PHYSICAL & DYNAMICAL METEOROLOGY

By

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SOME USEFUL CONSTANTS AND UNITS

For dry air the constant R in the equation $p = R\rho T$ is $2{\cdot}8703 \times 10^3$, when p is in millibars and ρ in gm/cm^3; when C.G.S. units are used, and p is in dynes/cm^2, R for dry air is $2{\cdot}8703 \times 10^6$.

For water-vapour the corresponding values of R are $4{\cdot}62 \times 10^3$ and $4{\cdot}62 \times 10^6$.

The universal gas-constant is $83{\cdot}15 \times 10^6$ ergs.

The density of dry air at 0° C and 1000 mb is 1·27617 gm/cm^3.

Specific heats of dry air: $c_p = 0{\cdot}2396$; $c_v = 0{\cdot}1707$; $\gamma = c_p/c_v = 1{\cdot}403$.

Mean molecular weight of atmospheric air = 28·9.

Latent heat of evaporation of water = $594{\cdot}9 - 0{\cdot}51t$, where t is the temperature in °C (see Table VII, p. 406).

Latent heat of fusion of ice = 79·7 calories.

Specific heat of ice = 0·5.

Number of gas molecules per cm^3 at 0° C and 1000 mb = $2{\cdot}66 \times 10^{19}$.

Stefan's constant $\sigma = 1{\cdot}38 \times 10^{-12}$ g-cal cm^{-2} sec^{-1}
$= 0{\cdot}826 \times 10^{-10}$ g-cal cm^{-2} min^{-1}
$= 5{\cdot}77 \times 10^{-5}$ ergs cm^{-2} sec^{-1}
$= 5{\cdot}77 \times 10^{-12}$ watt cm^{-2}.

1 gramme-calorie = $4{\cdot}186 \times 10^7$ ergs = 4·184 joules.

1 watt = 1 joule/sec = 10^7 ergs sec^{-1} = 14·32 cal min^{-1}.

1 kilowatt = 10^{10} ergs sec^{-1} = $1\frac{1}{3}$ horse power (approx).

1 kilowatt-hour = $3{\cdot}6 \times 10^{13}$ ergs.

1 kilowatt per (dekametre)^2 = $1{\cdot}43 \times 10^{-2}$ g-cal cm^{-2} min^{-1}.

1 foot = 0·3048 m; 1 m = 3·281 feet.

The equatorial semi-axis of the earth = 6378·2 km.

The polar semi-axis of the earth = 6356·5 km.

The area of the earth's surface is approximately $5{\cdot}1 \times 10^8$ km^2.

Angular velocity of the earth's rotation = $7{\cdot}29 \times 10^{-5}$ sec^{-1}.

The difference between geocentric and geographical longitude is $700'' \sin 2\phi$.

INTRODUCTION

METEOROLOGY, the science of things in the atmosphere, is concerned with the measurement and co-ordination of pressure, temperature, density, and humidity of the air, and of the motion of air relative to the earth. It seeks to explain the motions observed in terms of the changes of pressure, temperature and humidity, brought about directly or indirectly by the effects of incoming solar radiation. Meteorology in its widest sense includes the study of such phenomena as atmospheric electricity and terrestrial magnetism, but we shall not discuss these subjects in this book.

Instruments have been devised for measuring the factors mentioned, as well as some others such as incoming solar radiation, duration of bright sunlight, rate of fall of rain, etc. We may say of most of these instruments that in principle they are relatively simple, though in practice there may be considerable difficulties in using them in such a way as to give reliable results. It is almost invariably a tedious and difficult matter to obtain accurate meteorological observations. The physicist in the laboratory, wishing to measure the temperature of a fluid, merely plunges a thermometer in the fluid and reads the scale. But the meteorologist who wishes to measure the temperature of air out of doors has to take precautions to shade his thermometer from the direct rays of the sun and radiation from surrounding objects, and at the same time to provide for adequate ventilation. His problem is therefore much more complicated than that of the physicist. He may devise an electrically recording instrument which will measure the temperature of the air to 0·01° C, and then find that the record indicates a constantly fluctuating temperature. The measurement of air temperature to such a high degree of apparent accuracy is probably a waste of time, as the air does not know its own temperature to that degree of accuracy. A current of air, as we know it in the natural state in the atmosphere, is far from being homogeneous. The most casual glance at the record of wind direction and velocity provided by an anemometer shows that at a given point the motion is never steady, but the fact that the temperature is equally unsteady at a given point of observation is frequently overlooked.

The most serious of all difficulties of meteorology are probably those which arise from the fact that the air contains widely varying quantities of water-vapour. This ever-changing content is a potent source of supply of thermal energy to the atmosphere, but is a continual stumbling-block to the student of physical meteorology on account of the difficulty of applying physical reasoning to a medium of variable constitution.

A further difficulty arises from the paucity of observations over certain parts of the globe, and in certain types of weather. Relatively few observations have been made of wind, temperature, and humidity in the free atmosphere over the oceans, or in the polar regions, and even over the continents direct observation of upper air conditions in cyclonic depressions is practically impossible.

In the present work it is not proposed to discuss in detail the difficulties of observation, or the details of the methods which have been devised for obtaining observations of different meteorological factors. The reader who desires such information should consult the *Meteorological Observer's Handbook* (H.M.S.O.), the *Dictionary of Applied Physics*, **3**, and other works referred to in the subsequent pages.

Great as are the difficulties of accurate observation in meteorology, the difficulties of theoretical discussion are even greater. It is not possible to isolate a portion of the atmosphere, and to discuss the physical processes which take place in that portion, on account of the translation of pressure systems and of the variation of wind with height, both of which factors make it impossible to specify with certainty what is taking place in a given mass of air. It is customary to speak of the atmosphere as a heat engine, working between a source of heat at the equator and a source of cold at the poles. The most formidable obstacle to progress is ignorance of the laws of transfer of heat within this so-called engine.

CHAPTER I

THE FACTS WHICH CALL FOR EXPLANATION. A SKETCH OF THE SURFACE DISTRIBUTION OF THE METEOROLOGICAL ELEMENTS OVER THE GLOBE

§ 1. *Temperature*

THE distribution of temperature over the globe is here shown in four charts, for January and July, for the Northern and Southern hemispheres, the isotherms being drawn for intervals of 10° F (figs. 1–4). Looking first at the January (summer) chart of the Southern hemisphere, we are impressed by the fact that this is almost entirely an ocean hemisphere, having the Antarctic continent centrally situated. The isotherms run almost symmetrically over the oceans, with sudden swerves over the coasts of the continents. The land is everywhere warmer than the oceans in the corresponding latitudes. The July (winter) chart for the Southern hemisphere shows the same general symmetry of the isotherms over the oceans, but temperatures over the continents are now lower than the temperatures over the oceans in corresponding latitudes.

Coming to the January (winter) chart of the Northern hemisphere, we are impressed by the much greater extent and less symmetrical distribution of th land masses. The isotherms are by no means as symmetrical as those in t Southern hemisphere. The lowest temperatures are those recorded in No east Siberia, and there is another region of intense cold centred over C...n-land. The region immediately surrounding the North Pole is left free of isotherms, since the available observations are too few to yield reliable mean values. It is seen that the temperatures over the continents are lower than those over the oceans in corresponding latitudes, while in middle latitudes where westerly winds predominate (*vide* p. 13) the temperature is higher over the western coasts of the continents than over the eastern coasts. In the July (summer) chart for the Northern hemisphere the run of the isotherms is somewhat irregular. There is a long belt of high temperature (above 90° F) running across North Africa and South-west Asia, to the north of India, and a centre of high temperature over the South-west of the United States. The centre of extreme cold in North-east Siberia has now disappeared, and this region is warmer than any other region in the same latitude except the extreme North-west of Canada. The land is warmer than the sea in most latitudes, the difference being especially well marked over the North Pacific Ocean and the adjacent land masses. Greenland still interposes a tongue of cold into the now relatively warm area of the extreme north of the Atlantic Ocean.

A comparison of figs. 1 and 3, and of figs. 2 and 4 indicates that in January the highest surface temperatures observed in either hemisphere form a belt in

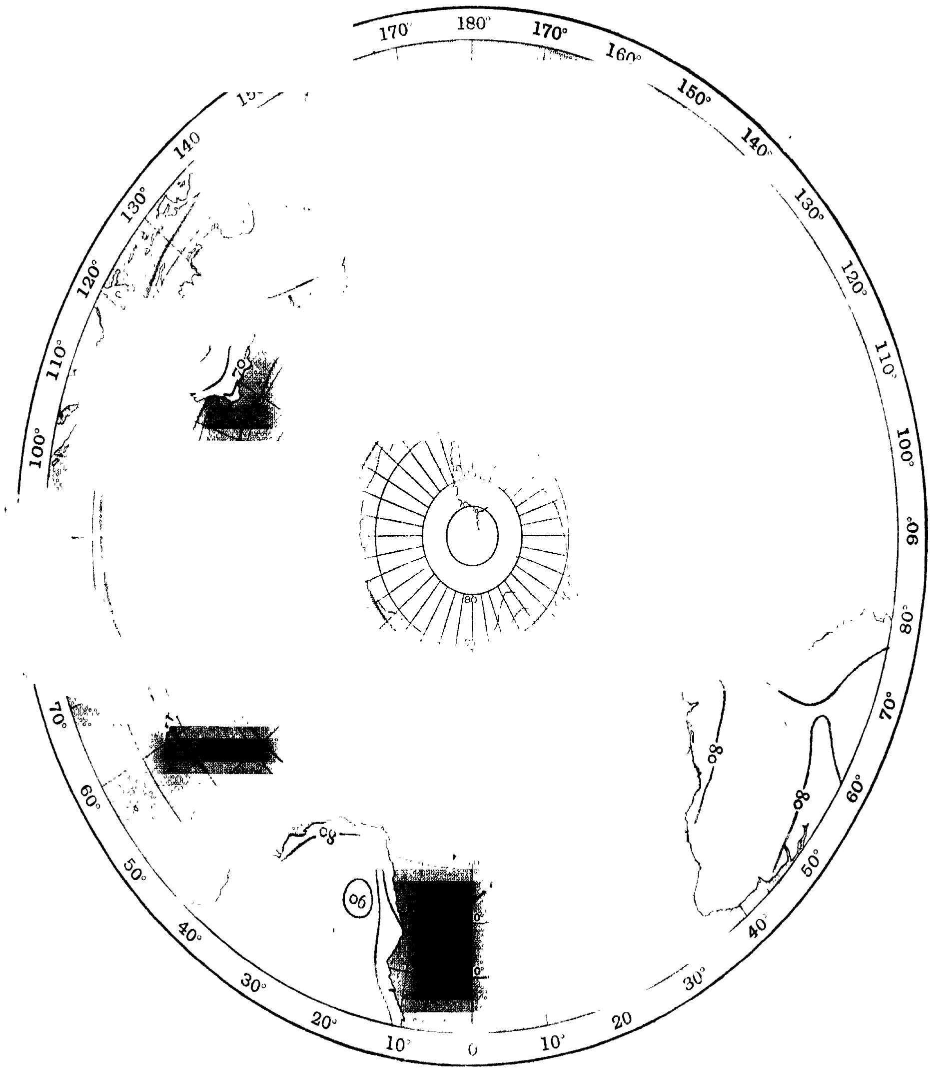


Fig. 1. Normal temperature at sea level, Southern hemisphere, January.

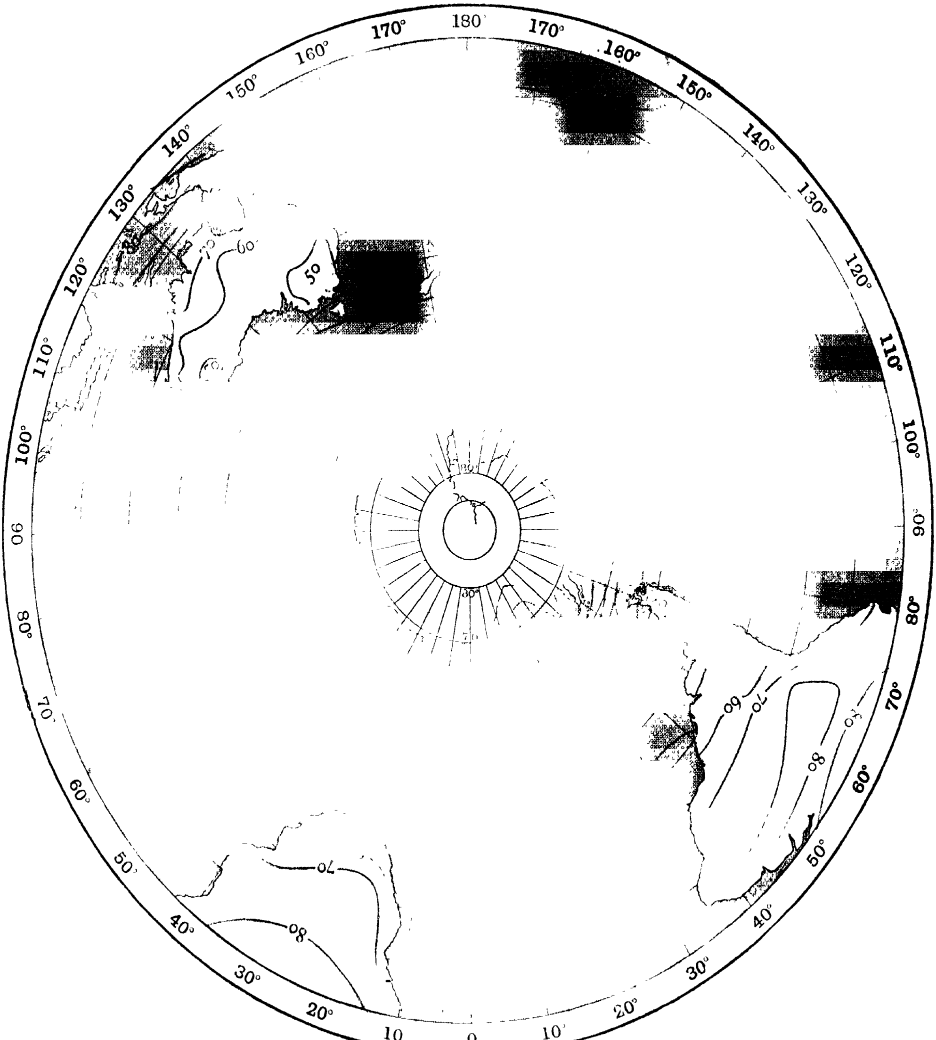


Fig. 2. Normal temperature at sea level, Southern hemisphere, July.

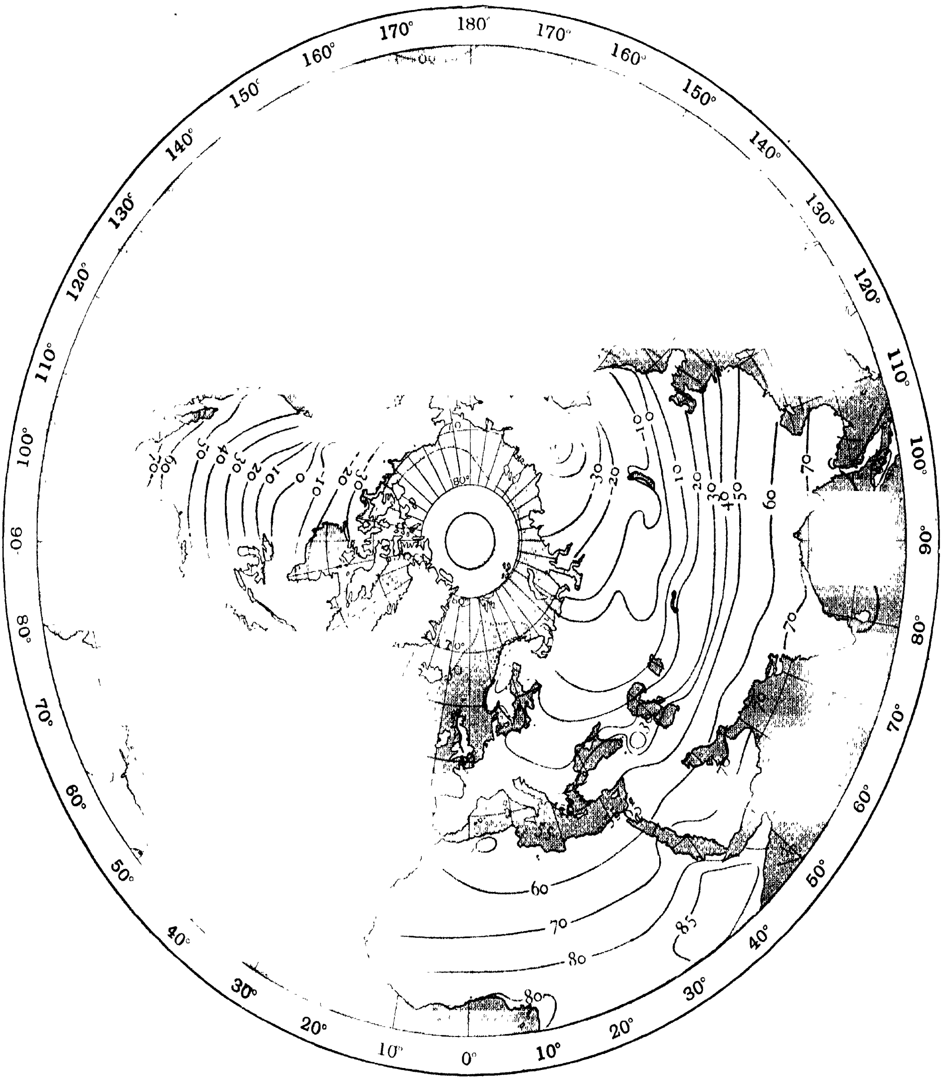


Fig. 3. Normal temperature at sea level, Northern hemisphere, January.

Fig. 4. Normal temperature at sea level, Northern hemisphere, July.

latitude 30° S approximately, while in July the belt of highest surface temperatures is in latitude 35° N.

One further point of interest arises from the comparison of the January and July charts. The temperatures at the equator show only slight variations in the course of the year, but in say latitude 60° in the Northern hemisphere the annual variation may be anything between 10° and 50° according to the longitude. Thus the average rate of decrease of temperature with increase of latitude is much greater in winter than in summer, so that any phenomenon which depends on the variation of temperature in the horizontal will be more intense in winter than in summer.

We have recounted above the main facts which are to be gathered from an inspection of the temperature charts for January and July for the two hemispheres. The other months are in the main intermediate between the two extreme months. Similar charts for all months of the year are given in Shaw, *Manual of Meteorology*, **2**, and should be consulted for further details.

§ 2. *Pressure*

The mean distribution of pressure for January and July for the Northern and Southern hemispheres is represented in figs. 5–8. The chart for the Southern hemisphere in January shows a belt of low pressure in latitude 60–70°, north of which pressure increases up to a belt of high pressure centred about latitude 30°, having three centres of highest pressure over the oceans, with relatively lower pressure over the land masses of South America, South Africa and Australia, corresponding to three shallow centres of low pressure over the equator in the same longitudes. The July chart for the Southern hemisphere indicates the same general features, but the sub-tropical belt of high pressure is now farther north, and is more intense and more continuous around the globe.

For the Northern hemisphere the January chart is far more complicated than either chart for the Southern hemisphere. In the latter the sub-tropical belt of high pressure appears as the most striking and persistent feature, but in the January chart for the Northern hemisphere the sub-tropical belt of high pressure is only traceable over North Africa, the North Atlantic Ocean and the eastern half of the North Pacific Ocean. The dominating features of this chart are the region of high pressure centred over Siberia, and the regions of low pressure centred south of Greenland and over the Aleutian Islands in the North Pacific, respectively. The July chart shows well-marked anticyclones centred over the North Atlantic and North Pacific Oceans in latitude 35°, and a centre of low pressure to the north of India.

The outstanding feature shown by the pressure charts is the winter development of high pressure over the continents and low pressure over the oceans, and the summer development of low pressure over the continents and high pressure over the oceans. In winter the continents are loaded with air at the expense of the oceans, and in summer the converse occurs. These phenomena are more marked in the Northern than in the Southern hemisphere on account of the greater extent of the land masses in the Northern hemisphere, and it will be seen later that these phenomena are produced by the annual variation of temperature over the continents.

For the distribution of pressure in the other months of the year the reader is referred to Shaw, *Manual of Meteorology*, **2**.

If we examine the Northern hemisphere charts of monthly mean pressures month by month, we find that the areas on these charts which show changes of 5 mb or more from one month to the next are definitely localised in certain restricted regions. The outstanding region is the Asiatic continent. In September a long tongue of pressure over 1015 mb but under 1020 mb extends from the Azores anticyclone across Europe and Asia nearly to the Pacific coast, while the depression over India is shallow with an extensive region of pressure just below 1005 mb. By October the latter region has disappeared and the pressure over Central Asia surpasses 1025 mb in the central region of the anticyclone. During the next three months the anticyclone grows steadily, reaching its maximum in January, when the pressure at the centre surpasses 1035 mb. During this period the Icelandic and Aleutian lows both develop steadily, reaching their maximum intensity at about the same time, while the subtropical high-pressure belt over the North Atlantic shows little marked modification, beyond an initial weakening, and the sub-tropical belt of high pressure in the North Pacific becomes weaker in November and December and then becomes rather more accentuated.

The growth of the Siberian anticyclone in the interval from September to October is remarkably rapid, pressure rising by an amount varying from 5 to 10 mb over more than one half of the area of Asia. During the same period a fall of about the same magnitude occurs in the region of the Aleutian low, but only over a very much smaller area. The figures suggest that the Siberian anticyclone is fed in part by the hollowing of the Aleutian low, but mainly by the denudation of a very wide area of its share of air. This area probably extends into the Southern hemisphere.

By March the Siberian anticyclone has again diminished in intensity, so that the central pressure is slightly in excess of 1025 mb, while the Icelandic and Aleutian lows have filled up slightly. Between March and April there is a marked diminution in the intensity of all three features, and the diminution is continued in May, when a shallow depression appears over India. In June the depression has extended over nearly all Asia, and the anticyclone has disappeared. The depression maintains its intensity during July and August, during which months the Aleutian low ceases to have a separate existence on the monthly charts.

Since the pressure at the earth's surface measures the total mass of air in the column which extends from the ground to the top of the atmosphere, we could, by integration over the whole hemisphere, find the mass of air above a hemisphere of the earth's surface. This computation would be incorrect, if carried out with the pressure reduced to mean sea level, in that it would, in effect,

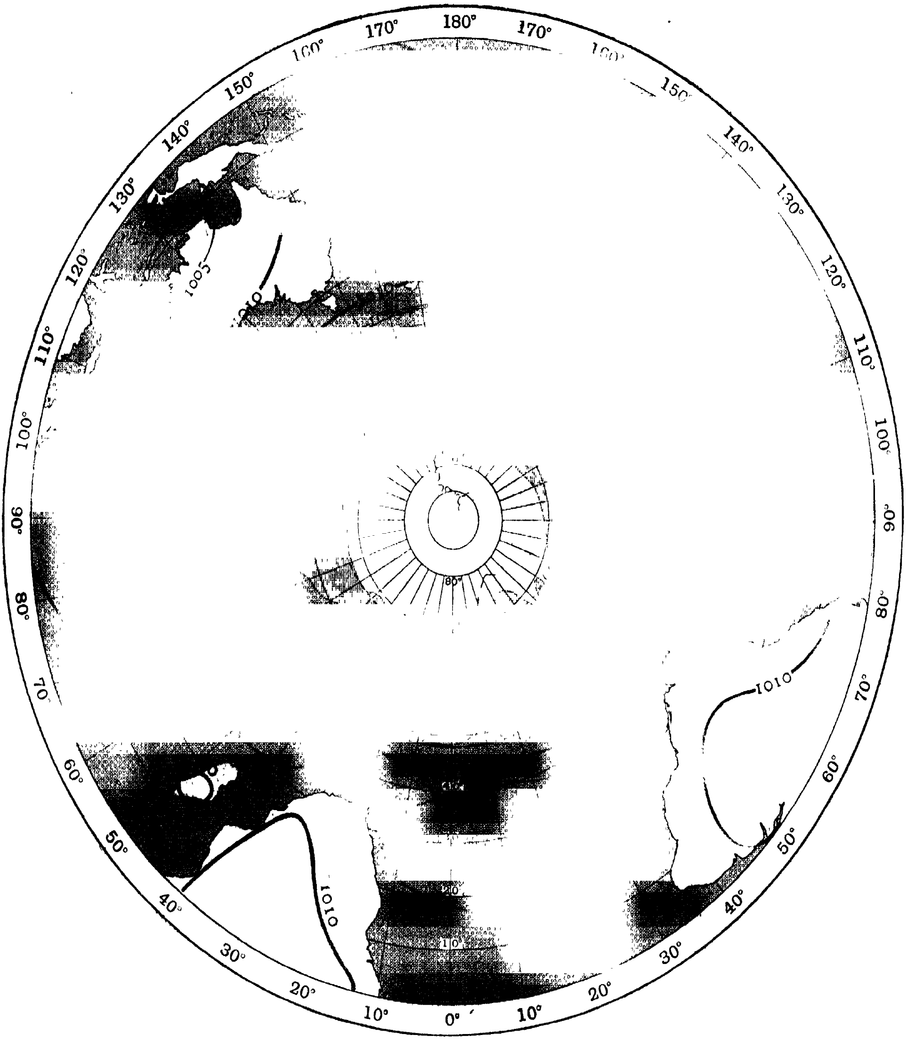


Fig. 5. Normal pressure at sea level, Southern hemisphere, January.

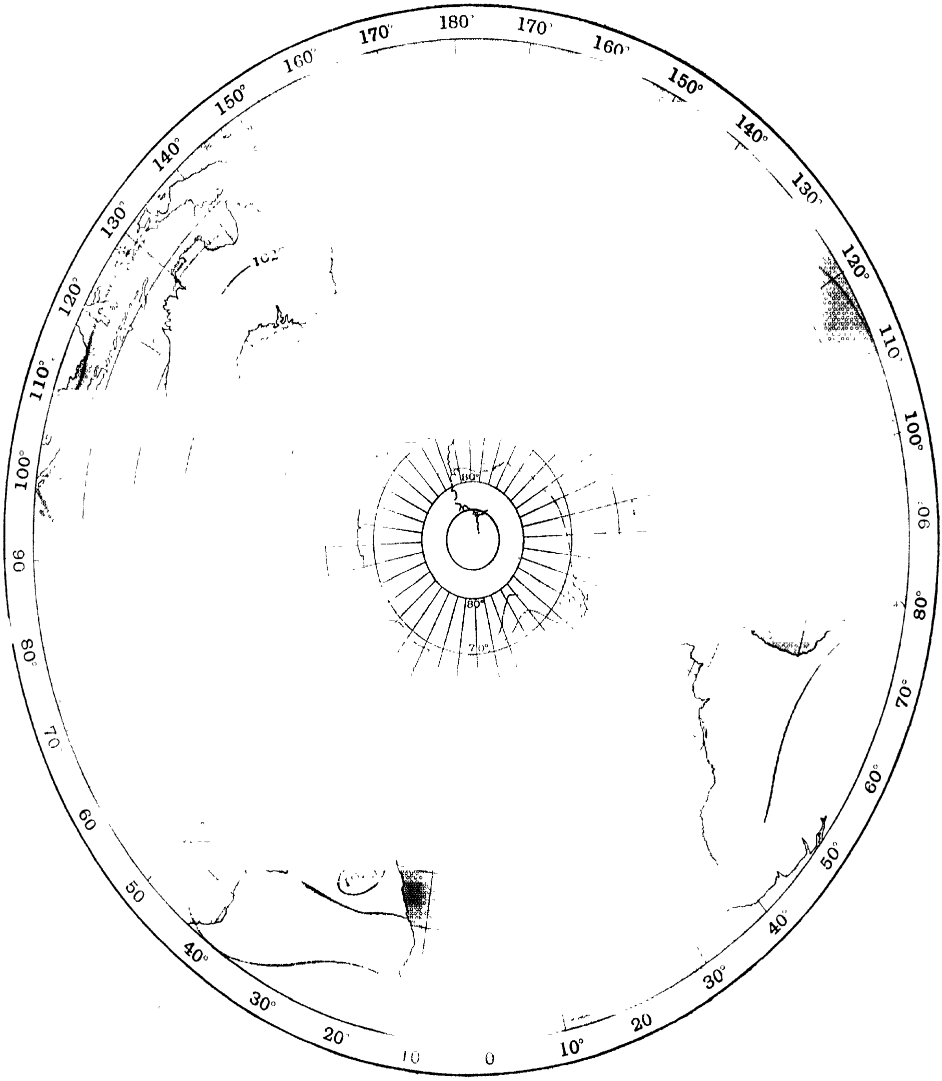


Fig. 6. Normal pressure at sea level, Southern hemisphere, July.

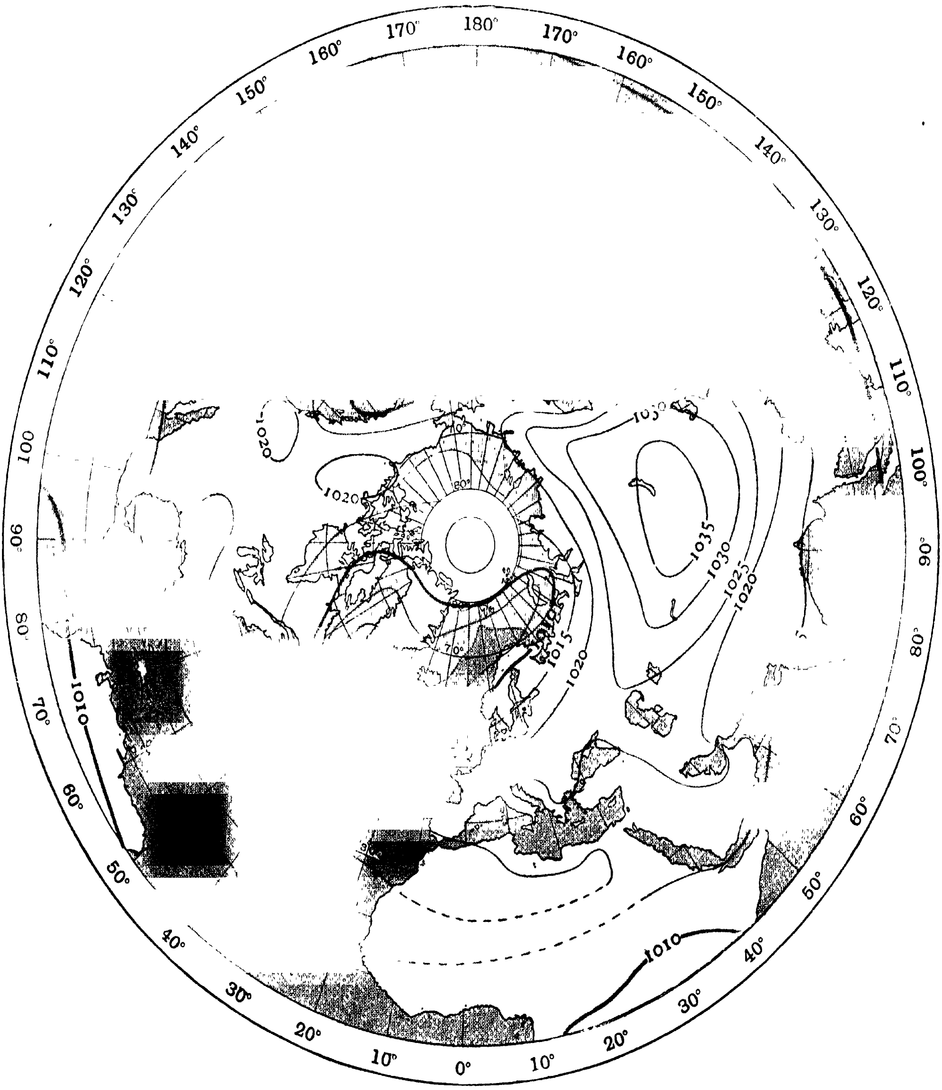


Fig. 7. Normal pressure at sea level, Northern hemisphere, January.

Fig. 8. Normal pressure at sea level, Northern hemisphere, July.

replace the land above sea level by air, the error being greatest in winter, on account of the greater density of the air at low levels at that time. But when we take the differences from month to month, the figures can be accepted as giving an idea of the order of magnitude of the changes which take place from month to month in the amount of air in a hemisphere. Fig. 9, which reproduces the result of a computation carried out by Sir Napier Shaw, shows that the amount of air in the Northern hemisphere has a maximum at mid-summer, and a minimum at mid-winter, the total range of variation being about ten billion tons. (The total amount of air over a hemisphere is about 2700 billion tons.) Shaw made a correction for temperature in the figures which he computed, and the corrected figures are shown in fig. 9.

It is worthy of note that the time of most rapid flow of air from the Southern to the Northern hemisphere is in September to November.

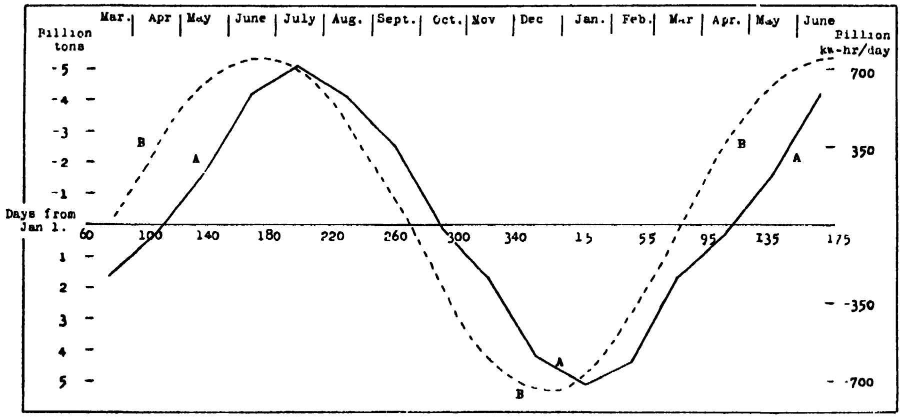


Fig. 9. Annual variation of the mass of air in the Northern hemisphere, shown reversed in curve *AA*, and the corresponding incoming solar energy, shown in curve *BB*.

§ 3. *Wind*

The distribution of winds over the globe can be most simply described by means of the distribution of pressure. There is a relation between the wind and the pressure distribution known as Buys Ballot's law, which may be stated as follows: "In the Northern hemisphere an observer who stands with his back to the wind will have lower pressure to his left than to his right; in the Southern hemisphere the contrary holds". In terms of the isobars on the chart this is equivalent to saying that the wind blows round the isobars keeping low pressure to the left and high pressure to the right in the Northern hemisphere, and the reverse in the Southern hemisphere. In actual practice it is found that the surface wind, while blowing in the general sense thus indicated, blows slightly across the isobars into low pressure, at an angle of 20–30°.

By the use of Buys Ballot's law we can readily interpret the charts of pressure distribution in terms of the prevailing winds. Take first the charts for the

Southern hemisphere. The region south of the sub-tropical high-pressure belt is a region of winds blowing from a westerly direction. North of the sub-tropical high-pressure belt the winds blow from a generally easterly direction, but from a direction somewhat south of east. These winds are the South-east trade winds. The equator is marked by a shallow belt of low pressure, known as the Doldrums, a region of calms and light variable winds. This region is slightly south of the equator in the Southern summer, and north of the equator in the Northern summer, its mean position being north of the equator. Conditions in the Northern hemisphere vary widely from winter to summer. In winter there is a clockwise circulation of winds round the Siberian anti-cyclone, giving over India and China the winds known as the North-east monsoon; while over the Atlantic north of the sub-tropical anticyclone there is a region of prevailing westerly winds, where the average conditions are disturbed by the depressions of middle latitudes. With the approach of summer the Siberian anticyclone disappears and its place as the chief controller of conditions over Asia is taken by a depression centred to the north of India. From the centre of this depression there is a continuous increase of pressure southward to the centre of the sub-tropical high-pressure belt of the South Indian Ocean, and over the whole of the Indian Ocean blows a broad current of wind which starts in the sub-tropical belt of the Southern hemisphere as a south-easterly wind, blows across the equator, and finally appears as a south-westerly wind blowing into the southern edge of the depression over India, where it is known as the South-west monsoon. This air current passes over some thousands of miles of ocean, and so reaches India as a warm and very moist wind. When it reaches India the configuration of the land forces it to rise over the coastal ridges of mountains, so giving rise to copious rainfall. The Asiatic depression, with the associated monsoon wind, develops in June and persists until late September, when the depression fills up, the winds die away, and the rain ceases. The economic importance to India of the monsoon rainfall tends to focus attention on the south-westerly winds which blow over India in summer, but it should be noted that the influence of the summer Asiatic depression extends to China, where the southerly wind which blows around the eastern edge of the depression is known as the Southerly monsoon of the China seas, and to the Mediterranean, where the northerly wind which blows around the western edge of the depression is known as the Etesian wind.

Figs. 10 and 11 indicate the prevailing winds over the globe for January and July. These charts should be examined in conjunction with the charts showing the pressure distribution reproduced in figs. 5–8.

§ 4. *Rainfall*

Rainfall is so extremely variable with place that it is difficult to treat it in a simple manner. It is only over a very restricted portion of the globe that the rainfall of a single year will resemble at all closely the "normal" rainfall. The outstanding case in which this occurs is India, where the maximum rainfall of

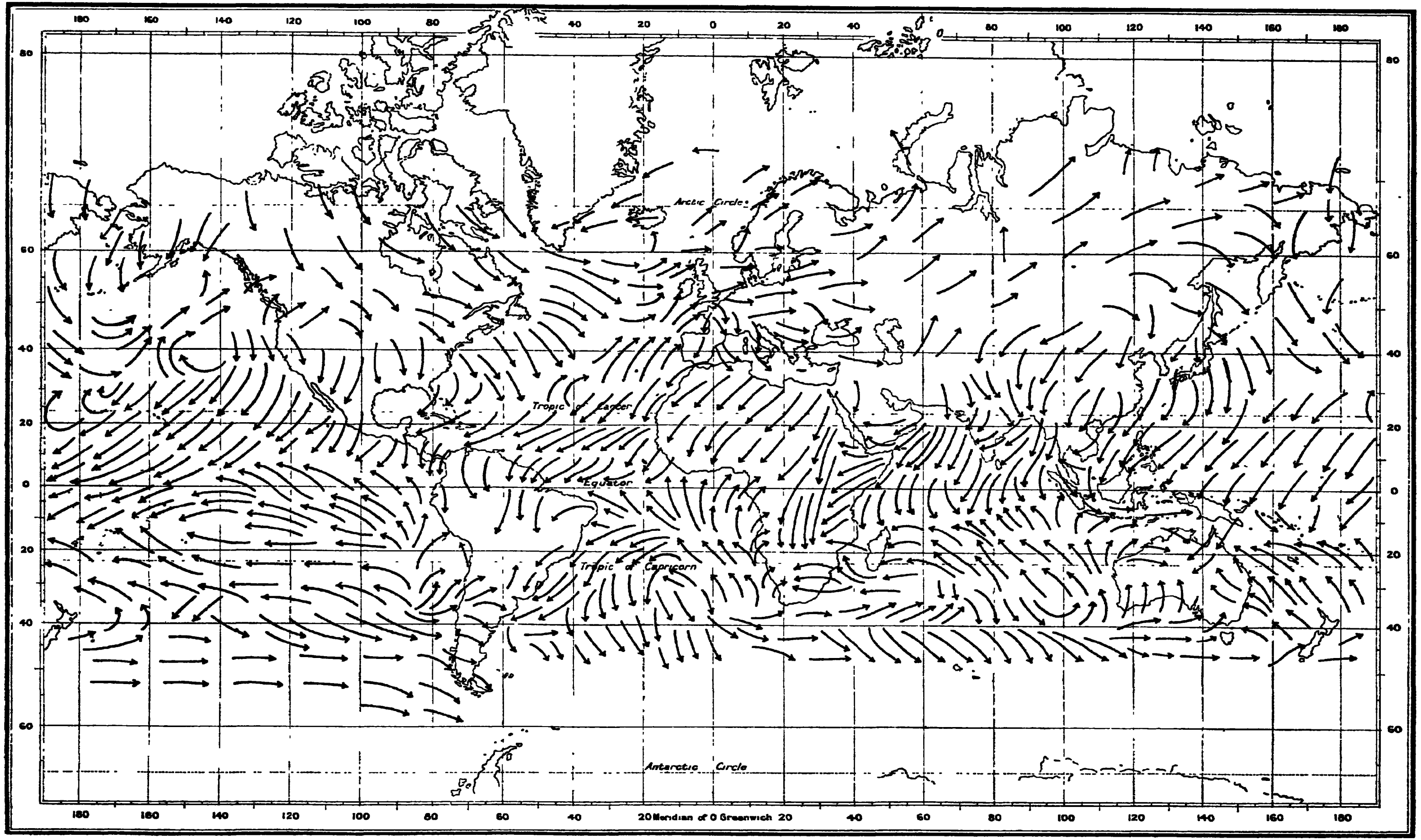


Fig. 10. Winds of the globe, January.

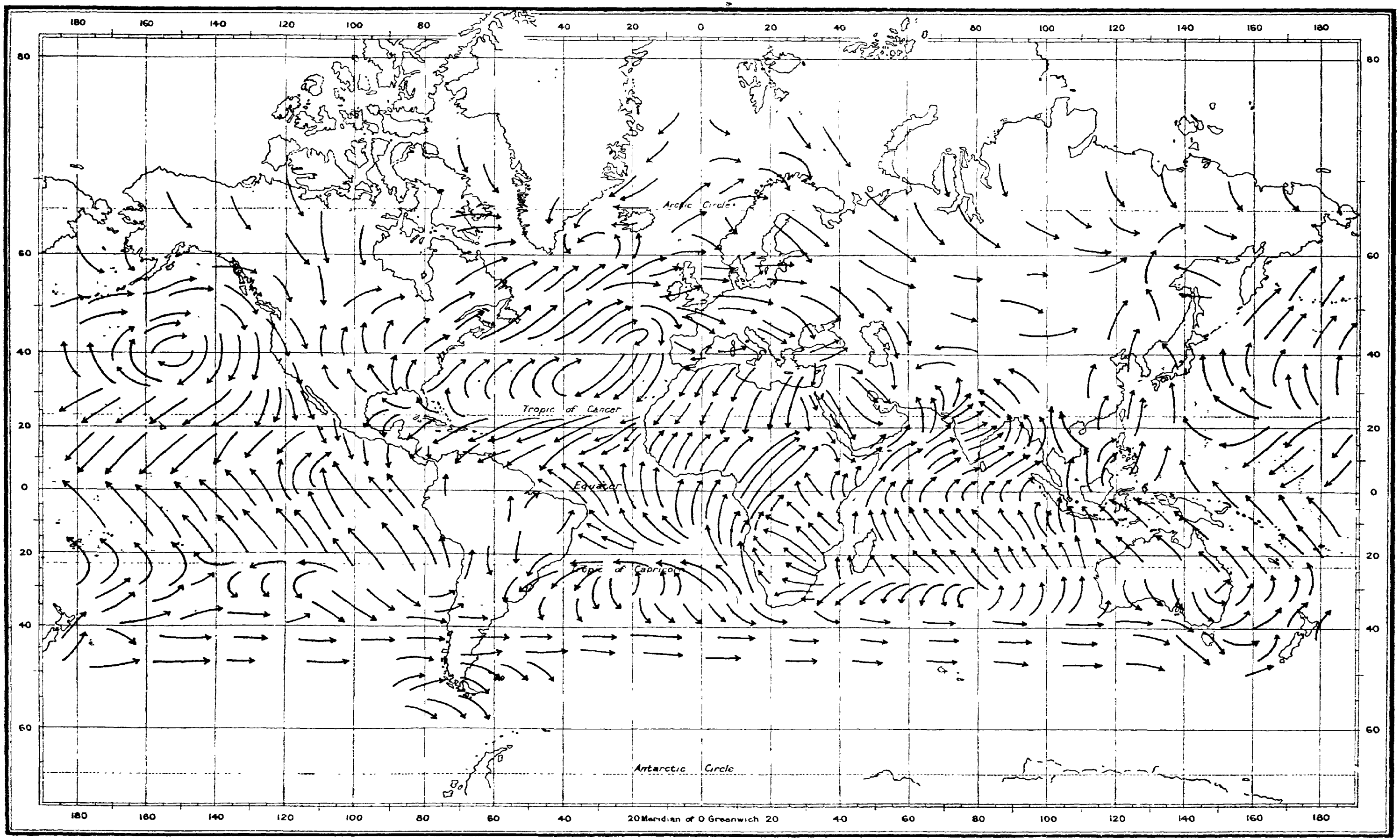


Fig. 11. Winds of the globe, July.

the year always occurs in June to September, in association with the South-west monsoon winds. Within the tropics rain usually has a well-marked maximum, and in some places two maxima; in other words there will be one or two rainy seasons each year. Thus Batavia in Java has its rainiest month in February; Colombo has two rainy months, May and October; Singapore has most rain during November, December and January; Bombay has its maximum rainfall in July, but Madras in November.

In the British Isles the averages taken over many years point to a definite maximum of rainfall in a particular month of the year. This month is October at Greenwich, December at Aberdeen, and July at Edinburgh, but it would be extremely hazardous to apply this result to forecast the rain of any particular year.

The Russian Meteorological Atlas gives two very interesting charts, one showing for the whole of Europe the month of maximum rainfall, and another showing for the same area the month of minimum rainfall. There is a clearly-marked tendency for the month of maximum rainfall to occur in May or June at stations in South-east Europe, but as we go farther westward over the continent the time of maximum rainfall becomes steadily later, and falls in October to December over a large part of the British Isles. The details of the annual variation of rainfall for individual stations can be most readily studied by means of tables of rainfall such as are to be found in Kendrew's *The Climate of the Continents* (Oxford, 1927) or other textbooks of climatology.

§ 5. *The general circulation and the local circulations*

In the preceding paragraphs we have given a brief outline of the average distribution of some meteorological elements as represented by monthly means over a large number of years, and have given a very brief account of the distribution of winds over the globe in the months of January and July. It is scarcely necessary to emphasise the fact that on any one day in either of these months the actual situation, as represented by the distribution of pressure, temperature and wind, may vary considerably from the mean values shown in figs. 1–11. But some of the main features of these charts remain constant from day to day, and so it is found more convenient to adopt the mean situation for the month, or in other words, the *general circulation* for the month, as a standard of reference, and to regard the situations which occur from day to day as deviations from the general circulation. The most important of these deviations are the travelling depressions and anticyclones of middle latitudes, and the tropical cyclones of low latitudes, and to these we may give the name of *local circulations*.

It is clear that the general circulation is not independent of the local circulations. It is made up of the mean conditions over a long period, and so implicitly contains the integrated effect of a large number of local circulations which in part, but only in part, cancel one another. The mean circulation is in fact a climatological normal in precisely the same sense as the mean temperature at a given station.

We shall return to the problems concerned with tropical cyclones and the depressions and anticyclones of middle latitudes in later chapters. In Chapter xx will be found a fuller description of the winds of the globe, taking account of the fact that the atmosphere is three-dimensional, and that the nature of the wind-circulation is different at different heights in the atmosphere.

§ 6. *Surfaces of separation between winds of different origin*

In the charts of winds given in figs. 10 and 11 it can readily be seen that currents of air originating in polar and equatorial regions flow along the surface of the earth. It is possible here and there to draw lines separating the currents of different origin. This was done by T. Bergeron (*Meteorologische Zeitschrift*, July 1930) for a part of the earth. Some of these lines of separation could be readily drawn in figs. 10 and 11, but charts on such a small scale could only show these lines incompletely. For the moment we are only concerned to show that the system of prevailing winds brings into juxtaposition masses of air of widely different temperatures, and that these masses are separated by lines which appear as discontinuities on a chart of this scale. We shall find later that such lines of separation, when they occur in association with the depressions of middle latitudes, are of great importance in determining the growth of depressions and the distribution of cloud and rainfall.

§ 7. *The observed distribution of temperature in the vertical*

The manner in which temperature varies with height is of fundamental importance in determining the processes of weather. On the average, temperature decreases with height at the rate of approximately 6° C per kilometre, or about 3° F per 1000 feet, from the ground up to a considerable height, but eventually a limit is reached at which the normal decrease with height ceases, and above this limit the temperature remains constant, or even increases slightly at first. The atmosphere is thus divided into two thermally distinct regions, the lower, known as the *troposphere*, in which the temperature decreases steadily with height, and an upper region, the *stratosphere*, in which the temperature remains constant or increases slightly with height. The surface of separation is known as the *tropopause*, and is at a height which varies from about 18 km at the equator, to about 11 km over Southern Europe, and to about 6 km at the poles. The height at the poles cannot be specified with accuracy, mainly on account of the paucity of observations in polar regions. The rate of diminution of temperature with height, known as the *lapse-rate*, does not diminish steadily to zero at the tropopause, but retains approximately its normal value right up to the limit of the tropopause, and then suddenly changes to zero or even changes sign.

The temperature within the stratosphere is usually reputed to increase steadily from the equator towards the poles, the generally accepted view being as represented in fig. 12, which is due to Ramanathan*. According to this diagram the coldest air in the whole atmosphere forms a limited ring round the equator in the lower stratosphere. The increase of temperature with height in the stratosphere shown in this diagram is definitely established by observations in low latitudes. The details of the diagram are less certain in high latitudes, though it probably represents the actual situation with reasonable

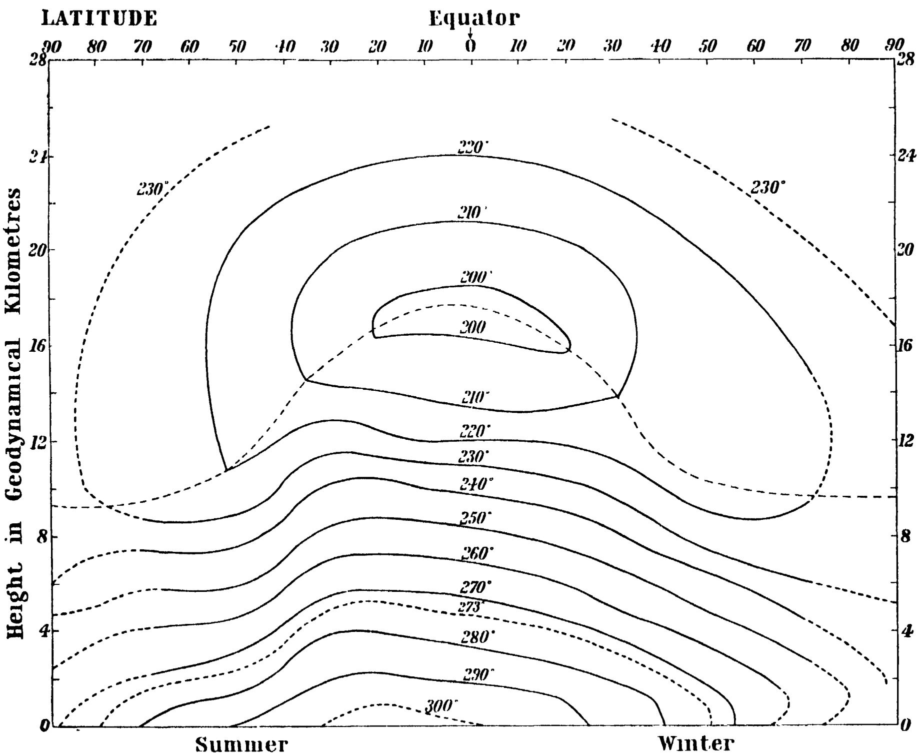


Fig. 12. Distribution of temperature in the atmosphere.

accuracy in summer. It is probable, however, that in the polar regions in winter, in the absence of any incoming solar radiation, the distribution of temperature varies widely from that shown in fig. 12, and that lower temperatures occur there than over the equator, possibly with temperature decreasing with height in the stratosphere.

L. H. G. Dines† has also shown that over the British Isles temperature increases with height in the lower stratosphere to a height of 3 km above the tropopause, followed by a slow decrease of temperature with height in the next 5 km. Thus while fig. 12 may be regarded as a useful first sketch of the

* As modified by Samuels, *Monthly Weather Review*, Sept. 1929.

† *Mem. R. Met. Soc.* **2**, No. 18, 1928.

temperature distribution in the atmosphere, it may be expected that careful study of the details of observations made in different regions will show that the actual temperature distribution is more complicated than is suggested by fig. 12, and possibly deviates widely from symmetry about the earth's axis.

Over North-west Europe the stratosphere is on the average lower and warmer than the normal over low surface pressures, and higher and colder than the normal over high surface pressure. The correlation of the surface pressure with the height and temperature of the tropopause is not very high. It is far from being clear that the correlation is similar over other regions of the globe. The seasonal variation of the temperature of the stratosphere varies considerably with place. Over Europe the stratosphere is higher and warmer in summer than in winter, but over Canada and India it is higher and colder in summer than in winter.

Table 1

Monthly mean temperatures for England on the absolute scale

Height in kilo-metres	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Amp. °C	Date min.
	200° +													
14	16	17	19	21	22	23	22	21	19	17	16	15	2·8	27
13	16	17	19	21	22	23	23	21	19	18	17	16	2·5	14
12	17	18	19	20	21	22	22	21	20	19	18	17	2·2	29
11	17	17	17	19	20	21	22	22	21	20	19	18	2·9	54
10	20	20	20	22	24	25	26	26	26	24	23	21	4·1	56
9	24	23	24	26	29	32	34	33	33	31	28	25	5·2	44
8	30	29	30	32	36	38	41	41	41	38	35	32	6·0	35
7	37	36	37	39	42	45	47	48	47	45	41	38	6·3	30
6	43	43	44	46	49	52	55	55	54	51	49	45	6·6	31
5	50	49	50	52	56	59	61	62	61	58	55	52	6·6	36
4	57	56	57	59	62	65	67	68	67	64	61	58	6·2	35
3	63	62	63	65	68	71	73	74	73	70	67	64	5·2	37
2	67	66	67	70	73	76	78	79	78	75	72	69	5·4	37
1	71	71	73	76	79	82	83	83	81	79	75	72	6·1	35
0	76	76	77	82	85	88	89	89	86	83	80	77	6·2	38

Table 1* gives on the absolute scale the monthly mean temperatures up to 14 km for each month of the year for England. Fitting a sine-curve to the data for each height gives the amplitude, in degrees C, given in the last column but one, and the date of minimum given in the last column, measured in days from Jan. 1st. The amplitude diminishes with height up to 3 km, and then increases, attaining its highest value at about 6 km, beyond which it steadily diminishes until at 11 km the annual range is less than half the annual range at the surface. The date of occurrence of minimum temperature is sensibly constant in the first 6 km, is then steadily retarded up to 10 or 11 km, and at still greater heights becomes earlier with increasing height. It is not claimed that these figures are strictly representative, as they are based upon means of a small number of observations, but the general nature of the changes with height of both the annual range and of the time of maximum agrees with results derived for other places.

* From W. H. Dines, *M.O. Geophysical Memoir*, No. 13.

Table I gives rather a wrong idea of what occurs at the tropopause. On account of the varying height of the tropopause the effect of taking the mean of a number of observations at fixed heights is to mask the sudden change at the tropopause, and to replace it by a gradual transition. Further, the table is based on selected observations mainly restricted to fair-weather conditions.

Individual observations show considerable irregularities near the ground, where the lapse-rate in the lowest layers may be several times the "normal" value on a sunny afternoon, and where at night the normal decrease of temperature may be replaced by a condition in which temperature steadily increases from the ground upward, giving what is known as an *inversion* in the lower layers, above which there is a return to the more normal decrease of temperature with height. At cloud levels and at surfaces of separation of warm and cold currents irregularities in the form of inversions may occur within a narrow range of height. The variations of the lapse-rate at the ground and at cloud levels are discussed later in Chapter VI, in connection with the effects of radiation and turbulence, which afford a rational explanation of the observed phenomena. Excluding these we are left with four outstanding problems in connection with the distribution of temperature in the free air:

1. The approximate constancy of the mean lapse-rate at all heights in the troposphere and in all latitudes.
2. The sudden nature of the change at the tropopause.
3. The approximate constancy of temperature at all heights within the lower stratosphere.
4. The decrease of temperature from pole to equator within the stratosphere.

These problems are discussed in later chapters, but it cannot be said that an adequate explanation has yet been given of any of them.

The temperature at the base of the stratosphere varies from about 196° A over the equator to about 222° A (in summer) and 216° A (in winter) over England. Within the limits of height investigated instrumentally (by meteorographs) the variations of temperature with height in the stratosphere have been found to be relatively small except in the tropics. Above these limits there is a further increase of temperature, and at heights of about 50 km the temperature is again as high as at the surface. This result was deduced by Lindemann and Dobson* from observations of meteors, and is to be explained as an effect of absorption of incoming ultra-violet light by ozone. Whipple† has shown that the travel of sounds from explosions bears out the supposition that the temperature at 50 km is at least 300° A.

For discussions of data of observation up to and within the stratosphere the reader is referred to the following papers, which do not, however, exhaust the available material:

* Lindemann and Dobson, *Proc. Roy. Soc.* A, **102**, 1922, pp. 411–36. See also later papers by Dobson, and by Dobson and others, *Proc. Roy. Soc.* A, **110**, **114**, **120**, **122** and **129**.

† *Q.J. Roy. Met. Soc.* **57**, 1931, p. 331, and **58**, 1932, p. 471.

GOLD, International Kite and Balloon Ascents, *M.O. Geophysical Memoir*, No. 5. (European data.)
GOLD and HARWOOD, *Report to British Association*, Winnipeg, 1909.
RYKATCHEW, *Met. Zeit.* **28**, pp. 1–16. (Pavlovsk.)
WAGNER, *ibid.* pp. 261–5. (Pavia.)
PEPPLER, *Beitr. Phys. fr. Atmos.* **4**. (Pavlovsk and Kutschino.)
J. PATTERSON, *Upper Air Observations in Canada*, Ottawa, 1915.
Monthly Weather Review, Jan. 1918. (Fort Omaha, Indianapolis, Huron, Avalon.)
Travaux Scientifiques de l'Obs. de Met. de Trappes, **4**, 1909, Paris. (North Atlantic.)
VAN BEMMELEN, *K. Mag. en Met. Obs. te Batavia*, 1916, p. 27. (Batavia.)
Annals Astr. Obs. Harvard College, **68**, pt 1, p. 68. (St Louis.)
HILDEBRANDSSON, *Geografiska Annaler*, 1920, p. 110. (North Atlantic.)
BERSON, *Ergebnisse K. Preuss. Aer. Obs. Lindenberg*, 1910, p. 82. (East African Observations made by Berson in 1908.)
HARWOOD, *Mem. Ind. Meteor. Dept.* **24**, pt 6, 1924. (India.)

§ 8. *Correlation between different variables in the free air*

A large amount of work has been done by W. H. Dines* in correlating the variations of pressure and temperature at different heights in the free air, and of the temperature and height of the stratosphere, using European data. The main results of his researches can be readily summarised in the form of a table, in which the following symbols are used:

P_0 the barometric pressure at M.S.L.
P_9 the barometric pressure at 9 km.
T_0 the temperature at the surface.
T_n the temperature at n km.
T_m the mean temperature from 1 km to 9 km.
T_{0-4} the mean temperature from the surface to 4 km.
V the total water-vapour content of the atmosphere.
H_c the height of the tropopause.
T_c the temperature at the tropopause.

	P_0	P_9	T_m	H_c	T_c	V	T_0	T_{0-4}	T_4	T_8
P_0	—	0·68	0·47	0·68	−0·52	0·08	0·16	0·34	—	—
P_9	0·68	—	0·95	0·84	−0·47	—	0·28	—	0·82	—
T_m	0·47	0·95	—	0·79	−0·37	—	—	—	—	—
H_c	0·68	0·84	0·79	—	−0·68	0·39	0·30	0·66	0·64	0·74
T_c	−0·52	−0·47	−0·37	−0·68	—	—	—	—	—	—
V	0·08	—	—	0·39	—	—	—	0·73	—	—
T_0	0·16	0·28	—	0·30	—	—	—	—	—	—
T_{0-4}	0·34	—	—	0·66	—	0·73	—	—	—	—
T_4	—	0·82	—	0·64	—	—	—	—	—	—
T_8	—	—	—	0·74	—	—	—	—	—	—

A number of interesting conclusions can be drawn from the above table. High pressure at 9 km is accompanied by

(*a*) high pressure at the ground,
(*b*) warm air from 0 to 9 km,
(*c*) high tropopause, and
(*d*) low temperature at the tropopause.

This is effectively equivalent to saying that the stratosphere is high and cold

* W. H. Dines, *M.O. Geophysical Memoirs*, Nos. 2 and 13.

over high surface pressures, and low and warm over low surface pressures. Again high mean temperature in the lowest 9 km is associated with high pressure and high and cold stratosphere. Indeed, the highest coefficient in the table above is that of 0·95 between T_m and P_9. High values of V, the total water-vapour content of the atmosphere, are associated with high stratosphere and high mean temperature in the lowest 4 km. There is practically no correlation between pressure and temperature at the surface.

In the same paper Dines gives the correlation coefficients between temperature and pressure at the same heights. The coefficients for heights 0–13 km were as follows:

0	1	2	3	4	5	6	7	8	9	10	11	12	13
0·11	0·42	0·66	0·77	0·84	0·85	0·86	0·86	0·86	0·71	0·32	−0·19	−0·36	−0·28

Thus the correlation between temperature and pressure at the ground is negligible, but increases to a maximum of 0·86 at levels of 6–8 km, then diminishes and becomes negative in the stratosphere.

Schedler* also computed a number of correlation coefficients similar to those given above, but in the majority of cases his values are definitely lower than those given by Dines.

Dines also gave a table of the standard deviations of temperature, pressure and density at different heights from the ground up to 13 km. The standard deviations of temperature were least in the summer quarter and greatest in the autumn quarter. They increased from the surface up to a level between 5 km and 7 km, and then decreased, but again increased at 12 and 13 km. The standard deviations of pressure were least in the summer quarter and greatest in the winter quarter. They showed no marked change from the surface up to a level of about 10 km, after which they definitely decreased. The standard deviations, expressed as percentages for each level, increased from the ground upward, to three times the surface value at 13 km, but expressed in gm/m^3 they showed a decrease from the ground up to about 6 km, and an increase beyond this level.

Some discussion of synchronous variations of temperature and pressure at different heights will be found later in Chapter XVIII, § 189.

§ 9. *The diurnal variation of meteorological factors*

(*a*) TEMPERATURE

The nature of the diurnal variation at the ground is sufficiently well known to require little discussion at the present stage. Surface temperature shows a clearly defined maximum at about two hours after noon, and a minimum shortly before sunrise. The amplitude of the diurnal variation differs for different localities, and for different times of the year, being much greater on the average in summer than in winter in England. Shaw gives in the *Manual of Meteorology*, **2**, pp. 84, 85, a chart for each hemisphere showing the varia-

* Schedler, *Beitr. Phys. fr. Atmos.* **7**, p. 88.

tion of diurnal range of temperature over t continent there is an extensive region wit for the year is 17° C (30° F) or more. Ove is only a small fraction of this, and amo degrees F. But the annual mean values n the diurnal variation at different times of gives the mean daily range for a few ty months of the year (in °C).

	Jan.
Arctic (1894–6)	0·7
Aberdeen	1·5
Calcutta	9·4
Batavia	4·7
Cape of Good Hope	5·8
Cape Evans (77° S, 167° E)	2·8
Paris (Parc St Maur)	3·3
Eiffel Tower	1·1

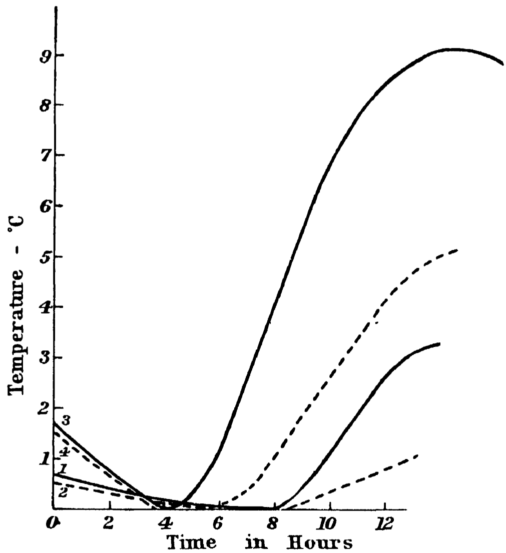


Fig. 13. Diurnal range of tempe
of the

The actual diurnal range observe will depend largely on the weather a clear calm night will yield a r average, while a cloudy day follow variation of temperature during

In the free air the daily rang height. Fig. 13 gives a comparis and the top of the Eiffel Towe diurnal variation falls off rapic how rapidly. Observations m

ans certain that the observations are not to
of the sun shining on the instrument. In
ı conclusively that with increasing height
time of maximum is retarded. This is in
the top of the Eiffel Tower, as shown in
tions appear to indicate that in summer at
e maximum occurs at mid-day, while in
these heights is at about 8 h. The diurnal

	Summer	Winter
	4·0° C	1·8° C
	0·56	0·46

aper by J. Durward* on the "Diurnal
:ity and Cloudiness".

.RIATION OF PRESSURE

has been the subject of very comprehensive
Simpson.§ In the tropics the characteristic
ble wave whose maxima occur at 10 h and
shows up clearly in a barograph trace on
ame features are shown at places in the
he individual days do not, except in quiet
iation, on account of the disturbance due
:lones; and averages over a year or more
characteristic form of the curve. This
ɔe vaguely described as a kind of wave
advance of the sun.
iation of pressure shows a combination
vhose times of maximum occur at the
latitude, though this time varies with
the times of maxima at stations north
:en 10 h 30 m and 13 h 30 m G.M.T.
uly the diurnal variations of pressure
of Meteorology, **2**) for Batavia, the
metres above M.S.L.). The time scale
ıberdeen and Ben Nevis, and local

ᐠTION OF WIND

ɔf the surface wind over the land
ɔn, and a minimum in the early
round the maximum wind occurs

263, and **128**, 1919, p. 379.
lémoires pt 1, and 1906, pt 1.

about midnight, and the minimum at about 10 h. Fig. 15
variation for the base and top of the Eiffel Tower.

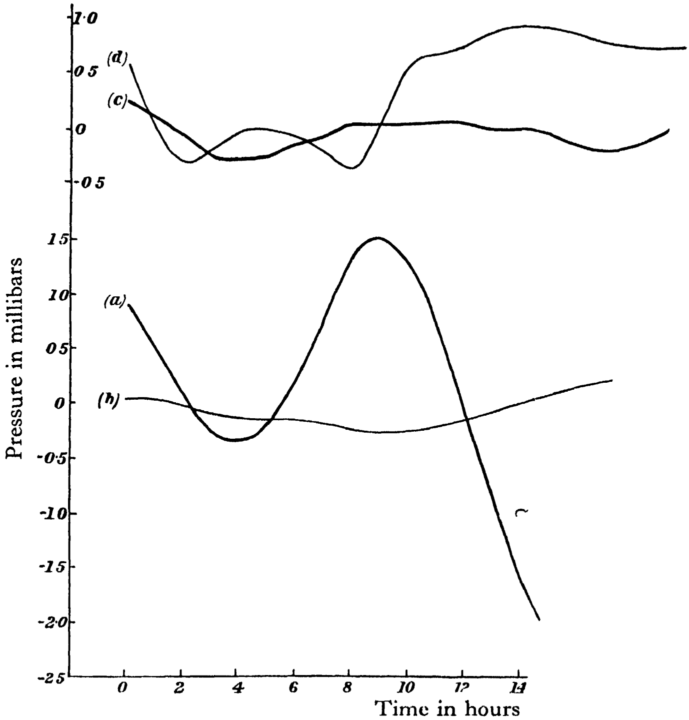


Fig. 14. Diurnal variation of pressure at Batav
and on Ben Nevis. (*a*) Batavia; (*b*) Arctic; (

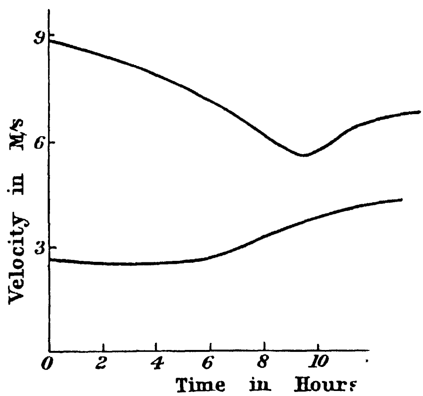


Fig. 15. Diurnal variation of win
the Eiff

The nature of the transition from o
At intermediate heights two maxima
midnight. With increasing height t

,eases, and finally the night maximum only remains. The e two maxima are of equal intensity will depend on the rface wind. Some actual observations bearing on this have y Hellmann,* who set up three anemometers at heights of tres above the ground in a flat meadow at Nauen. The results rvations are given in the paper referred to. Hellmann found winds were blowing all three anemometers showed a pro-ı in the early afternoon, and a minimum during the night. ere light the anemometer at 2 metres showed the usual day ıt minimum; the anemometer at 16 metres showed two ıidday and midnight, with minima in the morning and nemometer at 32 metres showed the same distribution of the midnight maximum being the greater. At 16 metres ust about equal in winter, but in summer there was a aximum to exceed the night maximum. Thus with light ıich the night maximum has become equal to the day winter, and is between 16 and 32 metres in summer. ight is greater than 32 metres at all seasons.

rmed by an examination of the records of wind at anemometer placed 41 metres above the ground. imum always occurred in the middle of the day, er. With light winds, in winter the maximum is in summer there is a weak maximum in the middle ximum in the middle of the night.

bservations made at Leafield† indicates that in s show a maximum during the day for light (6–9 m/s) winds, but a maximum at midday rong (> 9 m/s) winds. In summer light winds ate winds a day and a night maximum, and ım.

ed the diurnal variation of wind at greater lloon ascents. Durward‡ used 1736 ascents ade by the Meteorological Service with the March 1917 and September 1918. The feet agreed well with those derived from f the Eiffel Tower, yielding a maximum tween 9 a.m. and 10 a.m. At 2000 and ıe in the morning and one about 9 p.m. ward, 1101 reached 6000 feet. When all o appreciable diurnal variation at 4000 adrants differed considerably from one ed a minimum between 9 h and 10 h, ximum between those hours. Easterly

'. *Roy. Met. Soc.* **57**, 1931, p. 434.

winds showed a diurnal variation similar to the surface winds, while northerly winds showed no strongly marked feature, giving a weak minimum at night.

Tetens* analysed the upper wind observations at Lindenberg. In summer the winds at 4 km show a maximum in the early morning (2–4 h) and a minimum at about 14 h. At 3 km the main maximum occurs at about 6 h, and the minimum at about 14 h, but there is a secondary maximum about 16 h. At still lower levels there are complicated systems of maxima and minima, but at the ground the maximum occurs soon after noon. In winter the main maximum at 4 km and 3 km occurs from 16 h to 18 h, and the main minimum at 8 h.

At places near the sea coast there is usually a pronounced tendency for a wind to blow from the sea to the land in the morning, and from the land to the sea in the late evening, giving the phenomena known as the land and sea breezes. These are considered more fully later in Chapter XIV.

Over the sea the diurnal variation of the wind is much less marked than over the land. Gallé† states that during the months May to October the maximum velocity of the South-east trade winds of the Indian Ocean occurs during the night hours. No detailed observations at one place are available, and it is not possible to describe with certainty the diurnal variation of winds over the sea.

(*d*) Diurnal variation of cloudiness

The diurnal variation of cloud amount depends to some extent on the wind direction, and to a very considerable extent on the type of cloud. At Kew Observatory, Richmond, and at certain stations in the Rhine Valley, Brunt‡ showed that there is a marked tendency for cloud amount to diminish in the evening. A similar investigation by Dines and Mulholland§ for Valentia Observatory, South-west Ireland, showed a general tendency for the cloudiness to be greatest at 7 h, and to diminish steadily during the day, but the number of clear skies did not increase, though the number of overcast skies diminished. At Batavia,‖ 6° S of the equator, cloudiness attains its maximum during the evening or early night, and its minimum about 4 a.m. local time. At Helwan¶ cloudiness attains its maximum during the afternoon and its minimum during the evening.

(*e*) Diurnal variation of vapour-pressure and relative humidity

At inland stations vapour-pressure shows no very marked diurnal variation. At Kew, for example, during a month selected at random (July 1928), the mean vapour-pressure showed a maximum of 14·3 mb at about 18 h, and a minimum of 12·6 mb at about noon. On account of the small variation of vapour-pressure relative humidity shows a variation in the opposite sense to temperature.

* *Lindenberg Arbeiten*, **14**, 1922.
† *K. Ned. Met. Inst.* No. 102.
‡ *Professional Notes*, M.O., Nos. 1 and 14.
§ *Ibid.* No. 36.
‖ *Observations Batavia*, **38**, 1915.
¶ *Survey Dept. Cairo, Meteorological Report*, 1910.

CHAPTER II

SOME STATICAL AND THERMAL RELATIONSHIPS

§ 10. *The earth*

THE earth is a spheroid whose equatorial radius is 6378·2 km (3963 miles), and whose polar radius is 6356·5 km (3950 miles), so that it only differs slightly from a sphere. In what follows we shall in general neglect the deviation from the spherical form, except where this is explicitly stated.

The earth rotates on its axis once in 24 sidereal hours, and the rate of angular rotation, usually denoted by ω, is $7{\cdot}29 \times 10^{-5}$ radians per second.

Horizontal distances on the earth's surface will be given in kilometres, miles, nautical miles, or any other unit which may be convenient. The relation between these units is added here for convenience:

$$1 \text{ km} = 0{\cdot}62137 \text{ mile},$$

$$1 \text{ mile} = 1{\cdot}6093 \text{ km}.$$

The nautical mile, as used in hydrographical surveying, is identical with the geographical mile, and is defined as the length of a meridian arc of one minute of latitude. Its length varies with latitude on account of the spheroidal shape of the earth. If we denote the latitude by ϕ, the length of the geographical mile is

$$(6076{\cdot}8 - 31{\cdot}1 \cos 2\phi) \text{ feet}$$

or

$$(1852{\cdot}2 - 9{\cdot}5 \cos 2\phi) \text{ metres}.$$

Thus its length is 6045·7 feet (1842·7 metres) at the equator, and 6107·9 feet (1861·7 metres) at the poles.

The acceleration of gravity (plus the centrifugal acceleration due to the earth's rotation, see § 94 below), usually denoted by g, varies with latitude and with height above mean sea level. To a high degree of approximation the value of g at mean sea level in latitude ϕ may be represented by the formula

$$980{\cdot}617\,(1 - 0{\cdot}00259 \cos 2\phi) \text{ cm/sec}^2.$$

At a height z above mean sea level the value of g in the free air is equal to the mean sea level value multiplied by $E^2/(E+z)^2$, or to a sufficient degree of approximation by $(1 - 2z/E)$, E being the radius of the earth.

Thus the value of g at height z in latitude ϕ is

$$\begin{aligned} g &= 980{\cdot}617\,(1 - 0{\cdot}00259 \cos 2\phi)\,(1 - 2z/E) \\ &= 980{\cdot}617\,(1 - 0{\cdot}00259 \cos 2\phi)\,(1 - 3{\cdot}14 \times 10^{-7} z), \text{ where } z \text{ is in metres}, \\ &= 980{\cdot}617\,(1 - 0{\cdot}00259 \cos 2\phi)\,(1 - 9{\cdot}57 \times 10^{-8} z), \text{ where } z \text{ is in feet}. \end{aligned}$$

If the isostatic compensation for elevation were perfect, the above formulae would also hold at places on mountains. If there were no such compensation, the value of g would be given by

$$g = 980{\cdot}617\,(1 - 0{\cdot}00259 \cos 2\phi)\,(1 - 5z/4E)$$
$$= 980{\cdot}617\,(1 - 0{\cdot}00259 \cos 2\phi)\,(1 - 1{\cdot}96 \times 10^{-7}z), \text{ where } z \text{ is in metres,}$$
$$= 980{\cdot}617\,(1 - 0{\cdot}00259 \cos 2\phi)\,(1 - 5{\cdot}97 \times 10^{-8}z), \text{ where } z \text{ is in feet.}$$

It is probable that the isostatic compensation is only partial, so that the variation of g with height should be represented by a term intermediate between $(1 - 2z/E)$ and $(1 - 5z/4E)$. The International Meteorological Tables give the latter form, which should therefore be used in reductions of pressure to mean sea level.

§ 11. *Geopotential*

The geopotential at a point at a height z above the surface of the earth is the potential energy of unit mass placed at that point. It is equal to the work done in lifting unit mass from the surface (mean sea level) up to that point, and is therefore equal to $\int_0^z g\,dz$. If we may neglect the variation of g with height the geopotential is equal to gz, where g now represents the value appropriate to the latitude.

In the C.G.S. system the unit of geopotential, which is equal to the potential of unit mass raised through unit distance in a field of force of unit strength, is $1 \text{ cm} \times 1 \text{ cm/sec}^2$. V. Bjerknes has advocated the use of a unit 10^5 times the C.G.S. unit; and has suggested for this unit the name "dynamic metre". The name "leo" has also been suggested for this unit, and would be definitely preferable, as it avoids the confusion which is inevitable if we use metres and dynamic metres side by side. But all modern writers use the name "dynamic metre". The dynamic metre has dimensions L^2T^{-2}, the dimensions of (velocity)^2.

The vertical height interval separating two surfaces whose geopotentials differ by one dynamic metre is $10^5/g$ cm, and is thus approximately $10^5/981$ cm, or 1·02 metres.

If g_s is the surface value of g, then at height z in the free air

$$g = g_s \frac{E^2}{(E+z)^2} = g_s\,(1 + z/E)^{-2},$$

$$\int_0^z g\,dz = g_s z/(1 + z/E).$$

The error involved in using the surface value of g is equivalent to neglecting z/E by comparison with unity. Since E is over 6000 km, even at a height of 20 km the error is only of the order of 1/300.

The variation of g with latitude is represented by the factor $(1 - 0{\cdot}00259 \cos 2\phi)$. The maximum error introduced by the adoption of a mean value for g in all latitudes is only 0·26 per cent, or approximately 1/400. At first sight it may appear that such a small error is negligible, but since the pressure equivalent of a column of mercury of given length is proportional to g, an error of 1/400th in the value of g at the surface will lead to an error of about $2\frac{1}{2}$ mb in the estimate of pressure, an amount which is by no means negligible.

Table I, p. 404, gives the conversion from metres to dynamic metres, for different latitudes and for different heights.

§ 12. *Composition of the atmosphere*

The atmosphere is a mixture of gases, of which oxygen and nitrogen account for about 99 per cent, other constituents being argon, carbon dioxide, hydrogen, neon, helium, krypton and xenon, together with varying quantities of water-vapour. Apart from the water-vapour the other constituents are present in such unvarying proportions, except for local pollution due to factory chimneys and similar sources, that we may treat dry air as a uniform mixture, and leave entirely out of consideration all question of its constitution. It is only when we come to the nature of the atmosphere at very high levels which are inaccessible to direct observation that we need consider seriously the precise constitution of the air.

But while we may assume the homogeneity of the air as regards its constitution, we may not assume homogeneity of temperature or content of water-vapour. In respect to these two factors we shall find that the atmosphere shows remarkable variation from point to point, so that it is not a simple matter to decide the limits within which accuracy of observation is possible or desirable.

§ 13. *The fundamental gas equation*

If we denote pressure, absolute temperature, density and specific volume by p, T, ρ and v respectively, Boyle's law states that for a perfect gas

$$p = R\rho T \qquad \text{......(1)}$$

or

$$pv = RT \qquad \text{......(2),}$$

where R is the gas constant appropriate to that gas. If the pressure is expressed in millibars and ρ in kilogrammes per cubic metre, or v in cubic metres per kilogramme, the value of R for dry air is 2·8703. If p is in millibars and ρ in grammes per cubic centimetre, or v in cubic centimetres per gramme, the value of R for dry air is $2{\cdot}8703 \times 10^3$. If p is measured in dynes per square centimetre and ρ in grammes per cubic centimetre, or v in cubic centimetres per gramme, the value of R is $2{\cdot}8703 \times 10^6$.

Van der Waals gave a modified form

$$(p + a/v)(v - b) = RT \qquad \text{......(3),}$$

which holds more closely than form (2). Over the range of temperatures with which we are normally concerned in meteorology the error involved in using equation (2) instead of equation (3) is negligible, and we shall use Boyle's law in the forms (1) or (2) above without further modification.

§ 14. *The density of water-vapour and of damp air; virtual temperature*

The specific gravity of water-vapour as compared with that of dry air at the same temperature and pressure is 0·6221, which may be taken as 5/8 (0·625) for all practical purposes. If the pressure of water-vapour be e and its absolute temperature be T, the density is

$$\frac{5}{8}\frac{e}{RT} \qquad \text{......(4)},$$

where R is the constant for dry air.

If now p is the total pressure of damp air, in which the partial pressure of the water-vapour is e, the partial pressure of the dry air alone is $p-e$. The density of the mixture is therefore

$$\frac{p-e}{RT}+\frac{5}{8}\frac{e}{RT}$$

or

$$\frac{p}{RT}\left(1-\frac{3e}{8p}\right) \qquad \text{......(5)}$$

or p/RT', where T' is the virtual temperature defined by

$$T'=T/(1-\tfrac{3}{8}e/p) \qquad \text{......(6)}.$$

Thus the virtual temperature of the damp air is the temperature at which dry air of the same pressure would have the same density as the damp air.

To a high degree of approximation, in dealing with damp air, we may use equations (1) or (2) in the form

$$p=R'\rho T \quad \text{or} \quad pv=R'T,$$

where $R'=R/(1-\tfrac{3}{8}e/p)=R\,(1+\tfrac{3}{8}e/p)$ approximately.

But this equation must only be used at temperatures which are sufficiently high to permit of the air remaining always unsaturated.

