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THE ELEMENTS OF
ACTUARIAL SCIENCE

THE ELEMENTS OF ACTUARIAL SCIENCE

BY

R. E. UNDERWOOD, M.B.E.

ACTUARY OF THE LIVERPOOL VICTORIA FRIENDLY SOCIETY

FELLOW OF THE INSTITUTE OF ACTUARIES

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PREFACE TO FOURTH EDITION

LITTLE alteration has been made for this edition. The section on "Well-known Mortality Tables" has been brought up to date, but examples elsewhere based on what are now unusual rates of interest and out of date rates of tax have not been changed.

R. E. U.

PREFACE TO SECOND EDITION

THIS work is intended as a text book for the subject of Actuarial Science as defined in the syllabus of the examinations of the Institute of Chartered Accountants. The monetary and other tables given at the end of the volume have been inserted by the kind permission of the Institute of Actuaries, from whose publications they have been extracted. In addition, I am to a very considerable extent indebted to the *Institute of Actuaries Text Book*, Part I (new edition), for the list of existing interest tables given at the end of Chapter IV.

Except in the case of Chapter I, the notation is that universally employed by actuaries throughout the world.

The call for a new edition has given opportunity for a complete revision and amplification of certain points.

R. E. U.

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THE ELEMENTS OF ACTUARIAL SCIENCE

CHAPTER I

THE THEORY AND USE OF LOGARITHMS

[A brief note on indices appears at the end of the chapter, for the benefit of those unfamiliar with the theory. This should be read first.]

1. GIVEN that $x^n = y$ then n is the logarithm of y to the base x . This is expressed by the equation $\log_x y = n$.

Similarly, if $t^m = s$, then m is the logarithm of s to the base t .

2. The operation of multiplication may be reduced to that of addition with the aid of logarithms.

It will be observed from article 1 that a logarithm is an index, i.e. the index to which the base must be raised to produce the given number.

If we consider the equation $x^n = y$, we observe that when x and y are given we can find the value of n .

Therefore, if we keep x constant, we can find a value of n , such that $x^n = y$, no matter what be the value of y .

In other words, given the base x , we can express a , b , and c (say) in the form x^a , x^b , and x^c respectively.

A use for logarithms thus becomes apparent, for $a \times b \times c = x^n \times x^m \times x^p = x^{n+m+p}$, and we see, by article 1, that $n + m + p$ is the logarithm of $a \times b \times c$ to the base x (i.e. since $x^{n+m+p} = a \times b \times c$).

Hence, by adding together the logarithms of two or more quantities, we obtain the logarithm of the product of those quantities.

EXAMPLE.—Find by logarithms the product of 16 and 8.

We know that $16 = 2^4$ and $8 = 2^3$

$$\therefore \log_2 16 = 4 \text{ and } \log_2 8 = 3$$

$$\therefore \log_2 16 + \log_2 8 = 7 = \text{logarithm of required product.}$$

Hence, *Answer* $= 2^7 = 128$.

It should be noted that it is of no theoretical consequence to which base the logarithms are taken, provided that the same base is employed throughout the calculation.

3. The operation of division may be reduced to that of subtraction with the aid of logarithms.

Suppose that in article 2 we had desired the value of $\frac{a \times b}{c}$,

we should have had $\frac{a \times b}{c} = \frac{x^n \times x^m}{x^p} = x^{n+m-p}$
and we see that $n + m - p$ is the logarithm of $\frac{a \times b}{c}$
(i.e. since $x^{n+m-p} = \frac{a \times b}{c}$).

Hence, by deducting the logarithm of the denominator from the logarithm of the numerator we obtain the logarithm of the fraction.

EXAMPLE.—Find by logarithms the value of $\frac{27 \times 9 \times 81}{3 \times 243}$

The expression may be written $\frac{3^3 \times 3^3 \times 3^4}{3 \times 3^5}$, and we have

$$\begin{array}{rcl} \log_3 27 & = & 3 \\ \log_3 9 & = & 2 \\ \log_3 81 & = & 4 \\ \hline \text{Total} & = & 9 \end{array}$$

Less—

$$\begin{array}{rcl} \log_3 3 & = & 1 \\ \log_3 243 & = & 5 \\ \hline \text{Total} & = & 6 \end{array}$$

$$\text{Difference} = \underline{3} = \text{logarithm of result.}$$

$$\therefore \text{Answer} = 3^3 = 27.$$

4. The operations of involution and evolution may be reduced to those of multiplication and division with the aid of logarithms.

If we put $a = x^n$ as in the previous article, and raise both sides of the equation to the r^{th} power, we obtain

$$a^r = x^{rn}$$

$$\text{hence } \log_x (a^r) = rn$$

$$\text{But } n = \log_x a$$

$$\therefore rn = r \log_x a$$

$$\therefore \log_x (a^r) = r \log_x a \quad . \quad . \quad . \quad (1)$$

Therefore, the logarithm of the r^{th} power of a number equals r times the logarithm of that number.

EXAMPLE.—Find by logarithms the value of 125^3 .

$$\text{We have } 125 = 5^3$$

$$\therefore \log_5 125 = 3$$

$$\text{and } 3 \log_5 125 = 9 = \text{the logarithm of the result.}$$

$$\therefore \text{Answer} = 5^9 = 1,953,125.$$

5. If we put $r = \frac{1}{r}$ in equation (1), we have $\log_x \left(a^{\frac{1}{r}} \right) = \frac{1}{r} \log_x a$.

Therefore, the logarithm of the r^{th} root of a number equals $\frac{1}{r}$ times the logarithm of that number.

EXAMPLE.—Find by logarithms the cube root of 64.

We have $64 = 2^6$

$\therefore \log_2 64 = 6$

and $\frac{1}{3} \log_2 64 = 2 = \text{the logarithm of the result.}$

$\therefore \text{Answer} = 2^2 = 4.$

6. The logarithm of 1 is 0 whatever the base.

This follows from the equation $x^0 = 1$, in which x can be given any value.

7. The logarithm of the base itself is unity.

This follows from the equation $x^1 = x$.

COMMON LOGARITHMS

8. In order that logarithms may be conveniently used in practice, it is essential to have prepared tables giving at a glance the logarithm of any number to a given base. It has been found that for arithmetical work the base 10 is the most convenient, for in this case we have—

$$10^0 = 1$$

$$10^1 = 10$$

$$10^2 = 100$$

$$10^3 = 1000, \text{ etc., etc.}$$

Therefore, the logarithm to base 10 of any number less than 10 and greater than unity will lie between 1 and 0, i.e. it will be fractional.

Similarly, the logarithm to base 10 of any number less than 100 but not less than 10 will be less than 2, but not less than 1, i.e. it will be 1 or 1 plus a fraction, and the logarithm to base 10 of any number less than 1,000 but not less than 100 will be less than 3 but not less than 2, i.e. it will be 2 or 2 plus a fraction, and so on.

From these considerations we obtain a very convenient rule—

To find the whole number in the logarithm to base 10 of any given number not less than unity, deduct one from the number of digits in the given number.

9. The whole number in a logarithm is called the *characteristic*, the fractional part of the logarithm being called the *mantissa*.

EXAMPLE.—What is the characteristic of the logarithm to base 10 of 99,651?

There are 5 digits, therefore the characteristic is 4. The actual logarithm is 4 plus a fraction.

10. In the case of numbers *less than unity* we have—

$$10^0 = 1$$

$$10^{-1} = .1$$

$$10^{-2} = .01$$

$$10^{-3} = .001, \text{ etc., etc.}$$

Therefore, the logarithm to base 10 of any number not less than .1 but less than 1 will be not less than - 1 but less than 0, i.e. it will be equal to - 1 or - 1 plus a fraction.

Similarly, the logarithm to base 10 of any number not less than .01 but less than .1 will be not less than - 2, but less than - 1, i.e. it will be - 2 or - 2 plus a fraction, and the logarithm to base 10 of any number not less than .001 but less than .01 will be not less than - 3, but less than - 2, i.e. it will be - 3 or - 3 plus a fraction, and so on.

We thus obtain a further convenient rule as follows—

To find the characteristic in the logarithm to base 10 of any given number which is less than unity, add one to the number of ciphers immediately after the decimal point. The resulting characteristic is always negative.

11. It will be seen from this rule that in such cases the characteristic is taken to the next higher unit, the mantissa, which is always positive, reducing the logarithm to the correct figure. For example, $\log_{10} .09965 = \bar{2}.9985$.

In order to show that the characteristic is negative, whilst the mantissa is positive, the minus sign is placed *over* the characteristic instead of before it.

In the above logarithm, the figures $\bar{2}.9985$ indicate $\bar{2} + .9985$, which equals -1.0015 . This arrangement is adopted to avoid the necessity of having exceptions to the convenient rules given above for finding the characteristic.

EXAMPLE.—Find the characteristic of $\log_4 404$. We have—

$$4^4 = 256$$

$$4^5 = 1024$$

$\therefore$ logarithm is between 4 and 5 and characteristic is 4.

12. The mantissae (or fractional parts) of logarithms to base 10 are the same for the logarithms of all numbers which have the same significant figures.

In other words, if a given number is multiplied or divided by 10 or any power of 10, the mantissa will remain unaltered.

A consideration of the effect on the logarithm of multiplying the number by 10^n (where n is any integral quantity, positive or negative) will make this clear.

The effect is to add to the logarithm the quantity $n \log_{10} 10$.

But by article (6) $\log_{10} 10 = 1$,

$\therefore$ the effect is to add n to the logarithm.

But n is an integral quantity, either positive or negative, $\therefore$ the characteristic only is affected, the mantissa remaining unchanged.

We are thus able to tabulate the mantissae within a given range of significant figures and use them in combination with the characteristics to obtain the logarithm of any number.

NAPIERIAN OR NATURAL LOGARITHMS

13. Logarithms were first discovered by Napier, who showed how a complete set could be calculated to base e ,

where $e = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \dots$

The series e can be obtained by expanding the expression $\left(1 + \frac{1}{n}\right)^n$ by the binomial theorem as follows—

$$\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1 - \frac{1}{n}}{2} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{3} + \dots$$

If n be made infinitely great, all the factors involving n in the denominator will become negligible, and the series will become—

$$1 + 1 + \frac{1}{2} + \frac{1}{3} + \dots \quad (2)$$

The series e therefore represents the expansion of $\left(1 + \frac{1}{n}\right)^n$

when n is infinitely great. If the index of the power be changed to nx , we have—

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^{nx} &= 1 + x + \frac{nx(nx-1)}{[2]n^2} + \frac{nx(nx-1)(nx-2)}{[3]n^3} + \dots \\ &= 1 + x + \frac{x\left(x - \frac{1}{n}\right)}{[2]} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{[3]} + \dots \end{aligned}$$

If n be made infinitely great, this series becomes—

$$1 + x + \frac{x^2}{[2]} + \frac{x^3}{[3]} + \dots \quad (3)$$

Series (3) has been obtained by raising series (2) to the power of x . Hence—

$$e^x = 1 + x + \frac{x^2}{[2]} + \frac{x^3}{[3]} + \dots \quad (4)$$

Let $bx = x$, so that

$$e^{bx} = 1 + bx + \frac{b^2x^2}{[2]} + \frac{b^3x^3}{[3]} + \dots$$

Also let $e^b = a$, so that $b = \log_e a$, and—

$$e^{bx} = a^x = 1 + x \log_e a + \frac{x^2 (\log_e a)^2}{[2]} + \frac{x^3 (\log_e a)^3}{[3]} + \dots \quad (5)$$

We have thus produced an expansion of a^x in ascending powers of x . This series is called the *exponential theorem*.

In order to enable us to calculate actual values of logarithms, we require to expand $\log_e (1+x)$ in ascending powers of x . To do this, we select the expression $(1+x)^y$ and expand it, firstly by the exponential theorem and secondly by the binomial theorem. We can then equate

the coefficients of y in the two series and thus obtain a value. For details of the method and its limitations the student is referred to a standard work on algebra. The resulting series is—

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (6)$$

This is known as the *logarithmic series*, and from it and its modifications actual values of logarithms to base e can be calculated. The series is not in itself of great value from a practical point of view, as unless x is very small, it does not converge very rapidly. If, however, we write $x = \frac{1}{n}$ in formula (6), we get—

$$\log_e \left(1 + \frac{1}{n}\right) = \log_e (n + 1) - \log_e n = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots$$

Also, writing $x = -\frac{1}{n}$, and changing signs, we have—

$$-\log_e \left(1 - \frac{1}{n}\right) = \log_e n - \log_e (n - 1) = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \dots$$

Adding these two equations, we have—

$$\log_e (n + 1) - \log_e (n - 1) = 2 \left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} \right) \dots \quad (7)$$

This formula enables us to obtain actual values conveniently. For instance, putting $n = 3$, we get—

$$\log_e 4 - \log_e 2 = 2 \log_e 2 - \log_e 2 = \log_e 2$$

This gives us the logarithm of 2 and its powers.

Similarly, putting $n = 9$, we obtain $\log_e 10 - \log_e 8$, and as the latter is known, being $3 \log_e 2$, we find $\log_e 10$ —and so for other values.

EXAMPLES

Prove that $\log_e(1 + x - 2x^2) = x - \frac{5}{2}x^2 + \frac{7}{3}x^3 - \frac{17}{4}x^4 \dots$

Putting $x = x - 2x^2$ in formula (6), we have—

$$\begin{aligned}\log_e(1 + x - 2x^2) &= x - 2x^2 - \frac{x^2 - 4x^3 + 4x^4}{2} + \\ &\quad \frac{x^3 - 6x^4 + 12x^5 - 8x^6}{3} - \frac{x^4 \text{ etc.}}{4} \\ &= x - (2 + \frac{1}{2})x^2 + (2 + \frac{1}{2})x^3 \\ &\quad - (2 + 2 + \frac{1}{4})x^4 + \text{etc.} \\ &= x - \frac{5}{2}x^2 + \frac{7}{3}x^3 - \frac{17}{4}x^4 + \text{etc.}\end{aligned}$$

Q.E.D.

Prove that $e^{3x} - e^{-3x} = 3x \left(2 + 3x^2 + \frac{27}{20}x^4 + \dots \right)$

Putting $x = 3x$ and $x = -3x$ respectively in formula (4), we obtain the two series—

$$\begin{aligned}e^{3x} &= 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \frac{81x^4}{24} + \frac{243x^5}{120} + \dots \\ \text{and } e^{-3x} &= 1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6} + \frac{81x^4}{24} - \frac{243x^5}{120} + \dots\end{aligned}$$

Subtracting, we obtain—

$$\begin{aligned}e^{3x} - e^{-3x} &= 6x + 9x^3 + \frac{81}{20}x^5 + \dots \\ &= 3x \left(2 + 3x^2 + \frac{27}{20}x^4 + \dots \right)\end{aligned}$$

Q.E.D.

14. Having calculated a complete set of natural logarithms (i.e. to base e), it is desired, for practical convenience, to transfer them to base 10.

Let $x^n = w$ so that $n = \log_x w$,
and let $y^m = w$ so that $m = \log_y w$.

Taking the logarithm to base x of both sides of the equation $y^m = w$, we have—

$$m \log_x y = \log_x w$$

$$\therefore m = \frac{\log_x w}{\log_x y} = \frac{n}{\log_x y}$$

From this we see that to transfer a logarithm from base x to base y we multiply the logarithm by $\frac{1}{\log_x y}$, which is called the **modulus**.

15. The modulus of common logarithms is $\therefore \frac{1}{\log_e 10}$ since they are first calculated to base e and then transferred to base 10.

NOTE.—When it is understood that a certain base is intended, the suffix is omitted. For instance, in theoretical work we write $\log x$ for $\log_e x$, and in practical work $\log 15$ for $\log_{10} 15$.

EXAMPLES

Use the ordinary logarithm table to calculate $\log_{17} 1347$.

$$\log_{10} 1347 = 3.1293$$

To transfer this to base 17 we must multiply by

$$\frac{1}{\log_{10} 17} = \frac{1}{1.2304}$$

$$\therefore \log_{17} 1347 = \frac{3.1293}{1.2304} = 2.545 = \text{Answer.}$$

Use the ordinary logarithm table to calculate $\log_{100}846$.

If x be the required logarithm, then $100^x = 846$, whence $10^{2x} = 846$. From this we see that the logarithm to base 10 will be twice the logarithm to base 100.

$$\text{But } \log_{10}846 = 2.9274$$

$$\therefore \log_{100}846 = 1.4637 = \text{Answer.}$$

Given $\log_{10}2 = .30103$ and $\log_{10}70 = 1.84509$ find $\log_{\sqrt{2}}7$ and $\log_7\sqrt{2}$

$$\text{Let } x = \log_{\sqrt{2}}7$$

$$\text{Then } 2^{\frac{x}{2}} = 7 \text{ and } \frac{x}{2} \log 2 = \log 7$$

$$\therefore x = \frac{2 \times .84509}{.30103} = 5.61465 = \log_{\sqrt{2}}7$$

$$\text{Also, since } \log_a b = \frac{1}{\log_b a}$$

$$\therefore \log_7\sqrt{2} = \frac{1}{\log_{\sqrt{2}}7} = .17815 = \text{Answer.}$$

16. Tables of common logarithms are usually tabulated to seven places of decimals. In view of the principles explained in articles 8 to 12 inclusive, only the decimal parts of the logarithms are shown, the characteristics being obtained by the rules.

As an example of the use of a seven-figure table, we will evaluate the following fraction—

$$\frac{4921821 \times .9932113}{.07766459 \times 666232.1}$$

Taking the mantissae from the tables and calculating the characteristics by the rules, we have—

$$\begin{array}{rcl}
 \log 4921821 & = & 6.6921259 \\
 \log .9932113 & = & \overline{1}.9970416 \\
 \text{Total} & & \hline
 & & 6.6891675 \\
 \\
 \log .07766459 & = & \overline{2}.8902231 \\
 \log 666232.1 & = & 5.8236256 \\
 & & \hline
 \text{Total} & & 4.7138487 \\
 & & \hline
 \text{Difference} & = & 1.9753188 \\
 \text{Antilog difference} & = & 94.47542 = \text{Answer.}
 \end{array}$$

17. When obtaining the total of the logarithms of the numerator it will be observed that 1 is carried over from the first place of decimals, which is added to the positive characteristic 6 and makes + 7, from which is deducted the negative characteristic 1, leaving 6 as the result. Similarly, with the denominator we carry 1 and add it to 5, making 6, from which is deducted the negative 2, leaving 4 as the result. Having obtained the figures in the answer, the decimal place is determined by using the rules conversely.

For instance, in this example, as the characteristic of the logarithm is 1, there must be two figures in the whole number of the natural number.

The student should familiarize himself with the use of a 7-figure logarithm table by working out examples.

18. For many practical purposes a 5-figure table is sufficient, but usually one limited to four places of decimals does not give very accurate results.

19. A 4-figure logarithm table will be found at the end of this volume (Table I), and this has been used in the

following examples. Tables of antilogarithms are sometimes given for convenience, but it is quite easy to obtain the natural number corresponding to a given logarithm by looking for the logarithm in the table and extracting the corresponding natural number. A table of antilogarithms is given in Table II, but it has been thought more instructive to work the examples entirely from Table I.

20. EXAMPLES.—Multiply 14,835 by 177,933.

$$\begin{array}{r} \log 14835 = 4.1715 \\ \log 177933 = 5.2503 \\ \hline 9.4218 \\ \text{Antilog} = 2,641,000,000. \end{array}$$

In this example we are only able to obtain the logarithms of 1484 and 1779, owing to the limitations of the table, these being the numbers given, taken to the *nearest* fourth figure. The characteristics allow for the actual number of digits, and we really obtain the logarithms of 14840 and 177900.

The table gives at a glance the mantissa corresponding to three figures, for instance, we find 14 in the left-hand column and pass along the line to the column headed 8, from which we extract the figures 1703. We then carry the eye along to the column headed "Difference," where we find that a difference of a whole unit makes an increase in the mantissa of 30, from which we calculate that a difference of .4 of a unit makes an increase of 12. The complete mantissa is therefore $1703 + 12 = 1715$.

When taking the antilogarithm, we look amongst the mantissae in the table for 4218 and find 4216 against 264, which constitute accordingly the first three figures of the result. We then see from the difference column that an

increase of 16 in the mantissa corresponds to an increase of 1 in the third figure of the natural number. Hence an increase of 2 in the mantissa corresponds to an increase of $\frac{1}{8}$ of 1 or $\cdot 1$ (i.e. 1 in the fourth figure) in the natural number. Hence the answer is 2641. It should be noted that when using a four-figure table accuracy cannot be obtained by using the differences to obtain more than four figures in the answer. For instance, it would not have produced an exact result to have taken $\frac{1}{8}$ of 1 as $\cdot 125$.

Find the fifth root of $\cdot 19788$.

$\log \cdot 19788 = \bar{1}\cdot 2965$, which we cannot conveniently divide by 5, since it is partly negative and partly positive.

We therefore write

$$\begin{aligned}\bar{1}\cdot 2965 &= \bar{5} + 4\cdot 2965 \\ \text{which, divided by 5,} &= \bar{1} + \cdot 8593 \\ &= \bar{1}\cdot 8593 \\ \text{Antilog} &= \cdot 7233\end{aligned}$$

The characteristic being $\bar{1}$, it follows that the natural number must be a decimal fraction with *no* ciphers after the decimal point.

Evaluate the expression $\left(\frac{1479}{1579}\right)^{\cdot 1361}$

$$\log \left(\frac{1479}{1579}\right)^{\cdot 1361} = \cdot 1361 (\log 1479 - \log 1579)$$

$$\log 1479 = 3\cdot 1700$$

$$\log 1579 = 3\cdot 1984$$

$$\text{Difference} = \overline{1}\cdot 9716$$

which must be multiplied by $\cdot 1361$. This, again, can be done by logarithms.

In order to take the logarithm of $\bar{1}\cdot 9716$, it must first

be written in the form $-.0284$, and omitting the negative sign for the moment, we have—

$$\log .0284 = \bar{2}.4533$$

$$\log .1361 = \bar{1}.1338$$

$$\text{Total} = \underline{\bar{3}.5871}$$

$$\text{Antilog} = .003865.$$

Remembering the negative sign, we have $-.003865$ as the **logarithm** of the required value.

$$\text{But } -.003865 = \bar{1}.996135.$$

Whence—

$$\text{Antilog } \bar{1}.9961 = .9910 = \text{Answer}.$$

21. The student should work out for himself a number of examples with the aid of Table I, until he is perfectly familiar with the use of logarithmic tables for all ordinary arithmetical work. Table II should also be used, as it will be found to reduce the labour.

BRIEF NOTE ON INDICES

$$(a) x^n \times x^m = x^{n+m}$$

$$\text{and similarly } y^r \times y^s \times y^t = y^{r+s+t}$$

That is, the index in the product is the **sum** of the separate indices.

It should be particularly noted that this rule does not enable us to obtain the product of such expressions as $x^n \times y^r$, since x and y are different quantities.

$$(b) x^n \div x^m = x^{n-m}$$

$$\text{or, writing it differently, } \frac{x^n}{x^m} = x^{n-m}$$

That is, the index in the product is the **difference** of the indices.

$$\text{Similarly } \frac{x^n \times x^m}{x^r \times x^s} = x^{n+m-r-s}$$

(c) From (b) it follows that $\frac{1}{x^m} = x^{-m}$, whence a quantity may be transferred from the numerator of a fraction to the denominator and *vice versa*, it only being necessary to change the sign of the index.

$$(d) (x^n)^m = x^{nm}$$

That is, the index in the result is the **product** of the indices.

$$(e) (x^n)^{\frac{1}{m}} = x^{\frac{n}{m}}$$

This follows from (d), by putting $m = \frac{1}{m}$.

CHAPTER II

COMPOUND INTEREST

1. Interest is the consideration for the use of money. It is usually quoted in the form of a rate per cent. per annum to be calculated on the principal sum lent. It is, however, merely a matter of custom to quote the rate *per annum*, since it would serve equally well to quote the rate per half-year or per quinquennium. Further, although the rate per cent. is customarily quoted in practice, the rate per unit is found more convenient in theory.

2. When interest is paid periodically on the amount of the original capital only, it is called *Simple* Interest. Where, however, the interest, as it falls due, is added to the original capital and interest is paid on the total, it is called *Compound* Interest. Simple Interest needs no further explanation, and we may proceed to the consideration of Compound Interest.

NOMINAL AND EFFECTIVE RATES OF INTEREST

3. Let i be the rate of interest per unit per annum. Then a capital sum of 1 will amount in one year, with interest, to $1 + i$. During the second year, the interest earned on capital and interest, i.e. on the amount outstanding at the beginning of the year, will be $i(1 + i)$. Hence 1 will, in two years, amount with compound interest to $(1 + i) + i(1 + i) = (1 + i)^2$.

Similarly, the interest for the third year will be $i(1 + i)^2$, and the total accumulated capital and interest at the

end of the third year, known as the *amount*, will be $(1 + i)^2 + i(1 + i)^2 = (1 + i)^3$.

Following the same argument, the *amount* of 1 with compound interest for n years will be $(1 + i)^n$.

EXAMPLE.—Find the amount of £1,563 with compound interest at 5% per annum for 20 years.

We have $i = .05$

$\therefore$ the amount = 1563×1.05^{20} , and using the logarithm table at the end of the book, we have

$$\begin{array}{rcl} \log 1563 & = & 3.1939 \\ \log 1.05 & = & .0212 \\ \therefore 20 \log 1.05 & = & .4240 \\ \hline \text{Total} & & 3.6179 \\ \text{Antilog} & = & 4149 \end{array}$$

Answer, £4,149.

Table I is too restricted in figures to give us a more exact answer.

4. In practice, interest is frequently payable more often than once a year.

Let j be the rate of interest per unit *per annum* payable m times a year.

Then a capital sum of 1 will amount in $\frac{1}{m}$ th of a year to $1 + \frac{j}{m}$. During the second $\frac{1}{m}$ th of the year the interest earned will be $\frac{j}{m} \left(1 + \frac{j}{m}\right)$. Hence 1 will, by the end of $\frac{2}{m}$ ths of a year, amount to $\left(1 + \frac{j}{m}\right) + \frac{j}{m} \left(1 + \frac{j}{m}\right) = \left(1 + \frac{j}{m}\right)^2$.

Following the same argument, the sum of 1 with

compound interest at rate j per annum payable m times a year, will amount *in one year* to $\left(1 + \frac{j}{m}\right)^m$.

Hence, although j is nominally the rate of interest *per annum*, the effective yield per annum amounts to $\left(1 + \frac{j}{m}\right)^m - 1$.

5. In order that no misunderstanding may arise as to which yield is referred to, the effective yield is spoken of as the **effective rate of interest**, and is symbolized by i .

The nominal yield is spoken of as the **nominal rate of interest**, and is symbolized by j .

6. When interest is payable annually, the nominal and effective rates of interest are equal, but when it is payable more often than annually, the effective rate exceeds the nominal rate. This is so because the effective rate allows for the compounding of interest **during the year**, which is, of course, not allowed for in the nominal rate.

7. The expressions "interest convertible m^{th} ly" and "interest with m^{th} ly rests," mean interest payable m times a year.

8. It will be seen from article 4 that the actual yield (i.e. the effective rate of interest per annum) corresponding to a nominal rate j convertible m^{th} ly, is $\left(1 + \frac{j}{m}\right)^m - 1$.

This is expressed by the important equation—

$$i = \left(1 + \frac{j}{m}\right)^m - 1 \quad . \quad . \quad . \quad . \quad (1)$$

which gives the relation between the nominal and effective rates of interest.

9. It is sometimes required to express the nominal rate of interest j in terms of the effective rate i .

We have, from equation (1),

$$\begin{aligned}\left(1 + \frac{j}{m}\right)^m &= 1 + i \\ \therefore 1 + \frac{j}{m} &= (1 + i)^{\frac{1}{m}} \\ \therefore j &= m \left\{ (1 + i)^{\frac{1}{m}} - 1 \right\} \quad . \quad . \quad (2)\end{aligned}$$

NOTE.—The symbol $j_{(m)}$ is used to denote the nominal rate of interest per unit per annum convertible m times a year, corresponding to the effective rate i .

EXAMPLE I.—Find the effective rate of interest per annum corresponding to a nominal rate of 5% per annum convertible half-yearly.

Using equation (1), and remembering that $j = .05$ and $m = 2$, we have—

$$\begin{aligned}i &= \left(1 + \frac{.05}{2}\right)^2 - 1 \\ &= 1.025^2 - 1 \\ &= .050625.\end{aligned}$$

It is important to avoid too much reliance on formulae, as they sometimes hide the real problem. In the above example the real problem is to find the amount of 1 with compound interest at $2\frac{1}{2}\%$ per **period** for two periods, from which, by deducting the original capital of 1, we derive the **effective** yield. We might, therefore, proceed directly by using the method of article 3, remembering that it holds good for intervals other than periods of a **year**. Thus, the amount is 1.025^2 , and the yield $1.025^2 - 1$, as above.

EXAMPLE II.—Find the nominal rate of interest per annum with quarterly rests corresponding to an effective rate of 4·0604%.

Using equation (2), we have—

$$\begin{aligned} j_{(4)} &= 4\{1.040604^{\frac{1}{4}} - 1\} \\ \log 1.041 &= .0174 \\ \therefore \frac{1}{4} \log 1.041 &= .0044 \\ \text{Antilog} &= 1.010 \\ \therefore j_{(4)} &= 4(1.010 - 1) \\ &= 4 \times .01 \\ &= .04 \end{aligned}$$

10. The amount of 1 with compound interest at the effective rate i for n years is $(1 + i)^n$. Expressing i in terms of j and m by means of formula (1), we have the amount of 1 in n years with compound interest at the nominal rate j convertible m times a year, as—

$$\left(1 + \frac{j}{m}\right)^{mn} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In practice this formula is the same as $(1 + i)^n$, since we merely consider the amount of 1 in mn periods, with interest at the effective rate of $\frac{j}{m}$ per period.

EXAMPLE III.—Find the amount of 1 in 10 years with compound interest at 4% per annum with quarterly rests.

We have, from formula (3)—

$$\left(1 + \frac{.04}{4}\right)^{4 \times 10} = 1.01^{40} = 1.486.$$

Alternatively, we might have avoided formula (3) by considering the problem as one for 40 periods with interest

at the effective rate of $\cdot 01$ per period, thus obtaining $1\cdot 01^{40}$ directly.

It is instructive to note that the result could also have been obtained by first finding the effective rate *per annum* corresponding to a nominal rate of 4%, convertible quarterly, as follows—

From formula (1) we have $i = 1\cdot 01^4 - 1$,

$4 \log 1\cdot 01 = \cdot 0172$ and antilog $\cdot 0172 = 1\cdot 0404$,
whence $i = 1\cdot 0404 - 1 = \cdot 0404$.

It will be observed that in taking the antilogarithm of $\cdot 0172$ we have obtained five figures, as the usual four figures are not sufficient to produce a difference between i and j . A five-figure table was used.

We now find the value of $(1 + i)^n$ where $i = \cdot 0404$ and $n = 10$, that is: $1\cdot 0404^{10}$

$\log 1\cdot 0404 = \cdot 0172$ (using the value from which it was obtained)

$$\therefore 10 \log 1\cdot 0404 = \cdot 1720$$

Antilog $= 1\cdot 486$ as before.

11. We have shown that $1 + i$ is the amount of 1 with interest for one year at the effective rate i per unit per annum. Therefore, conversely, the present value of $1 + i$ due a year hence is 1.

By simple proportion, therefore, we have the present value of 1 due a year hence as $\frac{1}{1 + i}$.

For convenience we write—

$$\frac{1}{1 + i} = v \quad . \quad . \quad . \quad . \quad . \quad (4)$$

12. Similarly, the present value of 1 due n years hence $= \frac{1}{(1 + i)^n}$, which we write v^n .

of discount, but rather to speak of "discounting at a given rate of interest."

