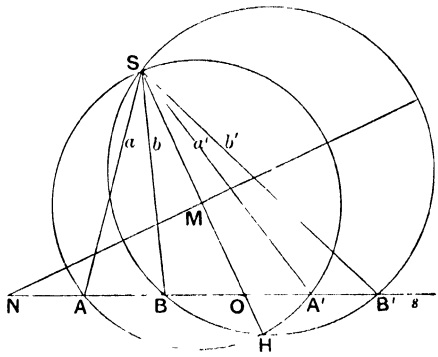
These theorems are particular cases of an important theorem proved in Art 117

58 In every involution pencil one pair of conjugate rays is at right angles, and if more than one pair of conjugate rays are at right angles every pair of conjugate rays is at right angles

Let aa', bb' be the pairs of conjugate rays, which determine the involution Let any transversal s meet the rays in A A', B, B' Let O be the centre of this involution Describe a circle through ASA' to meet SO in H Then since

$$OS \cdot OH = OA \cdot OA' = OB \cdot OB',$$

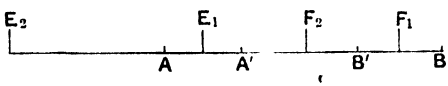
the circle through SBB' will also pass through H Hence conjugate points of the involution are determined by circles through S and H



If a and a' and also b and b' are at right angles, the centres of the circles SAA' and SBB' will lie on s and the centre of every circle through S and H will also be on this line In this case every pair of conjugate rays will be at right angles

If aa' and bb' are not at right angles, take M the middle point of SH and erect a perpendicular through M to meet s at N Describe a circle with centre N through S and H to meet s in C and C' Then s is a diameter of this circle and C and C' subtend a right angle at S

59 Four collinear points A, A', B, B' determine three involutions viz



(1) that in which AA', BB' are pairs of conjugate points

(2) that in which AB, $A'B'$ are pairs of conjugate points

(3) „ „ „ AB', BA' „ „ „ „ „

In two cases the involutions will not be overlapping, viz in the figure in cases (1) and (2) The double points of these involutions are real

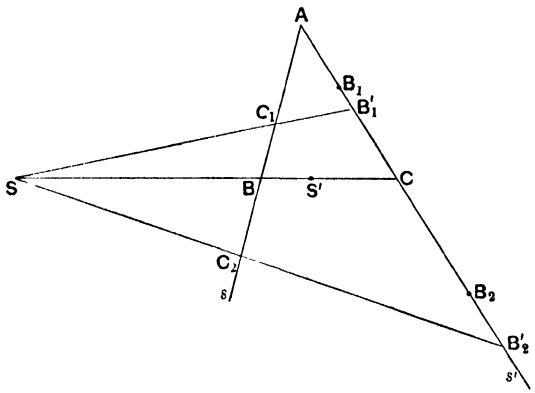
In one case the involution will be overlapping, viz in the figure in case (3) The double points of this involution are imaginary

E_1 and F_1 the common harmonic conjugates of AA' and BB', *i e* the double points of (1), are a pair of real conjugate points of (2) and of (3) (Example 1, page 105)

E_2 and F_2 the common harmonic conjugates of AB, $A'B'$, *i e* the double points of (2), are a pair of real conjugate points of (1) and of (3)

E_3 and F_3 the common harmonic conjugates of AB', BA', *i e* the imaginary double points of (3), are a pair of imaginary conjugate points of (2) and of (1)

60 Every pair of involutions in the same plane are in perspective if both the involutions have real or both imaginary double points, the perspective is real if one has real and the other imaginary double points, the perspective is imaginary



Let the involutions be situated on two lines s and s' which intersect in A and let B and C be the points of the two involutions which correspond to A Let C_1C_2 be any pair of conjugate points of the involution on AB and B_1B_2 any pair of conjugate points of the involution on AC Then, if c_1, c_2, b_1, b_2 be the ratios respectively of C_1, C_2, B_1, B_2 referred to ABC, $c_1\ c_2 = K$ and $b_1\ b_2 = K'$, where K and K' are constants for different pairs of conjugate points of the involutions (Art 53)

On BC take two points S and S' such that their ratios are

$$+\frac{1}{\sqrt{KK'}} \text{ and } -\frac{1}{\sqrt{KK'}}$$

These points are harmonic conjugates of B and C. Join S to C_1 and C_2 to meet AC' in B_1' and B_2'. Let the ratios of these points be b_1' and b_2'.

Then, by Menelaus' Theorem,

$$\frac{1}{\sqrt{KA'}}c_1b_1'=1 \quad \text{and} \quad \frac{1}{\sqrt{KK'}}c_2b_2'=1$$

Therefore $$\frac{c_1c_2b_1'b_2'}{KK'}=1 \quad \text{and} \quad b_1'b_2'=\frac{KK'}{c_1c_2}=K'$$

Hence B_1' and B_2' are a pair of conjugate points of the involution on AC.

Similarly, if S' be joined to C_1' and C_2' to meet AC in B_1'' and B_2'', these are a pair of conjugate points of the involution on AC'.

Hence if any pair of conjugate points of either involution be joined to S or S' their connectors will meet the other base in a pair of conjugate points of the involution on that base, and the two involutions are in perspective with either S or S' for centre of perspective.

It should be noticed that as long as the involutions are both of the same kind the points S and S' are real.

If E and F are the double points of the involution on AB and K and L the double points of the involution on AC, the ratios of these points are $\sqrt{K}$, $-\sqrt{K}$, $\sqrt{K'}$, $-\sqrt{K'}$.

Therefore, since $\frac{1}{\sqrt{KK'}}\sqrt{K}\sqrt{K'}=1$, S, L and K are collinear,

$\frac{1}{\sqrt{KK'}}(-\sqrt{K})(-\sqrt{K'})=1$, S, F and L „ „

$-\frac{1}{\sqrt{KK'}}(\sqrt{K})(-\sqrt{K'})=1$, S', E and L „ „

$-\frac{1}{\sqrt{KK'}}(-\sqrt{K})(\sqrt{K'})=1$, S', F and K „ „

Hence if the double points of the involutions are real the double points of one are projected into the double points of the other from either S or S'.

Since $\frac{1}{\sqrt{KK'}}(\sqrt{K})(-\sqrt{K'})=-1$, AS, CL, BL are concurrent,

$\frac{1}{\sqrt{KK'}}(-\sqrt{K})(\sqrt{K'})=-1$, AS, CF, BK „ „

$-\frac{1}{\sqrt{KK'}}(\sqrt{K})(\sqrt{K'})=-1$, AS', CE, BK „ „

$-\frac{1}{\sqrt{KK'}}(-\sqrt{K})(-\sqrt{K'})=-1$, AS, CF, BL „ „

The correlative theorem is as follows:

Every pair of involution pencils in the same plane are in perspective if both the pencils have real or both imaginary double rays, the perspective is real if one has real and the other imaginary double rays, the perspective is imaginary.

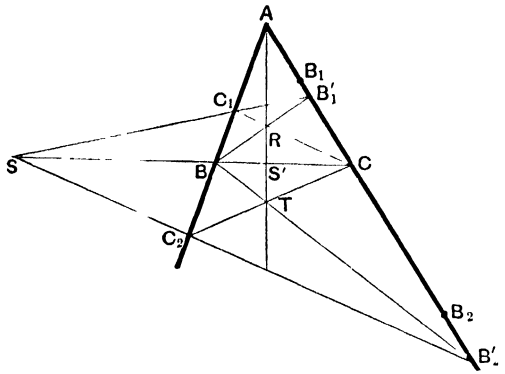
Let the centres of the involutions be B and C and let BA and CA be the rays corresponding to BC in the two involutions. Let a pair of rays of the involution,

centre C, meet AB in C_1 and C_2 and a pair in the involution centre B meet AC in B_1 and B_2. With the notation of the correlative theorem draw two lines AS and AS' such that the ratios of S and S', the points where they meet BC, are $+\frac{1}{\sqrt{KK'}}$ and $-\frac{1}{\sqrt{KK'}}$. Let CC_1' and CC_2 meet AS' in R and T. Join BR and BT to meet AC in B_1' and B_2'. Since AS, CC_1', BB_1' are concurrent,

$$-\frac{1}{\sqrt{KK'}}\, c_1b_1' = -1,$$

and since AT, CC_2, BB_2' are concurrent, $-\frac{1}{\sqrt{KK'}}\, c_2b_2' = -1$,

$$\frac{c_1b_1'c_2b_2'}{KK'} = 1, \qquad b_1'b_2' = K'$$



Therefore the lines BB_1' and BB_2' are conjugate rays of the involution whose centre is B. Hence the involutions are in perspective with AS' as axis. Similarly they are in perspective with AS as axis.

61 Analytical formulae connected with an involution

If the notation of Art. 14 be used and B and C are taken as the points of reference and x_1x_2, x_3x_4, x_5x_6, the ratios of A_1A_2, A_3A_4, A_5A_6, are given by

$$a'\lambda^2 + 2h'\lambda + b' = 0,$$
$$a''\lambda^2 + 2h''\lambda + b'' = 0,$$
$$a'''\lambda^2 + 2h'''\lambda + b''' = 0,$$

the following results are obtained.

(1) The condition that BC, A_1A_2, A_3A_4 should be three pairs of conjugate points of an involution is

$$x_1x_2 = x_3x_4$$

(2) The condition that A_1A_2, A_3A_4, A_5A_6 should be three pairs of conjugate points of an involution is

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1+x_2 & x_3+x_4 & x_5+x_6 \\ x_1x_2 & x_3x_4 & x_5x_6 \end{vmatrix} = 0$$

(3) The condition (1) that BC, A_1A_2, A_3A_4 should be three pairs of conjugate points of an involution may also be written

$$b'a''-b''a'=0$$

(4) The condition (2) that A_1A_2, A_3A_4, A_5A_6 should be three pairs of conjugate points of an involution may also be written

$$\begin{vmatrix} a' & h' & b' \\ a'' & h'' & b'' \\ a''' & h''' & b''' \end{vmatrix}=0$$

(5) The condition that A_1, $A_2(x_1, x_2)$ should be a pair of conjugate points of the involution determined by B, $A_3(0, x_3)$ and C, $A_4(\infty, x_4)$ is

$$x_1x_2-(x_1+x_2)x_4+x_3x_4=0$$

(6) In each case the double points of the involution are given by making $x_1=x_2=x$, where x stands for the ratio of either of the required double points

(7) If $A_a(a)$, $A_b(a')$ be the conjugates of C and B in an involution, the double points of the involution are given by

$$x^2-2xa'+aa'=0$$

EXAMPLES

Involution in connexion with its base

(1) If C' and C'' are the common harmonic conjugates of AB and $A'B'$, then AA', BB', CC'' are conjugate points of an involution

In this case
$$(ABCC'')=-1=(A'B'C''C'),$$
$$(ABCC'')=(AB'C''C),$$
AA', BB', CC'' form an involution

(2) If A and A' be any fixed pair of conjugate points of an involution and P and P' any other pair of conjugate points of the involution, then the common harmonic conjugates of AP and $A'P'$ and of AP' and $A'P$ form, for different pairs of conjugate points P and P', an involution of which A and A' are a pair of conjugate points

(3) AB, AB, CD are an involution on a straight line and $(AB, CD)=-1$. If $(PA, AB)=-1$, prove that $(PB, CD)=-1$

(4) If A, B, C, X, Y, P are six points on a straight line such that
$$(ABCP)=(ABCX)\ (ABCY),$$
prove that AB, XY, CP are three pairs of conjugate points of an involution

(5) If LL', NN' be two pairs of conjugate points of an involution of which P and Q are the double points, then

$$\frac{LP}{NP}\cdot\frac{LQ}{NQ}=\frac{LN'}{NN'}\cdot\frac{LL'}{NL'},$$

and
$$(LNPN')=(LNQN')$$

(6) PP', QQ', RR' are points on a straight line such that PP', QR, $Q'R'$ and also PP', QR', $Q'R$ are pairs of conjugate points of an involution Prove that P, P' are the double points of the involution of which QQ, RR' are pairs of conjugate points

In this case $(PP'QQ)=(P'PRR')$ and $(PP'QQ)=(P'PR'R)$,

$$(P'PRR')=(P'PR'R)=-1=(PP'QQ)$$

(7) If E and G are any two fixed points and A and A' are any pair of conjugate points of a given involution, then the harmonic conjugates of E and of G with respect to AA' describe two projective ranges

Let E' and G' be the harmonic conjugates of E and G with respect to AA' and M and N the double points of the involution Then

$$(AA'MN)=-1\,,\quad (AA'EE')=-1\,,\quad (AA'GG')=-1$$

Therefore MN, EE', GG' are pairs of conjugate points of an involution

Therefore $$(NMEG)=(MNE'G'),$$

and the ranges described by E and G' are projective This proof assumes N and M to be real

(8) The harmonic conjugates of a variable point with regard to four given pairs of conjugate points of an involution have a constant anharmonic ratio

(9) Q and Q' are the harmonic conjugates of any point P with respect to AA' and BB' respectively Prove

(a) if $AA'BB'$ is harmonic, that AA', BB', QQ' form an involution

(b) if AA', BB', QQ' form an involution, then $AA'BB'$ is harmonic, and if R and R' be the harmonic conjugates of Q' and Q with respect to AA' and BB' respectively, then R and R' coincide

(c) if another point P' determines Q and Q' as its harmonic conjugates with respect to BB' and AA' respectively, then AA', BB', PP' form an involution

If Q and Q' be taken as origin, so that the ratio of Q is 0 and that of Q' ∞, let the ratios of P, P', A, A', B, B' be p, p', a, a', b, b' respectively

Since $QPAA'$ is harmonic, $$p(a+a')=2aa' \qquad (1)$$

Since $Q'PBB'$ is harmonic, $$(b+b')=2p \qquad (2)$$

(a) If $AA'BB'$ is harmonic,

$$(a+a')(b+b')=2aa'+2bb'=4a'a \text{ from (1) and (2)}$$

$$aa'=bb'$$

AA', BB', QQ' form an involution

(b) If AA', BB', QQ' form an involution, $aa'=bb'$

from (1) and (2), $(a+a')(b+b')=4aa'=2aa'+2bb'$

Let r and r' be the ratios of R and R' Then since $(QR'BB')$ and $(Q'RAA')$ are harmonic $r'(b+b')=2bb'$ and $(a+a')=2r$

$$r'(a+a')(b+b')=4rbb'$$

$$r'aa'=rbb',\qquad r=r'$$

(c) In this case $$(a+a')=2p'$$

Therefore $$pp'=\frac{(a+a')(b+b')}{4}=aa'$$

Hence P and P' are a pair of conjugate points of the involution in part (a)

(10) If AA', BB', CC' are three pairs of conjugate points of an involution and AA', BB' form a harmonic range, then the harmonic conjugate of C with respect to AA' and the harmonic conjugate of C' with respect to BB' coincide

(11) If X_1Y_1, X_2Y_2, X_3Y_3 be three segments dividing AB harmonically and C_1, C_2, C_3 be their middle points, and P be any collinear point, then

$$(PX_1^2+PY_1^2)\,C_2C_3+(PX_2^2+PY_2^2)\,C_3C_1+(PX_3^2+PY_3^2)\,C_1C_2$$

is constant for all positions of P

The given expression equals

$$\begin{vmatrix} 1 & 1 & 1 \\ PC_1 & PC_2 & PC_3 \\ PX_1^2+PY_1^2 & PX_2^2+PY_2^2 & PX_3^2+PY_3^2 \end{vmatrix}$$

$$=\tfrac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ PX_1+PY_1 & PX_2+PY_2 & PX_3+PY_3 \\ (PX_1+PY_1)^2 & (PX_2+PY_2)^2 & (PX_3+PY_3)^2 \end{vmatrix}$$

$$-\begin{vmatrix} 1 & 1 & 1 \\ PX_1+PY & PX_2+PY_2 & PX_3+PY_3 \\ PX_1\ \ PY_1 & PX_2\ \ PY_2 & PX_3\ \ PY_3 \end{vmatrix}$$

The second of these determinants is zero, since relation (2) Art 61 holds when $x_1, x_2,$ are the distances of three pairs of conjugate points of an involution from any origin P

Hence the expression $=\tfrac{1}{2}\Pi(PX_1+PY_1-PX_2-PY_2)$

$=\tfrac{1}{2}\Pi(X_2X_1+Y_2Y_1)$, which is independent of P

(12) If X_1X_2 and Y_1Y_2 be any two pairs of conjugate points of an involution and $\dfrac{\{(A_1Y_1Y_2B_1)-(A_1Y_1X_2B_1)\}^2}{(X_1X_2Y_1Y_2)}$ is constant, prove that for different positions of A_1, B_1 describes one or other of two ranges projective with that described by A_1

(13) If three pairs of conjugate rays of an involution pencil contain equal angles, prove that all pairs of conjugate rays of the pencil are at right angles

Involution in connexion with the quadrangle and triangle

(14) The perpendiculars drawn from any point to the pairs of opposite sides of a complete quadrangle form a pencil in involution

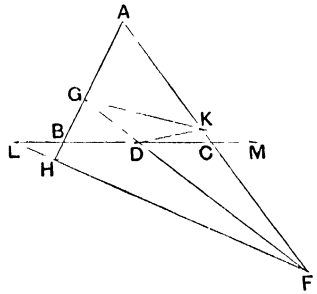
Consider the points in which the sides of the quadrangle meet the line at infinity. These points form an involution Hence the pencil formed by drawing through any point lines parallel to the sides of the quadrangle is an involution pencil Hence the pencil formed by the perpendiculars from any point on the sides is also an involution pencil, the angles between corresponding rays of the two pencils being equal

(15) If three parallel lines are drawn through the diagonal points of a quadrangle, show that their harmonic conjugates, with respect to the pairs of sides of the quadrangle, which meet at these diagonal points, are concurrent See Art 57

(16) If AA', BB', CC' be the pairs of opposite vertices of a quadrilateral and one pair of opposite sides AB and $A'B'$ subtend equal angles at a point P, at which the vertices subtend an involution with real double rays, then every other pair of opposite sides subtend equal angles at P

Join A, B, C, A', B', C' to S by a, b, c, a', b', c' Then aa', bb', cc' are pairs of conjugate rays of an involution Since $APB = A'PB'$ the double rays are the common bisectors of the angles APA' and BPB' Since these are at right angles they must be the bisectors of the angle CPC' Hence the theorem is true

(17) Prove that if ABC be a triangle and D the middle point of the base BC and HDK be a straight line cutting AB in H and AC' in K, and FDG be a straight line cutting AB in G and AC' in F and HF and GK cut BC' in L and M, then D is the middle point of LM



Consider the quadrangle $GKFH$ LM, BC are conjugate points of an involution of which D is a double point D bisects the distance between BC' Therefore

$$(BCD\infty) = -1$$

Therefore $(LMD\infty) = -1$

Therefore D bisects LM

(18) Three pairs of lines are drawn through the vertices A, B, C of a triangle to meet in two points P and Q, show that any line cut in involution by the three pairs of lines passes through a fixed point

(19) If the sides BC, CA, AB of a triangle are cut respectively by a transversal in P', Q', R', and if P, Q, R, which are points lying on the same transversal, form with P', Q', R' three conjugate pairs of an involution, prove that AP, BQ, CR are concurrent

This is the converse of the involution property of a quadrangle

(20) Perpendiculars are drawn from the vertices of a triangle and from three collinear points on the sides to a straight line lying in the plane of the triangle Prove that the perpendiculars meet the straight line in a system of points in involution

(21) Through any point O in the plane of a triangle ABC are drawn OA', OB' OC' bisecting the supplements of the angles BOC, COA, AOB, and meeting BC, CA, AB in A', B', C' respectively Show that A', B', C' are collinear and that the six lines OA, OB, OC, OA', OB', OC' form a pencil in involution

Since $\frac{AC'}{BC'} = \frac{OA}{OB}$ the first part follows from Menelaus' Theorem and the second from the involution property of the quadrilateral

(22) A, B, C, D, E are five coplanar points Construct a line parallel to AB which shall be cut by AC and BC, AD and BD, AE and BE in pairs of points of an involution

(23) Find the double points of an involution in the case when the transversal, cutting a quadrangle, passes through the intersection of a pair of opposite sides of the quadrangle

Involution in connexion with the circle

(24) Prove that if circles be drawn through pairs of points of an involution and through a common external point they will meet in a second fixed point

(25) Show that if four circles cut a line in pairs of corresponding points of an involution they have a common orthogonal circle

Let the circles meet the base in $A A'$, BB', CC', DD' Then if O be the centre of the involution, $OA \cdot OA' = OB \cdot OB' = K^2$ Hence a circle with centre O and radius K will cut the circles orthogonally If the double points of the involution are imaginary, the circle will be imaginary

(26) Find in an overlapping involution the pair of conjugates which are harmonic conjugates of a given pair of conjugates

Let $A A'$ be the pair of conjugates whose harmonic conjugates are required and BB' any other pair of conjugates Describe circles on $A A'$ and BB' as diameters to intersect at O Then AOA' is a right angle The bisectors of the angle AOA' will meet the base in the required points

(27) If four circles pass through a given point O, the connectors of this point O to their six other points of intersection form an involution pencil

The lines through the six points of intersection of the circles, perpendicular to the lines connecting these points to O, pass three by three through four points A, B, C, D one on each of the circles These six lines perpendicular to the connectors to O therefore form a quadrangle, and the lines through O are therefore perpendicular to the sides of a quadrangle and form an involution pencil

(28) If four circles touch a given line then six common tangents meet the given line in three pairs of conjugate points of an involution

This is the correlative of Example 27

(29) The three circles which can be described on the three diagonals of a quadrilateral as diameters intersect in a pair of points

Let $A A'$, BB', CC' be the ends of the diagonals Describe circles on $A A'$, BB' as diameters Let them intersect in P and Q Then PA, PA' and PB, PB' are mutually at right angles Hence, the involution formed by joining P to $A A'$, BB', CC' has therefore two pairs of conjugate rays at right angles Therefore the third pair PC and PC' are at right angles Therefore the circle described on CC' as diameter will pass through P Similarly it passes through Q

Since the centres of three such circles are collinear, the middle points of the diagonal are collinear

(30) The orthocentres of the four triangles, which can be formed by the four sides of a quadrilateral, lie on the line joining the points of intersection of the three circles described on its diagonals as diameters

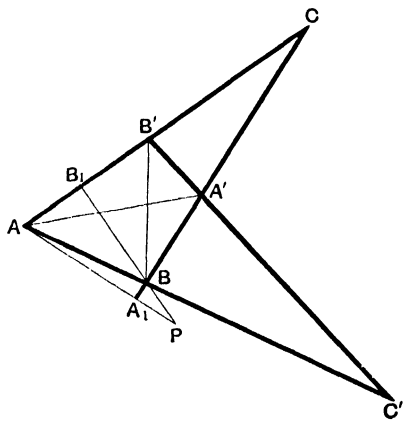
Let AA', BB', CC' be the diagonals of the quadrilateral

Let the circle on AA' meet CB in A_1 and the circle on BB' meet CA in B_1

Then AA_1C and AB_1B are right angles and AA_1 and BB_1 meet in P the orthocentre of ABC'

Since AA_1BB_1 are concyclic, $PA \cdot PA_1 = PB \cdot PB_1$

Therefore the tangents from P to the circles on AA' and BB' are equal Hence as will be proved, Art 83, P is on the line joining the points of intersection of the circles



CHAPTER IX

COPLANAR FIGURES PROBLEMS OF THE FIRST DEGREE

62 In any plane two figures may be either

(1) identical,
(2) in plane perspective,
(3) projective,
or (4) not projective

In case (2) every pair of corresponding points are collinear with some point S and every pair of corresponding lines intersect on some line s

It is not sufficient that all pairs of corresponding points should be collinear with some point S for, if it were, any point might be taken as S and lines might be drawn through S to meet the lines in the two figures Any pairs of points upon these lines might then be taken as pairs of corresponding points, but although all pairs of corresponding points would thus be collinear with S, the figures would not be in perspective with S as centre of perspective, for in this case the points of intersection of corresponding lines would not be corresponding points Also the lines joining corresponding points would not intersect on a given line s In this connexion, Art 63 should however be noticed

In case (3) the term projective implies that the figures can be deduced from each other by a series of projections It was shown in Art 21 that the characteristics of two such figures are

(a) To every point corresponds a point and to collinear points correspond collinear points

(b) The anharmonic ratio of four collinear points is the same as the anharmonic ratio of the corresponding points

(c) To every line corresponds a line and to concurrent lines correspond concurrent lines

(*d*) The anharmonic ratio of four concurrent lines is the same as the anharmonic ratio of the corresponding lines

(*e*) The line joining two points corresponds to the line joining the corresponding points The point of intersection of two lines corresponds to the point of intersection of the corresponding lines

In this chapter the difference between the above cases will be considered and also the displacements in one plane of a figure by which it may be brought into one of the above relationships with another figure

63 *Given two figures of points viz A, B, C, and A′, B′, C′, situated in the same plane, in which AA′, BB′, CC′, are regarded as corresponding points, and the lines joining pairs of corresponding points as corresponding lines, then if all pairs of corresponding lines intersect in collinear points, the two figures are in plane perspective*

Consider any three pairs of corresponding points A, B, C and A', B', C' Since

$$BC\ B'C',\ CA\ C'A',\ AB\ A'B$$

are collinear, therefore AA', BB', CC' are concurrent (Art 13 (*a*))

Similarly, if D and D' be any other pair of corresponding points, AA', BB', DD' are concurrent Therefore not only do all pairs of corresponding lines intersect on a given line but all pairs of corresponding points are collinear with a given point, viz AA' BB' Hence the figures are in plane perspective

Given two figures of lines, viz a, b, c and a′, b′, c′, situated in the same plane, in which aa′, bb′, cc′, are regarded as corresponding lines, and the points of intersection of pairs of corresponding lines as corresponding points, then if all pairs of corresponding points are collinear with the same point, the two figures are in plane perspective

Consider any three pairs of corresponding lines a, b, c and a', b', c' Since

$$bc\ b'c',\ ca\ c'a,\ ab\ a'b'$$

are concurrent, therefore aa', bb', cc' are collinear (Art 13 (*a*))

Similarly, if d and d' be any other pair of corresponding lines, aa', bb', dd' are collinear Therefore not only are all pairs of corresponding points collinear with a given point but all pairs of corresponding lines intersect on a given line, viz aa' bb' Hence the figures are in plane perspective

Hence, *if in two projective figures corresponding lines intersect in collinear points or corresponding lines are concurrent at a fixed point, the figures are in plane perspective*

64 *Given any two lines k and l intersecting in E, any two lines k' and l' intersecting in E', S any point on EE', and AA', BB', CC', DD' pairs of points collinear with S, A and B on k, C and D on l, A' and B' on k', C' and D' on l', then two figures can be constructed in plane perspective in which the points A, B, C, D, E correspond to the points A', B', C', D', E' and the lines k and l to the lines k' and l'*

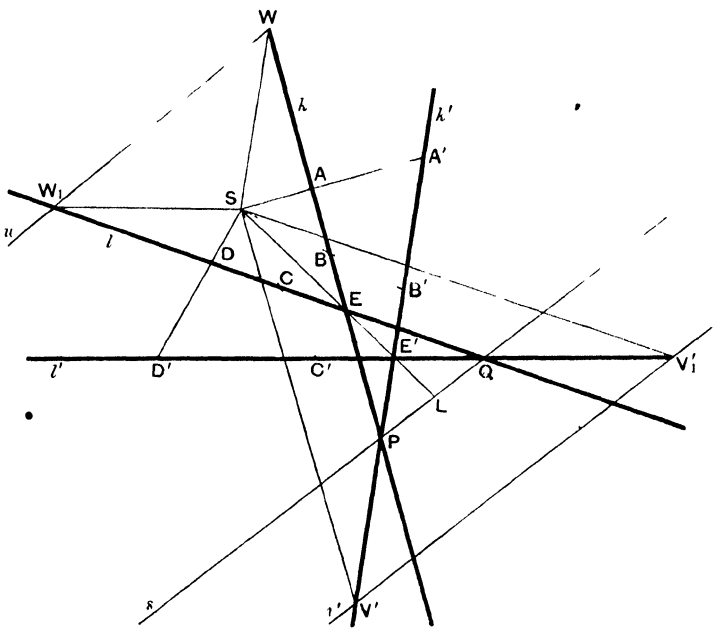
Let k and k' meet in P and l and l' in Q Let PQ be s Then it will be shown that the perspective which has S for centre, s for axis, and E and E' for corresponding points meets the requirements of the case

(1) The lines k and k' which meet on s and pass through E and E' are corresponding lines Similarly l and l' are corresponding lines

(2) Hence A and A', B and B', C and C', D and D' which are on these lines and are collinear with S are corresponding points

The point G' corresponding to any point G may be constructed as follows Join G to S to meet s in H Then G' is such that $(SHGG') = (SLEE')$ where SEE' meets s in L (Art 28)

The line g' corresponding to any given line g may be constructed by joining gs to S by a line h and taking g' such that $(shgg') = (slkk')$ where l is the connector of kk' to S (Art 28)



In connexion with the above it is seen that

(1) if a line SFF' meet k and k' in F and F', the points F and F' may be determined one from the other as corresponding points of the ranges determined by EBA and $E'B'A'$ respectively,

(2) if W and V' are the vanishing points of these ranges, parallels through W and V' to k' and k will intersect in S,

(3) if W_1 and V_1' are the vanishing points of the ranges on l and l' parallels through W_1 and V_1' to l' and l will intersect in S,

(4) the three lines s, WW_1 (w) and $V'V_1'$ (v) are parallel

65 To place any two given Quadrangles $ABCD$ and $A'B'C'D'$ in Plane Perspective (Figure, page 113)

Let AB be k, $A'B'$, k', CD, l and $C'D'$, l' Let kl and $k'l'$ be E and E' Regarding EBA and $E'B'A'$ as determining two projective ranges on k and k' find the vanishing points W and V'

Similarly, regarding ECD and $E'C'D'$ as determining two projective ranges on l and l' find the vanishing points W_1 and V_1'

Place the second figure so that $V'V_1'$ is parallel to WW_1 Through W and W_1 draw parallels to k' and l' meeting in S Through V' and V_1' draw parallels to k and l meeting at S' Keeping WW_1 and $V'V_1'$ parallel move the second figure till S' coincides with S

Consider the two quadrangles $SV'E'V_1'$ and $EWSW_1$ Five pairs of the lines joining these points are parallel, viz

$$\begin{array}{ll} SV' & EW, \\ V'E' & WS, \\ E'V_1' & SW_1, \\ V_1'S & W_1E, \\ V'V_1' & WW_1 \end{array}$$

Hence the lines joining the remaining pair of points are parallel, *i e* SE and SE' Therefore S, E, E' are collinear (Art 13 (*b*), page 21)

In the ranges on k and k' three pairs of corresponding points are collinear with S, viz EE', $W\infty'$ and $V'\infty$ Hence the ranges are in perspective at S Similarly, the ranges on l and l' are in perspective

Hence by the last article the quadrangles are in perspective

The above affords a solution of the following problems

Draw two transversals on which the sides of two given quadrangles determine equal corresponding segments

Determine two points at which the corresponding vertices of two given quadrilaterals determine equal angles

66 Properties of Figures projective or in Plane Perspective

The figures constructed from $ABCD$ and $A'B'C'D'$ in Art 64 are in perspective If one of them be displaced in its plane the two figures represent the most general case of two figures which are projective for they comply with the conditions laid down in Case (3), Art 62 In fact, the figures may be constructed without reference to S or s Corresponding points on k and k' may be constructed as corresponding points of the ranges determined by EAB and $E'A'B'$ Similarly, pairs of corresponding points may be constructed on l and l' Any line in the first figure will meet k and l in a pair of points and the corresponding line is the connector of the corresponding points on k' and l' The point corresponding to a given point in one figure may be found as the point of intersection of a pair of lines corresponding to any pair of lines through the given point Hence

Any two quadrangles determine two projectively related figures, and conversely,

The relationship of any two projective figures is completely determined if two corresponding quadrangles are given

Hence from Art 65 it is seen that

Any two projective figures may be superposed so as to be in plane perspective

Art 65 also enables us to

Project any straight line in a given figure into the line at infinity and any three points into a triangle of given dimensions

For in the figure DA may be projected into the line at infinity and CBE into any triangle

If the points A, B, C, D coincide with the points A', B', C', D' the two figures must completely coincide Therefore

If two projective figures have four non-collinear self-corresponding points the figures must completely coincide

If two projective figures have three collinear self-corresponding points they must be in perspective

In Figure, page 113, suppose E', B', A' to coincide with E, B, A respectively Then k' coincides with k and every point on this line corresponds

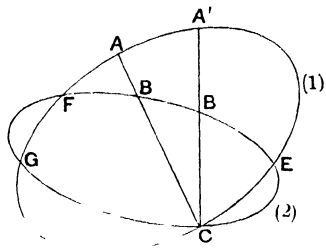
to itself This line is the axis of perspective s S is determined as the point of intersection of the lines DD' and CC' joining two pairs of corresponding points on the two corresponding lines l and l'

67 *Two projective coplanar figures have in general three double or self-corresponding points*

This proof involves the definition of a conic given, Art 92

If A, A' are a pair of fixed corresponding points and P, P' be any other pair of corresponding points, AP and $A'P'$ are a pair of corresponding rays of two pencils, and, for different positions of P and P', the locus of Q their point of intersection is a conic (1) through A and A' and every point on this conic is such that, if it be joined to A and A', the rays so formed are corresponding lines of the two figures

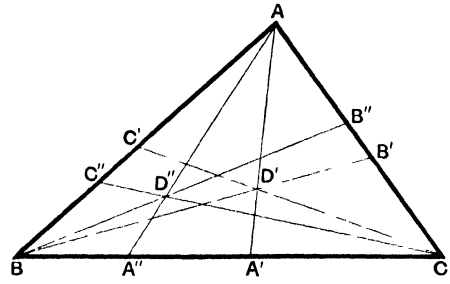
Take AA' and BB' two fixed pairs of corresponding points and let AB and $A'B'$ meet at C Generally C is not a self-corresponding point Describe by the method explained above the two conics (1) and (2) through AA' and through BB' respectively, which are such that the lines obtained by joining points on them to A, A' and B, B' are corresponding lines These conics will intersect in C' and in three other points E, F and G Since the lines AE, BE correspond to $A'E$ and $B'E$, the point E in the two figures corresponds to itself Similarly F and G are self-corresponding points Hence there are three self-corresponding or double points



Two of the three points E, F, G may be a pair of conjugate imaginary points

Particular case of the preceding

If the conic (1) break up into a pair of straight lines AA' and s, let AA' and PP' meet at S Then AP and $A'P'$ will meet at some point U on s Let R be any point on AP and R' the corresponding point, which will lie on $A'P'$ Hence since $(APRU)=(A'P'R'U)$, RR' will pass through S Also every point on s will correspond to itself and therefore every pair of corresponding lines must intersect on s Hence by Art 62 the figures are in plane perspective S and s being the centre and axis of perspective



Analytical relation between two superposed projective figures

Let A, B, C be the three self corresponding points of the two figures and let D and D' be two corresponding points If AD, BD, CD' meet the opposite sides of ABC in A', B', C', the ratios of A', B', C' determine the position of D' and may be termed its coordinates

Let x_0', y_0', z_0' be the coordinates of D' referred to ABC, and x_0'', y_0'', z_0'' the coordinates of D'' also referred to ABC

Let P' and P'' be any two other corresponding points of the figures whose coordinates are $x'y'z'$ and $x''y''z''$

Then
$$\frac{x'}{x_0'} = (A\ \ BCP'D') = (A\ \ BCP''D'') = \frac{x''}{x_0''},$$

$$x'' = x' \frac{x_0''}{x_0'} = x'\ k \quad \text{(suppose)},$$

so
$$y'' = y' \frac{y_0''}{y_0'} = y'\ l \qquad ,,$$

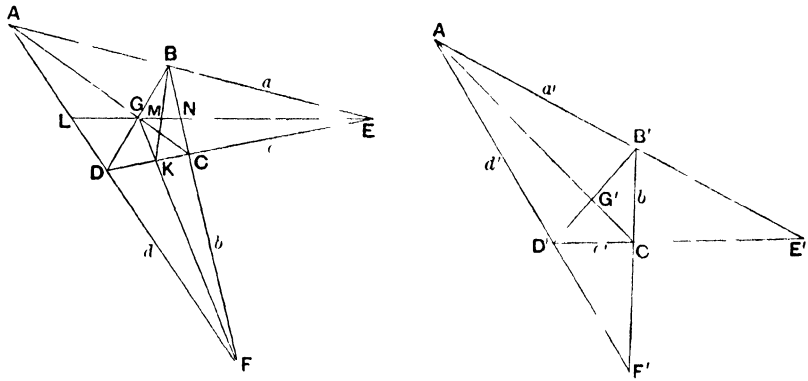
$$z'' = z' \frac{z_0''}{z_0'} = z'\ m \qquad ,,$$

where
$$klm = 1 \tag{1}$$

Hence the equation of the projection of any curve can be found at once By varying k, l, m subject to condition (1) different projective figures are obtained

68 The Quadrangle

Certain properties of a quadrangle follow at once from the fact that any quadrangle (or quadrilateral) is projective with any other quadrangle (or quadrilateral), Art 65 and Art 27



(a) Consider any two quadrangles $ABCD$ and $A'B'C'D'$ or any two quadrilaterals $abcd$ and $a'b'c'd'$

These are projective

Hence any range or pencil in one figure must have the same anharmonic ratio as the corresponding range (or pencil) in the other figure But the second figure may be looked upon as any quadrangle or quadrilateral Hence the value of the anharmonic ratio in question must be independent of the relative positions of A, B, C, D or those of a, b, c, d

Thus in the figure the anharmonic ratio of the range $DCKE$ must not depend on the positions of A, B, C, D, it is -1

Join K to B to meet GE in M and let EG meet d and b in L and N. Then $(LGME)$ must have some definite numerical value. This may be proved as follows

$$(LGME) = 1 - (EGML) = 1 - \frac{(EGMN)}{(EGLN)}$$

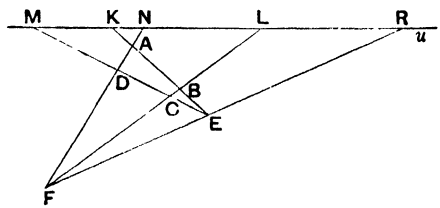
$$= 1 - \frac{1 - (KMGN)}{(EGLN)} = 1 - \frac{1 - (-1)}{(-1)} = 3$$

Hence *the anharmonic ratio of any range or pencil derived by point and line construction from four points or four lines is independent of the position of these points and lines*

(*b*) If any two points N and L be taken on any pair of sides AD and BC of a quadrangle $ABCD$, and points N', L' on the sides $A'D'$ and $B'C'$ of any other quadrangle $A'B'C'D'$, such that

$$(ADFN) = (A'D'F'N')$$

and $$(BCFL) = (B'C'F'L'),$$



and the lines NL and $N'L'$ be u and u', then the quadrangle $ABCD$ and the line u are projective with the quadrangle $A'B'C'D'$ and the line u'

Consequently if u and u' meet the other sides of the quadrangle in M, K, and M', K' respectively,

$$(ABEK) = (A'B'E'K'), \quad (DCEM) = (D'C'E'M'),$$

and $$(KLMN) = (K'L'M'N')$$

Also since A', B', C', D' may be any four points, all the anharmonic ratios of ranges in the figure must be uniquely expressible in terms of $(FDAN)$ and $(FCBL)$ the two anharmonic ratios, which determine the position of u

Thus let FE meet u in R and let $(FDAN) = m$ and $(FCBL) = n$

Projecting from E on u,

$$m = (FDAN) = (RMKN),$$

$$n = (FCBL) = (RMKL),$$

$$(RMLN) = \frac{m}{n} \quad \text{and} \quad (RKLN) = \frac{1-m}{1-n}$$

But $$(LNKM) = \frac{(LNRM)}{(LNRK)} = \frac{(RMLN)}{(RKLN)} = \frac{m(1-n)}{n(1-m)}$$

Projecting from F on u,

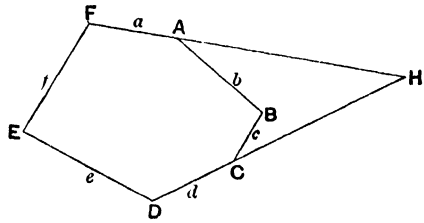
$$(ECDM) = (RLNM) = \frac{1}{1-(RMLN)} = \frac{n}{n-m},$$

and $$(EBAK) = (RLNK) = \frac{1}{1-(RKLN)} = \frac{1-n}{m-n}$$

In a similar manner, if another line be drawn to meet AD and CB in N_1 and L_1 and $(FDAN_1)$ and $(FCBL_1)$ be m_1 and n_1, the anharmonic ratios of all ranges and pencils in the resulting figure may be expressed in terms of m, n, m_1, and n_1

(c) *If a, b, c, d, e, f be the sides of a hexagon and three pairs of quadrilaterals abcd, defa, bcde, efab, and cdef, fabc be formed out of the sides, then the points of intersection of the connectors of the middle points of the diagonals of these quadrilaterals taken in pairs are collinear*

Let *ab*, *bc*, *cd*, *de*, *ef*, *fa* and *ad* be *A*, *B*, *C*, *D*, *E*, *F* and *H*



Then for a point *P* inside the hexagon *ABCDEF* on the connector of the middle point of the diagonals of *abcd*,

$$PAH - PBC = PHC - PAB$$

(Addendum 7),

and for a point *P* on the connector of the middle points of the diagonals of *adef*,

$$PFH + PED = PHD + PEF \text{ (Addendum 7)}$$

If *P* be the point of intersection of these lines, then

$$PFH - PAH + PED + PBC = PHD - PHC + PEF + PAB$$

Therefore $$PFA + PED + PBC = PDC + PEF + PAB$$

Hence the point of intersection is on the straight line (Addendum 7) which is the locus of points at which the sum of the areas subtended by the two groups of alternate sides of the hexagon are equal The symmetry of the result proves that the two other points lie on the same line

69 *To draw a straight line to cut five given straight lines in such a way that a point can be found at which the four segments subtend given angles*

Let the five lines be *a*, *b*, *c*, *d*, *e* and let α, β, γ, δ be the angles to be subtended by the segments

Take any point *P* on *d* and through *P* draw a line *l* to cut *abc* in three points which with *P* make a range of anharmonic ratio, the same as that of a pencil containing angles α, β, γ (Example 6, Chapter VI)

Describe a conic to touch *abcd* and *l* (Art 93) All tangents to this conic cut *abcd* in a range of the same anharmonic ratio as that in which it is cut by *l* (Art 93)

Take any point *Q* on *e* and through *Q* draw a line *m* to cut *abc* in three points which with *Q* make a range of the same anharmonic ratio as that of the pencil containing the angles α, β, $\gamma+\delta$

Describe a conic to touch *abce* and *m* All tangents to this conic cut *abce* in a range of the same anharmonic ratio as that in which it is cut by *m*

Now the two conics have three common tangents *a*, *b*, *c* Therefore they have a fourth common tangent *s* Draw *s* Let it be cut by *a*, *b*, *c*, *d*, *e* in *A*, *B*, *C*, *D*, *E* On *AB* and *BC* describe circles containing angles α and β Let them intersect in *S* Then *AB* and *BC* subtend angles α and β at *S*

Since the anharmonic ratio *ABCD* is the same as that of a pencil α, β, γ, *CD* subtends an angle γ at *S* Similarly *CE* subtends an angle $\gamma+\delta$ at *S* Therefore *DE* subtends an angle δ at *S* Therefore *s* and *S* are the required line and point

From the preceding the solution of the following problem can be obtained

To construct the perspective of four points so that it may be the same as the figure formed by four other given points

Let five of the lines joining the first four points be a, b, c, d, e, and those joining the corresponding points of the second set be a', b', c', d', e' Let the angles between the latter be α, β, γ, δ

Find S and s as in the preceding for the lines a, b, c, d, e and the angles α, β, γ, δ If S and s be taken as centre of perspective and vanishing line (Art 23) the perspective figure will be similar to that formed by $a'b'c'd'e'$ By choosing the axis of perspective parallel to S at a proper distance from s the perspective figure can be made of the right linear dimensions (Art 24)

70 The Correlative of Art 64 is as follows

Given any two points K and L connected by a line e and any two points K' and L' connected by e', s any line through ee', and aa', bb', cc', dd' pairs of lines intersecting on s, a and b passing through K, c and d passing through L, a' and b' passing through K', c' and d' passing through L, then two figures can be constructed in plane perspective in which the lines a, b, c, d, e correspond to the lines a', b', c', d', e' and the points K and L to the points K' and L'

Denote the lines KK' by p and LL' by q Let pq be S Then it will be shown that the perspective which has s for axis and S for centre and e and e' for corresponding lines meets our requirements

(1) The points K and K' which are collinear with S and lie on e and e' are corresponding points Similarly L and L' are corresponding points

(2) Hence a and a', b and b', c and c', d and d' which pass through these points and intersect on s are corresponding lines

The line corresponding to any line g may be constructed as follows Join gs to S by h Then g' is such that $(shgg')=(slee')$, where l is the line joining the point see' to S

The point G in one figure corresponding to the point G' in the other may be determined as the point of intersection of two lines corresponding to two lines through G'

PROBLEMS OF THE FIRST DEGREE

71 Under this head are included theorems and problems

(1) concerning loci the solutions of which are straight lines,

(2) the correlative problems in which the envelopes of straight lines are points,

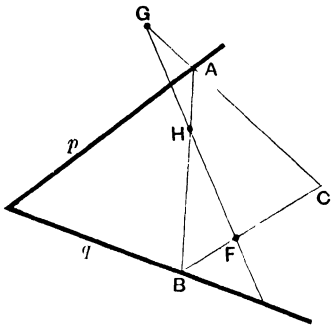
(3) problems which have unique solutions

The general method of solving questions coming under the heading (1) is to obtain the points of the required locus as the points of intersection of corresponding rays of two projective pencils Then if it can be shown that the pencils are also in perspective the required locus is a straight line

Problems coming under the heading (2) are treated by means of the correlative method Two projective ranges are obtained and if it can be shown that they are also in perspective, it is known that the lines joining corresponding points all pass through a fixed point, which is the envelope of the lines joining pairs of corresponding points

1 The vertices A and B of a triangle ABC move along two fixed lines p and q and the sides respectively pass through three fixed collinear points H, G and F

Find the locus of C



For different positions of the triangle ABC, the pencil formed by FC is projective with the range B and is therefore projective with the range A and with the pencil GC Hence the pencils formed by FC and GC are projective But if FC lies along FG then GC lies along GF Hence GF is a self corresponding ray of the two pencils and therefore the pencils are in perspective Hence the locus of C is a straight line

2 Find the locus of C in the above question if G and F are collinear with pq instead of with H

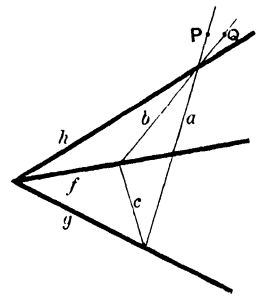
As in the last question the pencils described by FC and GC are projective

They are in perspective because the line joining F, G, pq is a self corresponding ray

3 Two fixed points A and B are taken on two fixed lines which intersect

The sides a and b of a triangle abc pass through two fixed points P and Q and the vertices respectively move along three fixed concurrent lines h, g and f

Find the envelope of c



For different positions of the triangle abc, the range described by fc is projective with the pencil b and is therefore projective with the pencil a and with the range gc Hence the ranges formed by fc and gc are projective But if fc lies at fq then gc lies at qf Hence qf is a self-corresponding point of the two ranges and therefore the ranges are in perspective Hence the envelope of c is a point

Find the envelope of c in the above question if g and f are concurrent with PQ instead of with h

As in the last question the ranges described by fc and gc are projective

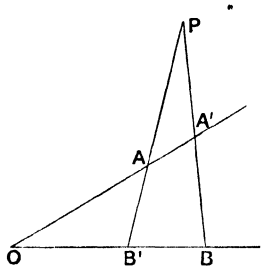
They are in perspective because the point of intersection of f, g, PQ is a self corresponding point

Two fixed lines a and b are taken passing through two fixed points situated

at O and on these lines variable points A' and B' are taken such that

$$OA+OB=OA'+OB'$$

Find the locus of the point of intersection of AB' and BA'



Since $OA'+OB'$ is constant the distance AA' equals the distance BB' and therefore the ranges formed by A' and B' are projective Hence the pencils BA' and AB' are projective

When the point A' coincides with A, the point B coincides with B', therefore the line AB is a self-corresponding ray of the two pencils

Therefore the pencils are in perspective and the locus of P is a straight line

4 In the preceding if

$$OA'-OB'=OA-OB$$

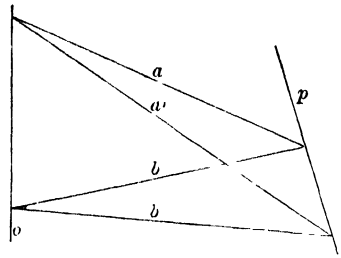
find the locus of the intersection of AB and $A'B'$

As in the previous case the pencils AB' and BA' are projective and in perspective

on a line o and through these points variable lines a' and b' are drawn such that

$$\widehat{oa}+\widehat{ob}=\widehat{oa'}+\widehat{ob'}$$

Find the envelope of the lines connecting the points ab' and ba'



Since $\widehat{oa'}+\widehat{ob'}$ is constant the angle $\widehat{aa'}$ equals the angle $\widehat{bb'}$ and therefore the pencils formed by a' and b' are projective Hence the ranges ba' and ab' are projective

When the line a' coincides with a, the ray b coincides with b', therefore the point ab is a self corresponding point of the two ranges

Therefore the ranges are in perspective and the envelope of p is a point

In the preceding if

$$\widehat{oa'}-\widehat{ob'}=\widehat{oa}-\widehat{ob}$$

find the envelope of the line joining ab' to $a'b$

As in the previous case the ranges ab' and ba' are projective and in perspective

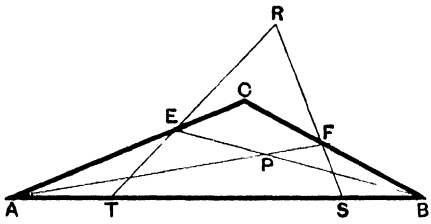
5 On the base AB of a triangle ABC two points T and S are taken such that AT and BS are in a given ratio Through T and S lines are drawn to a fixed point R so as to meet AC and BC in E and F Find the locus of the point of intersection of EB and FA

Because $AT=K\ SB$ the ranges described by T and S are projective, and as these are projective, the ranges described by E and F are projective

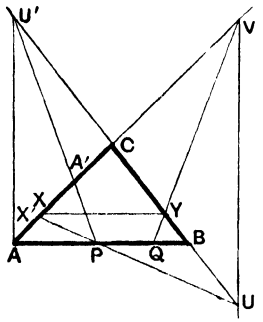
Hence the pencils BE and AF are projective

When T coincides with A, S coincides with B and E and F coincide with A and

B Hence the line AB is a self corresponding ray of the pencils described by BE and AF Therefore the pencils are in perspective and the locus of P is a straight line



6 P, Q are given points on the base AB of a triangle ABC Find points XY on the sides AC, BC or their productions, such that if PX meet BC in U and QY meet AC in V, XY shall be parallel, and UV perpendicular, to AB



Take any point X on AC Draw XY parallel to AB to meet CB in Y Join YQ meeting AC in V Draw VU perpendicular to AB to meet CB in U Join UP to meet AC in X' Then the ranges X and X' are projective

Take X at A, Y is then at B, V is at A, U is at U', X' is at A' Take X at C, Y is then at C, V is at C, U is at C, X' is at C

Therefore C is a self-corresponding point of the superposed ranges The required point Z the other self-corresponding point is found from the relation

$$(AC'ZX) = (A'CZX'),$$

whence

$$\frac{CZ}{C'Z}\cdot\frac{AX}{CX} = \frac{A'Z}{CZ}\cdot\frac{A'X'}{CX'}$$

or

$$\frac{AZ}{A'Z} = \frac{AX}{CX}\cdot\frac{CX'}{A'X'}$$

CHAPTER X

PROJECTIVE FORMS IN RELATION TO THE CIRCLE —ANHARMONIC PROPERTY POLE AND POLAR CIRCLES IN SELF-PERSPECTIVE

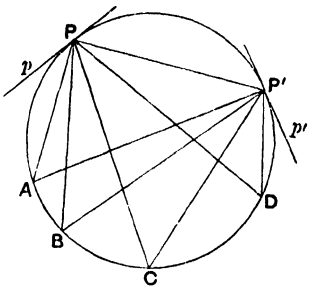
72 The Circle

Introductory A circle is usually defined as a curve such that the distance of every point on it from a fixed point is constant It immediately follows (see Art 73) that it is the locus of the points of intersection of corresponding rays of two directly equal pencils In Art 92 a conic is defined as the locus of the points of intersection of pairs of corresponding rays of two projective pencils Hence a circle is a particular case of a conic Also since equal pencils are projected into projective pencils, the projection of a circle is a conic and since the process of projection is reversible a conic may be projected into a circle It would be possible to consider the projective properties of a conic and hence infer those of a circle One method, however, of approaching the subject is to deduce by projection the properties of the conic from those of the circle

In Art 73 the anharmonic property of the circle is deduced from the usual definition In the remainder of this chapter the properties of the circle are deduced solely from this anharmonic property As this property is common to all conics the proofs (Arts 74 to 81) hold equally for all conics They have been stated only for the circle as it is thought that in this form they will be more easily grasped by the student in the first instance In Chapter XI theorems and proofs special to the circle are given and in Chapter XIII and succeeding chapters the general case of the conic is considered

73 The Anharmonic Property of the Circle

The anharmonic ratio of the pencil formed by joining four fixed points on a circle to a variable point on the circle is constant



Let A, B, C, D be any four fixed points on the circle and P and P' any two positions of the variable point

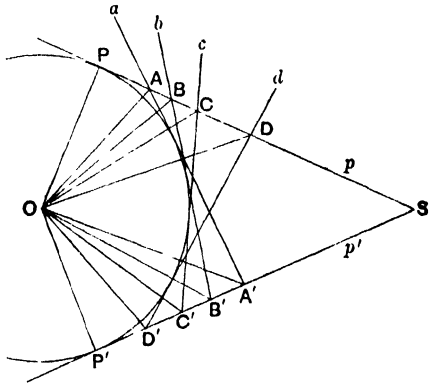
Then the angles APB and $AP'B$ which stand on the same arc AB are equal

Similarly the angles BPC and $BP'C$ are equal and likewise the angles CPD and $CP'D$

Hence the pencils $(P\ ABCD)$ and $(P'\ ABCD)$ contain equal angles between pairs of corresponding rays and are therefore equi-anharmonic and projective

The tangent p at P corresponds to the ray PP' of the pencil with vertex P', and the tangent p' at P'

The anharmonic ratio of the range formed by the intersections of four fixed tangents to a circle with a variable tangent is constant



Let a, b, c, d four fixed tangents to the circle meet two other tangents p and p' in A, B, C, D and A', B', C', D' respectively

Let O be the centre of the circle Then the angles AOA' and BOB' (Addendum 10) are equal and therefore the angles AOB and $A'OB'$ are equal

Similarly the angles BOC and $B'OC'$ are equal and likewise the angles COD and $C'OD'$

Hence the ranges $(p\ abcd)$ and $(p'\ abcd)$ are such that corresponding segments subtend equal angles at O, and are therefore equi-anharmonic and projective

The point of contact P of the tangent p corresponds to the point pp' of the range on p', and the

corresponds to the ray PP' of the pencil with vertex P

Conversely

The locus of the points of intersection of corresponding rays of two directly equal pencils is a circle, which passes through the vertices of the two pencils

Let PA, PB be the two rays of one pencil corresponding to the two rays $P'A$, $P'B$ of the other Then since the angles APB and $AP'B$ are equal, the circle through A, P, P' also passes through B

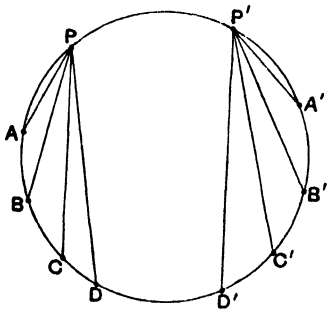
Similarly this circle passes through the point of intersection of any other pair of corresponding rays

point of contact P' of the tangent p' corresponds to the point pp' of the range on p

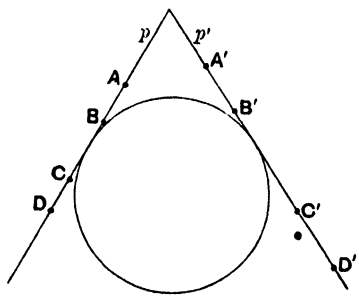
If two projective ranges on given bases are such that pairs of corresponding points subtend at a fixed point on a bisector of the angle between the bases an angle complemental to half the bisected angle, then the connectors of pairs of corresponding points envelope a circle of which the fixed point is the centre

Let p and p' be the bases of the ranges, S their point of intersection, O the fixed point, and AA' any pair of corresponding points Draw OP and OP' perpendiculars from O to p and p' Then since O is on the bisector of the angle $\widehat{pp'}$ a circle can be described with centre O to touch p and p' at P and P' Also since the angle AOA' equals the angle POS, the line AA' touches this circle (Addendum 10) Similarly the line joining any other pair of corresponding points on p and p' touches this circle

74 Projective Ranges on a circle, and Projective Systems of Tangents to a Circle



If through the vertices P and P' of two projective pencils a circle be described meeting the rays of the pencils in A, B, C, D and



If $ABCD$ and $A'B'C'D'$ be two projective ranges on bases p and p' and if tangents a, b, c, d and a', b', c', d' be drawn from

A', B', C', D' then these systems of points are such that if they are joined to any other points on the circle, the pencils so formed are projective Such systems of points are termed *projective ranges on the circle*

To any point considered as belonging to one range there corresponds one and only one point of the other

If A, B, C, D be four points on a circle, their anharmonic ratio, *i e* the anharmonic ratio of the pencil formed by joining them to any point on the circle, is denoted by $(ABCD)$

these points to a circle which touches p and p', then these systems of tangents are such that if they are cut by any other tangents, the ranges so formed are projective Such systems of tangents are termed *projective systems of tangents to the circle*

To any tangent considered as belonging to one system of tangents there corresponds one and only one tangent of the other system

If a, b, c, d be four tangents to a circle, their anharmonic ratio, *i e* the anharmonic ratio of the range which they determine on any tangent to the circle, is denoted by $(abcd)$

Involutions of Points on, and of Tangents to, a Circle

Similarly, if a circle be described through the vertex S of an involution pencil to meet pairs of conjugate rays in AA', BB', CC' these pairs of points are said to be conjugate points of an involution on the circle

The pencil formed by joining four of these points to any point on the circle is projective with the pencil formed by joining their four conjugate points to any point on the circle, for each of these pencils is projective with the pencil formed by joining the same points to a point on the circle

Two pairs of conjugate points

Similarly, if a circle be described to touch the base s of an involution and the tangents to the circle from pairs of conjugate points be aa', bb', cc' these pairs of tangents are said to be conjugate rays of an involution of tangents to the circle

The range formed by the intersections of four of these tangents with any tangent to the circle is projective with the range formed by the intersections of the four conjugate tangents with any tangent, for each of these ranges is projective with the range formed by the intersections of these tangents with a tangent to the circle

Two pairs of conjugate rays—

on the circle determine the involution and the conjugate of a given point in a given involution can be found from the fact that the range on the circle formed by any four points must be projective with the range formed by the conjugate points

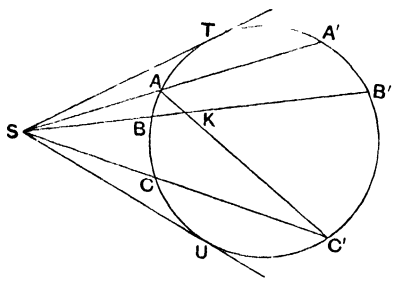
Conversely, if this condition is satisfied the points form an involution on the circle

tangents to the circle—determine the involution and the conjugate of a given ray—tangent—in a given involution can be found from the fact that the pencil of tangents to the circle formed by any four tangents must be projective with the pencil formed by the conjugate rays—tangents

Conversely, if this condition is satisfied the tangents form an involution of tangents to the circle

75 Involution on a Circle and Involution of Tangents to a Circle

If through any point S chords be drawn to intersect a circle in AA′, BB′, CC′ these pairs of points are pairs of conjugate points of an involution on the circle



Join AC' to meet BB' in K

Then

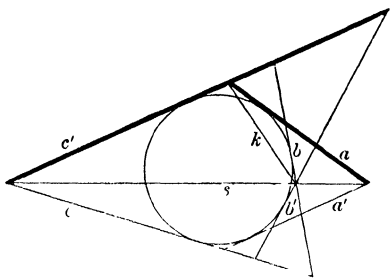
$$(BA'B'C') = (A\ \ BA'B'C') = (BSB'K)$$

by taking intersections of the pencil on BB'

Similarly

$$(B'ABC) = (C'\ \ B'ABC) = (B'KBS)$$

If from points on a line s pairs of tangents aa′, bb′, cc′ be drawn to a circle, these pairs of tangents intersect any other tangent in pairs of conjugate points of an involution



Join ac' to bb' by k

Then

$$(ba'b'c') = (a\ \ ba'b'c') = (bsb'k)$$

by connecting points of the range to bb'

Similarly

$$(b'abc) = (c'\ \ b'abc) = (b'kbs)$$

But $(BSB'K) = (B'KBS)$

Therefore

$$(BA'B'C') = (B'ABC)$$

Therefore the three pairs of points AA', BB', CC' are pairs of conjugate points of an involution on the circle

If S be a point external to the circle two tangents can be drawn from S to touch the circle at T and U T and U are the double points of the involution determined by the two pairs of conjugate points AA', BB'

Since in this case

$$(BUB'T) = (B'UBT),$$

the points of contact T and U of the tangents from any point S to the circle and the points of intersection B and B' of any chord through S are four points on the circle which form a harmonic range, *i e* the pencil formed by joining these points to any point on the curve is harmonic

Since the relation between pairs of harmonic conjugates is reciprocal, the tangents at B and B' intersect on the chord TU

But $(bsb'k) = (b'kbs)$

Therefore

$$(ba'b'c') = (b'abc)$$

Therefore the three pairs of points of intersection of the tangents aa', bb', cc' with any tangent are pairs of conjugate points of an involution

If the line s meets the circle in two points, two tangents t and u can be drawn at these points t and u are the double rays of the involution system of tangents determined by the pairs of conjugate rays aa', bb'

Since in this case

$$(bub't) = (b'ubt),$$

the tangents t and u at the points of intersection of a chord s with the circle and the tangents b and b' from any point on s are four tangents to the circle which form a harmonic system of tangents, *i e* a system of tangents which intersects any other tangent in a harmonic range

Since the relation between pairs of harmonic conjugates is reciprocal, the chord of contact of the tangents b and b' passes through tu

Hence the Fundamental Property of Harmonic Ranges of Points and Harmonic Pencils of Tangents to a Circle is obtained, viz

If through the point of intersection of the tangents at any points T and T' on a circle, a chord be drawn to meet the circle in A and A', then the range $TT'AA'$ is harmonic

If from any point on the chord of contact of any two tangents t and t' to a circle, two other tangents a and a' be drawn to the circle, then the pencil of tangents $tt'aa'$ is harmonic

Conversely

If AA', BB', CC' are pairs of conjugate points of an involution on a circle, the lines AA', BB', CC' are concurrent

The involution is determined by two pairs of conjugate points AA', BB' Let AA', BB' meet at S Join C to S to meet the circle in C'' Then C'' is a conjugate of C in the given involution and must therefore coincide with C'

Hence CC' passes through S

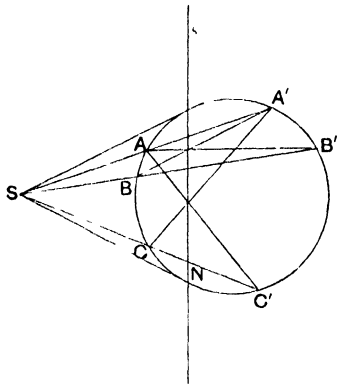
If aa', bb', cc' are pairs of conjugate rays of an involution pencil of tangents to a circle, the points aa', bb', cc' are collinear

The involution is determined by two pairs of conjugate tangents aa', bb' Let the connector of aa' to bb' be s Let c meet s in C and let c'' be the tangent from C to the circle Then c'' is a conjugate of c in the given involution and must therefore coincide with c'

Hence cc' lies on s

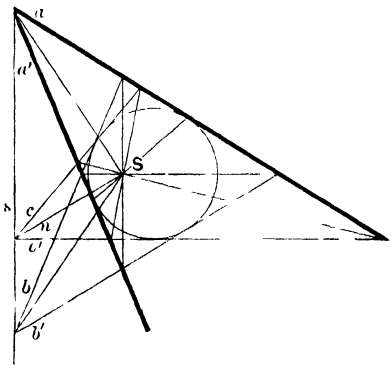
76 Pole and Polar

If through any fixed point a variable chord of a circle be drawn, the locus of the harmonic conjugates of the fixed point with regard to the points of intersection of the variable chord with the circle is a straight line



Draw through S three chords meeting the circle in AA', BB', CC' The first two chords may be regarded as fixed and the last as variable The six points form an involution on the circle and therefore the pencil formed by joining A, A', B, B', C, C' to A' is

If from a variable point on a fixed straight line pairs of tangents be drawn to a circle, the envelope of the harmonic conjugate of the fixed line with regard to the pair of variable tangents is a fixed point



From any three points on s draw pairs of tangents to the circle aa', bb', cc' The first two pairs may be regarded as fixed and the last as variable The six tangents form an involution of tangents and therefore the range formed by the intersections of a'

projective with the pencil formed by joining the conjugate points A', A, B', B, C', C to A

Since these pencils have a common self-corresponding ray in AA' they are also in perspective (Art 34)

Therefore the corresponding rays intersect on a line s This line is a fixed line since it passes through the points AB' BA' and AB $A'B'$

By the harmonic property of the quadrangle, s meets AA', BB', CC' in points which are harmonic conjugates of S with regard to AA', BB', CC' respectively

Hence, if CC' meets s in N, N is the point whose locus is required and its locus is the straight line s

If CC' be taken to coincide with AA' it is seen that the tangents at A and A' (and therefore those at B and B', and C and C') intersect on s

The line s is called the *polar* of S

By comparison of the work in the two columns it will be seen that S is the pole of s

From the preceding the following construction is obtained

Construction of the polar of a given point

Through P the given point draw two chords PBB' and PAA'

with a, a', b, b', c, c' is projective with the range formed by the intersections of a with the conjugate rays a', a, b', b, c', c

Since these ranges have a self-corresponding point in aa' they are also in perspective (Art 34)

Therefore the corresponding points are collinear with a point S This point is a fixed point since it is the intersection of the lines ab' $a'b$ and ab $a'b'$

By the harmonic property of the quadrilateral, the connectors of S with aa', bb', cc' are harmonic conjugates of s with regard to aa', bb', cc' respectively

Hence, if n be the connector of cc' to S, n is the line whose envelope is required and its envelope is the point S

If cc' be taken to coincide with aa' it will be seen that the points of contact of a and a' (and therefore those of b and b', and c and c') are collinear with S

The point S is called the *pole* of s

By comparison of the work in the two columns it will be seen that s is the polar of S

From the preceding the following construction is obtained

Construction of the pole of a given line

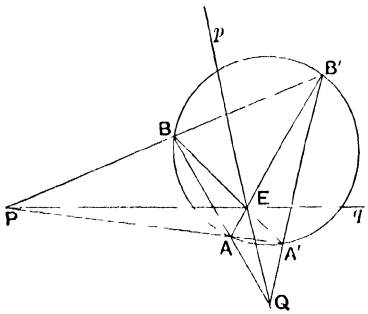
From two points on the given line p draw pairs of tangents aa' and bb'

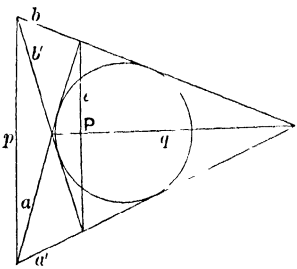
The polar p of P is the line joining AB' BA' (or E) to AB $A'B'$ (or Q)

If P is a point external to the circle, the polar may be constructed as the chord of contact of the tangents from P

The pole P of p is the point of intersection of ab' ba' (or q) and ab $a'b'$ (or e)

If p meets the circle in real points the pole may be constructed as the point of intersection of the tangents at these points





Conjugate points and lines with respect to a circle

If P and Q are two points such that the polar of one passes through the other, the points are called *conjugate points with respect to the circle*

If the line joining two conjugate points P and Q meets the circle in real points these points are harmonic conjugates of P and Q

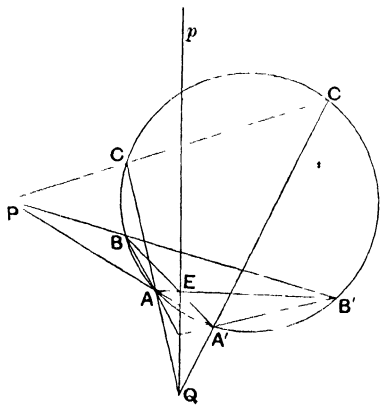
If p and q are two lines such that the pole of one lies on the other, the lines are called *conjugate lines with respect to the circle*

If tangents can be drawn to the circle from the point of intersection of two lines p and q these tangents are harmonic conjugates of p and q

If p the polar of P passes through Q, then q the polar of Q passes through P

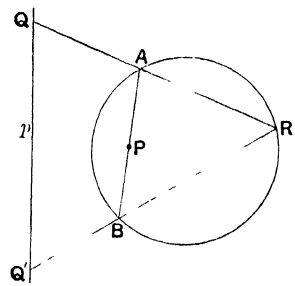
If PQ meets the circle in real points A and A' the theorem is obvious since $PQAA'$ is harmonic If PQ does not meet the circle the theorem may be proved as follows

Through P draw chords PAA', PBB' Then p is the line joining BA' AB' to AB $A'B$ Take Q any point on p Join QA and QA' to meet the circle in C and C' Then CC' must pass through P For if it does not let PC' meet the circle again in C'' Then $C''A'$ must pass



through Q for CA and $C''A'$ must intersect on p Therefore C' and C'' coincide Hence q the polar of Q is the line joining P to CA' AC'' and therefore passes through P

Construction for pairs of points on a given straight line conjugate with respect to a circle

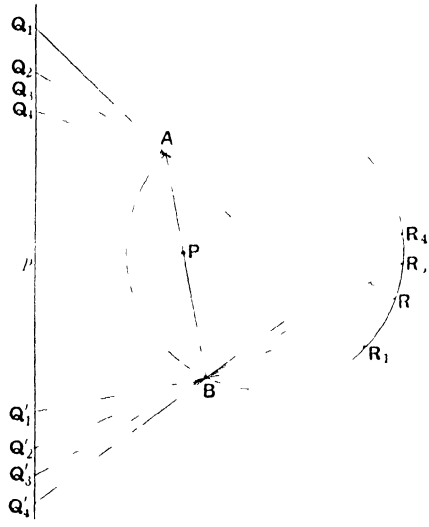


Let p be the given line and P its pole Through P draw any fixed chord AB Let Q be any point on p To find its conjugate join Q to A to meet the circle in R Let RB meet p in Q' Then Q and Q' will be shown to be conjugate points

Join PR to meet the circle in R' Then the polar of P is the line joining AR' BR to BR' AR Therefore since p is the polar of P, AR' and BR' must pass through Q' and Q respectively Therefore QP is the polar of Q' and PQ' is the polar of Q Hence Q and Q' are conjugate points with respect to the circle

Conversely it follows that, *if Q and Q', any two conjugate points with regard to a circle, be joined to any point R on the circle by lines, which meet the circle in A and B, then the line AB passes through the pole of the line QQ'*

77 *A circle determines by means of pairs of collinear conjugate points a definite involution on every straight line in its plane*



If four points Q_1, Q_2, Q_3, Q_4 be taken on any straight line p, then conjugates Q_1', Q_2', Q_3', Q_4' with respect to the circle may be constructed by means of the last Article In the figure, the range $Q_1Q_2Q_3Q_4$ is projective with the range $R_1R_2R_3R_4$ on the circle and this is projective with the range $Q_1'Q_2'Q_3'Q_4'$

But since Q_1 and Q_1' are conjugate points and the polar of each passes through the other, they mutually correspond Hence the two projective ranges form an involution

If the involution has real double points these points are the points of intersection of the line with the circle

The correlative of the preceding is as follows

A circle determines by means of pairs of concurrent conjugate lines a definite involution pencil at every point in its plane

If p_1, p_2, p_3, p_4 are any four straight lines passing through any point S, their poles P_1', P_2', P_3', P_4' lie on s the polar of S Let s meet p_1, p_2, p_3, p_4 in P_1, P_2, P_3, P_4 Then P_1P_1', P_2P_2', P_3P_3', P_4P_4', form an involution of conjugate points Hence if p_1', p_2', p_3', p_4' be the lines joining P_1', P_2', P_3', P_4' to S, the pairs of conjugate lines p_1p_1', p_2p_2', p_3p_3', p_4p_4', are pairs of conjugate rays of an involution pencil

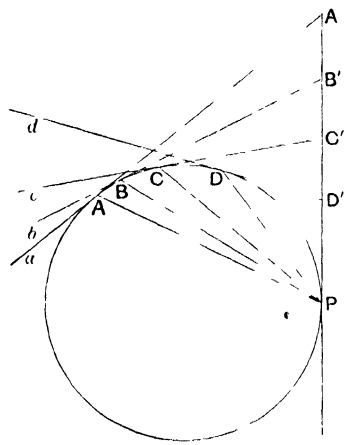
Since the tangents from S to the circle, if they are real, are harmonic conjugates of each of these pairs of conjugate lines, they are the double rays of the involution

The anharmonic ratio of a range of four collinear points is equal to that of the pencil formed by their polars

If Q_1, Q_2, Q_3, Q_4 are four collinear points situated on a line p, their polars, which all pass through the pole of p, meet p in points Q_1', Q_2', Q_3', Q_4', which are conjugates of Q_1, Q_2, Q_3, Q_4 Hence the range $Q_1Q_2Q_3Q_4$ is projective with the range $Q_1'Q_2'Q_3'Q_4'$ and therefore with the pencil formed by the polars of Q_1, Q_2, Q_3, Q_4

The anharmonic ratio of four points on a circle is equal to that of the four tangents at these points

Let A, B, C, D be any four points on the circle and a, b, c, d the tangents at these points Let P be any other point on the circle and let p the tangent at this point meet a, b, c, d in A', B', C', D'



Then the anharmonic ratio of A, B, C, D is that of the pencil $(P\ ABCD)$ and that of the tangents a, b, c, d is $(A'B'C'D')$ Since A', B', C', D' are the poles of PA, PB, PC, PD, these anharmonic ratios are equal

78 Properties of Pole and Polar

The following is a restatement of the theorems of the last two Articles together with certain important deductions

(*a*) The line joining two conjugate points, if it meets the curve in real points, is divided harmonically by the curve

Two conjugate lines are harmonic conjugates of the two tangents, if real, from their point of intersection to the curve

For the polar of a point is the locus of points which are harmonic conjugates of that point with respect to the points of intersection of any chord through that point with the curve

For the pole of a line is the envelope of lines which are harmonic conjugates of that line with respect to the tangents from any point on that line to the curve

(*b*) The envelope of the polars of all points on a straight line is the pole of the line

The locus of the poles of all lines through a given point is the polar of the point

For the pole of a given line is on the polars of all points on the given line and therefore all the polars must contain the pole

For the polar of a given point passes through the poles of all lines through the given point and therefore all the poles must lie on the polar

(*c*) The pole of the connector of two points is the point of intersection of their polars

The polar of the intersection of two lines is the connector of their poles

(*d*) The polar of a point on the curve is the tangent at the point

The pole of a tangent to the curve is its point of contact

(*e*) Any point on a tangent is a conjugate of the point of contact of the tangent

Any line through the point of contact of a tangent is a conjugate of the tangent

(*f*) The polar of a point determines on the curve the points of contact—real or imaginary—of the tangents from the point to the curve

The pole of a line determines the tangents—real or imaginary—to the curve at the points in which the line intersects the curve

For each point of contact is a conjugate of the point and therefore the polar passes through both points of contact

For each tangent is conjugate to the line and therefore passes through the pole of the line

(*g*) The connectors of a pair of conjugate points to the pole of their connector are conjugate lines

The intersections of a pair of conjugate lines with the polar of their point of intersection are conjugate points

(*h*) Two conjugate lines cut the curve in four points forming a harmonic range

For each conjugate line passes through the pole of the other

The tangents from two conjugate points to the curve form a harmonic pencil of tangents

For each conjugate point lies on the polar of the other

(*i*) The intersections of the polars of three given points form a triangle which is called *the polar*, or *conjugate triangle*, of the triangle formed by the three given points

The connectors of the poles of three given lines form a triangle which is called *the polar*, or *conjugate triangle*, of the triangle formed by the three given lines

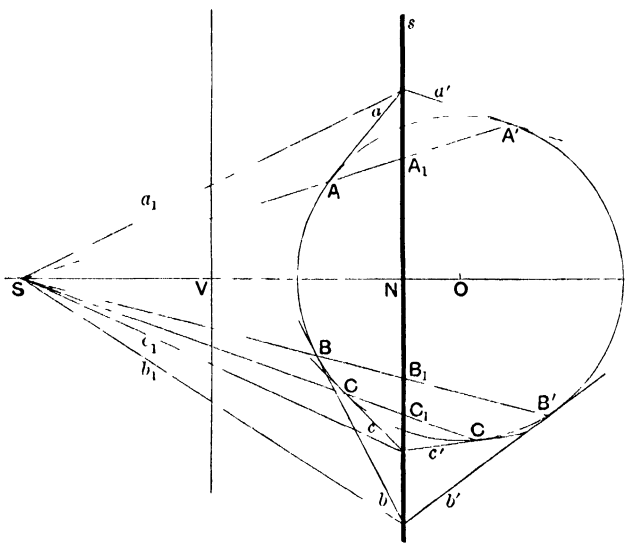
(*j*) A triangle such that each vertex is the pole of the opposite side is termed *a self-polar*, or *self-conjugate triangle*

A triangle such that each side is the polar of the opposite vertex is termed *a self-polar*, or *self-conjugate triangle*

79 Circle in Perspective with itself

Any circle may be looked upon as in harmonic perspective with itself, any point being the centre of perspective and the polar of that point the axis

Any circle may be looked upon as in harmonic perspective with itself, any line being the axis of perspective and the pole of that line the centre



Take any point S Through S draw chords SAA_1A', SBB_1B', SCC_1C' to meet s the polar of S in A_1, B_1, C_1 Then the ranges SA_1AA', SB_1BB', SC_1CC' are harmonic

Therefore, if S be taken as centre, s as axis of perspective, and the anharmonic ratio of the perspective be -1, A corresponds to A', B to B' and C to C' and vice versâ Hence the circle is in self-perspective

Take any line s Draw pairs of tangents aa', bb', cc' from points on s Then, if a_1, b_1, c_1 are the lines joining aa', bb', cc' to S the pole of s, the pencils sa_1aa', sb_1bb', sc_1cc' are harmonic

Therefore, if s be taken as axis, S as centre of perspective, and the anharmonic ratio of the perspective be -1, a corresponds to a', b to b' and c to c' and vice versâ Hence the circle is in self-perspective

As a consequence, it is seen that

(a) tangents at A and A', viz a and a', intersect on s	the points of contact of tangents a and a', viz A and A', are collinear with S
(b) corresponding chords AB and $A'B'$ intersect on s	corresponding points ab and $a'b'$ are collinear with S
(c) corresponding chords AB' and $A'B$ intersect on s	corresponding points ab' and $a'b$ are collinear with S
(d) tangents from S touch the circle on s	tangents where s meets the circle intersect at S

From S draw a line perpendicular to s to meet s in N On SN take a point V such that $SV=VN$ Then the line parallel to s through V is the vanishing line

The two columns above simply give a restatement of the same theorem looked at from two slightly different points of view and form a good instance of the Principle of Duality

80 Complete Inscribed Quadrangle and Complete Circumscribed Quadrilateral of a Circle

Any four points A, B, C, D on a circle determine

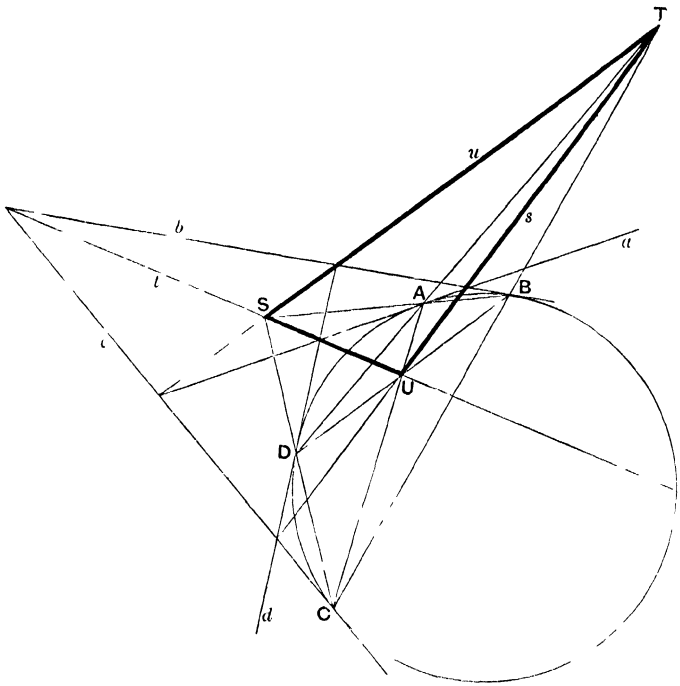
(1) three points S, T, U, as the points of intersection of their connectors taken in pairs

Any four tangents a, b, c, d to a circle determine

(1) three lines s, t, u, as the lines joining the points of intersections of the tangents taken in pairs

(2) three lines s, t, u, as the connectors of the points S, T, U in pairs

(2) three points S, T, U, as the intersections of s, t, u in pairs



The lines s, t, u, are the polars of S, T, U with regard to the circle (Art 76) Hence the tangents a, b, c, d at the points A, B, C, D intersect in pairs on s, t, u

As each side of the triangle STU is the polar of the opposite vertex it is a self-polar or self-conjugate triangle

The points S, T, U, are the poles of s, t, u (Art 76) Hence the points of contact A, B, C, D are collinear in pairs with S, T, U

As each vertex of the triangle stu is the pole of the opposite side it is a self-polar or triangle

It follows that

(a) The diagonal points triangle STU of the quadrangle $ABCD$ is the same as the diagonal triangle stu of the quadrilateral $abcd$

(*b*) From the quadrilateral *abcd*

$$\left.\begin{array}{llll} ac, & bd, & T, & S \\ ad, & bc, & U, & S \\ cd, & ba, & T, & U \end{array}\right\} \text{ are harmonic ranges}$$

(*c*) From the quadrangle *ABCD*

$$\left.\begin{array}{llllll} AC & u, & BD & u, & S, & T \\ AD & t, & BC & t, & S, & U \\ DC & s, & AB & s, & T, & U \end{array}\right\} \text{ are harmonic ranges}$$

(*d*) If the circle meets t in P, P' and s in Q, Q'

$$\left.\begin{array}{llll} P, & P', & S, & U \\ Q, & Q', & T, & U \end{array}\right\} \text{ are harmonic ranges}$$

In the figure, u does not meet the circle in real points One side of every self conjugate triangle with regard to a circle does not meet the curve in real points

81 Construction of a Self-Polar- or Self-Conjugate—Triangle of a Circle

The triangle formed by two conjugate points and the pole of their connector as vertices is a self-conjugate triangle

Given one vertex of such a triangle the number of triangles is infinite, for any two conjugate points on its polar may be taken as the remaining two vertices

Given two vertices, which must be conjugate points, the triangle is completely determined, for the third vertex must be the pole of their connector

The triangle formed by two conjugate lines and the polar of their point of intersection as sides is a self conjugate triangle

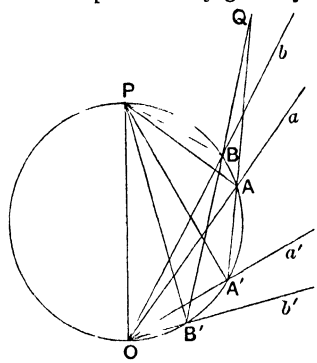
Given one side of such a triangle the number of triangles is infinite, for any two conjugate lines through its pole may be taken as the remaining two sides

Given two sides, which must be conjugate lines, the triangle is completely determined, for the third side must be the polar of their point of intersection

The limiting form of a self polar *triangle*, if one vertex is given, *is the tangent from that point to the circle* If the two conjugate points (which are vertices) are collinear with the centre of the circle, one of the vertices of the triangle is at infinity, and two sides are parallel Every self conjugate triangle with respect to a circle must be obtuse angled

EXAMPLES

(1) If from a fixed point perpendiculars be drawn to the pairs of conjugate rays of a pencil in involution, the lines joining their feet taken in pairs pass through a fixed point

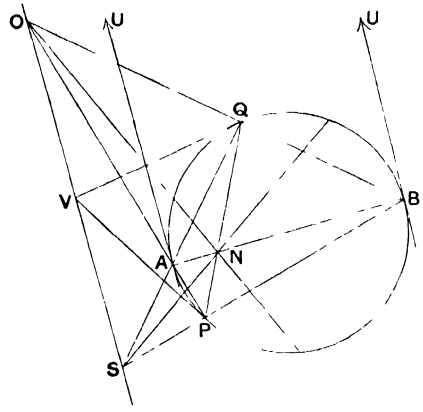


Let aa', bb' be conjugate rays of the involution pencil whose vertex is O and P any fixed point A circle described on OP as diameter will pass through P, A', B, B' the feet of the perpendiculars from P

$P\ AA'BB'$ is an involution pencil, since its rays are perpendicular to a, a', b, b' Hence AA', BB' are conjugate points of an involution on the circle Therefore (Art 75) AA', BB' pass through a fixed point Q

(2) A fixed point O external to a given circle is joined to the extremities A and B of any diameter, and OA, OB meet the circle again in P, Q Show that the tangents at P, Q intersect on a fixed line parallel to the polar of O

Consider the inscribed quadrangle $AQBP$ and the circumscribed quadrilateral formed by the tangents at these points In the figure the points U, V, O, S are collinear and form a harmonic range, therefore, since U is at infinity, V bisects OS Therefore V describes a line parallel to NS which being the polar of O is fixed



(3) A, B, C being three given points on a circle and M any other point on the circle, AM, BM are produced to meet the tangent at C in P, Q, respectively, show that the difference of the reciprocals of CP, CQ is constant

Through A and B draw lines parallel to the tangent at C to meet the circle in A' and B'. Let AB' and $A'B$ meet the tangent at C in O and O'. Then O and O' correspond respectively to the points at infinity in the projective ranges described by P and Q, and C is a self-corresponding point.

$$OP \cdot O'Q = K^2 = OC \cdot O'C,$$

$$(CP - CO)(CQ - CO') = OC \cdot O'C,$$

$$CP \cdot CQ = CO \cdot (CQ - CP) \text{ since } CO' = -CO,$$

$$\frac{1}{CO} = \frac{1}{CP} - \frac{1}{CQ}$$

(4) If a variable chord CD of a circle passes through the middle point O of a fixed chord AB then the locus of the intersections of the lines AD, CB, is a straight line parallel to AB.