§ 15. *Vapour-pressure and its control by temperature*

In a closed space which contains a supply of water the amount of water-vapour is independent of the presence of other gases. In other words, the pressure of the water-vapour is independent of the pressure of the dry air. This law is known as Dalton's law. For a given temperature there is a maximum limit to the vapour-pressure, attained when the air is "saturated". The saturation vapour-pressure increases with temperature, and an equation connecting these two variables can be derived by the use of the second law of thermodynamics. This equation is derived in § 53 below.

Fig. 16 shows the variation of vapour-pressure with temperature. This diagram represents the saturation vapour-pressure over water for temperatures above freezing point, and over ice for temperatures below freezing point. The vapour-pressure is slightly greater over water than over ice at temperatures below freezing point. For example, at 250° A (−23° C) the pressure over ice is 0·77 mb, that over water 1·77 mb. According to the Smithsonian Meteorological Tables the vapour-pressure over ice falls to 0·021 at $-55°$ C, to 0·0054 mb at $-65°$ C, and to 0·0026 mb at $-70°$ C. The saturation vapour-pressure is greater when the water is present in the form of drops. A table of saturation vapour-pressures at different temperatures is given on p. 405.

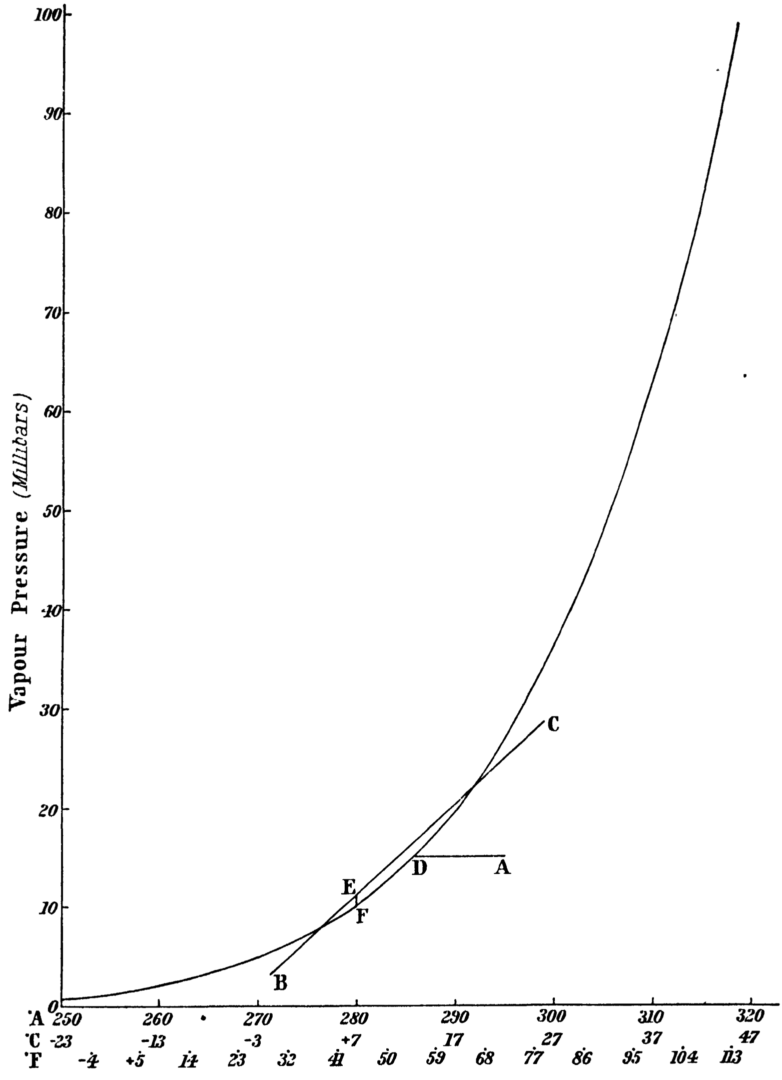


Fig. 16. Saturation vapour-pressure as a function of temperature.

The figures quoted above refer to saturated air. When air is unsaturated, the vapour-pressure is less than the saturation value, and the ratio of the actual vapour-pressure to the saturation value gives the relative humidity, which is most conveniently expressed as a percentage figure. Another way of specifying the state of air as regards content of water-vapour is by means of the "humidity mixing ratio", usually denoted by x, which expresses the relative proportions by weight of dry air and water-vapour in a given volume of the damp air. This is equivalent to saying that damp air is made up of x grammes of water-vapour to every gramme of dry air. Then

$$x = 0{\cdot}622\,\frac{e}{p-e} = 0{\cdot}622\,\frac{e}{p} \text{ approximately} \qquad \text{......(7).}$$

The value of x remains unchanged for a given mass of air, so long as there is no evaporation or condensation, or mixing with other air of different humidity. So long as x remains constant the relative humidity will rise or fall as the temperature falls or rises, since the vapour-pressure remains constant while the saturation vapour-pressure varies in the same sense as the temperature. The humidity mixing ratio x is so useful in the discussion of the thermodynamics of moist air that it merits some further consideration. If we call the specific gravity of water-vapour ϵ $(=0{\cdot}622)$, then

$$\frac{x}{\epsilon} = \frac{e}{p-e}, \quad 1 + \frac{x}{\epsilon} = \frac{p}{p-e}.$$

Let v be the specific volume of the dry air alone, then

$$(p-e)\,v = RT,$$
$$ev = xRT/\epsilon,$$
$$pv = (1 + x/\epsilon)\,RT \qquad \text{......(8).}$$

The density of the dry air alone is $1/v$, and the density of the mixture of dry air and water-vapour is $(1+x)/v$. If ρ be the density of the mixture, then

$$\rho = (1+x)/v = \frac{p}{RT}\,\frac{1+x}{1+x/\epsilon} = \frac{p}{RT'} \qquad \text{......(9).}$$

(See equation (6).) Hence

$$T' = T/(1 - \tfrac{3}{8}e/p) = T\,\frac{1 + x/\epsilon}{1+x} \qquad \text{......(10).}$$

If we prefer to define the density as $p/R'T$, then

$$R' = R/(1 - \tfrac{3}{8}e/p) = R\,\frac{1 + x/\epsilon}{1+x} \qquad \text{......(11).}$$

The value of x for a given value of e and p is readily computed. In fig. 17, p. 58, will be found a series of lines which show the values of x for saturation at different temperatures and pressures. The numbers there given are the numbers of grammes of water-vapour per kilogramme of dry air, and these numbers must be divided by 1000 in order to obtain x; e.g. for 10 grammes read 0·01.

§ 16. *Pressure and its variation with height*

In practice the meteorologist measures pressure by means of a mercury barometer, and deduces the pressure from the height of a column of mercury. The pressure is in fact equal to the weight of a column of mercury of unit cross-section and of the height of the observed column. To reduce the pressure to a form comparable with a fixed standard, allowance is made for the variation of gravity with latitude, and for the expansion or contraction of the mercury and the scale, by methods which are described in the *Meteorological Observer's Handbook*, or any descriptive textbook.

The pressure with which we are concerned is, however, not a length. Its dimensions are those of force/area, MLT^{-2}/L^2, or $ML^{-1}T^{-2}$. It may be measured in terms of any convenient unit of length, such as the inch or millimetre, or, preferably, the scale may be graduated so as to give pressure in terms of a unit such as the millibar, which is defined as a pressure of 1000 dynes per square centimetre.

By definition the pressure at any point is the weight of a vertical column of air of unit cross-section, whose base is centred at the given point, and which extends to the top of the atmosphere. Let p, ρ and T be the pressure, density and absolute temperature at height z in the atmosphere, and let $p+dp$ be the pressure at height $z+dz$. Then by definition, dp is the weight of a disc of air of unit cross-section and of height dz. Hence

$$dp = -g\rho\, dz \quad \text{or} \quad \frac{\partial p}{\partial z} = -g\rho \qquad \text{......(12).}$$

This is the fundamental statical equation of meteorology, and will be continually required in any physical discussion. It could be integrated if ρ could be expressed as a function of z. This is not possible, and it is convenient to transform the equation by the use of equation (1) above,

$$p = R\rho T.$$

Substituting for ρ in equation (12) we find

$$\frac{1}{p}\frac{\partial p}{\partial z} = -\frac{g}{RT} \qquad \text{......(13).}$$

This equation is readily integrable when T is constant, or can be represented as a function of z. The variation of g with height can be neglected in this connection. If T is constant, equation (13) yields on integration

$$\log_e p_0 - \log_e p = gz/RT \qquad \text{......(14),}$$

p_0 being the pressure at $z=0$. This may be written

$$z = \frac{RT}{g}(\log_e p_0 - \log_e p)$$

$$= \frac{RT}{g \log_{10} e}(\log p_0 - \log p) \qquad \text{......(15).}$$

If pressure is measured in millibars and height in feet,

$$z = 221 \cdot 1\, T (\log p_0 - \log p) \qquad \text{......(16).}$$

If pressure is measured in millibars and height in metres,

$$z = 67 \cdot 4\, T (\log p_0 - \log p) \qquad \text{......(17).}$$

Equation (14) or any of the three corresponding forms (15), (16) or (17) will give the relation of height and pressure in an atmosphere which has a uniform temperature T.

In an atmosphere in which the temperature decreases uniformly with height at a rate β, the temperature at height z may be represented by the expression $T_0-\beta z$. Substituting this for T in equation (13) we find

$$\frac{1}{p}\frac{\partial p}{\partial z}=-\frac{g}{R(T_0-\beta z)} \qquad \text{......(18).}$$

On integration this yields

$$\log p=\frac{g}{R\beta}\log(T_0-\beta z)+\text{const.}$$

or

$$\log\frac{p}{p_0}=\frac{g}{R\beta}\log\frac{T_0-\beta z}{T_0},$$

or

$$\frac{T_0-\beta z}{T_0}=\left(\frac{p}{p_0}\right)^{\frac{R\beta}{g}}=\left(\frac{p}{p_0}\right)^{0\cdot 293\beta} \qquad \text{......(19),}$$

if β is measured in degrees C per 100 metres.

§ 17. *Barometric altimetry*

(*a*) Isothermal atmosphere

Equations (16) or (17) above may be used to determine the height z at which a particular pressure p is observed, assuming some standard value of the temperature T. The ordinary altimeter is an aneroid barometer, with a scale graduated in inches of mercury or in millibars, with a movable height scale engraved on the outer edge of the dial, the graduation being based on the assumption that the mean temperature is 50° F or 10° C. Variations in ground pressure are allowed for by rotating the height scale until its zero is opposite the actual ground pressure. This method is strictly correct, since according to equation (14), which may be written

$$z=\frac{RT}{g}\log_e\frac{p_0}{p},$$

the height interval from pressure p_0 to pressure p is always the same for the same value of T, whether the ground pressure is p_0 or not.

The most important correction which has to be made to this type of altimeter is that for the variation of the mean temperature from the standard value of 50° F or 10° C. The atmosphere is never strictly isothermal through any considerable range of height, and it is therefore not strictly permissible to use equation (14). It is, however, frequently used, giving T the mean value of the temperature as observed from the ground up to the level z. Then z is proportional to the mean absolute temperature T. The rule for correction for variations from the standard temperature 10° C is readily seen to be: "For every 1° C increase of temperature above the standard value, increase the height as estimated by the altimeter by 1/283 of this value; for decrease of temperature below the standard value subtract the correction from the estimated height".

Allowance can be made for the humidity of the atmosphere by taking the virtual temperature instead of the ordinary temperature.

The use of equation (14) with the mean observed temperature for T is not strictly permissible, since the atmosphere is never even approximately isothermal except through a very restricted range of height, but its use can be justified as theoretically sound if we use for T the harmonic mean of the observed temperatures, taken at equal intervals of height.

(*b*) I.C.A.N. ATMOSPHERE

The International Commission for Air Navigation has put forward a specification of a standard atmosphere with a view to uniformity in the estimation of aircraft performances. Up to 11 km this standard atmosphere is specified by equation (19) above, with $T_0 = 288°$ A, and the lapse-rate equal to 6·5° C per kilometre of height. Equation (19) then becomes

$$\frac{p}{p_0} = \left(\frac{288 - 0{\cdot}0065z}{288}\right)^{5{\cdot}256} \qquad \ldots\ldots(20),$$

where z is now measured in metres.

Above 11 km the temperature is assumed constant at −56·5° C. For heights above 11 km we therefore use equation (14) which now takes the form

$$\log \frac{p_{11{,}000}}{p} = \frac{z - 11{,}000}{14{,}600} \qquad \ldots\ldots(21).$$

If the altimeter height is z' and the true height z, at a level where the pressure is p and the temperature T, and if p_0 is the sea-level pressure, a correction can be deduced to z', assuming the lapse-rate to have the standard value 6·5° C per kilometre. Equation (20) becomes

$$\frac{p_0}{p} = \left(\frac{T + {\cdot}0065z}{T}\right)^{5{\cdot}256} = \left(\frac{288}{288 - {\cdot}0065z'}\right)^{5{\cdot}256}.$$

By simple algebra we find

$$z - z' = \frac{z'(T - 288 + 0{\cdot}0065z')}{288 - 0{\cdot}0065z'},$$

or

$$z - z' = z'\left(\frac{T}{288 - 0{\cdot}0065z'} - 1\right) \qquad \ldots\ldots(22).$$

§ 18. *The specific heat of air*

The specific heat of a substance is defined as the thermal capacity per unit mass, or, in other words, the ratio of the quantity of heat required to raise the temperature of unit mass of it by a given amount to the quantity of heat required to raise the temperature of unit mass of water by the same amount. In practice the unit of heat adopted, the gramme-calorie, is the amount of heat required to raise the temperature of 1 gramme of water by 1° C. Thus if the specific heat of a substance is c, the amount of heat required to raise the temperature of 1 gramme of it by t degrees is ct calories.

The specific heat thus defined only has a definite meaning when the conditions under which the heating takes place are defined with precision. If the substance in question is a gas, and it is allowed to expand when heated, work is done against the external pressure, and some of the heat supplied is used in doing this work. The specific heat is thus greater when expansion is possible than it is when the volume is kept constant. In connection with gas it is customary to speak of two specific heats, that at constant volume, and that at constant pressure. These are denoted usually by the symbols c_v and c_p respectively. The ratio of these two quantities is also a physical constant of much importance; c_p/c_v is usually denoted by the symbol γ. It is possible to determine γ independently of the determination of c_p and c_v, since γ enters into a number of physical relations. For example, the velocity of sound in a gas whose temperature is T is $\sqrt{\gamma RT}$, where γ and R are the appropriate constants for that gas.

The specific heats of dry air have been determined experimentally by a great number of physicists. A full table of such determinations will be found in Partington and Shilling's book on the *Specific Heats of Gases*. In the same work will be found a table of the values of the specific heats which the authors regard as the most reliable. The values of the constants for dry air given in these tables are

$$c_p = 0{\cdot}2396, \quad c_v = 0{\cdot}1707, \quad \gamma = 1{\cdot}403.$$

The units in which c_p and c_v are given are gramme-calories. In the present work we shall adhere to these values.

The specific heats of a gas are related by a simple equation which we shall now derive. Suppose a quantity of heat dQ is supplied to unit mass of air, whose initial absolute temperature, pressure and specific volume are T, p and v respectively. Some of the heat is used in raising the temperature by an amount dT, while the remainder is used in doing work against the external pressure during the process of expansion. If the specific volume is increased by an amount dv, the work done in expanding against the pressure is $p\,dv$. This is equivalent to an amount of heat $Ap\,dv$, where A is the reciprocal of the mechanical equivalent of heat. It follows that

$$dQ = c_v dT + Ap\,dv \qquad \text{......(23).}$$

This equation states that

Heat added = Increase in internal energy
+ work done against external pressure.

The internal energy of unit mass of air is taken to be $c_v T$, whether the volume remains constant or not, during the series of changes of state under consideration. This question will be found discussed in detail in any textbook of heat, e.g. Preston's *Heat*, Chapter VIII, p. 662.

By Boyle's law $\quad pv = RT$ and $p\,dv + v\,dp = R\,dT$,

hence

$$dQ = (c_v + AR)\,dT - Av\,dp \qquad \text{......(24).}$$

If the expansion takes place under the condition that the pressure remains constant,

$$dQ = c_p dT = (c_v + AR)\,dT \qquad \text{......(25).}$$

Hence $c_p = c_v + AR$ or $c_p - c_v = AR$ ......(26).

Equation (24) may therefore be written

$$dQ = c_p dT - Av\,dp \qquad \text{......(27).}$$

§ 19. *The adiabatic equation*

If a mass of air expands or contracts without addition or subtraction of heat, the quantity $dQ = 0$. Equation (27) then becomes

$$c_p dT - Av\,dp = 0,$$

$$c_p dT - \frac{ART}{p}\,dp = 0,$$

$$c_p dT/T - (c_p - c_v)\,dp/p = 0 \qquad \text{......(28).}$$

If we write $c_p/c_v = \gamma$, equation (28) becomes

$$\gamma dT/T - (\gamma - 1)\,dp/p = 0 \qquad \text{......(29),}$$

which yields on integration

$$T^\gamma/p^{\gamma-1} = \text{constant} \qquad \text{......(30).}$$

This equation may be transformed into another form by substitution for T

$$\frac{(pv)^\gamma}{p^{\gamma-1}} = \text{constant},$$

or

$$pv^\gamma = \text{constant} \qquad \text{......(31).}$$

Note the values $\frac{c_p}{AR} = \frac{c_p}{c_p - c_v} = \frac{\gamma}{\gamma - 1} = 3{\cdot}49.$

§ 20. *Potential temperature*

The potential temperature of dry air is defined as the temperature it would attain if brought adiabatically to a standard pressure (which may conveniently be taken as 1000 mb). Let the standard pressure be p_0, and let the temperature and pressure of a mass of air be initially T and p respectively. When the air is brought to a pressure p_0 without addition or subtraction of heat, let it take up a temperature θ; i.e. let θ be the potential temperature of the air. Then by equation (30) above

$$\frac{T^\gamma}{p^{\gamma-1}} = \frac{\theta^\gamma}{p_0{}^{\gamma-1}}, \quad \text{or} \quad \theta = T\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} \qquad \text{......(32)}$$

or

$$\log\theta - \log T = \frac{\gamma - 1}{\gamma}(\log p_0 - \log p)$$

$$= 0{\cdot}288\,(\log p_0 - \log p) \qquad \text{......(33).}$$

This equation may be used to evaluate θ. Alternatively we may use equation (32) in the form

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = T\left(\frac{p_0}{p}\right)^{0{\cdot}288} \qquad \text{......(34).}$$

A table of values of $\left(\frac{p_0}{p}\right)^{0\cdot 288}$ for $p_0 = 1000$, for a range of values of p from 10 to 1090, is shown in Table II at the end of this volume (p. 404). By the use of :his table θ is readily computed on a slide-rule. The potential temperature can ılso be read directly from a tephigram (see p. 73).

§ 21. *The adiabatic lapse-rate; vertical stability for dry air*

The variation of pressure in the vertical direction in the atmosphere, when :here is no acceleration in the vertical, is given by the relation

$$\partial p/\partial z = -g\rho$$

ılready deduced in § 16. The effect of motion on the truth of this relation will e considered later in Chapter VIII, in connection with the derivation of the equations of motion, but we shall find that the statical relation above is true o a very high degree of approximation.

To determine whether the atmosphere is stable or not, consider what ıappens to any small element of air which is displaced through a small vertical listance dz from its initial height z. If in its new position it is subjected to orces which tend to restore it to its original level, the atmosphere is said to be n stable equilibrium; if in its new position it is subjected to forces which tend o displace it still further from its original position, the atmosphere is said to e in unstable or labile equilibrium; while if the element of air in its new osition is subject to no forces tending either to restore it to its original position, or to displace it still further from its original position, the equilibrium is aid to be neutral.

Since a moving element of air will at all stages of its ascent take up automatically the pressure of its immediate surroundings, the relative densit he moving element and of its environment will be determined by the ab emperatures.

Let p, ρ and T be the pressure, density and absolute temperature at a t height z in the atmosphere, and let p, ρ' and T' be the correspon bles for a small element of air which is moving vertically upwar vhen it is at the level z. In moving from a height z to hei z, l lement of air have its pressure changed to $p+dp$, an mperatu e to $T'+dT'$; at this height $z+dz$ the environment will the same pressure $p+dp$, but its temperature will be $T+dT$. If the moving mass of air neither eceives nor loses heat during its displacement, it follows from equation (29) bove that

$$\frac{dT'}{T'} = \frac{\gamma-1}{\gamma}\frac{dp}{p},$$

nd

$$\frac{\partial T'}{\partial z} = \frac{\gamma-1}{\gamma}\frac{T'}{p}\frac{\partial p}{\partial z} = -\frac{\gamma-1}{\gamma}g\rho\frac{T'}{R\rho T} = -\frac{g}{R}\frac{\gamma-1}{\gamma}\frac{T'}{T} = -\frac{gA}{c_p}\frac{T'}{T} \quad \text{......(35).}$$

Thus the temperature of a mass of air moving in the vertical decreases with eight at a rate equal to $\frac{gA}{c_p}\frac{T'}{T}$ per unit of height. If the temperature T' of the moving air does not differ appreciably from the temperature T of the environment, the rate of decrease of temperature with height is gA/c_p. If, for example, we desire to consider the changes of temperature in an element of air displaced vertically from its original position as part of the normal environment, we have to take $T'=T$, and the rate of change of temperature with height is then given accurately by gA/c_p. We shall represent the dry adiabatic lapse-rate by the symbol Γ. Substituting for g, A and c_p, we find that

$$\Gamma=\frac{gA}{c_p}=0{\cdot}0000986 \qquad \text{......(36).}$$

This is equal to the rate of decrease of temperature per unit of height; i.e. per centimetre. It is more convenient to express the result as 0·986° C per 100 metres or to a sufficient degree of approximation as 1° C per 100 metres. This is known as the *dry adiabatic lapse-rate.* It is the rate of decrease of temperature with height of a mass of air moving through an environment whose temperature does not differ by a finite amount from its own. The qualification as to the equality of the temperatures of moving air and environment is of importance; when there is a finite difference of temperature between the moving air and its environment, the factor T'/T must be taken into account in the rate of decrease of temperature with height.

In a dry atmosphere whose lapse-rate had the value Γ at all heights, any element of mass displaced in the vertical would have the temperature of its environment at all stages of its ascent. In any atmosphere whose lapse-rate differed from the dry adiabatic rate, any element displaced from its original level would initially have its temperature diminished at a rate Γ and would deviate from the temperature of its environment. When the displacement had ...tended through a finite range of height, it would be necessary to take into ...nt the factor T'/T in the original equation for the diminution of tem- ...e with height. (See also § 128 below.)

...e temperature in the atmosphere be T at height z, and $T+dT$ at $z+dz$, and a small element of air be displaced from height z to height ...s temperature in its displaced position will be $T+dT'$, where ... In the displaced position this air will be heavier or lighter than me... nvironment according as its temperature is lower or higher than that of its ...nment, i.e. according as

$$dT > \text{ or } < dT',$$

or according as

$$\frac{\partial T}{\partial z} > \text{ or } < -\frac{gA}{c_p},$$

i.e. as

$$-\frac{\partial T}{\partial z} < \text{ or } > \Gamma \qquad \text{......(37).}$$

If it is heavier than its new environment, it will tend to sink back to its original level, so that it is initially stable. If it is lighter than its new environment, it will tend to move further from its original position, and the original condition is then unstable. If it has the same density as the new environment, it will stay in its displaced position, and the atmosphere is then in "neutral" equilibrium. The argument needs a slight re-arrangement if the displacement is downward, but leads to the same result.

Thus a dry atmosphere is in stable, unstable, or neutral equilibrium, according as the lapse-rate is less than, greater than, or equal to the dry adiabatic lapse-rate.

From the definition of potential temperature it is seen that the above condition for stability can be stated as follows: The atmosphere is in stable, unstable, or neutral equilibrium according as the potential temperature increases, decreases, or is constant with increasing height. This result can be readily derived algebraically as follows: Differentiating equation (34) with respect to z, we find

$$\frac{1}{\theta}\frac{\partial\theta}{\partial z}=\frac{1}{T}\frac{\partial T}{\partial z}-\frac{\gamma-1}{\gamma}\cdot\frac{1}{p}\frac{\partial p}{\partial z}$$

$$=\frac{1}{T}\left(\frac{\partial T}{\partial z}+\frac{\gamma-1}{\gamma}\cdot\frac{g}{R}\right).$$

But since $\gamma=c_p/c_v$, and $c_p-c_v=AR$,

$$\frac{\gamma-1}{\gamma}\frac{g}{R}=\frac{g}{R}\frac{c_p-c_v}{c_p}=\frac{gA}{c_p}=\Gamma \qquad \text{......(38).}$$

Hence

$$\frac{1}{\theta}\frac{\partial\theta}{\partial z}=\frac{1}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right) \qquad \text{......(39).}$$

Combining the condition for stability given in (37) above with the relationship shown in equation (39), we find the condition for stability may be stated in the form: The atmosphere will be in stable, unstable, or neutral equilibrium according as

$$\partial\theta/\partial z >,\quad <,\quad \text{or}\quad =0 \qquad \text{......(40),}$$

i.e. according as the potential temperature increases, decreases, or remains constant with height.

In deriving equation (35) above it was assumed that a mass of air of temperature T' moved through an environment whose temperature was T, and it was found that the rate of change of temperature of the moving air with height was proportional to T'/T. This formula will apply to the discussion of the penetration of a limited mass of air through its environment. It is not here suggested that convection in the atmosphere is to be regarded as simple penetration of this kind. The general question of convection is discussed later in § 128, p. 216, and the aim of the present discussion is to establish the results shown in (37) and (40) above, that the limiting condition for stability of dry air is the dry adiabatic lapse-rate, whose magnitude is Γ or gA/c_p, which to a high degree of approximation is equal to 1° C per 100 metres.

§ 22. *The approximate computation of potential temperature*

Equation (35) above shows that the rate of change of temperature of a mass of air moving upward or downward through the atmosphere depends on the ratio of the temperatures of the moving air and its environment. For this reason it is not possible to compute the potential temperature of air by adding to its absolute temperature 1° C for every 100 metres of height, since the rate of increase of temperature will be greater than the dry adiabatic lapse-rate as soon as there is a difference of temperature between the descending air and its environment. The error is small, however, if the height is relatively small, and it is important to obtain an estimate of the limit within which it is possible to apply this approximate method.

The atmosphere will be assumed to have a lapse-rate β per unit height. If at a height z the temperature is T, and the pressure p, then by equation (19) above

$$\frac{T}{T+\beta z}=\left(\frac{p}{p_0}\right)^{\frac{R\beta}{g}}=\left(\frac{p}{p_0}\right)^{0\cdot 288\frac{\beta}{\Gamma}} \qquad \ldots\ldots(41),$$

where Γ is the dry adiabatic lapse-rate.

Let a small element of air be brought down from pressure p at height z to pressure p_0 at $z=0$, and let its final temperature at $z=0$ be θ'. Then by equation (34)

$$\frac{\theta'}{T}=\left(\frac{p_0}{p}\right)^{0\cdot 288}=\left(\frac{T+\beta z}{T}\right)^{\frac{\Gamma}{\beta}} \qquad \ldots\ldots(42),$$

$$\begin{aligned}\theta' &= T\left(\frac{T+\beta z}{T}\right)^{\frac{\Gamma}{\beta}}=T\left(1+\frac{\beta z}{T}\right)^{\frac{\Gamma}{\beta}}\\ &= T\left\{1+\frac{\Gamma}{\beta}\frac{\beta z}{T}+\frac{1}{2}\frac{\Gamma}{\beta}\left(\frac{\Gamma}{\beta}-1\right)\left(\frac{\beta z}{T}\right)^2+\text{etc.}\right\}\\ &= T+\Gamma z+\text{term in } z^2 \text{ and higher powers.}\end{aligned}$$

The term in z^2 will amount to 1° C when

$$\tfrac{1}{2}\Gamma(\Gamma-\beta)\frac{z^2}{T}=1,$$

or $z=3$ km approximately. At this height the later terms of the series are negligible. At greater heights the series converges more slowly and the approximation becomes poor. But up to a height of about 3 km we can with safety use a formula

$$\theta'=T+\Gamma z=T+\frac{z\text{ (in metres)}}{100} \qquad \ldots\ldots(43).$$

If the surface pressure differs appreciably from the usually accepted standard pressure 1000 mb, θ' will not be the potential temperature, and a further change of pressure from p_0 to 1000 mb must be considered. Let θ be the true potential temperature. Then

$$\begin{aligned}\frac{\theta}{\theta'} &= \left(\frac{p_0}{1000}\right)^{-0\cdot 288}=\left(1-\frac{1000-p_0}{1000}\right)^{-0\cdot 288}\\ &= 1+0\cdot 288\times 10^{-3}(1000-p_0) \text{ approximately,}\end{aligned}$$

and

$$\begin{aligned}\theta &= (T+\Gamma z)\{1+0\cdot 288\times 10^{-3}(1000-p_0)\}\\ &= (T+\Gamma z)\{1+0\cdot 0003(1000-p_0)\} \qquad \ldots\ldots(44).\end{aligned}$$

This equation will give the potential temperature with all the accuracy required in most cases up to heights of the order of 2–3 km. Above that height the accuracy of the formula rapidly diminishes, and at 10 km the error may amount to as much as 10° C or more. At heights of 20 km it may yield potential temperatures nearly 100° C too low. It should be noted that in all cases where the atmosphere is stable, having a lapse-rate less than the dry adiabatic, the potential temperature exceeds the absolute temperature at the level where $p=1000$ mb by more than 1° C per 100 metres of height. In all cases where the height exceeds about 3 km it is advisable to use the accurate equation (34) above in conjunction with Table II, p. 404, or the tephigram of fig. 22.

§ 23. *Stability in a moist atmosphere (unsaturated)*

If the atmosphere contains water-vapour, the above argument requires very careful reconsideration. If the composition of the damp air is uniform, the proportion of water-vapour being uniform throughout, then it is only necessary to define the specific heats c_p and c_v as the constants appropriate to the damp air. Let x be the humidity mixing ratio, and let c_p' be the specific heat of water-vapour. Then the specific heat at constant pressure of the damp air will be

$$c_p''=\frac{c_p+xc_p'}{1+x} \quad \text{or} \quad c_p=\frac{x(c_p-c_p')}{1+x}.$$

For values of x which can occur in the atmosphere the difference between this expression and c_p is small, and the adiabatic lapse-rate which fixes the limit for stability will be for all practical purposes the same as for a dry atmosphere, 1° C per 100 metres.

We have seen that the gas constant for damp air is R', or $R/(1-\frac{3}{8}e/p)$, where R is the gas constant for dry air. Then the density of the damp air is $p/R'T$. So long as there is neither evaporation nor condensation the value of R' remains unchanged. For under these conditions the humidity mixing ratio x remains constant, and since

$$x=0{\cdot}622\frac{e}{p-e}=0{\cdot}622\frac{e/p}{1-e/p},$$

it follows that this involves that e/p also remains constant, and so R' is constant.

Now suppose two masses of different water-vapour content are separated by a horizontal surface of separation, the different variables being distinguished by the use of the suffixes 1 and 2 for the lower and upper respectively. At the boundary the upper mass must have the lower density, if there is to be equilibrium. Hence

$$R_1'T_1<R_2'T_2.$$

If the lower has the greater water-vapour content, or the higher value of R', then it follows that $T_1<T_2$, or the upper mass has the higher temperature. This is in accord with the well-known fact that an inversion is frequently found at the upper boundary of a cloud. It appears from the analysis above that the inversion is a statical necessity, if stability is to exist. A cloud layer which had a layer of dry air above it could not remain in equilibrium unless the dry air were warmer than the cloud layer immediately below it, and a mass

of rising air which is damper than its environment must come to rest with warmer air above it. If the rising air is nearly saturated, while the general environment has relative humidity 60 per cent, the inversion required is about 1° F, if the temperature is about 50° F. This is the usual order of magnitude of the observed inversions at the upper surfaces of clouds.

If the upper mass is damper than the lower mass, or $R_1' < R_2'$, it is possible, though not necessary, for T_1 to be less than T_2, i.e. for the damp mass to be colder than the dry air beneath it. This is in keeping with the observed fact that floating clouds are often slightly colder than the surrounding air at the same level, though observed differences appear to be slightly greater than can be explained by the above analysis.

The special case discussed above emphasises the need for careful consideration of the distribution of water-vapour in any discussion of stability. In the example considered the condition for stability is not a lapse but an inversion. The question is, however, amenable to discussion in quite general terms.

Let the air be initially in equilibrium in the vertical, and let R be the gas-constant, e the vapour-pressure, p the total pressure, and T the absolute temperature at height z. Let a small element of mass originally at height z be displaced to a height $z+dz$, where its pressure is p', no heat being communicated to or extracted from it in the process. The argument of § 21 can be repeated here with R' substituted for R, and the specific heat at constant pressure of the damp air substituted for c_p. This specific heat we have already seen to be

$$\frac{c_p + xc_p'}{1+x} = c_p'' \text{ say.}$$

Then the temperature of the element of air which we are considering, after its displacement from height z to $z+dz$, will be

$$T' = T - \frac{gA}{c_p''}dz \qquad \text{......(45).}$$

Its density at its new level will be

$$\frac{p'}{R'T'} = \frac{p'}{R'T}\left(1 + \frac{gA}{c_p''T}dz\right) \qquad \text{......(46)}$$

if we neglect dz^2 and higher powers. The density of its environment will be

$$\frac{p'}{R'T}\left(1 - \frac{1}{R'}\frac{\partial R'}{\partial z}dz - \frac{1}{T}\frac{\partial T}{\partial z}dz\right) \qquad \text{......(47).}$$

Note that since R' is a function of height its variation with height must be considered. If the atmosphere is stable, the density given by (46) must be greater than that given by (47), for dz positive. Also

$$\frac{gA}{c_p''T} > -\frac{1}{R'}\frac{\partial R'}{\partial z} - \frac{1}{T}\frac{\partial T}{\partial z},$$

or

$$-\frac{\partial T}{\partial z} < \frac{gA}{c_p''} + \frac{T}{R'}\frac{\partial R'}{\partial z} \qquad \text{......(48).}$$

If R' is constant at all heights, the second term on the right-hand side of equation (48) disappears, and the limiting lapse-rate is

$$gA/c_p'' \quad \text{or} \quad gA\frac{1+x}{c_p+xc_p'}.$$

Since

$$x = 0{\cdot}622\frac{e}{p-e},$$

it follows that the value of x will seldom exceed about 0·04 and the value of c_p'' may be assumed to be c_p without serious error.

Returning to equation (48) above, we may now assume that the first term on the right-hand side is in practice equal to the dry adiabatic lapse-rate. We can find the order of magnitude of the term $\frac{1}{R'}\frac{\partial R'}{\partial z}$ as follows: To a high degree of approximation

$$R' = R\,(1+\tfrac{3}{8}e/p).$$

The larger scale variations of R' with height will be due to large variations of e with height. Again

$$\frac{\partial R'}{\partial z} = \tfrac{3}{8}R\frac{\partial}{\partial z}\left(\frac{e}{p}\right) = \frac{3}{8}\frac{R}{p}\frac{\partial e}{\partial z} \text{ approximately,}$$

and to a high degree of approximation

$$\frac{1}{R'}\frac{\partial R'}{\partial z} = \frac{3}{8}\frac{1}{p}\frac{\partial e}{\partial z}.$$

Inequality (48) may thus be written

$$-\frac{\partial T}{\partial z} < 0{\cdot}0001 + \frac{3}{8}\frac{T}{p}\frac{\partial e}{\partial z} \qquad \text{......(49).}$$

We shall assume $T/p = 300/1000 = 0{\cdot}3$ as sufficiently near for the purpose of evaluating the order of magnitude of the terms. The two terms on the right-hand side of equation (48) or (49) will then be equal when

$$\frac{\partial e}{\partial z} = 0{\cdot}0001 \times \frac{8}{0{\cdot}9} = 0{\cdot}001 \text{ mb/cm} = 1 \text{ mb/10 metres.}$$

It is evident that the effect of a decrease of e with height, corresponding to dry air above damp air, makes the limiting condition for stability a lapse-rate less than the dry adiabatic, and in special cases even an inversion. When e increases with height, the limiting condition for stability is a lapse-rate greater than the dry adiabatic.

If T_v is the virtual temperature, then by its definition

$$RT_v = R'T$$

and the limiting condition for stability is given by

$$-\frac{1}{T_v}\frac{\partial T_v}{\partial z} = -\frac{1}{T}\frac{\partial T}{\partial z} - \frac{1}{R'}\frac{\partial R'}{\partial z} < \frac{gA}{c_p''T} \qquad \text{......(50).}$$

If we may assume that c_p and c_p'' defined above are equal, this condition is that the lapse-rate of the virtual temperature shall equal the dry adiabatic lapse-rate multiplied by T_v/T. Since this factor is in practice very nearly unity, condition (50) effectively means that the condition for stability is that the lapse-rate of virtual temperature shall be equal to the dry adiabatic lapse-rate.

§ 24. *The effect of vertical motion on the lapse-rate*

Let a thin layer of air be raised or lowered adiabatically, from a level where the pressure is p to a level where the pressure is p'. Then since the amount of air in the layer is unchanged, and the potential temperature of the air is also unchanged, $\partial\theta/\partial p$ remains unaltered by the transfer. For if $\Delta\theta$ and Δp be the differences of potential temperature and pressure at the top and bottom of the layer, neither of these quantities is altered by the motion, and thus $\partial\theta/\partial p$, which is the limiting value of $\Delta\theta/\Delta p$ when the layer is made indefinitely shallow, is also unaltered. Hence $\frac{1}{\theta}\frac{\partial\theta}{\partial p}$ is also unaltered by the motion.

By equation (39) above

$$\frac{1}{\theta}\frac{\partial\theta}{\partial z}=\frac{1}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right).$$

Then
$$\frac{1}{\theta}\frac{\partial\theta}{\partial p}=\frac{1}{\theta}\frac{\partial\theta}{\partial z}\cdot\frac{\partial z}{\partial p}=-\frac{1}{gp}\frac{1}{\theta}\frac{\partial\theta}{\partial z}=-\frac{R}{gp}\left(\frac{\partial T}{\partial z}+\Gamma\right).$$

Hence $\frac{1}{p}\left(\frac{\partial T}{\partial z}+\Gamma\right)$ remains unchanged by adiabatic motion. If the values of z, p, T in the new position be indicated by accented letters,

$$\frac{1}{p'}\left(\frac{\partial T'}{\partial z'}+\Gamma\right)=\frac{1}{p}\left(\frac{\partial T}{\partial z}+\Gamma\right) \qquad \text{......(51).}$$

In other words the difference between the lapse-rate and the dry adiabatic is proportional to the pressure, and the ascent of a layer will cause the lapse-rate to approach more closely to the dry adiabatic, while descent will cause the lapse-rate to deviate still further from the dry adiabatic.

The discussion above can be immediately extended to the case when the column of air changes its horizontal extent, by spreading outward or by convergence. The only change in the argument consists in assuming that the horizontal area is initially S, and changes to S'. For a given mass of air it will now be $S\Delta p$ which will remain constant, instead of Δp. The result can be written down as follows, without further discussion:

$$\frac{1}{S'p'}\left(\frac{\partial T'}{\partial z'}+\Gamma\right)=\frac{1}{Sp}\left(\frac{\partial T}{\partial z}+\Gamma\right) \qquad \text{......(52).}$$

§ 25. *The dry adiabatic lapse-rate in geodynamic units*

The value of the adiabatic lapse-rate is proportional to g, and therefore differs slightly from place to place. Let position relative to the earth's surface be specified by means of the geopotential V, measured in dynamic metres. Then since the separation of two surfaces of unit difference of geopotential in dynamic metres is $10^5/g$ cm, it follows that

$$dV=dz\times g/10^5.$$

The adiabatic lapse-rate is therefore given by

$$-\frac{\partial T}{\partial V}=-\frac{\partial T}{\partial z}\times\frac{10^5}{g}=\frac{gA}{c_p}\frac{10^5}{g}=\frac{10^5 A}{c_p}=\frac{2{\cdot}392\times10^{-3}}{0{\cdot}2396}=0{\cdot}998\times10^{-2},$$

or
$$-\frac{\partial T}{\partial V}=1^\circ \text{ C per 100 dynamic metres} \qquad \text{......(53)},$$

with an accuracy of 0·17 per cent. This value is not dependent on the local value of gravity.

§ 26. *The variation of density with height; the auto-convection gradient*

It has been shown that the condition that the atmosphere should be in stable equilibrium is that the lapse-rate should be less than the dry adiabatic lapse-rate gA/c_p, or approximately 1° C per 100 metres. In an incompressible fluid the condition for stability is that the density shall not increase with height, and it is important to realise that in a compressible atmosphere the condition for stability is not merely that the density shall decrease with height.

The general equation for the variation of density with height is readily derived by the use of the statical equation

$$\frac{\partial p}{\partial z}=-g\rho \qquad \text{......(54)}$$

and the gas-equation
$$p=R\rho T \qquad \text{......(55)}.$$

Differentiating the second of these equations, we find

$$\frac{1}{p}\frac{\partial p}{\partial z}=\frac{1}{\rho}\frac{\partial \rho}{\partial z}+\frac{1}{T}\frac{\partial T}{\partial z}=-\frac{g\rho}{R\rho T}=-\frac{g}{RT}.$$

Hence
$$\frac{1}{\rho}\frac{\partial \rho}{\partial z}=-\frac{1}{T}\left(\frac{\partial T}{\partial z}+\frac{g}{R}\right) \qquad \text{......(56)}.$$

The condition that the density shall not change with height is

$$-\frac{\partial T}{\partial z}=\frac{g}{R}=\frac{981}{2{\cdot}87\times10^6}=3{\cdot}42\times10^{-4}\ {}^\circ\text{C/cm}$$
$$=3{\cdot}42^\circ \text{ C per 100 metres.}$$

The density will increase or decrease with height according as the lapse-rate is greater or less than g/R. The limiting lapse-rate which gives density not changing with height is g/R, whereas the dry adiabatic lapse-rate is $\frac{g}{R}\frac{\gamma-1}{\gamma}$. Thus the limit we are now considering is $\gamma/(\gamma-1)$ or 3·49 times the dry adiabatic lapse-rate. The name auto-convection gradient has been suggested for this limit, because it was thought that convection must set in automatically when density increases with height. There appears to be no justification for such a supposition, since in a compressible atmosphere ordinary statical theory shows that the limiting lapse-rate for stability is the dry adiabatic lapse-rate.

"Auto-convection" is an unhappy name, and the idea has no place in accepted meteorological theory. The question of convection requires careful consideration, and some further discussion of this topic will be found later in § 128. In the idealised statical theory unstable equilibrium of any magnitude can exist, in the absence of some small initial disturbance. But in the atmosphere there is never perfect equilibrium, and the occurrence of lapse-rates much in excess of the dry adiabatic lapse-rate is to be accounted for by other reasons than the absence of small disturbances.

In some ways it would be convenient, in discussing stability, to use a conception of "potential density", which would denote the density which the air under consideration would take if brought adiabatically to some standard pressure. The equation connecting p and ρ in adiabatic changes is

$$\frac{p}{\rho^\gamma} = \text{constant} \qquad \text{......(57).}$$

Hence, if ρ_i denote the potential density corresponding to a standard pressure p_0,

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_i^\gamma} \qquad \text{......(58).}$$

It is readily shown that if θ be the potential temperature,

$$\frac{1}{\rho_i}\frac{\partial \rho_i}{\partial z} = -\frac{1}{\theta}\frac{\partial \theta}{\partial z} = -\frac{1}{T}\left(\frac{\partial T}{\partial z} + \Gamma\right) \qquad \text{......(59),}$$

so that the condition for stability is that the potential density shall diminish with increasing height.

Density increasing with height in the layers near the ground is not uncommon on sunny days, and is the cause of most types of mirage.

§ 27. *The effect of mixing two masses of damp air*

Taylor has given an interesting application of fig. 16 above, following the lines of the earlier treatment of von Bezold. These two writers give a theorem which states that if the state of a mass of damp air as to temperature and vapour-pressure be represented by points in the diagram of fig. 16, then any mixture of the two masses will be represented by a point on the straight line joining the points which represent the two components. The theorem is not mathematically exact, and we give below a slight modification of Taylor's proof, in order to show to what extent the result can be in error.

Let m_1, T_1, e_1 and y_1 represent the total mass, the absolute temperature, the vapour-pressure and the mass of water-vapour per unit mass of moist air, in the first component of the mixture; and let similar variables with the subscript 2 represent the second mass. Let T and y represent the temperature and mass of water-vapour per unit mass of damp air in the final mixture.

The total mass of the mixture is $m_1 + m_2$, and the total mass of water-vapour in the mixture is $m_1 y_1 + m_2 y_2$. Hence the value of y is

$$\frac{m_1 y_1 + m_2 y_2}{m_1 + m_2},$$

and since the total pressure remains constant throughout the mixing, the final vapour-pressure is given by

$$e = \frac{m_1 e_1 + m_2 e_2}{m_1 + m_2} \qquad \text{......(60).}$$

Let T be the temperature of the mixture, and let $T_1 > T_2$. The mass m_1 is reduced from a temperature T_1 to a temperature T. It gives up an amount of heat

$$c_p m_1 (1 - y_1)(T_1 - T) + c_p' m_1 y_1 (T_1 - T),$$

where c_p and c_p' are the specific heats at constant pressure of dry air and water-vapour respectively. The second mass gains an amount of heat

$$c_p m_2 (1 - y_2)(T - T_2) + c_p' m_2 y_2 (T - T_2).$$

These two amounts should balance exactly, since the changes take place at constant pressure. Hence

$$T = \frac{c_p (m_1 T_1 + m_2 T_2) + (c_p' - c_p)(m_1 y_1 T_1 + m_2 y_2 T_2)}{c_p (m_1 + m_2) + (c_p' - c_p)(m_1 y_1 + m_2 y_2)} \qquad \text{......(61).}$$

In both numerator and denominator the terms in y will be small as a rule by comparison with the other terms, and an approximation is the mean temperature defined by

$$T_m = \frac{m_1 T_1 + m_2 T_2}{m_1 + m_2} \qquad \text{......(62).}$$

It is readily seen that equation (61) reduces to equation (62) when $y_1 = y_2$, or when $T_1 = T_2$, and a little reduction shows that in general

$$T - T_m = -\frac{c_p' - c_p}{c_p + (c_p' - c_p)\,y}(y_1 - y_2)(T_1 - T_2)\frac{m_1 m_2}{(m_1 + m_2)^2} \qquad \text{......(63).}$$

If the difference between T and T_m be neglected, the theorem is proved, and if in fig. 16 the points B and C represent the masses m_1 and m_2, the point representing the mixture will lie on the straight line BC, at a point E, where

$$BE/EC = m_2/m_1.$$

The vapour-pressure will be given accurately by the point E, as is seen from equation (60). The temperature derived in this way will require a slight correction, which is given by equation (63). The magnitude of the correction is most readily seen by means of an example.

Let $m_1 = 2$, $m_2 = 1$, $y_1 = 0{\cdot}0035$, $y_2 = 0{\cdot}02$, $T_1 = 28°$ F, $T_2 = 75°$ F, which correspond with the positions of the points B and C. For our present purpose we may take $(c_p' - c_p)/c_p = 1$, and we may also neglect $\frac{(c_p' - c_p)\,y}{c_p}$ by comparison with unity. The result of the computation is

$$T - T_m = 0{\cdot}172° \text{ F}.$$

For other distributions of masses the error is proportional to $m_1 m_2/(m_1 + m_2)^2$. This factor is always less than unity, and is greatest when the two masses are equal, when it is equal to $\frac{1}{4}$. For all practical purposes the error is of no importance whatsoever and we may with confidence take T_m as the temperature of the mixture.

The above discussion is similar to that given by von Bezold and Taylor, except that it does not neglect the effect of the presence of water-vapour on the density and specific heat of damp air.

The practical value of the above theorem is considerable. It is readily seen from the diagram that by mixing masses of widely different temperatures, neither of which need be near saturation, it is possible to obtain a mixture which will be saturated or supersaturated. In the latter case the superfluous water-vapour will be condensed, and may produce fog or rain. If the vapour-pressure computed by the use of the diagram exceeds the saturation value for the assigned temperature, as say at the point E, condensation must occur, and the vapour-pressure must come down to the value corresponding to the point F. The formation of fog at a surface of discontinuity in the atmosphere is to be explained, at least in part, by the direct effect of the mixing of masses of air of different temperatures and humidities.

The diagram of fig. 16 can also be used to evaluate the fall of temperature which is necessary in order to produce saturation, given any initial conditions. If, for example, the air is initially represented by the point A, as the air is cooled the point moves horizontally to the left, and when it reaches the point D, the air is saturated. D is thus the dew-point, and any further cooling leads to condensation, or at least to supersaturation, which in the atmosphere leads automatically in most cases to the condensation of water-vapour in the form of water drops.

CHAPTER III

THE ASCENT OF DAMP AIR. THE SATURATED ADIABATIC LAPSE-RATE

§ 28. *Statement of the general problem*

THE main problem to be discussed in connection with the thermodynamics of moist air is the variation of temperature produced by changes of pressure, which in the atmosphere are associated with vertical motion. When damp air ascends it must eventually attain saturation, and further ascent produces condensation, at first in the form of water drops, and as ice in the later stages, when the air has attained levels in which the temperature is below freezing point.

Hertz* and Neuhoff† discussed the problem on the assumption that the products of condensation are carried up with the ascending current, and partake of the changes of temperature of the air. At any stage the amount of liquid water or snow depends both on the temperature of the mixture at that stage and on the initial condition of the air. The process is strictly reversible. For if the damp air and water drops or snow are again brought downward, the evaporation of water drops or snow uses up at each stage the same amount of latent heat as is liberated by condensation on the upward path.

It was pointed out by von Bezold‡ that the actual process in the atmosphere is not reversible, as some of the condensed water falls out as rain drops. If all the condensed water or ice falls out as rain or snow the changes of temperature are slightly altered, since there is no longer any interchange of heat between the air and the water or ice. The process which then takes place was named by von Bezold pseudo-adiabatic.

It has been shown by Fjeldstad§ and others that the differences which arise according as we regard the products of condensation as falling out or being retained are so small as to be negligible in practice. We shall therefore follow the method of Hertz and Neuhoff in what follows, retaining the products of condensation with the ascending air, since this leads to easier mathematical treatment than in the pseudo-adiabatic process visualised by von Bezold. The results so derived will be applied to the discussion of the changes of temperature of ascending damp air, without further consideration whether the water drops and ice fall out, except at the hail stage. The errors which arise on account of the uncertainties as to what happens to the products of condensation

* *Met. Zeit.* **1**, 1884, pp. 421–31; also in English in Hertz, *Misc. Papers* (Macmillan).

† *K. Preuss. Met. Inst.* **1**, 1900, p. 271; also in Abbe, *Mechanics of Earth's Atmosphere*, **3**, 1910.

‡ *Sitz. Berl. Akad.* 1888.

§ *Geofys. Publikasjoner*, **3**, No. 13.

are likely to be much smaller than the uncertainties of some of the other assumptions. The theoretical treatment must assume that there is no exchange of heat between the ascending air and the environment. This is equivalent to assuming that it is legitimate to neglect the effects of radiation, which are probably small during the relatively small time of ascent, and also the effects of turbulence, which are certainly not negligible, though it is impossible to correct for them in any way.

In the work of Hertz the changes of state of 1 gramme of mixture of air and water-vapour were considered. Neuhoff considered the changes of state of 1 gramme of dry air together with the appropriate admixture of water in its various forms—vapour, liquid and solid. This method is preferable in that it leads to simpler mathematical treatment. We shall therefore follow Neuhoff in the following pages.

The initial conditions from which we start are defined by the pressure, temperature and humidity mixing ratio of unsaturated air. The analysis aims at the development of equations to represent the changes which take place as the air ascends and expands with diminishing pressure. Initially the water is present in the form of vapour, but in the later states the water may be present in the form of vapour, liquid or solid. The notation and constants used in the present chapter are given below:

x = humidity mixing ratio, or weight in grammes of water-vapour associated with 1 gramme of dry air.

y = weight in grammes of liquid water associated with 1 gramme of dry air.

z = weight in grammes of ice associated with 1 gramme of dry air.

$\xi = x+y+z$. This is constant throughout the process.

Q or dQ = an amount of heat in calories.

T = absolute temperature.

p = total pressure.

e = partial pressure of water-vapour.

c_p = specific heat of dry air at constant pressure = 0·2396.

c_v = specific heat of dry air at constant volume = 0·1707.

γ = ratio of specific heats of dry air = 1·403.

c_p' = specific heat of water-vapour = 0·4652.

c = specific heat of liquid water = 1·013 (value at 0° C).

c_i = specific heat of ice = 0·5.

ϵ = specific gravity of water-vapour = 0·622.

R = constant of the gas-equation for dry air = $2{\cdot}8703 \times 10^6$ in C.G.S. units.

v = specific volume of dry air.

ρ = true density of the gaseous mixture = $(1+x)/v$.

L = latent heat of evaporation of water = $594{\cdot}9 - 0{\cdot}51\ (T-273)$.

L_e = latent heat of fusion of ice = 79·7.

L_ι = latent heat of sublimation of ice = 677 (Fjeldstad's value).

A = reciprocal of the mechanical equivalent of heat = $2{\cdot}392 \times 10^{-8}$.

$M = \log_{10} e = 0{\cdot}4343$.

Four stages of dynamic cooling of moist air due to ascent have to be considered: (*a*) the air unsaturated; (*b*) saturated air and water drops at temperatures above 0° C; (*c*) the freezing stage at 0° C in which the water drops are changed to ice; and (*d*) saturated air and ice at temperatures below 0° C.

These four stages are considered in turn below, the aim being to develop a system of equations which will give the temperature and vapour-pressure corresponding to different values of the total pressure.

§ 29. *Stage (a). Unsaturated air*

During the stage when no condensation takes place the constitution of the mixture of air and water-vapour is unchanged, and the humidity mixing ratio has the constant value ξ. The specific heat of the damp air is $\frac{c_p+\xi c_p'}{1+\xi}$ and the specific volume of the mixture is $v/(1+\xi)$. Thus equation (27), p. 38 yields, with $dQ=0$,

$$(c_p+\xi c_p')\,dT-Av\,dp=0 \qquad \ldots\ldots(1).$$

Since $v\,(p-e)=RT$, and $ev=RT\xi/\epsilon$, it follows that

$$pv=RT\,(1+\xi/\epsilon) \qquad \ldots\ldots(2).$$

Substituting for v from equation (2), equation (1) becomes

$$(c_p+\xi c_p')\,dT-ART\,(1+\xi/\epsilon)\,dp/p=0 \qquad \ldots\ldots(3),$$

$$\frac{c_p+\xi c_p'}{1+\xi/\epsilon}\frac{dT}{T}-AR\frac{dp}{p}=0 \qquad \ldots\ldots(4).$$

From this we find readily by integration that

$$pT^{-m}=\text{const.} \qquad \ldots\ldots(5),$$

where
$$m=\frac{1}{AR}\left(\frac{c_p+\xi c_p'}{1+\xi/\epsilon}\right)=3{\cdot}49\,\frac{1+1{\cdot}94\xi}{1+1{\cdot}608\xi} \qquad \ldots\ldots(6).$$

Equation (5) may also be written in the form

$$\log p-m\log T=\text{const.} \qquad \ldots\ldots(7).$$

Equations (5) and (7) give the relation between p and T for purely adiabatic changes. They are similar in form to the equations already derived for dry air in the preceding chapter, and should be compared with equation (30), p. 38, in which the value of m is equal to that given by equation (6) above when $\xi=0$.

The adiabatic lapse-rate of unsaturated air has already been discussed earlier in § 23, p. 43.

§ 30. *Stage (b). Saturated air above* 0° C, *or the rain stage*

Once saturation is attained further cooling leads to condensation, and the quantity x of water-vapour per gramme of dry air becomes variable. Let y denote the weight in grammes of the amount of liquid water per gramme of dry air. Then $x+y=\xi$ remains constant and equal to the amount of water-vapour before condensation sets in. A part of the work of expansion as the air ascends is now supplied by the latent heat set free from the condensing vapour.

The amount of heat necessary to produce an increase of temperature dT, and a change of pressure $d(p-e)$ is as shown in equation (27), p. 38,

$$dQ' = c_p dT - \frac{ART\, d(p-e)}{p-e} \qquad \text{......(8).}$$

If the change in temperature by an amount dT brings about a change dx in the amount of water-vapour, and a change of dy in the amount of liquid water, we must take account of the latent heat liberated by the process of condensation, as well as the changes of temperature of the water-vapour and liquid water.

The process we are considering is reversible, and we shall make use of the condition that there is no change of entropy taking place (see § 37 of the next chapter). The entropy of 1 gramme of dry air is ϕ_1, where

$$\phi_1 = c_p \log T - AR \log (p-e) + \text{const.} \qquad \text{......(9).}$$

The entropy of x grammes of water-vapour $+(\xi - x)$ grammes of liquid water at temperature T is shown in § 46, p. 80, to be

$$\phi_2 = \xi c \log T + Lx/T.$$

The total entropy of the $(1+x)$ grammes of moist air $+(\xi - x)$ grammes of liquid water is therefore

$$\phi = \phi_1 + \phi_2 = (c_p + \xi c) \log T + Lx/T - AR \log (p-e) \quad \text{......(10).}$$

This quantity remains constant for any given mass of moist air, for which the constant value is fixed by the initial conditions. Using the subscript 0 to indicate initial conditions, we write at once,

$$\log \frac{p-e}{(p-e)_0} = \frac{c_p + \xi c}{AR} \log \frac{T}{T_0} + \frac{M}{AR}\left(\frac{Lx}{T} - \frac{L_0 x_0}{T_0}\right) \qquad \text{......(11),}$$

where the logarithms are now taken to base 10, and M is the modulus, or $\log_{10} e$, and is equal to 0·4343. The coefficient of $\log T/T_0$ may be written

$$m' = \frac{c_p}{AR}\left(1 + \frac{\xi c}{c_p}\right) = 3{\cdot}49\,(1 + 4{\cdot}23\xi) \qquad \text{......(12).}$$

Equation (11) gives the form of the saturation adiabatics, assuming all the condensed water remains in suspension.

But since $\quad x = \epsilon e/(p-e), \quad$ or $\quad p - e = \epsilon e/x,$

we can eliminate $p-e$ from (11), and so derive the equation

$$\log x + \frac{M}{AR}\cdot\frac{Lx}{T} + m' \log T - \log e = \text{const.} \qquad \text{......(13).}$$

Alternatively, the relations between x, e and $p-e$ may be used to eliminate x from equation (12),

$$\log (p-e) - m' \log T = \frac{M}{AR}\,\frac{Lx}{T} + \text{const.}$$

$$= \frac{M}{AR}\,\frac{L}{T}\,\frac{\epsilon e}{p-e} + \text{const.} \qquad \text{......(14).}$$

We now have three alternative equations, (12), (13) or (14), to use in the computation of the form of the adiabatics. Each of these equations involves three variables. The last of the three equations is normally used. It may be written in the form

$$\log(p-e) - 3{\cdot}93\,\frac{Le}{T}\,\frac{1}{p-e} - m'\log T = \text{const.} \qquad \ldots\ldots(15).$$

It is seen that ξ only enters into the index m'. Since ξ is small, m' only varies slowly with ξ, and it is therefore permissible to use a mean value of ξ. The details of the computation of the tables are best studied in the paper by Fjeldstad referred to above.

§ 31. *Stage (c). The freezing or hail stage*

Hertz and Neuhoff assumed that as soon as the temperature falls below the freezing point the liquid drops begin to freeze. This does not normally occur in the atmosphere, and sharply defined cumulus clouds consist of supercooled water drops. The stage in which supercooled water is present can be treated by means of equation (15) above, the rain stage being carried beyond the freezing point.

The analysis of the freezing or hail stage is given below. During this stage the liquid drops freeze, while the temperature remains constant, the work done in the expansion against external pressure being accounted for by the liberation of the latent heat of fusion of the ice.

The total water content ξ is now made up of three parts: x water-vapour, y liquid water, and z ice,

$$\xi = x + y + z.$$

Let the suffix 0 denote the conditions at the beginning of the freezing stage, and let the suffix 1 denote the conditions at the end of this stage.

Then initially $\quad x = x_0, \quad y = y_0, \quad z = 0, \quad \xi = x_0 + y_0,$

and finally $\quad x = x_1, \quad y = 0, \quad z = z_1, \quad \xi = x_1 + z_1.$

If L_e is the latent heat of fusion of ice, then since there is no change of temperature during this stage, the amount of heat required to change x to $x+dx$ and z to $z+dz$ at any point of the hail stage is

$$dQ = 0 = -Av\,d(p-e) + L\,dx - L_e\,dz \qquad \ldots\ldots(16).$$

But $v(p-e) = RT_0 = \text{const.}$ during this stage ($T_0 = 273^\circ$ A). Hence

$$-v\,d(p-e) = (p-e)\,dv = RT_0\,dv/v.$$

Equation (16) then becomes

$$ART_0\,\frac{dv}{v} + L\,dx - L_e\,dz = 0 \qquad \ldots\ldots(17).$$

It is to be noted that the only terms which enter into (16) are (*a*) the evaporation of water, (*b*) the freezing of water, and (*c*) the expansion against external pressure. Since the temperature is constant there is no term in dT to be considered.

Integrating equation (17) we find

$$\frac{ART_0}{M}\log\frac{v}{v_0}+L(x-x_0)-L_e(z-z_0)=0 \qquad \text{......(18).}$$

Also the change is isothermal,

$$\frac{p-e}{(p-e)_0}=\frac{v_0}{v},$$

and e remains unchanged throughout the process.

Initially $z=0$, and finally $y=0$. Since

$$x=0\cdot622\frac{e}{p-e}, \qquad z_1=\xi-x_1=\xi-0\cdot622\frac{e}{(p-e)_1}.$$

Hence

$$\frac{ART_0}{M}\log\frac{(p-e)_1}{(p-e)_0}=-0\cdot622L\left(\frac{e}{(p-e)_0}-\frac{e}{(p-e)_1}\right)-L_e\left(\xi-0\cdot622\frac{e}{(p-e)_1}\right)$$

$$=0\cdot622(L+L_e)\frac{e}{(p-e)_1}-L_e\xi-0\cdot622\frac{Le}{(p-e)_0} \qquad \text{......(19).}$$

This equation may be re-arranged so that terms representing the initial and final stages of the process of freezing come on opposite sides of the equation:

$$\log(p-e)_1-\frac{0\cdot4343\times0\cdot622}{ART_0}(L+L_e)\frac{e}{(p-e)_1}$$

$$=\log(p-e)_0-\frac{0\cdot4343\times0\cdot622}{ART_0}\frac{Le}{(p-e)_0}-L_e\xi\times\frac{0\cdot4343}{ART_0} \qquad \text{......(20).}$$

Inserting the values of the constants in this equation, we find

$$\log(p-e)_1-\frac{59\cdot60}{(p-e)_1}=\log(p-e)_0-\frac{52\cdot55}{(p-e)_0}-1\cdot846\xi \qquad \text{......(21).}$$

Given the initial values of p and e we can deduce the final values of p, and so determine the range of pressure involved, and hence the depth of this stage, if we are definitely concerned with the ascent of air and consequent dynamical cooling.

§ 32. *Stage (d). Below* 0° C. *The snow stage*

At temperatures below 0° C there will be present only ice and enough water-vapour to produce saturation. This stage differs from stage (*b*) only in that we now have ice instead of water, and the specific heat and latent heat of sublimation of ice must be substituted for the corresponding constants of water. The analysis is otherwise identical and equation (11) may be copied down here with the necessary changes of the constants:

$$\log(p-e)-\frac{c_p+\xi c_i}{AR}\log T-\frac{MxL_i}{ART}=\text{const.} \qquad \text{......(22).}$$

Substituting $x=0{\cdot}622\,\frac{e}{p-e}$, and writing $B=\frac{ML_i}{AR}\times 0{\cdot}622=2664$ and $m_2=\frac{c_p+\xi c_i}{AR}=3{\cdot}49\,(1+2{\cdot}09\xi)$, we may write (22) in the form

$$\log(p-e)-\frac{Be}{(p-e)\,T}-m_2\log T=\text{const.} \qquad \ldots\ldots(23).$$

Fjeldstad showed (*loc. cit.*) that there was reason to regard the latent heat of sublimation of ice as constant, and equal to 677 g-cal, and this value has been adopted above.

§ 33. *The Hertz diagram*

Equations (7), (14), (21) and (23) give the relations between p, T and x or e, and enable us to draw the adiabatic lines for the change of state of any mass of air for which the initial values of these variables are known. Very complete diagrams have been constructed by Hertz, Neuhoff and Fjeldstad, on the assumption that all the condensed water and ice remain in suspension. In fig. 17 is reproduced a modified Hertz diagram, based on the computations of Fjeldstad. In this diagram the pressure in millibars is represented on a logarithmic scale along the horizontal axis. Temperature is represented on a linear scale along the vertical axis. There are three series of lines running across the diagram. The steepest series of lines are the adiabatic lines for dry air. The less steep series of continuous lines are the adiabatic lines for saturated air. The broken lines are the lines of equal values of the humidity mixing ratio x, and the numbers marked against these lines indicate the number of grammes of water-vapour which will saturate 1 kilogramme of the air. It will be noted that the dry adiabatic lines are very nearly straight lines; they would be perfectly straight if temperature had been represented on a logarithmic scale. The saturated adiabatic lines are more curved than the dry adiabatics, and the slopes of the two series of lines approach equality as the temperature diminishes. Note for example how small is the difference of slope of the two families of curves near the bottom right-hand side of the diagram.

The use of the diagram can be most readily understood from the example indicated by a broken line which starts at the top right-hand side of the diagram. This line traces out the history of a mass of air which is initially at a pressure of 1020 mb, with temperature 25° C and relative humidity 50 per cent, and which is set in upward motion through the atmosphere. The diagram shows that at pressure 1020 mb and temperature 25° C it requires 20 grammes of water-vapour to saturate the air, and as the relative humidity is 50 per cent, the kilogramme of dry air which we are considering has 10 grammes of water-vapour initially. During the early stages of its ascent the air will follow the dry adiabatic through the point A which represents its initial condition. It will follow this line until it reaches the vapour-content line of 10 grammes, at which point of its ascent it will have just reached saturation. Its further ascent will be along a saturated adiabatic. At the different stages of its ascent the water-vapour content of the air will be given by the figures shown against the broken lines, the remainder of the initial 10 grammes of water-vapour being carried up as liquid water. When it attains the level where the temperature is 0° C the water begins to freeze, and the temperature remains at 0° C until all

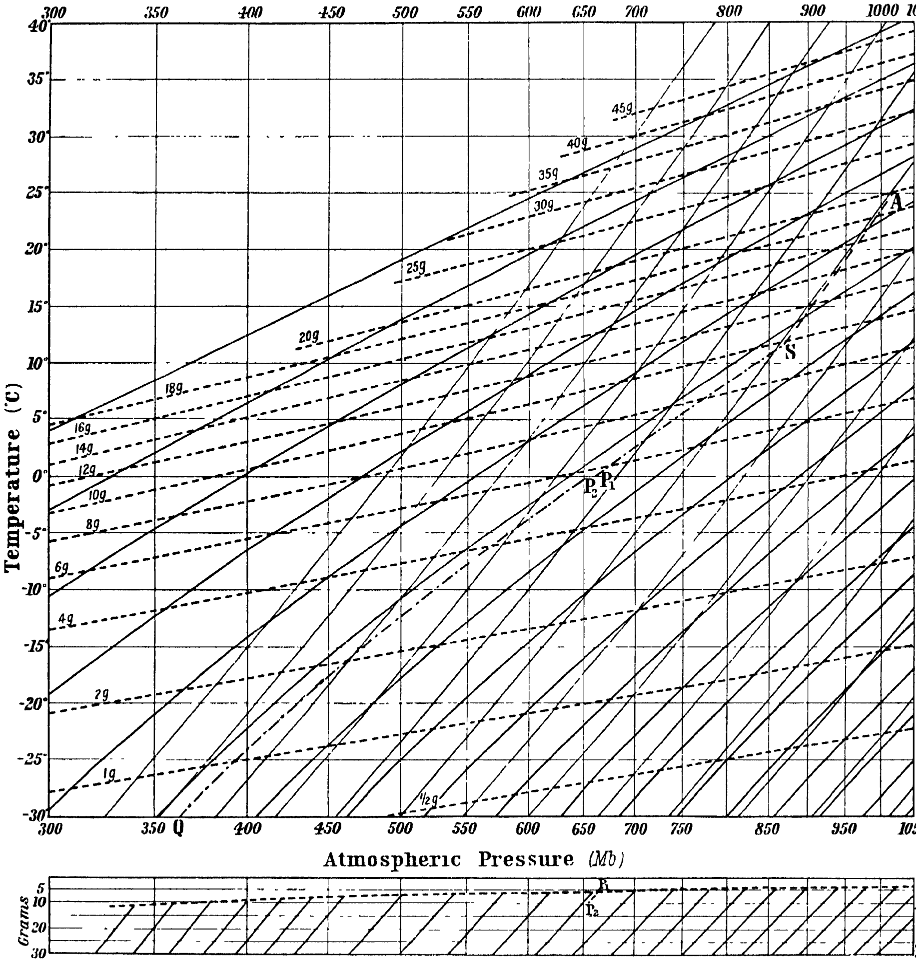


Fig. 17. The Hertz diagram.

the water is frozen. The latent heat liberated is used up in expanding the mixture. The mass of air we are considering reaches saturation at a pressure of 870 mb and temperature 11·5° C, represented on the diagram by the point *S*. The further ascent is along the saturated adiabatic through *S*, until the point P_1 is reached at which the temperature has just fallen to 0° C. At this stage of its ascent the air contains a little less than 6 grammes of water-vapour, and a little more than 4 grammes of liquid water. The small diagram at the bottom of the figure is used to estimate the range of pressure corresponding to the freezing stage. Take on the broken line of this small diagram the pressure corresponding to the pressure at which the air first reaches 0° C; in our example this is at the point P_1. Through this point take a line parallel to the sloping lines of the lower diagram and follow it down to the point at which it meets the line of water content equal to the original water content of the air, which in our example is 10 grammes. The latter point is at P_2 in our example. The pressure at P_2 gives the limit of the freezing stage, and indicates the pressure at which the air will begin to cool below 0° C, from the representative point P_2 in the diagram. After P_2 the representative point follows the saturated adiabatic P_2Q, and reaches −30° C at a pressure of 362 mb, at which stage the original 10 grammes of water-vapour are represented by about 0·7 gramme of water-vapour, and about 9·3 grammes of ice.

If saturation is not attained until the level of 0° C has been surpassed, the freezing stage will not occur, and the adiabatic lines will cross the line of zero temperature without a break such as we have shown at P_1P_2. But if saturation is attained at a higher temperature than 0° C, then the path followed by the adiabatics will consist of four parts—the dry adiabatic AS, the saturated adiabatic SP_1, the freezing stage P_1P_2 and the saturated adiabatic P_2Q.

§ 34. *The effect of the loss of precipitated water and ice*

The equations which relate the changes of pressure in damp air to the variations of the other meteorological factors have been derived above in full, on the assumption that the products of condensation are retained with the air, none being precipitated as rain or snow. The mathematical treatment has been reproduced in detail rather for the sake of completeness than for its direct utility, since the practical problems of meteorology can be much more readily handled by the use of the diagram of fig. 17, than by means of the equations.

One questionable assumption in the treatment given above is that relating to the non-precipitation of the products of condensation. In the atmosphere the phenomena are not always in keeping with this assumption. Rain drops fall, while cloud particles float, or rather fall so slowly that their rate of fall is unimportant, and it appears that in nature the phenomena are intermediate between those assumed as the basis of the above treatment and the pseudo-adiabatic conditions discussed by von Bezold, in which all products of condensation are precipitated. The difference in the mathematical treatment of von Bezold and that of Hertz, Neuhoff and Fjeldstad arises through the variability of ξ in von Bezold's equations, leading to a term

$$\int (c_p + \xi c) \frac{dT}{T},$$

instead of $\qquad (c_p + \xi c) \int \frac{dT}{T} \quad \text{or} \quad (c_p + \xi c) \log T.$

Neuhoff (*loc. cit.*) investigated the difference between the results derived by his method and that of von Bezold, and found that while the pseudo-adiabatic changes always lead to higher pressures for the same temperatures when compared with the true adiabatic changes, the differences are so small as to be negligible, except near 0° C. Thus air initially saturated at a temperature of 20° C and pressure 760 mm attains a temperature of 0° C at a pressure of 465 mm in the pseudo-adiabatic system and 463 mm in the adiabatic system; it reaches a temperature of −18° C at a pressure of 316 mm in the pseudo-adiabatic system, and 304 mm in the adiabatic system. The difference of 12 mm is almost entirely due to the hail stage, since on the adiabatic system the mixture first attains 0° C at a pressure of 463 mm, and begins to fall below 0° C at a pressure of 452 mm. It is in fact immaterial to the form of the saturation lines whether we regard the products of condensation as precipitated or retained, except at 0° C; the diagram of fig. 17 may be treated as adiabatic or pseudo-adiabatic, and if we wish to use it to discuss pseudo-adiabatic changes we simply omit the hail stage. In this event the representative point on the diagram crosses the line of zero temperature without side-stepping, and its further course is represented by an approximate continuation of the line SP_1 toward lower pressure. The adiabatics above and below 0° C are not strictly continuous in direction, there being a slight diminution of the lapse-rate just above zero, but the change of direction is so slight as to be scarcely noticeable in the diagram. The neglect of the hail stage means that the air reaches the bottom of the diagram at higher pressure in the pseudo-adiabatic system, at a point to the right of the point Q.

The course of events described above is strictly reversible if none of the condensed liquid or solid products of condensation are precipitated, for then the air in descending is compressed and heated dynamically, and evaporation goes on continually until the air has returned to the state in which it was just saturated, with no liquid or solid content. Further descent beyond this stage brings the air to a temperature at which it is unsaturated, and consequently its further course is along the dry adiabatic which it followed in its initial ascent. If any of the water, ice or snow has fallen out of the air in which it was formed the result will be that on the reverse journey the point at which the air ceases to be saturated is attained at an earlier stage of the downward journey, and a larger part of the downward journey is along a dry adiabatic, resulting in the air reaching its initial pressure at a higher temperature than it had at the beginning. If for example the air which we took from 1020 mb and 25° C up to a pressure of 362 mb and temperature of −30° C should shed all its water or ice, and then be taken back to a pressure of 1020 mb, it would go the whole way along a dry adiabatic, and would attain the level of 1020 mb with a temperature of about 54° C, or 29° above the temperature which it had initially at that pressure. If we imagine air starting at sea level and being forced to rise over a ridge of mountains, the height to which the air is forced to ascend being sufficient to produce considerable condensation and precipitation, the air on descending to its original level on the other side of the mountains would appear as a warm current (Föhn). This is but one of many important applications of the results derivable from the Hertz diagram.

There is a second questionable assumption underlying the mathematical treatment given above. It is that the conditions are strictly adiabatic, and that there are no exchanges of heat between the ascending air and its environment. This assumption is also inherent in von Bezold's pseudo-adiabatic treatment. It is equivalent to assuming that the effects of radiation and of turbulent mixing with the environment are both negligible. It is not possible to estimate arithmetically the effect of either, otherwise it would be possible to correct for them. But while it appears *a priori* that the effects of radiation are not likely to be very great, it is certain that on occasion the effects of turbulent mixing are very considerable, and that these effects produce a much greater uncertainty in the computations than the niceties of estimation of the amount of water or ice lost by precipitation.

A further source of error lies in the uncertainty as to the height at which freezing actually begins. If freezing is delayed to below 0° C, the representative point in fig. 17 follows the same saturated adiabatic to a point below P_1 and the freezing stage occurs at some lower temperature. In the atmosphere freezing does not always occur when the temperature falls to 0° C, and clouds of liquid droplets occur at temperatures as low as − 40° C.

§ 35. *The lapse-rate of ascending damp air*

The lapse-rate with height of damp air when displaced upward from its normal position can be readily derived without the use of the concept of entropy which is the basis of the derivation of equation (11) above. The condition of the air will be specified as in § 30 above, by the pressure p, the density ρ, the temperature T, the vapour-pressure e and the humidity mixing ratio x, and the amount of liquid water present at any stage will be denoted by $\xi - x$, where ξ is constant for any given element of mass of the damp air, being equal to the original vapour content of the mass when just saturated.

When unit mass of dry air, with the appropriate admixture of water-vapour and liquid water, is displaced upward through a distance dz, the increase in potential energy is balanced by the loss in internal energy of the air and water-vapour, and of any liquid water present. The internal energy of x grammes of water-vapour *plus* $(\xi - x)$ grammes of liquid water is equal to the internal energy of ξ grammes of liquid water *plus* the latent heat of x grammes of water-vapour, and is therefore equal to

$$\xi c T + Lx.$$

The change in this due to a vertical displacement dz is

$$\xi c\, dT + d\,(Lx).$$

The change in internal energy of 1 gramme of dry air is $c_p dT$. These two quantities are balanced by the increase in gravitational potential energy

$$g\,(1+\xi)\,dz.$$

The equation which expresses the balance is

$$(c_p + \xi c)\frac{dT}{dz} + \frac{d}{dz}(Lx) + Ag\,(1+\xi) = 0 \qquad \text{......(24).}$$

For the consideration of the displacement of an element of air from its initial position, ξ becomes equal to x.

Substituting $x=\epsilon e/(p-e)$, and differentiating Lx, we find, after a slight approximation,

$$-\frac{\partial T}{\partial z}=\Gamma\times\frac{X+p}{Z+p}, \qquad \text{......(25)},$$

where

$$\left.\begin{aligned} X&=\frac{\epsilon}{AR}\frac{Le}{T} \\ Z&=\frac{\epsilon}{c_p}\left\{e\left(c+\frac{dL}{dT}\right)+L\frac{de}{dT}\right\} \end{aligned}\right\} \qquad \text{......(26)}.$$

The details of the derivation of (25) have been given by Brunt*. This equation has been used to derive the lapse-rates for saturated air shown by the isopleths in fig. 18.

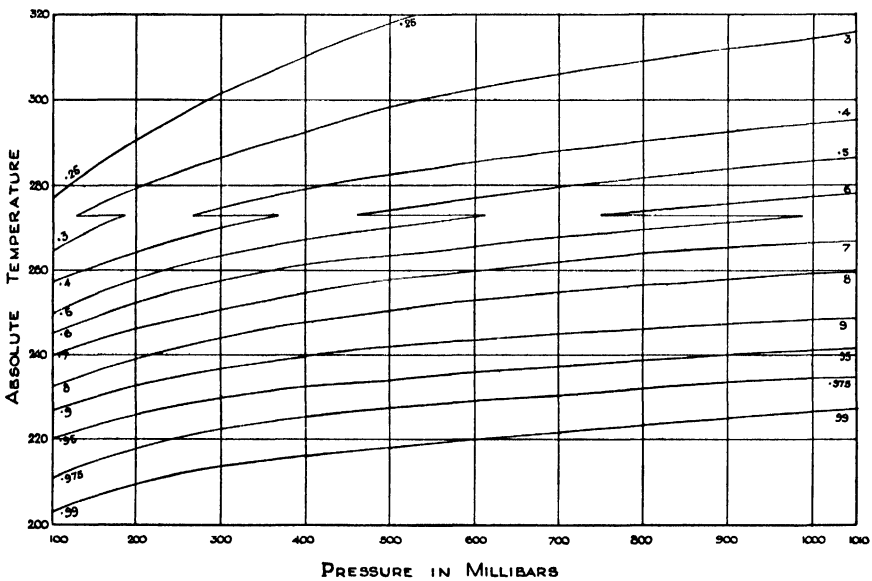


Fig. 18. Isopleths of the saturated adiabatic lapse-rate.

The isopleths give the ratio of the saturated adiabatic lapse-rate to the dry adiabatic lapse-rate for different pressures and temperatures, and can therefore be used for any planetary atmosphere. The numbers shown against the lines in fig. 18 give this ratio, which for the earth's atmosphere can be interpreted as giving the saturated lapse-rate in degrees C per 100 metres.

Equation (24) above can also be derived by differentiation of equation (11), p. 54, with respect to T, and eliminating de/dT by the use of the Clausius-Clapeyron equation. (See § 52 below.)

* *Q.J. Roy. Met. Soc.* **59**, 1933, p. 351.

§ 36. *Stability of saturated air*

f a small element of saturated air is displaced upward from its original posiion at a level z to a level $z+dz$, its temperature will fall by an amount $\left(\frac{\partial T}{\partial z}\right)_s dz$, where $\left(\frac{\partial T}{\partial z}\right)_s$ is the saturated adiabatic lapse-rate appropriate to the onditions of the air, as derived from equation (24) above. The stability or nstability of the air is to be determined by the relative density of the displaced ir and its environment. The displaced air has a temperature of

$$T+\left(\frac{\partial T}{\partial z}\right)_s dz,$$

ınd is surrounded by air whose temperature is

$$T+\frac{\partial T}{\partial z}dz.$$

3ut the density of the air will depend on the humidity as well as on the tem-)erature of the environment.

If the whole environment is also saturated, the condition for stability is)bviously

$$\frac{\partial T}{\partial z}>\left(\frac{\partial T}{\partial z}\right)_s \quad \text{or} \quad -\frac{\partial T}{\partial z}<-\left(\frac{\partial T}{\partial z}\right)_s \qquad \text{......(27)},$$

.e. the lapse-rate must be less than the saturated adiabatic for that level.

If the environment is unsaturated at the level $z+dz$, its density will be ;reater than that of saturated air at the same temperature and pressure, and nstability will enter when the lapse-rate is rather less than the saturated .diabatic. In this case, stability requires that the lapse-rate shall be less than he saturated adiabatic by a finite amount. In practice it is not possible to take ıccount of the details of the variation of water-vapour content of the atmo-phere, and the result derived above is most usefully stated in the following orm: An atmosphere whose lapse-rate is greater than the saturated adiabatic vill be unstable for saturated air, and the degree of instability will be the ;reater the drier the environment. This appears at first sight contradictory,)ut a little consideration shows that it is in keeping with the fact that dry air is lenser than damp air at the same temperature and pressure. It is to be used n the following way. Suppose the atmosphere is at all heights unsaturated, ınd that a small mass of air can be set in upward motion, until it attains aturation. The result we have derived above shows that once saturation is ıttained the air will be in an unstable position with a lapse-rate rather less han the saturated adiabatic above it, and that the further ascent of the aturated air will be facilitated by the fact that the air through which it moves s relatively dry.