15. From formula (5) we see that $1 - v = d$,

$$\therefore 1 - \frac{1}{1+i} = d$$

$$\text{and } \frac{i}{1+i} = d \quad . \quad . \quad . \quad . \quad (6)$$

It will be observed that this formula expresses the fact that the present value of i due a year hence is d . In other words, the interest on a sum of 1 is i if payable at the end of the year, or d if payable at the beginning of the year.

By the use of this formula we can obtain the rate of discount corresponding to a given rate of interest. It is therefore of no consequence whether discounting is based on a rate of interest or on a rate of discount, so long as the rates used are equivalents.

EXAMPLE IV.—What is the rate of interest per annum corresponding to a rate of discount of 5%?

Using formula (6) we have—

$$\frac{i}{1+i} = .05$$

$$\therefore i = .05 + .05 i$$

$$\therefore .95 i = .05$$

$$\text{and } i = \frac{.05}{.95} = .052632 \text{ (approx.)}$$

To prove the accuracy of this answer we may find the amount of the present value at 5% discount of 1 due a year hence.

At 5% discount the present value of 1 due a year hence is .95. Then $.95 \times 1.052632$ should and does equal 1.

16. It is well known that the difference between the rate of interest and the corresponding rate of discount is

one year's interest on the rate of discount, and it is interesting to prove this.

The first formula in article 15 shows that

$$1 - \frac{1}{1+i} = d$$

Multiplying both sides by i we have

$$i - \frac{i}{1+i} = id$$

But from formula (6) we have $\frac{i}{1+i} = d$

$$\therefore i - d = id$$

which is the algebraical equivalent of the above proposition.

17. The theory of discount is similar to the theory of interest. For instance, a little consideration will show that the present value of 1 due n years hence at the *effective* rate of discount d is $(1 - d)^n$.

Hence we have $(1 - d)^n = \frac{1}{(1+i)^n} = v^n$

18. If the nominal rate of discount per unit per annum be called f , and it be convertible m times a year, we have

$$1 - d = \left(1 - \frac{f}{m}\right)^m$$

$$\text{and } (1 - d)^n = \left(1 - \frac{f}{m}\right)^{mn}$$

ANNUITIES-CERTAIN

19. An **annuity-certain** is a series of periodical payments dependent on the lapse of time only, that is the payments do not depend on the happening of any contingency.

An **immediate annuity** is an annuity with the first periodical payment due at the end of the first period.

An **annuity due** is an annuity with the first periodical payment due at once.

PRESENT VALUES

20. An immediate annuity-certain of 1 per annum payable annually for n years, therefore, consists of 1 due a year hence, 1 due two years hence, and so on up to 1 due n years hence.

But the present value of 1 due a year hence is v , and the present value of 1 due two years hence is v^2 , and so on up to—

the present value of 1 due n years hence is v^n , and the present value of the annuity is the sum of the present values of each of the payments.

Let $a_{\overline{n}|}$ = the present value of an immediate annuity-certain of 1 per annum payable annually for n years. (NOTE.—The sign $\overline{}$ indicates that n is a term **certain** and not a life.)

$$\text{Then } a_n = v + v^2 + v^3 + \dots + v^n \quad . \quad . \quad (7)$$

21. Using the well-known formula for the sum of a series which is in Geometrical Progression, we have

$$a_n = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{\frac{1}{v}-1} = \frac{1-v^n}{1+i-1} = \frac{1-v^n}{i}$$

That is, $a_{\overline{n}} = \frac{1 - v^n}{i}$ (8)

22. When dealing with a similar **annuity-due** it must be remembered that the **number** of payments remains unaltered, the difference lying in the dates when the payments are due.

As the first payment is at once, its present value is 1 and the present values of its successive payments will be $v, v^2, \dots, v^{n-1}$ respectively. There will be no value v^n , since the last payment is at the beginning of the year, that is at the end of the $(n-1)^{th}$ year.

Hence the present value of a **certain annuity-due** of 1 per annum payable annually for n years is $1 + a_{\overline{n}|i}$.

For brevity an annuity-due is symbolized by " a ," so that

$$a_{\overline{n}|} = 1 + a_{\overline{n-1}|} \quad . \quad . \quad . \quad . \quad (9)$$

23. If the annuity is payable more often than yearly, two cases arise—

(1) Where interest is convertible with the same frequency as the annuity payments, and

(2) Where interest is convertible with a different frequency from the annuity payments.

24. In case (1) we merely have to consider periods instead of years. Suppose the annuity to be 1 per annum, payable m times a year for n years, and interest to be at the nominal rate $j_{(m)}$.

Then the annuity is a series of payments of $\frac{1}{m}$ payable at intervals of $\frac{1}{m}$ of a year for $m \times n$ periods, with interest at the **effective** rate of $\frac{j}{m}$ per period.

Adapting formula (8), and writing the present value of the annuity as $a_{\overline{n}|}^{(m)}$, to signify that it is payable m times a year, we have—

$$\begin{aligned} a_{\overline{n}|}^{(m)} &= \frac{1}{m} \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\frac{j}{m}} \\ &= \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{j} \quad . \quad . \quad . \quad (10) \end{aligned}$$

To show how simple is the application of this formula in practice, we will calculate the present value of an immediate annuity-certain of 1 per annum payable half-yearly for 10 years, with interest at 4% per annum with half-yearly rests.

Using formula (10) we have

$$a_{\overline{10}|}^{(2)} = \frac{1 - 1.02 \cdot 20}{.04} = 8.175$$

This is of course formula (8) multiplied by the periodical payment of $\frac{1}{2}$, each period being half a year.

Had the annuity been *due* instead of *immediate* we should

have taken $\frac{1}{2} + a_{\overline{9}|}^{(2)}$ or generally $\frac{1}{m} + a_{\overline{n - \frac{1}{m}}|}^{(m)}$

25. Passing to case (2), the present value of an immediate annuity-certain of 1 per annum, payable p times a year for n years with interest at the effective rate i , is the sum

of the series $\frac{1}{p} \left(v^{\frac{1}{p}} + v^{\frac{2}{p}} + \dots + v^n \right)$

$$\therefore a_{\overline{n}|}^{(p)} = \frac{1}{p} \cdot \frac{v^{\frac{1}{p}} (1 - v^n)}{1 - v^{\frac{1}{p}}} = \frac{1 - v^n}{p \left(v^{-\frac{1}{p}} - 1 \right)} =$$

$$\frac{1 - v^n}{p \left\{ (1 + i)^{\frac{1}{p}} - 1 \right\}}$$

$$\text{That is, } a_{\overline{n}|}^{(p)} = \frac{1 - v^n}{p \left\{ (1 + i)^{\frac{1}{p}} - 1 \right\}} \quad \cdot \quad \cdot \quad \cdot \quad (11)$$

It will be observed that this is formula (8), with the exception that in place of i we have the nominal rate of interest convertible p times a year corresponding to the effective rate i . This latter is conveniently written $j_{(p)}$, so that formula (11) becomes—

$$a_{\overline{n}|}^{(p)} = \frac{1 - v^n}{j_{(p)}} \quad . \quad . \quad . \quad . \quad (12)$$

26. We can always find the effective rate i corresponding to any nominal rate, so that formula (12) covers the case where the annuity is payable p times a year and interest is at the nominal rate j convertible m times a year. It is, however, an interesting exercise to produce a formula for this special case.

Putting $(1 + i) = \left(1 + \frac{j}{m}\right)^m$ in formula (11), we have

$$a_{\overline{n}|}^{(p)} = \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{p \left\{ \left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1 \right\}} \quad . \quad . \quad . \quad . \quad (13)$$

27. The student is particularly warned against learning by heart the many complicated formulae given in this chapter. The line of argument on which they are based should, however, be mastered, and when this has been done it will readily be appreciated how little it is necessary to know by heart.

EXAMPLE V.—Find the present value of an immediate annuity-certain of £10 per annum payable quarterly for 20 years, with interest at the effective rate of 5% per annum.

This is a case where the ordinary formula is applicable but with $j_{(4)}$ substituted for i in the denominator (see formula (12)).

$$\text{Now } j_{(4)} = 4 (1.05^{\frac{1}{4}} - 1)$$

$$\text{and } \log 1.05 = .0212$$

$$\therefore \frac{1}{4} \log 1.05 = .0053$$

$$\text{Antilog } .0053 = 1.0123 \text{ (taken to 5 figures for increased accuracy)}$$

$$\therefore j_{(4)} = 4 (1.0123 - 1) = .0492$$

$$\text{Also } v^{20} = 1.05^{-20} = .3767$$

$$\text{Therefore } a_{\overline{20}|}^{(4)} = \frac{1 - .3767}{.0492} = \frac{.6233}{.0492} = 12.67$$

and value of £10 per annum = £126.7.

NOTE.—This example suffers from the inadequacy of the four-figure logarithm table, and if five figures had not been used in one place the result would have been quite inaccurate.

28. Formula (12) might be written

$$\frac{1 - v^n}{i} \times \frac{i}{j_{(p)}} = a_{\overline{n}|} \times \frac{i}{j_{(p)}}.$$

In some works a table of the value of $\frac{i}{j_{(p)}}$ for various values of i and p is given. When such a table is available, the calculation becomes simply the multiplication of $a_{\overline{n}|}$ by $\frac{i}{j_{(p)}}$.

The above example could be calculated approximately by the following simple method—

A payment of 1 in 3 months' time will amount, with interest, in following 9 months to	1·0375
A payment of 1 in 6 months' time will amount, with interest, in following 6 months to	1·025
A payment of 1 in 9 months' time will amount, with interest, in following 3 months to	1·0125
A payment of 1 in 12 months' time will amount, with interest, in following 0 months to	1·
∴ Total amount of four payments, with interest, by end of year	<u>4·075</u>

∴ An immediate annuity of 1 per annum payable quarterly is the equivalent of an immediate annuity of $\frac{4·075}{4} = 1·01875$ payable annually. This is, of course, only

approximate, since the amount of the first quarterly payment should have been $1·05^{\frac{1}{4}}$ instead of $1 + \frac{1}{4} \times .05$ and so on. The present value will, therefore, be $10 \times 1·01875 \times a_{\overline{20}|5\%} = 126·96$, which compares very favourably with the true value as calculated in Example V.

AMOUNTS

29. Hitherto we have dealt only with the present values of annuities-certain, but we require also to know the **amounts** to which they will accumulate in a given number of years. The theory is similar in every way to that investigated for present values.

30. Following the lines of article 20, the payment which is made in one year's time will amount to $(1 + i)^{n-1}$ by the end of the n^{th} year, and the payment which is made in two years' time will amount to $(1 + i)^{n-2}$ by the end of the n^{th} year, and so on.

Let $s_{\overline{n}|i}$ = the amount of an immediate annuity-certain of 1 per annum payable annually for n years.

Then $s_{\overline{n}|i} = (1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \dots + (1+i) + 1$.

Summing the series, we get

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \quad (14)$$

31. The **amount** of an annuity-certain is the accumulated value at the end of a given number of years, whereas the **present value** is the value now. It therefore follows that if we discount the **amount** down to the present date we obtain the **present value**. In symbols—

$$a_{\overline{n}|i} = v^n s_{\overline{n}|i} \quad (15)$$

Applying this formula to formula (14), we have—

$$a_{\overline{n}|i} = v^n \cdot \frac{(1+i)^n - 1}{i} = \frac{1 - v^n}{i}$$

We have thus produced formula (8).

32. The relationship referred to in the last article enables us to solve all the problems relating to amounts of annuities without again returning to first principles.

The converse of the principle mentioned in article 31 clearly holds, that is that the present value accumulated at interest equals the amount, the interest being taken at the nominal or effective rate of interest applicable to the case.

From Formula (10), for instance, we obtain—

$$\begin{aligned} s_{\overline{n}|i}^{(m)} &= \left(1 + \frac{j}{m}\right)^{mn} \times \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\frac{j}{m}} \\ &= \frac{\left(1 + \frac{j}{m}\right)^{mn} - 1}{\frac{j}{m}} \quad (16) \end{aligned}$$

annuity-certain of 15 per annum payable half-yearly for 30 years, with interest at 6% with half-yearly rests.

$$\text{Amount} = s_{\overline{60}|}^{3\%} \times 7.5$$

This is really an example of formula (14), there being 60 periodical payments of 7.5. If we had used formula (16) we should have multiplied the value by 15 instead of 7.5, otherwise the work would have been identical.

$$s_{\overline{60}|}^{3\%} = \frac{1.03^{60} - 1}{.03} = 162.03$$

and $7.5 \times 162.03 = 1215.2$.

33. In the case of an annuity-due, each payment of the annuity is made one year (or one period) sooner than in the case of an immediate annuity. Each payment is therefore accumulated at interest for one more year (or period). Therefore, the amount of a certain annuity-due in n years is

$$(1 + i) s_{\overline{n}|} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

34. If we work out the value of formula (20) by reference to formula (14), we obtain a most convenient formula for the amount of an annuity-due, which should be carefully noted—

$$\begin{aligned} (1 + i) s_{\overline{n}|} &= (1 + i) \left\{ \frac{(1 + i)^n - 1}{i} \right\} = \frac{(1 + i)^{n+1} - (1 + i)}{i} \\ &= \frac{(1 + i)^{n+1} - 1}{i} - 1. \end{aligned}$$

It will be seen that this last expression $= s_{\overline{n+1}|} - 1$, which enables us to obtain the amount of an annuity-due for n years by deducting unity from the amount of an immediate annuity for $(n + 1)$ years. In symbols—

$$(1 + i) s_{\overline{n}|} = s_{\overline{n+1}|} - 1 \quad . \quad . \quad (21)$$

DEFERRED ANNUITIES

35. To find the present value of an immediate annuity-certain for n years deferred m years (i.e. becoming an **immediate** annuity at the end of m years, with the first payment at the end of $m + 1$ years), we deduct from the present value of an annuity payable for $m + n$ years the present value of an annuity for the m years during which it is not payable. Writing ${}_m|a_{\overline{n}|}$ for the present value, we have—

$${}_m|a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|} \quad . \quad . \quad . \quad (22)$$

36. It is to be noted that the **amount** of a similar deferred annuity is simply $s_{\overline{n}|}$, since nothing happens to affect the value during the first m years.

PERPETUITIES

37. If in formula (8) we make n infinitely great, the value of the term v^n becomes infinitely small and vanishes, leaving the formula $a_{\infty} = \frac{1}{i}$ (23)

where a_{∞} stands for the present value of an immediate **perpetuity** of 1 per annum payable yearly.

38. Similarly, from formula (10) we have—

$$a_{\infty}^{(m)} = \frac{1}{j}$$

and from formula (12)—

$$a_{\infty}^{(p)} = \frac{1}{j(p)}$$

If we make n infinitely great in formula (13), we obtain—

$$a_{\infty}^{(p)} = \frac{1}{p \left\{ \left(1 + \frac{j}{m} \right)^{\frac{m}{p}} - 1 \right\}}$$

which is the present value of an immediate perpetuity of 1 per annum payable p times a year, on the basis of interest at the nominal rate j convertible m times a year.

NOTE.—It is interesting to observe that the formulae for the present values of perpetuities may be obtained from general reasoning. For instance, a perpetual annuity of i per annum may be produced by the investment of a sum of 1, and therefore a perpetual annuity of 1 per annum may be produced by the investment of a sum of $\frac{1}{i}$. In other words, $\frac{1}{i}$ is the present value of the perpetuity of 1 per annum. The remaining formulae may be obtained by similar reasoning.

39. In order to facilitate compound interest calculations, tables of most of the functions in general use have been calculated at varying rates of interest and published. For example, a volume prepared by Lieut.-Col. W. H. Oakes contains the functions v^n , $a_{\overline{n}|}$, $(1+i)^n$, and $s_{\overline{n}|}$ for each rate of interest between $\frac{3}{4}$ and 10 per cent. per annum, proceeding by intervals of one-eighth, and for all values of n from 1 to 100 inclusive. The very low rates of interest are given for use when interest is not convertible yearly. For instance, $\frac{3}{4}$ per cent. is often useful when interest is 3 per cent. per annum convertible quarterly.

40. For purposes of illustration, tables of the functions mentioned in article 39 are given at the end of this volume for rates of interest 3, 4, and 5 per cent. (see Tables III and IV).

EXAMPLES

41. The following further examples will serve to illustrate the subject discussed in this chapter and bring out some of the difficulties met with in practice.

NOTE.—If the student is not familiar with decimalizing shillings and pence at sight, he should here read the note at the end of this chapter.

EXAMPLE VII.—What annual sum, if accumulated at compound interest at 4% per annum, will amount to £100 in 20 years?

Let x = the annual sum required.

$$\text{Then } x \times s_{\overline{20}|4\%} = 100$$

$$\therefore x = \frac{100}{s_{\overline{20}|}}$$

obtaining the value of $s_{\overline{20}|}$ at 4% from Table IV, we have

$$x = \frac{100}{29.7781} = 3.3582$$

Answer, £3 7s. 2d.

EXAMPLE VIII.—Find the present value of £5 per annum payable half-yearly for 30 years, with interest at 6% per annum with half-yearly rests.

We have value =

$$2.5 \times a_{\overline{60}|3\%} \text{ at } 3\% \text{ per period.}$$

But we cannot take $a_{\overline{60}|}$ direct from Table IV, as it is outside its range. We therefore proceed as follows—

$$a_{\overline{60}|} = a_{\overline{50}|} + v^{50} a_{\overline{10}|}$$

All the values are now in the tables, and we therefore have

$$\begin{aligned} a_{\overline{80}|} &= 25.7298 + .22811 \times 8.5302 \\ &= 25.7298 + 1.945 \\ &= 27.6748 \end{aligned}$$

$$\text{and } 2.5 \times a_{\overline{80}|} = 69.187$$

EXAMPLE IX.—What annual sum payable in arrear, if accumulated at compound interest at 4% per annum for 15 years, and then at 5% per annum for 10 years, will amount to £100?

Let x = the annual sum required.

We must now make an **equation of value**, that is we must equate the values at any one time. Although one time might be found more convenient than another, it is of no **theoretical** importance what time is chosen.

For instance, we have—

$$x \left(a_{\overline{15}|}^{4\%} + v_{4\%}^{15} \times a_{\overline{10}|}^{5\%} \right) = v_{4\%}^{15} \times v_{5\%}^{10} \times 100,$$

where the affixes indicate the rates of interest applicable to the respective functions. When constructing this and similar equations it is convenient to consider the first period as being for **all purposes** a 4% period and the second a 5% period.

A more convenient time to equate the values, however, will be the end of the period, when we have—

$$x \left(s_{\overline{15}|}^{4\%} \times 1.05^{10} + s_{\overline{10}|}^{5\%} \right) = 100$$

$$\text{whence } x (20.0236 \times 1.62889 + 12.5779) = 100$$

$$\therefore x (32.6 + 12.5779) = 100$$

$$\therefore x = \frac{100}{45.1779} = 2.213$$

Answer, £2 4s. 3d.

EXAMPLE X.—It is often desired to find the rate of interest involved in a particular transaction. For instance, at what rate of interest will an immediate annuity of £10 per annum, payable yearly for 10 years, amount to £122·882?

$$\text{We have } 10s_{\overline{10}|} = 122\cdot882$$

$$\text{so that } s_{\overline{10}|} = 12\cdot2882$$

Referring to the tables of $s_{\overline{n}|}$ and glancing along the line for $n = 10$, we observe the following values—

$$\text{at } 4\% \quad s_{\overline{10}|} = 12\cdot0061$$

$$\text{at } 5\% \quad s_{\overline{10}|} = 12\cdot5779$$

$$\text{Difference} = \underline{\underline{+ \cdot 5718}}$$

$$\text{Also, at } 4\% \quad s_{\overline{10}|} = 12\cdot0061$$

$$\text{and given value} = 12\cdot2882$$

$$\text{Difference} = \underline{\underline{+ \cdot 2821}}$$

It then becomes a matter of proportion.

As a difference in value of ·5718 corresponds to an increase of 1%, then a difference in value of ·2821 corresponds to an increase of $\frac{\cdot 2821}{\cdot 5718} = \cdot 4934 = \cdot 5$ approximately.

∴ The rate of interest required is approximately $4\frac{1}{2}\%$.

As a matter of fact $4\frac{1}{2}\%$ is exactly right, the small error in the actual result being chiefly due to the fact that the function $s_{\overline{n}|}$ does not increase in arithmetical progression, as is assumed by the method adopted. A very close result could be obtained by the application of the Calculus of Finite Differences.

EXAMPLE XI.—It is often possible to obtain the rate of interest by an algebraical method.

At what rate of interest does a sum of money double itself at compound interest in 14 years?

We have $(1 + i)^{14} = 2$

Then $(1 + i) = 2^{\frac{1}{14}} = 1.051$

and $i = .051$ or approximately 5%.

This result could also have been obtained by glancing along the line for $n = 14$ in the table of $(1 + i)^n$, from which it will be seen that at 5% $(1 + i)^{14} = 1.97993$ or nearly 2.

It could also have been obtained approximately by the use of the rough rule that to find the number of years in which money doubles itself, divide 72 by the rate of interest. Using this conversely, we have the rate of interest as $\frac{72}{14} = 5.1$.

THE FORCE OF INTEREST

42. The nominal rate of interest corresponding to a given effective rate i is expressed by the equation—

$$j = m \left((1 + i)^{\frac{1}{m}} - 1 \right)$$

Expanding by binomial theorem, this becomes—

$$\begin{aligned} &= i + \frac{1-m}{m} \frac{i^2}{2} + \frac{(1-m)(1-2m)}{m^2} \frac{i^3}{3} \dots \\ &= i + \left(\frac{1}{m} - 1 \right) \frac{i^2}{2} + \left(\frac{1}{m^2} - \frac{3}{m} + 2 \right) \frac{i^3}{3} \dots \quad (24) \end{aligned}$$

Now, if i be constant, the greater the value of m , the smaller will be the corresponding value of j (since a small nominal value will produce a large effective yield if convertible very frequently). There is, however, a limit beyond which j will not decrease. To find this limit, we

If, now, we make m infinitely great, we have—

$$\begin{aligned} f &= d + \frac{d^2}{2} + \frac{2d^3}{3} + \dots \\ &= -\log_e(1-d) = \text{the Force of Discount} \\ &= \delta^1 \text{ (say)} \quad \dots \quad (27) \end{aligned}$$

45. If, in equation (27), we put $-\log_e(1-d) = \log_e(1+i)$, we have—

$$\delta^1 = -\log_e(1-d) = \log_e(1+i) = \delta$$

In words, the force of discount is equal to the force of interest.

46. By means of the above equations many algebraical relationships may be established. For instance—

$$\delta = \log_e(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \dots$$

$$\text{also } \delta = -\log_e(1-d) = d + \frac{d^2}{2} + \frac{d^3}{3} + \dots$$

Adding these two equations together, and neglecting powers of i and d greater than the first, we have—

$$2\delta = i + d$$

$$\text{and } \delta = \frac{i + d}{2} \text{ approximately.}$$

This formula only differs from the true value by—

$$\frac{1}{2} (d^2 - i^2) + \frac{1}{3} (d^3 + i^3) + \frac{1}{4} (d^4 - i^4) + \dots$$

If it be desired to express the error in terms of δ , we may proceed as follows—

$$i = e^\delta - 1 = \delta + \frac{\delta^2}{2} + \frac{\delta^3}{3} + \dots$$

$$d = 1 - e^{-\delta} = \delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} - \dots$$

$$\therefore \frac{i + d}{2} = \delta \text{ and the error is } \frac{\delta^5}{3} + \frac{\delta^5}{5} + \dots \text{ in excess.}$$

This formula gives a result correct to at least four places of decimals for all values of i not greater than $\cdot 07$.

47. The value of δ corresponding to a given nominal rate of interest may be found as follows—

$$\delta = \log_e (1 + i) = m \log_{10} \left(1 + \frac{j}{m} \right) \div \log_{10} e$$

48. Turning now to annuities, we have—

$$s_{\overline{n}|}^{(p)} = \frac{i}{j^{(p)}} s_{\overline{n}|} = \frac{i}{p \left((1 + i)^{\frac{1}{p}} - 1 \right)} s_{\overline{n}|}$$

Writing $\bar{s}_{\overline{n}|}$ for the amount of an annuity payable continuously, and bearing in mind the analysis of article 42, we have—

$$\bar{s}_{\overline{n}|} = \frac{i}{\log_e (1 + i)} s_{\overline{n}|} = \frac{(1 + i)^n - 1}{\log_e (1 + i)} \quad \quad (28)$$

This is the case where interest is at the effective rate i per unit per annum.

THE AVERAGE INTEREST YIELD ON THE LIFE FUND OF AN ASSURANCE OFFICE

53. The Life Offices Association and the Board of Trade have agreed on the following formula for calculating the above, which is required to be stated in the Fifth Schedule of the Assurance Companies Act, 1909, at each valuation.

Let A be the total fund at the beginning of the year.

„ B „ „ „ end of the year.

„ I „ total interest received during the year.

Now B will include a full year's interest, which must be deducted to obtain the net fund at the end of the year. The average fund invested throughout the year will therefore be—

$$\frac{A + B - I}{2}$$

On this mean fund the interest earned is I .

Therefore, the rate of interest per unit of invested capital per annum yielded by the fund will be—

$$I \div \frac{A + B - I}{2} = \frac{2I}{A + B - I} = i \quad . \quad . \quad (30)$$

THE EFFECT OF INCOME TAX

54. If it be required to find the yield net of income tax, we must, in the above formula, write—

$$I - T \text{ for } I,$$

where T is the total tax deducted and/or paid during the year.

As in practice the bulk of the tax is deducted at the

source, we are justified in deducting only the net interest from the Fund *B* in order to obtain the amount invested.

55. If, however, it is the *gross* yield which is required, we must remember that the Fund *B* only includes the *net* interest, and our formula becomes—

$$i = \frac{2I}{A + B - (I - T)}$$

56. In practice it is usual to use formula (30) with *I* as the *net* interest in numerator and denominator, thus obtaining the net yield. This is then increased in the proportion which the gross interest bears to the net interest. It is the gross yield which is required to be returned under the Assurance Companies Act.

57. Where the valuation of liabilities is made less often than yearly (usually quinquennially), the above formula will not give good results for the average yield over the valuation period. It is usual, therefore, to calculate a separate yield for each year in the inter-valuation period, and to take the mean of the separate results.

58. For the purpose of the Act it is necessary to use the total interest received less any interest paid out. Thus it would be correct to deduct from the receipts, interest paid on an overdraft at the bank. Similarly, as assurance companies are permitted to retain tax deducted by them from annuity payments, subject to certain limitations, this tax should be deducted from the amount of tax on interest. As a rule this would be done by the accountant, the figure being shown net in the accounts.

NOTE ON DECIMALIZING SHILLINGS AND PENCE

(a) In order to solve compound interest problems readily it is necessary to be able to change shillings and

pence into decimals and *vice versa* at a glance. As a florin is one-tenth of a pound it is represented by £.1, from which follows the following easy rule: To turn shillings into decimals of a pound, divide the number of shillings by two. Thus, 1s. = £.05 and 19s. = £.95.

(b) Conversely, to turn decimals of a pound into shillings, multiply the figure in the first place of decimals by two. Thus, £.9 = 18s. If the figure in the second place of decimals is 5 or more there will be one more shilling to add. Thus, £.35 = 7s., and £.36 = 7s., plus a few pence.

(c) As regards pence, we may first consider the following rough rule: To turn pence into decimals of a pound, multiply the number of pence by 4, adding 1 for every 24 obtained. The result is inserted in the third place of decimals or, if greater than 9, in the second and third places. Thus, 6d. = £.025, and 4d. = .017 (adding 1 since 16 is more than half way towards 24). This produces a slightly closer result.

(d) Conversely, to obtain pence from decimals of a pound, deduct 1 for every 25 and divide the result by 4. Thus, £.038 = 9d. (2 were deducted as 38 is nearer 50 than 25).

(e) Combining the results of (a) and (c) above, we are able to turn any sum of money into decimals of a pound at sight. Thus, 19s. 3d. = £.95 + £.013 = £.963 and 3s. 10d. = .192.

(f) Conversely, combining the results of (b) and (d), we are able to obtain at sight the shillings and pence corresponding to any decimals of a pound. Thus, £.691 = 13s. 10d. (almost) and £.733 = 14s. 8d.

(g) The above methods give the nearest figure in the third place of decimals or the nearest penny as the case may be. Often, however, greater accuracy is required in practice, as it may be required to multiply the figures by

large amounts in order to obtain a desired result. The rule as to shillings is, of course, exact, and it is the pence which require further consideration. The exact value of one penny is £·00416̄, hence two pence equal £·0083̄, and so on. The following table will be useful. If the values for 1d., 3d., 6d., and 9d. be remembered, the remainder are easily obtained from them.

Pence.	Decimals of £1.	Pence.	Decimals of £1.
1	·00416̄	7	·02916̄
2	·0083̄	8	·03̄
3	·0125	9	·0375
4	·016̄	10	·0416̄
5	·02083̄	11	·04583̄
6	·025	12	·05

CHAPTER III

SINKING FUNDS

1. A **SINKING Fund** is a fund accumulated with a view to the ultimate replacement of capital.

For instance, a leasehold house is purchased under which the lease expires in n years, and it is desired to set up a sinking fund by annual payments to replace the capital sunk by the end of the term of the lease. Assuming that it is possible to invest the annual payments of x (say) at the effective rate i throughout the n years, and that the capital is unity, we have

$$x s_{\overline{n}|i} = 1$$

$$\therefore x = \frac{1}{s_{\overline{n}|i}} \quad \dots \dots \dots (1)$$

That is, an annual payment of $\frac{1}{s_{\overline{n}|i}}$ if accumulated with interest at rate i , will replace a capital of 1 in n years.

2. In some cases it would be more convenient to effect a **Capital Redemption Assurance**, or, as it is sometimes called, a **Leasehold Redemption Assurance**, with an assurance company, at an annual premium. In this case the first payment would fall due at once, instead of in one year's time, and, if $P_{\overline{n}|i}$ be the premium, we have

$$P_{\overline{n}|i} (s_{\overline{n+1}|i} - 1) = 1$$

$$\therefore P_{\overline{n}|i} = \frac{1}{s_{\overline{n+1}|i} - 1} \quad \dots \dots \dots (2)$$

EXAMPLE I.—A leasehold house is purchased for £500. The lease has only five years to run and it is desired to set

up a sinking fund by annual payments to replace the capital out of the rent within the five years. Assuming that the annual payments are invested at 3% interest net of income tax, calculate their amount and draw up a schedule showing the position of the fund in each year.

Using formula (1), the annual payment will be—

$$\frac{500}{s_{\frac{3\%}{5}}^5} = \frac{500}{5.3091} = 94.178$$

SCHEDULE

Year. (1)	Fund at beginning of year. (2)	Interest added at end of year. (3)	(2) + (3) plus annual payment. (4)
1	—	—	94.178
2	94.178	2.825	191.181
3	191.181	5.735	291.094
4	291.094	8.733	394.005
5	394.005	11.820	500.003

Had the problem been to arrange the sinking fund by means of a Capital Redemption Assurance, at a premium calculated on a net 3% basis, we should have used formula (2), since premiums are always payable in advance. This arrangement would, of course, have affected the schedule, since the fund at the beginning of the first year would have been the first premium then paid, a year's interest thereon would be added at the end of the first year, and so on.

3. The principles laid down in the last two articles are quite simple, and no difficulty will be found in applying them to cases where the payments are other than annual and interest is convertible more frequently than yearly.