(5) If a variable chord PQ of a circle passes through a fixed point L and perpendiculars PM, QN are drawn to the polar of L, then $\frac{1}{PM} + \frac{1}{QN}$ is constant.

(6) The polar of O meets the circle in P, Q and PR is any chord of the circle through P. Through O a line is drawn parallel to PR meeting the circle in U, V. Show that QR bisects UV.

Let M be the point $QR \cdot UV$. The range $PQUV$ on the circle is harmonic,

$$(R \cdot PQUV) = -1,$$

$$(\infty \cdot MUV) = -1,$$

M bisects UV.

CHAPTER XI

SPECIAL PROPERTIES OF THE CIRCLE — INVERSE POINTS COAXAL CIRCLES CIRCLES IN PERSPECTIVE WITH EACH OTHER

82 Inverse Points with respect to a Circle

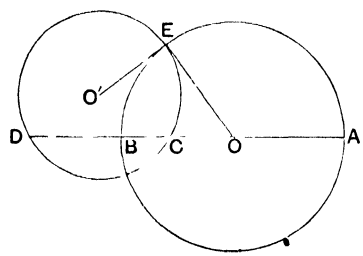
The theorems which have hitherto been proved for the circle are common to all conics and the same proofs hold for the conic in general as for the circle Some of these theorems may be proved for the circle in a manner not applicable to the general case, and there are certain theorems true for the circle which do not hold for the conic The more important of these are given in this chapter

Inverse Points Definition *Two conjugate points collinear with the centre of a circle are termed inverse points*

If two circles cut each other or [illegible] *each determines inverse points upon every diameter of the other*, and conversely,

If one circle pass through inverse points with respect to another circle, the two circles cut orthogonally

Let O be the centre of one circle and let a diameter of this circle meet it in A, B and the second circle in C, D



In the first case, since the circles are orthogonal, the line OE joining O to either of the points of intersection of the circles is a tangent to the second circle, and therefore

$$OC \cdot OD = OE^2 = OA^2$$

Therefore $(ABCD)$ is harmonic and C and D are inverse points with respect to the circle whose centre is O

In the second case, since C and D are a pair of inverse points with respect to the circle whose centre is O,

$$OC \cdot OD = OB^2 = OE^2$$

Therefore OE is a tangent to the second circle and the two circles cut orthogonally

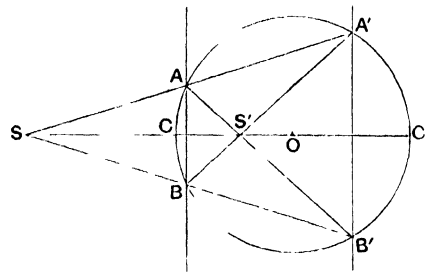
Cor I To describe a circle through two given points to cut a given circle orthogonally Let the given points be A and B Take C the inverse of A or of B with respect to the circle Then ABC is the orthogonal circle

Cor II To describe a circle through a given point to cut two given circles orthogonally Let the given point be A Take B and C the inverse points of A with respect to the two circles Then ABC is the required orthogonal circle

Cor III If C and D be conjugate points with respect to a circle (X), and a circle (Y) be described on CD as diameter, this circle (Y) will cut the circle (X) orthogonally For let CD meet (X) in A and B Then $(CDAB)$ is harmonic and since CD is a diameter of (Y) A and B are inverse points with respect to (Y)

Special Construction of Pole and Polar for a Circle

Let O be the centre of the circle and S the point whose polar is required Join OS to meet the circle at C and C' and draw two chords SAA' and SBB' equally inclined to SO



Then by symmetry AB' and BA' will intersect at some point S' on SO and AB and $A'B'$ will be perpendicular to SO

Hence S is the polar of S will be the line through S' perpendicular to SO, Art 76 Also because $(SS'CC')$ is harmonic and O is the middle point of CC',

$$OS \cdot OS' = OC^2,$$

while the points S, S', being conjugate points collinear with the centre, are inverse points

Hence *the polar of a given point with respect to a circle is the line through its inverse point perpendicular to the connector of the given point to the centre*

Also

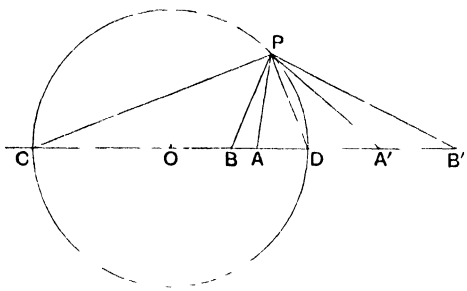
The pole of a given line is the inverse point of the foot of the perpendicular from the centre on the given line

If AA', BB' be two pairs of collinear inverse points, with respect to a circle, and P be any point on the circle, then

(1) *the ratio $AP\ A'P$ is constant for different positions of P,*

(2) *the angles APB and $A'PB'$ are equal or supplemental* (Cf Art 53)

Let the line $BAA'B'$ meet the circle at C and D Take P any point on the circle Then the ranges $CDAA'$ and $CDBB'$ are harmonic Therefore the pencils $(P\ CDAA')$ and $(P\ CDBB')$ are harmonic But CP, PD are at right angles, and are therefore common bisectors of the angles APA' and BPB'



Hence (1) $AP\ A'P = AD\ A'D =$ a constant,

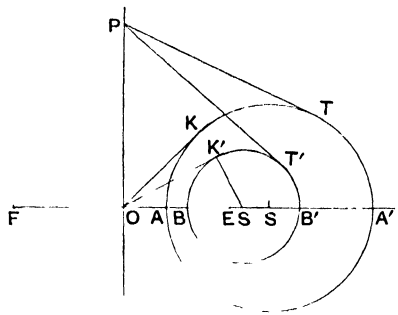
(2) the angles APB and $A'PB'$ are equal or supplemental

83 Coaxal Circles

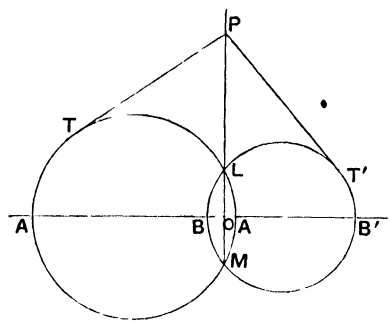
If AA', BB', CC', are pairs of conjugate points of an involution, the circles described on AA', BB', CC', as diameters are said to form a coaxal system of circles

There are two cases to be considered according as the involution has, or has not, real double points

If the double elements are real points E and F, let O be the centre of the involution



If the double elements are imaginary, let O be the centre of the involution



If PO be a line through O perpendicular to the base of the involution, this line is termed *the radical axis* of the circles and it will be shown that the tangents from any point P on this line to the circles of the system are equal

Since

$$OE^2 = OA \ \ OA' = OB \ \ OB',$$

the tangents, OK and OK', from O to any two of the circles are equal If R and R' be the radii of the circles, S and S' their centres and PT and PT' the tangents from a point P on the radical axis, then

$$PT^2 = PS^2 - R^2 = PO^2 + OS^2 - R^2$$
$$= PO^2 + OK^2$$

Similarly

$$PT'^2 = PO^2 + OK'^2$$

Therefore $PT = PT'$

All circles described through E and F, which are inverse points with respect to all the circles of the system, will (Art 82) cut every circle of the coaxal system at right angles

The circles through E and F form a system of coaxal circles of the nature described on the right-hand side, E and F taking the place of L and M

If PO be a line through O perpendicular to the base of the involution, this line is termed *the radical axis* of the circles and it will be shown that the tangents (if real) from any point P on this line to the circles of the system are equal

Take two points L and M vertically above and below O such that

$$OL^2 = OM^2 = OA \ \ OA' = OB \ \ OB'$$

Then all circles of the system will pass through L and M Hence if PT and PT' be the tangents to two circles for a point P on the radical axis,

$$PT^2 = PL \ \ PM$$

and

$$PT'^2 = PL \ \ PM$$

Therefore $PT = PT'$

All circles with regard to which L and M are inverse points (Art 82) will cut all circles of the coaxal system at right angles L and M will be common inverse points of such a system of circles

The circles which have L and M for common inverse points form a coaxal system of the kind described on the left-hand side, L and M taking the place of E and F

The points E and F are termed the *common inverse points* or *the limiting points* of the system of coaxal circles

The Radical Axis is sometimes defined as the locus of points from which the tangents to a system of coaxal circles are equal If the circles intersect in real points it may be constructed as the line joining

these points If the circles do not intersect in real points, let t and t' be two common tangents to two of the circles which touch them at P, Q and P', Q', respectively Then, if M and M' are the middle points of PQ and $P'Q'$, the radical axis is the line MM' Two circles completely determine a coaxal system, for two pairs of conjugate points AA' and BB' completely determine an involution A system of coaxal circles is also determined by a point—a limiting point—and a circle The radical axis in this case is termed the radical axis of the point and the circle

The three radical axes of three circles taken in pairs are concurrent

Let the circles be denoted by (1), (2) and (3) Consider P the point of intersection of the radical axes of (1) and (2), and of (2) and (3) At this point the tangent to (1) equals the tangent to (2) and therefore the tangent to (3) Hence the radical axis of (1) and (3) passes through P

To describe a circle to cut three given circles orthogonally

Let S be the point of intersection of the three radical axes of the circles taken in pairs Then if S be external to the circles the tangents from S to the three circles are equal and, if a circle be described with centre S and radius equal to the length of these tangents, this circle will cut the three circles orthogonally If S be an internal point the radius of the circle is an imaginary quantity

84 Coaxal Circles in Self-Perspective

Every circle of a coaxal system, with real limiting points, is in harmonic perspective with itself, either of the limiting points being the centre, and its polar the axis, of perspective

This theorem follows at once from Art 79 when it is borne in mind that the limiting points have the same polars with respect to each circle of the coaxal system The polar of each is the line through the other perpendicular to the line of centres

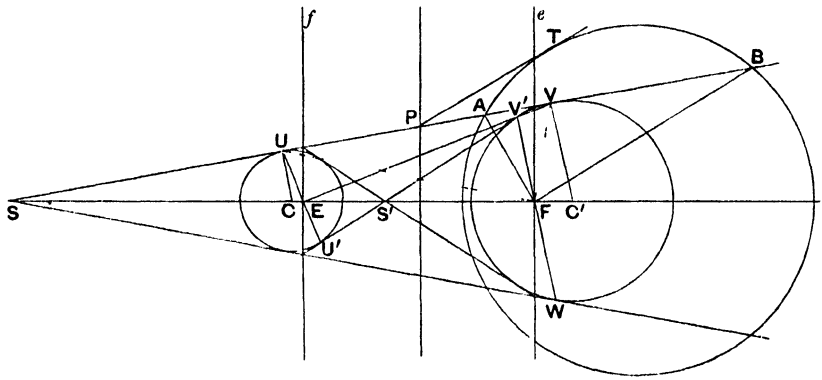
The following are some of the more important properties of a system of coaxal circles with real limiting points In these theorems the *centres of similitude of a pair of circles* occur Generally a centre of similitude of two curves is a centre of perspective of the curves, which has the line at infinity as one of the corresponding axes of perspective (Art. 29) The centres of similitude of two circles are the two points, which divide the distance between the centres in the ratio of the radii of the circles (see example 22, page 159) When the common tangents

to two circles are real, their points of intersection which lie on their line of centres are centres of similitude (Art 87)

(1) If U and V be the points of contact of a common tangent to two of the circles of the system, the circle described on UV as diameter cuts the circles orthogonally and will therefore pass through the limiting points E and F, if these points are real Hence the angles UFV and UEV are right angles

(2) If UV meets a circle of the system in A and B, then $(UVAB)$ is harmonic For if UV meets the radical axis in P, then $PV^2 = PT^2 = PA \cdot PB$, where T is the point of contact of the tangent from P to the circle AB

Since UFV is a right angle FV bisects the angle AFB



(3) If UE and VE meet the circles again in U' and V', then $U'V'$ is a common tangent to the circles and UV and $U'V'$ intersect on e the polar of E This follows from the fact that the circles are in self-perspective with E and e as centre and axis of perspective

(4) Hence EF, f, e are the connectors of the points of intersection of the common tangents to the two circles and therefore form a common self conjugate triangle These lines are the same for every pair of circles of the coaxal system and therefore form a common self-conjugate triangle for the whole system of coaxal circles

(5) Hence S and S', the points of intersection of pairs of common tangents to any pair of circles of the system which lie on the line of centres, are harmonic conjugates of E and F (Art 80)

(6) Hence the circle described on SS' as diameter, *i e* the circle of similitude of any pair of circles of the system, is itself a circle of the coaxal system

(7) If two chords through E meet any pair of circles of the coaxal system in A, A' and B, B', then the points $AB \cdot A'B'$ and $AB' \cdot A'B$ are on the line e

(8) If U, V, U', V' be joined to C and C', the centres of the circles, the joining lines are respectively perpendicular to UV and $U'V'$

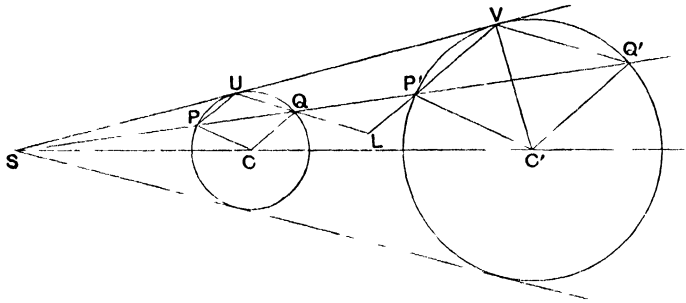
Hence if r and r' be the radii of the circles $\dfrac{CS'}{C'S'} = -\dfrac{r}{r'} = -\dfrac{CS}{C'S}$

Hence the range $SS'CC'$ is harmonic Therefore S and S' are the common harmonic conjugates of EF and CC'

(9) If a chord $SPQP'Q'$ be drawn to meet the circles in P, Q and P', Q'

$$\frac{SC}{CP} = \frac{SC'}{C'P'}$$

Hence the triangles SPC and $SP'C'$ are similar and $\frac{SP}{SP'} = \frac{?}{?}$



Similarly, $\frac{SU}{SV} = \frac{?}{?}$ Hence from the triangles SUP and SVP' it is seen that the lines PU and $P'V$ are parallel Similarly the lines UQ and VQ' are parallel

(10) The angle SQU = angle $SQ'V$ = angle UVP' Therefore U, V, Q, P' are concyclic and if UQ and $P'V$ meet at L, $LQ \cdot LU = LP' \cdot LV$ Hence the tangents from L to the two circles are equal and L is a point on the radical axis

(11) Since the lines UP, VP' are parallel, the pencils formed by joining points P and P' collinear with S to U and V respectively are projective Hence the ranges P and P' on the two circles are projective Similarly, since UQ and VP' intersect on the radical axis, the ranges described by Q and P' on the circles are projective

(12) The ratio of the tangents from any point on a circle of a coaxal system to two other given circles of the system is constant For (Addendum 13) if P be the point, C, C_1, C_2 the centres of the circles, M and N the points where the radical axis and the perpendicular from P meet the line of centres and PT_1 and PT_2 the tangents, then

$$PT_1^2 = 2\ MN \cdot CC_1$$

and

$$PT_2^2 = 2\ MN \cdot CC_2,$$

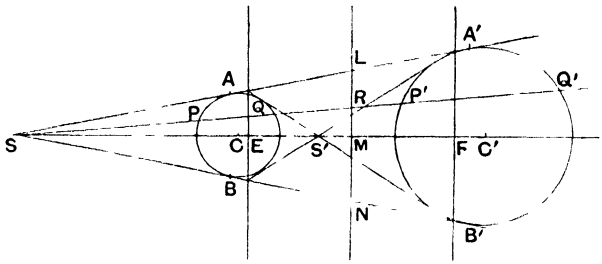
$$\frac{PT_1}{PT_2} = \sqrt{\frac{CC_1}{CC_2}}$$

85 Two circles in Perspective with each other

Any two circles may be looked upon as in perspective with each other, either of the centres of similitude being the centre of perspective, and the axis of perspective either their radical axis or the line at infinity

In Art 72 it was shown that the perspective of a circle is in general a conic A conic is determined by five points, since it is the

locus of points of intersection of corresponding rays of two projective pencils (Art 92) If a perspective is such that five points on a circle correspond to five points on a conic, the conic and circle must be in perspective Hence if in a perspective five points on one circle correspond to five points on another circle, the circles must be in perspective, since a circle is a particular form of a conic



Given two triangles APB and $A'P'B'$ such that the lines joining vertices meet at a point S, the two triangles determine a perspective of which the centre is S and the axis the line joining the points of intersection of pairs of corresponding sides

Let the tangents* from S, a centre of similitude of any two given circles, touch the circles in A, B and A', B' Draw $SPQP'Q$ any chord through S to meet the circles in P, Q and P', Q respectively Consider the perspective in which AA', BB', PP' are pairs of corresponding points

S is the centre of perspective and the axis the line at infinity since the lines AP, $A'P'$, PB, $P'B'$, AB, $A'B'$ are parallel in pairs (Art 84 (9)) In this perspective A, B, A', B' each count as two points on the circles and therefore five points on one circle correspond to five points on the other Therefore the circles are in perspective

Similarly, if the points A, P, B are taken to correspond to the points A', Q', B', a perspective is obtained in which S is the centre of perspective and the radical axis is the axis of perspective, for (Art 84 (10)) AP, $A'Q'$, PB, $Q'B'$, AB, AB' intersect in pairs on the radical axis

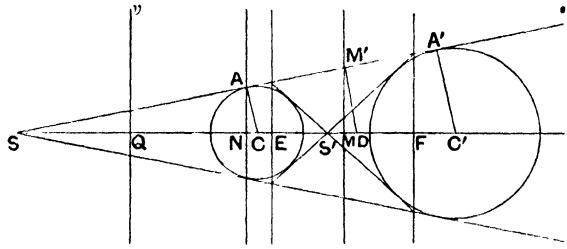
If the line at infinity is taken as the axis of perspective, the anharmonic ratio of the perspective is $(S\infty PP')$, which is constant

Therefore $\dfrac{SP}{SP'} = (S\infty PP') = (S\infty AA') = \dfrac{SA}{SA'} = \dfrac{r}{r'}$

In this case, both the [illegible] lines are at infinity

* The case when these tangents are imaginary is considered in example 22, page 159

If the radical axis is taken as the axis of perspective, and the line SPQ' meets the radical axis in R, $(SRPQ')$ is constant



Hence if M' be the middle point of AA',

$$(SRPQ') = (SM'AA') = -\frac{SA}{SA'} = -\frac{r}{r'}$$

If the line of centres meets the vanishing line v in the first figure in Q and the radical axis in M, then

$$\frac{SQ}{MQ} = (SMQ\infty) = (SM'AA') = -\frac{r}{r'}$$

Hence since $M'D$ in the figure equals $\frac{r + r'}{2}$,

$$2\,\frac{SQ}{SM} = \frac{r}{\frac{r + r'}{2}} = \frac{AC}{M'D} = \frac{SN}{SM}$$

Therefore $$SQ = QN$$

The line v is the radical axis of the point S and the circle, whose centre is C. It is half-way between S and N its inverse point with respect to the circle. The points E and F are corresponding points.

Similarly S' may be taken as the centre of Perspective and the line at infinity or the radical axis as the axis of Perspective.

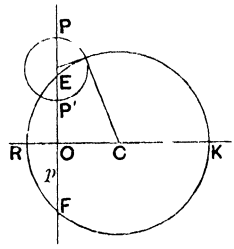
Hence it is seen that *if the perspective of a circle be formed with the radical axis of the circle and centre of perspective as vanishing line, the perspective is a circle having the axis of perspective for its radical axis with the given circle.*

86 Construction of the Involution determined on a given line by a circle

In Art. 77 it was proved that, if P be any point on a straight line p and the polar of P with respect to a circle meet this straight line in P', then, for different positions of P, the points P and P' are pairs of

conjugate points with respect to the circle and determine an involution on the given line p

(a) If the line p meet the circle in real points E and F, these are the double points of the involution and P and P' are harmonic conjugates of these points If O be the foot of the perpendicular from C, the centre of the circle, on the line p, O corresponds to the point at infinity in the involution and is the centre of the involution



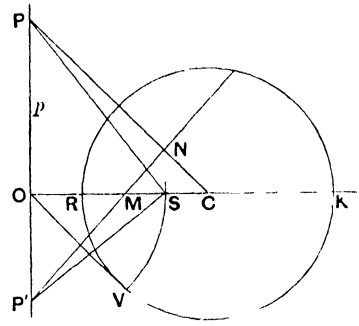
Therefore $OP \cdot OP' = OE^2 = OF^2$

A circle described on PP' as diameter cuts the given circle at right angles (Art 82) If CO meet the circle in R and K, then

$$OP \cdot OP' = -OE \cdot OF = -OR \cdot OK$$

(b) If the line p meet the circle in imaginary points, and the perpendicular from C, the centre of the circle, meet the line in O, then O is the centre of the involution and

$$OP \cdot OP' = \text{a constant}$$



Let the polar of P meet OC in M and CP in N, then since the triangles $OP'M$ and OCP are similar

$$\frac{OM}{OP'} = -\frac{OP}{OC}$$

$$OP \cdot OP' = -OM \cdot OC = -OR \cdot OK$$

But $OR \cdot OK = OV^2$, where OV is the tangent from O to the circle

Hence if a circle be described with centre O and radius OV to meet OC at S, this circle cuts the given circle orthogonally, and the pairs of conjugate points PP', QQ' of the involution subtend right angles at S

S is one of the pairs of common harmonic conjugates of MC and of RK

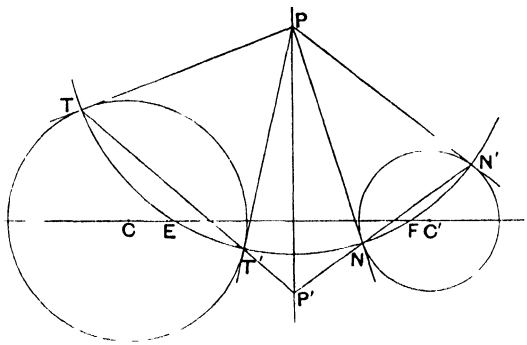
87 Common Involution chords of a pair of circles

Every pair of circles determines the same involution (a) *on their radical axis and* (b) *on the line at infinity*

(a) If the circles intersect in real points L and M, the involution on the radical axis LM in the case of each circle consists of pairs of

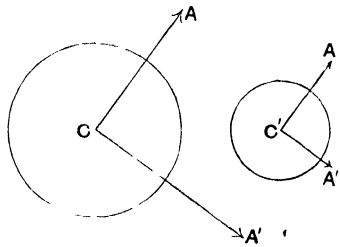
points which are harmonic conjugates of L and M, and therefore both circles determine the same involution on the radical axis

If the circles do not intersect in real points, let P be any point on the radical axis The tangents from P to the two circles are equal and a circle with centre P and radius equal to these tangents intersects both the circles at right angles Let TT', NN' be the chords of intersection of this orthogonal circle with the two given circles Since the radical



axes of the three circles taken in pairs are concurrent, the chords TT', NN' intersect at some point P' on the radical axis of the two given circles But TT', NN' are the polars of P with respect to the given circles Hence P and P' are conjugate points with respect to both circles and, since P may be any point on the radical axis, both circles determine the same involution on the radical axis

(*b*) Let C and C' be the centres of the two circles Then C and C' are the poles of the line at infinity with respect to these circles and pairs of conjugate lines through C and C' meet the line at infinity in pairs of conjugate points of the involutions determined on that line by the two circles



Draw any two lines at right angles through C, viz CA and CA' These are a pair of conjugate lines with respect to the first circle

Draw parallel lines through C' These are conjugate lines with respect to the second circle But parallel lines meet the line at infinity in the same points Hence CA, CA' and $C'A$, $C'A'$ determine the same pair of conjugate points on the line at infinity Since CA and CA' may

be any pair of conjugate lines, the involutions determined by the two circles on the line at infinity are the same The imaginary double points of this involution are termed *the circular points at infinity*

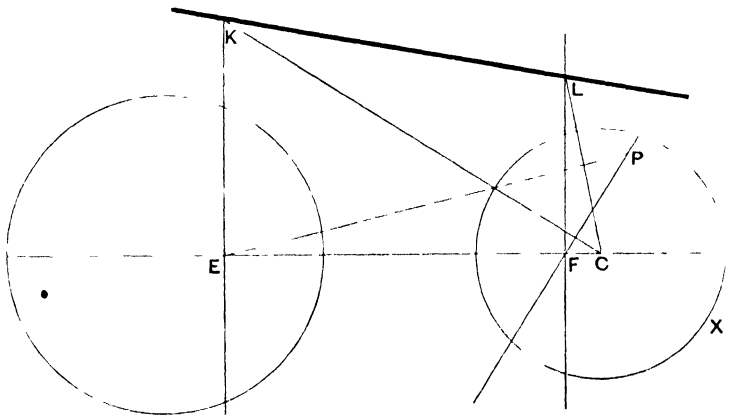
88 (*a*) *The envelope of the polars of a fixed point with respect to a system of coaxal circles is a point*

Let O be any fixed point By Art 83 a circle can be described to cut three given circles orthogonally If a point be substituted for one of the circles the construction enables a circle to be described through a given point to cut two given circles orthogonally Describe such a circle X through O to cut two—and therefore all—circles of the coaxal system orthogonally Let P be the diametrically opposite point of O on the circle X

Then O and P are conjugate points with respect to all circles of the system (Art 82) Therefore the polars of O with respect to all circles of the system pass through P, which is therefore their envelope

(*b*) *The locus of the pole of a fixed line with respect to a system of coaxal circles with real limiting points is a conic through these points*

Let E and F be the limiting points of the coaxal system and let the fixed line meet the perpendiculars through E and F to the line of centres in K and L Then EK and FL are the polars of F and E with respect to all the circles of the system



The polar of K with respect to any circle Λ of the system, whose centre is C, is a line through F perpendicular to CK

The polar of L with respect to the circle X is a line through E perpendicular to CL. The point of intersection P of these lines is the pole of KL with respect to the circle X.

For different positions of C on the line of centres the pencils KC and LC are projective. But the pencils EP and FP are projective with these pencils, and therefore are themselves projective. Therefore it will be seen on reference to the definition of a conic section (Art. 72) that the locus of P is a conic through E and F.

(c) *To find the locus of the points whose polars with respect to three given circles are concurrent*, or

To find the locus of the common conjugates of three given circles

The common [illegible] of a point P with respect to two given circles is the diametrically opposite point of the circle through P orthogonal to the two given circles. If this is the [illegible] of P with respect to a third circle, the orthogonal circle must likewise be orthogonal to the third circle. Hence the locus of P for common conjugates of three circles is the common orthogonal circle of the three circles. This circle has its centre at the point of intersection of the three radical axes and radius equal to the tangents from this point to the circles. As every circle determines on the line at infinity the same involution, every point on the line at infinity has the same [illegible] with regard to the three circles and is therefore part of the locus.

EXAMPLES

General

(1) To determine the position of the points the ratios of whose distances from three fixed points are given.

To find a point P such that

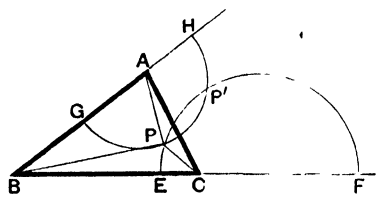
$$\frac{CP}{BP}=k, \quad \frac{AP}{BP}=l,$$

where ABC is any triangle, construct two points E and F on BC such that

$$\frac{BE}{CE}=-\frac{BF}{CF}=+\frac{1}{k}$$

and two points G and H on BA such that

$$\frac{BG}{AG}=-\frac{BH}{AH}=+\frac{1}{l}$$



Describe circles on EF and on GH as diameters to intersect in P and P'. Then P and P' are the required points. (Art. 82.)

(2) To find a point at which three collinear segments subtend equal angles

Let the segments be AB, $A'B'$, $A''B''$ Take E and F common harmonic conjugates of AA', BB', and G and H common harmonic conjugates of AA''' and BB''' Then the circles described on EF and GH as diameters meet at the two required points

(3) To find the points at which corresponding segments of two projective ranges, determined by points A, A', A'' and B, B', B'', whose self-corresponding points are imaginary, subtend equal angles (Cf Art 43)

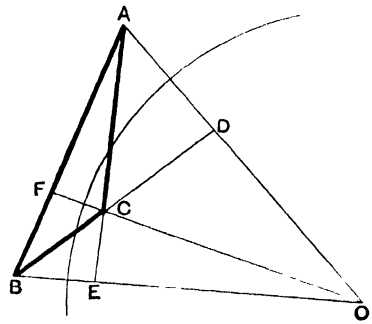
The points obtained in the last example give the solution of the problem

(4) If P, A, B, C, D are five points on a straight line and a circle, which touches the line at P, meets again the circles described on PA, PB, PC, PD, as diameters in $A'B'C'D'$, then $P(A'B'C'D')=(ABCD)$

Draw through P a diameter of the circle which touches the line at P to meet it at Q Join PA' Through A' draw a line perpendicular to PA' This line passes through A and Q Similarly BB', CC', DD' pass through Q

Hence $(ABCD)=(Q\ ABCD)=(Q\ A'B'C'D')=(P\ A'B'C'D')$

(5) Prove that a circle can be found with respect to which a given triangle is self-polar, and that the circle is real when the triangle is obtuse angled



Let D, E, F be the feet of the perpendiculars from the vertices on the opposite sides, and O the orthocentre of the triangle ABC Then

$$OE\ OB=OC\ OF=OD\ OA=K^2$$

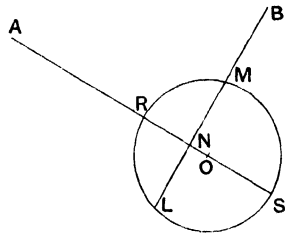
suppose Hence BC, CA, AB are the polars of A, B, C with respect to a circle centre O and radius K This is the required circle

If the triangle is acute angled OE OB is negative and the circle is imaginary but its centre, the orthocentre of the triangle, is real There is only one circle with regard to which a given triangle is self-conjugate

(6) The pedal triangle of ABC is DEF ABC is self polar with regard to a circle X Prove that the polars of DEF with respect to the circle X form a triangle similar to ABC and of twice its dimensions

F and C are conjugate points with regard to the circle X, therefore the polar of F is a line through C perpendicular to FC, that is, parallel to AB Hence the polar triangle of DEF is the triangle formed by drawing through ABC lines parallel to opposite sides

(7) Show that the square of the distance between a pair of conjugate points with respect to a circle is equal to the sum of the powers of the points

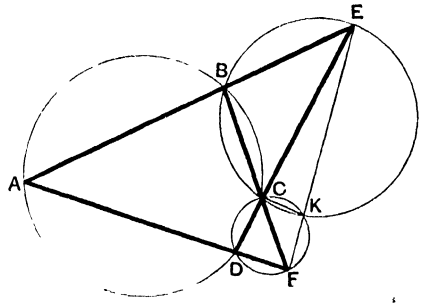


Let the polar of A meet the circle in L and M Take B any point on LM Join A to O the centre of circle O to meet LM in N Then

$$\begin{aligned} AB^2 &= AN^2 + BN^2 \\ &= AN^2 + NM^2 + BN^2 - NM^2 \\ &= AM^2 + BM\ BL \\ &= AR\ AS + BM\ BL \end{aligned}$$

This theorem may be stated in a different manner, viz

If a quadrangle be inscribed in a circle, the square of the distance between two of its diagonal points external to the circle equals the sum of the square of the tangents from these points



Describe circles round EBC and FDC meeting in K K will be on EF for $CKE + CKF = ABC + ADC =$ two right angles.

Then $EF^2 - EK\ EF + FE\ FA$
$= EC\ ED + FB\ FC$

(8) A and B being two given points in the plane of a given circle, find a point D on the circle, such that if DA, DB cut the circle again in E, F, the quadrangle $ABFE$ is inscribable in a circle

Coaxal Circles

(9) If a system of circles be such that the circles of the system have a common orthogonal circle and their centres are collinear, the system is coaxal

The perpendicular from the centre of the orthogonal circle upon the line of centres is the radical axis

(10) If the circles of one system cut orthogonally the circles of a second system, both systems are coaxal

Consider two circles of one system They determine a radical axis Since the circles of the other system cut these circles orthogonally, their centres are on this radical axis Hence their centres are collinear and as they have an orthogonal circle they form a coaxal system Similarly the other system is coaxal

(11) To draw through a given point a circle of a non-intersecting coaxal system of circles

Let E and F be the limiting points of the system Describe a circle through P, the given point, E and F, and let the tangent to this circle at P meet the line of centres at C Then the circle whose centre is C and radius CP is the required circle

(12) To describe a circle of a given coaxal system to touch a given straight line

Let the given line p meet the radical axis at P Describe a circle orthogonal to the given system with centre P to meet p in Q and Q' If the tangents at Q and Q' to this circle meet the line of centres in C and C', the two circles with centre C and C' and radii CQ and $C'Q'$ comply with the given conditions

(13) To draw a circle coaxal with two given circles (X and Y) to touch another given circle (Z)

Describe a circle of the system to meet Z in two real points P, Q Let PQ meet the radical axis in R From R draw tangents RL, RM to Z Then the circles of the coaxal system through L and M will touch Z at these points

(14) A variable circle passes through a fixed point O and cuts a given circle in Q, R, and the tangent at O meets QR in P Show that the locus of P is a straight line

Since $PQ \cdot PR = PO^2$ = the square of the tangent from P to the fixed circle, P lies on the radical axis of O and this circle

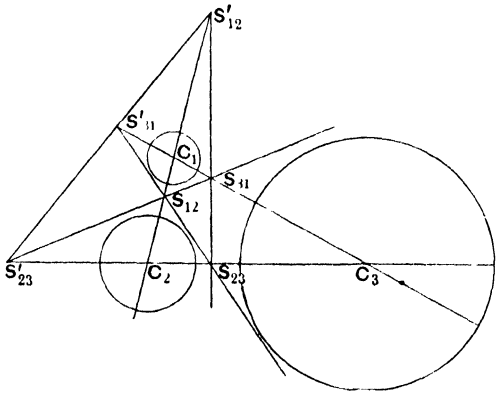
Centres of Similitude

(15) If two circles touch two other circles, the chords of contact of either pair and their radical axis pass through a centre of similitude of the other pair

(16) Find a point where three circles subtend equal angles

(17) O and O' are the centres of two circles and T is one of their centres of similitude A straight line $TRSR'S'$ is drawn to meet the circles in R, S and R', S', the points being in the order given Show that the rectangles $TR \cdot TS'$ and $TS \cdot TR'$ are equal and constant

(18) The six centres of similitude of three circles lie three by three on four straight lines termed the axes of similitude



Let the centres of the three circles be C_1, C_2, C_3, and their radii r_1, r_2, r_3 Let the internal centres of similitude be S_{12}, S_{23}, S_{31} and the external centres

S'_{12}, S'_{23}, S'_{31} as in the figure. Then if $C_1C_2C_3$ be taken as triangle of reference, the ratios of S_{12}, S_{23}, S_{31} are

$$-\frac{r_1}{r_2},\quad -\frac{r_2}{r_3},\quad -\frac{r_3}{r_1},$$

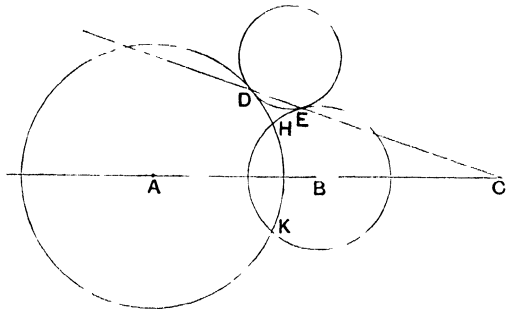
and those of S'_{12}, S'_{23}, S'_{31} are

$$\frac{r_1}{r_2},\quad \frac{r_2}{r_3},\quad \frac{r_3}{r_1}.$$

Hence the product of the ratios of S'_{12}, S'_{23}, S'_{31} is unity, as likewise are the products of the ratios of

$$S_{12}, S_{23}, S'_{31},\quad S'_{12}, S_{23}, S_{31},\quad S_{12}, S'_{23}, S_{31}.$$

(19) Two circles with centres A and B intersect in H and K, a third circle touches them at D and E and DE cuts AB in C. Show that the circle with centre C which passes through H and K cuts the third circle orthogonally.



The chord of contact DE passes through a centre of similitude of the circles with centres A and B (Example 15) and therefore $CD \cdot CE = CH^2$. Hence the square of the tangent from C to the third circle equals CH^2. Therefore the circle, centre C, radius CH, is orthogonal to the circle DE.

(20) Any line l meets the line joining the centres C and C' of two circles in L. l' is the common conjugate of l with regard to the two circles, *i.e.* the connector of P and P' the poles of l with respect to the two circles. l' meets CC' in L'. Prove that L and L' are conjugate points of an involution, and that whether or no there are common tangents to the circles, S and S', the centres of similitude of the circles, are the double points of this involution and are the common harmonic conjugates of C, C' and L, L'.

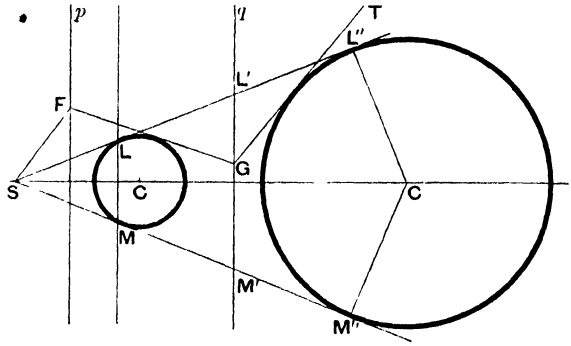
[Cf. Ex. 10, Ch. VII, and (c) Art. 106.]

Circles in Perspective

(21) S is a fixed point and p, q two given parallel straight lines, p being the radical axis of S and a fixed circle. FG is any tangent to this circle, meeting p in F and q in G, and GT is drawn parallel to SF. Prove that the envelope of GT is a circle and give a geometrical construction for finding its centre.

Take S as centre of perspective, q as axis and p as vanishing line. Then GT corresponds to GF. Hence it envelopes the curve which is the perspective of the circle. But the perspective of a circle is a circle when the vanishing line is the

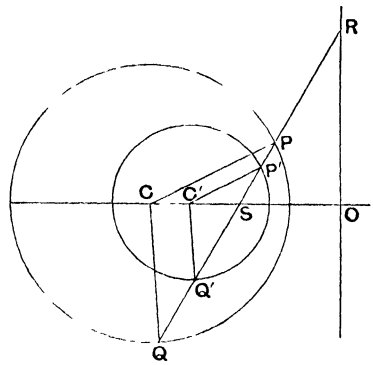
radical axis of the centre of perspective and the circle (Art 85) Hence GT touches a circle, which touches the tangents from S to the given circle and has for radical axis with the given circle the line q



If SL and SM the tangents from S meet q in L' and M', and L'', M'' be taken such that $LL' = L''L'$ and $MM' = M''M'$, then SL and SM touch the envelope at L'' and M'', and its centre C'' is the point of intersection of the perpendiculars through M'' and L''

(22) To find the centre and axis of perspective of two circles one of which is situated entirely within the other

Through C and C'' the centres of the circles, whose radii are r and r', draw any pair of parallel lines to meet the circles in P and P'' and let PP' meet CC'' in S and the radical axis in R



Then $\dfrac{CS}{C''S} = \dfrac{PS}{P''S} = \dfrac{CP}{C''P''} = \dfrac{r}{r'}$ Therefore S is a centre of similitude of the circles and is a fixed point (Art 84)

(1) Take S as centre and the line at infinity as axis of perspective

Then $(S\infty PP') = \dfrac{SP}{SP'} = \dfrac{r}{r'}$ Hence the circles are in perspective

(2) Let $SP'P$ meet the circles again in Q and Q' Then $\dfrac{SQ}{SQ'} = \dfrac{r}{r'}$

Take S as centre and the radical axis as axis of perspective

Then $$\frac{RP}{RQ'} = \frac{RP'}{RQ} = \frac{PP'}{QQ} = \frac{SP' - SP}{SQ - SQ'}$$

But $$(SRPQ') = \frac{SP}{SQ'}\,\frac{RP}{RQ'} = \frac{SP}{SQ}\,\frac{SP' - SP}{SQ - SQ'} = \left(\frac{r}{r'} - 1\right)\left(\frac{r'}{r} - 1\right) = -\frac{r}{r'}$$

Therefore the circles are again in perspective

A second centre of perspective S' may be found by producing CP to meet the larger circle in P_1 and determining S' from P_1 and P'

CHAPTER XII

PROJECTIVE THEOREMS FOR THE CIRCLE —CARNOT'S THEOREM PASCAL'S THEOREM DESARGUES' THEOREM

89 The proofs of Carnot's, Pascal's, Desargues' Theorems, and their correlatives for the circle are very similar to those for the conic, but in some cases they take a slightly different form and the proofs are simpler When proved for the circle they may be deduced for the conic by projection In this chapter, certain proofs for the circle are given, but the full discussion of these theorems is left over till they are considered for the conic in Chapter XIV

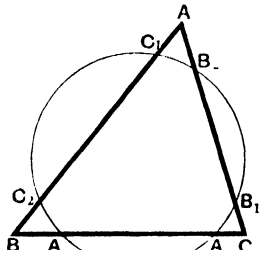
Carnot's theorem for the circle and its correlative

If a circle meet the sides BC, CA, AB of a triangle ABC in three pairs of points,

$$A_1A_2 \quad B_1B_2, \quad C_1C_2,$$

then

$$\frac{BA_1}{CA_1}\,\frac{BA_2}{CA_2}\,\frac{CB_1}{AB_1}\,\frac{CB_2}{AB_2}\,\frac{AC_1}{BC_1}\,\frac{AC_2}{BC} = 1$$

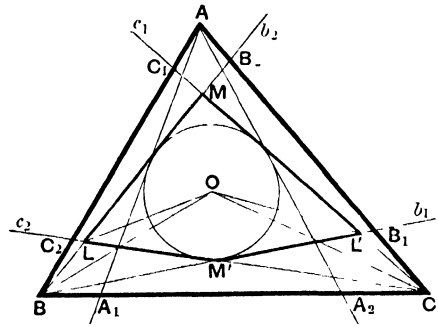


If the tangents to a circle from the vertices A, B, C of a triangle ABC meet the opposite sides in three pairs of points,

$$A_1A_2, \quad B_1B_2, \quad C_1C_2,$$

then

$$\frac{BA_1}{CA_1}\,\frac{BA_2}{CA_2}\,\frac{CB_1}{AB_1}\,\frac{CB_2}{AB_2}\,\frac{AC_1}{BC_1}\,\frac{AC_2}{BC_2} = 1$$



Since the two chords BC_2C_1 and BA_1A_2 are drawn through the point B to meet the circle in A_1, A_2 and B_1, B_2,

$$BA_1 \; BA_2 = BC_1 \; BC_2$$

Similarly

$$AC_1 \; AC_2 = AB_1 \; AB_2,$$
$$CB_1 \; CB_2 = CA_1 \; CA_2$$

Therefore

$$\frac{BA_1 \; BA_2}{CA_1 \; CA_2} \cdot \frac{CB_1 \; CB_2}{AB_1 \; AB_2} \cdot \frac{AC_1 \; AC_2}{BC_1 \; BC_2} = 1$$

Let the points BB_2 CC_1, BB_2 CC_2, BB_1 CC_1, BB_1 CC_2 be denoted by M, L, L', M', respectively, and let O be the centre of the circle

By Addendum (11)

$$\frac{BL \; BL'}{CL \; CL'} = \frac{OB^2}{OC^2}$$

If the sides of the triangle are denoted by a, b, c and the tangents by a_1, a_2, b_1, b_2, c_1, c_2, then from consideration of the triangles $BL'C$ and BLC

$$\frac{\sin \widehat{ac_1} \; \sin \widehat{ac_2}}{\sin \widehat{ab_1} \; \sin \widehat{ab_2}} = \frac{BL \; BL'}{CL \; CL'} = \frac{OB^2}{OC^2}$$

Similarly

$$\frac{\sin \widehat{ba_1} \; \sin \widehat{ba_2}}{\sin \widehat{bc_1} \; \sin \widehat{bc_2}} = \frac{OC^2}{OA^2}$$

$$\frac{\sin \widehat{cb_1} \; \sin \widehat{cb_2}}{\sin \widehat{ca_1} \; \sin \widehat{ca_2}} = \frac{OA^2}{OB^2}$$

Therefore

$$\frac{\sin \widehat{ac_1} \; \sin \widehat{ac_2}}{\sin \widehat{bc_1} \; \sin \widehat{bc_2}} \cdot \frac{\sin \widehat{ba_1}}{\sin \widehat{ca_1}} \cdot \frac{\sin \widehat{ba_2} \; \sin \widehat{cb_1}}{\sin \widehat{ca_2} \; \sin \widehat{ab_1}} \cdot \frac{\sin \widehat{cb_2}}{\sin \widehat{ab_2}} = 1 \qquad (1)$$

But from the triangles CC_1A and CC_1B

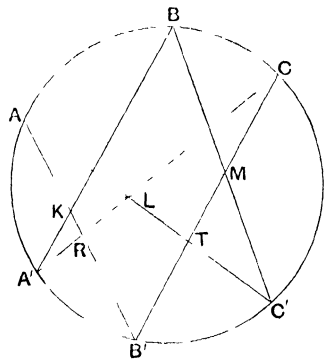
$$\frac{\sin \widehat{bc_1}}{\sin \widehat{ac_1}} = \frac{AC_1}{BC_1} \cdot \frac{\sin A}{\sin B}$$

Substituting this and the similar values in (1) it follows that

$$\frac{BA_1 \; BA_2}{CA_1 \; CA_2} \cdot \frac{CB_1 \; CB_2}{AB_1 \; AB_2} \cdot \frac{AC_1 \; AC_2}{BC_1 \; BC_2} = 1$$

90 Pascal's Theorem for the circle and its correlative (Brianchon's Theorem)

If a hexagon be inscribed in a circle the three pairs of opposite sides intersect in three collinear points



In the figure let $AC'BA'CB'$ be any inscribed hexagon, and let K, L, M be the points of intersection of pairs of opposite sides Let AB' meet $A'C$ in R, and $B'C$ meet AC' in T

By the anharmonic property of the circle (Art 73)

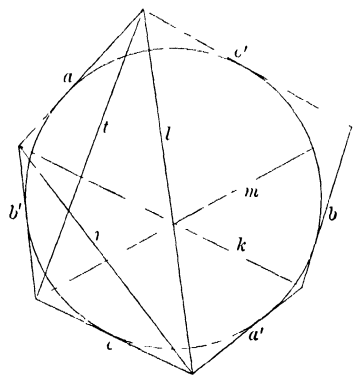
$$(A'\ B'CBA) = (C'\ B'CBA)$$

Therefore taking intercepts on AB' and CB',

$$(B'RKA) = (B'CMT)$$

Since these ranges have a self-corresponding point at B', the lines RC, KM, and TA, which join corresponding points, are concurrent Therefore the points K, L, M are collinear

If a hexagon be circumscribed to a circle the connectors of the three pairs of opposite vertices are concurrent



In the figure let $ac'ba'cb'$ be any circumscribed hexagon, and let k, l, m be the connectors of pairs of opposite vertices Let ab' $a'c$ be r, and $b'c$ ac' be t

By the anharmonic property of tangents to a circle (Art 73)

$$(a'\ b'cba) = (c'\ b'cba)$$

Therefore joining these ranges to ab' and cb',

$$(b'rka) = (b'cmt)$$

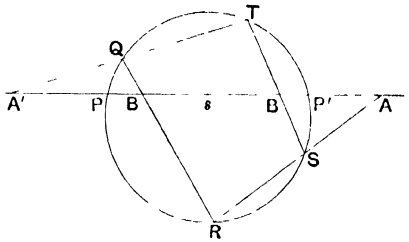
Since these pencils have a self-corresponding ray in b', the points rc, km, and ta, which are the intersections of corresponding rays, are collinear Therefore the lines k, l, m are concurrent

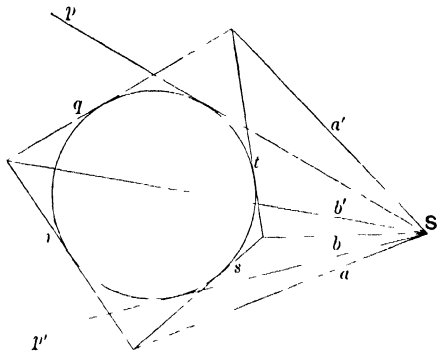
91 Desargues' Theorem for the circle and its correlative

Desargues' Theorem for a circle takes two different forms

(1) *If a quadrangle be inscribed in a circle, the pair of points in which this circle is met by any transversal are a pair of conjugate points of the involution determined on that transversal by the sides of the quadrangle*

If a quadrilateral be circumscribed to a circle, the pair of tangents from any point to the circle are a pair of conjugate lines of the involution determined at that point by the vertices of the quadrilateral





Let Q, R, S, T be any four fixed points on a circle and let any transversal s meet the circle in P, P' and the lines QR, ST, QT, RS in B', B, A' A

Then

$$(Q\ PRP'T) = (S\ PRP'T)$$

by the anharmonic property of the circle

But $(Q\ PRP'T) = (PB'P'A')$,

and $(S\ PRP'T) = (PAP'B)$

Therefore

$$(PB'P'A') = (PAP'B)$$

Therefore AA', BB', PP' are three pairs of conjugate points of an involution

Let q, r, s, t be any four fixed tangents to a circle, let the tangents from any point S to the circle be p, p' and let the lines joining S to qr, st, qt, rs be b', b, a', a

Then

$$(q\ prp't) = (s\ prp't)$$

by the anharmonic property of tangents to a circle

But $(q\ prp't) = (pb'p'a')$,

and $(s\ prp't) = (pap'b)$

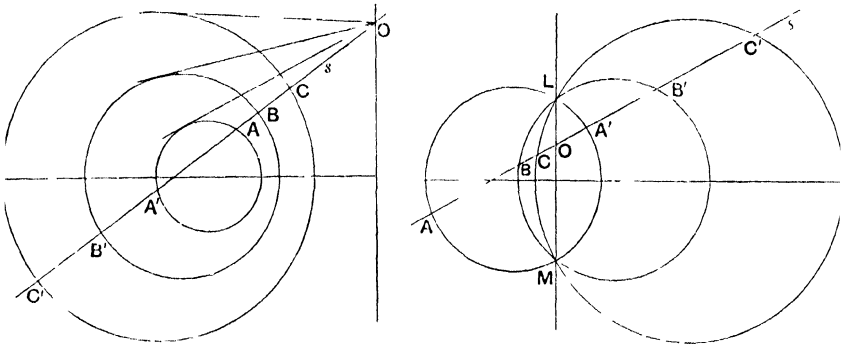
Therefore

$$(pb'p'a') = (pap'b)$$

Therefore aa', bb', pp' are three pairs of conjugate rays of an involution pencil

(11) It will be shown hereafter (Art 101) that every circle may be regarded as passing through two imaginary fixed points at infinity and that a circle may be considered to be a conic through these two fixed points at infinity It follows therefore from the general form of Desargues' theorem (Art 101) that for the circle it may be stated in a second form as follows

Every transversal cuts a system of coaxal circles in pairs of conjugate points of an involution



Let the transversal s meet the circles in AA', BB', CC', and the radical axis in O

If O is external to the circles the tangents from O to the circles are equal, and the squares of these tangents are equal to $OA\ \ OA'$, $OB\ \ OB'$, $OC\ \ OC'$, Hence $OA\ \ OA' = OB\ \ OB' = OC\ \ OC' =$ and the pairs of points AA', BB', CC', are pairs of conjugate points of an involution of which O is the centre

If O is inside the circles, the radical axis must meet the circles in two real points L and M Then

$$OL\ \ OM = OA\ \ OA' = OB\ \ OB = OC\ \ OC' =$$

Hence in this case also AA', BB', CC', are pairs of points of an involution of which O is the centre

In the former case the double points of the involution are real and in the latter they are imaginary

EXAMPLE

Through ABC the vertices of a triangle parallel lines are drawn cutting the circumcircle in $A'B'C'$ Show that the lines joining A', B', C' to any other point on the circumcircle cut BC, CA, AB, respectively, in points which lie on a straight line parallel to AA'

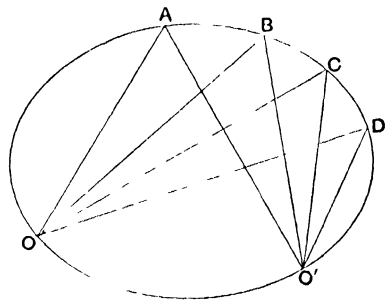
Let P be the point on the circumcircle Considering the Pascal hexagon $A'P$, PC', $C'C$, CB, BA, AA', it is seen that two of the points are on a line parallel to AA'. By considering the two similar hexagons the result is obtained

CHAPTER XIII

PROJECTIVE FORMS IN RELATION TO THE CONIC—DEFINITION, CONSTRUCTION AND ANHARMONIC PROPERTY OF THE CONIC GENERAL PROPERTIES OF THE CONIC PROPERTIES OF THE FOCI OF A CONIC ALTERNATIVE DEFINITIONS OF A CONIC

92 Definition A Conic Section, or shortly, a Conic, may be defined in several different ways, according to the point of view from which it is considered. *A Conic* is here defined as *the locus of the points of intersection of pairs of corresponding rays of two projective pencils*. In the particular case in which the projective pencils are in perspective the conic becomes a pair of straight lines.

(*a*) *A conic may be described through any five points.*



Let O, O', A, B, C be any five points in a plane. Join O and O' to A, B, C. Then OA, OB, OC and $O'A$, $O'B$, $O'C$ determine two projective pencils (Art. 35), the corresponding rays of which intersect in a conic, which passes through A, B, C. Since there is a ray of the pencil, whose vertex is O, which corresponds to $O'O$ looked upon as a ray of the pencil whose vertex is O', a pair of corresponding rays intersect at O. Similarly a pair of corresponding rays intersect at O'. Hence the conic passes through A, B, C, O and O'.

If three of the five points are collinear—say A, B, C,—the two pencils are in plane perspective. In this case the conic breaks up into the straight line ABC and the line OO' joining the vertices of the pencils.

The conic* obtained as the intersection of the pencils $(O\ ABC\ )$ and $(O'\ ABC\ )$ is completely determined when O, O', A, B, C are given, for (Art. 35) any number of pairs of corresponding rays may be constructed.

* There is an exception to this when four of the points O, O', A, B, C are collinear.

(*b*) *The anharmonic ratio of the pencil formed by joining any four fixed points on a conic to a variable point on the conic is constant*

Let A, B, C, D be any four points of intersection of corresponding rays of the projective pencils, whose vertices are O and O', so that A, B, C, D are any four points on the conic through $OO'ABCD$ Then the anharmonic ratios of the pencils $(O\ ABCD)$ and $(O'\ ABCD)$ are equal

But the conic is completely determined by the five points O', A, B, C, D Hence O may be looked upon as a variable point on the conic, and the anharmonic ratio of the pencil formed by joining any point O on the conic to any four fixed points A, B, C, D is constant

(*c*) *Through five given points, no four of which are collinear, only one conic can be described*

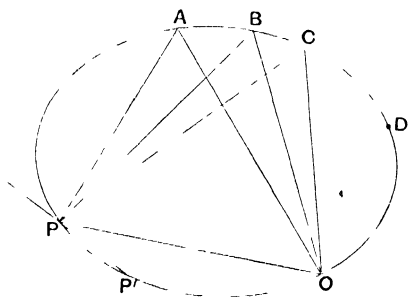
Let the five points be A, B, C, O, O' If two different conics pass through these points, draw through O' a line to meet one conic in D the other in D' Then the anharmonic ratios of the pencils $(O\ ABCD)$ and $(O\ ABCD')$ being equal to that of the pencil $(O'\ ABCD)$ are equal to each other Hence the points D and D' must coincide

(*d*) *No straight line can meet a conic in more than two points unless the conic is a pair of straight lines*

If a line meets a conic in three points A, B, C, join A, B, C to any other two points O and O' on the conic Then the pencils $(O\ ABC\ )$ and $(O'\ ABC\ )$ are in perspective and the conic becomes the line ABC and the line OO'

(*e*) *There is one and only one tangent at every point on a conic**

Let A, B, C, D, O be the five points which determine the conic, and P any other point on the conic Then the pencil formed by joining P, A, B, C to any other point P' on the conic has the same anharmonic ratio as $(O\ PABC)$ Let the point P' move indefinitely near to P Then the line PP' becomes the tangent at P, and it follows that the tangent at P together with PA, PB, PC forms a pencil of the same anharmonic ratio as the pencil $(O\ PABC)$

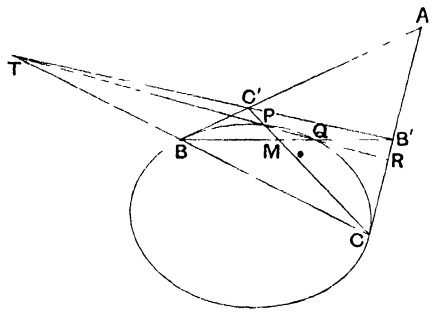


* There is an exception to this if the conic is a pair of straight lines

Hence, since three rays of this pencil and its anharmonic ratio are given, the fourth ray, which is the tangent at P, is uniquely determined

(f) *Given two tangents to a conic and their points of contact, to construct the tangent at any point on the conic*

Let the tangents at B and C to a conic meet at A Take P and Q any two points on the conic and let CP, BQ intersect at M and meet AB and AC in C' and B'



Then

$$(B\ BPQC) = (C\ BPQC)$$

Therefore

$$(C'PMC) = (BMQB') = (B'QMB)$$

Hence $C'B'$, PQ and CB are concurrent at some point T Let P and Q coincide Then $C'B'$, BC, and the tangent at P are concurrent Hence from the harmonic property of the quadrangle, if TP meet $A'C$ in R, the range $(ARB'C)$ is harmonic

Hence to construct the tangent at any point P, join BP to meet AC at B' and construct the point R on AC, which is the harmonic conjugate of A with respect to CB'

Then PR is the tangent at P

(g) *Any four fixed tangents to a conic determine a range of constant anharmonic ratio on all other tangents to the conic*

Let AC and AB be two given tangents to the conic Take points P_1, P_2, P_3, P_4 on the conic Let BP_1, BP_2, BP_3, BP_4 meet AC in B_1', B_2', B_3', B_4' and let the tangents at P_1, P_2, P_3, P_4 meet AC in R_1, R_2, R_3, R_4 Then the ranges $(AR_1B_1'C)$, $(AR_2B_2'C)$, $(AR_3B_3'C)$, $(AR_4B_4'C)$ are all harmonic Hence (Art 14, Ex (3)) $(R_1R_2R_3R_4) = (B_1'B_2'B_3'B_4')$ = the anharmonic ratio of the pencil formed by joining P_1, P_2, P_3, P_4 to any point on the conic The tangents at P_1, P_2, P_3, P_4 therefore determine on any other tangent a range of a constant anharmonic ratio, namely that of the pencil formed by joining P_1, P_2, P_3, P_4 to any point on the conic

93 Correlative Definition The correlative to the definition given in the last Article is as follows *The envelope of the lines which join pairs of corresponding points of two projective ranges is the correlative of a conic* It will be shown that the correlative of a conic is a conic, and this will be assumed in the meantime in the statement of certain theorems in this article

If the ranges are in perspective the conic becomes a pair of points

In the same way that properties of the conic were deduced from the preceding definition the following may be proved from the correlative definition

(*a*) *A conic may be described to touch any five lines*

(*b*) *The anharmonic ratio of the range formed by the intersection of any four fixed tangents to a conic with a variable tangent is constant*

(*c*) *Only one conic can be described to touch five given lines, no three of which are concurrent*

(*d*) *Through no point can more than two tangents be drawn to a conic*

(*e*) *Each tangent has one and only one point of contact*

The correlative of a conic is a conic

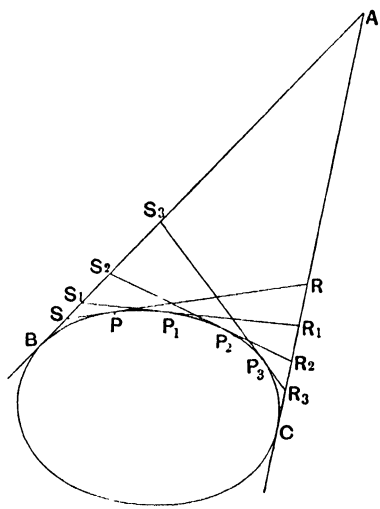
Describe a conic through any five points B, C, P_1, P_2, P_3 and draw the tangents at these points Let the tangents at P_1, P_2, P_3 meet the tangents BA and CA in S_1, S_2, S_3 and R_1, R_2, R_3, respectively Describe the correlative of a conic to touch these five lines Draw SR any other tangent to the conic

Since

$$(SS_1S_2S_3) = (RR_1R_2R_3)$$

the line SR is a tangent to the correlative of the conic

Hence any tangent to the conic is also a tangent to the correlative of a conic, which must therefore be a conic



The anharmonic property of the conic which has already been proved may be summarised as follows

(*a*) **The locus of the points of intersection of corresponding rays of two projective pencils is a conic through the vertices of the pencils**

(*a*) **The envelope of the lines joining corresponding points of two projective ranges is a conic touching the bases of the ranges**

Or conversely

(*b*) The anharmonic ratio of the pencil formed by joining any four fixed points on a conic to a variable point on the conic is constant

(*b*) The anharmonic ratio of the range formed by the intersections of any four fixed tangents to a conic with a variable tangent to the conic is constant

94 Classification of conics

Conics are divided into different classes according to the nature of their intersections with the line at infinity There is a point at infinity on a conic determined by each pair of parallel corresponding rays of the generating pencils On reference to Art 38 it will be seen that two projective pencils may have two pairs of parallel corresponding rays, one pair of such rays, or no pair

(i) If the projective pencils have two pairs of parallel corresponding rays the conic is termed *a hyperbola*

(ii) If they have one pair of such rays the conic is termed *a parabola*

(iii) If they have no such pair the conic is termed *an ellipse*

There are two particular cases of the preceding

(iv) If the projective pencils are oppositely equal the conic is termed *a rectangular hyperbola*

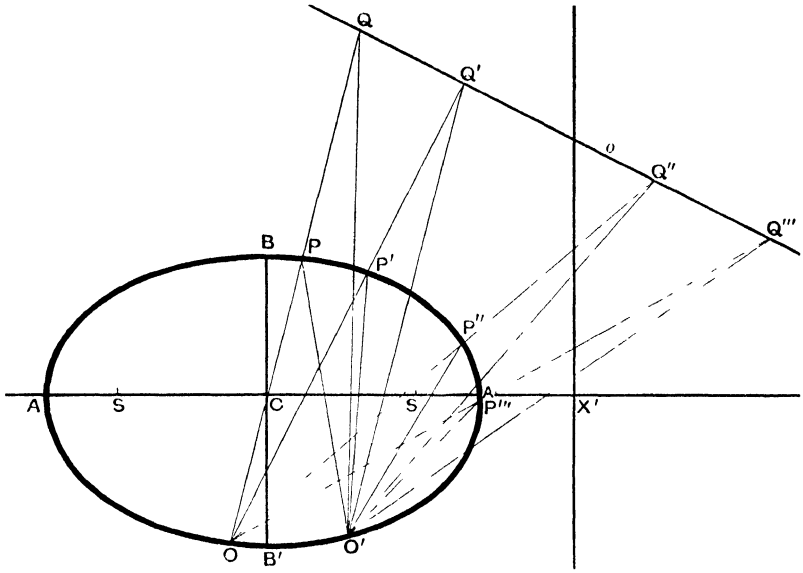
(v) If the projective pencils are directly equal the conic is *a circle*

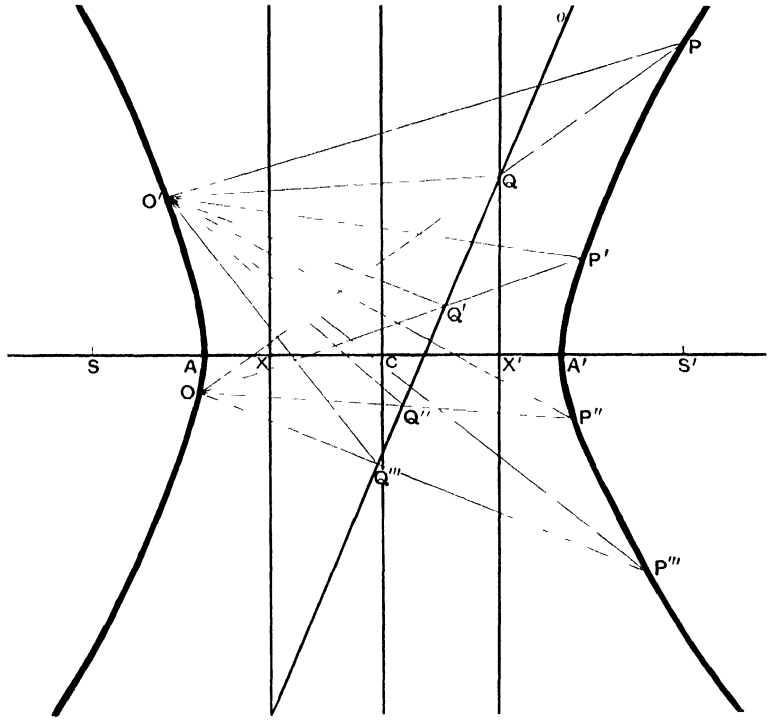
In Art 93 it was shown that the correlative of a conic is a conic The proof there given does not apply to the case when the conic breaks up into a pair of straight lines Hence to the above should be added

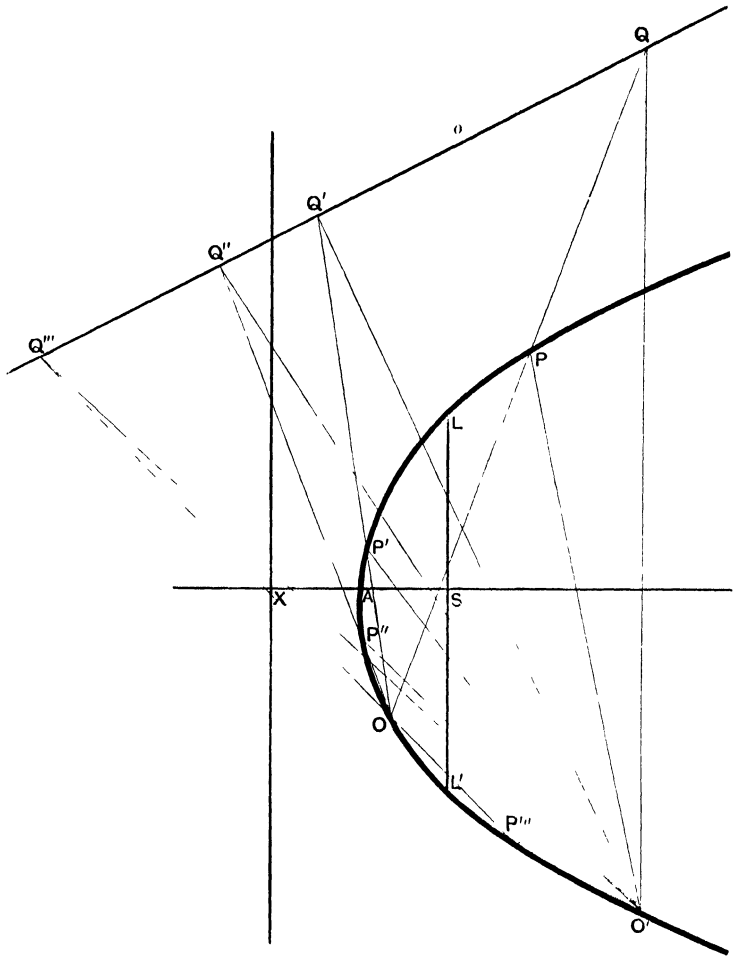
(vi) If the projective pencils are in plane perspective the conic becomes *a pair of straight lines*, viz the axis of perspective and the connector of the vertices

If the ranges, which determine the correlative curve, are in plane perspective, the conic becomes *a pair of points*, viz the centre of perspective and the point of intersection of the bases

In Art 37 it was shown how pencils of this nature could be constructed The following are the figures of the three kinds of conics constructed by the method there explained as the loci of the points of intersection of projective pencils


Q
Q'
o
Q''
Q'''
B
P
P'
P''
A
S
C
S
A
P'''
X'
O
B'
O'


o
P
O'
Q
P'
Q'
S
A
X
C
X'
A'
S'
O
Q''
P''
Q'''
P'''



In the figures, which represent respectively an Ellipse, a Hyperbola and a Parabola, O and O' are the vertices of two pencils which have o for axis of perspective. Pairs of corresponding rays intersect in the points Q, Q', Q'', Q'''. The pencil vertex O' has been rotated round its vertex so that its rays in the new position intersect the corresponding rays of the pencil vertex O in the points P, P', P'', P'''. These are points on the conics, as also are O and O'. In the case of the Ellipse and Hyperbola (see Art. 96), C is the centre, S and S' are the foci, AA' is the major or transverse axis and the perpendiculars through C, S, and S' are the minor or conjugate axis and the two directrices.