The question of the stability of saturated air must always be discussed with egard to *upward* motion. If a mass of saturated air be set in downward motion t is heated at the dry adiabatic rate, unless it carries water drops in suspension.

In the latter case the evaporation of the water drops keeps the rate of heating due to compression down to the saturated adiabatic rate.

If all condensed water drops are immediately precipitated, so that the saturated air carries no water drops, then saturated air is stable for downward motion when the lapse-rate is less than the dry adiabatic, but unstable for upward displacements unless the lapse-rate is less than the saturated adiabatic. The saturated adiabatic is the critical lapse-rate for both upward and downward motion only when the air carries with it a supply of water drops, which can evaporate when the air is dynamically heated by descent.

CHAPTER IV

THERMODYNAMICS OF THE ATMOSPHERE

§ 37. *The concept of entropy*

BEFORE we consider the thermodynamical concept of entropy, we shall find it useful to recall some of the main principles of thermodynamics. Heat, being a form of energy, can be made to do work, the work equivalent of the unit of heat, or, to give it its more familiar title, the mechanical equivalent of heat, amounting to $4 \cdot 18 \times 10^7$ ergs per gramme-caloric. This figure is usually denoted by J, and its reciprocal by A. ($A = 2 \cdot 392 \times 10^{-8}$.)

The state of a gas at any instant, *assuming its constitution is unaltered*, is completely specified by any two of the variables p, v, T, which denote the pressure, specific volume and absolute temperature, respectively. If, in the discussion of meteorological problems, portions of the atmosphere could be treated as isolated entities, there would be no need to introduce anything further. In nature, however, there is no such isolation, and there is a continuous mutual interaction between each portion of the atmosphere and the surrounding portions. When heat is added to a portion of the atmosphere a part of the heat is used in warming the air, and the remainder in expanding the air against the pressure of its surroundings. The name of "working substance" which it is convenient to give to the portion of any gas or mixture of gases on which our attention is at the moment concentrated is itself a recognition of the importance of the environment in the changes of state of that mass.

The state of the working substance at any instant may be represented by a point in a diagram in which the ordinate and abscissa are the pressure and specific volume. Such a diagram is known as a p-v diagram or "indicator diagram". Any cycle of changes of the working substance may be represented by a continuous line in this diagram. In fig. 19 the work done by the body in going from a state denoted by $P(p, v)$ to the state denoted by a neighbouring point P' $(p + dp, v + dv)$ is $p\,dv$, and the total amount of work done in going

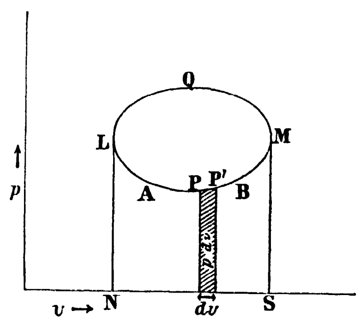


Fig. 19. Indicator diagram.

from A to B is $\int_A^B p\,dv$. If the working substance goes through a complete (Carnot) cycle of changes, starting at A, and eventually returning to A, the total amount of work done by the working substance is $\int p\,dv$, taken round the circuit, which is equal to the area enclosed by the curve. This amount of work done by the working substance is negative when the circuit is performed in a counter-clockwise sense, as in the diagram (fig. 19) when the circuit is performed in the direction *APBQA*. For the work done by the working substance along the path *LABM* is equal to the area *LAMSN*, and the work done in the path *MQL* is equal to the area $-MQLNS$. The total work done by the working substance is therefore negative and equal to the area enclosed by the loop, when the circuit is performed in the counter-clockwise sense. In this case work is done on the working substance by the surrounding medium. When work is done *by* the working substance, the circuit is performed in the reverse or clockwise sense, and energy is given out to the environment.

If we take a cycle bounded by two isothermals and two adiabatics, fig. 20, no heat is given to, or taken away from, the working substance in the parts of the cycle bounded by the adiabatics *AD*, *BC*, but in the parts of the cycle bounded by the isothermals *AB*, *CD* heat must be given to or taken away from the working substance in order to maintain the temperature constant. During the isothermal stages when heat is given to or taken away from the working substance, the temperature, and consequently the internal thermal energy, of the working substance remains constant. The thermal energy supplied is passed on as work done upon the medium. The changes of temperature of the working substance take place during the adiabatic portions of the cycle, when no heat energy is supplied or extracted, and the changes of internal energy of the working substance are accounted for by the work done by the medium on the working substance during compression, or by the working substance on the medium during expansion.

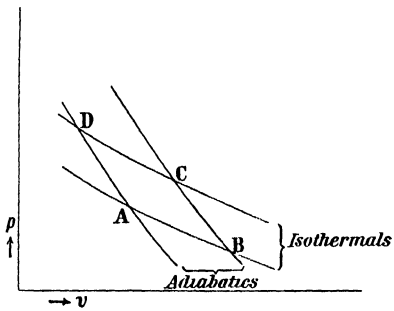


Fig. 20. Cycle bounded by isothermals and adiabatics.

When the working substance is expanding adiabatically it is doing work against the surroundings, and drawing upon its store of internal energy to do this. If in an adiabatic change the temperature falls from T_1 to T_2 the decrease of internal energy* is $c_v(T_1 - T_2)$, and this represents the work done in expanding against the pressure of the surroundings. When the working substance is expanding or contracting isothermally, an amount of heat must be given to or taken away from it, in just sufficient quantity to maintain the temperature constant.

An adiabatic is most conveniently specified by the corresponding potential temperature θ (see p. 38). The equation to the adiabatic may be written (see equations (31) and (32), p. 38)

$$pv^\gamma = D\theta^\gamma \qquad \ldots\ldots(1),$$

* See p. 37, after equation (23).

where D is a constant depending on the standard pressure with respect to which θ is defined. $(D=R^{\gamma}/p_0^{\gamma-1}.)$

If the working substance is kept at temperature T while it moves from the adiabatic θ_1 to the adiabatic θ_2, the amount of work done by the working substance is

$$\int p\,dv = RT\int dv/v = \frac{RT}{M}\log\frac{v_2}{v_1} = \frac{RT}{M}\log\frac{p_1}{p_2}$$

$$= \frac{RT}{M}\cdot\frac{1}{\gamma-1}\left(\gamma\log\frac{v_2}{v_1} - \log\frac{p_1}{p_2}\right) = \frac{RT}{M(\gamma-1)}\log\frac{p_2 v_2^{\gamma}}{p_1 v_1^{\gamma}}$$

$$= \frac{\gamma RT}{M(\gamma-1)}\log\frac{\theta_2}{\theta_1} = \frac{c_p T}{AM}\log\frac{\theta_2}{\theta_1} \qquad \text{......(2)},$$

where the logarithms are to base 10, and M is the modulus 0·4343.

The amount of heat which must be given to the working substance in order to enable it to perform this work is $\frac{c_p T}{M}\log\frac{\theta_2}{\theta_1}$. The quantity of heat necessary in order to enable the working substance to go isothermally from one adiabatic to another is thus proportional to the absolute temperature of the isothermal, i.e. to T.

Now consider the heat necessary to take the working substance from a point A, defined by p, v, T, θ, to a point B defined by p', v', T', θ', along any specified path in the indicator diagram. A point in the indicator diagram is completely specified by any two of the variables p, v, T, or θ. Divide the path into any number of very small arcs of which PQ is one. Let PR and QR be the adiabatic through P and the isothermal through Q respectively. Then the amount of heat which must be communicated to the working substance in order to take the representative point along PRQ is

$$\frac{c_p T}{M}\log\frac{\theta_Q}{\theta_P} \qquad \text{......(3)}.$$

This differs from the amount of heat required to take the point along the direct arc PQ only by the area of the small triangle PRQ, which is a small quantity of the second order.

Thus if θ, θ_1, θ_2, θ_3, θ_4, ... be the potential temperatures, and T, T_1, T_2, T_3, T_4, ... the absolute temperatures, at successive points along the curve AB, the total quantity of heat which must be communicated to the working substance in order to take it along the line AB is equal to

$$\frac{c_p}{M}\left\{T_1\log\frac{\theta_1}{\theta} + T_2\log\frac{\theta_2}{\theta_1} + T_3\log\frac{\theta_3}{\theta_2} + \ldots + T'\log\frac{\theta'}{\theta_n}\right\} \qquad \text{......(4)},$$

or in the limit in which the small arcs are infinitesimally small the amount of heat which must be given to the working substance is

$$c_p\int T\frac{d\theta}{\theta} \qquad \text{......(5)}.$$

Either expression shows that the total amount of heat is dependent on the path followed. Thus the total quantity of heat does not of itself determine the final state of the system.

It will be seen, however, that if the quantity of heat Q communicated to the working substance in the small arc QR is divided by T, then

$$\frac{Q}{T}=\frac{c_p}{M}\log\frac{\theta_Q}{\theta_P} \quad \text{and} \quad \Sigma\frac{Q}{T}=\frac{c_p}{M}\log\frac{\theta_B}{\theta_A} \qquad \text{......(6)}.$$

Thus $\Sigma\frac{Q}{T}$ depends only on the initial and final states of the working substance. For this reason the quantity $\Sigma\frac{Q}{T}$, or, to take the more general expression $\int\frac{dQ}{T}$, has come to be regarded as a fundamental concept in thermodynamics. It is called the *Entropy*, and is usually denoted by the Greek letter ϕ. It is defined by the equation

$$d\phi=\frac{dQ}{T} \quad \text{or} \quad \phi=\int\frac{dQ}{T} \qquad \text{......(7)}.$$

Conversely, the amount of heat communicated in a process which produces a change of entropy $d\phi$ is dQ, where

$$dQ=Td\phi \qquad \text{......(8)}.$$

We might therefore represent the changes of a system, in so far as the communication of heat is concerned, by an indicator diagram in which the ordinate and abscissa are T and ϕ. In an elementary complete cycle, represented by an elementary rectangle of sides $d\phi$ and dT, the total amount of heat communicated is $dTd\phi$, and in any closed cycle the total heat communicated is equal to the area enclosed in the T-ϕ diagram.

Since in any reversible process the change of entropy in going from A to B is

$$\phi_B-\phi_A=c_p\log\frac{\theta_B}{\theta_A},$$

we may therefore define the entropy ϕ by the equation

$$\phi=\frac{c_p}{M}\log\theta \qquad \text{......(9)},$$

where θ is the potential temperature. We may, if we choose, add a constant on the right-hand side of this equation, but as we are never concerned with the absolute magnitude of the entropy, but only with its changes, the addition of a constant is of no particular significance.

It is of fundamental importance to bear in mind the difference between reversible and irreversible physical processes, with regard to the changes of entropy. If the working substance is taken from a state A to a state B, and then brought back again to its original state, then the entropy will return to its original value at A, when the cycle is completed, if, and only if, the process is reversible. If any irreversible action occurs, as for example the conduction of some of the heat to a colder body, or the condensation and falling out of water drops, then the process ceases to be reversible, and the entropy of the whole system is increased thereby.

Consider the changes in entropy when the body goes through a complete (Carnot) cycle. The working substance takes in an amount of heat Q_1 at a temperature T_1, the heat being used in expansion against the pressure of the environment. The entropy increases by an amount Q_1/T_1. During the isothermal compression at temperature T_2 the working substance gives out an amount of heat Q_2 and loses an amount of entropy Q_2/T_2. It follows from equation (6) above that

$$Q_1/T_1 = Q_2/T_2.$$

For the amount of heat involved in the transfer of the working substance isothermally from the adiabatic θ_1 to another adiabatic θ_2 is $\frac{c_p}{M} T \log \frac{\theta_2}{\theta_1}$. Thus Q/T is the same for all isothermal transfers from one adiabatic to another. In any reversible cycle the gain of entropy along one isothermal is equal to the loss of entropy along the other isothermal, so that the initial and final values of the entropy are equal. This statement is readily generalised to the statement that in any reversible cycle the entropy returns to its original value.

The value of the concept of entropy lies in the fact that it is a function depending only on the state of the working substance, and independent of the sequence of changes by which that state was attained. In this respect it should be contrasted with the quantity of heat $\int dQ$ which must be given to the working substance to take it from the state A to the state B. The substance can be taken from the one state to the other by an infinite number of possible sequences, and to each of these possible sequences will correspond a different value of $\int dQ$, but the change of entropy in going from A to B will be the same for all the possible paths, provided that the changes which take place are all reversible.

Though entropy does not in fact give any additional information when we know already the pressure and temperature of each constituent of the system, it is frequently of great value in physical discussion. The fundamental variables of the observer are undoubtedly pressure and temperature, but one is tempted to say that entropy is one of the fundamental variables of nature. Entropy is frequently described as a mathematician's device, enabling him to integrate his differential equations; but it is probably much truer to describe it as a physicist's device for sorting out the most important physical factors involved in any particular process. An example of this was given in § 30, in the derivation of equation (10) for the changes of state of saturated air. The argument which was used was based on the fact that in a reversible process the entropy is a one-valued function of the state of the working substance. In general, if $d\phi$ can be expressed as a function of any variables x_1, x_2, x_3, etc. in the form

$$d\phi = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 + \text{etc.} \qquad \text{......(10)},$$

the right-hand side is a complete differential. Thus having specified the quantity of heat dQ involved in an elementary change of state we may divide it by

T and so find $d\phi$, which must be a complete differential. It might conceivably have been possible to derive equation (10), p. 54 without making use of the concept of entropy, if it were possible to state with certainty the order in which the different processes of condensation, heating and expansion take place. The use of entropy takes charge of this troublesome stage of the problem by sorting out the essential from the non-essential facts. The quantity of heat dQ involved in the elementary change discussed in § 30, when expressed as a function of T and x, is not a perfect differential, the physical reason being that in a complete cycle $\int dQ$ is not zero, being equal to the amount of work done, so that $\int dQ$ is not a single-valued function of the state of the working substance. A variety of different paths in the indicator diagram join any two points selected, and though the final state is clearly defined by the position of the appropriate point in the diagram, different paths represent the conversion of different amounts of heat into work done on the environment, so that $\int dQ$ is not specified by the initial and final points of the path.

§ 38. *The effect of conduction or mixing on the total entropy*

The effect of transfer of heat by conduction or mixing is to increase the total entropy of the system. For suppose a quantity of heat Q is conveyed from a body at temperature T_1 to a body at temperature T_2. (T_2 must obviously be lower than T_1 if the transfer is to be physically possible.) The first body loses an amount of entropy Q/T_1 and the second gains an amount of entropy equal to Q/T_2. The net gain of entropy of the system is

$$Q/T_2 - Q/T_1 = \frac{Q}{T_1 T_2}(T_1 - T_2) \qquad \text{......(11)},$$

which is positive since $T_1 > T_2$.

It should be noted, however, that while the total entropy of a system is increased by mixing, that is, the mean entropy of the unit mass within the system is increased by mixing, the effect of mixing masses of air of different potential temperatures is that the potential temperature of the mixture is the mean potential temperature of the original masses unmixed. For let a quantity m_1 of air at temperature T_1, pressure p_1, and potential temperature θ_1 be mixed with a quantity m_2 of air of temperature T_2, pressure p_2, and potential temperature θ_2, without any gain or loss to the environment. Let the resultant pressure of the final mixture be p. We may suppose the mixing to be carried out in two steps. First the two masses are brought adiabatically to the pressure p, when they will have temperatures $\theta_1 \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}$ and $\theta_2 \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}$ respectively $\left(\frac{\gamma-1}{\gamma} = 0{\cdot}288\right)$. Next let the two masses mix. The internal energy of the mixture will now be the sum of the internal energies of the constituents of the mixture, and the final temperature of the mixture will therefore be

$$\frac{m_1\theta_1 + m_2\theta_2}{m_1 + m_2}\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}.$$

The potential temperature of the mixture will therefore be $\frac{m_1\theta_1+m_2\theta_2}{m_1+m_2}$, which is equal to the mean potential temperature of the constituents.

§ 39. *Formulae for entropy*

Provided we are dealing with a substance whose constitution is not variable, and whose specific heat at constant pressure is c_p, we may define entropy by either of the equations

$$\phi=\frac{c_p}{M}\log\theta \qquad \ldots\ldots(12),$$

$$\phi=\frac{c_p}{M}\log T-\frac{AR}{M}\log p \qquad \ldots\ldots(13).$$

An arbitrary constant may be added to the right-hand side of either of these equations.

We may start from any convenient point for the zero of entropy. If, for example, we wish to take T_0 and p_0 as the starting points for temperature and pressure,

$$\phi=\frac{c_p}{M}\log\frac{T}{T_0}-\frac{AR}{M}\log\frac{p}{p_0}.$$

Convenient values are $T_0=100°$, and $p_0=1000$ mb. Then the equations are

$$\phi=2{\cdot}303\times10^7\log\frac{T}{100}-0{\cdot}661\times10^7\log\frac{p}{1000} \qquad \ldots\ldots(14).$$

The entropy is then given in C.G.S. units, the appropriate values having been inserted for c_p and R to ensure this. Since 1 joule $=10^7$ ergs, the entropy in joules per gramme is given when the factors 10^7 are omitted in the above expression,

$$\phi=2{\cdot}303\log\frac{T}{100}-0{\cdot}661\log\frac{p}{1000} \qquad \ldots\ldots(15)$$

measured in joules per gramme.

§ 40. *Efficiency of a heat engine*

If a reversible engine working between temperatures T_1 and T_2 takes in a quantity of heat Q_1 at temperature T_1, and rejects a quantity of heat Q_2 at temperature T_2, the changes from T_1 to T_2 and from T_2 to T_1 being adiabatic at potential temperatures θ and θ', then the efficiency of the engine is the ratio $(Q_1-Q_2)/Q_1$, which measures the fraction of the heat put into the engine at the higher temperature which is converted into work.

But since the quantities Q_1 and Q_2 are proportional to the absolute temperatures at which the isothermal stages of the cycle are performed (equation (3) above), the efficiency is also equal to

$$(T_1-T_2)/T_1.$$

For the perfect reversible engine the efficiency increases as the temperature

T_2 of the cold source diminishes. If the cold source were at absolute zero, the efficiency of the engine would be unity.

The condition for reversibility might be written

$$Q_1/T_1 = Q_2/T_2.$$

The engine is not reversible when heat is lost by some such agency as conduction to some external source, or precipitation of condensed water. The amount of heat finally rejected is then less than in the absence of such agencies, and

$$Q_2/T_2 < Q_1/T_1.$$

It follows from the definition of entropy given above that in such a case the entropy of the working substance is finally greater than at the beginning of the cycle.

§ 41. *Entropy-temperature diagrams or T-ϕ diagrams*

If a working substance is taken through a series of changes in a reversible manner, the state of the working substance, being thus at each instant fixed by the temperature and entropy, can be represented in a diagram with temperature and entropy as ordinates. It is convenient to measure entropy along the vertical axis, and temperature along the horizontal axis. The amount of heat communicated to the working substance in order to take it from the state represented by the point P to the state represented by the point P' is $Td\phi$, or the area $PNN'P'$ (fig. 21) and the total amount of heat given to the working substance in going from state A to state B is the area $ARSB$, or $\int_A^B Td\phi$. The total amount of heat is positive in the transit from A to B as drawn in the diagram, since the entropy increases steadily. If the path is followed in the reverse order from B to A, $\int_B^A Td\phi$ is negative, and heat is taken from the working substance.

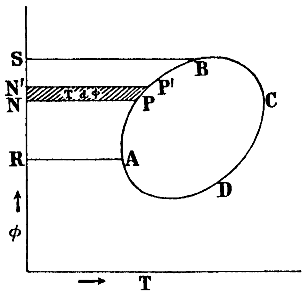


Fig. 21. T-ϕ diagram.

If the representative point executes a closed curve, indicating that the working substance, after going through a complete cycle, has returned to its original state, the total amount of heat communicated to the working substance is equal to the area enclosed by the closed curve $ABCD$, and it is readily seen that if the cycle is performed in the clockwise direction, i.e. in the direction $ABCD$, the net amount of heat communicated to the working substance is negative, or, in other words, heat is extracted from the working substance. When the cycle is performed in the counter-clockwise direction, as along $ADCB$, the net amount of heat communicated to the working substance is positive. In the first case work is done on the environment by the working substance, and in the second case work is done by the environment on the working substance. The amount of work done in either case is measured by the area enclosed by the curve in the T-ϕ diagram.

Examples of T-ϕ diagrams corresponding to certain particular cases are given in any textbook of thermodynamics, and the reader is referred to such books for further information as to the general use of T-ϕ diagrams. The T-ϕ diagram has one outstanding advantage in that the isothermals and adiabatics are straight lines in these diagrams, being parallel to the axes of co-ordinates, instead of being curved as in the usual p-v diagram. But in all thermodynamic diagrams the saturated adiabatics are curved lines.

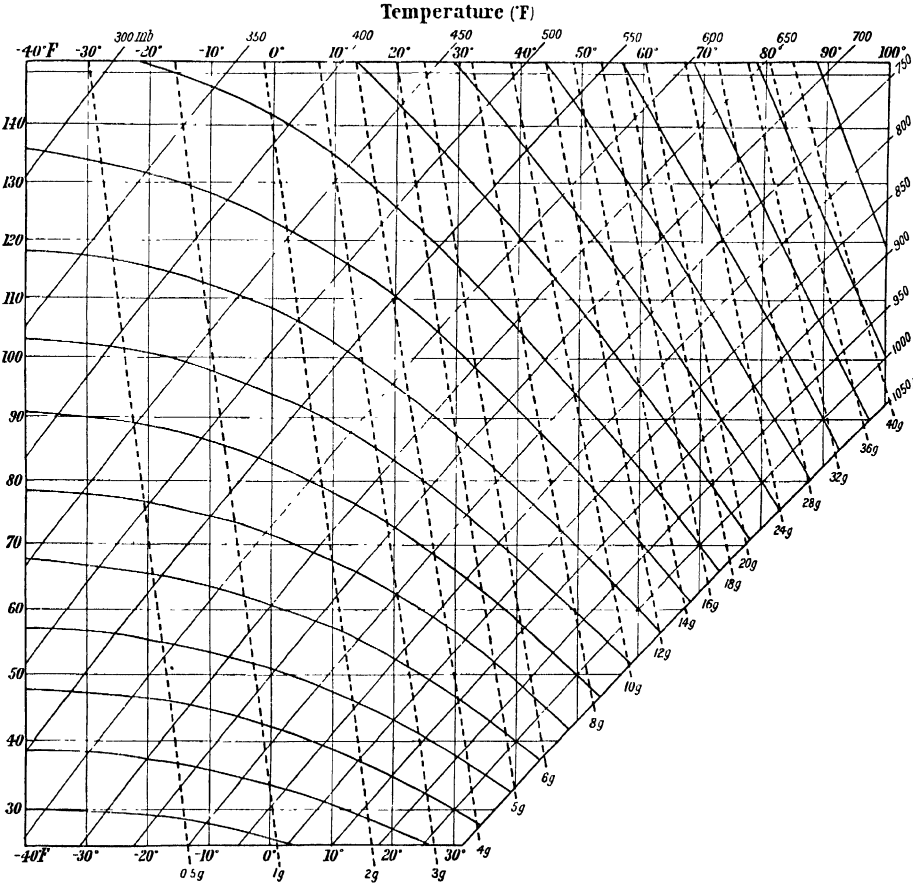


Fig. 22. Tephigram.

§ 42. *The tephigram*

The T-ϕ diagram has been adapted for meteorological use by Shaw in the *Manual of Meteorology*, **3**. On the tephigram (T-ϕ-gram) the absolute temperature is measured along the horizontal axis, and the entropy along the vertical axis (see fig. 22). Shaw identifies entropy with $c_p \log \theta/M$, where θ is the potential temperature, and the linear entropy scale is actually a logarithmic scale of potential temperature. The adiabatics or pseudo-adiabatics for saturated air are shown as curved lines, and the pressure lines are shown sloping downward from left to right, while the broken lines indicate the number of grammes of water-vapour necessary to saturate 1 kilogramme of dry air at the temperature shown. The values of entropy read off for points on the saturated adiabatics are values of the entropy of dry air at the corresponding temperature, evaluated by the formula

$$\phi = \frac{c_p}{M} \log \theta + \text{const.}$$

$$= \frac{c_p}{M} \log \frac{T}{T_0} - \frac{R}{M} \log \frac{p}{p_0} \qquad \text{......(16)},$$

where c_p is taken in dynamic units, and $c_p/M = 2{\cdot}303 \times 10^7$. It should be borne in mind that in all discussion of tephigrams hitherto published, the effect of

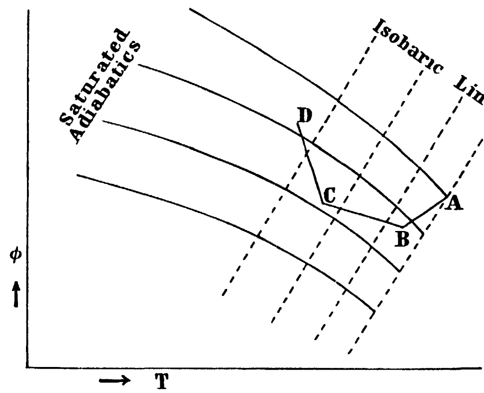


Fig. 23. Tephigram, schematic.

water-vapour on the entropy and density of the air has been disregarded. Nor should it be overlooked that the discussions of adiabatic descent or ascent of air also neglect the possible effects of radiation, absorption and turbulence, whose magnitude has not been estimated in a single instance.

Subject to the above limitations the tephigram makes it possible to see at a glance the state of the air as regards stability. The dry adiabatics are horizontal lines, while the isotherms are vertical lines (see fig. 22). When the lapse-rate exceeds the dry adiabatic the curve in the tephigram slopes downward to the left. When the lapse-rate is less than the dry adiabatic the curve slopes upward to the left, while an inversion gives a line sloping upward to the right. Isothermal conditions are represented by a portion of the curve running vertically, while a lapse-rate between the saturated and the dry adiabatic will slope upward to the left at an angle less than the slope of the saturated adiabatic.

Thus if *ABCD* in fig. 23 represents the observed conditions in the atmosphere, *AB* represents a super-adiabatic lapse-rate, *BC* a lapse-rate less than the dry adiabatic, and *CD* a lapse-rate less than the saturated adiabatic.

Figs. 22, 23 have been drawn so that temperature increases from left to right, as in the Hertz diagram, and in accordance with recent British practice.

§ 43. *The use of the Neuhoff diagram or the tephigram to evaluate the energy liberated during the ascent of air*

In the Neuhoff diagram the co-ordinates are temperature on a linear scale and pressure on a logarithmic scale. Let the conditions at height z in the atmosphere be represented by p, ρ and T. Let an element of air of unit mass, which is set in motion in a manner that we need not at the moment specify, have density ρ' and temperature T' when it is at height z. It will have the same pressure as its environment at the same level.

Whether the moving air or its environment is damp or not will not be considered, and any portion of air will be regarded as satisfying the same gas-equation

$$p = R\rho T,$$

R being the appropriate constant for the air.

The difference involved in neglecting the effect of water-vapour on density is slight.

The acceleration of the element of moving air is

$$g\frac{\rho-\rho'}{\rho'} \quad \text{or} \quad g\frac{T'-T}{T} \qquad \ldots\ldots(17).$$

In moving through a distance dz it does an amount of work dW, where

$$dW = \frac{g\,(T'-T)\,dz}{T} = \frac{g\rho\,(T'-T)\,dz}{\rho T} = -\frac{R\,(T'-T)\,dp}{p} \qquad \ldots\ldots(18),$$

or

$$dW = -R\,(T'-T)\,d\log p \qquad \ldots\ldots(19).$$

In fig. 24 (*a*), if *AQC* be the line indicating the changes of temperature of the moving air (this may be either a dry or saturated adiabatic, or partly one, partly the other), and if *ABCD* represents the conditions in the environment, then the moving air at *Q* is surrounded by air whose condition is represented by *P* at the same pressure. Let *P'Q'* be a subsequent position of *PQ*, separated by a very small vertical distance $d\log p$. Then

$$dW = R \times \text{area } PQQ'P'.$$

Thus the amount of work done, or energy released, in an elementary displacement is equal to R times the area $PQQ'P'$ drawn as shown in the figure.

If the air at *A* is in an unstable condition, and is set in vertical motion along the adiabatic *AQC*, it will attain a level where it can again be in equilibrium at *C*, and the amount of energy liberated per gramme of ascending air is R times the area enclosed between the curve of observed temperatures and the curve of temperature of the ascending air. The latter will be an adiabatic line of some kind. It may be a dry or a saturated adiabatic, or it may be a dry adiabatic along the lower part of its course and a saturated adiabatic along the remainder of its course. The proof given above is perfectly general, and applies to any of these possibilities.

A result which is precisely similar can be derived for the tephigram. By definition

$$\phi = c_p \log T - AR \log p + \text{const.} \qquad \text{......(20)},$$

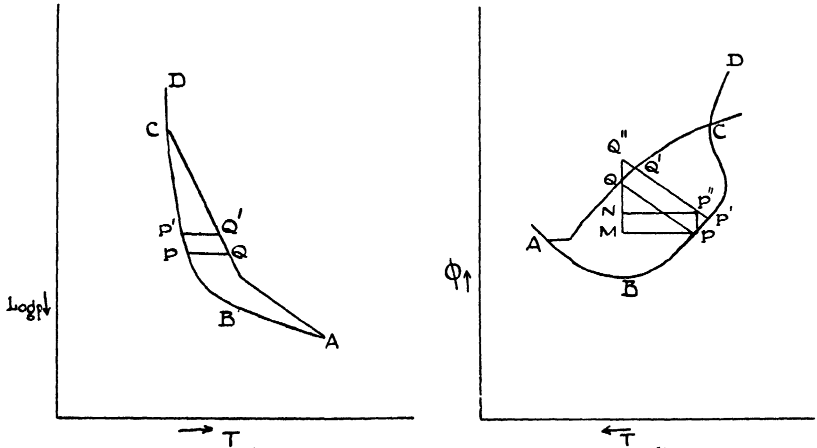
or

$$d\phi = c_p \frac{dT}{T} - AR \frac{dp}{p} \qquad \text{......(21)}.$$

As in equation (18)

$$dW = -(T' - T) R \frac{dp}{p}.$$

In fig. 24 (*b*) the line *ABCD* represents the condition of the environment, or, in other words, it is the curve which shows the actual observations of temperature, and *AQB* represents the change of state of the moving mass. *PP″* is



(*a*) (*b*)

Fig. 24. Energy in the Neuhoff diagram and the tephigram.

the change of entropy when the pressure changes by dp while the temperature remains constant. Thus

$$PP'' = R \frac{dp}{p},$$

$$dW = -(T' - T) R \frac{dp}{p} = \text{area } PP''NM = \text{area } PP''Q''Q$$

$$= \text{area } PP'Q'Q,$$

since the ends of the strip, the two small triangles *PP′P″* and *QQ′Q″*, may be neglected as small quantities of the second order. Thus the energy released by the ascent of the air from *Q* to *Q′* is equal to the area *PQQ′P′*, and by integration it follows that the energy released by the motion of the ascending air from *A* to *C*, where it is again in equilibrium with its environment, is measured by the area *APCQA*.

The use of the tephigram in the way suggested is to be regarded as an approximation, as it takes no account of the water-vapour in the atmosphere, in the estimate of the buoyancy of a moving mass. No method has been devised which can take account of the water-vapour in the atmosphere for all purposes. The tephigram in effect takes account only of the density of the dry air, though it takes account of the effect of the presence of water-vapour on the temperature changes which are produced by the ascent of the moist air. Strictly speaking, equilibrium of a mass of air in the atmosphere is determined by its density and not by its entropy, and the tephigram therefore fails to give a criterion which will apply to all cases. In practice, the error will be very small for motions on a large scale, and no serious error is to be feared if the tephigram is used in the manner suggested.

A clear distinction should be drawn between the entropy of damp air, as derived from the formulae which have been developed above, and the entropy which is represented in the tephigram. We may write (see § 46 below)

$$\begin{aligned}\text{Entropy of moist air} &= \text{Entropy of dry air}\\ &\quad + \text{Entropy of water-vapour}\\ &= c_p \log T - AR \log (p-e)\\ &\quad + xc \log T + Lx/T.\end{aligned}$$

It is an approximation to the first part of the entropy, given by

$$c_p \log T - AR \log p$$

which is represented in the tephigram, and the water-vapour is disregarded as having no effect on the stability of the atmosphere.

The tephigram gives precisely the same results as the Neuhoff diagram, both for estimating the stability of the atmosphere, and for estimating the amount of energy which can be released by the ascent of air. The use of the word entropy in this connection may mislead the uninitiated, and a word of warning is given here, that all methods which base their criteria on the neglect of the density effect of water-vapour must of necessity lead to the same results.

It has been suggested that the entropy of any mass of air will indicate the height at which it will be in equilibrium with its environment, but the entropy of dry air alone, which Shaw calls realised entropy, increases with height, and so the entropy measured by the tephigram as realised entropy is not constant for a given mass of air. Ascending air will come to rest when two conditions are realised, namely, the pressure and the density of the ascending air must both attain equality with those factors for the environment. It is assumed that the ascending air will at all stages take up the pressure of the environment. This is readily taken into account by plotting temperature against pressure, or against a function of temperature and pressure such as entropy. The graph of observed conditions is plotted on a diagram on which the adiabatics of dry and saturated air are shown, so that it is possible to determine the height at which the ascending air disturbed from its position as part of the normal environment will again attain the same temperature as the environment. This is not strictly the level at which the ascending air will attain equilibrium. For equilibrium is determined by equality of density, and density is partly determined by the water-vapour content. Neither the tephigram nor the Hertz or Neuhoff diagram can take account of the effect of water-vapour on density, though it would be possible to construct a diagram which would do so. This could be done by showing in the graph, not the observed temperatures, but temperatures at which the air would, when saturated, have the same density as the actual air of the environment*. But such a diagram would of necessity give the same general results as the tephigram or Neuhoff diagram, since the main large-scale variations of density are produced by variations of temperature, the main effect of the variations of water-vapour content being to change the height at which the air when disturbed from its equilibrium position will attain saturation.

To put the matter into a few words, there is no special virtue in any one of the standard methods of plotting temperature observations, and whether we adopt the tephigram, the T-log p diagram of Neuhoff, or any other method, is more a matter of personal inclination than a matter of real scientific significance. The tephigram and the Neuhoff diagram have the great advantage that, as shown in fig. 24 above, it is possible to evaluate on these diagrams the amount of energy which is liberated by the ascent of unit mass of air, by the direct computation of an area. This advantage is definitely of importance, but any diagram which will enable us to carry out this computation will have all the virtues of any other.

The tephigram has one great advantage, from the point of view of mere convenience, that the isothermals and dry adiabatics are straight lines at right angles to each other, and as a result the diagram is usually more compact than is the Neuhoff diagram, on which the area between the pseudo-adiabatics and the curve of observed temperatures is usually a long and narrow area.

To most of its users the tephigram is a diagram in which the co-ordinates are temperature and pressure, along axes which are not rectangular, but are so arranged that isothermals and dry adiabatics are vertical and horizontal respectively. That the vertical co-ordinate is entropy is not in practice taken into account by the average user of the tephigram.

§ 44. *The representation of geodynamic height on the tephigram*

The tephigram can be also used for the representation of height in the following way. Let the geopotential at any height z be V. Then

$$dV = g\,dz \qquad \text{......(22).}$$

Also
$$d\phi = c_v \frac{dT}{T} - \frac{R\,dp}{p} = c_p \frac{dT}{T} + \frac{Rg\rho\,dz}{R\rho T} = c_p \frac{dT}{T} + \frac{dV}{T} \qquad \text{......(23).}$$

Hence
$$V = \int T\,d\phi + c_p T + \text{const.} \qquad \text{......(24).}$$

* Such a temperature might be called the "saturated virtual temperature".

If V_0 be the geopotential at A (z_0, T_0), and V at B, then

$$V - V_0 = \int_A^B T d\phi + c_p (T - T_0) \qquad \text{......(25).}$$

In fig. 25, $\int_A^B T d\phi$ is the area between the observed curve AB and the line of zero temperature. The term $c_p (T - T_0)$ is the heat required to raise the temperature of the air from T_0 to T at constant pressure, and is therefore equal to $\int_A^C T d\phi$. V is thus represented by the singly-shaded area in the figure.

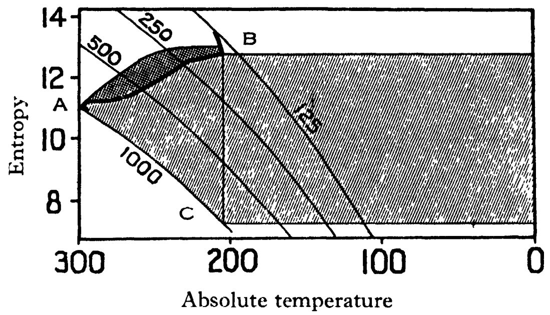


Fig. 25. Representation of geodynamic height on a tephigram.

§ 45. *Latent heat of evaporation*

The latent heat of evaporation of water is defined as the quantity of heat which is required to convert 1 gramme of liquid water into water-vapour at the same temperature. The heat is used partly in raising the internal energy of the water, and partly in doing the work of expansion against the external pressure.

Let v_1 = specific volume of liquid water.
v_2 = specific volume of water-vapour.
E_1 = internal energy of 1 gramme of liquid water.
E_2 = internal energy of 1 gramme of water-vapour.
e = vapour-pressure.
p = total pressure.
L = latent heat.
A = reciprocal of mechanical equivalent of heat.
ϵ = ratio of densities of water-vapour and dry air.

If the water is evaporated under the pressure of its own vapour, then it is seen that the latent heat of L is given by

$$L = E_2 - E_1 + Ae (v_2 - v_1) \qquad \text{......(26).}$$

The specific volume of liquid water, v_1, is negligible by comparison with that of water-vapour, v_2, and equation (26) then reduces to

$$L = E_2 - E_1 + ART/\epsilon \qquad \text{......(27).}$$

The question arises whether equation (27) still holds when the evaporation takes place into air.

Let a volume V be occupied by air and vapour at a total pressure p, and let an additional quantity m of water be evaporated, thereby increasing the volume to $V+v$. Then by definition

$$mL = mE_2 - mE_1 + Apv \qquad \text{......(28).}$$

In the final stage let p_1 be the partial pressure of the air and water-vapour originally present, and p_2 the partial pressure of the added vapour. Then

$$p = p_1 + p_2,$$

$$p_1 (V+v) = pV \text{ by Boyle's law,}$$

$$(p_1 + p_2)(V+v) = p(V+v).$$

Hence

$$p_2 (V+v) = pv.$$

But p_2 and $V+v$ are the partial pressure and volume of a mass m of water-vapour at the temperature T. Hence

$$pv = p_2 (V+v) = mRT/\epsilon \qquad \text{......(29).}$$

Substituting this value of pv in equation (28), we again obtain equation (27), which is therefore true in the case of evaporation into air.

§ 46. *The entropy of damp air*

In discussing the thermodynamics of moist air we shall require to write down the entropy of 1 gramme of water-vapour at a given temperature T. By the second law of thermodynamics, as we have seen in the earlier parts of the present chapter, the entropy will be independent of the manner in which the water-vapour is formed, as it depends only on the final state of the vapour. The entropy is the same when the vapour is formed by evaporation at the temperature T as it is when the vapour is first formed at a lower temperature T', and then subsequently heated to the temperature T. The first alternative affords a simple method of computing the entropy. We then have

Entropy of 1 gramme of water-vapour at temperature T

$$= \text{entropy of 1 gramme of liquid water at temperature } T + \frac{\text{Latent heat at temperature } T}{T}$$

$$= \int \frac{c\,dT}{T} + \frac{L}{T},$$

where c is the specific heat of liquid water at temperature T, and L is the latent heat at the same temperature. If we assume a mean value c of the specific heat, then the entropy may be written

$$\phi = c \log T + L/T \qquad \text{......(30).}$$

We have already made use of this expression in § 30 in the discussion of the thermodynamics of ascending moist air.

If we have to consider moist air whose humidity mixing ratio is x, then the entropy of $(1+x)$ grammes of the moist air is equal to the entropy of 1 gramme of dry air, *plus* the entropy of x grammes of water-vapour. It therefore amounts to

$$c_p \log T - AR \log (p-e) + xc \log T + Lx/T$$

or

$$(c_p + xc) \log T - AR \log (p-e) + Lx/T \qquad \ldots\ldots(31).$$

It has been assumed that the specific heat of water has a mean value c; this assumption is not strictly necessary, and the term which we have written in the last equation as $xc \log T$ might have been written as $x \int \frac{c\,dT}{T}$. Actually the error involved in writing $c \log T$ instead of the integral is usually very small, as the changes of entropy of moist air are predominatingly due to the changes in the value of the term Lx/T. Moreover, the expression which has been derived above for the entropy of moist air will usually only be used over a range of temperature which does not involve a wide variation in the value of c. Over such a restricted range we may take the mean value of c appropriate to that range.

§ 47. *The thermodynamics of the wet- and dry-bulb hygrometer*

When it has attained a steady state the wet-bulb thermometer is not gaining or losing heat, so that the heat required to evaporate water from the bulb must be supplied by the cooling of the air which comes into contact with the wet bulb. It is usually assumed that the air in contact with the wet bulb becomes saturated at the temperature of the wet bulb. This assumption is not justifiable *a priori*, and it must be judged by the results which follow from it rather than on strictly theoretical grounds.

We shall use the following notation:

T = absolute temperature of the dry-bulb thermometer.
T' = absolute temperature of the wet-bulb thermometer.
x = humidity mixing ratio of the normal air.
x' = humidity mixing ratio of air saturated at the wet-bulb temperature T'.
e = vapour-pressure of the normal air.
e' = vapour-pressure of air saturated at temperature T'.
c_p = specific heat at constant pressure of dry air.
c_p' = specific heat at constant pressure of water-vapour.
L = latent heat of water-vapour at temperature T.
L' = latent heat of water-vapour at temperature T'.
ϵ = ratio of densities of water-vapour and dry air at same temperature and pressure.
p = total pressure.

In the original air of the environment 1 gramme of dry air is associated with x grammes of water-vapour at temperature T. The assumption which we have mentioned above means that $(1+x)$ grammes of the original air, in cooling from T to T', yield up sufficient heat to evaporate $(x'-x)$ grammes of water, and that the result is to produce $(1+x')$ grammes of air saturated at temperature T'. The evaporation takes place at the temperature T' of the wet bulb, and the equation which gives the heat exchange is

$$(c_p + xc_p')(T - T') = L'(x' - x) \qquad \ldots\ldots(32).$$

The first factor on the left-hand side of this equation is very frequently reduced to its first term c_p, since c_p'/c_p is about 2, and x seldom exceeds 0·025. Equation (32) then becomes

$$c_p(T - T') = L'(x' - x)$$

or

$$T + \frac{L'x}{c_p} = T' + \frac{L'x'}{c_p} \qquad \ldots\ldots(33).$$

This is the form used by August and Apjohn, and also by Normand* in his discussion of the thermodynamics of the wet-bulb temperature. But equation (32), when expanded without any approximation, reads

$$c_pT + L'x + c_p'x(T - T') = c_pT' + L'x' \qquad \ldots\ldots(34).$$

If T'' is the temperature of absolutely dry air whose wet-bulb temperature is T', then equation (34) must hold for $T = T''$ and $x = 0$, so that

$$c_pT'' = c_pT' + L'x' \qquad \ldots\ldots(35).$$

Combining equations (34) and (35) we find

$$c_pT + L'x + c_p'x(T - T') = c_pT' + L'x' = c_pT'' \qquad \ldots\ldots(36).$$

The temperature T'' thus defined is known as the *equivalent temperature*, and it is clear from equation (35) that T'' is a function of the wet-bulb temperature only.

When in equation (32) we substitute $x = \epsilon e/(p - e)$, and $x' = \epsilon e'/(p - e')$, we find

$$\frac{\epsilon(e' - e)pL'}{(p - e)(p - e')} = (T - T')\left(c_p + \epsilon c_p' \frac{e}{p - e}\right) \qquad \ldots\ldots(37).$$

But $\epsilon c_p'/c_p = 1{\cdot}21$, and when e is not a large fraction of p we may neglect its variation from unity. The right-hand side of equation (37) then becomes $c_p(T - T')p/(p - e)$, and the equation may be written

$$e' - e = Bp(T - T') \qquad \ldots\ldots(38),$$

where

$$B = (1 - e'/p)\frac{c_p}{L'\epsilon} \qquad \ldots\ldots(39).$$

For the detailed application of this formula to psychrometry, reference should be made to a paper by Whipple†, in which this form was first given. The usual historic form of the equation has unity in place of the factor $1 - e'/p$.

§ 48. *Some thermodynamical propositions relating to the wet-bulb temperature*

In § 47 above we have stated the assumptions which are made in Normand's work on the thermodynamics of the wet-bulb temperature. It is necessary to bear these very clearly in mind in what follows, if confusion between assump-

* *Mem. Indian Met. Dept.* **23**, 1921, p. 1.
† *Proc. Phys. Soc.* **45**, 1933, p. 307.

tion and deduction is to be avoided. Normand deduced the following propositions from the assumptions:

Proposition I. The heat content of the air is equal to the heat content of the same air saturated at the wet-bulb temperature, *minus* the heat content of the additional liquid water required so to saturate it.

Proposition II. The entropy of air is equal to the entropy of the same air saturated at the wet-bulb temperature, *minus* the entropy of the additional liquid water required so to saturate it.

The first of these propositions appears a self-evident truth when stated as above. It is in fact no more than a re-statement of the assumptions made above as to what happens at the wet bulb, and equation (32) above is the mathematical statement of these assumptions. Any attempt at a mathematical proof of proposition I, which would of necessity use equation (32) in some form, would be to argue in a circle.

Proposition II is not quite in the same category. The process which we visualise is not reversible, since heat is taken from air at a temperature between T and T', and taken up by water at a temperature T', so that there is a gain of entropy. The simplest method of approaching this proposition is to evaluate the gain of entropy. The system which we have to consider consists initially of $(1+x)$ grammes of moist air, plus $(x'-x)$ grammes of liquid water, and at the final stage consists of $(1+x')$ grammes of moist air at temperature T'. An amount of heat $(c_p+xc_p')(T-T')$ is taken from the air and absorbed by water at a temperature T', being used up in evaporation. The gain of entropy by the water is equal to

$$(c_p+xc_p')\,\frac{T-T'}{T'}.$$

The loss of entropy of the initial damp air is

$$(c_p+xc_p')\log\frac{T}{T'}.$$

The total gain of entropy of the system is

$$(c_p+xc_p')\left(\frac{T-T'}{T'}-\log\frac{T}{T'}\right) \qquad \text{......(40).}$$

But $$\log T/T'=\log\left(1+\frac{T-T'}{T'}\right)=\frac{T-T'}{T'}-\frac{1}{3}\left(\frac{T-T'}{T'}\right)^3+\ldots \text{ etc.}$$

Substituting in equation (40) above, we find that the net gain of entropy is

$$(c_p+xc_p')\left[\frac{1}{3}\left(\frac{T-T'}{T'}\right)^3+\text{higher powers of }\frac{T-T'}{T'}\right] \qquad \text{......(41).}$$

Thus the net gain of entropy of the system is very closely equal to $\frac{1}{3}\left(\frac{T-T'}{T'}\right)^2$ times the loss of entropy of the original air. This is a small fraction, and may be neglected, and the process which we have been considering may therefore be treated as isentropic. Proposition II may thus be taken as verified to a high degree of approximation.

Proposition I has a number of interesting applications to the thermodynamics of moist air. It follows from the original assumption on which the derivation of equation (32) was based that $(1+x)$ grammes of moist air at temperature T, having a wet-bulb temperature T', can be cooled to the temperature T' by the evaporation into it of the necessary $(x'-x)$ grammes of liquid water. Being then saturated, it cannot be cooled beyond T', and the wet-bulb temperature may accordingly be defined as the lowest temperature to which air can be cooled by the evaporation of water into it.

If we now start off with a given mass of air whose temperature is T, and whose wet-bulb temperature is T', then we know that T' is the lowest temperature to which the air can be cooled by evaporation. But the cooling down to the temperature T' obviously need not be carried out in one step. Suppose that an amount of water is evaporated into the air, but that it is not sufficient to cool it down to T'. In the intermediate stage then reached, the maximum cooling down to T' will still be attainable by the further evaporation of the right amount of additional water. The limit attainable by cooling will still be the same, and hence the wet-bulb temperature is still T'. Hence it follows that if air is cooled by the evaporation of water into it, the wet-bulb temperature remains unaltered during the process, and the air cannot be cooled below the wet-bulb temperature by this means, since it will be saturated when it reaches that temperature, and any further evaporation will be impossible.

There is implied in the above discussion the assumption that the evaporation all takes place at the temperature T', or that the liquid water is introduced into the system at a temperature T'. Any error introduced by the variations in the temperature of the water from T' will be slight, on account of the smallness of the specific heat of water by comparison with its latent heat.

This result can be derived mathematically from equation (32), though the general argument used above, if it has been properly stated, should be more convincing than any equation. The heat content of $(1+x)$ grammes of moist air is

$$c_p T + L'x + xc_p'(T - T') + xcT' \qquad \text{......(42).}$$

By equation (32) this is equal to

$$(c_p T' + L'x') + xcT' \qquad \text{......(43).}$$

Thus the heat content of the air, *minus* the heat content of the water vapour when reduced to liquid water at temperature T', is a function of the wet-bulb temperature T' only. Now evaporate a further quantity x_2 of water. The total heat content of the air will then amount to

$$(c_p T' + L'x') + (x + x_2)\, cT' \qquad \text{......(44).}$$

It is seen at this stage that the part of this expression which remains when the heat content of the equivalent amount of water is subtracted is unaltered. It follows therefore that the wet-bulb temperature is unchanged.

It can also be deduced from equation (43) that when portions of air of different temperatures, but having the same wet-bulb temperatures, are mixed, the mixture has that same wet-bulb temperature. If for example two masses

m_1 and m_2, whose humidity mixing ratios are x_1 and x_2, each having the wet-bulb temperature T', are mixed together, it follows from (43) that the total heat content of the mixture is

$$(m_1+m_2)(c_pT'+L'x')+(m_1x_1+m_2x_2)\,cT' \qquad \text{......(45)}$$

and the heat content of 1 gramme of the mixture is

$$(c_pT'+L'x')+\frac{m_1x_1+m_2x_2}{m_1+m_2}\,cT' \qquad \text{......(46).}$$

Hence the wet-bulb and the equivalent temperature of the mixture are both the same as for the separate constituents of the mixture.

§ 49. *Variation of dry- and wet-bulb temperatures, and of dew-point in air ascending adiabatically*

Normand has further shown that the dry adiabatic through the dry-bulb temperature T, the saturated adiabatic through the wet-bulb temperature T', and the dew-point line through the dew-point T'', all meet in a point. For if air saturated at T' be taken up adiabatically from B (fig. 26) until the humidity mixing ratio reaches the value x at the point D, the air represented by D will have $(x'-x)$ grammes of liquid water mixed with it. Let this water be removed as rain, and let the air then return to ground level, following the dry adiabatic DA. The process has been adiabatic, and therefore the initial entropy of the air saturated at T' is equal to the entropy of the air at A, *plus* the entropy of the water removed at D. Hence it follows from the second proposition of p. 83 that the wet-bulb temperature is the same in the initial and final conditions. Further, since T' is the wet-bulb temperature of air with temperature A and humidity mixing ratio x, it follows that A represents the initial dry-bulb temperature T. The line DC represents the temperatures and pressures at which x grammes of water-vapour will saturate the air, and so D must be the dew-point of the initial air. Thus the proposition is proved. Hence if unsaturated air be raised adiabatically, it will attain saturation at the level where the dry adiabatic through T meets the saturated adiabatic through T'. This gives a simple method of finding the saturated adiabatic which ascending air will eventually follow, and provides a very simple method of completing the tephigram or Neuhoff diagram.

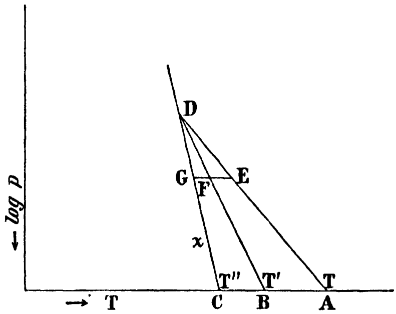


Fig. 26. Normand's temperature-height diagram.

If in fig. 26 EFG is drawn horizontally to intercept the three lines through D, the air, when it has reached the height appropriate to this level, will have dry-bulb temperature, wet-bulb temperature and dew-point temperature represented by the points E, F and G respectively.

It follows directly from the results which we have stated above that all masses of air which have at ground level the same wet-bulb temperature though varying dry-bulb temperature, will ultimately follow the same saturated adiabatic when they ascend.

The proposition whose proof we reproduce above from Normand (*loc. cit.*) is not mathematically exact. The temperature at which the water is removed from the system at D is not equal to the wet-bulb temperature but the error involved is very slight, less in fact than the errors of observation of temperature would introduce.

§ 50. *The wet-bulb potential temperature*

It has been shown that the wet-bulb temperature has the property of remaining invariant during the evaporation of water into the air. Also when a mass of damp air rises or falls in the atmosphere, the wet-bulb temperature follows a saturated adiabatic during the whole of the ascent or descent, whether there is actual condensation taking place or not. Consider a mass of air rising through the atmosphere, having not yet attained saturation. Let the ascent be checked at some stage, and let some additional water be evaporated into it. Up to the time of the addition of the water, the wet-bulb temperature followed a saturated adiabatic. During the evaporation the wet-bulb temperature remained constant, and during any subsequent ascent, the wet-bulb temperature must follow the same saturated adiabatic, since the evaporation did not move the wet-bulb temperature from that adiabatic. If then we introduce the concept of a wet-bulb potential temperature, θ', which we define as the wet-bulb temperature that the air will take up when brought adiabatically to a standard pressure, then the wet-bulb potential temperature so defined will have the property that it remains invariant during all adiabatic and pseudo-adiabatic changes, whether accompanied by condensation or evaporation or not. Loss or gain of water has no effect on the wet-bulb potential temperature, which only varies when heat is given to the air from outside itself, either by mixing or by radiation or conduction, or is taken from it by the same agencies. We may contrast with this property of the wet-bulb potential temperature the property of normal ordinary potential temperature, which remains invariant for adiabatic changes of pressure only so long as there is neither condensation nor evaporation taking place. The wet-bulb potential temperature is therefore a much more potent instrument in determining the differences of origin of different air masses.

A direct application of the wet-bulb potential temperature to the phenomenon of the Föhn was mentioned by Normand in his paper to which reference has frequently been made in the present chapter. It follows from the preceding discussion that if a mass of air rises over a mountain range, and deposits most of its original water-vapour content on the windward side as rain, when after passing over the crest it is again brought down on the other side to its original level, it will be warmed at the dry adiabatic rate. The dry-bulb temperature will accordingly be much higher on the lee side than on the windward side of the mountain, but from the preceding discussion it follows that the wet-bulb temperature will be the same on the two sides. Thus the wet-bulb temperature might be used as a criterion for establishing the original identity of the warm dry air on the lee side with the cooler damp air on the windward side of the mountain.

Normand quotes the following records of conditions at Berkeley, California, during a short-lived Föhn wind, in which an easterly wind from the Berkeley Hills replaced the usual westerly sea winds for a period of about half an hour. The temperature rose 12° F in less than 10 minutes, while the relative humidity dropped below 50 per cent with remarkable suddenness. The variations are shown in the small table below:

Time (hours)	$7\frac{1}{2}$	$7\frac{3}{4}$	8	$8\frac{1}{4}$	9
Air temperature (° F)	52	63	56	56	61
Relative humidity (%)	99	49	90	95	85
Wet-bulb temperature (° F)	52	53	54	55	58·5

The wet-bulb temperature rose during the morning at the normal diurnal rate, showing no sign of an abrupt change with the onset of the easterly wind. It may be concluded that the easterly wind might have had the same origin as the westerly wind, but had been subjected to a series of adiabatic changes by ascent and subsequent descent.

When air subsides its temperature increases, and it is marked by extreme dryness when the subsidence has proceeded through a considerable range of height. When rain falls through the subsiding air and is wholly or partly evaporated, the humidity of the subsiding air may be maintained at a high level. The air is at the same time cooled as a result of the evaporation, and so subsiding air through which rain has fallen is no longer distinguishable by high temperature. From what has been said above, however, it follows that the wet-bulb potential temperature of the subsiding air is unaltered, and the best criterion for identifying subsiding air is therefore its wet-bulb potential temperature.

The computation of the wet-bulb potential temperature is extremely simple, as it can be read off the tephigram or Hertz or Neuhoff diagram by interpolating a saturated adiabatic through the plotted wet-bulb temperature, and following it down to the standard pressure of 1000 mb.

§ 51. *Equivalent temperature and equivalent potential temperature*

In § 47 above the following equation was derived:

$$c_pT + L'x + c_p'x\,(T - T') = c_pT' + L'x' = c_pT'' \qquad \ldots\ldots(47).$$

This equation may be taken as defining the equivalent temperature T'' as the temperature of absolutely dry air which has the wet-bulb temperature T' of the air under consideration. The total heat content of $(1+x)$ grammes of moist air at temperature T and wet-bulb temperature T' may be written in any of the following forms:

$$c_pT+L'x+c_p'x(T-T')+xcT'=c_pT+Lx+xcT=c_pT'+L'x'+xcT'$$
$$=c_pT''+xcT' \qquad ......(48).$$

If then we regard the heat content of $(1+x)$ grammes of moist air as made up of the heat content of x grammes of liquid water at temperature T' *plus* the remainder of any of the expressions in (48), we may regard that remainder as measuring the "effective" heat content of the moist air. The part xcT' can never become effective in the same sense that the remainder can, since this amount will be carried away by the liquid water when this is condensed. What we have here called the effective heat is the only part that can become available for any other purpose, and since this part of the heat content is measured by the equivalent temperature, we should expect that the equivalent temperature should have important applications in the thermodynamics of the atmosphere.

If the mass of air under consideration could be carried upward adiabatically until all its water-vapour had been condensed and precipitated, and then brought down again to its original level and to its original pressure, its temperature when it returned to its original pressure would be T'', if we could assume that the condensed water had all been carried away at a temperature T'. The difference is not negligible. A simple example will serve to show the order of magnitude of the difference. From the tephigram of fig. 22 it is seen that a mass of air saturated at 53° F and 1000 mb will, when raised sufficiently high to lose all its water-vapour, assume a potential temperature of 97° F, so that when brought back down to a pressure of 1000 mb it will have a temperature of 97° F. But the computation of T'' from equation (47) above yields $T''=89°$ F.

The use of the equivalent potential temperature has not yet been thoroughly justified in practice. Most writers on the subject use numbers of approximations in their equations, and no analysis of the subject can be regarded as satisfactory until the accuracy of the approximations has been justified. Computation of the equivalent temperature corresponding to different points on a saturated adiabatic in fig. 22 shows that the equivalent potential temperature computed on the basis of equation (47) is not constant along the saturated adiabatic.

The name "effective heat content" suggested above is not a generally accepted term, and is introduced here in order to avoid repetition of a long definition. It would have been preferable to call it "realised heat", but that would have been liable to confusion in view of the use by Shaw of the term "realised entropy" to indicate the entropy of the dry air alone.

The effective heat content can be measured by either the wet-bulb temperature or by the equivalent temperature T'', which is a function of T' only, as is seen from equation (47). A simple table could be drawn up to give the equivalent temperature in terms of the wet-bulb temperature only, for a given pressure, or, simpler still, the wet-bulb thermometer could be graduated so as to give, for a selected pressure, the equivalent temperature by direct reading. But it must be noted that the relation between T'' and T' involves the pressure, and a table which gave the relation between T'' and T' at one standard pressure would not hold at another pressure.

If a mass of moist air is carried upward adiabatically, its dry-bulb and wet-bulb temperatures will follow a dry adiabatic and a saturated adiabatic respectively. Imagine that at the same time a second mass of completely dry air, whose wet-bulb temperature is initially T', the same as that of the first mass, at the ground, is also taken upward adiabatically. Its wet bulb will follow the same saturated adiabatic as that of the other mass, and its dry-bulb temperature will follow a dry adiabatic. But the temperature of the second mass is the equivalent temperature of the first mass. Thus when a mass of moist air is taken upward adiabatically from the ground, its equivalent temperature follows a dry adiabatic, which from our previous consideration we can state to be that dry adiabatic which at great heights is asymptotic to the saturated adiabatic that passes through the surface wet-bulb temperature.

We are thus led to define an equivalent potential temperature θ'', which is the equivalent temperature attained by a mass of air brought adiabatically to a standard pressure. It should be noted that θ'' is the temperature of absolutely dry air which has the wet-bulb temperature θ'. The variable θ'' is related to the entropy of moist air in a manner similar to the relation of the entropy of dry air to its potential temperature, by a relationship which we shall now derive.

It has been shown in § 48 that if S is the entropy of 1 gramme of water at temperature T', ϕ the entropy of $(1+x)$ grammes of moist air at temperature T, and ϕ_s the entropy of $(1+x')$ grammes of air saturated at temperature T', then

$$\phi=\phi_s-(x'-x)\,S \qquad \text{......(49)},$$

or

$$\phi=c_p \log T'-AR\log(p-e)+L'x'/T'+xS+\text{const.} \qquad \text{......(50)}.$$

But $L'x'/c_pT'$ is normally a fairly small fraction of unity on account of the small value of x', and we may write

$$\log(1+L'x'/c_pT')=L'x'/c_pT'$$

neglecting the third and higher powers of $L'x'/c_pT'$. Substituting this in equation (50), we find

$$\phi=c_p\log\left(T'+\frac{L'x'}{c_p}\right)-AR\log(p-e)+xS+\text{const.} \qquad \text{......(51)}$$

$$=c_p\log T''-AR\log(p-e)+xS+\text{const.} \qquad \text{......(52)}.$$

But since, as explained above, θ'' is the potential temperature of absolutely dry air at temperature T'', equation (52) becomes, approximately,

$$\phi=c_p\log\theta''+xS+\text{const.} \qquad \text{......(53)}.$$

This is the generalised form of the entropy-potential temperature relationship.

§ 52. *The Clausius-Clapeyron equation*

It is possible to relate the latent heat of evaporation to the rate of change of the saturation vapour-pressure with temperature. The relationship is most readily derived by considering a Carnot cycle, as represented in fig. 27. The pressure

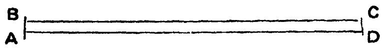


Fig. 27. Cycle for the Clausius-Clapeyron equation.

e is the vapour-pressure, and the lines of equal e are therefore isothermal lines. Let AB be the isothermal of T, the abscissae A and B representing the specific volumes of 1 gramme of water and of water-vapour respectively. Let CD be the isothermal $T+dT$, the points C and D representing water-vapour and water respectively, as before.

Let the latent heat of vaporisation be L at temperature T, and $L+dL$ at temperature $T+dT$. Now take 1 gramme of water round the cycle $ABCDA$, starting at A. To take the representative point from A to B, an amount of heat L must be communicated to the water, which at B is 1 gramme of water-vapour, under a pressure e. Next the water-vapour must be taken from B to C, and during this transit it must be kept saturated at each stage. Let s_2 be the specific heat of water-vapour when constrained to change its state in this way. Then an amount of heat $s_2 dT$ must be communicated to the water-vapour during the path BC. Next, by extraction of a quantity of heat $L+dL$ the water-vapour is condensed, yielding 1 gramme of liquid water, the representative point moving from C to D, along the isothermal $T+dT$. Then by extraction of a quantity of heat $c\,dT$, the water is brought back to the original point A. Here c is the specific heat of liquid water at the temperature T. Let v_1 and v_2 be the specific volumes of liquid water and water-vapour respectively.

The total amount of heat communicated to the working substance is equal to the work done by the substance on the environment, and since the cycle has been performed in the counter-clockwise sense, the work done on the environment is negative, and equal in magnitude to the area of the cycle in the diagram. Hence

$$L+s_2 dT-L-dL-c\,dT+A\,(v_2-v_1)\,de=0$$

or

$$s_2-c-\frac{dL}{dT}+A\,(v_2-v_1)\,\frac{de}{dT}=0 \qquad \ldots\ldots(54).$$

But since the cycle is reversible, the change of entropy in going round it is zero. Hence

$$\frac{L}{T}+s_2\frac{dT}{T}-\frac{L+dL}{T+dT}-c\,\frac{dT}{T}=0,$$

$$\frac{s_2-c}{T}\,dT=-\frac{L}{T}+\frac{L}{T}\left(1+\frac{dL}{L}\right)\left(1+\frac{dT}{T}\right)^{-1}$$

$$=\frac{dL}{T}-\frac{L\,dT}{T^2},$$

or

$$s_2-c=\frac{dL}{dT}-\frac{L}{T} \qquad \ldots\ldots(55),$$

or
$$s_2 - c = T\frac{d}{dT}\left(\frac{L}{T}\right).$$

Comparing (54) and (55) we find

$$\frac{L}{T} = A\,(v_2 - v_1)\,\frac{de}{dT} \qquad \text{......(56),}$$

where e is the saturation vapour-pressure of water-vapour at the temperature T.

Equation (56) has a further use than a purely theoretical one. It may now be used to derive dT/de. For the boiling point of water is the temperature at which the saturation vapour-pressure is equal to the external atmospheric pressure, and hence dT/de will give the rate of change of the boiling point with change of pressure. Neglecting v_1 and putting $v_2 = R'T/e$, we find

$$\frac{dT}{de} = \frac{AT}{L}\,\frac{R'T}{e} = \frac{AR'T^2}{Le} \qquad \text{......(57),}$$

which with the appropriate values of T, e and L will give the value of dT/de.

The use of the vapour-pressure method, with an instrument known as the hypsometer, demands very accurate thermometers. To obtain height above sea level correct to within 10 feet (3 metres) by means of the pressure deduced from the boiling point of water requires measurement of temperatures within 0·01° C.

§ 53. *The functional relation of saturation vapour-pressure to temperature*

In the preceding sections we have deduced a relationship between the latent heat and the saturation vapour-pressure of water-vapour. If L is a constant or a known function of the temperature T, the equation (56) above can be integrated. In practice the specific volume of liquid water is negligible by comparison with the specific volume of water-vapour. We have then from the gas-equation

$$ev_2 = R'T \qquad \text{......(58),}$$

where R' is the appropriate constant for water-vapour. Substituting this in equation (56), and neglecting v_1, we find

$$\frac{L}{T} = \frac{AR'T}{e}\cdot\frac{de}{dT} \qquad \text{......(59),}$$

or
$$\frac{L}{T^2}\,dT = \frac{AR'}{e}\,de.$$

For temperatures below freezing point, and for vapour over ice, L may be treated as a constant, whose value is 677 gramme-calories, or $677 \times 4{\cdot}18 \times 10^7$ ergs. Integrating the last equation, we find

$$AR' \log e = \text{const.} - L/T \qquad \text{......(60).}$$

An equation of this form was suggested by Young in 1820, from a consideration of experimental results. When the appropriate values of the constants are inserted, equation (60) gives for the saturation pressure over ice

$$\frac{1}{273} - \frac{1}{T} = 1 \cdot 629 \times 10^{-4} \log \frac{e}{e_0},$$

where e_0 is the value of e at 273° A. This equation gives a remarkably close fit to the observations. A graphical comparison is shown in the *Manual of Meteorology*, **3**, fig. 92.

Whipple* integrated equation (59) on the assumption that

$$L = L_v - (S_w - S_v)\, t,$$

where t is the Centigrade temperature, and L_v the latent heat at 0° C. He found the following results for temperatures below 0° C:

(*a*) For vapour-pressure over water,

$$\log \frac{e}{e_0} = 10 \cdot 78 \frac{t}{273 + t} - 5 \cdot 01 \log \frac{t + 273}{273};$$

(*b*) For vapour-pressure over ice,

$$\log \frac{e}{e_0} = 9 \cdot 95 \frac{t}{273 + t} - 0 \cdot 445 \log \frac{t + 273}{273};$$

(*c*) For relative humidity r over ice,

$$\log \frac{r}{100} = 4 \cdot 56 \log \frac{t + 273}{273} - 0 \cdot 83 \frac{t}{t + 273}.$$

In these formulae e_0 is the saturation vapour-pressure at 0° C.

Whipple found that the results given by his equations gave very close agreement with the vapour-pressures observed by Washburn. The close agreement shows that over the range of temperatures considered the variation of $S_w - S_v$ with temperature is negligible.

For temperatures above 0° C covered by meteorological observations the variations of L are slight. If these variations are neglected, equation (59) can again be integrated to give the saturation vapour-pressure over water. The result is quoted from Shaw, *Manual of Meteorology*, **3**, p. 239,

$$\frac{1}{273} - \frac{1}{T} = 1 \cdot 844 \times 10^{-4} \log \frac{e}{e_0}.$$

Shaw shows that the fit of the observations to the formula is again very close.

We shall not in practice require to use these formulae, as it will be found more convenient to use the tabulated values of the vapour-pressures, but the results are of very great interest, in that they demonstrate the power of the second law of thermodynamics to measure relationships which otherwise would appear to be incapable of theoretical treatment.

* *Monthly Weather Review*, 1927, p. 131.

CHAPTER V

RADIATION

§ 54. *Radiation of light and heat*

THE phenomena of radiation consist in the transmission of energy from one body to another through the intervening medium. The transmission takes place along straight lines, and with a velocity which, though large compared with the velocities of motion of matter in bulk with which we are familiar, is yet determinable by careful experiment. The process of transfer by radiation is analogous to wave motion, and a wide range of wave-lengths is possible. We can build into one spectrum the visible rays known as light rays, the infra-red or heat rays, X-rays, and the Hertzian waves used in wireless telegraphy; but in meteorology we are only concerned with a relatively narrow range of wave-length, as will be seen when we come later to discuss solar and terrestrial radiation. The units used in measuring wave-lengths are $\mu = 10^{-3}$ mm $= 10^{-4}$ cm, $\mu\mu = 10^{-6}$ mm $= 10^{-7}$ cm, and the Ångström unit 10^{-8} cm.

A clear distinction can be drawn between the phenomena of radiation and molecular conduction. All bodies radiate energy, the amount radiated depending on their temperature. But as any radiating body may also receive radiation from surrounding bodies, its net gain or loss of energy can only be determined when the temperature of each portion of the environment is known. By contrast with this, the conduction of heat requires a slope of temperature along which the heat travels from high to low temperature, and the net flow of heat by conduction can be evaluated when the slope of temperature is known. Radiation requires no such slope of temperature, and does not of necessity warm the medium through which it passes. It is true that it is possible to have conduction of heat through a medium without warming the medium, but this demands the very special condition of a constant gradient of temperature across the medium.

From the point of view of meteorology the distinction which is usually drawn between light and heat is a convenience rather than a fact of physical significance. The effect of the absorption of light, like that of heat, is to warm the medium which absorbs it, the absorbed energy being converted into molecular motion. The distinction which we draw between light and heat is mainly due to the limitation of the susceptibility of the human eye to one octave of wave-length; but it is a convenient distinction to have in mind, since air (and to a less extent water) is almost completely transparent to light rays.

§ 55. *Kirchhoff's law*

The rate at which a body sends out radiation from unit surface may be called the "emissive power", and the rate at which unit surface absorbs radiation falling upon it may be called "absorptive power". These two quantities are closely related, the relation as expressed by Kirchhoff being that "at a given temperature the ratio between the absorptive and emissive power for a given wave-length is the same for all bodies". It should be noted that two distinct physical facts are involved in this statement. In the first place it gives a qualitative rule which connects the radiation and absorption for a given substance; if a body emits radiation of a given wave-length at a given temperature, it will also absorb radiation of the same wave-length at that temperature. In the second place it gives a quantitative rule which establishes a relationship between different bodies.

If E_λ, A_λ are respectively the emission and absorption of a particular wave-length λ for a particular body at temperature T, and $E_\lambda/A_\lambda = k_\lambda$, then Kirchhoff's law states that k_λ is a function of λ and T only, and is independent of the nature of the body. This does not preclude the possibility of both E_λ and A_λ being zero, and the body being transparent to radiation of wave-length λ. Many bodies, notably gases and vapours, only absorb in limited ranges of wave-length, and their emission is therefore limited to the same ranges of wave-length.

It also follows from Kirchhoff's law that a good absorber is a good radiator, and a bad absorber a bad radiator. Highly polished surfaces, which reflect nearly all short-wave radiation falling upon them, are poor radiators of short waves. But a body which is transparent to one range of wave-lengths may readily absorb other wave-lengths. Thus water-vapour is nearly transparent to the short-wave radiation coming from the sun, but absorbs readily the radiation emitted at terrestrial and atmospheric temperatures within a wide range of wave-lengths.

§ 56. *Black-body radiation*

Experiment shows that there is an upper limit to the amount of radiation which can be emitted by unit surface of a body at a given temperature. A body which radiates for every wave-length the maximum amount of radiation possible at a given temperature is known as a "black body" or "perfectly black", or a "perfect radiator". The nearest approach to black-body radiation is the radiation which passes out through a small cavity in a solid body at a uniform temperature.

§ 57. *Grey radiation*

A body which emits for each wave-length a fixed proportion of the black-body radiation at the same temperature is called a "grey" body, and its radiation is known as "grey" radiation.

§ 58. *Planck's law*

The distribution of the energy in the spectrum of a black body has been represented by Planck by means of a formula

$$E_\lambda = \frac{c_1 \lambda^{-5}}{e^{\frac{c_2}{\lambda T}} - 1} \qquad \text{......(1)},$$

where E_λ is the energy emitted per unit area per unit time within unit range of wave-length centred on λ, and c_1 and c_2 are constants. The equation may be written

$$\frac{E_\lambda}{T^5}=\frac{c_1(\lambda T)^{-5}}{e^{\frac{c_2}{\lambda T}}-1}=f(\lambda T) \qquad \ldots\ldots(2).$$

The function $f(\lambda T)$ is zero for $\lambda=0$ and for $\lambda=\infty$, and has a maximum at some finite value of λT. If this value of λT be a, then the wave-length at which E_λ is a maximum for a given temperature is given by Wien's law

$$\lambda_m=a/T \qquad \ldots\ldots(3).$$

The value of a as given by Lummer and Pringsheim* is 2940, when λ is measured in μ, and T in degrees absolute. A slightly lower value, 2920, was given by Paschen†. Assuming the value given by Lummer and Pringsheim, we find that the maximum intensity in the spectrum of the radiation from a black body at a temperature of 294° A is at 10μ, while for a body at a temperature of 200° A the maximum is at about 15μ. These are roughly the limits of temperature within the earth's atmosphere. Conversely, by observing the wave-length of the maximum intensity of the radiation we may determine the temperature of the body which emits the radiation, by the use of the equation

$$\lambda_m T=2940 \qquad \ldots\ldots(4).$$

Further, by measuring the maximum value of E_λ we can determine T by using the relationship

$$\left(\frac{E_\lambda}{T^5}\right)_{\max}=\text{const.}=f(a).$$

These two methods have been employed to determine the temperature of the sun, and give approximate agreement, which may be taken as evidence that the sun radiates approximately as a black body. It is found that the maximum intensity in the solar spectrum is at about $0{\cdot}5\mu$, corresponding to an effective temperature of 5600° A.

Since E_λ/T^5 is a function of λT, if a diagram is drawn to represent E_λ/T^5 as a function of λ for any one temperature, it can be used to represent E_λ for any other temperature by making the scale of λ inversely proportional to T, and taking the scale of E_λ proportional to T^5. Thus in fig. 28 the emission E_λ is plotted against λ for a variety of temperatures, the appropriate horizontal scale being used according to the temperatures shown against the various scales. The unit μ used in the scale of wave-length is 1/1000 mm, or 10^{-6} metre.

In fig. 28 the common horizontal scale is in reality λT, the vertical scale measuring E_λ/T^5. It will be seen that if the curves of radiation intensity be drawn for two different temperatures, on the same absolute scale of wave-length and intensity, the curve for the higher temperature lies completely above the curve for the lower temperature. This is most readily seen from equation (1), which shows that the value of E_λ for any selected value of λ in-

* *Verh. Deut. Phys. Ges.* **1**, 1889, pp. 23 and 215.
† *Ann. der Physik*, **6**, 1901, p. 657.

creases as T increases. For as T increases the denominator decreases so that the expression as a whole increases. Hence a curve representing the variation of E_λ with λ rises along its whole length as the temperature increases.

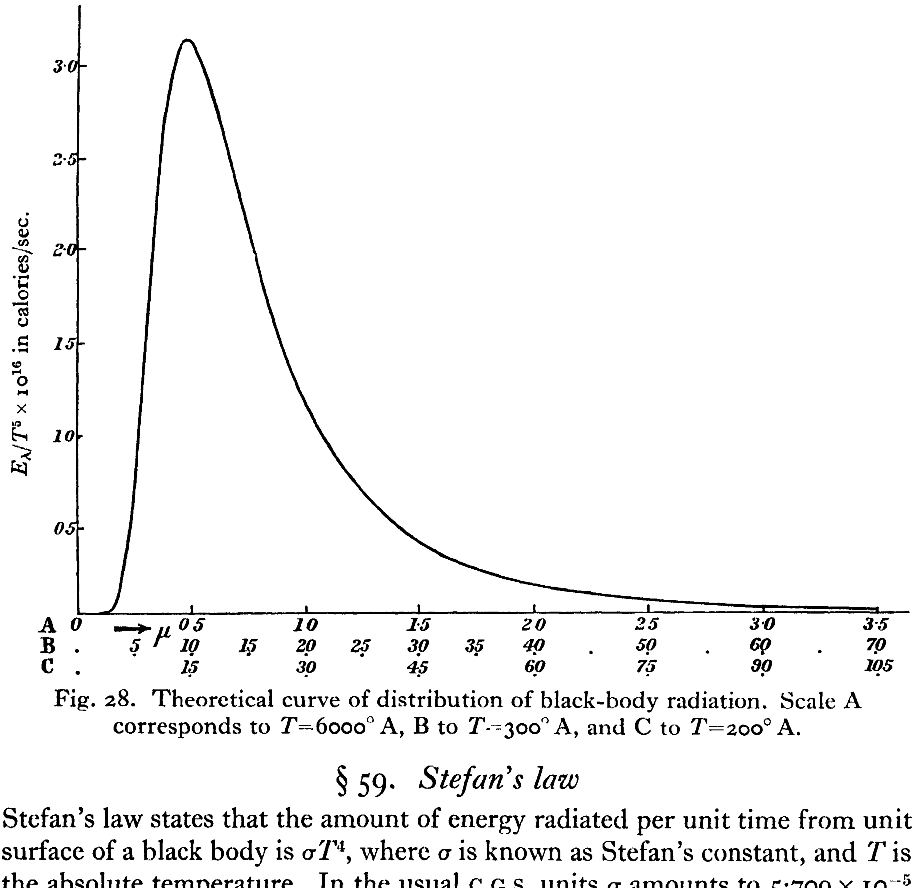


Fig. 28. Theoretical curve of distribution of black-body radiation. Scale A corresponds to $T=6000°$ A, B to $T=300°$ A, and C to $T=200°$ A.

§ 59. *Stefan's law*

Stefan's law states that the amount of energy radiated per unit time from unit surface of a black body is σT^4, where σ is known as Stefan's constant, and T is the absolute temperature. In the usual C.G.S. units σ amounts to $5{\cdot}709 \times 10^{-5}$ ergs per cm² per second, or $5{\cdot}709 \times 10^{-12}$ watts per cm², or 82×10^{-12} gramme-calories per cm² per minute.

This result may be derived from equation (1), from which it follows that

Total radiation from a perfect radiator at temperature T

$$= \int_0^\infty E_\lambda d\lambda = c_1 \int_0^\infty \frac{\lambda^{-5}}{e^{\frac{c_2}{\lambda T}} - 1} d\lambda$$

$$= c_1 T^4 \int_0^\infty \frac{(\lambda T)^{-5}}{e^{\frac{c_2}{\lambda T}} - 1} d(\lambda T)$$

$$= c_1 T^4 \int_0^\infty \frac{x^{-5}}{e^{c_2/x} - 1} \cdot dx$$

$$= \sigma T^4,$$

where σ is a constant.

§ 60. *Range of wave-lengths in radiation from bodies at different temperatures*

The distribution of complete or black-body radiation with wave-length is shown in fig. 28, with a uniform scale for λT. It can readily be seen that on account of the steep slope of the curve on the side of low wave-lengths the amount of radiation below a limit $\lambda T = 1000$ is negligible, being considerably less than 0·1 per cent of the total radiation. The slope of the curve is far less steep on the side of high wave-lengths, and the limit is not so readily set. Only 0·1 per cent of the total radiation will be neglected if we stop at an upper limit set by $\lambda T = 54,000$, and 1 per cent will be neglected if the upper limit is set by $\lambda T = 24,000$. For our present purposes we shall assume that the spectrum does not extend beyond the limit (set by $\lambda T = 24,000$) which contains 99 per cent of the total radiation.

Thus black-body radiation at 6000° A, approximately the temperature of the sun, is contained within the limits of $0{\cdot}17\mu$ and 4μ, and has its maximum intensity in the blue-green at about $0{\cdot}5\mu$. Black-body radiation at terrestrial temperatures of roughly 300° A will be contained within the limits 3μ and 80μ, having its maximum intensity at 10μ; and black-body radiation at stratospheric temperatures of about 200° A will be contained within the limits 4μ and 120μ, having its maximum intensity at 15μ. Thus black-body radiation at atmospheric and terrestrial temperatures is in the far infra-red part of the spectrum. Since this range of wave-lengths is widely separated from that of incoming solar radiation, it is customary to call it "long-wave" radiation, to distinguish it from direct solar radiation, which is called "short-wave" radiation.