4. A sinking fund may be set up by a single payment, in which case the single payment will clearly be the present

value of the capital sum required to be redeemed. If such single payment be written $A_{\overline{n}|}$, we have

$$A_{\overline{n}|} = v^n \quad . \quad . \quad . \quad . \quad . \quad (3)$$

5. Sometimes more than one rate of interest will enter into a sinking fund transaction. For instance, if an assurance company be willing to issue Capital Redemption Policies for a term of $m + n$ years on the basis of interest at the effective rate of i per unit per annum for the first m years and i' thereafter, then the premium required would be $P_{\overline{m+n}|}$, where

$$P_{\overline{m+n}|} \left\{ \left(s_{\overline{m+1}|}^{(i)} - 1 \right) (1 + i')^n + \left(s_{\overline{n+1}|}^{(i')} - 1 \right) \right\} = 1$$

$$\therefore P_{\overline{m+n}|} = \frac{1}{\left(s_{\overline{m+1}|}^{(i)} - 1 \right) (1 + i')^n + \left(s_{\overline{n+1}|}^{(i')} - 1 \right)} \quad . \quad (4)$$

6. The accumulated amount at the end of t years of the premiums paid under a Capital Redemption Assurance Policy originally effected for a term of n years (where n is greater than t) is called the *Policy Value*, and is written ${}_tV_{\overline{n}|}$.

We therefore have—

$${}_tV_{\overline{n}|} = P_{\overline{n}|} (s_{\overline{t+1}|} - 1) \quad . \quad . \quad . \quad . \quad (5)$$

7. Formula (5) is obtained by what is known as the **Retrospective** method. If the question be looked at from a **Prospective** point of view, it is evident that the value of the policy is the present value of the capital to be redeemed less the present value of the premiums still to be paid, i.e.

$${}_tV_{\overline{n}|} = A_{\overline{n-t}|} - P_{\overline{n}|} (1 + a_{\overline{n-t-1}|}) \quad . \quad . \quad . \quad (6)$$

This is algebraically identical with formula (5) and

represents the value of the policy immediately before the $(t + 1)^{\text{th}}$ premium falls due. The value at any time intermediate between two payments may be obtained by discounting the value at the end of the year down to the time at which the value is required.

REDEMPTION OF LOANS

8. In the case of loans by local authorities, mortgages, and other similar transactions, it is frequently the practice to repay by means of *equal* periodical instalments of principal and interest spread over a term of years. If x be the annual instalment which will repay a loan of 1, with interest, in n years, we have—

$$xa_{\overline{n}|} = 1$$

$$\text{and } x = \frac{1}{a_{\overline{n}|}} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This result is obtained by forming an equation of payments. It is clear that when a man lends a sum of money he expects to obtain its equivalent in return. In other words, the present value of the periodical repayments of principal and interest must be equal to the capital advanced.

9. The first periodical instalment contains interest on the full amount of the loan and the balance is applied to repay capital. The second periodical payment contains interest on the amount of the loan outstanding at the beginning of the second period only and the amount available to repay capital is therefore greater than on the first occasion. Similarly, as the payments proceed, the amount required for interest decreases and the amount available for capital repayment increases.

10. EXAMPLE II.—As an example of articles 8 and 9, let us calculate the amount of the equal half-yearly instalments

required to repay a loan of £25,729 16s. in 25 years at 6% interest, net of income tax, with half-yearly rests. If x be the half-yearly instalment, we have—

$$x = \frac{25729.8}{a_{\overline{50}|}^{3\%}} = 1,000$$

If a schedule showing the incidence of the payments be required, it could be conveniently arranged as follows—

SCHEDULE
Half-Yearly Payment, £1,000

Half-year.	Capital outstanding at beginning of half-year.	Interest on col. (2) at 3%.	1,000 less col. (3) = repayment of capital.
(1)	(2)	(3)	(4)
1	25729.800	771.894	228.106
2	25501.694	765.051	234.949
3	25266.745	758.002	241.998
:	:	:	:
:	:	:	:
:	:	:	:
49	1913.470	57.404	942.596
50	970.874	29.126	970.874

11. In practical work it is sometimes desired to calculate certain of the values which appear in the schedule without the trouble of constructing the whole schedule. It is therefore useful to analyse the payments. As $a_{\overline{n}|}$ is the present value of an annual payment of 1 for n years, it follows that an annual payment of 1 for n years will repay a loan of $a_{\overline{n}|}$ with interest at the effective rate i per unit.

The amount of the loan being $a_{\overline{n}|}$, which equals $\frac{1-v^n}{i}$, the interest contained in the first instalment will be

$i \times \frac{1 - v^n}{i} = 1 - v^n$. Consequently the capital repayment will be v^n . Deducting this capital repayment from $a_{\overline{n}|}$ we get the capital outstanding at the beginning of the second period as $\frac{1 - v^{n-1}}{i}$. The interest for the second period will therefore be $1 - v^{n-1}$, and the capital repayment v^{n-1} . Following out the argument, we see that our schedule may be analysed as follows—

SCHEDULE
Periodical payment, 1

Period. (1)	Capital outstanding at beginning of period. (2)	Interest on col. (2) at rate i . (3)	1 less col. (3) = repayment of capital. (4)
1	$a_{\overline{n} }$	$1 - v^n$	v^n
2	$a_{\overline{n-1} }$	$1 - v^{n-1}$	v^{n-1}
:	:	:	:
:	:	:	:
:	:	:	:
$n - 1$	$a_{\overline{1} }$	$1 - v^1$	v^1
n	$a_{\overline{1} }$	$1 - v$	v

12. The method of article 11 was used to calculate the values for the half-years 49 and 50 in the schedule in article 10. It will be found that to get a satisfactory result when the figures are so large it is necessary to calculate v to seven places of decimals.

13. A simple relationship exists between $\frac{1}{a_{\overline{n}|}}$ and $\frac{1}{s_{\overline{n}|}}$ by means of which the one can easily be obtained from the other. This obviates the necessity of tabulating both, and both tables will rarely be found in one work.

the annual payment provides $\frac{1}{s_{\overline{n}|i}}$, or the annual payment which will accumulate with interest to 1 in n years, and i , the annual interest on the loan of 1. If, therefore, we calculate $\frac{1}{s_{\overline{n}|i'}}$ at rate i' we shall obtain the required annual sum which will accumulate to 1 at rate i' in n years, and to this we add i , the annual interest required.

17. EXAMPLE III.—An advance of £1,000 is to be made to yield the lender an effective rate of interest of 5% per annum. Calculate the equal annual instalment of principal and interest which will be required to repay the loan in ten years, on the assumption that capital repayments will be reinvested at 3% per annum with yearly rests.

The annual instalment for a loan of 1 will be $\frac{1}{s_{\overline{n}|i}} + i$, where $\frac{1}{s_{\overline{n}|i}}$ is based on 3% interest, and $i = .05$.

That is: $\frac{1}{11.4639} + .05 = .087231 + .05 = .137231$.

Therefore, the annual instalment to repay £1,000 = 137.231.

18. EXAMPLE IV.—If in Example III we had been told that the annual instalment was .137231 per unit of capital, and that capital repayments were to be invested on the basis of 3% interest, we could have determined the yield to the lender as follows—

It is clear that $\frac{1}{s_{\overline{n}|3\%}} + i = .137231$

$$\therefore \frac{1}{11.4639} + i = .137231$$

$$\therefore i = .05$$

19. If a schedule be required on the lines of article 10, it would first be calculated on the basis of 3% (i.e. the rate of reinvestment), and the amounts in the interest column increased by the additional interest (i.e. in this case 2% on the original capital).

REDEEMABLE SECURITIES

20. When capital is raised by means of a Redeemable Security it is provided that interest be paid for a stated number of years and that the capital be then repaid. Sometimes the repayment of the capital is at the option of the issuers, they retaining the right to continue to pay interest instead of repaying the capital. If the market price of a security **redeemable at the option of the issuers** is less than the redemption price it is to be assumed that the issuers will cancel their stock by buying it in the open market rather than pay the higher price. In such circumstances, therefore, it will not be treated as redeemable for the purposes of valuation.

21. If a security redeemable *for certain* at the end of n years, and paying a dividend in the meantime at rate g per unit of nominal capital per annum, be bought to yield the effective rate i , the price paid must be x , where

$$x = v^n + ga_{\overline{n}|i} \quad . \quad . \quad . \quad (9)$$

22. Formula (9) expresses the fact that the value of the security is the present value of the dividend payments plus the present value of the capital payment, both present values being calculated at the rate of interest it is desired to secure on the investment.

EXAMPLE V.—Find the value to yield an effective rate of 5% of a security redeemable at par in 10 years time and paying an annual dividend of 7%.

Assuming that the first dividend is due in one year's time the value may be written—

$$100v^{10} + 7a_{\overline{10}|}$$

Inserting the values of the functions at 5%, we have

$$\begin{aligned} 61.391 + 54.0519 \\ = 115.4429. \end{aligned}$$

23. In Example V it is evident that if the purchaser used the full dividend of 7% each year he would find himself at the end of the 10 years with a capital of 100, instead of 115.4429. This could be allowed for by crediting himself each year with 5% on his invested capital only (i.e. 5.77215), and accumulating the balance of the dividend at 5% interest to the end of the 10 years, when it would amount to 15.4429 and, with the 100 repaid, replace his capital in full. The balance of the dividend will be $7 - 5.77215 = 1.22785$, and we therefore have—

$$1.22785 \times s_{\overline{10}|} = 15.4429$$

24. It is not always possible in practice to reinvest the balance of the dividend at the same rate of interest as that yielded by the security. We may therefore consider Example V on the assumption that the excess of the dividend over the yield can only be invested at the effective rate of 3% per annum.

EXAMPLE VI.—Let x = the price to be paid per unit so that the investment may still yield 5%.

Then a sinking fund of $\frac{x-1}{s_{\overline{n}|}}$ per annum calculated at 3% must be set up to replace that part of the capital in excess of the redemption price of 1. The purchaser will now receive back his full outlay at the end of the term, and in the meantime he will receive an annuity of the

dividend less the outlay for the sinking fund. The formula for the value of the security to yield 5% therefore becomes

$$x = xv^n + \left(g - \frac{x-1}{s_n} \right) a_n \quad . \quad . \quad . \quad (10)$$

where v^n and a_n are calculated at the rate of interest to be realized and s_n at the rate of interest at which the sinking fund can be accumulated.

Inserting the values for the example in question, we have

$$\begin{aligned} x &= .61391x + \left(.07 - \frac{x-1}{11.4639} \right) 7.7217 \\ &= .61391x + .540519 - .67356(x-1) \\ \therefore 1.05965x &= 1.214079 \\ \therefore x &= 1.14574. \end{aligned}$$

The accuracy of this result can be proved by first calculating the sinking fund, namely,

$$\frac{.14574}{s_{10}} = \frac{.14574}{11.4639} = .01272.$$

Deducting this from the dividend of .07, we get .05728, which is exactly 5% on the outlay of 1.14574.

25. If in the last example the redemption price had been **higher** than the purchase price the annual dividend would have been **increased** by the sinking fund payment instead of **decreased**. This could be effected in practice by writing up the value of the security in the books.

26. The problems connected with redeemable securities are many and various, but it will be found that in every case they depend on the application of the principles described in the last few articles.

27. No satisfactory formula has yet been devised for finding the yield on a redeemable security, and it is necessary to proceed by trial and interpolation.

EXAMPLE VII.—A security of the nominal amount of £100 redeemable at the end of ten years at 105 is purchased for 98. If dividends are paid at the nominal rate of 6% per annum with half-yearly rests, what is the yield on the investment?

Forming an equation of value, we have—

$$98 = 3a_{\overline{20}|} + 105 v^{20}$$

and the rate of interest which establishes this equality is the rate of interest required.

Clearly the yield lies between 3% and 4% per half-year, and valuing on these two bases with the aid of Tables III and IV, we have—

$$\text{Value at 3\%} = 3 \times 14.8775 + 105 \times .55368 = 44.6325 + 58.136.$$

$$\text{Value at 4\%} = 3 \times 13.5903 + 105 \times .45639 = 40.7709 + 47.921.$$

$$\text{Value at 3\%} = 102.7685$$

$$\text{Value at 4\%} = 88.6919$$

$$\text{Difference} = 14.0766$$

and the difference between the value at 3% and the 98 at which we are aiming is 4.7685.

∴ by simple proportion the excess of the yield over 3% will be—

$$\frac{4.7685}{14.0766} = .3388$$

$\therefore$ The yield is a nominal rate of $2 \times (3 + \cdot 3388) = 6\cdot 6776$ convertible half-yearly.

To test the accuracy of the result, we will calculate the value of $3a_{\overline{20}|} + 105v^{20}$ at $3\cdot 3388\%$.

$$\begin{aligned}
 v^{20} &= 1\cdot 033388^{-20} \text{ and } \log 1\cdot 033 = \cdot 0140 \\
 \therefore -20 \log 1\cdot 033 &= -\cdot 2800 \\
 &= \bar{1}\cdot 7200 \\
 \text{Antilog} &= \cdot 5248 = v^{20} \\
 a_{\overline{20}|} &= \frac{1 - \cdot 5248}{\cdot 033388} = \frac{\cdot 4752}{\cdot 03339} \\
 \log \cdot 4752 &= \bar{1}\cdot 6769 \\
 \log \cdot 03339 &= \bar{2}\cdot 5236 \\
 &\hline
 &= 1\cdot 1533 \\
 \text{Antilog} &= 14\cdot 23 = a_{\overline{20}|}
 \end{aligned}$$

We $\therefore$ have—

$$\begin{aligned}
 &3 \times 14\cdot 23 + 105 \times \cdot 5248 \\
 &= 42\cdot 69 + 55\cdot 104 \\
 &= 97\cdot 794.
 \end{aligned}$$

The result is remarkably close to the value of 98 required. If it were thought necessary to obtain an even closer result, this could be obtained by interpolating between the values at 3% and at $3\cdot 3388\%$ respectively in the same way as was done between the values at 3% and 4% .

28. The method described in the previous paragraph produces the yield to a purchaser on the assumption that it is possible to invest the sums required to write up or write down the security to its redemption value at the rate of interest which is yielded by the investment. This is a reasonable assumption, as presumably the yield is that obtainable on the market on similar securities. There may be circumstances, however, in which the assumption is not

reasonable, or in which it is inapplicable to the conditions available to a particular investor.

29. For instance, an investor who bought but £100 of nominal stock at a premium would find it difficult to invest the small annual amounts required for the sinking fund at the same rate as is yielded by his investment, but could quite well place those amounts in the Post Office Savings Bank at $2\frac{1}{2}\%$. On the other hand, a large investor, such as a life assurance office, might consider it preferable to assume that the sinking fund was invested at the average rate yielded by its assets, rather than at the rate yielded by the particular transaction.

30. It will be observed that these considerations lead to the use of a pre-determined rate of interest for the sinking fund and this is convenient, since once the periodical charge for the sinking fund is known, the whole difficulty of determining the yield disappears.

EXAMPLE VIII.—A security of the nominal amount of £100 redeemable at par at the end of 15 years is purchased for £110. If dividends are payable at 4% per annum with yearly rests, the first being due in one year's time, what is the yield on the transaction, on the assumption that the premium of £10 is replaced by a sinking fund calculated at 3% ?

The annual dividend will be 4.0000

The annual charge for the sinking fund will be—

$$\frac{10}{s_{\overline{15}|}} \text{ at } 3\% = \frac{10}{18.5989} = ... \underline{\underline{.5377}}$$

∴ The net return on the purchase money of £110

will be the difference — 3.4623

The yield % is thus 3.148 or nearly £3 3s.

31. This method may be used to find the approximate yield on a redeemable security, provided the market yield on similar securities is known.

EXAMPLE IX.—Calculate the approximate yield in the case of the purchase detailed in Example VII, on the assumption that the market yield on similar securities is about $6\frac{1}{2}\%$ per annum convertible half-yearly.

The half-yearly dividend will be 3.0000

The half-yearly charge for sinking fund will be—

$$\frac{7}{s_{20}|} \text{ at } 3\frac{1}{4}\% = \frac{7}{27.5642} = ... \underline{\underline{.2540}}$$

∴ The net return on the purchase money of £98
will be the sum = ... 3.2540

The yield % is thus 3.320 per half-year, comparing with the true half-yearly yield of 3.3388. This is quite a good approximation considering that the sinking fund is worked at $6\frac{1}{2}\%$ instead of at the correct rate of 6.6776%.

EFFECT OF INCOME TAX

32. The effect of Income Tax is nowadays a matter of considerable importance in connection with Compound Interest problems. Each case requires separate consideration, but a few examples will serve to show the general principles on which to proceed.

33. EXAMPLE VIII.—A man is in receipt of a gross income of £100 per annum payable half-yearly, the first payment being due at once. He decides to invest each payment as it falls due to yield a gross rate of interest of 5% per annum with half-yearly rests. Assuming that

income tax is deducted from each payment at the rate of 6s. in the pound, calculate the amount of his savings in 20 years.

In the first place it is evident that the income may be regarded as £50 per half-year less £15 tax, or £35 net.

In the second place, although payments are to be invested at £2 10s. % interest per half year, when tax is deducted the net yield will be £2 10s. less 15s. = £1 15s.

Remembering that the first payment is due at once, the accumulated amount of the savings by the end of the 40 half years will therefore be—

35 (s_{41} - 1) on the basis of an effective rate of interest of 1.75%.

$$\begin{aligned}
 \text{Now } s_{41} &= \frac{1.0175^{41} - 1}{.0175} \\
 &= \frac{2.049 - 1}{.0175} \\
 &= \frac{1.049}{.0175} \\
 &= 59.94 \\
 \therefore 35 (s_{41} - 1) &= 35 \times 58.94 \\
 &= 2062.9
 \end{aligned}$$

34. EXAMPLE IX.—A leaseholder raises a mortgage of £750 on his house, it being arranged that he shall pay 5% gross interest on the first £500 of his mortgage and 6% gross interest on the excess. Repayment is to be made by means of equal half-yearly instalments of principal and interest. Determine how long it will take him to repay the mortgage, on the assumption that the half-yearly payment is £30 after deduction of income tax at 6s. in the pound.

In order to understand this problem it is first necessary to consider how it is intended to deal with the half-yearly payment from the income tax point of view. The payment at the end of the first half-year is typical and will be calculated as follows—

Interest at $2\frac{1}{2}\%$ on £500	£12 10 -
Interest at 3% on £250	7 10 -
Total	20 - -
Less tax at 6s. in the £	6 - -
Net	14 - -
Repayment of Capital	16 - -
Half-yearly Payment	<u>£30 - -</u>

It will be observed that the repayment of capital is arrived at by deducting the net interest from the net £30 to be paid.

It is clear that the mortgagor will not start repaying the £500 at 5% until he has repaid the whole of the first £250 of the mortgage. The first problem, therefore, is to determine how long this will take.

The first £500 costs £12 10s. less tax £3 15s., or £8 15s. net per half-year. This leaves £30 less £8 15s. = £21 5s. available for the service of the remaining £250, on which the net half-yearly interest is £3 - 18s or £2 2s. $\frac{1}{2}\%$. Equating the capital with the present value of the instalments, we have—

$$21.25 a_{\overline{n}|2.1\%} = 250$$

$$\therefore a_{\overline{n}|} = 11.8$$

By reference to Archer's Tables, we see that both at 2% and at $2\frac{1}{8}\%$ the value of $a_{14|}$ is about 12, from which it is clear that in *about* 14 half-years, or 7 years, this part of the loan will be repaid. If, however, greater accuracy be required, we may proceed as follows—

$$a_{14|}^{2\frac{1}{8}\%} = 12\cdot00 \text{ and } a_{14|}^{2\%} = 12\cdot11.$$

Difference = $\cdot11$ produced by a difference of $\cdot125$ in the rate of interest.

$\therefore$ a difference of $\cdot025$ in the interest makes a difference of $\frac{1}{5} \times \cdot11 = \cdot02$ in the value. That is, $a_{14|}^{2\cdot1\%} = 12\cdot02$.

$$\text{Also } a_{13|}^{2\frac{1}{8}\%} = 11\cdot26 \text{ and } a_{13|}^{2\%} = 11\cdot35.$$

Difference = $\cdot09$ produced by a difference of $\cdot125$ in the rate of interest.

$\therefore$ a difference of $\cdot025$ in the interest makes a difference of $\frac{1}{5} \times \cdot09 = \cdot02$ in the value. That is, $a_{13|}^{2\cdot1\%} = 11\cdot28$.

$$\therefore a_{14|}^{2\cdot1\%} - a_{13|}^{2\cdot1\%} = 12\cdot02 - 11\cdot28 = \cdot74.$$

A difference of $\cdot74$ is produced by a reduction in the term of 1.

$\therefore$ a difference $(12\cdot02 - 11\cdot8)$ is produced by a reduction in the term of $\frac{\cdot22}{\cdot74} = \cdot3$.

$$\therefore \text{Term} = 14 - \cdot3 = 13\cdot7 \text{ half-years.}$$

$$= 6\cdot85 \text{ years.}$$

Repayment of remaining £500.

An annuity for 13·7 half-years means a payment of 1 at the end of each half-year for 13 half-years, with a final payment of $\cdot7$ at the end of the 14th half-year. There will,

therefore, be a balance of $\cdot 3 \times 21\cdot 25 = 6\cdot 375$ available from the 14th payment to be used in reduction of the debt of £500, leaving $500 - 6\cdot 375 = 493\cdot 625$ still to be repaid. For this purpose the whole sum of £30 per half-year will be available and interest will be at the net rate of £2 10s. less 15s. = £1 15s. %. Equating the debt with the present value of the payments, we have—

$$30 a_{\overline{n}|} = 493\cdot 625$$

$$\therefore a_{\overline{n}|}^{1\cdot 75\%} = 16\cdot 454$$

Tables are available in "Archer" at $1\frac{3}{4}\%$, and we have—

$$a_{\overline{19}|}^{1\frac{1}{4}\%} = 16\cdot 05$$

$$a_{\overline{20}|}^{1\frac{1}{4}\%} = 16\cdot 75$$

$$\text{Difference } \cdot 7$$

A difference of $\cdot 7$ is produced by an increase in the term of 1.

$\therefore$ a difference $(16\cdot 454 - 16\cdot 05)$ is produced by an increase in the term of $\frac{\cdot 404}{\cdot 7} = \cdot 58$.

$\therefore$ Term is $19 + \cdot 58 = 19\cdot 58$ half-years or 9·79 years.

$\therefore$ The Total Loan will be repaid in $7 + 9\cdot 79 = 16\cdot 79$ years.

35. EXAMPLE X.—Required to find the value of a debenture for £100 redeemable in 25 years at par, to yield a gross rate of interest of £8 11s. 6d. % convertible half-yearly, income tax being deducted from the dividends at 6s. in the pound and dividends being payable half-yearly at the rate of 6% per annum.

Deducting 6s. in the pound from the gross yield of £8 11s. 6d. % we get 6%, or 3% per half-year; also the

half-yearly dividend of £3 less 18s. tax leaves a net dividend of £2 2s. We therefore have—

Value = $100 v^{50} + 2 \cdot 1 a_{\overline{50}|}$, both functions being calculated at 3%.

Using Tables III and IV, the value becomes—

$$\begin{aligned} & 100 \times \cdot 22811 + 2 \cdot 1 \times 25 \cdot 7298 \\ &= 22 \cdot 811 + 54 \cdot 03258 \\ &= 76 \cdot 84358 \\ &= £76 \text{ 16s. 11d. to next higher penny.} \end{aligned}$$

CHAPTER IV

THE CONSTRUCTION OF COMPOUND INTEREST TABLES

1. TABLES of compound interest functions could be calculated by the use of the formulae given in previous chapters. It would, however, be laborious to calculate each individual value separately, and a better method is to start with a convenient **initial value** and calculate subsequent values from it by means of a **working formula**. It is also useful to have a **check formula** to provide an independent check on the whole work when completed.

2. To construct a table of $(1 + i)^n$ for all values of n from 1 to n . Each value may be obtained from the previous one by multiplying by $(1 + i)$, and we therefore have as a working formula—

$$(1 + i)^n = (1 + i)^{n-1} \times (1 + i) \quad . \quad . \quad (1)$$

If the value of i is such that multiplication by it can be effected at sight, formula (1) is very convenient.

EXAMPLE I.—Construct a table of $(1 + i)^n$ at 7% for all values of n from 1 to 10 correct to the nearest fourth place of decimals.

n (1)	$(1 + i)^n$ (2)	$i(1 + i)^n$ (3)
1	1.070000	.074900
2	1.144900	.080143
3	1.225043	.085753
4	1.310796	.091756
5	1.402552	.098179
6	1.500731	.105051
7	1.605782	.112405
8	1.718187	.120273
9	1.838460	.128692
10	1.967152	—

The initial value will be 1·07 and the work will be confined to that shown in Example I. It will be well to work to the nearest sixth place of decimals in order to be sure of the accuracy of the fourth place.

The above work is done in steps. The initial value having been inserted in column (2), the value in column (3) is calculated by multiplying column (2) by ·07. The value for $n = 2$ in column (2) is then obtained by adding together the values in columns (2) and (3) for $n = 1$. We thus obtain $(1 + i) + i(1 + i) = (1 + i)^2$, which is the value required, and so on. The work could, of course, be done in one column by the use of a multiplying machine.

3. We now have to consider a check formula. The successive values inserted in column (2) are $(1 + i)$, $(1 + i)^2$, $(1 + i)^3$ $(1 + i)^{10}$, and if we add these together we get $(1 + i) s_{\overline{10}|}$, which is 14·783599 as calculated by the usual formula from first principles. This, then, is the total to which column (2) should add up. It actually adds up, however, to 14·783603, which is a sufficiently good check for the purpose. This method of checking all the values in the column by casting should be carefully noted.

4. If the value of i did not lend itself to mental multiplication and a multiplying machine were not available, we could proceed by the use of logarithms. From formula (1) we have—

$$\log (1 + i)^n = \log (1 + i)^{n-1} + \log (1 + i) \quad (2)$$

Starting with $\log (1 + i)$, by the continuous addition of the constant $\log (1 + i)$, we obtain the successive values of $\log (1 + i)^n$.

5. The easiest method of effecting the constant addition is to write the value of $\log (1 + i)$ at the bottom of a card.

By placing the card so that the figures of $\log (1 + i)$ are immediately **above** the particular value to which they are to be added, the addition can be conveniently effected. The card is then moved down one space and $\log (1 + i)$ added to the next value and so on.

6. The work would be arranged as follows—

n (1)	$\log (1 + i)^n$ (2)	antilog col. (2) $= (1 + i)^n$ (3)
1	.029384	1.070000
2	.058768	1.144900
3	.088152	1.225044
:	:	:
:	:	:
:	:	:

The value of $\log 1.07$ to seven places of decimals is .0293838, so that by calling it .029384 we shall, after five successive additions, have added 1 too many in the sixth place. Observing this at the outset, it will be well to adjust every fifth result in column (2) by deducting the 1, in order to avoid an accumulated error which, in a long table, might become considerable. As after two additions the error will be 6 in the seventh place, it will be best to deduct 1 from the third result and from every fifth thereafter, thus preserving the nearest sixth place.

7. The check described in article 3 can be applied to column (3) of the above table. It will be well, however, in order to avoid carrying on an error for any distance, to effect a periodical check on column (2) as the work proceeds. This can be effected in two ways. In the first place, we may insert every tenth value (say) in advance, by independent calculation, thus stopping any error from proceeding far. In the second place, we may cast every ten (say)

values of column (2) and check that the total is equal to $\Sigma n \times \log(1 + i)$, where Σn is written for the sum of the ten values of n in the group.

8. To construct a table of v^n for all values of n from 1 to n . A convenient formula suggests itself, namely—

$$v^{n-1} = v^n \times (1 + i) \quad . \quad . \quad . \quad (3)$$

When using this formula the initial value is calculated from first principles and is v^n , where n is the longest term to be included in the table. The arrangement and the work are identical with that shown in the example of article 2, the headings of the columns being n , v^n , and iv^n respectively.

9. The successive values as calculated will be v^n , v^{n-1} , v^{n-2} . . . v , and the present value of this series is $a_{\overline{n}|}$. As a check, therefore, the total of the figures calculated should be compared with the value of $a_{\overline{n}|}$ as independently calculated by the usual formula.

10. The method of inserting in advance every tenth value, independently calculated, might be followed with advantage. Also, if it be desired to check short sections of the work, this could be done by casting. For instance, the sum of the values v^n to v^{n-i} inclusive should equal $a_{\overline{n}|} - a_{\overline{n-i}|}$.

11. If the value of i be not convenient for multiplication, and a multiplying machine be not available, we may proceed by the use of logarithms, using the working formula—

$$\log v^n = \log v^{n-1} + \log v \quad . \quad . \quad . \quad (4)$$

The work would be as described in article 4 for $(1 + i)^n$.

12. It would, of course, be equally correct to take the logarithms of formula (3) and write—

$$\log v^{n-1} = \log v^n + \log(1 + i) \quad . \quad . \quad (5)$$

In this case, instead of starting with $n = 1$, we should

start with n = the longest term required, and work backwards by adding $\log (1 + i)$. The usual checks should be applied.

13. To construct a table of $s_{\overline{n}|}$ for all values of n from 1 to n . A suitable working formula is a little less obvious in this case, but it is very easily found. We have—

$$\begin{aligned} s_{\overline{n}|} &= (1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \dots + 1 \\ s_{\overline{n-1}|} &= (1+i)^{n-2} + (1+i)^{n-3} + \dots + 1 \\ \therefore s_{\overline{n}|} - s_{\overline{n-1}|} &= (1+i)^{n-1} \\ \therefore s_{\overline{n}|} &= s_{\overline{n-1}|} + (1+i)^{n-1} \end{aligned} \quad (6)$$

14. By the use of formula (6) a complete table of $s_{\overline{n}|}$ can be calculated. It is only necessary to add to $s_{\overline{n-1}|}$ the value of $(1+i)^{n-1}$, and we obtain $s_{\overline{n}|}$, and so on.

EXAMPLE II.—Construct a table of $s_{\overline{n}|}$ at the effective rate of $2\frac{1}{2}\%$ for all values of n from 11 to 20 inclusive, to the nearest first place of decimals. It is to be assumed that we have already calculated a complete set of values of $(1+i)^n$, and as these are useful for many purposes

		n	$(1+i)^n$
		$\vdots$	$\vdots$
		9	1.249
		10	1.280
11	12.484	11	1.312
12	13.796	12	1.345
13	15.141	13	1.379
14	16.520	14	1.413
15	17.933	15	1.448
16	19.381	16	1.485
17	20.866	17	1.522
18	22.388	18	1.560
19	23.948	19	1.599
20	25.547	20	1.639
		21	1.680
		$\vdots$	$\vdots$

when tables are under construction, it is convenient to have them tabulated on a separate slip of paper which can be fastened by weights in the required position for any particular calculation. The schedule on page 74 illustrates the process.

15. Having fastened the table of $(1 + i)^n$ in the right position, the initial value, $s_{\overline{1}|i}$, is then calculated by the usual formula and inserted in the table. To this the value of $(1 + i)^n$ on the same line is added mentally and the result inserted as $s_{\overline{2}|i}$, and so on. The work is done to three places of decimals to ensure the first place being accurate. The insertion of every tenth value in advance obtained by independent calculation is desirable when constructing a long table, in order to avoid the carrying forward of an error, and the whole column should be checked by casting.

16. The cast of a complete column of $s_{\overline{n}|i}$ for all values of n from 1 to n gives—

$$\begin{aligned}
 & s_{\overline{1}|i} + s_{\overline{2}|i} + s_{\overline{3}|i} + \dots + s_{\overline{n}|i} \\
 &= \frac{(1+i) - 1}{i} + \frac{(1+i)^2 - 1}{i} + \dots + \frac{(1+i)^n - 1}{i} \\
 &= \frac{1}{i} \left\{ (1+i) + (1+i)^2 + \dots + (1+i)^n - n \right\} \\
 &= \frac{1}{i} \left(s_{\overline{n+1}|i} - 1 - n \right) \\
 \therefore \sum_{n=1}^{n=n} s_{\overline{n}|i} &= \frac{1}{i} \left\{ s_{\overline{n+1}|i} - (n+1) \right\} \quad . \quad . \quad . \quad (7)
 \end{aligned}$$

where $\sum_{n=1}^{n=n}$ means the sum for all values from 1 to n inclusive.