In the Parabola S is the focus, A the vertex, AS the axis and LSL' the latus rectum and the perpendicular through X, the directrix.

95 Summary of Projective Properties of a Conic

The projective properties of circles given in Chapter x, which depend on the anharmonic property of the circle, hold for the more general case of the conic The proofs there given are in such a form as to be applicable in the case of the conic with the mere substitution of the word "conic" for "circle" The student is referred to this chapter for the proofs of the following important properties of the conic

(*a*) The properties of projective ranges on a conic and of projective systems of tangents Art 74

(*b*) Construction of an involution of points on a conic and of tangents to a conic Art 74

(*c*) *If through any point S chords be drawn to intersect a conic in AA', BB', CC', these pairs of points are pairs of conjugate points of an involution on the conic*

If from points on a line s pairs of tangents aa', bb', cc', be drawn to a conic these pairs of tangents intersect any other tangent in pairs of conjugate points of an involution

Conversely

If AA', BB', CC', are pairs of conjugate points of an involution on a conic, the lines AA', BB', CC', are concurrent Art 75

If aa', bb', cc', are pairs of conjugate rays of an involution of tangents to a conic, the points aa', bb', cc', are collinear Art 75

(*d*) *If through the point of intersection of the tangents at any two points T and T' on a conic, a chord be drawn to meet the conic in A and A', then the range TT'AA' is harmonic* Art 75

If from any point on the chord of contact of any two tangents t and t' to a conic, two other tangents a and a' be drawn to the conic, then the system of tangents tt'aa' is harmonic Art 75

(*e*) *If through any point S a variable chord SAA' of a conic be drawn the locus of the harmonic conjugate of S with regard to AA' is a straight line s* Art 76

If from a variable point on a straight line s pairs of tangents a, a' be drawn to a conic the envelope of the harmonic conjugate of s with regard to aa' is a fixed point S Art 76

(*f*) The line s is called the polar of S Art 76

The point S is called the pole of s Art 76

(*g*) Construction for the polar of a given point Art 76

Construction for the pole of a given line Art 76

(*h*) If P and Q are two points such that the polar of one passes through the other, the points are called *conjugate points* with respect to the conic Art 76

If p and q are two straight lines such that the pole of one lies on the other, the lines are called *conjugate lines* with respect to the conic Art 76

(*i*) *If p the polar of P p[illegible] q Q, then q the polar of Q passes through P* Art 76

(*j*) The construction for pairs of conjugate points, with respect to a conic, on a given straight line, also the converse construction for the pole of a line Art 76

(*k*) *A conic determines by means of pairs of collinear conjugate points a definite involution on every straight line in its plane, the double points of which are the points of intersection of the line and conic* Art 77

And the correlative

A conic determines by means of pairs of concurrent conjugate lines a definite involution pencil at every point in its plane, the double rays of which are the tangents from the point to the conic Art 77

(*l*) *The anharmonic ratio of a range of four collinear points is equal to that of the pencil formed by their polars* Art 77

(*m*) *The anharmonic ratio of four points on a conic is equal to that of the four tangents at these points* Art 77

See also Art 92 (*g*)

(*n*) Properties of pole and polar Art 78

(*o*) *Any conic may be looked upon as in harmonic perspective with itself with any point as centre of perspective and the polar of that point as the axis* Art 79

(*p*) Properties of complete inscribed quadrangle and complete circumscribed quadrilateral of a conic Art 80

(*q*) Construction of a self-polar triangle of a conic Art 81

96 Centre, asymptotes, conjugate diameters, axes, and foci of a conic

Centre It has been proved (Art 22) that the points at infinity in a plane may be regarded as lying on a straight line termed the line at infinity Every line in the plane of a conic has a definite point associated with it which is the pole of the line with respect to the conic (Art 95 (*n*)) The pole of the line at infinity with respect to a conic is termed *its centre* If C be the centre of a conic and any chord be drawn through it

to meet the conic in P and P' and the line at infinity at ∞, then $(PP'C\infty) = -1$ Hence PP is bisected at C Also the tangents at P and P' intersect on the line at infinity and are therefore parallel

Asymptotes If the generating pencils of a conic have a pair or pairs of parallel corresponding rays, each such pair determines a point of the curve on the line at infinity In a hyperbola there are two such points Since the centre is the pole of the line at infinity, the lines through the centre to these points are tangents to the curve, whose points of contact are at infinity These lines are termed the *asymptotes* of the hyperbola

In the case of the parabola there is one point at infinity, which must be situated on the line at infinity A line which meets a conic in only one point must touch the curve Hence a parabola touches the line at infinity Since the pole of a tangent is its point of contact the centre of a parabola is the point at infinity on the curve A straight line which passes through the point at infinity on a parabola is termed a diameter Such a diameter meets the curve at one point at a finite distance

The asymptotes of an ellipse are imaginary

A conic whose centre is at a finite distance, namely an Ellipse or a Hyperbola, is termed a Central Conic

Conjugate diameters If a diameter PCP' be drawn through the centre C of a central conic, the tangents p and p' at P and P are parallel and meet at some point, ∞, on the line at infinity The polar of ∞ is the diameter PCP' If ∞C meet the curve in Q and Q' then the lines PCP' and QCQ are termed conjugate diameters and are such that each is parallel to the tangents at the ends of the other If through ∞ a chord $\infty RMR'$ be drawn to meet the conic in R and R' and PCP' in M, the range $\infty MRR'$ is harmonic and therefore M bisects RR' But this chord is parallel to QCQ' Hence chords of a conic parallel to a diameter are bisected by its conjugate diameter

Axes Every pair of conjugate diameters is such that each diameter passes through the pole of the other Therefore pairs of conjugate diameters are pairs of conjugate rays of an involution of which the centre is the vertex Such an involution has either one pair of rays at right angles or all pairs of corresponding rays are at right angles (Art 58) In the general case when there is one pair of conjugate rays at right angles, these are termed the axes of the conic They are

such that the tangents at the points in which each meets the curve are at right angles to the axis in question If every pair of conjugate rays are at right angles the conic—as can be easily proved—is a circle

In the case of the ellipse both axes meet the curve in real points and the distances of these points from the centre of the curve are termed the semi-major and semi-minor axes In the case of the hyperbola one axis, termed the transverse axis, meets the curve in real points The second axis termed *the conjugate axis* does not meet the curve in real points, but it is convenient to speak of the distance of the centre from the points, where it is met by a parallel to an asymptote drawn through an end of the transverse axis, as a semi-axis of the curve

In the case of the parabola, if a tangent meet the curve at P and the line at infinity at ∞, the polar of ∞ is the diameter through P Hence if a chord be drawn through ∞ to meet the curve in R and R' and the diameter in M, $(\infty MRR')$ is harmonic and M bisects R, R' This chord, since it passes through ∞, is parallel to the tangent at P Hence chords of a parabola parallel to a tangent are bisected by the diameter through the point of contact of the tangent

If a line be drawn perpendicular to the diameters of the parabola, it will meet the curve in two points P and P' Let the tangents at P and P' meet at T and through T draw a diameter to meet the curve in A The tangent at A will be parallel to PP' For, since PP' the polar of T passes through the point at infinity on PP', the polar of the point at infinity on PP' is the diameter through T and therefore the tangent at A, and the line PP' are parallel Hence in every parabola there is one point the tangent at which is perpendicular to the direction of the diameters This point A is called the vertex and the corresponding diameter the axis of the parabola

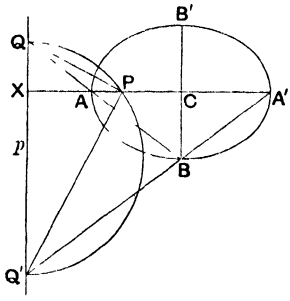
Foci of a conic* It has been shown Art 95 (*k*) that a conic determines on every line p in its plane an involution by means of pairs of collinear conjugate points and at every point P in its plane an involution pencil by means of pairs of concurrent conjugate lines The double rays of this involution pencil, if real, are the tangents from P to the conic Such an involution pencil has always one pair of conjugate rays at right angles (Art 58) If the involution pencil has more than one pair of conjugate lines at right angles every pair of conjugate rays is at right angles and every pair of lines at right angles through the vertex P form a pair of conjugate rays of the involution (Art 58) In

* In Art 97 a definition of a conic depending on the focus and directrix is given

this case P the vertex of the involution pencil is termed *a focus of the conic*, and its polar *the corresponding directrix*

A focus must be situated on an axis of the conic, for if the focus be joined to the centre of the conic, the pole of this diameter is at infinity on the perpendicular to the diameter through the focus Hence the tangents at the ends of the diameter are perpendicular to the diameter, which is therefore an axis The polar of a focus must also be parallel to an axis, that is, a directrix is parallel to an axis of the conic

To find the foci of a central conic



Ellipse

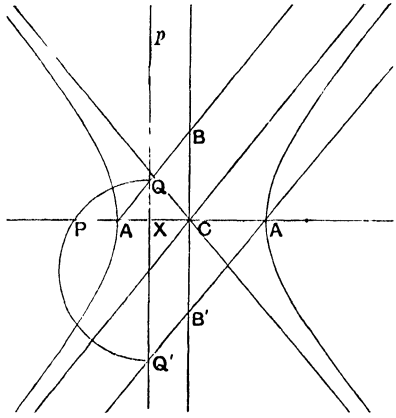
Let P be a focus of an ellipse, whose axes are AA' and BB' and centre C Let the polar p of P meet AA' in X Since the axis AA' passes through P the lines AB and $A'B$ determine a pair of conjugate points Q and Q' on p (Art 96 (n)), which since P is a focus subtend a right angle at P X is the centre of the involution on p

By similar triangles

$$XQ = XA \frac{CB}{CA}$$

and

$$XQ' = XA' \frac{CB'}{CA'}$$



Hyperbola

Let P be a focus of a hyperbola, whose transverse axis is AA' and whose centre is C Let the polar p of P meet AA' in X Draw AQ and $A'Q'$ parallel to an asymptote of the hyperbola to meet p in Q and Q' and the perpendicular through C in B and B' Since the axis passes through P, Q and Q' are a pair of conjugate points of the involution on p, which since P is a focus subtend a right angle at P X is the centre of the involution

By similar triangles

$$XQ = XA \frac{CB}{CA}$$

and

$$XQ' = XA' \frac{CB'}{CA'}$$

$$XQ \cdot XQ' = -XA \cdot XA' \frac{CB^2}{CA^2},$$

$$XP^2 = (XC^2 - AC^2)\frac{CB^2}{CA^2}$$

$$= (XC^2 - CP \cdot CA)\frac{CB^2}{CA^2},$$

since $(APAA')$ is harmonic

$$(CA - CP)^2 = CA(CX - CP)\frac{CB^2}{CA^2},$$

$$CX - CP = CX\frac{CB^2}{CA^2} = \frac{CB^2}{CP},$$

$$CX \cdot CP - CB^2 = CP^2,$$

$$CP^2 = CA^2 - CB^2$$

$$AQ \cdot XQ' = XA \cdot YA' \frac{CB^2}{CA^2},$$

$$XP^2 = (CA^2 - CX^2)\frac{CB^2}{CA^2}$$

$$= (CA \cdot CP - CA^2)\frac{CB^2}{CA^2},$$

since $(XPAA')$ is harmonic

$$(CP - CA)^2 = CX(CP - CA)\frac{CB^2}{CA^2},$$

$$CP - CX = CA\frac{CB^2}{CA^2} = \frac{CB^2}{CP},$$

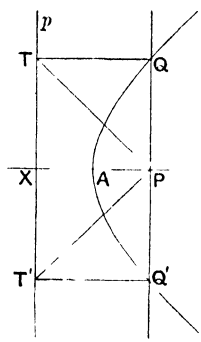
$$CP^2 - CA \cdot CP = CB^2,$$

$$CP^2 = CA^2 + CB^2$$

Hence for the ellipse, if $CA > CB$, there are two real foci at equal distances from C, and the foci are therefore situated on the major axis For the hyperbola there are two foci on the transverse axis, that is on the axis which meets the curve in real points

To find the focus of a parabola

Let P be a focus of the parabola and p its polar, then P must be on the axis and p perpendicular to it Let the axis meet the curve in A and p in X Draw QPQ' perpendicular to the axis and QT and $Q'T'$ parallel to the axis to meet p in T and T'



Then, since $(Q'QP\infty) = -1$,

$$PQ = PQ',$$

and since $(APA\infty) = -1$,

$$XA = AP$$

Join TP and $T'P$ Then T and T' are conjugate points and, since P is a focus, TP and $T'P$ are at right angles Therefore $QPAT$ and $QPXT'$ are squares and

$$QP = AP = 2\ AP$$

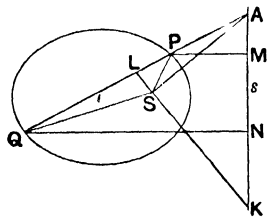
Hence the focus of a parabola is at such a distance from the vertex that the ordinate at the point is double this distance

To find the focus of a circle

From the construction for the foci of an ellipse, it follows that the two foci of a circle coincide at the centre This also follows from the fact that conjugate lines through the centre of a circle are at right angles

Principal properties of the foci of a conic

(*a*) *If S be a focus, s the corresponding directrix, P any point on the conic, and PM the perpendicular from P on the directrix, then the ratio $\frac{SP}{PM}$ is constant*



Let the chord PQ, joining any two points on the conic, meet s in A Join SA, SP, SQ Draw LSK perpendicular to SA to meet PQ and s in L and K respectively Then SA and SL, since they are at right angles, are conjugate lines through S

Therefore ASK is a self conjugate triangle Hence SL is the polar of A and $(ALPQ)$ is harmonic Therefore, since SL and SA are at right angles, they are the bisectors of the angle PSQ (Art 16)

Therefore $\frac{SQ}{SP}=\frac{QA}{PA}=\frac{QN}{PM}$, where PM and QN are perpendiculars on s Hence

$$\frac{SP}{PM}=\frac{SQ}{QN}=\text{constant}=e \text{ (suppose)}$$

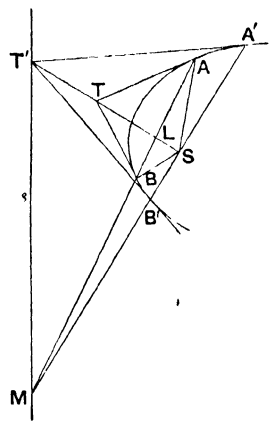
(*b*) *In an ellipse the sum of the distances, and in a hyperbola the differences of the distances, of any point on the curve from the foci is constant*

Let S and S' be the foci and PM and PM' the perpendiculars from P on the directrices It follows for the ellipse that $SP+SP'=e(PM+PM')=e\ MM'$ and for the hyperbola that $SP-SP'=e(PM-PM')=e\ MM'$, where MM' is the perpendicular distance between the directrices

(*c*) *The tangents from a focus to a conic pass through the circular points at infinity*

The tangents from a focus are the double rays of the involution of conjugate lines through the focus Since these conjugate lines are in the case of a focus at right angles, the double rays are the connectors of the focus to the circular points at infinity (Art 87)

(*d*) *A pair of tangents to a conic from any point subtend equal angles at a focus*



Let TA, TB be any pair of tangents to the conic Let S be a focus, and s its polar Let ST meet s in T' Let the chord of contact, $A'B'$, of the tangents from T' meet s in M Then $ST'T'$ is at right angles to $MBSA'$

Since the triangle SMT' is self conjugate, AB passes through M, and, if ST meet AB in L, $(MLBA)$ is harmonic

Hence SL, SM being at right angles are the bisectors of ASB

Therefore the angle BST=the angle AST

(*e*) *If a variable tangent meet two fixed tangents in A and B and S be a focus of the conic, then the angle ASB is constant*

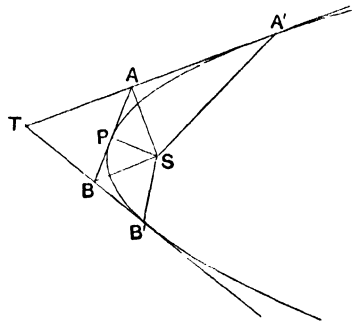
Let P be the point of contact of the variable tangent and A' and B' the points of contact of the fixed tangents TA and TB

Then $PSB=\frac{1}{2}\ B'SP$, and $PSA=\frac{1}{2}\ PSA'$

$$BSA=\tfrac{1}{2}\ B'SA'$$

Hence BSA is constant

Since the angle ASB is constant, the ranges described by A and B on TA' and TB' are projective Hence, if $AA_1A_2A_3$ and $BB_1B_2B_3$ are two projective ranges on TA' and TB', and a conic be described to touch the lines TA', TB', AB, A_1B_1, and A_2B_2, the line A_3B_3 will also touch this conic and, at either of its foci S and S', the segments AA_1 and BB_1, A_1A_2 and B_1B_2, A_2A_3 and B_2B_3, will subtend equal angles



Hence given two projective ranges on different bases two points can be found at which (1) corresponding segments subtend equal angles and (2) the connectors of pairs of corresponding points subtend a constant angle (See Art 39 (*a*))

Summary From the preceding, it is seen that a focus of a conic is

(1) A point through which pairs of conjugate lines are at right angles

(2) A fixed point such that the ratio of its distance from any point on the curve to the perpendicular distance of this point from the polar of the fixed point is constant

(3) A point of intersection of tangents from the circular points at infinity to the conic

(4) A point at which the intercepts by a variable tangent on any two fixed tangents subtend a constant angle

97 Alternative Definitions of the Conic and the deduction of the Anharmonic Property

(A) **Definition** *A conic is the curve obtained from a circle by conical projection or by plane perspective* (See Art 19) As the processes of Projection and Plane Perspective are always reversible, it follows that every conic can be projected into, or its plane perspective formed is, a circle

The anharmonic property has already been proved for the circle (Art 73) To prove it for any conic it is only necessary to form the corresponding figure by projection or plane perspective in such a way that the circle is replaced by a conic Then, since the theorems hold for the circle and the values of anharmonic ratios are unaltered by projection or plane perspective, the theorems hold for the conic

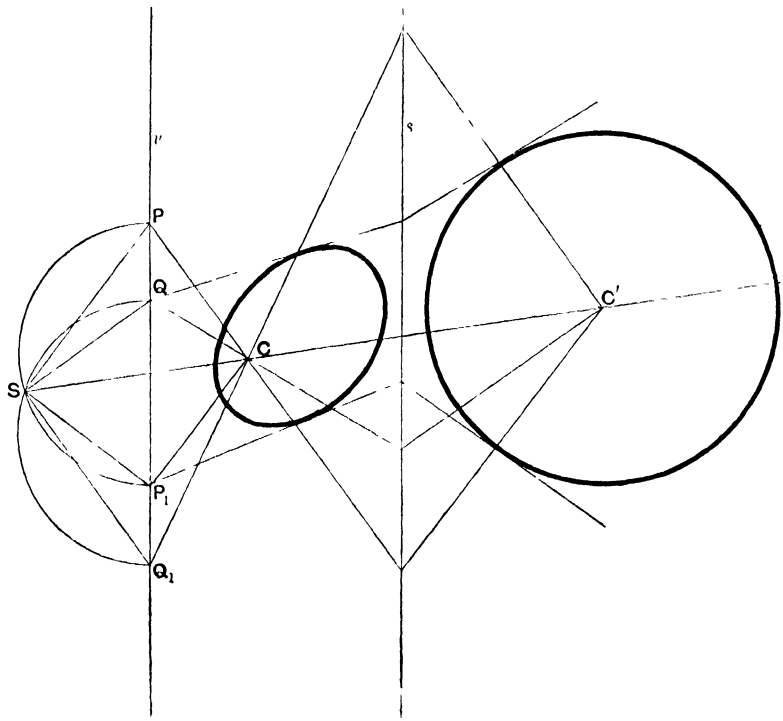
On reference to Art 73 it will be noticed that the converse theorems are only true in the case of the circle for particular cases of projective ranges and pencils In fact in order that the locus or envelope may be a circle—which is a particular form of conic—certain special conditions must be fulfilled In forming the new figure from the one containing the circle the special forms of range and pencil are replaced by the most general forms so that the theorems are true for the conic in the form stated at the end of Art 92

Any conic can be projected into a circle the centre of which is the projection of a given internal point

Let C be the point which is to be projected into the centre of the circle and c its polar with respect to the conic Draw two pairs of conjugate lines through C The angles between these will overlap, since the tangents from this point to the conic are imaginary Project these angles into right angles and the line c into the line at infinity (Art 25) The projection of C is the centre of the conic obtained by this projection The two pairs of conjugate lines through C are projected into two pairs of conjugate diameters of the conic obtained by the projection Since they are at right angles this conic must be a circle (Art 96)

From the above it is seen that a conic may be projected into a circle and any line, which does not meet the conic in real points, into the line at infinity

To form the perspective of a conic so that the corresponding curve may be a circle of which the centre is the point corresponding to a given internal point



Let C be the given point which is to correspond to the centre of the circle in the perspective figure Take its polar as vanishing line w Let the polars of P and Q any two points on w meet w in P_1 and Q_1 Then CP, CP_1 and CQ, CQ_1 are pairs of conjugate lines through C On PP_1 and QQ_1 describe semicircles intersecting in S Take S as centre of perspective and any line parallel to w as the axis s

In the corresponding figure the point C', which corresponds to C, is the centre of the conic and the lines corresponding to CP, CP_1 and to CQ, CQ_1 are pairs of conjugate lines through C' They are therefore pairs of conjugate diameters and, since they are at right angles, the conic is a circle

(B) **Definition** *A conic is the locus of a point which moves so that the ratio of its distance from a fixed point, termed the focus, to its distance from a fixed line, termed the directrix, is constant*

This ratio is termed the *eccentricity* of the conic It is less than unity for the Ellipse, greater than unity for the Hyperbola and equal to unity for the Parabola

It will be proved in the first place that a focus and directrix defined as above comply with the definition of Art 96

(*a*) *A point S and a line s, such that the distance of S from any point on a conic is in a constant ratio to the distance of the same point from s, are a focus and directrix of the conic*

Lemma If any chord PQ of the conic meet s in A, then SA is the external bisector of the angle PSQ

Let PM and QN be the perpendiculars from P and Q on the directrix Then

$$\frac{SQ}{SP} = \frac{QN}{PM} = \frac{QA}{PA}$$

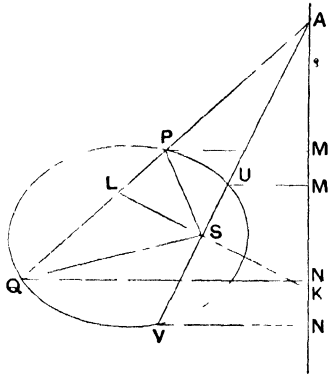
Hence, by Euclid VI A, SA is the external bisector of PSQ

Let SA meet the conic in U and V and let UM' and VN' be perpendiculars on the directrix Then

$$\frac{SV}{SU} = \frac{VN'}{UM'} = \frac{AV}{AU},$$

$$(ASUV) = -1,$$

and, since AS is any line through S, S and s, the corresponding directrix, are pole and polar



Draw a line through S perpendicular to SA to meet AQ at L

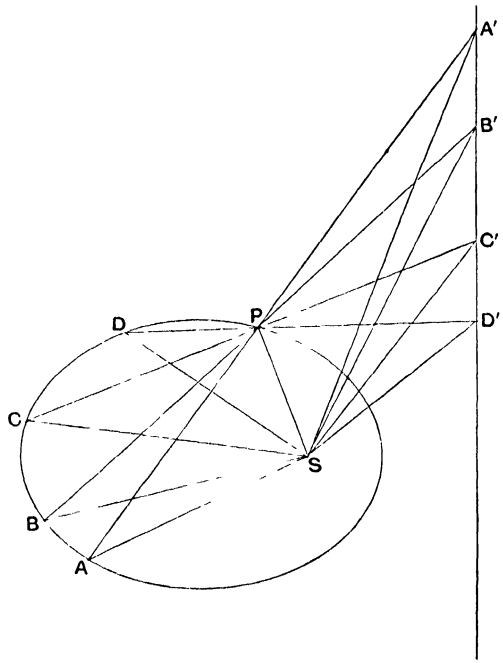
Then SL and SA by the lemma are the internal and external bisectors of PSQ, and the range $(ALPQ)$ is harmonic Also $(ASUV)$ is harmonic Therefore SL is the polar of 1 Hence SL and SA are conjugate lines and they are at right angles Similarly any other pair of conjugate lines through S may be proved to be at right angles Therefore S is a focus and s its polar is the corresponding directrix

(*b*) To prove the anharmonic property of a conic directly from the focal definition

The anharmonic ratio of the pencil formed by joining any four fixed points on a conic to a variable point on the conic is constant

Let S be a focus of the conic and A, B, C, D four fixed points on it Join A, B, C, D to any point P and let the joining lines meet the directrix in A', B', C', D' Join S to A, B, C, D, A', B', C', D'

Then, by the lemma to (*a*), SA', SB', SC', SD' are the external bisectors of the angles ASP, BSP, CSP, DSP Since the angles $A'SB'$, $B'SC'$, $C'SD'$ are half the angles ASB, BSC, CSD respectively, and these angles are constant, $(S\ A'B'C'D')$ is constant for different positions of P



Therefore
$$(P.ABCD)=(P\ A'B'C'D')$$
$$=(S\ A'B'C'D')$$
$$=a \text{ constant}$$

98 *Analytical treatment of the inscribed quadrangle and circumscribed quadrilateral of a conic*

In the projection of the figure of Art 80, it should be noticed, that if the triangle STU' be taken as triangle of reference, and the ratios of the points in which one of the tangents– say the tangent at A— meets its sides be denoted by a, b, c, then the ratios of the points in which the other tangents meet the sides of the triangle can be written down at once

Those for the tangent at	A	are	a,	b,	c,
„	„	„	„	B	„	a,	$-b$,	$-c$,
„	„	„	„	C	„	$-a$,	$-b$,	c,
„	„	„	„	D	„	$-a$,	b,	$-c$,

where $abc=1$

Also, if A determines, by its connectors with the vertices of the triangle of reference, ratios x, y, z on the opposite sides of the triangle of reference, then similar quantities for B, C, D may be written down

Those for A are x, y, z,
„ „ B „ $t, -y, -z$,
„ „ C „ $-t, -y, z$,
„ „ D „ $-x, y, -z$,
where $xyz = -1$

There is the additional condition that each of the points A, B, C, D is situated on one of the tangents at A, B, C or D

Involution on a Conic

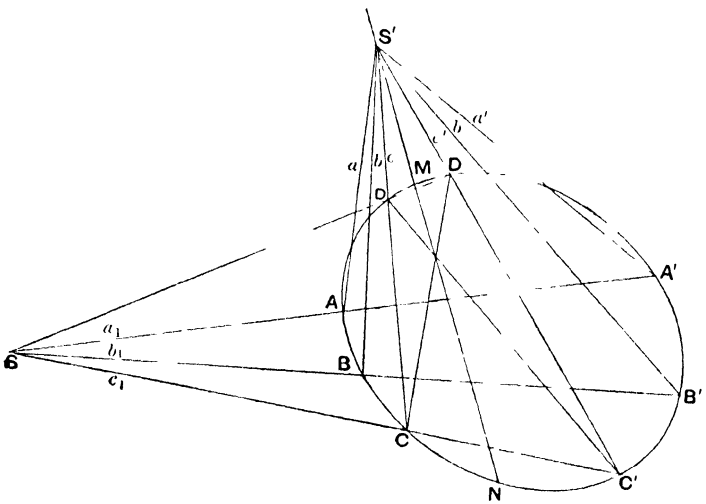
It has been proved (Art 95 (*c*) and Art 75) that, if through a point S chords SAA', SBB', SCC' are drawn to meet a conic in the pairs of points AA', BB', CC', these pairs of points are conjugate points of an involution on the conic, and conversely that, if AA', BB', CC' are pairs of conjugate points of an involution on the conic, AA', BB', and CC' are concurrent The following theorem further illustrates the matter

(1) *If through any point S chords a_1, b_1, c_1, be drawn meeting a conic in AA', BB', CC', , and AA', BB', CC', be joined to any point S'*

(*a*) *on the conic,*

(*b*) *on the polar of S,*

the pencil aa', bb', cc', so formed is an involution pencil



The first part has already been proved, it is therefore only necessary to prove (*b*) Let S' be any point, not on the conic, such that the pencil aa', bb', cc', forms

an involution Let $S'C$ or c meet the conic in D, and join SD to meet the conic again at D' Then D, D' are a pair of conjugate points of the involution AA', BB', CC', Hence $S'C'$ and $S'D'$ are both conjugates of c in the involution aa', bb', Therefore S', D', C' are collinear and, from the quadrangle $CC'D'D$, S' must be situated on the polar of S, i e S and S' must be conjugate points with respect to the conic

When this is the case $S'S$ and $S'MN$, the polar of S, are harmonic conjugates of aa', bb', cc', and are therefore the double rays of an involution pencil, of which these lines are pairs of conjugate rays

Conversely *an involution pencil is cut by a conic in pairs of conjugate points* AA', BB', CC', *of an involution, when the vertex is situated* (1) *on the conic or* (2) *on the polar of* AA' BB' *with respect to the conic*

EXAMPLES

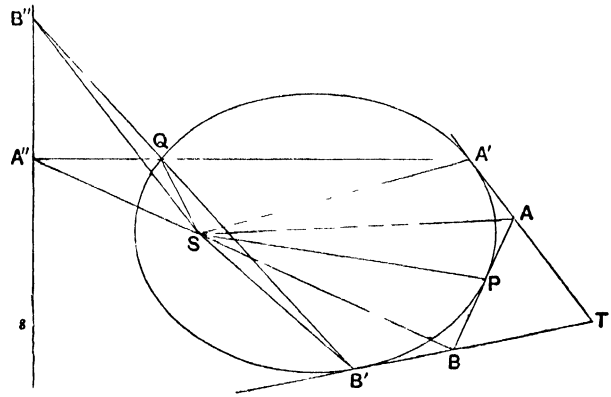
(1) Given six points, explain fully how with ruler and compass to construct the centre and axis of perspective so that the perspectives of the six points shall be one at the centre and the other five on the circumference of a circle

(2) Tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix

(3) The circumcircle of a triangle circumscribed to a parabola passes through the focus

(4) Prove that if any point P on a conic be joined to two points B and C on the conic by lines, which meet the tangents at B and C in C', B', and $C'B'$ meets CB in T, then TP is the tangent at P (Art 92)

(5) If the tangents at any two points A', B' on a conic meet the tangent at any point P in A and B, and the lines joining A' and B' to any point Q on the curve meet the directrix corresponding to a focus S in A'', B'', then the angles ASB and $A''SB''$ are equal



In the figure $ASP = \frac{1}{2}\ A'SP$
$BSP = \frac{1}{2}\ B'SP$

Therefore $ASB = \frac{1}{2}\ A'SB'$

But SA'' is the external bisector of QSA'

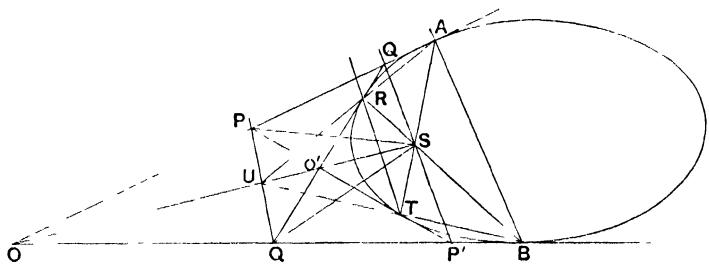
and SB'' „ „ „ „ QSB'

Therefore $A''SB'' = \frac{1}{2}\ A'SB' = ASB$

(6) Prove from Example 5 that the anharmonic ratio of four points on a conic is equal to that of the four tangents at the points

(7) If CA and CB be the semi-transverse and conjugate axes of a hyperbola and points E and F be taken on the transverse axis at distances $\sqrt{CA^2 - CB^2}$ from the centre, prove that the pairs of tangents from these points to the curve are at right angles

(8) A point at which pairs of corresponding points of two projective ranges on fixed bases subtend a constant angle is a focus of the conic enveloped by the connectors of pairs of corresponding points of the two ranges



Let OA, OB be the given bases, which touch the envelope at A and B Let S be the point at which connectors of pairs of corresponding points subtend a constant angle Let AS and BS meet the envelope at R and T and let the tangents at these points meet the bases in Q, Q' and P, P' Then the angles ASO, QSQ', PSP', OSB are all equal to some angle α

Let RQ PT be O' and PQ' OO' be U, then by the property of inscribed quadrangle and circumscribed quadrilateral

PQ', AR, OO' and BT' are concurrent at U,

AT, QP', RB and OO' „ „ „ S

Then the angle $Q'SR$

$$= \alpha - RSQ = USB - BSP' = USP' = \alpha - USP = USA - USP = PSA$$

Therefore, since $Q'SR = USP' = PSA$, the angles between three pairs of conjugate rays through S are equal and therefore (Example 13, Chapter VIII) the involution pencil at S is orthogonal and S is a focus

CHAPTER XIV

PROJECTIVE FORMS IN RELATION TO THE CONIC —CARNOT'S THEOREM, PASCAL'S THEOREM, DESARGUES' THEOREM

99 Carnot's, Pascal's and Desargues' Theorems, together with the anharmonic property, are the fundamental theorems for the Conic In this chapter the first three of these theorems are deduced from the anharmonic property of the conic, but on account of their importance alternative proofs are given in each case

Carnot's Theorem and its Correlative

If a conic meet the sides BC, CA, AB of a triangle in three pairs of points

$$A_1, A_2,\ B_1, B_2,\ C_1, C_2,$$

then

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2} = 1$$

Converse *If the above relation holds, a conic can be described through the six points A_1, A_2, B_1, B_2, C_1, C_2*

If the tangents from the vertices A, B, C of a triangle meet the opposite sides in three pairs of points A_1, A_2, B_1, B_2, C_1, C_2,

then

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2} = 1$$

Converse *If the above relation holds, a conic can be described to touch the six lines AA_1, AA_2, BB_1, BB_2, CC_1, CC_2*

The expression $\dfrac{BA_1}{CA_1}\cdot\dfrac{BA_2}{CA_2}\cdot\dfrac{CB_1}{AB_1}\cdot\dfrac{CB_2}{AB_2}\cdot\dfrac{AC_1}{BC_1}\cdot\dfrac{AC_2}{BC_2}$,

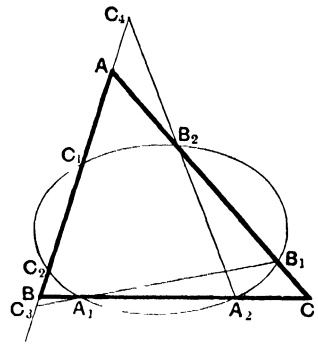
where A, B, C are the vertices of any triangle and A_1, A_2, B_1, B_2, C_1, C_2 are any three pairs of points situated on the sides BC, CA, AB of the triangle respectively, will be termed Carnot's Expression The value of this expression was shown (Art 31) to be unaltered by projection If the ratios of the distance of A_1, A_2 from the vertices B and C be denoted by a_1, a_2, the ratios of the distances of B_1, B_2 from the vertices

C and A by b_1, b_2, and the ratios of the distances of C_1, C_2 from the vertices A and B by c_1, c_2, the expression may be written

$$a_1a_2b_1b_2c_1c_2$$

For the sake of brevity Carnot's Expression will often be quoted in this form, and it should be noticed that, if, for a given triangle ABC, the value of Carnot's Expression for the points A_1, A_2, B_1, B_2, C_1, C_2 is given, and also the position of five of the points, then the sixth point is uniquely determined

Proof from the anharmonic property of a conic



In the figure, let B_1A_1 meet AB in C_3 and B_2A_2 meet AB in C_4

Then by the anharmonic property of a conic

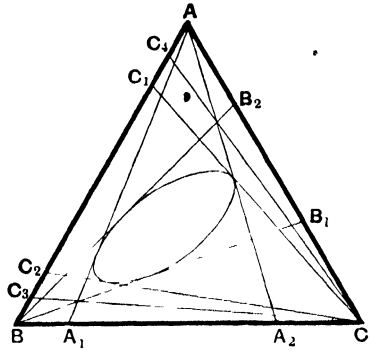
$$(B_1\ C_1C_2B_2A_1) = (A_2\ C_1C_2B_2A_1)$$

Taking intercepts on AB

$$(C_1C_2AC_3) = (C_1C_2C_4B)$$

Hence AB, C_1C_2, C_3C_4 are pairs of conjugate points of an involution and therefore (Art 53)

$$\frac{AC_1}{BC_1}\frac{AC_2}{BC_2} = \frac{AC_3}{BC_3}\frac{AC_4}{BC_4} \quad (1)$$



In the figure, let AA_1 BB_1 and BB_2 AA_2 be projected from C on AB into the points C_3 and C_4 respectively

Then by the anharmonic property of tangents to a conic the intersections of CC_1, CC_2, AA_1, BB_2 with AA_2 and BB_1 form two projective ranges

Projecting these ranges from C on AB

$$(C_1C_2AC_4) = (C_1C_2C_3B)$$

Hence AB, C_1C_2, C_3C_4 are pairs of conjugate points of an involution and therefore (Art 53)

$$\frac{AC_1}{BC_1}\frac{AC_2}{BC_2} = \frac{AC_3}{BC_3}\frac{AC_4}{BC_4} \quad (1)$$

But by Menelaus' Theorem

$$\frac{AC_3}{BC_3} = \frac{1}{\dfrac{BA_1}{CA_1}\dfrac{CB_1}{AB_1}}$$

and

$$\frac{AC_4}{BC_4} = \frac{1}{\dfrac{BA_2}{CA_2}\dfrac{CB_2}{AB_2}}$$

Substituting these values of $\frac{AC_3}{BC_3}$ and $\frac{AC_4}{BC_4}$ in (1),

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2} = 1$$

But by Ceva's Theorem

$$\frac{AC_3}{BC_3} = \frac{-1}{\dfrac{BA_1}{CA_1}\dfrac{CB_1}{AB_1}}$$

and

$$\frac{AC_4}{BC_4} = \frac{-1}{\dfrac{BA_2}{CA_2}\dfrac{CB_2}{AB_2}}$$

Substituting these values of $\frac{AC_3}{BC_3}$ and $\frac{AC_4}{BC_4}$ in (1),

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2} = 1$$

As in the case of the circle, if the sides of the triangle be denoted by a, b, c, and the tangents from the vertices by a_1, a_2, b_1, b_2, c_1, c_2, the correlative of Carnot's Theorem may be stated in the form

$$\frac{\sin\widehat{ac_1}}{\sin\widehat{ab_1}}\cdot\frac{\sin\widehat{ac_2}}{\sin\widehat{ab_2}}\cdot\frac{\sin\widehat{ba_1}}{\sin\widehat{bc_1}}\cdot\frac{\sin\widehat{ba_2}}{\sin\widehat{bc_2}}\cdot\frac{\sin\widehat{cb_1}}{\sin\widehat{ca_1}}\cdot\frac{\sin\widehat{cb_2}}{\sin\widehat{ca_2}} = 1$$

Converse

In this case, Carnot's expression for the points A_1, A_2, B_1, B_2, C_1, C_2 equals unity

If a conic be described through the five points A_1, A_2, B_1, B_2 and C_1, it will meet C_1C_2 in a point C_2' such that Carnot's expression for the points A_1, A_2, B_1, B_2, C_1 and C_2' equals unity

Hence the points C_2 and C_2' must coincide

In this case, Carnot's expression for the points A_1, A_2, B_1, B_2, C_1, C_2 equals unity

If a conic be described to touch the five lines AA_1, AA_2, BB_1, BB_2 and CC_1, a second tangent from C to this conic will meet BC in a point C_2' such that Carnot's expression for the points A_1, A_2, B_1, B_2, C_1 and C_2' equals unity

Hence the lines CC_2 and CC_2' must coincide

Proof by Conical Projection or Plane Perspective

Let the sides BC, CA, AB of a triangle meet any conic in A_1A_2, B_1B_2, C_1C_2 respectively

Let the tangents to any conic from the vertices A, B, C of a triangle meet the opposite sides in A_1A_2, B_1B_2, C_1C_2 respectively

Form by conical projection or plane perspective a figure in which the conic is replaced by a circle (Art 97)

The value of Carnot's expression for the new figure will be the same as that for the given figure (Art 31)

But in the new figure, since Carnot's theorem holds for the circle, Carnot's expression is unity (Art 89)

Hence Carnot's theorem holds for the conic

Form by conical projection or plane perspective a figure in which the conic is replaced by a circle (Art 97)

The value of Carnot's expression for the new figure will be the same as that for the given figure (Art 31)

But in the new figure, since the correlative of Carnot's theorem holds for the circle, Carnot's expression is unity (Art 89)

Hence the correlative of Carnot's theorem holds for the conic

100 Pascal's Theorem and its Correlative (Brianchon's Theorem)

If a hexagon be inscribed in a conic, the three points of intersection of pairs of opposite sides are collinear

Converse

If two triangles are in perspective, a conic can be described through the six points of intersection of the non corresponding sides of the triangles

If a hexagon be circumscribed to a conic, the connectors of the three pairs of opposite vertices are concurrent

Converse

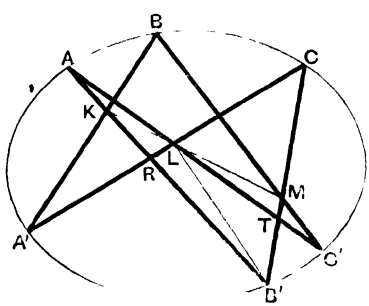
If two triangles are in perspective, a conic can be described to touch the six lines which join the non-corresponding vertices of the triangles

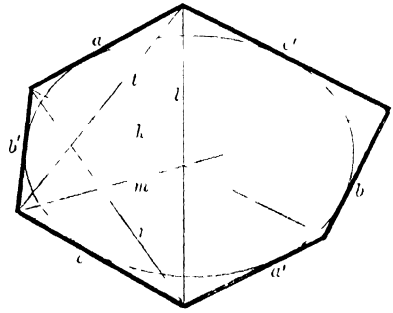
Proof by the Anharmonic Property of the Conic

This proof will be seen to be identical with that given for the circle in Art 90

Let $AB'CA'BC'$ be any hexagon inscribed in a conic, and K, L, M the points of intersection of pairs of opposite sides

Let $ab'ca'bc'$ be any hexagon circumscribed to a conic, and k, l, m the connectors of pairs of opposite vertices





Let AB' meet $A'C$ in R and let $B'C$ meet AC' in T Join LB'

Then

$$(A' \ ABCB') = (C' \ ABCB')$$

by the anharmonic property of the conic

Therefore, taking intercepts on AB' and CB',

$$(AKRB') = (TMCB')$$

Therefore since B' is a self-corresponding point, AT, KM and RC are concurrent

Therefore K, L, M are collinear

Converse

Let the line joining ab' to $a'c$ be r and that joining $b'c$ to ac' be t

Then

$$(a' \ abcb') = (c' \ abcb')$$

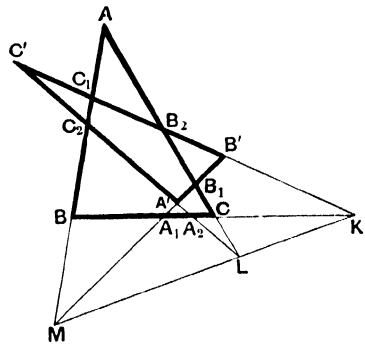
by the anharmonic property of tangents

Therefore, projecting from ab' and cb',

$$(akrb') = (tmcb')$$

Therefore since b' is a self-corresponding ray, at, km, and rc are collinear

Therefore k, l, m are concurrent

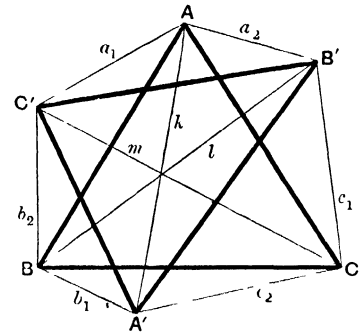


Let the corresponding sides of the triangles ABC, $A'B'C'$ intersect in the collinear points K L, M, and the non-corresponding sides in the points A_1, A_2, B_1, B_2, C_1, C_2

Describe a conic through the five points C_1, B_1, B_2, A_1, A_2

Let the line AB, which passes through C_1, meet the conic again in C_2'

Since $C_1C_2'B_1B_2A_1A_2$ is a hexagon inscribed in a conic, $C_2'A_2$ must pass through L



Let the connectors of the corresponding vertices of the triangles $ABC, A'B'C$ be the concurrent lines k, l, m, and those of the non-corresponding vertices $a_1, a_2, b_1, b_2, c_1, c_2$

Describe a conic to touch the five lines c_1, b_1, b_2, a_1, a_2

Let the second tangent from C to this conic be c_2'

Since $c_1c_2'b_1b_2a_1a_2$ is a hexagon circumscribed to a conic, $c_2'b_1$ must be on the line k

Therefore LA_2 passes through C_2', and C_2' must coincide with C_2 the point of intersection of AB and A_2L

Hence the six points A_1, A_2, B_1, B_2, C_1, C_2 are on a conic

Therefore c_2' must pass through the point kb_1 Therefore the tangent c_2' must be the line c_2

Hence the six lines a_1, a_2, b_1, b_2, c_1, c_2 all touch a conic

(*a*) Proof by Conical Projection or Plane Perspective

Let $AB'CA'BC'$ be any hexagon inscribed in a conic and K, L, M the points of intersection of the three pairs of opposite sides

Form by projection or plane perspective a corresponding figure, wherein the conic is replaced by a circle, and the line LM becomes the line at infinity* (Art 97)

The theorem then depends for its proof on the following, which is proved in the addendum 3 (*a*)

"If two pairs of opposite sides of a hexagon inscribed in a circle are parallel, then the third pair of opposite sides are also parallel"

Let $ab'ca'bc'$ be any hexagon circumscribed to a conic and k, l, m the connectors of the three pairs of opposite vertices

Form by projection or plane perspective a corresponding figure, wherein the conic is replaced by a circle, and the point lm becomes the centre of the circle* (Art 97)

The theorem then depends for its proof on the following, which is proved in the addendum 3 (*b*)

"If the lines joining two pairs of opposite vertices of a hexagon circumscribed to a circle pass through the centre, the line joining the third pair of vertices likewise passes through the centre"

(*b*) Proof by Carnot's Theorem

Let A_1, A_2, B_1, B_2, C_1, C_2 be the vertices of the hexagon and let the pairs of opposite sides intersect as in the figure in A', B', C' It is required to prove that A', B', C' are collinear

Consider the triangle A_1A_2, B_1B_2, C_1C_2 whose vertices are A, B, C

Since A', B_2, C_1 are collinear, by Menelaus' Theorem

$$\frac{BA'}{CA'} = \frac{1}{\dfrac{CB_2}{AB_2}\dfrac{AC_1}{BC_1}}$$

Let A, B', C, A', B, C' be the vertices of the hexagon and let AA', BB', CC' meet the sides of the triangle ABC in A'', B'', C'' It is required to prove that AA', BB', CC' are concurrent

Let the tangents from A, B, C meet the sides of the triangle ABC in A_1A_2, B_1B_2, C_1C_2 as in the figure

Since AA'', BB_1, CC_2 are concurrent, by Ceva's Theorem

$$\frac{BA''}{CA''} = -\frac{1}{\dfrac{CB_1}{AB_1}\dfrac{AC_2}{BC_2}}$$

* If the line LM meets the circle in real points in the first case or the point lm in the second case is external to the conic, the construction of the corresponding figure is imaginary and this method does not give a proof of the theorem

Similarly

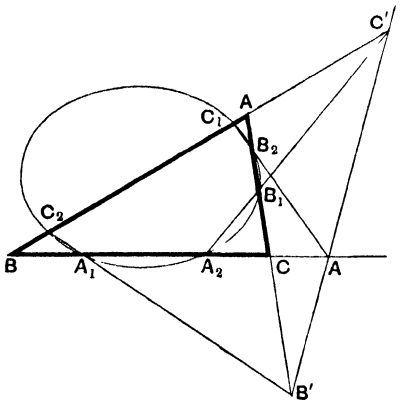
$$\frac{AC'}{BC'} = \frac{1}{\dfrac{BA_2}{CA_2}\dfrac{CB_1}{AB_1}}$$

and

$$\frac{CB'}{AB'} = \frac{1}{\dfrac{BA_1}{CA_1}\dfrac{AC_2}{BC_2}}$$

Therefore

$$\frac{BA'}{CA'}\frac{AC'}{BC'}\frac{CB'}{AB'} = \frac{1}{\dfrac{AC_1}{BC_1}\dfrac{AC_2}{BC_2}\dfrac{BA_1}{CA_1}\dfrac{BA_2}{CA_2}\dfrac{CB_1}{AB_1}\dfrac{CB_2}{AB_2}}$$



The expression on the right hand side is by Carnot's theorem equal to unity Hence by the converse of Menelaus' theorem A', B', C' are collinear

Similarly

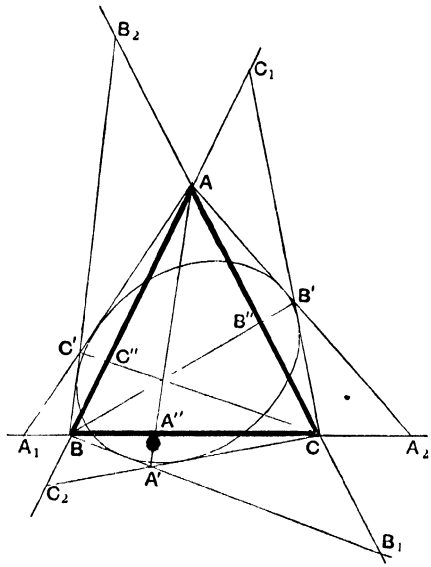
$$\frac{CB''}{AB''} = -\frac{1}{\dfrac{BA_2}{CA_2}\dfrac{AC_1}{BC_1}}$$

and

$$\frac{AC''}{BC''} = -\frac{1}{\dfrac{CB_2}{AB_2}\dfrac{BA_1}{CA_1}}$$

Therefore

$$\frac{BA''}{CA''}\frac{AC''}{BC''}\frac{CB''}{AB''} = \frac{-1}{\dfrac{AC_1}{BC_1}\dfrac{AC_2}{BC_2}\dfrac{BA_1}{CA_1}\dfrac{BA_2}{CA_2}\dfrac{CB_1}{AB_1}\dfrac{CB_2}{AB_2}}$$



The expression on the right-hand side is by the correlative of Carnot's theorem equal to -1 Hence by the converse of Ceva's theorem AA', BB', CC'' are concurrent

The proof of the converse theorems by means of Carnot's theorem is left as an exercise for the student

(*c*) Proof by Projective ranges on the Conic

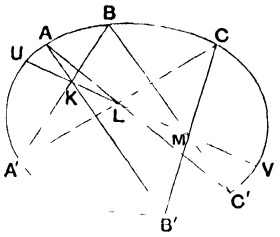
Let the inscribed hexagon be $AB'CA'BC''$ and let K, L, M be the points BA' AB', $A'C$ AC'' and BC'' CB'

Let the circumscribed hexagon be $ab'ca'bc'$ and let k, l, m be the lines ba' ab', $a'c$ ac' and bc' cb' It is

It is required to prove that these points are collinear

The two groups of three points A, B, C and A', B', C' determine two projective ranges on the conic (Art 95 (*a*))

Consider the pencils B $A'B'C'$ and B' ABC They have a self-corresponding ray in BB' and are therefore in perspective Hence every pair of corresponding rays of these pencils intersects on KM If KM meets the conic in U and V*, BU and BV correspond to $B'U$ and $B'V$ Therefore U and V are self-corresponding



points of the projective ranges determined on the conic by the pencils with vertices B and B', that is by the pairs of corresponding points AA', BB', CC'

Hence, when the ranges ABC and $A'B'C'$ are given, U and V are fixed, as is also the line KM

Similarly by considering the pencils A' ABC and A $A'B'C'$ it is seen that the point L (AC' CA') is on UV Hence K, L, M are collinear

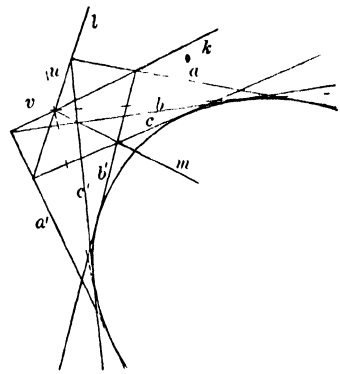
This proof should be compared with that given (Art 36) for the corresponding theorem in which the vertices of the hexagon lie on a pair of straight lines

* This proof only holds when the line KM meets the conic in real points

required to prove that these lines are concurrent

The two groups of three tangents a, b, c and a', b', c' determine two projective systems of tangents to the conic (Art 95 (*a*))

Consider the ranges b $a'b'c'$ and b' abc They have a self corresponding point bb' and are therefore in perspective Hence every pair of corresponding points of these ranges is collinear with km If u and v* are tangents from km to the conic, bu and bv correspond to $b'u$ and $b'v$ Therefore u and v are the self-corre-



sponding rays of the two projective pencils of tangents to the conic determined by the ranges on b and b' and therefore by the pairs of corresponding tangents aa', bb', cc'

Hence, when the systems of tangents abc and $a'b'c'$ are given, u and v are fixed, as is also the point km

Similarly by considering the ranges a' abc and a $a'b'c'$ it is seen that the line l (aa' aa') passes through uv Hence k, l, m are concurrent

This proof should be compared with that given (Art 36) for the corresponding theorem in which the sides of the hexagon pass through two points

* This proof only holds when the tangents from km to the conic are real

101 Desargues' Theorem and Correlative

Any transversal is cut by a system of conics, through four fixed points, in pairs of conjugate points of an involution

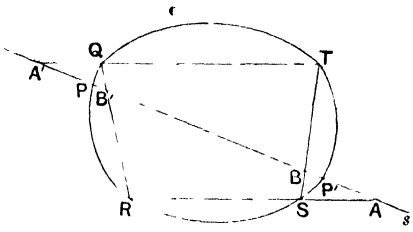
Converse *Conics described through the vertices of a triangle and through pairs of conjugate points of an involution range intersect in a fourth fixed point*

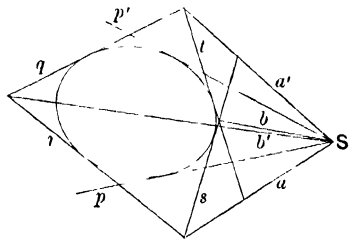
The pairs of tangents from any point to a system of conics, touching four fixed lines, are pairs of conjugate rays of an involution

Converse *Conics described so as to touch the sides of a triangle and also a pair of conjugate rays of an involution pencil touch a fourth fixed line*

Proof by the Anharmonic property of the Conic

This proof should be compared with that for the circle in Art 91 (1)





Let Q, R, S, T be any four fixed points and let a conic through them meet any transversal s in P and P' Let QR, ST, QT and RS meet s in B', B, A' and A

Then $(Q\ PRP'T) = (S\ PRP'T)$

But $(Q\ PRP'T) = (PB'P'A')$

and $(S\ PRP'T) = (PAP'B)$

Therefore

$$(PB'P'A') = (PAP'B)$$

Therefore AA', BB', PP' are pairs of conjugate points of an involution

If any other conic be described through Q, R, S, T, it will meet s

Let q, r, s, t be the four fixed lines and let the tangents from any point S to a conic, which touches these lines, be p and p' Let the lines joining qr, st, qt and rs to S be b', b, a' and a

Then $(q\ prp't) = (s\ prp't)$

But $(q\ prp't) = (pb'p'a')$

and $(s\ prp't) = (pap'b)$

Therefore

$$(pb'p'a') = (pap'b)$$

Therefore aa', bb', pp' are pairs of conjugate rays of an involution pencil

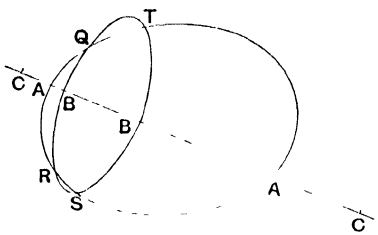
If any other conic be described to touch q, r, s, t, the tangents

in a pair of points of the same involution, namely that determined by the two pairs of conjugate points A, A' and B, B'

The points where SQ and RT meet s are also a pair of conjugate points of the same involution (Art 56)

Converse

Let the vertices of the triangle be Q, R, S and let the involution be determined by the pairs of conjugate points A, A' and B, B'



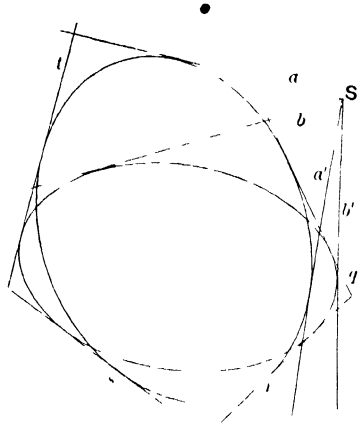
Describe two conics through the points A, A', Q, R, S and B, B', Q, R, S respectively to intersect in some point T

Describe a third conic through Q, R, S, T and C, some given point on the transversal

This conic meets the transversal in C' the conjugate of C in the involution determined by A, A' and B, B' Since only one conic can be from S will be a pair of conjugate rays of the same pencil, namely that determined by the two pairs of conjugate rays a, a' and b, b'

The lines joining S to sq and rt are also a pair of conjugate rays of the same involution (Art 56)

Let the sides of the triangle be q, r, s and let the involution be determined by the pairs of conjugate lines a, a' and b, b'



Describe two conics to touch the lines a, a', q, r, s and b, b', q, r, s respectively Draw the fourth common tangent, t, to these conics

Describe a third conic to touch q, r, s, t and c, some given line through S, the centre of the involution

This conic will touch c' the conjugate to c in the involution determined by a, a' and b, b' Since only one conic can be

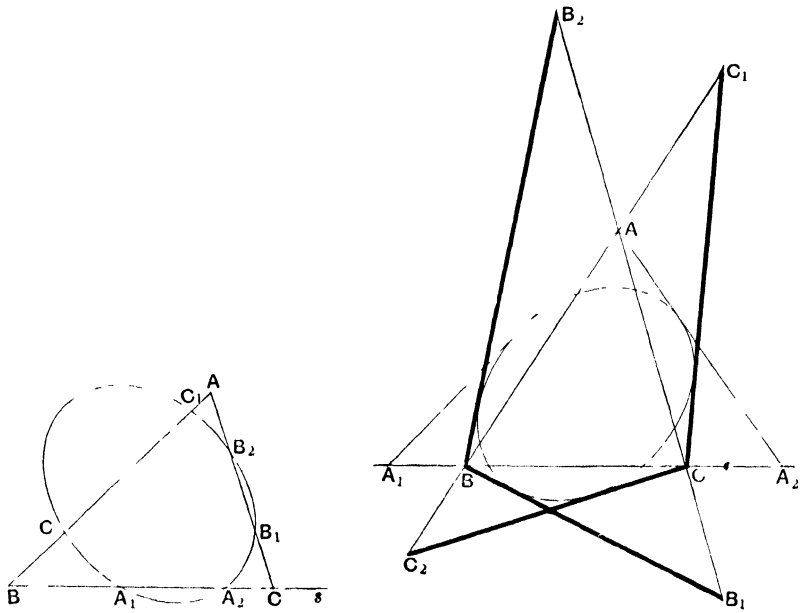
described through five points, this conic may be looked upon as the conic through Q, R, S, C and C' Since this conic passes through T, the theorem is proved

described to touch five lines, this conic may be looked upon as the conic touching q, r, s, c and c' Since this conic touches the line t, the theorem is proved

(*a*) **Proof by Projection or Plane Perspective**

Desargues' Theorem is related to the theorem, already proved in Art 91 (11), that a system of coaxal circles is cut by every transversal in pairs of conjugate points of an involution The connexion may be shown by an imaginary projection as follows Project one of the conics through the four points into a circle and the connector of two of the four given points into the line at infinity* These two points then become the circular points at infinity (Art 87) and, since the projections of all the conics pass through these points, the conics all become circles (see page 189) The circles all pass through two other points and therefore form a system of coaxal circles Since these circles are cut by every transversal in pairs of conjugate points of an involution, the conics in the original figure are all cut in a similar manner by every transversal

(*b*) **Proof by Carnot's Theorem**



* This involves an imaginary projection the use of which has not been justified

Let C_1, C_2, B_1, B_2 be the four fixed points through which the conics are described. Let one of the conics meet the transversal s in A_1, A_2 and another conic (not shown in the figure) meet s in A_1', A_2'. Let B_1B_2 and C_1C_2 intersect at A and meet s in C and B.

Then by Carnot's theorem

$$\frac{BA_1}{CA_1}\frac{BA_2}{CA_2} = \frac{1}{\frac{CB_1}{AB_1}\frac{CB_2}{AB_2}\frac{AC_1}{BC_1}\frac{AC_2}{BC_2}} = \frac{BA_1'}{CA_1'}\frac{BA_2'}{CA_2'}$$

Therefore $(BCA_1A_1') = (CBA_2A_2')$.

Hence B, C, A_1, A_2, A_1', A_2' are pairs of conjugate points of an involution. Since B, C and A_1, A_2 may be regarded as fixed, any other conic through B_1, B_2, C_1, C_2 determines on BC pairs of points, which are conjugates in the same involution.

Let BB_2, BB_1, CC_2, CC_1 be the four fixed lines which are touched by the conics. Let the pair of tangents from any point A to one conic meet BC in A_1, A_2 and those from A to another conic (not shown in the figure) meet BC in A_1', A_2'.

Then by the correlative to Carnot's theorem

$$\frac{BA_1}{CA_1}\frac{BA_2}{CA_2} = \frac{1}{\frac{CB_1}{AB_1}\frac{CB_2}{AB_2}\frac{AC_1}{BC_1}\frac{AC_2}{BC_2}} = \frac{BA_1'}{CA_1'}\frac{BA_2'}{CA_2'}$$

Therefore $(BCA_1A_1') = (CBA_2A_2')$.

Hence B, C, A_1, A_2, A_1', A_2' are pairs of conjugate points of an involution. Since B, C and A_1, A_2 may be regarded as fixed, any other conic touching BB_1, BB_2, CC_1, CC_2 determines by tangents from A pairs of lines, which are conjugates in a given involution.

Carnot's, Pascal's and Desargues' Theorems for a pair of lines and their correlatives for a pair of points

In Art. 94 it was pointed out that a pair of straight lines could be regarded as a conic and a pair of points as the correlative of a conic. On reference to the earlier chapters it will be seen that the theorems proved in this chapter are true, as might be expected, for a pair of lines, and their correlatives for a pair of points. Thus

Carnot's theorem for a pair of lines follows at once from Menelaus' theorem (Art. 13 (*d*)).

Pascal's theorem for a pair of lines becomes the theorem proved in Art. 36.

Desargues' theorem for pairs of lines becomes the Involution Property of a quadrangle proved in Art. 56.

Carnot's theorem for a pair of points follows at once from Ceva's theorem (Art. 13 (*c*)).

Brianchon's theorem for a pair of points becomes the theorem proved in Art. 36.

The correlative of Desargues' theorem for a pair of points becomes the Involution Property of a quadrilateral proved in Art. 56.

The anharmonic property of the conic and also the involution property hold in these particular cases.

Properties of the Conic and Circle involving the imaginary

The consideration of the properties of the circle and conic, which depend on the imaginary, has been postponed for later consideration, but in connexion with the solution of problems it is useful to be able to quote certain of the results which may be obtained. A short summary of these is therefore given.

It has been proved (Art 95 (*k*)) that every conic and every circle determines on each line in its plane a definite involution Such an involution has two real, two coincident, or two imaginary double points If the two double points are real they are the points of intersection of the line and conic (Art 95 (*c*)) If they are coincident the line touches the conic and the pair of coincident double points become the point of contact If the double points are imaginary, which is the case when the pairs of conjugate points of the involution overlap, the determination of the double points involves $\sqrt{-1}$ and graphically the line does not meet the conic Such a pair of imaginary points, termed *conjugate imaginary points*, may be regarded, from a certain point of view, as on the curve Most theorems which are true for a pair of real points on the conic or circle, have an interpretation when a pair of such imaginary conjugate points are substituted for these real points

The statement that the imaginary (or real) double points of an involution lie on a conic is equivalent to saying that the conic in question is such that the pairs of conjugate points of the involution are pairs of conjugate points with regard to the conic The student is strongly advised to bear in mind this point of view, which removes many difficulties

Correlatively a conic or circle determines at every point in its plane an involution pencil by means of pairs of conjugate lines passing through a point P (Art 95 (*k*)) The double rays of this involution pencil, if real, are tangents from P to the conic and, if imaginary, they are the imaginary tangents from P Thus the statement that a conic touches two such imaginary lines may be interpreted as meaning that the pairs of conjugate rays of the involution, of which these lines are double rays, are conjugate lines with respect to the conic

It has been shown that conics are divided into different classes according to the nature of their intersections with the line at infinity All circles determine on the line at infinity the same involution, pairs of conjugate points of which are determined by pairs of orthogonal diameters (Art 87) Hence, with the assumption of the previous paragraph, every circle meets the line at infinity in the same pair of imaginary points, termed the *circular points at infinity*, and every pair of orthogonal lines determines on the line at infinity a pair of points which are harmonic conjugates of these circular points at infinity Conversely every conic through the circular points at infinity is a circle

It is usual to assume that any pair of points in a plane may be projected into the circular points at infinity On reference to Art 60 it will be seen that any two involutions with real double points may be projected into each other and likewise any pair of involutions with imaginary double points Hence any pair of conjugate imaginary points may be projected into the circular points at infinity The projection of an involution with real double points into an involution with imaginary double points is a process most difficult to justify involving as it does the change of the other points from real to imaginary points Yet it is a fact that in most theorems a pair of real points may be replaced by the circular points at infinity or by any other pair of conjugate imaginary points Probably the true explanation of this lies, not in the fact that real points may be projected into imaginary points, but in the fact that every conic is met by some lines in real and by some lines in imaginary points, and that the properties of the involutions determined on these lines are similar

The usual method for projecting two real points A and B into the circular points at infinity is as follows

"Describe a conic through these points Project this conic into a circle and the line AB into the line at infinity (Art 97)" This can only be done by an imaginary projection "Then the projections of A and B are the circular points at infinity and every conic through these points is a circle"

The circular points at infinity possess another remarkable property Pairs of conjugate lines through a focus are at right angles Hence the tangents to a conic from a focus, which are the double rays of the involution pencil formed by these conjugate lines, pass through the circular points at infinity Therefore the foci of a conic are the points of intersection of tangents to the conic from the circular points at infinity One pair are real and are the real foci (Art 96) of the conic The other pair of points of intersection are imaginary and are a pair of imaginary foci of the conic, which are situated on the minor or conjugate axis

CHAPTER XV

DEDUCTIONS FROM THE ANHARMONIC PROPERTY OF THE CONIC, FROM THE POLE AND POLAR PROPERTIES OF THE CONIC, AND FROM CARNOT'S THEOREM

102 Anharmonic Property of the Conic

Particular Cases

In dealing with deductions from the anharmonic property of the conic and its correlative an important special case should be noticed In an anharmonic ratio, say $(ABCD)$, it is often possible to arrange for one of the elements D to be at infinity, in which case the anharmonic ratio reduces to the simple ratio $\frac{AC}{BC}$

Similarly in the case of a pencil $abcd$ whose anharmonic ratio is $\frac{\sin \widehat{ac}}{\sin \widehat{bc}} \frac{\sin \widehat{ad}}{\sin \widehat{bd}}$, if d be the internal bisector of the angle $\widehat{ab}$, $\frac{\sin \widehat{ad}}{\sin \widehat{bd}} = -1$, and if d be the external bisector of the angle $\widehat{ab}$, $\frac{\sin \widehat{ad}}{\sin \widehat{bd}} = +1$ In the former case the anharmonic ratio reduces to $-\frac{\sin \widehat{ac}}{\sin \widehat{bc}}$ and in the latter to $\frac{\sin \widehat{ac}}{\sin \widehat{bc}}$

Deductions and Examples

Parabola

(1) If two fixed points A and B on a parabola be joined to a variable point P on the curve and PA and PB meet the diameter through a fixed point C in A' and B', then $CA' \cdot CB'$ is constant

If ∞ be the point at infinity on the diameter through C, then

$$(P \cdot C\infty \cdot A'B') = \text{constant}$$

Two fixed tangents to a parabola are cut proportionally by tangents to the curve

If ∞ represent the line at infinity, which is a tangent, p and q the two fixed tangents, and a, b, c any three tangents, then $(p \cdot abc\infty) = (q \cdot abc\infty)$

Conversely, the lines joining corresponding points of two similar ranges envelope a parabola

(2) If a diameter of a parabola meets the curve in P, the two tangents at A and B in A' and B' and their chord of contact at C, then $PC^2 = PA' \ PB'$

Let ∞ be the point at infinity on the diameter, then

$$(A \ ABP\infty) = (B \ ABP\infty),$$

and taking intercepts on the diameter

$$(A'CP\infty) = (CB'P\infty),$$

$$\frac{A'P}{CP} = \frac{CP}{B'P}, \qquad CP^2 = PA' \ PB'$$

Hyperbola

(3) Parallels to the asymptotes through any point on a hyperbola cut any semi diameter so that it is a mean proportional between the segments on it measured from the centre

Let A be the end of the diameter, O the centre, P the point on the curve, P' and P'' the points where lines through P parallel to the asymptotes meet OA, α and α' the points at infinity on the curve

Then

$$(\infty \ AP\infty\alpha') = (\alpha' \ AP\infty\alpha'),$$

or taking intercepts on OA,

$$\frac{AO}{P'O} = \frac{P''O}{AO}$$

A variable tangent to a hyperbola meets the asymptotes in two points the product of whose distances from the centre is constant

Let a and b be any two tangents and a_1 and a_2 the asymptotes Then

$$(a_1 \ aba_1a_2) = (a_2 \ aba_1a_2)$$

Taking intercepts on the asymptotes the required result is obtained

(4) If through a fixed point O a variable line be drawn to meet two fixed lines in A and B, and a point P be taken on this line such that $OP^2 = OA \ OB$, then the locus of P is a hyperbola, whose asymptotes are parallel to the given lines

Converse of (3)

If points A and B be taken on two fixed lines such that, if O be the point of intersection of the lines, $OA \ OB$ is constant, then the envelope of the line AB is a hyperbola, having OA and OB for asymptotes

Converse of (3)

(5) If two fixed points A and B on a hyperbola be joined to a variable point P and the joining lines meet a parallel to an asymptote through a point on the curve C in A' and B', then the ratio $CA' \ CB'$ is constant

(6) The lines joining two fixed points on a hyperbola to a variable point on the curve intercept a constant length on either asymptote

In example (5) if A and B be joined to another point P' on the curve so as to meet the parallel to the asymptote in A'', B'', $\frac{CA'}{CB'} = \frac{CA''}{CB''} = \frac{A'A''}{B'B''}$ If C be taken at infinity, $A'A'' = B'B''$

Conic

(7) If a variable tangent p to a conic meet the tangents a and b, drawn at the ends A and B of a diameter, in A' and B', then the product $AA' \ BB'$ is constant and equal to the square of the semi diameter parallel to the tangents

Let a second tangent p' meet a and b in A'', B'' Then $(a.pp'ab)=(b\ pp'ab)$ Therefore $(A'A''A\infty)=(B'B''\infty B)$ Hence $AA'\ BB'=AA''.BB''=$ a constant

(8) If four points A, B, C, D on a conic be joined to a point P by lines which meet the conic again in A', B', C', D' respectively, then $AB\ C'D'$, $AC\ B'D'$, $AD\ B'C'$, $BC\ A'D'$, $BD\ A'C'$ and $DC\ A'B'$ are collinear with P

Let $AB\ C'D'$ be N and $BD\ A'C'$ be K If N, P, K are collinear $(C'\ D'CA'B)$ must equal $(B\ AB'DC')$ But $(AB'DC')=(A'BD'C)=(D'CA'B)$

(9) A, B, C, D are four points on a conic, P any point on the conic and O a point not on the conic OA, OB cut the conic again at A', B' Prove that

$$(O.ABCD)=(P\ ABCD)\times(P\ A'B'CD)$$

Let OA, OB and $A'B$ meet DC in A'', B'', B''' respectively Then

$$(O\ ABCD)=(A''B''CD)=(DC'B''A''),$$

$$(P\ ABCD)=(A'\ ABCD)=(A''B'''CD)=(DCB'''A''),$$

$$(P\ A'B'C'D)=(B\ A'B'CD)=(B'''B''CD)=(DC'B''B'''),$$

$$(P\ ABCD)(P\ A'B'C'D)=(DCB'''A'')(DC'B''B''')=(DC'B''A'')$$
$$=(O\ ABCD)$$

(10) If POP', QOQ', ROR', SOS' be four concurrent chords of the same conic, and conics be drawn through O, P, Q, R, S and O, P', Q', R', S' respectively, prove that these conics touch at O

Let OT, OT' be the respective tangents to the two conics at O The tangents OT and OT' will coincide if $(TQRS)=(T'Q'R'S')$, that is if $(O\ OQRS)=(O\ OQ'R'S')$, that is if $(P\ OQRS)=(P'\ OQ'R'S')$, that is if $(P'QRS)=(PQ'R'S')$ Since PP', QQ', RR', SS' are conjugate points of an involution this condition is satisfied

(11) Prove that, if three pencils of rays are projective with one another, the three conics generated by the intersections of corresponding rays of the pencils taken two by two have three common points

Let S_1, S_2, S_3 be the vertices of the pencils The two conics determined by the pencils S_1, S_2 and by the pencils S_2, S will intersect in S_2 and in three other points If d_1, d_2, d_3 be the rays of the pencils which meet at one of these points, the points d_1d_2 and d_2d_3 coincide Therefore d_1, d_2, d_3 intersect at the same point and the conic determined by the pencils S_1, S_3 passes through this point

103 Involution and Pole and Polar

Particular Cases

A conic determines on every line in its plane an involution, the conjugate points of which are such that the polar of each passes through the other The double points of the involution are the points in which the line meets the conic (Art 95 (*k*) and Art 77)

Consider a diameter of a parabola One of the double points is at infinity, therefore the other bisects the distance between every pair of conjugate points Hence, if the tangents at the end of a chord PP'

of a parabola meet at T and a diameter is drawn through T to meet the chord in V and the parabola in Q, $TQ = QV$ Again, if P be any point and p its polar with regard to a hyperbola, and a line be drawn through P parallel to an asymptote to meet the curve in Q, and p in V, then $PQ = QV$

Deductions

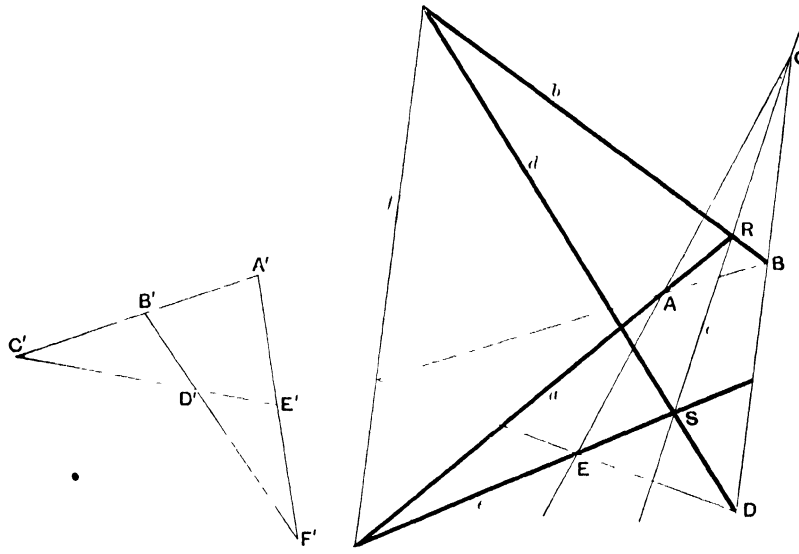
(*a*) *The lines joining the vertices of a triangle to the corresponding vertices of its polar or conjugate triangle meet in a point*

Let ABC be the triangle and $A'B'C'$ its conjugate Join CC' to meet AB in D and $A'B'$ in D' Let AB $A'B'$ be S Then the polars of A', B', D' and S are BC, AC, SC and DC respectively Therefore $(A'B'D'S) = (BASD) = (ABDS)$ Hence the ranges $A'B'D'S$ and $ABDS$ are projective and, since they have a self-corresponding point in S, they are in perspective Therefore AA', BB', CC' are concurrent

The correlative theorem is

The points of intersection of the sides of a triangle with the corresponding sides of its conjugate triangle are collinear

(*b*) *If two quadrilaterals are such that five of the vertices of the one are conjugates of five of the vertices of the other with respect to a conic, then the remaining vertices are also conjugate points*



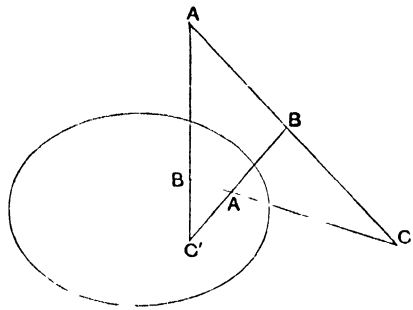
Let A', B', C', D', E', F' be the vertices of one of the quadrilaterals, the points A', B', C', and also the points C', D', E', being collinear Construct a, b, c, d, e the polars of A', B', C', D', E' These pass through two points R and S the poles of $A'B'C'$ and of $C'D'E'$ respectively, and c is the line joining RS The polar of F' is the line joining a e to b d Denote this line by f If a point C be taken anywhere

on c and lines CAE and CBD be drawn to meet a, b, c, d in A, B, C, D, then $ABCDE$ has the given relationship to $A'B'C'D'E'$ The theorem will therefore be true, if f passes through the point of intersection of AB and DE This is the case since the triangles ARB and ESD are in perspective

The correlative theorem is

If two quadrangles are such that five of the sides of the one are conjugate lines of five of the sides of the other with respect to a conic, then the remaining sides are also conjugate lines

(c) *If the extremities of each of two diagonals of a complete quadrilateral are conjugate points with respect to a given conic, the extremities of the third diagonal are also conjugate points with respect to the same conic*



Let A, A' and B, B' be the ends of the two diagonals which are conjugate points Let $AB\ A'B'$ be C and $AB'\ BA'$ be C'

Consider the polar triangle of $A'B'C'$ This triangle is in perspective with $A'B'C'$ But the polar of A' passes through A, and the polar of B' passes through B Hence AB is the axis of perspective of the two triangles and therefore the polar of C' passes through C Hence the theorem is true

If P be the pole of ABC with regard to the conic, the theorem may at once be proved from the involution property of the quadrangle $A'B'C'P$

(d) *If four conics are described through three fixed points the lines joining them at other points of intersection to any one of these three points form a pencil in involution*

Let the three fixed points of intersection be A, B, C and let the conics be (1), (2), (3), (4) Denote the point of intersection of (1) and (2) by P_{12}, that of (2) and (3) by P_{23}, and so on

Let the tangents to (2) at A and C meet at T_2 Let T_2B meet the conic (2) in Q_2 Then $(ACBQ_2)$ is a harmonic range on the conic (2) Therefore the pencil $(P_{12}\ ACBQ_2)$ is harmonic

Similarly, if Q_2P_{12} meet (1) in Q_1 the pencil $(P_{12}\ ACBQ_1)$ is harmonic

Similarly, through P_{23}, P_{34}, P_{13}, P_{14}, P_{24} draw chords which are harmonic conjugates of the connectors of these points to B with respect to their connectors to A and C

Since there is only one harmonic conjugate of B with respect to AC on each of these conics, these lines must pass three by three through four points Q_1, Q_2, Q_3, Q_4

Hence the points P_{12}, P_{23}, lie on the sides of the quadrangle $Q_1Q_2Q_3Q_4$ and are such that their connectors to B are harmonic conjugates of the sides on which they lie with respect to their connectors to A and C

Hence, if BP_{12}, BP_{23}, . meet AC in B_{12}, B_{23}, , B_{13}, B_{23}, are harmonic conjugates with respect to A and C of the points, in which the sides of the

quadrangle $Q_1Q_2Q_3Q_4$ meet AC But the points where the sides of $Q_1Q_2Q_3Q_4$ meet AC form an involution Therefore B_{12}, B_{23}, B_{34}, form an involution Hence their connectors to B form an involution

(*e*) *The poles of a pencil of lines, which pass through a fixed point, with respect to a fixed triangle, lie on a conic which circumscribes the triangle, and the tangents to this conic at the vertices of the triangle intersect the opposite sides in the points, where they are met by the polar of the given point*

Let the variable line p pass through a fixed point O and meet the sides of the triangle ABC in A', B', C' Let A'', B'', C'' be the harmonic conjugates of A', B', C' with respect to the vertices of the triangle with which they are collinear Then P the pole of p is the point of intersection of AA'', BB'', CC'' For different positions of p the ranges A' and C' are projective These ranges are projective with those described by A'' and C'', which are therefore projective with each other Hence the pencils AA'' and CC'' are projective and therefore the locus of P is a conic through A and C, which by symmetry also passes through B

The tangent to the conic at A may be constructed by joining O to A to meet BC in A_1 and taking A_1' the harmonic conjugate of A_1 with respect to BC, in which case the line AA_1' is the tangent at A Hence the connector of the points, where the tangents at A, B and C meet the opposite sides of the triangle, is the polar of O

Conversely the following theorem is obtained

If P be any point on a conic circumscribing a triangle ABC, the polar of P, with respect to the triangle ABC, passes through the pole, with respect to the conic, of the line joining the points, where the tangents at A, B and C to the conic meet the opposite sides of the triangle ABC

EXAMPLES.