§ 61. *The spectral distribution of solar radiation*

The outer surface of the sun may be regarded as approximately a black body at a temperature of about 5600° A. Its radiation should therefore be contained within the limits of wave-length $\cdot 15\mu$ to 4μ. An appreciable amount of this radiation should be contained in the ultra-violet range, below $0{\cdot}4\mu$, while a still larger amount is contained in the range beyond $0{\cdot}7\mu$, in the infra-red region. Roughly half the total energy in the sun's radiation is contained within the range 0·4 to $0{\cdot}7\mu$, so that about half of the total solar energy is in the form of light.

Close investigation of the intensity of the radiation of different wave-lengths indicates that there are many dark lines in which little or no radiation is present. Many of these dark lines are absorption lines due to the absorption, in the outer atmosphere of the sun, of radiation coming from deeper layers. There are, however, some dark lines due to absorption in the earth's atmosphere. The blue end of the spectrum is sharply limited at $0{\cdot}3\mu$, even when photographed from considerable heights above the earth's surface. Wigand in 1913 succeeded in photographing the solar spectrum from a height of 9 km, and found the same sharp limitation at $0{\cdot}3\mu$ which had been noted in spectra photographed from the earth's surface. It was first suggested by Hartley that the absence of all wave-lengths below 0·3μ was due to absorption by ozone. This question was studied in greater detail by Fabry and Buisson in 1912, and these writers brought forward evidence that the ozone is localised at heights inaccessible to direct observation. More recent studies of the day to day variation in the amount of ozone absorption by Dr G. M. B. Dobson and his collaborators* have brought out some striking relationships between the amounts of ozone at heights of about 20–50 km and the distribution of pressure at the earth's surface.

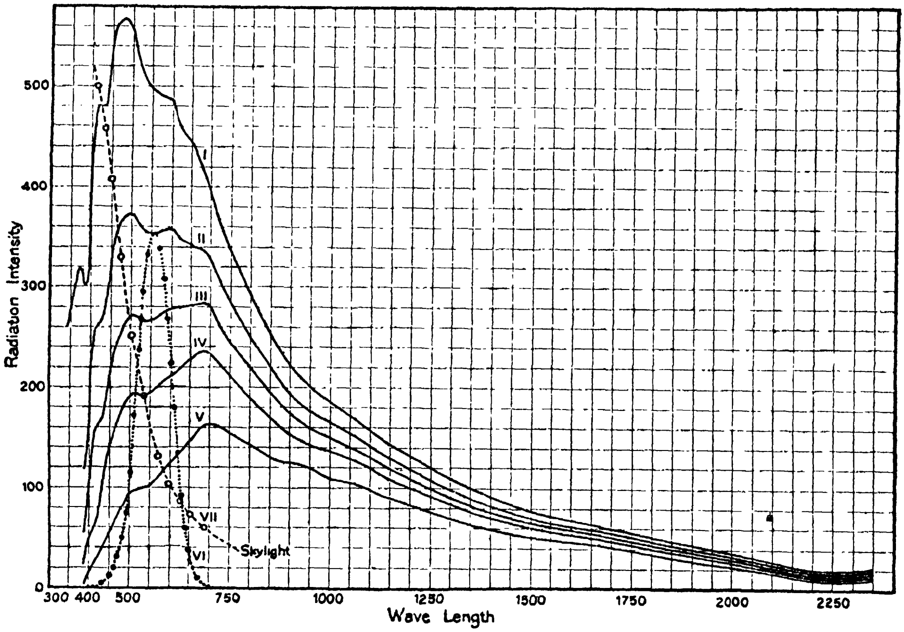


Fig. 29. Observed intensity distribution of solar radiation.

Within the limits of the appreciable solar spectrum the main lines of atmospheric absorption are those due to oxygen at 0·69μ and 0·76μ, those due to water-vapour at 0·72μ, 0·81μ, 0·93μ, 1·13μ, 1·42μ, 1·89μ, and the two wide bands due to water-vapour centred at about 2·01μ and 2·05μ. Except for the two lines due to oxygen referred to above, the simple gases of the atmosphere do not absorb any radiation, and carbon dioxide does not absorb within the limits of wave-length of the solar beam. The oxygen lines are so narrow that they represent only a very minute loss of energy from the solar beam.

Fig. 29 shows in curve II the spectral distribution of solar radiation as received at Washington with zenith sun and cloudless sky, and in curve I the distribution as estimated outside the atmosphere in the latitude of Washington, after allowing for the absorption and scattering of the atmosphere. Curves III, IV, V give the distributions for solar altitudes 30°, 19·3°, and 11·3° (air-masses

* *Vide* references, footnote, p. 20 above.

2, 3, 5), while curve VI gives the relative brightness of the parts of the spectrum.

According to Fowle*, on a clear day, with the sun in the zenith, the total loss from the solar beam due to absorption in the atmosphere down to sea level is only about 6 to 8 per cent of the incident radiation. Thus on a clear day by far the greater part of the solar radiation passes unchanged through the earth's atmosphere, and is either absorbed by the earth's surface, or is reflected upward by various parts of the earth's surface. Clouds will also reflect a large part of the incident radiation. It is estimated that 0·43 of the solar radiation which reaches the outer limit of the earth's atmosphere is lost by reflexion or absorption, so that only 0·57 of the incident radiation becomes available for heating the earth's atmosphere and surface. We shall return to this aspect of the question in § 64 below.

§ 62. *The solar constant*

The beam of incoming solar radiation is liable to suffer loss by reflexion at the surfaces of clouds, by absorption by the gaseous constituents of the atmosphere and by scattering by small particles suspended in the atmosphere. All three sources of loss are diminished when observations are carried out at a high level station, and a correction can be made for the residual loss by absorption and scattering in the uppermost layers of the atmosphere. The most exhaustive observations of the intensity of the incoming solar radiation are those carried out by the Smithsonian Institution at the high level observatories on Table Mountain, California, Mt. Montezuma, Chile, and Mt. St Catherine, Sinai.

The intensity of the solar radiation at the outer limit of the earth's atmosphere is known as the "solar constant". The mean value deduced from a long series of observations is 1·94 gramme-calories per cm^2 per minute, or 136 kilowatts per square dekametre. The measured values show a variation from day to day, with a total range of a few per cent of the whole. The reality of these variations is by no means generally accepted, though there is some evidence that high or low values occur at the same time in the observations made at Mount Wilson and at Calama. Notable depressions in the measured values occurred in 1902 and 1912, associated with the volcanic eruptions of Pelée, Santa Maria, and Colima in 1902, and of Katmai in 1912. Large variations also followed earlier volcanic eruptions, but those mentioned gave the most marked depressions in the measured values of the solar constant. It should be added that as methods of observation have been improved, the magnitude of the variations in the day to day observations of the solar constant has diminished, and this is in itself an argument against the reality of these variations. Pettit gives (in the *Third Report on Solar and Terrestrial Relationships*) a curve showing the variation in the ratio of the intensity of ultra-violet radiation at about $0{\cdot}3\mu$ to the intensity in the green at about $0{\cdot}5\mu$, and this curve shows very clearly an annual variation, suggesting that the effects of the earth's atmosphere have not been entirely eliminated. It is possible that the residual

* *Ann. Astroph. Obs.* 4, p. 274.

error is mainly attributable to the estimate of the ultra-violet intensity, but the result leaves us in some doubt as to whether the variations of atmospheric turbidity have been entirely eliminated in the evaluation of the solar constant.

§ 63. *The variation of insolation with season and latitude*

The amount of solar radiation reaching unit area of any part of the earth's surface in one day depends upon

(*a*) the solar constant,
(*b*) the transparency of the atmosphere,
(*c*) the latitude of the place, and
(*d*) the time of the year.

No simple formula will summarise these effects. Angot*, starting from the assumption that the atmosphere is transparent, has calculated a table giving for intervals of latitude, for each month of the year, the insolation or total amount of radiation reaching unit surface of the earth per month. In this table the unit is the amount of energy that would be received on unit area on the equator in one day, at the equinox, with the sun at its mean distance, the atmosphere being assumed completely transparent. This unit amounts to 458·4 times the solar constant, or 889 gramme-calories per cm^2, taking the solar constant to be 1·94 gramme-calories per cm^2 per minute. Angot's table is reproduced in the table below. It will be noted that the figures for the Southern hemisphere are not exactly equal to those for the Northern hemisphere shifted through 6 months, on account of the varying distance of the earth from the sun. The totals for the whole earth show a maximum in Dec.-Jan., and a minimum in June–July, the times of perihelion and aphelion respectively.

Table 2

Calculated insolation reaching the earth

Lat.	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Year
N 90	0·0	0·0	1·9	17·5	31·5	36·4	32·9	21·1	4·6	0·0	0·0	0·0	145·4
80	0·0	0·1	5·0	17·5	30·5	35·8	32·4	20·9	7·4	0·6	0·0	0·0	150·2
60	3·0	7·4	14·8	23·2	30·2	33·2	31·1	24·9	16·7	9·0	3·8	1·9	199·2
40	12·5	17·0	23·1	28·6	32·4	33·8	32·8	29·4	24·3	18·4	13·4	11·1	276·8
20	22·0	25·1	28·6	30·9	31·8	32·0	31·8	30·9	28·9	25·8	22·5	20·9	331·2
Equat.	29·4	30·4	30·6	29·6	28·0	27·1	27·6	28·6	30·1	30·2	29·5	28·9	350·3
S 20	33·8	32·2	29·0	24·9	21·2	19·6	20·5	23·7	27·7	31·1	33·3	34·1	331·2
40	34·8	30·4	23·9	17·4	12·5	10·4	11·6	15·8	21·9	28·5	33·6	36·0	276·8
60	33·0	25·3	16·0	8·1	3·3	1·7	2·7	6·5	13·6	22·6	31·1	35·3	199·2
80	34·2	20·5	6·3	0·3	0·0	0·0	0·0	0·0	3·8	16·0	31·0	38·1	150·2
90	34·7	20·7	3·2	0·0	0·0	0·0	0·0	0·0	1·0	15·6	31·5	38·7	145·4

§ 64. *The earth's albedo*

On p. 99 we quoted as the total reflecting power of the earth, or the earth's albedo, the figure 0·43, a value due to Aldrich†. This figure is based on the assumption that with a cloudless sky a fraction 0·08 is reflected from the

* *Ann. Bur. Cent. Met.*, Paris, 1883, 1ère partie, pp. B 136–61.

† *Smithsonian Misc. Coll.* **69**, No. 10, 1919; also *Ann. Astroph. Obs.* **4**, 1922, p. 379.

earth's surface, and 0·09 from the atmosphere, giving a total loss of 0·17. For totally overcast skies the reflexion factor is taken as 0·78, and for a mean cloud amount 0·52, which is estimated to be the mean cloud amount of the earth, Aldrich computes that a fraction 0·43 of the incoming radiation is reflected by the atmosphere, the clouds and the earth's surface.

Ångström* gave a linear relationship between the albedo a and the cloud amount c, in the form

$$a = 0{\cdot}70c + 0{\cdot}17\,(1-c).$$

The difference between the albedo of the whole earth deduced from Ångström's equation and the value estimated by Aldrich is very slight, and the value 0·43 for the whole earth may be taken as a reasonably accurate value.

Different natural surfaces have widely different factors of reflexion for solar radiation†. Thus snow reflects from 70 to 80 per cent; water an amount varying from 2 per cent with the sun at an altitude of 47° to 71 per cent with the sun at an altitude of $5\frac{1}{2}°$; grass reflects from 10 to 33 per cent; rock 12 to 15 per cent; dry mould 14 per cent, and wet mould 8 to 9 per cent. Thus when the ground is not snow-covered the differences in reflective power are not of great importance. The greatest variation which has to be taken into account is the difference between clear and clouded skies, since with overcast skies the reflexion factor is 0·78, whereas the mean reflexion factor of the earth's surface is of the order of 0·1, except when the ground is covered with snow, when it becomes 0·7 to 0·8.

§ 65. *The coefficient of absorption*

The law of absorption of radiation as usually stated is known as Beer's law. If I_0 is the intensity of the incident radiation, the intensity of the radiation after it has passed over a path containing m units of mass of the absorbing substance per cm² of cross-section is reduced to I, where

$$I = I_0 e^{-km}$$

and k is defined as the coefficient of absorption.

The reduction of intensity in a path containing dm of absorbing substance is given by

$$dI/I = -k\,dm.$$

This equation might equally well be taken as defining the coefficient of absorption.

In practice, the exponential form is not as convenient as the form

$$I = I_0 \times 10^{\cdot 0{\cdot}4343km}$$

and it is also convenient to substitute another symbol α' for $0{\cdot}4343k$, so that the equation reads

$$I = I_0 \times 10^{-\alpha' m}.$$

The two symbols k and α' might be distinguished by calling k the "Napierian" coefficient of absorption, and α' the "decimal" coefficient of absorption.

* *Beitr. Geophys. Leipzig*, **61**, 1926, h. 1.
† Ångström, *Geog. Ann.* 1925, h. 4.

In the following pages we shall use either k or α', taking the one which happens to be most convenient in each problem to be considered.

§ 66. *Absorption of long-wave radiation in the atmosphere*

It was stated in § 61 that within the limits of the wave-lengths included in the beam of radiation from the sun, the amount of atmospheric absorption is relatively slight, being limited to certain bands mainly due to water-vapour, but in part also to ozone and oxygen. Radiation from, and absorption by, bodies at atmospheric and terrestrial temperatures, as we have seen from fig. 28, will lie within a totally different range of wave-lengths, say 3μ to 100μ. Within these limits there is no appreciable absorption by any of the simple gases of the atmosphere. Ozone, however, has a strong absorption band at 9–10μ. Carbon dioxide* has only one absorption band above 4μ, a narrow intense band centred at $14{\cdot}7\mu$, and extending from 12μ to about $16{\cdot}3\mu$. By far the most important absorbing constituent of the atmosphere is water-vapour, which shows very marked absorption within a very wide range of wave-lengths. It was for the latter reason that many writers treated water-vapour as a "grey" radiator, which absorbed all wave-lengths in the same proportion. It was first clearly demonstrated by G. C. Simpson† that this assumption led to erroneous conclusions, and that it was necessary to take account of the actual observed nature of the absorption spectrum of water-vapour in order to explain atmospheric phenomena.

§ 67. *The absorption spectrum of water-vapour*

The absorption spectrum of water-vapour has been the subject of many researches, of which the most notable is that of Hettner‡, which we shall adopt as the basis of the following discussion. Earlier workers had found marked absorption bands centred at $2{\cdot}7\mu$ and $6{\cdot}7\mu$, and a wide band extending from about 14μ upwards. Rubens and Hettner§ mention the occurrence of very strong absorption at 50μ, $58{\cdot}5\mu$, 66μ and 79μ.

Hettner's observations have been used to compute the decimal coefficient of absorption α', up to a wave-length of 34μ, and the results are shown graphically in fig. 30. If the amount of precipitable water per cm^2 of cross-section of the path is m, the transmission is $10^{-\alpha' m}$, where α' is the ordinate read off from fig. 30. In fig. 30 the inset diagram is a reproduction, with an enlarged vertical scale, of the portion of the spectrum between 9μ and 19μ. The diagram can be used to compute the fraction of the incident radiation which will be transmitted through any given amount of water-vapour. Thus at $19{\cdot}5\mu$, $\alpha' = 50$, and the fraction of the incident radiation transmitted through 1 mm of precipitable water is 10^{-5}, while a fraction 10^{-10} is transmitted through 2 mm of precipitable water.

* Rubens and Aschkinass, *Ann. Phys. u. Chem.* **64**, 1898, p. 584.
† *Memoirs R. Met. Soc.* **3**, No. 21.
‡ *Ann. Phys.* **55**, 1918, p. 476.
§ *Berlin Sitzber. Ak. Wiss.* 1916, p. 167.

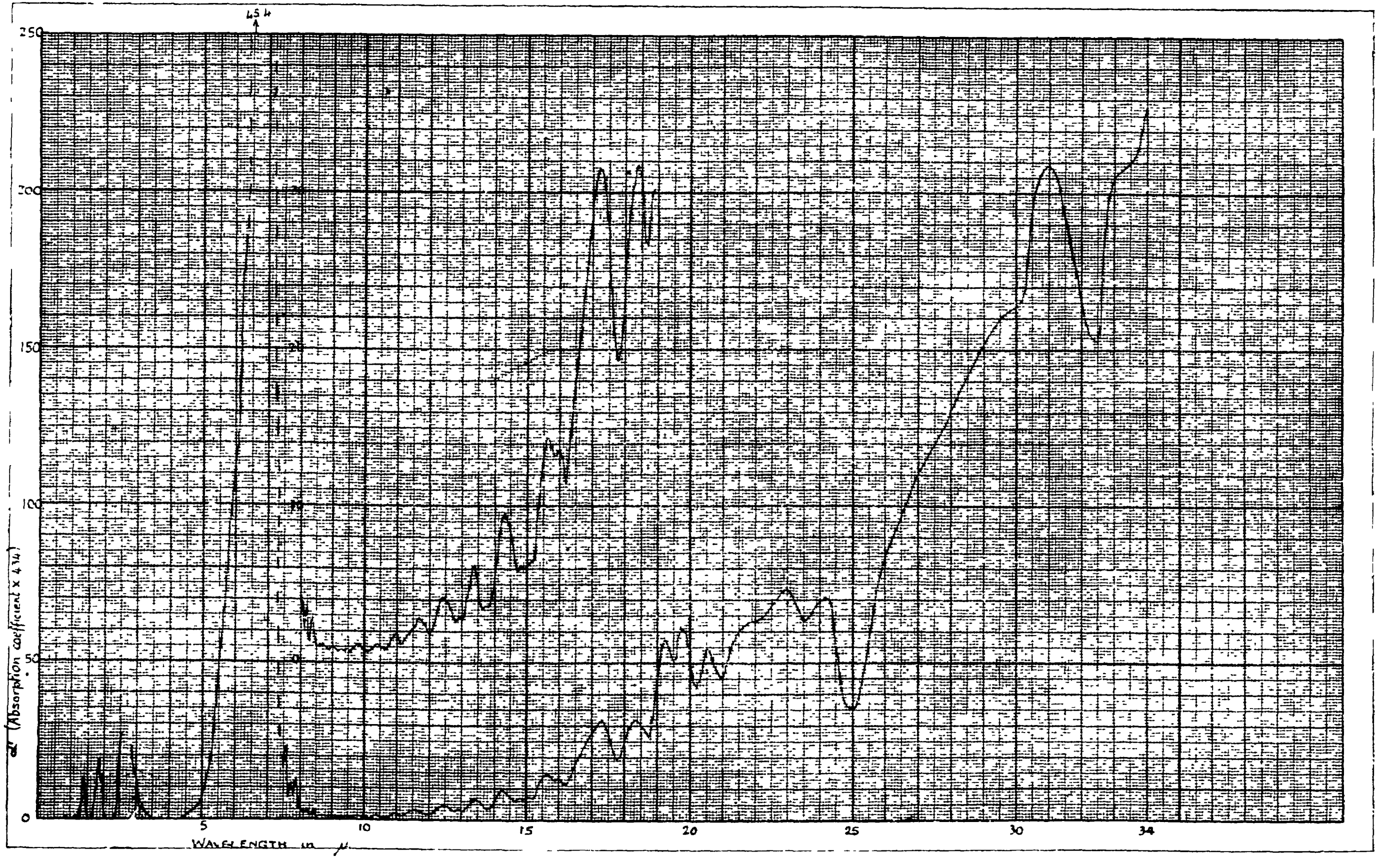


Fig. 30. The absorption spectrum of water-vapour.

Apart from some very narrow lines around 1μ, the main regions of absorption shown by fig. 30 are:

(*a*) Bands centred at $1{\cdot}37\mu$, $1{\cdot}84\mu$, and $2{\cdot}66\mu$;

(*b*) A very intense band centred at $6{\cdot}26\mu$;

(*c*) A wide band beginning at about 9μ, rising rapidly, and except for a series of oscillations, maintaining the same tendency for increasing absorption with increasing wave-length up to the limit of 34μ at which Hettner's observations terminate. There is reason to suppose that this intense absorption continues up to about 80μ, and possibly still further.

Hettner found no appreciable absorption between $3\frac{1}{2}\mu$ and $4\frac{1}{2}\mu$. Between $8\frac{1}{2}\mu$ and $9\frac{1}{2}\mu$ the absorption measured by Hettner is so small that it is questionable whether it is to be regarded as real. Also Fowle, in his observations of the transmission of radiation through water-vapour, found no evidence of absorption in this region, and it is probable that when the density of the water-vapour is as small as in the earth's atmosphere, the region $8\frac{1}{2}\mu$ to $9\frac{1}{2}\mu$ is effectively transparent.

Simpson (*loc. cit.* fig. 1) gave a curve of absorption by 0·3 mm of precipitable water. His results are readily derived by the use of fig. 30 above, with $m = 0{\cdot}03$ cm. Only one-tenth of the incident radiation will pass through the column for all wave-lengths for which α' is greater than about 33. Thus all radiation of wave-length greater than about 19μ, or between $5{\cdot}5\mu$ and 7μ, will be absorbed practically completely by the water-vapour in the column. Simpson assumed that the absorption was negligible in the range $8\frac{1}{2}\mu$ to 11μ, and taking the absorption by CO_2 in the region 13μ to 17μ as added to the effect of the absorption by water-vapour, he was able to reduce the main features of atmospheric absorption by a column containing 0·3 mm of precipitable water in the form of vapour to

(*a*) effectively complete absorption from $5{\cdot}5\mu$ to 7μ, and from 14μ upwards;

(*b*) complete transparency from $8{\cdot}5\mu$ to 11μ and below about 4μ; and

(*c*) intermediate regions of incomplete absorption in the ranges of wave-length 7–$8{\cdot}5\mu$ and 11–14μ.

Simpson's simplifications of the water-vapour spectrum enabled him to give for the first time a reasonable explanation of a number of observed phenomena, and the final justification of his hypotheses is their success in application.

Apart from the question of the variation in the form of the absorption curve of fig. 30 in going from the conditions in Hettner's experiments to those in the stratosphere, we are also concerned with the question of the transparency of the band of wave-lengths from $8\frac{1}{2}\mu$ to 11μ. Slightly varying limits have been assigned to this band by different writers, but all appear agreed as to its reality. Fig. 30 indicates the presence of lines at $9{\cdot}25\mu$, $9{\cdot}8\mu$, $10{\cdot}4\mu$ and 11μ in the absorption spectrum of steam. The density of water-vapour in the atmosphere is far below that of the steam in Hettner's experiments, and it is probable that under atmospheric conditions the width of these absorption lines is very small, so that their effect is negligible. This suggestion is confirmed by the observations of Abbot and Aldrich, who found no absorption by water-vapour in the atmosphere between the limits 9μ and 12μ, even by columns containing 3 cm of precipitable water in the form of vapour. The reality of the transparency appears to be established by direct observation, independently of any theoretical considerations.

A detailed discussion of infra-red spectra, from the theoretical and observational standpoint, will be found in Schaeffer and Matossi, *Das Ultra Rote Spektrum.*

§ 68. *The effect of pressure on the water-vapour spectrum*

In Hettner's experiments the water-vapour was in the form of steam at 400° A, and thus had far greater density than the water-vapour in the atmosphere. The question arises as to the direct applicability of Hettner's results to the water-vapour in the atmosphere, in view of the difference in vapour-pressure in Hettner's apparatus and in normal atmospheric conditions.

The work of K. Ångström*, Eva von Bahr†, Becker‡ and others has shown that the effect of increasing pressure is to widen the absorption lines giving diminished intensity of absorption in the centre of the band. It may therefore be anticipated that in the atmosphere the absorption bands of water-vapour will be narrower than those shown by the measurements of Hettner. It is not possible to say with certainty how far the width of the bands shown in fig. 30 is due to the width of the slit used by Hettner, and it is at least possible that the intense absorption shown in fig. 30 is due, in part at least, to a background of continuous absorption. Albrecht§ has suggested that in the conditions which hold in the stratosphere the bands shown in fig. 30 should be replaced by a series of very narrow lines of intense absorption, separated by transparent regions, but neither theory nor experiment is yet sufficiently advanced to enable us to say with any certainty what modifications should be made in fig. 30 to make it strictly applicable to atmospheric conditions. In the absence of any definite knowledge, it is probably best to assume that Hettner's results shown in fig. 30 can be applied to the earth's atmosphere without correction. Some further discussion of the difficulties raised above, with further references to various original researches, will be found in a paper by Brunt||.

§ 69. *The absorption spectrum of liquid water*

The absorption spectrum of liquid water has been investigated in some detail by Rubens and Ladenberg¶ up to wave-lengths of 18μ, using films of water and glycerine, and by Aschkinass** using pure water, up to a wave-length of 7μ. The curve of absorption coefficients plotted against wave-length has the same general form as the curve in fig. 30, except that the values are much

* *Arkiv Mat. Ast. Och. Fys.* **4**, No. 30, 1908.
† *Ann. Phys.* **29**, 1909, p. 780; *ibid.* **33**, 1910, p. 585.
‡ *Zeit. Phys.* **34**, 1925, p. 255.
§ *Met. Zeit.* Nov. 1931.
|| *Q.J. Roy. Met. Soc.* **58**, 1932, p. 389.
¶ *Verhandl. Phys. Ges.* **11**, 1909, p. 16.
** *Wied. Ann.* **55**, 1895, p. 401.

greater. Using the same notation as is used above, referring to a unit of 1 cm of liquid water we obtain the values of α' plotted in fig. 31 which are therefore directly comparable with those in fig. 30. The only striking differences of form between figs. 30 and 31 are the intensity of the band at 3μ in the liquid water spectrum, and the relatively smaller intensity of the band at 6μ by comparison with the intensity in the region beyond 12μ. It is seen that even for those wave-lengths for which it is most transparent to long waves, water of a thickness of 1 mm only transmits 10^{-20} of the incident unreflected radiation, and

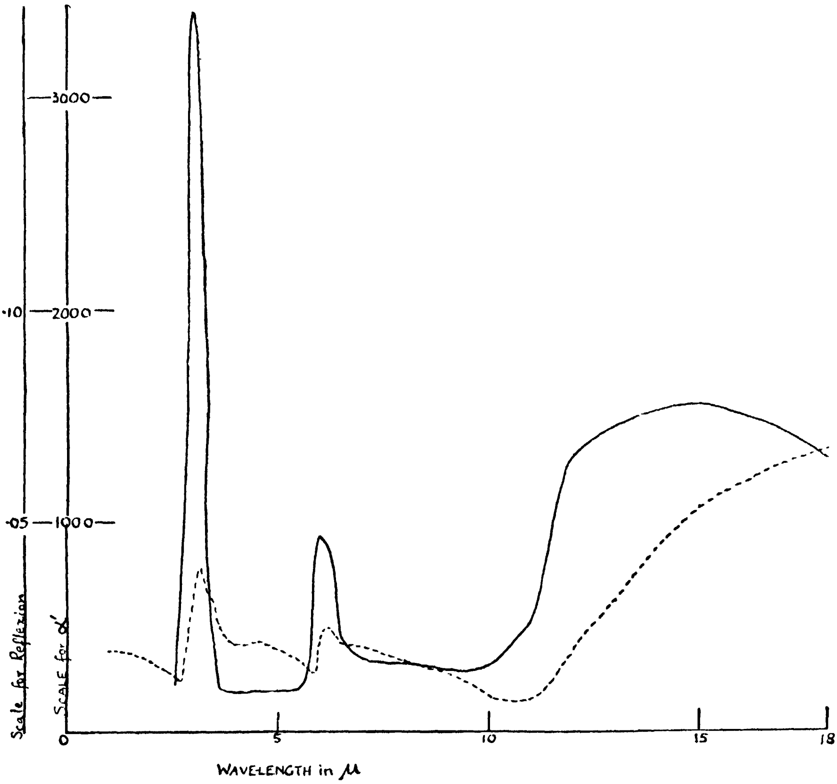


Fig. 31. The absorption spectrum of liquid water (continuous line), and the reflecting power of liquid water (broken line).

that a layer of water of 0·1 mm thickness only transmits 1/100th. In the regions of greater absorption such as the band at 6μ a layer of thickness 0·02 mm (20 microns) only transmits 1/100th of the unreflected radiation, and even a layer of 10 microns only transmits 1/10th of the incident radiation.

The broken curve in fig. 31 gives the reflecting power of a water surface for normal incidence. In the wave-lengths at which atmospheric radiation is most intense, say 5μ to 15μ, the value is everywhere small, so that only a few per cent of the incident radiation is reflected. Hence we may assume that films or drops of water of a thickness of 0·1 mm or more are within a few per cent of being black-body radiators.

The absorption curve of fig. 31 is based on observations on thin plane films of water, but presumably the results may be applied to water drops also. Thus individual water drops of $\frac{1}{2}$ mm in diameter may be treated as black-body radiators for all practical purposes. Even individual small drops of 10μ in diameter such as are found in fogs and clouds, may be regarded as almost equivalent to black-body radiators, and a layer of fog or mist containing the equivalent of $\frac{1}{2}$ mm of water per cm^2 of cross-section may be treated as a perfect black body within the limits of a few per cent. It is, however, clear that in a fog near the ground there will be a certain amount of diffuse scattering or reflexion of long-waved radiation produced in the same manner as the scattering of light in the sky, by particles whose dimensions are less than the wave-lengths of the radiation. W. H. Dines* found that the radiation from a fog was equivalent to the complete black-body radiation at the same temperature, but his measurements could not separate the true radiation from the fog particles from the radiation originating at the ground and undergoing diffuse scattering in the fog.

According to the *Meteorological Glossary*, the amount of suspended water in a cloud or fog is of the order of a few grammes of water per $(\text{metre})^3$. Thus 1 $(\text{metre})^3$ contains say 2 cc of liquid water, and the length of path which contains 0·1 mm of liquid water per cm^2 of cross-section is 50 metres. Thus a layer of cloud or fog a few metres thick can be treated as a black-body radiator.

The computation of the thickness of drop-laden air which is capable of functioning as a black body neglects the effect of reflexion and scattering by the drops. A figure of a few per cent was quoted above for the reflecting power of water (in sheets), but it is not strictly justifiable to use the same figure for a layer of a few metres in thickness, containing small water drops, since any portion of a beam of radiation passing through these few metres would meet a large number of reflecting surfaces, and the number would be greater the smaller the drops. Thus a cloud or fog is nearer to being a black body when the drops are large than when the drops are small. There appears to be no method of estimating the effect mentioned.

The coefficient of absorption of liquid water for wave-lengths less than 2μ, such as occur in the direct solar beam, are everywhere small, and the values derived for wave-lengths less than $0{\cdot}5\mu$ can be attributed to Rayleigh scattering by the molecules. The following table gives the values of α' for a range of wave-lengths below $1{\cdot}2\mu$ referred to 1 cm of water:

Wave-length (μ)	0·779	0·865	0·945	1·19
α'	0·118	0·128	0·234	0·842

Thus the absorptive power of water is very small for light waves. In a cloud or fog, the incident solar rays are scattered by irregular reflexion by the drops, and the total path through the water drops in the upper layers of the cloud or fog will be many times greater than the equivalent depth of liquid water. It is therefore possible that there is an appreciable amount of absorption of solar rays in the upper region of a cloud sheet or fog.

* M.O., *Geoph. Mem.* No. 18, 1921.

§ 70. *Diffuse reflexion and scattering*

The scattering of light by small obstacles was studied in great detail by the late Lord Rayleigh*. When a beam of radiation passes through a transparent medium containing in suspension small particles whose refractive index differs from that of the medium, some of the radiation is taken from the direct beam, and sent out in all directions from the particles. Rayleigh showed that the incident beam I_0 is reduced to $I_0 e^{-sx}$ in a distance x, s being the "coefficient of scattering". The coefficient s is proportional to λ^{-4}, and is therefore much greater for blue than for red light. Hence blue light is much more readily scattered than red light. Rayleigh explained the blue colour of the sky as due to the effect of scattering by the molecules of the air. He also showed that true scattering can only take place when the diameter of the obscuring particles is less than the wave-length of the incident light. Thus tobacco smoke, which consists of liquid drops whose diameter is about $0{\cdot}2\mu$, scatters light of all wave-lengths, but scatters the blue light much more than the red light, and so appears blue by scattered light; while the sun appears red when viewed through a smoke fog or haze, the light from the blue end of the spectrum being removed from the solar beam by scattering.

When the diameter of the particles is larger than the wave-length of the incident light true scattering no longer occurs, and the effect of the particles is of the nature of diffuse reflexion, which is equally effective for all wave-lengths. The last statement is borne out by the fact that the light reflected from a cloud when the sun is behind the observer is pure white, and that the sun appears white when viewed through a fog of water drops, whose diameter is of the order of 10μ.

Solar radiation is scattered by molecules or other sufficiently small particles in the air, and suffers loss by diffuse reflexion by larger particles. The larger particles such as cloud droplets occur in greater quantities at low levels than at high levels in the atmosphere, and hence observations of the intensity of solar radiation are made at stations at high level, in places removed as far as possible from the effects of low cloud and of local sources of pollution. At such stations the observations of the intensity of solar radiation are only affected by scattering, and possibly a small amount of absorption, in the upper atmosphere. Schuster showed that on clear days the loss of radiation from the solar beam of incoming radiation at Mount Wilson could be accounted for by the effect of molecular scattering alone.

The importance of the effects included in the terms scattering and diffuse reflexion will be seen when we come later to discuss the effects of cloud sheets on temperature (p. 131). The magnitude of the effects of scattering upon radiation of short wave-lengths is shown by the low amount of ultra-violet light measured in large towns as compared with the amounts measured in the country.

* *Scientific Papers*, **1**, p. 92. *Vide* also L. V. King, *Phil. Trans.* A, CCXII, 1913, p. 375.

It is important to realise the difference in the effects of small particles or molecules and cloud droplets. The former, by scattering a larger amount of light of the shorter wave-lengths, diminishes the proportion of these wave-lengths in the direct beam, and so causes the direct beam to be richer in red rays, and the scattered light to be richer in blue rays, than the original incident light. On the other hand, cloud droplets do not produce any change in the relative intensities of the different wave-lengths, and when a cloud appears red the colour is not to be explained as a result of the cloud droplets showing a selective scattering effect, but as due to the loss of the blue light by scattering from the beam which illuminates the cloud.

The scattering of light by volcanic dust is capable of producing a decided attenuation of the incoming solar radiation, and may give rise to marked temperature effects.

CHAPTER VI

RADIATION IN THE TROPOSPHERE

§ 71. *General survey of radiation phenomena in the atmosphere*

THE incoming solar radiation suffers depletion in at least five ways:

(1) Absorption by oxygen* at heights well above 100 km, and by the ozone layers at heights of about 20–50 km.

(2) Absorption by the other constituents of dry air.

(3) Scattering by molecules of dry air and water-vapour.

(4) Absorption by water-vapour.

(5) Scattering and diffuse reflexion and absorption by solid and liquid particles suspended in the atmosphere.

Item (1) will be discussed more fully in the next chapter. Item (2) is not of any considerable importance, as the atmospheric gases at ordinary atmospheric temperatures absorb practically no radiation. Item (3) has been estimated to be sufficient in itself to account for the blue colour of the sky. Schuster† and Fowle‡ have shown that scattering by the molecules of dry air is in itself sufficient to account for the loss of solar radiation in the uppermost layers of the atmosphere, within which water-vapour absorption is negligible. The corresponding coefficients of transmission can be evaluated by the use of the formulae of Rayleigh and King. Item (4), the absorption by water-vapour, has already been mentioned in § 66, and is discussed in some detail below. The amount of the loss due to the phenomena grouped under item (5) is highly variable, according to the conditions prevailing at the time of observation. The amount of dust in the lower atmosphere varies with time and place, and may comprise not only particles small enough to exert an effect which is greater for blue light than for red light in accordance with Rayleigh's theory, but also larger particles which do not come within the scope of Rayleigh's theory, and produce diffuse reflexion and absorption rather than scattering. Again, droplets of water in fog and cloud, being large by comparison with the wave-length of the incident radiation, also produce reflexion rather than true scattering. A cloud layer is capable of reflecting back to the sky about four-fifths of the incident radiation, and therefore the main factor in determining the net inflow of solar radiation is cloud amount.

The radiation which reaches the earth's surface is in part reflected by the sea, or diffusely reflected and scattered by the different parts of the earth's surface, such as trees, grass and other details of the earth's structure. Over

* *Vide* Chapman, *Q.J. Roy. Met. Soc.* **60**, 1934, p. 135.
† *Nature*, **38**, 1909, p. 97.
‡ *Astroph. Journ.* **38**, Nov. 1913.

land, the portion which is not reflected or scattered is absorbed by the earth's surface, except for the fraction which is used in evaporating water from the surface. Over the sea, the amount lost by reflexion back to the sky is great, and varies from about 2 to over 70 per cent; the amount of energy used in evaporation is probably greater than over land; and as the residual penetrates to considerable depths before it is completely absorbed, the changes of temperature of the surface of the sea are much less than the changes of temperature of the surface of the land.

Considerable variations are, however, possible in the temperature of the sea surface where there are large variations in depth, from quite shallow water to deep water. Where the water is shallow the radiation which penetrates through warms the sea floor, and this in turn warms the shallow water above it to a higher temperature than that of deep water. The high temperature of shallow pools on the sea shore bear out this view in a striking manner.

While the sun is high, the surface of the earth is continually heated by the incoming radiation, and the earth's surface radiates back to the sky an amount approximately in accordance with Stefan's law. The radiation from the earth's surface is long-waved radiation, much of which is readily absorbed by the water-vapour in the atmosphere. The water-vapour radiates in turn, both upward and downward, an amount of energy dependent on its temperature, and the result is to yield a complexity of streams of radiation which require careful consideration. The phenomena are further complicated by convection currents which carry upward streams of warm air, and so act as carriers of heat.

The amount of incoming radiation from the sun reaches its maximum when the sun is on the meridian at noon, but the rise in temperature of the earth's surface continues for some time after this, since the loss of energy by radiation is not at noon sufficient to balance the incoming radiation. The balance between incoming and outgoing radiation is reached some time after noon, but after that time the loss by radiation from the earth's surface exceeds the incoming radiation, and the temperature steadily falls during the rest of the day, and during most of the night. The fall is only checked by the beginning of incoming radiation shortly before sunrise. The diurnal variation of temperature should thus show a maximum shortly after noon, and a minimum shortly before sunrise. The curve of variation will not be a pure sine-curve, but its form will vary with the time of the year, though its main features will remain unchanged*.

The air temperature will follow the same general course as the soil temperature, but with a lag in the times of occurrence of the maximum and minimum. The relationship cannot be simple, on account of the complexity of the phenomena of radiation and turbulence, but we should expect to find the diurnal variation of temperature decreasing with height, as is actually the case. The transfer of heat upward through the atmosphere is discussed in §§ 76, 132 and 135 below.

* See H. L. Wright, *Mem. R. Met. Soc.* **4**, No. 31; also fig. 13, p. 23 above.

§ 72. *Water-vapour as a controlling agent in atmospheric absorption and radiation*

Before proceeding to discuss the various streams of radiation which must be considered, we must form a mental picture of the radiative mechanism of the atmosphere. The main result of § 71 above is that so far as true absorption in the atmosphere is concerned the incoming solar beam passes through the atmosphere almost undiminished. The light reflected and scattered by molecules of dry air and water-vapour, water drops, etc., will remain short-wave radiation. The long-wave radiation from the atmosphere itself, and the absorption in the atmosphere of long-wave radiation from the earth's surface, are so nearly completely due to water-vapour that we may, at least in a preliminary survey, neglect the radiation and absorption of all other gaseous constituents of the atmosphere. This is equivalent to regarding the atmosphere as consisting of water-vapour only, so far as radiative effects are concerned. The effect of the addition of dry air to the water-vapour is to weight the water-vapour with an added specific heat, without modifying any of its other properties. Since the establishment of a balance between incoming and outgoing radiation depends on the temperatures of the radiating bodies, it may be concluded that the temperature distribution at which equilibrium will be finally established will not be affected by the addition of the dry air to the water-vapour atmosphere, but that the time taken to establish any given change of temperature will be thereby increased. The mathematical treatment of the problem is not much simplified by the suggestion made here, since the water-vapour atmosphere will be endowed with a specific heat whose magnitude will be subject to wide variations with height, and with time at any given level, but it will nevertheless be worth bearing in mind the relative functions of dry air and water-vapour in the radiative processes of the atmosphere.

§ 73. *The heat balance of the atmosphere*

On the average the intensity of the incoming solar beam at the outer limit of the atmosphere is 1·94 gramme-calories per cm^2 per minute, upon a surface placed at right angles to the beam. Of this amount 6 to 8 per cent is absorbed in the atmosphere, 9 per cent is reflected back from the atmosphere, 8 per cent is reflected back from the earth's surface, and a considerable fraction is scattered by the molecules of the atmosphere, or suffers diffuse reflexion by particles in suspension in the atmosphere. The short-wave radiation which reaches the ground thus comprises two categories: (*a*) the direct solar beam, and (*b*) the radiation diffused from the sky. Ångström* has estimated that if Q is the total short-wave radiation $(a+b)$, and D the diffuse radiation from the sky (*b*), the fraction D/Q has a minimum of 0·25 in May, and a maximum of 0·8 to 0·9 in winter (at Stockholm). He further estimates that on a completely overcast day the total Q amounts to one-fourth of its value on a clear day, and that on

* *Vide* Radiation and Climate, *Geog. Ann.* 1925, p. 122

a day when the maximum possible number of hours of sunshine is N, and the actual number of hours of sunshine is n, the total value of Q is given by

$$Q = Q_0\,(0{\cdot}25 + 0{\cdot}75n/N),$$

where Q_0 is the total incoming radiation on a clear day.

Over land the short waves which reach the earth's surface are in part reflected by the surface. The loss by reflexion does not usually exceed about 10 per cent, except over a snow surface, which is capable of reflecting about 80 per cent of the incident short-wave radiation. A fraction, whose magnitude is not readily estimated, is used in evaporating water from the surface of grass, leaves, etc., and in high latitudes a portion is used in melting ice and snow, and the remainder is absorbed.

Over the sea the phenomena are somewhat different. The fraction reflected is very slight when the sun is high, but increases rapidly as the sun's zenith distance increases. Schmidt* has estimated that on the average for the whole year the percentage r of the total incoming radiation which is reflected at a water surface in different latitudes is as shown in the following table:

Lat.	0	10	20	30	40	50	60	70	80	90
r	3·3	3·5	3·6	4·2	4·6	5·3	6·2	8·0	11·5	13·5

Ångström† has given estimates of evaporation based on observations extending over ten days of August 1905, over Lake Vassijäure, the period being selected so that there was practically no difference in the temperature of the air and water, and no progressive change in the temperature of the water, during the period. His estimate of the evaporation amounted to a little less than 2 mm of water per day, corresponding to nearly 120 gramme-calories per cm^2, or approximately one-third of the incoming net radiation from sun and sky. Lake Vassijäure being in latitude 68° N, we thus find that of the incoming radiation from sun and sky, 8 per cent is reflected from the water surface, and 33 per cent is used in evaporation, the remaining 59 per cent being absorbed by the water and re-radiated to the atmosphere as long waves. Ångström's figure for evaporation is rather less than that of 2·3 mm per day estimated by Wallén for Lake Hjälmaren during August, but it is in close agreement with Witting's estimate for Botten Bay, which is in almost the same latitude as Lake Vassijäure, while Lake Hjälmaren is farther south. It is of interest to compare these estimates with the estimate given by Wust‡ of the total evaporation during the year from sea and land, his respective figures being 267 and 112, in units of 1000 km^3 of water. The total surface of the oceans being $3{\cdot}68 \times 10^{18}$ cm^2, over which there has to be distributed 267×10^{18} cm^3 of evaporation, the depth of water evaporated per annum is $267 \div 3{\cdot}68$, or 72·6 cm, yielding an average of 2 mm per day over the whole earth. The area of the earth's land surface is $1{\cdot}45 \times 10^{18}$ cm^2, so that the total evaporation per annum amounts to $112 \div 1{\cdot}45$ cm, or 77·2 cm, or an average of about 2·1 mm per day. At this stage we do not wish to emphasise these actual magnitudes,

* *Ann. Hydrog. u. Marit. Met.* 1915, h. 3–4.
† *Geog. Ann.* 1920, h. 3.
‡ *Zeit. Ges. Erdkunde*, Nos. 1–2, 1922.

but the general agreement of the figures given by different workers appears to justify the adoption of Ångström's figures as giving the order of magnitude of the consumption of heat in evaporation.

The part of the incoming solar radiation which reaches the earth's surface and is not reflected or used up in evaporation is absorbed by the surface. As a result the temperature of the earth's surface increases steadily while the amount of incoming radiation is increasing during the morning. The maximum temperature of the earth's surface occurs some time after noon, the lag after noon depending on the nature of the surface. Thus at a depth of 1 cm in sand the maximum occurs at 12h 30m, while just inside the soil under a grass covering the maximum occurs some three hours after noon. The surface of the earth radiates to the atmosphere, its radiation being of long wave-length, and therefore readily absorbed by the water-vapour in the atmosphere. The water-vapour in turn sends out long-wave radiation, a portion of which is directed downward to the earth's surface, where it is absorbed. A considerable fraction of the radiation from the earth's surface falls within the band of transparency of water-vapour ($8\frac{1}{2}\mu$ to 11μ), and that the radiation within this band of wave-lengths will go upward through the atmosphere without being absorbed.

The net outward flow of long-wave radiation from the earth, which is the difference between the radiation from the earth's surface and the long-wave radiation of the atmosphere, is of the same order of magnitude by night and by day. But as it is much easier to observe this net radiation by night than by day, on account of the absence at night of the diffuse radiation of short wave-length from the sky, night observations have been more frequent than day observations. Moreover, the nocturnal cooling is bound up with the practical problem of forecasting the occurrence of frost, a problem of considerable economic importance.

Observations of the radiation of the atmosphere at night have been carried out by Dines, Ångström, Asklöf and others, and the amount of this radiation is found to be of the order of three-fourths of the full black-body radiation at the temperature of the surface. Since the earth sends out practically the full black-body radiation appropriate to its own temperature, it follows that the net flow of radiation outward from the earth (outgoing – incoming) is of the order of one-fourth of the black-body radiation. This net outflow of radiation is mainly the radiation of the earth's surface in the transparent band from $8\frac{1}{2}\mu$ to 11μ.

In discussing radiation of long waves at ordinary terrestrial temperatures, we may treat as black-body radiators the solid ground, the sea surface, the surface of snow, a cloud sheet thick enough to cast a shadow, and a fog. These assumptions are found to fit the facts with great accuracy.

§ 74. *The equations of radiative transfer of heat*

In the first place we shall consider the variation of the upward and downward beams of radiation of a particular wave-length λ. Let the upward and downward beams of radiation of this wave-length be A_λ and B_λ at a level where the temperature is T, and let E_λ be the black-body radiation of this wave-length at temperature T. Let the level be specified by the amount τ of the water-vapour *above* this level. The beams entering and leaving a thin layer containing an amount of water-vapour $d\tau$ are shown in fig. 32. The upward beam

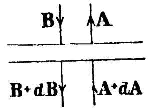


Fig. 32. Diagram of radiative transfer.

loses an amount $k_\lambda d\tau A_\lambda$ by absorption, but gains an amount $k_\lambda d\tau E_\lambda$ from the upward radiation of the layer itself. Hence

$$\frac{dA}{d\tau} = k_\lambda (A_\lambda - E_\lambda) \qquad \text{......(1)},$$

$$\frac{dB}{d\tau} = k_\lambda (E_\lambda - B_\lambda) \qquad \text{......(2)}.$$

In the atmosphere, under normal conditions, when the temperature decreases steadily with height, the upward beam A_λ is greater than E_λ at all heights, except at the ground, when the two are equal for all wave-lengths radiated and absorbed by water-vapour. For long-wave radiation B_λ is zero at the outer limit of the atmosphere, and remains less than E_λ at all lower levels. The difference between A_λ or B_λ and E_λ will be small at all levels for those values of λ for which k_λ is great, and will be greatest at all levels for those values of λ for which k_λ is small, i.e. for those wave-lengths to which water-vapour is transparent.

It should be noted that, for a parallel beam, equations (1) and (2) are true whether the radiation is black body, grey, or selective.

The nature of the beams ΣA and ΣB will thus be very complicated, and neither beam can be represented as the radiation of water-vapour of a particular temperature with any great accuracy.

Some simplification of the problem is, however, possible along the lines of Simpson's discussion as amplified by Brunt*. But at this stage, the serious difficulty that arises through the diffuse nature of the streams of radiation will be left out of account. The upward and downward streams of radiation are not, in fact, parallel streams of radiation, but consist of an infinite number of beams in widely varying directions. For reasons stated in § 83 below, a reasonable approximation is obtained by treating the beams as parallel, and doubling the coefficient k_λ. For the present we shall limit our discussion to that of parallel beams without considering too closely the appropriate magnitude of the constant k in equations (1) and (2) above.

* *Proc. Roy. Soc.* A, **124**, 1929, p. 201.

§ 75. *An equation for radiative transfer*

The maximum intensity of the radiation from the earth's surface is at about 10μ, and that of black-body radiation from a body at the temperature of the stratosphere is at about $12\frac{1}{2}\mu$. Thus the existence of the band from $8\frac{1}{2}\mu$ to 11μ, in which water-vapour is effectively transparent, is of predominating importance in meteorology. As soon as radiation within this band of wave-lengths leaves the earth's surface, it is effectively lost as far as the atmosphere is concerned, if the sky is clear. The treatment which follows here attempts to take these facts into account.

We shall assume with Simpson that a column of air which contains 0·3 mm of precipitable water as vapour will completely absorb all radiation of wave-lengths between $5{\cdot}5\mu$ and 7μ, and of wave-lengths greater than 14μ. We shall neglect the partial absorption between 4μ and $5{\cdot}5\mu$, between 7μ and $8{\cdot}5\mu$, and between 11μ and 14μ. It is considered that a reasonable first approximation to the facts is obtained by making this approximation; and the success with which Simpson applied these simplifications justifies their use as a first approximation.

The name "*W*-radiation" has been suggested (Brunt, *loc. cit.*) for radiation restricted to the wave-lengths within which water-vapour absorbs and radiates, and the term will be adopted here in order to save continual re-statement. It is also assumed that dry air does not absorb or radiate long waves to an appreciable extent.

The length l of the column of air which contains 0·3 mm of precipitable water, or 0·03 gramme of water-vapour per cc, is readily deduced. Let e be the vapour-pressure in mb, ρ_w the density of the water-vapour, T the temperature, and R' the gas-constant for water-vapour. Then

$$e = R'\rho_w T, \quad \text{where} \quad R' = 4{\cdot}62 \times 10^3.$$

By the definition of l $\qquad l\rho_w = 0{\cdot}03,$

$$l = 0{\cdot}03/\rho_w = 0{\cdot}03 \times 4{\cdot}62 \times 10^3 \times T/e = 139T/e \text{ cm} = 1{\cdot}39T/e \text{ metres} \quad \ldots\ldots(3).$$

If we take $T = 275°$ A, $\qquad l = 380/e$ metres.

In the lower atmosphere e is normally of the order of 10 mb, and is often considerably greater than this. Hence l is normally of the order of 40 metres, and it is legitimate to treat the layer of thickness l as having uniform temperature equal to the mean temperature of the layer, so far as the computation of the radiation from the layer is concerned. We shall adopt this method as the basis of computation of the upward and downward streams of radiation in the atmosphere.

Beginning at the ground we divide the whole atmosphere into layers of varying thickness such that each contains 0·3 mm of precipitable water. The thickness of each layer is $139T/e$ cm, where T and e represent the mean absolute temperature and the mean vapour-pressure in millibars, within each layer. The thickness of each layer is thus proportional to the appropriate value of T/e. Let AB in fig. 33 be the upper boundary of the rth layer. The amount

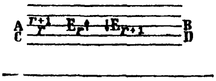


Fig. 33. Diagram of transfer of W-radiation.

of W-radiation moving upward across AB is equal to E_r, the radiation from the rth layer, since no radiation reaching AB can by hypothesis have originated below CD, the lower boundary of the rth layer. Similarly the amount of W-radiation moving downward across AB is equal to E_{r+1}, the radiation from the $(r+1)$th layer. Let the net upward flux of radiation across AB be F_r. Then

$$F_r = E_r - E_{r+1} \qquad \text{......(4).}$$

If T_r be the mean temperature of the rth layer, then with the usual notation

$$F_r = E_r - E_{r+1} = -\Delta E_r = -\frac{\Delta E}{\Delta T}(T_{r+1} - T_r)$$

$$= -\frac{\Delta E}{\Delta T}\left\{\frac{1}{2}\left(\frac{\Delta T}{\Delta z} l\right)_{r+1} + \frac{1}{2}\left(\frac{\Delta T}{\Delta z} l\right)_r\right\} \qquad \text{......(5),}$$

$$= -\frac{\partial E}{\partial T}\frac{\partial T}{\partial z} l = -\frac{139 T}{e}\frac{\partial E}{\partial T}\frac{\partial T}{\partial z} = -j\frac{\partial T}{\partial z} \qquad \text{......(6).}$$

The use of the differential coefficient $\partial E/\partial T$ in place of the corresponding finite-difference ratio is justified by the slow rate of change of this quantity with T, as shown in the next paragraph. The use of the differential coefficient $\partial T/\partial z$ in place of the corresponding finite-difference ratio is justified if we neglect small quantities of the first order, and also by the fact that in practice $\partial T/\partial z$ is determined by the measurement of the difference of temperature at points separated by a finite difference of level. In this way we obtain the expression $-j\,\partial T/\partial z$ for the net upward flux of heat, j being a quantity which normally changes only very slowly with height, as is seen from a consideration of the factors involved in it.

The value of j depends upon the assumption that 0·3 mm of precipitable water absorbs completely the W-radiation passing through it. Should the estimate of 0·3 mm be too small, the value of j is proportionately too small. Thus any uncertainty as to the true value of j is due only to uncertainty as to the truth of the estimate of 0·3 mm for the quantity of precipitable water which will completely absorb W-radiation. The value of $\partial E/\partial T$ can be readily deduced from a table of values of E given by Simpson (*loc. cit.* p. 8):

Temperature	200° A	220° A	270° A	295° A
$\frac{\partial E}{\partial T} \times 10^3$	1·6	2·0	3·0	3·5

E is measured in gramme-calories per cm^2 per min.

Within these limits of temperature $\partial E/\partial T$ appears to be almost accurately a linear function of temperature

$$\frac{\partial E}{\partial T} \times 10^3 = 3{\cdot}0 + 0{\cdot}02\,(T - 270) \qquad \text{......(7).}$$

The vapour-pressure e varies from day to day at a given place, but shows no marked diurnal variation. It varies to some extent with height. According to Hann* the relative values of e at the ground, at 0·5 km and at 1 km are 1·0, 0·83 and 0·68. But observations frequently show a much slower decrease with height than this. Thus Steiner† found in his discussion of kite ascents made at Rostock, that vapour-pressure showed extremely little variation with height up to 600 metres.

If we assume a temperature of 275° A and a vapour-pressure of 5 mb, corresponding to relative humidity of about 70 per cent, the value of j in equation (6) is found to be

$3 \times 10^{-3} \times 139 \times 275 \div 5 = 23$, the units being the gramme-calorie and minute.

We can use this figure to compute the upward flux of radiation corresponding to any given value of $\partial T/\partial z$. Let us assume the lapse-rate of temperature to be adiabatic, so that $\partial T/\partial z = -10^{-4}$. Then the vertical flow of radiation is $2{\cdot}3 \times 10^{-3}$ gramme-calories per minute. To compare this with the average amount of radiation coming in from the sun, we take the solar constant to be 2 gramme-calories per square centimetre per minute. Allowing for averaging over day and night, and over all latitudes, and assuming Aldrich's value of the albedo 0·43, we find that the amount of incoming energy which has to be transported out again by radiation, turbulence, or otherwise, is about 0·275 gramme-calories per square centimetre per minute. Thus with the assumptions made above in evaluating j, and assuming an adiabatic lapse-rate, except for the radiation contained within the transparent band from $8\frac{1}{2}\mu$ to 11μ, which passes out through the atmosphere without absorption, less than 1/100th of the incoming energy can be disposed of by passing outward as W-radiation through the lowest layers. While it is true that the lapse-rate may from time to time exceed many times the adiabatic, this only occurs during a part of the day, and moreover the value of j deduced above is based on a low figure for vapour-pressure. With larger vapour-pressures j is diminished. It can therefore be accepted that under normal conditions only a very small fraction of the incoming solar radiation passes back through the lowest layers of the atmosphere as W-radiation.

It will be noted that the flow of energy by radiation is always from high temperature to low temperature, so that radiation tends to produce isothermal conditions. There is resemblance between the results here derived and the equations for conduction of heat in solids.

The net flux of heat across unit area of a horizontal surface at height $z + dz$ is

$$-j\frac{\partial T}{\partial z} - \frac{\partial}{\partial z}\left(j\frac{\partial T}{\partial z}\right)dz.$$

A disc of air of unit horizontal area and thickness dz will therefore gain a quantity of heat $-\frac{\partial}{\partial z}\left(j\frac{\partial T}{\partial z}\right)dz$ per minute. If c_p be the specific heat of air at

* *Lehrbuch der Meteorologie*, p. 243.

† *Wiss. Abhandl. Luftwarte Rostock*, 1926, h. 1.

constant pressure, and the unit of time t be 1 second, this gain of heat must be equal to $60\rho c_p \frac{\partial T}{\partial t}\,dz$. Thus the equation of radiative transfer of heat may be written

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{60}\frac{\partial}{\partial z}\left(j\frac{\partial T}{\partial z}\right) = \frac{139}{60}\frac{\partial}{\partial z}\left(\frac{T}{e}\frac{\partial E}{\partial T}\frac{\partial T}{\partial z}\right) \qquad \text{......(8).}$$

We have seen that in the lowest layers of the atmosphere j will vary slowly with height, while $\partial T/\partial z$ is shown by observation to be subject to variation within wide limits, a lapse-rate 10 times the dry adiabatic lapse-rate being by no means uncommon near the ground. Thus in an investigation of the transfer of heat by radiation in the lowest layers we may treat j as constant in equation (8), giving T and e the values corresponding to the level under discussion. Equation (8) then reduces to

$$\frac{\partial T}{\partial t} = \frac{139}{60\rho c_p}\frac{T}{e}\frac{\partial E}{\partial T}\frac{\partial^2 T}{\partial z^2} = K_R\frac{\partial^2 T}{\partial z^2} \qquad \text{......(9).}$$

This equation is similar to that for conduction of heat in a solid with a constant K_R replacing the thermometric conductivity κ, where

$$K_R = \frac{j}{60\rho c_p} = \frac{139}{60\rho c_p}\frac{T}{e}\frac{\partial E}{\partial T} \qquad \text{......(10).}$$

This value of K_R is adjusted to give a unit of time of 1 second in equation (9).

Again, taking $T = 275°$ A, $e = 5$ mb, we find

$$K_R = \frac{23}{60\rho c_p} = \frac{23}{60 \times 0{\cdot}00125 \times 0{\cdot}24} = 1{\cdot}3 \times 10^3 \qquad \text{......(11).}$$

For the reasons given above, in discussing j, this estimate of K_R is likely to be rather higher than the normal values, on account of the low value assumed for e. For larger values of e, K_R is reduced in inverse proportion. The constant K_R might, on the analogy with the conduction of heat in solids, be called the *radiative diffusivity*. Even allowing for the fact that the value of $1{\cdot}3 \times 10^3$ is likely to be rather high, it is very much in excess of the corresponding coefficient for the molecular conduction of heat, whose value is about 0·16 in the same units.

The derivation of equation (6) is based on the assumption that there is a complete layer of air whose water-vapour content is equivalent to 0·3 mm of precipitable water, above and below the level across which the flux of radiation is to be stated. It cannot therefore be applied without further consideration to heights above the ground of less than l as defined above.

§ 76. *The diurnal variation of temperature as affected by radiation*

We have seen in equation (9) above that the transfer of heat by radiation is analogous to the conduction of heat in a solid, but with a much larger coefficient of radiative diffusivity substituted for the usual coefficient of conduction. The theory as given above is not strictly applicable down to the surface. Let us as a first approximation neglect this limitation, and consider how the diurnal variation of temperature will vary with height, if the transfer of heat is entirely by radiation in accordance with equation (9).

Let the temperature at the earth's surface be given by a pure diurnal sine-curve

$$T = T_0 + A \sin qt \qquad \text{......(12).}$$

The solution of equation (8) which agrees with this relation at the ground is

$$T = T_0 - \beta z + Ae^{-bz} \sin (qt - bz) \qquad \text{......(13),}$$

where b is a constant given by

$$b^2 = q/2K_R \quad \text{and} \quad q = 2\pi/(24 \times 60 \times 60).$$

The term βz is included to allow for the mean lapse-rate β from the ground upwards.

Equation (13) gives the diurnal variation of temperature at any height z. The amplitude Ae^{-bz} falls off exponentially, and the time of maximum at height z occurs at bz/q seconds later than at the surface.

The observations on the Eiffel Tower may be compared with the results shown in equation (13). Taking $K_R = 650$, corresponding to a vapour-pressure of 10 mb, we find $b = 0{\cdot}000236$. At the top of the Eiffel Tower

$$z = 300 \text{ metres} = 3 \times 10^4 \text{ cm}, \quad \text{and} \quad bz = 7{\cdot}08.$$

Hence the ratio of the diurnal range at the top and bottom of the Tower should be e^{-7} or about 10^{-3}. The observed ratio is between 1/3 and 2/3, according to the time of year. We therefore conclude that radiation is only of slight importance in the spread of heat upward to any considerable distance above the ground, as has indeed already been seen in § 75. As will be seen later in Chapter XIII, turbulence is a far more effective agent in effecting the transfer of heat upward through the atmosphere.

At very small heights above the surface of the ground, turbulence is unable to develop effectively, and the transfer of heat is there mainly by radiation. Hence on sunny days, with a large amount of incoming radiation, the surface heat is transferred only slowly up to small heights above the ground, and the result is the formation of very large lapse-rates in the immediate neighbourhood of the ground surface. Many efforts have been made to study these large lapse-rates in detail, but the instrumental difficulties are serious. The theoretical difficulties are equally serious. Efforts to give a detailed theory have been made by Malurkar and Ramdas*, and by Brunt†, but neither of these efforts is wholly successful, owing to the difficulty of dealing with the discontinuous nature of the phenomena near the surface of the earth.

§ 77. *Limitations of the discussion of* §§ 75 *and* 76

The difficulties in the way of a theoretical discussion of radiative phenomena are threefold: (*a*) In the first place, since the absorption is mainly due to water-vapour, it is important to be able to specify the distribution of water-vapour

* *Indian Journ. Phys.* **6**, pt 6, 1932. † *Proc. Roy. Soc.* A, **130**, 1930, p. 98.

with height. This cannot be done with any degree of accuracy, since the distribution is irregular at any instant, and is subject to wide variations from one hour to the next. (*b*) In the second place, the radiation which has a general upward (or downward) component of motion is not a parallel beam, but is made up of an infinite number of beams distributed through all possible directions in space, whose intensities follow no simple law. In a first approximation it is necessary to simplify the treatment to a parallel beam, though it is not necessary to assume that the beam is normal to the earth's surface. If the direction of the parallel beam makes an angle ψ with the vertical, equations (1) and (2), p. 115, only require modification by substitution of $d\tau \sec\psi$ for $d\tau$, or, what comes to the same thing, the substitution of $k \sec\psi$ for k. (*c*) In the third place, the coefficient of absorption k is a complex function of the wave-length, and is not capable of representation by any simple mathematical law.

The third of these difficulties led to the development given above in § 75, following the lines adopted by Simpson. It is realised that the representation of the transport of heat by radiation as strictly analogous to the transport of heat by conduction in a solid is only a first approximation. It fails to take account of the parts of the inward and outward streams of radiation which are of such wave-lengths as to pass through the atmosphere with only very slight absorption. But referring back to equation (9), p. 119, we see that the coefficient K_R aims at taking account only of those parts of the streams of radiation which can produce changes of temperature, and the neglect of the wave-lengths which are slightly absorbed is therefore of no practical importance. Roberts* has computed the outward stream of radiation through the atmosphere with certain simplifying assumptions, and representing the stream by a formula

$$A_1 \frac{\partial T}{\partial z} + A_2 \frac{\partial^2 T}{\partial z^2} + A_3 \frac{\partial^3 T}{\partial z^3} + \dots,$$

has assumed that the coefficient A_1 is the radiative diffusivity. He finds, as is indeed obvious *a priori*, that the contributions to A_1 due to wave-lengths which are only very slightly absorbed, are very great, and concludes from this that the transport of heat by radiation is not analogous to transport by conduction in a solid. The treatment of § 75 above assumes that the transport of heat by radiation is in part analogous to conduction, and is in part a steady stream passing through the atmosphere without change. The real physical difference between the two points of view is not great, but the method of § 75 has the advantage that it leads to equations which can be applied to concrete problems.

A much more serious limitation of our method is the increase of l, the depth of the layer containing 0·3 mm of precipitable water, at heights where the vapour-pressure is small. It is not then permissible to treat the layer as having uniform temperature equal to the mean temperature of the layer.

* *Proc. Roy. Soc. Edin.* **50**, 1930, p. 225.

§ 78. *The interchange of long-wave radiation between the atmosphere and the earth's surface. Nocturnal radiation with clear skies*

It has been stated above in § 73 that the total amount of long-wave radiation from the atmosphere to the earth's surface is of the order of $\frac{1}{2}$ to $\frac{3}{4}$ of the black-body radiation at the temperature of the earth's surface. Methods have been developed for measuring the net loss of heat from the earth's surface by long waves. Some of these methods are only applicable at night, in the absence of direct and diffuse solar radiation, but other methods, notably that of W. H. Dines*, can be used both by day and by night. The observations are, however, easier to carry out at night, and hence the net exchange of long-wave radiation between the atmosphere and the earth's surface has been more extensively studied by night than by day. The net outward flow of long waves by day cannot differ as to order of magnitude from that which occurs at night, since the water-vapour content of the atmosphere is not subject to any considerable diurnal variation.

Simpson showed (*loc. cit.*) that the net outward flow of long waves from the earth's surface can be placed between certain limits on the basis of his assumptions as to the nature of the water-vapour spectrum. He compared the actual observations of W. H. Dines and L. H. G. Dines† with the conclusions he was able to draw on theoretical grounds, and found close agreement, the mean observed values falling well within the prescribed limits.

It is possible to obtain an empirical formula which appears to represent the observed nocturnal radiation with great fidelity. In a medium in which the transfer of heat is by conduction, with a coefficient of diffusivity κ, the diurnal changes of temperature are transported in accordance with equation (13), p. 120. The net flow of heat at any level is proportional to $\kappa \frac{\partial T}{\partial z}$, and as is readily seen by differentiating the equation for T, this involves its being proportional to $\sqrt{\kappa}$. Now in § 75 we have seen that κ is replaced by the radiative diffusivity K_R which is inversely proportional to e, and though the equations we have developed are not strictly applicable down to the earth's surface, all these relations taken together suggest that the net radiation should be a function of $\sqrt{e}$.

It was shown by Brunt‡ that the observations made by W. H. Dines and L. H. G. Dines§ of the incoming radiation from the six zones of sky centred on zenith distances $7\frac{1}{2}°$, $22\frac{1}{2}°$, $37\frac{1}{2}°$, $52\frac{1}{2}°$, $67\frac{1}{2}°$ and $88\frac{1}{2}°$, could all be represented with great accuracy by a formula

$$R/\sigma T^4 = a + b\sqrt{e} \qquad \text{......(14)},$$

where R stands for the measured radiation, and σT^4 is the total black-body radiation at the surface temperature T. The constant a differed for different zones, but b was practically constant for all six zones. The total radiation from

* *Met. Mag.* Oct. 1920.

† *Mem. R. Met. Soc.* **2**, No. 11.

‡ *Q.J. Roy. Met. Soc.* **58**, 1933, p. 389.

§ *Mem. R. Met. Soc.* **2**, No. 11.

the whole sky could also be represented by equation (14) with a high degree of accuracy. For the whole hemisphere and each of the six zones separately the coefficient of correlation between $R/\sigma T^4$ and $\sqrt{e}$ was between 0·94 and 0·97. The correlated data were monthly mean values.

	Correlation coefficient	$R/\sigma T^4$
Zone 1	0·96	$0{\cdot}473+0{\cdot}065\sqrt{e}$
„ 2	0·96	$0{\cdot}477+0{\cdot}065\sqrt{e}$
„ 3	0·97	$0{\cdot}495+0{\cdot}063\sqrt{e}$
„ 4	0·97	$0{\cdot}515+0{\cdot}065\sqrt{e}$
„ 5	0·94	$0{\cdot}537+0{\cdot}072\sqrt{e}$
„ 6	0·96	$0{\cdot}749+0{\cdot}058\sqrt{e}$
Hemisphere	0·97	$0{\cdot}526+0{\cdot}065\sqrt{e}$

The uniformly high values of the correlation coefficients is striking. The effect of varying vapour-pressure, as shown by the coefficient of $\sqrt{e}$ in the regression equation, is remarkably uniform in the first four zones, and does not deviate widely in the fifth zone. In the sixth zone, however, the total radiation is much nearer to complete black-body radiation than in any higher zone, and the variation with vapour-pressure is less than in any higher zones. It is possible that the observations in the lowest zone do not represent the true atmospheric radiation alone, and that the effect of radiation from trees and other obstacles is partly superposed upon the true atmospheric radiation.

The regression equation for the whole hemisphere,

$$R/\sigma T^4 = 0{\cdot}526 + 0{\cdot}065\sqrt{e} \qquad \ldots\ldots(15),$$

where e is the vapour-pressure in millibars, may be taken as reproducing with a very high degree of accuracy the monthly mean values of the radiation as observed by Dines. The equation shows that the atmosphere always radiates more than half the total black-body radiation at the temperature of the surface. The mean value of $\sqrt{e}$ was roughly 3·3 in the observations in question, and the corresponding mean value of $R/\sigma T^4$ was 0·74, indicating that with clear skies the net loss of heat by radiation of long waves is approximately one-fourth of the black-body radiation at the corresponding temperature. It is of interest to compare this figure with the fraction of the radiation from the earth's surface, treated as a black body, which is contained within the limits $8\frac{1}{2}\mu$ and 11μ of the transparent band; the latter fraction is 0·155 at 280° A, 0·166 at 292° A, and 0·174 at 299° A, indicating that about two-thirds of the net loss is accounted for by the radiation in the band $8\frac{1}{2}\mu$ to 11μ alone.

It has also been shown (Brunt, *loc. cit.*) that a similar formula will fit the individual observations of $R/\sigma T^4$ made by Asklöf*, though the correlation coefficient between the computed and observed values of $R/\sigma T^4$ is 0·83, rather less than was found for the monthly mean values of Dines. The fit to Asklöf's observations was, however, much closer than could be obtained by the use of the formula

$$R/\sigma T^4 = A - B10^{-\gamma e} \qquad \ldots\ldots(16)$$

suggested by Ångström. With this formula the correlation coefficient between the computed and observed values of $R/\sigma T^4$ was 0·46.

* *Geog. Ann.* **2**, 1920, p. 253.

A formula of the type $R/\sigma T^4 = a + b\sqrt{e}$

will give a close fit to the observations of Ångström* at Bassour in Algeria at a height of 1160 metres above mean sea level and at various mountain stations in America; it can also be adjusted to give a remarkably close fit to the mean values of atmospheric radiation at Lindenberg quoted by Robitsch†, giving a coefficient of 1·0 between the computed and mean observed values of $R/\sigma T^4$; it also gives a close fit to certain observations made by Boutaric‡ at Montpellier and Pic du Midi, and the observations at Poona summarised by Ramanathan and Desai§. Details of the fitting of the formula to the various series of observations will be found in the paper by Brunt referred to above.

It is curious that while it is possible to find values of the coefficients a and b in equation (14) above which will yield a good fit to any series of observations of radiation of long waves from the earth to the atmosphere, the values of a and b differ for different series of observations. The following table shows the values of a and b, and the correlation coefficient between the ratio $R/\sigma T^4$ and $\sqrt{e}$, for different series of observations, reduced to refer to vapour-pressures in millibars in each case.

	a	b	r	Range of e in mb
Dines (Benson)	0·52	0·065	0·97	7–14
Asklöf (Upsala)	0·43	0·082	0·83	2–8
Ångström (Bassour)	0·48	0·058	0·73	5–15
Boutaric (France)	0·60	0·042	—	3–11
Robitsch (Lindenberg)	0·34	0·110	1·0	3–22
Ramanathan and Desai (Poona)	0·26	0·120	0·93	8–18
Mean	0·44	0·080		

The precise reason for the wide differences in the values of a and b is by no means obvious, but it is probable that it is mainly due to instrumental causes. The methods of observation adopted by the different observers whose results are represented in the above table were widely different, and the constants of some of these instruments were possibly not known with certainty. It is also probable that all the instruments do not treat radiation from all zones of the sky in the same manner.

The formula which we have given above

$$R/\sigma T^4 = a + b\sqrt{e}$$

is to be regarded as an empirical formula, for which no strictly theoretical justification has been advanced. The term in $\sqrt{e}$ is regarded as to some extent taking account of the variation of conditions with height. It is not considered in any way justifiable to interpret its meaning when $e=0$ as indicating that $a\sigma T^4$ is the radiation from dry air alone. If this were accepted, then the radiation from the dry air would exceed that from the water-vapour in the atmosphere, and all the available evidence points to the falsity of such a supposition.

* *Smithsonian Inst. Misc. Coll.* **65**, No. 3, 1918.
† *Lindenberg Arbeiten*, **15**, p. 203.
‡ *La Météorologie*, **4**, 1928, p. 289.
§ *Beitr. z. Geophys.* **35**, h. 1, p. 68.

§ 79. *Nocturnal radiation: conditions within the surface layers of the ground*

Since the earth's surface radiates effectively as a black body, the amount of heat which it sends into the atmosphere is independent of the nature of the ground. The amount of energy which the earth's surface gains from the atmosphere depends upon the distribution of temperature and humidity in the lower layers of the atmosphere. Thus the net outward radiation at night from the earth's surface depends only on atmospheric conditions, and on the temperature of the earth's surface. But the temperature changes which are produced in the earth by a given amount of radiation will depend upon the readiness with which the loss of heat from the surface is compensated by conduction of heat upwards from lower layers of the earth. The precise nature of this dependence can be readily ascertained.

We shall neglect the effect of the transfer of heat by conduction from the air to the earth's surface, as there is no obvious method of allowing for this. Further, we are here concerned mainly with conditions during night inversions and in such conditions the transfer of heat is from air to ground, tending to prevent the fall of temperature of the ground. Thus any formula which we derive for estimating the drop of temperature at the ground will give an over-estimate, or a limit to the maximum fall of temperature at the surface.

The surface layers of the earth will be assumed to have a specific conductivity of heat κ_1. Values of κ_1 have been derived by Johnson and Davies* and by Wright†, the mean of their determinations being $\kappa_1 = 4 \cdot 7 \times 10^{-3}$ in C.G.S. units. The equation of conduction of heat through the ground is then

$$\frac{\partial T}{\partial t} = \kappa_1 \frac{\partial^2 T}{\partial z^2} \qquad \ldots\ldots(17),$$

where z is the depth measured positive downwards.

If the net loss of heat by radiation from the ground to the atmosphere is R_N, then

$$R_N = \kappa_1 \rho_1 c_1 \left(\frac{\partial T}{\partial z}\right)_{z=0} \qquad \ldots\ldots(18),$$

where ρ_1 and c_1 are respectively the density and specific heat of the ground. For the loss of heat from the surface outwards is equal to the flow of heat from below to the surface. Further, we have seen that to a first approximation the net radiation R_N remains constant throughout the night, since

$$R_N = \sigma T^4 (1 - a - b\sqrt{e}) \qquad \ldots\ldots(19).$$

The vapour-pressure has only a slight diurnal variation, and since the fall of temperature during the night is only a relatively small fraction of T, we may assume as a first approximation that R_N is constant. By equation (18) this involves a constant value of $\partial T / \partial z$ at $z = 0$.

* *Q.J. Roy. Met. Soc.* **53**, 1927, p. 45. † *Mem. R. Met. Soc.* **4**, No. 31.

Differentiating equation (17) once with respect to z and replacing $\partial T/\partial z$ by S, we find

$$\frac{\partial S}{\partial t}=\kappa_1\frac{\partial^2 S}{\partial z^2} \qquad \text{......(20).}$$

This now has to be solved with the boundary condition

$$S=-\frac{R_N}{\rho_1 c_1 \kappa_1} \text{ at } z=0.$$

The appropriate solution is given in any textbook on the subject, in the form

$$S=\frac{2}{\sqrt{\pi}}\frac{R_N}{\rho_1 c_1 \kappa_1}\int_{\frac{z}{2\sqrt{\kappa_1 t}}}^{\infty} e^{-u^2}du \qquad \text{......(21).}$$

Integrating equation (21) we readily find

$$T=T_1-\frac{2}{\sqrt{\pi}}\frac{R_N}{\rho_1 c_1 \kappa_1}\left\{\sqrt{\kappa_1 t}.e^{-\frac{z^2}{4\kappa_1 t}}-z\int_{\frac{z}{2\sqrt{\kappa_1 t}}}^{\infty} e^{-u^2}du\right\} \qquad \text{......(22).}$$

Thus the temperature at $z=0$ is given by

$$T=T_1-\frac{2}{\sqrt{\pi}}\frac{R_N}{\rho_1 c_1 \sqrt{\kappa_1}}\sqrt{t} \qquad \text{......(23).}$$

In this equation R_N is to be measured in gramme-calories per second and t in seconds. It is readily seen that we can take R_N to indicate the net radiation in gramme-calories per minute, and t the time in hours.

Before we consider the application of this result to any practical problem, we must consider the conditions under which it has been derived. Referring back to equation (21) we find that at $z=0$, S or $\partial T/\partial z$ is constant at all times except at $t=0$, when it is zero, through the lower limit of the integral becoming infinite. Thus the solutions represented by equations (22) and (23) correspond to the case when the outward radiation is initially zero at $t=0$, and then instantaneously jumps up to the value R_N. Such a change in physical conditions cannot happen in nature, but something approaching it probably happens at sunset, on a clear evening. It is known that the incoming short-waved radiation from sun and sky becomes negligibly small just before sunset, and that just before this happens there is a very rapid fall in the amount of the incoming radiation. Thus while the change from a net flow of radiation inwards through a complete balance, to a net flow outwards at the rate R_N, does not take place instantaneously, it takes only a very short time to be accomplished, and we should expect to find that equation (23) should give a reasonable approximation to the fall of temperature of the ground during a clear night, t being measured from sunset, except for small values of t, when the infinite rate of fall of temperature given by equation (23) at $t=0$ would in practice be replaced by a steep but finite rate of fall.

A nearer approximation to the conditions pre-supposed in the deduction of the above equations occurs on an occasion when the sky, after being overcast with low cloud, suddenly clears. It is shown later in § 80 that there is only a very slight loss of heat from the ground when the sky is overcast with low cloud, and so the clearing of the cloud brings suddenly on the net loss R_N appropriate to a clear night. On such occasions, therefore, we may use equation (23) with fair confidence to forecast the fall of temperature in a time t after the disappearance of the cloud, assuming that there is no wind and therefore no convection of heat to or from the ground.

There do not appear to be any published observations available which make it possible to test the hypotheses adopted above with regard to the lapse-rate of temperature in the ground, and some doubt inevitably remains as to the possible effects of the conditions prevailing during the daytime upon the temperature changes during the night.

Returning to equation (23) we find that the fall of temperature after sunset is (approximately)

$$\frac{2}{\sqrt{\pi}} \frac{R_N}{\rho_1 c_1 \sqrt{\kappa_1}} \sqrt{t} = \frac{2}{\sqrt{\pi}} \frac{\sigma T^4 (1 - a - b\sqrt{e})}{\rho_1 c_1 \sqrt{\kappa_1}} \sqrt{t} \qquad \text{......(24).}$$

The minimum temperature of the night is derived by making t here equal to the number of hours from sunset to sunrise. For dry soil we can adopt as approximate values

$$\rho_1 = 2\cdot5, \quad c_1 = 0\cdot2, \quad \kappa_1 = 4\cdot7 \times 10^{-3}.$$

Johnson gives (*loc. cit.* Fig. 15a) the mean curve for clear days and nights in June, and taking the mean value of T at 287° A, and assuming the factor $1 - a - b\sqrt{e}$ to be equal to its mean value at Benson in June, 0·225, we find that the fall of temperature in the seven hours from sunset to sunrise should be

$$\frac{2 \times 0\cdot56 \times 0\cdot225 \times 2\cdot65}{\sqrt{\pi} \times 2\cdot5 \times 0\cdot2 \times \sqrt{4\cdot7 \times 10^{-3}}} \text{ or } 11^\circ \text{ C.}$$

The mean fall of temperature shown by Johnson's curve is about 9° C, which is in very close agreement with the theoretically derived value, but is lower than the latter, as we should expect, since the effect of wind will be to diminish the fall of surface temperature by the downward transfer of heat through the inversion to the ground.

The figures for the specific heat and diffusivity of dry soil used above are Patten's estimates*. If we assume that the corresponding values given by Patten for light soil containing 20 per cent of water will be appropriate for December, giving a value of $\rho_1 c_1 \sqrt{\kappa_1}$ five times greater than for dry soil, the mean fall of temperature between sunset and sunrise during clear nights in December as calculated from equation (23) is 3·3° C, as compared with the value 2·9° C which Johnson found by observation. Both the June and December values are mean temperatures as observed in a Stevenson screen, and only a rough mean value of the factor $(1 - a - b\sqrt{e})$ has been used in each case. The agreement in each of the two months is sufficiently close to show that the main physical factors have been taken into account.