17. In Example II a complete set of values starting at

$n = 1$ has not been obtained. We therefore modify formula (7) as follows—

$$\begin{aligned} & s_{11} + s_{12} + \dots + s_{20} \\ &= \frac{1}{i} \left\{ (1+i)^{11} + (1+i)^{12} + \dots + (1+i)^{20} - 10 \right\} \\ &= \frac{1}{i} \left\{ s_{21} - s_{11} - 10 \right\} \end{aligned}$$

The cast of the s_{n-1} column comes to 188·004, and the result by the above formula is 187·992 or ·012 less. This error is due to $(1+i)^n$ having been taken to the nearest third place of decimals which, as it happened, amounted to taking the next higher figure nearly all through the column. The check, however, is a good one, since it proves the calculations to be accurate to the first place of decimals, which is all that is required.

18. When a prepared table of $(1+i)^n$ is not available it is usually more convenient to proceed by another method.

We know that $(1+i)s_n = s_{n+1} - 1$ (see formula (21) of Chapter II).

$$\therefore s_{n+1} = (1+i)s_n + 1 \quad \dots \quad (8)$$

EXAMPLE III.—Construct a table of s_{n-1} at the effective rate of 3% for all values of n from 1 to 10 inclusive, to the nearest second place of decimals.

n (1)	s_{n-1} (2)	$i s_{n-1}$ (3)
1	1·0000	·0300
2	2·0300	·0609
3	3·0909	·0927
4	4·1836	·1255
5	5·3091	·1593
6	6·4684	·1941
7	7·6625	·2299
8	8·8924	·2668
9	10·1592	·3048
10	11·4640	—

19. Having inserted the initial value for $n = 1$, the corresponding value $s_{\overline{n}|}$ is mentally calculated (or a machine may be used where necessary) and inserted in column (3). The figures in columns (2) and (3) are then mentally added and the result plus one inserted on the next line in column (2), and so on. The usual check values should be inserted in advance. As a final check we have the cast of column (2) as 60·2601, which should be equal to $\frac{1}{\cdot 03} (s_{\overline{60}|} - 11)$ by formula (7). The actual figure is 60·26, which exactly agrees to two places of decimals.

20. To construct a table of $a_{\overline{n}|}$ for all values of n from 1 to n .

$$\begin{aligned} \text{We have } a_{\overline{n}|} &= v + v^2 + \dots + v^{n-1} + v^n \\ \text{and } a_{\overline{n-1}|} &= v + v^2 + \dots + v^{n-1} \\ \therefore a_{\overline{n}|} - a_{\overline{n-1}|} &= v^n \\ \text{and } a_{\overline{n}|} &= a_{\overline{n-1}|} + v^n \end{aligned} \quad (9)$$

		n	v^n
		$\vdots$	$\vdots$
		$\vdots$	$\vdots$
		49	·1463
		50	·1407
50	21·4822	51	·1353
51	21·6175	52	·1301
52	21·7476	53	·1251
53	21·8727	54	·1203
54	21·9930	55	·1157
55	22·1087	56	·1112
56	22·2199	57	·1069
57	22·3268	58	·1028
58	22·4296	59	·0989
59	22·5285	60	·0951
60	22·6236	61	·0914
		62	·0879
		63	·0845
		$\vdots$	$\vdots$
		$\vdots$	$\vdots$

21. The application of this formula is similar in all details with that of formula (6).

EXAMPLE IV.—Construct a table of $a_{\overline{n}|}$ at the effective rate of 4% for all values of n from 50 to 60 to the nearest second place of decimals. (See table on page 77.)

22. The initial value $a_{\overline{50}|}$ is of course calculated by the usual formula. It should be noted that the movable slip is placed so that v^{51} is in a line with $a_{\overline{50}|}$ in accordance with the requirements of formula (9).

23. In the case of a complete table of $a_{\overline{n}|}$, we have

$$\begin{aligned}\sum_{n=1}^{n=\infty} a_{\overline{n}|} &= a_{\overline{1}|} + a_{\overline{2}|} + \dots + a_{\overline{n}|} \\ &= \frac{1}{i} \left(n - (v + v^2 + \dots + v^n) \right) \\ \therefore \sum_{n=1}^{n=\infty} a_{\overline{n}|} &= \frac{1}{i} \left(n - a_{\overline{n}|} \right) \quad . \quad . \quad . \quad . \quad (10)\end{aligned}$$

24. In Example IV the cast of the column $a_{\overline{n}|}$ should equal $a_{\overline{50}|} + a_{\overline{51}|} + \dots + a_{\overline{60}|}$

$$\begin{aligned}&= \frac{1}{i} \left(11 - a_{\overline{60}|} + a_{\overline{49}|} \right) \\ &= \frac{11 - 22.6235 + 21.3415}{.04} \\ &= \frac{9.7180}{.04} = 242.95.\end{aligned}$$

The actual cast of the column amounts to 242.9501.

25. If a prepared table of v^n be not available, a more convenient formula than number (9) can be devised. We have—

$$\begin{aligned}a_{\overline{n}|} &= v + v^2 + \dots + v^n \\ \therefore (1+i)a_{\overline{n}|} &= 1 + v + \dots + v^{n-1} \\ &= 1 + a_{\overline{n-1}|} \\ \therefore a_{\overline{n-1}|} &= (1+i)a_{\overline{n}|} - 1 \quad . \quad . \quad . \quad (11)\end{aligned}$$

26. The application of formula (11) is very similar to that of formula (8), except that as we are obtaining $a_{\overline{n}|i}$ from $a_{\overline{n-1}|i}$ we must clearly start with the longest term required and work backwards to the shortest.

EXAMPLE V.—Construct a table of $a_{\overline{n}|i}$ at the effective rate of $2\frac{1}{2}\%$ for all values of n from 1 to 10 inclusive, to the nearest third place of decimals.

n (1)	$a_{\overline{n} i}$ (2)	$i a_{\overline{n} i}$ (3)
10	8.75206	.21880
9	7.97086	.19927
8	7.17013	.17925
7	6.34938	.15873
6	5.50811	.13770
5	4.64581	.11615
4	3.76196	.09405
3	2.85601	.07140
2	1.92741	.04819
1	.97560	

27. The initial value $a_{\overline{10}|i}$ having been calculated and inserted in column (2), the first value in column (3) is obtained by dividing one-tenth of the initial value by 4 (i.e. to obtain $2\frac{1}{2}\%$). The second value in column (2) is then obtained by mentally adding $a_{\overline{10}|i}$ to $i a_{\overline{10}|i}$ and deducting 1.

28. The cast of column (2) should = $\frac{1}{.025} (10 - a_{\overline{10}|i})$ (see formula (10)). This equals 49.9174468 (taking $a_{\overline{10}|i}$ to eight decimals), and the actual cast is 49.91733.

29. Tables of $\frac{1}{s_{\overline{n}|i}}$ and $\frac{1}{a_{\overline{n}|i}}$ may be calculated by taking the reciprocals of previously prepared tables of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ by the use of a table of reciprocals.

30. As an alternative, tables of these functions may be

constructed by means of logarithms. The following example illustrates the method and at the same time shows how to construct a table of $s_{\overline{n}|}$ by means of logarithms.

EXAMPLE VI.—Construct a table of $\frac{1}{s_{\overline{n}|}}$ at 5% interest, to the nearest third place of decimals.

Working formula : $\log s_{\overline{n}|} + \log (1 + i) = \log (s_{\overline{n+1}|} - 1) \dots$ see formula (8).

$$\log 1.05 = .02119$$

n (1)	$\log s_{\overline{n} }$ (2)	(2) + $\log$ 1.05 (3)	Antilog (3) + 1 $= s_{\overline{n+1} }$ (4)	- Col. (2) $= \log$ $s_{\overline{n} }^{-1}$ (5)	Antilog (5) $= s_{\overline{n} }^{-1}$ (6)
1	.00000	.02119	2.05000	.00000	1.00000
2	.31175	.33294	3.15248	1.68825	.48781
3	.49865	.51984	4.31009	1.50135	.31721
:	:	:	:	:	:
:	:	:	:	:	:

31. Having inserted the initial value in column (2), this is added to $\log 1.05$ and the result inserted in column (3). The antilogarithm plus 1 is inserted in column (4), the first value being $s_{\overline{2}|}$. The logarithm of this value is inserted against $n = 2$ in column (2) and so the work proceeds. The usual checks should be applied, including the insertion of every 10th value in advance. Column (5) is completed by taking the values in column (2) negatively, so as to obtain $-\log s_{\overline{n}|}$ and column (6) is the antilog of column (5). The final column could be checked by reciprocation and subsequent addition.

32. In the Appendix will be found a set of compound interest tables based on 6% interest, showing the whole construction, which is fully explained. This should be

carefully followed. The student should also construct several tables for himself.

33. For purposes of reference, a list of important existing interest tables is given below—

LIEUT.-COL. W. H. OAKES: "Loans Payable by Drawings and Debenture Interest Tables."

Here are to be found to three places of decimals the half-yearly rate of interest realized on a debenture bearing interest at 3 to 9% per annum payable half-yearly, to be redeemed at par in any number of half-years from 1 to 60, and bought at a discount of from 1 to 30 per cent.

LIEUT.-COL. W. H. OAKES: "Tables of Compound Interest."

The values of $(1+i)^n$, v^n , $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ are given to five places, for all values of n from 1 to 100, at $\frac{1}{4}\%$ to 10% proceeding by intervals of $\frac{1}{8}\%$.

T. K. STUBBINS: "Annuity Tables for Building Societies and General Use."

Four places of decimals are shown, and the present values of annuities of 1 per month, quarter, or half-year, payable monthly, quarterly, or half-yearly, are given at rates of interest from 3% to 8%, with intervals of $\frac{1}{8}\%$, and convertible with corresponding frequency. In the monthly table n is given up to 300, in the quarterly to 200, and in the half-yearly to 100.

LIEUT.-COL. W. H. OAKES: "Tables for finding the Half-yearly Rate of Interest from $1\frac{1}{4}\%$ upwards, realized on Stock or Bonds, bearing $1\frac{1}{2}\%$, $1\frac{3}{4}\%$, 2, $2\frac{1}{4}\%$, $2\frac{3}{4}\%$, and 3 per cent. Half-yearly Interest, issued at any premium and redeemable at par in any number of half-years not exceeding 60."

J. A. ARCHER: "Tables for the Repayment of Loans."

These tables give (a) the values of $\frac{1}{a_{\overline{n}|i}}$ for all values of n from 1 to 60 at rates of interest proceeding by $\frac{1}{8}\%$ from 2 to 5 per cent.; (b) the values of $\frac{1}{a_{\overline{n}|i}}$ for even values of n from 2 to 120 at rates of interest proceeding by $\frac{1}{16}\%$ from 1 to $2\frac{1}{2}\%$ per cent.; (c) the values of $s_{\overline{n}|i}$ for all values of n from 1 to 60 at rates proceeding by $\frac{1}{8}\%$ from 2 to $3\frac{1}{2}\%$ per cent., also at 4 and 5 per cent.; (d) the values of $\frac{1}{s_{\overline{n}|i}}$ for even values of n from 1 to 120 at rates of interest proceeding by $\frac{1}{8}\%$ from 1 to $1\frac{1}{2}\%$ per cent., also at 2 and $2\frac{1}{2}\%$ per cent. All the

results are given to seven places of decimals and also for (a) and (b) in money to the nearest farthing per cent.

S. SPITZER: "Tabellen für die Zinses-Zinsen und Renten-Rechnung."

These tables give (a) the values of $(1+i)^n$, v^n , $s_{\overline{n}|i} - 1$, $a_{\overline{n}|i}$ and $\frac{1}{a_{\overline{n}|i}}$ for all values of n from 1 to 100, at rates of interest proceeding by $\frac{1}{2}$ and $\frac{1}{4}$ from 0 to 6 per cent., and thence by $\frac{1}{2}$ and $\frac{1}{4}$ to 10 per cent., also at $3\frac{1}{2}$ per cent. (being 4 on 105), and at the rates of interest corresponding to the various rates of discount; (b) the values of $\frac{1}{1+a_{\overline{n}|d}}$ at rates of interest corresponding to the above rates of discount, with the addition of 4%. Eight places of decimals are given.

D. M'KIE: "Tables of Compound Interest and Annuities."

Forty-eight rates commencing at $\frac{1}{100}$ % and proceeding by intervals of $\frac{1}{10}$ % for from 1 to 120 periods, and carried to nine decimal places.

Inwood's Tables of Interest and Mortality for the purchasing of Estates and Valuation of Properties including logarithms of natural numbers, and logarithmic interest and annuity tables. 32nd Edition, 1928.

Included in a wide range of interest and mortality functions are the amounts and present values of one pound and one pound per annum for periods from 1 to 100 and rates of interest proceeding by $\frac{1}{2}$ % intervals from 1% to 5%, by $\frac{1}{4}$ % intervals from 5% to 10%, and by 1% intervals from 10% to 15%.

Also $\frac{1}{s_{\overline{n}|i}}$ is given for the same periods at rates proceeding by $\frac{1}{2}$ % from 1% to $4\frac{1}{2}$ %, by 1% from 5% to 8% and at 10%.

Nominal and effective rates of interest are included for a range varying from 1% to 100% interest and values of $\frac{i}{j(m)}$ for from 1% to 10%.

CHAPTER V

THE MORTALITY TABLE

[A BRIEF note on Probabilities appears at the end of the chapter, for the benefit of those unfamiliar with the theory. This should be read first.]

1. THE mortality table is the means by which are calculated the probabilities of life and of death.

ELEMENTARY FUNCTIONS

2. If it were possible to trace the lives of a large number of persons from birth to death and to record the number of survivors on each birthday, we should obtain a column of figures similar to that headed l_x in the mortality table at the end of this volume (*see* Table V).

3. It should be noted that by this method of constructing a mortality table (or life table, as it is sometimes called) we obtain the numbers, out of a given number born, who attain each **exact age** until all have died. The number living at the **exact age** x is written l_x .

4. The difference between the number living at the exact age x and the number living at the exact age $x + 1$ represents the deaths which have occurred between the exact ages x and $x + 1$. These deaths are written d_x , so that

$$d_x = l_x - l_{x+1} \quad . \quad . \quad . \quad . \quad (1)$$

5. It will be observed that whereas l_x is the number living at the exact age x , d_x is not the number dying at the exact age x , but the number dying between the exact ages of x and $x + 1$. The persons included in l_x are, therefore, all aged exactly x , whereas the persons included in d_x are of varying ages from x to $x + 1$.

6. If we assume that the deaths are spread evenly over

the year of age, it follows that **on the average** the deaths aged x last birthday (i.e. d_x) occur at age $x + \frac{1}{2}$. This assumption, though convenient, is not absolutely accurate, especially at the older ages, where the rate of mortality is increasing rapidly.

7. If we assume that **births** are spread evenly throughout each year the **population** aged between x and $x + 1$ will, **on the average**, be aged exactly $x + \frac{1}{2}$. The population aged x last birthday is written L_x to distinguish it from l_x , the number living at exact age x . We may therefore write, **approximately**,

$$L_x = l_{x + \frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

8. The **rate of mortality** at age x is the ratio of the number dying between the ages of x and $x + 1$ to the number living at exact age x . In other words, the rate of mortality at age x is the probability that a person aged exactly x will die within one year.

9. If we write q_x for the rate of mortality at age x , we have

$$q_x = \frac{d_x}{l_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

10. This probability may be obtained very simply as follows—

Amongst l_x persons aged exactly x the deaths within one year will be d_x .

$\therefore$ The average number of deaths per person at risk within one year will be $\frac{d_x}{l_x}$. This **average** is the probability required

11. The probability that a person aged exactly x will live one year is, from the same reasoning, $\frac{l_{x+1}}{l_x}$.

If, therefore, we write this probability as p_x , we have

$$p_x = \frac{l_{x+1}}{l_x} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

12. A life must either survive the year or die within it. Therefore, the sum of the probabilities of death and of survivance must together make certainty, which is, of course, unity. In symbols—

$$p_x + q_x = 1 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

13. Out of the l_x people who attain the exact age x , the number of survivors at age $x + n$ is l_{x+n} . If, therefore, we write ${}_np_x$ for the probability of a life aged x surviving n years, we have

$${}_np_x = \frac{l_{x+n}}{l_x} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

14. On the lines of article 12, and writing ${}_nq_x$ for the probability of a life aged x dying within n years, we have

$${}_nq_x = 1 - {}_np_x = \frac{l_x - l_{x+n}}{l_x} \quad . \quad . \quad . \quad (7)$$

15. Formula (7) is also obtained by considering that out of the l_x persons alive at exact age x there will be $l_x - l_{x+n}$ who die within n years, whence the probability of any given life dying within n years is $\frac{l_x - l_{x+n}}{l_x}$.

16. We have seen that, given a complete column of l_x , we can calculate therefrom d_x , q_x , and other functions. In practice, however, the column l_x is not that normally obtained from the rough data. For instance, from a census we obtain the population in groups of ages, but the lives enumerated in the group $x + 1$ to $x + n$ are not, as in the case of the l_x column, the survivors of those in

the group $x - n + 1$ to x . The census being taken on a given date, each group of lives is entirely independent. Further, the lives enumerated in the group $x + 1$ to $x + n$ are not even the survivors of the lives enumerated at a previous census, owing to the effects of immigration and emigration.

17. One of the many methods of avoiding these difficulties is, very briefly, to obtain a complete column showing the **population** at each age. The deaths per annum are then obtained from the records of deaths, and the rate of mortality obtained by means of the formula

$$q_x = \frac{d_x}{L_x + \frac{1}{2} d_x} \quad (8)$$

18. This formula is derived from formulae (2) and (3). From formula (2) we have $L_x = l_x + \frac{1}{2} d_x$, where $l_x + \frac{1}{2} d_x$ is the number of lives who survive to the exact age $x + \frac{1}{2}$ out of l_x lives aged exactly x . This number will be the original l_x reduced by one-half a year's deaths, so that we have

$$\begin{aligned} l_x + \frac{1}{2} d_x &= l_x - \frac{1}{2} d_x \\ \text{whence } L_x &= l_x - \frac{1}{2} d_x \\ \text{and } l_x &= L_x + \frac{1}{2} d_x \end{aligned}$$

By substituting this value for l_x in formula (3) we obtain formula (8).

19. Other methods than the above are frequently used, but that described will serve to illustrate the essential difference between the rough data and the mortality table. This difference lies in the fact that the mortality table represents a **stationary** population, whereas this is practically never found in practice. In the mortality table there is no cause of exit other than death, whereas in almost any data on which a mortality table could be based some other form of exit is to be found. It therefore follows

that in the mortality table the numbers living at each age are the survivors of those at each preceding age, a state of affairs not commonly found in the data for a mortality experience.

20. Returning to the construction of the table, we will assume that a complete table of q_x has been calculated for all ages and properly graduated. By graduation we mean the **smoothing** of the curve so as to remove irregularities due to the smallness of the data. As the data are never infinitely great these irregularities are always to be found, but the greater the volume of data the smaller the irregularities.

21. It now remains to choose a suitable radix. The figures in the l_x column have no meaning in themselves, in that they do not represent any known population. Their value, however, lies in their relation to one another. It is therefore of no importance, other than as a convenience, exactly what figures we use. Having arbitrarily chosen a certain number of lives as the number alive at a certain age, the remaining values must be calculated from that number. The arbitrary number chosen is called the **radix**.

22. In the case of Table V, the radix chosen is 100,000 lives at exact age 10. Now $q_{10} = \frac{d_{10}}{l_{10}}$. If, therefore, we multiply l_{10} , as selected, by q_{10} , as already calculated, we obtain d_{10} , which we insert in the d_x column parallel to the l_x column. Deducting d_{10} from l_{10} , we obtain l_{11} , which in its turn is multiplied by q_{11} , thus producing d_{11} . By continuing this process we obtain the remaining values of l_x and d_x above age 10.

23. From formula (5) we have $p_x = 1 - q_x$, by means of which we can easily fill in all values of p_x from p_0 .

upwards. Remembering then that $p_9 = \frac{l_{10}}{l_9}$ and that therefore $\frac{l_{10}}{p_9} = l_9$, we are able to complete the l_x column backwards to age 0. The remaining values of d_x are then obtained by subtraction.

24. The symbol l_0 means, by definition, the number alive at the exact age 0, or, in other words, the number of births. The column l_x in Table V therefore represents the experience of 127,283 lives born amongst the class of persons from whom the data were obtained, and after allowance has been made for disturbance due to migration and any other factors which may have affected the data in question.

25. Data for the construction of a mortality table can be obtained from numberless sources, and each source usually requires somewhat different treatment of the rough data according to the circumstances of the case. The data for the table considered above were obtained from the experience of assured lives.

26. From formula (6) we see that ${}_np_x = \frac{l_{x+n}}{l_x}$. It is instructive to obtain this formula in another way.

The probability of a life aged exactly

$$\begin{array}{llll} x \text{ living 1 year is } & p_x & & \\ x + 1 & \text{,,} & \text{,,} & p_{x+1} \\ x + 2 & \text{,,} & \text{,,} & p_{x+2} \end{array}$$

and so on.

If we multiply together these probabilities we obtain the probability of a life aged exactly x living n years, i.e.

$${}_np_x = p_x \times p_{x+1} \times p_{x+2} \cdots \times p_{x+n-1}$$

The accuracy of this can be proved by writing p_n in terms of l_x as follows

$${}_np_x = \frac{l_{x+1}}{l_x} \times \frac{l_{x+2}}{l_{x+1}} \times \frac{l_{x+3}}{l_{x+2}} \dots \times \frac{l_{x+n}}{l_{x+n-1}} = \frac{l_{x+n}}{l_x}$$

THE EXPECTATION OF LIFE

27. The **expectation of life** at a given age is the average after lifetime at that age according to the mortality table in question. This function does not represent the after lifetime which a person may expect, but the **average** after lifetime of the class of people of whom he forms one.

28. Of l_x people alive at the exact age x , there are l_{x+1} who enjoy a complete year's life in the year of age x to $x+1$. Also there are l_{x+2} who enjoy a year of life in the year $x+1$ to $x+2$, and so on. The total after lifetime of the l_x people is therefore $l_{x+1} + l_{x+2} + \dots$ up to the limiting age in the table (written l_ω), after which all are dead.

If, therefore, we write e_x for the expectation of life at the exact age x , we have

$$e_x = \frac{l_{x+1} + l_{x+2} + \dots + l_\omega}{l_x} \quad (9)$$

29. It will be observed that in formula (9) no allowance is made for the fractional parts of years lived in the year of death, it having been assumed that each life dies at an exact age. We have therefore obtained what is called the *curtate* expectation of life. If we assume that each life dies on the **average** in the middle of the year, we must add to the numerator of formula (9) half a year for each

death at each age. The addition per person aged x will therefore be

$$\frac{1}{2} \left\{ (l_x - l_{x+1}) + (l_{x+1} - l_{x+2}) + \dots \right\} = \frac{1}{2} l_x$$

Writing e_x° for what is known as the **complete** expectation of life, we have

$$e_x^\circ = \frac{1}{2} + e_x \quad (10)$$

30. If, for purposes of brevity, we write

$$N'_x = l_{x+1} + l_{x+2} + \dots$$

the last two formulae become

$$e_x = \frac{N'_x}{l_x} \quad (11)$$

$$\text{and } e_x^\circ = \frac{1}{2} + \frac{N'_x}{l_x} \quad (12)$$

31. When dealing with population statistics it is convenient to have a symbol for the total population at age x and all higher ages. We therefore write

$$T_x = \sum L_x \quad (13)$$

It should be particularly noted that the first term in the series represented by N'_x is l_{x+1} , whereas the first term in the series represented by T_x is L_x .

32. It will be useful to consider a few examples of the application of the principles laid down above, with the aid of Table V.

EXAMPLE I.—Given the following data, choose a suitable radix and calculate as many values of l_x and d_x as the data will allow

$$q_{10} = \cdot 00409$$

$$q_{11} = \cdot 00370$$

$$q_{12} = \cdot 00347$$

$$q_{13} = \cdot 00342$$

$$q_{14} = \cdot 00342$$

$$q_{15} = \cdot 00365$$

If we select 100,000 as the radix at age 10, we have

x	l_x	$q_x \times l_x = d_x$
10	100,000	409
11	99,591	368
12	99,223	344
13	98,879	338
14	98,541	337
15	98,204	358
16	97,846	

It will be observed that the figures differ slightly from those in the mortality table. This is due to the fact that the table was constructed by an entirely different method from the above.

EXAMPLE II.—What are the probabilities, on the basis of Table V, of the happening of the following events—

- (1) That a man aged 20 will live to be 50.
- (2) That a man aged 20 will die in his 50th year.
- (3) That a man aged 20 will not live to be 50.
- (4) That a man aged 20 will not die in his 50th year.

We have

$$(1) {}_{30}p_{20} = \frac{l_{50}}{l_{20}} = \frac{72795}{96061} = .7577 \text{ (by logs).}$$

That is, the probability is roughly 3 to 1 in his favour.

(2) The number out of l_{20} starting who die in the year of age 49 to 50 is

$$l_{49} - l_{50} = 73896 - 72795 = 1,101 \text{ (i.e. } d_{49}\text{)}$$

$\therefore$ The probability of an individual aged 20 dying in his 50th year is

$$\frac{1101}{96061} = .01146$$

(3) As a man can only do two things, namely, live to be 50 or not so live, this probability must be 1 minus the probability calculated in the first section of the question, namely,

$$1 - .7577 = .2423$$

We could, of course, calculate it as follows—

The number who do not live to be 50 is $l_{20} - l_{50} = 23266$
 $\therefore$ the probability that an individual will not so live is

$$\frac{23266}{96061} = .2422.$$

This differs in the fourth place of decimals owing to the use of four-figure logarithms.

(4) This is the complement of the probability of section (2), i.e. $1 - .01146 = .98854$.

This could also be obtained in several other ways, such as adding up all values of d_x from $x = 20$ to the end of the table, but excluding d_{49} , and dividing the result by l_{20} . In this connection it is useful to observe that the l_{20} people alive at age x must all die by the time the limiting age of the table is reached. The record of how the deaths occur is to be found in the d_x column. Hence $l_x = d_x + d_{x+1} + \dots$, a very useful point to remember. In this example we require $d_{20} + d_{21} + \dots$ (which we will write Σd_{20}), less d_{49} . But

$$\Sigma d_{20} - d_{49} = l_{20} - d_{49} = 96,061 - 1101 = 94,960$$

$$\therefore \text{required probability is } \frac{94960}{l_{20}} = .9885.$$

EXAMPLE III.—What is the probability that a life aged 60 will not die between the ages of 70 and 75?

The probability that he *will* so die is

$$\begin{aligned} \frac{d_{70} + d_{71} \dots + d_{74}}{l_{60}} &= \frac{l_{70} - l_{75}}{l_{60}} \\ &= \frac{37977 - 25602}{58842} = \frac{12375}{58842} = .2104 \text{ (by logs)} \end{aligned}$$

$\therefore$ the probability that the life will *not* so die is $1 - .2104 = .7896$.

EXAMPLE IV.—What is the average after lifetime between the ages of 60 and 70 of lives now aged 30?

This may be described as the deferred temporary expectation of life and is obtained as follows—

${}_{30|10}e_{30}$ is the symbol which expresses the function required, the ${}_{30|}$ meaning that it is deferred for 30 years, and the $|_{10}$ expressing that it is limited to 10 years after the deferred period. We have

$$\begin{aligned} {}_{30|10}e_{30} &= \frac{l_{61} + \dots + l_{70}}{l_{30}} = \frac{N'_{60} - N'_{70}}{l_{30}} \\ &= \frac{783065 - 303504}{89685} = 5.346 \end{aligned}$$

(N'_x is given in Table VI.)

To this value we must add the adjustment to make the expectation **complete**. As the expectation does not cover the remainder of life, nor begin at once, the adjustment will not be $\frac{1}{2}$, but the method of article 29 will apply. We have

$$\begin{aligned} &\frac{\frac{1}{2} \{ (l_{60} - l_{61}) + (l_{61} - l_{62}) + \dots + (l_{69} - l_{70}) \}}{l_{30}} \\ &= \frac{1}{2} \cdot \frac{(l_{60} - l_{70})}{l_{30}} = \frac{1}{2} \cdot \frac{58842 - 37977}{89685} = \frac{20865}{179370} = .1163 \end{aligned}$$

The complete expectation is, therefore, $5.346 + .1163 = 5.4623$.

THE CENTRAL DEATH RATE

33. Before closing this chapter it will be well to consider two other measures of mortality which are in common use in addition to q_x .

34. From article 8 we see that q_x is based on the number alive at the exact age x , namely l_x . This, however, is not the number who are exposed to a year's risk between the ages of x and $x + 1$, but merely the number who *start* the year of age. The number exposed to risk of death starts at l_x and gradually decreases by deaths until it becomes l_{x+1} by the end of the year. On the average, therefore, the number exposed to risk throughout the year of age is $l_x + \frac{1}{2}$, and the ratio of the deaths to the number exposed to risk is a mortality rate which is sometimes useful. This rate is called the **central death rate** and is written m_x , so that

$$m_x = \frac{d_x}{l_x + \frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Applying formula (2), this becomes

$$m_x = \frac{d_x}{L_x} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

35. It thus appears that the central death rate is the rate obtained by dividing the deaths by the population, and in certain circumstances this is a more convenient function to calculate than is q_x . As, however, q_x is usually required as well, it is important to investigate the relationship between m_x and q_x .

36. On the assumption of a uniform distribution of

deaths throughout each year of age, we may write $l_x + \frac{1}{2} = \frac{1}{2} (l_x + l_{x+1})$. Applying this to formula (14), we have

$$m_x = \frac{d_x}{\frac{1}{2} (l_x + l_{x+1})} = \frac{2 d_x}{l_x + l_{x+1}} = \frac{2 q_x}{1 + p_x}$$

The last step is obtained by dividing numerator and denominator by l_x . As $p_x = 1 - q_x$ this gives us

$$m_x = \frac{2 q_x}{2 - q_x} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

37. From formula (16) we have

$$\begin{aligned} 2 m_x - q_x m_x &= 2 q_x \\ \therefore q_x (2 + m_x) &= 2 m_x \\ \therefore q_x &= \frac{2 m_x}{2 + m_x} \quad . \quad . \quad . \quad (17) \end{aligned}$$

THE FORCE OF MORTALITY

38. We have hitherto considered the rate of mortality and the central death rate, both of which measure mortality occurring between the ages of x and $x + 1$. There is, however, another measure of mortality, known as the **force of mortality**, which measures the rate of mortality **per annum** at the **precise** age x .

39. Lives remain only for a moment at the precise age x , and are then older than x . With a stationary population, however, such as the l_x column of the mortality table, as l_x is constant the numbers who attain the precise age x each moment must exactly equal the numbers who cease to be that precise age. The force of mortality at age x is the rate **per annum** at which deaths occur amongst this constantly changing crowd of lives who are constantly aged precisely x .