(1) A, A', B, B' are any four points on a conic Find the points P, P' on the conic separating harmonically A and A' and also B and B', and show that the line joining the points is the same for all conics passing through the four points

The line determining these points is the polar of AA' BB'

(2) AA', BB', CC' are three pairs of points homographically related on a conic If O, O' be the self conjugate points of the system, show that the pole of OO' and the points OB $O'A'$ and OA $O'B'$ are collinear

From the data $(O\ \ OO'BA) = (O'\ \ OO'B'A') = (O'\ \ O'OA'B')$

Hence the pencils $(O\ \ OO'BA)$ and $(O'\ \ O'OA'B')$ are projective, but $O'O$ is a self corresponding ray Therefore the pencils are in perspective Hence the given points are collinear

(3) Two lines SP and SQ at right angles are drawn through the focus of a conic, intersecting the directrix in P and Q R is any point on the conic and PR and QR meet the conic again in L and M Prove that LM is a focal chord

PS and QS are conjugate lines (Art 96) and therefore P and Q are conjugate points Hence by the converse of the construction for conjugate points (Art 95 (*j*)) the chord LM must pass through the pole of PQ, that is through the focus S

(4) PQ is a chord of a conic O is a point on PQ and M is a point on the polar of O A parallel through O to MQ cuts MP at B and the polar at A Show that B bisects OA

Let MA meet PQ in T Then $(TOPQ)$ is harmonic Therefore the pencil $(M\ TOPQ)$ is harmonic Taking intercepts on OBA, it is seen that $OB=BA$

(5) M, N are a fixed pair of conjugate points with regard to a conic, O a variable point on the conic, OM, ON meet the conic again in P, Q respectively, prove that P, Q are a pair of conjugate points of a definite involution

PQ passes through the pole of the line MN which is a fixed point

(6) The sides BC, CA, AB of a triangle ABC, self conjugate with respect to a given conic, intersect any line p in A', B', C', also O is the pole of p Prove that OA, OA', OB, OB', OC, OC' form a pencil in involution

OA is the polar of A', OB of B', OC of C' Hence the theorem is true

(7) Given two points A and B on a conic, find two other points P and Q on the conic such that A and B shall lie on a circle of which PQ is a diameter

Draw through A two pairs of orthogonal rays AC, AC' and AD, AD' Join CC', DD' to meet in K Every chord through K will determine on the conic a pair of conjugate points of an involution which subtend a right angle at A Similarly from B a point L can be found such that every chord through L will subtend a right angle at B Hence the line KL meets the conic in the required points P and Q

(8) Through a given point in the plane of a conic draw a chord such that it subtends a right angle at a given point on the conic Show that in general there is one and only one solution

Describe on the conic the involution formed by pairs of lines at right angles through the given point on the conic Let the chords which determine this involution pass through O Join O to the given point, which is not on the conic The line so obtained determines the required chord

(9) The anharmonic ratio of four diameters of a conic is equal to that of their four conjugates

This is a particular case of Art 95 (l) The diameter a' conjugate to a diameter a is obtained by joining the pole of a, which is at infinity, to the centre of the conic The theorem may also be proved from the fact that conjugate diameters are parallel to supplemental chords

(10) Find the locus of the centre of a conic which circumscribes a given quadrangle

Draw diameters through the middle points of the sides These form a pencil whose anharmonic ratio is equal to that of their conjugate diameters The anharmonic ratio of these lines is given, since the conjugate diameters are parallel to the sides of the quadrangle Hence the locus is a conic through the middle points of the sides

(11) A, A', B, B', C, C' are three pairs of points on a conic The double points of the involutions determined by (B, B', C, C'), (C, C', A, A'), (A, A', B, B') are P, P', Q, Q', R, R' respectively Show that the involutions determined by (A, A', P, P'), (B, B', Q, Q'), (C, C', R, R') have a common pair of conjugate points

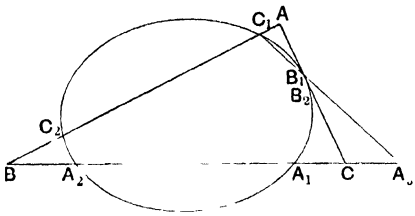
If S, T, U are the vertices of the triangle formed by AA', BB', CC', this triangle has PP', QQ', RR' for its polar triangle and therefore the sides of STU meet the corresponding sides of PP', QQ', RR' in three collinear points The line on which these points are situated determines a common pair of conjugates of the three involutions

(12) If a triangle is inscribed in a conic, the points in which two of the sides are cut by any straight line drawn through the pole of the third side are a pair of conjugate points with respect to the conic

Draw p a chord through the pole R of the side BC of the inscribed triangle ABC to meet the conic in P and P', and the sides AB and AC in Q and Q' Then the range $(BCPP')$ is harmonic Project this range from A upon p Then $(QQ'PP')$ is harmonic and therefore Q and Q' are conjugate points Thus it may be shown that the connectors of four pairs of conjugate points on AB and AC pass through R and therefore, since these pairs of conjugate points form two projective ranges, the connectors of all such pairs pass through R

104 Carnot's Theorem

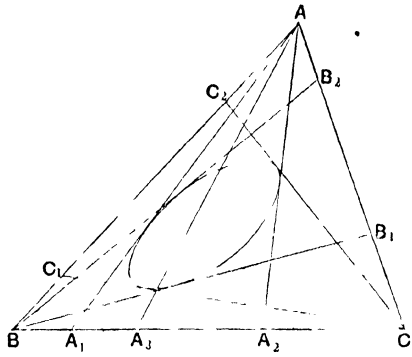
Particular Cases



Carnot's Theorem (Art 99) may be put into a slightly different form

If in the figure C_1B_1 meet BC in A_1 then by Menelaus' Theorem

$$\frac{AC_1}{BC_1}\,\frac{CB_1}{AB_1} = \frac{1}{\dfrac{BA_1}{CA_1}}$$



The correlative of Carnot's Theorem (Art 99) may be put into a slightly different form

If in the figure the line joining A to BB_1 CC_1 meet BC in A_1 then by Ceva's Theorem

$$\frac{AC_1}{BC_1}\,\frac{CB_1}{AB_1} = -\frac{1}{\dfrac{BA_1}{CA_1}}$$

Hence the relation

$$\frac{AC_1}{BC_1}\,\frac{AC_2}{BC_2}\,\frac{BA_1}{CA_1}\,\frac{BA_2}{CA_2}\,\frac{CB_1}{AB_1}\,\frac{CB_2}{AB_2} = 1$$

becomes

$$\frac{AC_2}{BC_2}\,\frac{BA_1}{CA_1}\,\frac{BA_2}{CA_2}\,\frac{CB_2}{AB_2} = \frac{BA_3}{CA_3}$$

(i) If A be situated on the conic, C_1B_1 becomes the tangent at A Hence the following is obtained

If one vertex A of a triangle ABC be on a conic and the sides AB and AC meet the conic in C_2, B_2 and the side BC meet the conic in A_1, A_2 and the tangent at A in A_3, then

$$\frac{AC_2}{BC_2}\,\frac{CB_2}{AB_2}\,\frac{BA_1}{CA_1}\,\frac{BA_2}{CA_2} = \frac{BA_3}{CA_3}$$

or

$$\frac{AC_2}{BC_2}\,\frac{CB_2}{AB_2}\,\frac{BA_2}{CA_2} = (BCA_3A_1)$$

(ii) If the three vertices of the triangle are all situated on the conic and the tangents at these points meet the opposite sides in A_3, B_3, C_3 the theorem becomes

$$\frac{BA_3}{CA_3}\,\frac{CB_3}{AB_3}\,\frac{AC_3}{BC_3} = 1$$

Hence A_3, B_3, C_3 are collinear

(iii) If the conic break up into a pair of lines the theorem is still true

If one of the lines is the line at infinity, Menelaus' theorem is obtained, viz

$$\frac{AC_1}{BC_1}\,\frac{BA_1}{CA_1}\,\frac{CB_1}{AB_1} = 1$$

Hence the relation

$$\frac{AC_1}{BC_1}\,\frac{AC_2}{BC_2}\,\frac{BA_1}{CA_1}\,\frac{BA_2}{CA_2}\,\frac{CB_1}{AB_1}\,\frac{CB_2}{AB_2} = 1$$

becomes

$$\frac{AC_2}{BC_2}\,\frac{BA_1}{CA_1}\,\frac{BA_2}{CA_2}\,\frac{CB_2}{AB_2} = -\frac{BA_3}{CA_3}$$

If BC be a tangent to the conic, A_3 becomes the point of contact Hence the following is obtained

If one side BC of a triangle ABC touch a conic at A_3 and the tangents from B and C meet the opposite sides in B_2, C_2 and the tangent from A meet BC in A_1, A_2, then

$$\frac{AC_2}{BC_2}\,\frac{CB_2}{AB_2}\,\frac{BA_1}{CA_1}\,\frac{BA_2}{CA_2} = -\frac{BA_3}{CA_3}$$

or

$$-\frac{AC_2}{BC_2}\,\frac{CB_2}{AB_2}\,\frac{BA_2}{CA_2} = (BCA_3A_1)$$

If the three sides of the triangle are all tangents to the conic and their points of contact are A_3, B_3, C_3 the theorem becomes

$$\frac{BA_3}{CA_3}\,\frac{CB_3}{AB_3}\,\frac{AC_3}{BC_3} = -1$$

Hence AA_3, BB_3, CC_3 are concurrent

If the conic break up into a pair of points the theorem is still true

If one of the points is the centroid of the triangle, Ceva's theorem is obtained, viz

$$\frac{AC_1}{BC_1}\,\frac{BA_1}{CA_1}\,\frac{CB_1}{AB_1} = -1$$

Deductions

(*a*) **Newton's Theorem** *If through any two points O and O′ two pairs of parallel chords OAB, OCD* and *O′A′B′, O′C′D′ are drawn to meet a conic in A, B, C, D and A′, B′, C′, D′, then*

$$\frac{OA}{OC}\frac{OB}{OD} = \frac{O'A'}{O'C'}\frac{O'B'}{O'D'}$$

The lines OAB, $O'A'B'$ meet at a point at infinity U, and the lines OCD, $O'C'D'$ at a point at infinity V

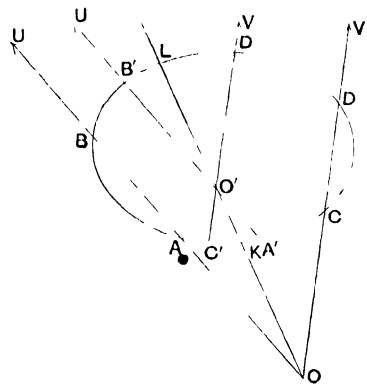
Let OO' meet the conic in K and L

From the triangle $OO'U$

$$\frac{OA}{UA}\frac{OB}{UB}\frac{UA'}{O'A'}\frac{UB'}{O'B'}\frac{O'L}{OL}\frac{O'K}{OK} = 1$$

But $\dfrac{UA'}{UA}\dfrac{UB'}{UB} = 1$, since U is at infinity

Therefore $\dfrac{OA}{O'A'}\dfrac{OB}{O'B'} = \dfrac{OL}{O'L}\dfrac{OK}{O'K}$



Similarly, from the triangle $OO'V$,

$$\frac{OC}{OD}\frac{OD}{O'C'}\frac{O'D'}{} = \frac{OL}{O'L}\frac{OK}{O'K}$$

Therefore

$$\frac{OA}{OC}\frac{OB}{OD} = \frac{O'A'}{O'C'}\frac{O'B'}{O'D'}$$

The above proof assumes that the line OO' meets the conic in a pair of real points K and L If OO' does not meet the conic in real points, O and O' must both be external to the conic Take any point O'' within the conic Then the theorem holds for the pairs of parallel chords through O and O'' and for pairs of parallel chords through O' and O'' Hence it holds for parallel chords through O and O'

Deductions from Newton's Theorem

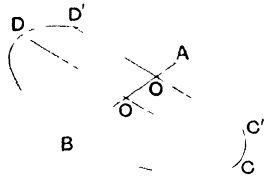
(1) If d and d' are the semi-diameters, and t and t' the tangents, parallel to OAB and OCD respectively, then

$$\frac{OA}{OC}\frac{OB}{OD} = \frac{d^2}{d'^2} = \frac{t^2}{t'^2}$$

(2) In the case of an Ellipse, if O be the centre of the curve, OA and OC a pair of conjugate diameters, and O' be taken on OA, then in the figure

$$\frac{OD^2}{OA^2} = \frac{O'D'^2}{BO'\ O'A} = \frac{O'D'^2}{OA^2 - OO'^2},$$

$$\frac{O'D'^2}{OD^2} + \frac{OO'^2}{OA^2} = 1$$



(3) In the case of a Parabola, let UMU' be an ordinate to a diameter PM, and S the focus Draw the focal chord QQ' parallel to UU' Then by

Newton's Theorem, since the diameter meets the curve at infinity,

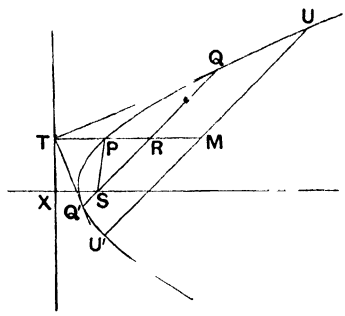
$$\frac{UM}{PM}\cdot\frac{U'M}{} = \frac{QR}{PR}\cdot\frac{Q'R}{}$$

But in the figure, if AT be the directrix, $RP=PT=SP$. Also, since QTQ' is a right angle, R is centre of circle through TQQ',

$$QR^2=RT^2=4\ SP^2,$$

$$\frac{QR}{PR}\cdot\frac{Q'R}{}=4\ SP,$$

$$UM^2=4\ SP\ PM$$



(*b*) *If a circle and a central conic intersect in four points, the pairs of chords of intersection are equally inclined to the axes of the conic.*

Let the conic and the circle intersect in A, B, C, D and let the chords AB and CD intersect in O. Let d and d' be the semi diameters of the conic parallel to AB and CD respectively.

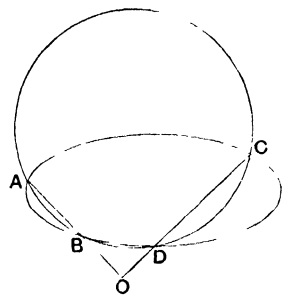
Then from the conic

$$\frac{OA}{OC}\cdot\frac{OB}{OD}=\frac{d^2}{d'^2},$$

and from the circle

$$\frac{OA}{OC}\cdot\frac{OB}{OD}=1$$

Therefore $\dfrac{d^2}{d'^2}=1$ and $d=d'$



Hence the semi diameters of the conic parallel to AB and CD are equal, and therefore equally inclined to the axes of the conic, as are also the chords of intersection of the circle and conic.

Centre of Curvature. If a circle be described to meet a conic in three coincident points P, the circle is termed *the circle of curvature* at P and its centre *the centre of curvature*. From the preceding it is seen that the tangent to the conic at P and the common chord of the circle and conic are equally inclined to the axes of the conic. The centre of curvature is obviously on the normal at P. Hence the following construction is obtained for the centre of curvature at any point P.

Draw PT the tangent at P and PL a chord equally inclined to the axis. Let M be the middle point of PL, then C the point of intersection of the normal at P and the perpendicular through M to PL is the centre of curvature.

EXAMPLES

(1) If A_1, B_1, C_1 be the points where three concurrent lines through A, B, C meet the opposite sides of a triangle, and a conic through A_1, B_1, C_1 meet the sides in A_2, B_2, C_2, then the lines AA_2, BB_2, CC_2 are concurrent.

(2) If tangents are drawn to a conic from the vertices of a triangle to meet the opposite sides in A_1, A_2, B_1, B_2, C_1, C_2, then a conic can be described through these six points.

(3) If a conic cut the sides of a triangle ABC in $A_1, A_2, B_1, B_2, C_1, C_2$, then a conic can be described to touch the six lines $AA_1, AA_2, BB_1, BB_2, CC_1, CC_2$

(4) If a conic touch the sides of a triangle ABC at A', B', C', and A'', B'' be the harmonic conjugates of A' and B' with respect to BC and CA, a conic can be described to touch the sides of the triangle at A'', B'', C''

(5) If a straight line meet a hyperbola in A and B, and its asymptotes in A' and B', then AA' and BB' are equal

(6) If a conic meet the sides of a triangle ABC in $A_1, A_2, B_1, B_2, C_1, C_2$, and A_1B_1, A_2B_2 intersect in C_3, B_1C_1 and B_2C_2 in A_3, C_1A_1 and C_2A_2 in B_3, then the lines AA_3, BB_3, CC_3 are concurrent

(7) The vertices A, B, C of a triangle are joined to any point P, and the lines PA, PB, PC meet the opposite sides in A_1, B_1, C_1. A conic is described through A, A_1, B_1, C_1, meeting BC in A_2, and the tangent to the conic at A meets BC in A_3. Prove that the range BCA_2A_3 is harmonic and that A_2, A_3 are conjugate points of a definite involution.

(8) If P, Q, R, S be the points of contact of the sides AB, BC, CD, DA of a quadrilateral circumscribed to a conic,

$$\frac{AP}{BP}\cdot\frac{BQ}{CQ}\cdot\frac{CR}{DR}\cdot\frac{DS}{AS}=1$$

(9) Extend Carnot's theorem to the case of a polygon meeting the sides of a conic.

(10) If a hexagon $ABCDEF$ circumscribe a conic and the sides AB, BC, CD, DE, EF, FA touch the conic at A', B', C', D', E', F', and the ratios

$$\frac{AA'}{BA'}, \quad \frac{BB'}{CB'} \text{ etc. be denoted by } a, b, c, d, e, f,$$

prove that $$abcdef=1$$

(11) Show that, if from any two points the vertices of a triangle are projected upon the opposite sides, the six points so obtained lie on a conic, and find by the ruler only the points in which it cuts again the rays of projection.

(12) Through a point P on a conic chords PA, PB are drawn, and chords MCC', MDD' are drawn, respectively parallel to these, through another point M, not on the conic. On PA, PB lengths PQ, PR are taken inversely proportional to

$$\frac{MC}{PA}\cdot\frac{MC'}{PA}, \quad \frac{MD}{PB}\cdot\frac{MD'}{PB}$$

Prove that the normal at P to the conic is a diameter of the circle PQR.

(13) Prove that, if a circle intersects a parabola in four points, the pairs of connectors of these points are equally inclined to the axis of the parabola, and obtain a construction for the centre of curvature of any point on the parabola.

CHAPTER XVI

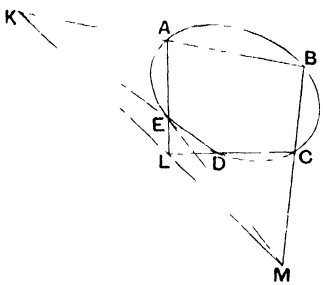
DEDUCTIONS FROM PASCAL'S THEOREM, DESARGUES' THEOREM, AND THE COMPLETE INSCRIBED QUADRANGLE AND CIRCUMSCRIBED QUADRILATERAL OF A CONIC CONICS IN SELF-PERSPECTIVE THE RECTANGULAR HYPERBOLA

105 Pascal's and Brianchon's Theorems

Particular Cases

Pascal's and Brianchon's Theorems are true for any six points on any conic and for any six tangents to any conic There are certain particular cases which should be noted

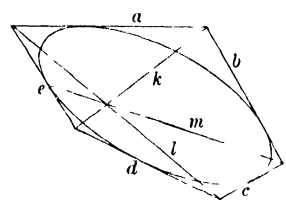
(1) If two of the six points coincide, one of the sides of the hexagon is replaced by a tangent Hence the following is obtained



If ABCDE be a pentagon inscribed in a conic and BC meets the connector of AB DE ($\equiv K$) *and CD EA* ($\equiv L$) *in M, then the line ME is the tangent at E*

Hence given any five points on a conic the tangents at any of these points can be constructed

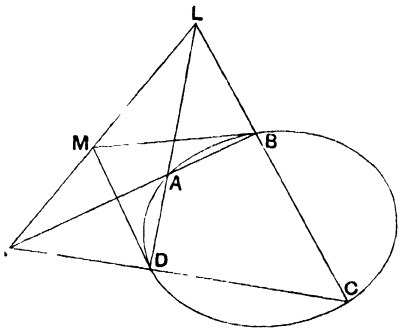
If two of the six tangents coincide, one of the vertices of the hexagon is replaced by the point of contact of a tangent Hence the following is obtained



If abcde be a pentagon circumscribed to a conic and m is the connector of bc to the point of intersection of ab de ($\equiv k$) *and cd ea* ($\equiv l$), *then m passes through the point of contact of e*

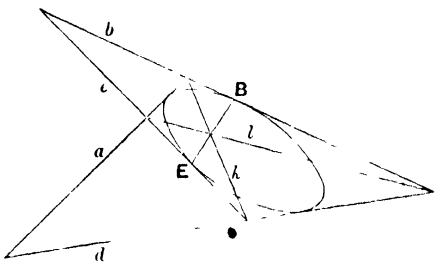
Hence given any five tangents to a conic the points of contact of any of these tangents can be constructed

(ii) If two pairs of the six points coincide, the properties of the complete quadrangle inscribed in a conic are obtained, viz



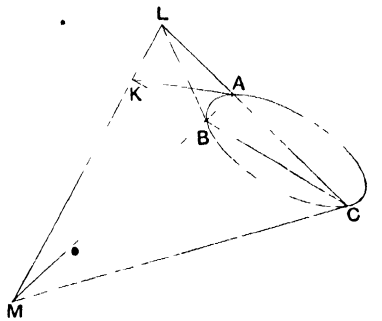
If a quadrangle be inscribed in a conic, the points of intersection of the tangents at the vertices lie two by two on the sides of the diagonal points triangle of the quadrangle

If two pairs of the six tangents coincide, the properties of the complete quadrilateral circumscribed to a conic are obtained, viz



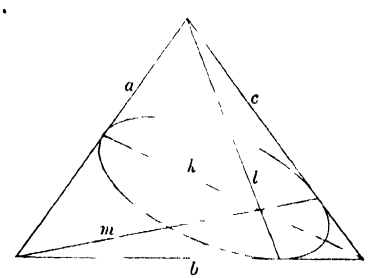
If a quadrilateral be circumscribed to a conic, the connectors of the points of contact of the tangents pass two by two through the points of intersection of the diagonals of the quadrilateral

(iii) If three pairs of the six points coincide, the following theorem is obtained



If a triangle be inscribed in a conic, the tangents at the vertices intersect the opposite sides in three collinear points

If three pairs of the six tangents coincide, the following theorem is obtained



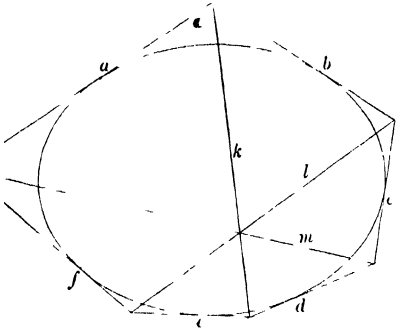
If a triangle be circumscribed to a conic, the lines joining the points of contact to the opposite vertices are concurrent

The Parabola In the case of the parabola the point at infinity on any diameter is on the curve and the line at infinity is a tangent to the curve Consequently the theorems of this article may be expressed in a simpler form for the parabola The deduction of these results is left as an exercise for the reader

Pascal's Hexagons for six points If any six points are taken on a conic, 60 Pascal Hexagons can be formed by taking the six points in different orders The properties of such hexagons are given in Art 146

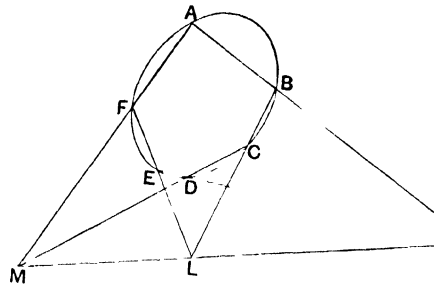
Deductions

(*a*) *The locus of the poles of tangents to one conic with respect to a second conic is a conic*



Let a, b, c, d, e, f be any six tangents to the given conic Then by Brianchon's theorem, k, l, m the three connectors of pairs of opposite vertices of the hexagon formed by these lines are concurrent Let A, B, C, D, E, F be the poles of a, b, c, d, e, f with respect to the second conic Then, since k, l, m are concurrent, K, L, M the points of intersection of pairs of opposite sides of the hexagon $ABCDEF$ are collinear Hence by the converse of Pascal's theorem the points A, B, C, D, E, F lie on a conic If f be regarded as a

The envelope of the polars of points on one conic with respect to a second conic is a conic

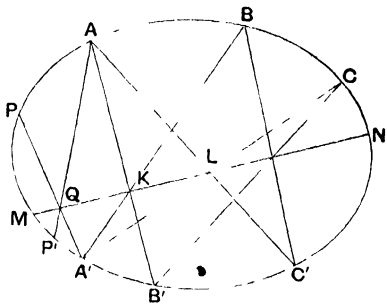


Let A, B, C, D, E, F be any six points on the given conic Then by Pascal's theorem, K, L, M the three points of intersection of pairs of opposite sides of the hexagon formed by these points are collinear Let a, b, c, d, e, f be the polars of A, B, C, D, E, F with respect to the second conic Then, since K, L, M are collinear, k, l, m the connectors of pairs of opposite vertices of the hexagon $abcdef$ are concurrent Hence by the converse of Brianchon's theorem the lines a, b, c, d, e, f touch a conic If F be regarded

variable line, it follows that the locus of its pole F is the conic $ABCDE$

as a variable point, it follows that the envelope of its polar f is the conic $abcde$

(*b*) *Construction of Projective Ranges on a Conic and of their self-corresponding points*



Two projective ranges on a conic are determined, when three pairs of corresponding elements are given By Pascal's theorem the self-corresponding elements can be found

Let A, B, C, A', B', C' be the determining elements Then the pencils $(A\ A'B'C')$ and $(A'.ABC)$ have a self-corresponding ray in AA' and are therefore in perspective The axis of perspective is the Pascal line of the hexagon $AC'BA'CB'$, that is the line KL in the figure If Q be any point on KL, the lines AQ and $A'Q$ meet the conic in points P and P', which are corresponding points of the ranges ABC and $A'B'C'$ If the line KL meet the conic in M and N, these points correspond to themselves and are the two self-corresponding points of the ranges ABC and $A'B'C'$ on the conic

Given the self-corresponding points, M and N, of two projective ranges, and a pair of corresponding points, A and A', the ranges can be constructed Take any point Q on MN and let QA' and QA meet the conic in P and P' respectively, then P and P' are corresponding points

Conversely, if points Q on a fixed line MN be joined to any pair of fixed points A and A', and QA, QA' meet the conic in P' and P, these points are corresponding points of two ranges having M and N for self-corresponding points and A and A' for corresponding points

By the correlative method the self-corresponding elements of two projective systems of tangents may be constructed

EXAMPLES

(1) If through the points A and B, where a tangent to any hyperbola meets the asymptotes, lines parallel to the asymptotes be drawn to intersect in Q, and O be the centre of the hyperbola, OQ passes through the point of contact of the tangent and is bisected at that point

This is a particular case of (iii)

(2) If through the points of contact of two tangents to a parabola, which intersect in T, lines parallel to these tangents be drawn to intersect in V, TV is a diameter of the parabola and it is bisected by the chord of contact

This is a particular case of (iii)

(3) P and Q are two points on a conic and pairs of chords PA, PB are drawn through P, and QC, QD through Q Show that if PA and QC meet in R and PB and QD in T, then the lines AD, BC, RT are concurrent

$PACQDB$ is a hexagon inscribed in a conic and therefore by Pascal's theorem the triangles ACR and DBT are in perspective

(4) Three straight lines are drawn parallel to the sides of a triangle, show that their six points of intersection with the sides of the triangle lie on a conic

Converse of Pascal's theorem with the line at infinity for axis of perspective

(5) The hexagon which is formed by the six points of intersection of alternate pairs of sides of a Pascal hexagon is a Brianchon hexagon

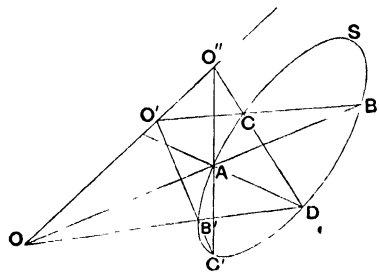
If the vertices of the hexagon formed by the six points of intersection of alternate pairs of sides be A, B, C, D, E, F, the triangles ACE and BDF are in perspective, the axis of perspective being the Pascal line of the given hexagon Hence AD, BE, CF are concurrent and the hexagon is a Brianchon hexagon

(6) If through any two fixed points C and C' on a conic chords CB', CA', $C'A$, $C'B$ be drawn and CA, $C'B$ intersect in W and $C'B'$, CA' in W', then AB', BA', WW' are concurrent

Consider the Pascal hexagon $ACA'BC'B'$

(7) If the sides of a given triangle meet a conic in six points, the triangle formed by joining the six points in pairs is in perspective with the given triangle

(8) O, O', O'' are three fixed collinear points and A is any point on a fixed conic S AO meets S in B, BO' meets S in C, CO'' meets S in D Prove that AD will always pass through a fixed point



Range described by A is projective with

range described by B,

hence with

range described by C,

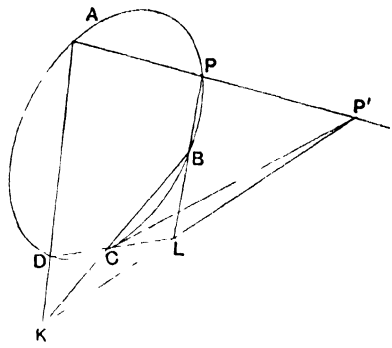
hence with

range described by D

Let OD meet the conic in B' and $O'B'$ meet the conic in C' Then $O''C'$ will pass through A, for $ABCDB'C'$ is a Pascal hexagon Therefore A and D mutually correspond Therefore they are conjugate points of an involution on the conic and consequently AD always passes through a fixed point

(9) If the conics of a system pass through four points A, B, C, D, and a fixed chord through A meet one of the conics in P and the tangent to the same conic at one of the points (C) in P', then P and P' describe projective ranges

Consider the Pascal hexagon $APBCCD$ in which two vertices are coincident at C Then, in the figure, K is fixed and DC and AP are fixed lines

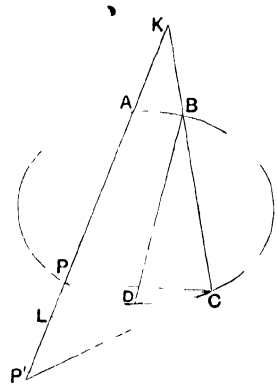


The range P' is projective with the range L on CD This is projective with the pencil BL and this is projective with the range P

Therefore the ranges P and P' are projective

(10) A system of conics passes through four real points A, B, C, D Prove that the harmonic conjugate of AC with regard to AB and AD is cut by the conics and the tangents at C in an involution, whose double points are real

Let AP the harmonic conjugate of AC with regard to AB and AD meet a conic of the system in P and the tangent at C in P' Let CB and CD meet AP in K and L Then since the pencil $(A\ BDCP)$ is harmonic, $(C\ BDCP)$ is harmonic Therefore $(KLP'P)$ is harmonic and P and P' are a pair of conjugate points of an involution of which K and L are the double points



(11) A quadrangle $ABCD$ is inscribed in a parabola through two of its vertices C and D straight lines are drawn parallel to the axis, meeting DA, BC in P and Q, show that PQ is parallel to AB

If ∞ be the point at infinity on the parabola, the hexagon $B\,A\,D\infty\,\infty\,C$ is inscribed in the parabola Therefore the points $BA\ \infty\infty$, $AD\ \infty C$, $CB\ D\infty$ are collinear Hence PQ and AB are parallel

106 Desargues' Theorem

Particular Cases

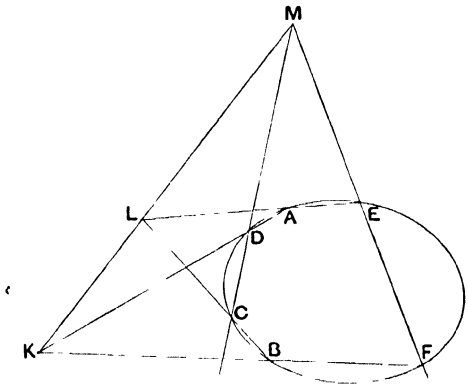
(1) If in Desargues' theorem (Art 101) the transversal be drawn through one of the four points of intersection of the system of conics, it does not meet the system of conics in an involution properly so called In this case, however, the following theorems hold

(*a*) *A system of conics through four fixed points determines on any two transversals, drawn through two of these points, two ranges which are in perspective*

Let $A, B\ C, D$ be the points of intersection of the conics Through

A and B draw any two transversals AE and BF to meet one of the conics in E and F

Let AE CB be L,
AD BF be K,
and DC EF be M



Then K, L, M are collinear by Pascal's theorem But the points L and K are fixed as is the line DC, and therefore M is a fixed point Hence, since EF passes through the fixed point M, the ranges E and F are in perspective

(*b*) *A system of conics through four fixed points determines two projective ranges on any two transversals, drawn through one of these points*

If a third transversal be drawn through one of the other points of intersection of the conics, the range determined by the conics on this transversal is in perspective with each of the ranges on the given transversal, which are therefore projective with each other

From (*a*) and (*b*) the following theorems are obtained

Given four points on a conic A, B, C, D and two fixed lines AE, BF each passing through one of them, and meeting the conic in E and F, the line EF passes through a fixed point

Given four points on a conic A, B, C, D and two fixed lines through one of them, AE, AF, meeting the conic in E and F, the envelope of EF is a conic

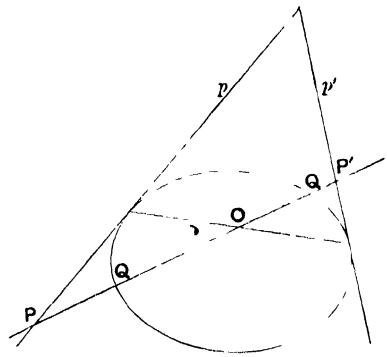
The correlatives of theorems (1) (*a*) and (1) (*b*) are as follows

If a system of conics be described to touch four fixed lines, the pencils formed by the tangents to the conics from two points, each on one of the fixed lines, are in perspective

If a system of conics be described to touch four fixed lines, the pencils formed by the tangents to the conics from two points, both on one of the fixed lines, are projective

(ii) *If a system of conics be described through two given points, Q and Q', to touch two given lines p and p', the chords of contact of the conics with p and p' pass through one or other of two fixed points, and the poles of QQ' lie on one or other of two fixed lines*

Consider the quadrangle formed by the points in which a conic of the system meets p and p' Let QQ' meet p and p' in P and P' and their chord of contact in O Then QQ', PP' are two pairs of conjugate points of the involution determined by the conic on QQ' Of this involution O is a double point and, since this is a given involution, O is one or other of two given points



The second part of the theorem is the correlative of the first

(iii) *The polar of a given point with respect to a conic passes through the conjugate of the given point with respect to every quadrangle inscribed in the conic* (See Art 57)

Let O be the given point, $ABCD$ any inscribed quadrangle, and G and E the points AB DC and AC BD Join A, B, C, D to O to meet the conic again in A', B', C', D', respectively Let AB and $C'D'$ meet in K and let OK meet DC in K' and the polar of O in M Then $OMKK'$ is harmonic and therefore $(G\ OMAC)$ is harmonic

By Example (8), Art 102, BD and $C'A'$ meet in a point N on OK Let AC meet OK in N' Then N and N' are harmonic conjugates of O and M Therefore $(E\ OMAD)$ is harmonic Hence M is the conjugate of O with respect to the quadrangle $ABCD$

If the connector of the given point O to its conjugate M with regard to the quadrangle meets a conic in real points P and P', the points O and M are double points of the involution determined by conics circumscribing the quadrangle Therefore $OMPP'$ is harmonic and O, M are conjugate points with regard to the conic

Deductions

(a) *If three conics circumscribe the same quadrangle a common tangent to any two is cut harmonically by the third conic*

The points of contact are the double points of the involution determined by the conics on the common tangent

If through the intersection of a pair of common chords of two conics a tangent be drawn to one of them, this tangent is cut harmonically by the other conic

This theorem is a particular case of the preceding, one conic being replaced by a pair of lines The tangent from the point of intersection of the common chords may be looked upon as a common tangent

If two conics have double contact with each other, any tangent to one is cut harmonically, at the points where it meets the other, at its point of contact and at its point of intersection with the common chord (Cf Art 131)

The preceding theorem takes this form when the two common chords coincide

The correlatives of the preceding should be noticed they are as follows

If three conics are inscribed in the same quadrilateral, the tangents to two of them at a point of intersection are harmonic conjugates of the tangents from this point to the third

The chord joining the points of intersection of two pairs of common tangents to two conics, the tangent to one of them at a point where this chord meets it, and the pair of tangents from this point to the other conic, form a harmonic pencil

If two conics have double contact with each other, the tangent at any point (P) on one is a harmonic conjugate of the line joining its point of contact (P') to the intersection of the common tangents, with respect to the two tangents from that point (P) to the other conic (Cf Art 131)

(*b*) There is an important group of theorems the proofs of which depend on the correlative of Desargues' Theorem and on the properties of orthogonal involution pencils (Art 58) In involution ranges there is no property strictly correlative to these properties of an involution pencil and consequently these theorems have no correlatives

Director Circle and Directrix

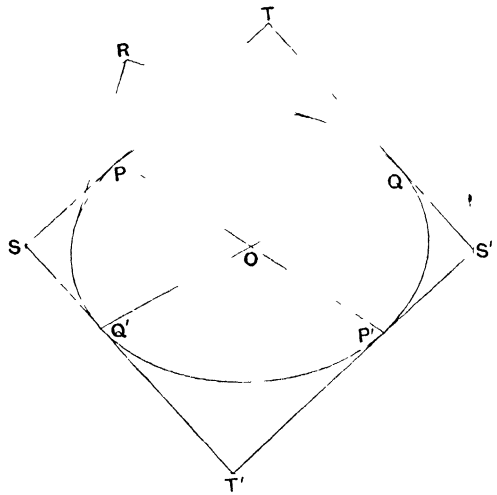
The locus of the points of intersection of tangents to a conic, which are at right angles, is for an ellipse or hyperbola a concentric circle (real or imaginary) termed the director circle, and for a parabola the directrix

Ellipse and Hyperbola

Draw any two perpendicular tangents TP and TQ to touch the conic at P and Q Join P and Q to O, the centre of the conic, to meet

the curve again in P' and Q', and let the tangents at P' and Q' intersect in T'. The four tangents at P, Q, P', Q' form a rectangle. Two of the pairs of opposite vertices are T and T', and S and S', and the third pair consists of the points at infinity in the directions TQ and TP.

Take any other point R, the tangents from which to the conic are at right angles, and consider the involution determined at R by the conic and the circumscribed quadrilateral $TS'T'S$.



Two pairs of conjugate rays of this involution are at right angles, viz. the pair of tangents from R to the conic and the connectors of R to the points at infinity on TP and TQ. Therefore every pair of conjugate rays are at right angles. Therefore the angle TRT' is a right angle, and the locus of R is a circle described on TT' as diameter. This circle has O for its centre.

If the conic be an ellipse, and A, A' and B, B' be the ends of the major and minor axes, Q, Q' and P, P' may be taken to coincide with these points, and it follows that $OT^2 = OA^2 + OB^2$.

If the conic be a hyperbola and OA and OB be its semi-transverse and semi-conjugate axes, the tangents to the conic from points on the transverse axis at distances $\pm\sqrt{OA^2 - OB^2}$ are at right angles (Ex. (7), page 185). Hence in this case $OT^2 = OA^2 - OB^2$.

If the hyperbola is rectangular, the director circle becomes the centre of the hyperbola, and if the hyperbola is oblique the circle is imaginary.

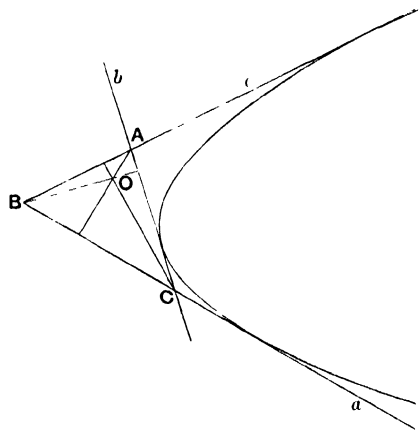
Parabola Let T and R be any two points the tangents from which to the parabola are at right angles, and let TP, TQ be the tangents from T Let ∞ represent the line at infinity Consider the involution pencil determined at R by the quadrilateral TQ, TP, ∞, ∞ and the parabola The tangents from R are at right angles as are also the lines joining R to the points at infinity on TP and TQ Hence the involution pencil is orthogonal Therefore RT is perpendicular to the line joining R to the point at infinity on the parabola, that is to its axis Since T may be taken on the axis, RT is the directrix

Pair of Points If the conic become a pair of points (Art 94) the director circle is the circle described on the line joining the points as diameter

(1) Properties of the Directrix of a Parabola

(1) *The directrix of a parabola passes through the orthocentre of every circumscribed triangle* or *The directrix of every parabola inscribed in a triangle passes through its orthocentre*

Let ABC be a triangle circumscribed to a parabola, O its orthocentre, a, b, c its sides and ∞ the line at infinity Then $abc\infty$ is a circumscribed quadrilateral of the parabola Consider the involution subtended by it at O A pair of rays of this involution are the lines joining O to bc and $a\infty$ These are parallel to OA and a and are therefore at right angles Similarly the lines joining O to ba and $c\infty$ are at right angles Therefore the involution pencil is orthogonal and the tangents from O to the parabola are at right angles Therefore O is on its directrix



(2) *The directrix of a parabola inscribed in a quadrilateral passes through the orthocentre of each of the four triangles formed by the sides of the quadrilateral and therefore the four orthocentres are collinear*

This follows from (1)

(3) *The circles described on the diagonals of a quadrilateral as diameters meet the line of orthocentres in the same pair of points and are therefore coaxal*

Consider either of the points of intersection of two of the circles The involution subtended at this point by the vertices of the quadrilateral is orthogonal Hence the circle described on the third diagonal passes through the point and it is a point on the directrix of the inscribed parabola Hence it is on the line of orthocentres of the quadrilateral

From (3) it follows that the middle points of the diagonals of the quadrilateral, being the centres of a system of coaxal circles, are collinear, and that their line of collinearity is perpendicular to the line of orthocentres, which is the radical axis of the coaxal circles

(11) **Properties of the Director Circle of a Conic**

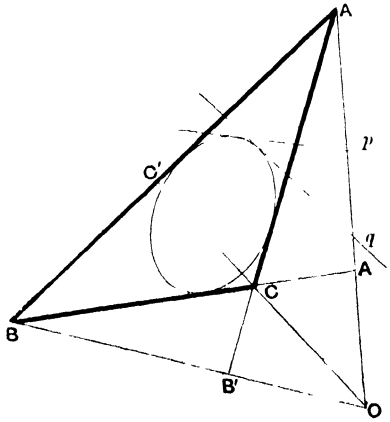
(1) *The director circles of a system of conics inscribed in a quadrilateral are coaxal*

(2) *The circles described on the diagonal of the quadrilateral as diameters are circles of the system*

(3) *The radical axis is the directrix of the inscribed parabola, and is also the line of orthocentres of the quadrilateral*

Consider one of the points of intersection of two of the director circles The system of tangents to the conics from this point forms an orthogonal involution, and therefore the director circles of all the conics pass through the point, including the circles described on the diagonals as diameters and the directrix of the inscribed parabola The same applies to the second point of intersection of the circles Hence (1), (2) and (3) are true

(4) *The self polar circle of a triangle cuts the director circles of all conics inscribed in the triangle orthogonally*



Let ABC be the triangle, a, b, c its sides, O its orthocentre, and AA', BB', and CC' the perpendiculars from the vertices on the opposite sides

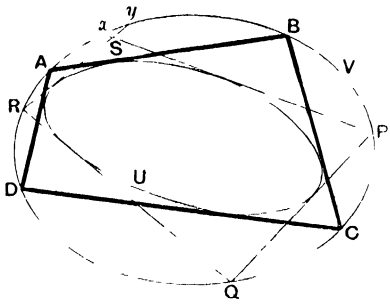
Then O is the centre of the circle with regard to which the triangle ABC is self-conjugate and $OB\ OB'$ is equal to the square of its radius (Example (5) Art 88)

Draw any two straight lines p and q Describe a conic S to touch a, b, c, p, q and systems of conics to touch a, b, c, p and a, b, c, q The point O is on the radical axis of the director circles of both these systems and therefore the tangents from O to the director circles of both systems are equal These must be equal to the tangents from O to the director circle of S and are therefore all equal

The pair of points A, B constitute a conic inscribed in abc and the circle described on AB is its director circle This circle passes through B' Hence the square of the tangent from O to this circle equals $OB\ OB$ Hence the squares of the tangents from O to the director circles of all conics inscribed in abc are equal to $OB.OB'$ Therefore the circle with regard to which the triangle is self-conjugate cuts the director circles of all conics inscribed in abc orthogonally

Corr This gives another proof of (1) (3), for, since the four self-conjugate circles of the triangles of a quadrilateral cut the director circles of the inscribed conics of the quadrilateral orthogonally, these latter are coaxal (Art 88, Example (10)), and the orthocentres of the triangles are collinear and lie on the radical axis

(c) *If one quadrangle can be inscribed in a conic so that two pairs of opposite sides touch a second conic, an infinite number of quadrangles can be constructed in a similar manner*



Let $ABCD$ be a quadrangle described about a conic U and inscribed within a conic V Let $PQRS$ be a second quadrangle described about U so that P, Q, R are on V and let PS, RS meet V in x, y respectively

It is necessary to show that S, x and y coincide

The lines joining P (any point on V) to the vertices of the circumscribed quadrilateral $ABCD$ of the conic U are pairs of conjugate rays of an involution of which the tangents from P to the conic U are conjugate rays (correlative of Desargues' Theorem) Therefore $(P\ AC,\ BD,\ Qx)$ is an involution pencil and $AC,\ BD,\ Qx$ form an involution on V Similarly $(R\ AC,\ BD,\ Qy)$ is an involution pencil and $AC,\ BD,\ Qy$ form an involution on V

Since these involutions on the conic V are determined by $AC, BD,$

and a point can have only one conjugate in a given involution, x and y, which are conjugates of Q, must coincide and the quadrangle $PQRS$ is therefore inscribed in the conic V

(*d*) *If two points are conjugate with regard to two conics of a system of conics through four fixed points, they are conjugates with regard to all conics of the system*

The polars of a point P with respect to the conics all pass through P', the conjugate of P with regard to the quadrangle formed by the four fixed points ((m), Art 106) Hence P' must be the common conjugate of P with respect to the two conics and is a common conjugate of P with respect to all the conics of the system

This may also be deduced from Desargues' Theorem

(*e*) *The polars of any two fixed points with respect to a system of conics through four fixed points form two projective pencils*

Let P and Q be any two fixed points, l the line joining them, and P_0, Q_0 their common conjugates with respect to the system of conics ((*d*) above) Consider one of the conics which meets l in real points A and A' The polars of P and Q will be two lines p', q' which pass through P_0 and Q_0 respectively and meet l in a pair of points P', Q' which are harmonic conjugates of P and Q with respect to A, A'

Consider a second conic of the system, which meets l in B, B',

If two lines are conjugate with regard to two conics of a system of conics touching four fixed lines, they are conjugates with regard to all conics of the system

The poles of a line p with respect to the conics all lie on a line p', the conjugate of p with respect to the quadrilateral formed by the four fixed lines (Corl (m), Art 106) Hence p' must be the common conjugate of p with respect to the two conics and is a common conjugate of p with respect to all the conics of the system

This may also be deduced from the correlative of Desargues' Theorem

The poles of any two fixed lines with respect to a system of conics touching four fixed lines form two projective ranges

Let p and q be any two fixed lines, L their point of intersection and p_0, q_0 their common conjugates with respect to the system of conics ((*d*) above) Consider one of the conics to which the tangents from L are real lines a and a' The poles of p and q will be two points P', Q' which lie on p_0 and q_0 respectively and their connectors to L are a pair of lines p', q' which are harmonic conjugates of p and q with respect to a, a'

Consider a second conic of the system, the tangents to which from

a second pair of polars p'', q'' of P and Q are obtained These pass through P_0, Q_0 and meet l in points P'', Q'', which are harmonic conjugates of P, Q with respect to BB'

Proceeding in a similar manner, on l are obtained two ranges of points $P'P''P'''$ and $Q'Q''Q'''$ which are harmonic conjugates of P and Q with respect to AA', BB', CC', where the latter points, by Desargues' theorem, are pairs of conjugate points of an involution

Hence (Example (7), Chapter VIII) the ranges $P'P''P'''$ and $Q'Q''Q'''$ are projective Therefore the pencils $p'p''p'''$ and $q'q''q'''$ through P_0 and Q_0 are projective

L are b, b', a second pair of poles P'', Q'' of p and q are obtained These lie on p_0, q_0 and their connectors to L are p'', q'', which are harmonic conjugates of p, q with respect to bb'

Proceeding in a similar manner, through L are obtained two pencils of rays $p'p''p'''$ and $q'q''q'''$ which are harmonic conjugates of p and q with respect to aa', bb', cc', where the latter rays, by the correlative of Desargues' theorem, are pairs of conjugate rays of an involution

Hence (correlative of Example (7), Chapter VIII) the pencils $p'p''p'''$ and $q'q''q'''$ are projective Therefore the ranges $P'P''P'''$ and $Q'Q''Q'''$ on p_0 and q_0 are projective

EXAMPLES

(1) Prove that an asymptote to a hyperbola is cut by the three pairs of opposite sides of a complete quadrangle inscribed in the hyperbola in three segments with a common middle point

The double points of the involution determined on the asymptote are the points of contact of the two conics which can be described through the vertices of the quadrangle to touch the asymptote One of these is the point at infinity at which the asymptote touches the hyperbola Hence the other double point bisects the distances between all pairs of conjugate points

(2) A circle has double contact with a hyperbola and from P a point on the latter are drawn PM, parallel to an asymptote to meet the chord of contact in M, and PT to touch the circle at T Prove that $PM = PT$

Let PM meet the circle in A and A' Then P is the centre and M a double point of the involution determined by the hyperbola and circle on PM Therefore $PM^2 = PA \cdot PA' = PT^2$

(3) Given eight points in a plane and any straight line not passing through one of them, prove that in general two conics can be drawn, one passing through four points selected from the eight and one through the other four, so as to intersect each other in two points on the line

The two points on the line are the common conjugates of the involutions determined on the line by the two quadrangles

(4) Four points A, B, C, D are taken on a conic Any straight line through D meets the conic again in D' and the sides of the triangle in A', B', C'' Prove that $(A'B'C'D')$ is equal to $(ABCD)$

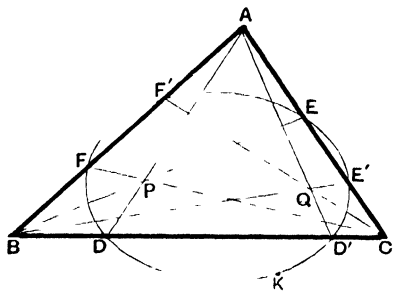
Let the tangent at A meet DD' in A_1 Then if P be any point on the conic $(P\ ABCD)=(A\ ABCD)=(A_1C'B'D)$ But $(A_1C''B'D)=(A'B'C''D')$ by Desargues' theorem, for $AABC$ is an inscribed quadrangle of which two points coincide at A

(5) If the four points of intersection of two parabolas are concyclic their axes are perpendicular

The parabolas and the circle determine an involution on the line at infinity of which the points of contact of the parabolas are the double points These points are therefore harmonic conjugates of the circular points at infinity They therefore subtend a right angle at any point and the axes of the parabolas are consequently at right angles

(6) Two parabolas whose axes are perpendicular intersect in four concyclic points

(7) ABC is a triangle and D, E, F are three fixed points on BC, CA, AB, such that AD, BE, CF are concurrent A conic passing through DEF and any fourth point meets BC, CA, AB again in D', E', F' Prove that $E'F'$, $F'D'$, $D'E'$ each pass through a fixed point, and that the point of concurrency of AD', BE', CF' lies on a fixed conic circumscribing the triangle ABC



Since the conic passes through four fixed points and the sides of the triangle each pass through one of these, the ranges E' and F'' are in perspective (Art 106 (1) (α)) $E'F''$ therefore passes through a fixed point The pencils BE', CF'' are projective Therefore Q describes a conic through B and C and by symmetry through A

(8) Given two quadrangles in a plane prove that in general through any point P in the plane may be drawn three straight lines such that on each the mates of P in the two involutions determined on it by pairs of sides of the two quadrangles coincide

Let P be the given point and $ABCD$ and $A'B'C'D$ the given quadrangles Describe conics through P, A, B, C, D and P, A', B', C', D' Let these conics intersect in Q, R, S Consider the line PQ P, Q are conjugate points of the involution determined on PQ by the sides of the first quadrangle P, Q are likewise conjugate points of the involution determined on PQ by the sides of the second quadrangle Therefore PQ is one of the required lines Similarly PR and PS are the other two required lines

(9) Two conics intersect in four points A, B, C, D Through A and B two chords AEE', BFF' are drawn to intersect the conics in E, F and E', F' respectively Then EF, $E'F'$ intersect on CD See Art 106 (1) (α)

(10) If the tangents to a conic S at points U and V meet at P and two lines be drawn through P to meet the conic in W and X, the conic S' described through U, V, W, X to touch PW at W will touch PX at X.

Since $UVWX$ is a common inscribed quadrangle of the two conics, the tangents at U and V to S and at W and X to S' will intersect on the line joining UX VW to UW VX, this line being the common polar of UV WX. Hence P is on this line and, since the tangent at W to S' meets it at P, the tangent at X to S' likewise passes through P and is the line PX.

(11) If two conics S and S' intersect in U, V, W, X and P the pole of UV with respect to S coincides with the pole of WX with respect to S', then the pole of UV with respect to S' coincides with the pole of WX with respect to S.

Let the pole of UV with respect to S' be R' and the pole of WX with respect to S be R. Then P, R, R' are collinear being on the common polar of UV WX.

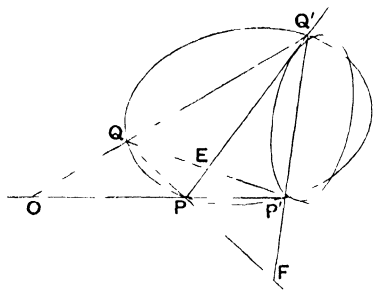
Let this line meet S, S', and UV and WX in *AA', BB', QQ'. These points form an involution.

Since P is the harmonic conjugate of Q and Q' with respect to AA' and BB', the range $AA'BB'$ is harmonic. (Example 9 (b), Art. 61.)

Hence R and R' the harmonic conjugates of Q' and Q with respect to AA' and BB' coincide. (Example 9 (b), Art. 61.)

(12) If from a fixed point O a series of pairs of tangents be drawn to a system of conics which touch four given lines and these tangents meet a fixed conic of the system (to which O is external) in variable points P, P' and Q, Q', show that the points of intersection of PQ, $P'Q'$ and of PQ' and $P'Q$ are fixed points.

Let PQ, $P'Q'$ meet at F, and PQ', QP' at E. Then EF is the polar of O with respect to the fixed conic, and is a fixed line.



The tangents from O to the conics form an involution pencil of which OP, OQ and the pair of tangents from O to the fixed conic are pairs of conjugate rays. But OE, OF are common harmonic conjugates of these pairs of rays and are therefore the double rays of the involution. Therefore E and F are fixed.

(13) A quadrilateral has one pair of opposite angles each a right angle. Show that the inscribed parabola has the line joining these angular points for directrix and the point of intersection of the other diagonals for focus.

Since pairs of tangents to the parabola from points on the directrix intersect at right angles, the directrix is the line joining the vertices of the quadrilateral at which the right angles are situated. The focus is the pole of the directrix and therefore the opposite vertex of the diagonal triangle is the focus.

* This proof assumes that A, A', B, B' are real.

(14) Through a point O straight lines OP, OQ are drawn at right angles to one another to meet two given lines AB, AC—not passing through O—in P and Q. Prove that the envelope of PQ is a conic having O for focus and touching AB, AC in the points in which they are met by the straight line through O perpendicular to OA.

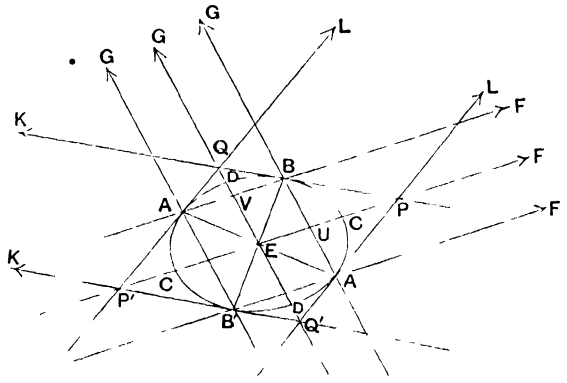
Since the pencils OP, OQ are projective, the ranges P, Q are projective and the envelope of PQ is a conic touching AB, AC at the points of the ranges corresponding to A, that is at the points where the line perpendicular to OA meets these lines.

Let OQ, OP meet AB, AC in P', Q'. Then $PP'QQ'$ is a circumscribed quadrilateral of the conic. Therefore by the property of the complete quadrilateral circumscribed to the conic, PQ and PQ' are conjugate lines. But as these are any pair of conjugate lines through O and they are at right angles, O is a focus of the conic.

(15) A, B, C, D are any four points on a given conic and the two conics, which pass through A, B, C, D and touch a directrix of the given conic are drawn. Show that the portion of the directrix, intercepted between the points of contact, subtends a right angle at the corresponding focus.

If P and Q are the points at which the two conics touch the directrix, they are conjugate points with regard to these conics and therefore with regard to all conics through A, B, C, D, Art. 106 (*d*). Hence P and Q are conjugate points with regard to the given conic. Since S the focus is the pole of the directrix, SP and SQ are a pair of conjugate lines, and, being conjugate lines through a focus, are at right angles.

107 Complete inscribed quadrangle and circumscribed quadrilateral, and conics in self-perspective



Particular Cases

If the ends AA', BB' of two diameters of a conic be taken as the vertices of an inscribed quadrangle, and the circumscribed quadrangle be constructed, then

(1) One of the vertices of the diagonal points triangle (E) becomes the centre of the conic and the other two (G and F) are at infinity

Let the finite point of intersection of the tangents at A, A', B, B' be P, P', Q, Q'

Let EP and EQ meet BA and BA' in U and V

(2) EU and EV are conjugate diameters of the conic, for, since $BAUG$ is harmonic and G is at infinity, each bisects chords parallel to the other

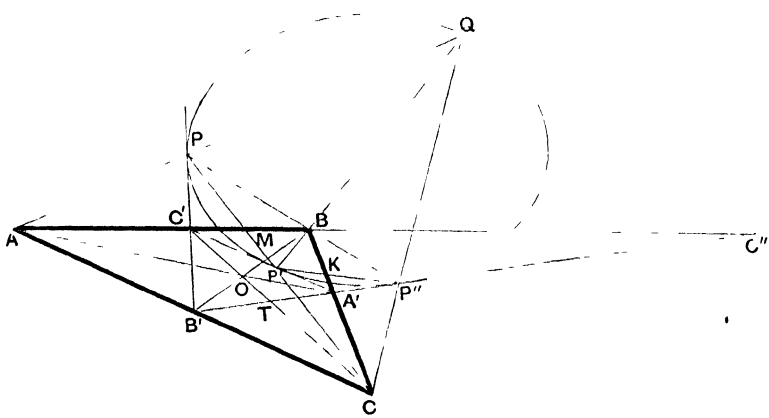
(3) The supplemental chords AB and $A'B$ are parallel to EV and EU and are therefore parallel to conjugate diameters

(4) Since the lines EP and EQ are conjugate diameters, any tangent meets a pair of parallel tangents in points whose connectors to the centre are a pair of conjugate diameters

Certain theorems in connexion with the conic can be conveniently proved from the fact that a conic is in self-perspective, if the perspective is harmonic, and any vertex and the opposite side of a self-conjugate triangle are the centre and axis of perspective

Deductions

(*a*) *If ABC be a triangle self-conjugate with regard to a conic and any tangent to the conic meets the sides AB and AC in C' and B' and the line connecting A to the point CC' BB' meets the side BC in A', then AC' and AB' are tangents to the conic*



Let the point BB' CC' be O and let $A'B'$ meet AB in C'' and CC' in T Then from the quadrangle $ABA'B'$ the range $(B'A'TC'')$ and consequently the pencil $(C'$ $B'A'CB)$ are harmonic Therefore in the harmonic perspective, centre C' and axis AB, $C'B'$ and $C'A'$ are corresponding lines But in this perspective the conic

corresponds to itself and therefore, since $C''B'$ is a tangent, $C'A'$ is also a tangent Similarly $B'A'$ touches the conic

From the preceding the following theorems are obtained

(*b*) *If a triangle be inscribed in a conic in such a way that its sides pass through the vertices of another triangle, which is self-conjugate with respect to the conic, the two triangles are in perspective*

In the preceding the points of contact of $B'C'$, $C'A'$, $A'B'$ form such an inscribed triangle Q is the centre of perspective

(*c*) *If two triangles are inscribed in, and in perspective with, a given triangle, a conic can be described to touch their six sides and to have the given triangle for a self-conjugate triangle*

Describe a conic, which has the given triangle for a self conjugate triangle, to touch one side of each of the other triangles This conic will touch them six sides

(*d*) *If a triangle be circumscribed to a conic so that its vertices lie on the sides of a triangle which is self-conjugate with regard to the conic the triangles are in perspective* This is the converse of (*a*)

(*e*) *If two triangles are circumscribed to and in perspective with a given triangle, a conic can be described to pass through their six vertices and to have the given triangle for a self-conjugate triangle*

Describe a conic which has the given triangle for a self conjugate triangle and passes through one of the vertices of each of the other triangles This conic will pass through them six vertices

(*f*) *If a triangle ABC' be self-conjugate with regard to a conic and a triangle $A'B'C'$ be circumscribed to the given triangle so as to be in perspective with it, then if one vertex of the latter triangle is on the conic the other two are likewise on the conic*

Let A' be on the conic, then the vertices B' and C' may be constructed as harmonic perspectives of A' with respect to the vertices and sides of the triangle

EXAMPLES

(1) Two triangles are inscribed in a given triangle and are in perspective with it Show that a conic can be circumscribed to the first triangle with respect to which the other two triangles are self-conjugate

The required conic is the conic through the vertices of the triangle and the two centres of perspective

(2) A parabola is drawn touching the sides AB, BC, CD, DA of the cyclic quadrilateral $ABCD$ Show that its directrix passes through the intersection of AC and BD

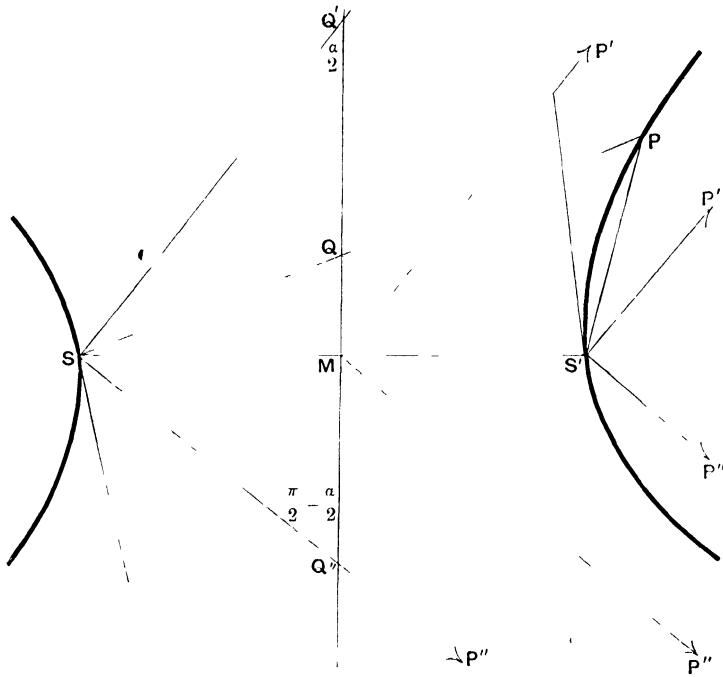
Let AB CD and BC DA be E and F respectively Then since $ABCD$ is inscribed in a circle the circle round BEC and CFD will intersect in the focus of the parabola and (Example (7), Chapter XI) on the line EF But AC BD is a vertex of a self-conjugate triangle of the parabola of which EF is the opposite side, therefore, as the focus is on EF, the directrix passes through AC BD

(3) Prove that a conic may be described to touch the eight sides of two quadrilaterals which have the same diagonal triangle

(4) EFG is a triangle self conjugate with regard to a conic Prove that, if through E any two lines, harmonic conjugates of EF and EG, be drawn to meet the conic in K, L, M, N, then EFG is the diagonal points triangle of the quadrangle $KLMN$

Rectangular Hyperbola

108 *A rectangular hyperbola is the locus of the points of intersection of corresponding rays of two oppositely equal pencils* (Art 94 (iv))



Construction

Let S and S' be any two fixed points and M the middle point of SS' Through M draw a line MQ perpendicular to SS' On MQ take any point Q Join SQ, $S'Q$ and through S' draw a line $S'P$ making any given angle α with $S'Q$ Then SQ and $S'P$ are two corresponding rays of oppositely equal pencils and (by definition) P then point of intersection for different positions of Q describes a rectangular hyperbola

Asymptotes

If a ray SQ be drawn to make an angle $\frac{\alpha}{2}$ with MQ it will be parallel to the corresponding ray of the other pencil It therefore determines a point at infinity on the curve and is parallel to an asymptote

If SQ' be drawn perpendicular to SQ it will likewise be parallel to its corresponding ray and therefore SQ'' is the direction of the other asymptote Hence the asymptotes are at right angles This is the property from which the hyperbola is named

To the ray SS' of the pencil, centre S, corresponds a ray through S' making an angle α with $S'S$. Hence the tangent to the locus at S' makes an angle α with $S'S$. The tangent at S to the locus likewise makes an angle α with SS'. Hence the tangents at S, S' are parallel and therefore SS' is a diameter and M is the centre of the rectangular hyperbola.

Construction of the tangent

Since the asymptote MP' makes an angle $\frac{\pi}{2} - \frac{\alpha}{2}$ with SS' and the tangent at S' makes an angle α with SS', both the tangent at S' and MS' make equal angles $\frac{\pi}{2} - \frac{\alpha}{2}$ with the asymptote. Hence, to construct the tangent at any point P of the curve, join P to the centre M and draw through P a line making the same angle with the asymptote, that PM makes with it.

It follows that pairs of conjugate diameters of a rectangular hyperbola are equally inclined to the asymptotes.

Axes

If the angle α is taken to be $\frac{\pi}{2}$ the tangents at S and S' are both perpendicular to SS', and SS' is the transverse axis of the rectangular hyperbola.

If from any point P on a rectangular hyperbola a perpendicular PN is drawn to the axis and S and S' are the ends of the axis, then $PN^2 = SN \cdot S'N$.

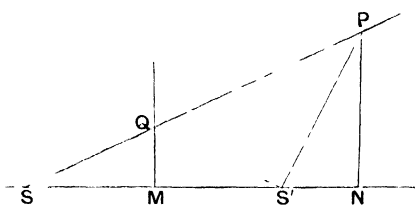
In the figure $QS'P$ is a right angle, and therefore the angle

$$PSN = QS'M = S'PN.$$

Therefore the triangles PSN and $S'PN$ are similar, and

$$\frac{PN}{SN} = \frac{S'N}{PN}.$$

Therefore $PN^2 = SN \cdot S'N$,



also since $SN = MN + MS'$ and $S'N = MN - MS'$,

$$MN^2 - PN^2 = S'M^2.$$

By considering the special case when α is zero, it is seen that the pair of lines MQ and SS', which are at right angles, form a rectangular hyperbola. Hence any pair of lines at right angles may be looked upon as a rectangular hyperbola.

If two conics described through four points are rectangular hyperbolas, all conics through these points are rectangular hyperbolas.

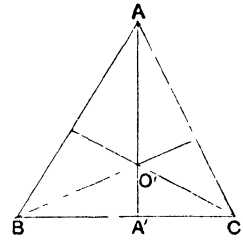
The conics determine an involution on the line at infinity. Since two of the conics are rectangular hyperbolas, two pairs of the conjugate points on this line are determined by orthogonal rays. Hence all pairs of conjugate points are determined by orthogonal rays. Therefore all conics through the four points have their asymptotes at right angles, and are rectangular hyperbolas.

All conics through the vertices of a triangle and its orthocentre are rectangular hyperbolas.

Any side and the perpendicular from the opposite vertex constitute a rectangular hyperbola. Therefore there are two rectangular hyperbolas through the four points, and all conics through these points are rectangular hyperbolas.

If a rectangular hyperbola passes through the vertices of a triangle it also passes through the orthocentre

Let ABC be the triangle and let the given rectangular hyperbola meet AA' the perpendicular to BC in O. Then through A, B, C, O there are two rectangular hyperbolas, viz. the given one and that made up of the lines AO, BC. Therefore every conic through these points is a rectangular hyperbola. Therefore BO and AC which constitute a conic through these points are a rectangular hyperbola and are at right angles. Therefore O is the orthocentre of the triangle.



A system of rectangular hyperbolas, described with any two points S and S' for the ends of a diameter, determines the same involution on the perpendicular diameter.

Let C be the middle point of SS'. Then, if the generating pencil be rotated through an angle α the direction of one of the asymptotes is inclined at an angle $\frac{\alpha}{2}$ to CP, the perpendicular diameter through C. Hence the diameter CQ conjugate to CP is inclined at an angle $\frac{\pi}{2} - \alpha$ to CS'.

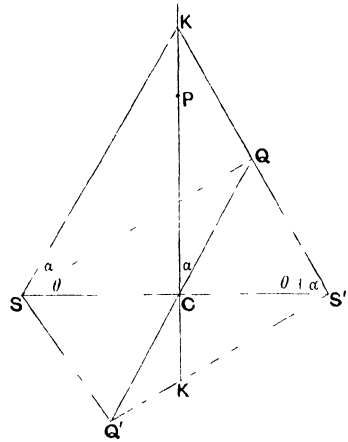
Let Q and Q' be the points in which this diameter meets the rectangular hyperbola. Let $S'Q$ and $S'Q'$ meet CP in K and K'. Then, since QQ' passes through the pole of CP, K and K' are a pair of conjugate points of the involution determined on CP by the rectangular hyperbola.

In the figure if CSQ be θ, then $CS'Q$ is $\theta + \alpha$. The angle KSC = angle $KS'C$. Hence angle $KSQ = \alpha$. Hence a circle passes through K, Q, C, S. Therefore the angle SQK = angle SCK = a right angle.

Hence $QS'Q'$ is a right angle and the circle on KK' as diameter passes through S and S'.

Therefore $CK \cdot CK' = -CS^2$.

The involution determined by K and K' of which C is the centre is therefore the same whatever α may be and is therefore the same for all rectangular hyperbolas of the system.



EXAMPLES

(1) If the normal at P to a rectangular hyperbola meets the conic again in P' then PP' subtends a right angle at the other extremity of the diameter through P.