The chief difficulty in the way of using formula (23) for forecasting the night minimum temperature lies in the uncertainty of the value of the

* Washington, D. C., *Bull. U.S. Dept. Agri. Bur. Soils*, No. 59, 1909.

coefficient $\rho_1 c_1 \sqrt{\kappa_1}$. It has already been mentioned that the addition of 20 per cent of water to dry soil increases this factor five-fold.

Fig. 34 shows two examples of thermographs for clear nights: (*a*) on Salisbury Plain; and (*b*) at a station in Malaya, at a height of 1090 feet above sea level. The agreement of the form of the night portion of these curves with the parabolic form predicted on theoretical grounds is in each case very close.

Equation (23) indicates that the amplitude of the variation of temperature is inversely proportional to $\rho_1 c_1 \sqrt{\kappa_1}$. This factor therefore indicates the nature

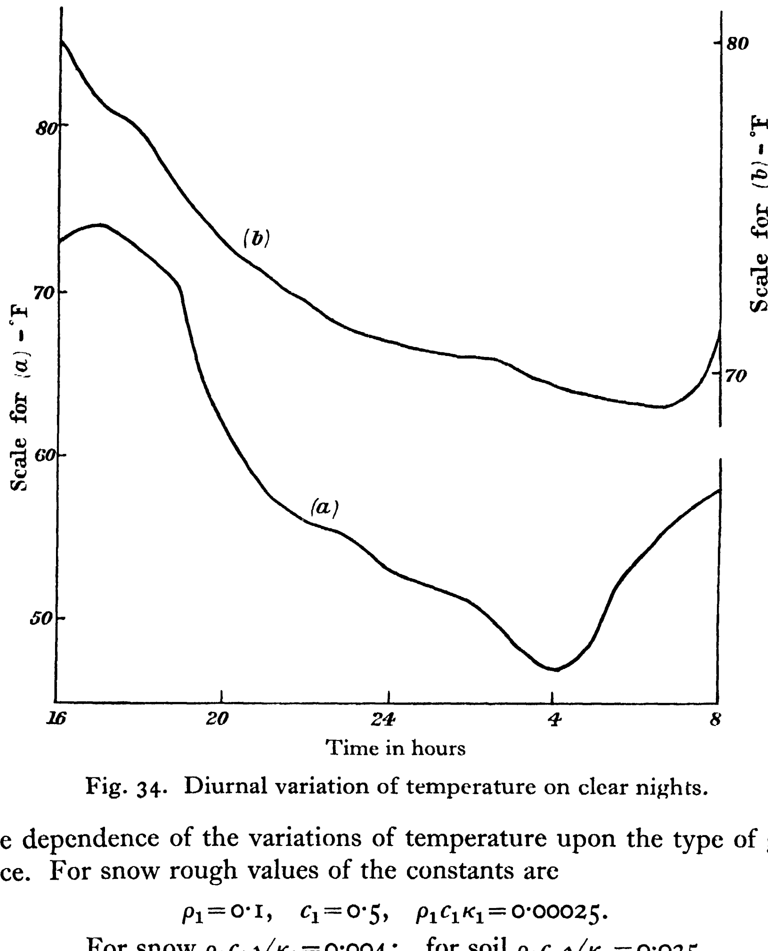


Fig. 34. Diurnal variation of temperature on clear nights.

of the dependence of the variations of temperature upon the type of ground surface. For snow rough values of the constants are

$$\rho_1 = 0{\cdot}1, \quad c_1 = 0{\cdot}5, \quad \rho_1 c_1 \kappa_1 = 0{\cdot}00025.$$

$$\text{For snow } \rho_1 c_1 \sqrt{\kappa_1} = 0{\cdot}004; \quad \text{for soil } \rho_1 c_1 \sqrt{\kappa_1} = 0{\cdot}035.$$

The temperature variation over the surface of snow may thus be many times that over a surface of soil, if radiation and conduction are the only factors which are active. This accords with experience, the lowest temperatures which are measured being associated with a snow covering over the ground. While snow causes extremely low temperatures at its surface it is at the same time a protection to anything which it covers, in that it conducts heat so slowly that any object covered by it retains its own heat. During any prolonged spell of clear weather with snow on the ground the temperature does not recover during the day, since 80 per cent of the solar radiation is reflected upwards at the snow surface, leaving only a small fraction to be absorbed and to become available for raising the surface temperature.

This is in accordance with the extraordinary results shown by Simpson* in his discussions of the diurnal range of temperature on the Barrier in the Antarctic, where the small diurnal change in daily insolation corresponding to the sun oscillating between 10° and 35° above the horizon produced a diurnal variation of temperature with an average amplitude of 20° F.

It has been assumed that the transfer of heat in the soil can be represented by a single uniform set of constants ρ_1, c_1 and κ_1, which have the same values at all depths below the surface. This is known to be a very rough approximation to the facts (*vide* Wright, *loc. cit.*), but it appears to be a sufficiently good approximation to enable us to derive the form of the curve of change of the surface temperature.

§ 80. *Nocturnal radiation with cloudy skies*

When the sky is overcast with cloud the radiation conditions are enormously modified, on account of the fact that the cloud sheet absorbs and radiates effectively as a black body. The effect can be readily understood from a consideration of the rate of change of the intensity of the beam of radiation of one wave-length λ, in the vertical direction.

In § 74 it was found that the upward beam of radiation of wave-length λ is equal to the black-body radiation E_λ at the ground, and is greater than E_λ at heights above the ground; while the downward beam is zero at the outer limit of the atmosphere, and in general is less than E_λ at all levels in a clear atmosphere. This statement is obviously true in a cloudless atmosphere, if the temperature decreases steadily with height. A cloud sheet whose thickness is at least of the order of say 50 metres can be assumed to radiate like a black body (*vide* § 69). The presence of a sheet of cloud therefore modifies very seriously the flow of radiation. At the base of the cloud the upward moving radiation of all wave-lengths is completely absorbed, and the downward moving radiation leaving the cloud base is equal to the complete black-body radiation at the temperature of the cloud base. Similarly at the top of the cloud, all the downward moving radiation is completely absorbed, and the upward moving radiation leaving the top of the cloud is equal to the complete black-body radiation at the temperature of the top of the cloud.

There are three important features to be observed in relation to the presence of a sheet of cloud:

(*a*) *At the top of the cloud.* The downward moving radiation of long wave-length will be only *W*-radiation, and at each level less (to a varying extent in different wave-lengths) than the corresponding black-body radiation. It will

* *British Antarctic Expedition*, 1910–13, *Meteorology*, **1**, p. 56.

be completely lacking in radiation of wave-lengths corresponding to the transparent bands. The long-wave radiation leaving the top of the cloud will therefore exceed the downward beam from above, in all wave-lengths, so that there is a net loss of heat at the upper surface. The net loss will be greater, the lower the cloud. The upper surface of the cloud acts in effect as an elevated ground surface.

(*b*) *At the base of the cloud.* The above argument is simply reversed, the upward beam coming into the cloud from below exceeding the black-body radiation in all wave-lengths. There is therefore a net gain of heat by the lower surface of the cloud, and the net gain will obviously increase as the height of the cloud increases.

(*c*) *At the ground.* The chief effect of the interposition of a cloud sheet between the ground and outer space is to put into the downward beam radiation of the wave-lengths corresponding to the transparent band, which are not present when the sky is clear. The net loss of heat by the ground is therefore considerably diminished, approximately by an amount equal to the radiation from the cloud in the wave-lengths of the transparent band, and the net nocturnal radiation from the ground will bear a rough proportionality to the height of the cloud.

Ångström* has given the results of observations of (*a*) net radiation received from the ground and the intervening atmosphere, by a horizontal black body, and (*b*) the net radiation lost by a horizontal black body to the atmosphere above, the observations being carried out in a free balloon. The figures for the first of these were as follows:

Height (metres)	975	1350	2000	3000	4000
Net gain of radiation	0·015	0·000	0·008	0·019	0·036

The net radiation is in gramme-calories per cm^2 per minute. The observations were commenced at about 10 p.m. on 3 July 1922, so that there was in all probability an inversion at the ground, and the radiation leaving the ground would be initially less than the equivalent black-body radiation at the temperature of air a short distance above the ground. Thus the conditions were not similar to those we have supposed, and the net radiation measured is at all levels less than would have been measured if there had been no inversion. From 1350 m upwards there is, however, a very rough proportionality of net radiation to excess of height above 1350 m. Similar results were obtained during the night of 5–6 June 1923, when, probably with an inversion at the ground, negative values of net gain of radiation were observed up to nearly 1200 m, above which the values were positive, becoming 0·036 at 2000 m and 0·055 at 2750 m, the greatest height at which measurements were obtained. On this occasion there was a varying amount of cloud about 2600 m to 2850 m. Above this level observations were made of the net loss of radiation of a black body to the atmosphere above it. This value was 0·226 at 3300 m, diminishing

* *Beitr. Phys. fr. Atmos.* **14**, h. 1–2, 1928.

to 0·216 at 4400 m. We shall make further use of these results later in discussing the effects of radiation on cloud sheets.

Ångström's observations are sufficient to show that our conclusion of a rough proportionality between $(\Sigma A - \Sigma E)$ and the height is justified as to order of magnitude; and we therefore conclude that in a similar way $(\Sigma E - \Sigma B)$ is proportional to depth below the cloud level. The special form of the last relation which we require is that the net loss of radiation from the ground is roughly proportional to the height of the cloud. Asklöf's observations* on cloudy nights can be applied to check this conclusion roughly. Asklöf's observations on clear nights, and those on cloudy nights, were made in the period March to June, 1918. The net radiation from the ground on clear nights was between about 0·15 and 0·20 gramme-calorie per cm². The net radiation on overcast nights gave the following average values:

Cloud	Net radiation	Average height, Upsala
Nb., St., or St.-Cu.	0·023	1·5 km
A.-Cu.	0·039	2·8 ,,
Ci.-St.	0·135	6·4 ,,
Clear sky	0·169	—

Thus the net loss of heat from the ground with a sky covered with high cloud is almost as great as with clear skies, while when the sky is covered with low cloud, the net loss of heat from the ground is only about one-seventh the value observed for clear skies. Consequently the fall of temperature during the night with a sky overcast with high cloud should be nearly as great as with a clear sky, but should be only of the order of one-seventh of this amount, if the sky is overcast with low cloud. N. K. Johnson† reproduces a chart showing temperature variations during the night of 15–16 March 1923, in which the drop of temperature during the night is shown to be about 1° F, the sky being overcast with St.-Cu. and Nb. This is of about the right order of magnitude to agree with Asklöf's observations.

Ångström has also suggested that the net loss of radiation from the ground may be represented by

$$R_m = (1 - 0{\cdot}09m)\, R_0 \qquad \text{......(25)},$$

where R_m is the observed net loss of radiation from the ground when m tenths of the sky are covered with cloud, R_0 being the net loss of radiation to a clear sky in the same circumstances of temperature and humidity; but it is clear that such a rule cannot deal with clouds of different height, and a different formula, with a constant appropriate to the height, should be employed for each type of cloud.

During the daytime the net loss of heat from the ground by long-wave radiation is practically the same as it would be at night with the same atmospheric conditions of temperature and humidity. But the surface of the earth gains heat by absorption of the short-wave radiation in the direct beam of sunlight, or in the diffuse solar radiation from the sky. When the sky is overcast,

* *Geog. Annaler, Stockholm*, **2**, 1920, p. 253.
† M.O., *Geoph. Mem.* No. 46.

the amount of diffuse radiation which reaches the earth's surface is very much diminished, but it is still sufficient to outweigh the net loss by long-wave radiation from the ground. W. H. Dines and L. H. G. Dines* give tables of mean monthly values of diffuse radiation from overcast skies which can be directly compared with their mean values of net loss by long waves, and for each month the gain from short waves outweighs the loss by long waves. We therefore conclude that the surface temperature should rise during the day, even with overcast skies, but the rise will be small when the sky is overcast with a thick sheet of low cloud. The record for an overcast day and night, 15–16 March 1923, reproduced by Johnson (*loc. cit.*) indicates a rise of only about 2° F during the day on the 15th.

§ 81. *Conditions which favour the nocturnal cooling of the ground*

In the derivation of equation (23), which represents the nocturnal cooling of the ground by radiation alone, the effect of convection of heat to the ground was assumed negligible. If the wind remains strong and turbulent during the night the cooling of the surface of the earth by radiation is impeded, as fresh air is then being continually brought into contact with the ground, and a flow of heat from the air to the ground prevents the temperature of the ground from falling rapidly. The outward stream of radiation from the ground is not appreciably affected by this, and the effect of the convection is to distribute the loss of heat through a much deeper layer of air, so that an inversion of temperature cannot form.

The conditions which favour the formation of inversions at the ground are:

(*a*) clear skies, or only high cloud;

(*b*) absence of wind;

(*c*) low vapour-pressure in the atmosphere;

(*d*) low thermal conductivity and specific heat of the ground.

These four conditions are in fact well known from experience. Their effects are to some extent stated mathematically in some of the preceding sections. As to the last of these conditions, it has been shown that the higher value of the coefficient $\rho_1 c_1 \sqrt{\kappa_1}$ in winter when the ground is waterlogged accounts for the smaller diurnal variation of temperature in winter than in summer, and far outweighs the effect of lower vapour-pressure and longer nights in winter.

Our lack of knowledge of the physical constants of soils, and in particular, of the precise nature of the effect of a grass covering, makes it at present impossible to use the equation given above to forecast night minimum temperatures. There is a considerable literature dealing with the what is known as the "frost problem", which is concerned with the forecasting of night minimum temperatures, and a vast number of empirical formulae have been proposed by various writers. Details of such formulae will be found in papers by Ångström†, Ellison‡, Warren Smith§, and Pick||.

* *Mem. R. Met. Soc.* **2**, No. 11.

† *Geog. Ann.* 1920, h. 1; 1921, h. 3; 1923, h. 4.

‡ *Monthly Weather Review*, **51**, 1928, p. 485.

§ *Ibid.* Supplement No. 16.

|| *Met. Mag.*, various papers during 1927 to 1930.

CHAPTER VII

RADIATIVE EQUILIBRIUM AND THE STRATOSPHERE

§ 82. *Statement of the problem*

THE most striking problem to which we have to apply considerations of radiative processes is the existence of an upper region of the atmosphere, known as the stratosphere, within which the temperature is practically constant, or even increases slowly with height, while below it, in the troposphere, there is a definite lapse-rate whose mean value is about 6° C per km at all heights and in all latitudes. The change from a definite fall of temperature with height to constant temperature is abrupt, and any satisfactory theory must be capable of explaining both the sudden change of lapse-rate at the tropopause, and the occurrence of constant or increasing temperature above that level.

It was first suggested by Gold* and Humphreys† that the conditions within the stratosphere can be explained as due to a complete balance between radiation and absorption, leading to a constancy of temperature, dynamical disturbances of the temperature at these levels being regarded as negligible on account of the stability of the air at these levels preventing convection or turbulence on any considerable scale. Both of these writers regarded the stratosphere as controlled by the long-wave radiation coming up from the lower levels of the atmosphere. Strictly speaking, account should be taken of the incoming beam of solar radiation of short wave-length, as well as of the outgoing beam of radiation of long wave-length. But as the incoming beam passes down through the atmosphere with relatively little attenuation, we shall leave out of consideration, at least in a first survey of the question, the possible absorption of the direct solar beam in the atmosphere, on its way downward. In effect this means that we shall regard the earth's surface as the source of heat of the atmosphere.

Humphreys points out that the temperature changes in the stratosphere between winter and summer are such as to justify the assumption that the conditions in the upper air are determined by radiation from the lower atmosphere, since the ratios of the absolute temperatures in winter and summer are approximately the same at all heights. He therefore proceeds to consider the effect of radiation from the earth and the lower atmosphere, leaving out of consideration the absorption of incoming solar radiation in the stratosphere. The radiation which comes up from below is effectively (says Humphreys) the radiation from a black body at a temperature of 259° A, and he therefore replaces the earth and the lower atmosphere by a black body at this temperature, sending out radiation of the appropriate amount. The curvature of

* *Proc. Roy. Soc.* A, **82**, 1909, p. 43. † *Astroph. Journ.* **29**, 1909, p. 14.

the earth may be neglected in the discussion of the narrow range of heights with which we are concerned, so that we may replace the spherical black body by an infinite plane black body.

Now consider two such surfaces, parallel and directly facing each other, at a finite distance apart, each having a temperature T_2. If any object is placed midway between the planes, in what is effectively an enclosed space (since the lateral boundaries of the surfaces are very distant), it will take up the temperature T_2. If now one of the parallel planes is removed, the state of the object, which is kept in the same position, is akin to the state of the stratosphere, and the problem is now to find the temperature at which the object will be in equilibrium. Since it now absorbs and radiates only half the amount it absorbed and radiated when it was placed between the two planes, it follows that if the radiation is proportional to T^n, its final temperature T_1 will be given by the equation

$$2T_1^n = T_2^n, \quad \text{or} \quad T_1 = 2^{-\frac{1}{n}} T_2.$$

There is therefore a minimum temperature T_1 below which the temperature of the upper atmosphere may not fall on account of the radiation from the lower atmosphere. Assuming the radiation of the upper atmosphere to be roughly continuous, and therefore to follow approximately Stefan's law, we find for the temperature of the stratosphere $T_1 = 2^{-\frac{1}{4}} T_2$; and since the effective temperature T_2 is about 259° A, it follows that $T_1 = 218°$ A. This value is in close agreement with the mean observed temperature of the stratosphere, which is estimated as 219° A for Europe.

At first sight Humphreys' proposed solution of the problem appears complete and accurate, but on closer examination it appears to offer several points for criticism. What Humphreys *proves* is that an isothermal state is possible in the first system he postulates, i.e. when the atmosphere is irradiated in both directions by radiation of its own temperature, and there is no net flow of radiation in any direction. The second system which he considers, with one radiating plane only, is irradiated from one side only by radiation of a higher temperature than its own, having a net outflow of heat upwards, and it is not possible to deduce what will happen in the second system from what happens in the first system. Thus the extension of the argument of constancy of temperature in an enclosed space to the unenclosed space outside one plane is not justifiable. Humphreys does not prove that the amount of radiation which will reach the element of mass exposed to one plane only will be the same at all points, and it is in fact clear that this cannot be so in practice in the system he considers. The thermodynamical system which he considers is also open to objection. He regards the stratosphere, whose temperature is everywhere T_1, as irradiated from below by a beam of intensity $2T_1^n$, which is equivalent to the net outward flow of radiation necessary to compensate the inward flow of solar radiation. The stratosphere absorbs an amount C of this beam, and re-radiates it, $\frac{1}{2}C$ outward to space, and $\frac{1}{2}C$ downward to the earth. The net outward flow is thus diminished by $\frac{1}{2}C$, which the earth gains at the expense of the outward beam. A further objection will be seen later when we come to discuss the problem mathematically.

§ 83. *The difficulties in the way of mathematical treatment*

It has been shown that the radiation from water-vapour is distributed through a wide range of wave-lengths, but does not extend through the whole range of wave-lengths of black-body radiation at the same temperature; and that the radiative and absorptive power of a given mass of water-vapour varies with the wave-length. These facts are represented in fig. 30 above. A brief study of the tables of water-vapour radiation given by Simpson (*loc. cit.*) shows that the total radiation from a column of water-vapour cannot be represented by any simple mathematical expression, such as is necessary if we are to develop any system of equations to represent the flow of radiation through the atmosphere. It may be added, however, that the total W-radiation at temperature T is closely proportional to $T^{3\frac{1}{2}}$.

A further difficulty lies in the fact that the radiation which traverses the atmosphere is not one-dimensional, i.e. it does not travel upward or downward as a parallel beam, but is diffuse. This is in practice an almost insuperable difficulty, since there is no simple law relating the intensity of the infinitely numerous beams of which the radiation present in the atmosphere at any instant is composed, with the directions in which they are travelling. A black body sends out radiation in all directions from each element of its surface, the intensity being the same in all directions. A beam inclined at an angle θ to the vertical in the atmosphere travels a distance $h \sec \theta$ before it reaches a height h, and thus the law relating the intensity to the direction θ must vary with the height h. Emden* has shown that in an isothermal atmosphere of infinite mass the total absorptive power of a layer for diffuse black-body radiation is twice the absorptive power for normal radiation incidence. In other atmospheres he finds that no simple law is derivable, and he finally concludes that a close approximation to actual phenomena is obtained by assuming the rule that in a given thin layer the absorption and radiation are equal and proportional to the amount of water-vapour, or other absorbing medium in the layer. This is in agreement with the method followed by Schwarzschild†, and by Milne‡ in some of his work on radiation in stellar atmospheres. Roberts§ showed that the diffuse radiation from the earth's surface follows approximately the simple law of absorption having $1\frac{1}{2}$ times the coefficient of absorption for a parallel beam.

Gold‖ has discussed radiative processes in the atmosphere, allowing for the diffuseness of the radiation. His treatment is by no means easy to follow on account of its brevity. In the later stages of his paper Gold assumes that in the atmosphere in convective equilibrium $T^4 \propto p$, or $T \propto p^{\frac{1}{4}}$, as an approxi-

* *Sitzber. Math.-Phys. Kl. Akad. Wiss. Munchen*, 1913, h. 1.

† *Göttingen Nachrichten*, 1906, p. 41.

‡ *Phil. Mag.* **144**, 1922, p. 871.

§ *Proc. R.S. Edin.* **50**, 1930, p. 225.

‖ *Proc. Roy. Soc.* A, **82**, 1909, p. 43.

mation to the theoretical law $T^{3.5} \propto p$, which holds for an adiabatic atmosphere. We have seen on p. 38 that in an adiabatic atmosphere $p/T^{\frac{\gamma}{\gamma-1}}$ is constant. Gold's treatment is therefore strictly applicable to an atmosphere in which $\gamma=4/3$, i.e. a triatomic atmosphere. Gold also assumes that the amount of water-vapour per unit mass of air may be represented by $\frac{\alpha}{q-p}$, where α and q are constants, and p is the barometric pressure. The assumptions made as to the relation of temperature and water-vapour content to pressure are necessary in order to render the mathematical treatment possible, and in view of the uncertainties involved in these assumptions, the severity of the mathematical treatment of diffuseness of radiation is scarcely justified. Emden (*loc. cit.*) has shown that if the radiation be treated as a parallel beam, with the coefficient k doubled, the results derived by Gold can be obtained substantially in the form in which they were given by Gold.

We are therefore faced with a problem whose rigorous solution is impossible, on account of the fact that the physical facts cannot be summarised in such a form as to lead to tractable mathematical expressions. We shall briefly consider the question on the basis of the assumption that the inward and outward long-wave radiation can be treated as parallel beams, realising that at best this method can only be expected to yield an outline of the actual processes in the atmosphere.

§ 84. *Radiative equilibrium and the stratosphere*

In a first consideration of the question of the radiative processes in the atmosphere we shall assume that there is no absorption of the direct solar beam on its way through the atmosphere, so that we are limited to the consideration of long-wave radiation. The inward and outward radiation will be assumed to be parallel beams, and the radiation and absorption will be assumed to be entirely due to water-vapour.

The total amount of water-vapour above any given level is represented by τ, so that τ is to be regarded as the variable which fixes the level of any point in the atmosphere. At a given level, let A and B be the upward and downward beams of radiation, and let E be the total black-body radiation at the temperature at that level. Then the equations which determine the changes of A and B with height are given in § 74 above, as follows:

$$\frac{dA}{d\tau}=k(A-E) \qquad \text{......(1)},$$

$$\frac{dB}{d\tau}=k(E-B) \qquad \text{......(2)}.$$

Equations (1) and (2) can be readily integrated when E is constant, i.e. when the temperature is assumed to remain constant with height. The integrals are then

$$A=E+Ce^{k\tau} \qquad \text{......(3)},$$

$$B=E+De^{-k\tau} \qquad \text{......(4)}.$$

Equations (1) to (4) above are true for each individual wave-length include in the beams considered, and we might have written these equations wi suffixes λ to A, B, E, k, C and D, in order to indicate this fact.

We can apply these equations to consider the scheme put forward b Humphreys to explain the isothermal condition of the stratosphere. Tl radiation is to be regarded as grey radiation (see § 57, p. 94), with a consta k for all wave-lengths, and A, B and E in the equations will measure the tot energy in the beams. Since no long-wave radiation enters the atmosphe from above, the value of B at $\tau = 0$ is zero. Hence in equation (4)

$$D = -E$$

and the equation may be written

$$B = E(1 - e^{-k\tau}) \qquad \text{......(5)}.$$

We have yet to apply the condition of radiative equilibrium, which may b written in two forms:

(*a*) $$A - B = A_0 = \text{const.} \qquad \text{......(6)}.$$

This is the mathematical statement of the condition that there is no accumu lation of heat at any level, so that the net flow of heat is the same across ar horizontal surface.

(*b*) $$k(A + B) = 2kE \quad \text{or} \quad A + B = 2E \qquad \text{......(7)}.$$

This is equivalent to the statement that the amount of heat absorbed by ar element of the medium is equal to the amount radiated in the same time.

The two statements are equivalent to each other, both physically and math matically, and one of them could be derived from the other by the use equations (1) and (2). From equation (6) it follows that

$$\frac{d}{d\tau}(A - B) = 0.$$

From equations (1) to (4) it follows that

$$A - E = E - B = Ce^{k\tau} = -De^{-k\tau} = Ee^{-k\tau}$$

which cannot be satisfied for any value of the constant C; or in other word isothermal radiative equilibrium is impossible in an atmosphere such as v have supposed, in which the radiation from outside the atmosphere ente from below only.

We might, however, proceed as Gold did, instead of as above, and assumir an initially isothermal atmosphere, find the level above which the atmosphe gains as much heat by absorption as it loses by radiation; i.e. we seek th value of τ, say τ_1, at which the equation

$$A - B = A_0$$

will hold as well as at $\tau = 0$. From equations (3) and (5) we find, in the initi stage while isothermal conditions still hold,

$$A_0 = A - B = Ce^{k\tau_1} + Ee^{-k\tau_1} = C + E,$$

and $$C(e^{k\tau_1} - 1) = E(1 - e^{-k\tau_1})$$

or $$C = Ee^{-k\tau_1} = A_0 - E,$$

whence $$A_0 = E(e^{-k\tau_1} + 1) \qquad \text{......(8)}.$$

Hence equation (3) becomes

$$A = E\,(1 + e^{-k(\tau_1 - \tau)}) \qquad \ldots\ldots(9),$$

and at the level of $\tau = \tau_1$

$$A = 2E,$$

or the upward beam of radiation is equal to twice the intensity of black-body radiation at the temperature of the isothermal region. Equation (8) determines the lower limit of the region within which the total gain by absorption is equal to the total loss by radiation. We have not imposed the condition that radiative equilibrium should subsist at all points. The gain of energy per unit mass per unit time is

$$k\,(A + B - 2E).$$

Substituting in this expression from equations (5) and (9), we find that the net gain of heat per unit time is

$$kE\,\{e^{-k(\tau_1 - \tau)} - e^{-k\tau}\} \qquad \ldots\ldots(10).$$

When $\tau = \frac{1}{2}\tau_1$, the expression (10) vanishes. For values of τ less than $\frac{1}{2}\tau_1$ it s negative, and for values of τ greater than $\frac{1}{2}\tau_1$, it is positive. Thus the lower half of the region will be heated and the upper half cooled. The gain of heat in the lower half will be exactly equal to the loss of heat in the upper half of the region. Thus even if isothermal conditions could be established momentarily, they could not persist, and the effects of long-wave radiation and absorption would be to establish a decrease of temperature with height.

The amount of heat gained or lost would not be negligible. For at $\tau = \tau_1$, expression (10) becomes

$$kE\,(1 - e^{-k\tau_1}),$$

which is an appreciable fraction of the total black-body radiation, $2kE$; this fraction approaches $\frac{1}{2}$ if τ_1 is large, but in any case the salient fact is that the whole of the lower half of the region becomes warmed at the expense of the upper half. Thus it is impossible for the upper region of the atmosphere to be kept in an isothermal condition by radiative equilibrium when it is only irradiated from below by long-wave radiation. The argument only breaks down when $k = 0$, or the stratosphere has no optical thickness. Radiative processes could not then destroy an isothermal state once this was established, but if we rule out radiation and absorption, all hope of any explanation of the condition of the stratosphere appears to vanish.

The argument as stated above assumes that water-vapour radiates as a grey body within the limits of wave-length in which it radiates, but does not require that the radiation should include all the wave-lengths contained in black-body radiation at the same temperature. It is not therefore likely to lead to results which are fundamentally wrong, and we may accept the results as substantially true, no matter what minor modifications may be necessary.

The method which we have used above is the simplest, if we are to make use of the assumption of grey radiation, for the problem of the isothermal atmosphere. The variable which denotes height is the optical thickness τ. Difficulties arise, however, when the atmosphere to be discussed is not isothermal, since it then becomes necessary to introduce some form of relationship between τ and the temperature.

The result which we have here obtained is substantially that derived by Gold, that there is in the atmosphere an upper layer within which the net loss of heat by radiation is balanced by the net gain of heat by absorption. It turns out, however, that the net balance is due to the fact that the loss of heat on balance by the outermost part of the layer is equal to the gain of heat by the inner part. Thus if isothermal conditions could be established in the upper atmosphere, they could not persist, and the outer layers would cool, while the lower layers would heat up, until a state of things was established in which a balance of heat exchanges could be maintained permanently. It must not be assumed, however, that the state which would then be set up would be convective equilibrium. Some light is thrown on this point by Emden's work.

§ 85. *Emden's solution for grey radiation*

Since we have seen that it is not possible to have radiative equilibrium and isothermy, the next step is to consider the result of using the condition of radiative equilibrium only. In this we shall follow Emden, assuming grey radiation.

From equations (1) and (2) of p. 136, we find, by addition and subtraction,

$$\frac{d}{d\tau}(A+B)=k(A-B) \qquad \text{......(11)},$$

$$\frac{d}{d\tau}(A-B)=k(A+B-2E)=0 \qquad \text{......(12)},$$

by the condition of radiative equilibrium.

From the second of these equations,

$$A-B=\text{const.}=A_0 \qquad \text{......(13)}.$$

By substitution in (11), $\frac{d}{d\tau}(A+B)=kA_0.$

Hence $A+B=kA_0\tau+C.$

Since at $\tau=0$, $B=0$, it follows that $C=A_0$, and

$$A+B=2E=A_0(1+k\tau) \qquad \text{......(14)}.$$

From (13) and (14), by addition and subtraction,

$$\left.\begin{aligned}A&=A_0(1+\tfrac{1}{2}k\tau)\\ B&=\tfrac{1}{2}A_0k\tau\end{aligned}\right\} \qquad \text{......(15)}.$$

Equation (14) gives the relation between E and the variable τ. If the radiation is grey, then $E=\sigma T^4$, and by putting this value for E in the equation above we find

$$\sigma T^4=\tfrac{1}{2}A_0(1+k\tau) \qquad \text{......(16)}.$$

In the upper stratosphere τ is never very great, so that this relationship should lead to a slow decrease of temperature with height, the boundary temperature being given by

$$\sigma T^4=\tfrac{1}{2}A_0 \qquad \text{......(17)}.$$

The above argument does not require that the beams of radiation should extend through the whole range of wave-lengths included in black-body radiation at the same temperature, but it does require that the coefficient of absorption k should be the same for all wave-lengths. It does not therefore seem likely that the results given are seriously in error, except for the substitution of σT^4 for E. This will no longer be true, though E will still increase with temperature. Actually it is found by examination of the figures given by Simpson for what we have called W-radiation that this is proportional to $T^{3\frac{1}{2}}$, and not to T^4, and equations (16) and (17) could be modified by the substitution of $T^{3\frac{1}{2}}$ for T^4.

The above discussion shows that isothermal conditions are only possible in the stratosphere when $k=0$, or when the stratosphere has zero optical density, the result already derived in § 84.

We are therefore led to the conclusion that the temperature of the stratosphere cannot remain constant with increasing height, still less increase with increasing height, if its temperature is controlled by long-wave radiation from the lower atmosphere. There is, however, one further result of interest which can be derived from the analysis of the above paragraph. Equation (17) states that the value of E, and therefore the temperature, will increase as the net flow of radiation outwards increases. It is seen from equation (6) of § 75 above that the flow of radiation outwards at a particular temperature is proportional to the lapse-rate, and inversely proportional to the vapour-pressure. Though it was not stressed at the time, this result is true for each separate wave-length of W-radiation, and is thus not dependent on any obviously doubtful assumption. Gold* has suggested that the condensation of water-vapour governs very closely the actual lapse-rate in the troposphere. It has been shown that the saturated lapse-rate at a given temperature diminishes with the pressure, and is therefore greater when that temperature occurs at low levels in the atmosphere than when it occurs at high levels. If then Gold's view (which is partly confirmed by the examination of the tephigrams reproduced by Sir Napier Shaw, in the *Manual of Meteorology*, **3**) is accepted, we should expect the lapse-rate at a given temperature to be greater when that temperature occurs at low levels than when it occurs at high levels. Thus the net outward flow across a given isothermal surface is greater where that surface is low in the atmosphere than where it is high, and so the net outward flow of heat across the upper troposphere is greater in high latitudes than at the equator. But equation (17) above indicates that the temperature in the upper stratosphere increases with the net outward flow, and so the temperature in the stratosphere should increase from the equator towards higher latitudes. This argument might be affected by an increase of vapour-pressure at a given temperature in the same sense as the increase of lapse-rate, in such a way as to diminish the ratio of the two, and lead to a decrease with increasing latitude of the net outflow of radiation across the upper troposphere. This is a very improbable occurrence, since the relative humidity is more or less uniform through the

* M.O., *Geoph. Mem.* No. 5.

whole atmosphere, and certainly shows no definite tendency, so far as observations are available to test it, to increase from equator to pole.

It is thus established that while it is not possible to explain any vertical distribution of temperature, except a slow decrease of temperature with height, by the effect of long-wave radiation from the lower atmosphere, it is possible to explain rationally the latitude variation of temperature in the stratosphere by this effect, taking into account the controlling effect of the condensation of water-vapour on the actual lapse-rate in the troposphere.

The next logical step is to consider what other factors can exert a control over the temperature distribution in the stratosphere, but before doing so we shall give very briefly an outline of the results derived by Gold.

§ 86. *An outline of Gold's treatment*

In an atmosphere in convective equilibrium $T/p^{(\gamma-1)/\gamma}$ is constant (see § 19, p. 38), the potential temperature being the same at all levels. For atmospheric air $(\gamma-1)/\gamma$ is approximately equal to $1/3{\cdot}49$. Grey radiation is therefore proportional to p very nearly, since it is proportional to $T^{3\frac{1}{2}}$ very accurately. Thus in an atmosphere in convective equilibrium grey radiation may be assumed, following Gold, to vary directly as p. Gold further assumed that the amount of water-vapour in the atmosphere may be represented by an expression $\alpha/(q-p)$, where p is the pressure, and α and q are constants.

The outstanding feature of Gold's treatment lay in the completeness with which he allowed for the diffuseness of the long-wave radiation. We shall not follow this aspect of Gold's work, believing that the other uncertainties which underlie the work make it superfluous to go into very elaborate mathematical analysis, and we shall follow Schwarzschild and others in taking the radiation to be effectively in parallel beams, allowing for the diffuseness by doubling the coefficient of absorption. Emden* showed that such a method will yield all the results derived by Gold substantially in the same form as Gold obtained them.

Equations (1) and (2) become, when we take account of the variation of E with p,

$$\frac{dA}{dp} = kA - kE_0\frac{p}{P} \qquad \text{......(18)},$$

$$\frac{dB}{dp} = -kB + kE_0\frac{p}{P} \qquad \text{......(19)},$$

where E_0 is the radiation at the surface pressure P. The coefficient k is not the same as was used above in this and earlier chapters, but represents the absorption by a layer corresponding to unit difference of pressure.

Emden has shown that the solution of equations (1) and (2) appropriate to the condition that $B=0$ at $p=0$, and $A=E_0$ at $p=P$, is

$$A = -\frac{E_0}{P}\frac{q-P}{\alpha+1}\left(\frac{q-P}{q-p}\right)^{\alpha} - \frac{E_0}{P}\frac{\alpha q}{\alpha+1}\frac{q-p}{q} + \frac{E_0 q}{P} \qquad \text{......(20)},$$

$$B = \frac{E_0}{P}\frac{q}{\alpha-1}\left(\frac{q-p}{q}\right)^{\alpha} - \frac{E_0}{P}\frac{\alpha q}{\alpha-1}\frac{q-p}{q} + \frac{E_0 q}{P} \qquad \text{......(21)}.$$

* *Loc. cit.*

In a layer of thickness dp we have

$$\text{Absorption} - \text{emission} = k\,dp\,(A + B - 2E).$$

Emden found that this quantity is positive or negative according as p is $<$ or $> \frac{1}{4}P$, a result given by Gold. Thus the absorption will exceed the emission above the level where the pressure is one-fourth the surface pressure. Thus, if the convective state extends initially to the top of the atmosphere, the absorption of radiation will warm the upper layers until the convective state no longer holds, and eventually convection will not be able to surpass the level at which the pressure is one-fourth of the surface pressure. Emden further states that it is possible to derive in the same way Gold's result that in the levels where p is between $\frac{1}{4}P$ and $\frac{1}{2}P$ the difference between absorption and emission will be very slight, so that there will be little or no convection between these levels, the effect of the difference between absorption and emission being only sufficient to produce small convective streams. We shall not follow the details of Emden's analysis, for which reference should be made to the original paper. It should be noted, however, that the use of the approximate method of Schwarzschild, treating the radiation as parallel beams, yields substantially the same result as Gold's much more severe analysis.

Reference has already been made to a difficulty in Gold's treatment, to which attention was first drawn by Milne, that although Gold has shown that in the region above $p = \frac{1}{4}P$ there is on the whole a balance between absorption and radiation, there is a net loss of heat in the upper part of this region, and a corresponding gain of heat in the lower part of the region, which would destroy the isothermal state (see p. 138 above). Both Milne and Gold assume that convection and turbulence will carry heat upward, but it will be shown later, in Chapter XIII, that when the atmosphere is in a steady state the effect of turbulence is to carry heat downward.

Gold's method is too difficult to use, to justify our going into full details here, and the reader who wishes to study it further is advised to consult the original paper, which represents the most elaborate frontal attack yet made on the problem of radiative equilibrium in the atmosphere.

§ 87. *The effect of absorption of solar radiation in the stratosphere*

The incoming beam of solar radiation enters the atmosphere as a parallel beam, and remains effectively a parallel beam down to the earth's surface. The effects of diffuse reflexion and scattering are small in the upper layers of the atmosphere, but increase in magnitude as the beam reaches the lower levels of the atmosphere. The range of wave-lengths in the direct solar radiation extends from about $0{\cdot}3\mu$ to about 4μ, and is therefore definitely separated from the range of wave-lengths of terrestrial and atmospheric radiation. We cannot express the coefficient of absorption of the earth's atmosphere for these rays as a function of the wave-length. The best we can do is to assume that a coefficient k' exists such that the amount of absorption from a beam of intensity S

in a path whose optical thickness is $d\tau$ is $k'd\tau S$. As water-vapour is the main source of absorption, we shall assume that τ here represents the amount of water-vapour above the level under consideration. The intense absorption by oxygen and ozone at very high levels is left out of consideration for the present.

Let S be the intensity of the incoming solar beam at level τ, and S_0 the corresponding value at $\tau = 0$. Let ψ be the zenith distance of the sun, or the angle between the incoming beam and the vertical. Then in a layer of optical thickness $d\tau$, the length of the path is equal to the vertical path in a layer $d\tau \sec\psi$, and the amount of energy absorbed in the layer is $k'Sd\tau\sec\psi$. The actual amount of radiation incident on unit horizontal surface is $S\cos\psi$. Since the incoming beam is only diminished by absorption (scattering being neglected), and is not reinforced by emission of short waves,

$$dS = -k'Sd\tau\sec\psi,$$

$$S = S_0 e^{-k'\tau\sec\psi} \qquad \text{......(22).}$$

Equations (1) and (2) of § 84 still hold, but the condition for radiative equilibrium must be re-written in the form

$$A - B = S\cos\psi = \cos\psi\, S_0 e^{-k'\tau\sec\psi} \qquad \text{......(23)}$$

and the condition that the total absorption by any layer must equal the total emission is now

$$k(A+B) + k'S_0 e^{-k'\tau\sec\psi} = 2kE \qquad \text{......(24).}$$

By addition and subtraction we find

$$A = E + \frac{k\cos\psi - k'}{2k} S_0 e^{-k'\tau\sec\psi} \qquad \text{......(25),}$$

$$B = E - \frac{k\cos\psi + k'}{2k} S_0 e^{-k'\tau\sec\psi} \qquad \text{......(26).}$$

But at the outer boundary $\tau = 0$, and $B = 0$. It follows from (26) that, at $\tau = 0$,

$$E = \frac{k\cos\psi + k'}{2k} S_0 \qquad \text{......(27).}$$

Equations (25) and (26) are deduced from the two preceding equations, which state that the net flow of radiation across any horizontal surface is zero, and that the absorption and emission by any layer shall balance. A and B must in addition satisfy the equations

$$\frac{dA}{d\tau} = k(A - E) \qquad \text{......(28),}$$

$$\frac{dB}{d\tau} = k(E - B) \qquad \text{......(29).}$$

We may therefore use either of equations (28) or (29) in conjunction with (25) and (26) to determine the form of A, B and E. We have not four independent equations connecting A, B and E, since the equation obtained by subtracting (29) from (28) can also be derived by differentiating (23), and using (24) to eliminate S from the resulting equation.

Differentiating equation (25), we find

$$\frac{dA}{d\tau}=\frac{dE}{d\tau}-\frac{k'\sec\psi}{2k}(k\cos\psi-k')\,S_0e^{-k'\tau\sec\psi}$$

$$=k(A-E)\quad\text{from (28)}$$

$$=\frac{k\cos\psi-k'}{2}S_0e^{-k'\tau\sec\psi}\quad\text{from (25).}$$

Hence
$$\frac{dE}{d\tau}=\frac{S_0}{2k}(k\cos\psi-k')(k'\sec\psi+k)\,e^{-k'\tau\sec\psi}$$

$$=\frac{S_0\sec\psi}{2k}(k\cos\psi-k')(k\cos\psi+k')\,e^{-k'\tau\sec\psi}\quad\text{......(30).}$$

Integrating this equation, we find

$$E=\text{const.}-\frac{S_0}{2kk'}(k\cos\psi-k')(k\cos\psi+k')\,e^{-k'\tau\sec\psi}.$$

When $\tau=0$, E is given by (27). Substituting, we find

$$E=\frac{S_0\cos\psi}{2k'}(k\cos\psi+k')-\frac{S_0}{2kk'}(k\cos\psi-k')(k\cos\psi+k')\,e^{-k'\tau\sec\psi}\quad\text{...(31).}$$

Equation (30) is of special interest. $\frac{dE}{d\tau}$ is positive, negative, or zero according as $k\cos\psi$ is $>$, $<$, or $=k'$, or according as $k'\sec\psi$ is $<$, $>$, or $=k$. It is thus possible to have isothermal conditions in the atmosphere if $k'\sec\psi=k$, and to have temperature increasing outwards if $k'\sec\psi>k$. In virtue of the fact that the incoming solar radiation is a parallel beam, and that it may have a low angle of incidence ψ, it is possible, if k' were not too small by comparison with k, that all the heat in the atmosphere should be concentrated in the outer layers. It is not possible to assign any numerical value to k'/k with any degree of certainty, and we cannot in fact say whether there is any probability that this effect is of any importance in the heat economy of the stratosphere.

Again, the angle ψ represents the zenith distance of the sun, which varies within wide limits in the course of the day. S_0 takes its normal value during the day, and becomes zero at night. We ought obviously to give ψ some kind of mean value, and presumably it should be a mean value capable of satisfying the condition that the net flow of radiation across any horizontal surface should be zero. (Equation (23).) Clearly no value of ψ can do so for all values of τ, and the best we can do is to satisfy the condition of zero net flux at $\tau=0$, since such a value should also satisfy the condition to a rough degree of approximation for small values of τ. But the mean value of $\cos\psi$ defined in this way is proportional to the mean value of the vertical component of incoming solar radiation, which we have already given in Table 2, p. 100. The table shows that at the equinoxes $\cos\psi$ has a very marked maximum at the equator, and falls off to about 7 per cent of the maximum value at the poles. At the summer solstice, the actual maximum is at the North pole, with a very flat secondary maximum in latitude 40°, followed by a steady fall from about latitude 30° N down to latitude 70° S, becoming zero on the Antarctic Circle.

These results suggest that if the absorption of incoming solar radiation by the same atmospheric constituents which absorb the outgoing long-wave radiation is the important factor in the distribution of temperature in the upper air, then there should be a marked annual variation in the form of the temperature distribution in the outer layers of the atmosphere. At the solstices the highest boundary temperatures (at $\tau = 0$) should occur at the summer pole, while at the equinoxes, the highest boundary temperatures should occur at the equator (see equation (27)). There is a certain amount of evidence that such a change in the latitude variation of temperature in the stratosphere actually takes place. It is generally supposed that the stratosphere is warmer over the poles than over the equator, at all times of the year, but the observational evidence for this is meagre, and it is much more probable that only in summer is the stratosphere warmer over the poles than in lower latitudes, and that in winter it is colder over the poles than in lower latitudes*. But the evidence in either direction is as yet quite insufficient to allow of the formulation of any simple statement of the phenomena.

Equation (30) shows that the temperature will decrease or increase with height according as $k \cos \psi >$ or $< k'$, it being remembered that τ diminishes with increasing height. Since at the equinoxes $\cos \psi$ has its maximum at the equator, we should expect at these times that the equatorial region should show the most marked tendency for temperature to decrease with height, while the tendency to increase of temperature with height should be shown in high latitudes. At the solstices, $\cos \psi$ has its maximum at the summer pole, which should then show the least tendency to increase of temperature with height, within the stratosphere.

Some of the inferences drawn above from the supposition that the absorption of the incoming solar beam in the upper atmosphere is a factor of importance should be capable of being put to the test by comparison with observations, when observations of the right kind become available. But too much stress must not be laid on the details of the results we have described. The transport of heat in the atmosphere is not entirely by radiation. Convection, advection, and the evaporation and condensation of water-vapour all take a part in the transport of heat. It is clear that if radiation alone were effective, then during the long Arctic winter, when no direct radiation reaches the polar region, temperature should fall with great rapidity to a much lower value than has yet been recorded. The transport of heat by the winds checks this fall of temperature, and equalises conditions over the earth's surface to a greater extent than any other factor.

* See Rolf, *Medd. Stat. Met. Hydr. Anstalt, Stockholm*, **5**, No. 5, 1932, which gives results of sounding balloons at Abisko.

§ 88. *Correction of the results for the reflexion of short waves*

The treatment given above makes no allowance for the reflexion of the solar beam in part by the surface of the earth, of the sea, and of cloud sheets. Let the amount of radiation reflected in this manner be βS_0. The reflected waves are sent back towards the upper atmosphere as diffuse radiation, and their absorption may therefore be taken into account by assuming a coefficient of absorption of $2k'$. The amount of absorption from the returning beam so calculated will clearly be very small, and may be neglected. This is in accordance with the earlier discussion above, where we regarded the index $k' \sec \psi$ as important only in virtue of the factor $\sec \psi$, which may have large values for rays approaching horizontal incidence.

Equations (24), (28) and (29) still hold, but a further term βS_0 is now added to the left-hand side of (23), and accordingly a term $-\frac{1}{2}\beta S_0$ to the right-hand side of (25) and $\frac{1}{2}\beta S_0$ to the right-hand side of (26). Proceeding as before we find

$$\frac{dE}{d\tau} = \frac{S_0 \sec \psi}{2k} (k^2 \cos^2 \psi - k'^2)\, e^{-k'\tau \sec \psi} - \tfrac{1}{2}\beta\, S_0 k \cos \psi \quad \ldots\ldots(32),$$

$$E = \frac{S_0 \cos \psi}{2k'} (k \cos \psi + k') - \frac{S_0}{2kk'} (k^2 \cos^2 \psi - k'^2)\, e^{-k'\tau \sec \psi} - \tfrac{1}{2}\beta\, S_0 k\tau \cos \psi \quad \ldots\ldots(33).$$

Equation (27), which gives the value of E at $\tau = 0$, is not changed, and the effect of the variations of ψ on the boundary temperature is unaltered. The value of $dE/d\tau$ at the outer boundary of the atmosphere where τ is very small is positive or negative according as

$$k^2 (\cos^2 \psi - \beta \cos^2 \psi) > \quad \text{or} \quad < k'^2.$$

Thus the temperature in the upper regions of the atmosphere will decrease or increase with height according as

$$k'^2 \sec^2 \psi < \quad \text{or} \quad > k^2 (1 - \beta) \quad \ldots\ldots(34).$$

Thus in the cloudy regions of the earth, where β is greater than in the clear regions, the temperature should show a more marked tendency to increase with height in the stratosphere. The effect of the oceans with clear skies should resemble the effect of clouds, but should be less marked. Thus in a given latitude, the stratosphere should be warmest in the cloudless continental regions, cooler in the cloudless oceanic regions, and coolest in the cloudy regions, whether continental or oceanic; and the cooler the stratosphere, the greater the tendency for the temperature to increase with height.

These results are in general accordance with observation. At the equator (Batavia, 6° S), where incoming radiation is subject to less variation in the course of the year than elsewhere, the lowest temperatures in the stratosphere occur at the cloudiest time of the year, in January. Again at Agra (27° N), the lowest temperatures at the base of the stratosphere occur in October, at the end of the monsoon period of cloudy skies, while the highest temperatures occur in January, when the cloud amount is least. The highest surface temperatures at Agra occur in June, and the lowest in January. The magnitude of the inversion in the lower stratosphere at Agra is greater in the cloudy month of October than in January, when the cloud amount is very much less. Dines* has shown that over England the highest temperatures at the base of the stratosphere occur in June and July, and the lowest temperatures in January to March, following closely the same course as the annual variation at the ground. The mean cloud amount over England is practically the same in January and July, being very slightly greater in January.

§ 89. *Summary of the results derived above*

It has been seen that if conditions in the stratosphere were entirely due to the absorption and emission of long-wave radiation, whose main source is the surface of the earth, the temperature in the stratosphere would decrease slowly outward; but on account of the greater transmission of radiation through the troposphere in high than in low latitudes, the temperature of the stratosphere might be expected to increase from the equator to higher latitudes. The effort to take account of absorption of incoming radiation leads to no very practical results, on account of our lack of specific knowledge on three points: (*a*) the absorbing power of the upper atmosphere for short-wave radiation from the sun, (*b*) the amount of water-vapour actually present in the upper atmosphere, and (*c*) the effect of low temperatures and pressures on the absorption bands of water-vapour. It was found, however, that if the absorption of incoming solar radiation is an important factor in the heat economy of the stratosphere, the most probable region in which we should look for an increase of temperature with height would be in high latitudes of both hemispheres at the equinoxes, and in high latitudes in the winter hemisphere at the solstices, and at all times in cloudy rather than clear regions in any latitude. This course of events should be bound up with an annual variation of the form of the curve of the latitude variation of temperature in the stratosphere. It is by no means clear that some such change does not take place, at least in part. The observational data are insufficient to decide this definitely. The inversion which forms at the ground in the polar regions in winter spreads upward to considerable heights, and makes it impossible to separate out the direct effect of the cooling of the ground from the purely atmospheric radiational effects which we have discussed above.

It cannot be said that any completely satisfactory theory of the conditions in the stratosphere has yet been evolved. The upward increase of temperature in the stratosphere, which is particularly noticeable in the tropics and subtropics in Ramanathan's diagram, reproduced in fig. 12, p. 18 above, remains unexplained, at least on a purely radiational basis. It is indeed possible that the distribution in the lower stratosphere, which shows so markedly over the equator in fig. 12, is largely the result of convection in the upper troposphere

* See Table 1, p. 19.

producing a marked drop of temperature below that which would otherwise occur. The diagram suggests the possibility that at heights of 20 to 25 km the temperature of the stratosphere is everywhere uniform, and that the conditions in the lower stratosphere may be very largely determined by dynamical considerations.

In our first consideration of radiative equilibrium in the stratosphere, account was taken only of the absorption and emission of long-wave radiation, assuming that only the water-vapour in the atmosphere could radiate and absorb. The later discussion of the effects of the absorption of incoming solar radiation in the upper atmosphere was also implicitly based on the assumption that the absorbing agent is water-vapour, or at least some constituent whose distribution in the upper atmosphere is everywhere similar to that of water-vapour. The latter is involved in our use of the same optical thickness τ for long and short waves.

The latest observations of ozone by Götz, Meathem and Dobson* indicate that the ozone occurs at much lower heights than were originally estimated. It is now suggested that the mean height of the ozone is about 22 km and that an appreciable fraction of the ozone is to be found below the level of the tropopause. The volume ratio of ozone to air increases with height up to a level of about 35 km. It therefore appears possible to explain an increase of temperature with height within the stratosphere as an effect of absorption of short-wave radiation by ozone. Moreover the amount of ozone increases from equator to pole, and the explanation of the increase of temperature in the stratosphere from equator to pole may perhaps be explained, in part at least, as the effect of ozone absorption.

It remains for observation to show whether the temperature at heights of the order of 25 km in the stratosphere really shows an increase from equator to pole. Until that is conclusively settled, we cannot be certain that the low temperatures at the base of the stratosphere are not purely dynamical effects due to convection in the upper troposphere.

Once it is accepted that conditions in the stratosphere are controlled by radiation, while conditions in the troposphere are controlled mainly by convection, it follows that the height of the tropopause must be closely proportional to the difference between the temperatures at the ground and at the tropopause. This is very definitely confirmed by observation.

Emden developed his system of equations for the distribution of temperature in terms of the distribution of water-vapour, and then by using Hann's relation between vapour-pressure and height, eliminated the water-vapour distribution from his equations, so obtaining a relation between temperature and height. But the resulting temperatures gave more than ten-fold saturation in parts of the atmosphere. His results were not valid, therefore, as was shown by Hergesell†, who first clearly drew attention to the necessity that the distributions of temperature and water-vapour should be mutually consistent.

* *Proc. Roy. Soc.* A. **145**, 1934, p. 416.
† *Lindenberg Arbeiten*, **15**.

Hergesell unfortunately committed a somewhat similar error in the later stages of his own discussion of the same problem*, and the result which he derived, that the temperature would be almost completely uniform in an atmosphere in radiative equilibrium, is based on a doubtful argument. The reader who wishes to look into the details of the argument is referred to the original papers.

§ 90. *Absorption by ozone at high levels*

Fabry and Buisson† have given an account of the methods by which was established the now generally accepted view that the sharp termination of the solar spectrum at 0·2890μ is due to absorption by ozone at levels of 20 to 50 km above the earth's surface. Even when photographed from heights of 9 km above the earth's surface, the spectrum showed no extension beyond this point. Laboratory observations have shown that ozone has an intense absorption band extending from 0·23μ to 0·32μ, with a pronounced maximum at 0·255μ. This is known as the Hartley band, from its discoverer. Ozone also has other absorption bands, one in the red and orange, with a maximum at 0·61μ, another in the infra-red, centred at 4·8μ, and a third at 9 to 11μ. Absorption by the Hartley band accounts for the sharp limitation of the solar spectrum on the ultra-violet side, and it is estimated that from 4 to 6 per cent of the total energy in the incoming solar beam is taken from it by this means, though the total amount of ozone is so small that if concentrated in a uniform layer at normal temperature and pressure, it would only make a layer 3 mm in thickness.

The details of the methods of observation of the quantity and amount of the ozone are given in papers by Dobson and others‡ (see footnote, p. 20). The mean height is about 22 km, and the amount of ozone measured may vary 25 per cent above or below the mean value. The ozone amount shows a marked annual variation, with a maximum at the spring equinox, and a minimum at the autumnal equinox; it also shows a marked variation with latitude, the amount increasing from equator to pole. Perhaps the most remarkable feature of all is the occurrence of very high values in the western quadrants of depressions, and of very low values in the western quadrants of anticyclones. It appears that, so far as is at present known, the ozone amount in the rear of depressions is greater than the normal amount anywhere over the earth's surface. Why the ozone should show this association with depressions is as yet a mystery, whose solution may reverse many of the ideas at present accepted concerning the dynamics of the atmosphere.

The energy absorbed by the ozone is in the ultra-violet, but at normal atmospheric temperatures the ozone could only re-radiate this energy in the relatively weak band between 9 and 11μ. It could only maintain radiative equilibrium, and re-radiate at the same rate as it absorbs, if it had a very much

* See Pekeris, *Gerlands Beiträge*, **28**, 1930, p. 377.
† *Mem. Sci. Phys.*, *Acad. Sci. Paris*, Fasc. XI, 1930.
‡ See also *Nature*, **127**, 1931, p. 668; *Proc. Roy. Soc.* A. **145**, 1934, p. 416.

higher temperature than would allow it to persist without dissociating. We are therefore forced to conclude that the re-radiation is performed mainly by another constituent of the atmosphere, most probably water-vapour. E. H. Gowan* has shown that with certain reasonable assumptions, a temperature of 400° A would be not unlikely. This is in agreement with the conclusions drawn by Whipple† from the study of the travel of sound from explosions, and explains the occurrence of zones of abnormal audibility at the earth's surface.

A theory of the formation of ozone has been given by Chapman‡, who attributes the formation of ozone to the dissociation of oxygen by ultra-violet radiation, and the subsequent dissociation of ozone to the same cause. Chapman's theory leads to the result that above about 80 km the ratio of atomic oxygen to molecular oxygen should increase without limit, and that above 60 km the ratio of ozone to molecular oxygen must decrease indefinitely with increasing height. In the extreme outer layers of the atmosphere oxygen tends to exist in the atomic rather than in the molecular form.

We may refer in passing to two zones of ionisation in the high atmosphere, one at 100 km, and known as the Kennelly-Heaviside layer, and another at 200 km, known as the Appleton layer. The ionisation in both may probably be ascribed to ultra-violet radiation, the lower being due to the ionisation of nitrogen molecules, and the upper to the ionisation of atomic oxygen§.

Chapman¶ has discussed the absorption by oxygen in the range from $0{\cdot}13\mu$ to $0{\cdot}175\mu$. It appears that within this range oxygen absorbs even more strongly than ozone does in the Hartley band from $0{\cdot}23\mu$ to $0{\cdot}32\mu$. Chapman states that this absorption takes place at heights well above 100 km.

§ 91. *The heat balance of the atmosphere*

When allowance has been made for the earth's albedo the mean inflow of solar radiation to the earth's surface, averaged over all latitudes, and over day and night, is 0·278 gramme-calorie per cm² per minute. Since the earth is not getting any hotter or colder on the average, this amount must be re-radiated out to space. Before leaving the subject of radiation we must make some effort to form a picture of the processes by which the earth and its atmosphere balance the account between incoming and outgoing radiation.

Fig. 35, reproduced from Simpson's paper**, shows (curve I) the effective incoming solar radiation in hundredths of gramme-calories per cm² per minute in different latitudes, with a mean value of 0·278. Here allowance has been made for the reflexion of solar radiation from the atmosphere, the earth's surface, and the appropriate cloud amount for each latitude. Simpson has computed for each latitude the total amount of outgoing radiation with clear and cloudy skies. To follow the details of his computation we have to bear in

* *Proc. Roy. Soc.* A, **120**, 1928, p. 655; **128**, 1930, p. 531.
† *Q.J. Roy. Met. Soc.* **57**, 1931, p. 331.
‡ *Mem. R. Met. Soc.* **3**, No. 26, 1930.
§ *Vide* Chapman, Bakerian Lecture, *Proc. Roy. Soc.* A, **132**, 1931, p. 353.
¶ *Q.J. Roy. Met. Soc.* **60**, 1934, p. 127.
** *Mem. R. Met. Soc.* **3**, No. 21, 1928.

mind the three categories into which he divided radiation at terrestrial and atmospheric temperatures:

(*a*) Wave-lengths in which water-vapour is transparent to radiation—$8\frac{1}{2}\mu$ to 11μ.

(*b*) Wave-lengths in which water-vapour amounting to 0·3 mm of precipitable water will completely absorb all radiation—$5\frac{1}{2}\mu$ to 7μ, and $> 14\mu$.

(*c*) Wave-lengths in which water-vapour absorption is intermediate between (*a*) and (*b*)—7μ to $8\frac{1}{2}\mu$, and 11μ to 14μ.

With a clear sky the radiation which gets out into space will belong in part to each of these categories. The radiation of category (*a*) will have originated at the earth's surface, and its amount can therefore be computed from the known mean temperatures of the earth's surface in different latitudes. Simpson assumes that the stratosphere contains at least 0·3 mm of precipitable water, and therefore the radiation of category (*b*) which gets out into space will have originated in the stratosphere, and its amount can therefore be computed from radiation tables such as are given by Simpson (*loc. cit.*). Of the radiation of category (*c*) which gets out into space we cannot make any equally direct statement. It will be intermediate in amount between the radiation of this kind at the temperature of the stratosphere and that at the temperature of the earth's surface, and we shall not be far out if we follow Simpson and take a value roughly intermediate between the two. Fig. 36, reproduced from Simpson's memoir, shows the method which he followed, assuming ground temperature 280° A, and stratosphere temperature 218° A. In the horizontally hatched region from $8\frac{1}{2}\mu$ to 11μ the radiation to space is that at the surface temperature. From $5\frac{1}{2}\mu$ to 7μ, and above 14μ, the radiation to space is equal to the vertically hatched area which measures the radiation at 218° A. In the intermediate regions, from 7μ to $8\frac{1}{2}\mu$, and from 11μ to 14μ, the amount of radiation is the diagonally hatched area. The upper boundaries of these areas cannot be specified with absolute accuracy, but as Simpson says, "...there is very little latitude; for any reasonable curve will divide the area *JDEK* into two nearly equal parts". Thus a reasonable approximation is obtained by taking the area *DPQK* equal to the mean of *JPQK* and *DPQE*. On this basis, Simpson was able to compute the total radiation passing out into space through cloudless skies.

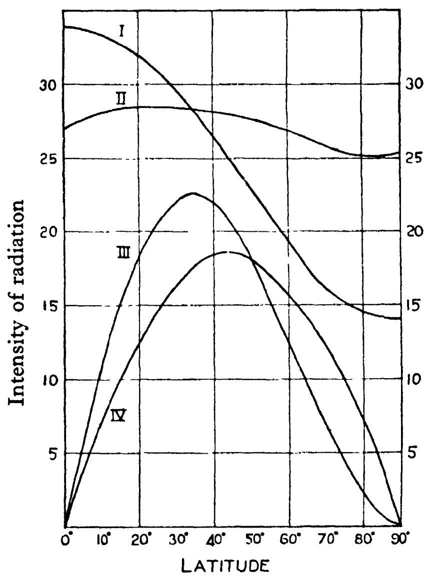


Fig. 35. Solar and terrestrial radiation.

When the sky is overcast, since for all practical purposes a cloud sheet radiates like a black body, the only difference in the computation is that the surface of the earth is replaced by the upper surface of the cloud, and the temperature of the earth's surface has to be replaced by the temperature of the cloud, which Simpson assumes to be 261° A in all latitudes.

Simpson takes no account of the variation of cloud amount with latitude in his second memoir, but uses a mean cloud amount of 5/10 for all latitudes, so that his final figure for the total radiation from the whole earth is the mean of the figures for clear and cloudy skies. The values so obtained give a mean

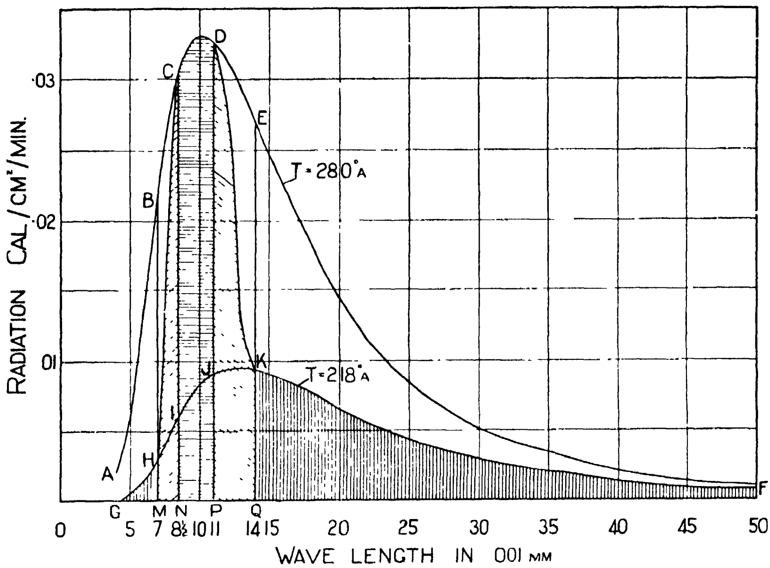


Fig. 36. The heat balance of the atmosphere.

outgoing radiation of 0·271 cal. per cm² per min., which is within a few per cent of the mean incoming radiation estimated by using Aldrich's value of the albedo. A correction can be made so as to bring the mean incoming and the mean outgoing radiation into agreement, and the values plotted in curve II of fig. 35 are the values so corrected.

A comparison of curves I and II in fig. 35 shows that from the equator to about latitude 35° the outgoing radiation is less than the incoming radiation, while in higher latitudes, the outgoing radiation exceeds the incoming radiation. There is therefore a horizontal transfer of heat from low to high latitudes, through the agency of the general circulation. In fig. 35 curve III gives the total horizontal heat transfer across circles of latitude, and curve IV the horizontal heat transfer per cm. of circle of latitude.

There is one feature of Simpson's work which should be emphasised, that while he makes use of Hettner's observations to show that 0·3 mm of precipitable water as vapour will suffice to absorb practically completely all radiation of certain wave-lengths, and is transparent to certain other wave-lengths, he makes no use of the actual values of the coefficients of absorption evaluated from Hettner's observations, which we have represented graphically in fig. 30 above. This is a point which should not be overlooked, as it lends great weight to the close agreement between the theoretical deductions and the observations which we find in Simpson's "Further Studies".

The outstanding result of this research by Simpson is that the amount of energy radiated from the earth and its atmosphere to space varies surprisingly little with latitude. It emphasises the heat transporting function of the general circulation, and puts this on a quantitative basis. In his third memoir, Simpson* has worked out similar data, using this time monthly, and not annual, means of the temperatures. The maps which he reproduces in his memoir form a new basis of attack upon the problems associated with the general circulation of the atmosphere, and it is to be hoped that they will be applied to this problem in the near future.

We cannot here enter into the details of a further application of these ideas by Simpson† to discuss the variations of climate due to the variations of solar radiation. It is readily seen that an increase in solar radiation *might* be compensated by an increase in cloud amount, which might lead to increased precipitation, and possibly to the occurrence of ice ages.

In a paper by Abbot‡, an effort has been made to develop a more elaborate analysis than Simpson's, though along the same lines. Unfortunately, this paper contains several errors, and the results cannot be accepted as valid.

§ 92. *The vertical transport of heat in the atmosphere*

In Chapter VI, § 75, it was shown that the net outflow of heat by radiation is proportional to $T\frac{dE}{dT}\frac{1}{e}\frac{dT}{dz}$, where E is the total W-radiation at temperature T. It was shown that in the lower atmosphere at ordinary temperatures, the fraction of the net outflow of energy represented by the net outflow of W-radiation is very small. Where the vapour-pressure is small, as in very high latitudes, and still more, at great elevations above the earth's surface, the thickness of the layer containing 0·3 mm of precipitable water is so great that it is no longer strictly permissible to assume that the radiation from such a layer is equal to the radiation from a layer of equal thickness whose temperature is the mean temperature of the actual layer. Nevertheless, it must still be a first approximation to the radiation from such a layer, and we shall assume that it is at least sufficiently near to the truth to give an idea of the order of magnitude of the net outflow from the atmosphere.

An idea of the change with height of the net outward flux of heat by W-radiation can be readily got by taking say the July observations of temperature for the British Isles given in Table 1, p. 19. Unfortunately it is not possible

* *Mem. R. Met. Soc.* **3**, No. 23, 1929.
† *Ibid.* **3**, No. 21, 1928; *Mem. and Proc. Manchester Lit. and Phil. Soc.* **74**, Jan. 1930.
‡ *Smithsonian Misc. Coll.* **82**, No. 3, 1928.

to assign a mean vapour-pressure to each of these heights, since the observations of humidity are not as reliable as those of temperature. We shall assume a mean relative humidity of 60 per cent at all heights, so that the vapour-pressure is 60 per cent of the saturation vapour-pressure at the corresponding temperature. The following table shows the computation of the net outflow of heat at each level:

	T	e_s	e	$dE/dT \times 10^3$	$dT/dz \times 10^4$	Net flux
Ground	289	18·2	10·9	3·4	0·6	$7{\cdot}5 \times 10^{-4}$
2 km	278	8·7	5·2	3·1	0·5	$11{\cdot}5 \times 10^{-4}$
4 km	267	3·7	2·2	2·9	0·6	29×10^{-4}
6 km	255	1·26	0·76	2·6	0·7	85×10^{-4}
9 km	234	0·14	0·09	2·2	0·75	384×10^{-4}
10 km	226	0·056	0·034	2·1	0·6	1170×10^{-4}

The net outflow of heat by W-radiation is $\frac{139}{e} T \frac{dE}{dT} \frac{dT}{dz}$. The total outflow from the ground of W-radiation is about 0·31 gramme-calorie at the temperature assumed above, and so the net flow which we have computed in the table amounts at low levels to only a small fraction of the total, but becomes at higher levels an increasingly great portion of the total flow. But as the flow of heat outward increases steadily with height, it is clear that either the upper troposphere must be continually cooling, or the main transport of heat in the troposphere is carried out by some other mechanism than radiation. The mechanism is obviously convection, which takes large quantities of heat upwards to middle levels of the atmosphere. In low latitudes, in particular, the increase of the vapour-pressure e would more than counterbalance the increase of $T(dE/dT)$, and the water-vapour in the atmosphere acts as a blanket on the outward flow of radiation, and by keeping the energy at low levels, gives the general circulation of the atmosphere time to carry it away to high latitudes. The increase in the outflow of heat with height is slow at first, but increases rapidly above 4 km, showing that the effects of condensation are greater in the middle levels of the atmosphere, as we might have anticipated from the greater frequency of clouds at those levels.

The absence of any condensation in the atmosphere would involve a direct cooling of the middle and upper troposphere, leading to a diminution of the lapse-rate in the upper troposphere, and an increase in the middle troposphere, until some kind of equilibrium was attained. There should therefore be a very definite difference in the vertical temperature distribution in regions of condensation and those of continued clear weather, if other factors could be allowed for. Note that this certainly is not true of the difference between the cyclone and anticyclone according to W. H. Dines. The anticyclone is warmer in the upper troposphere relative to the cyclone than it is at the ground, and it is certain that dynamical factors are more important than radiational factors in the anticyclone.

CHAPTER VIII

THE GENERAL EQUATIONS OF MOTION

§ 93. *The general equations of motion in spherical polar co-ordinates*

THE equations of motion in three dimensions can be most readily derived by an extension of the equations for two dimensions. In two dimensions the position of a point P (fig. 37*a*) may be fixed by means of the length r and the

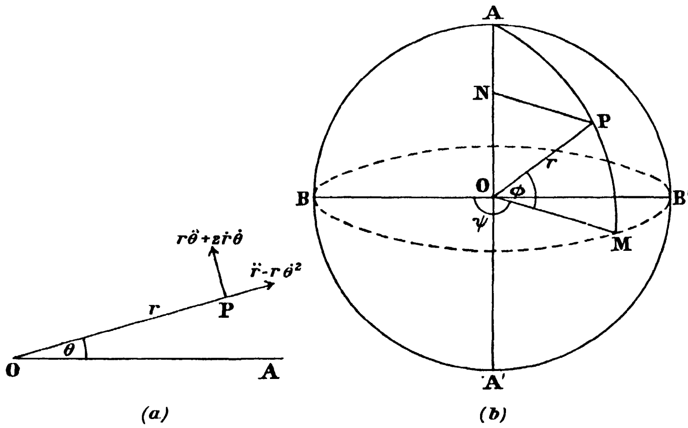


Fig. 37. Polar co-ordinates.

angle AOP, or θ, O being a fixed point, and OA a fixed direction. The accelerations of the point P are

$$\ddot{r}-r\dot{\theta}^2 \text{ along } OP$$

and

$$r\ddot{\theta}+2\dot{r}\dot{\theta} \quad \text{or} \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \qquad \text{......(1)}$$

at right angles to OP, where the dots represent differentiation with respect to time t.

For the case of motion relative to the earth we adopt a system of spherical co-ordinates. In fig. 37*b* let O be the centre of the earth, and AOA' the axis of rotation of the earth. The equatorial plane is shown in the figure by a dotted circle BMB'. We require to fix the position of the point P. Let the sphere whose centre is at O and radius OP cut the equatorial plane in the circle BMB', and the axis of the earth at the points A, A'. Through P draw the great circle $APMA'$ cutting the equatorial section in M. This great circle is the meridian through P. Let the angle between its plane and the plane $ABA'B'$ fixed in space be ψ. Also let the angle POM be denoted by ϕ. Then

ϕ is the geocentric latitude of P, and r, ϕ and ψ are the spherical co-ordinates of P. If λ be the longitude of P, taken as positive when measured to East, then

$$\dot{\psi}=\dot{\lambda}+\omega \qquad \text{......(2),}$$

where ω is the angular velocity of rotation of the earth. For $\dot{\psi}$ is the rate of rotation in space of the meridian plane through P, and $\dot{\lambda}$ is its rate of rotation relative to the standard meridian plane fixed in the earth, and ω is the rate of rotation in space of the standard meridian plane.

In the diagram (fig. 37*b*) PN is drawn perpendicular to AA', and all the points P, A, A', B, B' and M are on the sphere whose centre is O, and whose radius is r. The motion of P is made up of

(i) a motion in the meridian plane APA', and

(ii) the effect of the rotation of this plane about the axis AA', or in other words, that part of the motion of P due to the rotation of PN about N.

Each of these effects is derivable directly from the polar equations for two dimensions. From (i), P has accelerations $\ddot{r}-r\dot{\phi}^2$ along OP and $r\ddot{\phi}+2\dot{r}\dot{\phi}$ directed towards North along the tangent at P to the meridian PA. From (ii), P has accelerations $-r\cos\phi\,.\,\dot{\psi}^2$ along NP and $r\cos\phi\,.\,\ddot{\psi}+2\dot{\psi}\dfrac{d}{dt}(r\cos\phi)$ along the tangent to the parallel of latitude drawn at P towards East.

Note that under (ii) we are only concerned with the accelerations which involve the rotational terms in ψ, and the radial term is therefore not included. Collecting the terms under (i) and (ii), and resolving the acceleration along NP into two components, directed along OP and to North respectively, we find the accelerations of P along three rectangular axes to be as follows:

Along OP,

$$\ddot{r}-r\dot{\phi}^2-r\cos^2\phi\,.\,\dot{\psi}^2 \qquad \text{......(3).}$$

Perpendicular to OP and to East,

$$r\cos\phi\,.\,\ddot{\psi}+2\dot{\psi}\,(\dot{r}\cos\phi-r\sin\phi\,.\,\dot{\phi}) \qquad \text{......(4).}$$

Perpendicular to OP and to North,

$$r\ddot{\phi}+2\dot{r}\dot{\phi}+r\sin\phi\cos\phi\,.\,\dot{\psi}^2 \qquad \text{......(5).}$$

These expressions give the accelerations in three directions of any body whose co-ordinates are r, ϕ, ψ. The accelerations are relative to a system of axes fixed in space. To obtain the motion relative to the earth, we need only substitute

$$\left.\begin{aligned}\dot{\psi}&=\dot{\lambda}+\omega\\ \ddot{\psi}&=\ddot{\lambda}\end{aligned}\right\} \qquad \text{......(6).}$$

It will be seen that ψ only enters into the accelerations in the form $\dot{\psi}$ or $\ddot{\psi}$.

Now take axes x, y and z at P, the axis of x being horizontal, and directed towards East, the axis of y being horizontal and directed towards North, and the axis of z being vertical, or along OP continued. These axes are "moving axes" whose directions change with the motion of P. Let the total forces acting on unit mass at P have components X', Y' and Z' along the axes thus specified. Then we may equate these components of force to the expressions which we have derived above for the accelerations of P. In rewriting these expressions we substitute for $\dot{\psi}$ and $\ddot{\psi}$ from (6)

$$r\cos\phi\,.\,\ddot{\lambda}+2\,(\dot{\lambda}+\omega)\,(\dot{r}\cos\phi-r\sin\phi\,.\,\dot{\phi})=X' \qquad \ldots\ldots(7),$$

$$r\ddot{\phi}+2\dot{r}\dot{\phi}+r\sin\phi\cos\phi\,(\dot{\lambda}+\omega)^2=Y' \qquad \ldots\ldots(8),$$

$$\ddot{r}-r\dot{\phi}^2-r\cos^2\phi\,(\dot{\lambda}+\omega)^2=Z' \qquad \ldots\ldots(9).$$

These equations can also be readily derived by the use of Lagrange's equations. The reader unfamiliar with the use of Lagrange's equations may, however, find it useful to follow the above derivation of these equations from simple beginnings.

Equations (7), (8) and (9) must meet all cases of motion, including the special case of a particle at rest on the earth's surface. For a particle at rest,

$$\dot{r}=\dot{\phi}=\dot{\lambda}=\ddot{r}=\ddot{\phi}=\ddot{\lambda}=0.$$

Substituting these values in (7), (8) and (9), we find

$$X'=0,$$

$$Y'=r\sin\phi\cos\phi\,.\,\omega^2,$$

$$Z'=-r\cos^2\phi\,.\,\omega^2.$$

Thus it is not possible for a particle to remain permanently at rest on a spherical earth, without the action of a force Y' whose magnitude is given by the second of these equations; it requires a force $r\sin\phi\cos\phi\,.\,\omega^2$ directed towards North to keep it in equilibrium. In the absence of such a force any body free to move on the earth's surface would move towards the equator.

It has hitherto been assumed that the earth may be regarded as a sphere, and that the radius vector OP will always represent the vertical at P. Actually the earth is not a sphere, but a spheroid, whose polar axis is slightly less than the equatorial axis. The angle between the radius vector OP and the vertical at P fixed by the normal to the geoidal surface is $700''\sin 2\phi$. This very small angle is the difference between the geocentric and the astronomical latitudes. If now we alter our system of axes, so that the z-axis becomes the true vertical at P, and the y-axis becomes the true horizontal drawn towards North, while the x-axis remains as before, we have to rotate our original system of axes about the x-axis through an angle which even at its maximum is only slightly greater than one minute of arc. But there can be no component of force Y' along the new y-axis. The term $r\sin\phi\cos\phi\,.\,\omega^2$ is then in fact compensated by a component of gravitational attraction. On account of the spheroidal form of the earth, the gravitational attraction of the earth on a material point outside its own mass is not directed exactly towards the centre of the earth, and when the force of attraction is resolved along the vertical direction and the horizontal line directed to North, there is a component along the latter direction which is exactly equal to the constant $r\sin\phi\cos\phi\,.\,\omega^2$ of equation (8). The component along the true vertical at the point under consideration is combined with the constant term $-r\cos^2\phi\,.\,\omega^2$ of equation (9), and the sum is what is usually known as g. We cannot by direct observation separate the two parts of g.

Let the external forces acting at P, other than gravitation, have components X, Y, Z along the axes as modified; the direction of the axis of z is then the true vertical. Then we may write equations (7), (8) and (9) in the form

$$r\cos\phi.\ddot{\lambda}+2(\dot{\lambda}+\omega)(\dot{r}\cos\phi-r\sin\phi.\dot{\phi})=X \qquad \text{......(10)},$$

$$r\ddot{\phi}+2\dot{r}\dot{\phi}+r\sin\phi\cos\phi.\dot{\lambda}(\dot{\lambda}+2\omega)=Y \qquad \text{......(11)},$$

$$\ddot{r}-r\dot{\phi}^2-r\cos^2\phi.\dot{\lambda}(\dot{\lambda}+2\omega)=Z-g \qquad \text{......(12)}.$$

The quantities X, Y, Z are related to X', Y', Z' by the relations

$$X=X' \qquad \text{......(13)},$$

$$Y=Y'-r\cos\phi\sin\phi.\omega^2 \qquad \text{......(14)},$$

$$Z=Z'+g+r\cos^2\phi.\omega^2 \qquad \text{......(15)}.$$

We have made use of the fact that the earth is a spheroid in order to account for the second term on the right-hand side in equation (14), but the angle between the true vertical and the radius vector from the earth's centre is so small that we need not take the difference into account in what follows, and we may now without risk of obtaining erroneous results assume equations (10), (11) and (12) to apply to motion relative to a spherical earth.

Equation (10) can be readily integrated when $X=0$, or the motion is under "no forces". Multiplying the equation throughout by $r\cos\phi$, it becomes

$$\frac{d}{dt}\{r^2\cos^2\phi(\dot{\lambda}+\omega)\}=0 \qquad \text{......(16)},$$

from which it follows that

$$r^2\cos^2\phi(\dot{\lambda}+\omega)=\text{const.} \qquad \text{......(17)}.$$

This equation states that the moment of momentum about the axis of the earth remains constant in any motion during which there is no West-East component of force. Hence if the West-East component of velocity of an element of mass is known in any one latitude, equation (17) enables us to compute it for any other latitude attained by that element, provided there is no West-East component of force acting on the element. It cannot be too strongly emphasised that the question whether a specified element of mass originally in some specified latitude will subsequently attain some other specified latitude has to be answered from other considerations. Neglect to take account of this has frequently led writers on meteorology into giving estimates of velocity in the earth's atmosphere which are far in excess of observed velocities.

§ 94. *Expression of the general equations of motion in Cartesian co-ordinates*

We have already defined a convenient set of axes of co-ordinates as follows:

x horizontal and drawn to East,

y horizontal and drawn to North,

z vertical.

Left-handed axes are chosen because for two dimensions they reduce to the type of axes normally used in discussing two-dimensional problems.