40. The force of mortality is written μ_x , but without the use of the differential calculus it is not possible to investigate formulae for its value. It is an important function in actuarial work, but for the present it will suffice to give a rough formula which will enable an approximate value to be calculated from the life table

$$\mu_x = \frac{l_{x-1} - l_{x+1}}{2l_x} \quad . \quad . \quad . \quad (18)$$

41. From formula (18) we can obtain an interesting **approximate** relationship between μ_x and m_x . We have

$$\begin{aligned} \mu_x &= \frac{l_{x-1} - l_{x+1}}{2l_x} = \frac{(l_{x-1} - l_x) + (l_x - l_{x+1})}{2l_x} \\ &= \frac{\frac{1}{2}(d_{x-1} + d_x)}{l_x} \end{aligned}$$

But $\frac{1}{2}(d_{x-1} + d_x)$ is approximately $d_{x+\frac{1}{2}}$

$$\therefore \mu_x = \frac{d_{x+\frac{1}{2}}}{l_x}$$

If now we put $x = x + \frac{1}{2}$ we get

$$\mu_{x+\frac{1}{2}} = \frac{d_x}{l_{x+\frac{1}{2}}} = m_x \quad . \quad . \quad . \quad (19)$$

Formula (19) expresses the fact that **approximately** the force of mortality at a given age is equal to the central death rate at an age one half of a year younger.

BRIEF NOTE ON PROBABILITIES

If an event can happen in a ways and fail to happen in b ways, and each of these ways is equally likely, then the total number of ways being $a + b$, the probability of the event happening is $\frac{a}{a+b}$, and the probability of it failing to

happen is $\frac{b}{a+b}$. As the event must either happen or fail to happen, the total of the probabilities must represent certainty. But the total of the probabilities is $\frac{a+b}{a+b} = 1$, from which we see that certainty is represented by unity. It is also to be observed that, having calculated the one probability, the other may be obtained by subtraction from unity. In the above illustration, therefore, the probability that the event will not happen is $1 - \frac{a}{a+b} = \frac{b}{a+b}$ as before. It should be noted, however, that this method requires care in its general application. There may be more than two things that can happen, as, for instance, where the event can partly happen and partly fail. It is safer, therefore, to observe that if there are n things that can happen and the probabilities of $n-1$ of them have been calculated, the n th probability may be obtained by deducting the total of the calculated probabilities from unity.

The probability of an event happening may be looked upon in the light of a sum in proportion. For instance, if out of $a+b$ ways the event happens in a ways, then out of 1 way the event will happen in $\frac{a}{a+b}$ ways. Or it may be looked upon as an average, the probability of the event happening being the number of times it happens on the average per way. By its nature the probability is fractional. The theory of probability is further developed in Chapter VI.

CHAPTER VI

JOINT LIFE PROBABILITIES

1. WHEN dealing with probabilities it is important to remember that all the probabilities connected with an event must add up to unity, which is certainty.

2. When two events are associated together, it is necessary, in order to obtain the probability of their both happening, to multiply together the probabilities of each event happening.

3. As an example of the application of articles 1 and 2, we will investigate the probabilities applicable to two lives aged x and y respectively for a period of 1 year. They are as follows—

- (a) That x and y will both live 1 year. $= p_x \times p_y$
- (b) That x will live but y will die during the year $= p_x \times q_y$
- (c) That y will live but x will die during the year $= q_x \times p_y$
- (d) That x and y will both die during the year $= q_x \times q_y$

These four probabilities represent all that can happen within one year and should add up to 1. If their values be set down in terms of l_x and l_y it will be found that the total is actually 1

4. The probability that both x and y will survive n years is

$${}_n p_x \times {}_n p_y = \frac{l_{x+n}}{l_x} \times \frac{l_{y+n}}{l_y}$$

This may be written $\frac{l_{x+n} : y+n}{l_x : y}$ for convenience in analysis.

Another convenient way of writing the expression and one which is consistent with the general scheme of notation is

$$\frac{{}_n l_{xy}}{l_{xy}}$$

We may therefore write

$${}_n p_{xy} = \frac{l_x + n : y + n}{l_{xy}} = \frac{{}_n l_{xy}}{l_{xy}} \quad . \quad . \quad . \quad (1)$$

5. It will be observed that we write l_{xy} for $l_x \times l_y$. We also write p_{xy} for $p_x \times p_y$. This is clearly correct notation, since p_{xy} is the probability that the joint lives will survive one year and $p_x \times p_y$ is the probability that both x and y will survive one year—and these two probabilities mean the same thing.

6. It is important to note, however, that $q_x \times q_y$ is **not** equal to q_{xy} . The expression $q_x \times q_y$ represents the probability that **both** x and y will die within one year, whereas the expression q_{xy} represents the probability that the joint lives will fail within one year. This latter is a totally different thing, being, in fact, the probability that either or both of the lives will so fail.

7. The probability that x will live n years and y will not is

$${}_n p_x \times {}_1 q_y = {}_n p_x (1 - {}_n p_y) = {}_n p_x - {}_n p_{xy}$$

8. The probability that at least one of any number of lives will fail within n years is the complement of the probability that none will fail. In the case of three lives aged x , y , and z respectively, we express this as follows

$${}_1 q_{xyz} = 1 - {}_n p_{xyz}$$

9. The probability that at least one of two lives aged x and y respectively will live n years is ${}_n p_x \times {}_1 q_y + {}_1 q_x \times {}_n p_y + {}_n p_{xy}$. This probability is more often expressed

as the probability that the **last survivor** of two lives will live n years and is symbolized ${}_np_{\overline{xy}}$, the bar over the ages indicating that it is a last survivor and not a joint life probability.

It will be observed that to obtain this probability we have to remember—

(a) That when an event happens to x , an event must also happen to y , and that these two events are associated together, the probabilities therefore being multiplied by one another.

(b) That more than one set of events can occur, each set having a separate probability.

(c) That these sets of events cannot be associated together, in that, for example, x cannot both live n years and also die within the n years.

(d) That the total probability is the sum of the values of each set of unassociated probabilities.

It is also to be noted that if we added one more probability, namely, that both x and y will die within n years, we should have included all the sets of events which could happen, and the total would add up to unity.

Writing the value in terms of p in the above expression, we have

$${}_np_{\overline{xy}} = {}_np_x (1 - {}_np_y) + (1 - {}_np_x) {}_np_y + {}_np_{xy}$$

$$\therefore {}_np_{\overline{xy}} = {}_np_x + {}_np_y - {}_np_{xy} \quad . \quad . \quad . \quad . \quad (2)$$

This is an important and convenient formula.

10. If we write ${}_nq_{\overline{xy}}$ for the probability that neither of two lives will live n years (or that the survivor of two lives will die within n years) we have

$${}_nq_{\overline{xy}} = 1 - {}_np_{\overline{xy}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This probability is, of course, the probability that both x and y will die within n years, therefore

$$|nq_{\overline{xy}} = |nq_x \times |nq_y \quad . \quad . \quad . \quad . \quad . \quad (4)$$

11. It is interesting to establish the identity of formulae (3) and (4).

$$\begin{aligned} |nq_x \times |nq_y &= (1 - {}_np_x)(1 - {}_np_y) \\ &= 1 - {}_np_x - {}_np_y + {}_np_{xy} = 1 - {}_np_{\overline{xy}} \end{aligned}$$

12. Apart from some of the above probabilities, which are so common in practice as to be worth remembering, there are many which occur from time to time and can be solved on general principles. Each case requires a little thought to secure that no probabilities in the case are overlooked.

13. As an example, we will calculate the probability that a life aged exactly 30 will die in his 60th year, another life aged exactly 40 being still alive at his death. In symbols we have

$$\frac{d_{59}}{l_{30}} \left({}_{30}p_{40} + \frac{1}{2} \frac{d_{60}}{l_{40}} \right) = \frac{d_{59}}{l_{30}} \left(\frac{l_{70}}{l_{40}} + \frac{1}{2} \frac{d_{60}}{l_{40}} \right)$$

This consists of the probability of 30 dying between 59 and 60, multiplied by the probability that 40 will not die in 30 years, modified as follows. The probability is not that 40 will be alive at the end of the year of age, but **at the time of the death of x** . There is therefore a probability not allowed for in the expression ${}_{30}p_{40}$, namely the probability that 40 will die in the 30th year but after the death of 30. On the usual assumption that deaths are evenly distributed throughout the year, there is an even chance that 30 will die before 40 in that year and that 40 will

die before 30. Therefore we must add $\frac{1}{2} \cdot \frac{d_{69}}{l_{40}}$ for the probability that 40 will die in his 69th year, but after the death of 30 in that year.

Evaluating the expression on the basis of Table V, we have

$$\frac{1682}{89685} \left(\frac{37977}{82277} + \frac{1}{2} \cdot \frac{2388}{82277} \right) = \frac{1682}{89685} \times \frac{39171}{82277} = .008928$$

which is the required probability.

14. It is useful to observe that $\frac{d_{59}}{l_{30}}$ is the probability that 30 will live to be 59, less the probability that he will live to be 60. That is

$$\begin{aligned} \frac{d_{59}}{l_{30}} &= {}_{30}p_{30} - {}_{30}p_{60} \\ &= \frac{l_{59}}{l_{30}} - \frac{l_{60}}{l_{30}} = \frac{d_{59}}{l_{30}} \end{aligned}$$

CHAPTER VII

LIFE ANNUITIES AND ASSURANCES

1. A **life annuity** is a series of periodical payments dependent on the continued existence of a given life or lives.

2. The present value of a sum of 1 due for certain at the end of n years is v^n . If, however, the sum is only payable in the event of a life now aged x being still alive, we must, to find the present value, multiply the v^n by the probability of its being received, namely ${}_np_x$.

3. Writing ${}_nE_x$ for the present value of 1 due at the end of n years if a life now aged x be still alive we have

$${}_nE_x = v^n \times {}_np_x \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This benefit is known as a "Pure Endowment."

4. In the case of a life annuity payable at the end of each year so long as x remains alive, we have a series of Pure Endowments. Writing a_x for the present value of such an annuity, we have

$$a_x = v p_x + v^2 {}_2p_x + v^3 {}_3p_x + \dots \quad . \quad . \quad . \quad (2)$$

$$= \frac{v l_{x+1}}{l_x} + \frac{v^2 l_{x+2}}{l_x} + \frac{v^3 l_{x+3}}{l_x} + \dots \quad . \quad (3)$$

COMMUTATION COLUMNS

5. To evaluate either formula (2) or (3) we must calculate every individual term until we reach the end of the table, when $l_{x+n} = 0$. This is rather a lengthy process, and the work may be greatly reduced by means of an ingenious device.

6. Multiplying each term of formula (3), both numerator and denominator, by v^x , we have

$$a_x = \frac{v^{x+1}l_{x+1} + v^{x+2}l_{x+2} + v^{x+3}l_{x+3} + \dots}{v^x l_x}$$

If, now, we write $D_x = v^x l_x$, we have

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \dots}{D_x} \quad (4)$$

7. Formula (4) provides a ready means of obtaining the value of an annuity at any age by means of a previously prepared column of D_x . Such a column, based on 4% interest, is given in Table VII, the mortality being that of Table V.

8. The advantage of the use of a function which is constant for any given value of x , such as D_x , may be further developed. It will be seen that the numerator of the expression for the value of a_x , no matter what the value of x , is ΣD_{x+1} , where $\Sigma D_{x+1} = D_{x+1} + D_{x+2} + \dots$. If we write $\Sigma D_{x+1} = N_x$, we have the convenient expression

$$a_x = \frac{N_x}{D_x} \quad (5)$$

9. Having given complete columns of D_x and N_x for a given mortality table and rate of interest, we can obtain a_x by a single division. D_x , N_x and other similar columns are called "Commutation Columns."

10. It will be observed that D_{x+1} is the first term in the series of which N_x is the sum. Some writers, however, have used the symbol N_x as ΣD_x . It is necessary, therefore, before using a column of N_x with which one is unfamiliar, to apply a simple test to determine its actual

meaning. Against the age before the limiting age of the table we find that $D_x = N_x$ when $N_x = \Sigma D_x$, but in the more usual case, where $N_x = \Sigma D_{x+1}$, we find that at the same age $N_x = 0$. In the case of the more modern tables, such as the O^M , a special symbol $\mathbb{N}_x$ (hollow N_x) is used for ΣD_x , the old symbol N_x retaining the meaning of ΣD_{x+1} . In the following pages N_x will be used to signify ΣD_{x+1} .

TEMPORARY AND DEFERRED ANNUITIES

11. Commutation columns provide a ready means of calculating the values of temporary and deferred annuities. Writing $|_n a_x$ for the value of an immediate annuity payable annually for n years, if a life aged x shall live so long (that is a **temporary** life annuity), we have

$$|_n a_x = \frac{N_x - N_{x+n}}{D_x} \quad . \quad . \quad . \quad . \quad (6)$$

The symbol more commonly used for this benefit is $a_{x:\overline{n}|}$.

12. If we write ${}_n|a_x$ for a similar life annuity deferred n years, we have

$${}_n|a_x = \frac{N_{x+n}}{D_x} \quad . \quad . \quad . \quad . \quad (7)$$

It should be noted that in the case of an **immediate** annuity deferred n years the first payment is at the end of $n+1$ years from the outset, that is, n years later than it would have been had it not been deferred.

13. It is useful to establish the relationship between the value of a temporary annuity, a life annuity, and a deferred annuity. Clearly, if we deduct from the present value of a life annuity the present value of a deferred

annuity, we obtain the present value of a temporary annuity. In symbols—

$${}_n a_x = a_x - {}_n | a_x \quad . \quad . \quad . \quad . \quad (8)$$

Formula (8) is sometimes useful in avoiding the calculation of both the temporary and deferred annuity values.

14. As in the case of annuities-certain, the symbol a_x is used to denote the present value of an **annuity due** on a life aged exactly x .

15. For some calculations the value of ΣN_x is required and this is written S_x , so that

$$S_x = \Sigma N_x \quad . \quad . \quad . \quad . \quad (9)$$

A column of S_x is given in Table VII.

ASSURANCES

16. Let A_x be the present value of a sum of 1 payable at the **end** of the year of death of a life now aged exactly x . Of the l_x persons now aged x , the deaths in the first year will be d_x . The cost **per person** of paying 1 for each death amongst l_x persons is $\therefore \frac{d_x}{l_x}$. If this sum is payable at

the end of the year of death, the present value will be $v \frac{d_x}{l_x}$.

This result might have been obtained, as in the case of annuities, by multiplying the sum to be received by the probability of it being received, and discounting the result to the present time.

Similarly, in respect of the second year we have the present value of the cost per person as $v^2 \frac{d_{x+1}}{l_x}$.

We therefore have

$$A_x = \frac{v d_x + v^2 d_{x+1} + v^3 d_{x+2} + \dots}{l_x} \quad . \quad (10)$$

17. If we multiply the numerator and denominator of formula (10) by v^x we get

$$A_x = \frac{v^{x+1} d_x + v^{x+2} d_{x+1} + v^{x+3} d_{x+2} + \dots}{v^x l_x}$$

Writing $C_x = v^{x+1} d_x$ we have

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \dots}{D_x} \quad (11)$$

18. Writing $M_x = \Sigma C_x$, formula (11) becomes

$$A_x = \frac{M_x}{D_x} \quad (12)$$

By means of prepared columns of M_x and D_x the value of A_x can be calculated for any value of x .

19. The value of ΣM_x is sometimes required and this is written R_x . A column of R_x is given in Table VII.

20. It will be observed that A_x is the single premium for an assurance of 1 payable at the end of the year of death of x . It is, of course, the **net** single premium, since it contains no **loading** for office expenses such as would be included in the single premium quoted by an assurance office.

21. The single premium for a temporary assurance for n years on a life aged x , is written $A_{x:\overline{n}|}^1$, the 1 over the x signifying that the risk covered is x 's death before n years, i.e. x dies first. On analogy with this symbol ${}_nE_x$ is sometimes written $A_{x:\overline{n}|}^{\frac{1}{n}}$, meaning that the risk covered is that of the term expiring before x 's death, i.e. the term expires first.

22. Using commutation columns, we have

$$A_{x:\overline{n}|}^1 = \frac{M_x - M_{x+n}}{D_x} \quad (13)$$

This function may also be written ${}_nA_x$.

23. Similarly, the single premium for a deferred assurance of 1 payable at the end of the year of the death of x is

$${}_n|A_x = \frac{M_{x+n}}{D_x} \quad . \quad . \quad . \quad . \quad (14)$$

and a deferred temporary assurance

$${}_n|m A_x = \frac{M_{x+n} - M_{x+n+m}}{D_x} \quad . \quad . \quad . \quad (15)$$

24. Formula (13) should be carefully distinguished from that for the single premium for an endowment assurance, which is symbolized $A_{x:\overline{n}|}$. An endowment assurance provides two distinct benefits, namely (1) a temporary assurance for n years, (2) a pure endowment payable at the end of n years if the life survive. The symbol $A_{x:\overline{n}|}$ may, therefore, be interpreted in the same way as the symbol A_{xy} , namely a sum of 1 payable on the failure of the joint status $x:\overline{n}|$. In commutation symbols

$$A_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x} \quad . \quad . \quad . \quad (16)$$

25. A double endowment assurance provides a temporary assurance of 1 and a pure endowment of 2, so that

$$A_{x:\overline{n}|} + {}_nE_x = \frac{M_x - M_{x+n} + 2D_{x+n}}{D_x} \quad . \quad . \quad . \quad (17)$$

CONVERSION TABLES

26. It is instructive to consider the present value of 1 payable at the end of the year of death of x from a different point of view. If there be an annuity payable at the end of each year which x commences and an annuity payable at the end of each year which he completes, the difference

will be a sum of 1 payable at the **end** of the year of his death. We thus establish a valuable connection between the present values of annuities and assurances, for

$$A_x = v (1 + a_x) - a_x \quad . \quad . \quad . \quad (18)$$

27. Putting $v = 1 - d$ in equation (18) we get

$$A_x = 1 - d (1 + a_x) \quad . \quad . \quad . \quad (19)$$

which is the most important of many formulae which may be obtained by algebraical transformation of formula (18).

It is interesting to note that formula (19) may itself be obtained by general reasoning. If a sum of 1 be payable at the end of the year of death, it could instead be paid at once in consideration of the payment of i at the end of each year up to and inclusive of the year when the capital sum becomes payable. Now d , payable at the beginning of each year, is the equivalent of i payable at the end (see article 15) and $d (1 + a_x)$ is therefore the present value of i payable at the end of each year which the life commences. Deducting this value from the capital sum 1, we obtain the present value of 1 payable at the end of the year of death.

28. Formula (19) shows that for a given rate of interest there is a definite relationship between a_x and A_x , so that if one be given the other may be obtained. It is not necessary to know the table of mortality on which either is based. Tables have accordingly been constructed, similar in principle to logarithmic tables, with a_x for the argument instead of the natural number and $1 - d (1 + a_x)$ for the result instead of the logarithm. A separate table must be constructed for each rate of interest. These tables are known as **Single Premium Conversion Tables**, and an abbreviated table is given as Table IX. It should be observed that as the annuity value increases

the assurance value decreases. Consequently, the differences tabulated must be **deducted** and not added as in the case of logarithms. If the assurance value has been calculated and the annuity value is required, this may be obtained by the inverse process as in the case of taking an antilogarithm.

29. It is important to observe that formula (19) holds good for all cases where the sum assured is certainly payable one year after the last payment of the annuity. If, therefore, we enter the table with a_{xy} we obtain A_{xy} . Similarly, if we enter the table with $a_{x:\overline{n-1}|}$ we obtain $A_{x:\overline{n}|}$. The annuity is here taken for a term of $n - 1$ years, since the sum assured will be payable at the end of the year of death of x or at the end of the n^{th} year. By taking $n - 1$ we fulfil the required condition. If we follow the argument by means of which formula (18) was obtained, we see why $a_{x:\overline{n-1}|}$ must be used, for

$$A_{x:\overline{n}|} = 1 - d(1 + a_{x:\overline{n-1}|}) \quad . \quad . \quad . \quad (20)$$

30. It is evident that conversion tables may be used for annuities certain but not for temporary assurances, since in the latter case the sum assured is not **certainly** payable at all, but only in the event of the death occurring within a stated period.

31. The present value of the **annual** premiums payable under a policy of assurance must equal the single premium. If, therefore, we write P_x as the annual premium for an assurance of 1 payable at the end of the year of death of a life now aged x , we have, remembering that premiums are payable in advance,

$$\begin{aligned} P_x(1 + a_x) &= A_x \\ \therefore P_x &= \frac{A_x}{1 + a_x} \quad . \quad . \quad . \quad (21) \end{aligned}$$

32. In the case of an endowment assurance, we must remember that if there are n yearly premiums payable there will be 1 at once and $n - 1$ others at intervals of 1 year, whence

$$\therefore P_{x\bar{n}} = \frac{A_{x\bar{n}}}{1 + a_{x\bar{n}-1}} \quad (22)$$

33. Another class of policy often met with in practice is the whole life policy with the premiums limited to t (say) from the outset. Writing ${}_tP_x$ for the annual premium, we have

$$\therefore {}_tP_x = \frac{A_x}{1 + a_{x:\overline{t-1}|}} \quad (23)$$

34. Formulae (21) to (23) may be expressed in commutation functions as follows

$$\begin{aligned} P_x \times \frac{N_{x+1}}{D_x} &= \frac{M_x}{D_x} \\ \therefore P_x &= \frac{M_x}{N_{x+1}} \end{aligned} \quad (24)$$

$$\text{Also } P_{x+n} \times \frac{N_{x+1} - N_{x+n+1}}{D_x} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

$$\therefore P_{x[n]} = \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+n-1}} \quad (25)$$

$$\text{and } {}_tP_x \times \frac{N_{x-1} - N_{x+t-1}}{D_x} = \frac{M_x}{D_x}$$

$$\therefore P_x = \frac{M_x}{N_{x+1} - N_{x+t-1}} \quad (26)$$

35. From formula (19) we have

$$\begin{aligned}
 A_x &= 1 - d(1 + a_x) \\
 \therefore P_x(1 + a_x) &= 1 - d(1 + a_x) \\
 \therefore (P_x + d)(1 + a_x) &= 1 \\
 \text{and } P_x &= \frac{1}{1 + a_x} - d \quad . \quad . \quad (27)
 \end{aligned}$$

The formula $(P_x + d)(1 + a_x) = 1$ may be established by general reasoning as follows. An annual premium of P_x payable at the beginning of each year which x commences will provide a sum of 1 payable at the end of the year of death, and the present value of the payments is $P_x(1 + a_x)$. If to this single payment be added the present value of the interest on the capital sum of 1 up to the end of the year of death, the total will be equal to the sum of 1 payable at once. But $d(1 + a_x)$ is the present value of the interest—see article (27)—and the total is therefore $(P_x + d)(1 + a_x)$. This leads us to an interpretation of formula (27). A sum of $1 + a_x$ is the equivalent of 1 per annum payable in advance during the lifetime of x , so that a sum of 1 is the equivalent of $\frac{1}{1 + a_x}$ similarly payable. This latter annual payment consists of two parts, namely, an annual premium to replace the capital at death, and the annual interest payable in advance until death. If, therefore, we deduct the annual interest (which is d when payable in advance), we obtain the annual premium, so that $P_x = \frac{1}{1 + a_x} - d$.

36. Formula (27) has the same properties as formula (19), and we are therefore able to construct what is called an

Annual Premium Conversion Table, by means of which we can obtain the annual premium corresponding to a given annuity value. A separate table must, of course, be prepared for each rate of interest, as in the case of Single Premium Conversion Tables. An abbreviated table is given in Table X.

37. It should be noted that the Annual Premium Conversion Table applies in the same cases as the Single Premium Conversion Table and that where the one does not apply neither does the other.

38. A few examples of the principles laid down in this chapter will help to make matters clear.

EXAMPLE I.—Calculate by means of the commutation columns in Table VII the present value on the basis of 4% interest of a temporary immediate annuity to a life now aged 30 payable for 20 years, but deferred 10 years.

$${}_{10|20}a_{30} = \frac{N_{40} - N_{60}}{D_{30}} = \frac{259,398 - 52,850.7}{27652} = \frac{206,547.3}{27652} \\ = 7.470 = \text{Answer.}$$

EXAMPLE II.—What is the present value at 4% interest, and on the basis of the mortality of Table V, of an annuity-due payable to a life aged 40 for 10 years, if he shall live so long?

$$a_{40:\overline{10}|} = \frac{N_{39} - N_{49}}{D_{40}} = \frac{276,536 - 138,501}{17138} = \frac{138,035}{17138} \\ = 8.052 = \text{Answer.}$$

EXAMPLE III.—What is the present value, on the same

basis, of an annuity-due deferred 10 years and payable for the remainder of the life of a person now aged 40?

$${}_{10}|a_{40} = a_{40} - a_{40} \overline{10}| = 16.136 - 8.052 = 8.084 = \text{Answer.}$$

The value of a_{40} is obtained by adding 1 to a_{40} as given in Table VIII. Example II gives the temporary annuity value.

EXAMPLE IV.—A life aged exactly 33 paying an annual premium of £1 14s. on a whole life policy effected some years back desires to commute all future premiums by payment of a lump sum down. What should he pay on the basis of 4% interest and the mortality of Table V, assuming that a premium is now due and unpaid?

$$1.7 (1 + a_{33}) = 1.7 \times 17.612 = 29.9404 = \text{Answer.}$$

EXAMPLE V.—Calculate by means of conversion tables the single and annual premiums for an endowment assurance on a life now aged 20, payable at the end of 30 years or on previous death, on the basis of 4% interest and the mortality of Table V.

$$\begin{aligned} \text{In the first place we require } a_{20} \overline{29}| &= \frac{N_{20} - N_{40}}{D_{20}} \\ &= \frac{818,165 - 138,501}{43841} = \frac{679,664}{43,841} = 15.503 \end{aligned}$$

Turning to the single premium conversion table, under 4% for $a = 15$, we have $A = .38462$.

For difference .5 we have a difference in A of .01923

"	.003	"	"	"	"	.00012
						Total
						.01935

$$\therefore A_{20} \overline{30}| = .38462 - .01935 = .36527$$

From the annual premium conversion table we have, for $a = 15$, $P = \cdot 02404$.

When $a = 15$, for a difference of

$\cdot 5$	we have a difference in P of	$\cdot 00189$
$\cdot 003$	" " " " "	$\cdot 00001$
Total		$\cdot 00190$

(The difference of $\cdot 003$ is obtained by shifting the decimal point for the difference of $\cdot 3$.)

$$\therefore P_{20 \overline{30}} = \cdot 02404 - \cdot 00190 = \cdot 02214.$$

To check accuracy of results (on assumption that $a_{20 \overline{29}}$ is correct), we have $\cdot 02214 \times 16 \cdot 503 = \cdot 36538$, which differs from $A_{20 \overline{30}}$ by only $\cdot 00011$.

It should be noticed that the differences in the annual premium table do not depend on the rate of interest, as in the case of the single premium table. A consideration of the formulae involved will explain why this is the case.

CHAPTER VIII

- (a) SPECIAL ANNUITIES AND ASSURANCES. (b) POLICY
VALUES. (c) SELECT AND OTHER TABLES.
(d) WELL-KNOWN MORTALITY TABLES

[THE following are brief notes on matters of interest connected with life contingencies which it is unnecessary for present purposes to develop more fully.]

(a) Special Annuities and Assurances.

PERIODICAL PAYMENTS

1. We have hitherto only considered life annuities payable yearly, but in practice they are often payable more frequently.

2. The dates at which the payments are to be made are of importance, not only on account of interest, but also because the probabilities of survivance are affected. We can, however, obtain an approximate formula which is independent of age, mortality, and interest, and is sufficiently accurate for most practical purposes.

3. The present value of a yearly annuity on a life aged x , with the first payment to be made at once, is $1 + a_x$.

If the first payment is to be made in one year's time, the present value is a_x . Hence, by proportion, if the first payment is to be made in $\frac{1}{m}$ th of a year, the present value will be $a_x + \frac{m-1}{m}$.

4. In the case of an immediate annuity of 1 per annum payable m times a year, the present value consists of the sum of the present values of m yearly annuities of $\frac{1}{m}$

per annum the respective first payments under which are due $\frac{1}{m}, \frac{2}{m}$, etc., of a year hence. If, therefore, we write $a_x^{(m)}$ for the present value of an immediate annuity of 1 per annum payable m times a year on a life aged x , we have

$$\begin{aligned} a_x^{(m)} &= \frac{1}{m} \left(\frac{1}{m} |a_x + \frac{2}{m} |a_x + \dots + \frac{m}{m} |a_x \right) \\ &= \frac{1}{m} \left(a_x + \frac{m-1}{m} + a_x + \frac{m-2}{m} + \dots + a_x + \frac{m-m}{m} \right) \\ &= \frac{1}{m} \left(ma_x + \frac{1}{m} \times \frac{(m-1)m}{2} \right) \\ \therefore a_x^{(m)} &= a_x + \frac{m-1}{2m} \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

5. In the case of periodical premiums, no matter how often per annum they are payable, it is still a condition that the present value of future premiums must equal the present value of the benefit. Writing $P_x^{(m)}$ for the premium per unit per annum payable m times a year for a whole life assurance on x , we have

$$\begin{aligned} P_x^{(m)} \left(\frac{1}{m} + a_x^{(m)} \right) &= A_x \\ \therefore P_x^{(m)} &= \frac{A_x}{\frac{1}{m} + a_x^{(m)}} \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

6. The argument on which formulae (1) and (2) are based holds good for joint life and last survivor assurances, and the form of the formulae therefore also holds good. It does not, however, apply to deferred and temporary annuities and assurances, for obvious reasons.

COMPLETE ANNUITIES

7. The life annuities hitherto considered cease with the payment immediately preceding the date of death. Complete, or apportionable, annuities are, however, common in practice, and we must investigate a formula to express their value.

8. Let us first consider the simple case of a yearly annuity of 1 per annum. Making the usual assumption that the deaths are evenly distributed throughout the year, and that therefore, on the average, they all occur in the middle of the year, it follows that, on the average, the amount payable at death will be $\frac{1}{2}$. If this were payable at the end of the year of death its present value would be $\frac{1}{2} A_x$. As a matter of fact, however, the amount is payable at death, which is, on the average, half a year sooner than the end of the year of death. The value is, therefore, $\frac{1}{2} A_x (1 + i)^{\frac{1}{2}}$. The expression $\frac{1}{2} (1 + i)^{\frac{1}{2}}$ represents the amount which, if discounted from the end of the year to the middle of the year, will provide $\frac{1}{2}$ at the average date of death. Writing $\ddot{a}_x$ for the present value, we have—

$$\ddot{a}_x = a_x + \frac{1}{2} A_x (1 + i)^{\frac{1}{2}} \quad . \quad . \quad . \quad (3)$$

This formula is only approximate, but it will serve for present purposes.

9. If the annuity be payable m times a year the average amount payable at death will be $\frac{1}{2m}$, which, as before, will be payable, on the average, in the middle of the year of death. Hence

$$\ddot{a}_x^{(m)} = a_x^{(m)} + \frac{1}{2m} A_x (1 + i)^{\frac{1}{2}} \quad . \quad . \quad . \quad (4)$$

JOINT LIFE ANNUITIES

10. In the case of several of the older well-known mortality tables extensive tables of joint life annuity values have been published dependent on two lives, and it is convenient to have a rule by means of which annuity values dependent on three lives may be obtained. If $x < y < z$, obtain from the joint life tables the value of $a_{y,z}$. Look for this value in the **single** life table, thus obtaining the age w such that $a_w = a_{y,z}$. An approximate value for $a_{x,y,z}$ can then be obtained by obtaining the value of $a_{x,w}$ from the two-life tables.

11. In the case of the mortality table given in Table V and of several modern tables based on life office experience, the rate of mortality has been based on what is known as "Makeham's Law" or some modification of it. Briefly, this means that a mathematical relationship has been established between the rate of mortality at one age and that at another.

12. The great property of a table based on Makeham's law is that it renders unnecessary the tabulation of extensive tables of joint life annuities based on the many combinations of two or more lives together which arise in practice. All that is required is a complete set of functions based on two, three, four, etc., lives (as the case may be) of **equal** ages. Each function may therefore be tabulated in a single column.