(2) A and B are the ends of a diameter of a rectangular hyperbola and CD the perpendicular diameter. P is any point on the rectangular hyperbola. Prove that, if AP meets CD in Q, the angle PBQ is constant.

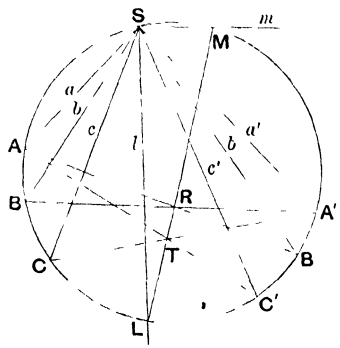
This follows from the construction of a rectangular hyperbola.

CHAPTER XVII

CONSTRUCTION OF SELF-CORRESPONDING ELEMENTS. PROBLEMS OF SECOND DEGREE. CONSTRUCTION OF CONICS UNDER VARIOUS CONDITIONS

109. To construct the self-corresponding elements—when real—of two superposed projective forms

To construct the self-corresponding elements of two superposed projective pencils $(S\,abc\ldots)$ *and* $(S\,a'b'c'\ldots)$

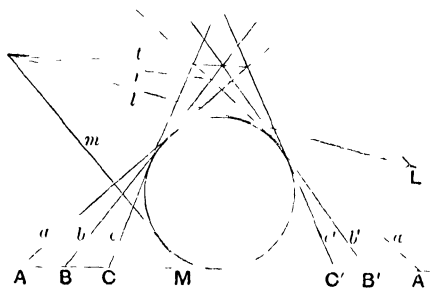


(1) Through S describe any circle or conic to meet three pairs of corresponding rays of the pencils in A, B, C and A', B', C'

Let R and T be respectively the points AB' $A'B$ and AC' $A'C$

Let RT meet the circle or conic in M and L. Then since M and L are the self-corresponding points of the ranges $ABC\ldots$ and $A'B'C'\ldots$ on the conic (Art. 105 (*b*)),

To construct the self-corresponding elements of two superposed projective ranges $(s\,ABC\ldots)$ *and* $(s\,A'B'C'\ldots)$



Describe any circle or conic to touch s and let a, b, c and a', b', c' be the tangents to this curve from three pairs of corresponding points of the ranges.

Let l and t be respectively the lines ab' $a'b$ and ac' $a'c$

Let the tangents from lt to the circle or conic be m and l. Then since m and l are the self-corresponding tangents of the projective systems of tangents $abc\ldots$ and

SM and SL are the self-corresponding rays of the pencils

(ii) Take any circle and let any tangent s to the circle cut the pencils in A, B, C and A', B', C' Construct by the right-hand side of (i) the self-corresponding points L and M of these ranges

The rays joining these points to S will be the self-corresponding rays of the pencils, viz l and m

$a'b'c'$ to the conic (Art 105 (*b*)), sm and sl are the self-corresponding points of the ranges

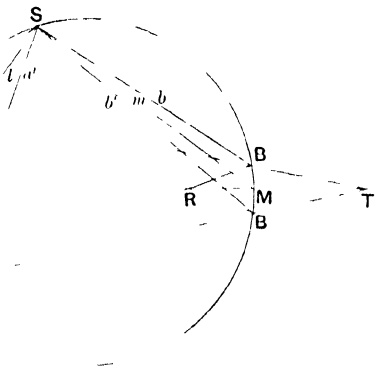
Take any circle and project from any point S on the circle the ranges into pencils (S abc) and (S $a'b'c'$) Construct by the left-hand of (i) the self-corresponding rays l and m of these pencils

These rays will meet s in the self-corresponding points of the ranges, viz L and M

110 To construct the double elements—when real—of an involution form

To construct the double elements of an involution pencil

(S $aa'bb'$)



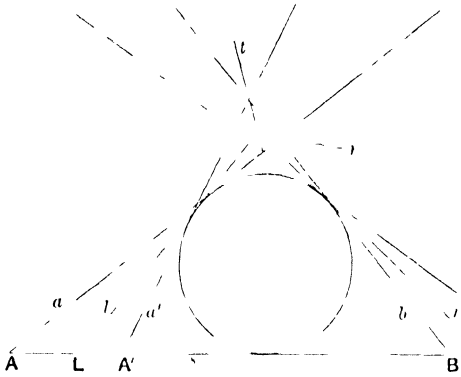
(i) Through S describe any circle or conic to meet two pairs of corresponding rays in A A', B B'

Let R and T be the points AB' $A'B$ and AB $A'B'$

Let the line RT' meet the circle or conic in L and M

To construct the double elements of an involution range

(s $AA'BB'$)



Describe any circle or conic to touch s and let the tangents from two pairs of corresponding points of the ranges be a, a', b, b'

Let r and t be the lines ab' $a'b$ and ab $a'b'$

Let the tangents from rt to the circle or conic be l and m

Then since L and M are the double points of the involution on the circle or conic (Art 75), SL and SM are the double rays of the involution pencil

(ii) Take any circle and let s any tangent to this circle cut the involution in $AA'BB'$

Construct by the right-hand side of (i) the double elements L and M of this involution range

The rays joining these points to S will be the double rays of the involution pencil

Then since l and m are the double rays of the involution system of tangents to the circle or conic (Art 75), sl and sm are the double points of the involution

Take any circle and project from S any point on the circle the range into a pencil (S $aa'bb'$)

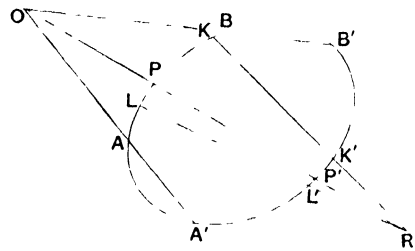
Construct by the left-hand side of (i) the double elements l and m of this involution pencil

The points in which these rays meet s will be the double points of the involution range

111 To find the common pair of conjugates—when real—of two involutions situated on a conic (cf Art 55)

Let the involutions be determined by the pairs of conjugate points AA', BB' and KK', LL' Let AA' BB' be O and LL' KK' be R Join OR

Then the points P, P', if real, in which OR meets the conic are the common pair of conjugates of the two involutions (Art 95 (c))



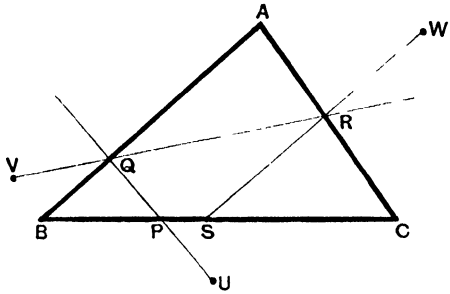
By the correlative method the two tangents, which are a pair of conjugates in two involutions of tangents to the conic, may be constructed

112 Method of false positions In attempting to solve a geometrical problem it frequently happens that the solution of the problem is found to depend on finding the configuration for which two points coincide Such points are frequently situated on some given line or conic and in this case it often happens that for different positions of one of the points it may be proved that the second point describes a range projective with that described by the first In such cases the problem can immediately be solved Any three positions of the first point may be taken and the three corresponding positions of the second The projective ranges are then completely determined

and their self-corresponding points can (by Arts 109 and 110) be at once constructed These self-corresponding points give the required configuration The following are instances of the application of this method

(*a*) *To inscribe a triangle in a given triangle so that its sides may pass through three given points*

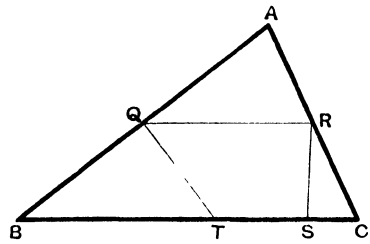
Let ABC be the given triangle and U, V, W the given points Through U draw a transversal to meet BC and BA in P and Q Join V to Q to meet AC in R Join W to R to meet BC in S



Then the range described by P is projective with the range described by Q This is projective with the range described by R, which is projective with the range described by S Therefore P and S describe two superposed projective ranges, and the self corresponding points of these give the solution of the problem

(*b*) *To inscribe a square in a given triangle in such a way that one of its sides may lie along a side of the triangle*

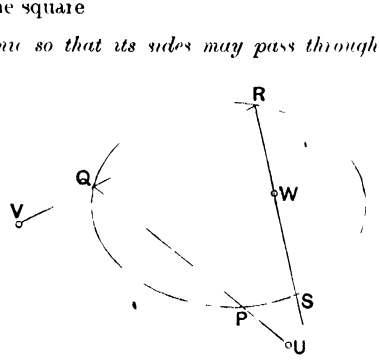
Take any point Q on the side BA of the triangle Through Q draw a parallel to BC to meet AC in R Through R draw a perpendicular to QR to meet BC in S Draw QT making an angle of 45 with QR



Then, for different positions of Q, the range described by Q is projective with the range described by R and therefore with the range described by S It is also projective with the range described by T Hence the ranges described by T and S are projective The self-corresponding points of these ranges each give the position of a vertex of the square

(*c*) *To inscribe a triangle in a given conic so that its sides may pass through three given points*

Let U, V, W be the given points Through U draw a chord to meet the conic in P and Q Join Q to V to meet the conic again in R Join R to W to meet the conic in S



Then the range described by P is projective with the range described by Q This is projective with the range described by R, which is projective with the range described by S

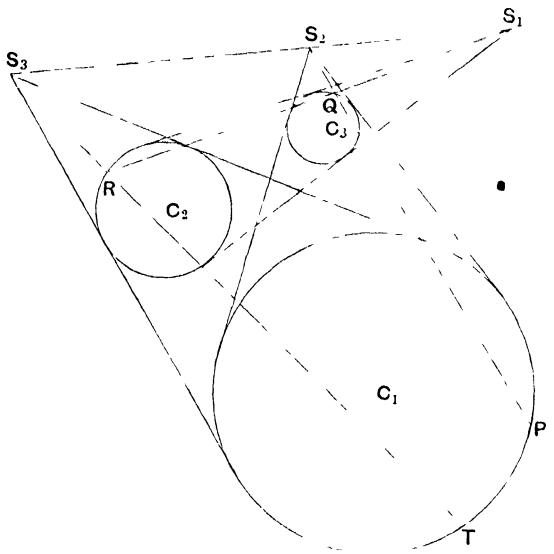
Hence the ranges described by P and S are projective and their self corresponding points give possible vertices of the required triangle

The correlative construction gives the solution of the following problem

To circumscribe a triangle to a given conic so that its vertices shall lie on three given lines

The above and likewise the correlative solution may be extended to the case in which the triangle is replaced by a polygon, whose sides pass through fixed points or whose vertices lie on fixed lines

(*d*) *To describe a circle to touch three given circles*



It has been proved

(1) that the six centres of similitude of three circles lie three by three on four straight lines termed the axes of similitude (Ex 18, Chapter XI)

(2) that if a circle touch two given circles its points of contact are collinear with a centre of similitude of the circles (Ex 15, Chapter XI)

Construct S_1, S_2, S_3 three collinear centres of similitude of the three given circles Take P any point on the circle centre C_1 Join P to S_2 to meet circle centre C_3 at Q Join Q to S_1 to meet circle centre C_2 at R Join R to S_3 to meet circle centre C_1 at T

Then (Art 84 (11)) for different positions of P the ranges T, R, Q, P on the circles are projective Determine the self-corresponding points of the ranges T and P Let P', Q', R' be the positions of P, Q, R, when P' is one of these points

Since the triangles $P'Q'R'$ and $C_1C_2C_3$ are in perspective $P'C_1$, $Q'C_2$, $R'C_3$ meet at a point O This point is the centre of the circles, which by the converse of (2) touch the given circles in pairs at T', Q', R' It is therefore the centre of a circle which touches the three given circles

(e) *Given two straight lines l, l' and two parabolas S, S', find P on l and P' on l' such that the tangents from P to S are parallel to the tangents from P' to S'*

The pairs of tangents from the points on l to S determine conjugate points of an involution on the line at infinity (Art 75)

Similarly the pairs of tangents from points on l' to S' determine conjugate points of an involution on the line at infinity

The pair of common conjugates of these two involutions determine the points, where the required tangents meet the line at infinity *i e* they determine the directions of the required tangents

113 Construction of a Conic

In Art 94 it was shown how conics of various kinds can be constructed by means of the anharmonic properties of the conic Conics may, however, be constructed by means of any of the other three fundamental theorems or by means of their correlatives The methods are as follows

Construction by means of points

(A) To construct a conic through five given points

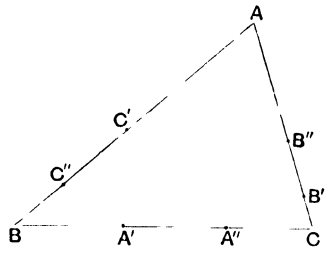
(i) *By Carnot's Theorem*

Let A, A', B, B'', C' be the five points Through C'' draw any straight line meeting $B'A''$ and $B'B''$ in B and A respectively, and let C be the point of intersection of $A'A''$ and BB''

Then C''', the point in which the conic meets AB, is determined uniquely by the relation

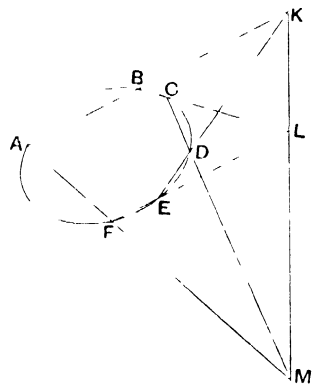
$$\frac{BA'}{CA'}\cdot\frac{BA''}{CA''}\cdot\frac{CB}{AB}\cdot\frac{C'B''}{C'B}\cdot\frac{AC'}{AB''}\cdot\frac{BC'''}{BC'}=\frac{BC'''}{AC'''}$$

Hence any number of points on the conic may be found



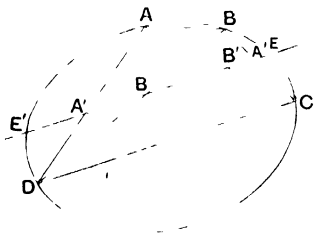
(ii) *By Pascal's Theorem*

Let A, B, C, D, E be the given points Through E draw any straight line EL Let AB and CD intersect in K, and BC and EL in L Join KL to meet DC in M Then EL and AM intersect in a point F on the conic



(iii) *By Desargues' Theorem*

Let A, B, C, D, E' be the given points Through E' draw any transversal meeting AD and BC in A', A'' and BD and AC in B', B'' Then in the involution determined by the two pairs of conjugates $A'A''$, $B'B''$ construct E'' the conjugate of E' (Art 56) Then E'' is the point in which the transversal is met by the conic



The above are unique solutions

(B) *To construct a conic to pass through three given points and to touch a given line at a given point*

This is a particular case of (A) The necessary modifications are as follows

(i) Let A' and A'' coincide, BC becomes the given line which is touched by the conic at A'

(ii) Let B and C coincide, BCL becomes the given line which is touched by the conic at B

(iii) Let B and C coincide, BC becomes the given line which is touched by the conic at B

The above are unique solutions

(C) *To construct a conic to pass through one given point and to touch two given lines at given points*

This is likewise a special case of (A) To obtain the construction the following modifications are necessary in (B)

(i) Let B' also coincide with B

(ii) Let D also coincide with E

(iii) Let B and D coincide and likewise A and C Then B' and B'' are a pair of conjugate points of the involution and AB meets the transversal in a double point Hence E'' the conjugate of E' can be found

The above are unique solutions

(D) *To construct a conic to pass through four given points and to touch a given line*

Let A, B, C, D be the given points They determine on the given line an involution Let E and F be the double points of this involution A conic through A, B, C, D and E or F will by Desargues' theorem touch the given line at E or F

In this case there are two solutions

(E) *To construct a conic to pass through three given points and to touch two given lines*

Let the given points be Q, Q', Q'' and the given lines p, p' Consider a conic touching p and p' and passing through Q and Q' It is seen (Art 106 (ii)) that the chord of contact of this conic with p and p' passes through one or other of two fixed points on QQ' Similarly, by considering a conic touching p and p' and passing through Q and Q'', it is seen that the chord of contact passes through one or other of two fixed points on QQ'' Therefore the chord of contact is one or other of four fixed lines

In this case there are four solutions

(F) *To construct a conic to pass through two given points, to touch a given line at a given point and to touch a second line*

Let the given points be B, C, A the point at which the given line a must be touched and l the second given line

The points at which the required conic will touch l are the double points of the involution determined by A, A, B, C on l

In this case there are two solutions

Constructions by means of lines

By using the correlative methods the following problems may be solved

(A) *To construct a conic to touch five given lines* (One solution)

(B) *To construct a conic to touch three given lines and a given line at a given point* (One solution)

(C) *To construct a conic to touch a given line and to touch two given lines at given points* (One solution)

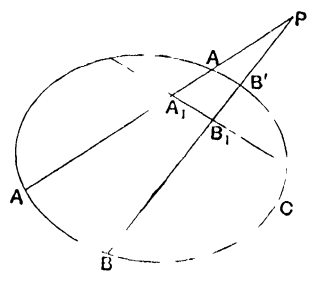
(D) *To construct a conic to touch four given lines and pass through a given point* (Two solutions)

(E) *To construct a conic to touch three given lines and pass through two given points* (Four solutions)

(F) *To construct a conic to touch two given lines, a given line at a given point and to pass through a second point* (Two solutions)

114 Constructions involving pole and polar, and conjugate points

(*a*) *To construct a conic such that a given point may be the pole of a given line with respect to it and such that it* (1) *passes through three given points*, (2) *passes through two given points and touches a given line*, (3) *passes through one given point and touches two given lines*, or (4) *touches three given lines*



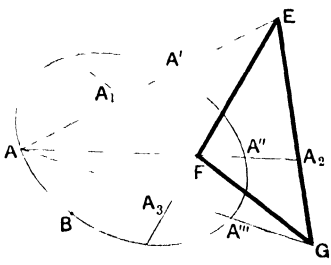
(1) Let P be the given point, p its polar, and A, B, C the given points on the conic. Join P to A and B to meet p in A_1 and B_1. Take A' and B' the harmonic conjugates of A and B with regard to PA_1 and PB_1. The conic through $ABCA'B'$ is the required conic.

(2) If the conic touches a line c instead of passing through C. Proceed as in (1) but describe a conic through $AB\,A'B'$ to touch c.

(3) and (4) are the correlatives respectively of (2) and (1).

(1) and (4) have one solution while (2) and (3) have two solutions.

(*b*) *To construct a conic such that a given triangle may be self conjugate with respect to it and such that it* (1) *passes through two given points*, (2) *passes through a given point and touches a given line*, or (3) *touches two given lines*.



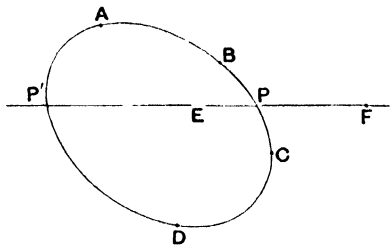
(1) Let EFG be the given triangle and A and B the given points. Join A to E, F, G to meet the opposite sides in A_1, A_2, A_3, take A', A'', A''' the harmonic conjugates of A with respect to A_1E, A_2F, A_3G. The conic through A, A', A'', A''' and B is the required conic.

(2) Proceed as in (1) but describe a conic through A, A', A'', A''' to touch the given line b.

(3) is the correlative of (1).

(1) and (3) have one solution while (2) has two

(*c*) *To construct a conic through four given points such that a pair of given points may be conjugate with regard to it**



Let A, B, C, D be the given points and E and F the pair of conjugate points

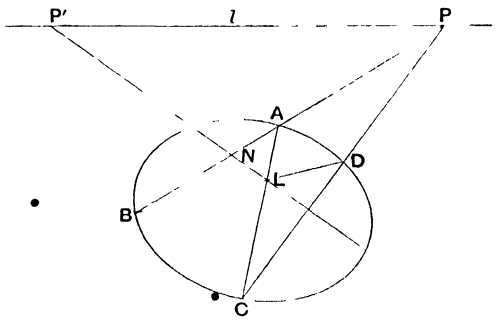
A, B, C, D determine on EF an involution of which P and P', the points where EF meets the conic, are a pair of conjugate points If E, F are conjugate points with respect to the conic, P and P' are harmonic conjugates of E and F Therefore P and P' are a pair of conjugate points of the involution of which E and F are double points P and P' are therefore determined as the common pair of conjugates of two involutions They can be found by Art 55 Hence six points on the conic are given

A similar construction holds when the conic instead of passing through four points satisfies other conditions

The correlative theorem is

To construct a conic to touch four given straight lines such that a given pair of lines are conjugate with regard to it

(*d*) *To describe a conic through three given real points and through the two double points (real or imaginary) of a given involution**



Let l be the line on which the given involution is situated and let A, B, C be the three given points Let AB meet l in P and let P' be the point of the involution conjugate to P On AB take N the harmonic conjugate of P with respect to AB Join $P'N$ to meet AC in L Let BL and PC meet in D Then since $P'N$ is the polar of P, D is a point on the curve Similarly other points D_1 and D_2 may be determined by considering the points where AC and BC meet l The conic through A, B, C, D, D_1, D_2 is the required conic

The correlative theorem is

To describe a conic to touch three real lines and to touch the two double rays (real or imaginary) of an involution pencil

* See also Chapter XXII

(*e*) *Given a conic determined either by five points or by five tangents to construct*

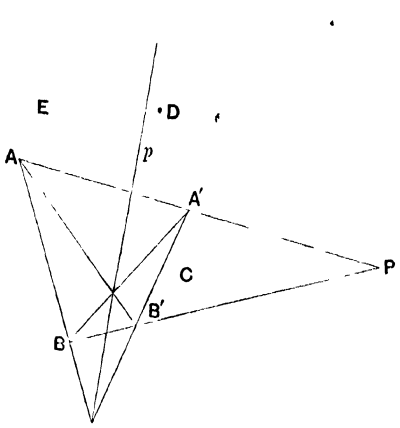
(1) *The polar of a given point,*

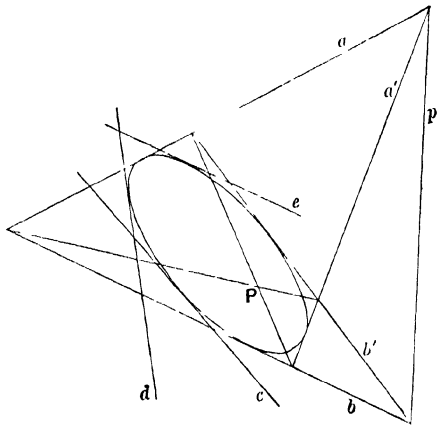
(2) *The pole of a given line,*

(3) *The points of intersection of a given line with the conic,*

(4) *The tangents from a given point to the conic*

If five points on a conic are given, the five tangents at these points may be constructed by Pascal's theorem, and if five tangents are given their points of contact can be constructed by Brianchon's theorem Hence the conic may be regarded as determined either by five points or by five tangents





(1) Let the conic be determined by A, B, C, D, E and let P be the given point Join A and B to P and find by Pascal's theorem A' and B' the other points in which these lines meet the conic Then p the line joining AB' BA' to AB $A'B'$ is the polar of P

(2) Let the conic be determined by a, b, c, d, e and let p be the given line Through the points where a and b meet p draw by Brianchon's theorem the other tangents a' and b' to the conic Then P the point of intersection of ab' ba' and ab $a'b'$ is the pole of p

(3) Let the conic be determined by A, B, C, D, E and let p be the given line Take any pair of points P and Q on the line p and determine their polars These will meet p in a pair of points P' and Q' which are conjugates of P and Q with respect to the conic Determine on p the double points of the involution PP', QQ' These double points are the points of intersection of the line with the conic

(4) Let the conic be determined by a, b, c, d, e and let P be the given point Take any pair of lines p and q through the point P and determine their poles The lines p' and q' joining these points to P are conjugates of p and q with respect to the conic Determine in the pencil vertex P the double rays of the involution pp', qq' These double rays are the pairs of tangents from P to the conic

(*f*) *To describe a conic through* 4, 3, 2, 1 *or* 0 *given points, so that* 1, 2, 3, 4 *or* 5 *pairs of given points may be conjugates with regard to it*

Let the given points be A, B, C, D, and let the connectors of the pairs of conjugate points be a, b, c, d, e. The conic in each case has to pass through points on these lines, which are harmonic conjugates of the pairs of conjugate points, that is through pairs of conjugate points of involutions on these lines. Such involutions will be called the involutions a, b, c, d or e.

(1) Draw two conics through A, B, C and through two pairs of conjugate points of involution e. These meet at a point P. By Art. 101, converse, the conic through A, B, C, D and P is the required conic.

(2) Draw two conics through A, B, C and two pairs of conjugate points of involution e, and two other conics through A, B, C and two pairs of conjugate points of involution d. These pairs of conics intersect in two points P and P'. The conic A, B, C, P, P' is the required conic.

(3) Construct as in (2), a conic S through A, B, C, P, P' and a second conic S' substituting some point R' for C'. These conics intersect in four points A, B, C', D'. Any conic through these points passes through conjugate points of the involutions d and e. Determine from A, B, C' and involution c a point Q as in (1). Then $ABC'D'Q$ is the required conic.

(4) Determine from A, B, and the involutions c, d, e a conic S as in (3). Substitute a point R'' for B and determine a second conic S'. The conics S and S' intersect in four points A, B', C'', D''. Any conic through these points meets c, d, e in conjugate points of the given involutions. Determine T as in (1) from A, B'', C'' and the involution b. Then A, B'', C'', D'', T is the required conic.

(5) Determine for A and the involutions b, c, d, e a conic S as in (4). From any point R''' and the involutions b, c, d, e determine similarly a conic S'. The conics S and S' intersect in four points A''', B''', C''', D''', and any conic through these points passes through conjugate points of involutions b, c, d, e. Determine from A''', B''', C''' and the involution a a point E'''. Then the conic $A'''B'''C'''D'''E'''$ passes through conjugate points of the given involutions on a, b, c, d, e.

Particular Cases

115 Particular cases of the preceding frequently arise, but with a little ingenuity they may be solved by the methods already given, if it is borne in mind that the data in the left hand column below are only particular cases of those in the right hand column.

The direction of an asymptote	A point at infinity on the curve
An asymptote	A point on the curve and the tangent at the point
The centre	The pole of a given line
An axis	The polar of a given point, which point is at infinity on any perpendicular line
Two axes	A self-conjugate triangle
A focus	Two tangents which are double rays of an involution
Two foci	Four tangents
A diameter	A pair of conjugate lines
A pair of conjugate diameters	A self-conjugate triangle

If the conic is

(1) a parabola, a tangent is given,

(2) a rectangular hyperbola, a pair of conjugate points with respect to the conic are given, viz the circular points at infinity,

(3) a circle, a pair of points on the conic, which are the double points of an involution, are given

The following are special constructions for the parabola and hyperbola

Parabola

(1) Given two tangents a and b to a parabola and their points of contact A and B to construct the curve

Let O be the middle point of AB Join ab to O and let the point at infinity on this line be ∞ Through ab draw any line to meet $A\infty$ and $B\infty$ in D and E respectively Then AE DB $(\equiv F)$ is a point on the curve

This follows from the Pascal hexagon $AAFBB\infty$

(2) Given four tangents to construct a parabola

Let two of the tangents t and t' meet the other two tangents a and b in A and A' and in B and B' On a and b construct two similar ranges in which A and A' and B and B' are pairs of corresponding points

Then the connectors of the corresponding points envelope the parabola (Art 102, Ex (1))

Hyperbola

Given three points A, B, C on the curve and the directions of the asymptotes to construct the curve

Through A and C draw lines parallel to the asymptotes to meet in K Through K draw any line to meet AB and CB in R and S Parallels through R and S to the asymptotes meet in a point P on the curve

By Pascal's theorem

"

EXAMPLES

(1) Given five points on a conic determine by a linear construction the pole of the line joining two of them Art 105 (1)

(2) Construct a parabola to pass through four given points and show that two parabolas can be drawn in general to pass through four points

(3) Prove that two equilateral hyperbolas, if any, can be drawn touching four given lines

(4) Construct a hyperbola given three points on the curve and an asymptote

(5) Show how to construct a conic given a focus and three points on it, and prove that there are four such conics

(6) Given five points on a conic and any parallelogram, find a construction for the centre by means of ruler only

(7) Given five points on a hyperbola construct its asymptotes

(8) Given three tangents of a conic and one of the foci, construct the second focus and any number of tangents

CHAPTER XVIII

TRIANGLE LOCI AND ENVELOPES POLE AND POLAR LOCI AND ENVELOPES ANHARMONIC AND GENERAL LOCI AND ENVELOPES

116 In this chapter certain problems and theorems concerning loci and envelopes are considered Those at the commencement of Art 117 are in themselves of considerable importance and should be carefully studied

Triangle Loci and Envelopes

The anharmonic property of a conic and its correlative theorem together with the converse of these theorems give a solution of many problems concerning loci, in which the loci are conic sections The following are instances It is assumed in speaking of a triangle that A, B, C are the vertices opposite respectively to the sides a, b, c

1 If the three sides of a triangle a, b, c pass respectively through fixed points P, Q, R, and two vertices A and B move on fixed lines d and e, then the third vertex C describes a conic through P and Q (Cf Ex 1, Art 71)

The pencil described by b, which passes through Q is projective with that described by a, which passes through R, for these pencils are projective with the ranges described by A and B on d and e, which are themselves projective

2 If the sides a and b of a triangle pass through the fixed points P and Q and the vertices A and B move along fixed lines d and e and the side c touches a conic which also touches d and e, then the vertex C describes a conic, which passes through P and Q

The proof is identical with the preceding, except that the ranges described

If the three vertices of a triangle A, B, C lie respectively on three fixed lines p, q, r, and two sides a and b pass through fixed points D and E, then the third side c envelopes a conic touching p and q (Cf Ex 1, Art 71)

The range described by B on q is projective with that described by A on p, for these ranges are projective with the pencils described by a and b through D and E, which are themselves projective

If the vertices A and B of a triangle lie on fixed lines p and q and the sides a and b pass through fixed points D and E and the vertex C lies on a conic which also passes through D and E, then the side c envelopes a conic, which touches p and q

The proof is identical with the preceding, except that the pencils described

by A and B on d and e are projective because c is a tangent to a conic which touches d and e

3 Through two fixed points A and B lines are drawn to a variable point P on a fixed straight line If lines AC, BC are drawn through A and B such that the angles PAC, PBC have given values, the locus of C is a conic through A and B

The pencils described by AC and BC, through A and B, are projective

4 If in the preceding the lines AC and BC cut off equal or proportional intercepts on given lines, the locus of C is a conic through A and B

Two fixed straight lines a and b are intersected at A and B by a variable line p, which passes through a fixed point If a line c meet a and b in points A' and B' such that the lengths AA' and BB' have given values, the envelope of c is a conic touching a and b

The ranges described by A' and B' on a and b are projective

If in the preceding the lengths AA' and BB' subtend equal angles at two given points, the envelope of C' is a conic touching a and b

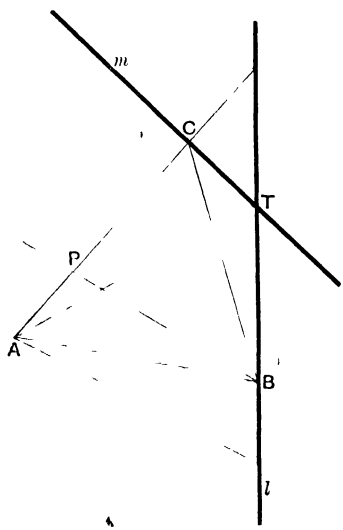
5 In a triangle given in species* if one vertex be fixed and another move along a straight line,

(1) the envelope of the side opposite the fixed vertex is a parabola, of which the fixed vertex is a focus,

(2) the locus of the point of intersection of a side which passes through the fixed vertex with a line, drawn in a fixed direction through the opposite vertex, is a conic

It should be noticed that the sides of triangles given in species determine three projective ranges on the line at infinity

(1) Let the vertex A be fixed and let B move along the straight line l Then CB determines on the line at infinity a range projective with that determined by AB, and therefore with the range described by B on l Hence the envelope of CB is a parabola touching the line l



Similarly the parabola touches the line m on which (Addendum 6 (a)) C' moves If m and l meet at T the points C, T, B, A are concyclic and therefore A is the focus of the parabola Example (3), Chapter XIII

(2) If a line through B drawn in a fixed direction meets AC at P, P is the point of intersection of rays of two projective pencils, whose vertices are at infinity in the given direction and at A Hence the locus is a conic through A

* In Addendum 6 some important theorems connected with triangles, given in species, are proved

6 Find the envelope of a side of a triangle given in species and inscribed in a given triangle

If the triangle $A'B'C'$ be inscribed in the triangle ABC, the circumcircles of $AC'B'$, $BC'A'$ and $CA'B'$ meet at a fixed point O, and the triangle $OA'B'$ is given in species (Addendum 6 *(b)*) Since O is fixed and A' and B' lie on two fixed lines the envelope of $A'B'$ is a parabola of which O is the focus

7 If a series of similar figures are described, three corresponding points of which are situated on the sides of a triangle, then the loci of all other points of the figures are straight lines, and the envelopes of all lines are parabolas which have the same point for focus

In the last example let P' be a variable point such that $A'B'C'P'$ is always similar to a given figure Then the triangle $OA'P'$ is given in species, O is fixed and A' moves along a fixed line Hence the locus of P is a fixed line

If P', Q' be any two points of the figure, $OP'Q'$ forms a triangle given in species of which one vertex O is fixed while the other vertices P', Q' move along straight lines Therefore the envelope of $P'Q'$ is a parabola of which O is the focus

117 Pole and Polar Loci and Envelopes

1 *Given four points on a conic the polar of a fixed point passes through a fixed point*

This is equivalent to proving that,

Every point has one common conjugate point with respect to a system of conics through four fixed points

Let A, B, C, D be the four fixed points through which the system of conics is described and S the point, the envelope of whose polars with respect to the conics is required Construct S' the conjugate of S with respect to the quadrangle $ABCD$ Then the polar of S with respect to every conic through A, B, C, D passes through S' Art 106 (iii)

If a conic be described through the points A, B, C, D, S, the tangent to this conic at S will

Given four tangents to a conic the pole of a fixed line lies on another fixed line

This is equivalent to proving that,

Every line has one common conjugate line with respect to a system of conics touching four fixed lines

Let a, b, c, d be the four fixed lines which are touched by the conics of the system and s the line, the locus of whose poles with respect to the conics is required Construct s' the conjugate of s with respect to the quadrilateral $abcd$ Then the pole of s with respect to every conic touching a, b, c, d lies on s' Correlative of Art 106 (iii)

If a conic be described to touch the lines a, b, c, d, s and S be the point of contact of this

cut the conics of the system in an involution of which S is a double point (by Desargues' theorem) Let S'' be the other double point of the involution

Then S and S'' are harmonic [illegible] of every pair of conjugates of the involution determined on SS' and are therefore conjugate points with respect to all the conics of the system, which meet SS'' in real points Hence S'' is the same point as S'

conic with s, the pairs of tangents from S to the conics of the system will form an involution of which s is a double ray (by Correlative of Desargues' theorem) Let s'' be the other double ray of the involution

Then s and s'' are harmonic conjugates of every pair of tangents from S to the conics of the systems and are therefore conjugate lines with respect to all conics of the system, the tangents to which from ss'' are real Hence s'' is the same line as s'

2 The Eleven points locus and its correlative

Given a system of conics through four fixed points (*a*) *the locus of the common conjugates of points on a fixed line and* (*b*) *the locus of the poles of the fixed line with respect to the conics of the system, is a conic which passes through* (i) *the six harmonic conjugates of the points where the fixed line meets each side of the quadrangle of fixed points with respect to the vertices on that side,* (ii) *the three vertices of the common self-conjugate triangle of the conics and* (iii) *the two points at which conics of the system touch the given line*

Let P be any point on the fixed line l and let L and L' be the poles of l with respect to two given conics (1) and (2) of the

Given a system of conics touching four fixed lines (*a*) *the envelope of the common conjugate lines of lines through a fixed point and* (*b*) *the envelope of the polars of the fixed point with respect to the conics of the system, is a conic which touches* (i) *the six harmonic conjugates of the lines joining the fixed point to the vertices of the quadrilateral with respect to the pairs of sides of the quadrilateral which meet at that vertex,* (ii) *the three sides of the common self-conjugate triangle of the conics and* (iii) *the two lines which are tangents to conics of the system at the fixed point*

Let p be any line through the fixed point L and let l and l' be the polars of L with respect to two given conics (1) and (2) of the

system The polars of P with respect to the conics (1) and (2) will be two straight lines LQ and $L'Q$ which pass through L and L' and meet at Q a point which is a common conjugate of P with respect to all the conics of the system (Art 117, 1)

As P moves along the line l, LQ and $L'Q$ describe two pencils through L and L' which are each equi-anharmonic with the range described by P and are therefore projective with each other

Hence the locus of Q is a conic through L and L', which is therefore the locus of common conjugates of points on l with respect to conics through the four fixed points

Since L and L' are the poles of l with respect to the conics (1) and (2), which are any two conics of the system, the poles of l with respect to every conic of the system lie on this conic, which is therefore the locus of poles of l with respect to all the conics of the system

(1) If the four points through which the conics of the system pass are A, B, C, D and l meet AB in R, and R be the harmonic conjugate of R with respect to A and B, then R' is a common [illegible] of R Therefore the locus passes through R' and the five other points similarly constructed

system The poles of p with respect to the conics (1) and (2) will be two points lq and $l'q$ which lie on l and l' and have for their connector q which is a common conjugate of p with respect to all the conics of the system (Art 117, 1)

As p rotates round L, lq and $l'q$ describe two ranges along l and l' which are each equi-anharmonic with the pencil described by p and are therefore projective with each other

Hence the envelope of q is a conic touching l and l', which is therefore the envelope of common conjugates of lines through L with respect to conics touching the four fixed lines

Since l and l' are the polars of L with respect to the conics (1) and (2), which are any two conics of the system, the polars of L with respect to every conic of the system are tangents to this conic, which is therefore the envelope of polars of L with respect to all the conics of the system

If the four lines which are touched by the conics of the system are a, b, c, d and the connector of L to ab be r, and r' be the harmonic conjugate of r with respect to a and b, then r' is a common conjugate of r Therefore the envelope touches r' and the five other lines similarly constructed

(ii) If EFG be the diagonal points triangle of $ABCD$ and l meet EF in T, then G will be a common conjugate of T, since the polars of T with respect to all conics of the system pass through G Hence the locus passes through G, and similarly through E and F

If efg be the diagonal triangle of the quadrilateral $abcd$ and the connector of L to ef be t, then g will be a common conjugate of t, since the poles of t with respect to all conics of the system lie on g Hence the envelope touches the line g, and similarly it touches e and f

(iii) The locus passes through the points where the two conics of the system touch the line l, for these points are poles of the line l with respect to conics of the system These points may be constructed as the double points of the involution determined by the quadrangle $ABCD$ on l

The envelope touches the two lines which are tangents at L to conics of the system, for the lines are polars of the point L with respect to conics of the system These lines may be constructed as the double rays of the involution determined by the quadrilateral $abcd$ at L

3 *Transversals are drawn across a quadrangle in such a way that the locus of one of the double points of the involutions, in which they are cut, is a straight line Prove that the locus of the other double point is a conic circumscribing the harmonic triangle of the quadrangle*

The double points of the involution in which the transversal is cut are harmonic conjugates of its intersections with conics described through the vertices of the quadrangle and are therefore common conjugate points of the system of conics Consequently, since one describes a straight line, the other describes the conic obtained in 2

4 *Prove that a conic, which circumscribes the harmonic triangle of a quadrangle and which passes through the double points of the involution determined by the quadrangle on a given transversal, passes also through the six points which are the harmonic conjugates with respect to every pair of vertices of the quadrangle, of the points where the transversal meets the side of the quadrangle containing that pair*

The conic is the locus obtained in 2

5 *The locus of the centres of a system of conics circumscribed to a quadrangle is a conic circumscribing its diagonal points triangle and passing through the six middle points of the sides*

(See Example 10, Art 103)

The locus of the centres of a system of conics inscribed in a quadrilateral is a straight line passing through the middle points of the diagonals

The centre of a conic is the pole of the line at infinity, which is a fixed line, consequently the above is a particular case of Art 117, 2

The centre of a conic is the pole of the line at infinity, which is a fixed line, consequently the above is a particular case of Art 117, 1

The theorem on the right-hand side may also be proved as follows

Let a, b, c, d be the four lines touched by the conics of the system Find the direction of the point at infinity on the parabola which touches a, b, c, d (*i e* the point of contact with the line at infinity of the conic touching *abcd* and ∞) The line through the centre of any conic of the system parallel to this direction is the locus of the centres For, by the correlative of Desargues' theorem, if I be the point at infinity on the parabola, this line and the line at infinity are the double rays of the involution formed by drawing pairs of parallel tangents from I to the conics and the middle points of the chords of contact of such tangents are the centres of the conics and lie on the line in question

Since the limiting forms of the conics are the three pairs of opposite vertices, the locus passes through the middle points of the diagonals

6 *The locus of the centres of a system of conics which pass through the vertices of a triangle and through its orthocentre is the nine-points circle of the triangle*

The system of conics is a system of conics through four fixed points and therefore this theorem is a particular case of 5 As two of the pairs of lines which determine the involution on the line at infinity are at right angles the double points of this involution, through which the locus passes, are the circular points at infinity The locus of centres is therefore a circle This locus passes through the middle points of the connectors of the four points and is therefore the nine-points circle It also passes through the feet of the perpendiculars which are the diagonal points of the quadrangle

The conics of the system are rectangular hyperbolas (Art 108)

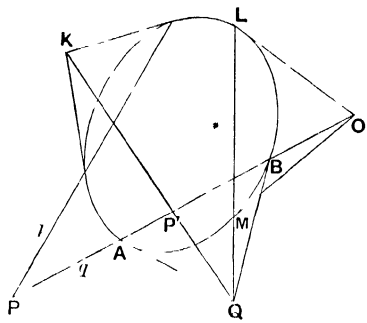
7 The locus of the centres of hyperbolas, which pass through three given points and have an asymptote given in direction, is a hyperbola, which has an asymptote in the given direction

This is a particular case of the locus of the centre of a system of conics through four points, one point being at infinity The middle points of the lines joining the point at infinity to the three given points are at infinity in the direction of the asymptote and therefore an asymptote of the locus is in that direction

8 Prove that the locus of the centre of a conic which passes through the centres of the inscribed and escribed circles of a triangle is the circumscribing circle

If A', B', C' be the centres of the escribed circles and O the centre of the inscribed circle, O is the orthocentre of the triangle $A'B'C'$, and by 6 the locus of the centres of conics through $A'B'C'$ and O is the nine points circle of $A'B'C'$, i.e. the circumcircle of ABC

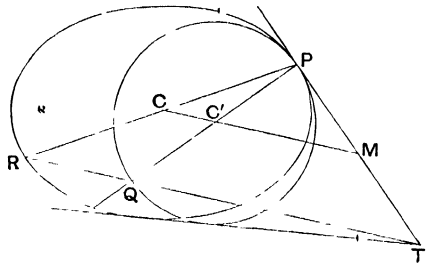
9 The connector of a pair of conjugate points with respect to a given conic passes through a fixed point and one of the pair lies on a given straight line, show that the locus of the other is a conic passing through (1) the fixed point, (2) the intersections of the given conic with the given straight line and with the polar of the fixed point, and (3) the pole of the given straight line



Let O be the point through which the variable chord q passes, meeting the conic in A and B, and the fixed line k (whose pole is K) in P Let Q be the pole of q which is on LM the polar of O

Then QK is the polar of P and the point P', where it meets the line q, is the conjugate of P with regard to the conic But the pencil q is projective with the range described by its pole Q on LM Therefore the pencils q and KQ are projective and the locus of P' is a conic through A and O It obviously passes through L and M At the points where k meets the conic P' coincides with P and therefore these points are also on the locus

10 A circle and a conic osculate at P and have diameters PQ, PR Show that the line QR, the tangent at P and the other common tangent are concurrent

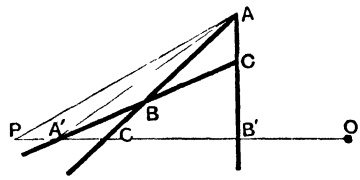


Let C and C' be the centres of the conic and circle Let the tangent at P meet the other common tangent at T and CC' at M Then from the nature of the locus of centres of conics touching four straight lines, M bisects PT Therefore QR, which is parallel to $CC'M$, passes through T

118 Loci and Envelopes by Anharmonic Ratios

1 *A variable line passes through a fixed point to find the locus of a point on the line which forms with the three points in which the line meets the sides of a given triangle a range of a constant anharmonic ratio*

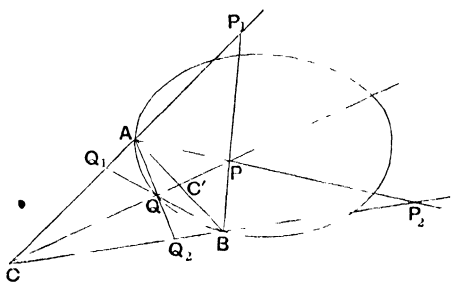
Through O the fixed point draw any transversal to meet the sides of the triangle in A', B', C' Let P be a point such that $(A'B'C'P)$ is constant Join A' and P to A Then the pencil $(A\ A'B'C'P)$ has the given constant anharmonic ratio Therefore since AC', AB' are fixed the pencils described by AP and AA' are projective But, since A' is situated on the fixed line BC, the pencils AA' and OA' are projective Hence the pencils AP and OA' are projective Therefore the locus of P is a conic through A and O By symmetry this conic passes through B and C



Hence, if a system of conics be described through four points, and through one of the points a transversal be drawn to meet one of the conics in P and the sides of the triangle formed by the other three fixed points in $A'B'C'$, then the anharmonic ratio of the range $(A'B'C'P)$ is constant for every transversal Also by taking a different conic of the system the value of this anharmonic ratio may be made equal to any given quantity

2 *If a variable transversal be drawn through one vertex C of a triangle ABC to meet the opposite side in C' and a conic through A, B in Q, find the locus of a point P on this transversal such that $(CC'QP)$ is constant*

Project Q and P from B on to AC into the points Q_1, P_1 respectively, and from A on to CB into the points Q_2, P_2 Then the pencils AQ, BQ are projective and therefore the ranges Q_1, Q_2 are projective But, since the anharmonic ratios of the ranges CAQ_1P_1 and CBQ_2P_2 are constant and C, A and B are fixed, the range Q_1 is projective with the range P_1 and the range Q_2 with the range P_2 Hence the ranges P_1 and P_2 are projective Therefore the pencils AP and BP are projective and the locus of P is a conic through A and B



3 *If a variable chord CQQ' of a conic be drawn through a fixed point C and a point P be taken in this chord such that $(CQPQ')$ is constant, then the locus of P is a conic*

Let CA, CB be the tangents from C to the conic and let the chord of contact meet AB in C'
Then

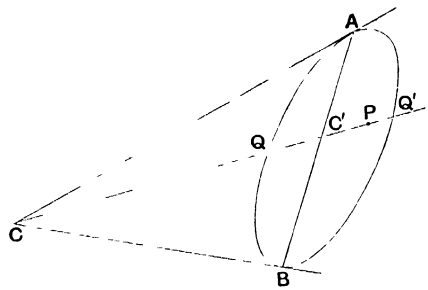
$$(CQPQ') = K \text{ (a constant)}$$

But $(CQC'Q') = 2$

Therefore

$$\frac{(CQPQ')}{(CQC'Q')} = (CQPC') = \frac{K}{2}$$

Hence by the last example P describes a conic through A and B



4 *A conic touches two sides CA and CB of a triangle ABC Any tangent is drawn to the conic meeting BC in P, AC in P' and AB in Q On the tangent a point N is taken such that $PP'QN$ is harmonic Find the locus of N*

Let CN meet AB in K and let AP BP' be L The ranges described by P and P' on CB and CA are projective, therefore the pencils AP, BP' are projective and the locus of L is a conic through A and B From the harmonic property of the quadrangle $ABPP'$, the pencil C $ABLQ$ is harmonic and therefore L lies on CN Hence, since $(CLKN)$ is harmonic and L describes a conic through A and B, the locus of N is also a conic through A and B

5 If a variable triangle ABC is such that two of its vertices A, B lie respectively on two fixed tangents to a given conic, while AB always touches the conic, and the sides BC, CA pass respectively through two fixed points P, Q, then C will lie on a fixed conic which passes through P and Q

6 Fixed straight lines OA, OB are cut in P, Q by a line which passes through a fixed point Prove that the centre of the circumcircle of the triangle OPQ lies on a hyperbola

Take M and N the middle point of OP, OQ and erect perpendiculars at M and N to meet at R R is the centre of the circumcircle of OPQ

Ranges P and Q are projective, therefore the ranges M and N are projective Hence the pencils formed by joining M and N to the points at infinity in directions perpendicular to OA and OB are projective Therefore the locus of R is a conic through the points at infinity perpendicular to OA and OB, *i e* it is a hyperbola

7 If S_1 and S_2 be two points which project three collinear points A_1, B_1, C_1 and three collinear points A_2, B_2, C_2 respectively into the same three collinear points, prove that S_1S_2 touches a conic

Let S_1S_2 meet the two bases in D_1, D_2 Then if D_1 is given D_2 is uniquely determined for $(A_1B_1C_1D_1) = (A_2B_2C_2D_2)$ Therefore for different positions of S_1 and S_2, the ranges D_1 and D_2 are projective Therefore D_1D_2 (or S_1S_2) touches a conic which also touches the two bases

8 A perpendicular is drawn to any tangent to a parabola at the point where it meets a fixed tangent prove that the envelope of this line is a parabola which touches the fixed tangent at the point where it is met by the directrix of the given parabola

Let a be the fixed tangent and e the line at infinity Let the variable tangent meet the fixed tangent and the line at infinity in P and P' If a perpendicular to the tangent be drawn through P, it will meet the line at infinity in a point Q which is the conjugate of P' in a definite involution Therefore the ranges P, Q' are projective and PQ' will envelope a conic which touches the fixed tangent and the line at infinity, *i e* a parabola The point where it touches a will be the point on a corresponding to ae At this point the variable tangent must be perpendicular to a and therefore this point is a point on the directrix

9 Prove that the locus of the point in which the perpendicular from a fixed point to a variable tangent to a parabola meets the corresponding diameter is a rectangular hyperbola passing through the fixed point, and find its asymptotes

Let the perpendicular from the fixed point O meet the tangent at P in N and the corresponding diameter in Q The perpendicular from the focus S will meet the diameter through P in a point M on the directrix The pencil OQ is projective with the pencil SM, therefore with the range M on the directrix, therefore with the pencil formed by joining M to the point at infinity on the axis Therefore the locus of Q is a conic through O and the point at infinity on the axis A line through O parallel to the directrix will meet a perpendicular tangent to the parabola at infinity Thus a second point at infinity is obtained on the curve Since these points are in perpendicular directions the locus is a rectangular hyperbola

10 A variable diameter of a conic meets a fixed line in P and through P a line is drawn parallel to the conjugate diameter Prove that the envelope of this line is a parabola

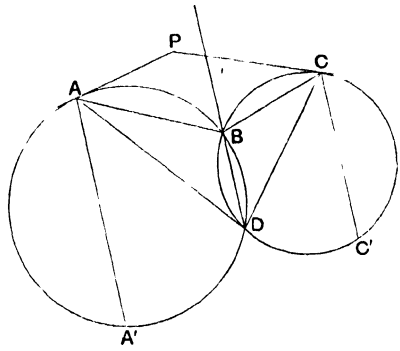
Let V be the pole of the fixed line Then the point at infinity I on a parallel through V to the conjugate diameter will be the pole of the diameter Hence the ranges described by P on the fixed line and by I on the line at infinity are projective But the line whose envelope is required is PI and the envelope is therefore a conic touching the line at infinity, that is a parabola

11 Two points P and Q are conjugate with regard to a conic P lies on a fixed straight line and PQ subtends a right angle at a fixed point Prove that the locus of Q is a conic passing through the fixed point

Let PM be the fixed line on which P lies and O the fixed point The polar QL of P will pass through a fixed point N, the pole of PM The range described by P is projective with the pencil described by its polar NQ Also the pencil OQ is projective with the pencil OP and therefore with the range P Hence the pencils OQ and NQ are projective and the locus of Q is a conic through O and N

12 A, B, C are three fixed points A circle is drawn through A, B and another through B, C If these two circles have a fixed radical axis, prove that their respective tangents at A, C meet on a fixed conic

Let D be the second point of intersection of the circles Through A and C draw AA', CC' parallel to BD Then the pencils AD and CD for different positions of D are projective But since angle $A'AD = BDA = BAP$, the pencils AD and AP are projective



Similarly the pencils CD and CP are projective Therefore the pencils AP and CP are projective, and the locus of P is a conic through A and C

13 Given a straight line along which the base of a triangle lies and an inscribed conic of the triangle, find the locus of the vertex if the base always subtends a right angle at a fixed point

Let ABC and $A'B'C'$ be two positions of the triangle Since the base always subtends a right angle at a fixed point the ends of the base are a pair of conjugate points of an involution determined by BC and $B'C'$ Hence the tangents from the ends of the base always intersect on the straight line AA', which is therefore the locus of the vertex (Arts 75 and 95 (*c*))

14 Through a fixed point O a straight line is drawn to meet a straight line l in P and to intersect the polar of P with respect to a fixed conic in Q Show that as P describes the straight line l, Q describes a conic passing through two fixed points independent of the line l chosen

Let O' be the pole of l, which will be situated on p the polar of P Then for different positions of P, the pencil described by $O'Q$ will be projective with the range described by P and consequently with the pencil OP Therefore the locus of Q, the point of intersection of corresponding rays of these pencils, is a conic through O and O It also passes through the points of contact of the tangents from O to the conic

15 A, B, C are three fixed points, s a fixed line and S a fixed conic passing through A, B, Q is any point on s, AQ meets the conic in R, show that the locus of the intersection of CQ, BR is a conic passing through B, C, the intersections of s and S and through D the other point of intersection of AC and S

Pencil CQ is projective with the pencil AR and therefore with the range R on the conic Therefore the pencils BR and CQ are projective and the locus of P is a conic through B and C It obviously passes through the intersections of s with S and through D

16 Two conics $AEFC$, $BEFD$ are such that the points C, D, F are collinear, L is a point on EF, AL meets the first conic again in Q and BL meets the second conic again in R, show that the locus of the point of intersection of CQ, DR is a straight line through E

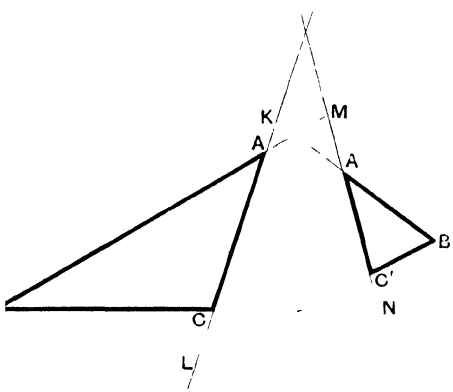
The pencils AL, BL are projective Therefore the ranges Q and R are projective, and the pencils CQ, DR are projective Since CDF is a self-corresponding ray the ranges are in perspective and the locus of U is a straight line which passes through E

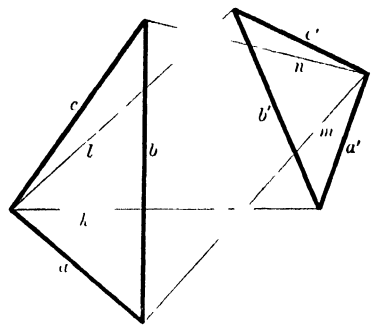
CHAPTER XIX

SIX POINTS ON A CONIC, SIX TANGENTS TO A CONIC

119 In Chapter XIII it was shown how a conic could be described through five given points or to touch five given straight lines In certain special cases a conic may be described to pass through more than five points or to touch more than five lines

(*a*) *If a conic can be described through the six vertices of two triangles, another conic can be described to touch the six sides, and correlatively, if a conic can be described to touch the six sides, another conic can be described through the six vertices of the triangles*





Let ABC and $A'B'C'$ be the two triangles and let AC meet $B'A'$ and $B'C'$ in K and L, and $A'C'$ meet BA and BC in M and N

If a conic circumscribes A, B, C, A', B', C',

$$(B'\ ACA'C') = (B\ ACA'C')$$

Therefore taking intercepts on AC and $A'C'$

$$(ACKL) = (MNA'C')$$

Hence AM, CN, KA', LC', AC and $A'C'$ envelope a conic, which touches the sides of the two triangles

Let abc and $a'b'c'$ be the two triangles and let the connectors of ac to $b'a'$ and $b'c'$ be k and l, and those of $a'c'$ to ba and bc be m and n

If a conic touches a, b, c, a', b', c',

$$(b'\ aca'c') = (b\ aca'c')$$

Therefore by joining the ranges to ac and $a'c'$

$$(ackl) = (mna'c')$$

Hence am, cn, ka', lc', ac and $a'c'$ all lie on a conic, which passes through the vertices of the two triangles

(*b*) *A conic can be described to touch the six sides of two triangles or to pass through their six vertices, when the triangles are* (1) *self-conjugate with regard to a conic,* (2) *constructed of two pairs of tangents to a conic and their respective chords of contact,* (3) *circumscribed to a triangle and in perspective with it or* (4) *inscribed in a triangle and in perspective with it*

Let ABC and $A'B'C'$ be the triangles Then in the figure (p 259)

In case (1), A, C, K, L are the poles of the lines joining B to C, A, C', A' respectively

In case (2), if B and B' be the poles of AC and $A'C'$, A, C, K, L are the poles of the lines joining B to A, C, A', C' respectively

Therefore in both cases

$$(B'\ ACA'C') = (B\ ACA'C')$$

and taking intercepts on AC and $A'C'$

$$(ACKL) = (MNA'C')$$

Hence a conic can be described to touch AM, CN, KA', LC', AC and $A'C'$, *i e* to touch the six sides of the triangles

Case (3) follows from Menelaus' theorem and the converse of the correlative of Carnot's theorem

Case (4) follows by (*a*) from the theorem on the right-hand side

Let abc and $a'b'c'$ be the triangles Then in the figure (p 259)

In case (1), a, c, k, l are the polars of the intersection of b with c, a, c', a' respectively

In case (2), if b and b' be the polars of ac and $a'c'$, a, c, k, l are the polars of the intersections of b with a, c, a', c' respectively

Therefore in both cases

$$(b'\ aca'c') = (b\ aca'c')$$

and by joining the ranges to ac and $a'c'$

$$(ackl) = (mna'c')$$

Hence a conic can be described through am, cn, ka', lc', ac and $a'c'$, *i e* to pass through the six vertices of the triangles

Case (3) follows by (*a*) from the theorem on the left-hand side

Case (4) follows from Ceva's theorem and the converse of Carnot's theorem

120 At the end of Art 101 a short statement was given in regard to the properties of the conic involving the "imaginary" As there stated it is often assumed that a pair of real points can be replaced by, or projected into, a pair of imaginary points, these points usually being the circular points at infinity No justification of this has been attempted and the consideration of this part of the subject is reserved for the present The theorems however of Art 119 afford instances of how this method may be applied As such the following are given

Particular case of (*a*)

If two of the vertices of one triangle be assumed to be the circular points at infinity, then the following theorem is obtained

A parabola can be inscribed in any triangle, inscribed in a circle, so that its focus may be at any given point on the circumference of the circle

If two of the vertices of one triangle be assumed to be the circular points at infinity, then the following theorem is obtained

The circle, which circumscribes a triangle formed by three tangents to a parabola, passes through its focus

Particular cases of (*b*) (1)

If two of the vertices of one of the self conjugate triangles are assumed to be the circular points at infinity, the conic will be a rectangular hyperbola

Hence *the conic inscribed in a triangle self conjugate with regard to a rectangular hyperbola and having a focus at the centre of the rectangular hyperbola is a parabola*

If two of the vertices of one of the self-conjugate triangles are assumed to be the circular points at infinity, the conic will be a rectangular hyperbola

Hence *the circle circumscribing a triangle self conjugate with regard to a rectangular hyperbola passes through the centre of the rectangular hyperbola*

Particular cases of (*b*) (2)

(i) If the two vertices of one triangle, in which the tangents meet the conic, be assumed to be the circular points at infinity, then the following theorem is obtained

The conic described to touch two tangents to a circle, and their chord of contact, and to have a focus at the centre, is a parabola

(i) If the two vertices of one triangle, in which the tangents meet the conic, be assumed to be the circular points at infinity, then the following theorem is obtained

The circle described through the points of contact of two tangents to a circle and through their point of intersection passes through the centre of the circle

(ii) If one of the chords of contact be the line at infinity, then the following theorem is obtained

The conic which can be described to touch two tangents to a hyperbola, their chord of contact and the two asymptotes is a parabola

(ii) If one of the chords of contact be the line at infinity, then the following theorem is obtained

If a hyperbola be described to pass through the points where any line meets a given hyperbola, to pass through the pole of this line, and to have its asymptotes parallel to those of the given hyperbola, then this hyperbola will pass through the centre of the given hyperbola

121 Certain of the theorems of Art 119 may be expressed in a somewhat different form

**If two conics are such that a triangle can be* (*a*) *inscribed in one and circumscribed to the other,* (*b*) *inscribed in one and self-conjugate with regard to the other, or* (*c*) *circumscribed to one and self-conjugate with*

* For similar theorems for a quadrangle see Art 134

regard to the other, then an infinite number of such triangles can be constructed

The proofs in each case are similar Let ABC be the given triangle

(a) Let (1) be the conic to which ABC is circumscribed Draw $B'C'$ any tangent to this conic to meet the other conic (2) in B', C' Let the other tangents from B' and C' to (1) meet at A' Then a conic can be described through A, B, C, A', B', C' But (2) passes through A, B, C, B', C' and is therefore this conic, which also passes through A'

(b) Let (1) be the conic with regard to which the triangle ABC is self-conjugate Take B' any point on the other conic (2), and let its polar with regard to (1) meet (2) in C' Construct the polar of C' with regard to (1) This passes through B' Let these polars of B' and C' with regard to (1) intersect at A' Then ABC and $A'B'C'$ are two triangles self-conjugate with regard to (1) Hence a conic will pass through A, B, C, A', B', C' But (2) which passes through A, B, C, B', C' must be this conic

(c) Let (1) be the conic with regard to which the triangle abc is self-conjugate Take b' any tangent to the other conic (2) and let a tangent from its pole B', with regard to (1), to (2) be c' Let a' be the polar of $b'c'$ with regard to (1) This passes through B' Then $a'b'c'$ is self-conjugate with regard to (1) A conic can then be described to touch the sides of the triangles abc and $a'b'c'$ As (2) touches five of these lines it must be this conic, and must therefore touch the sixth side a'

It should be noticed that two triangles are in perspective

(1) when one triangle is the polar triangle of the other with regard to a conic (Art 103 (a))

(2) when one triangle is self-conjugate with regard to a conic and the other is inscribed in the conic and circumscribed to the former triangle (Art 107 (b)), or correlatively

when one triangle is self-conjugate with regard to a conic and the other is circumscribed to the conic and inscribed in the former triangle (Art 107 (d)),

(3) when one of the triangles is inscribed in and the other is circumscribed to the same triangle and both are in perspective with it (Example 15, Ch vi)

122 A conic may be described to touch

(1) *The six lines joining the vertices of a triangle to the points where the opposite sides are met by a conic*

The converse of the correlative to Carnot's Theorem Art 99

A conic may be described to pass through

(1) *The six points where the tangents from the vertices of a triangle to a conic meet the opposite sides of the triangle*

The converse of Carnot's Theorem Art 99

(2) *The six lines joining the non-corresponding vertices of two triangles in perspective*

The converse of Brianchon's Theorem Art 100

(3) *The eight tangents to two conics at their four points of intersection* (Art 126 (*d*))

(4) *The eight sides of two quadrilaterals which have the same diagonal triangle* (Art 126 (*a*))

(5) *Any three pairs of lines through the diagonal points of a quadrangle, which are harmonic conjugates of the pairs of sides meeting at the diagonal points*

(6) *Three pairs of lines* m_1, n_1, m_2, n_2, m_3, n_3, *where* m_1, n_1 *are the common harmonic conjugates of* a_1, a_1' *and* a_2, a_2' a_1, a_2 *and* a_1', a_2' *being the tangents from a vertex of a common self-conjugate triangle of two conics to the conics—and* m_2, n_2 *and* m_3, n_3 *are the similar lines for the other vertices of the common self conjugate triangle*

(7) *The six harmonic conjugates of the six lines joining any point to the vertices of a quadrilateral, with respect to the pairs of sides of the quadrilateral meeting at each vertex* (Art 117 (2))

(8) *The six polars of a given point with respect to the six conics, which can be described through five out of six given points* (Art 135 (*b*))

The proofs of (5) and (6) are as follows

(5) Take the diagonal points triangle ABC as triangle of reference Let the ratios of l, m, s in the figure be a, b, c Then since these lines are concurrent $abc = -1$

The ratios of n, r, k in the figure are

(2) *The six points of intersection of the non corresponding sides of two triangles in perspective*

The converse of Pascal's Theorem Art 100

(3) *The eight points of contact of the common tangents to two conics* (Art 126 (*d*))

(4) *The eight vertices of two quadrangles which have the same diagonal points triangle* (Art 126 (*a*))

(5) **Any three pairs of points on the diagonals of a quadrilateral, which are harmonic conjugates of the ends of the diagonals*

(6) *Three pairs of points* M_1, N_1, M_2, N_2, M_3, N_3, *where* M_1, N_1 *are the common harmonic conjugates of* A_1, A_1' *and* A_2, A_2' $-A_1, 1_2$ *and* $A_1', 1_2'$ *being the points in which two conics are met by a side of their common self conjugate triangle—and* M_2, N_2 *and* M_3, N_3 *are similar points for the other sides of the common self conjugate triangle*

(7) *The six harmonic conjugates of the six points in which the sides of a quadrangle are met by any straight line, with respect to the pairs of vertices of the quadrangle situated on each side* (Art 117 (2))

(8) *The six poles of a given line with respect to the six conics, which can be described to touch five out of six given lines* (Art 135 (*b*))

(5) Take the diagonal triangle ABC as triangle of reference Let the ratios of K, S, N in the figure be a, b, c Then since these points are collinear $abc = +1$

The ratios of M, R, L in the figure

* This is practically a restatement of Art 103 (*c*)

therefore $-a$, $-b$, $-c$ If x_1, x_2 are harmonic conjugates of a, $-a$, then

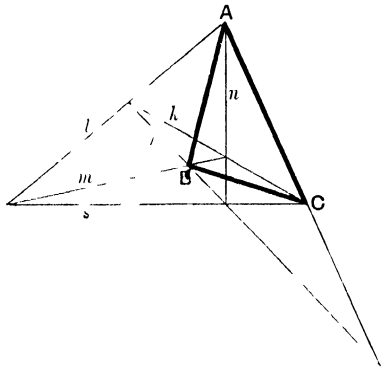
$$x_1 x_2 = a^2$$

Similarly $y_1 y_2 = b^2$,

$$z_1 z_2 = c^2$$

$$x_1 x_2 y_1 y_2 z_1 z_2 = a^2 b^2 c^2 = 1$$

by the converse of the correlative to Carnot's Theorem the six lines touch a conic



(6) If the ratios of the points determined by a_1, a_1' on the sides of the self conjugate triangle are a_1, a_1', then those of a_2, a_2' will be $-a_1$ and $-a_1'$

The common harmonic conjugates (x_1, x_2) of these are given by

$$x_1 = -x_2 \text{ and } x_1 x_2 = -a_1 a_1'$$

The condition that $x_1 x_2 y_1 y_2 z_1 z_2$ should touch a conic is $x_1 x_2 y_1 y_2 z_1 z_2 = 1$,

or $$a_1 a_1' b_1 b_1' c_1 c_1' = -1 \qquad (1)$$

But $$a_1^2 b_1^2 c_1^2 = -1,$$

and $$a_1'^2 b_1'^2 c_1'^2 = -1$$

$$(a_1 a_1' b_1 b_1' c_1 c_1')^2 = 1,$$

$$a_1 a_1' b_1 b_1' c_1 c_1' = 1 \text{ or } -1$$

Since the former does not generally hold, the latter is true and (1) is satisfied

are therefore $-a$, $-b$, $-c$ If x_1, x_2 are harmonic conjugates of a, a, then

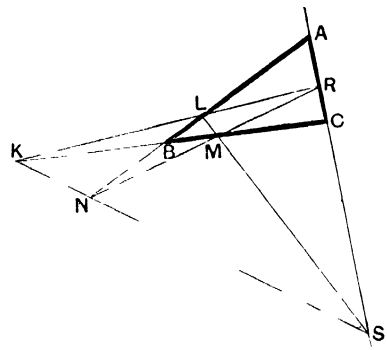
$$x_1 x_2 = a^2$$

Similarly $y_1 y_2 = b^2$,

$$z_1 z_2 = c^2$$

$$x_1 x_2 y_1 y_2 z_1 z_2 = a^2 b^2 c^2 = 1$$

by the converse of Carnot's Theorem the six points are on a conic



(6) If the ratios of l_1, l_1' referred to the self conjugate triangle be a_1, a_1', then those of l_2, l_2' will be $-a_1$, a_1'

The common harmonic conjugates (x_1, x_2) of these pairs of points are given by

$$x_1 = -x_2 \text{ and } x_1 x_2 = -a_1 a_1'$$

The condition that $x_1 x_2 y_1 y_2 z_1 z_2$ should lie on a conic is $x_1 x_2 y_1 y_2 z_1 z_2 = 1$,

or $$a_1 a_1' b_1 b_1' c_1 c_1' = -1 \qquad (1)$$

But $$a_1^2 b_1^2 c_1^2 = -1,$$

and $$a_1'^2 b_1'^2 c_1'^2 = -1$$

$$(a_1 a_1' b_1 b_1' c_1 c_1')^2 = 1,$$

$$a_1 a_1' b_1 b_1' c_1 c_1' = 1 \text{ or } -1$$

Since the former does not generally hold, the latter is true and (1) is satisfied

EXAMPLES

(1) A, B, C and D are any four points on a conic, AB and CD intersect in O, the tangents at A and B intersect at P and the tangents at C and D at Q. Show that a conic can be described through the points A, B, C, D, P, Q and that the tangents to this conic at P and Q are the straight lines OP, OQ.

By Art. 119 *b* (2), A, B, C, D, P, Q are on a conic.

Since PQ is the polar of O with respect to this conic—which passes through $ABCD$—OP and OQ are the tangents to it from O.

(2) On the sides BC, CA of a triangle ABC points A', A'' and B', B'' are taken, and a point C''' is taken on the side AB, $A'B'$ and $C'B''$ meet in H, $A''B''$ and BH meet in K, and $B'K$ meets AB in C'''. Prove that AA', AA'', BB', BB'', CC', CC'' are tangents to a conic.

Consider the hexagon $A'A''B''C'C''B'$. The pairs of opposite sides intersect in the collinear points B, K, H. Therefore the vertices are on a conic. Therefore the conic through $A'A''B'B''C'$ passes through C''', and a conic can be described to touch the lines joining these points to the opposite vertices.

(3) AB is a fixed chord of a conic and CD is a fixed tangent at C meeting AB in D, CP and CQ are any two lines meeting the conic in P and Q and cut by any chord l through D in P' and Q'. Show that a conic can be drawn through $ABPQP'Q'$.

Let l meet the conic in L and M and PQ in R; then P', Q' are conjugate points of the involution determined on l by conics through $ABPQ$. For they are conjugate points of the involution determined by $CCPQ$, and for both involutions D, R and L, M are pairs of conjugate points.

Hence a conic through A, B, P, Q, P' passes through Q.

(4) OPP', OQQ', OXX' are chords of a conic, prove that PQ, PQ', $P'Q$, $P'Q'$ and the tangents at X, Y' all touch a conic.

Let PQ', QP' be E' and PQ, $P'Q'$ be E. Describe a conic to touch PQ, PQ', $P'Q$, $P'Q'$ and the tangent at X. The two conics have OEE' for common self-conjugate triangle and are therefore in harmonic perspective with O as centre and EE' as axis. The tangents at X and X' correspond, and, since the second conic touches the tangent at X, it also touches the tangent at X'.

(5) From any point on the circumscribed circle of a triangle ABC perpendiculars are drawn to the sides BC, CA, AB to meet the circle again in D, E, F respectively. Prove that a conic can be drawn to touch the six straight lines AB, BC, CA, DE, EF, FD.

(6) A conic may be described through the six vertices of two triangles which are in harmonic perspective.

Let the triangles be ABC, $A'B'C'$ and S the centre of perspective. Take M and N harmonic conjugates of S with respect to AA', BB'. MN is then the axis of perspective. Describe a conic through AA', BB' and C'. This conic is in self-perspective (harmonic) with S and s as centre and axis. Therefore the conic passes through C'.

Conversely, if two triangles are in perspective and a conic can be described through their vertices the triangles are in harmonic perspective.