Let u, v, w be the component velocities along these three axes. We then have the following relations:

$$u = r\cos\phi\,.\,\dot{\lambda},$$
$$v = r\dot{\phi},$$
$$w = \dot{r}.$$

Also
$$\frac{du}{dt} = r\cos\phi\,.\,\ddot{\lambda} + \dot{r}\cos\phi\,.\,\dot{\lambda} - r\sin\phi\,.\,\dot{\phi}\dot{\lambda},$$
$$\frac{dv}{dt} = r\ddot{\phi} + \dot{r}\dot{\phi},$$
$$\frac{dw}{dt} = \ddot{r}.$$

By means of these relations we can re-write equations (10), (11) and (12) in the form

$$\frac{du}{dt} - v\dot{\lambda}\sin\phi + w\dot{\lambda}\cos\phi + 2\omega\,(w\cos\phi - v\sin\phi) = X \quad \text{......(18)},$$
$$\frac{dv}{dt} + w\dot{\phi} + u\dot{\lambda}\sin\phi + 2\omega u\sin\phi = Y \quad \text{......(19)},$$
$$\frac{dw}{dt} - u\dot{\lambda}\cos\phi - v\dot{\phi} - 2\omega u\cos\phi = -g + Z \quad \text{......(20)}.$$

But as the point moves over the earth, the axes of reference of equations (18), (19) and (20) rotate, and the rates of rotation are readily seen to be $\dot{\phi}$, $\dot{\lambda}\cos\phi$ and $\dot{\lambda}\sin\phi$. The first three terms on the right-hand side of these equations therefore give the accelerations along axes fixed in the earth, but instantaneously coinciding with the moving axes*. Hence when the motion is referred to axes *fixed in the earth*, the equations of motion become

$$\frac{du}{dt} + 2\omega\,(w\cos\phi - v\sin\phi) = X \quad \text{......(21)},$$
$$\frac{dv}{dt} + 2\omega u\sin\phi = Y \quad \text{......(22)},$$
$$\frac{dw}{dt} - 2\omega u\cos\phi = Z - g \quad \text{......(23)}.$$

The terms in ω represent the effect of the rotation of the earth. If u is positive, or the motion has a component directed towards East, the term $2\omega u\cos\phi$ effectively diminishes g, while when u is negative, or the motion has a component towards West, the rotational term effectively increases g. Thus westward moving air should behave as though it were denser than eastward moving air of the same temperature and pressure.

The velocity u in the second term on the left-hand side of equation (23) is

* See, for example, Routh's *Rigid Dynamics*, **1**, p. 205.

the horizontal velocity directed towards East. If axes other than those directed towards East and North are used, account must be taken of this in equation (23), u being replaced by the appropriate expression for velocity to East.

Equations (21), (22) and (23) indicate that the effect of the rotation of the earth can be taken into account by adding in the equations of motion accelerations $-2\omega(w\cos\phi - v\sin\phi)$, $-2\omega u\sin\phi$, $2\omega u\cos\phi$, parallel to the axes of x, y and z respectively. Alternatively we may say that the effect of the earth's rotation is a "deviating force" whose magnitude per unit mass has the components stated above. Since the direction cosines of the earth's axis are 0, $\cos\phi$, $\sin\phi$, it follows that the deviating force is perpendicular to the earth's axis. Further, when the components of the deviating force are multiplied by u, v, w, respectively, the sum of the products is zero, and hence the deviating force is perpendicular to the resultant velocity. The direction of the total deviating force on any particle is therefore parallel to the plane of the equator, and is perpendicular to the direction of motion of the particle.

§ 95. *The equations in hydrodynamical form*

In equations (21), (22) and (23) the terms du/dt, dv/dt, dw/dt represent the accelerations of an element of mass, or, in other words, they represent "differentiation following the motion". In the discussion of hydrodynamical and aerodynamical questions it is often more convenient to express by u, v, w the component velocities at a given point (x, y, z) at a time t. The complete form of equations (21), (22) and (23) may then be written

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+2\omega(w\cos\phi-v\sin\phi)=X \quad\text{......(24)},$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2\omega u\sin\phi=Y \quad\text{......(25)},$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-2\omega u\cos\phi=-g+Z \quad\text{......(26)}.$$

§ 96. *The special case of horizontal motion*

When the motion is purely horizontal, or $w=0$, the equations (21) and (22) reduce to

$$\frac{du}{dt}-2\omega v\sin\phi=X \quad\text{......(27)},$$

$$\frac{dv}{dt}+2\omega u\sin\phi=Y \quad\text{......(28)}.$$

The terms involving ω represent the effect of the rotation of the earth. These yield accelerations $2\omega v\sin\phi$ along the x-axis, and $-2\omega u\sin\phi$ along the y-axis. Thus the resultant acceleration is $2\omega\sin\phi\times$(resultant velocity), and is directed at right angles to the motion, and to the right-hand side of the line of motion. For if, in fig. 38, OP represents the total velocity V whose components are u, v, and OQ represents the total acceleration due to the ω terms, $QN = 2\omega v \sin \phi$, $ON = 2\omega u \sin \phi$; also since $\frac{QN}{ON} = \frac{v}{u}$, the angle QON = angle POM, and therefore OQ is at right angles to OP, and its magnitude is $2\omega \sin \phi \times V$.

The result as stated is true for all latitudes. In the Southern hemisphere the latitude ϕ is to be regarded as negative, and the acceleration can be more usefully described as $2\omega \sin \phi \times V$ to the left of the direction of motion, and at right angles to the latter.

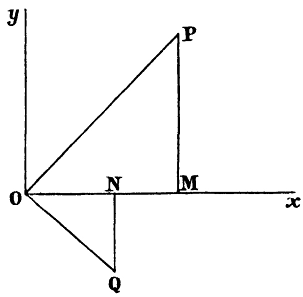


Fig. 38. The deviating force in two dimensions.

Thus, so far as horizontal motion is concerned, the effect of the rotation of the earth is to produce an acceleration directed at right angles to the motion, of magnitude $2\omega \sin \phi$ times the velocity. The equations (27) and (28) will always represent this fact whether the axes are drawn to East and North or not. In fact, the limitation of the direction of the axes is no longer necessary so far as these two equations are concerned. In contrast with this, it should be noted that equations (21), (22), (23) are only true for motion referred to axes drawn as specified above. For the term in $2\omega w \cos \phi$ in equation (21) represents an acceleration directed along the West-East line, of magnitude proportional to the vertical velocity; and if we use axes not directed to East and North, this acceleration must be resolved along the new axes, giving a component along both, instead of along one axis only, as in equation (21). Note also that equation (23) shows that there is a vertical acceleration proportional to the velocity along the West-East line.

§ 97. *The deviating force of the earth's rotation*

The acceleration $2\omega \sin \phi . V$, where V is the total velocity in the horizontal plane, is frequently referred to as the "deviating force due to the earth's rotation". Since it is always directed at right angles to the direction of motion, it cannot produce any change of velocity, but only changes of direction of motion. If, therefore, a small mass is set in motion in any latitude, with a total velocity V, it will continue to move with the same velocity, though the direction of its motion will not remain constant.

Let c be the radius of curvature of the path at the point P, where the velocity is V. The acceleration towards the centre of curvature is V^2/c, and this is equal to the deviating force $2\omega V \sin \phi$. Hence $c = V/2\omega \sin \phi$ is the radius of curvature of the path, and since the speed remains unaltered, the radius of curvature c also remains unaltered, and the path is a circle. It is assumed in the above that the variation of ϕ may be neglected. The circle is known as the *circle of inertia*. The same result is readily derived by the direct use of equations (27) and (28) above. Multiplying these equations by u and v respectively, and adding the products, we find

$$\frac{d}{dt}(u^2+v^2)=0, \quad \text{or} \quad u^2+v^2=V^2=\text{const.}$$

If initially the particle is at the origin (0, 0), and has components of velocity u_0, v_0, we may integrate equations (27) and (28) directly,

$$u-u_0-2\omega \sin\phi . y=0, \quad \text{or} \quad u=u_0+2\omega\sin\phi . y$$

$$v-v_0+2\omega\sin\phi . x=0, \quad \text{or} \quad v=v_0-2\omega\sin\phi . x.$$

Hence $$V^2=u^2+v^2=(u_0+2\omega\sin\phi . y)^2+(v_0-2\omega\sin\phi . x)^2$$

or $$\left(x-\frac{v_0}{2\omega\sin\phi}\right)^2+\left(y+\frac{u_0}{2\omega\sin\phi}\right)^2=\frac{V^2}{(2\omega\sin\phi)^2}.$$

The particle therefore moves in a circle with uniform velocity V, the circle having a radius $V/2\omega\sin\phi$. In deriving the equation to this path we have neglected the variation of latitude along the path. If account is taken of the variation of ϕ, the path deviates from a circle, though only very slightly, except at the equator. For a full discussion of the form of paths of particles moving over the earth's surface under no forces, the reader is referred to a paper by F. J. W. Whipple*.

The radius of the circle of inertia is proportional to the velocity of projection, and inversely proportional to sin ϕ. For a velocity V of 10 metres per second the radius of the circle is 69 km at the poles, 90 km in latitude 50°, 138 km in latitude 30°. A mass started moving horizontally poleward with a velocity of 10 metres per second in latitude 8° would execute approximately a circle whose radius would be equal to 4° of latitude.

§ 98. *The order of magnitude of the terms in the equations of motion*

It is important to keep in mind an estimate of the order of magnitude of the different terms in equations (21), (22) and (23). We shall take the component velocities u and v to be both 20 metres per second, giving a total velocity of 28 metres per second. This is distinctly greater than the mean velocity of motion observed in the earth's atmosphere, and is therefore a reasonably safe guide to the terms which may be neglected.

g is of the order of 1000 cm/sec²,

$2\omega u\sin\phi=2\omega v\sin\phi=0{\cdot}28\sin\phi$ cm/sec²,

$2\omega u\cos\phi=0{\cdot}28\cos\phi$ cm/sec².

The vertical component of velocity, w, is usually small, and of the order of magnitude of 1 metre/sec, and seldom exceeds 5 metres/sec, except possibly in thunderstorms. Thus in general it is possible in equation (21) to neglect $2\omega w\cos\phi$ by comparison with the term $2\omega v\sin\phi$, but care may be necessary

* *Phil. Mag.* **33**, 1917, p. 457.

in special cases. The term $2\omega u \cos\phi$ is normally negligible by comparison with g, but it will be seen that it may be of importance at a surface of discontinuity of density or velocity (see § 113 below).

§ 99. *The equation of continuity*

Let ρ represent the density of a fluid, which we regard as a function of the co-ordinates x, y, z, and of the time t. Consider the flow into a small parallelepiped whose edges are parallel to the axes of co-ordinates, and of length dx, dy, dz. The rate of flow of mass across a face set parallel to the yz plane is $\rho u\,dy\,dz$, and the net flow per unit time out of the parallelepiped across the two faces parallel to the yz plane is

$$\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz;$$

similar considerations apply to the other faces, and the result is that the net flow of mass per unit time out of the parallelepiped is

$$\left\{\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right\}dx\,dy\,dz.$$

This is equal to the rate of diminution of mass within the element of volume, and is therefore equal to

$$-\frac{\partial}{\partial t}(\rho\,dx\,dy\,dz) = -\frac{\partial\rho}{\partial t}\,dx\,dy\,dz.$$

Hence the condition that fluid is nowhere being annihilated or created within the region is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0 \qquad \ldots\ldots(29)$$

or

$$\frac{1}{\rho}\frac{d\rho}{dt}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \ldots\ldots(30),$$

where d/dt represents total differentiation following the fluid, or

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

In the special case of incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \ldots\ldots(31).$$

Equation (29) or (30) is known as the "equation of continuity". In polar co-ordinates this equation becomes

$$\frac{\partial\rho}{\partial t}+\frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}(\rho u)+\frac{1}{r}\frac{\partial}{\partial\phi}(\rho v)-\frac{\rho v}{r}\tan\phi+\frac{\partial}{\partial r}(\rho w)+\frac{2}{r}\rho w=0 \ldots\ldots(32).$$

§ 100. *The forces X, Y, Z*

So far no attempt has been made to specify the nature of the forces X, Y, Z. These forces were defined as the external forces acting on an element of mass of the body whose motion is being considered. They must clearly include the effects of pressure, friction and turbulence. The effect of pressure distribution can be readily introduced into our equations, and the detailed discussion of the effects of friction and turbulence will be postponed to a later chapter.

In a fluid the pressure may vary from point to point, but the variations are continuous. In three-dimensional space, defined by Cartesian co-ordinates x, y, z, the pressure p will be a continuous function of x, y, z.

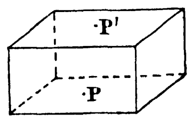


Fig. 39. The effect of pressure gradient.

Take a parallelepiped with edges dx, dy, dz parallel to the axes of coordinates, of which the z-axis is vertical. Let P, fig. 39, the centre of the lower face of the parallelepiped, be (x, y, z), and let P', the centre of the upper face, be $(x, y, z+dz)$. If p is the pressure at P, this may be regarded as the mean pressure over the lower face. The pressure at P' will be $p+\frac{\partial p}{\partial z}dz$, and this may be regarded as the mean pressure over the upper face. The total pressure on the lower face is $p\,dx\,dy$ acting upward, and the total pressure on the upper face acting downward is $\left(p+\frac{\partial p}{\partial z}dz\right)dx\,dy$. Thus the net pressure on the element of volume along the direction of the axis of z is $-\frac{\partial p}{\partial z}dx\,dy\,dz$, acting upward. This is equivalent to a force $-\frac{\partial p}{\partial z}$ per unit volume, or $-\frac{1}{\rho}\frac{\partial p}{\partial z}$ per unit mass. Similarly the resultant pressures parallel to the axes of x and y are respectively $-\frac{\partial p}{\partial x}$, $-\frac{\partial p}{\partial y}$ per unit volume, or $-\frac{1}{\rho}\frac{\partial p}{\partial x}$, $-\frac{1}{\rho}\frac{\partial p}{\partial y}$ per unit mass. Hence the parts of the forces X, Y, Z, which are due to the variation of pressure from point to point of the fluid, may be written $-\frac{1}{\rho}\frac{\partial p}{\partial x}$, $-\frac{1}{\rho}\frac{\partial p}{\partial y}$, $-\frac{1}{\rho}\frac{\partial p}{\partial z}$. The negative signs denote that the effect of the pressure distribution is equivalent to a force tending to push the fluid towards the region of lowest pressure, as might have been inferred from general considerations. The resultant force is at right angles to the isobaric surfaces. When the isobars are drawn for particular levels, the resultant of the two terms $-\frac{1}{\rho}\frac{\partial p}{\partial x}$, $-\frac{1}{\rho}\frac{\partial p}{\partial y}$ is normal to the isobars. The name "pressure gradient" is usually given to the resultant of $-\frac{\partial p}{\partial x}$, $-\frac{\partial p}{\partial y}$.

Should pressure be given as a function of spherical polar co-ordinates, r, ϕ, λ, it is a simple exercise to show that the components of the effective pressure forces to East, North and vertically upward are

$$-\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda},\quad -\frac{1}{\rho r}\frac{\partial p}{\partial\phi},\quad -\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

If all frictional and viscous forces may be neglected, equation (23) may be written

$$\frac{dw}{dt}-2\omega u\cos\phi=-g-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{......(33).}$$

If there is no motion, all the terms on the left-hand side of this equation vanish, and we obtain the statical equation of p. 34,

$$\frac{\partial p}{\partial r} = -g\rho \qquad \text{......(34).}$$

In equation (33) the term $2\omega u \cos\phi$ is usually negligible by comparison with g, but dw/dt may be considerable in very strong convection currents, and it is not then permissible to assume that the variation of pressure with height is given by equation (34).

We may sum up the results derived in the present chapter in the following equations, where X, Y, Z are the components of the forces not due to pressure and gravity:

$$\frac{du}{dt} + 2\omega\,(w\cos\phi - v\sin\phi) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + X \qquad \text{......(35),}$$

$$\frac{dv}{dt} + 2\omega u\sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} + Y \qquad \text{......(36),}$$

$$\frac{dw}{dt} - 2\omega u\cos\phi = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + Z \qquad \text{......(37),}$$

where d/dt may be replaced by

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

To these we add the equation of continuity in the form in which it is most commonly used:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{......(38).}$$

Equations (35), (36) and (37) are referred to axes drawn to East, to North, and vertically upward respectively. In practice it is frequently convenient to use axes oriented in some other direction. Let us, for example, take a new axis of x drawn at an angle β with the East-West line, as shown in fig. 40. Then the relation between the new co-ordinates x', y' and the original co-ordinates are as shown below:

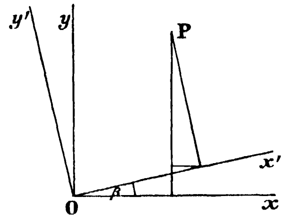


Fig. 40. Co-ordinate axes in any direction.

$$x = x'\cos\beta - y'\sin\beta,$$
$$y = x'\sin\beta + y'\cos\beta,$$
$$u = u'\cos\beta - v'\sin\beta,$$
$$v = u'\sin\beta + v'\cos\beta.$$

Substituting these values in equations (35), (36) and (37), we find the corresponding form referred to the axes x and y, with the axis of z vertical.

Only two points call for special care in the process of substitution. Equation (35) indicates that the effect of vertical motion is to give an acceleration $2\omega w\cos\phi$ along the West-East line. It will now be necessary to resolve this along the new axes of x' and y', yielding components $-2\omega w\cos\phi\sin\beta$ along the axis of y', and $2\omega w \cos\phi \cos\beta$ along the axis of x'. Similarly equation (37) indicates a vertical acceleration $2\omega \cos\phi \times$ velocity to East, which is now represented by $u' \cos\beta - v \sin\beta$. Hence equations (35), (36) and (37) are now replaced by the following, the accents being now omitted:

$$\frac{du}{dt} + 2\omega\,(w \cos\phi \cos\beta - v \sin\phi) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + X \qquad \text{......(39)},$$

$$\frac{dv}{dt} + 2\omega\,(w \cos\phi \sin\beta + u \sin\phi) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + Y \qquad \text{......(40)},$$

$$\frac{dw}{dt} - 2\omega \cos\phi\,(u \cos\beta - v \sin\beta) = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + Z \qquad \text{......(41)}.$$

These three equations agree with the preceding set of equations when $\beta = 0$.

§ 101. *Barotropic and baroclinic fluids*

The field of distribution of pressure in a fluid is conveniently given by a system of isobaric surfaces, which are such that on any one surface the pressure is the same at all points. If surfaces are drawn for each unit of pressure, the whole fluid considered is thereby divided into a set of isobaric sheets.

Similarly the distribution of density may be represented by a system of surfaces of equal density, or *isopycnic surfaces*, and if isopycnic surfaces are drawn for equal or unit intervals of density, the fluid is divided into a set of isopycnic sheets.

V. Bjerknes has discussed the properties of such surfaces*, and has distinguished the case in which the surfaces intersect from the case in which they do not intersect. In the former case the fluid is said to be *baroclinic*, in the latter case *barotropic*. It is thus seen that barotropy is a special case of degeneration of the two families of equiscalar surfaces into one family, the isobaric surfaces then being also isopycnic. The physical distinction between barotropic and baroclinic fluids is that in the barotropic fluid pressure is a function of density alone, while in the baroclinic fluid pressure is not determined by density alone, and is in part determined by other factors such as temperature or water-vapour content.

Instead of drawing surfaces of equal density we might draw surfaces of equal specific volume, to which Bjerknes gives the name of *isosteric* surfaces. Any isosteric surface is also isopycnic, but the spacing of surfaces separated by unit difference of density and of specific volume will of course be different. If we wish to consider only the form of these surfaces without specifying the variable, we may, following Bjerknes, call them "equisubstantial" surfaces.

When the fluid is baroclinic, the isobaric and equisubstantial surfaces have definite lines of intersection. The two sets of surfaces will divide the fluid into a system of unit tubes. In a barotropic fluid neither curves of intersection nor unit tubes can be specified. It may be added that the fluid of "classical" hydrodynamics, which is normally assumed to be homogeneous and incompressible, is obviously barotropic.

* *Geofys. Publ.* **2**, No. 3.

§ 102. *Circulation and vorticity*

The circulation around a closed curve is defined as the line-integral of the velocity around the curve. For an element ds of the curve, the tangential component of the velocity v is evaluated, and the sum of the products $v_T ds$ for the whole curve is evaluated. The quantity so defined, $\int v_T ds$, is known as the circulation around the circuit. The circulation, which we shall denote by C, is readily seen to be given by

$$C = \int (u\,dx + v\,dy + w\,dz) \qquad \ldots\ldots(42),$$

where the integral is taken around the closed curve.

The simplest form of circulation which we can consider is uniform motion around a circle of radius r, with tangential velocity v. The circulation is then seen to be

$$C = 2\pi r v.$$

Stokes* has shown that the motion of a small element of fluid for a short time may be regarded as made up of four parts:

(*a*) a motion of bodily translation,

(*b*) a motion of solid rotation of the whole element,

(*c*) a volume expansion or contraction, and

(*d*) a shear.

This classification, which is only true to the first order of small quantities, has permeated the whole mechanics of the continuum. The reader who desires to consider its complete analytical discussion is referred to Love's *Treatise on Elasticity*, or to Lamb's *Hydrodynamics*, Chapter III.

We shall here consider briefly the item (*b*) of the above classification, since a large part of meteorology is concerned with the nature and growth of circulation and rotation. The rotation as a whole of the small element, which is assumed to be sufficiently small to permit our assuming that its rotation may be regarded as uniform, is usually specified by the *vorticity*, which is equal to twice the angular velocity of rotation. Being a vector, it may be specified by the components about three rectangular co-ordinate axes. The adoption of twice the angular velocity as the vorticity is merely a useful convention which avoids the necessity for the continual recurrence of a factor 2 in a large number of hydrodynamical equations.

Consider first the simple case of a circular cylinder rotating as a solid with angular velocity $\frac{1}{2}\zeta$. Let the outer boundary have the radius R. The vorticity has everywhere the value ζ. The circulation around the boundary will be $2\pi R \times \frac{1}{2}\zeta R = \pi R^2 \zeta$, or: Circulation = area × vorticity. This result is also readily seen to be true for an area of any form when the fluid rotates as a solid.

In fig. 41 let O be any point within the circuit, and let PP' be a small element of the circuit. Let $PP' = ds$, and let the angle POP' be $d\theta$. The velocity

* *Collected Papers*, **1**, p. 80; or *Camb. Phil. Trans.* **7**, 1845.

of rotation of the point P about O is $\frac{1}{2}\zeta$. Hence the contribution to the circulation by the small element PP' is

$$PP' \times \tfrac{1}{2}r\zeta \times \cos POP' = \tfrac{1}{2}r^2\zeta \times P\hat{O}P' = \text{vorticity} \times \text{small area } POP'.$$

Integrating around a complete circuit will give the same result as before that the circulation is equal to area × vorticity.

The last relationship may be taken as the definition of vorticity, that it is such that the circulation around a small plane circuit is equal to the product of the component of the vorticity about an axis perpendicular to the element and the area of the element. This definition enables us to derive a number of general relationships of interest. When we come to the stage at which it is necessary to make actual computations we shall find it necessary to use Cartesian co-ordinates. The natural development of the formulae is by the use of polar co-ordinates, but the transformation to Cartesians is now readily made. In fig. 42 the small rectangle in the xy plane, whose sides are parallel to the co-ordinate axes, is assumed to be so small that the angular velocity

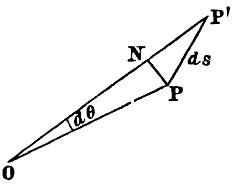


Fig. 41. Circulation and vorticity.

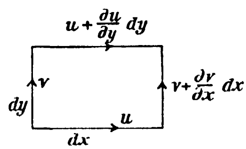


Fig. 42. Vorticity in Cartesian co-ordinates.

about the axis of z has the same value over the whole area. The value of the circulation is

$$C = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \zeta\,dx\,dy,$$

where ζ is the component of vorticity parallel to the axis of z. Hence

Similarly

$$\left.\begin{aligned} \zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ \xi &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \eta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \end{aligned}\right\} \qquad \ldots\ldots(43).$$

These equations give positive vorticity and positive circulation for counterclockwise motion. This is contrary to the usage of Lamb's *Hydrodynamics*, in which the axes used form a right-handed screw system. But in meteorology it is definitely advantageous to adopt the convention used above, since the rotation of the earth, in the Northern hemisphere, is equivalent to a counterclockwise rotation.

Any fluid motion which has vorticity is called "rotational" motion, and fluid motion which has no vorticity is known as "irrotational" motion. The simplest case of rotational motion is rotation as a solid. The earth's atmosphere rotates with the earth, and the mean vorticity of the atmosphere, apart from local variations, is 2ω, where ω is the angular velocity of the earth.

The simplest form of symmetrical motion which shall be irrotational is readily found as follows. Take a sector of the circular distribution. Let r_1 and r_2 be two radii, and v_1 and v_2 the corresponding velocities. Take the circulation around the circuit *PQRS* in fig. 43. The sides *PQ* and *RS* being at right angles to the flow do not contribute to the circulation. The total circulation around *PQRS* is therefore

$$(v_1 r_1 - v_2 r_2) \times \text{angle } POS,$$

and since this must be zero by hypothesis, $v_1 r_1 = v_2 r_2$, and the product vr must be constant. The circulation is in fact constant around all concentric circles. This distribution, in which the transverse velocity is everywhere inversely proportional to the distance from the centre, is called the "simple vortex", or the "*vr* vortex". It is, however, irrotational, and has everywhere zero vorticity, except at the centre, where the velocity is infinite.

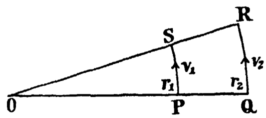


Fig. 43. Irrotational symmetrical motion.

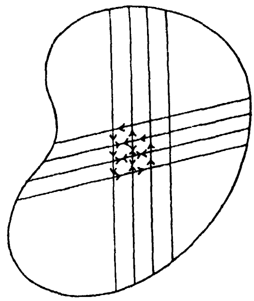


Fig. 44. Circulation and vorticity in any circuit.

It will be seen from the above discussion that it is possible to have cyclic motion without vorticity in the main part of the field. The so-called "*vr* vortex" is of the nature of an infinite cylinder in the vertical direction. It extends out to an infinite distance laterally, and has infinite velocity at the centre. In nature the *vr* vortex can be readily set up through a finite region if these infinities can be avoided. The infinities at the ends of the cylinder are avoided if the fluid has upper and lower boundaries, as for example in a bowl of water, in which the bottom of the bowl and the free surface of the water form the necessary boundaries. The infinite horizontal extent is also removed by the boundaries formed by the side of the bowl, and the infinity of velocity at the centre may be evaded in two ways, either by the core being empty, as is the case when water whirls rapidly in a bowl, or by the provision of an inner region in which there is a continuous, but not necessarily constant, distribution of vorticity.

Around any closed plane curve the circulation is equal to the integral $\int \zeta_n dS$, where dS is an element of surface, and ζ_n the component vorticity perpendicular to the element dS. This is readily seen from fig. 44. The circulation around the outer boundary is equal to the sum of the circulations around all the elementary circuits, or in other words to the sum of the products $\zeta_n dS$ corresponding to each element, and when the elements of area are taken sufficiently small, the sum may be replaced by the integral $\int \zeta_n dS$, which is therefore equal to the circulation around the outer circuit.

Fig. 44 can be applied to the general case in which the circuit is not a plane circuit, and the surface of which it is a boundary is no longer part of a plane. The elementary circuits into which the surface is divided must now be sufficiently small to be treated as plane elements. The circulation around a small circuit enclosing an area dS is $dS \times$ vorticity about the normal to dS. If the direction cosines of the normal to dS be l, m, n, and if the components of vorticity be ξ, η, ζ, the circulation about the elementary circuit is

$$(l\xi + m\eta + n\zeta)\, dS.$$

Hence the circulation C about the circuit which forms the outer boundary of the surface is given by

$$C = \int (u\,dx + v\,dy + w\,dz) = \iint (l\xi + m\eta + n\zeta)\, dS,$$

where dS is a small element of any surface bounded by the circuit. This equation defines the close relation which subsists between circulation and vorticity.

§ 103. *Vortex lines, filaments and sheets*

A line drawn from point to point so that its direction is everywhere that of the instantaneous direction of the axis of rotation of the fluid is called a "vortex line". If through every point of a small closed curve we draw the corresponding vortex line, these lines mark out a tube, which is called a "vortex tube", and the fluid within the tube is called a "vortex filament". The circulation around the boundary of any cross-section of the tube, taken normal to its length, is $\Omega\sigma$, where Ω is the total vorticity $\sqrt{\xi^2 + \eta^2 + \zeta^2}$, and σ is the area of the small cross-section.

A vortex filament may be surrounded by fluid in irrotational motion, but the field of the vortex filament must be regarded as permeating the whole fluid.

A surface of discontinuity of velocity may be regarded, from the point of view of pure kinematics, as a surface or sheet of infinite vorticity, the axis of the vorticity being along the lines of relative flow. If, for example, the motion be represented by components of velocity, u, v, w, on one side of the sheet, and u', v', w', on the other side, the components of vorticity are proportional to $u-u'$, $v-v'$, $w-w'$, and the vortex lines at any point are given by

$$\frac{dx}{u-u'} = \frac{dy}{v-v'} = \frac{dz}{w-w'}.$$

The concept of a surface of discontinuity as a vortex sheet is a purely mathematical idea, and does not of necessity assist in the discussion of meteorological problems.

§ 104. *The generation of circulation in the atmosphere*

Kelvin established a theorem that if

$$C=\int (u\,dx+v\,dy+w\,dz),$$

then

$$\frac{dC}{dt}=\int\left(\frac{du}{dt}\,dx+\frac{dv}{dt}\,dy+\frac{dw}{dt}\,dz\right) \qquad \text{......(44)},$$

where the integrals are both taken around a contour c composed of the same moving particles of fluid. By direct differentiation it is seen that

$$\frac{dC}{dt}=\int\left(\frac{du}{dt}\,dx+\frac{dv}{dt}\,dy+\frac{dw}{dt}\,dz\right)+\int\left(u\,\frac{d(dx)}{dt}+v\,\frac{d(dy)}{dt}+w\,\frac{d(dz)}{dt}\right).$$

Since

$$\frac{dx}{dt}=u,\quad \frac{dy}{dt}=v,\quad \frac{dz}{dt}=w,$$

this reduces to

$$\frac{dC}{dt}=\int\left(\frac{du}{dt}\,dx+\frac{dv}{dt}\,dy+\frac{dw}{dt}\,dz\right)+\frac{1}{2}\int\frac{\partial}{\partial s}\,(u^2+v^2+w^2)\,ds.$$

Since the velocity is a single-valued function of position, the second term on the right-hand side of this equation vanishes, and the equation reduces to (44) above.

Substituting in equation (44) the values of du/dt, dv/dt and dw/dt from equations (35), (36) and (37) above,

$$\frac{dC}{dt}=\int -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\,dx+\frac{\partial p}{\partial y}\,dy+\frac{\partial p}{\partial z}\,dz\right)+\int (X\,dx+Y\,dy+Z\,dz)$$

$$-\int g\,dz-2\omega\int\{(w\cos\phi-v\sin\phi)\,dx+u\sin\phi\,.\,dy-u\cos\phi\,.\,dz\}$$

$$=-\int\frac{dp}{\rho}+\int (X\,dx+Y\,dy+Z\,dz)-\text{terms in } 2\omega \qquad \text{......(45)}.$$

The terms in ω are readily evaluated as follows:

In fig. 45 consider the circuit f which is the projection of the original circuit c upon the plane of the equator. In the plane of the equator take axes x', y', where x' is parallel to the original axis of x, and y' is the intersection of the original yz plane with the equator. Let u', v' be the components of velocity in the equatorial plane of the original velocities u, v, w.

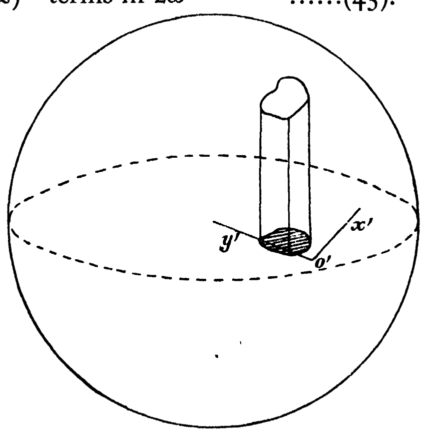


Fig. 45. Circulation in the earth's atmosphere.

Then $$dx' = dx, \quad dy' = \sin\phi . dy - \cos\phi . dz,$$
$$u' = u, \quad v' = \sin\phi . v - \cos\phi . w,$$
$$(w\cos\phi - v\sin\phi)\, dx + u\sin\phi . dy - u\cos\phi . dz = -v'dx' + u'dy'.$$

The terms in ω in (45) may therefore be written as

$$-2\omega \int (u'dy' - v'dx') \qquad \text{......(46)},$$

where the integral is taken around f, the projection of the original contour on the plane of the equator. We take the circulation to be positive in the counter-clockwise sense in the plane of the equator. It is seen that if F denote the area of the contour f, (46) reduces to

$$-2\omega \frac{dF}{dt} \qquad \text{......(47)},$$

and writing $$W = \int (X\,dx + Y\,dy + Z\,dz),$$

equation (45) becomes $$\frac{dC}{dt} = -\int \frac{dp}{\rho} + W - 2\omega \frac{dF}{dt} \qquad \text{......(48)}.$$

In (48) F is to be taken as positive when the circuit c is traversed in such a sense that the circuit f is traversed in the counter-clockwise direction.

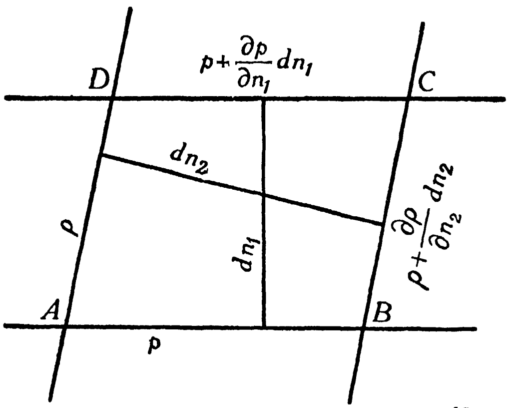


Fig. 46. Line and surface integrals for $-\int \frac{dp}{\rho}$.

The term $-\int \frac{dp}{\rho}$ taken around a closed circuit disappears if, and only if, the fluid is barotropic, and p is a function of ρ only. If p is a function of other variables, in particular of temperature, $-\int \frac{dp}{\rho}$ will not in general vanish around a closed circuit.

Let any surface be drawn through the circuit c as its boundary, and divide the area enclosed by c, by isobaric and isopycnic lines, into small parallelograms, such as $ABCD$, fig. 46. The value of $-\int \frac{dp}{\rho}$ taken around the circuit c is the sum of the values of $-\int \frac{dp}{\rho}$ taken around all the elementary circuits. Let the distances between opposite sides of the parallelogram $ABCD$ be dn_1 and dn_2, and let the angle ABC be χ.

The contributions to $-\int \frac{dp}{\rho}$ along AB and CD are zero; along BC and DA they are $\dfrac{-\frac{\partial p}{\partial n_1}dn_1}{\rho + \frac{\partial \rho}{\partial n_2}dn_2}$ and $\frac{1}{\rho}\frac{\partial p}{\partial n_1}dn_1$ respectively.

The value of $-\int \frac{dp}{\rho}$ taken around the small circuit $ABCD$ is, to the second order of small quantities,

$$\frac{1}{\rho^2}\,dp\,d\rho.$$

$ABCD$ is approximately a parallelogram, whose area, to the second order of small quantities, is

$$dn_1 dn_2/\sin\chi.$$

Hence the value of $-\int \frac{dp}{\rho}$ taken around $ABCD$ is, correctly to the second order of small quantities,

$$\frac{1}{\rho^2}\,dp\,d\rho = \frac{1}{\rho^2}\frac{\partial p}{\partial n_1}\frac{\partial \rho}{\partial n_2}\,dn_1 dn_2 = \frac{1}{\rho^2}\frac{\partial p}{\partial n_1}\frac{\partial \rho}{\partial n_2}\sin\chi \times \text{area } ABCD.$$

χ is positive when the rotation of the gradient of density into coincidence with the gradient of pressure is in the counter-clockwise direction. The total value of the integral $-\int \frac{dp}{\rho}$ taken around any circuit c may be regarded as equivalent to the sum of the contributions

$$\frac{1}{\rho^2}\frac{\partial p}{\partial n_1}\frac{\partial \rho}{\partial n_2}\sin\chi \qquad \text{......(49)}$$

per unit area of any surface bounded by the circuit. This expression was first given by Silberstein*.

The result thus derived shows that in any region where the surfaces of equal pressure and equal density intersect there is a tendency for a circulation to be set up about axes which lie along the intersections of these families of surfaces, in such a sense as to bring the gradient of density into line with the gradient of pressure.

The same result has been derived by V. Bjerknes† in a slightly different way. The whole of space is regarded as divided into a system of tubes by families of isobaric and isosteric surfaces which are separated respectively by unit difference of pressure and of density. The result expressed in (49) above is equivalent to saying that the rate of increase of circulation around the circuit c is measured by the number of unit tubes or solenoids which pass through the closed curve c. Here we prefer to follow the expression (49) for this effect, instead of the expression in terms of solenoids given by Bjerknes.

The result which is represented in equation (49) has been derived by a transformation of equations (35), (36) and (37) above. It may be represented

* *Bull. Acad. Sci. Cracow*, 1896; *Vectorial Mechanics* (Macmillan), p. 167.

† *Geofys. Publ.* **2**, No. 3.

by the diagram of fig. 48, which indicates that the result may be described as a tendency of the lighter air to seek lower pressure, and of the heavier air to seek higher pressure. This result might in fact have been inferred from equations (35), (36) and (37), in which the effect of the pressure distribution is represented by terms such as $-\frac{1}{\rho}\frac{\partial p}{\partial x}$. In a given gradient of pressure the effect is inversely proportional to the density, and so is greater for the lighter than for the denser air. The lighter air will therefore show a greater tendency to drift across the isobars into low pressure. Before considering further the interpretation of expression (49) we shall investigate some transformations of this expression into terms of other variables.

In meteorology density is not directly observed, and it is convenient to express the above result first in terms of the gradients of pressure and temperature, and finally of pressure and potential temperature, which is the most useful form.

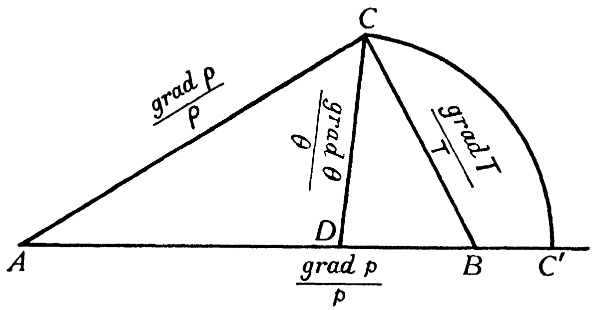


Fig. 47. The geometrical relations between the gradients of p, ρ, θ and T.

At any point in the atmosphere let p, ρ, T and θ be the pressure, density, absolute temperature, and potential temperature respectively. Then

$$p=R\rho T \qquad \text{......(50)},$$

where R is the gas-constant for dry air. Neglecting the effect of water-vapour, and taking logarithms in (50) and differentiating,

$$\frac{dp}{p}=\frac{d\rho}{\rho}+\frac{dT}{T} \qquad \text{......(51)}.$$

In the notation of vector analysis, equation (51) leads at once to

$$\frac{\text{grad } p}{p}=\frac{\text{grad } \rho}{\rho}+\frac{\text{grad } T}{T} \qquad \text{......(52)},$$

the vector relation between the gradients of pressure, density and temperature.

In fig. 47 AB, AC, CB represent $\frac{\text{grad } p}{p}$, $\frac{\text{grad } \rho}{\rho}$, and $\frac{\text{grad } T}{T}$, in magnitude and direction. The potential temperature θ, defined as the temperature which a mass of air at pressure p will take when brought adiabatically to a standard pressure p_0, is given by

$$\theta=T\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} \qquad \text{......(53)},$$

where γ is the ratio of the specific heats of air. From (53)

$$\frac{d\theta}{\theta}=\frac{dT}{T}-\frac{\gamma-1}{\gamma}\frac{dp}{p},$$

whence

$$\frac{\text{grad}\,\theta}{\theta}=\frac{\text{grad}\,T}{T}-\frac{\gamma-1}{\gamma}\frac{\text{grad}\,p}{p} \qquad \ldots\ldots(54).$$

In fig. 47, $DB=\frac{\gamma-1}{\gamma}AB$, and from (54) $\frac{\text{grad}\,\theta}{\theta}$ is represented by CD. Fig. 47 is a true vector diagram, connecting the total gradients of pressure, density, temperature and potential temperature. It can also be treated as referring to the components of these gradients in any plane.

In (49) above, the quantity $\frac{1}{\rho^2}\frac{\partial p}{\partial n_1}\frac{\partial \rho}{\partial n_2}\sin\chi$, which is the rate of increase of circulation per unit area, may be regarded as a vector perpendicular to $ABCD$, and is equal to the vector product

$$\frac{1}{\rho^2}\,\text{grad}\,p\times\text{grad}\,\rho \qquad \ldots\ldots(55),$$

where the symbol $\times$ denotes vectorial multiplication.

From (52) and (54)

$$\frac{\text{grad}\,p}{p}\times\frac{\text{grad}\,\rho}{\rho}=-\frac{\text{grad}\,p}{p}\times\frac{\text{grad}\,T}{T}=-\frac{\text{grad}\,p}{p}\times\frac{\text{grad}\,\theta}{\theta}\ \ldots\ldots(56).$$

From (55) and (56)

$$\frac{1}{\rho^2}\,\text{grad}\,p\times\text{grad}\,\rho=-\frac{1}{\rho T}\,\text{grad}\,p\times\text{grad}\,T=-\frac{1}{\rho\theta}\,\text{grad}\,p\times\text{grad}\,\theta\ \ldots\ldots(57).$$

It was shown above that in any region where the gradients of pressure and density are not parallel there is a growth of circulation or vorticity tending to rotate the density gradient towards coincidence with the pressure gradient. Since the density is a function of the pressure, it cannot be assumed that the gradient of density rotates (about A in fig. 47) without change of magnitude. Neglecting all effects due to radiation, conduction of heat, or mixing, we may assume that the changes of state of the fluid are adiabatic, so that each element retains its original potential temperature, and the gradient of potential temperature rotates with the fluid, with its scalar magnitude unchanged. Then in fig. 47 the point C revolves about D, moving towards C' on AB. If the inertia of the fluid carries C beyond C' to the other side of C', the forces called into play tend to check its motion, and to carry the point C back towards C'. If therefore the initial conditions be represented by fig. 47, the phenomena produced may be of the nature of an oscillation of the line CD about the mean position $C'D$.

The argument used above assumes that the pressure gradient remains unchanged during the motion. In the atmosphere this is a reasonable assumption to make, since we may regard the pressure distribution as superposed from above.

The phenomena of land and sea breezes afford an interesting and simple illustration of the results derived above. Suppose the atmosphere is initially at rest, with pressure uniform at mean sea level. When now the land becomes heated during the day, the rate of decrease of pressure with height is less over land than over the sea, and the isobaric surfaces slope downward from sea to land. The isosteric surfaces slope in the opposite direction, downward from land to sea. The gradients of pressure and density are as shown in fig. 49, and the tendency will be therefore to produce a circulation from sea to land. This is no more than the tendency of the cold air to push under the warm air, and in the idealised conditions we have supposed to exist the sea breeze will be normal to the shore line. So long as the land remains warmer than the sea the motion will continue. Fig. 49 only represents the initial motion of the air, and we are not concerned at this stage with the subsequent history of the air masses thus set in motion. At night when the land becomes colder than the sea the process is reversed.

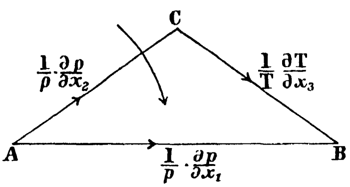


Fig. 48. The direction of apparent spin.

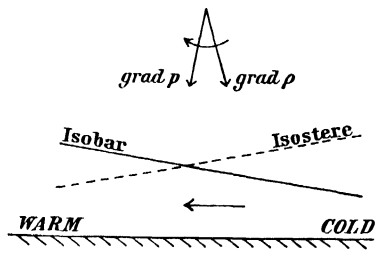


Fig. 49. Land and sea breezes.

The result stated in (49) above is a perfectly general result which appears to make no assumptions of any kind as to the nature of the fluid, and it therefore appears at first sight that the growth of cyclic motion in the atmosphere might be explained in this way. But in the atmosphere it is possible to have cyclic motion with no vorticity in the main body of fluid, as in the so-called simple vortex, and also to have vorticity without cyclic motion, as for example when there is a gradient of velocity in one direction. It is not at first sight easy to decide whether the apparent growth of circulation deduced above is purely of the nature of a shearing of the elements of fluid considered, with no tendency to produce cyclic motion.

It is important to bear in mind that the motions produced by the effect of the term $-\int\frac{dp}{\rho}$ may involve convergence or divergence which may lead to an effect in the opposite sense, as measured by $2\omega\frac{dF}{dt}$.

Take a specially simplified problem of motion in a horizontal plane, in a field of pressure represented by equidistant straight isobars, drawn parallel to the axis of x. The equations of motion may be written

$$\left.\begin{aligned}\frac{du}{dt}-2\omega\sin\phi\,.\,v&=-\frac{1}{\rho}\frac{\partial p}{\partial x}=0\\ \frac{dv}{dt}+2\omega\sin\phi\,.\,u&=-\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned}\right\}\qquad\ldots\ldots(58).$$

Consider the motion of an element of air of density ρ, whose component velocities at time $t=0$ are $u=u_0$, $v=v_0$. Multiplying the second of the above equations by $i\,(\sqrt{-1})$, and adding to the first, we find

$$\frac{d}{dt}(u+iv)+2\omega\sin\phi\,.\,i\,(u+iv)=-\frac{i}{\rho}\frac{\partial p}{\partial y} \qquad \text{......(59)}.$$

This equation is readily integrated, giving

$$u+iv=Ae^{-2\omega\sin\phi\, i(t-\beta)}-\frac{1}{2\omega\sin\phi}\frac{1}{\rho}\frac{\partial p}{\partial y},$$

or
$$u+iv=\left\{u_0+iv_0+\frac{1}{2\omega\sin\phi}\frac{1}{\rho}\frac{\partial p}{\partial y}\right\}e^{-2\omega\sin\phi\, it}-\frac{1}{2\omega\sin\phi}\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{......(60)}.$$

This equation indicates that the motion of the air as defined by the equations and conditions stipulated above will be an oscillation of the velocity about the geostrophic value $-\frac{1}{2\omega\sin\phi}\frac{1}{\rho}\frac{\partial p}{\partial y}$, except in the case when the initial velocity is equal to the geostrophic value, in which case the velocity remains constant. Equation (60) shows that the constant value about which the velocity oscillates is inversely proportional to the density ρ.

Now suppose that initially a mass of air is moving with the appropriate geostrophic velocity along the straight isobars postulated above, and that a difference of temperature is produced in any manner. The warmer and therefore the less dense portion of the air will acquire a velocity which oscillates about a higher mean value than that of the denser portion. The initial effect is obviously to cause the lighter air to move faster inwards across the isobars towards lower pressure, but the eventual result is to cause the lighter air to get further and further ahead downwind. Thus the result which we have expressed in (49) above is only applicable in the initial state, and the final result of a difference of density may be exactly opposite to the initial effect. It is thus not possible to explain the growth of large-scale horizontal circulations in the atmosphere by the reactions due to the gradients of pressure and density not being parallel to each other.

§ 105. *Velocity potential*

When the motion is irrotational, or the vorticity is zero,

$$\frac{\partial u}{\partial z}=\frac{\partial w}{\partial x},\quad \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x},\quad \frac{\partial w}{\partial y}=\frac{\partial v}{\partial z}.$$

These three equations are the conditions that

$$(u\,dx+v\,dy+w\,dz)$$

should be a complete differential. Call this $-d\phi$. Then

$$u=-\frac{\partial\phi}{\partial x},\quad v=-\frac{\partial\phi}{\partial y},\quad w=-\frac{\partial\phi}{\partial z}.$$

If the fluid is incompressible, the equation of continuity then reduces to the form

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad \text{......(61).}$$

In two-dimensional flow, we may also imagine a function $\psi(x, y)$ which gives the form of the stream lines. Along the stream lines

$$\frac{\partial\psi}{\partial x}\,dx+\frac{\partial\psi}{\partial y}\,dy=0 \qquad \text{......(62)}$$

and we may then take $\quad v=\frac{\partial\psi}{\partial x},\quad u=-\frac{\partial\psi}{\partial y}.$

The equation of continuity for two-dimensional flow is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

if the fluid is incompressible; this is automatically satisfied by the velocities deduced from ψ as shown above. The vorticity is given by

$$2\zeta=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} \qquad \text{......(63).}$$

If the flow is irrotational, $\quad \nabla^2\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0.$

In two-dimensional polars equation (63) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=2\zeta \qquad \text{......(64).}$$

The radial velocity is given by $\partial\psi/r\partial\theta$, and the transverse velocity by $\partial\psi/\partial r$. If ζ is a known function of r and θ, the equation is soluble.

We have only collected some of the more important results here. For further details see Lamb's *Hydrodynamics*, or Glauert's *Aerofoil and Airscrew Theory*.

§ 106. *The stresses in a viscous fluid*

If at a point P in the fluid we imagine three planes to be drawn perpendicular to the axes of x, y, z respectively, the stresses exerted across the first of these planes are

$$p_{xx},\quad p_{xy},\quad p_{xz}.$$

The stresses across the second plane are

$$p_{yx},\quad p_{yy},\quad p_{yz},$$

and the stresses across the third plane are

$$p_{zx},\quad p_{zy},\quad p_{zz}.$$

If the fluid is frictionless $\quad p_{xx}=p_{yy}=p_{zz}=-p.$

Taking moments about the centre of an element $dx\,dy\,dz$, we find

$$p_{xy}=p_{yx},\quad p_{yz}=p_{zy},\quad p_{zx}=p_{xz}.$$

These three relationships reduce the number of components of stress to six. The detailed development of these formulae will be found in Lamb's *Hydrodynamics* (6th edition, art. 326):

$$\left.\begin{aligned} p_{xx} &= -p - \tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x} \\ p_{yy} &= -p - \tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial v}{\partial y} \\ p_{zz} &= -p - \tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z} \end{aligned}\right\} \quad \ldots\ldots(65),$$

$$\left.\begin{aligned} p_{yz} &= p_{zy} = \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \\ p_{zx} &= p_{xz} = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ p_{xy} &= p_{yx} = \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad \ldots\ldots(66),$$

where μ is the coefficient of viscosity. A more convenient coefficient is $\frac{\mu}{\rho}=\nu$, called the kinematic coefficient of viscosity. On an element $dx\,dy\,dz$ of fluid the tractions in the direction of the x-axis are

$$\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z} = -\frac{\partial p}{\partial x}+\tfrac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\mu\nabla^2 u \ldots\ldots(67).$$

Hence in equations (39), (40) and (41) above the contributions to X, Y, Z due to the viscous forces are

$$\left.\begin{aligned} &\tfrac{1}{3}\nu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\nu\nabla^2 u \\ &\tfrac{1}{3}\nu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\nu\nabla^2 v \\ &\tfrac{1}{3}\nu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\nu\nabla^2 w \end{aligned}\right\} \quad \ldots\ldots(68).$$

When the fluid is incompressible the term in $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ is zero. In the atmosphere, which is compressible, the effect of this term is considerably smaller than that of the final term in the above equation, and the viscous forces may be regarded as accounted for with all the accuracy which is required in meteorological discussion by the terms $\nu\nabla^2 u$, $\nu\nabla^2 v$, $\nu\nabla^2 w$, respectively.

The effect of the terms such as $\nu\nabla^2 u$ is analogous to the term in the equation of heat transfer by diffusion, and indicates that in a viscous fluid there is a diffusion of fluid momentum due to the viscosity, of amount proportional to the coefficient of viscosity.

Lamb shows (*loc. cit.* pp. 579 *et seq.*) that in a viscous fluid kinetic energy is dissipated at a rate

$$\mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\} \quad \ldots(69),$$

per unit volume of the fluid. To obtain the total dissipation through a finite volume of the fluid, this expression should be integrated through the volume in question.

Lamb shows (*loc. cit.* p. 578) that the equations for the change in the components of vorticity ξ, η, ζ in a viscous fluid are

$$\left.\begin{aligned}\frac{d\xi}{dt}&=\xi\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z}+\nu\nabla^2\xi\\ \frac{d\eta}{dt}&=\xi\frac{\partial v}{\partial x}+\eta\frac{\partial v}{\partial y}+\zeta\frac{\partial v}{\partial z}+\nu\nabla^2\eta\\ \frac{d\zeta}{dt}&=\xi\frac{\partial w}{\partial x}+\eta\frac{\partial w}{\partial y}+\zeta\frac{\partial w}{\partial z}+\nu\nabla^2\zeta\end{aligned}\right\} \quad \ldots\ldots(70).$$

The first three terms on the right-hand side of each of these equations represent the change of vorticity when the vortex lines move with the fluid, the strengths of the vortices remaining constant. The last term in each equation represents the diffusion of vorticity produced by viscosity, following a law similar to the conduction of heat. It is evident from this analogy that vorticity cannot originate in the body of a fluid, but must be diffused outward from solid boundaries.

The value of ν for different temperatures is shown in a table on p. 406 below. For the purpose of rough computation it will usually suffice to adopt the value $\nu=0{\cdot}15$, in C.G.S. units.

CHAPTER IX

MOTION UNDER BALANCED FORCES: THE GRADIENT WIND

§ 107. *Conditions for steady motion*

WHEN the motion of air is purely horizontal the equations of motion are

$$\frac{du}{dt} - 2\omega v \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} + X \qquad \text{......(1)},$$

$$\frac{dv}{dt} + 2\omega u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} + Y \qquad \text{......(2)},$$

where X, Y are the components of forces other than those due to the pressure distribution. These equations state that the forces X, Y, together with the gradient of pressure and the deviating force due to the earth's rotation, balance the accelerations of the element of air in its path.

An important application of this is to the steady motion of air at heights sufficiently great to be removed from all effects due to friction and turbulence at the surface of the earth. The motion at any point is then independent of the time, or $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0$. Equations (1) and (2), in which we now put $X = Y = 0$, state that the acceleration of the air in its path will balance the combined effect of the pressure gradient and the deviating force, and we are at liberty to write the accelerations relative to the earth in any convenient form. The most convenient form is that which gives the components of acceleration along the tangent and the normal to the path, the latter being the centripetal acceleration. If V be the velocity at any point of the path where the radius of curvature is r, the total acceleration is made up of V^2/r directed towards the centre of curvature, and dV/dt or VdV/ds along the tangent to the path. The algebraic equivalence of these two expressions to du/dt and dv/dt is readily seen by writing $u = V\cos\psi$, $v = V\sin\psi$, and differentiating. The result follows immediately when $d\psi/ds$ is replaced by $1/r$.

In considering horizontal motion under balanced forces, it is the custom of most writers to omit the tangential acceleration VdV/ds, but it is not obvious that this is justifiable *a priori*. We can, however, justify it by an appeal to observations. Shaw and Lempfert found, in their study of the *Life-History of Surface Air Currents*, that these currents frequently flow for thousands of miles with no appreciable change of velocity. Also Durward*, in a study of pilot balloon observations made behind the British lines in France during the war of 1914–18, found that the changes in velocity downwind were very

* M.O., *Professional Notes*, No. 24.

slight. We shall therefore assume that the tangential acceleration is negligible, and that only the centripetal acceleration perpendicular to the path need be considered. The centripetal acceleration, of magnitude V^2/r, is directed towards the centre of curvature. This is balanced by the deviating force and the pressure gradient. The deviating force is along the same line as the resultant acceleration, being at right angles to the path. If the motion is exactly balanced, the pressure is along the same direction, and must be normal to the path. This means that the path must be tangential to the isobars, and the motion is around the isobars.

The wind which blows around the isobars, and has such a magnitude that it produces the right centripetal acceleration and deviating force to balance exactly the pressure gradient, is called the *gradient wind*. The conditions that it fulfils, which have been stated above, assume that the pressure distribution is not changing. It is therefore not justifiable to use the conception of a gradient wind in conditions when the pressure distribution is changing, without some consideration of the rate at which the pressure distribution is changing.

§ 108. *The gradient wind equation*

In fig. 50 let P be a point on the path of an element of air, and let PP' be the circle of curvature of the path at P, PP' being a small circle on the earth's surface. The points P, P' are taken at opposite points of a diameter, and N the

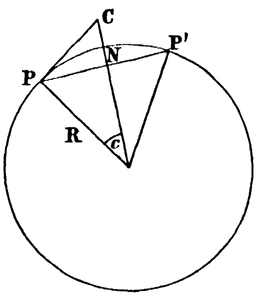


Fig. 50. Motion in a small circle.

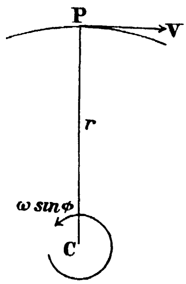


Fig. 51. Centrifugal force on an element of air.

midpoint of PP' is the centre of the small circle. The angular radius of the circle, or the angle subtended by NP at the centre of the earth, is equal to c. If R be the radius of the earth, the radius of the small circle is $R \sin c$.

The horizontal velocity of the element of air at P being V, the centripetal acceleration along PN is $V^2/R \sin c$, and the horizontal component of this along the line PC is $V^2/R \tan c$. Or, if r be the radius of curvature at P of the projection of the path of the air on the plane of the horizon at P, the horizontal acceleration along PC is V^2/r. The vertical component, directed downwards, is $V^2/R \sin c \times \sin c = V^2/R$.

The three terms in our equations of motion, which must be considered are

(*a*) the pressure gradient acting from high to low pressure, and at right angles to the isobars,

(*b*) the deviating force acting at right angles to the direction of motion, and towards the right (in the Northern hemisphere), and

(*c*) the centripetal acceleration, acting at right angles to the direction of motion, towards the concave side of the isobars.

The condition for steady motion is that these three items shall balance, or that (*c*) shall be equal to the resultant of (*a*) and (*b*). It is convenient to distinguish two cases, according as (*a*) and (*c*) are in the same or in opposite directions, i.e. according as the isobars have anticyclonic or cyclonic curvature. In the first case

$$\text{(Anticyclone)} \qquad 2\omega V \sin\phi - V^2/r = \frac{\text{pressure gradient}}{\rho} \quad \text{......(3)},$$

and in the second case

$$\text{(Cyclone)} \qquad 2\omega V \sin\phi + V^2/r = \frac{\text{pressure gradient}}{\rho} \quad \text{......(4)}.$$

Equation (4) may be regarded as included in (3), if we postulate that r shall be regarded as positive when it is directed to the right of the direction of motion, and negative when it is directed to the left of the direction of motion.

The solution of the appropriate one of equations (3) or (4) is called the *gradient wind*, and the equation is called the *gradient wind equation*. The gradient wind is such that the accelerations it calls forth are exactly balanced by the gradient of pressure. When the isobars are straight and parallel, the path of an element of air is along a great circle. The term in V^2/r in equations (3) and (4) then vanishes, and the value of V is defined by

$$2\omega V \sin\phi = \frac{\text{pressure gradient}}{\rho} \quad \text{......(5)}.$$

When the isobars are only slightly curved, it will be found that the term in V^2/r may be neglected by comparison with $2\omega V \sin\phi$. The value of V which is derived when the term V^2/r is neglected is called the *geostrophic wind*. If we denote the geostrophic wind by G, then the

$$\text{gradient of pressure} = 2\omega\rho G \sin\phi \quad \text{......(6)}.$$

We shall frequently find it convenient to express the gradient of pressure by the expression in (6). The geostrophic wind is thus the wind which, blowing around the isobars, would produce a deviating force of the right magnitude to balance exactly the pressure gradient. The use at any time of the symbol G, or references to the geostrophic wind, must not be taken to imply that the actual motion is along a great circle (or along straight isobars). The geostrophic wind is to be regarded as a first approximation to the gradient wind.

The gradient wind equation may be rewritten in the form

$$2\omega \sin\phi\,(V - G) = \pm V^2/r \quad \text{......(7)}$$

in which the upper or lower sign is used according as the original equation is

(3) or (4); the upper sign is used for anticyclonic isobars, and the lower sign for cyclonic isobars. From equation (7) it follows that G is an underestimate of the gradient wind in an anticyclone, and an overestimate in a cyclone.

In equations (3) and (4) the first term on the left-hand side is called the geostrophic component of the pressure gradient, and the second is called the cyclostrophic component. It is frequently assumed that in middle and high latitudes the first of these outweighs the second, so that the geostrophic wind may be regarded as a close approximation to the true wind, provided the pressure distribution is not changing. The validity of this assumption is discussed later, in § 112. In low latitudes, on account of the smallness of sin ϕ, the cyclostrophic component is the more important, and the geostrophic component is then negligible. Equation (3) is then replaced by

$$\frac{V^2}{r} = \frac{\text{pressure gradient}}{\rho} = \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{......(8)},$$

which gives the velocity required to produce a balance between the gradient of pressure and the centripetal acceleration. The left-hand side of this equation must always be positive, and steady cyclostrophic motion is only possible when $\partial p/\partial r$ is positive, or the pressure distribution is cyclonic. Equation (8) cannot be satisfied when $\partial p/\partial r$ is negative, or with an anticyclonic distribution of pressure. This conclusion is to be interpreted as indicating that closed anticyclonic isobars are not possible as stable systems in very low latitudes.

§ 109. *Comparison of the gradient wind with observed winds*

The first detailed comparison of the gradient wind with observed winds was made by E. Gold*, who found that at a height of 500 metres above the ground the direction of the wind was almost exactly along the mean sea level isobars, while the velocity was slightly below the computed gradient wind. The agreement was found to be sufficiently good to justify the adoption of the gradient wind as a close approximation to the actual wind, within the limits of the errors of observation. In Gold's investigation the term V^2/r was computed for each individual occasion, but r was taken as the curvature of the isobars, and not of the actual path of the air. The curvature of the path of the air is not the same as the curvature of the isobars, except when the conditions are unchanging. Gold found† that in a cyclone the radius of the curvature of the path exceeded that of the isobars, at points to the right-hand side of the path of the centre, and was less than that of the isobars at points to the left-hand side of the path of the centre; in an anticyclone the inequalities were reversed. But the main value of Gold's investigation is its confirmation that the direct use of the synoptic chart gives a reasonably close approximation to the observed wind.

The geostrophic wind is readily evaluated by the use of a scale graduated in miles per hour or metres per second, which is placed normally across the

* M.O. 190. *Barometric Gradient and Wind Force.* † *Ibid.* p. 42.

isobars at the place for which the geostrophic wind is required. The scale is graduated on a scale of reciprocals of the velocities, and the nearer the isobars the greater is the gradient of pressure, and the greater the geostrophic wind. The scale is graduated for a standard density, but the small correction required to allow for variations in density is readily made.

The computation of the gradient wind is more difficult, since it requires first an estimate of the curvature of the path of the air, and secondly the solution of a quadratic equation. The result of such a computation is somewhat unreliable on account of the uncertainty of any estimate of the curvature of the path, which is not of necessity the same as the curvature of the isobars, and no serious attempt has been made to use the gradient wind in practical meteorology. It is partly for this reason that the geostrophic wind has been adopted as the most convenient approximation to the wind at heights of 500 or 1000 metres.

Attention is again directed to the fact that the gradient wind has no definite meaning when the pressure distribution is changing, and the motion is therefore not under balanced forces. The principle of motion under balanced forces was first elaborated by Sir Napier Shaw, in the *Manual of Meteorology*, **4**, where it is maintained that the adjustment of the wind to the pressure gradient is automatic and complete. If this were so the motion would be under balanced forces, and the gradient wind equation would be applicable.

§ 110. *The solution of the gradient wind equation*

The gradient wind equation has been given in several forms in equations (3), (4) and (7) above. The equation is a quadratic, and thus has two solutions. When the isobars are straight the quadratic degenerates into a simple equation whose solution is $V=G$, and the wind becomes equal to the geostrophic wind. Any solution of the quadratic which has a real physical meaning must therefore give $V=G$ when the radius of curvature r is increased indefinitely.

We shall consider first the equation which is appropriate to anticyclonic isobars, which have high pressure on the concave side. The equation is

$$\frac{V^2}{r} - 2\omega \sin\phi\,(V-G) = 0 \qquad \text{......(9)}.$$

Solving this in the usual way, we find

$$V = r\omega \sin\phi \left\{1 \mp \sqrt{1 - \frac{2G}{r\omega \sin\phi}}\right\} \qquad \text{......(10)}.$$

Expanding the expression under the radical, we find

$$V = r\omega \sin\phi \left\{1 \mp \left(1 - \frac{G}{r\omega \sin\phi} - \frac{\frac{1}{2}G^2}{r^2\omega^2 \sin^2\phi} - \ldots\right)\right\} \text{......(11)}.$$

The upper sign gives $V = G + \frac{1}{2}G^2/r\omega \sin\phi + \ldots$ etc. ......(12);

which gives $V=G$ when $r=\infty$, and so yields a solution which is continuous near straight isobars. The lower sign gives

$$V=2r\omega \sin\phi - G - \tfrac{1}{2}G^2/r\omega \sin\phi - \text{etc.} \qquad \text{......(13).}$$

For indefinitely great radius of curvature r, equation (13) yields indefinitely great velocities, and therefore this solution of the gradient wind equation requires an infinite discontinuity near straight isobars, since in a system of straight isobars the wind is finite and equal to G. Such an arrangement would also require an infinite amount of energy to set it in motion, and this is not available in the earth's atmosphere.

Equation (13) requires that the velocity should decrease as the gradient increases, while equation (12) requires that the velocity and the gradient should increase together. When the gradient is zero, and G zero, equation (13) appears at first sight to demand a finite velocity. But if there is no gradient of pressure, no isobars can be drawn, and r in equation (13) cannot then be defined, so that the second solution then becomes indeterminate. The known facts of observation are in direct contradiction to the results here derived from a consideration of equation (13).

Thus the only solution of equation (9) which can extend out to a region of straight isobars is given by equation (10) with the negative sign before the radical. In an anticyclone, therefore, the only physically significant solution of the gradient wind equation is

$$V=r\omega \sin\phi \{1-\sqrt{1-2G/r\omega \sin\phi}\} \qquad \text{......(14).}$$

It should be noted that $V<r\omega \sin\phi$. Now the rate of rotation of the horizon about the vertical is $\omega \sin\phi$. If, in fig. 51, C is the centre of curvature of the isobar at the point P, and $CP=r$, the rate of rotation relative to the earth of the air at P about the point C is V/r, and thus the rate of rotation of the air in space is $(\omega \sin\phi - V/r)$ in a counter-clockwise direction. This is a positive quantity since $V<r\omega \sin\phi$. Thus we arrive at the result that the solution of the gradient wind equation which is significant represents a motion which is counter-clockwise in space. The anticyclone therefore rotates in space in the same direction as the earth, but as its rate of rotation is less than that of the earth, it appears to an observer on the earth to be a clockwise circulation. This fact will be reconsidered in some of its dynamical bearings later, in Chapter XIX.

The solution which we rejected as physically inappropriate, having the lower sign in equation (10), gives $V>r\omega \sin\phi$, and consequently represents a circulation which rotates faster than the earth beneath it, so that it is a clockwise rotation in space, and therefore in the opposite sense to the earth's rotation. There is no mechanism in the atmosphere capable of producing large-scale rotations in the opposite sense to the earth's rotation, and the second solution should therefore be regarded as an algebraic accident which has no physical significance.

For the cyclone, the gradient wind equation is the same as that for the anticyclone, with the sign of r changed. The solution of the equation is therefore

$$V=r\omega \sin\phi \{\sqrt{1+2G/r\omega \sin\phi}-1\} \qquad \text{......(15).}$$

This solution is continuous near straight isobars, being in fact the solution given in (14) with the sign of r changed. Since the circulation represented by (15) is counter-clockwise relative to the earth it is also counter-clockwise in space, and thus has in space the same sense of rotation as the earth. The rejected solution for a cyclone would have a negative sign in front of the radical in (15), and would therefore represent a clockwise rotation relative to the earth, the velocity being then greater than $r\omega \sin \phi$ (arithmetically). This solution would thus represent a circulation which has in space the sense of rotation which is opposed to that of the earth.

Thus the rejected solution of the gradient wind equation for cyclone or anticyclone represents a circulation whose direction is opposite to that of the earth beneath it; and we may be satisfied that in rejecting it we are rejecting a hypothetical system which cannot come into existence on a large scale, and which has never been observed in the earth's atmosphere.

A point of some dynamic interest in connection with the solutions derived above is that for the anticyclone V is always less than $r\omega \sin \phi$ while for the cyclonic system there is no limit to the value of V so far as the algebraic form of the equation is concerned.

§ 111. *Direct derivation of the gradient wind equation*

In fig. 52 let C be the centre of curvature of the isobar at P. Let the velocity at P be V, in the clockwise direction, so that we restrict the discussion for the

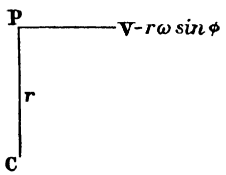


Fig. 52. Direct derivation of the gradient wind equation.

moment to the case of anticyclonic isobars. Let the angular velocity about C of the air at P be ζ, so that $V = r\zeta$. The angular velocity in space about C of the air at P is $(\omega \sin \phi - \zeta)$, while the angular velocity in space of the observer on the earth's surface at P about the point C is $\omega \sin \phi$. The condition from which we derive the equation of motion may be stated thus: "The gradient of pressure balances the acceleration of the air relative to the system in which the pressure is observed, or relative to the observer who measures the pressure".

Now the acceleration of the air at P relative to the observer at P on the earth's surface

= Acceleration of the air at P relative to C

– Acceleration of the observer at P relative to C ......(16)

$$= r(\omega \sin \phi - \zeta)^2 - r\omega^2 \sin^2 \phi = -r\zeta(2\omega \sin \phi - \zeta) \qquad \text{......(17).}$$

The acceleration is along PC, and must be balanced by the pressure gradient. Hence

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -r\zeta\,(2\omega\sin\phi - \zeta)$$

$$= -2\omega\sin\phi\,.\,V + V^2/r$$

or

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{\text{pressure gradient}}{\rho} = 2\omega\sin\phi\,.\,V - V^2/r \qquad \text{......(18)},$$

which is a repetition of equation (3) above.

The cyclonic counterpart is derived by drawing PC in the opposite direction to that drawn in the figure, and this has only the effect of changing the sign of r in the whole of the analysis, leading to equation (4) instead of (3). This may be regarded as an independent derivation of the magnitude and direction of the so-called "deviating force".

There is one interesting detail in the above derivation of the gradient wind equation which is worthy of note. In (16) one might be tempted to overlook the second term, which is equivalent to overlooking the need to refer the accelerations to the same system of co-ordinates as that in which the pressure gradients are measured. This is perhaps the only point in meteorology at which relativity of motion demands close attention. The omission of the second term in (16) is equivalent to the omission of the second term in (17), of which the remaining term is always positive, and we appear to reach the conclusion that pressure will increase outwards from the centre in all circulations, whether clockwise or counter-clockwise, and that centres of high pressure are impossible.

§ 112. *The effect of changing pressure distribution*

If $\omega\sin\phi = l$, the equations of motion for purely horizontal motion may be written

$$\frac{du}{dt} - lv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{......(19)},$$

$$\frac{dv}{dt} + lu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{......(20)}.$$

Multiplying the second equation by $i = \sqrt{-1}$, and adding, we obtain a single equation

$$\frac{d}{dt}(u + iv) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x} + i\frac{\partial p}{\partial y}\right) - il\,(u + iv),$$

or

$$u + iv = \frac{i}{l}\frac{d}{dt}(u + iv) + \frac{i}{l\rho}\left(\frac{\partial p}{\partial x} + i\frac{\partial p}{\partial y}\right) \qquad \text{......(21)}.$$

Differentiating equation (21) partially with respect to time, we find

$$\frac{\partial}{\partial t}(u + iv) = \frac{i}{l}\frac{\partial}{\partial t}\frac{d}{dt}(u + iv) + \frac{i}{l\rho}\left(\frac{\partial \dot{p}}{\partial x} + i\frac{\partial \dot{p}}{\partial y}\right) \qquad \text{......(22)},$$

where ρ is assumed constant and $\dot{p} = \dfrac{\partial p}{\partial t}$.

Substituting for $\frac{\partial}{\partial t}(u+iv)$ in equation (21)

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)(u+iv)+il\,(u+iv)$$
$$=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}+i\frac{\partial p}{\partial y}\right)-\frac{i}{l\rho}\left(\frac{\partial \dot{p}}{\partial x}+i\frac{\partial \dot{p}}{\partial y}\right)-\frac{i}{l}\frac{\partial}{\partial t}\frac{d}{dt}(u+iv) \quad \ldots\ldots(23).$$

This equation was given by Brunt and Douglas*, who showed that in general the third term on the right-hand side of the equation may be neglected by comparison with the others. The first term on the left-hand side represents the ordinary accelerations of the air in its path, and if, following the same authors, we neglect these terms, we are left with

$$u+iv=\frac{i}{l\rho}\left(\frac{\partial p}{\partial x}+i\frac{\partial p}{\partial y}\right)-\frac{1}{l^2\rho}\left(\frac{\partial \dot{p}}{\partial x}+i\frac{\partial \dot{p}}{\partial y}\right) \quad \ldots\ldots(24).$$

Here $\dot{p}$ is the rate of change of pressure, and to a rough degree of approximation the field of $\dot{p}$ may be taken as the field of isallobars obtained by plotting the barometric tendencies, which give the changes of pressure during three hours. The second term on the right-hand side of equation (24) thus represents a component of velocity of magnitude proportional to the gradient of the isallobars, and directed normally across the isallobars into the low values. The first term on the right-hand side is merely another way of expressing the geostrophic wind.

Thus we derive the result that the effect of changing pressure distribution can be allowed for by adding to the geostrophic wind a component blowing across the isallobars at right angles into the low values of isallobars, with a magnitude proportional to the gradient of the isallobars. A scale similar to a geostrophic wind scale can readily be made for measuring the magnitude of this component, or the ordinary geostrophic wind scale may be used for the purpose, the results directly read off being corrected by a factor appropriate to the scale of the charts, and the units in which they are plotted.

Brunt and Douglas found that the isallobaric component of wind frequently amounted to as much as 5 metres/sec, and they accounted for the rather unexpected course of development of the pressure distribution on a number of occasions by the effect of this component. Some further consideration will be given in a later chapter to the effect of this component on weather.

It was necessary to make a number of simplifying assumptions above in order to reduce the mathematical aspect of the question to a reasonably tractable form. If we refer back to equation (23) above, and only assume that the third term on the right-hand side may be neglected, while retaining the first term on the left-hand side, we may describe the resulting form of equation (23) as the standard form of the gradient wind equation, with the addition to the pressure gradient of a force directed around the isallobars, of a magnitude proportional to the gradient of the isallobars.

* *Mem. R. Met. Soc.* **3**, No. 22.

CHAPTER X

SURFACES OF DISCONTINUITY

§ 113. *The slope of a surface of discontinuity with steady motion*

It was first shown by Helmholtz* that currents of air at different temperatures, moving with different velocities, could remain in steady motion with a surface of discontinuity separating them, this surface being inclined to the horizontal at an angle whose magnitude depends upon the differences of temperature and velocity on the two sides of the surface. We shall first discuss briefly the relation of the discontinuities of temperature and velocity to the slope of the surface for the case of steady motion in the horizontal. This is admittedly a simple case, which may not be reproduced very frequently in practice, but its consideration will be of value as a preliminary to the discussion of more complicated cases. We shall, however, bear in mind that the analysis will not be strictly applicable to the more frequent phenomenon of warm air climbing up a sloping surface above cold air.

In fig. 53 let OX, assumed to be a straight line, be the intersection of the surface of discontinuity with the horizontal plane, and let OY be at right angles with OX in the horizontal plane, while the axis OZ is vertical. Let OS be the intersection of the surface of discontinuity with the plane YOZ, the angle YOS being α. Let ρ_1, u_1, and ρ_2, u_2, be the densities and velocities parallel to OX, of the air on the two sides of the surface of discontinuity. The velocities are taken to be both positive in the direction OX. Further let OX make an angle β with the East-West line, so that the components of velocity towards East are $u_1 \cos \beta$ and $u_2 \cos \beta$.

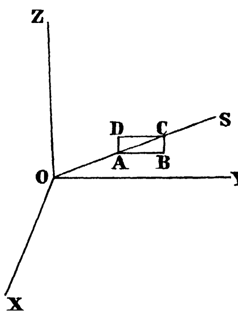


Fig. 53. Surface of discontinuity.

In the plane YOZ draw a small rectangle $ABCD$, whose sides are dy and dz. If p_a, p_b, p_c, p_d denote the pressures at A, B, C, D, respectively,

$$p_c - p_a = p_b - p_a + p_c - p_b = \left(\frac{\partial p}{\partial y}\right)_1 dy + \left(\frac{\partial p}{\partial z}\right)_1 dz,$$

$$= p_c - p_d + p_d - p_a = \left(\frac{\partial p}{\partial y}\right)_2 dy + \left(\frac{\partial p}{\partial z}\right)_2 dz,$$

since the pressure is continuous across the surface. Hence

$$\tan \alpha = \frac{dz}{dy} = -\frac{\left(\frac{\partial p}{\partial y}\right)_1 - \left(\frac{\partial p}{\partial y}\right)_2}{\left(\frac{\partial p}{\partial z}\right)_1 - \left(\frac{\partial p}{\partial z}\right)_2} \quad \text{......(1).}$$

* *Wissenschaftliche Abhandlungen*, **3.**

Equations (39), (40) and (41) of p. 166 will apply separately to each of the air masses. It is further assumed that the motion is purely horizontal and parallel to the axis of x, and that frictional forces need not be considered, so that $w=v=0$, and $X=Y=Z=0$. For steady linear motion, in each air mass separately

$$2\omega u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{......(2)},$$

$$g - 2\omega u \cos\phi \cos\beta = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{......(3)}.$$

By substitution for $\partial p/\partial y$, $\partial p/\partial z$ from (2) and (3) in (1), we find

$$\tan\alpha = -\frac{2\omega \sin\phi\,(\rho_1 u_1 - \rho_2 u_2)}{g(\rho_1-\rho_2) - 2\omega\cos\phi\cos\beta\,(\rho_1 u_1 - \rho_2 u_2)} \qquad \text{......(4)}.$$

When the vertical component of acceleration is neglected, the last equation reduces to

$$\tan\alpha' = -\frac{2\omega \sin\phi\,(\rho_1 u_1 - \rho_2 u_2)}{g(\rho_1-\rho_2)} \qquad \text{......(5)}.$$

The angle α' may be regarded for the moment as an approximation to α. When $\beta=\pi/2$ and the discontinuity runs along the North-South line, α and α' are equal.

If it is assumed that the differences of density are due to differences of temperature alone, so that at any point on the surface of discontinuity $\rho_1 T_1 = \rho_2 T_2$, equation (4) becomes

$$\tan\alpha = -\frac{2\omega \sin\phi\,(u_1 T_2 - u_2 T_1)}{g(T_2 - T_1) - 2\omega\cos\phi\cos\beta\,(u_1 T_2 - u_2 T_1)} \qquad \text{......(6)}.$$

§ 114. *Discontinuity of velocity alone; density uniform*

Here $\rho_1=\rho_2$, and equation (4) reduces to

$$\tan\alpha = \tan\phi \sec\beta \qquad \text{......(7)}.$$

When the intersection of the surface of discontinuity with the earth's surface runs West-East, and $\beta=0$, $\tan\alpha=\tan\phi$, and the surface of discontinuity is parallel to the earth's axis. When the surface of discontinuity runs North-South, $\beta=\pi/2$, and $\tan\alpha$ is infinite, so that the surface of discontinuity is vertical. These are two special cases of a general rule that when the discontinuity is one of velocity alone, the surface of discontinuity is parallel to the earth's axis. This result is readily derived from a consideration of the direction cosines of the polar axis of the earth referred to the co-ordinate system used above. They are $\cos\phi\sin\beta$, $\cos\phi\cos\beta$ and $\sin\phi$. The direction cosines of the normal to the surface of discontinuity are 0, $-\sin\alpha$ and $\cos\alpha$. The two are therefore at right angles since

$$-\sin\alpha\cos\phi\cos\beta + \sin\phi\cos\alpha = 0$$

from equation (6).

§ 115. *Discontinuity of density alone*

Let $u_1 = u_2 = u$, and $\rho_1 \neq \rho_2$. Equation (4) now becomes

$$\tan\alpha = \frac{2\omega \sin\phi . u}{2\omega\cos\phi\cos\beta . u - g}$$

$$= \frac{\tan\phi}{1 - g/2\omega\cos\phi\cos\beta . u}.$$

The value of α depends on the value of $g/2\omega\cos\phi . u$. The values of u which occur in the atmosphere are of the order of 10 metres per second or 10^3 cm/sec. In C.G.S. units, $g = 1000$, $\omega = 7{\cdot}3 \times 10^{-5}$,

$$\frac{g}{2\omega u} = \frac{10^3 \times 10^5}{2 \times 7{\cdot}3 \times 10^3} = 7 \times 10^3 \text{ approximately.}$$

Hence $\tan\alpha$ is a very small fraction of $\tan\phi$, which means that the lighter fluid will float on the heavier with a practically horizontal surface of separation.