13. Let it be required to find two equal lives aged w such that all joint mortality functions based on them will be equal to the same functions based on two unequal lives aged x and y respectively. For instance, such that $a_{ww} = a_{xy}$.

Firstly, let $\mu_x + \mu_y = z$.

Then the mean value of μ_x and μ_y will be $\frac{1}{2}z$.

Find the age w at which $\mu_w = \frac{1}{2}z$.

Then w is the age required.

14. As an example of article 13, let us find two equal ages equivalent to two unequal ages 23 and 37 respectively. From Table VI we have

$$\begin{array}{rcl} \mu_{23} & = & \cdot 00659 \\ \mu_{37} & = & \cdot 00901 \\ \hline \text{Total} & = & \cdot 01560 \\ \text{One-half} & = & \cdot 00780 \\ & = & \mu_{31} \text{ approximately.} \end{array}$$

Then $a_{31:31} = a_{23:37}$ and similarly for other functions.

To increase accuracy it is customary to obtain w to one or more places of decimals.

15. The method above described is based on the **Law of Uniform Seniority**, which is only applicable when Makeham's law, or some modification of it, has been used in the construction of the mortality table.

16. Having obtained the annuity value by the above method, the corresponding single and annual premiums can be obtained by means of a conversion table.

REVERSIONARY ANNUITIES

17. To obtain the present value of an annuity payable to x after the death of y , we deduct from the value of a whole life annuity to x , the value of that part of the annuity which will not be payable, namely, an annuity

payable whilst both x and y are alive. Writing the present value of the benefit $a_{y|x}$, we have

$$a_{y|x} = a_x - a_{xy} \quad . \quad . \quad . \quad . \quad (5)$$

(b) Policy Values.

18. The annual premium for an assurance on a life exceeds that required to cover the risk in the early years of the assurance. As the mortality increases, however, the excess becomes less and less until in the later years of the assurance the premium is insufficient to cover the risk. It is, therefore, necessary to save up the excess premium in the early years to meet the excess risk of the later years. This accumulation of excess premiums is known as the Policy Value or Reserve.

19. Although we may obtain the policy value by the accumulation method, allowing for interest and mortality, it is simpler to look at it prospectively. The present value of the sum assured less the present value of future premiums must be the sum held as a reserve against the policy.

20. If we write ${}_nV_x$ as the value of a whole life policy effected n years ago on a life then aged x , on the supposition that the $(n+1)^{\text{th}}$ premium is due but unpaid, we have

$${}_nV_x = A_{x+n} - P_x (1 + a_{x+n}) \quad . \quad . \quad (6)$$

21. Similarly, if we write ${}_nV_{x:\overline{t}|}$ as the value of a " t " year endowment assurance effected n years ago on a life then aged x , we have

$${}_nV_{x:\overline{t}|} = A_{x+n:\overline{t}|} - P_{x:\overline{t}|} (1 + a_{x+n:\overline{t}|}) \quad . \quad (7)$$

22. As an example, let us calculate the policy value on a 4% basis in the case of a whole life policy effected exactly 10 years ago for £100, on a life then aged exactly 30 at an annual premium of £1.662.

On the basis of Table V, the 4% functions being taken from Table VIII, the value is

$$\begin{aligned} & 100A_{40} - 1.662 (1 + a_{40}) \\ &= 37.937 - 1.662 \times 16.136 \\ &= 37.937 - 26.82 \\ &= 11.117. \end{aligned}$$

23. In the case of a single premium policy, the value will be the present value of the sum assured, there being no further premiums payable. This also applies to a limited payment policy under which all the premiums have been paid. If, however, a number of premiums still remain to be paid, the present value of these must clearly be deducted. As an example, let us calculate the reserve just before payment of the tenth premium on a whole life policy for £100, effected on a life aged 30 at entry, the premiums being limited to 25 in number.

It is first necessary to calculate the net premium per unit. This is $\frac{M_{30}}{N_{29} - N_{54}}$, which, on the basis of Table VII,

$$= \frac{8344.01}{501,999 - 92,569.6} = .02039$$

The premium for £100 is therefore 2.039, and the reserve—

$$\begin{aligned} & 100 A_{39} - 2.039 (1 + a_{39 \overline{15}}) \\ \text{Now } a_{39 \overline{15}} &= \frac{N_{39} - N_{54}}{D_{39}} = \frac{276,536 - 92,569.6}{17997} = 10.22 \\ \therefore \text{ Reserve} &= 37.058 - 2.039 \times 11.22 = 14.178 \end{aligned}$$

(c) Select and other Tables.

24. Lives accepted for life assurance are usually medically examined and therefore, as a body, may be expected to

experience lower rates of mortality in the earlier years of assurance than those of other bodies not similarly **selected**.

25. Lives purchasing annuities only do so if they feel reasonably satisfied that they are in good health, and may therefore similarly be expected to experience, as a body, lighter rates of mortality in the early years after purchase than those of other bodies not similarly **self-selected**.

26. Data obtained from the experience of lives assured, annuitants, and other bodies where for one reason or another a form of **selection** occurs, have been found to produce rates of mortality much lower in the first year of experience, and considerably lower afterwards—in some cases for many years. It has therefore been found necessary to construct tables of mortality showing the rates of mortality at each age at the outset and for each successive year from the outset. It is thus possible to obtain from these tables the rate of mortality experienced amongst lives t years after their entry at the age of x . These tables are called **Select Mortality Tables**, and the rate referred to would be written $q_{[x] + t}$, meaning the rate for a life aged $x + t$ who was selected at age x .

27. Premiums for life assurance are usually based on select mortality tables, thus giving the life assured the benefit of his selection by the company.

28. Ordinary mortality tables, such as that at the end of this volume, are known as **Aggregate Tables**, since they are constructed from lives aggregated together irrespective of the period of their selection, if any.

29. It is inconvenient to show the effect of selection in a mortality table throughout the period of life, and as the effect of selection wears off in time it is usual to tabulate select rates for a period of five or ten years from the age at which the selection takes place and then to throw

all the lives together into an aggregate table. For instance, if selection be shown for five years, the rates shown would be $q_{[x]}$, $q_{[x] + 1}$, $q_{[x] + 2}$, $q_{[x] + 3}$, $q_{[x] + 4}$, but after that would follow $q_{x + 5}$, etc., where $q_{x + 5}$ represents the rate of mortality experienced amongst lives aged $x + 5$ irrespective of whether they were selected five or more years before. The aggregate table into which the select rates run is called an **ultimate** table of mortality. The ultimate table differs from an ordinary aggregate table in that it contains no lives who have been selected within, say, five years.

30. Sometimes the ultimate table is tabulated by itself. In this case it is constructed from lives none of whom have been selected within, say, five years. Such a table is called a **Truncated Aggregate Table**.

(d) **Well-known Mortality Tables.**

31. The *Northampton* Table was the first to be used for assurance purposes. It was, however, constructed on faulty principles, and although it has had an extensive vogue it is now practically obsolete.

32. The *Carlisle* Table, constructed in 1815, has proved a very useful table. It was, however, based on small data, of which more than half were female, and its graduation is not good. The main reasons for its popularity are, firstly, that it has been found to produce rates of premium not far removed from the true ones, and, secondly, that very extensive tables of functions based upon it have been published. Apart from such old tables of life assurance premiums as have not been altered, it is not commonly used now outside Reversionary transactions, and even for valuing Reversions and Life Interests it is largely superseded by the a (f) and a (m) Tables. It has long been the custom to treat female lives as being three or more

years younger than they actually are when using the Carlisle Table. This makes a rough adjustment for superior female mortality.

33. There are at present ten tables called *English Life Tables*, in each of which males and females are tabulated separately. They are based on the following data, taken from England and Wales—

English Life Tables.	Data.
No.	
1	Census of 1841 and deaths for same year.
2	Census of 1841 and deaths for 1838 to 1844 inclusive.
3	Censuses of 1841 and 1851 and deaths for 1838 to 1854 inclusive.
4	Censuses of 1871 and 1881 and deaths for 1871 to 1880 inclusive.
5	Censuses of 1881 and 1891 and deaths for 1881 to 1890 inclusive.
6	Censuses of 1891 and 1901 and deaths for 1891 to 1900 inclusive.
7	Censuses of 1901 and 1911 and deaths for 1901 to 1910 inclusive.
8	Census of 1911 and deaths for 1910 to 1912 inclusive.
9	Census of 1921 and deaths for 1920 to 1922 inclusive.
10	Census of 1931 and deaths for 1930 to 1932 inclusive.

Prior to 1861 the *English Life Tables* were published in the Registrar-General's Annual Reports, but from 1861 onwards they have appeared in supplements to Annual Reports. From 1921 onwards these annual reports have been known as "Statistical Reviews" and the *English Life Table No. 9* was published in a volume described by the new title of "The Registrar-General's Decennial Supplement," a course which was followed in the case of the English Life Table No. 10.

Generally speaking the Tables show a continuous decline in the rate of mortality throughout except in the case of

old lives where the rates under the latest table showed a slight increase. These remarks apply to both sexes.

34. The most important of the older tables is the English Life No. 3, for which separate tables were published for persons, males and females. This table was formerly the standard for population statistics and was largely used for Industrial Assurance calculations. The English Life Table No. 10 has now largely superseded previous tables for Industrial Assurance and general population statistics, being the most up-to-date table yet published. Where, however, statistics are required for National purposes, reference is usually made to the Registrar-General's Annual Statistical Review.

35. In addition to the English Life Tables, three tables have been published under the title of *Healthy English Life Tables*. No. 1 was based on the census of 1851 and the deaths for 1849 to 1853 inclusive in sixty-three districts in England and Wales, in which the crude death-rate was not greater than $17\frac{1}{2}$ per 1,000 during the period 1841 to 1850. No. 2, was based on the censuses of 1881 and 1891, and the deaths for the intervening years, in 263 districts in which the corrected death-rate did not exceed 15 per 1,000. No. 3 was based on the censuses of 1891 and 1901 and corresponding deaths in 260 districts in which the corrected death-rate did not exceed 14 per 1,000. These tables were intended to serve as a standard for the mortality of the general population under healthy conditions.

36. On the occasion of the 1911 census the principle of selecting certain districts known to be healthy was abandoned. Instead, certain tables were constructed for London, Rural Districts, Urban Districts, and County Boroughs, and this procedure has been continued in the case of later Censuses.

37. The *Seventeen Offices Table* was the first table to be based on life assurance experience, the data being drawn from the first twenty-three years' experience of the life offices contributing. It was prepared by a Committee of Actuaries in 1838, and from 1872 to 1909 was the legal standard in this country for the valuation of policies in the case of a company being wound up. Under the Assurance Companies' Act, 1909, however, the standard is to be "according to such tables as the Court may direct." The table, though still used abroad, is practically obsolete in this country.

38. The *Institute of Actuaries Tables* were based on the experience of twenty life offices up to the 31st December, 1863. The tables constructed were—

H^M (Healthy males), H^F (Healthy females) ;

H^{MF} (Healthy males and females), D^{MF} (Diseased males and females).

The only table of importance is the H^M , which has been largely used for valuation purposes by British life offices.

39. In 1896 a set of monetary tables was published, based on the H^M data as reconstructed in the form of select tables by the late Dr. T. B. Sprague. The select table is commonly referred to as "Sprague's Select," and is symbolised as $H^{[M]}$.

40. The *British Offices Tables*, published in 1902, exhibit the experience of sixty British offices for the years 1863 to 1893.

41. The O^M (offices male) is an aggregate table based on male lives and is still used by some offices for valuation purposes. The reserves produced by this table slightly exceed those by the H^M table.

42. The $O^{M(5)}$ table is a truncated aggregate table based

on the same experience, but excluding the first five years of assurance. It is occasionally used for valuation purposes, either by itself or in conjunction with the O^M .

43. The $O^{[M]}$ table is a select table exhibiting the mortality at each age separately for each of the first ten years of assurance, the mortality then running into the ultimate table. It was largely used for the calculation of premiums, until superseded by the A 1924-29 Experience.

44. The $O^{[NM]}$ table is a select table based on the experience of male lives under non-participating assurances, and is used for certain premium calculations.

45. The $O^{[am]}$ and $O^{[af]}$ tables are based on the experience of male and female annuitants respectively, and were used for calculating the cost of annuities, for valuing annuities, and in many reversionary transactions until superseded by later tables.

46. The British Offices tables showed much lighter mortality than that of any previous experience, the rates for annuitants being specially noticeable.

47. The Institute of Actuaries has recently published the results of an investigation into the mortality of life office annuitants for the period 1900 to 1920, known as the $a(m)$ and $a(f)$ tables, from which it was evident that amongst this class mortality was much lighter during the period than under the older experience. This experience has largely superseded the out-of-date $O^{[am]}$ and $O^{[af]}$ tables. Figures have since been published which show that annuitants' mortality has continued to improve.

48. Five *Government Annuitants* experience tables have been constructed, the third one, constructed in 1883, being the best known. It was intended primarily to determine the rates of annuity to be granted by the Government, but was also used by life offices to value their annuity contracts

until it was superseded by the $O^{[a m]}$ and $O^{[a f]}$ tables. The fourth table, which included the experience of Government annuitants up to 1904, was published in 1912. The female experience proved even lighter than the $O^{[a f]}$, but the male experience was very similar to the $O^{[a m]}$.

The fifth table was published in 1924 and covered the same period as the latest Institute of Actuaries' table. The results were strikingly like those of the latter table, and the new Government tables have been adopted as the basis for Government annuities.

49. The latest experience of assured lives was published by the Institute of Actuaries in 1934 and is known as the A 1924-1929 Experience. The tables showed such marked improvement of mortality at all ages as to render the O^M experience quite obsolete. The experience of lives assured under endowment assurances was, as in the case of the O^M experience, lighter than in that of those assured under whole life policies. A striking feature was that there was little difference between lives assured with and without medical examination, the non-medical having lighter mortality for durations of assurance 5 and over but not so light at the early durations. Thus there was selection against the offices under non-medical business which had its effect on the experience of the early durations, but the exclusion of doubtful lives from acceptance without medical examination ultimately resulted in light mortality. The data were supplied by the contributing offices continuously throughout the period and it was thus possible to publish the tables whilst they still represented current experience. Monetary tables were published based on a table amalgamating the experience of all classes and offices and both medical and non-medical business. A further volume containing two tables was published in 1937.

CHAPTER IX

CONSTRUCTION OF TABLES BASED ON THE LIFE TABLE

1. THE principles laid down in Chapter IV hold good for the construction of tables generally.

2. We have seen how a table of l_x may be constructed when q_x has been previously obtained. If a table of $\log l_x$ be required each individual value must be obtained by entering the logarithm table and there is no practicable way of checking the results by summation. It will, therefore, be desirable, in such a case, for the work to be done in duplicate by different computers and the results called over. When checking a printer's proof, however, besides calling over the figures with the original copy, the totals of each column should be obtained and compared, as this is a valuable safeguard against an error being overlooked.

3. To obtain a table of $\log p_x$ we have

$$p_x = \frac{l_x + 1}{l_x}$$

$$\therefore \log p_x = \log l_{x+1} - \log l_x = \Delta \log l_x$$

By differencing a column of $\log l_x$, therefore, we obtain the column $\log p_x$.

4. It should be observed that in this instance the insertion of periodical values in advance will not help us, as the accuracy of one value does not depend on the accuracy of the previous one.

5. Remembering, however, that ${}_np_x = p_x \times p_{x+1} \times \dots \times p_{x+n-1}$, and that, therefore, $\log {}_np_x = \log p_x + \log p_{x+1} + \dots + \log p_{x+n-1}$, we may check the accuracy of all values of p_x from $x = x$ to $x = x + n - 1$ by comparing the total with $\log {}_np_x$, which is, of course, $\log l_{x+n} - \log l_x$.

6. Addition is always easier than subtraction, and it is sometimes found convenient to change the operation of subtraction to that of addition by the use of what is known as the "colog" of a quantity. The cologarithm of a quantity is the logarithm of the reciprocal of that quantity.

For instance, $\text{colog } l_x = \log \frac{1}{l_x} = -\log l_x$.

7. To construct a table of $\log q_x$, we have

$$\begin{aligned}\log q_x &= \log d_x - \log l_x \\ &= \log d_x + \text{colog } l_x\end{aligned}$$

If, therefore, we place side by side columns of $\log d_x$ and $\text{colog } l_x$ respectively, by adding the figures across we obtain a column of $\log q_x$. This would be arithmetically much easier than deducting the value of $\log l_x$ at each operation.

8. A column of $\text{colog } l_x$ is obtained by taking negatively a column of $\log l_x$. For example, $\log l_0 = 5.10477$, and $\text{colog } l_0$ is, therefore, $\bar{6}.89523$.

9. To construct a table of $\dot{e}_x$ we have

$$\begin{aligned}\dot{e}_x &= \frac{1}{2} + \frac{l_{x+1} + l_{x+2} + \dots}{l_x} \\ &= \frac{\frac{1}{2} l_x + l_{x+1} + l_{x+2} + \dots}{l_x} \\ &= \frac{\frac{1}{2} (l_x + l_{x+1}) + \frac{1}{2} (l_{x+1} + l_{x+2}) + \dots}{l_x} \\ &= \frac{L_x + L_{x+1} + \dots}{l_x} \\ &= \frac{T_x}{l_x}\end{aligned}$$

$$\therefore \log \dot{e}_x = \log T_x + \text{colog } l_x \quad . \quad . \quad . \quad (1)$$

It will be observed that $\text{colog } l_x$ is frequently useful and well worth the labour of calculating. It is convenient to tabulate it on a movable slip, so that it could be fastened in position as required.

10. In this instance, having tabulated $\log T_x$ and placed the column $\text{colog } l_x$ alongside, $\log e_x$ would be obtained by summing crossways. The antilogarithms of the results would be the required values of e_x .

11. As an example, let us construct a table of e_x by the use of four-figure logarithms for all values of x from 10 to 15 inclusive. The values of $\log T_x$ must be taken for granted as T_x is not tabulated in this book.

x (1)	$\log T_x$ (2)	$\text{colog } l_x$ (3)	(2) + (3) (4)	Antilog. (4) (5)
10	6.7012	$\bar{5}.0000$	1.7012	50.26
11	6.6925	$\bar{5}.0018$	1.6943	49.47
12	6.6835	$\bar{5}.0034$	1.6869	48.63
13	6.6745	$\bar{5}.0049$	1.6794	47.80
14	6.6654	$\bar{5}.0064$	1.6718	46.97
15	6.6561	$\bar{5}.0079$	1.6640	46.13

In such a case it would be usual in practice to insert the characteristics of the logarithms on the first line only, the position of the decimal place in the results being sufficiently obvious from the appearance of the series.

12. In the present volume the column T_x is not given, and it would therefore be easier to construct a table of e_x from the formula $e_x = \frac{N'_x}{l_x}$ and to add .5 to the result.

The method would be similar to that described in article 11.

13. If the column N'_x were not available, a table of e_x might be constructed as follows—

$$\begin{aligned}
 e_x &= \frac{l_{x+1} + l_{x+2} + \dots}{l_x} \\
 &= \frac{l_{x+1}}{l_x} \times \frac{l_{x+1} + l_{x+2} + \dots}{l_{x+1}} \\
 \therefore e_x &= p_x (1 + e_{x+1}) \quad \dots \quad (2)
 \end{aligned}$$

We can therefore obtain the expectation of life at any age from the expectation at the next higher age. Turning to the mortality table, we find that at age 101 $p_x = 0$, whence $p_{101}(1 + e_{102}) = e_{101} = 0$. We therefore commence our calculations with age 100 and working backwards obtain each value of e_x from the value at the next succeeding age. The following table illustrates the process—

x (1)	$\log p_x$ (2)	$\log (1 + e_{x+1})$ (3)	(2) + (3) $= \log e_x$ (4)	antilog (4) $= e_x$ (5)
100	1.3979	0.0000	1.3979	.2500
99	.6478	.0969	.7447	.5555
98	.6755	.1920	.8675	.7370
97	.7106	.2398	.9504	.8920
96	.7169	.2770	.9939	.9860
:	:	:	:	:
:	:	:	:	:

14. We have seen that $e_{101} = 0$, and therefore our initial value of $\log (1 + e_{101}) = 0$, whence we are able to complete the first line of the calculation. The next value in column (3) is then inserted, being $\log 1.250$, i.e. $\log (1 + e_{100})$, and so the work continues. After the first line the characteristics of the logarithms have been omitted as being unnecessary. The function $\log p_x$ is much in use when

calculating monetary tables and might usefully be written on a movable slip.

15. It will be observed that the accuracy of each value of e_x depends on the accuracy of all previous values. A useful check may, therefore, be obtained by inserting in advance every tenth (say) value of e_x , obtained by independent calculation. For this purpose the formula

$e_x = \frac{N'_x}{l_x}$ might be used, the values of N'_x being obtained

by summing the l_x column at the necessary intervals.

16. Passing now to functions involving the rate of interest we observe that, at each age, $D_{x+1} = vp_x D_x$, and therefore

$$\log D_{x+1} = \log vp_x + \log D_x \quad . \quad . \quad . \quad (3)$$

Having calculated the initial value of $\log D_x$ for the age x at which it is desired to start the table, this value is inserted in the column D_x . Parallel with this column the column $\log vp_x$ should be placed and the value for the initial age added to the initial value of $\log D_x$, thus producing $\log D_{x+1}$, and so on.

17. In this and similar calculations it should be observed that an accumulative error occurs. The work is limited to a certain number of decimal places, and the last figure is therefore approximate only. In the case of $\log e_x$ it will be sometimes larger and sometimes smaller than the true value (since it is taken to the nearest figure), and in a long table the errors will counteract one another and be unimportant. In the above calculation, however, we have, in preparing a table of $\log vp_x$ in advance, added the constant $\log v$, which is either slightly smaller or slightly larger than the true value. For instance, in working to four places of decimals, at 4% interest, the value of $\log v$ to the nearest fifth place of decimals would be $\bar{1}.98297$,

19. When commutation columns are available, we have the simple formula—

$$\log a_x = \log N_x + \text{colog } D_x \quad . \quad . \quad . \quad (5)$$

by means of which the table of a_x could be constructed. The work should be done in duplicate by different computers and compared.

20. Tables of A_x and P_x can similarly be constructed by means of the formulae

$$\log A_x = \log M_x + \text{colog } D_x \quad . \quad . \quad . \quad (6)$$

$$\log P_x = \log M_x + \text{colog } N_{x-1} \quad . \quad . \quad . \quad (7)$$

21. As an example of formula (7), let us construct a table of P_x for all values of x from 55 to 60. If we take the logarithms of M and N , the natural numbers of which are given in Table VII, by reference to Table I, we shall only be able to work to four figures, so that we can only assume our results to be accurate to two significant figures.

Age x (1)	$\log$ N_{x-1} (2)	colog N_{x-1} (3)	$\log$ M_x (4)	(3) + (4) (5)	$P_x =$ antilog (5) (6)
55	4.9665	5.0335	3.6168	2.6503	.04470
56	.9288	.0712	.6000	.6712	.04690
57	.8901	.1099	.5824	.6923	.04923
58	.8501	.1499	.5641	.7140	.05176
59	.8091	.1909	.5448	.7357	.05441
60	.7667	.2333	.5245	.7578	.05725

22. In column (2) above we have set out the values of $\log N_{x-1}$, and by taking the values negatively have produced column (3). This is easily done by deducting each digit in the decimal portion of column (2) from 9, except the last digit, which is deducted from 10. This simple method of subtracting a decimal quantity from unity is frequently useful.

23. The values of P_x obtained above actually only differ from the true values by as much as 2 in the fifth place of decimals in one case, the errors in the remaining cases being 1 or 0. In a simple calculation like the above we have, therefore, obtained very accurate results by the use of the four-figure logarithm table, but it will be observed that the errors in the fifth place of decimals might easily produce an error of 1 in the fourth place when extracting the results to the nearest fourth place of decimals.

24. To construct a table of deferred annuities, we have the formula

$$\log {}_n|a_x = \log N_{x+n} + \text{colog } D_x . . . (8)$$

For instance, if it be desired to calculate the present values of annuities at all ages from 20 to 54 inclusive of immediate annuities deferred after age 55, the column $\text{colog } D_x$ for $x = 20$ to $x = 54$ inclusive would be set down. The value of $\log N_{55}$ would then be written at the foot of a card and placed over the function $\text{colog } D_{20}$. The two values could then be added mentally and the sum placed in the adjoining column. The card would then be shifted down one line and the process repeated, and so on.

25. To construct a table of temporary life annuities a simple method, when the deferred annuities have also to be calculated, would be to use the formula

$${}_n|a_x = a_x - {}_n|a_x . . . (9)$$

26. By means of the principles laid down in this chapter it is possible to construct any tables which may be required, and the student should construct tables for himself in order to become thoroughly familiar with the work. Many other more complicated methods have been used by which labour has been saved and satisfactory checks provided, but for present purposes these may be safely disregarded.

TABLE I
TABLE OF LOGARITHMS

Nat. Nos.	0	1	2	3	4	5	6	7	8	9	Diff.
10	·0000	·0043	·0086	·0128	·0170	·0212	·0253	·0294	·0334	·0374	41
1	·0414	·0453	·0492	·0531	·0569	·0607	·0645	·0682	·0719	·0755	39
2	·0792	·0828	·0864	·0899	·0934	·0969	·1004	·1038	·1072	·1106	35
3	·1139	·1173	·1206	·1239	·1271	·1303	·1335	·1367	·1399	·1430	33
4	·1461	·1492	·1523	·1553	·1584	·1614	·1644	·1673	·1703	·1732	30
15	·1761	·1790	·1818	·1847	·1875	·1903	·1931	·1959	·1987	·2014	28
6	·2041	·2068	·2095	·2122	·2148	·2175	·2201	·2227	·2253	·2279	26
7	·2304	·2330	·2355	·2380	·2405	·2430	·2455	·2480	·2504	·2529	25
8	·2553	·2577	·2601	·2625	·2648	·2672	·2695	·2718	·2742	·2765	23
9	·2788	·2810	·2833	·2856	·2878	·2900	·2923	·2945	·2967	·2989	22
20	·3010	·3032	·3054	·3075	·3096	·3118	·3139	·3160	·3181	·3201	21
1	·3222	·3243	·3263	·3284	·3304	·3324	·3345	·3365	·3385	·3404	20
2	·3424	·3444	·3464	·3483	·3502	·3522	·3541	·3560	·3579	·3598	19
3	·3617	·3636	·3655	·3674	·3692	·3711	·3729	·3747	·3766	·3784	18
4	·3802	·3820	·3838	·3856	·3874	·3892	·3909	·3927	·3945	·3962	18
25	·3979	·3997	·4014	·4031	·4048	·4065	·4082	·4099	·4116	·4133	17
6	·4150	·4168	·4183	·4200	·4218	·4232	·4249	·4265	·4281	·4298	16
7	·4314	·4330	·4346	·4362	·4378	·4393	·4409	·4425	·4440	·4456	16
8	·4472	·4487	·4502	·4518	·4533	·4548	·4564	·4579	·4594	·4609	15
9	·4624	·4639	·4654	·4669	·4683	·4698	·4713	·4728	·4742	·4757	15
30	·4771	·4786	·4800	·4814	·4829	·4843	·4857	·4871	·4886	·4900	14
1	·4914	·4928	·4942	·4955	·4969	·4983	·4997	·5011	·5024	·5038	14
2	·5051	·5065	·5079	·5092	·5105	·5119	·5132	·5145	·5159	·5172	13
3	·5185	·5198	·5211	·5224	·5237	·5250	·5263	·5276	·5289	·5302	13
4	·5315	·5328	·5340	·5353	·5366	·5378	·5391	·5403	·5416	·5428	13
35	·5441	·5453	·5465	·5478	·5490	·5502	·5514	·5527	·5539	·5551	12
6	·5563	·5575	·5587	·5599	·5611	·5623	·5635	·5647	·5658	·5670	12
7	·5682	·5694	·5705	·5717	·5729	·5740	·5752	·5763	·5775	·5786	12
8	·5798	·5809	·5821	·5832	·5843	·5855	·5866	·5877	·5888	·5899	11
9	·5911	·5922	·5933	·5944	·5955	·5966	·5977	·5988	·5999	·6010	11
40	·6021	·6031	·6042	·6053	·6064	·6075	·6085	·6096	·6107	·6117	11
1	·6128	·6138	·6149	·6160	·6170	·6180	·6191	·6201	·6212	·6222	10
2	·6232	·6243	·6253	·6263	·6274	·6284	·6294	·6304	·6314	·6325	10
3	·6335	·6345	·6355	·6365	·6375	·6385	·6395	·6405	·6415	·6425	10
4	·6435	·6444	·6454	·6464	·6474	·6484	·6493	·6503	·6513	·6522	10
45	·6532	·6542	·6551	·6561	·6571	·6580	·6590	·6599	·6609	·6618	10
6	·6628	·6637	·6646	·6656	·6665	·6675	·6684	·6693	·6702	·6712	9
7	·6721	·6730	·6739	·6749	·6758	·6767	·6776	·6785	·6794	·6803	9
8	·6812	·6821	·6830	·6839	·6848	·6857	·6866	·6875	·6884	·6893	9
9	·6902	·6911	·6920	·6928	·6937	·6946	·6955	·6964	·6972	·6981	9
50	·6990	·6998	·7007	·7016	·7024	·7033	·7042	·7050	·7059	·7067	9
1	·7076	·7084	·7093	·7101	·7110	·7118	·7126	·7135	·7143	·7152	8
2	·7160	·7168	·7177	·7185	·7193	·7202	·7210	·7218	·7226	·7235	8
3	·7243	·7251	·7259	·7267	·7275	·7284	·7292	·7300	·7308	·7316	8
4	·7324	·7332	·7340	·7348	·7356	·7364	·7372	·7380	·7388	·7396	8
Nat. Nos.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE I (contd.)
TABLE OF LOGARITHMS (contd.)

Nat. Nos.	0	1	2	3	4	5	6	7	8	9	Diff.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8687	6
4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4
Nat. Nos.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE II
ANTILOGARITHMS

Logs.	0	1	2	3	4	5	6	7	8	9	Diff.
-00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	2
-01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	2
-02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	2
-03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	2
-04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	2
-05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	3
-06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	3
-07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	3
-08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	3
-09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	3
-10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	3
-11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	3
-12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	3
-13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	3
-14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	3
-15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	3
-16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	3
-17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	3
-18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	4
-19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	4
-20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	4
-21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	4
-22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	4
-23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	4
-24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	4
-25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	4
-26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	4
-27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	4
-28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	4
-29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	5
-30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	5
-31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	5
-32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	5
-33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	5
-34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	5
-35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	5
-36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	5
-37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	5
-38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	6
-39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	6
-40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	6
-41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	6
-42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	6
-43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	6
-44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	6
-45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	7
-46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	7
-47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	7
-48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	7
-49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	7
Logs.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE II (contd.)
ANTILOGARITHMS (contd.)