CHAPTER XX

THEOREMS CONCERNING TWO CONICS —COMMON CONJUGATE POINTS, CONICS IN SELF PERSPECTIVE, CONICS IN PERSPECTIVE WITH EACH OTHER, CONICS WITH DOUBLE CONTACT, HARMONIC LOCUS OF TWO CONICS

TWO CONICS

123 RECAPITULATION

Conjugate Points

A Conic determines on every straight line in its plane an involution, any pair of conjugate points of which are conjugate points with regard to the conic, while its double points, if real, are the points of intersection of the line and conic (Art 95(*k*))

The conjugate A' on a given line l of any point A on the line l with respect to a given conic may be constructed—

(1) as the point of intersection of the polar of A with the line l,

(2) as the harmonic conjugate of A with regard to the points E and F (if real), in which l meets the conic,

(3) by taking any chord PQ of the conic which passes through the pole of l, joining A to P to meet the conic at R, and joining R to Q to meet l at A'

(Art 95(*j*))

Conjugate Lines

A Conic determines at every point in its plane an involution pencil, any pair of conjugate rays of which are conjugate lines with regard to the conic, while its double rays, if real, are the tangents from the point to the conic

(Art 95(*k*))

The conjugate a' through a given point L of any ray a through L with respect to a given conic may be constructed—

(1) as the connector of the pole of a with the point L,

(2) as the harmonic conjugate of a with regard to the tangents e and f (if real) from L to the conic,

(3) by taking any pair of tangents p, q to the conic which intersect on the polar of L, constructing r the tangent from ap to the conic and constructing a' the line joining rq to L

(Corl of Art 95(*j*))

Common Conjugates of Conics through four points and of conics touching four straight lines

If a pair of points are conjugate points with regard to any two conics (including a pair of lines as a conic), they are conjugate points with regard to every conic through their four points of intersection (Art 117(1))

If a pair of lines are conjugate lines with regard to any two conics (including a pair of points as a conic), they are conjugate lines with regard to every conic which touches the four common tangents to these conics (Art 117(1))

Construction of Common Conjugates

(1) To construct the common conjugate of a given point A with respect to a system of conics through four given points

Construct the polars of A with respect to two of the conics Then point of intersection A' is the common conjugate of A

(2) To construct on a given line l a pair of points, which are common conjugates with respect to a system of conics through four fixed points

Construct on l the involutions determined on this line by two of the conics The pair of common conjugates of these involutions are the required points

Or

Construct the two conics of the system which touch the line l Then points of contact are the pair of common conjugates on l

If the points A, B, C, D, through which the conics pass, are determined as the intersections of a pair of lines AB and CD with a conic S, the following construction is useful

Draw the tangent p at any point P on the conic Join P to AB CD by a Take a' the harmonic conjugate of a with respect to AB and CD a' meets p in P' a common conjugate of P

This follows from the fact that every point on the tangent at P is a conjugate of P with regard to the conic

The connectors of a pair of points, constructed in this way, with the diagonal points of the quadrangle are harmonic conjugates of the pairs of sides which meet at these points

(1) To construct the common conjugate of a given line a with respect to a system of conics touching four given lines

Construct the poles of a with respect to two of the conics Their connector a' is the common conjugate of a

(2) To construct through a given point L a pair of lines, which are common conjugates with respect to a system of conics touching four fixed lines

Construct the involution pencils determined at L by two of the conics The pair of common conjugates of these involution pencils are the required lines

Or

Construct the two conics of the system which pass through L The tangents at L to these conics are the pair of common conjugates through L

If the lines a, b, c, d, which are touched by the conics, are determined as the tangents for a pair of points ab and cd to a conic S, the following construction is useful

Construct the point of contact P of any tangent p to the conic Construct A the point of intersection of p and ab cd Take A' the harmonic conjugate of A with respect to ab and cd p' the connector of A' to P is a common conjugate of p

This follows from the fact that every line through the point of contact of a tangent p is a conjugate of p with regard to the conic

The points of intersection of pairs of lines, constructed in this way, with the diagonals of the quadrilateral are harmonic conjugates of the pairs of vertices on these diagonals

Eleven points locus and its correlative

Given a system of conics through four fixed points the locus of the common conjugates of points on a fixed line is a conic which passes through eleven determined points

This important theorem is proved in Art 117 (2)

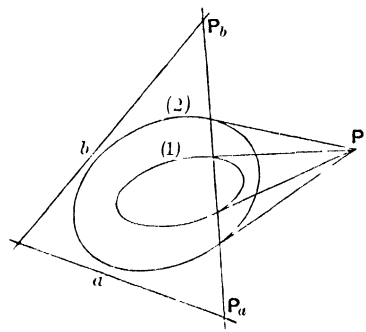
The polars of any two fixed points with respect to a system of conics through four fixed points form two projective pencils (Art 106 (*e*))

Given a system of conics touching four fixed lines the envelope of the common conjugates of lines passing through a fixed point is a conic which touches eleven determined lines

This important theorem is proved in Art 117 (2)

The poles of any two fixed lines with respect to a system of conics touching four fixed lines form two projective ranges (Art 106 (*e*))

124 *Every pair of real conics have a real common self-conjugate triangle except when the conics intersect in only two real points, in which case two vertices and the opposite sides of the triangle are imaginary*



The loci of the common conjugates of points on lines a and b with respect to the conics (1) and (2) are two conics

The common conjugate of the point ab must be on both of these conics These conics will therefore intersect in another real point P Let P_a and P_b be the common conjugates of P on the lines a and b Since P_a and P_b are conjugates of P with respect to (1), P_aP_b is the polar of P with respect to (1)

Similarly it is the polar of P with respect to (2) Therefore P_aP_b is a common polar of P

The chord P_aP_b may meet the conics

(1) in real points A_1, B_1 and A_2, B_2 such that A_1B_1 and A_2B_2 do not overlap

(2) in real points A_1, B_1 and A_2, B_2 such that A_1B_1 and A_2B_2 do overlap

(3) in real points A_1, B_1 and in two imaginary points

(4) in four imaginary points

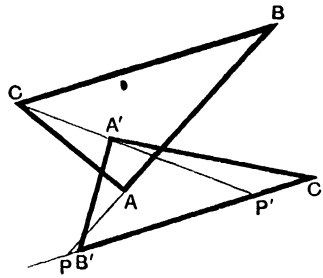
The vertices of the common self-conjugate triangle situated on P_aP_b are the common harmonic conjugates of A_1B_1 and A_2B_2, *i e* the double elements of the involution which they determine

In case (1) these points are real, (Art 48)
(2) these points are imaginary conjugate points, ,,
(3) these points are real, ,,
(4) these points are real ,,

In case (2) the conics intersect in two real and two imaginary points

Two conics which intersect in four distinct points can have only one common self-conjugate triangle

If possible let ABC and $A'B'C'$ be two common self-conjugate triangles Then B' and C' are conjugate points with respect to both conics Let AB meet $B'C'$ in P



Then the polar of P with regard to both conics is CA' Let CA' meet $B'C''$ in P' Then P and P' are common conjugate points with respect to the two conics Hence, since B', C' and P, P' are common conjugate points the conics must meet $B'C'$ in the same pair of points Similarly they meet the other five sides of the triangles in the same points Hence they have twelve points common and are therefore the same conic

125 Common involution chords of two conics

There are associated with every pair of conics two real lines on which the conics determine the same involution

If the conics have four real points of intersection, the lines in question are any two of their common chords

If the conics have no real points of intersection, consider the locus of the common of points on a line u, which passes through one of the vertices F of

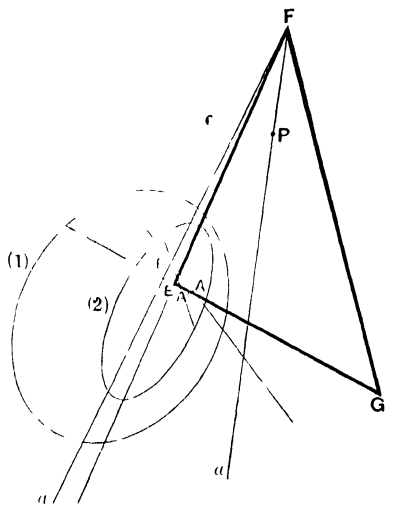
There are associated with every pair of conics two real points at which the conics determine the same involution

If the conics have four real common tangents, the points are any two of the points of intersection of these common tangents

If the conics have no real common tangents, consider the envelope of the common [illegible] of lines through a point A on one of the sides f of the common

the common self-conjugate triangle of the conics (1) and (2)

Let the poles of the line a with regard to (1) and (2) be A and A' These points are on EG, where FEG is the common self-conjugate triangle The polars of P any point on a with respect to (1) and (2) will meet in some point P'

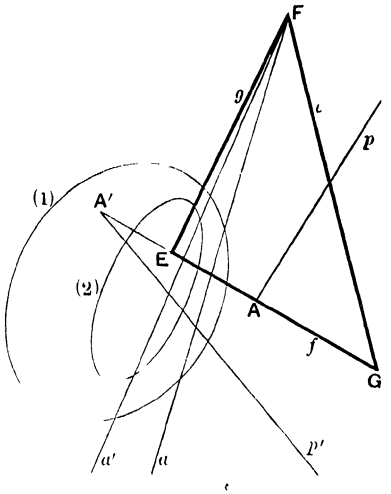


Since the points A, A' and also the points E, G, F lie on the conic, which is the locus of P', this conic will break up into a pair of lines, one of which is $EA'AG$ and the other FP'

The polars of P' with respect to (1) and (2) will pass through P Therefore, if FP' be a', the locus of P for different positions of P' on a' is the line a Hence the lines a and a' form an involution

self-conjugate triangle of the conics (1) and (2)

Let the polars of the point A with regard to (1) and (2) be a and a' These pass through F, where FEG is the common self-conjugate triangle The poles of p any line through A with respect to (1) and (2) will have as their connector some line p'



Since the lines a, a' and also the lines e, g, f are tangents to the conic, which is the envelope of p', this conic will break up into a pair of points, one of which is $ea'ag$ and the other fp'

The poles of p' with respect to (1) and (2) will be on p Therefore, if fp' be A', the envelope of p for different positions of p' through A' is the point A Hence the points A and A' form an involution

pencil If FG is taken as a, then EF will be a' Therefore FG and FE are conjugate elements of the involution There will be a pair of double rays of this involution, which will be the common harmonic conjugates of FE, FG and of FP, FP' For these double rays F, P, P' will be collinear On these double rays, p and p', the involution determined by the two conics will be the same If the intersections of the conics are real, these double rays p and p pass through the points common to (1) and (2), *i e* they are chords of intersection

The condition that p and p' should be real is that the lines joining P and P' to the corresponding vertex of the triangle EFG should not overlap the sides There always will be one vertex for which they do not overlap For, taking P a point inside the triangle, P' must lie inside one of the angles EFG, FGE or FEG If it lies inside EFG, then the pair of lines FP, FP' do not overlap EF, FG

If the conics have two real points of intersection, two vertices E and G of the triangle EFG are imaginary, and one chord of intersection is real This is one of the

If fg is taken as A, then ef will be A' Therefore fg and fe are conjugate points of the involution There will be a pair of double points of this involution, which will be the common harmonic conjugates of fe, gf and pf, $p'f$ For these double points f, p, p' will be concurrent At these double points, P and P', the involution determined (by pairs of conjugate lines) by the two conics will be the same Hence the double points P and P' are the points of intersection of the common tangents, if real, to the two conics

The condition that P and P' should be real is that the points where the lines p and p' meet the sides of the triangle efg should not overlap the vertices There will always be one side for which they do not overlap For, if x_1, y_1, z_1 be the ratios of the points where one line meets the sides of EFG and x_2, y_2, z_2 be the corresponding ratios for the other, then

$$x_1y_1z_1 = 1\ ,\quad x_2y_2z_2 = 1$$

Hence two or none of the ratios x_1, y_1, z_1 and x_2, y_2, z_2 must be negative, therefore one pair have always the same sign

If the conics have two real common tangents, two sides of the triangle efg are imaginary, and the points of intersection of these give one of the required points, which

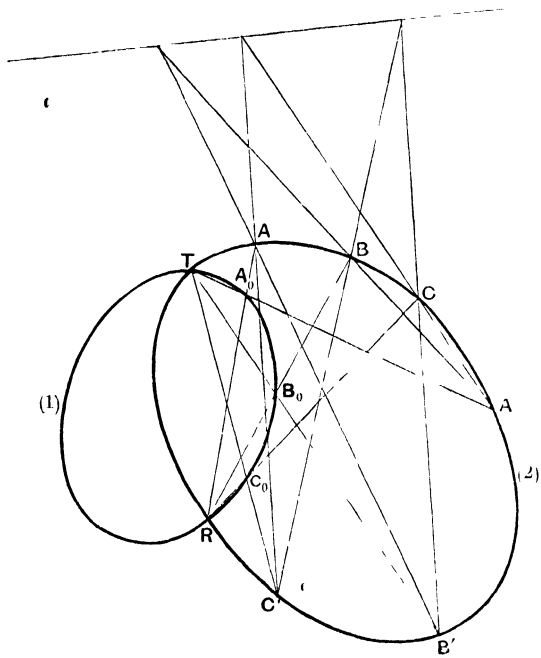
double rays of the involution a, a' determined at F The other double ray is the second chord

Hence every pair of conics have one pair of real lines on which the conics determine the same involution

is a double point of the involution A, A' The other double point is the second required point

Hence every pair of conics have a pair of real points at which the conics determine the same involution pencil

Given two real points of intersection of two conics, determining one chord of intersection, to construct a second chord on which the conics determine the same involution



Let R and T be the given points of intersection of the conics (1) and (2) Take any three points A_0, B_0 C_0 on (1) Let the connectors of these points to R and T meet the conic (2) in A, B, C and A', B', C' respectively Then A, B C and A' B', C' determine two projective ranges on (2) and the Pascal line of $AB'CA'BC'$ meets the conic (2) in the self-corresponding points of the two ranges These points if joined to R and T give two pairs of corresponding rays of the pencils (R $A_0B_0C_0$) and (T $A_0B_0C_0$) Therefore they are on the conic (1) If as in the figure the points of intersection of this Pascal line with the conics are imaginary, the Pascal line is the second chord, on which the conics determine the same involution

This theorem may be stated as follows

If R and T are two points of intersection of two conics and any two points A_0 and B_0 on one conic be projected upon the other conic into points A, B and A', B' from R and T respectively, then the locus of AB' BA' is a common chord of the conics

A second construction for this chord is given in Art 127

Construction of the common self-conjugate triangle of two conics

In certain cases the common self-conjugate triangle of two conics may be easily constructed

(1) If the conics intersect in four real points, the common self-conjugate triangle is the diagonal points triangle of the quadrangle formed by these four points (Art 95 (p))

(2) If the conics have four real common tangents the common self-conjugate triangle is the diagonal triangle of the quadrilateral formed by the four common tangents (Art 95 (p))

(3) If the conics intersect in two real points A and B, construct the poles K and K' of AB with respect to the conics Let KK' meet AB in R and the conics in P, Q and P', Q' Take T the harmonic conjugate of R with respect to A and B Then T is the pole of KK' with respect to both conics, and is therefore a vertex of the common self conjugate triangle The other vertices are the two imaginary common harmonic conjugates of PQ and $P'Q'$

(4) If the conics have two real common tangents, the correlative method to that employed in (3) may be used

(5) If the conics have no real points of intersection and no real common tangents, the general method must be employed

Certain of the points of intersection A, B, C, D may coincide

(1) If D and C coincide and A and B are real, the self conjugate triangle constructed as the diagonal points triangle of $ABC'D$ becomes in the limit the tangent SD, where S is the point in which AB meets the tangent at D The point S has a common polar with respect to the two conics, viz, the line joining D to the harmonic conjugate of S with respect to A and B

(2) If D and C coincide and A and B are imaginary, the line AB may be constructed, and S is thus determined The conjugate of S in the involution on AB, when joined to D, gives the polar of S

(3) If D and C coincide and also A and B, the conics have double contact If the tangents at A and D meet in S, then S and any pair of harmonic conjugates of A and D on AD form a common self conjugate triangle of the conics

(4) If B, C, D all coincide so that the conics have contact of the third order, the self conjugate triangle in the limit coincides with the point B

EXAMPLES

(1) Given three common points of two conics and a common pair, A and B, of conjugate points, make a geometrical construction for the fourth common point

Let K, L, M, N be the points of intersection of the conic of which N has to be found If A, B be conjugate points with respect to the two conics, they are conjugate points with respect to every conic through K, L, M, N and therefore with respect to each pair of lines through these points Hence, if the line AB meets KL and MN in A and B', $(ABA'B') = -1$ Therefore B' is found and the line MN is obtained Similarly the line KN may be found and therefore N is known

(2) Three of the intersections of two conics are given and two other points on each to find the remaining point of intersection of the conics

Let A, B, C be the three given points of intersection Construct R and R' the poles of BC and S and S' the poles of AB Let SS' meet BC in K and RR' meet AB in L Then KA and LC intersect in D the fourth point of intersection of the conics

(3) Two of the intersections of two conics are given and three other points on each to find the remaining points of intersection of the conics

Let A, B be the given points of intersection P, Q, R the three given points on conic (1) and U, V, W those on (2) Join P, Q, R to A and B and find by Pascal's Theorem the points P', Q', R' and P'', Q'', R'' where these lines meet conic (2) Then the points $P'Q''$ $Q'P''$ and $P'R''$ $P''R'$ determine the second common chord of intersection Determine the points where this line meets either conic These are the two other points of intersection of the conics

(4) If two conics touch at D and a chord DRR' be drawn through D to meet the conics in R and R', show that the locus of the point of intersection of the tangents at R and R' is a straight line through the other points of intersection of the conics

(5) If two conics intersect in A, B, C, D, and through A and B two lines a and b be drawn to intersect on the line joining the points of intersection of the tangents to the conics at A and B, then a third conic can be described to pass through A, B, C, D and touch a and b at A and B

Draw a conic through A, B, C, D to touch a at A Then in the harmonic perspective in which AB CD is the centre and its polar the axis, this conic will correspond to itself and will touch b, the line corresponding to a, at B, the point corresponding to A

Conics in Self-Perspective

126 *Each of two conics may be looked upon as in harmonic perspective with itself, the centre and axis of perspective being a vertex and the opposite side of their common self-conjugate triangle*

In Art 95 (*o*) it was proved that any conic may be looked upon as in perspective with itself, if any point and its polar are taken as the centre and axis of perspective and the anharmonic ratio of the perspective is -1

In Art 124 it was shown that every pair of conics have a common self-conjugate triangle except when they intersect in two real and two imaginary points, in which case only one vertex and side are real Hence, if a vertex and the opposite side of this triangle be taken as centre and axis of perspective and the anharmonic ratio of the perspective be -1, each of the conics is in perspective with itself

As a particular case, the two pairs of sides of a quadrilateral are each in self-perspective with a vertex and opposite side of the diagonal triangle for the centre and axis of perspective and, correlatively, the two pairs of opposite vertices of a quadrangle are in self-perspective with a vertex and opposite side of the diagonal points triangle for centre and axis

Deductions

(*a*) *If two quadrilaterals have the same diagonal triangle, a conic can be described to touch their eight sides*

Describe a conic to touch the four sides of one of the quadrilaterals and one of the sides of the other The two quadrilaterals and the conic may be looked upon as all in self-perspective with any vertex of the diagonal triangle and the opposite side as centre and axis of perspective Considering one vertex and the opposite side as centre and axis of perspective, it is seen that because the conic touches one side of the second quadrilateral it must touch a second Similarly by considering the other two possible perspectives it follows that this conic also touches the other two sides of the second quadrilateral

The correlative theorem is as follows

If two complete quadrangles have the same diagonal points triangle, a conic may be described through their eight vertices

(*b*) *All conics with respect to which a given triangle is self-conjugate and which pass through a given point pass through three other fixed points*

The three other points are obtained from the given point by the three systems of self-perspective derived from the self-conjugate triangle

(*c*) *To construct a conic with a given self-conjugate triangle to pass through two fixed points*

From each of the two fixed points construct three other points by the three systems of self-perspective A conic through the two given points and these six points is the required conic

(*d*) *A conic can be described to touch the four pairs of tangents which can be drawn to two conics at their four points of intersection*

Describe a third conic to touch the four tangents to one conic (1) and one of the tangents to the other conic (2) This conic and the two given conics have a common self-polar triangle In the three systems of self-perspective the tangents to the two given conics at their points of intersection are corresponding lines Hence, as the third conic touches one of the four tangents to the conic (2), it must touch them all

The correlative theorem is as follows

A conic can be described through the four pairs of points of contact of the four common tangents to two conics

This may be proved in a similar manner

EXAMPLES

(1) If, through a given point S any fixed chord be drawn, show that tangents to any given conic, whose points of contact are collinear with S, determine conjugate points of an involution on the fixed chord

(2) Show that pairs of common tangents to conics through four given points determine an involution on any chord through one of the points of intersection of a pair of common chords of the system

(3) Prove that if two variable chords SAA', SBB' be drawn to a conic through a fixed point S, then the lines AB, $A'B'$ determine corresponding points of an involution on any chord through S

(4) If two conics are inscribed in the same quadrilateral, the eight points of contact lie on a third conic, and the polars of a common chord of the first two conics with respect to all three are collinear

Conics in Perspective with each other

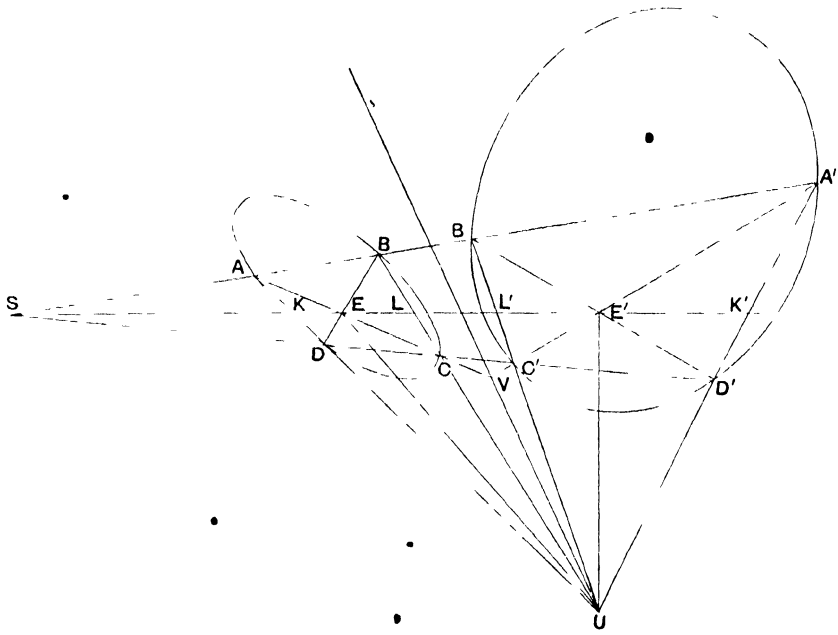
127 *Each of any two conics may be looked upon as the perspective of the other, either of the points at which the conics subtend the same involution being the centre of perspective, and either of the chords, on which they determine the same involution, being the axis of perspective*

Through S a common involution point draw any chord to meet the conics in A, B, B' and A', and let the polars of S meet at U Join U to A, B, B', A' to meet the conics again in D, C, C', D' Let AC BD and $A'C'$ $B'D'$ be E and E' Then since SE and SE' are the polars of U and S is a common involution point S, E, and E' are collinear Let

SEE' meet UA, UB, UB' and UA' in K, L, L' and K' Then since $(UKDA)$, $(ULCB)$, $(UL'C'B')$, $(UK'D'A')$ are all harmonic, D, C, C', D' are all collinear with S Let AC and $A'C$ meet in V and let UV be s

Consider the perspective with centre S, axis s and A and A' for corresponding points In this perspective A, E, C and D correspond to A', E', C' and D' Therefore B corresponds to B' and EU to $E'U'$ Hence the conics have four pairs of corresponding points

Also EU and $E'U$ are the polars of S Hence if a pair of conjugate points with respect to the conic $ABCD$ be taken on EU, they will be



projected from S into a pair of points on UE' which are conjugate with respect to the conic $A'B'C'D'$ Therefore the involutions determined by the conics on these lines are projected into each other from S

Hence the perspective of the conic $ABCD$ is the conic $A'B'C'D'$

If the common tangents from the common involution point are real, the proof may be put into the following simpler form

Let the common tangents PP', TT' to the two conics intersect at S Through S draw a chord meeting the conics in Q and Q' Then PQT and $P'Q'T'$ are two triangles in perspective and therefore the

corresponding sides will intersect on a line s. Take S for centre of perspective and s for axis. Then for the two conics

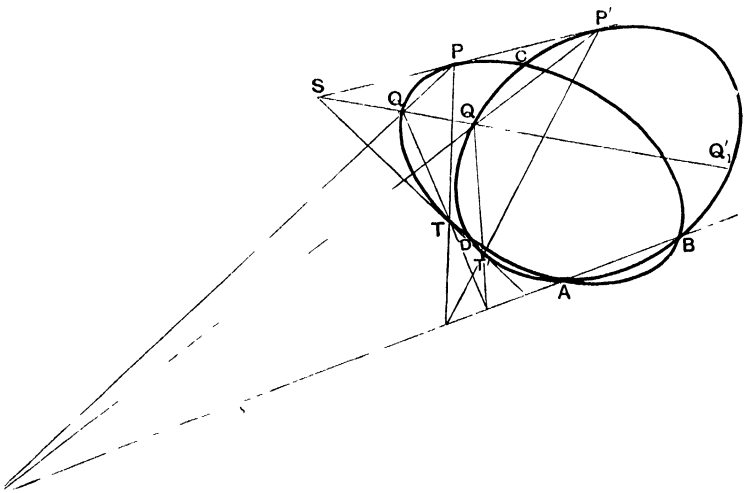
Two points at P correspond to two points at P'

Two points at T correspond to two points at T'

One point at Q corresponds to one point at Q'

Hence, since five points determine a conic, the conics are in perspective.

If Q and Q_1' in the figure are taken as corresponding points the conics are also in perspective, and the axis of perspective is the line joining the points of intersection of the corresponding sides of the triangles PQT and $P'Q_1'T'$. Thus there are two perspectives with S as centre.

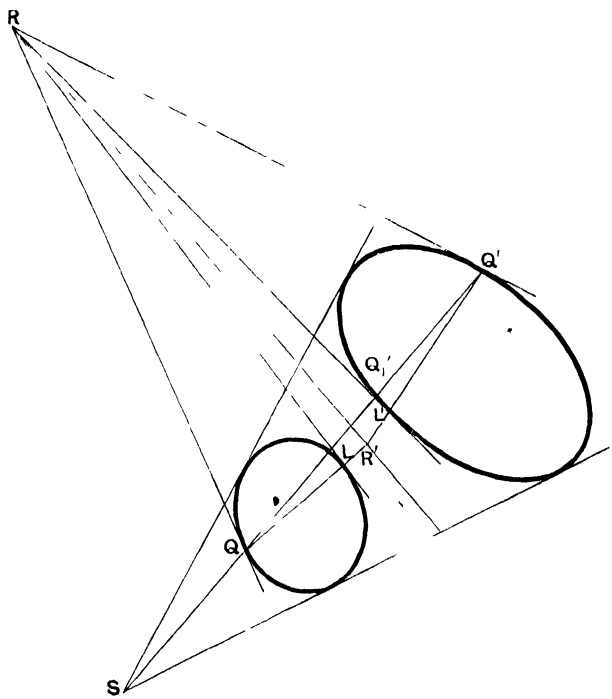


Determination of the axes of perspective

If the conics have four common tangents and intersect in four real points as in the figure, let the point Q correspond to the point Q'. As the point Q' moves along one conic, the point Q will move along the other. When Q' is at D, Q will be at the point where DS meets the other conic. When Q' reaches A, Q will be at the same point. Hence A is a self-corresponding point in the perspective. Similarly B is a self-corresponding point. Therefore s the axis of perspective passes through A and B and is therefore a common chord.

If Q_1' is taken as corresponding to Q, the points D and C are self-corresponding points and DC is the axis of perspective

If the conics do not intersect in real points, let the point Q correspond to the point Q' Then the tangents at Q and Q', being corresponding lines, will intersect at a point R on s Draw the other tangents RL and RL' from R to the two conics These are corresponding lines Therefore the chords of contact QL and $Q'L'$ are also corresponding lines and intersect in R' on s R and R' are a pair of common conjugate points on s with respect to the conics Similarly any number of pairs of common conjugate points may be determined on s Hence the two conics determine on s the same involution, and s is one of the common involution chords of the conics, and passes through two of their imaginary points of intersection



Similarly, if Q, Q_1' be taken as corresponding to each other, a second real chord passing through two other imaginary points of intersection of the conics is obtained as the axis of perspective

Generally there are six points of intersection of the four common tangents and six common chords joining the four points of intersection of the two conics

Corresponding to each of the six centres of perspective there are two axes of perspective, making in all twelve perspectives

Each of the six centres of perspective has two axes of perspective and each of the six axes has two centres of perspective

The chords of contact of the common tangents to two conics pass four by four through the three vertices of their common self-conjugate triangle, that is, through the diagonal points of the quadrangle formed by their four points of intersection

In the figure, page 278, the chords of contact PT, $P'T'$ of the common tangents intersect on the axis of perspective corresponding to S, *i e* on the chord AB, when AB is the axis of perspective They must also intersect on CD, when S and CD are taken as the centre and axis of perspective They, therefore, pass through $CD\ AB$

If the other pair of common tangents are drawn to intersect in S', where S' is the other centre of perspective which has AB and CD for axes of perspective, it follows in a similar way, that the chords of contact of this pair of common tangents pass through the point $CD\ AB$

128 Common inscribed quadrangle and circumscribed quadrilateral of two conics

If A, B, C, D be the points of intersection of two conics (1) and (2) and a, b, c, d their four common tangents, then

(1) *the common self conjugate triangle EFG may be constructed as the diagonal points triangle of the quadrangle $ABCD$ or as the diagonal triangle of the quadrilateral $abcd$,*

(2) *the chords of contact of the common tangents a, b, c, d with the two conics pass four by four through the three vertices E, F, G of the common self-conjugate triangle, and correlatively*

(3) *the points of intersection of the tangents at A, B, C, D lie four by four on the sides e, f, g of the common self conjugate triangle*

A, B, C, D determine a common self-conjugate triangle EFG as their diagonal points triangle, and a, b, c, d determine a common self conjugate triangle efg as their diagonal triangle As there can be only one common self conjugate triangle (Art 124), e, f, g are the sides of EFG

The chords of contact of a, b, c, d with conic (1) pass through the vertices of the diagonal triangle of $abcd$, that is through E, F, G (Art 80 (*a*))

Similarly the chords of contact of a, b, c, d with conic (2) pass through E, F, G

The tangents at A, B, C, D to conic (1) intersect in pairs on the sides of the diagonal points triangle of $ABCD$, i.e. on e, f, g (Art 80 (a))

Similarly the tangents at A, B, C, D to conic (2) intersect in pairs on e, f, g

Since the conics are in self perspective with any vertex of the self conjugate triangle for centre and the opposite side for axis of perspective all the pairs of corresponding points on the sides of the common self-conjugate triangle are harmonic conjugates with respect to the vertices on that side, and form an involution with these points as double points

Let the vertices of $abcd$ on the sides FG, GE and EF of the triangle EFG be E_1, E_2, F_1, F_2, G_1, G_2 respectively

The tangents at the points where a line through G_1 meets the two conics intersect on AD or BC This follows from the fact that the conics are in perspective with each other, with any vertex G_1 of the circumscribed quadrilateral as centre of perspective and either of the common chords through the vertex of the common self-conjugate triangle opposite to the side on which G_1 is situated, as axis

A system of conics with a common self-conjugate triangle

A system of conics with a given common self conjugate triangle EFG is a doubly infinite system For a conic can be described through any two points A and A' to have EFG for a self conjugate triangle From A three points B, C, D can be deduced by harmonic perspective Any conic through $ABCD$ has EFG for a self conjugate triangle Similarly from A' three other points B', C', D' can be deduced by harmonic perspective The two quadrangles $ABCD$ and $A'B'C'D'$ have a common self conjugate triangle EFG and their vertices- Art 126 (a)- -lie on a conic

Similarly a conic can be described to touch any two lines a and a' and have efg for self-conjugate triangle From a and a' two quadrilaterals can be obtained by harmonic perspective and their eight sides a, b, c, d, and a', b', c', d' are all tangents to the conic

Two conics can likewise be described to pass through any point A, to touch any line a and to have EFG for a self conjugate triangle From A three other points B, C, D can be deduced by harmonic perspective Every conic through these points has EFG for common self conjugate triangle Two conics can be described through A, B, C, D to touch the line a (Art 113 (D)) These conics also touch the three lines b, c, d obtained by harmonic perspective from a Such a pair of conics have a, b, c, d for common tangents and A, B, C, D for their points of intersection

If a system of conics has a common self conjugate triangle, any straight line through a vertex of this triangle is cut by the system of conics in pairs of conjugate points of an involution

If a line through a vertex S of the common self conjugate triangle meet the opposite side in N and the conics in pairs of points AA', BB', CC', , then

$$(SNAA') = (SNBB') = (SNCC') \quad -1$$

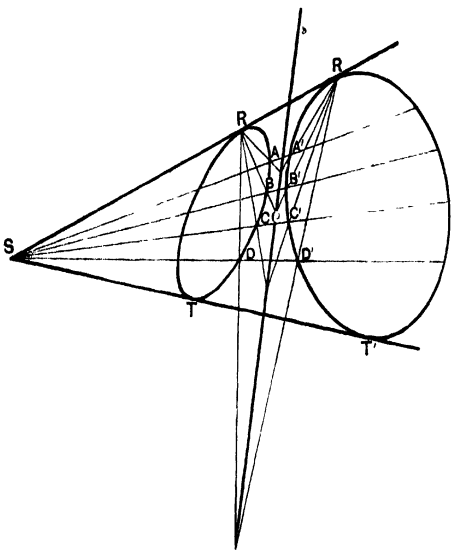
Hence AA', BB', CC', are conjugate points of an involution of which S and N are the double points

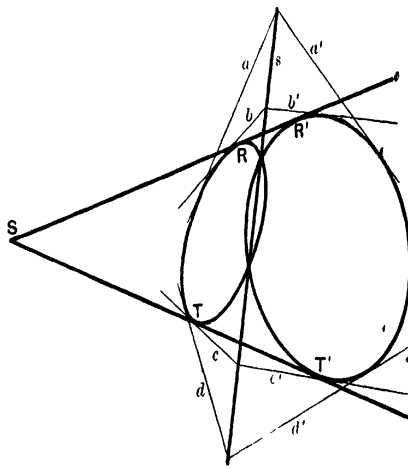
The correlative theorem is as follows

If a system of conics has a common self-conjugate triangle and pairs of tangents are drawn from any point on one of the sides to the system of conics, these pairs of tangents are conjugate lines of an involution pencil

129 *If through a point of intersection of common tangents to two conics a variable chord be drawn to meet one conic in A and the other in A', then the ranges formed by A and A' on the two conics are projective*

If from a variable point on a common chord of two conics a tangent a be drawn to one and a tangent a' to the other, then the systems of tangents to the two conics formed by a and a' are projective





Let the common tangents TT', RR' intersect at S Let SAA', SBB', SCC', SDD' be four positions of the variable chord Join A, B, C, D to R and A', B', C' D' to R'

Then the two conics are in perspective with each other, S being the centre of perspective and the points R, A, B, C, D corresponding to the points R', A', B', C', D' (Art 127)

Therefore the lines RA, RB, RC, RD correspond to $R'A'$,

Let s be a common chord of the two conics and from points on s draw tangents a, b, c, d and a', b', c', d' to the two conics Let S be a point of intersection of common tangents RR' and TT'

Then the two conics are in perspective with each other, s being an axis of perspective and the lines a, b, c, d corresponding to the lines a', b', c', d' (Art 127)

Hence the points of intersection of a, b, c, d with RR' correspond

$R'B'$, $R'C'$, $R'D'$ and the pencils $(R\ ABCD)$ and $(R'\ A'B'C'D')$ are projective

to the points of intersection of a', b', c', d' with RR' Hence since RR' is a tangent to both conics the two systems of tangents are projective

These theorems may also be deduced by conical projection or plane perspective from the corresponding theorems for the circle, which were proved in Art 84 (11) The theorems also hold when S and s are respectively a common involution point and a common involution chord

EXAMPLE

Through the point of intersection of a pair of common tangents to two conics C_1, C_2, a straight line is drawn cutting the conics in Q, R respectively A, B are fixed points on the conics C_1, C_2 respectively Show that the locus of intersection of AQ, BR is one or other of two conics

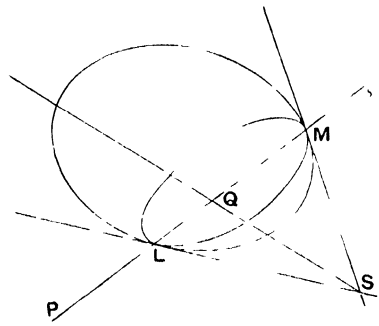
Conics having Double Contact

130 (*a*) *If two conics have double contact, each may be looked upon as its own perspective, the perspective being harmonic, and the centre and axis of perspective either* (1) *the pole of the chord of contact and the chord of contact or* (2) *any point on the chord of contact and its polar*

Case (1) is obvious at once, since the common chord of contact has the same pole with respect to both the conics

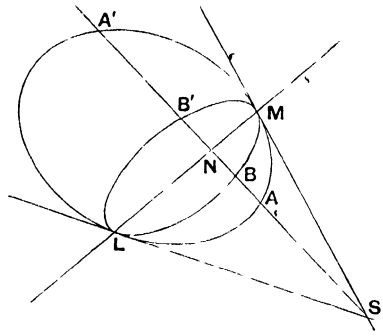
Case (2) follows from the fact that any point on the common chord of contact has the same polar with respect to both conics

Let the two conics touch at L and M and let S be the common pole of LM (s) with respect to the conics Take P any point on s and Q the harmonic conjugate of P with respect to L, M Then Q and S are points on the polar of P with respect to both the conics and therefore QS is the polar of P with respect to both conics



(*b*) *If two conics have double contact each may be looked upon as the perspective of the other, the centre and axis of perspective being the pole of the chord of contact and the chord of contact*

Let the two conics touch at L and M and let S be the pole of LM (s). Through S draw a line to meet one conic in A and A', the other in B and B', and the chord of contact LM in N.

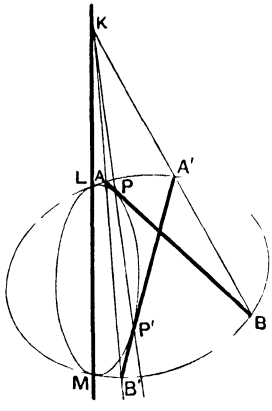


Take S and s as centre and axis of perspective and $(SNAB)$ as the anharmonic ratio of the perspective.

Then two points at M on one conic correspond to two points at M on the other, two points at L on one conic correspond to two points at L on the other, and one point at A on one conic corresponds to one point at B on the other. Hence the two conics are in perspective.

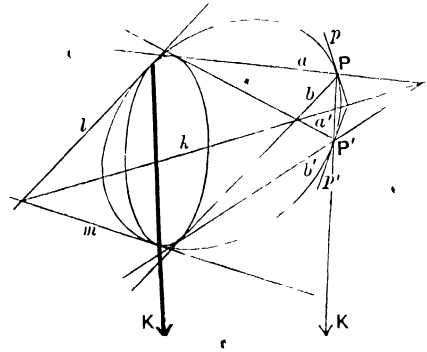
Similarly it may be proved that the conics are in perspective with M and SL, or L and SM as centre and axis of perspective.

131 *If two conics have double contact at L and M, and tangents at P and P' to one meet the other in A, B and A', B', then a pair of lines AB', BA' are concurrent with PP' at a point on LM.*



Let PP' meet LM in K. Then both the conics may be looked

If two conics have double contact, the common tangents being l and m, and the tangents from points P and P' on one conic to the other be a, b and a', b', then a pair of points ab', ba' and the point of intersection of the tangents at P and P' are collinear with lm.



Let the connector of lm and pp' be k, where p and p' are the

upon as in self-perspective with K as centre of perspective and its polar as axis (Art 130 (a))

To the tangent at P will correspond the tangent at P' Therefore to A and B will correspond B' and A' Therefore AB' and BA' are collinear with K

If two conics have double contact and any tangent to one of them at P meets the other at A and B, then the ranges described by P, A, B on the two conics are projective

If in the figure the points A, P, B are kept fixed, and $A'P'B'$ is variable the ranges described by A', P', B' on the conics are projective with the range described by K on LM and therefore with each other

The points of contact L and M of the conics are the self-corresponding points of the ranges described by A' and B'

Conversely

The lines joining corresponding points of two projective ranges on a conic envelope a conic having double contact with the given conic

Construct, Art 109, the self-corresponding points L and M of the two projective ranges

tangents at P and P' If PP' meets the common chord of contact at K, K is the pole of k with respect to both conics Take K and k as centre and axis of harmonic perspective

To the tangent at P corresponds the tangent at P' and to a and b correspond b' and a' Hence ba', ab', lm, pp' all lie on k

If two conics have double contact and the tangents to one from the point of contact of a tangent p to the other be a and b, then the systems of tangents p, a, b to the two conics are projective

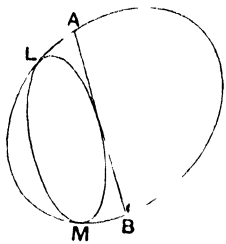
If in the figure the tangents a, p, b are kept fixed, and $a'p'b'$ is variable, the systems of tangents a', p', b' are projective with the pencil described by the line k and therefore with each other

The tangents l and m at the points of contact of the conics are the self-corresponding tangents of the two projective systems of tangents described by a' and b'

The locus of the points of intersection of corresponding tangents of two projective systems of tangents to a conic is a conic having double contact with the given conic

Construct, Art 109, the self- tangents l and m of the two projective systems of tangents

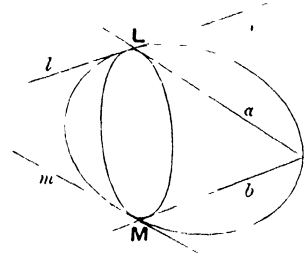
Draw the tangents l and m at L and M and describe a conic to touch l and m at L and M and to touch the line AB joining a pair of corresponding points on the conic (Art 113 (*C*))



By the last Article any tangent to this conic will meet the given conic in a pair of corresponding points of the two projective ranges determined by L and M as self-corresponding points and A and B as a pair of corresponding points

This conic is therefore the envelope of the chords joining corresponding points of the given projective ranges

Construct the points of contact L and M of the tangents l and m and describe a conic to touch l and m at L and M and pass through ab the point of intersection of a pair of corresponding tangents (Art 113 (*C*))



By the last Article any pair of tangents from a point on this conic will form a pair of corresponding tangents of the two systems of projective tangents determined by l and m as self-corresponding tangents and a and b as a pair of corresponding tangents

This conic is therefore the locus of the points of intersection of corresponding tangents of the two projective systems of tangents

The correlative of Desargues' Theorem for a system of conics with double contact is as follows

If a system of conics have double contact at L and M and E be the common pole of LM, then the pairs of tangents from any point P to the conics of the system are pairs of conjugate rays of an involution, of which PE is a double ray and of which PL and PM are a pair of conjugate rays

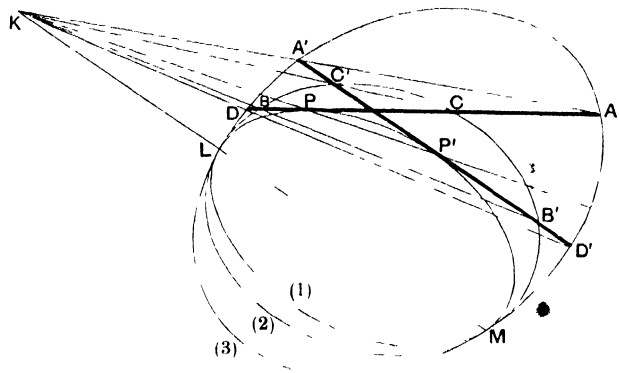
The other double ray of the involution is the tangent at P to the conic of the system which can be described through P (Art 113 (*C*))

From Art 125 the following property of two conics having double contact is obtained

If two conics touch at L and M and these points are joined to any two points A and B on the one conic by lines, which meet the other conic in A', B, and A'', B', respectively, then $A'B''$ and $A''B'$ intersect on the common chord of contact of the conics

13. *Given three conics all touching each other at the same two points, any tangent to one is cut by the other two in four points whose anharmonic ratio is constant*

(*a*) Let the three conics (1), (2), (3) touch at L and M, and let the tangents at P and P' to (1) meet (2) and (3) in A, C, B, D and A', C'', B', D'



Then AA' and DD' pass through the point K where PP' meets LM Similarly CC'' and BB' pass through K

Hence the ranges $ACBD$ and $A'C'B'D'$ are in perspective from K and therefore their anharmonic ratios are equal

The correlative theorem is

Given three conics all touching each other at the same two points, the two pairs of tangents from any point on one to the other two form a pencil of constant anharmonic ratio

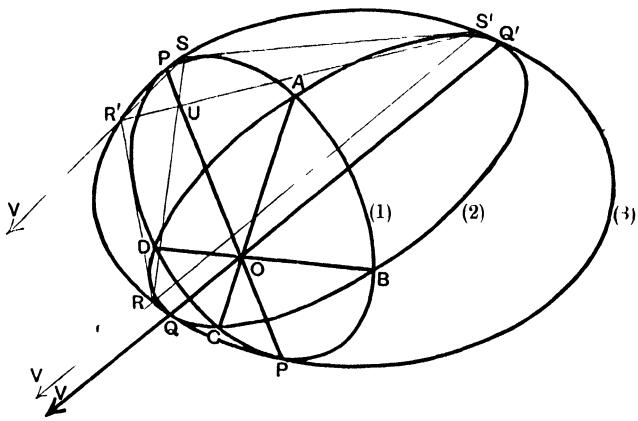
In most theorems concerning two or more conics having double contact at L and M, the pair of tangents at L and M may be substituted for one of the conics having double contact In the correlative theorems the points of contact L and M may themselves be looked upon as a conic of the system having double contact at L and M

(*b*) *If two conics* (1) *and* (2) *have each double contact with a third conic* (3), *the chords of contact of* (1) *and* (2) *with* (3) *and two of the chords of intersection of* (1) *and* (2) *meet in a point and form a harmonic pencil*

Let the points of intersection of the conics (1) and (2) be A, B, C, D and let PP' and QQ' be their chords of contact with (3) Let PP', QQ meet at O

Since (3) and (1) have double contact at P and P', every point on PP' has the same polar with respect to the two conics Similarly every point on QQ' has the

same polar with respect to (3) and (2) Therefore O the point of intersection of PP', QQ' has the same polar with respect to all three conics Therefore all the conics are in self-perspective with O as centre and the polar of O as axis of perspective But the points of intersection of corresponding curves are corresponding points, and, therefore, BD and AC' must pass through O



Draw a pair of common tangents to (1) and (2) to meet (3) in R, R' and S, S' Then (Art 131) SR and $S'R'$ will meet in a point U on OP and SR' and $S'R$ in a point V on OQ

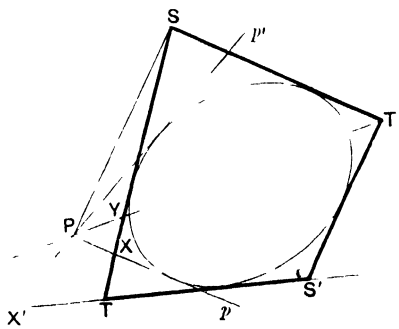
By construction U and V are conjugate points with respect to (3)

But U being on POP' has the same polar with respect to (1) and (3),

and V being on QOQ has the same polar with respect to (2) and (3)

Hence U and V are common conjugate points with respect to (1) and (2) Hence they are conjugate points with respect to all conics through A, B, C, D Hence they are conjugate points with respect to AC' and BD Therefore the pencil $(O\ ADUV)$ is harmonic

(c) *If a quadrangle $STS'T'$ be circumscribed to a conic and on any tangent p to the conic a point P be taken such that PS is the harmonic conjugate of p with regard to PT and PT', then the locus of P is a conic having double contact with the given conic*



Let p' be the second tangent from P to the conic Let p meet TS in A and let p' meet TS' in A' Join PT' to meet TS in Y and TS' in Y'

Then $(P\ TT'AS)=(P\ T'TA'S')$ by the correlative to Desargues' theorem Hence, since the first of these pencils is harmonic, the second is also harmonic, therefore $(TYYS)$ and $(Y'TX'S)$ are harmonic

Therefore since S and T are fixed the ranges Y and X are projective,

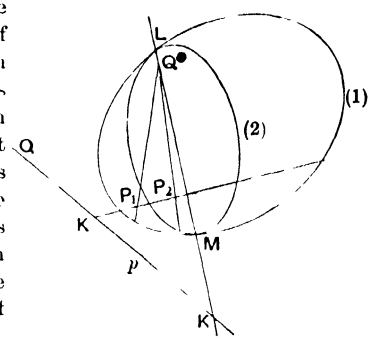
and since S' and T are fixed the ranges Y' and X' are projective

But since YY' passes through T', the ranges Y and Y' are projective

Therefore the ranges X and X' are projective and the tangents from P to the conic form two projective systems of tangents Therefore the locus of P is a conic having double contact with the given conic (Art 131)

(*d*) *To find the locus of common conjugates of points situated on a given straight line with respect to two conics having double contact*

Let the two conics (1) and (2) have double contact at L and M Consider the locus of common conjugates of points on a line p which meets LM in K Let P_1 and P_2 be the poles of p with respect to (1) and (2) They lie on the common polar of K Take Q any point on p Its polar with respect to (1) passes through P_1 and meets LM in a point Q' Since Q' is on LM its polar with respect to (2) passes through Q Therefore the polar of Q with respect to (2) passes through Q and is the line P_2Q Hence LM is part of the locus of common conjugates of points on p



Again, P_1P_2 is the polar of K with respect to (1) and (2) Hence the polars of every point of P_1P_2 with respect to (1) and (2) pass through K

Hence P_1P_2 the polar of K is part of the locus which consists of LM and P_1P_2 The above may be stated as follows

The polars of any point with respect to two conics which have double contact intersect on their chord of contact

(*e*) *To find the locus of the common conjugates of three conics two of which have double contact with the third*

Let the conics (1) and (3) have double contact at L and M

Let the conics (2) and (3) have double contact at L and M'

Consider a straight line p which meets LM in A and $L'M'$ in B

Let the common polar of A with respect to (1) and (3) meet ML in A' and $M'L'$ in A''

Let the common polar of B with respect to (2) and (3) meet ML in B'' and ML' in B'

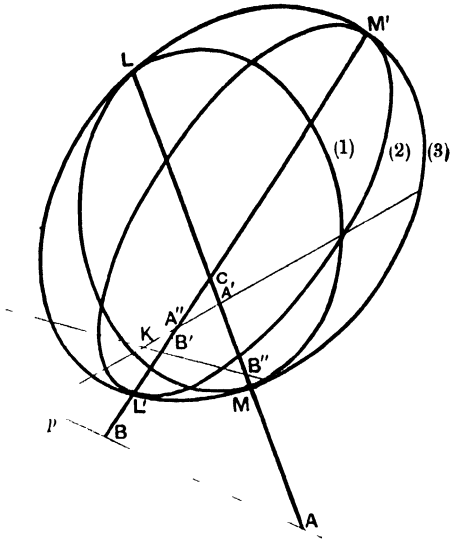
The locus of common conjugates of points on p with respect to (1) and (3) is LM and $A'A''$

The locus of common conjugates of points on p with respect to (2) and (3) is $L'M'$ and $B'B''$

These loci intersect in $C(LM\ L'M')$, $K(A'A''\ B'B'')$ and A'', B'' The point K is the pole of p with respect to (3) and is not a point on the locus

A'', B'' are common conjugates of A and B and are points on the locus For different positions of the line p, the loci of A'' and B'' are the lines LM and $L'M'$

C is a fixed point It has the same polar with respect to the three conics The polars of every point on its polar pass through C Hence the polar of C is part of the locus The locus, therefore, consists of three straight lines, viz LM, $L'M'$ and the common polar of C



EXAMPLES

(1) Two fixed conics touch at A and B LM is a chord of one touching the other at Q Prove that the line joining Q to the intersection of AM and BL passes through a fixed point

Let LM meet AB in Q' and let AM BL be P Then the conics are in self-perspective with Q' as centre of perspective and its polar QP as axis In this perspective MA and BL correspond as do the tangents at A and B Therefore PQ must pass through their point of intersection which is a fixed point

(2) Through a fixed point O is drawn a chord OPP' of a given conic, and on it is taken a point Q such that the range $OP'QP'$ has a constant anharmonic ratio Prove that the locus of Q is part of a conic having double contact with the given conic, and that if Q' be the point in which OP meets the other part of the conic, the product $(OQPP')(OQ'PP')$ is unity See Art 118 3

(3) A variable tangent to a parabola meets two fixed tangents in P, Q, and the diameters through P, Q meet the curve in L, M Determine the envelope of LM

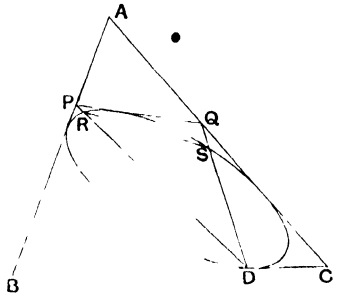
P and Q are two projective ranges on the fixed tangents $P\infty$, $Q\infty$ are two projective pencils, where ∞ is the point at infinity on the axis of the parabola

Therefore L, M are two projective ranges on the conic Therefore the envelope of LM is a conic which may be shown to be a parabola

(4) The triangle PQR is inscribed in a given conic If the directions of PQ and PR are given, show that QR envelopes a conic having the same asymptotes as the given conic

By Art 131 the envelope of QR is a conic having double contact with the given hyperbola The line QR may be constructed by taking any point P on the hyperbola and drawing lines in fixed directions to meet the curve in Q and R As P moves to infinity in the direction of one asymptote the points Q and R move to infinity in the direction of the other asymptote and at last coincide in a double point which is at infinity Hence the envelope has double contact with the hyperbola at infinity and has therefore the same asymptotes

(5) A fixed conic inscribed in the triangle ABC touches the side BC in D and P, Q are the points in which any variable tangent to the conic cuts AB, AC Prove that if DP, DQ cut the conic in R, S, the envelope of RS is a conic and find where it meets the given conic



The ranges described by P and Q are projective Therefore the ranges described by R and S are projective and the envelope of RS is a conic The self-corresponding points of the ranges coincide at D, at which point the conics have four point contact

(6) If a polygon be inscribed in a conic so that all its sides but one pass through fixed points, then the envelope of that side is a conic having double contact with the given conic (Art 131)

(7) If a polygon be circumscribed to a conic so that all its vertices but one lie on fixed lines, then the locus of that vertex is a conic having double contact with the given conic (Correlative of Ex (6))

(8) Two systems of conics have double contact with a given conic at pairs of given points Prove that one system of their chords of intersection forms an involution pencil

(9) Show that, in general, three systems of conics may be described to have double contact with each of two given conics and that the chords of contact of the conics of each system form an involution pencil with a vertex at one of the vertices of the common self-conjugate triangle of the given conics

(10) The chords of intersection of two conics of the same system, which have double contact with two given conics, pass through a point of intersection of common chords of these conics

(11) The tangents from any point P to a conic are p, p' and its connectors to two fixed points L, M on the conic are l, m If $(pp'lm)$ is constant, prove that the locus of P is a conic having double contact at L and M

(12) Three conics have double contact at L and M The tangents from any point P on one to the other two are aa', bb' and the connectors of P to L, M are l, m Prove that $(ablm) = K\ (a'b'lm)$, where K is constant

(13) A conic can be constructed to touch three given straight lines and have double contact with a given conic

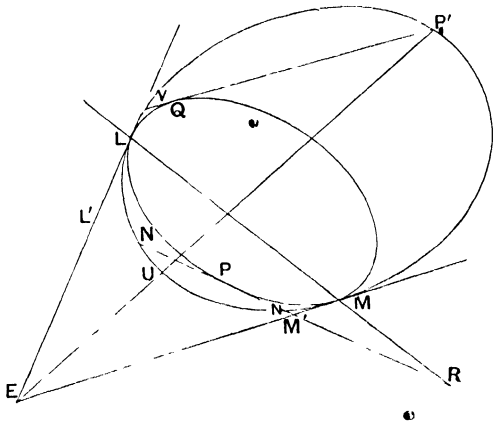
Let the lines a, b, c meet the conic in AA', BB', CC' Construct the self-corresponding elements L and M of the ranges ABC and $A'B'C'$ on the conic Then a conic touching the given conic at L and M and the line a will be the required conic For this conic touches the lines joining each pair of corresponding points of the ranges determined by L and M as self-corresponding points and A and A' as corresponding points

One conic can be described to touch two given lines at given points and to touch another given line (By particular case of Brianchon's theorem) There are, however, four ways of arranging the pairs of points AA', BB', CC' so as to determine different projective ranges on the conic, and there are therefore four solutions of the problem

(14) The points of contact of the tangents drawn from three points A, B, C to a conic S are the three pairs of points A_1A_2, B_1B_2, C_1C_2 respectively Show that the four chords of contact of the four conics that can be drawn through A, B, C to have double contact with S are the Pascal lines of the four hexagons

$$A_1B_1C_1\,A_2B_2C_2, \qquad A_1B_1C_2\,A_2B_2C_1,$$
$$A_1B_2C_1\,A_2B_1C_2, \qquad A_2B_1C_1\,A_1B_2C_2$$

(15) Two conics have double contact at L and M and E is the pole of LM (1) If any tangent to one at P meet the other in N and N' $(LMNN') = (E\ LMPN)$, and (2) if a tangent be drawn from any point P' on one to touch the other at Q, $(P'\ LMEQ)$ is constant



(1) Let NPN' meet EL, EM and LM in L', M', R Then
$(LMNN') = (L'RNN')$ and $(E\ LMPN') = (L'M'PN')$

It is necessary therefore to prove that

$$(L'N'RN)=(L'N'M'P)$$

But P and R are the double points of an involution of which N, N', L', M' are two pairs of conjugate points and therefore (Ex 5, Chapter VIII) this relation holds

(2) If $P'Q$ and PE meet the conic, which passes through P', in V and U, $(P'\ LMEQ)=(LMUV)$ and $(LMUP')=-1$ Therefore $(P'\ LMEQ)=-\frac{(LMUV)}{(LMUP')}$ $=-(LMP'V)$ This by Art 131 is constant

The Harmonic Locus of Two Conics

133 The following theorems are of importance in connexion with the theory of two conics

The envelope of a line, which meets two conics in two pairs of points which are harmonic conjugates, is a conic which touches the eight tangents which can be drawn to the two conics at their points of intersection and has the common self-conjugate triangle of the two conics for a self-conjugate triangle

The locus of a point, the two pairs of tangents from which to two conics are harmonic conjugates, is a conic, which passes through the eight points of contact of the common tangents to the two conics and has the common self-conjugate triangle of the two conics for a self-conjugate triangle

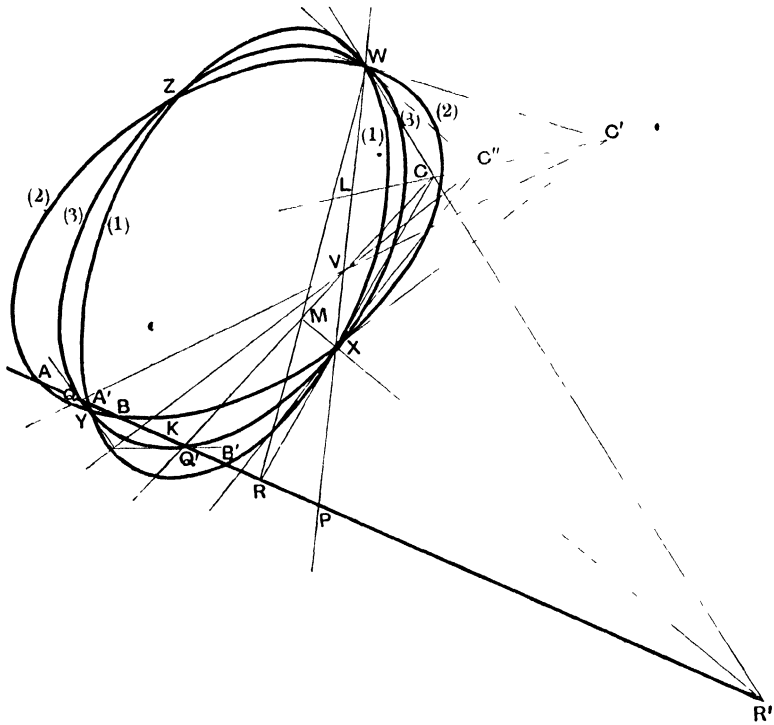
Conversely

If a conic be drawn to touch the eight tangents which can be drawn to two conics at their four points of intersection, any tangent to this conic is cut by the two conics in two pairs of points which are harmonic conjugates

If a conic be drawn through the eight points of contact of the four common tangents to two conics, the two pairs of tangents which can be drawn to the two conics from any point on this conic are pairs of harmonic conjugates

Let the two given conics (1) and (2) intersect in W, X, Y, Z Draw the chord WX and let the tangents to (1) at W and X intersect in C and those to (2) in C' Join CC' meeting XW in L On this line take C'' the harmonic of L with regard to $C'C$ Join WC'', XC'' and describe a conic through W, X, Y, Z to touch WC'' and XC'' at W and X respectively (Ex 5, Art 125) Call this conic (3) Take any point V on WX and let CV and $C'V$ meet (3) in Q' and Q Join QQ' to meet WX in P and $C''V$ in K Then the range $QQ'KP$ is harmonic

and, since the tangents at X and W to (3) meet at C'', $C''VK$ is the polar of P with respect to (3) Therefore $WXVP$ is harmonic and CVQ' and $C\,VQ$ are the polars of P with regard to (1) and (2) respectively



Let QQ' meet (2) in A and B and (1) in A', B' Then since the conics (1), (2), (3) pass through four fixed points W, X, Y, Z, the pairs of points AB, $A'B'$, QQ' are pairs of conjugate points of an involution Also Q and Q' are harmonic conjugates of P with respect to AB and $A'B'$ respectively Hence, by Ex 9 (*b*), Chapter VIII, A, B are harmonic conjugates of A', B'

Every chord cut harmonically may thus be constructed by taking different positions of V on the line WX, and the locus of Q and Q' is the conic (3) which passes through W, X, Y, Z and touches $C''X$ and $C''W$ at X and W

Consider the tangents CW, CX to the conic (1) Let CX and CW meet QQ' in R and R' Since the pencil (C' $WXVP$) is harmonic, $R'X$ and RW will intersect in some point M on CV, and, from the harmonic

property of the quadrangle $WCXM$, C, M will be harmonic [illegible] of Q, V Now for different positions of V on WX, C is fixed, and Q' describes a conic through W and X, therefore M will describe a conic through W and X (Ex 2, Art 118) Hence the pencils WM and XM are projective Therefore the ranges described by R and R' on CX and CW are projective Therefore RR' will envelope a conic which touches CW and CX

By symmetry the conic also touches the tangents WC' and XC' to (2) and likewise the tangents to (1) and (2) at Y and Z Since the conic touches the tangents at X, Y, Z, W to (1) and (2), it has the same self-conjugate triangle as (1) and (2)

The correlative theorem may be proved in a similar way

The converse theorems are obviously true as only one conic can be described to touch the eight tangents which can be drawn to two conics at their points of intersection and only one conic can be drawn through the eight points of contact of their common tangents

Alternative Proofs

(*a*) By Projection or Plane Perspective

If it be assumed that by an imaginary projection X and W, two of the points of intersection of the conics, may be projected into the circular points at infinity, the poles C and C' of XW with respect to the two conics become the centres of the circles into which the conics are projected The envelope touches WC, WC', XC, XC' Therefore, if W and X are the circular points at infinity, C and C' must become the foci of the envelope Hence the theorem becomes the following

The envelope of a chord which is cut by two circles in pairs of harmonic conjugates is a conic having the centres of the circles for foci

This follows from Addendum 14 by the converse of Art 139 (7)

(*b*) It may easily be proved that the locus is of such a nature that two tangents may be drawn to it from any point Hence there is reason to suppose that it is a conic (See Art 142)

134 If in the theorem of Art 133 a pair of straight lines be substituted for one of the conics and in the correlative one of the conics becomes a pair of points, the following theorems are obtained

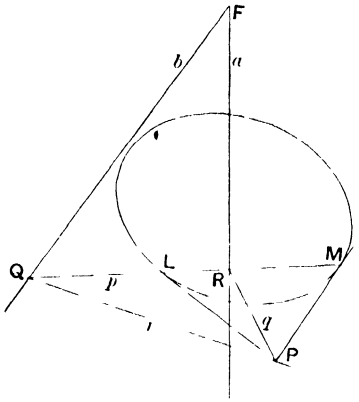
The envelope of a line, such that its points of intersection with a given conic are harmonic conjugates of its intersections with a pair of given lines, is a conic which touches the pair of given lines

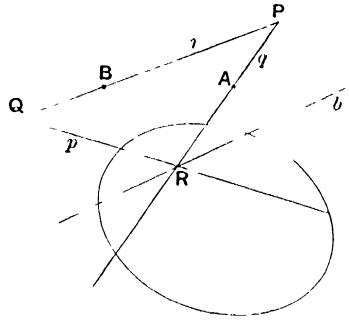
The locus of a point, such that the tangents from it to a given conic are harmonic [illegible] of its connectors with a pair of given points, is a conic which passes through the pair of given points

These theorems may be proved independently as follows

Let a and b be the given lines and p a tangent to the required envelope Then pa, pb (R and Q) are conjugate points with respect to the conic Hence if P, the pole of p, be joined to Q and R by r and q, pqr is a self-conjugate triangle

Let A and B be the given points and P a point on the required locus Then PA, PB (q and r) are conjugate lines with respect to the conic Hence, if the polar of P meets r and q in Q and R, PQR is a self-conjugate triangle





Hence r is the polar of R, and the range described by R on the fixed line a is projective with the pencil described by r through A the pole of a, and therefore with the range described by Q on b Hence the envelope of p is a conic touching a and b

Hence R is the pole of r, and the pencil formed by r, which passes through B, is projective with the range described by R on b the polar of B, and therefore with the pencil q through A Hence the locus of P is a conic through A and B

The right-hand side of the above may also be stated as follows

The locus of the points of intersection of tangents to a given conic from pairs of conjugate points of an involution is a conic, which passes through the double points of the involution

Particular Cases

If the lines a and b in the left-hand figure are the connectors of a point F to the circular points at infinity, the chord of intersection LM

of p with the conic will subtend a right angle at F, which will be a focus of the envelope Hence the following is obtained

The envelope of a chord of a conic which subtends a right angle at a fixed point is a conic having the fixed point for a focus

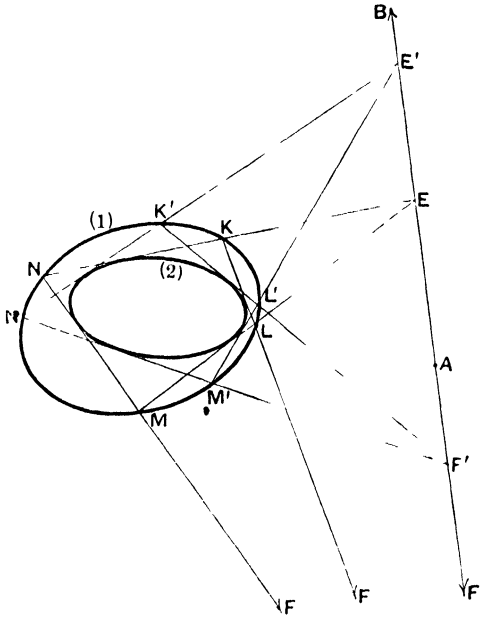
If the points A and B in the right-hand side figure are taken for the circular points at infinity the following is obtained

The locus of the points of intersection of tangents to a conic which are at right angles is a circle

If a be the polar of A, the connectors of ab to A and B are the tangents at A and B to the locus

From the preceding the following important theorem may be deduced

If a simple quadrilateral can be inscribed in one conic and circumscribed to another, an infinite number of quadrilaterals can be similarly constructed



Let $KLMN$ be the simple quadrilateral and let the two remaining points of intersection of the sides be E and F

From any point K' on the conic (1) draw a pair of tangents to (2) to meet EF in E', F' Take A and B the common harmonic [illegible]

of EF and $E'F'$ Then the lines joining A, B to K, L, M, N, K' are harmonic conjugates of the tangents from these points to (2) Hence this holds for every point on (1) Hence, if $K'F'$ meets (1) in L', the other tangent from L' passes through E' If this tangent meets (1) in M', the other tangent from M' to (2) passes through F' Let $E'K'$ and $F'M'$ meet at N' Then $N'E'$ and $N'F'$ are harmonic conjugates of $N'A$ and $N'B$ Therefore N' is on (1) Hence a quadrangle has been inscribed in (1) and circumscribed to (2)

If one of the quadrilaterals $KLMN$ is a parallelogram, all the other quadrilaterals will be parallelograms

135 (*a*) *If five conics be described to touch a given line* 0 *and four out of five given lines* 1, 2, 3, 4, 5 *and* Q_1, Q_2, Q_3, Q_4, Q_5 *be the points of contact with* 0 *of the conics which do not touch the lines* 1, 2, 3, 4, 5 *respectively, and* P_1, P_2, P_3, P_4, P_5 *be the points in which* 1, 2, 3, 4, 5 *meet* 0, *then* P_1Q_1, P_2Q_2, P_3Q_3, P_4Q_4, P_5Q_5 *are pairs of conjugate points of an involution**

Consider the conic (1) which touches 0, 2, 3, 4, 5

Let 5 meet 0, 1, 2, 3, 4 in F_0, F_1, F_2, F_3, F_4 respectively

Then from the conic (1) $(0\ 0234)=(5\ 0234)$,

therefore $$(Q_1P_2P_3P_4)=(F_0F_2F_3F_4) \tag{1},$$

similarly from the conic (2) $$(Q_2P_1P_3P_4)=(F_0F_1F_3F_4) \tag{2},$$

and from the conic (3) $$(Q_3P_1P_2P_4)=(F_0F_1F_2F_4) \tag{3}$$

From (1) and (2) $$\frac{(Q_1P_2P_3P_4)}{(Q_2P_1P_3P_4)}=\frac{(F_0F_2F_3F_4)}{(F_0F_1F_3F_4)},$$

$$(Q_1Q_2P_3P_4)(P_1P_2P_3P_4)=(F_1F_2F_3F_4) \tag{i}$$

Similarly from (1) and (3)

$$(Q_1Q_3P_2P_4)(P_1P_3P_2P_4)=(F_1F_3F_2F_4) \tag{ii},$$

and from (2) and (3) $$(Q_2Q_3P_1P_4)(P_2P_3P_1P_4)=(F_2F_3F_1F_4) \tag{iii}$$

But $$(F_2F_3F_1F_4)=-\frac{(F_1F_3F_2F_4)}{(F_1F_2F_3F_4)}$$

Therefore dividing (ii) by (i)

$$\frac{(Q_1Q_3P_2P_4)(P_1P_3P_2P_4)}{(Q_1Q_2P_3P_4)(P_1P_2P_3P_4)}=\frac{(F_1F_3F_2F_4)}{(F_1F_2F_3F_4)},$$

and $$\frac{(Q_1Q_3P_2P_4)}{(Q_1Q_2P_3P_4)}(P_2P_3P_1P_4)=(F_2F_3F_1F_4),$$

from (iii) $$(Q_1Q_3P_2P_4)=(Q_1Q_2P_3P_4)(Q_2Q_3P_1P_4),$$

$$\frac{Q_1P_2}{Q_3P_2}\ \frac{Q_2P_3}{Q_1P_3}\ \frac{Q_3P_1}{Q_2P_1}=1$$

* I am indebted to Mr S G Soal of the East London College for this theorem and the proof.

Hence (Art 53) Q_1P_1, Q_2P_2, Q_3P_3 are pairs of conjugate points of an involution

Similarly, Q_1P_1, Q_2P_2, Q_4P_4 are pairs of conjugate points of an involution, which must be the same involution

Similarly, by considering the lines 3 or 4 in place of 5, it is seen that Q_1P_1, Q_2P_2, Q_5P_5 are pairs of conjugate points of an involution

Therefore P_1Q_1, P_2Q_2, P_3Q_3, P_4Q_4, P_5Q_5 are five pairs of conjugate points of an involution

The correlative theorem is as follows

If five conics be described through a given point O and through four out of five given points 1, 2, 3, 4, 5 *and* q_1, q_2, q_3, q_4, q_5 *be the tangents at O to the conics which do not pass through* 1, 2, 3, 4, 5 *respectively, and* p_1, p_2, p_3, p_4, p_5 *be the connectors of O to the points* 1, 2, 3, 4, 5 *respectively, then* p_1q_1, p_2q_2, p_3q_3, p_4q_4, p_5q_5 *are pairs of conjugate rays of an involution*

(*b*) *The six polars of a given point with respect to the six conics which pass through five out of six given points touch a conic*

Consider the polars of the point O with respect to the six conics which pass through five out of the six points 1, 2, 3, 4, 5, 6

Let p_1, p_2, p_3, p_4, p_5, p_6 denote the polars of O with respect to the conics which do not pass through the points 1, 2, 3, 4, 5, 6 respectively

Consider the intersections of p_2, p_3, p_4, p_5 with p_1 and p_6

The conics whose polars are p_1 and p_2 have four points common, viz the points 3, 4, 5, 6 The polars of O with respect to all conics through 3, 4, 5, 6 will pass through the point of intersection of p_1 and p_2 which is the common conjugate of O with regard to all such conics (Art 117) The conic through O and 3, 4, 5, 6 has the tangent at O for the polar of O This tangent consequently passes through the point of intersection of p_1 and p_2

Similarly the points p_1p_3, p_1p_4, p_1p_5 are on the tangents at O to the conics which pass through O and through four of the six points other than 1, 3, 1, 4, 1, 5 respectively

Similarly the intersections of p_2, p_3, p_4, p_5 with p_6 lie on the tangents at O to the four conics which pass through O and through the six points other than 6, 2, 6, 3, 6, 4, 6, 5 respectively

But by the correlative of (*a*), the pencils formed by these two sets of four tangents are projective with the pencil formed by joining O to 2, 3, 4, 5 and therefore with each other Hence the ranges on p_1 and p_6 are projective and by the converse of the anharmonic property of tangents to a conic the lines p_1, p_2, p_3, p_4, p_5, p_6 all touch a conic

The correlative is as follows

The poles of a fixed line with respect to six conics which touch five out of six given lines lie on a conic

As a particular case, the following theorem is obtained

The centres of the six conics which touch five out of six given straight lines lie on a conic (Mr H M Taylor)

This may be proved directly as follows

Denote the sides of the hexagon by a, b, c, d, e, f, the centre of the conic touching b, c d, e, f by C_a and the line joining the middle points of the diagonals of c, d, e, f by L_{ab}

Then (Art 117 5) since L_{ab} is the locus of the centres of conics which touch c, d, e, f, C_a and C_b lie on L_{ab} and similarly C_d, C_e lie on L_{de}

Hence, by the converse of Pascal's Theorem, the condition that C_a, C_b, C_c, C_d, C_e, C_f should lie on a conic is that L_{ab} L_{de}, L_{bc} L_{ef}, L_{cd} L_{fa} should be collinear

This is the case by Art 68 (*c*)

EXAMPLES

(1) The envelope of a chord which meets a conic in a pair of points, which with its points of intersection with two fixed tangents to the conic form a range of constant anharmonic ratio, is a conic having double contact with the given conic at the points where it touches the fixed tangents

This is the converse of Art 132 (*a*) when one of the conics becomes a pair of tangents to the conic

(2) The locus of the point of intersection of two tangents to a conic, which form a pencil of constant anharmonic ratio with the connectors of the point with two given points on the conic, is a conic having double contact with the conic at the two fixed points

This is the correlative of the preceding

(3) The locus of the point of intersection of two tangents to a conic which form a pencil of constant anharmonic ratio with the connectors of the point to two fixed points on any fixed tangent to the conic is a conic having double contact with the given conic

The tangents determine on the fixed tangents two points which with the two given points on the tangent form a range of constant anharmonic ratio Therefore the points where the two tangents meet the fixed tangent form two projective ranges Hence the tangents from these points form two projective systems of tangents, which (Art 131) intersect on a conic having double contact with the given conic

(4) The envelope of a line which meets a conic in a pair of points, which form a range of constant anharmonic ratio with its points of intersection with two fixed lines which intersect on the conic, is a conic having double contact with the given conic

The correlative of the last theorem

(5) a and b are two lines conjugate with regard to a conic S Prove that, if the points of intersection of a transversal p with S and with a and b form a harmonic range, p must pass through the pole of a or of b with respect to S

(6) If triangles be constructed such that they are self conjugate with regard to one conic and two sides touch another conic, then the third side will envelope a conic

(7) If a conic (1) be circumscribed to a quadrangle $ABCD$ and a conic (2) be inscribed in the quadrangle, and a tangent be drawn to (2) to meet (1) in E and F, then the other tangents from E and F to (2) will intersect on the line joining AB CD to BC AD and their chord of contact will pass through BD AC

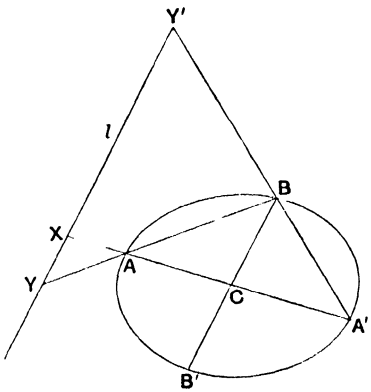
(8) If any tangent to a conic (1) meets a second conic (2), with which it has double contact at L and M, in K and N', and K and N' be joined to S any fixed point on LM, by lines which meet the conic (2) again in K' and N, then the lines KN and $K'N'$ meet LM in two fixed points A and A', which are such that the tangents from A and A' to (2) intersect where the polar of S meets (2)

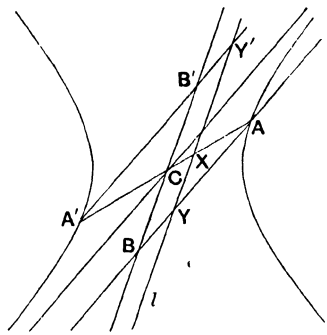
This may be proved by means of the theorems given in Arts 130 132

CHAPTER XXI

THE CONSTANT OF AN INVOLUTION FOCAL PROPERTIES AND CONFOCAL CONICS GEOMETRICAL CORRESPONDENCE CHARACTERISTICS OF SYSTEMS OF CONICS STEINER'S AND KIRKMAN'S THEOREMS

136 To find the constant of the involution determined on any straight line by a central conic





(1) Let the line l meet the central conic in imaginary points

Let C be the centre of the conic Draw BCB' parallel to l, and let the diameter conjugate to BCB' meet the conic in A and A' and l in X

Since l passes through the pole of AA', the pole of l is on AA'

Hence the lines joining A and A' to points on the curve determine conjugate points on l (Art

Let C be the centre of the conic Draw BCB' parallel to l, and let the diameter conjugate to BCB' meet the conic in A and A' and l in X

Since l passes through the pole of AA', the pole of l is on AA'

Hence the lines joining A and A' to points on the curve determine conjugate points on l (Art

95 (j)), and X is the centre of the involution on l

Join A and A' to B to meet l in Y and Y'

Then

$$XY = XA \frac{CB}{CA},$$

$$XY' = XA' \frac{CB}{CA'},$$

$$XY\ XY' = -XA\ XA' \frac{CB^2}{CA^2}$$

95 (j)) and X is the centre of the involution on l

Join A and A' to one of the points at infinity on the curve by lines which meet BCB' in B and B' and l in Y and Y'

Then

$$XY = XA \frac{CB}{CA},$$

$$XY' = XA' \frac{CB'}{CA'},$$

$$XY\ XY' = XA\ XA' \frac{CB^2}{CA^2}$$

If in the proof for the hyperbola l be moved parallel to itself, $\frac{CB^2}{CA^2}$ is constant, and it follows that CB is the distance from C of the two conjugate points of the involution determined on BCB' by the conic which are equidistant from C, and that $-CB^2$ is the constant of this involution

Hence lines parallel to an asymptote through the ends of a diameter meet its conjugate diameter in the pair of conjugate points of the involution on that diameter, which are equidistant from the centre

(2) Let the line l meet the central conic in real points

In the case of the ellipse, and in the case of the hyperbola, when the line meets the same branch of the hyperbola, the proof given in (1) holds

Thus for the ellipse

$$XY\ XY' = -XA\ XA' \frac{CB^2}{CA^2}$$

But $XA\ XA'$ is negative in this case

Hence there are two double points P and P' of the involution, which are on the curve, such that

$$XP^2 = -XA\ XA' \frac{CB^2}{CA^2}$$

[Cf Art 104 (a) 2]

Thus for the hyperbola

$$XY\ XY' = XA\ XA' \frac{CB^2}{CA^2}$$

But $XA\ XA'$ is positive in this case

Hence there are two double points P and P' of the involution, which are on the curve, such that

$$XP^2 = XA\ XA' \frac{CB^2}{CA^2}$$

In the preceding (1) and (2), if K be the constant of the involution determined on BCB', then for both the ellipse and hyperbola

$$XY \cdot XY' = -XA \cdot XA' \frac{K}{CA^2}$$

$$= \left\{1 - \frac{XC^2}{CA^2}\right\} K$$

$$\frac{XY \cdot XY'}{K} = 1 - \frac{XC^2}{CA^2}$$

Hence, if P_1, P_2 are a pair of conjugate points of an involution, X its centre, B_1, B_2 and A_1, A_2 pairs of [illegible] points on the diameter parallel to P_1P_2 and on its conjugate, and C the centre of the curve,

$$\frac{XP_1 \cdot XP_2}{CB_1 \cdot CB_2} + \frac{CX^2}{CA_1 \cdot CA_2} = 1$$

If P be a double point of the involution, for the ellipse

$$\frac{XP^2}{CB^2} + \frac{CX^2}{CA^2} = 1$$

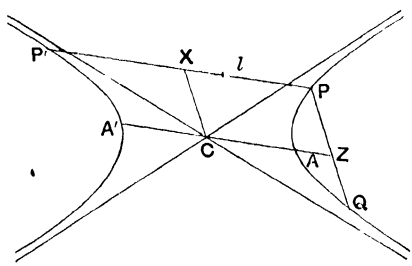
and for the hyperbola

$$-\frac{XP^2}{CB^2} + \frac{CX^2}{CA^2} = 1$$

The first of these results has been obtained from Carnot's Theorem (Art 104 (*a*) (2))

(3) Let the line l meet different branches of the hyperbola in P and P'

If X be the middle point of PP', XP^2 is the constant of the involution on PP' Draw ACA' the diameter parallel to PP', and PZQ parallel to CX to meet ACA' in Z



Then, since PQ and PP' are parallel to conjugate diameters,

$$\frac{CZ^2}{CA^2} - \frac{PZ^2}{CB^2} = 1,$$

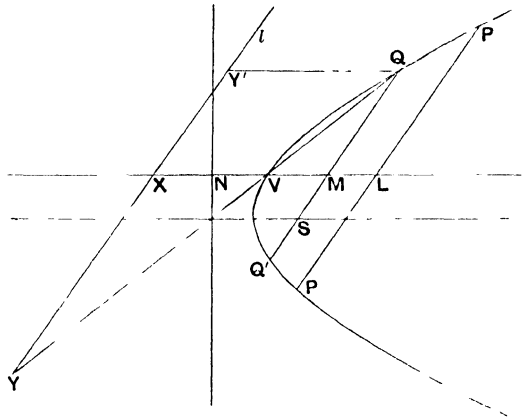
where $-CB^2$ is the constant of the involution on CX

Therefore $XP^2 = CZ^2 = \frac{CA^2}{CB^2}(CB^2 + CA^2)$ This is the constant of the involution on PP'

137 To find the constant of an involution determined on any straight line by a parabola

(1) Let the line l meet the parabola in imaginary points Take L its pole and draw a chord PP' of the parabola through L parallel to l

Draw a parallel chord through the focus S to meet the parabola in Q and Q'. Let M be the middle point of QQ' and let LM, which is a diameter of the parabola, meet the curve in V, l in X, and the directrix in N. Let ∞ denote the point at infinity on the parabola.