§ 116. *Approximate form of equation* (4) *or* (6)

The first point to be decided is the relative magnitude of the two terms in the denominator of equation (4) or (6). The difference between u_1 and u_2 is usually of the same order of magnitude as either of the two velocities, so that only a small error is involved in writing $T(u_1 - u_2)$ for $u_1 T_2 - u_2 T_1$, T being the mean of T_1 and T_2. The denominator then becomes

$$g(T_2 - T_1) - 2\omega\cos\phi\cos\beta . T(u_1 - u_2).$$

Let $\qquad u_1 - u_2 = 1 \text{ m/sec} = 10^2 \text{ cm/sec},$

and let $\qquad T_2 - T_1 = 1^\circ \text{ C}, \quad T = 275^\circ, \quad \omega\cos\phi = 4 \times 10^{-5},$

$$\frac{g(T_2 - T_1)}{2\omega\cos\phi . T(u_1 - u_2)\cos\beta} = \frac{10^3}{2 \times 4 \times 10^{-5} \times 275 \times 10^2 \cos\beta} = 450 \sec\beta$$

approximately.

The ratio decreases as the temperature difference decreases, or as the velocity difference increases, but for all practical purposes, where the differences of temperature and velocity are appreciable, and of the orders of magnitude which actually occur in the atmosphere, the first term in the denominator will always far exceed the second term, and equation (6) may be written

$$\tan\alpha = -\frac{2\omega\sin\phi\,(u_1 T_2 - u_2 T_1)}{g(T_2 - T_1)} \qquad \text{......(8).}$$

For most purposes it is legitimate to simplify it still further to

$$\tan\alpha = \frac{2\omega\sin\phi}{g}\,\frac{T(u_1 - u_2)}{T_1 - T_2} \qquad \text{......(9).}$$

This equation shows that the slope of the surface of discontinuity increases in direct proportion to the difference of the wind velocities on the two sides, and in inverse proportion to the difference of temperatures.

To obtain an idea of the slope of the surface of discontinuity the values of T_1, T_2, u_1, u_2 used in the preceding computation may be inserted in equation (9). Then

$$\tan \alpha = 2 \times 7{\cdot}3 \times 10^{-5} \sin \phi \times 275 \times 10^2 \times 10^{-3}$$
$$= 4 \times 10^{-3} \sin \phi$$

and $\alpha = 10'$ approximately. If Δu and ΔT represent the differences of velocities in metres per second and in degrees C respectively,

$$\tan \alpha = 4 \times 10^{-3} \sin \phi \times \Delta u / \Delta T.$$

Δu is usually of the order of 10, and ΔT may be about 5° C, or may be as small as ½° C. With the differences which usually occur α is never likely to exceed a few degrees, even in extreme cases, and in practice it is usually found to be a small fraction of a degree; except that when $\tan \alpha$ is negative, it is $180° - \alpha$ which is a small angle. The inclination of the surface of discontinuity to the horizontal is however a small angle, if the difference of temperature is such as to be appreciable.

The approximation involved in the neglect of the second term in the denominator of equation (4) or (6) is only of importance when the difference of temperature is very slight. There is a possible small value of $T_1 - T_2$ which will give a vertical surface of separation between the two currents, and between this small difference of temperature and zero difference there is a wide range of possible slope of the surface.

§ 117. *General nature of the results derived above*

Apart from the slight approximation involved in equation (9), the result is readily stated verbally. Imagine the observer placed within the colder current looking towards the warmer current. If he is situated somewhere on OY in fig. 53, then $T_1 < T_2$. If the velocities are measured positive to his left, i.e. along the positive direction of OX, then the condition that $\tan \alpha$ shall be positive, and that the cold air shall lie as a wedge under the warm air, is that the warm air shall move more rapidly than the cold air towards the observer's left. If the cold air moves the more rapidly towards the observer's left, $\tan \alpha$ is negative and the warm air lies as a wedge below the cold air. (In the Southern hemisphere for "left" read "right" above.)

Thus at the surface of separation of a cold easterly current with a warm westerly current to the south of it, or of a warm cold northerly wind with a warm southerly wind to west of it, the cold air will lie as a wedge under the warm air. Also at the surface of separation of a cool westerly wind with a warm easterly wind south of it, or of a cold northerly wind with a warm southerly wind east of it, the warm air lies as a wedge below the cold air. If the cold air is to lie in the form of a wedge beneath the warm air, one of the following must hold:

(*a*) If the cold air lies to the north of the warm air, it must have the greater velocity towards west.

(*b*) If the cold air lies south of the warm air, it must have the greater velocity towards east.

(*c*) If the cold air lies west of the warm air, it must have the greater velocity towards south.

(*d*) If the cold air lies east of the warm air, it must have the greater velocity towards north.

Thus it is possible to have a cone of cold air with a centre of high pressure and an anticyclonic circulation, surrounded by warm air on all sides if the anticyclonic circulation is stronger in the cold air than in the warm air around it. The same is obviously true of a long tongue of high pressure colder than its surroundings. The circulation in the cold air must be more vigorous than that in the surrounding warm air. Similarly a cone or tongue of cold air can be maintained with a sharp surface of discontinuity between it and the surrounding warm air when its circulation is cyclonic if the circulation in the cold air is less vigorous than that in the surrounding warm air.

An interesting result which appears from the above is that at the boundary between the westerly winds of middle latitudes and the trade winds the cooler westerly winds should flow over the warmer trade winds. There is also some interest in considering for this case the application of the accurate equation (6). Here $T_2 > T_1$, $u_1 - u_2 > 0$, and $u_1 T_2 - u_2 T_1 > 0$. Hence in equation (6) the two terms in the denominator are of opposite sign. For a given value of $u_1 - u_2$ there is a definite small value of $T_2 - T_1$ which gives an infinite value of $\tan \alpha$, or a vertical surface of separation. Thus as the difference of temperature decreases the surface of separation rises rapidly, passing through the vertical for a certain small value of $T_2 - T_1$, and then falling to a slope parallel to the earth's axis when the difference of temperature becomes zero. The boundary between a westerly current and an easterly current to the south of it is therefore liable to be very unstable when the difference of temperature is very small.

A further point to be noted from equation (9), which is sufficiently accurate for all practical purposes, is that the slope of the surface of discontinuity does not depend on the actual velocities u_1 and u_2. No appreciable change is produced by any changes in u_1 and u_2 provided the difference $u_1 - u_2$ remains constant. The winds on the two sides of the surface of discontinuity may therefore be in the same or in opposite directions.

§ 118. *Extension to motion in small circles, with a surface of discontinuity*

For steady motion with a curved surface of discontinuity the intersection of the surface of discontinuity with the horizontal plane must be an isobar. Let the radius of curvature be r, then the equation

$$2\omega \sin \phi \, . \, u = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

must be replaced by
$$2\omega \sin \phi \, . \, u \pm u^2/r = -\frac{1}{\rho} \frac{\partial p}{\partial r}.$$

The sign is + or − according as the curvature of the isobars is cyclonic or anticyclonic. The transformation of the equations in § 113 to meet the conditions now specified is readily made by adding a term $\pm u^2/r$ wherever the term $2\omega \sin\phi . u$ occurs. The result is

$$\tan\alpha = \frac{2\omega \sin\phi\,(T_2u_1 - T_1u_2) \pm \frac{1}{r}(T_2u_1^2 - T_1u_2^2)}{g\,(T_1 - T_2) + 2\omega \cos\phi \cos\beta\,(T_2u_1 - T_1u_2)} \quad \text{......(10).}$$

The second term in the denominator is negligible by comparison with the first when the change of temperature in crossing the surface is greater than about $\frac{1}{2}^\circ$ C, since in general the difference $T_2 - T_1$ is only a small fraction of T_1 or T_2, while the difference $u_1 - u_2$ is of the same order as either velocity. The numerator can therefore be written

$$T\,(u_1 - u_2)\left\{2\omega \sin\phi \pm \frac{u_1}{r} \pm \frac{u_2}{r}\right\} \quad \text{......(11).}$$

From § 110 it follows that algebraically it is always true that $u_1/r < \omega \sin\phi$ and $u_2/r < \omega \sin\phi$, so that the quantity in the second bracket is always positive. Hence the imposition of curvature on the motion cannot change the sign of $\tan\alpha$, though it increases $\tan\alpha$ in cyclonic, and decreases it in anticyclonic curvature.

§ 119. *The form of the isobaric surfaces*

We might use the general equations of motion to obtain the form of the isobaric surfaces by writing

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz \quad \text{......(12),}$$

and substituting for $\partial p/\partial x$, etc., then integrating the resulting equations in the general case. For steady motion this is unnecessary, and we may assume that the geostrophic relation is satisfied to a sufficient degree of approximation. The pressure increases from right to left across each current. Thus the surface of separation of two currents in opposite directions, with the cold current lying in a wedge under the warm current, will correspond to a trough of low pressure; while the surface of separation of two currents in opposite directions with the warm air lying as a wedge under the cold air will correspond to a ridge of high pressure. When the two currents of different temperatures are in the same direction but with different velocities the gradient of pressure will be in the same direction, but there will be a discontinuous change of the pressure gradient at the surface of separation.

§ 120. *The general equations for the slope of surfaces of discontinuity*

So far only stationary surfaces of discontinuity have been considered. When the assumption that the surfaces are stationary is abandoned it is necessary to use the full equations of motion (39), (40) and (41) of Chapter VIII,

p. 166, substituting the values of $\partial p/\partial y$, $\partial p/\partial z$ from these equations in the equation

$$\tan\alpha = -\frac{\left(\frac{\partial p}{\partial y}\right)_1 - \left(\frac{\partial p}{\partial y}\right)}{\left(\frac{\partial p}{\partial z}\right)_1 - \left(\frac{\partial p}{\partial z}\right)_2} \qquad \text{......(13).}$$

Putting $X = Y = Z = 0$, and so neglecting all forces other than the pressure forces and gravitation,

$$\tan\alpha = -\frac{(\rho_1\dot{v}_1 - \rho_2\dot{v}_2) - 2\omega\cos\phi\sin\beta\,(\rho_1 w_1 - \rho_2 w_2) + 2\omega\sin\phi\,(\rho_1 u_1 - \rho_2 u_2)}{g(\rho_1 - \rho_2) + (\rho_1\dot{w}_1 - \rho_2\dot{w}_2) - 2\omega\cos\phi\cos\beta\,(\rho_1 u_1 - \rho_2 u_2) + 2\omega\cos\phi\sin\beta\,(\rho_1 v_1 - \rho_2 v_2)} \qquad \text{......(14).}$$

For unaccelerated horizontal motion and a stationary surface $w = v = 0$, and $\dot{w} = \dot{v} = 0$, and equation (14) reduces to the earlier form given in equation (4). From equation (39), p. 166, since the pressure is continuous at the surface of discontinuity, and therefore

$$\left(\frac{\partial p}{\partial x}\right)_1 = \left(\frac{\partial p}{\partial x}\right)_2,$$

$$\rho_1\dot{u}_1 - \rho_2\dot{u}_2 = -2\omega\cos\phi\cos\beta\,(\rho_1 w_1 - \rho_2 w_2) + 2\omega\sin\phi\,(\rho_1 v_1 - \rho_2 v_2) \qquad \text{......(15).}$$

Equations (14) and (15) are extremely complicated and require for their strict application a complete knowledge of all three components of velocity and of acceleration. In practice considerable simplification is possible. In the denominator of (14) the term $g(\rho_1 - \rho_2)$ will by far exceed the other terms in magnitude, provided the difference in temperature of the two currents is not less than about $\frac{1}{2}°$ C. We shall therefore omit all the other terms in the denominator of equation (14), bearing in mind that by doing so we leave out of consideration all cases where the difference of temperature is very small. This is not a serious practical loss, as cases of small differences of temperature cannot be recognised on a synoptic chart. Also, as we only seek to obtain an idea of the general nature of the phenomena, we shall neglect the term in $\rho_1 w_1 - \rho_2 w_2$ in the numerator of (14) by comparison with $\rho_1 u_1 - \rho_2 u_2$. This is legitimate in view of the fact that the motion is very nearly horizontal. We then have

$$\rho_1\dot{v}_1 - \rho_2\dot{v}_2 = -2\omega\sin\phi\,(\rho_1 u_1 - \rho_2 u_2) - g\tan\alpha\,(\rho_1 - \rho_2) \qquad \text{......(16).}$$

From the same considerations equation (15) is reduced to

$$\rho_1\dot{u}_1 - \rho_2\dot{u}_2 = 2\omega\sin\phi\,(\rho_1 v_1 - \rho_2 v_2) \qquad \text{......(17).}$$

There is a simple kinematical relationship between the values of v and w on the two sides of the surface of discontinuity, if it may be assumed that the surface moves forward without appreciable change of shape, and that the density of either fluid is not changing. Let v_f be the forward horizontal velocity of the surface. Then if in fig. 54 ABC be a small right-angled triangle whose sides dx, dz are respectively parallel to the axes of x and z, the net flow into a wedge of unit length parallel to the x-axis, of which the triangle ABC is the vertical section, is zero in the absence of changes of density,

$$w\,dx = (v - v_f)\,dz,$$

$$v - v_f = w \cot\alpha, \quad \text{or} \quad w = (v - v_f)\tan\alpha \qquad \text{......(18).}$$

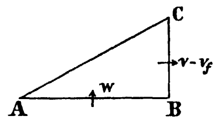


Fig. 54. Velocities at a surface of discontinuity.

From this it follows that

$$\rho_1 w_1 - \rho_2 w_2 = (\rho_1 v_1 - \rho_2 v_2)\tan\alpha + (\rho_1 - \rho_2)\,v_f \tan\alpha \qquad \text{......(19).}$$

If the surface moves forward at a uniform velocity, we find by differentiation of (19)

$$\rho_1 \dot{w}_1 - \rho_2 \dot{w}_2 = (\rho_1 \dot{v}_1 - \rho_2 \dot{v}_2)\tan\alpha \qquad \text{......(20),}$$

$$= -2\omega\sin\phi\,(\rho_1 u_1 - \rho_2 u_2)\tan\alpha - g\,(\rho_1 - \rho_2)\tan^2\alpha \qquad \text{......(21).}$$

Differentiating (16) and substituting from (17), we find

$$\rho_1 \ddot{v}_1 - \rho_2 \ddot{v}_2 = -2\omega\sin\phi\,(\rho_1 \dot{u}_1 - \rho_2 \dot{u}_2)$$

$$= -(2\omega\sin\phi)^2\,(\rho_1 v_1 - \rho_2 v_2) \qquad \text{......(22),}$$

which can be integrated giving

$$\rho_1 v_1 - \rho_2 v_2 = A\cos\{(2\omega\sin\phi)\,t - \epsilon\} \qquad \text{......(23).}$$

By substituting in equations (17) and (19) from (23) it can be shown that $\rho_1 u_1 - \rho_2 u_2$ and $\rho_1 w_1 - \rho_2 w_2$ also have the same form, so that the surface of discontinuity is stable for the special type of disturbance which does not involve deformations of the discontinuity itself.

If the surface is accelerated, so that v_f is not constant while the inclination α remains unaltered,

$$\rho_1 \dot{w}_1 - \rho_2 \dot{w}_2 = (\rho_1 \dot{v}_1 - \rho_2 \dot{v}_2)\tan\alpha + (\rho_1 - \rho_2)\,\dot{v}_f \tan\alpha \qquad \text{......(24).}$$

At the typical warm front surface of a depression (see Chapter XVIII) the motion of air is as shown in fig. 55(*a*). Here

$$v_1 - v_f < 0, \quad w_1 < 0, \qquad v_2 - v_f > 0, \quad w_2 \gtrless 0,$$

$$\rho_1 v_1 - \rho_2 v_2 < 0, \quad \rho_1 > \rho_2.$$

In equation (17) therefore $\rho_1 \dot{u}_1 - \rho_2 \dot{u}_2 < 0.$

Since ρ_1/ρ_2 is very nearly unity we may infer that the system of accelerations is as shown in fig. 55(*b*). Figs. 55(*c*) and (*d*) show the reverse system of velocities and accelerations*.

J. Bjerknes† has shown that when u_1, $\dot{u}_1$, u_2, $\dot{u}_2$, etc. refer to points at an infinitesimal distance apart, as for example when the sharp surface of dis-

* From J. Bjerknes, *Geofys. Publ.* **3**, No. 6.
† *Geofys. Publ.* **3**, No. 6.

continuity is replaced by a finite layer of rapid transition, the differences in equations (16), (17), and (19) may be replaced by differentials,

$$\frac{\partial}{\partial y}(\rho\dot{v}) = -2\omega\sin\phi\frac{\partial}{\partial y}(\rho u) - g\tan\alpha\frac{\partial\rho}{\partial y},$$

$$\frac{\partial}{\partial y}(\rho\dot{u}) = 2\omega\sin\phi\frac{\partial}{\partial y}(\rho v),$$

$$\frac{\partial}{\partial y}(\rho\dot{w}) = \tan\alpha\frac{\partial}{\partial y}(\rho v) + v_f\frac{\partial\rho}{\partial y}\tan\alpha.$$

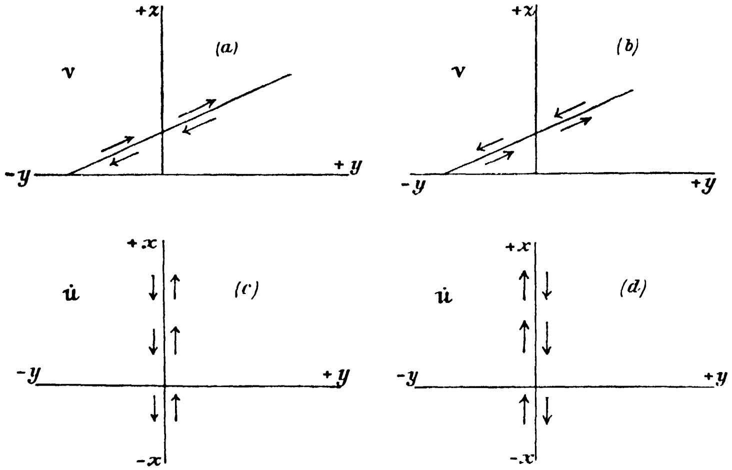


Fig. 55. Motion at a front.

If the surfaces are not accelerated, and the changes of density are negligible,

$$\frac{\partial}{\partial y}(\rho\dot{w}) = \tan\alpha\frac{\partial}{\partial y}(\rho v).$$

Here $\frac{\partial v}{\partial y}$ is <0. Hence $\frac{\partial}{\partial y}(\rho\dot{w}) < 0$,

$$\frac{\partial}{\partial y}(\rho\dot{v}) \lesseqgtr 0 \text{ as } \frac{\partial}{\partial y}(\rho u) \lesseqgtr -\frac{g}{2\omega\sin\phi}\frac{\partial\rho}{\partial y}\tan\alpha.$$

In the first case the warm air has an acceleration upward with reference to the cold wedge, whereas in the second case the warm air has a downward acceleration with reference to the cold wedge.

CHAPTER XI

WIND IN THE TROPOSPHERE: THE EFFECT OF HORIZONTAL GRADIENTS OF TEMPERATURE

§ 121. *The variation of wind with height*

IT will be adopted as the basis of the following discussion that in any layer which is above the reach of surface turbulence, the wind will approximate to the geostrophic wind computed from the gradient of pressure in that layer, provided the pressure distribution is not changing rapidly. The distribution of pressure at any level depends not only on the distribution of pressure at sea level but also on the distribution of temperature in the intervening layers.

Let p, T, ρ be the pressure, absolute temperature and density at a point (x, y, z); the axis of z is vertical, and the axes of x and y are any convenient horizontal axes. Let u and v be the components of the geostrophic wind parallel to the axes of x and y respectively. Then

$$\left.\begin{aligned} 2\omega\sin\phi\,.\,\rho u &= -\frac{\partial p}{\partial y} \\ 2\omega\sin\phi\,.\,\rho v &= \frac{\partial p}{\partial x} \end{aligned}\right\} \qquad \ldots\ldots(1).$$

Substituting

$$\rho = p/RT,$$

$$\left.\begin{aligned} \frac{2\omega\sin\phi}{R}\frac{u}{T} &= -\frac{1}{p}\frac{\partial p}{\partial y} \\ \frac{2\omega\sin\phi}{R}\frac{v}{T} &= \frac{1}{p}\frac{\partial p}{\partial x} \end{aligned}\right\} \qquad \ldots\ldots(2).$$

From the statical equation

$$\frac{\partial p}{\partial z} = -g\rho,$$

it follows that

$$\frac{1}{p}\frac{\partial p}{\partial z} = -\frac{g}{RT} \qquad \ldots\ldots(3).$$

Differentiating this equation with respect to x and substituting in the second of equations (2),

$$\frac{g}{R}\frac{1}{T^2}\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{p}\frac{\partial p}{\partial z}\right) = \frac{\partial^2}{\partial x\,\partial z}\log p = \frac{\partial^2}{\partial z\,\partial x}\log p = \frac{\partial}{\partial z}\left(\frac{1}{p}\frac{\partial p}{\partial x}\right) = \frac{2\omega\sin\phi}{R}\frac{\partial}{\partial z}\left(\frac{v}{T}\right),$$

$$\left.\begin{aligned} \frac{\partial}{\partial z}\left(\frac{v}{T}\right) &= \frac{g}{2\omega\sin\phi}\frac{1}{T^2}\frac{\partial T}{\partial x}, \\ \text{similarly}\qquad \frac{\partial}{\partial z}\left(\frac{u}{T}\right) &= -\frac{g}{2\omega\sin\phi}\frac{1}{T^2}\frac{\partial T}{\partial y} \end{aligned}\right\} \qquad \ldots\ldots(4).$$

Equations (4) may also be written in the form

$$\left.\begin{aligned}\frac{\partial u}{\partial z}&=\frac{u}{T}\frac{\partial T}{\partial z}-\frac{g}{2\omega\sin\phi}\frac{1}{T}\frac{\partial T}{\partial y}\\ \frac{\partial v}{\partial z}&=\frac{v}{T}\frac{\partial T}{\partial z}+\frac{g}{2\omega\sin\phi}\frac{1}{T}\frac{\partial T}{\partial x}\end{aligned}\right\}\qquad\ldots\ldots(5).$$

If for g we substitute $-\frac{1}{\rho}\frac{\partial p}{\partial z}$ in these equations, they can be readily reduced to the form

$$\left.\begin{aligned}\frac{\partial u}{\partial z}&=\frac{1}{2\rho T\omega\sin\phi}\left\{\frac{\partial p}{\partial y}\frac{\partial T}{\partial z}-\frac{\partial p}{\partial z}\frac{\partial T}{\partial y}\right\}\\ \frac{\partial v}{\partial z}&=\frac{1}{2\rho T\omega\sin\phi}\left\{\frac{\partial p}{\partial x}\frac{\partial T}{\partial z}-\frac{\partial p}{\partial z}\frac{\partial T}{\partial x}\right\}\end{aligned}\right\}\qquad\ldots\ldots(6).$$

The condition that u and v shall not vary with height, or that

$$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=0,$$

may be written

$$\frac{\partial p}{\partial x}:\frac{\partial p}{\partial y}:\frac{\partial p}{\partial z}=\frac{\partial T}{\partial x}:\frac{\partial T}{\partial y}:\frac{\partial T}{\partial z}\qquad\ldots\ldots(7).$$

This is the condition that the tangent planes to the isobaric and isothermal surfaces through any point shall coincide; it is the condition that the isobaric surfaces shall also be isothermal surfaces. This condition was first enunciated by W. H. Dines*.

Equations (4) can be integrated in the form

$$\left.\begin{aligned}\frac{u}{T}&=\frac{u_0}{T_0}-\frac{g}{2\omega\sin\phi}\int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial y}dz\\ \frac{v}{T}&=\frac{v_0}{T_0}+\frac{g}{2\omega\sin\phi}\int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial x}dz\end{aligned}\right\}\qquad\ldots\ldots(8),$$

where u_0, v_0 are the components of the geostrophic wind at height z_0, where the absolute temperature is T_0. Thus the geostrophic wind can be regarded as made up of two components, (*a*) a component equal to the geostrophic wind at level z_0 reduced in the ratio T/T_0, and (*b*) a thermal wind whose components are

$$-\frac{gT}{2\omega\sin\phi}\int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial y}dz \quad\text{and}\quad \frac{gT}{2\omega\sin\phi}\int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial x}dz.$$

The magnitude of the thermal wind will increase steadily with height in regions where the temperature gradient maintains its general direction unchanged. The signs of the components show that it blows around low temperature in the same sense that the geostrophic wind blows around low pressure, and that it keeps low temperature to its left.

Equations (8) above can be put in another form by substituting for u/T, v/T from equations (2), and clearing the factor $1/2\omega\sin\phi$. We then find

$$\left.\begin{aligned}\frac{\partial p}{\partial x}&=\frac{p}{p_0}\left(\frac{\partial p}{\partial x}\right)_0+g\rho T\int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial x}dz\\ \frac{\partial p}{\partial y}&=\frac{p}{p_0}\left(\frac{\partial p}{\partial y}\right)_0+g\rho T\int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial y}dz\end{aligned}\right\}\qquad\ldots\ldots(9).$$

* *Nature*, **99**, 1917, p. 24.

From these equations it is seen that the pressure gradient may be represented by two parts, the first of which diminishes with height, while the second increases with height in a region where the direction of the temperature gradient is the same at all levels.

§ 122. *Some applications of the above formulae*

Some special cases are worth considering in further detail. (*a*) If low temperature is associated with low pressure, the wind will increase with height. (*b*) If low temperature is associated with high pressure, the wind will decrease with height and may even be reversed at moderate heights if the horizontal gradients of temperature be large. (*c*) If the geostrophic wind blows from high temperature to low, it will veer with height. (*d*) If the geostrophic wind blows from low temperature to high, it will back with height.

These results are not easy to apply to individual cases, and they are perhaps more useful when inverted from the form given above. Thus (*c*) and (*d*) may be interpreted to indicate that a wind veering with height will bring up higher temperatures, and that a wind backing with height will bring up lower temperatures.

The same kind of reasoning can be applied to the larger scale movements of the atmosphere. Since the main latitude variation in the troposphere is an increase of temperature from pole to equator, the thermal wind in the troposphere is from West to East, so that the westerly component of wind should increase with height. This fact accounts for the tendency of westerly winds to increase with height, and of easterly winds to decrease with height. Within the stratosphere the temperature gradient is reversed, and the result is to cause a steady diminution with height of the westerly component of wind.

A useful summary of some investigations of wind in the troposphere and stratosphere will be found in a paper by Dobson*. On the whole the wind increases with height in the upper troposphere, and then falls off rapidly in the lower stratosphere.

By the use of equations (4) above it is readily possible to compute the horizontal temperature gradients for a given wind distribution, if the temperatures at different levels are known. For most cases it is sufficiently accurate to adopt mean values of the temperatures at different levels. The equations are written in the form

$$\left.\begin{aligned} d\left(\frac{u}{T}\right) &= -\frac{g}{2\omega \sin\phi}\frac{1}{T^2}\frac{\partial T}{\partial y}dz \\ d\left(\frac{v}{T}\right) &= \frac{g}{2\omega \sin\phi}\frac{1}{T^2}\frac{\partial T}{\partial x}dz \end{aligned}\right\} \quad \text{......(10).}$$

The differences $d\left(\frac{u}{T}\right)$ and $d\left(\frac{v}{T}\right)$ are computed for any convenient steps of dz, say $dz = 1$ km, and the corresponding values of $\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}$ are then obtained by a simple computation. An example of such a computation is shown in the

* *Q.J. Roy. Met. Soc.* **46**, 1920, p. 54.

table below. Here the height interval dz is 1 km. The observed components of velocity to East and North, u and v, are given in columns 3 and 4. The mean temperatures at different heights are given in column 2. The remainder of the computation is carried out in accordance with the equations above, the last column but one giving $10^5 q$, which is the horizontal temperature gradient in degrees C per 100 km. The last column gives the azimuth of the temperature gradient.

ght m	T	u	v	$100u/T$	Diff.	$\partial T/\partial y$ 10^5	$100v/T$	Diff.	$\partial T/\partial x$ 10^5	$10^5 q$	Azi
	246	17·7	−10·3	7·19			−4·15				
	250				2·03	1·45		−1·17	0·84	1·67	3
	252	13·0	−7·5	5·16			−2·98				
	255				1·92	1·44		−2·40	1·80	2·30	5
	259	8·4	−1·5	3·24			−0·58				
	262				0·82	0·65		−0·81	0·64	0·91	4
	265	6·4	0·6	2·42			0·23				
	267				−0·36	−0·30		−0·77	0·63	0·70	11
	270	7·5	2·7	2·78			1·00				
	273				1·22	1·05		0·09	−0·08	1·05	35
	276	4·3	2·5	1·56			0·91				

The magnitude of the effect is readily seen from this example. A horizontal temperature gradient of 1° C per 100 km produces in a range of height of 1 km a thermal wind of about 3 m per second. If this gradient of temperature persisted through 2 km of height, the thermal wind would amount to 6 m per second, assuming a mean temperature of about 270° A. The amount of the vectorial wind change produced in a given range of height by a given horizontal temperature gradient is very nearly inversely proportional to the absolute temperature, so that with a constant horizontal temperature gradient the vectorial rate of change of wind with height will itself increase with height, on account of the decrease of temperature with height.

It will be recalled that reference was made above (Chapter I) to the high correlation which W. H. Dines found to exist between pressure and temperature at different levels in the middle troposphere, yielding the result that on the average high temperature is associated with high pressure. This does not mean that the isothermal and isobaric surfaces are nearly coincident, so that the wind should change relatively little with height*. In the diagram which Dines gave of the mean distribution of upper air temperatures and pressures over anticyclones the average inclination of the isotherms is about three times as great as that of the isobars. The wind current which would correspond with this distribution would be a current increasing with height, while maintaining a steady direction. Direct observations show the frequent occurrence of large changes of wind with height at all levels, though the greatest changes are usually confined to the levels from 1 to 3 km.

An examination of the results derived by Cave† shows that solid currents, i.e. currents whose velocity and direction change little with height, are moderately frequent in the conditions in which observations are possible.

* See Douglas, *Mem. R. Met. Soc.* **3**, No. 29, especially pp. 160, 161.

† *The Structure of the Atmosphere in Clear Weather.*

Moreover it is difficult to see how the depressions of middle latitudes could persist if the wind varied very greatly with height. Of Cave's ascents, 200 in number, excluding 10 rather indefinite cases,

(*a*) Thirty-two showed a solid current, the wind attaining the gradient value and not increasing beyond this limit.

(*b*) Forty-nine showed a considerable increase of velocity with height, surpassing the gradient value. These winds, in the typical cases, were associated with cyclonic systems of some intensity to West or North. The directions were mainly grouped about SE and WNW, and the surface temperatures were in most cases such as to account for a thermal wind in the right direction.

(*c*) Twenty-seven showed a marked decrease of velocity with height, these winds were predominatingly easterly, and the fall off in velocity was explainable as the result of a decrease of temperature from South to North.

(*d*) Thirty-four showed marked change of wind direction at different heights. It is difficult to summarise these in any manner.

(*e*) Thirty-seven showed an upper wind blowing out of the centre of low pressure, frequently with reversals in a lower layer.

(*f*) Eleven were followed well into the stratosphere, and most of these showed a definite fall of velocity in the stratosphere, without any great change of direction. The decrease of velocity in the stratosphere indicates a reversal of the horizontal temperature gradients in going into the stratosphere.

Richardson and Munday*, in an investigation of the possibility of treating the atmosphere as a single layer, came to the conclusion that barotropy is not even an approximation to the truth in the atmosphere. These authors examined the results of a large number of sounding balloons, and came to the conclusion that the equations which represent the motions of the atmosphere cannot be summarised in the form of the equation for a single layer. We need not therefore be surprised that the motion of the atmosphere should deviate widely from the motion of a barotropic fluid.

C. K. M. Douglas† compared the variations of temperature predicted by equations (6) above with the actual observed changes of temperature, and found that the correlation coefficient between the observed and predicted changes was a little less than 0·5, both for 6 hour and 24 hour changes, but was greater when only the occasions of large changes of temperature were considered. The relatively low value of the correlation coefficient may be due partly to the uncertainty of the determination of wind velocity by the pilot balloon method, partly to large vertical movements of air masses, and partly to the deviation of the winds from the geostrophic value which is assumed as the basis of the theory developed above. The uncertainty of the wind determination by the pilot balloon method is probably the most serious source of error. It should be noted that equation (2) above assumes that the observed

* *Mem. R. Met. Soc.* **1**, No. 2.
† *Ibid.* **1**, No. 7.

wind is equal to the geostrophic wind at the same level. In Chapter IX it was shown that in steady conditions the geostrophic wind is an underestimate of the wind in an anticyclone and an overestimate in a cyclone, so that the assumption of geostrophic wind leads to an error which may be systematic if the observations are made in conditions which are predominatingly cyclonic or anticyclonic, though the error is not likely to be great in average conditions.

Douglas assumed that the air masses which he investigated moved without change of temperature. It then follows that

$$\frac{dT}{dt}=\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=0.$$

$\partial T/\partial x$ and $\partial T/\partial y$ could be computed from the observed winds by the method outlined above, and $\partial T/\partial t$ computed from the equation

$$\frac{\partial T}{\partial t}=-u\frac{\partial T}{\partial x}-v\frac{\partial T}{\partial y}.$$

But from equation (5), multiplying by v and u and subtracting,

$$v\frac{\partial u}{\partial z}-u\frac{\partial v}{\partial z}=-\frac{g}{2\omega\sin\phi}\frac{1}{T}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right).$$

Hence $$\frac{\partial T}{\partial t}=\frac{2\omega\sin\phi}{g}T\left(v\frac{\partial u}{\partial z}-u\frac{\partial v}{\partial z}\right) \qquad \text{......(11).}$$

Over a finite interval of time Δt the change of temperature ΔT is given by $\frac{\Delta T}{\Delta t}=\frac{2\omega\sin\phi}{g}T\times$ observed wind at height $z\times$ (change of component at right angles to observed wind at height z, between the levels $z-\frac{1}{2}\Delta z$ and $z+\frac{1}{2}\Delta z)\div\Delta z$. When the relationship is used in this form, the amount of computation is reduced to a minimum and the predicted temperature is readily evaluated.

Douglas suggested that the low correlation coefficients which he obtained were in part due to the vertical motion of the masses. This motion would lead to another term in the equation for $\partial T/\partial t$, a term involving the vertical component of velocity. But it is possible that the low coefficients were also in part due to the assumption that the air moved without change of temperature. It would appear safer to assume that the air moved adiabatically, though it is clear that even that assumption is only a very crude approximation. For adiabatic motion

$$\frac{d}{dt}\left(\frac{p^\gamma}{T^{\gamma-1}}\right)=0,\quad \frac{\gamma-1}{\gamma}\frac{1}{T}\frac{dT}{dt}=\frac{1}{p}\frac{dp}{dt},$$

$$\frac{\partial T}{\partial t}=-u\frac{\partial T}{\partial x}-v\frac{\partial T}{\partial y}+\frac{\gamma}{\gamma-1}\frac{T}{p}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right) \qquad \text{......(12).}$$

It is again assumed that the motion is purely horizontal, and the w term in the full equations is omitted. If the motion is along the isobars at each level

$$u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}=0,$$

and the equation reduces to

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} + \frac{\gamma}{\gamma-1}\frac{T}{p}\frac{\partial p}{\partial t}$$

$$= \frac{2\omega \sin\phi}{g} T\left(v\frac{\partial u}{\partial z} - u\frac{\partial v}{\partial z}\right) + \frac{\gamma}{\gamma-1}\frac{T}{p}\frac{\partial p}{\partial t} \qquad \ldots\ldots(13).$$

This equation would make it possible to take account of the changes of pressure which may be brought about by advection in the upper air.

Errors will arise in the forecasts of temperature by the above method on account of neglected physical processes such as radiation, turbulent mixing, vertical motion, which may be accompanied by condensation of water-vapour or evaporation of water drops falling through the air. It is therefore not surprising that the coefficient of correlation between the predicted and observed temperatures should come out rather low.

CHAPTER XII

THE GENERAL ASPECTS OF TURBULENCE

§ 123. *Stream-line and turbulent flow*

THE smooth motion of a fluid which can be represented by a stream-line function does not occur in nature except under certain definite conditions. When these conditions are not fulfilled the motion becomes irregular or "turbulent". We may define turbulence as an irregular motion which in general makes its appearance in fluids, gaseous or liquid, when they flow past solid surfaces, or even when neighbouring streams of the same fluid flow past or over one another. The existence of turbulence in the atmosphere is made visible by the trail of smoke from a chimney, or by the gusts and lulls in the trace of an anemometer. While the air in general has a "mean velocity" and a "mean direction", which are reasonably constant over intervals of from 10 to 100 minutes, the instantaneous velocity and direction may differ widely from the mean values. We have no clear idea why or how turbulence arises, or of the exact nature of the eddies which form in a fluid in turbulent flow, but something is known of the conditions which must be satisfied if a fluid is to flow without turbulence, or as "stream-line" motion; and some information has been accumulated concerning the magnitude of the eddy-velocities in certain cases.

The local variations of velocity in a fluid are associated with, and are presumably due to, the local variations of static pressure, which travel downstream with the mean velocity of the stream. The eddy which becomes visible as a dimple in the surface of a stream of water travels in this manner; at the same time it appears to be a simple rotational circulation, the motion being in its essentials two-dimensional. When a current of air moves over uneven ground we might be tempted to think of the eddies which form in it as cylindrical eddies, having their axes in the horizontal plane at right angles to the mean motion. This would require that the anemometer trace should show a ribbon of finite width for velocity, with a practically constant direction. Observation shows that this is not the case in practice, wide direction traces being associated with wide velocity traces.

If over a certain interval of time the mean velocity of flow remains constant, having components $\bar{u}$, $\bar{v}$, $\bar{w}$, the instantaneous deviations of the component velocities are known as the components of the eddy velocity. If the total velocity at any instant is represented by the components $\bar{u}+u'$, $\bar{v}+v'$, $\bar{w}+w'$, then u', v', w' are the components of the eddy velocity.

§ 124. *The Reynolds stresses*

The study of turbulence is largely the creation of Osborne Reynolds*, who carried out many beautiful experiments on turbulence in fluids, the motion being made visible by the introduction of colouring matter into the fluid. Reynolds found that turbulence occurred mainly when (*a*) the transverse variation of velocity exceeded a certain limit, and (*b*) the fluid was bounded by solid boundaries. The mathematical method outlined below follows Reynolds in assuming the fluid to be incompressible. This is not strictly true in the atmosphere, but it is probable that the turbulence in the atmosphere is in the main due to other causes than its compressibility, and that compressibility only modifies the motion to a slight degree.

The equations of motion of an incompressible viscous fluid, as shown in Chapter VIII, p. 179, are of the form

$$\rho\frac{du}{dt}=\rho X-\frac{\partial p}{\partial x}+\mu\nabla^2 u \qquad \text{......(1)},$$

or

$$\rho\frac{du}{dt}=\rho X+\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z} \qquad \text{......(2)},$$

which may be re-arranged in the form

$$\rho\frac{\partial u}{\partial t}=\rho X+\frac{\partial}{\partial x}(p_{xx}-\rho uu)+\frac{\partial}{\partial y}(p_{yx}-\rho uv)+\frac{\partial}{\partial z}(p_{zx}-\rho uw) \text{......(3)}.$$

Reynolds defines the mean velocities $\bar{u}$, $\bar{v}$, $\bar{w}$ as follows

$$\bar{u}=\frac{1}{\tau}\int_{t-\frac{1}{2}\tau}^{t+\frac{1}{2}\tau}u\,dt,\quad \bar{v}=\frac{1}{\tau}\int_{t-\frac{1}{2}\tau}^{t+\frac{1}{2}\tau}v\,dt,\quad \bar{w}=\frac{1}{\tau}\int_{t-\frac{1}{2}\tau}^{t+\frac{1}{2}\tau}w\,dt \qquad \text{......(4)}.$$

The mean values of u', v', w' are zero when the means are taken over an interval of time sufficiently long to permit of a large number of alternations of the velocities about their mean value. Then

$$\overline{uu}=\text{mean of }(\bar{u}\bar{u}+u'u'+2\bar{u}u')$$
$$=\bar{u}\bar{u}+\overline{u'u'}.$$

Similarly

$$\overline{uv}=\bar{u}\bar{v}+\overline{u'v'},$$
$$\overline{uw}=\bar{u}\bar{w}+\overline{u'w'}.$$

Substituting these values, we find

$$\rho\frac{\partial\bar{u}}{\partial t}=\rho X+\frac{\partial}{\partial x}(\bar{p}_{xx}-\rho\bar{u}\bar{u}-\rho\overline{u'u'})+\frac{\partial}{\partial y}(\bar{p}_{yx}-\rho\bar{u}\bar{v}-\rho\overline{u'v'})$$
$$+\frac{\partial}{\partial z}(\bar{p}_{zx}-\rho\bar{u}\bar{w}-\rho\overline{u'w'}) \qquad \text{......(5)},$$

with two similar equations. The equation of continuity gives

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}=0 \qquad \text{......(6)}.$$

* *Collected Papers*, **2**, p. 51.

We thus find that the equations of mean motion are of the same form as the exact equations (3) above, provided we add the additional stresses

$$\left.\begin{aligned} \widehat{xx} &= -\rho\overline{u'u'}, & \widehat{yx} &= -\rho\overline{u'v'}, & \widehat{zx} &= -\rho\overline{u'w'} \\ \widehat{yy} &= -\rho\overline{v'v'}, & \widehat{yz} &= -\rho\overline{v'w'}, & \widehat{zz} &= -\rho\overline{w'w'} \end{aligned}\right\} \quad \ldots\ldots(7).$$

These are the six eddy stresses of Osborne Reynolds. The notation is that of Karl Pearson, and the convention as to signs conforms to that of Love's *Theory of Elasticity*.

The equations derived above are not based on any assumptions as to the nature of turbulence. All that is necessary is that the interval τ over which the mean values $\bar{u}$, $\bar{v}$, $\bar{w}$ are taken should be sufficiently long to permit of a large number of oscillations of the wind components during that interval.

Note that
$$\left.\begin{aligned} \widehat{xx} &= -\rho\overline{u'u'} = -\rho\,\overline{(u-\bar{u})(u-\bar{u})} &&= -\rho\sigma_u^2 \\ \widehat{xy} &= -\rho\overline{u'v'} = -\rho\,\overline{(u-\bar{u})(v-\bar{v})} &&= -\rho r_{uv}\sigma_u\sigma_v \\ \widehat{xz} &= -\rho\overline{u'w'} = -\rho\,\overline{(u-\bar{u})(w-\bar{w})} &&= -\rho r_{uw}\sigma_u\sigma_w \end{aligned}\right\} \quad \ldots\ldots(8),$$

etc., where σ_u, σ_v, σ_w are the standard deviations of the wind components, and r_{uv}, r_{uw}, etc. are the coefficients of correlation between the eddy components of velocity.

§ 125. *Dynamical similitude; the Reynolds number*

If the geometrical conditions of two separate motions are precisely similar, as for example when geometrically similar bodies are surrounded by fluid or immersed in fluid, we can find certain conditions which must be satisfied in order that the fluid motions may likewise be similar. The conditions in the one case may be specified by the density ρ_1, the velocity v_1, the coefficient of kinematic viscosity ν_1, and a linear dimension l_1 of the body or bodies which bound the fluid in any way. The corresponding variables for the second fluid are ρ_2, v_2, ν_2, l_2. All characteristic lengths such as x will be measured in terms of the appropriate l, and the velocities being of the nature dx/dt, will also be proportional to l. We require to find the condition that the flow in the two cases shall be geometrically similar.

The ratios of the three forces due to the pressure gradient, friction and inertia must be the same at corresponding points. Since these forces are in equilibrium we need only consider two of them, say the friction and the inertia forces. One of the components of the inertia forces is $-\rho u\,\partial u/\partial x$. This must be proportional to $\rho v^2/l$, since the condition of geometrical similarity of flow demands that the u's should be proportional to the selected characteristic velocities v_1, v_2, etc., and the lengths x must be proportional to l. Again the frictional forces are of the nature of $\mu\,\partial^2 u/\partial x^2$, which must be proportional to $\mu v/l^2$. The ratio of the inertia forces to the frictional forces must be the same for the two fluids, if the conditions of flow are to be geometrically similar, and

$$\frac{\rho v^2}{l} \div \frac{\mu v}{l^2} = \frac{\rho v l}{\mu} = \frac{v l}{\nu} \quad \ldots\ldots(9)$$

must be the same for the two fluids. The two systems of flow will therefore be similar if

$$\frac{v_1 l_1}{\nu_1} = \frac{v_2 l_2}{\nu_2} = \text{const.} \qquad \ldots\ldots(10).$$

It is readily seen that this ratio has no dimensions. The ratio is known as the *Reynolds number*. When this number is small the viscous forces are great by comparison with the inertia forces, and turbulent motion is readily damped out. When the ratio is large the dynamic forces set up turbulence in spite of the viscous forces.

In his experiments on the flow of liquids in pipes Reynolds distinguished an "upper" and a "lower" Reynolds number. The lower gives the critical value above which initially turbulent flow entering the tube ceases to become laminar in the tube, while the upper gives the critical value at which initially laminar flow becomes turbulent.

It is found that the general mathematical treatment of turbulence is in practice so difficult that only special cases are amenable to discussion, in particular the cases where the Reynolds number is either very small or very large. A few cases of motion of solids in a viscous fluid have been worked out in detail when the shape of the solid is simple. One of the best known results is that derived by Stokes for the resistance of a viscous fluid to the motion of a sphere, which is found to be

$$6\pi\mu v r,$$

r being the radius of the sphere and v its velocity. When the sphere falls through the air, and v is its terminal velocity, the resistance is balanced by the buoyancy forces, $\frac{4}{3}\pi r^3 g\,(\rho_1 - \rho_2)$, where ρ_1 and ρ_2 are the densities of water and air, respectively. Neglecting ρ_2 by comparison with ρ_1, and putting $\rho_1 = 1$, we find

$$6\pi\mu v r = \frac{4}{3}\pi r^3 g, \quad \text{or} \quad v = \frac{2}{9}\frac{r^2 g}{\mu}.$$

This formula only holds for values of R small by comparison with unity. If r is in cm, $v = 1{\cdot}3 \times 10^6 r^2$, and the formula only holds for droplets with radius 0·001 mm or less.

Reynolds, in his experiments on the flow of liquids, found that turbulence set in when the Reynolds number reached 6400, but later experimenters have been able by careful adjustment to obtain non-turbulent flow with values of vl/ν as high as 25,000.

In the atmosphere, for motions reaching up to the tropopause, l is presumably to be taken as equal to the height of the tropopause, or perhaps the height of the homogeneous atmosphere. With velocities of 1 metre per second, vl/ν is then of the order of 10^9, and motion on such a scale must always be turbulent. But atmospheric motions on a much smaller scale than this must be turbulent, on account of the low value of the coefficient of viscosity of air.

For the consideration of motion in the atmosphere it is possible to give a slightly different form to Reynolds number. Let l be the linear dimension of a disturbance in the flow, and $\partial\bar{v}/\partial z$ the vertical gradient of velocity. Then instead of

$$R=\frac{lv}{\nu}$$

we may write

$$\frac{l^2}{\nu}\frac{\partial\bar{v}}{\partial z}=f(R).$$

This equation indicates that the linear dimension of any disturbance in the flow is given by

$$l^2=\frac{f(R)\,\nu}{\partial\bar{v}/\partial z}.$$

If $f(R)$ were known, we could estimate the linear dimension of the disturbance corresponding to any particular value of $\partial\bar{v}/\partial z$. If $\partial\bar{v}/\partial z$ be small then l is great, and the disturbances are large in extent, but a longer time will be required to set them up.

It is not possible to apply the Reynolds number in a straightforward manner to motions in the atmosphere. The analysis used by Reynolds to derive his criterion R presupposes that in the different states of flow compared the only differences are those of size of the boundary and the rate and scale of the flow, the nature of the fluid being unchanged. In the atmosphere there is an added complexity due to the lapse-rate of temperature. The existence of a large lapse-rate facilitates the formation of turbulence, while the existence of an inversion checks the formation of turbulence, thus suggesting that the value of Reynolds number alone is not sufficient to determine whether an atmospheric motion is turbulent or not. Practically all wind-tunnel experiments refer to conditions in which the lapse-rate is zero, and are therefore strictly comparable with each other, but not with atmospheric motions.

§ 126. *The partition of eddy energy*

Taylor* has given a simple proof that the eddying velocity transverse to the mean wind is on the average about equal to the eddying velocity along the direction of the mean wind. Let OA (fig. 56) represent the mean wind in velocity and direction, and let OC and OD represent the extremes in the gusts and lulls. With A as centre and AC as radius, draw the circle $CEDF$. Then if the wind is always represented by a vector OP with P always within the circle and capable of taking all positions within it, we have the relation

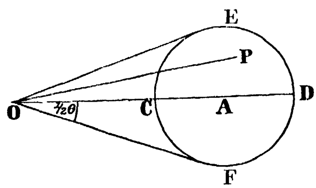


Fig. 56. Mean and eddy winds.

$$\frac{V_{\max}-V_{\min}}{V_{\max}+V_{\min}}=\sin\tfrac{1}{2}\theta,$$

θ being the angular width of the trace. Taylor found that this relationship was on the whole very closely confirmed in practice. We therefore conclude that

* *Aero. Res. Committee*, R. and M., No. 345.

if u', v', w' be the components of eddy velocity about the mean value $\bar{u}$ ($\bar{v}=\bar{w}=0$),

$$\overline{u'^2}=\overline{v'^2}.$$

Taylor also compared the magnitudes of v' and w' by observing the oscillations of a tethered balloon from some distance to leeward of its point of attachment, following the balloon with a pointer to which was attached a pencil whose motion was recorded on a piece of smoked glass. Taylor's records show complex curves which have no obvious axis of symmetry, and which could be roughly enclosed within a circle, thus showing on the average equal values of $\overline{v'^2}$ and $\overline{w'^2}$. This establishes a complete equipartition of eddying energy,

$$\overline{u'^2}=\overline{v'^2}=\overline{w'^2}.$$

Taylor's observations were made on a balloon tethered at 20 feet above the ground. Scrase* carried out somewhat similar observations by means of Taylor's bi-directional vane, the observations giving values of v' and w'. His observations were limited to occasions when the lapse-rate was small (a difference of $-0{\cdot}1°$ F to $0{\cdot}5°$ F in 17 metres) and the wind was from a direction between 270° and 310°, or between 160° and 200°, measured from North. The conditions in his observations were therefore reasonably comparable. The ratio of the v' component to the w' component was found to be 1·59 at 2 metres, diminishing to 1·20 at 18 metres. Scrase states that on the average at 2 metres the ratio of the lateral to the vertical diameter of the trace was about 1·5, indicating that $\overline{v'^2}$ is more than twice as great as w'^2. He also found that while the diameters of the traces increased rapidly during the first minute, the subsequent increase was slow. From this we infer that most of the eddies have periods of, at most, a few seconds, the later slower growth of the diameter of the trace being due to the passing of large individual eddies. Further, when the duration of the record was kept constant (1 minute), and the instrument was maintained at a height of 3·3 metres, the lateral width of the trace showed no variation with wind velocity, while the vertical diameter increased slowly with wind velocity. The mean ratio of the lateral to the vertical diameter was 1·46. It is to be noted that the observations made with the bi-directional vane are observations of extremes of wind in each direction. From this it may be inferred that the extremes of the eddy winds are closely proportional to the wind velocity. In a further series of experiments Scrase found that the lateral and vertical diameters of the trace increased with height above ground up to a height of about 1·4 metres, and then decreased slowly, the ratio of the horizontal to the vertical diameter decreasing from 1·63 at 0·5 metres above the ground to 1·2 at 18 metres. The departure from isotropic distribution of the eddy components observed at heights of a metre or so thus appears to diminish with increasing height, leading to a nearer approach to equipartition in the v, w components at heights of the order of about 18 metres. It should be noted that the analysis by which Taylor established the relation $\overline{u'^2}=\overline{v'^2}$ was applied to observations at 40 metres above the ground.

* M.O., *Geophys. Mem.* No. 52.

The results derived by Scrase are in reasonable agreement with some observations described by G. I. Taylor in a lecture before the Royal Meteorological Society*, which show that at 2 feet $v/w=3$, and at 8 feet $v/w=1\cdot4$. At 25 feet over grassland Taylor found a rough equality of v and w, but Scrase's lengthier series of measurements indicate that the departure from complete isotropy of v and w extends to a height of at least 20 metres.

Scrase (*loc. cit.*) has also described a series of observations of a light bi-directional vane photographed kinematographically, 16 photographs being taken per second. These records thus give the true mean value of the wind components, and not the extremes as yielded by the pen traces. The reproductions of readings taken from the photographs at intervals of 1/5 second, 3/5 second and 5 seconds respectively indicate that by far the greater part of the variability is associated with eddies of periods of about 1 second or less. The mean deviation from zero of the v component decreased from 0·24 for intervals of 1/5 second to 0·14 for intervals of 5 seconds. This is equivalent to a decrease of the eddying energy to $\left(\frac{0\cdot14}{0\cdot24}\right)^2$ or 1/3. Thus at least two-thirds of the eddying energy is associated with eddies of period of less than 5 seconds. Reducing the interval from 1/5 to 1/16 second increased the mean deviation from 0·115 to 0·132 for one short record analysed, equivalent to an increase of the eddying energy by one-third. The kinematograph records confirmed the results previously derived that the horizontal transverse variations exceeded the vertical variations. They also showed that there is a tendency for positive u' to be associated with positive w', so that the faster moving air goes upward, while the slower moving air comes downward, but the correlation between the signs of v' and w', and between the signs of v' and u' was not clearly marked. Other conditions being equal (contour and lapse-rate), the mean magnitudes of u', v' and w' at a given height are proportional to the mean velocity of the wind.

Möller† has also considered Scrase's results, using mean values of the eddy components over intervals of 6 seconds, and found a coefficient of correlation of $-0\cdot8$ between v' and w', again indicating a tendency for fast-moving air to rise, and for slow-moving air to fall. These results are strongly suggestive of a tendency for the eddying motion to be in planes strongly inclined to the horizontal.

Further observational details of wind structure will be found in *Geophysical Memoir*, No. 54, by the late M. A. Giblett and others. The variation of the u and v components at a height of 50 feet is represented vectorially, and it is suggested that the mean velocities over intervals of 80 seconds give diagrams elongated in the u direction, while the deviations of 5-second means from the 80-second means give a practically circular distribution. If this is accepted the eddies of longer period are not distributed at random, though the eddies whose periods are in a middle range, from 5 seconds to about 80 seconds, are

* *Q.J. Roy. Met. Soc.* **53**, 1927, p. 201.
† *Beitr. Phys. fr. Atmos.* **20**, 1933.

distributed at random. The effect is clearly marked in one record reproduced in the memoir, but not in the second (*loc. cit.* fig. 47), and it is perhaps hazardous to draw any definite conclusion from one record. On pp. 100 and 101 of the memoir are shown the coefficients of correlation between records of wind speed and direction at different points, or at the same time at different points, at varying intervals of time. An example of these is the correlation between wind speeds at one anemometer (Record 391, p. 101), at intervals of 5, 10, 20, 30, 40, 50 and 60 seconds. The correlation coefficient falls from unity at 0 second to a negative maximum $-0{\cdot}42$ at 50 seconds, appearing to indicate a pattern repeating itself at intervals of about 100 seconds. As the mean wind was about 34 miles per hour, this corresponds to a horizontal diameter of 5000 feet. The evidence is not altogether convincing, and it is possible that the moderate correlation at 50 seconds is a chance effect.

The investigations which have been very briefly summarised above all appear to point to a marked absence of isotropy of eddying velocities near the ground, with a nearer approach to isotropy at heights of the order of 20 metres.

Scrase showed that there are usually present large numbers of small scale eddies whose periods are of the order of one second, which give at a height of 1·5 metres mean components in the u, v, w directions respectively in the ratios 1·0 : 1·16 : 0·75, and at a height of 19 metres in the ratios 1·0 : 0·73 : 0·56. He found that the mean u' component showed very little variation from 1·5 to 19 metres, while the v' and w' components diminished in the same range to about two-thirds of their value at 1·5 metres. Scrase only summarises the v', w' components for eddies of periods of the order of a few minutes, but he states that he found the v' component greater than the w' component. Each of these increased from the ground up to a height of about 1–2 metres, and then decreased, the values at 18 metres being about two-thirds the values at 1·5 metres. Scrase found that mean values of the eddy velocity over intervals of an hour were proportional to the mean wind velocity, and that at a height of 13 metres the mean u', v' and w' components were in the ratios 1·0 : 0·76 : 0·39.

§ 127. *The nature of eddies*

So far no effort has been made to define clearly what is meant by an eddy, nor is it considered likely that any definition could be given which would be universally acceptable. The eddies which form at the edge of a stream flowing into a millpond are of the nature of vortices with vertical axes, but the name "eddy" as used in discussing motion in the atmosphere is not restricted to circular motions. We can only define an eddy as a physical entity which disturbs the uniform flow of air, and this definition will include rotating eddies, convection currents, and any other type of disturbance. There is a type of eddy which can be produced and made visible in the laboratory, that has become known largely through the work of H. Bénard, for which the name of "convection cell" is suggested as appropriate. Bénard* showed that when a shallow layer

* *Ann. Phys. Chem.*, Paris, **23**, 1901.

of fluid containing volatile constituents is cooled at the upper surface by evaporation, the whole mass breaks up into a number of separate cells in each of which there is an upward motion at the centre, diverging motion at the top, and descending motion in the outer regions (see fig. 57). The diameter of the cells is from three to four times the depth of the fluid, and in very steady conditions the cells become hexagonal.

The motion in the Bénard cells was investigated mathematically by Rayleigh*, who showed that it is possible for a fluid to remain in equilibrium with the density greater above than below, so long as the excess of density does not exceed the limit given by the following inequality

$$\frac{\rho_1 - \rho_0}{\rho_0} < \frac{27\pi^4 \kappa \nu}{4gh^3},$$

where κ is the coefficient of molecular conduction of heat, and ν the kinematic

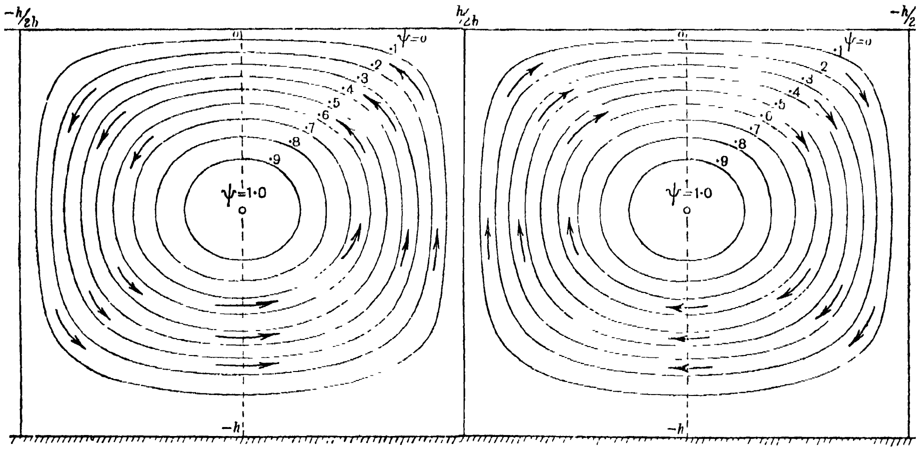


Fig. 57. Circulation in a Bénard cell.

coefficient of viscosity, ρ_1 the density of the fluid at the top, ρ_0 the density of the fluid at the bottom of the layer whose thickness is h. This result is confirmed when the cells are formed in a dish which is slightly tilted so that the fluid becomes shallow at one edge. The cells become smaller as the fluid becomes shallower, but a narrow belt of very shallow fluid remains free of cells.

The mathematical treatment of the problem was extended and clarified by Jeffreys†, who corrected certain of the boundary conditions used by Rayleigh, and further showed that when the fluid has a motion of steady translation the cells are drawn out into long strips, as had been suggested earlier by A. R. Low‡.

The simple Bénard cell is analogous to the ring vortex or the smoke ring sometimes produced by a locomotive. It is an interesting case of the application of the considerations of § 104 above. In the liquid we may regard potential and absolute temperature as identical for all practical purposes, so that

* *Phil. Mag.*, London, **32**, 1916, p. 529. † *Proc. Roy. Soc.* A, **118**, 1928, p. 195.
‡ *Nature*, **115**, 1925, p. 299.

stability requires that the density and pressure gradients should be parallel. In the unstable state the density is greatest at the top, and any disturbance will tend to produce circulations in vertical planes. As the flow of heat tends to maintain the unstable condition, the circulation persists. The Bénard cell is perhaps the clearest illustration of the possibility of generating vorticity in a fluid in which pressure is not a function of density alone. Frictional forces have little or nothing to do with the genesis of the circulation, which in fact dies away rapidly through the action of friction as soon as the transfer of heat dies away. If, for example, the dish in which the cells are formed by the evaporation of volatile constituents of a liquid is covered by a glass plate, the evaporation ceases and the circulation dies away rapidly.

The attention of English meteorologists was first directed to the work of Bénard by Low and Brunt*, who suggested that the existence and persistence of super-adiabatic lapse-rates in the atmosphere might be explained by the joint effects of radiation and turbulence taking the place of molecular conduction and viscosity in Rayleigh's inequality above. Low† also showed that it was possible for the unstable layer of fluid to be filled by several layers of cells, though it is not clear how far this occurs in practice. In the atmosphere the multiple cells, if they occur at all, are probably transitory.

Later developments of these ideas by Idrac‡, Mal§, and by Phillips and Walker|| have shown that a wide variety of forms of cellular structure can be obtained by suitable variation of the rate of shear in the original motion. Idrac showed that when the fluid is sheared a series of vortices were formed having their axes parallel to the direction of the shear. Mal investigated the detailed application of these ideas to explain cloud forms, and showed that, in some of the cases he investigated, particular cloud forms were associated with instability and wind-shear. Phillips and Walker extended the work of Mal, and described a number of laboratory experiments which showed that by suitable adjustment of the variation of velocity with height it was possible to obtain, in unstable air in a wind channel, polygons, transverse vortices, crossed vortices, and longitudinal vortices. The resemblance to cloud forms shown by the details of the motions found in the photographs reproduced in their paper is very remarkable. The work of Mal and of Phillips and Walker renders it very probable that a wide variety of cloud forms can be explained as the effect of instability combined with shear. In particular it should be remarked that long rolls of cloud may thus be explained, the direction of the roll being in the direction of the shear. The earlier view that such clouds were to be explained as Helmholtz gravitational waves would require the direction of the roll to be perpendicular to the direction of shear, and thus the test of observation of wind at different heights can distinguish between the two theories.

The present writer¶ has described a remarkable hailstorm which occurred in France in 1865, in which the cloud was described as resembling a huge net,

* *Nature*, **115**, 1925, pp. 299–301.
† *3rd Int. Congress App. Mech.*, Stockholm, 1930.
‡ *Comptes Rendus*, 1920, July–Dec. p. 42.
§ *Beitr. Phys. fr. Atmos.* **17**, 1930, p. 40.
|| *Q.J. Roy. Met. Soc.* **58**, 1932, p. 23.
¶ *Met. Mag.* **63**, 1928, p. 14.

and the hailstones produced damage in a series of areas which formed an irregular network at distances of 50 to 100 metres apart. This appears to suggest a more or less cellular distribution of convection currents, involving a shallow layer of violent instability.

If the phenomena observed in the laboratory are reproduced with any fidelity in the atmosphere, in which the instability set up as the result of the heating of the earth's surface by solar radiation is limited to the lowest 2000 feet or so, the convection currents induced should be at distances apart of the order of 5000 to 8000 feet. The fact that the typical cumulus forms in long rows immediately suggests an analogy with the forms depicted in the photographs of Mal and Phillips.

§ 128. *The nature of convection*

The comparison of the phenomena observed in viscous fluids in the laboratory with the forms of clouds suggests that what is loosely termed "thermal convection" is to be explained largely by means of the circulation shown in fig. 57. The horizontal extent of the convection cell may therefore be expected to be of the order of three times the depth of the unstable layer. The idea thus given of convection differs materially from the ascent of isolated bubbles of warm air which penetrate through the environment much as an ascending balloon does, the place of the ascending air being filled by a general drift inward of air from all sides. The most recent work on the form of clouds, such as that of G. T. Walker and his pupils, very definitely supports the idea that thermal convection is to be represented by the circulation in fig. 57 rather than by the ascending bubble. The acceptance of such a view of the mechanism of convection makes it practically impossible to specify a "temperature of the environment" through which the ascending air passes, such as is represented by T' in § 21 above. True polygonal cells in the laboratory have a circulation which can be represented accurately by fig. 57. Large cells in the atmosphere appear to have some rotation about a vertical axis superposed on the motion shown in fig. 57. Thus atmospheric convection cells are of rather more complex structure than the cells observed in the laboratory.

When the convective process is set up in still air the convection cell will be roughly polygonal in shape, and fig. 57 will represent any axial cross-section of the cell. When on the other hand the air has a general motion in a particular direction, the cells will be long strips, and fig. 57 will then represent the circulation in a vertical section at right angles to the direction of flow. The clouds then formed will be typical cumulus in long parallel strips. The same view of convection is put forward by Durst in his classification of eddies which is briefly described in § 129 below.

On this view the typical cumulus cloud is at the top of an ascending current, and the temperature within the cloud may be lower than that in the surrounding air, the cloud being partly supported by the ascending current. It is then not necessary to suppose the density within the cloud to be equal to the density outside at the same level, as was done by Kopp*; and Kopp's argument in favour of very great supersaturation within the cloud loses its validity.

§ 129. *Durst's classification of eddies*

Durst† classifies eddies into four main types, as follows:

Type I. Cumulo-nimbus present, and convection taking place in the upper layers of the troposphere (say about 2000 feet). Gusts are then spaced out at comparatively wide intervals, but the anemometer trace is irregular in both direction and velocity.

Type II. Instability in the lower troposphere (say below 2000 feet). The larger fluctuations of wind are now more rapid, but are still considerable in range in both velocity and direction.

Type III. Stable lapse-rate or inversion near the ground. The velocity and direction traces are both wide, but the variations are more rapid than in types I and II.

Type IV. With a strong inversion in the lowest layer, the fluctuations of wind almost disappear.

Durst notes that the tendency for the u-component to exceed the v-component is practically absent in type IV, while the eddies of the first three types, which are mainly on a larger linear scale, show that tendency to a marked degree.

Durst gives (*loc. cit.* fig. 81) a diagrammatic picture of the structure of the convectional eddies associated with rolls of cumulus clouds at about 2000 feet. He pictures the fronts at which gusts occur as forming a horseshoe pattern at the ground. The surfaces of which these are the intersections with the ground extend upward to the cloud level, and have ascending motion in front, and descending motion in the rear of each surface. Some of the anemometer records reproduced in the memoir appear to agree with this structure, in that they give rapid rise of wind followed by a decrease, this cycle of changes being reproduced at the passage of each cell over the anemometer.

§ 130. *Stability of motion*

Numerous writers have attacked directly the problem of determining the stability of laminar motion, and though the analysis used in some cases has been questioned by later writers it is generally regarded as probable that, for infinitely small disturbances, smooth flow is stable, and that a finite disturbance is required to produce turbulence in initially steady stream-line flow. In the atmosphere there is no difficulty in imagining the occurrence of finite disturbances, on account of the magnitude of the impediment to smooth flow produced by obstacles at the ground, and the effect of widely varying types of surface upon the flow of air.

* *Beitr. Phys. fr. Atmos.* **16**, 1930, p. 173.
† M.O., *Geophys. Mem.* No. 54, section III.

When a given state of motion becomes unstable two possible courses arise. The motion may become turbulent, so that its details become unpredictable, or it may change to another type of steady motion. The Bénard convection cell to which reference has been made above is a special case of the latter. In the initial state of unstable equilibrium of a liquid cooled from above or heated from below there is no motion, but when the equilibrium breaks down a steady motion sets in which is not turbulent. In his mathematical investigation of the conditions in the convection cell, Rayleigh assumed that the component velocities u, v, w, and the departure of the temperature from that corresponding to stable equilibrium, were all sufficiently small to permit of his neglecting all squares and products by comparison with terms of the first degree. He further assumed that these small quantities varied as $e^{ilx}e^{imy}e^{int}$. If in is real and positive the corresponding disturbance will increase exponentially with time. Rayleigh's treatment was essentially based on the assumption that the disturbance which had the largest rate of growth would eventually predominate. Taylor*, in his investigation of the motion of fluid between two rotating cylinders, followed substantially the same method, and found close agreement with his experimental investigations, in which the uniform circular motion of rotation about an axis broke down, and was replaced by toroidal ring flow combined with rotation about the common axis of the cylinders. In the Bénard-Rayleigh problem it is the density gradient which attains a critical value, but in Taylor's experiments it is the gradient of velocity whose critical value determines the change in the type of motion.

For the details of these investigations reference should be made to the original papers. They are mentioned here in order to emphasise the difference between a change to a new type of steady motion and a break-down of laminar flow leading to turbulent flow. In Bénard's experiments the motion in the cells is to be regarded as a steady motion, and in no sense as turbulent motion, and we should perhaps make it clear that in including a description of these phenomena in the present chapter we regard the cellular structure which may occur in the atmosphere as continually breaking down and re-forming.

In the next chapter will be found some further discussion of stability in relation to vertical distribution of density.

* *Phil. Trans. Roy. Soc.* A, **223**, 1922, p. 289.

CHAPTER XIII

TURBULENCE IN THE ATMOSPHERE: THE EDDIES AS DIFFUSING AGENCIES

§ 131. *Diffusion by eddies*

It is readily understood that in a convection current the air which rises from the ground on account of its excess of heat content will carry some of its excess heat into any higher layer with which it partially mixes. It will also carry some of its excess (or defect) of momentum in the same way. An eddy which moves from one place to another may therefore be regarded as an agency in a process of diffusion of heat, momentum, content of water-vapour or carbon dioxide, or other properties of the fluid.

The molecular diffusion of momentum in a given azimuth by viscosity, the molecular conduction of heat, and gaseous diffusion such as the diffusion of water-vapour into air, can all be represented by an equation of the form

$$\frac{dV}{dt} = k \frac{\partial^2 V}{\partial z^2},$$

where V represents component velocity, temperature, or water content per unit mass. For diffusion of momentum k has the value ν, the kinematic coefficient of viscosity; for conduction of heat, k has the value κ; and for the diffusion of water-vapour into air k has the value represented usually by D. The values of these coefficients are tabulated on p. 406.

§ 132. *The vertical transfer of heat by turbulence*

We here follow the main lines of the treatment of the diffusion of heat by turbulence given by G. I. Taylor*, as modified slightly by Brunt†. Let p, T, θ, ρ be the pressure, absolute temperature, potential temperature (referred to a standard pressure P), and density at any point. Let c_p be the specific heat of dry air at constant pressure. Then the quantity of heat required to raise the temperature of 1 gramme of air by an amount dT is $c_p dT$. The potential temperature θ is defined by

$$\theta = T\left(\frac{p}{P}\right)^{-\frac{\gamma-1}{\gamma}} \qquad \text{......(1)},$$

where γ is the ratio of the specific heats of air, and $(\gamma-1)/\gamma = 0{\cdot}29$ for dry air. For the present we shall neglect any effects due to the presence of water-vapour in the atmosphere, and the air will be regarded as dry.

* *Phil. Trans. Roy. Soc.* A, **215**, 1915, p. 1. † *Proc. Roy. Soc.* A, **124**, 1929, p. 201.

The effect of the motion of a large number of eddies in the atmosphere upon the distribution of heat is calculable. An eddy which is initially a normal sample of the air at its level moves vertically through the air, and eventually mixes with its surroundings at some new level. The effects of radiation and molecular conduction are neglected, being in any case much smaller in magnitude than the effects of mixing. It is now required to calculate the flux of heat across a large horizontal area A. The pressure p is taken to specify the vertical coordinate. The potential temperature at any point is $\theta(p, t)$ at time t. Let an eddy start from p_0 at time t_0, having the potential temperature $\theta(p_0, t_0)$ appropriate to that point, and let it reach a pressure p at time t. Let m be the mass of the eddy. Then provided $p_0 - p$ and $t_0 - t$ are both small

$$\theta(p_0, t_0) = \theta(p, t) + (p_0 - p)\frac{\partial\theta}{\partial p} + (t_0 - t)\frac{\partial\theta}{\partial t} \qquad \ldots\ldots(2).$$

The absolute temperature of the eddy at its new level will therefore be

$$\left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}}\left\{\theta(p, t) + (p_0 - p)\frac{\partial\theta}{\partial p} + (t_0 - t)\frac{\partial\theta}{\partial t}\right\} \qquad \ldots\ldots(3).$$

At each stage the eddy takes up the pressure of its surroundings, and in mixing it shares with its surroundings its excess or defect of thermal energy, which is measured by

mass of eddy × specific heat c_p × absolute temperature.

The motion of the eddy across the isobaric surface p is therefore equivalent to a flow of heat

$$mc_p\left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}}\left\{\theta(p, t) + (p_0 - p)\frac{\partial\theta}{\partial p} + (t_0 - t)\frac{\partial\theta}{\partial t}\right\}.$$

If we make the proviso that m shall be considered positive for upward moving eddies and negative for downward moving eddies, the net upward eddy flux of heat is

$$\Sigma mc_p\left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}}\left\{\theta(p, t) + (p_0 - p)\frac{\partial\theta}{\partial p} + (t_0 - t)\frac{\partial\theta}{\partial t}\right\} = c_p\left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}}\theta(p, t)\,\Sigma m$$

$$+ c_p\left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}}\frac{\partial\theta}{\partial p}\,\Sigma m\,(p_0 - p) + c_p\left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}}\Sigma m\,(t_0 - t)\frac{\partial\theta}{\partial t}.$$

There can be no resultant transfer of mass across the isobaric surface. Hence $\Sigma m = 0$. Also in $\Sigma m\,(t_0 - t)$, the factor $(t_0 - t)$ is always negative, and a given value of $t_0 - t$ is as likely to be associated with positive as with negative values of m. Thus the net upward eddy flux of heat reduces to the middle term above, or

$$c_p\left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}}\frac{\partial\theta}{\partial p}\,\Sigma m\,(p_0 - p) \qquad \ldots\ldots(4).$$

But
$$\frac{\partial\theta}{\partial p}=\frac{\partial\theta}{\partial z}\frac{\partial z}{\partial p}=-\frac{1}{g\rho}\frac{\partial\theta}{\partial z}=-\frac{1}{g\rho}\frac{\theta}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right),$$
where Γ is the dry adiabatic lapse-rate (see p. 39 above). Substituting
$$\theta=T\left(\frac{p}{P}\right)^{-\frac{\gamma-1}{\gamma}},$$
we find
$$\frac{\partial\theta}{\partial p}=-\frac{1}{g\rho}\left(\frac{p}{P}\right)^{-\frac{\gamma-1}{\gamma}}\left(\frac{\partial T}{\partial z}+\Gamma\right).$$
The eddy flux of heat across unit area per unit time is therefore
$$-\frac{c_p}{g\rho}\left(\frac{\partial T}{\partial z}+\Gamma\right)\Sigma m\,(p_0-p)=-K\rho c_p\left(\frac{\partial T}{\partial z}+\Gamma\right)\qquad\text{......(5)},$$
if $K=\dfrac{\Sigma m\,(p_0-p)}{g\rho^2}$, the summation extending over unit area and unit time.

Taylor gave the expression $-K\rho c_p\dfrac{\partial\theta}{\partial z}$, K being defined as $\overline{w\,(z_0-z)}$, where w is the vertical velocity, and z the vertical co-ordinate of an eddy which originated at z_0.

Expression (5) gives the net flux of heat across the isobaric surface p. The net gain of heat in the layer between p and $p+dp$ is
$$-\frac{\partial}{\partial p}K\rho c_p\left(\frac{\partial T}{\partial z}+\Gamma\right)dp=\frac{c_p}{g\rho}\frac{\partial}{\partial z}\left\{K\rho\left(\frac{\partial T}{\partial z}+\Gamma\right)\right\}dp.$$
Since the mass in a small cylinder of unit horizontal cross-section and height equivalent to dp is $-dp/g$, the last expression is equivalent to $\dfrac{dp}{g}c_p\dfrac{dT}{dt}$. Hence
$$\rho\frac{dT}{dt}=\frac{\partial}{\partial z}\left\{K\rho\left(\frac{\partial T}{\partial z}+\Gamma\right)\right\}\qquad\text{......(6)}.$$
On the left-hand side dT/dt is to be interpreted as implying differentiation following the isobaric surface p, which is equivalent to differentiation following the mean motion. The left-hand side may without serious error be reduced to $\rho\,\partial T/\partial t$, so that equation (6) becomes
$$\rho\frac{\partial T}{\partial t}=\frac{\partial}{\partial z}\left\{K\rho\left(\frac{\partial T}{\partial z}+\Gamma\right)\right\}\qquad\text{......(7)}.$$
The variations of ρ with height being relatively slow, we may without serious error write the equation in the form
$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial z}K\left(\frac{\partial T}{\partial z}+\Gamma\right)\qquad\text{......(8)}.$$
If further we neglect the variation of K with height, the equation is further simplified to the form
$$\frac{\partial T}{\partial t}=K\frac{\partial^2 T}{\partial z^2}\qquad\text{......(9)}.$$
Equations (7), (8) and (9) are to be regarded as approximations to equation (6), which is the equation directly derived by Taylor's analysis.

Equation (9) shows that there is an analogy between the diffusion of heat by eddies, and the ordinary diffusion of heat by molecular conduction, the coefficient K taking the place of the coefficient of thermal diffusivity in the equation. For this reason the quantity K is called the *eddy diffusivity*. It will be shown later that K is very much greater than the corresponding coefficient for molecular diffusion, and usually greater than the coefficient of radiative diffusivity (see § 75).

§ 133. *Richardson's treatment of diffusion by eddies*

Let Z stand for the property whose diffusion we wish to study, and let χ be the amount of Z per unit mass of air. χ will be a function of the height z, and of the time t. In a layer of thickness dz the amount of Z is $\chi\rho\, dz$ per unit of horizontal area. The upward flux of Z is the amount of Z which flows across unit horizontal surface. The rate of increase of Z in the layer dz is

$$-\frac{\partial}{\partial z}(\text{upward flux})=\frac{\partial}{\partial t}(\rho\chi) \qquad \ldots\ldots(10).$$

When an eddy moves from one level to another carrying its original quantity χ of Z with it, the new level gains an amount proportional to the difference of χ at the two levels, and if χ is uniform at all levels, there can be no gain or loss by transport. It thus appears reasonable to assume

$$\text{upward flux}=-c\frac{\partial\chi}{\partial z} \qquad \ldots\ldots(11),$$

where c will depend in part on the amount of air crossing the surface across which the flux is measured. It may also be a function of z, of χ, and of $\partial\chi/\partial z$, but it must remain finite when $\partial\chi/\partial z=0$, since by definition the flux must then be zero. Substituting in (10), we find

$$\frac{\partial}{\partial t}(\rho\chi)=\rho\frac{\partial\chi}{\partial t}=\frac{\partial}{\partial z}\left(c\frac{\partial\chi}{\partial z}\right) \qquad \ldots\ldots(12).$$

The limitations imposed upon χ are of importance. We must not use χ to represent properties whose measures are changed by delay or by transportation to a new level, and presumably χ must be such that its upward flux has a physical meaning. Properties which appear to satisfy these conditions are content of water-vapour, dust or carbon dioxide, etc., momentum in a fixed azimuth, deviation of temperature from an adiabatic distribution.

Richardson calls the quantity c the *eddy conductivity*. Its dimensions are $ML^{-1}T^{-1}$. Since $dp=-g\rho\, dz$, we might write (12) in the form

$$\rho\frac{\partial\chi}{\partial t}=g\rho\frac{\partial}{\partial p}\left(g\rho c\frac{\partial\chi}{\partial p}\right)=\rho\frac{\partial}{\partial p}\left(\xi\frac{\partial\chi}{\partial p}\right) \qquad \ldots\ldots(13),$$

where $$\xi=g^2\rho c.$$

Richardson calls ξ the *turbulivity*.

§ 134. *Application of Taylor's results to the atmosphere*

The equation of eddy transfer of heat will be used in the form (9) of p. 221, it being assumed that K does not vary with height. We then have

$$\frac{\partial T}{\partial t}=K\frac{\partial^2 T}{\partial z^2} \qquad \text{......(14).}$$

Equation (14) can be solved for a number of special cases in which the boundary conditions are specified. One of the most interesting of Taylor's applications of his formula was to discuss the changes in the distribution of temperature within a current of air which, after being heated during its passage over warm land, passed over a cold sea. Suppose the air on reaching the coast had a constant lapse-rate β, its surface temperature being T_0. Let the surface temperature of the sea be T_1. Then the air immediately in contact with the sea has its temperature suddenly lowered to T_1. The problem is then specified, so far as the boundary conditions are concerned. The solution of the equation which is appropriate is given in any textbook on the theory of the conduction of heat, and may be written

$$T=T_0-\beta z+(T_1-T_0)\left\{1-\frac{2}{\sqrt{\pi}}\int_0^{z/\sqrt{4Kt}} e^{-\mu^2}d\mu\right\} \qquad \text{......(15).}$$

The term which multiplies (T_1-T_0) in this equation is unity at $z=0$, and falls to 0·1 at a height given by $z/\sqrt{4Kt}=1{\cdot}2$. Taylor assumes that for all practical purposes this term has no effect beyond $z/\sqrt{4Kt}=1$. Thus in a time t the height to which the effect of the surface changes of temperature extends is given by

$$z/\sqrt{4Kt}=1, \quad \text{or} \quad z^2=4Kt \qquad \text{......(16).}$$

The observations which Taylor made of the vertical distribution of temperature above the Great Banks of Newfoundland showed a marked inversion in the lower layers, with a lapse-rate approaching the dry adiabatic at higher levels. From the height at which the inversion ceased, combined with an estimate of the time which the air had been flowing above the cold sea, Taylor deduced values of K of the order of 10^3 in C.G.S. units. He found K to be 1×10^3 with winds of force 2, and 3×10^3 with winds of force 3. It must be borne in mind that in inversions the atmosphere is very stable, and that the upward movement of eddies is thereby checked. Thus we should expect to find with large lapse-rates greater values of K than were found by Taylor during inversions over the Great Banks.