Logs.	0	1	2	3	4	5	6	7	8	9	Diff.
50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	7
51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	8
52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	8
53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	8
54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	8
55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	8
56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	8
57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	9
58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	9
59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	9
60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	9
61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	10
63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	10
64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	10
65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	10
66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	11
67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	11
68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	11
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	11
70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	12
71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	12
72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	12
73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	13
74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	13
75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	13
76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	13
77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	14
78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	14
79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	14
80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	15
81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	15
82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	15
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	16
84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	16
85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	16
86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	17
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	17
88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	18
89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	18
90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	19
91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	19
92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	19
93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	20
94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	20
95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	21
96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	21
97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	22
98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	22
99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	23
Logs.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE III

n	(1+i) ⁿ			v ⁿ			n
	3%	4%	5%	3%	4%	5%	
1	1.03000	1.04000	1.05000	.97087	.96154	.95238	1
2	1.06090	1.08160	1.10250	.94280	.92456	.90703	2
3	1.09273	1.12486	1.15763	.91514	.88900	.86384	3
4	1.12551	1.16986	1.21551	.88849	.85480	.82270	4
5	1.15927	1.21665	1.27628	.86261	.82193	.78353	5
6	1.19405	1.26532	1.34010	.83748	.79031	.74622	6
7	1.22967	1.31563	1.40710	.81309	.75992	.71068	7
8	1.26677	1.36857	1.47746	.78941	.73069	.67684	8
9	1.30477	1.42331	1.55133	.76642	.70259	.64461	9
10	1.34392	1.48024	1.62889	.74409	.67556	.61391	10
11	1.38423	1.53945	1.71034	.72242	.64958	.58468	11
12	1.42576	1.60108	1.79586	.70138	.62460	.55684	12
13	1.46853	1.66507	1.88565	.68095	.60057	.53032	13
14	1.51259	1.73168	1.97993	.66112	.57748	.50507	14
15	1.55797	1.80094	2.07893	.64186	.55526	.48192	15
16	1.60471	1.87298	2.18287	.62317	.53391	.45811	16
17	1.65285	1.94790	2.29202	.60502	.51337	.43630	17
18	1.70243	2.02582	2.40662	.58739	.49363	.41552	18
19	1.75351	2.10685	2.52695	.57029	.47464	.39573	19
20	1.80611	2.19112	2.65330	.55368	.45639	.37689	20
21	1.86029	2.27877	2.78596	.53755	.43883	.35894	21
22	1.91610	2.36992	2.92528	.52189	.42196	.34185	22
23	1.97359	2.46472	3.07152	.50669	.40573	.32557	23
24	2.03279	2.56330	3.22510	.49193	.39012	.31007	24
25	2.09378	2.66584	3.38635	.47761	.37512	.29530	25
26	2.15659	2.77247	3.55567	.46369	.36069	.28124	26
27	2.22129	2.88337	3.73346	.45019	.34682	.26785	27
28	2.28793	2.99870	3.92013	.43708	.33348	.25509	28
29	2.35657	3.11865	4.11614	.42435	.32065	.24295	29
30	2.42726	3.24340	4.32194	.41199	.30832	.23138	30
31	2.50008	3.37313	4.53804	.39999	.29646	.22036	31
32	2.57508	3.50806	4.76494	.38834	.28506	.20967	32
33	2.65234	3.64838	5.00319	.37703	.27409	.19967	33
34	2.73191	3.79432	5.25335	.36604	.26355	.19035	34
35	2.81386	3.94609	5.51602	.35538	.25342	.18129	35
36	2.89828	4.10393	5.79182	.34503	.24367	.17266	36
37	2.98523	4.26809	6.08141	.33498	.23430	.16444	37
38	3.07478	4.43881	6.38548	.32523	.22529	.15661	38
39	3.16703	4.61637	6.70475	.31575	.21662	.14915	39
40	3.26204	4.80102	7.03999	.30656	.20829	.14205	40
41	3.35990	4.99306	7.39199	.29763	.20028	.13528	41
42	3.46070	5.19278	7.76159	.28896	.19257	.12884	42
43	3.56452	5.40050	8.14967	.28054	.18517	.12270	43
44	3.67143	5.61652	8.55715	.27237	.17805	.11686	44
45	3.78160	5.84118	8.98501	.26444	.17120	.11130	45
46	3.89504	6.07482	9.43426	.25674	.16461	.10600	46
47	4.01190	6.31782	9.90597	.24928	.15828	.10095	47
48	4.13225	6.57055	10.40127	.24200	.15219	.09614	48
49	4.25622	6.83335	10.92133	.23495	.14634	.09156	49
50	4.38391	7.10668	11.46740	.22811	.14071	.08720	50

TABLE IV

n	s n			a n			n
	3%	4%	5%	3%	4%	5%	
1	1.0000	1.0000	1.0000	0.9709	.9615	.9524	1
2	2.0300	2.0400	2.0500	1.9135	1.8881	1.8594	2
3	3.0909	3.1216	3.1525	2.8286	2.7751	2.7232	3
4	4.1836	4.2465	4.3101	3.7171	3.6299	3.5460	4
5	5.3091	5.4163	5.5256	4.5797	4.4518	4.3295	5
6	6.4684	6.6330	6.8019	5.4172	5.2421	5.0757	6
7	7.6625	7.8983	8.1420	6.2303	6.0021	5.7864	7
8	8.8923	9.2142	9.5491	7.0197	6.7327	6.4632	8
9	10.1591	10.5828	11.0266	7.7861	7.4353	7.1078	9
10	11.4639	12.0061	12.5779	8.5302	8.1109	7.7217	10
11	12.8078	13.4884	14.2068	9.2526	8.7605	8.3064	11
12	14.1920	15.0258	15.9171	9.9540	9.3851	8.8633	12
13	15.6178	16.6268	17.7130	10.6350	9.9856	9.3936	13
14	17.0863	18.2919	19.5986	11.2961	10.5631	9.8986	14
15	18.5989	20.0236	21.5786	11.9379	11.1184	10.3797	15
16	20.1569	21.8245	23.6575	12.5611	11.6523	10.8378	16
17	21.7616	23.6975	25.8404	13.1661	12.1657	11.2741	17
18	23.4144	25.6454	28.1324	13.7535	12.6593	11.6896	18
19	25.1169	27.6712	30.5390	14.3238	13.1339	12.0853	19
20	26.8704	29.7781	33.0660	14.8775	13.5903	12.4622	20
21	28.6765	31.9692	35.7193	15.4150	14.0292	12.8212	21
22	30.5368	34.2480	38.5052	15.9369	14.4511	13.1630	22
23	32.4529	36.6179	41.4305	16.4436	14.8568	13.4866	23
24	34.4285	39.0826	44.5020	16.9355	15.2470	13.7986	24
25	36.4593	41.6459	47.7271	17.4131	15.6221	14.0999	25
26	38.5530	44.3117	51.1135	17.8768	15.9828	14.3752	26
27	40.7096	47.0842	54.6691	18.3270	16.3296	14.6430	27
28	42.9309	49.9676	58.4026	18.7641	16.6631	14.8981	28
29	45.2189	52.9663	62.3227	19.1885	16.9837	15.1411	29
30	47.5754	56.0849	66.4388	19.6004	17.2920	15.3725	30
31	50.0027	59.3283	70.7608	20.0004	17.5885	15.5928	31
32	52.5028	62.7015	75.2988	20.3888	17.8736	15.8027	32
33	55.0778	66.2095	80.0638	20.7658	18.1476	16.0025	33
34	57.7302	69.8579	85.0670	21.1318	18.4112	16.1929	34
35	60.4621	73.6522	90.3203	21.4872	18.6646	16.3742	35
36	63.2759	77.5983	95.8363	21.8323	18.9083	16.5469	36
37	66.1742	81.7022	101.6281	22.1672	19.1426	16.7113	37
38	69.1594	85.9703	107.7095	22.4925	19.3679	16.8679	38
39	72.2342	90.4091	114.0950	22.8082	19.5845	17.0170	39
40	75.4013	95.0255	120.7998	23.1148	19.7928	17.1591	40
41	78.6633	99.8265	127.8398	23.4124	19.9931	17.2944	41
42	82.0232	104.8196	135.2318	23.7014	20.1856	17.4232	42
43	85.4839	110.0124	142.9933	23.9819	20.3708	17.5459	43
44	89.0484	115.4129	151.1430	24.2543	20.5488	17.6628	44
45	92.7199	121.0294	159.7002	24.5187	20.7200	17.7741	45
46	96.5015	126.8706	168.6852	24.7754	20.8847	17.8801	46
47	100.3965	132.9454	178.1194	25.0247	21.0429	17.9810	47
48	104.4084	139.2632	188.0254	25.2667	21.1951	18.0772	48
49	108.5406	145.8337	198.4267	25.5017	21.3415	18.1687	49
50	112.7969	152.6671	209.3490	25.7298	21.4822	18.2559	50

TABLE V
THE LIFE TABLE

x	l_x	d_x	q_x	x	l_x	d_x	q_x
0	127283	14358	·11280	52	70458	1243	·01764
1	112925	3962	·03508	53	69215	1296	·01873
2	108993	2375	·02179	54	67919	1353	·01992
3	106588	1846	·01544				
4	104942	1325	·01263	55	66506	1414	·02123
				56	65152	1475	·02265
5	103617	1061	·01024	57	63677	1541	·02420
6	102556	852	·00830	58	62136	1612	·02593
7	101704	683	·00672	59	60524	1682	·02779
8	101021	557	·00551				
9	100464	464	·00462	60	58842	1755	·02983
				61	57087	1830	·03206
10	100000	408	·00409	62	55257	1906	·03451
11	99592	369	·00370	63	53351	1983	·03717
12	99223	346	·00347	64	51368	2059	·04007
13	98877	337	·00342				
14	98540	337	·00342	65	49309	2133	·04327
				66	47176	2204	·04672
15	98203	360	·00365	67	44972	2273	·05053
16	97843	384	·00393	68	42699	2334	·05466
17	97459	425	·00437	69	40365	2388	·05917
18	97034	465	·00478				
19	96599	508	·00526	70	37977	2434	·06410
				71	35543	2468	·06943
20	96061	548	·00572	72	33075	2490	·07528
21	95513	582	·00608	73	30585	2496	·08160
22	94931	609	·00643	74	28089	2487	·08856
23	94322	631	·00668				
24	93691	647	·00691	75	25602	2459	·09604
				76	23143	2412	·10422
25	93044	658	·00707	77	20731	2343	·11303
26	92386	664	·00720	78	18388	2255	·12262
27	91722	673	·00732	79	16133	2146	·13304
28	91049	678	·00746				
29	90371	686	·00759	80	13987	2018	·14426
				81	11969	1873	·15649
30	89685	691	·00771	82	10096	1712	·16958
31	88994	700	·00787	83	8384	1540	·18368
32	88294	709	·00803	84	6844	1361	·19886
33	87585	719	·00821				
34	86866	729	·00839	85	5483	1180	·21522
				86	4303	1002	·23285
35	86137	742	·00862	87	3301	830	·25145
36	85395	756	·00885	88	2471	671	·27155
37	84639	770	·00910	89	1800	527	·29277
38	83869	786	·00937				
39	83083	806	·00969	90	1273	402	·31579
				91	871	296	·33984
40	82277	823	·01001	92	575	209	·36348
41	81454	846	·01038	93	366	144	·39345
42	80608	871	·01081	94	222	93	·41891
43	79737	895	·01122				
44	78842	924	·01172	95	129	58	·44961
				96	71	34	·47888
45	77918	954	·01224	97	37	18	·48649
46	76964	986	·01281	98	19	10	·52631
47	75978	1021	·01345	99	9	5	·55555
48	74957	1061	·01415				
49	73896	1101	·01490	100	4	3	·75000
				101	1	1	1·00000
50	72795	1144	·01572	102	0	—	—
51	71651	1193	·01665				

TABLE VI
THE LIFE TABLE (contd.)

x	l_x	L_x	N'_x	e_x	x
0	-15920	120104	6018388	47-784	0
1	-07901	110944	5905463	52-796	1
2	-02366	107776	5796500	53-687	2
3	-01787	105785	5689912	53-881	3
4	-01379	104279	5584970	53-719	4
5	-01142	103087	5481353	53-401	5
6	-00925	102130	5378797	52-948	6
7	-00748	101362	5277093	52-387	7
8	-00607	100743	5176072	51-738	8
9	-00502	100232	5075608	51-022	9
10	-00428	99796	4975608	50-257	10
11	-00388	99408	4876016	49-460	11
12	-00359	99050	4776793	48-643	12
13	-00342	98708	4677916	47-810	13
14	-00340	98372	4579376	46-973	14
15	-00353	98023	4481173	46-132	15
16	-00378	97651	4383330	45-299	16
17	-00414	97246	4285871	44-476	17
18	-00458	96802	4188837	43-669	18
19	-00504	96315	4092268	42-877	19
20	-00550	95787	3996207	42-101	20
21	-00592	95222	3900694	41-339	21
22	-00629	94626	3805763	40-590	22
23	-00659	94007	3711441	39-849	23
24	-00682	93367	3617750	39-114	24
25	-00701	92715	3524706	38-382	25
26	-00716	92054	3432320	37-652	26
27	-00729	91386	3340598	36-921	27
28	-00742	90710	3249549	36-189	28
29	-00755	90028	3159178	35-458	29
30	-00768	89339	3069493	34-726	30
31	-00782	88644	2980499	33-991	31
32	-00798	87940	2892205	33-257	32
33	-00815	87225	2804620	32-521	33
34	-00833	86502	2717754	31-787	34
35	-00854	85766	2631617	31-051	35
36	-00876	85017	2546222	30-317	36
37	-00901	84254	2461583	29-584	37
38	-00928	83476	2377714	28-850	38
39	-00957	82680	2294631	28-113	39
40	-00990	81865	2212354	27-389	40
41	-01025	81031	2130900	26-661	41
42	-01064	80173	2050292	25-935	42
43	-01106	79289	1970555	25-214	43
44	-01153	78380	1891713	24-493	44
45	-01204	77441	1813795	23-778	45
46	-01260	76471	1736831	23-066	46
47	-01321	75468	1660853	22-360	47
48	-01388	74426	1585896	21-658	48
49	-01462	73346	1512000	20-961	49
50	-01542	72223	1493205	20-271	50
51	-01631	71054	1367554	19-587	51

TABLE VI (contd.)
THE LIFE TABLE (contd.)

x	μ_x	L_x	N'_x	${}_xq_x$	x
53	·01727	69637	1297096	18·909	53
53	·01833	68567	1227881	18·240	53
54	·01950	67242	1159962	17·579	54
55	·02077	65859	1093396	16·926	55
55	·02216	64415	1028244	16·282	55
57	·02369	62906	964567	15·648	57
58	·02536	61330	902431	15·023	58
59	·02719	59683	841907	14·410	59
60	·02920	57965	783065	13·808	60
61	·03140	56172	725978	13·217	61
62	·03381	54304	670721	12·638	62
63	·03645	52359	617370	12·072	63
64	·03934	50339	566002	11·519	64
65	·04251	48242	516693	10·979	65
66	·04599	46074	469517	10·452	66
67	·04979	43836	424545	9·940	67
68	·05396	41532	381848	9·443	68
69	·05853	39171	341481	8·960	69
70	·06353	36760	303504	8·492	70
71	·06901	34309	267961	8·039	71
72	·07502	31830	234886	7·602	72
73	·08160	29337	204301	7·180	73
74	·08881	26845	176212	6·773	74
75	·09671	24373	150610	6·383	75
76	·10536	21937	127467	6·008	76
77	·11485	19559	106736	5·649	77
78	·12523	17261	88348	5·305	78
79	·13662	15060	72215	4·976	79
80	·14909	12978	58228	4·663	80
81	·16275	11033	46259	4·365	81
82	·17772	9240	36183	4·082	82
83	·19412	7614	27779	3·813	83
84	·21209	6164	20935	3·559	84
85	·23177	4893	15452	3·318	85
86	·25334	3902	11149	3·091	86
87	·27697	2986	7848	2·877	87
88	·30286	2135	5377	2·676	88
89	·33123	1537	3577	2·487	89
90	·36230	1072	2304	2·310	90
91	·39635	723	1433	2·145	91
92	·43366	470	858	1·992	92
93	·47453	294	492	1·844	93
94	·51930	176	270	1·716	94
95	·56836	100	141	1·593	95
96	·62211	54	70	1·486	96
97	·68100	28	33	1·392	97
98	·74552	14	14	1·237	98
99	·81621	6	5	1·056	99
100	·89366	3	1	·750	100
101	·97851	0	—	·500	101
102	∞	—	—	·000	102

TABLE VII
COMMUTATION TABLE—4 PER CENT

<i>s</i>	<i>D_s</i>	<i>N_s</i>	<i>S_s</i>	<i>C_s</i>	<i>M_s</i>	<i>R_s</i>
0	127280	2204737	42280578	13808-0	37590-0	618157
1	108580	2086157	40075841	3863-0	23784-0	578587
2	100740	1995417	37979684	2111-3	20121-0	554783
3	94787	1900680	35984287	1407-0	18008-7	534682
4	89706	1810954	34083907	1089-1	16902-7	516653
5	85165	1725789	32272653	838-53	15513-57	500050
6	81051	1644738	30546864	647-46	14675-04	484586
7	77288	1587450	28902128	499-06	14027-58	469861
8	73814	1493636	27384676	391-35	13528-52	455634
9	70585	1423051	25841040	313-47	13137-17	442305
10	67557	1355494	24417989	265-03	12823-70	429168
11	64692	1290802	23062495	230-48	12558-67	416344
12	61974	1228828	21771693	207-80	12328-19	403786
13	59384	1169444	20542865	194-61	12120-39	391457
14	56904	1112540	19373421	187-12	11925-78	379337
15	54528	1059012	18260881	192-21	11738-66	367411
16	52240	1005772	17202899	197-13	11546-45	355673
17	50032	955740	16197097	209-79	11349-32	344126
18	47898	907842	15241357	220-71	11139-53	332777
19	45836	862006	14333515	231-84	10918-82	321637
20	43841	818165	13471509	240-48	10686-98	310719
21	41914	776251	12653344	245-58	10446-50	300032
22	40057	736194	11877093	247-09	10200-92	289585
23	38268	697926	11140699	246-17	9953-83	279384
24	36551	661375	10442973	242-70	9707-66	269430
25	34903	626472	9781598	237-33	9464-66	259723
26	33323	593149	9155126	230-29	9227-63	250258
27	31810	561339	8561977	224-43	8997-34	241030
28	30363	530976	8000638	217-40	8772-91	232033
29	28977	501999	7469662	211-50	8555-51	223260
30	27652	474347	6967663	204-86	8344-01	214704
31	26383	447964	6493316	199-54	8139-15	206360
32	25169	422795	6045352	194-33	7939-61	198221
33	24007	398788	5622557	189-50	7745-28	190281
34	22894	375894	5223769	184-74	7555-78	182536
35	21828	354066	4847875	180-80	7371-04	174980
36	20808	333258	4493809	177-13	7190-24	167609
37	19831	313427	4160551	173-47	7013-11	160419
38	18894	294533	3847124	170-28	6839-64	153406
39	17997	276536	3552591	167-88	6669-38	146666
40	17138	259398	3276055	164-83	6501-50	139967
41	16313	243085	3016657	162-92	6336-67	133396
42	15523	227562	2773572	161-28	6173-75	127059
43	14765	212797	2546010	159-35	6012-47	120885
44	14038	198759	2333213	158-19	5853-12	114873
45	13340	185419	2134454	157-04	5694-93	109020
46	12670	172749	1949035	156-07	5537-89	103325
47	12026	160723	1776286	155-39	5381-62	97787
48	11408	149315	1615563	155-27	5226-43	92405
49	10814	138501	1466248	154-92	5071-16	87178
50	10243	128258	1327747	154-79	4916-24	82107

NOTE.—*S_s* and *R_s* are shown to nearest unit only.

TABLE VII (contd.)
COMMUTATION TABLE—4 PER CENT (contd.)

x	D_x	N_x	S_x	C_x	M_x	R_x
51	9694.4	118563.6	1199489	155.21	4761.45	77191
52	9166.4	109397.2	1080926	155.49	4606.24	72430
53	8658.2	100739.0	971529	155.89	4450.75	67823
54	8169.4	92569.6	870790	156.48	4294.86	63373
55	7698.7	84870.9	778220	157.25	4138.38	59078
56	7245.4	77625.5	693349	157.72	3981.13	54939
57	6808.9	70816.6	615724	158.44	3823.41	50958
58	6388.7	64427.9	544907	159.37	3664.97	47135
59	5983.6	58444.3	480479	159.89	3505.60	43470
60	5593.6	52850.7	422035	160.42	3345.71	39964
61	5218.1	47632.6	369184	160.83	3185.29	36619
62	4856.5	42776.1	321551	161.07	3024.46	33433
63	4508.6	38267.5	278775	161.14	2863.39	30409
64	4174.1	34093.4	240508	160.88	2702.25	27545
65	3852.7	30240.7	206414	160.25	2541.37	24843
66	3544.2	26696.5	176174	159.21	2381.12	22302
67	3248.7	23447.8	149477	157.88	2221.91	19921
68	2965.9	20481.9	126029	155.88	2064.03	17699
69	2695.9	17786.0	105548	153.36	1908.15	15635
70	2438.9	15347.1	87762	150.30	1754.79	13727
71	2194.7	13152.4	72414	146.54	1604.49	11972
72	1963.8	11188.6	59262	142.16	1457.95	10367
73	1746.1	9442.5	48073	137.02	1315.79	8909
74	1541.9	7900.6	38631	131.27	1178.77	7594
75	1351.4	6549.2	30730	124.80	1047.50	6415
76	1174.6	5374.8	24181	117.71	922.70	5367
77	1011.7	4362.91	18806	109.94	804.99	4445
78	862.84	3500.07	14444	101.75	695.05	3640
79	727.93	2772.14	10943	93.102	593.299	2945
80	606.81	2165.33	8171	84.182	500.197	2351
81	499.30	1666.03	6006	75.129	416.015	1851
82	404.97	1261.06	4340	66.028	340.686	1435
83	323.36	937.70	3079	57.111	274.858	1094
84	253.81	683.89	2141	48.532	217.747	819
85	195.52	488.37	1457	40.459	169.215	602
86	147.54	340.83	969	33.035	128.756	432
87	108.83	232.001	623	26.312	95.721	304
88	78.332	153.669	396	20.453	69.409	208
89	54.866	98.803	242	15.446	48.956	138
90	37.310	61.493	144	11.329	33.510	89
91	24.547	36.946	82	8.0208	22.1811	56
92	15.581	21.365	45	5.4457	14.1603	34
93	9.536	11.829	24	3.6077	8.7146	20
94	5.562	6.287	12	2.2403	5.1069	11
95	3.108	3.159	6	1.3435	2.8666	6
96	1.645	1.514	3	.7573	1.5231	3
97	.824	.690	1	.3855	.7658	1
98	.407	.283	0	.2059	.3803	1
99	.185	.098	0	.0990	.1744	0
100	.079	.019	0	.0571	.0754	0
101	.019	—	—	.0183	.0183	0

NOTE.— S_x and R_x are shown to nearest unit only.

TABLE VIII
VALUE OF ANNUITIES AND SINGLE AND ANNUAL PREMIUMS—4 PER CENT

x	a_x	A_x	P_x	x	a_x	A_x	P_x
0	17-321	-29532	-01612	50	12-522	-47995	-03550
1	19-305	-21904	-01079	51	12-230	-48116	-03712
2	19-807	-19973	-00960	52	11-935	-50250	-03885
3	20-059	-19006	-00903	53	11-635	-51408	-04068
4	20-188	-18508	-00874	54	11-331	-52573	-04263
5	20-264	-18216	-00857	55	11-024	-53754	-04471
6	20-292	-18106	-00850	56	10-714	-54947	-04691
7	20-282	-18151	-00853	57	10-401	-56153	-04925
8	20-234	-18329	-00863	58	10-085	-57367	-05175
9	20-162	-18612	-00880	59	9-767	-58587	-05441
10	20-065	-18982	-00901	60	9-448	-59814	-05725
11	19-953	-19414	-00927	61	9-129	-61044	-06027
12	19-828	-19892	-00955	62	8-808	-62277	-06350
13	19-692	-20409	-00986	63	8-488	-63510	-06694
14	19-551	-20958	-01020	64	8-168	-64740	-07061
15	19-403	-21528	-01055	65	7-850	-65964	-07454
16	19-254	-22102	-01091	66	7-533	-67183	-07874
17	19-102	-22684	-01128	67	7-218	-68393	-08323
18	18-954	-23258	-01168	68	6-906	-69592	-08802
19	18-806	-23822	-01203	69	6-597	-70780	-09318
20	18-662	-24377	-01240	70	6-293	-71952	-09866
21	18-520	-24925	-01277	71	5-993	-73107	-10455
22	18-379	-25468	-01314	72	5-698	-74244	-11086
23	18-238	-26011	-01352	73	5-408	-75355	-11760
24	18-095	-26560	-01391	74	5-124	-76449	-12484
25	17-949	-27118	-01431	75	4-846	-77514	-13258
26	17-800	-27692	-01473	76	4-576	-78554	-14089
27	17-647	-28284	-01517	77	4-313	-79568	-14978
28	17-488	-28893	-01563	78	4-057	-80555	-15931
29	17-324	-29525	-01611	79	3-808	-81504	-16951
30	17-155	-30175	-01662	80	3-568	-82431	-18044
31	16-979	-30850	-01716	81	3-337	-83320	-19213
32	16-799	-31546	-01772	82	3-114	-84176	-20461
33	16-612	-32263	-01832	83	2-900	-85002	-21795
34	16-419	-33004	-01895	84	2-695	-85793	-23222
35	16-221	-33768	-01961	85	2-498	-86549	-24743
36	16-016	-34555	-02031	86	2-310	-87273	-26365
37	15-805	-35365	-02104	87	2-132	-87955	-28085
38	15-588	-36199	-02182	88	1-962	-88609	-29918
39	15-366	-37058	-02264	89	1-801	-89230	-31858
40	15-136	-37937	-02351	90	1-648	-89813	-33916
41	14-902	-38844	-02443	91	1-505	-90363	-36070
42	14-660	-39771	-02540	92	1-371	-90878	-38326
43	14-413	-40722	-02642	93	1-240	-91384	-40790
44	14-159	-41697	-02751	94	1-127	-91821	-43173
45	13-900	-42692	-02865	95	1-017	-92247	-45744
46	13-635	-43711	-02987	96	-921	-92613	-48214
47	13-364	-44752	-03115	97	-838	-92931	-50566
48	13-089	-45813	-03252	98	-697	-93474	-55085
49	12-807	-46894	-03396	99	-530	-94115	-61509
				100	-240	-95227	-76773
				101		-96154	-96154

TABLE IX
SINGLE PREMIUM CONVERSION TABLE
FOR FINDING BY INSPECTION THE VALUE OF A FROM THAT OF a

Value of a	Value of A when			Difference of a	Difference of A (subtractive)		
	$i = .03$	$i = .04$	$i = .05$		$i = .03$	$i = .04$	$i = .05$
0	.97087	.96154	.95238	.1	.00291	.00385	.00476
1	.94175	.92308	.90476	.2	.00583	.00769	.00952
2	.91262	.88462	.85714	.3	.00874	.01154	.01429
3	.88350	.84615	.80952	.4	.01165	.01538	.01905
4	.85437	.80769	.76190	.5	.01456	.01923	.02381
5	.82524	.76923	.71429	.6	.01748	.02308	.02857
6	.79612	.73077	.66667	.7	.02039	.02692	.03333
7	.76699	.69231	.61905	.8	.02330	.03077	.03810
8	.73786	.65385	.57143	.9	.02621	.03462	.04286
9	.70874	.61538	.52381	.01	.00029	.00038	.00048
10	.67961	.57692	.47619	.02	.00058	.00077	.00095
11	.65049	.53846	.42857	.03	.00087	.00115	.00143
12	.62136	.50000	.38095	.04	.00117	.00154	.00190
13	.59223	.46154	.33333	.05	.00146	.00192	.00238
14	.56311	.42308	.28571	.06	.00175	.00231	.00286
15	.53398	.38462	.23810	.07	.00204	.00269	.00333
16	.50485	.34815	.19048	.08	.00233	.00308	.00381
17	.47573	.30769	.14286	.09	.00262	.00346	.00429
18	.44660	.26923	.09524	.001	.00003	.00004	.00005
19	.41748	.23077	.04762	.002	.00008	.00008	.00010
20	.38835	.19231	.00000	.003	.00008	.00012	.00014
21	.35922	.15385		.004	.00012	.00015	.00019
22	.33010	.11538		.005	.00015	.00019	.00024
23	.30097	.07692		.006	.00017	.00023	.00029
24	.27184	.03846		.007	.00020	.00027	.00033
25	.24272	.00000		.008	.00023	.00031	.00038
26	.21359			.009	.00026	.00035	.00043
27	.18447						
28	.15534						
29	.12621						
30	.09709						