Then $V\infty$ is a chord of the parabola through the pole of l. Hence, if Q be joined to V and ∞ by lines which meet l in Y and Y', these points are conjugate points of the involution on l of which X is the centre.

Then $$XY = XV \frac{MQ}{MV} \text{ and } XY' = MQ$$

$$XY \cdot XY' = XV \frac{QM^2}{MV}$$

But since the tangents at Q and Q' intersect at right angles at N, a circle with centre M may be described to pass through Q, N, Q'. Therefore $QM = MN$ and, since $MV = VN$,

$$\frac{QM^2}{MV} = \frac{MN^2}{MV} = 4 \cdot VN = 4 \cdot SV$$

Therefore $$XY \cdot XY' = -4 \cdot SV \cdot VX$$

The negative sign is taken since XY and XY' are drawn in opposite directions.

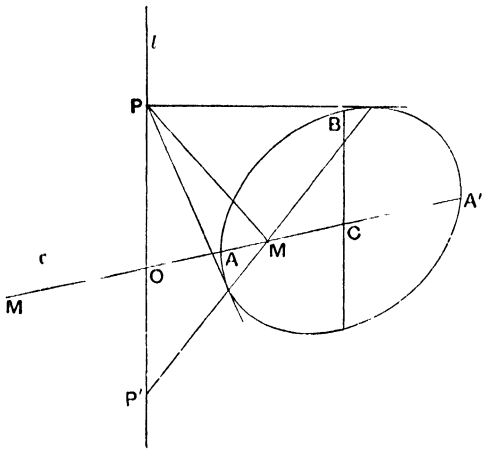
If P and P' are joined to the point at infinity on the curve, a pair of conjugate points are found on l, the product of whose distances from X is $-PL^2$.

Therefore $$PL^2 = 4 \cdot SV \cdot VL$$

(2) If l meets the curve in real points, P and P', the constant of the involution is PL^2 which equals 4 $SV\ VL$ (Art 104 (*a*) (3))

138 Gaskin's Theorem

The circle, which circumscribes any self-conjugate triangle of a conic, is cut orthogonally by the director circle of the conic



Let $PP'M$ be any self-conjugate triangle and let PP' be l Join M to the centre of the conic to meet l in O, the conic in A, and the circumscribed circle of $PP'M$ in M' Let CB be the diameter conjugate to CA

Then $CM\ \ CM' = CM(CO + OM') = CM\left(CO + \dfrac{OP\ \ OP'}{OM}\right)$,

since $OP\ \ OP' = OM\ \ OM'$,

$= CM\left(CO - \dfrac{CB^2}{CA^2}\ \dfrac{OM\ \ OC}{OM}\right)$

(by Art 136), since $OA\ \ OA' = OM\ \ OC$,

$= CM\ \ CO\left(1 + \dfrac{CB^2}{CA^2}\right)$

$= \dfrac{CM\ \ CO}{CA^2}(CA^2 + CB^2)$

$= CA^2 + CB^2$ since O and M are conjugate points

= sum of the squares of the semi-axes (Art 139)

EXAMPLES

(1) Show that the circumcircle of a triangle self-conjugate with respect to a rectangular hyperbola passes through the centre of the hyperbola

The director circle of the rectangular hyperbola is the centre

(2) The centre of the circumcircle of a triangle self-conjugate with regard to a parabola lies on the directrix of the parabola

(3) The feet of the perpendiculars drawn from any point on the director circle to the polar of that point and to the three sides of a self conjugate triangle are concyclic

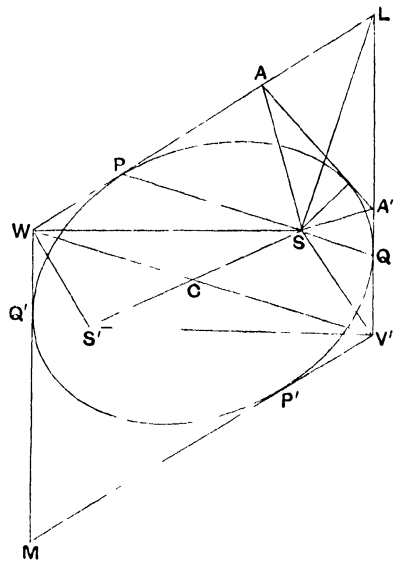
Let P be a point on the director circle, p its polar, and ABC a self conjugate triangle Let p meet the sides of ABC in A', B', C' Then PA, PA', PB, PB', PC, PC' are pairs of conjugate rays of the involution determined by the conic at P, of which the tangents from P are the double rays Since these are at right angles, the angles BPC and $B'PC'$ are equal Hence, Addendum 15, the feet of the perpendiculars from P on p and on the sides of ABC are concyclic

(4) If any two conics are rotated in any manner round their centres the locus of the centre of the circumcircle of their common self conjugate triangle is the radical axis of their director circles

Focal Properties of Conics

139 In Chapter XIII, Arts 96 and 97, some theorems connected with the foci of a conic were given In this article certain metrical properties of conics connected with their foci will be proved

If LW, $V'W$, LV', WM be any two pairs of parallel tangents to a conic, and AA' any other tangent, which meets WL and LV' in A and A', then A, A', W, ∞, ∞, V' determine on LW and LV' two projective ranges, and any other tangent to the conic meets these lines in pairs of corresponding points of these ranges (Art 93)



By Art 89 construct points S and S' such that corresponding points of the ranges subtend a constant angle α at S, and a constant angle α' at S' Then S and S' are the foci of the conic and are the same points for all pairs of tangents LW and LV' (Art 98, Example (8))

From the construction for these points

$$SW.SV' = WL.V'A' = \text{a constant},$$

and the angles AWS and $A'V'S$ are equal

Let P, Q, P', Q' be the points of contact of WL, LV', $V'M$ and MW, and C the middle point of WV', which is also

the middle point of SS', since $SV'S'W$ is a parallelogram Then the angles PSL and LSQ are equal to each other and to the angle ASA'

(1) *The tangents from any point to a conic are equally inclined to the focal distances of the point*

The angle PWS equals the angle $S'WQ$ (Art 39) and therefore equals the angle $S'WQ'$

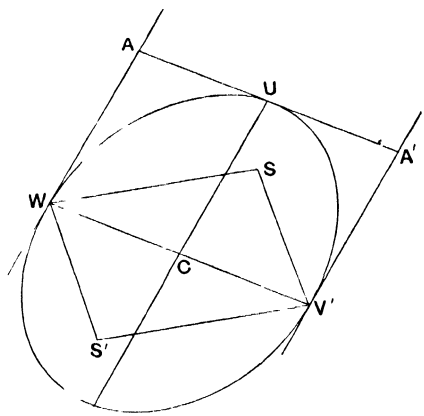
(2) *The tangent at any point to a conic is equally inclined to the focal distances of the point*

If in (1) the point W is on the conic, the theorem follows at once

(3) *If a pair of parallel tangents to a conic meet the tangents from a point W in A and B, then $WA \cdot WB$ equals the product of the focal distances of W*

If a tangent BB' parallel to AA' be drawn to meet WM and MV' in B and B', then $A'V' = WB$

If the tangents LW and LV' are parallel, the points P, W, Q' and the points Q, V', P' coincide in two points W and V' on the conic, which are the ends of a diameter If AUA' be drawn parallel to WV', it will touch the conic at a point U such that CU is the diameter conjugate to WCV'



(4) If CW and CU are at right angles, they will be the axes of the conic and CU will pass through S and S' (Art 96) SW will likewise equal $S'V'$

Then for an ellipse, since $WA \cdot V'A'$ equal $WS \cdot V'S$,

$$CU^2 = SW^2 = WC^2 + CS^2,$$
$$CS^2 = CU^2 - WC^2 = a^2 - b^2,$$

where a and b are the semi axes of the conic

Similarly it may be proved for a hyperbola that $CS^2 = a^2 + b^2$ (Cf Art 96)

In the more general case

(5) $SW \cdot SV' = WA \cdot VA' = CU^2$

= square of the semi-diameter conjugate to CW

(Cf Art 102, Example (7))

(6) *The sum of the squares of a pair of conjugate diameters is constant*

In the figure $(SW + SV')^2 = WS^2 + SV'^2 + 2\ WS \cdot SV'$

$$= 2\ SC^2 + 2\ CW^2 + 2\ CU^2$$
$$= 2(a^2 - b^2) + 2(CW^2 + CU^2)$$

But, Art 96 (*b*), $SW + SV' = 2a$,

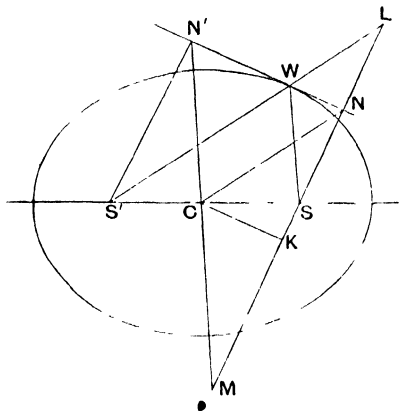
$$CW^2 + CU^2 = \tfrac{1}{2}(SW + SV')^2 - (a^2 - b^2)$$
$$= a^2 + b^2$$

The corresponding theorem for a hyperbola may be proved in a similar way

(7) *The locus of the foot of a perpendicular from a focus of an ellipse on a variable tangent is a circle concentric with the ellipse*

Let SN be the perpendicular from S a focus of an ellipse on the tangent at W. Produce SN to meet $S'W$ at L. Then, since $S'W$ and SW make equal angles with the tangent at W, the triangles SWN and LWN are equal. Therefore SN equals NL, and SW equals WL. Therefore

$$S'L = S'W + WS = 2a$$



Join C the centre of the conic to N. Then CN is parallel to $S'L$. Therefore $CN = \frac{1}{2}S'L = a$. Hence the locus of N is a circle concentric with the ellipse.

The theorem for the hyperbola may be proved in a similar manner.

(8) If N' be the foot of the perpendicular from S', the second focus, on the tangent at W and CN' meets NS at M, then CM and CN are equally inclined to the perpendicular CK from C on NS, and $MS = S'N'$.

Therefore $SN \cdot S'N' = SM \cdot SN = (NK + SK)(NK - SK) = NK^2 - SK^2$

$$= CN^2 - CS^2 = a^2 - (a^2 - b^2) = b^2$$

(9) *The area of the parallelogram formed by a pair of tangents parallel to a pair of conjugate diameters is constant.*

Let CR be the perpendicular from C on the tangent at W. Then, from the similar triangles SWN and $S'WN'$,

$$\frac{SN}{SW} = \frac{S'N'}{S'W} = \frac{\sqrt{SN \cdot S'N'}}{\sqrt{SW \cdot S'W}} = \frac{b}{CU} \text{ by (8) and (5)}$$

$$= \frac{SN + S'N'}{SW + S'W} = \frac{CR}{a}$$

Therefore $CU \cdot CR = a \cdot b$.

Confocal Conics

140 *Conics, which have the same pair of points for their real foci, are termed confocal conics.*

All conics of a confocal system are inscribed in the same quadrilateral.

By Art. 101 the tangents (imaginary)—from a focus of a conic to the conic pass through the circular points at infinity. Hence the connectors of these points to the two foci form a circumscribed quadrilateral for all conics having the same foci, *i.e.* for all confocal conics. One pair of opposite vertices of the quadrilateral are the two real foci, and another pair are the circular points at infinity. The third pair of vertices are a pair of imaginary foci of the conics, which are situated on the minor or conjugate axis. The common self-conjugate triangle of the conics consists of the two axes of the conics and the line at infinity. The vertices of this triangle are the common centre of the conics and the points at infinity on the axes.

Since confocal conics are inscribed in a quadrilateral one confocal of a given system can be drawn to touch a given straight line, and two confocals can be drawn through any given point (Art 113)

Every pair of confocal conics, which intersect in a real point, intersect at right angles

Consider the involution pencil determined at any point P by a system of confocal conics and their common circumscribed quadrilateral Let S and S' be the foci of the conics and Ω and Ω' the circular points at infinity Since $P\Omega$ and $P\Omega'$ are conjugate rays of the involution pencil, the double rays of the pencil, which are harmonic conjugates of these lines, are at right angles But these double rays are the tangents to the two conics of the system, which pass through P Hence these conics, which are the two conics of the system which pass through P, intersect orthogonally

The pairs of tangents, which can be drawn from any point to the conics of a confocal system, are equally inclined to the focal distances of the point

Since the double rays of the involution pencil determined at any point P by the confocal system are at right angles, the pairs of conjugate rays are equally inclined to the double rays (Art 16) Since the connectors of P to the foci are a pair of conjugate rays of this involution the result follows at once The theorem also follows from Art 139 (1)

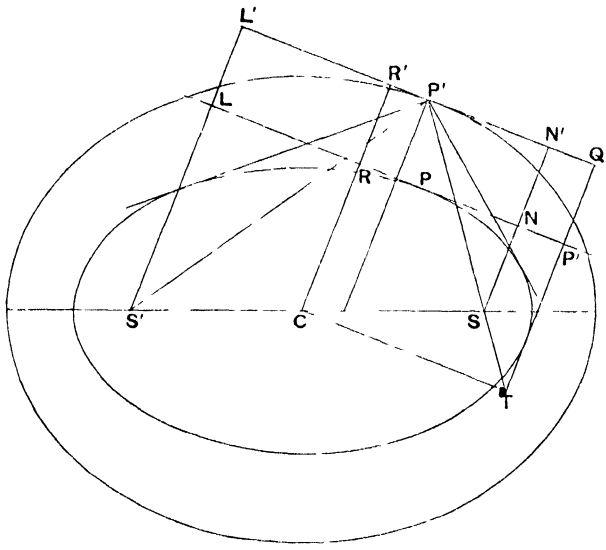
The locus of the pole of a given straight line, with respect to a system of confocal conics, is the normal at its point of contact to the conic of the system which touches the given straight line

Since the confocal system is inscribed in a quadrilateral the locus of the pole of a given straight line p with respect to the conics of the system is a straight line p' (Art 117 (1)) This line passes through the point of contact of the conic of the system, which touches the line p The lines p and p' are common conjugates with respect to all conics of the system and are therefore harmonic conjugates of all pairs of tangents from pp' to conics of the system They are therefore at right angles and are the double rays of the involution determined by the conics at pp' Hence they are the tangents to the two conics of the system, which intersect orthogonally at pp'

The envelope of the polars of a given point, with respect to a system of confocal conics, is a parabola, which touches the axes of the conics

Since the confocal system is inscribed in a quadrilateral, the envelope of the polar of a given point P is a conic (Art 117 (2)) This conic touches the sides of the common self conjugate triangle of the conics and is therefore a parabola, which touches the axes of the conics It also touches the tangents p and p' to the conics of the system through P Since p and p' are at right angles, as are also the axes of the conics, the directrix of the parabola is the line CP, where C is the centre of the conics If p and p' meet the axes of the conics in A_1, A_1' and B_1, B_1', the circles described on A_1B_1 and $A_1'B_1'$ as diameters intersect at C and in F the focus of the parabola

The difference of the squares of the perpendiculars from the centre on parallel tangents to two confocal conics is constant



Let the perpendiculars from the centre C and foci S and S' on two parallel tangents at P and P' meet the tangents in R, V, L and R', N', L' respectively

Then

$$CR'^2 - CR^2 = \left(\frac{S'L' + SN'}{2}\right)^2 - \left(\frac{S'L + SN}{2}\right)^2$$

$$\left(\frac{S'L - SN'}{2}\right)^2 - \left(\frac{S'L - SN}{2}\right)^2 + S'L'\ SN' - S'L\ SN$$

$$= \left(\frac{S'L' - SN'}{2}\right)^2 - \left(\frac{S'L' - SN'}{2}\right)^2 + b'^2 - b^2,$$

where b and b' are the semi-minor axes (Art 139 (8)),

$$= b'^2 - b^2$$

The locus of the point of intersection of orthogonal tangents to two confocal conics is a circle which has the same centre as the confocal conics

Let TQ be a tangent to one confocal, which meets the tangent at P' to the other orthogonally at Q, and let T be the foot of the perpendicular from C on this tangent

Let QT meet the tangent at P in P''

Then

$$CQ^2 = CR'^2 + CT^2 = CR^2 + b'^2 - b^2 + CT^2 = CP''^2 + b'^2 - b' = a^2 + b^2 + b'^2 - b^2 = a^2 + b'^2$$

Hence the locus of Q is a circle concentric with the conics

EXAMPLES

(1) C is the centre of two confocal ellipses, S a focus and A and A' the ends of their major axes. Prove that the lines perpendicular to AA' drawn through points P, such that $CP \cdot CS = CA \cdot CA'$, are the real chords of intersection of the ellipses.

(2) The locus of the common conjugates, with respect to two confocal ellipses, centre C, focus S and ends of the major axes A and A', of points on a line passing through a point P at a distance CP from the centre and parallel to the minor axis, is a straight line parallel to the minor axis, which meets the major axis at a point P' such that $CP \cdot CP' = \dfrac{CA^2 \cdot CA'^2}{CS^2}$. (Art. 125)

(3) If P and P' be two points collinear with a focus on two confocal ellipses centre C, foci S and S' and ends of major axes A and A', and these points be joined to A or A', the locus of the point of intersection of PA and $P'A'$ is two straight lines perpendicular to the major axis and passing through points on it at distances $\pm \dfrac{CA \cdot CA'}{CS}$ from the centre.

(4) Prove that the double rays of the pencils of parallel lines in (2) are the common chords of the two ellipses.

One to one correspondence

141 One to one correspondence may be defined as follows. *When two groups of geometrical forms (points, straight lines, curves) are so related, that any element of one group determines one and only one element of the other group, and the latter element determines the first element uniquely, the elements of the groups are said to have one to one correspondence.*

The following examples illustrate this principle.

(1) A ray OQR passing through a fixed point O determines points Q on a fixed line a, and R on another fixed line b, and these points determine rays AQ, BR at fixed points A, B. By construction there is one to one correspondence between AQ and BR.

(2) A point P on a conic determines rays AP, BP at fixed points A and B on the conic. The construction shows that there is a one to one correspondence between the rays AP, BP.

(3) A range of points A, B, C, ... at distances OA, OB, OC, ... from a fixed point O on their base, and a second range of points A', B', C', ... on the same base at distances from O such that $OA' = 2\,OA$, $OB' = 2\,OB$, $OC' = 2\,OC$, ... have one to one correspondence.

(4) A pencil of rays a, b, c, ... through a fixed point S, which make angles $\widehat{oa}$, $\widehat{ob}$, $\widehat{oc}$, ... with a fixed ray o through S, and a second pencil of rays a', b', c', ... through S, such that the angles $\widehat{oa'}$, $\widehat{ob'}$, $\widehat{oc'}$,

are double of the angles $\widehat{oa}$, $\widehat{ob}$, $\widehat{oc}$, , have not one to one correspondence For, if $\widehat{oa} = \alpha$ and $\widehat{ob} = \alpha + \frac{\pi}{2}$, the rays a' and b' are given by $\widehat{oa'} = 2\alpha$ and $\widehat{ob'} = 2\alpha + \pi$ Hence the rays a' and b' coincide, and one ray of the second pencil corresponds to more than one ray of the first

In Chapter VI, Art 44, it was proved that, if there is an algebraic one to one correspondence between the points of two ranges, these ranges are projective The following statement summarises the circumstances in which it is safe to infer that a one and one correspondence is algebraic

If in a one to one correspondence, as now defined, a point of one range—or a ray of one pencil—is obtained from the corresponding element of the other range—or pencil—by a geometrical construction involving

(1) *The determination of straight lines, as the connectors of points or by means of the construction of given angles, and*

The determination of points, as the intersections of straight lines or by the measurement of distances along straight lines,

(2) *The construction of conics by points or tangents,*

(3) *The determination of points on conics as their points of intersection with other conics or straight lines, and*

The determination of tangents to conics, as common tangents to conics, or as tangents from points to conics,

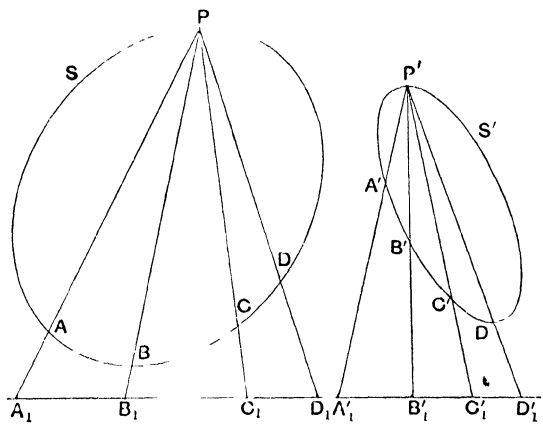
then the one to one correspondence is algebraic

This theorem or principle depends on the fact that a conic is an algebraic curve and its truth will be realised at once by the student who is acquainted with the elements of coordinate geometry The degree of a plane algebraic curve is determined by the number of points real, coincident, or imaginary in which a line meets the curve In Art 95 (*k*) it was proved that a conic determines on every straight line in its plane a definite involution the double points of which are two points real, coincident, or imaginary In Art 101 it was shown, that these points may be regarded as the points of intersection of the line with the conic Hence a conic is an algebraic curve of the second degree The class of a curve is defined by the number of tangents to the curve, which may be drawn to it from any point in its plane In Art 93 it was shown that a conic is a curve of the second class Conversely it may be proved that all algebraic curves of the second degree or of the

second class are conics The correspondence here defined will be termed *an algebraic one to one correspondence* or shortly when there is no fear of confusion *a one to one correspondence* In this case Art 44 applies and the ranges and pencils are projective Although the point does not at present arise the theorem now enunciated may be extended by substituting "any algebraic curve" for "conic" in the statement of it

The theorem that two ranges on linear bases between which there is an algebraic one to one correspondence are projective may be extended to ranges on conics between which there is an algebraic one to one correspondence The proof excludes the case of ranges obtained by measuring lengths along the curve Such ranges have not generally a one to one correspondence

If there is an algebraic one to one correspondence between two ranges on the same or on two different conics, the ranges are projective



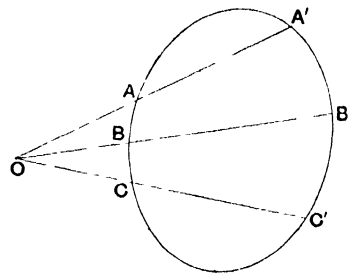
Let A, B, C, D and A', B', C', D' be corresponding points of ranges situated on conics S and S' Join A, B, C, D to any point P on S and A', B', C', D' to any point P' on S' Let the pencils so formed be cut by any line s in A_1, B_1, C_1, D_1 and A_1', B_1', C_1', D_1' respectively Since there is one and one algebraic correspondence between $ABCD$ and $A'B'C'D'$, there is a similar correspondence between A_1, B_1, C_1, D_1 and A_1', B_1', C_1', D_1' Hence (Art 44) these ranges are projective Therefore the ranges A, B, C, D and A', B', C', D' are also projective

If the corresponding elements of two ranges, or pencils, having algebraic one to one correspondence mutually correspond, it follows from Art 51 that they form an involution

The following are examples of the application of the method of one to one correspondence

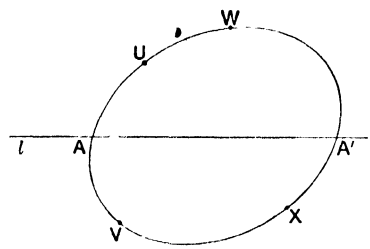
(1) *Involution Property of the Conic*

Through any point O draw two fixed chords to meet the conic in A, A' and B, B' These chords may be regarded as determining O Draw OCC' any other chord which may be considered as a variable chord When C is given C' is uniquely determined, and when C' is given C is uniquely determined, hence the ranges described by C and C' are projective Since C' corresponds to C'' and C'' to C, they form an involution



(2) *Desargues' Theorem*

Let U, V, W, X be the four fixed points, through which the system of conics are described, and l the given transversal On l take any point A Describe a conic through U, V, W, X and A Only one such conic can be described It will meet l in a point A' If a conic is described through A', U, V, X, W it will pass through A Hence A determines A' uniquely and A' determines A uniquely Therefore the ranges described by A and A' are projective Also A and A' mutually correspond and therefore the ranges described by A and A' form an involution



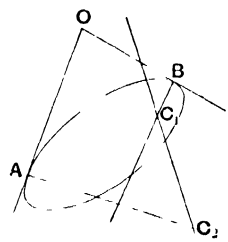
(3) *If a system of conics is described to touch two fixed lines at fixed points A, B, the envelope of the line joining the corresponding centres of curvature at A and B is a parabola*

The centres of curvature C_1 and C_2 are situated on the respective normals at A and B For each conic they are uniquely determined and therefore form two projective ranges on the normals

Therefore C_1C_2 envelopes a conic

In the limiting case when the conic becomes the straight lines OA, OB, C_1 and C_2 are both at infinity

In this case C_1C_2 becomes the line at infinity, and therefore the envelope touches the line at infinity and is a parabola

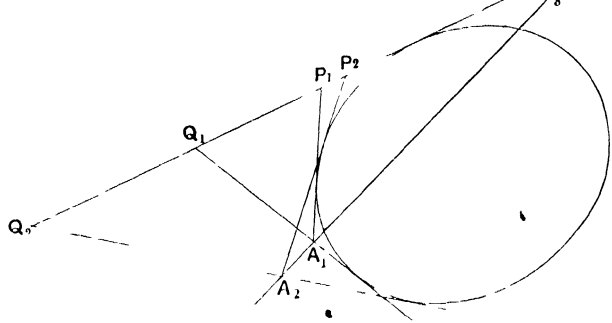


142 Determination of the nature of a locus from its points of intersection with a straight line

In the last article it was shown that, if the construction of corresponding elements of ranges or pencils, which have one to one

correspondence, depends on certain geometrical operations, it is safe to assume that this one to one correspondence is algebraic If the construction of points on a locus, or of tangents to an envelope, depends on the same operations it follows that the locus or envelope must be an algebraic curve In this case the degree of the locus may be inferred from the number of points in which any straight line, or a particular straight line, meets the locus, and the class of an envelope may be found from the number of tangents, which may be drawn to it from any point or from a particular point In the use of this method care should be exercised as geometrically it is often difficult to detect the imaginary points of intersection of a straight line with a locus or the imaginary tangents from particular points The following which are generalisations of theorems already proved, are given as illustrations of the method

(1) *To find the nature of the locus of the points of intersection of tangents to a given conic from corresponding points of two superposed projective ranges*



Let t be any tangent to the conic It will meet the base s on which the ranges are situated in some point A The point A may be looked upon as a point of either of the projective ranges and, according to the range to which it is regarded as belonging, it will correspond to a point A_1 or a point A_2 on s The points A_1 and A_2 will only coincide, if the ranges form an involution (Art 51) The tangents from A_1 will meet t in two points of the locus P_1, Q_1 The tangents from A_2 will meet t in two other points of the locus P_2, Q_2 Hence the locus meets t in four different points P_1, Q_1, P_2, Q_2 and is therefore of the fourth degree

This result should be compared with (i) Art 75, converse, and Art 95 (*c*), with (ii) Art 131, converse, and with (iii) Art 134, which are particular cases It is thus seen that

The locus of the points of intersection of tangents to a conic from pairs of corresponding points of

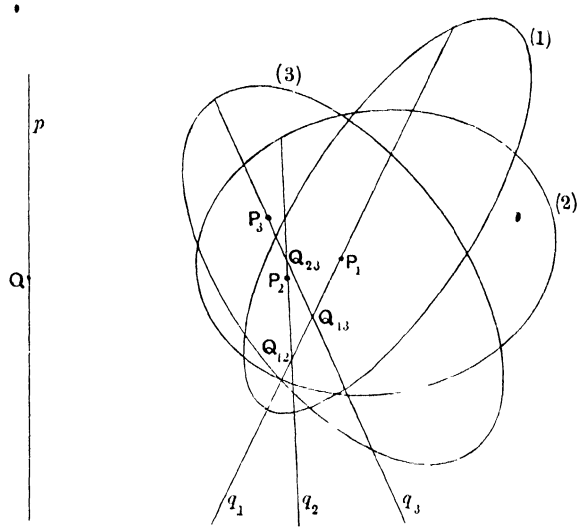
(i) *an involution on a tangent to the conic, is a straight line,*

(ii) *two superposed projective ranges on a tangent to the conic, is a conic having double contact with the given conic,*

(iii) *an involution, is a conic through the double points of the involution,*

(iv) *two superposed projective ranges, is a curve of the fourth degree*

(2) *The locus of the common conjugates of three conics is a cubic*



Let P_1, P_2, P_3 be the poles of any line p with regard to the conics (1), (2) and (3) respectively Let the polars of any point Q on p with respect to (1), (2) and (3) respectively be q_1, q_2, q_3 If q_1 and q_2 intersect in Q_{12} and q_1 and q_3 intersect in Q_{13} the loci of Q_{12} and Q_{13}, for different positions of Q on p, will be the loci of common conjugates of points on p with respect to (1), (2) and (1), (3) respectively These loci are conics (Art 117 (2)) They will intersect in four points of which P_1 is one At their points of intersection other than P_1 the three polars of points on p with respect to (1), (2) and (3) meet at a point Hence there are three points on p the polars of which with respect to (1), (2) and (3) meet at a point Therefore there are three points on p which have common conjugates with regard to the three conics Hence it may be inferred that the line p meets the locus of common conjugates in three points and that the locus is therefore a cubic

If C_{12}, C_{13}, C_{23} be the loci of common conjugates of points on p with respect to (1), (2), (1), (3) and (2), (3) respectively, these conics will pass through the three common conjugates of points on p, their other points of intersection in pairs being the points P_1, P_2 and P_3

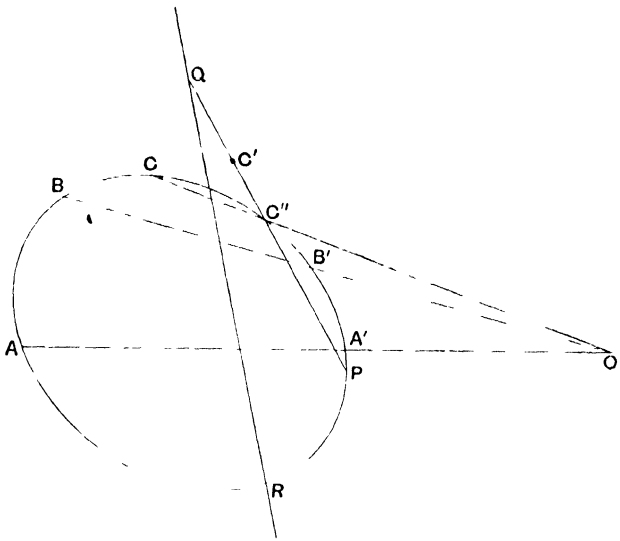
If the three conics pass through two points A and B, the line AB is part of the

locus Hence in this case the cubic becomes a straight line and a conic, which also passes through A and B

Hence *if three conics pass through two points A and B, the locus of their common conjugates is a conic through A and B together with the line AB*

This result should be compared with those of Arts 88 (*c*) and 132 (*e*) of which it is the general case

(3) *To find the nature of the locus of a point such that its connectors to three pairs of coplanar points form an involution pencil*



Let A, A', B, B', C, C'' be the three pairs of points The locus passes through each of these points For through any one of them C, a line can be drawn which is a conjugate of CC'' in the involution determined by A, A' and B, B' at C This line is a tangent at C to the locus

Describe a conic through any five of the six points, viz A, A', B, B' and C Let AA' and BB' intersect at O Join CO to meet the conic at C'' Let $C'C''$ meet the conic at P, and the polar of O with respect to the conic, *i e* the connector of AB' BA' and AB $A'B$, at Q

Consider the points of the locus, which are situated on PQ For any point on PQ the pencil determined by A, A', B, B', C, C' will generally be the same as the pencil determined by A, A', B, B', C, C'' Hence generally, if one is an involution pencil, so is the other There are however two exceptions At C'' the involution pencil determined by AA', BB', CC'' consists of the connectors of the first five points to C'' together with the tangent to the conic at C'' Therefore C'' is not a point on the locus The locus passes through C'' and the involution pencil determined at C' consists of the connectors of C' to A, A', B, B', C and the tangent to the locus at C' At all other points on PQ the pencils are the same

The pencil obtained by joining a variable point to AA', BB', CC''' will not be an involution pencil unless the variable point is situated on the conic or on the polar of O with respect to the conic (Art 98) Hence the only points beside C', in which PQ meets the locus, are P and Q Therefore the locus is met by the line PQ in three and only three points, viz C', P and Q, and is therefore a cubic curve

The same result may also be obtained by considering the points, in which the locus is met by the connector of any pair of the given pairs of points—say $1\,1'$ The locus meets this line in A and A' If T be any other point of the locus situated on $1\,1'$, the line AA' must be a double ray of the involution determined at T by B, B' and C, C' Let CC' meet AA' in O' Take O_1 and O_1' the harmonic conjugates of O and O' with respect to BB' and CC' respectively Then O_1O_1' meets AA' in T Since this construction for T is unique the locus intersects $1\,1'$ in only three points, viz A, A' and T, and is therefore a cubic curve

When the six points are on a conic and the connectors of the pairs of points are concurrent at a point O, the cubic becomes the conic through the six points and the polar of O with respect to the conic (Art 98)

If C' is situated on the conic, $C'C''$ is the tangent to the locus at C'' and $C'C''$ meets the locus in a third point at Q

The cubic curve obtained as the above locus is the most general form of cubic curve and from a projective point of view a cubic curve may with advantage be defined from this involution property The student who is desirous of continuing his studies in the projective properties of cubic curves is referred to the work by Schroter, *Theorie der ebenen Kurven dritter ordnung*, on this subject

If the left-hand side of the equations of the lines AC'', $C'B$, BA', $A'C$, CB', $B'1$, in the form $x\cos\alpha+y\sin\alpha-p=0$, be respectively denoted by u, w', v, u, w, v', and the sides of the hexagon so formed by a, c', b, a', c, b', respectively, then from the equality of the anharmonic ratios of an involution pencil, the equation of the cubic can be obtained in the form

$$a\ b\ c\ u\ v\ w = a'\ b'\ c'\ u'\ v'\ w'$$

The following examples refer to some of the properties of the cubic considered from this point of view

EXAMPLES

(1) If AB $A'B'$ and AB' BA' be denoted by C''', C'''' and points A'', A''' and B'', B''' be constructed in a similar manner, prove that the cubic passes through A'', A''', B'', B''', C'', C''', and that the pairs of points $1'\,1'''$, $B'B'''$, $C''C''''$ are related to the cubic in the same way as AA', BB', CC'

(2) Prove that the cubic passes through the points $C'''A''$ $C''''A'''$, $C''A'''$ $C'''A''$, and the four other points found in a similar manner

(3) If P be a point on the cubic and any pair of conjugate rays in the involution pencil $(P\ AA'BB'CC')$ meet the cubic in D and D', prove that the given cubic meets the cubic determined by AA', BB', DD' in nine determined points

(4) Prove that the tangents at C and C' are the conjugates of CC' in the involutions determined by AA', BB' at C and C'

(5) Construct the third point in which the tangent at any one of the six points meets the cubic

m to n correspondence

143 In the consideration of two ranges of points situated either on given straight lines or on conics it may happen that to a point of the first correspond n points of the second and that to a point of the second correspond m points of the first This is spoken of as m to n correspondence It may be defined as follows, viz

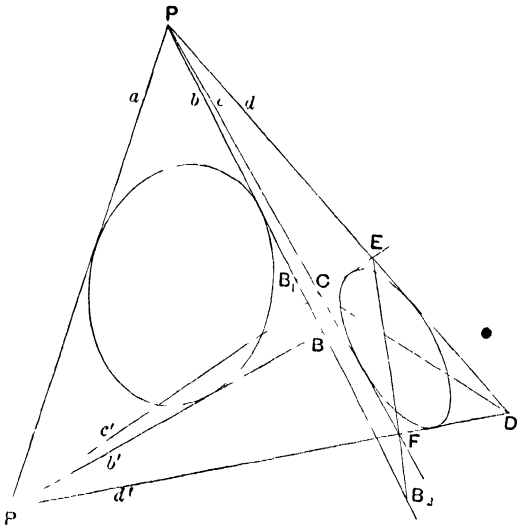
When two groups of geometrical forms (points, straight lines, curves) are so related that a set of m elements of one group corresponds to a set of n elements of the other, in such a way that any element of either set determines all the elements of the other set, then the elements of the groups are said to have m to n correspondence

Thus, if two systems or pencils of conics are described through two sets of four points A, B, C, D and A', B', C', D' respectively, a conic of either system is determined, if the tangent at A or A' to the conic is given If there is one to one correspondence between the tangents at A and A', there is one to one correspondence between the conics The conics of the two systems determine involutions P and Q on any straight line in that plane Any point P of one involution determines two points of the involution Q and any point Q determines two points of the involution P Hence there is two to two correspondence between the points P and Q

In the case of two superposed ranges with algebraic one to one correspondence, it was proved (Art 41) that there are 2, *viz* $1+1$, self-corresponding points Similarly, *in the case of two superposed ranges with m to n correspondence there are $m+n$ self-corresponding points* For, if x and x' are the distances of a pair of corresponding points from some fixed point on the base, there must be a relation $F(x, x')=0$ between these quantities, where $F(x, x)$ is a polynomial in x and x' Since this relation is algebraic and n points of the second range correspond to a given point of the first, it must be of degree n in x' Similarly, it is of degree m in x To obtain the self-corresponding points of the ranges x' must be made equal to x and the equation thereby obtained is of the $(m+n)$th degree Hence there are $m+n$ self-corresponding points of the given ranges

To find the locus of points the tangents from which to two given conics form a pencil of constant anharmonic ratio

Let a, b and c, d be the tangents from a point P to the two conics and let the anharmonic ratio $(abcd)$ be λ



Take any other point P' on a Draw tangents b', c' and d' to the conics, and let bb' be B, cc', C, dd', D, $c'd$, E and $d'c$, F

If the tangents from P' have the same anharmonic ratio as those from P, the points B, C, D or the points B, E, F must be collinear Consider the ranges determined on b by B and by EF and CD for different positions of P' The self corresponding points of these ranges give the positions of P' for which the pencil $ab'c'd'$ has the value λ Let the points of intersection of CD and EF with b be B_1 and B_2 If B_1 is given, P' is found by joining B_1 to CD EF, the fixed pole of a with respect to the second conic, by a line which meets c and d in C' and D P' is then the point of intersection of the second tangent from C' or D with a Therefore, disregarding the fixed point P, there is a two to one correspondence on b Hence there are three self corresponding points of the ranges on b and therefore three positions of P' independent of P Hence the line PP' is met in four points including the point P by the locus, which is therefore of the fourth degree

This result should be compared with that obtained in Art 133

Involution of the nth order

When a group of geometric forms may be divided into sets of n forms and any one of each such set determines the remaining $n-1$ elements of the set uniquely, then the different sets of forms are said to constitute an involution of the nth order

Thus

(1) A ray AP, through a fixed point A on a fixed conic, determines a point P on the conic the connector of P to a fixed point O, not on the conic, determines another point P' on the conic The construction shows that AP and AP' mutually determine each other and therefore form an involution of the second order The rays OP and AP are in one to two correspondence

(2) A conic of a pencil of conics through four fixed points A, B, C, D determines two points P, P' on a fixed straight line, and either P or P' determines by means of the conic P' or P The construction shows that P, P' are elements of an involution of the second order Also a tangent AT to the conic at A is in one to two correspondence with P and in one to one correspondence with the conic

(3) A cubic of a pencil of cubics through eight fixed points determines three points P, P', P'' on a fixed straight line, and any one of the points P, P', P'' determines by means of the cubic the other two points Hence P, P', P'' are elements of an involution of the third order

It follows from the definition of an involution that *if two involutions of the mth and nth order are in one to one correspondence, then there is an m to n correspondence between the elements, which constitute the involutions*

An involution of the nth order has $2(n-1)$ *united corresponding elements*

Let $x_1, x_2, x_3, \ldots, x_n$ be coordinates, which determine the positions of a set of n elements of the involution When any one of these is given, the other elements of the set are uniquely determined Consequently an algebraic relation of the form $F(x_1, x_2) = 0$ exists connecting the coordinates of any pair of elements If x_1 is given the other $n-1$ elements are determined and, since x_2 may be the coordinate of any one of these elements, the expression $F(x_1, x_2)$ must be of the $(n-1)$th degree in x_2 Similarly it is of the $(n-1)$th degree in x_1 To obtain the united corresponding elements of the involution x_1 and x_2 in the equation $F(x_1, x_2) = 0$ must be made equal to some quantity x The equation thus obtained is of degree $2(n-1)$ in x and therefore there are $2(n-1)$ united corresponding elements of an involution of the nth order

Characteristics of Systems of Conics

144 In the preceding pages numerous instances have occurred of conics which comply with certain conditions Such conditions may be classified as one-fold, two-fold, three-fold, four-fold and five-fold

Thus, if a conic is required to pass through a given point, or touch a given line, or have a given pair of points for conjugate points, it is said to have a one-fold condition imposed on it

If a conic is required to touch a given line at a given point, or to determine a given involution on a given line, or to have a given point for the pole of a given line, it is said to have a two-fold condition imposed on it

If a conic is required to be inscribed in, or circumscribed about, a given triangle, or to have a given triangle for a self-conjugate triangle, it is said to have a three-fold condition imposed on it, and so on

In every case a two-fold, three-fold, four-fold condition may be looked upon as a combination of two, three, four, etc one-fold conditions This is illustrated by the instances given above Conversely, a combination of two, three, four, etc one-fold conditions may be looked upon as a manifold condition

A conic may be described to satisfy five one-fold conditions or, what is the same thing, to satisfy a five-fold condition The number of conics which can be described to satisfy this number of independent conditions is a definite and finite number—depending on the nature of the conditions—which can in each case be ascertained

A system of conics, that is an infinite number of conics, can be described to satisfy a four-fold condition If an additional condition be imposed, either that the conics should (a) pass through a given point, or (b) touch a given line, the number of conics complying with the condition is a definite and finite number, which in the first case will be denoted by μ and in the second by ν The numbers μ and ν are termed the characteristics of the system

The characteristics of systems of conics through points and touching lines may be obtained from considering the number of conics passing through five points, through four points and touching one line, through three points and touching two lines, etc , which will be denoted by $(\ )$, $(\ /)$, $(\ //)$,

From Art 113,

$$(\ )=1, \qquad (\ /)=2, \qquad (\ //)=4,$$
$$(\ ///)=4, \qquad (\ ////)=2, \qquad (/////)=1$$

By the principle of duality it is clear that

$$(\ \ )=(/////),\quad (\ \ /)=(\ ////),\quad (\ \ //)=(\ ///)$$

Hence for the system of conics $(\ \ )$, etc the following values of μ and ν are obtained, viz

System of Conics	μ	ν
$(\ \ )$	1	2
$(\ \cdot/)$	2	4
$(\ //)$	4	4
$(\cdot///)$	4	2
$(////)$	2	1

Let C denote that a pair of [illegible] points with respect to the conic are given

Let c denote that a pair of conjugate lines with respect to the conic are given

Let I_2 denote that an involution determined by the conic on a straight line is given

Let $\imath_2$ denote that an involution pencil determined by the conic is given

Let $\frac{1}{2}$ denote that the conic touches a given line at a given point

Let p_2 denote that a given point and line are pole and polar with respect to the conic

Let p_3 denote that a given triangle is self-polar with respect to the conic

In the above the subscript letters denote the manifoldness of the condition imposed on the conic The relations c, $\imath_2$ are the correlatives of C and I_2, while $\frac{1}{2}$, p_2 and p_3 are their own correlatives

On reference to Chapter XVII it will be seen that

$$(\ \ \tfrac{1}{2})=1 \qquad (///\tfrac{1}{2})=1,$$
$$(\ \tfrac{1}{2}\tfrac{1}{2})=1 \qquad (/\tfrac{1}{2}\tfrac{1}{2})=1,$$
$$(\ \tfrac{1}{2}/)=2 \qquad (//\tfrac{1}{2}\ )=2$$

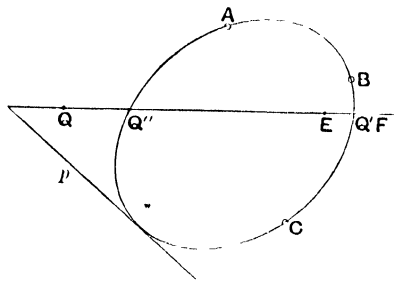
$$(p_2\ \ )=1 \qquad (p_2///)=1,$$
$$(p_2\ /)=2 \qquad (p_2\ //)=2$$

$$(p_3\ )=1 \qquad (p_3\ /)=2 \qquad (p_3//)=1$$

$$(I_2\ \ )=1 \qquad (\imath_2///)=1,$$

and from Chapter XXII $\qquad (I_2I_2\ )=1 \qquad (\imath_2\imath_2/)=1$

It may be proved that in the above (C) may be substituted for ( ) and (c) for (//) The proof is general, but for the sake of definiteness it is given for the number of conics, which may be described to pass through three given points, touch a line and have a given pair of points for conjugate points



Let the points be A, B, C, the line p and the pair of conjugate points E and F Take any point Q on EF, and let Q' be the harmonic conjugate of Q with regard to E and F Let λ be the number of conics which can be described through A, B, C, Q' to touch p There will then be λ points Q'' on EF corresponding to Q' Again, to any one of these points Q'' there will be λ points Q' Hence the correspondence between Q' and Q'' and therefore between Q and Q'' is λ to λ Hence there are 2λ cases in which Q will be the harmonic conjugate of Q' with respect to E and F But the conics will pass through these points in pairs, since $QQ'EF$ is harmonic Hence 2λ must be divided by 2 and the number of solutions is λ

It was proved in Chapter XVII that

$$(C\ \ )=1 \quad \text{and} \quad \left(c\,{}^{//}_{//}\right)=1,$$

which is consistent with the above result

By the above any ( ) may be replaced by a (C) and afterwards any other ( ) may be replaced by another (C), so that ( ) and (C) are in all cases interchangeable This also applies to (/) and (c)

These results likewise follow from Art 114 (f)

Similarly for (I_2) and (i_2) may be substituted ( ) and (//)

145 Limiting Forms of Conics

If a conic degenerate into a pair of straight lines, it is spoken of as a line-pair, and, if into a pair of points, as a point-pair Under certain circumstances it may become what is termed a line-pair-point The connection of these with the conic has been stated by Professor Cayley as follows

"A Conic is a curve of the second order and second class *quâ* curve of the second order it may degenerate into a pair of lines or line pair (but the class is then $=0$), *qua* curve of the second class it may degenerate into a pair of points, or

point-pair (but the order is then $=0$) The two lines of a line pair may be coincident, and we have then a coincident line pair, such a line pair (it may I think be postulated) ordinarily arises, not from a line pair the two lines of which become coincident, but from a proper conic, flattening by the gradual diminution of its conjugate axis, while its transverse axis remains constant or approaches a limit different from zero, the conic thus tends (not to an indefinitely extended but) to a terminated line*, in other words, the tangents to the Conics become more and more nearly lines through two fixed points, the terminations of the terminated line, and these terminating points, which continue to exist up to the instant when the conjugate axis takes its limiting value $=0$, are regarded as still existing at this instant, and the coincident line-pair as being in fact the point pair formed by the two terminating points Similarly, the two points of a point-pair may be coincident, and we have then a coincident point pair, such a point-pair (it must in like manner be postulated) ordinarily arises, not from a point pair the two points of which become coincident, but from a proper conic sharpening itself to coincide with its asymptotes and so becoming ultimately a pair of lines through the coincident point-pair, and the coincident point pair is regarded as being in fact the line pair formed by the same two lines through the coincident point-pair

"In accordance with the foregoing notions we may with propriety, and it will in the sequel be found convenient to speak of a point pair as a line terminated by two points in this line, and similarly to speak of a line pair as a point terminated (that is the pencil of lines through the point is terminated) by two lines through the point

"If in a point pair thus considered as a line terminated by two points the two points become coincident (the line continuing to exist as a definite line), or, what is the same thing, if in a line-pair thus considered as a point terminated by two lines the two lines become coincident (the point continuing to exist as a definite point), we have a line-pair point, viz this is at once a coincident line pair and a coincident point pair, it may also be regarded as the limit of a conic the axes of which, and the ratio of the conjugate to the transverse axis, all ultimately vanish it may be described as a line terminated each way at a point thereof, or as a point terminated each way at a line through it

"* A line is regarded as extending from any point A thereof to B, and then in the same direction, from B through infinity to A, it thus consists of two portions separated by these points, and considering either portion removed, the remaining portion is a terminated line"

A conic which is a line-pair considered as an envelope consists of two points united at the point of intersection of the line-pair, and a conic which is a point-pair regarded as a locus consists of two straight lines united in the connector of the point-pair

146 Properties of a System of Conics of characteristics μ and ν

(1) The locus of the pole of a given line is a curve of degree ν

Consider in how many points the locus meets the given line Where it meets it the pole of the given line is on the given line, which must therefore be a tangent at

that point to some conic of the system But there are ν such points Hence the locus meets the given line in ν points and is of degree ν

(2) The locus of the points of intersection of tangents from two fixed points to conics of the system is of degree 3ν

Let K, L be the two fixed points The number of points in which this line is met by the locus will determine the degree of the locus For a point P on KL to be on the locus a conic of the system must touch KL at P Then one pair of tangents intersect at P, another at K, and a third at L Hence for every conic which touches KL there are three points on KL Therefore the locus is of degree 3ν and K and L are each double points on the locus of order ν

(3) The envelope of the polars of a given point is a curve of order μ

This is the correlative of (1)

(4) The locus of a point such that its polar with respect to a fixed conic coincides with its polar with respect to a conic of the system is of degree $\mu+\nu$

Consider in how many points the locus meets a given line l Take two points P and P' on l such that p is the polar of P with regard to the fixed conic and the polar of P' with regard to a conic of the system

If P be fixed, p is fixed and by (1) the locus of its poles with respect to the system of conics is a curve of degree ν which determines ν positions of P' on l

If P' is fixed, by (3) its polars with respect to the system of conics envelope a curve of class μ From Q the pole of l with respect to the fixed conic μ tangents can therefore be drawn to this locus Therefore there are μ polars of P', which pass through Q, and their poles with respect to the fixed conic are on l Hence there are μ positions of P on l

Therefore the correspondence between P and P' is a μ to ν correspondence, and the locus meets l in $\mu+\nu$ points and is of degree $\mu+\nu$

(5) The number of conics which touch a fixed conic is $2(\mu+\nu)$

If a conic of the system touch the fixed conic, the point of contact has the same polar with respect to both conics and is therefore on the locus in (4) The points of contact are therefore the intersections of the fixed conic with the locus in (4) There are therefore $2(\mu+\nu)$ points of contact and the same number of conics touching the given one

(6) Verify that there are $2\nu-\mu$ line pairs and $2\mu-\nu$ point pairs in the system

EXAMPLES

(1) The locus of the foci of a system of conics touching four straight lines is a cubic passing through the circular points at infinity

This follows from (2)

(2) If in (2) one of the conditions is that the conic should touch KL, the degree of the locus is 2ν, K and L being double points of order ν

(3) A conic section touches three given straight lines and has one of its principal axes in a given direction, show that the locus of its foci is a cubic curve which passes through the angular points of the triangle formed by the given lines

Pascal Hexagons for six points on a conic

147 In Art 100 it was proved that, if any six points be taken on a conic, the three points of intersection of the pairs of opposite sides of the hexagon so formed lie on a straight line termed the Pascal line of the hexagon in question

Notation The vertices of the Pascal hexagon will be denoted by the numbers 1, 2, 3, 4, 5, 6 The Pascal line of this hexagon is the connector of 12 45, 34 61, and 56 23 If the points are written in reverse order the hexagon is unaltered, thus (346521) is the same as (125643) Also any number of points may be transferred from the beginning to the end of the cycle without altering the hexagon Thus (564123) is the same as (412356)

If three lines are selected by taking the numbers in any pairs, say 35, 41, 26, and a second selection is made in a similar way, say 21, 36, 54, and this system of lines is combined with the former, thus

$$\begin{cases}35\\21\end{cases} \quad \begin{cases}41\\36\end{cases} \quad \begin{cases}26\\54\end{cases}$$

in such a way that no combination contains the same number twice, then a hexagon may always be formed of which the points of intersection of the pairs of lines are the Pascal line

Thus in the instance in question, consider $\begin{cases}35\\21\end{cases}$ in the first place Write down 3, 5, 2, 1 as the 1st, 2nd, 4th, and 5th elements—thus (35 21) In the next combination there is 41 Place a 4 after the 1—thus (35 214) Corresponding to 41 there occurs the line 36 Interchange the 3 and 5—thus (53 214), then insert 6 as the third element In this way (536214) is obtained Then 26 and 54 are the remaining pair of opposite sides

(1) *With the same 6 points, which may be called the P points, as vertices 60 Pascal hexagons can be formed, and corresponding to these hexagons there are 60 different Pascal lines, which may be termed the p lines*

Taking one of the points as the first vertex, the second may be selected in 5 ways, the third in 4, the fourth in 3, the fifth in 2 and the sixth in 1 way Thus 120 hexagons are obtained, but as each occurs with its vertices in reverse order there are 60 hexagons

(2) *There are 15 connectors of the 6 points, termed c lines, which intersect in 45 points, termed T points, in addition to the 6 P points*

A connector c is obtained by joining any one of the 6 P points to any of the remaining 5 P points In this way 30 c connectors are obtained, but each of these connectors is counted twice, *e g* 12 and 21 are the same line Hence the number of connectors is 15 Each of the connectors c meets 4 of the other connectors at each of the P points situated on it These account for 8 of its 14 intersections with c lines There are therefore 6 other T points on each connector Hence, as each T point is counted twice, there are 45 T points

(3) *Through each of the 45 T points 4 Pascal lines pass, and each Pascal line passes through 3 T points*

Let the points P be denoted by the numbers 1, 2, 3, 4, 5, 6. Then the Pascal lines of the hexagons (123456), (123516), (126453) and (126543) all pass through the point 12 45

These hexagons are obtained by keeping one consecutive pair of elements 12 unaltered and in turn changing among themselves the elements of the opposite pair and those of the third or remaining pair

By Pascal's theorem each Pascal line passes through 3 T points

From this the number of Pascal lines can be independently obtained for $45 \times 4 = 180$ But by Pascal's theorem, a Pascal line goes through 3 T points and the number of Pascal lines is therefore $180 - 3 = 60$

(4) *If three alternate vertices of a Pascal hexagon are the same and the other vertices are deduced by interchanging the other three vertices in cyclic order, the three Pascal lines of the hexagons so obtained are concurrent in a point termed a G point*

Consider (123456) (163254) (143652)

Then Pascal lines are

12 45	16 25	14 65
23 56	63 54	43 52
34 61	32 41	36 21

Consider the points of these which are underlined

Three of these are the vertices of a triangle 12 63 45
The other three „ „ „ „ 43 52 61

The three points of intersection of pairs of corresponding sides of these triangles are

$$\begin{Bmatrix}12\\43\end{Bmatrix} \quad \begin{Bmatrix}63\\52\end{Bmatrix} \quad \begin{Bmatrix}45\\61\end{Bmatrix}$$

Since they are on the Pascal line of (125436) they are collinear Hence the connectors of the vertices of the triangles are concurrent, that is the three given Pascal lines are concurrent

If the first hexagon had been (163452), the other hexagons would have been (123654) and (143256), and then Pascal lines would have given another G point Hence *corresponding to any three fixed vertices there are two G points, which may be termed conjugate G points*

(5) *There are 20 G points at each of which 3 Pascal lines intersect*

Taking any vertex for the first of the fixed vertices, the second of the fixed vertices may be selected in 5 ways and the third in 4 ways This gives 20 arrangements of the fixed vertices But if the arrangements are read in opposite order the hexagons are not altered Hence this number must be divided by 2 For each order there are 2 G points Hence the total number of G points is 20

(6) *If on a Pascal line any pair of opposite elements are taken, and these are interchanged in turn with the pairs of adjacent elements, so that they themselves become adjacent elements, then the two Pascal lines so obtained are concurrent with the given Pascal line in an H point, and as this transposition may be accomplished in 3 ways, there are 3 H points on every Pascal line*

Consider ABC the Pascal line of (123456)

Take 12, 34, 56 three of the sides of this hexagon, each of which passes through a different point A, B, C, and which form a triangle (1)

Through each of A, B and C three other Pascal lines pass (3) If any one line be taken out of each of these groups, a triangle is formed which is in perspective with the triangle 12, 34, 56, with ABC for axis of perspective

Three such Pascal lines are

$$P(213546) \quad P(624351) \quad P(132465)$$

The first is obtained from (123456) by interchanging the 12 and the pair of opposite vertices The second and third are obtained in a similar manner by interchanging 34 and 56 and the pairs of opposite vertices The triangle formed by these lines may be termed the triangle (2)

Since the triangles (1) and (2) are in perspective, the lines joining their corresponding vertices are concurrent

But the point of intersection of

$$P(624351) \text{ and } P(132465) \text{ is } 24 \quad 51$$
$$P(132465) \text{ and } P(213546) \text{ is } 46 \quad 13$$
$$P(213546) \text{ and } P(624351) \text{ is } 62 \quad 35$$

Therefore the connectors of 34 56 to 24 51

56 12 to 46 13

12 34 to 62 35

are concurrent

But these lines are the Pascals of

$$(342651), \quad (564213), \quad (126435)$$

The second and third Pascal lines are obtained from the first by interchanging according to the rule the pair of opposite elements 2 and 1 From (5) and (6) it follows that *on every Pascal line there are* 3 H *points and one* G *point*

(7) *There are* 60 H *points, through each of which three Pascal lines pass, and three* H *points are situated on each Pascal line*

This follows from (6)

(8) *The* 60 p *lines may be divided into* 6 *systems of* 10 p *lines, such that the* p *lines of each system intersect each other in threes in the three* H *points which are situated on these lines so determining* 10 H *points*

Consider any three Pascal lines $P(123456)$, $P(214356)$, $P(623541)$, which intersect in an H point The other H points on these Pascal lines are the points of intersection of these Pascals with six other Pascals, viz

$$P(213465), \qquad P(132546), \qquad P(623514)$$
$$P(124365), \qquad P(632451), \qquad P(263514)$$

These form two triangles in perspective and the points of intersection of corresponding sides are three H points, which lie on the Pascal line $P(425136)$ These ten Pascal lines form the system

On each of the 10 p lines 3 H points have been formed and therefore there can be no further H points on these lines None of the other points of intersection of these 10 p lines give G points or T points Hence the three p lines which meet at a G point and the four p lines which meet at a T point all belong to different systems

The Pascal lines of any system may be obtained by writing down any Pascal and interchanging any number of times the elements in any two adjacent pairs The six Pascals through a G point and the conjugate G point thus determine the six different systems

(9) *There are* 20 g *lines each of which passes through one* G *point and three* H *points*

Let the triangle formed by the three Pascal lines

$$P(124635) \quad P(513462) \quad P(241356)$$

and termed triangle (3) be associated with the triangles (1) and (2) of (6)

Each Pascal line is obtained from the corresponding one in (2), by interchanging the elements of the pairs 12, 34, 56, and afterwards interchanging the other two pairs of elements

These lines intersect in the points

$$13 \quad 62$$

$$35 \quad 24$$

and $$46 \quad 51$$

Their connectors respectively to

$$34 \quad 56$$

$$56 \quad 12$$

$$12 \quad 34$$

are the Pascal lines of

$$(342651), \quad (564213), \quad (126435)$$

Hence the three triangles (1), (2) and (3) have a common centre of perspective, viz the H point of (6)

Hence the corresponding sides of (2) and (3) intersect on a straight line, *i e* the points

$$P(213546) \quad P(124635)$$

$$P(624351) \quad P(513462)$$

$$P(132465) \quad P(241356)$$

are collinear But these are 3 H points which are therefore collinear

The axis of perspective of the triangles (2) and (3) passes through the point of intersection of the axes of perspective of (1) and (2) and of (1) and (3) (Art 33 (ii)) The axis of perspective of (1) and (2) is $P(123456)$ The axis of perspective of (1) and (3) is the connector of 12 63, 34 52, 56 14, *i e* $P(143652)$ These intersect in a G point which is therefore collinear with the three H points already found

As there are twenty G points through each of which a g line passes, there are twenty g lines As there are 60 H points and each g line passes through 3 H points, only one g line passes through each H point

The g line through a point G on a Pascal line passes through H points on 3 Pascals obtained by interchanging pairs of opposite vertices The H points on these Pascals are obtained by interchanging the already interchanged vertices with the adjacent elements of the other pairs, so that they become adjacent elements

Thus the g line through the G point on (123456) passes through the following H points, viz

$$\begin{cases} 423156 \\ 241356 \\ 623514 \end{cases} \quad \begin{cases} 153426 \\ 135246 \\ 513462 \end{cases} \quad \begin{cases} 126453 \\ 124635 \\ 362451 \end{cases}$$

(10) *If a triangle be formed by the connectors of three pairs of the P points, the four Pascal lines through each vertex of this triangle intersect in threes in 4 G points and in threes in 4 H points*

Consider the triangle formed by 14, 23 and 56 and let 14 23, 23 56 and 56 14 be C, B and A

By (3) the Pascals through A, B and C are respectively

	A	B	C
(1)	P (563412)	P (234561)	P (145236)
(2)	P (562143)	P (234651)	P (146235)
(3)	P (563142)	P (231564)	P (146325)
(4)	P (562413)	P (231654)	P (145326)

The Pascals given in the horizontal columns intersect by (4) in 4 G points

The Pascal lines may be grouped as follows

	A	B	C
(1)	P (563412)	P (234651)	P (146325)
(2)	P (562143)	P (234561)	P (145326)
(3)	P (563142)	P (231654)	P (145236)
(4)	P (562413)	P (231564)	P (146235)

In this case the Pascals in each row intersect in an H point (6) If 56, 32, 41 are regarded as the pairs of elements and, when they stand consecutively, they are called adjacent, and when not so standing, opposite elements, the following rule is obtained to write down one of the above H points from another

In one Pascal, change the adjacent elements, in another the opposite elements, and in the third both the adjacent and the opposite elements

The same rule holds for obtaining the above G points one from another

Any one of the above 12 Pascals meets others of the 12 Pascals in 3 points at A, B and C, at 2 points at G points, and at 2 points in H points

There are therefore 4 of these Pascals which meet a given Pascal in other points—these Pascals pass two by two through the two vertices of the triangle ABC through which the given Pascal does not pass

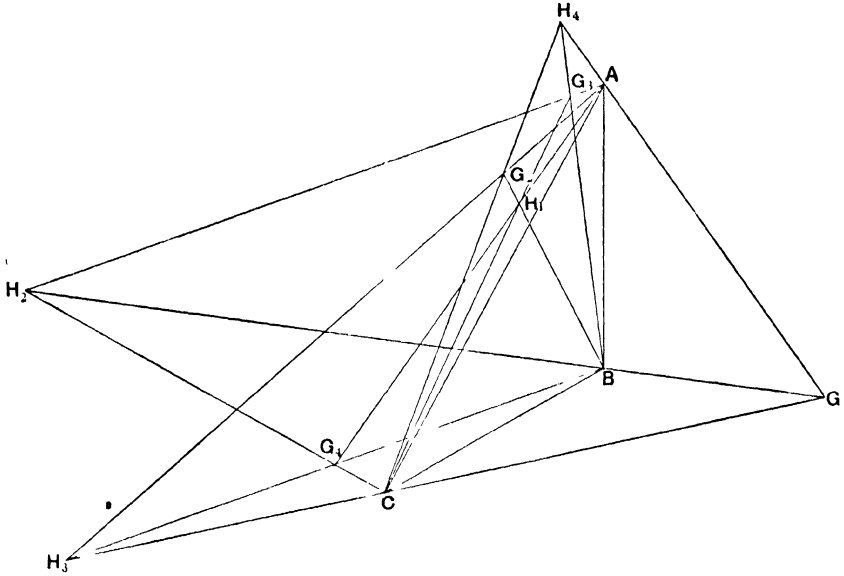
(11) *The four g lines of four G points on Pascals obtained by keeping one pair of adjacent elements fixed and interchanging in turn the elements of the opposite pair and of the third pair are concurrent at an I point*

Thus it has to be proved that the g lines of the G points on P (563412), P (562143), P (563142) and P (562413) are concurrent These are the G points of (10)

By (9) these g lines are determined by joining the respective G points to corresponding H points Thus to obtain these lines it is necessary to join the G point of

P (563412) to	P (562413)	P (564231)
P (562143)	P (563142)	P (561324)
P (563142)	P (562143)	P (561234)
P (562413)	P (563412)	P (564321)

But these H points are respectively the H points (4), (3), (2) and (1) of (10)



Hence the G points and H points form two quadrangles which are in perspective in three ways with the vertices of the triangle ABC and the opposite sides as centres and axes of perspective Hence (Example 4, Art 33) the lines G_1H_1, G_2H_2, G_3H_3, G_4H_4 meet at a point This is the required I point

Fifteen such triangles as ABC can be constructed and therefore the twenty g lines pass 4 by 4 through fifteen I points

(12) *The G points lie 4 by 4 on 15 lines*

Consider the 3 lines (a)

P (126453) P (451362) P (635124)

which meet at an H point

Take three T points on these lines (a), viz

(46 13) (26 15) (24 35)

which form a triangle (1) whose sides are

P (263514) P (246531) P (315462)

Take the three Pascal lines

P (361524) P (461253) P (134526)

They meet on the lines (a) and form a triangle (u) the vertices of which are H points

The corresponding sides of the two triangles (i) and (u) intersect in three collinear points

$$\begin{matrix} P\ (263514) \\ P\ (361524) \end{matrix} \qquad \begin{matrix} P\ (246531) \\ P\ (643521) \end{matrix} \qquad \begin{matrix} P\ (315462) \\ P\ (345261) \end{matrix}$$

which are three G points

The determining Pascals of these G points may be written

$$\begin{matrix} P\ (615243) & P\ (612534) & P\ (613452) \\ P\ (635142) & P\ (642135) & P\ (623154) \end{matrix}$$

The adjacent elements of the Pascals in the upper row are the combinations of 61, 52, 43 One other such combination may be formed, viz $P\ (614325)$ which gives a G point where it is met by $P\ (654123)$ By symmetry this G point must be on the line joining the other three G points whose determining Pascals are given above Three groups such as 61, 52, 43 can be selected from the given 6 P points in 15 ways and therefore there are 15 a lines

If $\equiv$ denotes the word "contains" it will be seen that there are

$$60\ p \text{ lines}, \quad 20\ q \text{ lines}, \quad 15\ a \text{ lines},$$
$$60\ H \text{ points}, \quad 20\ G \text{ points}, \quad 15\ I \text{ points},$$

and that $H \equiv 3\ p$, $G \equiv 1\ q$, $H \equiv 1\ q$, $G \equiv 1\ q + 3\ p$, $I \equiv 4\ q$,
$p \equiv 3\ H$, $q \equiv 1\ G$, $p \equiv 1\ G$, $q \equiv 1\ G + 3\ H$, $a \equiv 4\ G$

EXAMPLES

(1) In (10) three of the anharmonic ratios of the three pencils of Pascal lines through A, B, C are such that their product is unity

(2) Conjugate G points are conjugate points with respect to the conic

CHAPTER XXII

INVOLUTION IN CONNEXION WITH THE CONIC

148 Involutions Determined by Conics

It has been shown Art 95 (*e*) that given a conic every point in its plane has associated with it a definite line called the polar of the point If any point P be taken on a line (l) its polar will meet the line (l) in a point P' and for different positions of P, the points P and P' are pairs of points of an involution (Art 95 (*k*)) For different conics the involutions determined on l will generally be different If the line l meets the conic in real points, these points are the double points of the involution If there are no real double points of the involution, the line does not meet the conic in real points The involution, however, exists in the case of every line and every conic Theorems, therefore, which deal with the points of intersection of lines and conics may in some cases be extended to theorems dealing with conics and the involutions which they determine on lines

If a line passes through two real points of intersection of two conics, both conics determine the same involution on the line, viz, an involution made up of pairs of points which are harmonic conjugates of the two points in which the line meets the two conics Hence, if lines are discovered which do not meet conics in real points but on which the conics determine the same involution, there is a strong analogy between such lines and lines joining the points of intersection of the conics

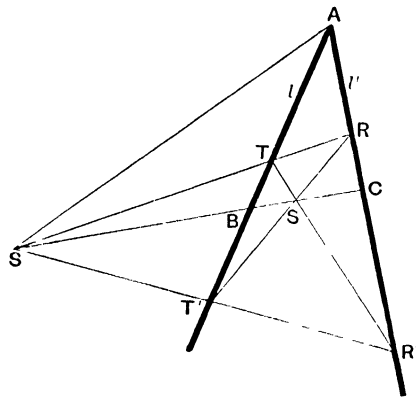
There are at least two straight lines—termed common involution chords—on which a pair of conics determine the same involution (Art 125, Chapter XX)

In this chapter certain theorems concerning conics and the involutions which they determine on straight lines will be proved

If a conic determines two involutions of the same kind on two straight lines l and l' which intersect at A and are in perspective at S and S', then ASS' is a self-conjugate triangle of the conic

(1) If the involutions have real double points, the conic passes through these points and ASS' is the diagonal points triangle of an inscribed quadrangle and is therefore a triangle

(2) If the involutions have imaginary double points, let B and C be the conjugates of A in the involutions, then BC is the polar of A, and S and S' are situated on BC. Since the involution on AB has imaginary double points, a pair of conjugate points T and T' which are harmonic conjugates of A and B can be found, for they are the common conjugates of two involutions, one with real and one with imaginary double points (Art. 55)



Join T and T' to S' to meet l' in R' and R.

Join T' to R' to meet BC in S''. Then, because $ABTT'$ is harmonic, $CBSS''$ is harmonic, therefore S'' coincides with S the second centre of perspective and $R'T'$ passes through S.

Similarly RT passes through S.

Consider the quadrangle $STS'T'$. The ends of the diagonals TT' and RR' are conjugate points with regard to the conic. Therefore S and S' are also conjugate points, and, since SS' is the polar of A, the triangle ASS' is self-conjugate.

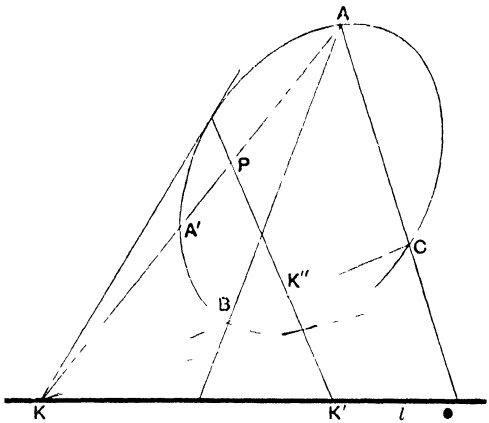
R and R' are harmonic conjugates of A and C, and S and S' may be found at once from T, T' and R, R'.

149 Methods for describing conics to determine given involutions on given lines

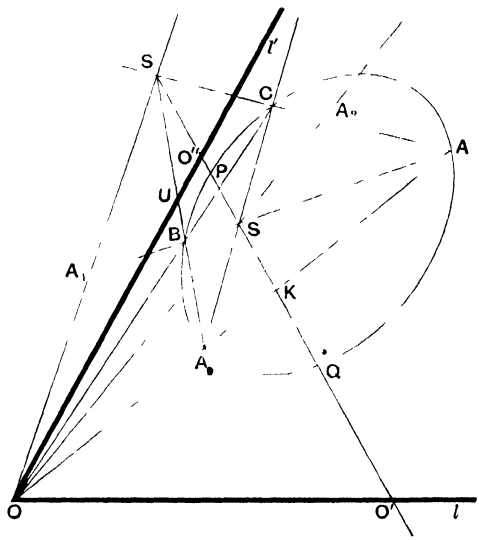
(1) *To describe a conic through three given points to determine a given involution on a given line*

Let A, B, C be the given points and l the given line. Let BC meet l in K. Take K'' the harmonic conjugate of K with respect to BC, and K' the conjugate of K in the given involution. Then $K'K''$ is the polar of K. Let Kl meet $K'K''$ in P. Take A' the harmonic conjugate of A with respect to KP. Then A' is a point

on the conic Similarly points B and C' may be constructed The required conic is the conic through A, B, C, A', B', C'



(2) *To describe a conic through a given point to determine given involutions on two given lines*



Let A be the given point and l and l' the lines on which the involutions are situated Let l and l' meet at O and let O' and O'' be the conjugates of O in the two given involutions

Then $O'O''$ is the polar of O Join OA to meet $O'O''$ in K Take A' the harmonic conjugate of A with respect to OK

Then A' is a point on the curve

Let S and S' be the two points on $O'O'$ which are centres of perspective for the two involutions (Art 60) Then the triangle OSS' is self-conjugate with regard to the conic (Art 148)

Join A to S and S' to meet the opposite sides of this triangle in A_1 and A_2

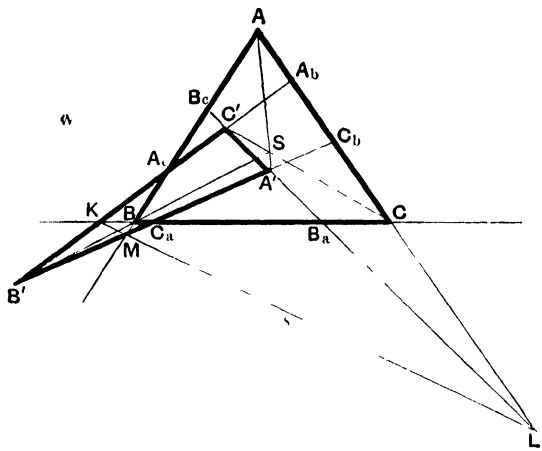
Let B and C be the harmonic conjugates of A with respect to A_1S and A_2S'

Then B and C are on the conic

Let BA meet l' in U The polar of U is the line joining its conjugate on l' to its harmonic conjugate with regard to BA' Let this line meet UA in R Let U' be the harmonic conjugate of A with respect to UR Then U' is also on the curve

Hence the five points A, A', B, C and U' determine the required conic

(3) *To find the necessary and sufficient condition that it may be possible to describe a conic* to determine three given involutions on three given straight lines*



Let A, B, C be the points of intersection of the given lines

Let the polar of A be A_cA_b

" " B " B_cB_a

" " C " C_aC_b

These points determine the involutions on the sides of the triangle ABC

If the polars meet at A', B', C', then the triangles ABC and $A'B'C'$ being polar triangles are in perspective In the figure S is the centre and KLM the axis of perspective Hence by the converse of Pascal's Theorem a conic can be described through the points of intersections of the non corresponding sides Therefore, A_b, A_c, B_a, B_c, C_a, C_b are on a conic

This condition may be shown to be sufficient as follows Let E_a, F_a, E_b, F_b, E_c, F_c be the pairs of common harmonic conjugates of BB_a, CC_a, CC_b, AA_b and AA_c, BB_c respectively Then (Art 59) they are pairs of conjugate points of the involutions determined by BC, B_aC_a, CA, C_bA_b, AB, A_cB_c respectively

Therefore $\dfrac{BA_c}{AA_c}\dfrac{BB_c}{AB_c} = \dfrac{BE_c}{AE_c}\dfrac{BF_c}{AF_c}$ (Art 53) and two similar relations hold for E_a, F_a

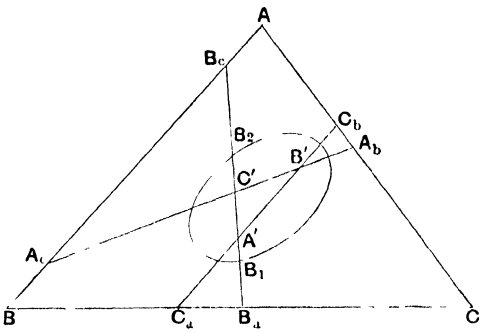
* This conic may be imaginary

and E_b, F_b But Carnot's theorem holds for the points A_c, B_c, C_a, B_a, C_b, A_b and therefore by substituting from these relations it follows that it holds for the points E_a, F_a, E_b, F_b, E_c, F_c Hence these six points lie on a conic real or imaginary The six points A_c, B_c, C_a, B_a, C_b, A_b can always be associated with the vertices of the triangle in such a way that the conic is real

(4) *To describe the conic, if real, to determine on given lines three given involutions which satisfy the condition found in* (3)

If any of the involutions have real double points, construct these points and apply the method given in (2)

If none of the involutions have real double points, let A_cA_b, B_aB_c, C_aC_b be the polars of A, B, C and let them intersect in A', B', C'



Then A' is the pole of BC, therefore A' and B_a are conjugate points Similarly C' and B_c are conjugate points

Therefore B_1 and B_2, the points in which B_cB_a meets the conic, are constructed as the common harmonic conjugates of $A'B_a$ and $C'B_c$ or as the double points of the involution of which these points are pairs of conjugates

Similarly, the points where A_cA_b and C_aC_b meet the conic are determined If none of the sides AB, BC, CA meet the conic in real points and the conic is real, the tangents from A, B, C must be real and therefore the double points of the three involutions are real Hence six points on the conic are given

If two triangles are in perspective, to construct the conic, if real, with respect to which they are pole and polar triangles

The condition set forth in (3) is satisfied, as the triangles are in perspective Therefore the conic described by (4) to determine on the sides of one triangle the involutions determined on them by the sides of the other triangle is the required conic

150 Anharmonic Property of a conic and the involution which it determines on a given line

If a conic determine an involution on a straight line and A_1, B_1 be the projections on the same straight line of any two fixed points on the conic from a variable point on the conic, then

$$\frac{\{(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)\}^2}{(X_1X_2Y_1Y_2)}$$

is constant, where X_1X_2, Y_1Y_2 may be any two pairs of conjugate points of the involution, or if X_1X_2, Y_1Y_2 are two fixed pairs of conjugate points of the involution, $(A_1X_1Y_2B_1) - (A_1Y_1X_2B_1)$ is constant

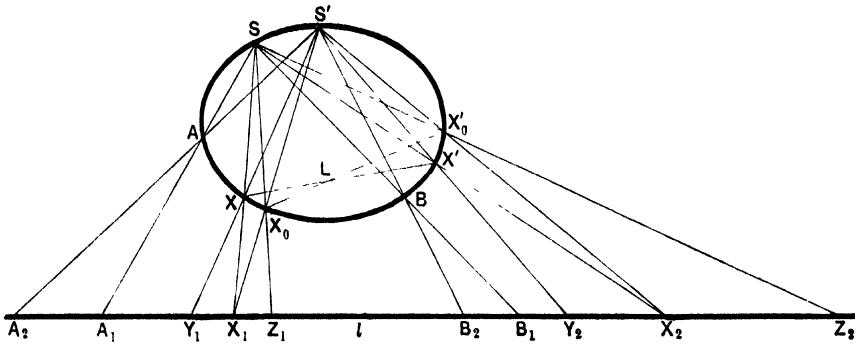
The proof is condensed by determining the positions of the points by means of ratios If X_1, X_2 are taken as the points of reference, *i e* if their ratio coordinates are 0 and ∞ respectively, Y_1, Y_2, A_1, B_1 may be denoted by y_1, y_2, a_1, b_1 Then

$$\frac{\{(A_1X_1Y_2B_1) - (A_1Y_1X_2B_1)\}^2}{(X_1X_2\,Y_1Y_2)}$$

$$= \frac{\left\{\begin{matrix} y_2 - a_1 & b_1 - a_1 \\ y_2 & b_1 \end{matrix} - \begin{matrix} \infty - a_1 & b_1 - a_1 \\ \infty - y_1 & b_1 - y_1 \end{matrix}\right\}^2}{\begin{matrix} y_1 & y_2 \\ y_1 - \infty & y_2 - \infty \end{matrix}}$$

$$= \frac{\left\{\begin{matrix} b_1 y_2 - a_1 & b_1 - y_1 \\ y_2 b_1 - a_1 & b_1 - a_1 \end{matrix}\right\}^2}{\begin{matrix} y_1 \\ y_2 \end{matrix}}$$

$$= \left\{\frac{y_1y_2 - a_1b_1}{(a_1 - b_1)\sqrt{y_1y_2}}\right\}^2$$



Let l be a line which does not meet the conic in real points and let L be its pole Take A and B any two fixed points on the conic and X and X' two points on the conic collinear with L The projections of any such pairs of points from any point on the conic upon l give a pair of conjugate points of the involution determined by the conic on l

Project A, B, X, X' from any point S on the conic into A_1, B_1, X_1, X_2 Take any point S' on the curve Join S' to X_1 X_2 to meet the conic in

X_0, X_0' Then, because X_1, X_2 are conjugate points, X_0, X_0' are collineal with L (Art 95 (*j*))

Project X_0, X_0' from S into Z_1, Z_2

Project A, B, X, X' from S' into A_2, B_2, Y_1, Y_2

Then Z_1Z_2, X_1X_2, Y_1Y_2 are pairs of conjugate points of the involution determined on l by the conic and A_1, B_1, X_1, X_2, Z_1, Z_2 are projective with A_2, B_2, Y_1, Y_2, X_1, X_2

Take $X_1(0)$ and $X_2(\infty)$ as origin and let small letters denote the ratios of the points denoted by the corresponding large letters, then

$$y_1y_2 = z_1z_2 = \text{constant for the involution} \qquad (1),$$

also $$(X_1X_2Z_1Z_2) = (Y_1Y_2X_1X_2),$$

$$\frac{z_1}{z_2} = \frac{y_1}{y_2} \qquad (2),$$

from (1) $z_1 = -y_1$, $z_2 = -y_2$

Again $$(A_1B_1X_1X_2) = (A_2B_2Y_1Y_2),$$

$$\frac{a_1}{b_1} = \frac{y_1 - a_2}{y_1 - b_2}\,\frac{y_2 - a_2}{y_2 - b_2} = \frac{y_1y_2 + a_2b_2 - y_1b_2 + a_2y_2}{y_1y_2 + a_2b_2 - y_1a_2 + b_2y_2}$$

$$(a_1 - b_1)(y_1y_2 + a_2b_2) = a_1(y_1a_2 + b_2y_2) - b_1(y_1b_2 + a_2y_2)$$
$$= y_1(a_1a_2 - b_1b_2) + y_2(a_1b_2 - b_1a_2) \qquad (3),$$

also $$(A_1B_1Z_1Z_2) = (A_2B_2X_1X_2),$$

$$\frac{a_2}{b_2} = \frac{z_1 - a_1}{z_1 - b_1}\,\frac{z_2 - a_1}{z_2 - b_1},$$

$$(a_2 - b_2)(z_1z_2 + a_1b_1) = z_1(a_1a_2 - b_1b_2) - z_2(a_1b_2 - b_1a_2)$$

Writing y_1 for $-z_1$ and y_2 for $-z_2$,

$$(a_2 - b_2)(y_1y_2 + a_1b_1) = -y_1(a_1a_2 - b_1b_2) + y_2(a_1b_2 - b_1a_2) \qquad (4)$$

Square (3) and (4) and subtract

$$(a_1 - b_1)^2(y_1y_2 + a_2b_2)^2 - (a_2 - b_2)^2(y_1y_2 + a_1b_1)^2$$
$$= 4y_1y_2(a_1a_2 - b_1b_2)(a_1b_2 - b_1a_2),$$

$$(a_1 - b_1)^2(y_1y_2 - a_2b_2)^2 - (a_2 - b_2)^2(y_1y_2 - a_1b_1)^2 = 0$$

$$\frac{(y_1y_2 - a_1b_1)^2}{(a_1 - b_1)^2} = \frac{(y_1y_2 - a_2b_2)^2}{(a_2 - b_2)^2}$$

Hence the given expression is constant

151 The expression

$$\frac{\{(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)\}^2}{(X_1X_2)(Y_1Y_2)}$$

will be denoted by $H\{*A_1*B_1 X_1X_2Y_1Y_2\}$ or, shortly, by $H\{A_1B_1\}$.

It is thus seen that, if from any point S on a conic two fixed points A, B are projected into A_1, B_1 on a line, on which the conic determines an involution of which X_1, X_2 and Y_1, Y_2 are any pair of conjugate points, then H is constant.

It should be noticed that

$$H(ABX_1X_2Y_1Y_2)=H(ABY_1Y_2X_1X_2),$$
$$H(ABX_1X_2Y_1Y_2)=H(BAX_1X_2Y_1Y_2),$$
$$H(ABX_1Y_2Y_1Y_2)=H(ABX_2X_1Y_1Y_2),$$
$$H(ABX_1Y_2Y_1Y_2)=H(ABX_1X_2Y_2Y_1).$$

Hence the expression consists of three pairs of elements, and

(a) the elements in any one, or more than one, of the pairs can be interchanged without altering the value of the expression,

(b) the elements in the second and third pairs, that is the pairs of conjugate points of the involution, may be similarly interchanged.

If $X_1\equiv X_2\equiv X$ and $Y_1\equiv Y_2\equiv Y$, then

$$H\{AB\}\equiv\left(\frac{1+(ABXY)}{1-(ABXY)}\right)^2=\{2(AXYB)-1\}^2$$

In this case, if two H's are equal, *either*

$$(AXYB)=(A'X'Y'B') \quad or \quad (AXYB)+(A'X'Y'B')=1,$$

i.e.
$$(AXYB)=1-(A'X'Y'B')-(B'X'Y'A'),$$

where the dashed letters refer to the second H.

Thus if the elements X_1X_2, Y_1Y_2 are the same in both H's

either $\quad (AXYB)=(A'XYB')$

or $\quad (AXYB)=(B'XYA')$

Therefore $\quad (AXYB)=(B'YXA')$

or $\quad (AXYB)=(A'YXB)$

Thus either XY, AB', BA' or XY, AA', BB' form an involution.

On reference to the figure it will be seen that if X_1X_2, Y_1Y_2 are fixed, the ranges described by A_1 and B_1 are projective. On reference to the analytical expression, viz $\left\{\dfrac{y_1y_2-a_1b_1}{(a_1-b_1)\sqrt{y_1y_2}}\right\}^2$, it will be seen that if this expression is constant B_1 will describe one or other of two ranges projective with that described by A_1 according as the + or − sign is given to the square root of the constant.

If $H\{AB\}$ is given and also either A or B, there are two positions of the other according as the + or − sign is taken.

* *Note* When dealing with a conic the points which are projected from any point on the conic are frequently substituted for A_1B_1 in this expression.

The expression $(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)$ may be denoted by $h\{A_1B_1X_1X_2Y_1Y_2\}$ or shortly by $h\{A_1B_1\}$ It is such that, if A_1, B_1 be the projections of any two fixed points on a conic from a variable point on the conic upon a fixed line on which the conic determines an involution of which X_1X_2, Y_1Y_2 are two fixed pairs of conjugates, it is constant, except in regard to sign If X_1, X_2, Y_1, Y_2 are unaltered in order it does not change its sign

It should be noticed that

(i) $$h\{ABX_1X_2Y_1Y_2\}=-h\{BAX_1X_2Y_1Y_2\},$$

$$h\{ABY_1Y_2X_1X_2\}=-h\{ABX_1X_2Y_1Y_2\}$$

Hence, if A and B be interchanged or if the pair of conjugates X_1X_2 be interchanged with the pair of conjugates Y_1Y_2, the sign of the expression is changed

(ii) $$h\{ABX_1X_2Y_1Y_2\}=(X_1X_2Y_1Y_2)\times h\{ABX_1X_2Y_2Y_1\},$$

$$h\{ABX_1X_2Y_1Y_2\}=(X_1X_2Y_1Y_2)\times h\{ABX_2X_1Y_1Y_2\}$$

(iii) The expression $\dfrac{h\{ABX_1X_2Y_1Y_2\}}{\sqrt{(X_1X_2Y_1Y_2)}}$ is such that it is constant when any other pair of conjugates of the involution are substituted for X_1X_2 and for Y_1Y_2

If $$\frac{h\{ABX_1X_2Y_1Y_2\}}{\sqrt{(X_1X_2Y_1Y_2)}}=\frac{h\{A'B'X_1'X_2'Y_1'Y_2'\}}{\sqrt{(X_1'X_2'Y_1'Y_2')}},$$

and X and Y and X' and Y' are the double points of the involutions, it follows that

$$(AXYB)=(A'X'Y'B')$$

If $$h\{ABX_1X_2Y_1Y_2\}=h\{A'B'X_1X_2Y_1Y_2\} \quad \cdot (1),$$

after dividing both sides by $\sqrt{(X_1X_2Y_1Y_2)}$ the double points may be substituted for X_1X_2 and for Y_1Y_2 and then

$$(AXYB)=(A'XYB')=(BYXA')$$

Therefore AB', XY, BA' form an involution

If in (1) the sign of one expression is changed, it follows that

$$AA',\ XY,\ BB' \text{ form an involution}$$

The analytical expression for $h\{A_1B_1X_1X_2Y_1Y_2\}$ is $\dfrac{y_1y_2-a_1b_1}{y_2(b_1-a_1)}$

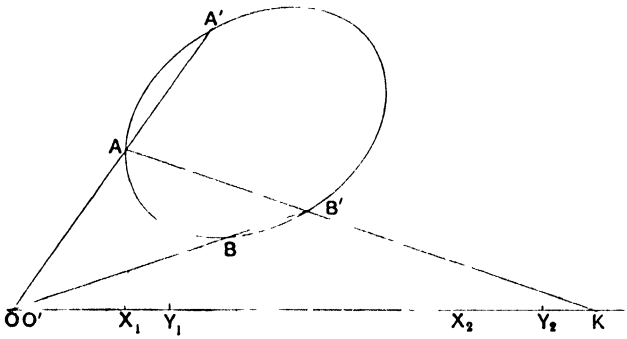
If the elements X_1, X_2, Y_1, Y_2 which determine the involution are given and $h\{AB\}$ is constant, A and B describe projective ranges, and if A is given, B is uniquely determined

If $\dfrac{h\{AB\}}{\sqrt{(X_1X_2Y_1Y_2)}}$ and the involution are given, the ranges A and B are projective, and if A is given, B is uniquely determined

152 Extension of the Involution Property of a Conic

If through any point O three chords be drawn the first two of which meet a conic in AA', BB', while the conic determines on the third an involution X_1X_2, Y_1Y_2, then

$$h\{ABX_1X_2Y_1Y_2\} = h\{A'B'Y_1Y_2X_1X_2\}$$



Let AB' meet the base of the involution at K

$h\{ABX_1X_2Y_1Y_2\}$ by projection from B' becomes $(KX_1Y_2O) - (KY_1X_2O)$

$h\{A'B'Y_1Y_2X_1X_2\}$ by projection from A becomes $(OY_1X_2K) - (OX_1Y_2K)$

But
$$\begin{aligned}(OY_1X_2K) - (OX_1Y_2K) &= (KX_2Y_1O) - (KY_2X_1O)\\ &= 1 - (KY_1X_2O) - 1 + (KX_1Y_2O)\\ &= (KX_1Y_2O) - (KY_1X_2O)\end{aligned}$$

Therefore
$$h\{ABX_1X_2Y_1Y_2\} = h\{A'B'Y_1Y_2X_1X_2\}$$

Deductions
$$\frac{h\{ABX_1X_2Y_1Y_2\}}{\sqrt{(X_1X_2Y_1Y_2)}} = \frac{h\{A'B'Y_1Y_2X_1X_2\}}{\sqrt{(Y_1Y_2X_1X_2)}}$$

and
$$H\{ABX_1X_2Y_1Y_2\} = H\{A'B'X_1X_2Y_1Y_2\}$$

Converse of extension of the Involution Property

If A, A', B, B' are points on a conic which determines an involution X_1Y_2, Y_1Y_2 on a given line, then if $h\{ABX_1X_2Y_1Y_2\} = h\{A'B'Y_1Y_2X_1X_2\}$ the three lines AA', BB' and $X_1X_2Y_1Y_2$ are concurrent

Let AA' meet Y_1Y_2 in O and let BB' meet X_1X_2 in O'

$h\{ABX_1X_2Y_1Y_2\}$ by projection from B' becomes $(KX_1Y_2O) - (KY_1X_2O)$

$h\{A'B'Y_1Y_2X_1X_2\}$ by projection from A becomes $(OY_1X_2K) - (OX_1Y_2K)$

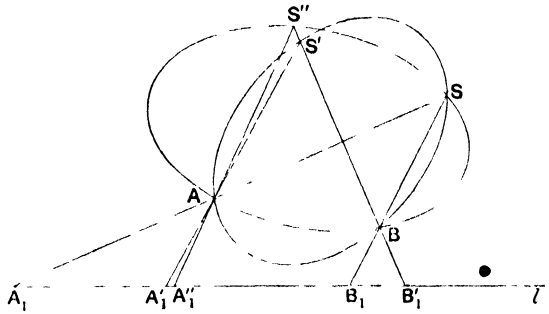
$$\begin{aligned}(KX_1Y_2O) - (KY_1X_2O) &= (OY_1X_2K) - (OX_1Y_2K)\\ &= (KY_1Y_2O) - (KY_1X_2O)\end{aligned}$$

Therefore $h\{KO'X_1X_2Y_1Y_2\}$ and $h\{KOX_1X_2Y_1Y_2\}$ have the same values and O and O' must coincide

If X_1, X_2, Y_1, Y_2 be any four fixed points on a straight line l, A_1, B_1 and A_1', B_1' any other two pairs of points on this line such that

$$h\{A_1B_1X_1X_2Y_1Y_2\}=h\{A_1'B_1'X_1X_2Y_1Y_2\},$$

then, if S and S' be any two points in the plane and A_1S $A_1'S'$ be A and B_1S $B_1'S'$ be B, a conic can be described through A, B, S, S' such that it determines the involution X_1X_2, Y_1Y_2 on l



If the conic which determines the given involution on l and passes through A, B, S does not pass through S', let it meet B_1BS' in S'' Join S'' to 1 to meet l in A_1''

Then $h\{A_1B_1\}=h\{A_1''B_1'\}$ But $h\{A_1B_1\}=h\{A_1'B_1'\}$,

$$h\{A_1''B_1'\}=h\{A_1'B_1'\}$$

Therefore A_1' and A_1'' coincide This proves the theorem

By joining A_1 to S' and A_1' to S a point A' is obtained in lieu of A, and, by joining B_1 to S' and B_1' to S a point B' is obtained in lieu of B

The condition given above likewise assures that a conic can be described through S, S', A', and B' to determine the given involution on l

153 Extension of Carnot's Theorem

If ABC be any triangle and the polars of A, B, C with respect to any conic meet the sides AB, BC, CA in A_cB_c, C_aB_a, A_bC_b as in the figure, then

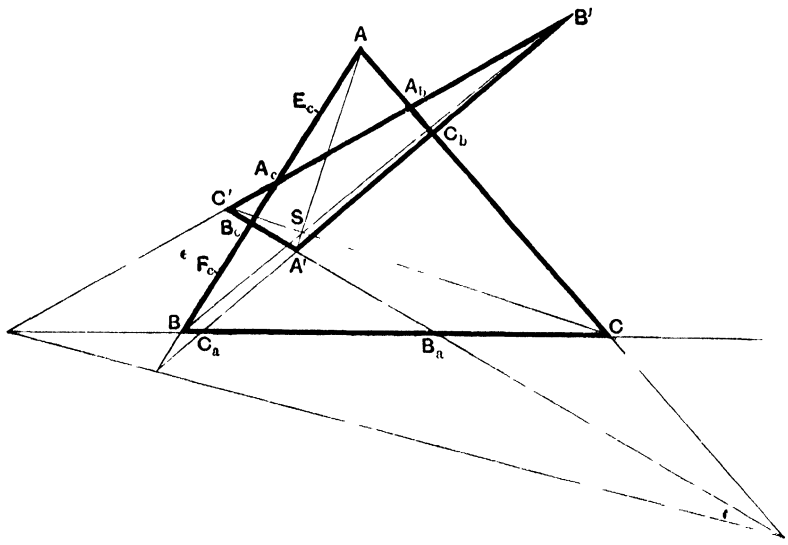
$$\frac{AA_c}{BA_c}\cdot\frac{AB_c}{BB_c}\cdot\frac{BC_a}{CC_a}\cdot\frac{BB_a}{CB_a}\cdot\frac{CA_b}{AA_b}\cdot\frac{CC_b}{AC_b}=1$$

Let the polars of A, B, C meet in A', B', C' Then the triangles ABC, $A'B'C'$ are in perspective (Art 103 (*a*))

Hence a conic can be described through A_b, A_c, B_a, B_c, C_a, C_b and the above expression is equal to unity by Carnot's Theorem

If the conic meets the sides of the triangle ABC in real points E_a, F_a, E_b, F_b, E_c, F_c the connexion between this result and Carnot's

theorem may be shown as follows Since conics through E_a, F_a, E_b, F_b meet AB in E_c, F_c and in A, B, and A_c, B_c are conjugates of the involution determined by A, B, E_c, F_c (Art 59), a conic can be described through E_a, F_a, E_b, F_b and A_c and B_c Hence Carnot's theorem holds for E_a, F_a, E_b, F_b and A_c and B_c, and A_c and B_c can be substituted for E_c and F_c



Similarly, the points where the polars meet the other sides may be substituted for the double points of the involutions on these sides

154 Extension of Desargues' Theorem

Each conic of a system of conics through four fixed points determines on every line in its plane an involution and the double points if real of the involutions on any line form an involution If one of the involutions is given by X_1X_2, Y_1Y_2, and the involution of double points by A_1B_1' and B_1A_1', then these points are connected by the relation

$$h\{A_1B_1X_1X_2Y_1Y_2\} = h\{B_1'A_1'Y_1Y_2X_1X_2\}$$

In the figure of page 345 let A, B, S, S' be the four given points through which the conics are described and let one of the conics determine on the line l an involution of which X_1X_2, Y_1Y_2 are two pairs of conjugate points

Then $$h\{A_1B_1X_1X_2Y_1Y_2\} = h\{A_1'B_1'X_1X_2Y_1Y_2\}$$
$$= h\{B_1'A_1'Y_1Y_2X_1X_2\},$$
$$\frac{h\{A_1B_1X_1X_2Y_1Y_2\}}{\sqrt{(X_1X_2Y_1Y_2)}} = \frac{h\{B_1'A_1'Y_1Y_2X_1X_2\}}{\sqrt{(Y_1Y_2X_1X_2)}}$$

If the involution X_1X_2, Y_1Y_2 has real double points X, Y, they may be substituted for X_1, X_2 and for Y_1, Y_2 in this expression and

$$(A_1XYB_1) = (B_1'YXA_1')$$

Thus X, Y are [illegible] points of the involution determined by A_1B_1' and B_1A_1'

Since

$$(A_1X_1Y_2B_1) - (A_1Y_1X_2B_1) = (A_1'X_1Y_2B_1') - (A_1'Y_1X_2B_1) \qquad (1),$$
$$(X_1A_1B_1Y_2) - (X_1A_1'B_1'Y_2) = (Y_1A_1B_1X_2) - (Y_1A_1'B_1'X_2)$$

Now A_1B_1' and B_1A_1' may be looked upon as determining an involution Then the above becomes

$$h\{X_1Y_2, A_1B_1' A_1'B_1\} = h\{Y_1X_2, A_1B_1', A_1'B_1\},$$
$$\frac{h\{X_1Y_2, A_1B_1', A_1'B_1\}}{\sqrt{(A_1B_1' A_1'B_1)}} = \frac{h\{Y_1X_2, A_1B_1', A_1'B_1\}}{\sqrt{(A_1B_1' A_1'B_1)}}$$

Hence for A_1B_1', $A_1'B_1$ may be substituted in this and consequently in (1) any two pairs of [illegible] points in the involution of double points. These double points when real are the intersections of conics of the system with the line l

Hence the following is obtained

If any three conics are described through four fixed points and the first two meet a given line in K and K', and L and L' and the third determines an involution on the line of which X_1X_2, Y_1Y_2 are any two pairs of conjugate points, then

$$h\{KLX_1X_2Y_1Y_2\} = h\{L'K'X_1X_2Y_1Y_2\}$$

The theorem may also be stated thus

If a system of conics be described through four fixed points, they determine on every line in their plane an involution Let KK', LL' be any two pairs of conjugate points of this involution Let X_1X_2, Y_1Y_2 be any two pairs of conjugate points of the involution determined by any one of the conics on the same line Then

$$h\{KLX_1X_2Y_1Y_2\} = h\{LK'X_1X_2Y_1Y_2\},$$

and
$$h\{X_1Y_2KK'LL'\} = h\{Y_1X_2KK'LL'\}$$

The above relation between the two involutions is reciprocal If KK', LL' is termed involution (1), and X_1Y_2, Y_1Y_2 involution (2), it is known that the double points of (2) when real are a pair of conjugate points of (1) Hence the double points of (1) when real are a pair of conjugate points of (2) But (2) may be the involution determined by any conic of the system Hence the double points of (1) are a pair of conjugate points of the involution determined on the line by any conic of the system

Hence, *if two conics be described through four points R, S, T, U to touch a given line l at X and Y, then X and Y are a pair of conjugate points of the involution determined by l on any conic through R, S, T, U* This is otherwise obvious when the double points of involution (2) are real

The extension of Desargues' theorem may also be expressed in another form

It has already been proved (Art 153) that Carnot's theorem holds when for the points of intersection of a side with the conic are substituted the points where the polars of the vertices, on that side, meet the side in question Hence the following is obtained

If a system of conics be described through four points C_b, A_b, B_a, C_a and any transversal meet the lines A_bA_b and B_aC_a in A and B, then the polars of A and B with respect to any conic of the system meet the transversal in a pair of conjugate points of the same involution as that in which the transversal is met by conics of the system which meet it in real points

EXAMPLES

(1) Show that, if a conic meet the sides AC, CB of a triangle in two pairs of points B_1, B_2, A_1, A_2 whose ratios are b_2b_1 and a_2a_1 and determines on the side AB an involution, the ratio of two pairs of conjugates of which are x_1x_2, y_1y_2, then

$$(x_1+x_2)\{1+y_1y_2a_1b_1a_2b_2\}=(y_1+y_2)\{1+x_1x_2a_1b_1a_2b_2\}$$

Let A_1B_1 and A_2B_2 meet AB in C_3 and C_4 Then we have $h\{AC_4\}=h\{C_3B\}$ Let c_3 and c_4 be the ratios of C_3 and C_4 Express in terms of ratios with A and B as origin and afterwards substitute

$$\frac{1}{a_1b_2} \text{ for } c_4 \text{ and } \frac{1}{a_2b_1} \text{ for } c_3$$

(2) Deduce Carnot's Theorem from (1)

(3) If X and X' be the ratios of the conjugates of A and B in the involution on AB in (1), deduce that

$$a_1a_2b_1b_2XX'=1$$

(4) Prove that if A, B are a pair of conjugate points of the involution on AB in (1) then

$$a_1a_2b_1b_2x_1x_2=-1$$

(5) Show that $a_1a_2b_1b_2x_1x_2y_1y_2=\dfrac{x_1+x_2-y_1+y_2}{\dfrac{1}{y_1}+\dfrac{1}{y_2}-\dfrac{1}{x_1}+\dfrac{1}{x_2}}$

(6) Prove that the general value of $h\{AB\,A_1X_2Y_1Y_2\}$ with any origin is

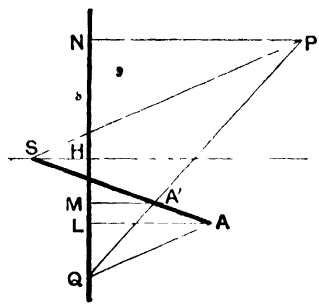
$$\frac{(a+b)(x_1x_2-y_1y_2)+(x_1+x_2)(y_1y_2-ab)-(y_1+y_2)(x_1x_2-ab)}{(b-a)}$$

ADDENDUM

NON-PROJECTIVE PROOFS OF PROPERTIES OF STRAIGHT LINES AND CIRCLES

The straight line

1 *Through A, one of three fixed collinear points A, A', S, a variable line AQ is drawn to meet a fixed line s at Q. A line through S parallel to AQ meets QA' in P. Then the locus of P is a line parallel to s.*



Let L, M, N, H be the feet of the perpendiculars from A, A', P and S respectively on s.

Then $$\frac{PN}{A'M} = \frac{PQ}{A'Q} = \frac{SA}{A'A},$$

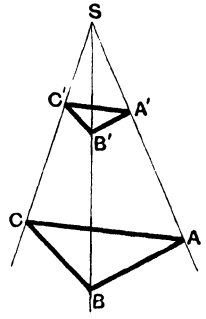
since SP and QA are parallel.

Therefore $$PN = A'M \frac{SA}{A'A},$$

and is constant.

2 (*a*) *If the three pairs of sides of two triangles are parallel, the lines joining the three pairs of corresponding vertices are concurrent.*

Let ABC and $A'B'C'$ be two triangles whose corresponding sides are parallel. Let AA' meet BB' in S and CC' in S_1. It will be shown that S and S_1 coincide.



In the similar triangles SAB and $SA'B'$,

$$\frac{SA'}{SA} = \frac{A'B'}{AB}$$

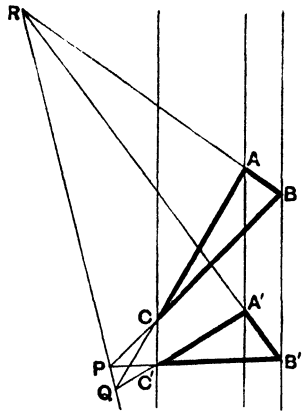
Similarly $$\frac{S_1A'}{S_1A} = \frac{C'A'}{CA}$$

But $$\frac{A'B}{AB} = \frac{C'A'}{CA}$$

Therefore $$\frac{SA'}{SA} = \frac{S_1A}{S_1A},$$

and S and S_1 coincide.

(*b*) *If the lines joining three pairs of corresponding vertices of two triangles are parallel, then the points of intersection of corresponding pairs of sides are collinear*



Let ABC and $A'B'C'$ be the two triangles whose corresponding vertices are on the parallel lines AA', BB', CC'

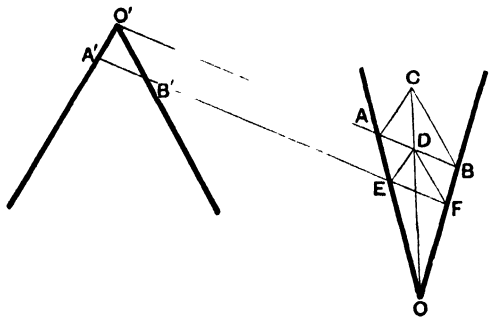
Let CB $C'B'$, AC $A'C'$ and BA $B'A'$ be P, Q and R

Then
$$\frac{BP}{CP}=\frac{BB'}{CC'},$$
$$\frac{CQ}{AQ}=\frac{CC'}{AA'},$$
$$\frac{AR}{BR}=\frac{AA'}{BB'}$$

Therefore
$$\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=1,$$

and by Menelaus' theorem P, Q, R are collinear

3 *To draw a straight line in a given direction so that the intercepts made on it by the sides of two given angles shall be equal*



Let the two given angles be AOB and $A'O'B'$ Through O' draw a line in the given direction to meet OA and OB in A and B Through A and B draw lines AC and BC, parallel respectively to $O'A'$ and $O'B'$, to meet at C Join OC to meet AB at D and through D draw DE and DF, parallel to CA and CB, to meet OA and OB at E and F The line EF is the required line

Let EF meet $O'A'$ and $O'B'$ in A' and B'

Then, since $\frac{OE}{EA}=\frac{OD}{DC}=\frac{OF}{FB}$, EF is parallel to AB Therefore the figure $O'A'ED$ is a parallelogram and $O'A'$ is equal to DE But the triangles $O'A'B'$ and DEF are similar, and, since $O'A'$ equals ED, they are equal Therefore $A'B'$ equals EF and EF, since it is parallel to AB, is the required line

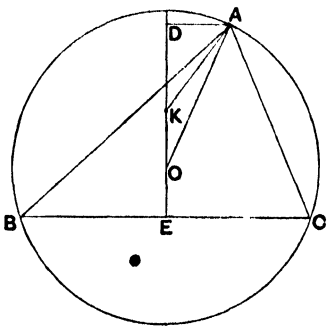
4 *To describe a triangle ABC given* (1) *the base BC,* (2) *the difference of base angles C − B, and* (3) *the product of sides AB AC*

Let the given difference of base angles $= \theta$ and let $\frac{bc}{a^2} = \lambda$, where a, b, c are the sides of the required triangle

Take any circle centre O and draw OA making an angle θ with a diameter OD

From OD cut off OK so that $\frac{OA}{OK} = 4\lambda$

Cut off KE equal to KA and draw BEC perpendicular to DO The triangle ABC will be shown to be similar to the required triangle which can therefore be constructed Draw AD perpendicular to OK, then



$$KE^2 = KA^2 = KD^2 + OA^2 - OD^2,$$
$$OD^2 - KD^2 = OC^2 - KE^2 = OE^2 + EC^2 - KE^2,$$
$$OK(OD + KD) = EC^2 - OK(OE + KE),$$
$$EC^2 = 2\,OK\;DE$$

Therefore since $AB\;AC = 2\;OA\;DE$ and $\frac{OA}{OK} = 4\lambda$,

$$\frac{AB\;AC}{BC^2} = \frac{2\;OA\;DE}{4\,BC^2} = \lambda$$

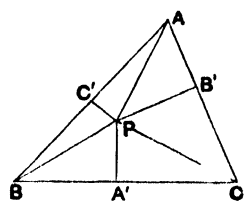
Also $$C - B = A\hat{O}D = \theta$$

Therefore the triangle ABC is similar to the one required

5 *If A′, B′, C′ be three points situated on the sides of the triangle ABC, the necessary and sufficient condition that the perpendiculars through A′, B′, C′ to the sides of the triangle should be concurrent is that*

$$BA'^2 + CB'^2 + AC'^2 = AB'^2 + BC'^2 + CA'^2$$

Let the perpendiculars through A' and C' meet at P Draw PB' perpendicular to AC



Then $BP^2 = BA'^2 + PA'^2$, $BP^2 = BC'^2 + PC'^2$,
$AP^2 = AC'^2 + PC'^2$, $AP^2 = AB'^2 + PB'^2$,
$CP^2 = CB'^2 + PB'^2$, $CP^2 = CA'^2 + PA'^2$,

Hence $BA'^2 + AC'^2 + CB'^2 = AB'^2 + CA'^2 + BC'^2$

Conversely, if the given relation holds, and B'' is the foot of the perpendicular from P on AC, then

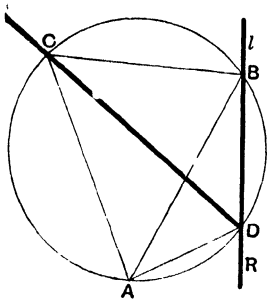
$$CB'^2 - AB'^2 = CB''^2 - AB''^2,$$

or
$$CA(CB' + AB') = CA(CB'' + AB''),$$
$$CB' + AB' = CB'' + AB''$$

Hence $2\;B''B'$ is zero and B' and B'' must coincide

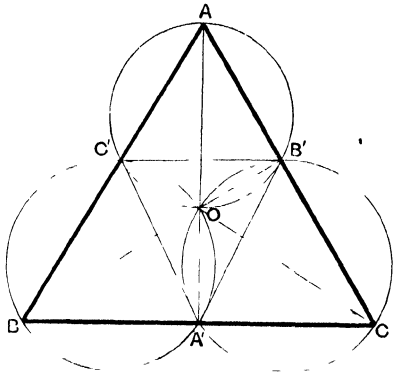
6 *Triangles given in species*

(*a*) *Given a triangle in species, if one vertex is fixed and another moves along a straight line, the locus of the third vertex is likewise a straight line*



Let A be the fixed vertex of the triangle ABC and let B move along the straight line l Describe a circle round ABC to meet l at D Since the angle $ADB = ACB$, the point D is fixed Join CD Since the angle $CDB = BAC$, the line CD is fixed Hence this line is the locus of C

(*b*) *A triangle given in species, and inscribed in a given triangle, may be divided into three triangles each of which is given in species and has one vertex fixed, while its other two vertices are on sides of the given triangle*



Let ABC be the given triangle and $A'B'C'$ the inscribed triangle given in species If circles be described through C, A', B' and B, A', C' to meet in O, the sum of the angles $C'BA'$, $C'OA'$, $A'CB'$ and $A'OB'$ is four right angles and therefore the sum of the angles $C'AB'$ and $C'OB'$ is two right angles Hence a circle can be described through A, C', O, B'

The angle

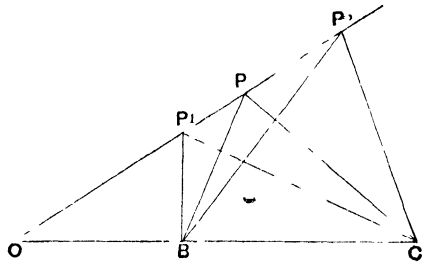
$$\begin{aligned} BOC &= BAC + ABO + ACO \\ &= BAC + C'B'O + OA'B \\ &= BAC + C'B'A' \end{aligned}$$

As these angles are given the angle BOC is constant and O is situated on a given circle through B and C Similarly, it is on a given circle through B and A and is therefore a fixed point

Also the angle $OC'A'$ equals the angle OBC and is therefore constant Hence the triangle $OC'A'$ is given in species One of its vertices O is fixed and the other two move along fixed lines AB and BC

7 *Areas*

(*a*) *If A_1 and A_2 be the areas of the triangles P_1BC and P_2BC and P be any point on the line P_1P_2, then the area PBC equals* $\dfrac{PP_2}{P_1P_2} A_1 + \dfrac{P_1P}{P_1P_2} A_2$



Let P_1P_2 meet BC in O, and let p_1, p_2, p be the perpendiculars from P_1, P_2, P respectively on BC

Then, if A be the area of PBC,

$$\frac{A_1}{A} = \frac{A_2}{p_1} = \frac{p}{p_2} = \frac{OP_1}{OP} = \frac{OP_2}{}$$

Hence it follows that

$$A = \frac{PP_2}{P_1P_2} A_1 + \frac{P_1P}{P_1P_2} A_2$$

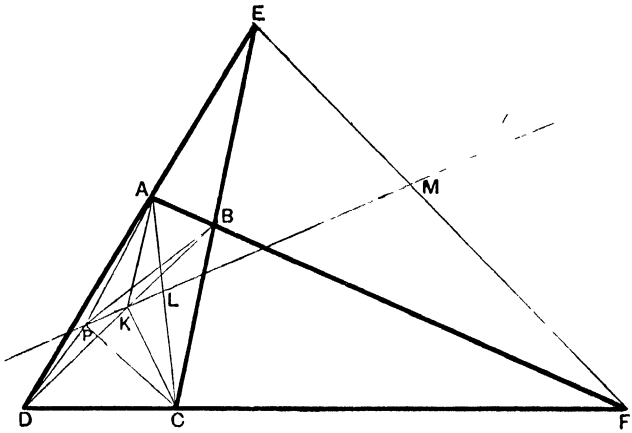
Conversely, if the above relation holds the point P is on the line joining P_1 to P_2

Note In all cases when an area passes through a zero value its sign must be changed

(*b*) *If B_1C_1, B_2C_2, B_3C_3, and D_1F_1, D_2F_2, D_3F_3, be any finite straight lines of limited length in a plane, the locus of a point such that $\Sigma PB_1C_1 = \Sigma PD_1F_1$ is a straight line*

This result may be easily deduced from (*a*)

8 *At any internal point on the connector of the middle points of the diagonals of a quadrilateral the sum of the areas subtended by one pair of opposite sides is equal to the sum of the areas subtended by the other pair*



Let K, L, M be the middle points of the diagonals BD, AC and EF of the quadrilateral

Take P a point on KML inside the quadrilateral

Then, denoting areas of triangles by their three vertices,

$$PDC + PKC = KDC + PDK,$$

and

$$PAB + PKB = ABK + PKA$$

But $PKC = PKA$, since L is the middle point of AC,

and $PKB = PDK$ „ K „ „ „ „ BD

$$PDC + PAB = KDC + KAB$$

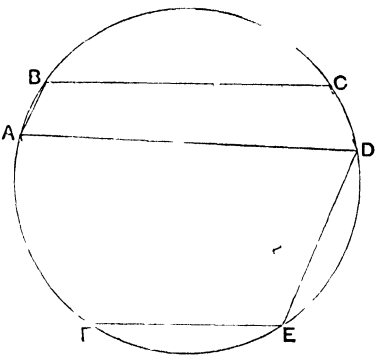
Similarly

$$PAD + PBC = KAD + KBC$$

But $KAB = KAD$ and $KDC = KBC$, since K is the middle point of BD

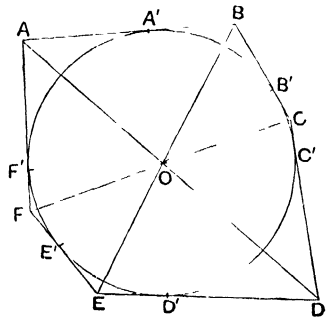
The Circle

9 (*a*) *If two pairs of opposite sides of a hexagon inscribed in a circle are parallel, then the remaining pair of opposite sides are also parallel*



In the hexagon $ABCDEF$, let the pairs of sides AB, ED and BC, EF be parallel Join AD Since AB and BC are parallel to ED and FE, the angles ABC and FED are equal But the angles ADC and ABC together equal two right angles as do the angles FAD and FED Therefore FAD and CDA are equal and the lines CD and AF are parallel

(*b*) *If a hexagon be circumscribed to a circle and two of the lines joining pairs of opposite vertices pass through the centre, then the line joining the third pair of opposite vertices also passes through the centre*



Let $ABCDEF$ be a circumscribed hexagon, the sides of which touch the circle as in the figure at A', B', C', D', E', F' Let AD and BE pass through the centre O Join FO, CO

Then the angle $F'OB' = 2\ AOB = 2\ EOD$

$$= E'OC',$$

also $$F'OF = E'OF$$

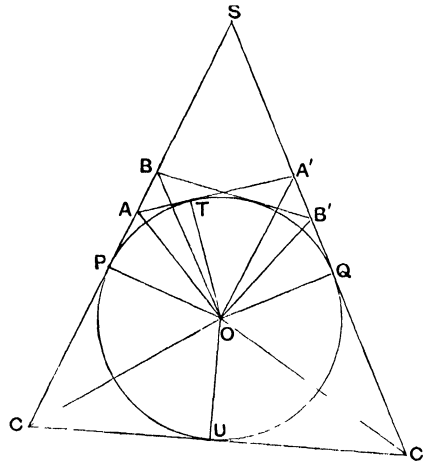
and $$B'OC = C'OC'$$

Therefore, since the angles at O are four right angles, the angle

$$FOC = 2 \text{ right angles}$$

Therefore FO and OC are in the same straight line

10 *If two tangents to a circle intersect two other tangents in A, A' and B, B' and O be the centre of the circle, then the angles AOA' and BOB' are equal or supplemental*



Let P, Q and T be the points of contact of the tangents SB, SA' and AA' Then the angle

$$AOT = \tfrac{1}{2} \text{ angle } POT$$

and the angle

$$A'OT = \tfrac{1}{2} \text{ angle } TOQ$$

Therefore the angle $IOA' = \frac{1}{2}$ angle POQ

and is constant for different tangents AA', BB', Hence

$$IOA' = BOB'$$

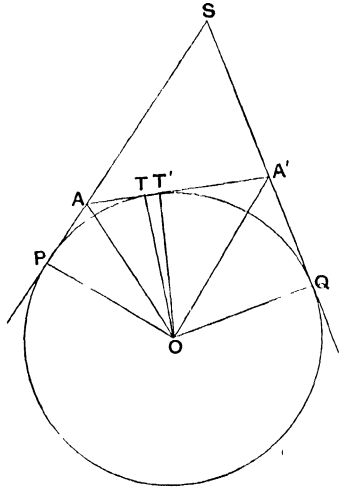
If a tangent CC' be drawn as in the figure,

$$\text{angle } COC' = \tfrac{1}{2}(\text{angle } POU + \text{angle } UOQ) = \tfrac{1}{2}(2\pi - \text{angle } POQ) = \pi - \tfrac{1}{2} \text{ angle } POQ$$
$$= \pi - \text{angle } IOA'$$

Therefore angle COC' + angle $AOA' = \pi$

Conversely

If SP, SQ be any two tangents to a circle of which O is the centre, and two lines OA, OA' containing an angle equal to half POQ be drawn through O to meet SP and SQ in A and A', then AA' is a tangent to the circle



Let the tangents from A and A' meet the circle at T and T'. Then

$$\text{angle } TOA = \tfrac{1}{2} \text{ angle } POT,$$

and $$\text{angle } T'OA' = \tfrac{1}{2} \text{ angle } T'OQ$$

$$\text{angle } TOA + \text{angle } T'OA' = \tfrac{1}{2}\{\text{angle } POT + \text{angle } T'OQ\}$$

But

$$\text{angle } TOA + \text{angle } TOT' + \text{angle } T'OA' = \tfrac{1}{2}\{\text{angle } POT + \text{angle } TOT' + \text{angle } T'OQ\}$$

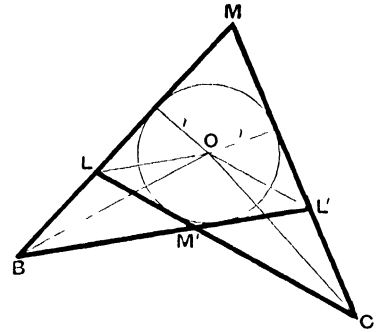
Therefore

$$\text{angle } TOT' = \tfrac{1}{2} \text{ angle } TOT$$

Hence the angle TOT' is zero, and T and T' coincide so that AA' is a tangent to the circle.

11. *If LMM' be a quadrilateral circumscribed to a circle of centre O, and B and C are the points of intersection of ML, $M'L'$ and of ML', LM' respectively, then*

$$\frac{BL}{BO^2} \cdot \frac{BL'}{} = \frac{CL}{CO^2} \cdot \frac{CL'}{}$$



Let r be the radius of the circle. Then the angles LOC and BOL' are equal or supplemental and the angles LOB and $L'OC$ are equal.

Hence considering the areas of the triangles BOL and COL',

$$\frac{BL}{OL} \cdot \frac{}{OB} \sin LOB = 1$$

$$= \frac{L'C}{OL'} \cdot \frac{}{OC} \sin COL',$$

$$\frac{BL}{CL'} = \frac{OL}{OL'} \cdot \frac{OB}{OC}.$$

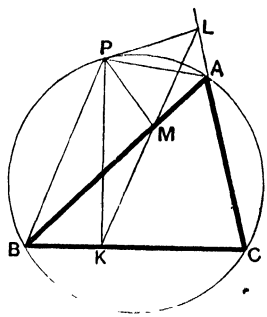
Similarly, from the areas of the triangles LOC and BOL',

$$\frac{BL'}{CL} = \frac{OL'}{OL}\,\frac{OB}{OC'}$$

$$\frac{BL}{CL'}\,\frac{BL'}{CL'} = \frac{OB^2}{OC^2}$$

12 *The feet of the perpendiculars to the sides of a triangle from any point on its circumcircle are collinear*

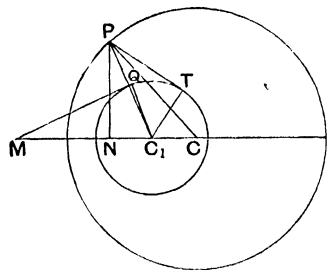
Let ABC be the triangle, P any point on its circumcircle and K, L, M the feet of the perpendiculars from P on the sides of ABC. Join M to K and L.



Angle $PMK = \pi$ − angle PBK
since P, M, K, B are concyclic
− angle PAC
since P, A, C, B are concyclic
$= \pi$ − angle PAL
$= \pi$ − angle PML
since P, M, A, L are concyclic

Therefore MK and ML are in the same straight line.

13 *If M be the point where the radical axis of two circles, whose centres are C and C_1, meets their line of centres, then the square of the tangent from any point P on the first circle to the second circle is equal to $2\ CC_1\ NM$, when N is the foot of the perpendicular from P on the line of centres.*



Let PT be a tangent to the circle centre C_1 from P a point on the circle centre C, and let R and r be the radii of the circles, then

$$PT^2 = C_1P^2 - r^2$$
$$= CC_1^2 - 2\ CN\ CC_1 + R^2 - r^2$$
$$CC_1^2 - 2\ CN\ CC_1 + CM^2 - C_1M^2 \text{ since the tangents from } M \text{ to the circles are equal}$$
$$= CC_1^2 - 2\ CN\ CC_1 + CC_1(CM + C_1M)$$
$$= 2\ CC_1\ CM - 2\ CC_1\ CN$$
$$= 2\ CC_1\ NM$$

14 *If a chord cut two circles in pairs of points which are harmonic conjugates, the locus of the feet of the perpendiculars from the centres of the circles on the chord is a circle.*

Let $AA'BB'$ be a chord which is cut harmonically by the two circles whose centres are C and C' and radii R and R'. Draw the perpendiculars CN and $C'N'$ to $AA'BB'$ and let NT be the tangent from N to the circle whose centre is C'. Join $C'N$ and $C'T$.

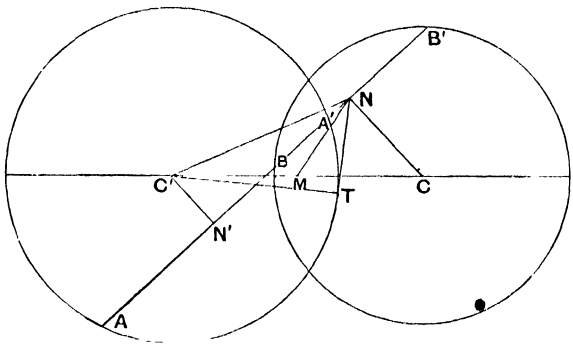
Since $AA'BB'$ is harmonic,

$$NB^2 = NA \cdot NA' = NT^2 = C'N^2 - R'^2$$

Also $$NB^2 = R^2 - CN^2$$

Therefore $$R^2 + R'^2 = CN^2 + C'N^2 = 2C'M^2 + 2MN^2,$$

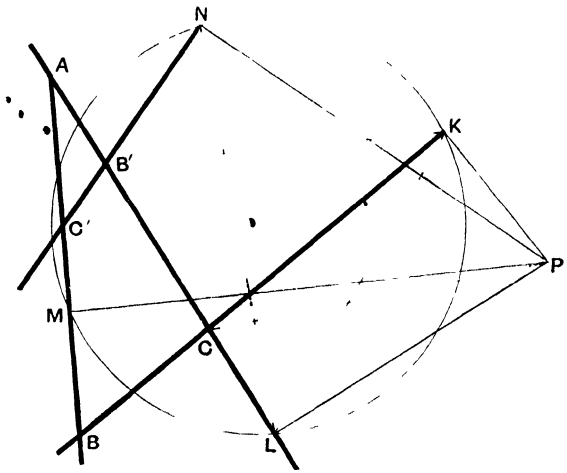
where M is the middle point of CC'



Therefore $MN^2 = \frac{R^2 + R'^2}{2} - C'M^2 = \text{a constant}$

Hence the locus of N is a circle whose centre is at M

15 *If two vertices of a quadrilateral subtend at a given point an angle equal to the angle which the two opposite vertices subtend at the same point, the feet of the perpendiculars from the given point on the sides of the quadrilateral are concyclic*



Let three of the sides of the quadrilateral be lines BC, CA, AB and let the fourth side meet these lines in A', B', C' respectively Let the feet of the perpendiculars from any point P on BC, CA, AB and $B'C'$ be K, L, M and N respectively

Then the angle $NLK = NLP - PLK = NB'P - PCK$

and the angle $NMK = NMP - PMK = NC'P - PBK$

Therefore $NLK - NMK = (NB'P - NC'P) - (PCK - PBK)$

$= B'PC' - CPB$

Hence, if the angles $B'PC'$ and CPB are equal, the angles NLK and NMK are also equal and the points K, L, M, N are concyclic

EXAMPLES

(1) If AA', YY', ZZ' be the diagonals of a quadrilateral circumscribing a circle, L, M, N the middle points, and O the centre of the circle, then

$$\frac{OX}{OL} \cdot \frac{OA'}{} = \frac{OY}{OM} \cdot \frac{OA'}{} = \frac{OZ}{ON} \cdot \frac{OZ'}{}$$

(2) If from the vertices of a triangle perpendiculars be drawn to any straight line, and from the feet of these perpendiculars three perpendiculars are drawn on the opposite sides of the triangle, prove that these perpendiculars are concurrent.

Use Addendum 5

(3) If lines be drawn from a point P on the circumcircle of a triangle, each making an angle α with the perpendiculars from P on the sides of the triangle, the three points in which they meet the sides of the triangle are collinear

(4) If ABC be a triangle of which O is the orthocentre, and through O any two straight lines are drawn at right angles to meet the sides CA and BA in E and F, and through B and C lines BN and CN are drawn parallel to OE and OF respectively, then F, N and E are collinear

INDEX OF TERMS AND DEFINITIONS

The reference numbers refer to pages

Algebraic correspondence, 74, 313
Anharmonic ratio 13
Asymptotes of a conic, 174
Auxiliary circle, 309 (7)
Axes of a conic, 174
Axial pencil, 4
Axis of perspective, 33
of similitude, 157

Carnot's expression, 186
Central line of perspective, 48
Centre of a conic, 173
an involution, 93
curvature, 210
perspective, 33
projection, 32
similitude, 146
Characteristics of systems of conics, 323
Circle, 124 169
Circular points at infinity, 153, 198
Coaxal circles, 144
Common harmonic conjugates, 83
inverse points, 145
involution chords, 151, 269
involution points, 269
Conditions for systems of conics, 323
Confocal conics, 309
Conic, 165, 179, 181
Conic Section, 165
Conics, classification of, 169
Conjugate diameters, 174
imaginary points, 198
lines of an involution pencil, 93
lines with respect to a pair of points, 84
lines with respect to the vertices of a quadrilateral, 101
lines with respect to a circle, 132
lines with respect to a conic, 173
points of an involution range, 91
points with respect to a pair of lines, 84
points with respect to the sides of a quadrangle, 100
points with respect to a circle, 132
points with respect to a conic, 173
triangles with respect to a circle, 136
triangles with respect to a conic, 173
Conjugates, harmonic, 24
Copolar triangles, 55
Correlative of a conic, 167
of a figure, 2
of a pair of points, 169
Correspondence, one to one, 312
one to one, algebraic, 313
m to n, 320
Cross ratio equivalent to anharmonic ratio
Cubic curve, projective definition, 319
Curvature, centre of, 210

Diagonal points triangle of quadrangle, 3
triangle of quadrilateral, 3
Director circle of a conic 220
Directrix of a conic, 176
Double points of an involution, 91
points of an involution of nth order, 322
rays of an involution pencil, 93
Duality, principle of, 2

Ellipse, 169
Envelope, 2
Equianharmonic, 74

Flat pencil, 4
Focus of a conic, 175

Harmonic conjugates, 24
pencil, 25
pencil of tangents to circle, 129
pencil of tangents to conic, 172
perspective, 45, 85
range, 24
range on circle, 129
range on conic, 172
Homological by affinity, 45
Homology 31
Hyperbola, 169
Hyperbola, rectangular, 169

Imaginary points, 92
Infinity, line at, 38
point at, on a line, 9
Inverse points with respect to circle, 142

Involution chords, common, 151, 269
determined by circle, 133
determined by conic, 173
of *n*th order, 321
pencil, 93
pencil of tangents to circle, 127
pencil of tangents to conic, 172
points, common, 269
range, 91
range on circle, 127
range on conic, 172

Limiting forms of conics, 325
Limiting points, 145
Line of planes, 4
Line pair, 325
Line pair point, 325
Locus, 2

Manifold conditions for systems of conics, 323
Maximum and minimum segments of projective ranges, 70
"Meets" equivalent to "points of intersection of"

Notation, 1

One to one correspondence, 312
Orthogonal circle, 146
"Orthogonal involution pencil" equivalent to an involution pencil whose conjugate rays are at right angles

Parabola, 169
Pascal lines, 328
Pedal triangle, 155
Pencil, 4
Pencils, equal, 19
Perspective, 10, 33
anharmonic ratio of, 44
Plane figure, 4
perspective, 31
Point pair, 325
Points, imaginary, 92
double, of an involution, 91
Polar, of a point with respect to a pair of lines, 84
of a point with respect to a triangle, 88
of a point with respect to a circle, 130
of a point with respect to a conic, 172
reciprocation, 5
triangles with respect to circle, 136
triangles with respect to conic, 173
Pole of perspective, 55
of a line with respect to a pair of points, 84
of a line with respect to a triangle, 88
of a line with respect to a circle, 130
of a line with respect to a conic, 172
Power of a point, 156

Principle of duality, 2
Projection, conical, 31
in a plane, 10
orthogonal, 31, 45
Projective expression, 11
pencils equivalent to pencils cut in projective ranges
pencils of tangents to circle, 126
pencils of tangents to conic, 172
pencils, superposed, 58
ranges, 10
ranges on circles, 126
ranges on conics, 172
ranges, superposed, 58
relationship, 11

Quadrangle, complete, 2
Quadrilateral, complete, 2

Radical axis, 145
Range of points, 4
Ranges, equal, 18
equianharmonic, 74
on circle, 126
on conic, 172
projective, 10
similar, 18
uniquely determined, 74
with one to one correspondence, 74
Ratio of a point on a line, 9
Ratios of a point on a plane, 116
Reciprocation, polar, 5
Rectangular hyperbola, 169
Row of points, 4

Section, plane of, 32
Self conjugate triangle with respect to circle, 136
with respect to conic, 173
Self corresponding points of ranges, 58
points of ranges on circle or conic, 21
rays of pencils, 58
rays of tangents to circle or conic, 21
Self perspective, circles in, 136
conics in, 274
Self polar triangle of circle, 136
of conic, 173
Sheaf, figures in, 4
Solid figure, 4
Similar figures, 15
Similitude, axes of, 157
centres of, 146
Superposed ranges, 58
pencils, 58

Triangles given in species, 248

Uniquely determined ranges, 74

Vanishing lines, 37
points, 11

INDEX OF THEOREMS

The reference numbers refer to pages

Angles, measurement of, 9
Anharmonic condition for collinearity of three points, 18
for concurrency of three lines, 17
Anharmonic property of the circle, 125
of the conic 168
of the conic, deductions from, 200
of the conic, extension of, 339
Anharmonic ratio, projective, 13
change of value with change of order of elements, 14
fourth point uniquely determined by, 15
values of, for coincident points, 16
sign of 17
of points on a circle equal to that of the tangents at the same points, 134
of collinear points equal to that of their polars with respect to a circle, 134
of points on a conic equal to that of the tangents at the same points, 173
of collinear points equal to that of their polars with respect to a conic, 173

Brianchon's theorem for a pair of points, 64
for a circle, 162
for a conic, 189
converse of, for a conic, 189
particular cases of, 212
deductions from, 214

Carnot's expression, 50
Carnot's theorem for the circle, 160
correlative of, for the circle, 160
and its converse for the conic, 186
correlative of, and its converse for the conic, 186
particular cases of, 207
deductions from 209
extension of, 345
Centres of conics touching 5 out of 6 lines lie on a conic, 299
which pass through four points lie on a conic, 206, 252
which touch four lines lie on a line, 252
Centres of curvature, 210
Centres of similitude, properties of, 157
of three circles are collinear, 157
Ceva's expression projective, 50
Ceva's theorem, 21, 51
secondary form of, 22
problems on, 78
Characteristics of systems of conics, 323
Chords, common involution, of two circles, 151
common involution, of two conics, 269
of contact of conics touching two lines and passing through two points pass through one or other of two fixed points, 219
of intersection of circle and conic equally inclined to the axes of the conic, 210
of intersection of two conics, 269
Circle, anharmonic property of, 125
projective ranges on and systems of tangents to, 126
involution in connexion with, 127
involution on, construction of, 128
pole and polar with respect to, 130
harmonic ranges on, 129
conjugate points with respect to, 132
construction of conjugate points with respect to, 133
involution determined by, at every point and on every line 133
anharmonic ratios of points on, and tangents to, equal, 134
anharmonic ratios of poles and polars equal, 134
in perspective with itself, 136
in perspective with another circle, 148
complete inscribed quadrangle and circumscribed quadrilateral of, 137
self polar triangle of, 139, 155
Carnot's theorem for, 160
correlative of Carnot's theorem for, 160
Pascal's theorem for, 162
Brianchon's theorem for, 162
Desargues' theorem for, 163
correlative of Desargues' theorem for, 163
Circles, coaxal see Coaxal circles
common involution chords of, 151
condition for intersection of, 46
intersect line at infinity in two fixed points, 152
in perspective with each other, 148
Circular points at infinity, properties of, 198
Coaxal circles, properties of, 144, 147, 156
in self perspective, 146
envelope of polars of fixed point with regard to, a point, 153
locus of pole of fixed line with regard to, a conic, 153
locus of common conjugates of, a circle, 154
cut by every transversal on an involution, 164
Collinearity of three points, anharmonic condition for, 18
of three points by ratios (Menelaus' theorem), 21
of centres of similitude of three circles, 157

Common conjugates, locus of, with respect to three circles, 154
with respect to three conics, 317
with respect to three conics through two points, 317
with respect to three conics, two of which have double contact with the third, 289
of points on a straight line with respect to two conics, 250
of points on a straight line with respect to two conics having double contact, 289
Common harmonic conjugates, construction of, 83
Common involution chords of two circles, 151
of two conics, 269
Common pair of conjugate elements of two involutions, 97
Concurrency of three straight lines, an harmonic condition for, 17
of three straight lines, by ratios (Ceva's theorem), 21
Confocal conics -properties of, 309
are inscribed in a quadrilateral, 309
Conic can be described through five points, 165
construction of tangent to, 167
correlative of, is a conic, 168
anharmonic property of, 168
anharmonic property of, extension of, 339
classification of, 169
construction of, 170
projective ranges on, 172
projective systems of tangents to, 172
involution of points on and of tangents to, 172
chords through a fixed point determine an involution on 172
involution property of, extension of, 344
tangents from a fixed line form an involution of tangents to, 172
harmonic ranges and systems of tangents, 172
properties of pole and polar 173
involutions determined at every point and on every line by, 173
in self perspective, 173
anharmonic ratio of points on, equal to that of tangents 173
anharmonic ratio of collinear points equal to that of polars, 173
foci, properties of see focus of a conic
alternative definitions properties deduced from, 179
inscribed quadrangle and circumscribed quadrilateral, 182
involution on, 183
Carnot's theorem for, and its converse, 186
Carnot's theorem, extension of, 345
Conic (*cont.*)
Carnot's theorem for, correlative of and its converse, 186
Pascal's theorem for, and its converse, 189
Brianchon's theorem for, and its converse, 189
Desargues' theorem for, and its converse, 194
correlative of Desargues' theorem for, and its converse, 194
Desargues theorem, extension of, 346
properties of, involving the imaginary, 197
properties in connexion with the circular points at infinity, 197
conditions that six or more points may lie on a conic, 259
conditions that six or more lines may touch a conic, 259
construction of, under various conditions, 240
Conical projection, nature of the figure obtained by, 10
two angles projected into given angle and a given line into the line at infinity, 41
relation of, to plane perspective, 42
particular cases of, 45
Conics having a triangle for a self conjugate triangle of one and an inscribed triangle of the other, 261
having a triangle for a self conjugate triangle of one and a circumscribed triangle of the other, 261
having a triangle for an inscribed triangle of one and a circumscribed triangle of the other, 261
Conics, limiting forms of, 325
Conics through four points, if two are rectangular hyperbolas all are rectangular hyperbolas, 233
conics through vertices and orthocentre of triangle are rectangular hyperbolas, 233
common conjugate points of, 225
polars of two fixed points form two projective pencils, 225
locus of common conjugates of points on a fixed line is a conic, 250
locus of poles of a fixed line is a conic, 250
locus of centres is a conic through the middle points of sides of quadrangle formed by the four points, 206, 252
Conics touching four lines, common conjugate lines, 225
poles of two fixed lines form two projective ranges, 225
envelope of common conjugates of lines through a fixed point is a conic, 250
envelope of the polars of a fixed point is a conic, 250

Conics touching four lines (*cont.*)
locus of the centre is a straight line through the middle points of the diagonals of the quadrilateral, 252
the director circles of, are coaxal, 223
Conics with common self conjugate triangle, 281
Conics with double contact, in self perspective, 283
in perspective with each other, 283
tangents to one determine projective ranges on another, 284
tangents from points on one to another form projective systems of tangents, 284
tangents to one, are cut in projective ranges by other conics of the system, 287
locus of common conjugates and of poles of a given line, 289
properties of, 287
Conics circumscribed to and inscribed in, a quadrangle (simple), 297
a triangle, 261
Conjugate lines with respect to a pair of points, 84
with respect to a quadrilateral 101
with respect to a circle, 132
with respect to a conic, 173
Conjugate points with respect to a pair of lines, 84
with respect to a quadrangle, 100
with respect to a circle, 132
with respect to a conic, 173
Conjugate vertices of quadrilaterals with respect to a conic, 203
extremities of diagonals of a quadrilateral with respect to a conic, 204
triangles with respect to a conic are in perspective, 203
Construction of a conic, 170
Correlative of a conic is a conic, 168
Correspondence, one to one, 312
one to one when algebraic, 313
one to one, when algebraic, is projective, 314
m to n, 320
Curvature, centre of, construction for, 210

Desargues theorem for pairs of straight lines (Involution property of the quadrangle), 98
for a circle, 163
for a conic and its converse, 194
extension of, 346
correlative of, for pairs of points (Involution property of the quadrilateral), 98
correlative of, for a circle, 163
correlative of, for a conic and its converse, 194
Desargues' theorem and its correlative, particular cases of, 217
deductions from, 219
Director circle of a conic, locus of points of intersection of orthogonal tangents, 220
of all conics inscribed in a quadrilateral are coaxal 223
cuts the circumcircles of all self conjugate triangles orthogonally (Gaskin's theorem), 306
feet of the perpendiculars from a point on, upon polar of the point and upon the sides of any self conjugate triangle are concyclic, 307
Directrix, the locus of the points of intersection of orthogonal tangents to a parabola is the directrix, 222
Distances, theorem concerning, 6

Equianharmonic ranges are projective, 74

False positions, method of 237
Foci of a conic, conjugate lines through, are at right angles 179
tangents from, pass through circular points at infinity, 179
determination of position of, 176
sum or difference of distances of point on conic from constant 178
pair of tangents subtend equal angles at, 178
intercepts by variable tangent on two fixed tangents subtend equal angles at 178
properties of, deduced from alternative definitions, 181
properties of 175, 307
Focus of a circle, its centre, 177
Focus of a conic, ratio of distance of a point on curve from, to distance from corresponding directrix, constant, 177
Focus of a parabola, determination of, 177
circumcircle of any circumscribed triangle passes through 184, 261

Gaskin's theorem, 306

Harmonic conjugates, construction of, 83
common, construction of, 83
Harmonic pencil, properties of, 26
internal and external bisectors of angle form a, 27
of tangents to a circle, 129
of tangents to a conic, 172
Harmonic perspective, 45
triangles in, 55
quadrangles and quadrilaterals in, 85
circles in 136
coaxal circles in, 146
conic in, 173
conics in, 274

Harmonic property of the quadrangle and quadrilateral, 51, 82
Harmonic range, properties of, 24
points dividing segment internally and externally in the same ratio form a, 25
on circle, 129
on conic, 172
Harmonic ranges and pencils in perspective, 81
Homology, 32, 34
determining elements of, 36
Hyperbola, properties of, 201
Hyperbola, rectangular, *see* Rectangular hyperbola

Inverse points with respect to a circle any circle through, cuts given circle orthogonally, 142
Involution chords common of two circles, 151
of two conics, 269
Involution defined, 91
Involution determined by four conics through three fixed points, 204
locus of a point at which six points determine an, 318
Involution of *n*th order defined, 321
of *n*th order has $2(n-1)$ double elements, 322
Involution pencil, defined, 93
has only one pair of conjugate orthogonal rays or all pairs are orthogonal, 101
formed by connectors of common point of intersection of four circles with their other points of intersection, 109
the feet of the perpendicular from any point on conjugate rays of, are collinear with a fixed point, 140
tangents from points on a fixed line to a circle form an involution, 128
tangents from points on a fixed line to a conic form an involution, 172
Involution property of circle, 128
of conic, 172
of conic, extension of, 344
Involution range relation connecting points forming 94
construction of, 95
construction of double points of, 236
summary of properties of, 96
determined by two pairs of points 102
determined on circle by concurrent lines, 128
determined on conic by concurrent lines, 172
Involution ranges and pencils Involution property of the quadrangle and quadrilateral, 98
construction of conjugate elements of, 99
construction of common conjugates of, on a conic, 237
Involution ranges and pencils (*cont.*)
every pair in perspective, 102
analytical formulae connected with, 104
determined by circle on every line and at every point in its plane, 133
determined by conic on every line and at every point in its plane, 173
constant of, determined by conic, 302
Involution ranges with respect to conic, centres of perspective of, and point of intersection of bases, form self conjugate triangle, 336
Involutions, conic constructed to determine, on given lines, 336

Limiting forms of conics, 325
Locus, *see* under nature of construction of the locus

Maximum and minimum segments of superposed projective ranges, 70
Menelaus' theorem, 21, 51
problems on, 78

Newton's theorem, 209
deduction from, 209

Orthogonal circle of three given circles, construction of, 146
of three given circles is the locus of common conjugates of the three circles, 154
Orthogonal circles determine inverse points on diameters, and converse, 142

Parabola, definition of, 169
determination of focus of, 177
properties of, 200
orthogonal tangents intersect on directrix, 184, 222
circumcircle of three tangents passes through focus, 184, 261
the orthocentre of triangle formed by three tangents lies on the directrix, 222
Parabolas, if the four points of intersection of two parabolas are concyclic the axes are perpendicular, 227
Parallel corresponding rays of two projective pencils construction of, 65
pair of harmonic conjugates of two pairs of parallel lines, 98
Pascal lines for six points on a conic, 328
Pascal's theorem for a pair of lines, 64
for a circle, 162
and its converse for a conic, 189
particular cases of, 212
deductions from, 214
Pencil of constant anharmonic ratio formed by tangents to two conics, locus of vertex of 321
Perspective, defined, 32
determining elements of, 36

Perspective (*cont*)
nature of figure obtained by, 40
two angles converted into given angles and a given line into the line at infinity by, 41
relation of, to conical projection, 42
constructed by anharmonic ratios, 44
particular cases of, 45
triangles in, 19, 51
quadrangles and quadrilaterals in, 20
circles in, 148, 158, 159
conics in, 276
conics having double contact in, 283
of a conic, to be a circle, 180
any two quadrangles may be placed in, 114
construction of figures in, 113
properties of circles in, 158
ranges and pencils in, 59
Perspective, harmonic, 45
straight lines in, 85
triangles in, 55
quadrangle and quadrilateral in, 85
circle in, 136
coaxal circles in, 146
conic in, 173
conics in, 274
Perspective, self, *see* Perspective, harmonic
Plane perspective *see* Perspective
Points, conditions that six or more may lie on a conic, 259
Polar of a given point with respect to a conic passes through the conjugate of the given point with respect to every quadrangle inscribed in the conic, 219
Polars of a fixed point with regard to coaxal circles, envelope of, a point 153
conics through four points, envelope of, a point, 225
conics touching four fixed lines, envelope of, a conic, 250
Polars of points on a conic with respect to another conic envelope a conic, 214
of points on a fixed line with respect to conics touching five out of six lines touch a conic, 299
Polars of two fixed points with regard to conics through four fixed points form two projective pencils, 225
Pole and Polar with respect to a pair of points or lines, 85
to a circle, 130
to a conic, 173
to a conic, deductions from, 202
Poles of a fixed line with respect to coaxal circles lie on a conic, 153
conics touching four lines lie on a line, 249
conics through four points lie on a conic, 250
Poles of tangents to a conic with respect to another conic lie on a conic, 214
Poles of a line through a fixed point with regard to a triangle lie on a conic, 205
Poles of a line (*cont*)
with respect to conics through five out of six points lie on a conic, 299
Projection of a conic, to construct as a circle, 180
Projective figures in perspective, 112
placed in plane perspective, 115
with self corresponding quadrangle are identical, 115
self corresponding elements of, 116
analytical equations for, 117
Projective pencils, construction of elements of, 61
parallel corresponding rays of, 65
lines cutting in equal segments, 69
self corresponding rays of, 70
construction of self corresponding rays of, 235
of tangents to a circle, 126
of tangents to a conic, 172
Projective ranges, uniquely determined ranges are, 74
equianharmonic ranges are, 74
range with algebraic one to one correspondence are, 74
product of distances of corresponding points from vanishing points, constant, 12
points at which corresponding segments subtend equal angles, 68
self corresponding points of, construction of, 235
summary of properties of, 69
maximum and minimum segments of superposed projective ranges, 70
summary of properties of superposed projective ranges, 73
on a circle, 126
on a conic, 172
on a conic, construction of, 215
on a conic connectors of corresponding points of, envelope a conic having double contact with given conic, 285

Quadrangle in perspective with a quadrangle, 20
perspectives of, 52
may be projected into any quadrangle, 43
may be placed in plane perspective with any quadrangle, 114
the foundation of relation between two projective figures, 115
anharmonic ratios of figures derived therefrom are numerical, 117
harmonic property of, 51, 82
properties of, derived from harmonic perspective, 85
transversal cutting sides of, 87
involution property of, 98
polars of given point with respect to pairs of opposite sides are concurrent, 200
inscribed in a circle, 137

Quadrangle (*cont*)
inscribed in a conic, 182
inscribed in one conic and circumscribed to another, 297
Quadrilateral, harmonic property of, 51, 82
properties of, from harmonic perspective, 85
poles of given line with respect to ends of diagonals are collinear, 88 (10)
middle points of diagonals are collinear 109
involution property of, 98
orthocentres of triangles formed by sides of, are collinear, 109, 222
circles described on diagonals as diameters are coaxal, with line of orthocentres for radical axis, 109
Quadrilaterals, points of intersection of connectors of middle points of diagonals collinear in certain cases, 119
circumscribed to a circle, 157
circumscribed to a conic, 182
with extremities of two diagonals for conjugate points with respect to a conic have ends of third diagonal for conjugate points, 204
two, with same diagonal triangle have an inscribed conic, 275
with five pairs of vertices for conjugate points with respect to a conic have the sixth pair for conjugate points, 203

Radical axes of three circles concurrent, 146
axis of a system of coaxal circles 145
axis, tangents from points on, to circles equal 145
axis of two circles a common involution chord 151
Rectangular hyperbola, properties of 232
circumcircle of triangle self conjugate with respect to, passes through centre, 261

Self conjugate triangle of a circle, 137, 139, 155
of a conic 173
of a conic cuts director circle orthogonally, 306
of a conic, properties of, 230
conics with common, 281
common, for two conics, 268
Self corresponding elements of ranges and pencils, 58
points of projective ranges, 70, 235
rays of projective pencils 70, 235
Self perspective, *see* Perspective, harmonic
Self polar *see* Self conjugate
Superposed projective ranges self corresponding points, 58, 70
maximum and minimum segments, 70
points at which corresponding segments subtend equal angles, 72
summary of properties of, 73
construction of self corresponding points, 235
tangents from corresponding points of to a conic intersect in a quartic, 316
Superposed projective pencils self corresponding rays, 58, 70
construction of self corresponding rays 235
Systems of conics, characteristics of, 323

Tangents, condition that six or more lines may touch a conic, 259
Triangle, perspectives of, 52
to draw a transversal through a given point to meet the sides of, in a range of given anharmonic ratio, 76
properties of pole and polar of, 87
triangle and conjugate triangle with respect to a conic are in perspective, 203
loci, 247
locus of the poles of a line through a fixed point with regard to, 205
inscribed in one conic and circumscribed in another, 261
inscribed in one conic and self conjugate with regard to another, 261
circumscribed to one conic and self conjugate with regard to another, 261
Triangles in perspective, 19, 51
in perspective, particular cases of, 54
copolar in two ways, 55
given in species, loci connected with, 248

Uniquely determined ranges are projective, 74

Vanishing lines, 37

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