TABLE X

ANNUAL PREMIUM CONVERSION TABLE

FOR FINDING BY INSPECTION THE VALUE OF P FROM THAT OF Q

Value of a	Value of P when				Value of a	Difference of P (subtractive)										$\Delta a = -8$	$\Delta a = -7$	$\Delta a = -6$	$\Delta a = -5$	$\Delta a = -4$	$\Delta a = -3$	$\Delta a = -2$	$\Delta a = -1$	$\Delta a = 0$	$\Delta a = +1$	$\Delta a = +2$	$\Delta a = +3$	$\Delta a = +4$	$\Delta a = +5$	$\Delta a = +6$	$\Delta a = +7$	$\Delta a = +8$	$\Delta a = +9$			
	$i = -03$	$i = -04$	$i = -05$	$i = -06$																																
0	-97087	-86154	-95238	-104322	0	-09091	-16697	-22077	-28571	-33333	-37500	-41176	-44444	-47368	-50000	-52333	-54333	-56000	-57333	-58333	-59000	-59333	-59666	-59833	-59900	-59933	-59966	-59983	-59990	-59993	-59996	-59998	-59999	-60000		
1	-47087	-46154	-45238	-44322	1	-02381	-04545	-06522	-08333	-10000	-11538	-12963	-14286	-15517	-16666	-17733	-18733	-19666	-20533	-21286	-22000	-22666	-23286	-23866	-24400	-24886	-25322	-25711	-26066	-26386	-26666	-26911	-27111	-27286	-27444	-27586
2	-30421	-29487	-28571	-27655	2	-01075	-02083	-03030	-03921	-04762	-05551	-06286	-07017	-07692	-08311	-08886	-09411	-09886	-10311	-10686	-11066	-11444	-11822	-12200	-12577	-12955	-13333	-13711	-14086	-14466	-14844	-15222	-15599	-15977	-16355	-16733
3	-22087	-21154	-20238	-19322	3	-00610	-01190	-01744	-02273	-02778	-03261	-03723	-04167	-04592	-05000	-05411	-05822	-06233	-06644	-07055	-07466	-07877	-08286	-08692	-09099	-09506	-09911	-10317	-10722	-11128	-11533	-11938	-12344	-12749	-13155	-13561
4	-17087	-16154	-15238	-14322	4	-00392	-00769	-01132	-01481	-01818	-02143	-02456	-02759	-03051	-03344	-03633	-03922	-04211	-04500	-04786	-05072	-05359	-05644	-05929	-06214	-06499	-06783	-07067	-07351	-07636	-07921	-08206	-08491	-08776	-09061	-09346
5	-13754	-12821	-11905	-10989	5	-00274	-00538	-00794	-01042	-01282	-01515	-01742	-01961	-02174	-02386	-02599	-02811	-03022	-03233	-03444	-03655	-03866	-04077	-04286	-04496	-04706	-04916	-05126	-05336	-05546	-05756	-05966	-06176	-06386	-06596	-06806
6	-11373	-10440	-09524	-08608	6	-00201	-00397	-00587	-00772	-00953	-01128	-01299	-01465	-01626	-01786	-01946	-02106	-02266	-02426	-02586	-02746	-02906	-03066	-03226	-03386	-03546	-03706	-03866	-04026	-04186	-04346	-04506	-04666	-04826	-04986	-05146
7	-09387	-08654	-07738	-06822	7	-00154	-00305	-00452	-00595	-00735	-00872	-01006	-01136	-01261	-01386	-01511	-01636	-01761	-01886	-02011	-02136	-02261	-02386	-02511	-02636	-02761	-02886	-03011	-03136	-03261	-03386	-03511	-03636	-03761	-03886	-04011
8	-08199	-07265	-06349	-05433	8	-00122	-00241	-00358	-00473	-00585	-00692	-00802	-00907	-01010	-01113	-01216	-01319	-01422	-01525	-01628	-01731	-01834	-01937	-02040	-02143	-02246	-02349	-02452	-02555	-02658	-02761	-02864	-02967	-03070	-03173	-03276
9	-07087	-06154	-05238	-04322	9	-00099	-00196	-00291	-00385	-00476	-00566	-00654	-00741	-00826	-00911	-01000	-01089	-01178	-01267	-01356	-01445	-01534	-01623	-01712	-01801	-01890	-01979	-02068	-02157	-02246	-02335	-02424	-02513	-02602	-02691	-02780
10	-06178	-05245	-04329	-03413	10	-00082	-00162	-00241	-00319	-00395	-00470	-00544	-00616	-00688	-00761	-00833	-00906	-00979	-01052	-01125	-01198	-01271	-01344	-01417	-01490	-01563	-01636	-01709	-01782	-01855	-01928	-02001	-02074	-02147	-02220	
11	-05421	-04487	-03571	-02655	11	-00068	-00136	-00203	-00268	-00333	-00396	-00459	-00520	-00581	-00642	-00703	-00764	-00825	-00886	-00947	-01008	-01069	-01130	-01191	-01252	-01313	-01374	-01435	-01496	-01557	-01618	-01679	-01740	-01801	-01862	
12	-04780	-03846	-02930	-02014	12	-00058	-00116	-00173	-00229	-00285	-00339	-00393	-00446	-00498	-00551	-00604	-00657	-00710	-00763	-00816	-00869	-00922	-00975	-01028	-01081	-01134	-01187	-01240	-01293	-01346	-01399	-01452	-01505	-01558	-01611	
13	-04230	-03297	-02381	-01465	13	-00051	-00101	-00150	-00199	-00246	-00294	-00340	-00386	-00432	-00478	-00524	-00570	-00616	-00662	-00708	-00754	-00800	-00846	-00892	-00938	-00984	-01030	-01076	-01122	-01168	-01214	-01260	-01306	-01352	-01398	
14	-03754	-02821	-01905	-00989	14	-00044	-00088	-00131	-00173	-00215	-00257	-00298	-00338	-00378	-00418	-00459	-00500	-00541	-00582	-00623	-00664	-00705	-00746	-00787	-00828	-00869	-00910	-00951	-00992	-01033	-01074	-01115	-01156	-01197	-01238	
15	-03337	-02404	-01488	-00572	15	-00039	-00077	-00115	-00152	-00189	-00226	-00262	-00298	-00333	-00368	-00404	-00440	-00476	-00512	-00548	-00584	-00620	-00656	-00692	-00728	-00764	-00800	-00836	-00872	-00908	-00944	-00980	-01016	-01052	-01088	
16	-02970	-02036	-01121	-00205	16	-00034	-00068	-00102	-00135	-00168	-00200	-00232	-00264	-00295	-00326	-00357	-00388	-00419	-00450	-00481	-00512	-00543	-00574	-00605	-00636	-00667	-00698	-00729	-00760	-00791	-00822	-00853	-00884	-00915	-00946	
17	-02643	-01709	-00794	-00000	17	-00031	-00051	-00081	-00111	-00141	-00170	-00200	-00229	-00258	-00287	-00316	-00345	-00374	-00403	-00432	-00461	-00490	-00519	-00548	-00577	-00606	-00635	-00664	-00693	-00722	-00751	-00780	-00809	-00838	-00867	
18	-02351	-01417	-00501	-00000	18	-00027	-00045	-00072	-00100	-00127	-00154	-00181	-00208	-00235	-00262	-00289	-00316	-00343	-00370	-00397	-00424	-00451	-00478	-00505	-00532	-00559	-00586	-00613	-00640	-00667	-00694	-00721	-00748	-00775	-00802	
19	-02087	-01154	-00238	-00000	19	-00025	-00049	-00074	-00098	-00122	-00146	-00169	-00192	-00215	-00238	-00261	-00284	-00307	-00330	-00353	-00376	-00399	-00422	-00445	-00468	-00491	-00514	-00537	-00560	-00583	-00606	-00629	-00652	-00675	-00698	
20	-01849	-00916	-00000	-00000	20	-00023	-00045	-00067	-00089	-00111	-00132	-00154	-00175	-00196	-00217	-00238	-00259	-00280	-00301	-00322	-00343	-00364	-00385	-00406	-00427	-00448	-00469	-00490	-00511	-00532	-00553	-00574	-00595	-00616	-00637	
21	-01633	-00699	-00000	-00000	21	-00021	-00041	-00062	-00082	-00102	-00121	-00141	-00160	-00179	-00198	-00217	-00236	-00255	-00274	-00293	-00312	-00331	-00350	-00369	-00388	-00407	-00426	-00445	-00464	-00483	-00502	-00521	-00540	-00559	-00578	
22	-01435	-00502	-00000	-00000	22	-00019	-00038	-00056	-00074	-00093	-00111	-00129	-00146	-00164	-00181	-00199	-00217	-00235	-00253	-00271	-00289	-00307	-00325	-00343	-00361	-00379	-00397	-00415	-00433	-00451	-00469	-00487	-00505	-00523	-00541	
23	-01254	-00321	-00000	-00000	23	-00018	-00035	-00052	-00069	-00085	-00102	-00118	-00135	-00151	-00168	-00184	-00201	-00218	-00234	-00251	-00268	-00284	-00301	-00317	-00334	-00350	-00367	-00383	-00400	-00416	-00433	-00449	-00466	-00482	-00499	
24	-01087	-00154	-00000	-00000	24	-00016	-00032	-00047	-00063	-00078	-00094	-00109	-00124	-00139	-00154	-00169	-00184	-00199	-00214	-00229	-00244	-00259	-00274	-00289	-00304	-00319	-00334	-00349	-00364	-00379	-00394	-00409	-00424	-00439	-00454	
25	-00931	-00000	-00000	-00000	25	-00015	-00029	-00044	-00058	-00072	-00087	-00101	-00115	-00128	-00141	-00154	-00168	-00181	-00195	-00208	-00222	-00235	-00249	-00262	-00275	-00288	-00302	-00315	-00328	-00342	-00355	-00368	-00382	-00395	-00408	
26	-00791	-00000	-00000	-00000	26	-00014	-00027	-00041	-00054	-00068	-00081	-00094	-00107	-00120	-00133	-00146	-00159	-00172	-00185	-00198	-00211	-00224	-00237	-00250	-00263	-00276	-00289	-00302	-00315	-00328	-00341	-00354	-00367	-00380	-00393	
27	-00659	-00000	-00000	-00000	27	-00012	-00025	-00037	-00050	-00062	-00075	-00087	-00100	-00112	-00124	-00136	-00148	-00160	-00172	-00184	-00196	-00208	-00220	-00232	-00244	-00256	-00268	-00280	-00292	-00304	-00316	-00328	-00340	-00352	-00364	
28	-00536	-00000	-00000	-00000	28	-00011	-00022	-00033	-00043	-00054	-00065	-00075	-00086	-00096	-00106	-00116	-00126	-00136	-00146	-00156	-00166	-00176	-00186	-00196	-00206	-00216	-00226	-00236	-00246	-00256	-00266	-00276	-00286	-00296	-00306	
29	-00431	-00000	-00000	-00000	29	-00010	-00021	-00031	-00041	-00051	-00061	-00071	-00081	-00091	-00101	-00111	-00121	-00131	-00141	-00151	-00161	-00171	-00181	-00191	-00201	-00211	-00221	-00231	-00241	-00251	-00261	-00271	-00281	-00291	-00301	
30	-00313	-00000	-00000	-00000	30	-00010	-00021	-00031	-00041	-00051	-00061	-00071	-00081	-00091	-00101	-00111	-00121	-00131	-00141	-00151	-00161	-00171	-00181	-00191	-00201	-00211	-00221	-00231	-00241	-00251	-00261	-00271	-00281	-00291	-00301	

APPENDIX

CONSTRUCTION OF SIX PER CENT TABLES ACCURATELY TO FIVE PLACES OF DECIMALS

FOR the better illustration of the methods described in Chapter IV of this work for the construction of tables of compound interest functions, a set of tables based on 6% interest has been specially constructed, all the working being shown hereunder. The following is a full account of the procedure—

(a) $(1 + i)^n$.

(1) Insert values for $n = 10, 20, 30, 40$, and 50 in advance, as obtained by independent calculation.

(2) Insert initial value in column (2), i.e. 1.0600000.

(3) Apply working formula $(1 + i)^n = (1 + i) \times (1 + i)^{n-1}$

To do this, calculate $i (1 + i)$, that is $.06 \times 1.06$, by mental multiplication and insert result as the first entry in column (3). The mental addition of the values in columns (2) and (3) will then give $(1 + i) (1 + i) = (1 + i)^2$, being the second value to be inserted in column (2)—and so on to complete the two columns. As each 10th value is reached it should be checked that the result is accurate. An error of 1 or 2 in the last place of decimals is unimportant, as the work should be done to two more places than are required in the final result. To preserve accuracy, continue the work by reference to the accurate figures inserted in advance, thus at once abandoning any error which has accumulated.

(4) Apply check formula $\sum_{n=1}^{n=50} (1+i)^n = s_{\overline{n}|i} - 1$.

The total of column (2), namely 307·7560597, should be equal to $s_{\overline{51}|} - 1$ obtained by independent calculation, which will be found to be 307·7560589. This is sufficiently accurate.

(b) $s_{\overline{n}|}$.

(1) Proceed as in (a) paragraphs (1) and (2).

(2) Apply working formula $s_{\overline{n}|} = s_{\overline{n-1}|} + (1+i)^{n-1}$. To do this, sum the values in line 1 of columns (2) and (4), thus obtaining the next value to be inserted in column (4), and so on to complete the column. It will be observed that the value produced for $n = 50$ differs from the check value by as much as 8 in the last place of decimals. This is of no importance, as the work is done to two more places than are required and is due to the errors in the last 10 values of $(1+i)^n$ tending in the same direction.

(3) We may now apply the check formula—

$$\begin{aligned} \sum_{n=1}^{n=50} \frac{(1+i)^n - 1}{i} &= \frac{1}{i} (s_{\overline{n+1}|} - \overline{n+1}) \\ &= \frac{1}{\cdot 06} (308\cdot 75605886 - 51) \\ &= \frac{257\cdot 75605886}{\cdot 06} \\ &= 4295\cdot 9343143 \end{aligned}$$

This result is remarkably close to the total of column (4), and is a satisfactory check of the accuracy of the work to five places of decimals.

(c) v^n .

(1) In order to illustrate another method, this table has been constructed with the aid of a table of seven-figure logarithms.

(2) Insert values for $n = 10, 20, 30, 40$, and 50 in advance, as obtained by independent calculation.

(3) Insert the initial value $\log v = -\log 1.06 = 1.9746941$.

(4) The working formula is—

$$\log v^n = \log v^{n-1} + \log v$$

This is most easily applied by writing the value of $\log v$ at the foot of a card, holding the card immediately over the initial value and adding the two values, thus obtaining $\log v^2$, and so on to complete the column.

(5) If a table of the values of v were all that were required, the values of the logarithms thus obtained would be sufficiently accurate. As, however, it is proposed to obtain a table of $a_{\overline{n}|i}$ from these values, a slight adjustment is desirable. The value of $\log v$ to ten places of decimals is 1.9746941347 , so that by using 1.9746941 the error after three values have been summed will amount to a trifle over 1. To obtain the nearest seventh place accurately, therefore, we must, *as we proceed with the work of column (2)*, increase the 2nd, 5th, and 8th values in each set of 10, by unity. This reduces the error in the seventh place, which would otherwise accumulate by each 10th value to 3.47 to an error of .47, and even this error is discarded by the use of the previously inserted 10th values.

(6) Column (3) is now completed by taking the antilogarithm of each value in column (2). The total of the results, namely 15.7618591, compares very favourably

with the true result of 15.7618606, being $a_{\overline{50}|}$ by independent calculation.

(d) $a_{\overline{n}|}$.

(1) Insert in advance the initial value and every 10th value as obtained by independent calculation.

(2) The working formula is—

$$a_{\overline{n}|} = a_{\overline{n-1}|} + v^n$$

Add, therefore, to the initial value, the value of v^n from the previous column, thus obtaining $a_{\overline{2}|}$, and so on to complete the column.

(3) The check formula is :

$$\begin{aligned}\Sigma a_{\overline{n}|} &= \frac{50 - a_{\overline{50}|}}{.06} = \frac{50 - 15.76186064}{.06} = \frac{34.23813936}{.06} \\ &= 570.635656.\end{aligned}$$

This is a good rough check, the size of the error being due to the errors which accumulate with a method of continuous addition of factors themselves not quite accurate.

CONSTRUCTION OF 6 PER CENT TABLES

n (1)	$(1+i)^n$ (2)	$i(1+i)^n$ (3)	s_n (4)	n (5)
1	1.0600000	.0636000	1.0000000	1
2	1.1236000	.0674160	2.0600000	2
3	1.1910160	.0714610	3.1836000	3
4	1.2624770	.0757486	4.3746160	4
5	1.3382256	.0802935	5.6370930	5
6	1.4185191	.0851111	6.9753186	6
7	1.5036302	.0902178	8.3938377	7
8	1.5938480	.0956309	9.8974679	8
9	1.6894789	.1013687	11.4913159	9
10	1.7908477	.1074509	13.1807949	10
11	1.8982986	.1138979	14.9716426	11
12	2.0121965	.1207318	16.8699412	12
13	2.1329283	.1279757	18.8821377	13
14	2.2609040	.1356542	21.0150660	14
15	2.3965582	.1437935	23.2759700	15
16	2.5403517	.1524211	25.6725282	16
17	2.6927728	.1615664	28.2128799	17
18	2.8543392	.1712604	30.9056527	18
19	3.0255996	.1815360	33.7599919	19
20	3.2071355	.1924281	36.7855912	20
21	3.3995636	.2039738	39.9927287	21
22	3.6035374	.2162122	43.3922903	22
23	3.8197496	.2291850	46.9958277	23
24	4.0489346	.2429361	50.8155773	24
25	4.2918707	.2575122	54.8645119	25
26	4.5493829	.2729630	59.1563826	26
27	4.8223459	.2893408	63.7057655	27
28	5.1116867	.3067012	68.5281114	28
29	5.4183879	.3251033	73.6397981	29
30	5.7434912	.3446095	79.0581862	30
31	6.0881007	.3652860	84.8016774	31
32	6.4533867	.3872032	90.8897781	32
33	6.8405899	.4104354	97.3431648	33
34	7.2510253	.4350615	104.1837547	34
35	7.6860868	.4611652	111.4347800	35
36	8.1472520	.4888351	119.1208668	36
37	8.6360871	.5181652	127.2681188	37
38	9.1542523	.5492551	135.9042059	38
39	9.7035074	.5822104	145.0584582	39
40	10.2857179	.6171431	154.7618666	40
41	10.9028610	.6541717	165.0476835	41
42	11.5570327	.6934220	175.9505445	42
43	12.2504547	.7350273	187.5075772	43
44	12.9854820	.7791289	199.7580319	44
45	13.7646109	.8258767	212.7435139	45
46	14.5904876	.8754293	226.5081248	46
47	15.4659169	.9279550	241.0986124	47
48	16.3938719	.9836323	256.5645293	48
49	17.3775042	1.0426503	272.9584012	49
50	18.4201543	—	290.3359046	50
Total	307.7580597	—	4295.9343167	

CONSTRUCTION OF 6 PER CENT TABLES

n (1)	$\log v^n$ (2)	Antilog (2) = v^n (3)	a_n (4)	n (5)
1	T-9746941	-9433960	-9433960	1
2	T-9493883	-8899966	1-8333926	2
3	T-9240824	-8396192	2-6730118	3
4	T-8987765	-7920935	3-4651053	4
5	T-8734707	-7472583	4-2123636	5
6	T-8481648	-7049605	4-9173241	6
7	T-8228589	-6650570	5-5823811	7
8	T-7975531	-6274124	6-2097935	8
9	T-7722472	-5918985	6-8016920	9
10	T-7469413	-5583948	7-3600671	10
11	T-7216354	-5267875	7-8868746	11
12	T-6963296	-4969695	8-3838441	12
13	T-6710237	-4688389	8-8526830	13
14	T-6457178	-4423009	9-2949839	14
15	T-6204120	-4172650	9-7122489	15
16	T-5951061	-3936463	10-1058952	16
17	T-5698002	-3713643	10-4772595	17
18	T-5444944	-3503438	10-8276033	18
19	T-5191885	-3305130	11-1581163	19
20	T-4938826	-3118047	11-4699212	20
21	T-4685767	-2941553	11-7640765	21
22	T-4432709	-2775051	12-0415816	22
23	T-4179650	-2617972	12-3033788	23
24	T-3926591	-2469785	12-5503573	24
25	T-3673533	-2329985	12-7833558	25
26	T-3420474	-2198100	13-0031658	26
27	T-3167415	-2073679	13-2105337	27
28	T-2914357	-1956301	13-4061638	28
29	T-2661298	-1845567	13-5907205	29
30	T-2408239	-1741101	13-7648312	30
31	T-2155180	-1642548	13-9290860	31
32	T-1902122	-1549574	14-0840434	32
33	T-1649063	-1461862	14-2302296	33
34	T-1396004	-1379115	14-3681411	34
35	T-1142946	-1301052	14-4982463	35
36	T-0889887	-1227407	14-6209870	36
37	T-0636828	-1157931	14-7367801	37
38	T-0383770	-1092388	14-8460189	38
39	T-0130711	-1030555	14-9490744	39
40	Z-9877653	-0972222	15-0462969	40
41	Z-9624594	-0917190	15-1390159	41
42	Z-9371536	-0865274	15-2245433	42
43	Z-9118477	-0816296	15-3061729	43
44	Z-8865418	-0770090	15-3831819	44
45	Z-8612360	-0726501	15-4558320	45
46	Z-8359301	-0685378	15-5243698	46
47	Z-8106242	-0646583	15-5890281	47
48	Z-7853184	-0609984	15-6500265	48
49	Z-7600125	-0575458	15-7075721	49
50	Z-7347066	-0542884	15-7618806	50
Total	—	15-7618391	570-6356489	

EXERCISES AND EXAMPLES

CHAPTER I

1. Evaluate the following by 4-figure logs—

(a) 1428×9133 ; (b) $17 \times .6238 \times 1981$;

(c) $46.42 \times .003581 \times .0001327$; (d) $\frac{13421 \times 139}{.005714 \times 18691}$;

(e) $1.923^{3.219}$; (f) $\left(\frac{1234}{5678}\right)^{.9101}$

2. Using logs, find the value of x from the relation—

$$\frac{(31.67)^2 - x^2}{.3485} + \frac{x^2}{.0306} = 3174$$

3. Given $\log 2 = .3010$ and $\log 3 = .4771$ —

(a) Find the value of $\log \frac{375}{1024} + 2 \log \frac{4}{27} - 3 \log \frac{5}{8}$

(b) Solve the equation $\frac{2^x 3^{x+1}}{5^x} = .005$ correct to the nearest tenth.

4. Find p , q , and r given that $\log_{\sqrt{5}} 125 = p$; $\log_q 256 = 8$; and $\log_s r = 4$.

5. Find correct to three places of decimals the value of $\log_s 7$, being given $\log 7 = .8451$ and $\log 3 = .4771$.

6. Find (a) $\log \sqrt[3]{5}$ to base .008; and (b) $\log \sqrt{128}$ to base $\sqrt{4}$.

7. Expand $\log_e (1 + 3x + 4x^2)$ in ascending powers of x omitting fifth and higher powers.

8. Prove that $(m+n)^{-y} + (m+n)^{+y} = 2 + y^2 \{\log_e (m+n)\}^2 + \frac{y^4}{12} \{\log_e (m+n)\}^4 + \dots$

CHAPTER II

1. Given $\log 1.04 = .0170333$ and $\log 1.009853 = .0042583$, find the nominal rate of interest convertible quarterly, corresponding to an effective rate of 4% per annum.

2. Find the present value at 6% interest, with half-yearly rests, of £20 due 15 years hence. (Use 4-figure logs.)

3. What is the discount on £250 due six months hence (a) on the basis of a Bank Rate of 3%; (b) at an effective rate of interest of 3% per annum; (c) at an effective rate of discount of 3% per annum?

4. (a) What is the present value at 3% of an immediate annuity-certain for 10 years of £100 for the first four years, £50 for the next 4 years, and £25 for the last two years?

(b) What would be the present value of the above at the effective rate of 3% per annum if the annuities were payable half-yearly?

(c) What would be the present value of the above at 6% if both interest and annuities were payable half-yearly?

(d) What would be the present value of the above at the nominal rate of 2.9778% convertible half-yearly, the annuities being payable yearly?

5. (a) What is the present value of an annuity of £20 per annum payable yearly for 10 years, the first payment being due in six years' time, followed by an annuity of £50 per annum for a further 10 years, on the basis of an effective rate of interest of 4%?

(b) What would be the amount of the same?

6. The dividends on a life assurance company's shares are increased 5% every quinquennial valuation. If a valuation has just taken place and a certain shareholder's dividends for the next five years are fixed at £80 per annum, what is the value of his holding in the company, on a 4% basis?

7. What is the amount of an annuity of £100 per annum for 10 years—

(a) If the annuity is payable half-yearly in arrear and interest is allowed at 8% per annum with half-yearly rests?

(b) If the annuity is payable quarterly in arrear and interest as in (a)?

(c) If the annuity is payable half-yearly in advance and interest is allowed at the nominal rate of 9.878% per annum convertible quarterly?

8. (a) One shilling per week is deposited at 3% compound interest. Assuming that this differs but little from continuous depositing, calculate the approximate amount on deposit after 10 years, given that $\log_e 1.03 = .0295588$.

(b) On a similar assumption, what is the present value of 1s. per week payable in perpetuity?

9. (a) The funds of a life assurance company were £650,000 at the beginning of a year and £850,000 at the end of the year. The gross interest received was £36,000. Calculate the gross effective yield.

(b) What would be the gross effective yield if the gross interest were reduced by £5,000 in respect of income tax?

CHAPTER III

1. What sum should be set aside at the end of each year to provide £250 at the end of seven years, assuming accumulation throughout at 3% compound interest?

2. Calculate the annual premium for a capital redemption assurance of £1,000 effected for a period of 10 years, on a net 4% basis for the first five years, the interest being decreased to 3% for the last five years.

3. Assuming a 3% basis, (a) calculate the value at the end of five years of a sinking fund policy to repay £1,000, 10 years from the outset; (b) prove the result by an alternative method.

4. What is the present value of an annuity-certain of £100 per annum payable half-yearly for 10 years, to yield a gross effective rate of interest of 4% per annum, where income tax is at the rate of 5s. in the £, and is chargeable in full on each half-yearly payment of annuity?

5. A foreign government loan of £100,000 bearing interest at $5\frac{1}{4}\%$ per annum payable yearly, is redeemed at 110% by a yearly annuity of fixed amount, including capital and interest, extending over five years. Find (a) the amount of each yearly payment, and (b) the amount of the loan paid off by the end of the second year.

6. What annual instalment of principal and interest will repay a loan of £2,000 in 20 years, if the lender is to receive 4% on his loan and to replace the capital by effecting a sinking fund policy on a 3% basis?

7. A man pays £1,000 for a leasehold house yielding him a net rent of £86 2s. 8d. yearly until the lease expires in 20 years' time. If he effects a capital redemption policy on a 3% basis, what yield does he secure on his outlay?

8. A company can exercise the following methods of redeeming its debentures, interest on which at 5% is payable annually—

(a) It can repay them at the expiration of five years at 102%, but cannot exercise the option at any other time.

(b) Any debentures not so repaid must be redeemed at par at the expiration of 10 years.

What is the maximum price that a purchaser can now give in order to realize in any event not less than 4% on his investment?

9. Find the value to yield an effective rate of 4%, of a 5% debenture redeemable in 10 years' time at par, on the assumption that a sinking fund policy, based on 3% interest, is effected to make good the capital outlay on redemption.

10. (a) What is the price of an 8% debenture redeemable at 105 in five years' time, to yield a gross effective rate of 4% per annum? Assume dividends payable yearly, the first in one year's time, and allow for income tax at 5s. in £.

(b) What would be the gross yield if the debenture were bought at 113?

CHAPTER IV

No examples. Actual construction with appropriate checks is essential.

CHAPTER V

1. Given the columns l_x and d_x in Table V for ages 90 onwards, calculate the values of p_x , q_x , e_x , and L_x for each of the said ages.

2. Given the following rough data from a census and the corresponding death returns, calculate the value of q_x for each of the ages given—

Age last Birthday	Population	Deaths
25	46,348	332
26	45,997	338
27	45,633	349
28	45,250	355

3. (a) In what way does the use of the affix x differ in the symbols l_x and d_x ?

(b) What assumption underlies the approximate formula $L_x = l_x + \frac{1}{2}d_x$?

4. Calculate on the basis of Table V, the probability that—

(a) A life aged exactly 25 will survive to age 60.

(b) A life aged exactly 60 will not reach his 65th birthday.

(c) A life aged exactly 70 will reach age 71 but not age 72.

5. (a) What is the "Central Death Rate"?

(b) How does it differ from the "Rate of Mortality"?

(c) Find q_x in terms of m_x .

6. From Tables V and VI, calculate—

(a) m_{40} , and test its accuracy by comparison with q_{40} .

(b) $m_{40\frac{1}{2}}$, and test its accuracy approximately by comparison with a value taken from the μ column.

7. (a) Define the "Expectation of Life."

(b) What is the most probable age at death of a life now exactly aged 27? (Use Table V.)

(c) Define the "Force of mortality," and express its value approximately in terms of l_x .

CHAPTER VI

Calculate from Table V the probabilities that of two lives aged exactly 30 and 25—

1. Both will survive 25 years.
2. Both will survive 15 years and both die in 16th year.
3. The life aged 30 will die within 10 years and the life aged 25 will attain age 80.
4. Both will die within 15 years.
5. Either or both will die within 15 years.
6. At least one will live 50 years.
7. Neither will live 50 years.

CHAPTER VII

1. Explain clearly the nature of the functions D_x and N_x , and check the values of D_{60} and N_{90} appearing in Table VII.

2. (a) Check the values of a_{35} , A_{35} , and P_{35} in Table VIII from the functions in Table VII.

(b) Check the values of A_{50} and P_{50} from the value of a_{50} , using Tables IX and X.

3. Using Table VII throughout, calculate the present value of—

(a) An immediate annuity of £150 payable yearly on a life aged 70.

(b) The same, deferred five years.

(c) A similar annuity-due deferred five years.

4. (a) Using Table VII, calculate the net annual premium for a 10-year endowment assurance for £100 on a life aged exactly 30.

(b) What would it be if the premiums were limited to five in number?

5. Calculate $A_{\overline{35}|}^1$ at 4% from Tables III and V and compare the result with that obtained from Table VII.

6. Using Table VII, calculate (a) the single, and (b) the annual premiums for a double endowment assurance on a life aged 30, under which £200 is payable at the end of 25 years, with £100 at previous death, and (c) check the results solely from the D and N columns and the use of Tables IX and X.

CHAPTER VIII

[NOTE. Base your replies on Tables VII to X inclusive.]

1. What is the present value of an annuity payable on a life aged 55, if—

(a) The first payment is due at once?

- (b) The first payment is due in six months' time?
- (c) The first payment is due in nine months' time?
- (d) The first payment is due in one month's time and the annuity is payable monthly thereafter?

2. Calculate (a) the quarterly, and (b) the half-yearly, rates for a whole life assurance on a life aged 35.

3. What is the present value of a complete annuity payable on a life aged 70, if it is payable (a) yearly, (b) half-yearly, and (c) quarterly?

4. (a) What two equal ages are, for the purpose of obtaining annuity values, the equivalent of ages 60 and 25, and (b) what three are the equivalent of ages 30, 65, and 70?

5. A husband aged x desires to provide a pension of £1 per annum for his wife aged y , after his death, by surrendering a certain amount of his unapportionable pension. Write down an expression for the amount which he must surrender per annum.

6. What is the policy value of an endowment assurance for 10 years, effected five years ago on a life then aged 30 for £100, at a net annual premium of 8.418?

7. What is the policy value at the end of 20 years of a single premium 30-year endowment assurance for £100, effected on a life aged 25?

8. (a) Would you use "Select" or "Aggregate" mortality tables for the construction of life assurance premiums? Give reasons.

(b) What is an "Ultimate" table of mortality and how does it differ from an "Aggregate" table?

9. (a) From what data was the English Life Table No. VIII constructed?

(b) What replaced the continuation of the series of Healthy English Life Tables when the No. VIII Table was published?

10. (a) Which table is the recognized standard for valuation of the liabilities of a life office?

(b) What is a "truncated aggregate" table and for what purpose is it used?

11. Which are the two latest tables of annuitant experience published and what special features do they exhibit?

CHAPTER IX

No examples. Actual construction with appropriate checks is essential.

ANSWERS TO EXERCISES AND EXAMPLES

NOTE. In approximate calculations, such as those where 4-figure logarithms are employed, small differences in the results may arise from the order in which the operations are performed.

CHAPTER I

- (1) (a) 13,040,000; (b) 21,010; (c) .00002204; (d) 17,470; (e) 8.206; (f) .2493. (2) 3.149. (3) (a) 2.5175; (b) $x = -6$. (4) $p = 6$, $q = 2$, $r = 81$. (5) 1.771. (6) (a) 1.8889; (b) 5.25. (7) $3x - \frac{x^2}{2} - 3x^3 + \frac{31x^4}{4}$. (8) Formula (5).

CHAPTER II

- (1) 3.9412%. (2) 8.26. (3) (a) 3.75; (b) 3.6675; (c) 3.775. (4) (a) 574.603; (b) 578.878; (c) 518.476; (d) 574.603. (5) (a) 358.517; (b) 955.745. (6) 2,244. (7) (a) 1488.905; (b) 1503.645; (c) 1735.965. (8) (a) 30.251; (b) 87.96. (9) (a) 4.918%; (b) 4.901%.

CHAPTER III

- (1) 32.626. (2) £83 6s. 11d. (3) (a) 463.119; (b) see para. 7. (4) 644.525. (5) (a) £25,407 1s. 10d.; (b) £37,100 nominal (nearest £). (6) 152.263. (7) 5%. (8) 106.096. (9) £111 16s. 2d. (10) (a) 118.052; (b) £5 6s. 8d.%

CHAPTER IV

No examples.

CHAPTER V

- (1) Text and Table VI. (2) .00714, .00732, .00762, .00781 (3) (a) p. 83; (b) p. 84. (4) (a) .632; (b) .162; (c) .065. (5) pp. 94 and 95. (6) (a) p. 95; (b) p. 96. (7) (a) p. 89; (b) age 73, when d_x is a maximum; (c) pp. 95 and 96.

CHAPTER VI

- (1) .581. (2) .000941. (3) .0124. (4) .0152. (5) .232. (6) .388. (7) .612.

CHAPTER VII

(1) p. 104. (2) See text. (3) (a) 943·894; (b) 402·796; (c) 485·912. (4) (a) 8·418; (b) 15·051. (5) ·02434. (6) (a) 70·892; (b) 4·788; (c) $a_{\overline{30} \cdot \overline{0\frac{1}{2}}\frac{1}{4}} = 13·807$, and convert, adding premium for extra endowment.

CHAPTER VIII

(1) (a) 12·024; (b) 11·524; (c) 11·274; (d) 11·482. (2) (a) ·501% per quarter; (b) ·995% per half-year. (3) (a) 6·660; (b) 6·726; (c) 6·760. (4) (a) 52·8 and 52·8; (b) 63·5 and 63·5. (5) $a_{x/y} \div a_x$. (6) 44·173. (7) 69·380. (8) (a) p. 123; (b) p. 124. (9) (a) p. 125; (b) p. 126. (10) (a) and (b) p. 127. (11) pp. 128–129.

CHAPTER IX

No examples.

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