The relation

$$z^2=4Kt$$

for the height to which turbulence is effective in a time interval t was derived on the supposition that the surface temperature underwent a sudden change. But the result is only slightly modified when the rate of change of temperature at the surface is uniform. Let the surface temperature diminish n° per unit time, so that the surface temperature is

$$T_0-nt.$$

Then at time t the temperature at height z is

$$T_0-\beta z-nt\left\{\left(1+\frac{z^2}{2Kt}\right)\left(1-\frac{2}{\sqrt{\pi}}\int_0^{z/\sqrt{4Kt}}e^{-\mu^2}\,d\mu\right)-\frac{2}{\sqrt{\pi}}\frac{z}{\sqrt{4Kt}}e^{-z^2/4Kt}\right\}\quad\ldots(17).$$

The term which multiplies nt in this expression is unity at the surface, and 0·1 at $z/\sqrt{4Kt}=0\cdot8$. The height attained in time t is therefore rather less than in the case of a sudden fall of the surface temperature, but we may still take as an approximation

$$z^2=4Kt.$$

This method gives only the order of magnitude of K, but at this stage of the development of the subject an estimate of the order of magnitude of the effect is of value, in that it shows that turbulence must be a much more effective agent than molecular diffusion in the transport of heat in the vertical. Even in quiet conditions, such as those over the Great Banks of Newfoundland investigated by Taylor, the mean value of K from the surface to 1000 feet is of the order of 10^3, and therefore at least as great as K_R, the coefficient of radiative diffusivity. In more normal conditions K for the same levels is of the order of 10^5, and the effect of eddies on the transfer of heat in the vertical direction is then enormously greater than that of radiation, which is itself enormously more effective than molecular conduction.

At quite low levels, a few inches from the ground, K is of the order of 10^{-1}. It increases with height at first, probably up to about 300 or 500 metres above the ground, after which there is a slow decrease.

A more elaborate discussion of the equations of heat transfer with the boundary conditions assumed above will be found in Riemann-Weber, *Die Differential- und Integral-Gleichungen der Mechanik und Physik*, **2**, p. 220.

§ 135. *The effect of turbulence on the diurnal variation of temperature*

If we assume that the transfer of heat in the vertical is brought about entirely by eddies, we can readily find the change with height in the form of the curve of diurnal variation of temperature, for the case when K is constant with height. Let the temperature at the ground be given by

$$T=T_0+A\sin qt\qquad\ldots\ldots(18).$$

The solution of the equation

$$\frac{\partial T}{\partial t}=K\frac{\partial^2 T}{\partial z^2}\qquad\ldots\ldots(19),$$

which has this boundary condition at $z=0$, is

$$T=T_0-\beta z+Ae^{-bz}\sin(qt-bz)\qquad\ldots\ldots(20),$$

where b is a constant defined by $b^2=q/2K$. The term $-\beta z$ is included on the right-hand side of the equation to allow for the mean lapse-rate during the period.

Now the diurnal variation of temperature at the ground can be represented with reasonable accuracy by a single sine-term of period 24 hours. Thus

$$q=2\pi/24\times60\times60=7\cdot3\times10^{-5}.$$

Then the diurnal variation at any height z is in accordance with equation (20) above. The amplitude Ae^{-bz} falls off exponentially, the ratio of the amplitudes at any two heights z_1 and z_2 being $e^{-b(z_1-z_2)}$. The lag in the occurrence of maximum temperature from z_1 to z_2 is $b(z_2-z_1)$, and from this, or from the ratio of amplitudes, the value of b can be readily derived. Taylor compared this theory with observations made on the Eiffel Tower at heights of 18, 123, 197 and 302 metres above the ground. By evaluating K for a number of stages from the ground upward, he found a definite tendency for K to increase with height in summer, and to decrease with height in winter, the mean value of K for the whole year, deduced from a comparison of the amplitudes at 18 and 302 metres, being 10^5. It is therefore seen that the mean value of K for the Eiffel Tower is much in excess of the values which Taylor derived from observations of inversions over the sea. The highest values of K at the Eiffel Tower occur in summer, when large lapse-rates are most frequent. In winter the lapse-rates are much smaller, and turbulence is much less active. Presumably the eddies in winter are on a smaller scale, and do not disturb the atmosphere to such heights as in the summer.

Theoretically it should be possible to compute K also from the lag in the time of maximum between any two levels, but in practice this method is unreliable on account of the difficulty in estimating accurately the extent of the lag. It is also open to doubt whether the afternoon temperatures at the top of the tower are altogether reliable, on account of the effect of the solarisation of the tower itself. It is not likely that the total range of temperature is very seriously affected by this, but it is probable that the time of maximum is appreciably affected.

Let us see how the diurnal variation of temperature should diminish with height if we adopt the value $K=10^5$. We then have $b^2=\pi/10^5\times 24\times 60\times 60$ and $b=2.10^{-5}$ approximately. The diurnal variation at a height z bears to the diurnal variation at the ground the ratio e^{-bz}. If we measure z in metres this becomes $e^{-2.10^{-3}z}$. The diurnal variation thus falls to $1/e$ (or 0·37) of the surface value at $z=500$ metres, and to $1/e^3$ (or 0·05) at 1500 metres. This is in fairly close agreement with observation. The Lindenberg observations give for the amplitude of the 24 hour variation of temperature the following values:

Height in km	0	0·5	1·0	1·5	2·0
Amplitude °C	3·0	1·1	0·7	0·5	0·2

These figures agree fairly well with our conclusions based on a value $K=10^5$, up to 500 metres, but beyond this level the agreement is less accurate, and appears to indicate an increase in the value of K at heights above 500 metres. The comparison must not be pushed too far, however, as the data are a little uncertain above 500 metres, and the effect of the condensation of water-vapour begins to be important at heights of about 1 km and above.

Similar arguments to those used above might be applied to the second, third, and higher harmonics of the diurnal variation of temperature, the only difference in the analysis being that the value assigned to q will now be 2, 3, etc.

times the value previously used. Since b is proportional to $\sqrt{q}$, its value will increase with the order of the harmonic, and thus e^{-bz} will diminish with increasing order of the harmonic. Thus the diurnal variation of temperature, if transferred upward purely by the effects of turbulence, should approach more and more closely with increasing height to a sine-curve of period 24 hours.

One other feature of the equations of eddy transfer of heat requires to be emphasised. The net upward flux of heat across unit horizontal surface is

$$-K\rho c_p \left(\frac{\partial T}{\partial z} + \Gamma\right) \qquad \text{......(21)},$$

where Γ is the dry adiabatic lapse-rate. If the lapse-rate $-\partial T/\partial z$ is less than Γ, or if the atmosphere is stable, this expression is negative, and the flow of heat produced by the eddies is downward. The direction in which heat is transferred by eddies is downward or upward according as the atmosphere is stable or unstable. If the atmosphere is stable, churning it up will cause the bottom to grow warm and the top to grow cold, until the lapse-rate becomes equal to the dry adiabatic, after which further churning can produce no result.

§ 136. *Taylor's discussion of the eddy transfer of momentum*

If the wind varies with height it is natural to suppose that an eddy originating at one height and travelling to another in which the velocity is different will transfer horizontal momentum to its new level. Let U_z, V_z be the average components of velocity at height z parallel to rectangular axes x, y in a horizontal plane, and let u', v', w' be the three components of the eddy velocity, so that the three components of the total velocity of the air at x, y, z are

$$U_z + u', \quad V_z + v', \quad w'.$$

The rate of transfer of x-momentum upward across a horizontal surface is

$$\iint \rho\,(U_z + u')\,w'\,dx\,dy = \iint \rho u' w'\,dx\,dy \qquad \text{......(22)},$$

and of y-momentum $\iint \rho\,(V_z + v')\,w'\,dx\,dy = \iint \rho v' w'\,dx\,dy$ ......(23),

each integral being taken over the area in question. The terms $\iint \rho U_z w'\,dx\,dy$, $\iint \rho V_z w'\,dx\,dy$ vanish, since it is assumed that there is no net transfer of mass across the surface. Relations (22) and (23) should be compared with equations (5) and (7) of Chapter XII and it is then seen that what Reynolds calls the eddy stresses are the rates of transport into unit volume of momenta parallel to the co-ordinate axes.

The immediate problem, which is in fact the central problem to be faced in the discussion of turbulence, is that of putting expressions (22) and (23) into a form which involves only the mean motions U_z, V_z, W_z, and their derivatives relative to x, y and z. Schmidt and Prandtl assume that each eddy conserves the momentum of the layer in which it originates, so that

$$U_z + u' = U_{z_0}, \quad V_z + v' = V_{z_0},$$

where z_0 is the height at which the eddy originated. These equations may be written

$$u' = U_{z_0} - U_z = \frac{\partial U}{\partial z}(z_0 - z) + \frac{1}{2}\frac{\partial^2 U}{\partial z^2}(z_0 - z)^2 + \ldots \text{etc.}$$

If it is assumed that $z_0 - z$ is small so that the first term of the infinite series predominates,

$$u' = \frac{\partial U}{\partial z}(z_0 - z).$$

The integral for the eddy transfer of momentum upward then becomes

$$U\iint \rho w'\,dx\,dy + \iint \rho\,(z_0 - z)\,w'\,\frac{\partial U}{\partial z}\,dx\,dy.$$

Since there is no net transfer of mass across the horizontal plane, the first term is zero, and the second term may be written

$$\iint \rho\,(z_0 - z)\,w'\,\frac{\partial U}{\partial z}\,dx\,dy = -K\rho\,\frac{\partial U}{\partial z} \text{ per unit surface} \quad ......(24),$$

where K is readily seen to have the same meaning as was previously given to it in discussing the transfer of heat in § 132.

The net rate of gain of momentum by unit volume at height z is

$$\frac{\partial}{\partial z}\left(K\rho\,\frac{\partial U}{\partial z}\right) \quad ......(25).$$

This is the method which has been followed by Schmidt and other writers. The coefficient which Schmidt* calls the *Austausch* is equal to $K\rho$ in our notation. Prandtl follows substantially the same method, and defines the mean values of $z - z_0$ as the *Mischungsweg* or "path of mixing".

Taylor starts from a different standpoint. He makes it clear that the irregularities of motion in a turbulent fluid are associated with irregularities of distribution of static pressure, and emphasises the possibility of these pressure differences influencing the horizontal momentum of the moving eddies. In his paper of 1915 he restricts the motion to two-dimensional flow, the mean motion being parallel to the axis of x, and the turbulent flow being restricted to the xz plane. When the motion is thus restricted to two dimensions the *vorticity* of any element of the fluid is not affected by the local variations of pressure, and it is the constancy of vorticity, and not the constancy of momentum, of a moving element which we must apply in order to evaluate the net loss of momentum per unit volume of the turbulent fluid.

From (22) above it follows that the net gain of momentum per unit volume

$$= -\frac{\partial}{\partial z}\iint u'w'\,dx\,dy = I, \text{ say.}$$

We neglect the variations of density ρ, and treat the fluid as incompressible. Then

$$-I = \rho\iint\left(u'\,\frac{\partial w'}{\partial z} + w'\,\frac{\partial u'}{\partial z}\right)dx\,dy \quad ......(26).$$

The equation of continuity is $\quad \dfrac{\partial u'}{\partial x} + \dfrac{\partial w'}{\partial z} = 0 \quad ......(27).$

* W. Schmidt, *Der Massenaustausch in freier Luft und verwandte Erscheinungen.* Hamburg, 1925.

The condition that the moving element retains its original vorticity may be written

$$\frac{\partial}{\partial z}(U_z+u')-\frac{\partial w'}{\partial x}=\frac{\partial U_0}{\partial z} \qquad \text{......(28).}$$

Substituting in (26) the values of $\partial w'/\partial z$ and $\partial u'/\partial z$ given by equations (27) and (28), we find

$$-I=\rho\iint\left\{-u'\frac{\partial u'}{\partial x}+w'\frac{\partial w'}{\partial x}+w'\left[\left(\frac{\partial U}{\partial z}\right)_{z_0}-\left(\frac{\partial U}{\partial z}\right)_z\right]\right\}dx\,dy$$

$$=\tfrac{1}{2}\rho\iint\frac{\partial}{\partial x}(w'^2-u'^2)\,dx\,dy+\rho\iint w'\left[\left(\frac{\partial U}{\partial z}\right)_{z_0}-\left(\frac{\partial U}{\partial z}\right)_z\right]dx\,dy.$$

The first term integrates out and vanishes, since it may be assumed that $\overline{u'^2}$ and $\overline{w'^2}$ do not vary over the horizontal area. It follows that

$$-I=\rho\iint w'\left\{(z_0-z)\frac{\partial^2 U}{\partial z^2}+\tfrac{1}{2}(z_0-z)^2\frac{\partial^3 U}{\partial z^3}+\ldots\right\}dx\,dy \qquad \text{......(29).}$$

This equation is rigidly true for all disturbances, but if z_0-z is sufficiently small so that within this limit the changes in $\frac{\partial^2 U}{\partial z^2}$ are small by comparison with itself, we may write down the first term only of the series

$$I=-\rho\iint w'(z_0-z)\frac{\partial^2 U}{\partial z^2}\,dx\,dy=\rho\overline{w'(z-z_0)}\frac{\partial^2 U}{\partial z^2} \qquad \text{......(30),}$$

where $\overline{w'(z-z_0)}$ is the mean value of $w'(z-z_0)$ taken over the horizontal area. Or, returning to the notation of equation (24),

$$I=K\rho\frac{\partial^2 U}{\partial z^2} \qquad \text{......(31),}$$

I represents the rate of loss of momentum by eddy transfer from unit volume of the turbulent medium.

It is of interest to compare equation (31) with equation (25). The latter may be written

$$I=\frac{\partial}{\partial z}\left(K\rho\frac{\partial U}{\partial z}\right)=K\rho\frac{\partial^2 U}{\partial z^2}+\rho\frac{\partial K}{\partial z}\frac{\partial U}{\partial z} \qquad \text{......(32),}$$

neglecting the variations of ρ with height. Equation (31) only contains the first term on the right-hand side of the above equation. Since the terms on the right-hand side of equation (32) represent the net difference between the transfers into and out of unit volume, whereas the first term, according to Taylor's analysis, represents the net gain or loss of momentum by unit volume, it follows that the second term on the right-hand side of (32) must represent the rate at which momentum is destroyed by the action of the local differences of pressure.

Taylor's proof of equation (31), which has been reproduced above from his original paper, makes no assumption that K is constant at all heights, and equation (29) can therefore be used for K varying with height. Taylor's discussion of the nature of the distribution of temperature and wind-velocity with height, based on the assumption that K is constant, shows that the theory predicts results in reasonable agreement with observation. The assumption of constant K simplifies the mathematical treatment, and is in any case the obvious assumption to make in a first attempt to compare the theory with observations. Some further details of the variation of wind with height with different assumptions as to the nature of the variation of K with height will be found in § 141 below.

In view of the importance of the result shown in equation (31) it is of interest to consider a restatement of this proof given by Taylor in a recent paper*. The motion is again restricted to two dimensions, x and z. If we write η for the vorticity $\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$, we may write the equation of motion in the x-direction

$$-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\tfrac{1}{2}u^2+\tfrac{1}{2}w^2\right)=\frac{\partial u}{\partial t}+2w'\eta \qquad \text{......(33).}$$

This equation is rigidly true for each element of mass. Again neglecting variations of ρ, we may write it

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2+\tfrac{1}{2}w^2\right)-2w'\eta \qquad \text{......(34).}$$

If we suppose the eddying motion to be on the average uniform in the direction of x, the second term on the right-hand side vanishes when we take mean values

$$\frac{\partial U}{\partial t}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}-\overline{2w'\eta} \qquad \text{......(35).}$$

Thus the term $-\overline{2w'\eta}$ represents the rate of increase of mean velocity as a result of the eddying motion. If the eddy has retained the vorticity which it had as a result of the mean motion when at height z_0, then at height z it still has a vorticity $(z_0-z)\frac{\partial}{\partial z}\left(\frac{1}{2}\frac{\partial U}{\partial z}\right)$ in excess of the normal vorticity η at that level

$$2w'\eta=2w'\left\{\eta_0+\tfrac{1}{2}(z_0-z)\frac{\partial^2 U}{\partial z^2}\right\},$$

and taking mean values over a large horizontal area

$$\overline{2w'\eta}=\overline{w'(z_0-z)}\frac{\partial^2 U}{\partial z^2} \qquad \text{......(36).}$$

Substituting in equation (35) we find

$$\frac{\partial U}{\partial t}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}-\overline{w'(z_0-z)}\frac{\partial^2 U}{\partial z^2}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\overline{w'(z-z_0)}\frac{\partial^2 U}{\partial z^2},$$

$$\frac{\partial U}{\partial t}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+K\frac{\partial^2 U}{\partial z^2} \qquad \text{......(37).}$$

Hence the rate of gain of momentum due to the eddying motion is

$$K\rho\frac{\partial^2 U}{\partial z^2} \text{ per unit volume.}$$

* *Proc. Roy. Soc.* A, **135**, 1932, p. 685.

§ 137. *Extension to three dimensions*

In the latest paper to which reference was made above Taylor has extended the vorticity-transport theory to three dimensions, but as Taylor remarks the results which he derives are so complicated as to be of little practical use. The lengthy expressions which he derives for the eddy transport of momentum reduce to the result of the earlier discussion (equation (31) above), in the special case of two-dimensional flow parallel to the xz plane. When the turbulent flow is parallel to the plane of yz the results reduce to Prandtl's form, shown in equation (32) above.

In three-dimensional flow we can no longer assume that the moving eddy retains its original components of vorticity, since the vorticity will be affected by the local variations of pressure. The mathematical conditions become far more complex, and there is no simple condition which can take the place of the constancy of vorticity which was the basis of the two-dimensional treatment.

The momentum-transport theory, as developed by Prandtl, Schmidt, and others, is directly applicable to three-dimensional motion, since the u and v components of velocity are treated separately and independently, and the expressions $\frac{\partial}{\partial z}\left(K\rho\,\frac{\partial U}{\partial z}\right)$ and $\frac{\partial}{\partial z}\left(K\rho\,\frac{\partial V}{\partial z}\right)$ are derived together by the same argument.

In applications to the study of turbulent motions in the atmosphere we are faced with the difficulty that observations show that the turbulent motion is in three dimensions, and that near the ground the cross-wind or v component is greater than either of the other two components. The use of Taylor's expression $K\rho\,\frac{\partial^2 U}{\partial z^2}$, or of Prandtl's expression $\frac{\partial}{\partial z}\left(K\rho\,\frac{\partial U}{\partial z}\right)$, cannot be held to be a complete representation of the conditions which we know to exist.

Many writers on this subject have been attracted by the analogy between eddy viscosity and ordinary viscosity. The analogy appears to be a true one for two-dimensional motion, but is no longer strictly true for three-dimensional motion, and we are left in some doubt as to how close this analogy really is. There is in fact no mathematical theory of turbulence in three dimensions which is in a form applicable to motion in the atmosphere. The serious obstacle to the dynamical study of turbulence is the difficulty of visualising the nature of a single eddy. It is, moreover, probable that eddies should be divided into a number of classes, each of which might behave differently with regard to the transfer of mass, momentum, vorticity, etc. It is possible to form a mental picture of the working of a simple convection bubble, and to visualise it as an agent in the transfer of heat and momentum in the vertical direction. But a vortex ring which mixes with its surroundings can impart no vorticity to those surroundings, since any two diametrically opposite arcs have vorticity of opposite signs, and will cancel each other when complete mixing takes place.

Prandtl's form $\frac{\partial}{\partial z}\left(K\rho\frac{\partial U}{\partial z}\right)$ is applicable to motion in a tube in which the turbulence is in a plane at right angles to the direction of the mean motion, or the vortices have axes parallel to the mean motion. In some ways the atmosphere appears more nearly analogous to the motion in a tube than to the two-dimensional motion discussed by Taylor, but the conditions in the atmosphere are complicated by the change in the direction of the mean wind with height. Taylor's equations for the transport of vorticity in three dimensions are so complicated that one is forced to conclude that the atmospheric problem is not capable of dynamical solution, and that the line of approach suggested by Taylor in another paper, which starts from a more purely statistical basis, is the most hopeful. This method is discussed in §§ 147–51 below.

It should be noted that a frictional term such as $K\,\partial^2 U/\partial z^2$ tends to annihilate existing differences of velocity, no matter how K varies with height; while a frictional term $\frac{\partial}{\partial z}\left(K\frac{\partial U}{\partial z}\right)$ will not of necessity do so, but may in fact lead to an accentuation of the existing differences of velocity.

§ 138. *Comparison of the momentum-transport and the vorticity-transport theories*

In the paper referred to above Taylor compares the results derived for the transport of heat and momentum on the momentum-transport theory and on the vorticity-transport theory respectively, by considering the distribution of temperature and velocity in the wake behind a cylindrical obstacle. Taylor shows that on the Prandtl theory the distribution of temperature and of velocity across the wake should follow the same law, as is indeed obvious *a priori*, whereas on the vorticity-transport theory these distributions should differ.

If η_0 is the value of the vorticity η at the edge of the wake, and $\xi=\eta/\eta_0$, then the distribution of velocity on either theory should be given by

$$\frac{u}{u_0}=(1-\xi^{\frac{3}{2}})^2 \qquad \text{......(38).}$$

The distribution of temperature should be given by

$$\frac{\theta}{\theta_0}=(1-\xi^{\frac{3}{2}})^2 \qquad \text{......(39),}$$

on the momentum-transport theory, and by

$$\frac{\theta}{\theta_0}=1-\xi^{\frac{3}{2}} \qquad \text{......(40)}$$

on the vorticity-transport theory. Measurements of the distribution of temperature and velocity in the wake behind a heated obstacle carried out at the National Physical Laboratory showed that the distribution of velocity agreed very closely with equation (38) above, and that the distribution of temperature agreed closely with equation (40) above, but failed to agree even approximately with equation (39). Taylor claims that it is thus established that, at least in these conditions, the vorticity-transport theory is a better representation of the facts than the momentum-transport theory.

The argument is not completely convincing. The velocity u is represented by

$$\frac{u}{u_0} = \phi(x) f(\eta)$$

and $\phi(x)$ is taken as $x^{\frac{1}{2}}$. Taking

$$f(\eta) = e^{-a\eta^2} \qquad \ldots\ldots(41)$$

we shall have a solution of the equation which fits the observational data with slightly greater accuracy than equation (40). The adoption of equation (41) requires that K should be constant, on either the momentum-transport or the vorticity-transport theory. The temperature observations can be fitted by a similar equation with a constant K, which is, however, different from the K for momentum. It must be remembered that the corresponding constants for molecular transport are different, being κ the coefficient of thermal diffusivity, and ν the kinematic coefficient of viscosity. (See also § 139 below.)

§ 139. *Prandtl's theory of the Mischungsweg*

Reynolds showed that the effect of turbulent motion might be taken into account by the addition to the equations of motion of certain shearing stresses of the type

$$\widehat{xz} = -\rho\overline{u'w'},$$

where u' and w' are the components of eddy velocity, or the deviations of the instantaneous velocities u, w from their true mean values $\bar{u}$, $\bar{w}$. The problem consists essentially in finding a method of expressing $\overline{u'w'}$ in terms of measurable quantities. The assumption which Prandtl* makes is that an eddy consists of a mass of fluid originally a normal sample of the fluid at level z, which moves to a new level $z+l$, and there mixes with its environment. Its eddy velocity just before mixing is $\bar{u}(z) - \bar{u}(z+1)$, which to a first approximation is $-l\frac{\partial \bar{u}}{\partial z}$. The parameter l is called the *Mischungsweg*, or mixing length. It is to be noted that the theory assumes that mixing is a discontinuous process.

Prandtl further assumes that w' must be of the same order of magnitude as u' and proportional to it, so that

$$\widehat{xz} = \pm\rho l^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2 \qquad \ldots\ldots(42)$$

except for a possible factor of proportionality on the right-hand side. This factor may be taken as unity, thereby adding somewhat to the indefinite nature of the parameter l. In order to determine which sign is appropriate, take a case of mean velocity increasing with height. Then upward motion, with w' positive, involves negative u', and *vice versa*, and the equation must be written

$$\widehat{xz} = +\rho l^2 \left|\frac{\partial u}{\partial z}\right| \frac{\partial u}{\partial z} \qquad \ldots\ldots(43).$$

* *Vide* Prandtl, *The Physics of Solids and Fluids* (Blackie), pp. 277–83; or Prandtl, *Abriss der Strömungslehre* (Vieweg).

Schmidt's *Austausch* coefficient A may then be written

$$A = \rho l^2 \left| \frac{\partial u}{\partial z} \right| \qquad \ldots\ldots(44).$$

Von Kármán has investigated the conditions under which disturbances can be similar to each other when the states of flow in the neighbourhood of two points differ only in the scale of the disturbed velocities, and has found from the form of the differential equation that l must be proportional to $\frac{\partial \bar{u}}{\partial z} \Big/ \frac{\partial^2 \bar{u}}{\partial z^2}$. If we assume that $\widehat{xz}$ is constant in a given region (say near a solid boundary), this form for l leads to an equation which can be readily integrated. Let $\widehat{xz}/\rho = C^2$, since $\partial \bar{u}/\partial z$ is positive. Then from (42)

$$C^2 = k^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^4 \Big/ \left(\frac{\partial^2 \bar{u}}{\partial z^2}\right)^2 \quad \text{or} \quad \frac{\partial^2 \bar{u}}{\partial z^2} \Big/ \left(\frac{\partial \bar{u}}{\partial z}\right)^2 = -\frac{C}{k},$$

when $l = k \frac{\partial \bar{u}}{\partial z} \Big/ \frac{\partial^2 \bar{u}}{\partial z^2}$. This equation yields on integration

$$k \Big/ \frac{\partial \bar{u}}{\partial z} = Cz + C',$$

$$\frac{\partial \bar{u}}{\partial z} = \frac{k/C}{z + C_1},$$

$$\bar{u} = k/C \log_e (z + C_1) + C_2 \qquad \ldots\ldots(45).$$

The application of von Kármán's formula leads to difficulties when $\partial \bar{u}/\partial z$ or $\partial^2 \bar{u}/\partial z^2$ is zero, but it is possible to overcome most of these difficulties by a slight complication of the formula.

It is of interest to consider the application of von Kármán's formula to motion near a boundary, where

$$u \propto z^q.$$

Then

$$l \propto z \quad \text{or} \quad l = cz \qquad \ldots\ldots(46)$$

independently of the value of q.

Experiments indicate that $c = 0{\cdot}4$. The form of equation (46) is a reasonable one from the point of view of dimensional analysis, since z, the distance from the boundary, is the only length available in the problem.

If this result is applied to flow in a pipe as above, z being the distance from the boundary, then since $\partial \bar{u}/\partial z$ is positive near the boundary,

$$C^2 = \frac{\widehat{xz}}{\rho} = c^2 z^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2,$$

and this equation may be written (with $c = 0{\cdot}4$)

$$2{\cdot}5 C \frac{dz}{z} = du,$$

$$u = 2{\cdot}5 C \log_e \frac{z}{C'} \qquad \ldots\ldots(47).$$

If this extends to the centre of the pipe, where $z=r$,

$$u_{max}-u=2\cdot 5C\log_e\frac{r}{z} \qquad \ldots\ldots(48),$$

$$u_{max}-u=5\cdot 75\sqrt{\frac{\widehat{xz}}{\rho}}\log_{10}\frac{r}{z} \qquad \ldots\ldots(49).$$

Equation (49) gives a very close fit to the observations* in a tube, from the boundary right up to the centre. This is all the more remarkable since the equation is based on the assumption that $\widehat{xz}/\rho$ is constant. Thorade suggests that the variations in $\widehat{xz}/\rho$ are compensated by variations in l, which keep the formula correct within the limits of observational error. The most remarkable feature of equation (49) is that it holds for flow with widely varying Reynolds numbers, and brings within the ambit of one formula all the experimental data available.

The maximum velocity u_{max} naturally depends on the roughness of the wall. If k is the linear dimension of the inequalities of the wall, the velocity u will depend on the ratio z/k. The Göttingen experiments show that the constant C' of equation (47) is equal to $k/30$. When this value is substituted, equation (47) becomes

$$u=5\cdot 75\sqrt{\frac{\widehat{xz}}{\rho}}\log\frac{30z}{k},$$

or

$$u=\sqrt{\frac{\widehat{xz}}{\rho}}(5\cdot 75\log z/k+8\cdot 5) \qquad \ldots\ldots(50).$$

Prandtl has pointed out that for the flow in a pipe Taylor's vorticity-transport theory does not give an interpretation of the flow. In Taylor's notation, neglecting variation of ρ,

$$\frac{\partial}{\partial y}(\widehat{xz})=K\rho\frac{\partial^2 u}{\partial z^2}=\rho l^2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}=0.$$

This must be zero if the stress $\widehat{xz}$ is constant. It follows that either $l=0$, or $\partial^2 u/\partial z^2=0$. The solution $l=0$ is not consistent with the existence of transport of heat or momentum. If $l\neq 0$, $\partial^2 u/\partial z^2=0$, and

$$u=a+bz \qquad \ldots\ldots(51),$$

where a and b have arbitrary values which have no connection with the value of $\widehat{xz}$ at the boundary. The reason for the failure of Taylor's equation in this case is the assumption of two-dimensional motion in the plane of xz, whereas the flow near the boundary is, as shown by Fage, three-dimensional, more nearly transverse to the mean motion than in normal planes parallel to the direction of mean motion.

Prandtl also pointed out (*loc. cit.*) that in the wake of a long cylindrical body there would be a tendency to form vortices having their axes parallel to the mean motion, so that, as shown by Taylor, the vorticity-transport theory

* Prandtl, V. D. I., *Zeit. Vereins deut. Ingen.* **77**, 1933, Nr. 5, p. 105; or a synopsis by Thorade, *Ann. Hydr. u. Mar. Met.* 1933, Abb. 31, Nr. 2, p. 239.

would fit the facts more closely than the momentum-transport theory. There is therefore no difficulty in appreciating why the turbulence in the wake of a long cylindrical body obeys a different law from the turbulence near a boundary. Prandtl showed that near a boundary vortices with their axes parallel to the direction of mean motion would be independent of local pressure differences, and the flow of heat and of momentum should therefore be similar.

An excellent account of the work of Prandtl, von Kármán, and others of the Göttingen school is given in a paper by MacColl in the *Journal of the Royal Aeronautical Society*, August, 1930.

Rossby has considered the application of the idea of the mixing length in atmospheric problems, and has somewhat generalised von Kármán's method of applying dynamical similitude to these problems, showing that this method only applies strictly when $\partial^3 u/\partial z^3$ and higher derivatives all vanish. It is probable that near the ground the curvature of the velocity-height curve is sufficiently small to ensure this.

Prandtl's theory of the *Mischungsweg* has inspired a large volume of work on the Continent, and it appears to provide a method of correlating experimental observations, more particularly in the field of aerodynamics. The *Mischungsweg* l should be regarded, however, in the light of a correlation factor rather than a truly physical quantity. Indeed all meteorological observations tend to show that mixing is a continuous process rather than an explosively discontinuous one.

§ 140. *Eddy diffusivity and Austausch for different properties*

The eddy diffusivity and the *Austausch* coefficient can be defined separately for momentum, temperature, etc. In § 136 above K was defined as the mean value of $w'l'$, where l' is the vertical distance traversed by the eddy since it was last a normal specimen of its immediate environment, for the property which we are considering. It is clear that if we approach the subject from this point of view we may regard l' as differing for momentum, heat, etc. In the first place consider the diffusion of momentum.

Taylor has shown that $\overline{w'l'}$ may be written in several other forms, and that l' may be replaced by a distance l which the observed element has moved since some initial time t_0, provided t_0 is sufficiently early to ensure that each observed element has at least once since t_0 been a normal portion of its immediate surroundings. Then

$$\overline{w'l'} = \overline{w'l} - \overline{w'(l-l')} = \overline{w'l} \qquad \ldots\ldots(52)$$

since there is no correlation between w' and $l-l'$; also

$$\overline{w'l'} = \overline{w'l} = \overline{\frac{dl}{dt}\,l} = \frac{1}{2}\frac{d}{dt}\overline{l^2} \qquad \ldots\ldots(53).$$

For the diffusion of heat let the corresponding values of l be represented by l''. Then

$$\overline{w'l''} = \overline{w'l'} + \overline{w'(l''-l')} \qquad \ldots\ldots(54).$$

If inequalities of heat were diffused more slowly than inequalities of momentum, then l'' would generally be measured from an earlier instant than l', and there would be no correlation between w' and $l''-l'$. In this case

$$\overline{w'l''}=\overline{w'l'}=\overline{w'l} \qquad \text{......(55).}$$

If, however, the inequalities of heat were diffused more rapidly than inequalities of momentum, then l'' would generally be measured from a later instant than l', and $l'-l''$ would in general have the same sign as w'. Hence

$$\overline{w'l''}=\overline{w'l'}-\overline{w'(l'-l'')},$$

$$\overline{w'l''}=\overline{w'l'}-\text{a positive quantity},$$

and

$$\overline{w'l''}<\overline{w'l'} \qquad \text{......(56).}$$

Thus the value of K appropriate to momentum should be at least as great as that appropriate to heat or any other diffusing property, and may conceivably be very much greater. Taylor* has quoted some figures showing that in the sea K may be at least nineteen times as great for momentum as for salinity.

§ 141. *The variation of wind with height*

(*a*) THE LAYER NEAR THE GROUND (0–10 METRES)

Observations of the flow of liquids in pipes have shown that near the boundary the mean velocity in turbulent flow can be represented by a fractional power law of the distance from the boundary, i.e.

$$u \propto z^{\frac{1}{n}} \qquad \text{......(57).}$$

The constant n is equal to 7 for a wide range of conditions (say up to values of 50,000 for Reynolds number), but increases as the Reynolds number increases, being approximately 10 when Reynolds number is 10^6. A full account of the researches into these questions will be found in Wien-Harms, *Handbuch der Experimental Physik*, **4**, pt 4, Chapter IV. It may be also noted in passing that Stanton's† observations of velocity distribution in air have been shown by von Kármán‡ to fit a 1/7th power law with great accuracy.

In the layer of the atmosphere near the ground the conditions approach those near the boundary of the tube containing the fluid, the deviating force being negligible by comparison with the turbulent stresses and the pressure gradient. The distribution of velocity with height should therefore approximate to a fractional power law.

Sutton§ has computed the value of $1/n$ for observations at Leafield, and has found that it varies in summer from about 1/6 at midnight to about 1/14 in the afternoon, and in winter from 1/8 at night to about 1/12 in the afternoon. The lapse-rate appears to have a very marked effect on the value of $1/n$, the largest values appearing in inversions and the smallest values in large lapse-rates. A similar investigation by Barkat Ali‖ of the winds at Agra for the range

* *Brit. Ass. Rep., London*, 1931.
† *Proc. Roy. Soc.* A, **85**, 1911, p. 355.
‡ *Zeit. Angewandt. Math. u. Mech.* **1**, 1921, p. 239.
§ *Q.J. Roy. Met. Soc.* **58**, 1932, p. 74.
‖ *Q.J. Roy. Met. Soc.* **58**, 1932, p. 285.

6 feet to 72 feet shows still larger diurnal and seasonal variations of $1/n$, values as high as 0·9 occurring at Agra in the small hours of the morning.

The table reproduced below (from *Geoph. Mem.* No. 54, Table XXI, p. 65) gives the ratio of the wind speed at 150 feet to that at 50 feet, for different values of the wind speeds at 150 feet and of the lapse-rate. It is seen that the ratio is nearly unity in large lapses, but increases to nearly 2 in very large inversions. For high wind velocities the ratio never differs very much from 1·2. The values of $1/n$ corresponding to the ratios 2 and 1·2 are respectively 0·63 and 0·166. The results shown in this table are of practical importance in observational meteorology, and show that it is not possible to draw up a table of wind velocities for different heights for the Beaufort scale numbers. A different table would be required for each value of the lapse-rate, in order to yield comparable values of the Beaufort numbers.

Ratio of wind speed at 150 feet to that at 50 feet in relation to wind speed at 150 feet and vertical temperature gradient

/ind :ed at ɔ 'eet .p.h.	Vertical temperature difference 143 feet–4 feet (°F)												
	−5·0	−4·0	−3·0	−2·0	−1·0	0·0	1·0	2·0	3·0	4·0	5·0	6·0	7·0
ɔ–14	1·01	1·02	1·03	1·04	1·07	1·15	1·23	1·35	1·50	1·60	1·73	1·85	1·98
5–19	—	1·05	1·07	1·09	1·12	1·17	1·23	1·32	1·40	1·49	1·58	—	—
ɔ–24	—	—	1·16	1·16	1·18	1·20	1·20	1·21	—	—	—	—	—
5–29	—	—	1·17	1·15	1·14	1·16	1·18	1·20	—	—	—	—	—

A logarithmic law for the variation of wind with height has been given by various writers. E. H. Chapman* has shown that a number of sets of observations can be represented by the formula

$$\frac{V}{V_{10}} = a \log z + b \qquad \ldots\ldots(58)$$

and Hellmann† has suggested a slightly different formula

$$V = a \log (z + c) + b \qquad \ldots\ldots(59).$$

This law acquires a special interest in view of the wide generality of equation (50) above for the flow in pipes. So far the application of equation (50) to the atmosphere has not been discussed from the standpoint of the relation to turbulent conditions of the constants required to fit equation (58) or (59) to actual observations.

(*b*) THE LAYERS FROM 10 TO 1000 METRES, K BEING ASSUMED CONSTANT AND THE MOTION STEADY

Except in the narrow region in the immediate neighbourhood of the ground the deviating force due to the earth's rotation has to be taken into consideration. The variation of wind with height for these conditions has been investigated by Taylor, whose analysis is reproduced below.

* M.O., *Professional Notes*, No. 6.
† *Preuss. Akad. Wiss. Berlin*, **10**, 1917, p. 174.

It has been shown that if K is constant the rate of eddy transfer of x-momentum to unit volume is $K\rho\frac{\partial^2 u}{\partial z^2}$. Equations (1) and (2) of § 107 then become

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+2\omega\sin\phi.v&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+K\frac{\partial^2 u}{\partial z^2}\\ \frac{\partial v}{\partial t}-2\omega\sin\phi.u&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+K\frac{\partial^2 v}{\partial z^2}\end{aligned}\right\}\quad\ldots\ldots(60).$$

The motion being steady $\partial u/\partial t$ and $\partial v/\partial t$ are zero. Multiplying the second of these equations by $i\,(\sqrt{-1})$, and adding to the first, we find

$$K\frac{\partial^2}{\partial z^2}(u+iv)=\frac{1}{\rho}\left(\frac{\partial p}{\partial x}+i\frac{\partial p}{\partial y}\right)-2i\omega\sin\phi\,(u+iv)\quad\ldots\ldots(61).$$

Now let $2\omega\sin\phi\,.Gi$ represent the pressure gradient, G being the geostrophic wind. For convenience we take the axis of x to be tangential to the isobar, so that G is along the axis of x. Further let $V=u+iv$. Then equation (61) may be written

$$\frac{d^2V}{dz^2}-(1+i)^2B^2(V-G)=0\quad\ldots\ldots(62),$$

where

$$B^2=\omega\sin\phi/K\quad\ldots\ldots(63).$$

Within the limits of height with which we are concerned G may be treated as a constant. The solution which is derived below is also true when G is a linear function of the height z. The solution of equation (62) may be written

$$V-G=C_1e^{(1+i)Bz}+C_2e^{-(1+i)Bz}\quad\ldots\ldots(64).$$

Since the velocity must not become infinite at great heights, $C_1=0$. Equation (64) then reduces to the second term, which we now write

$$V-G=Ce^{-(1+i)Bz+i\gamma}\quad\ldots\ldots(65),$$

where C and γ are both real constants, whose values are to be determined from the boundary conditions. We assume with Taylor that at the surface the direction of slip is in the direction of strain, i.e. that V and $\partial V/\partial z$ are parallel at $z=0$. Then at $z=0$,

$$\frac{\partial V}{\partial z}=-CB(1+i)e^{i\gamma}=-\sqrt{2}CBe^{i\left(\gamma+\frac{\pi}{4}\right)}=\sqrt{2}CBe^{i\left(\gamma+\frac{5\pi}{4}\right)},$$

since $1+i=\sqrt{2}e^{i\pi/4}$, and $-1=e^{i\pi}$. At the surface let V be inclined at an angle α to the isobar, and to G. Then

$$V=De^{i\alpha}\quad\ldots\ldots(66),$$

where D and α are real. Equations (65) and (66) are identical. Hence

$$\gamma+\frac{5\pi}{4}=\alpha\quad\text{or}\quad\gamma=\alpha-\frac{5\pi}{4}.$$

Also at $z=0$,

$$\begin{aligned}V=G+Ce^{i\gamma}&=G+C(\cos\gamma+i\sin\gamma)\\ &=D(\cos\alpha+i\sin\alpha).\end{aligned}$$

Equating the real and imaginary parts and eliminating D, we readily find

$$C=\sqrt{2}G\sin\alpha.$$

Hence the complete solution of the equation (62) which is appropriate to the conditions stipulated is

$$V-G=\sqrt{2}G\sin\alpha\,.\,e^{-Bz+i(\alpha+3\pi/4-Bz)} \qquad \ldots\ldots(67)$$

or $$u+iv-G=\sqrt{2}G\sin\alpha\,.\,e^{-Bz}\left\{\cos\left(\alpha+\frac{3\pi}{4}-Bz\right)+i\sin\left(\alpha+\frac{3\pi}{4}-Bz\right)\right\}.$$

Hence
$$\left.\begin{aligned}u&=G-\sqrt{2}G\sin\alpha\,.\,e^{-Bz}\cos\left(\alpha-\frac{\pi}{4}-Bz\right)\\ v&=-\sqrt{2}G\sin\alpha\,.\,e^{-Bz}\sin\left(\alpha-\frac{\pi}{4}-Bz\right)\end{aligned}\right\} \qquad \ldots\ldots(68).$$

If we plot the wind at all heights on a plane diagram, as in fig. 58, where O is the origin, Og is the axis of x, and $Og=G$, then

$$gP=\sqrt{2}G\sin\alpha\,.\,e^{-Bz}, \quad\text{and}\quad \angle PgL=\alpha+\frac{3\pi}{4}-Bz.$$

If OS is the surface wind,

$$\angle SOg=\alpha,\quad \angle SgL=\alpha+\frac{3\pi}{4},\quad \angle SgO=\frac{\pi}{4}-\alpha \quad\text{and}\quad \angle OSg=\frac{3\pi}{4}.$$

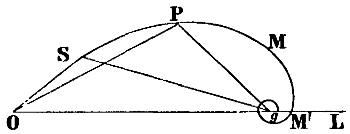


Fig. 58. The variation of wind with height; the equiangular spiral.

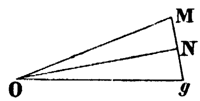


Fig. 59. The height at which the geostrophic wind is attained.

The wind at a height z is made up of the geostrophic wind G together with an added component whose magnitude is $\sqrt{2}G\sin\alpha\,.\,e^{-Bz}$, acting in a direction making an angle $\alpha+3\pi/4-Bz$ with the geostrophic wind. The point P therefore sweeps out an equiangular spiral of angle $\pi/4$.

At the ground ($z=0$) the velocity is $G(\cos\alpha-\sin\alpha)$. If this is to be positive α must be less than $\pi/4$, and when the surface wind is nearly zero α must be very nearly equal to $\pi/4$. Normal daytime values of α are more nearly $\pi/8$, the wind blowing in across the isobars at about this angle in steady conditions. From fig. 58 we should expect that with increasing height the wind would veer, slowly at first, and then more rapidly, the velocity increasing and attaining the geostrophic value at a height represented by M, where $OM=Og$. In fig. 59 let N be the midpoint of Mg. Then since ONg is a right angle,

$$Mg=2Ng=2G\cos NgO=-2G\cos MgL=-2G\cos\left(\alpha-\frac{\pi}{4}-BH\right),$$

where H is the height corresponding to M. Then

$$\sqrt{2}G\sin\alpha\,.\,e^{-BH}=-2G\cos\left(\alpha-\frac{\pi}{4}-BH\right)\sin\alpha\,.\,e^{-BH}=\sqrt{2}\cos\left(\alpha-\frac{\pi}{4}-BH\right) \qquad \ldots\ldots(69).$$

Hence if α be known, and H can be obtained from observations, B can be evaluated by the use of equation (69). At a greater height the wind attains the gradient direction, corresponding to the point M' in fig. 58. The height H' corresponding to this point is given by

$$\alpha+\frac{3\pi}{4}-BH'=0 \qquad \text{......(70).}$$

Beyond this height the wind veers still further, and its velocity diminishes.

Observations made by Dobson* at Upavon enabled Taylor to compute values of H and H', the mean values being 300 metres and 800 metres respectively. From these the values of K were deduced, yielding 6×10^4, 5×10^4 and 3×10^4 for strong, moderate and light winds respectively, the values deduced from H and H' agreeing very closely. These values of K are greater than those obtained by Taylor from the observations in inversions over the sea, referred to on p. 224 above, but are in fair agreement with the values deduced by Taylor from the observations of temperature on the Eiffel Tower.

§ 142. *The internal friction due to turbulence*

The effect of turbulence on the mean motion is allowed for in equation (60) by the addition of terms $K\rho\,\partial^2u/\partial z^2$, $K\rho\,\partial^2v/\partial z^2$. These terms may be regarded as representing a virtual frictional force, and they may be combined into one term $K\rho\,\partial^2V/\partial z^2$. Differentiating equation (67) twice, we find that this term amounts to

$$2\sqrt{2}KB^2\rho G\sin\alpha\,.\,e^{-Bz+i(\alpha+5\pi/4-Bz)}=2\sqrt{2}\rho\omega\sin\phi\,.\,G\sin\alpha\,.\,e^{-Bz+i(\alpha+5\pi/4-Bz)} \qquad \text{......(71).}$$

A comparison of this equation with equation (60) shows that the virtual friction at the ground makes an angle of $\pi/4$ with the direction of the wind reversed. This should be contrasted with Guldberg and Mohn's assumption† that the frictional effect is opposite to the wind direction. The magnitude of the frictional effect decreases with height in proportion to e^{-Bz}. Since the term $K\rho\,\partial^2V/\partial z^2$ balances the pressure gradient and the deviating force, it is possible to evaluate it from observations, if a synoptic chart is available. A series of values of the virtual frictional force at the top and bottom of the Eiffel Tower was evaluated by Akerblom. From these values, or rather from the ratios of the values at the top and bottom of the tower, it is possible to evaluate B directly, and from this to deduce the value of K. The mean values of K so derived were $6{\cdot}8\times10^4$ in winter, and $9{\cdot}3\times10^4$ in summer‡.

§ 143. *The variation of wind with height when K is variable*

Equation (61), or its equivalent (62), was derived without any assumption as to the constancy of K with height. It can be solved in finite terms in a few cases of K varying with height. The two simplest cases are those in which B varies inversely as z or as z^2.

* *Q.J. Roy. Met. Soc.* **40**, 1914, p. 123.

† See Abbe, *Mechanics of the Earth's Atmosphere*, Smithsonian Misc. Coll., 1910.

‡ For details of the computation see a paper by Brunt, *Q.J. Roy. Met. Soc.* **46**, 1920, p. 175.

(*a*) Let $B=c/z$, so that $K\propto z^2$. Then so long as G is either constant or a linear function of z the solution is readily shown to be

$$V-G=Az^{m_0}\{\cos(m_1\log z)+i\sin(m_1\log z)\} \qquad \ldots\ldots(72),$$

where m_0+m_1 is the solution of the equation

$$m(m-1)-(1+i)^2c=0$$

or

$$m(m-1)-2ic=0,$$

the solution in which m_0 is negative being taken.

If again we plot the wind vector as in fig. 58 the curve obtained is an equiangular spiral, but the approach to the geostrophic wind is in this case much slower than when K is a constant.

(*b*) Let $B=c/z^2$, so that $K\propto z^4$.

The differential equation now becomes

$$\frac{d^2}{dz^2}(V-G)-\frac{c^2}{z^4}(1+i)^2(V-G)=0.$$

The solution of this equation is

$$V-G=Az\{e^{c(1+i)/z}+e^{-c(1+i)/z}\}=Az\left\{\sinh\frac{c}{z}\cos\frac{c}{z}+i\cosh\frac{c}{z}\sin\frac{c}{z}\right\}\ldots\ldots(73).$$

At great heights where c/z is small this equation reduces to

$$V-G=2Ap(1+i) \qquad \ldots\ldots(74).$$

Thus the wind never attains the geostrophic value.

When K is assumed to be proportional to z the solution of the differential equation is possible, but involves the use of ber and bei, and ker and kei functions, which are Bessel functions of a complex variable. This problem has been solved by Takaya* on the assumption made by Taylor that the slip at the ground is in the direction of the strain, and by Möller† on the assumption that the wind at the ground is zero. Both obtain results which resemble the observed phenomena.

It has been shown by R. N. Apte‡ that if K be assumed to vary with height according to the law

$$K=K_0(1+cz)^{\frac{2}{2n+1}},$$

where n is a positive integer, then the surface velocity is $G(\cos\alpha-\sin\alpha)$, just as in the case where K is constant. This result is of interest in that it may explain the very close agreement which Taylor obtained between observed and calculated values of α, on the assumption of K constant, which is not true.

* *Mem. Imp. Mar. Obs., Kobe, Japan*, **4**, No. 1.
† *Met. Zeit.* 1931, h. 2.
‡ *Journ. Indian Math. Soc.* **15**, 1924, p. 183.

§ 144. *The height to which the effects of surface turbulence extends*

We can obtain some idea of the height to which the effects of surface turbulence are appreciable by considering equations (68) and (20) above. The former shows that the vectorial effect of surface turbulence upon the wind is measured by

$$\sqrt{2}G \sin\alpha . e^{-Bz}.$$

We assume $K=10^5$, so that $B=2{\cdot}4\times10^{-5}$, and $\alpha=22\tfrac{1}{2}°$. The height at which e^{-Bz} is approximately 0·05 is given by

$$z=3/B \text{ cm}=4\times10^4\times3 \text{ cm}=1200 \text{ m}.$$

Thus the effect of turbulence on the wind is practically inappreciable beyond a height of a little more than 1 km. The height thus deduced is inversely proportional to B, and therefore directly proportional to $\sqrt{K}$.

Again from equation (20) the effect of surface turbulence on the temperature at different heights is given by a term e^{-bz}, and if we again assume the effect to become negligible when the exponential term reaches the same limit 0·05, we find that the corresponding height bears to that deduced for the wind the ratio $\sqrt{2\sin\phi}:1$. Thus below latitude 30° the effect upon the wind is appreciable to greater heights than the effect upon temperature, while above latitude 30° the effect upon temperature is appreciable to the greater height. For a given value of K the height to which the wind is affected by surface turbulence increases as $1/\sqrt{\sin\phi}$, and it appears from the equation that with approach to the equator the effects of surface turbulence should become appreciable at ever increasing heights.

§ 145. *Skin friction at the ground*

For fluid flowing through a pipe Stanton* showed that with high values of lV/ν the velocity V of the fluid near the wall of the pipe is about 0·6 of the velocity in the middle, and that the mean velocity V_0 is about 0·85 of the velocity in the middle, so that

$$V=0{\cdot}7V_0.$$

The skin friction for the highest values of lV/ν which Stanton was able to obtain could be expressed by the formula

$$F=0{\cdot}002\rho V_0^2,$$

which is equivalent to

$$F=0{\cdot}004\rho V^2 \qquad \text{......(75)}.$$

(l represents the linear dimension of the system, and ν is the coefficient of kinematic viscosity.)

Taylor† has suggested that the law of dynamic similarity is applicable to skin friction, so that the friction at the ground may be represented by

$$F=\kappa\rho V^2,$$

where V is the velocity at the ground, and κ is therefore comparable with 0·004.

* *Coll. Res. N.P.L.* **9**, 1913, plate I, p. 6. † *Proc. Roy. Soc.* A, **92**, 1916, p. 196.

If F_x, F_y be the components of the wind friction acting on unit area of the ground

$$F_x=\int_0^\infty K\rho\,\frac{d^2u}{dz^2}\,dz,\quad F_y=\int_0^\infty K\rho\,\frac{d^2v}{dz^2}\,dz,$$

$$F_x+iF_y=\int_0^\infty K\rho\,\frac{d^2}{dz^2}(u+iv)\,dz$$

$$=K\rho\frac{d}{dz}(u+iv),\quad \text{at } z=0$$

$$=KB\rho\sqrt{2}G\sin\alpha\,(1+i)\,e^{-Bz+i(\alpha+3\pi/4-Bz)},\quad \text{at } z=0$$

$$=2KB\rho G\sin\alpha\,.\,e^{-Bz+i(\alpha+\pi-Bz)},\quad \text{at } z=0,$$

$$F=\sqrt{F_x^2+F_y^2}=2KB\rho G\sin\alpha \qquad \text{......(76)},$$

and the direction is exactly opposed to the surface wind as shown by the inclination to the isobars, $\alpha+\pi$.

From (75) and (76) and the relation

$$V_s=G(\cos\alpha-\sin\alpha)$$

it follows that $$\kappa=\frac{2KBG\sin\alpha}{G^2(\cos\alpha-\sin\alpha)^2}=\frac{2KB\sin\alpha}{G(\cos\alpha-\sin\alpha)^2} \qquad \text{......(77)},$$

or $$\kappa=2K\sin\alpha\,\frac{\left(\frac{3\pi}{4}+\alpha\right)G}{H'V_s^2} \qquad \text{......(78)},$$

where H' is the height at which the wind attains the gradient direction, as defined by equation (70). From Dobson's observations over Salisbury Plain Taylor deduced the following results:

	K	α	H'	G	V	κ
Light winds	$2{\cdot}8\times10^4$	$13°$	600 m	460 cm/s	330 cm/s	0·0023
Moderate winds	$5{\cdot}0\times10^4$	$21\frac{1}{2}$	800	910	590	0·0032
Strong winds	$6{\cdot}2\times10^4$	20	900	1560	950	0·0022

The coefficient κ does not increase with wind velocity, and the use of equation (75) is thus justified. The mean value of κ may be taken as 0·0025, as compared with 0·004 in a pipe. The ratio of the values of lV/ν in the two cases is more than 10^5, and the same law thus appears to hold over this very wide range.

It would appear as a corollary that the surface wind should be about 0·7 times the geostrophic value for light winds, and 0·6 times the geostrophic value for strong winds.

Equation (77) may also be written in the form

$$\kappa=\frac{2\omega\sin\phi}{BG}\,\frac{\sin\alpha}{(\cos\alpha-\sin\alpha)^2},$$

or $$\frac{1}{BG}=\frac{\kappa}{2\omega\sin\phi}\,\frac{(\cos\alpha-\sin\alpha)^2}{\sin\alpha}=\frac{20{\cdot}4}{\sin\alpha}(\cos\alpha-\sin\alpha)^2 \qquad \text{......(79)}.$$

Equation (79) gives a relationship between B, G and α, and since

$$1/B^2G^2 = K/\omega \sin \phi G^2,$$

it follows that K/G is a function of α. Taylor has given a table of the corresponding values of K/G and α, from which the following is an extract:

α in °	4	6	8	10	12	14	16	18	20	24	30	36
K/G	3·54	1·35	0·635	0·338	0·192	0·116	0·069	0·042	0·027	0·0094	0·0017	0·00016

Taylor also gave a series of curves showing the variation of wind with height for different values of α, which he then transformed into a series of curves showing the variation of V/G for given values of z/G, as α varies from 6° to 30°. From these he deduced curves for the diurnal variation of wind at different heights assuming α to vary from 10° at midday to 30° at midnight, and obtained the result that from the ground up to the height for which $z/G = 1$ the maximum wind velocity should occur at about midday, but that at heights above that at which $z/G = 15$ there should be a minimum at midday and a maximum at midnight, while at intermediate heights there should be two maxima.

These results are in good agreement with observation. At the top of the Eiffel Tower the diurnal variation of wind velocity is the reverse of the surface variation, the maximum occurring in the night and the minimum in the middle of the day. Hellmann* set up three anemometers at heights of 2, 16 and 32 metres above the ground in a flat meadow at Nauen, with a view to investigating the height at which the change in the nature of the diurnal variation took place. His results may be briefly summarised as follows: With light winds the anemometer at 2 metres showed a maximum in the early afternoon and a minimum at night, while those at 16 and 32 metres showed two maxima, one about midday and the other about midnight, with minima in the early morning and afternoon. At 16 metres, in winter the two maxima were about equal, while in summer the night maximum tended to exceed the day maximum; but at 32 metres the night maximum was greater both in winter and in summer. Thus with light winds the height at which the night maximum is equal to the day maximum is below 16 metres in winter, and between 16 and 32 metres in summer. With strong winds the day maximum was the stronger up to 32 metres. Hellmann confirmed these results by an examination of the records at Potsdam, where the anemometer is fixed at a height of 41 metres above the ground. With strong winds the maximum always occurs in the middle of the day, while with light winds there is in winter a minimum in the middle of the day and a maximum in the middle of the night; with light winds in summer there is a weak maximum in the middle of the day and a stronger maximum in the middle of the night. Thus the height at which the reversal in type of the diurnal variation takes place is greater than 41 metres for strong winds, but less than 41 metres for light winds.

The analysis of the wind velocities at 13 metres and at 95 metres at Leafield, given by Heywood†, confirms these results. Heywood classifies the

* *Met. Zeit.* Jan. 1915. † *Q.J. Roy. Met. Soc.* **57**, 1931, p. 433.

winds into three groups, light winds (<6 m/s), moderate winds (6–9 m/s), and strong winds ($\geqslant 9$ m/s), the classification being based on the mean wind for 24 hours at 95 metres above the ground. In summer with strong winds there is only one maximum, shortly after midday, even at 95 metres, while in winter the night maximum is appreciable at 13 metres, and is still a little lower than the day maximum at 95 metres. With moderate winds, in summer the night maximum is absent at 13 metres, and is slightly lower than the day maximum at 95 metres; while in winter the night maximum is just appreciable at 13 metres, and at 95 metres there is a maximum in the middle of the night and a minimum in the middle of the day. With light winds in summer there is a faint maximum in the middle of the night at 13 metres, and a night maximum and a day maximum at 95 metres, while in winter the day maximum is weaker than the night maximum at 13 metres, and is replaced by a day minimum at 95 metres. It should be noted that Heywood's strong winds are stronger than Hellmann's "strong winds", and so the height of transition from day maximum to night maximum which he deduces is higher than the corresponding estimate given by Hellmann and quoted by Taylor.

From the above brief description of the observations available it will be seen that the changes with height of the diurnal variations of wind are in reasonable agreement with the theory deduced by Taylor, based on the variations of α during the day.

§ 146. *Sakakibara's transverse component of eddy viscosity*

Sakakibara* has suggested that the downwind and crosswind components of the virtual friction, which Taylor denotes by $K\frac{\partial^2 u}{\partial z^2}$ and $K\frac{\partial^2 v}{\partial z^2}$ should be replaced by $K_1\frac{\partial^2 u}{\partial z^2}+K_2\frac{\partial^2 v}{\partial z^2}$ and $K_1\frac{\partial^2 v}{\partial z^2}-K_2\frac{\partial^2 u}{\partial z^2}$, where K_1 and K_2 are different constants. Multiplying the second of these terms by i (or $\sqrt{-1}$), and adding to the first, we find $(K_1+iK_2)\frac{\partial^2}{\partial z^2}(u+iv)$, which should be inserted in equation (61) in lieu of $K\frac{\partial^2}{\partial z^2}(u+iv)$. This is equivalent to assuming the eddy frictional effect to be at a constant angle $\tan^{-1}\frac{K_2}{K_1}$ with the direction of $\frac{\partial^2}{\partial z^2}(u+iv)$, instead of parallel to it as in Taylor's theory.

It is difficult to see any physical justification for such an assumption. The results derived are in fair agreement with observations, but it must be remembered that the theory yields equations which have three adjustable arbitrary constants, and with that number of available constants it is not difficult to fit any reasonably smooth curve to a formula.

* *Geoph. Mag. Tokyo*, **1**, No. 130, 1928.

§ 147. *Diffusion by continuous movement*

The analysis of § 137 above leads to a conviction that any effort at discussion of atmospheric turbulence from a purely dynamical standpoint is likely to be fruitless, on account of the impossibility of visualising the individual eddy, and the dynamical processes involved in its movement from place to place. A totally different line of approach, also due to G. I. Taylor*, is available, and will be briefly described here.

Consider a condition in which the turbulence in a fluid is uniformly distributed, so that the average conditions are the same at every point. Let u be the velocity of any particle parallel to the axis of x. Let R_ξ be the coefficient of correlation between the velocities of the particles considered at time t, and at an interval of time ξ later. Then by definition

$$[u_t u_{t+\xi}] = [u_t^2]\, R_\xi,$$

where the square brackets denote mean values taken over a long period. Integrating with respect to ξ we find

$$[u^2] \int_0^t R_\xi d\xi = \int_0^t u_t u_{t+\xi} d\xi = u_t \int_0^t u_{t+\xi} d\xi = [u_t X] \qquad \text{......(80)},$$

where X is the distance travelled by the particle in time t. Thus

$$[u^2] \int_0^t R_\xi d\xi = [u_t X] = \frac{1}{2}\frac{d}{dt}[X^2] \qquad \text{......(81)}$$

and

$$[X^2] = 2\,[u^2] \int_0^t \int_0^t R_\xi d\xi\, dt \qquad \text{......(82)}.$$

This remarkable equation reduces the problem of diffusion to that of the determination of the mean square velocity, and of the correlation between the velocity of a particle at one instant and at a time ξ later.

When t is so small that there is not sufficient time for the coefficient of correlation to fall appreciably from unity the double integral becomes $\frac{1}{2}t^2$, so that the equation (82) reduces to

$$[X^2] = [u^2]\, t^2 \quad \text{or} \quad \sqrt{[X^2]} = \sqrt{[u^2]}\, t \qquad \text{......(83)},$$

or, in other words, the deviation of the particle from its initial position is proportional to the time.

We should anticipate that in a turbulent fluid R_ξ should fall off to zero for large values of ξ. It might remain positive, or might oscillate between positive and negative values, but in either case there should be a finite interval of time T_1 such that at the end of this interval there should be no correlation with the velocity at the beginning. In this case suppose that $\lim_{t\to\infty} \int_0^t R_\xi d\xi$ is finite and equal to I. Then at any time $T\,(> T_1)$ after the beginning of the motion equation (81) becomes

$$\frac{d}{dt}[X^2] = 2\,[u^2]\, I, \quad \text{or} \quad [X^2] = 2\,[u^2]\, IT \qquad \text{......(84)},$$

so that $[X^2]$ increases at a uniform rate. Thus a continuous eddying motion

* *Proc. Lond. Math. Soc.* **20**, 1922, p. 196.

may be expected to have the property that the standard deviation of X is proportional to the square root of the time.

It will also be noted from equation (80) that for $T > T_1$

$$[Xu] = [u^2]\, I = \text{const.} \qquad \ldots\ldots(85).$$

Thus though X increases with the time, the product $[Xu]$ is constant. It follows that X must be positively correlated with u, but the correlation coefficient decreases with increasing X. Let this correlation coefficient be r_{Xu}. Then

$$r_{Xu} = \frac{[Xu]}{\sqrt{[X^2]\,[u^2]}} = \frac{I\sqrt{[u^2]}}{\sqrt{[X^2]}} = \frac{I}{\sqrt{2IT}} = \sqrt{\frac{I}{2T}} \qquad \ldots\ldots(86)$$

for large values of T.

Consider the application of the above results to the diffusion of smoke emitted at a point, and carried away downwind. At short distances from the point of emission, at which the correlation coefficient R_ξ is still nearly unity, X is proportional to the time, and therefore the outline of the smoke should be a cone. At greater distances we should use the result shown in equation (84), according to which the deviation is proportional to the square root of the time, and the outline is therefore a paraboloid. These anticipations are in accordance with observation, and appear to bear out Taylor's deduction that $[Xu]$ becomes constant after a certain interval of time, this interval being long enough to allow $\int_0^T R_\xi d\xi$ to become practically constant.

Richardson* has also derived an equation similar to equation (84) above. He showed that the increase in the standard deviation of a set of particles in time T was equal to $2KT$, where K is the coefficient of eddy diffusivity as defined by Taylor. Comparing this with equation (84) above, we find

$$K = [u^2]\, I = [u^2] \int_0^\infty R_\xi d\xi \qquad \ldots\ldots(87).$$

§ 148. *The form of* R_ξ

The direct detailed investigation of R_ξ is not in general a feasible proposition, since it would require a series of anemometers strung in a line downwind. If such a line of anemometers could be set up it would only be utilisable when the wind was directed along that line. Some general results can, however, be derived from simple considerations. If the turbulence were of the nature of indestructible whirls moving downwind, the correlation between the velocity of a selected element at a time t and at an interval ξ later would be perfect for all values of ξ, though the correlation would not of necessity be linear. This supposition is far from the truth, which must be more closely represented by the statement that eddies are continually being destroyed by mixing, others being created to take their place. Let us assume for the moment that all the eddies are of the same size, and let the correlation between the velocity of a

* *Phil. Trans. Roy. Soc.* A, **221**, 1920, p. 1.

selected element at the beginning and at the end of an interval ξ be R_ξ. The part of $u^2_{t+\xi}$ which is due to correlation with u_t is $R_\xi^2 u_t^2$, and the part of $u^2_{t+\xi}$ which is independent of u_t is $(1-R_\xi^2)\,u_t^2$. In other words the part of the total eddying energy, at the end of the interval, which is due to the eddies which have persisted throughout the interval is $\frac{1}{2}R_\xi^2 u_t^2$, and the remaining portion $\frac{1}{2}(1-R_\xi^2)\,u_t^2$ is due to eddies which have come into existence during the interval. Considering the same process as taking place during a further interval ξ subsequent to the first, we arrive at the result that of the eddying energy at the end of the double interval 2ξ a portion $\frac{1}{2}R_\xi^4 u_t^2$ is due to eddies originally in existence. Thus

$$R_{2\xi}=R_\xi^2 \quad \text{or} \quad R_\xi=e^{-a\xi}.$$

This result depends on the assumption that all the eddies are of the same size, with a uniform rate of creation and destruction, and is only true with that assumption. The constant a in the last equation must obviously depend on the size of the eddy, and is no longer true when there are present eddies of varying sizes.

Returning to equation (82) and substituting $R_\xi=e^{-a\xi}$, we find

$$\begin{aligned}[X^2]&=2\,[u^2]\int_0^T\int_0^t e^{-a\xi}\,d\xi\,dt\\ &=2\,[u^2]\int_0^T\frac{1}{a}(1-e^{-at})\,dt\\ &=2\,[u^2]\left\{\frac{T}{a}-\frac{1}{a^2}(1-e^{-aT})\right\} \qquad \text{......(88).}\end{aligned}$$

Since T is sufficiently great to permit of our neglecting $\int_T^\infty R_\xi d\xi$, aT must be large by comparison with unity, and e^{-aT} is negligible by comparison with unity, and $1/a^2$ by comparison with T/a, so that the equation reduces to

$$[X^2]=2\,[u^2]\,T/a \qquad \text{......(89).}$$

§ 149. *Langevin's derivation of the equation* $[X^2]=2KT$

The following modification of Langevin's derivation, which I owe to O. G. Sutton, is of some interest in connection with the above discussion. Assume, with Taylor, that $[u^2]$ is independent of time. The motion of any particle in an agreed x-direction is given by

$$m\frac{d^2x}{dt^2} \quad = \quad f\frac{dx}{dt} \quad + R_x \qquad \text{......(90).}$$

(inertia term) (frictional term) (total effect of other particles)

This may be written

$$\tfrac{1}{2}m\frac{d}{dt}\left(\frac{d}{dt}x^2\right)+m\left(\frac{dx}{dt}\right)^2=\tfrac{1}{2}f\frac{d}{dt}(x^2)+xR_x.$$

Now take means over a large number of particles. If the motion be random, $[xR_x]$ is zero. Let

$$\left[\frac{d}{dt}(x^2)\right]=\frac{d}{dt}[x^2]=\zeta, \text{ say} \qquad \text{......(91)}.$$

Then

$$\tfrac{1}{2}m\frac{d\zeta}{dt}+\tfrac{1}{2}f\zeta=m[u^2],$$

and this equation may be integrated, yielding

$$\zeta=\frac{2m}{f}[u^2]+Ce^{-\frac{tf}{m}} \qquad \text{......(92)}.$$

If m/f is small we may neglect the exponential term and retain the term in m/f only. Then

$$\zeta=\frac{2m}{f}[u^2] \qquad \text{......(93)},$$

and integrating this equation between limits 0 and T, taking $x=x_0$ at time $t=t_0$,

$$[x^2]-[x_0{}^2]=\frac{2m}{f}[u^2]\,T \qquad \text{......(94)},$$

or

$$\sigma^2=2KT, \quad \text{where} \quad K=\frac{m\,[u^2]}{f} \qquad \text{......(95)}.$$

When the particles under consideration are molecules

$$m[u^2]=RT/N, \quad f=6\pi\eta r,$$

where N is the number of molecules per unit volume, η is the coefficient of viscosity, and r is the radius of the molecule. The formula then reduces to that of Einstein.

The equation $\sigma^2=2KT$ is of peculiar interest in that it has been derived by a number of different writers from widely different standpoints, with naturally different definitions of the constant K. These are defined as follows:

Taylor $K=[u^2]\int_0^\infty R_\xi\,d\xi$,

Langevin $K=\dfrac{m\,[u^2]}{f}$,

Hesselberg* $K=\tfrac{1}{4}P\,[u^2]$,

where P is the period of the eddies.

It will be noted that Taylor's form of the equation requires that $\int_0^\infty R_\xi\,d\xi$ shall be constant. This is clearly satisfied by that treatment of turbulence which regards the eddies as analogous to molecules, and treats the dynamics of turbulence essentially as "collision dynamics". These theories regard the eddy as preserving its identity and its motion until it reaches a certain point at time t_0, when it suddenly mixes with its new environment. Implicitly it is assumed that

$$R_\xi=1 \quad \text{for} \quad \xi\leqslant t_0,$$
$$R_\xi=0 \quad \text{for} \quad \xi>t_0.$$

* *Ann. Hydrog. u. Mar. Met.* Oct. 1929.

Such assumptions make $\int_0^t R_\xi d\xi = t_0$ for all values of $t \geqslant t_0$, and $= t$ for all values of $t < t_0$.

The outstanding feature of Taylor's treatment by correlation methods is that it permits of our regarding mixing as a process which is continuous, and not intermittently explosive.

§ 150. *Sutton's extension of Taylor's theory*

It was suggested by O. G. Sutton* that R might be of the form

$$R_\xi = \left(\frac{a}{[u]\,\xi}\right)^n \qquad \text{......(96),}$$

where a is a constant length and $[u]$ is the mean wind speed; n is a real quantity whose exact value will be defined more closely later. A precise mathematical expression for R_ξ is

$$R_\xi = \left(\frac{a}{a+[u]\,\xi}\right)^n \qquad \text{......(97).}$$

As $\xi \to 0$, $R_\xi \to 1$, which is clearly an essential condition; on the other hand, $R_\xi \to 0$ as $\xi \to \infty$.

If the length a is small compared with $[u]\,\xi$, a condition that will in general be satisfied, then by substitution from (97) in (82) above and integrating, we find for σ, the standard deviation of the particles from their mean position after time T, the following relation

$$\sigma^2 = \frac{2a^n}{(1-n)(2-n)} \frac{[u^2]\,T^{2-n}}{[u]^n} \qquad \text{......(98).}$$

If the velocities are now assumed to follow the Maxwellian law of distribution, then there is a simple relation between $[u^2]$ and $[u]^2$. This is

$$[u^2] = \tfrac{1}{2}\pi\,[u]^2.$$

Hence

$$\sigma^2 = \frac{\pi a^n}{(1-n)(2-n)}([u]\,T)^{2-n} = \tfrac{1}{2}C^2\,([u]\,T)^{2-n} \text{ if } C^2 = \frac{2\pi a^n}{(1-n)(2-n)} \qquad \text{......(99).}$$

This expression should be compared with the expression derived from the theory as developed by Taylor,

$$\sigma^2 = 2KT.$$

In Sutton's analysis the quantity C takes the place of the eddy diffusivity K, and the value of the theory may be judged by the variability or otherwise of the quantity C. It will be seen later that C is only slightly variable, so that it is on the whole a more useful parameter than K.

When $n = 0$, or $R_\xi = 1$ for all values of ξ, the motion is stream-line motion, and

$$\sigma^2 = ([u]\,T)^2$$

as might be expected.

Sutton takes equation (99) above to define the turbulent motion, and seeks solutions of the equation of continuity which satisfy this equation and certain boundary conditions.

* *Proc. Roy. Soc.* A, **135**, 1932, p. 143.

Let $\chi(x, y, z)$ be the density of a diffusing property, say density of smoke in grammes per cm³ at the point (x, y, z). Take axes moving with the mean velocity of the wind, and let the axis of x be downwind, the axis of y transverse to the mean wind direction in the horizontal plane, and the axis of z vertical. A single puff of smoke is produced at the origin at time $t=0$. Then χ is a function which must satisfy the following properties:

(1) $\chi \to 0$ as $t \to \infty$.

(2) $\chi \to 0$ as $t \to 0$ except at the origin.

(3) The square of the standard deviation of the particles from their mean position must be $\frac{1}{2}C^2 (ut)^{2-n}$.

(4) The total amount of matter is constant $(=Q)$ at any instant.

A function which satisfies these conditions is

$$\chi = \frac{Q}{\pi^{\frac{3}{2}} C^3 (ut)^{3m/2}} \exp\left(-\frac{r^2}{C^2 (ut)^m}\right) \qquad \ldots\ldots(100),$$

where u may now be taken to be the mean velocity, and where

$$r^2 = x^2 + y^2 + z^2, \quad \text{and} \quad m = 2 - n.$$

It is obvious from inspection that conditions (1) and (2) are satisfied. Also since

$$\sigma_x^2 = \int_0^\infty x^2 \chi \, dx \Big/ \int_0^\infty \chi \, dx = \tfrac{1}{2}C^2 (ut)^{2-n},$$

and

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2,$$

while

$$Q = \int_0^\infty 4\pi r^2 \chi \, dr,$$

it is readily verified that conditions (3) and (4) are satisfied. Differentiating χ with respect to t, we find

$$\frac{\partial \chi}{\partial t} = \frac{Q}{\pi^{\frac{3}{2}} C^3 (ut)^{3m/2}} e^{-\frac{r^2}{C^2 (ut)^m}} \left\{ \frac{mur^2}{C^2 (ut)^{m+1}} - \frac{3mu}{2ut} \right\}.$$

Thus at a distance r_0 from the centre of a puff of smoke the density χ is zero at time $t=0$ and at time $t=\infty$, and has a maximum at

$$t = \frac{1}{u} \left(\frac{2r_0^2}{3C^2}\right)^{1/m}.$$

Equation (100) may be tested against observations of puffs of smoke. Sutton assumes that the visible outline of the puff is determined by the presence of a minimum amount of smoke, say M, in a tube of unit cross-section from the eye. Then at the edge of the visible cloud

$$M = \int_{-\infty}^{+\infty} \chi \, dy = \frac{Q}{\pi^{\frac{3}{2}} C^3 (ut)^{3m/2}} \exp\left(-\frac{\zeta^2}{C^2 (ut)^m}\right) \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{C^2 (ut)^m}\right) dy$$

$$= \frac{Q}{\pi C^2 (ut)^m} \exp\left(-\frac{\zeta^2}{C^2 (ut)^m}\right),$$

where

$$\zeta^2 = y^2 + z^2.$$

Hence
$$\frac{\zeta^2}{(ut)^m} = mC^2 \left[\frac{1}{m} \log_e \frac{Q}{\pi M C^2} - \log_e ut\right] \qquad \text{......(101)},$$

and if d is the diameter of the visible outline at time t

$$\frac{d^2}{(ut)^m} = \text{const.} - 4mC^2 \log_e ut \qquad \text{......(102)}.$$

If then we plot $d^2/(ut)^m$ against $\log ut$, for a given m, we should obtain a straight line, if the theory is in accordance with the facts. The slope of the line being $4mC^2$, it is possible to compute C without considering the value of M.

Taking $m = 1{\cdot}75$, Sutton found that after the initial expansion due to the explosion the form of a cloud due to a shell-burst satisfies the above equation, giving values of C varying from 0·022 to 0·088, at heights of about 1 km to $5\frac{1}{2}$ km above the ground, the relation between C and the height being

$$C = 0{\cdot}302 - 0{\cdot}075 \log_{10} z$$

to a high degree of approximation.

Thus observation bears out the assumptions of Sutton that a value of m between 1 and 2, or a value of n less than unity, will describe the turbulent motion in the atmosphere. The equations derivable from the standard Fickian equation require $m = 1 = n$. Thus the analysis of shell-bursts indicates that the phenomena can be more closely interpreted by the assumptions underlying Sutton's development, than by the method first put forward by Taylor. The method of analysis of continuous movements developed by Taylor in 1922, and summarised briefly in § 147 above, appears therefore to afford the most potent instrument for the investigation of turbulence. It is capable of allowing for the observed fact that the rate of separation of two individual elements by turbulence is a function of their distance apart, whereas Taylor's first method, described in § 136 above, implies that the rate of separation is independent of the distance apart.

§ 151. *Extension of Sutton's method to a cloud emitted from a point or line*

In equation (100) is given the density distribution due to a single puff discharged at a point. This can be readily extended to give the distribution due to a cloud of smoke produced by continuous emission at a point, the smoke travelling downwind, and at the same time being diffused by the action of eddies.

The origin is now taken at the point of emission, and the mean wind being U, the centre of a puff emitted at $t = 0$ will at time t be at $x = Ut$. In the equation (100) above we now substitute

$$r^2 = (x - Ut) + y^2 + z^2 \qquad \text{......(103)}.$$

The part of the density distribution due to an emission $Q\,dt$ in time dt of a puff now centred at $x = Ut$ will be

$$d\chi = \frac{Q\,dt}{\pi C^2 (Ut)^m} \exp\left(-\frac{(x-Ut)^2+y^2+z^2}{C^2 (ut)^m}\right) \qquad \text{......(104).}$$

The whole density distribution χ is got by integrating $d\chi$ for all the eddies in existence, i.e. by integrating with respect to t from $-\infty$ to $+\infty$. In practice the density at a given point is only appreciably affected by puffs whose centres are at a distance from the point not greater than about half the width of the cloud. Further the width of a cloud is relatively small by comparison with its length. We are therefore only concerned with puffs whose ages differ from x/U by a small quantity, not greater than some limiting value which we need not for the moment specify. Let

$$t = \frac{x}{U}(1+\tau) \qquad \text{......(105)}$$

and integrate the expression for $d\tau$ from $-\tau_1$ to $+\tau_1$, where τ_1 is small by comparison with unity.

The total density distribution is then given by

$$\chi = \int d\chi = \frac{x}{U}\int_{-\tau_1}^{+\tau_1} \frac{Q\,d\tau}{\pi C^2 x^m (1+\tau)^m} \exp\left(-\frac{x^2\tau^2+y^2+z^2}{C^2 x^m (1+\tau)^m}\right) \text{......(106).}$$

But since τ is small by comparison with unity we shall neglect it in the denominator of the exponential term. The result of this is that y and z only enter the integrand in the form $e^{-(y^2+z^2)/C^2x^m}$, and the term

$$\exp\left(-\frac{y^2+z^2}{C^2 x^m}\right)$$

may be removed outside the sign of integration. Writing χ in the form

$$\chi = f(x) \exp\left(-\frac{y^2+z^2}{C^2 x^m}\right) \qquad \text{......(107)}$$

and making use of the fact that the total amount of matter crossing any plane at right angles to the axis of the cloud is independent of the distance x of the plane from the source of the cloud, we readily find $f(x)$ to be $Q/\pi C U x^m$.

Sutton has extended the results summarised above to an atmosphere in which turbulence is non-isotropic. The reader who desires further information on these points is referred to the original paper.

Reference should also be made to a paper by O. F. T. Roberts*, in which the scattering of smoke was investigated on the assumption that "every point in the atmosphere possesses a coefficient of eddy diffusion K which may vary from point to point, and which may further differ for diffusion in different directions at any point, analogous to the diffusion of heat in crystalline substances". The results derived by Roberts for an atmosphere in which K is isotropic can be obtained from Sutton's results by taking $m=1$, and $C^2 = 4K/U$.

* *Proc. Roy. Soc.* A, **104**, 1923, p. 640.

§ 152. *Stability and the criterion of turbulence*

Richardson* has given a criterion to determine whether turbulence will increase or decrease. The principle on which the criterion is based is that the flow will remain turbulent if the rate of supply of energy by the Reynolds stresses is at least as great as the work which has to be done to maintain the turbulence against gravity.

At a level z, where pressure, density and temperature are p, ρ, T respectively, let an eddy arrive from level $z-l'$, at which it had the normal temperature at that level, $T-l'\frac{\partial T}{\partial z}$. At its new level its temperature will be $T-l'\left(\frac{\partial T}{\partial z}+\Gamma\right)$, where Γ is the adiabatic lapse-rate. Its excess of density over the normal environment at that level will be

$$\frac{l'\rho}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right),$$

and the downward force upon it due to this excess of density will be

$$\frac{gl'\rho}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right) \text{ per unit volume,}$$

or

$$\frac{gl'}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right) \text{ per unit mass.}$$

The rate of upward flow of fluid per unit horizontal area per unit time is w', and hence the amount of work done against gravity per unit volume per unit time is

$$\frac{g\rho\overline{w'l'}}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right),$$

or

$$\frac{gK_T\rho}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right),$$

where K_T is the appropriate constant for diffusion of heat, defined by

$$K_T=\overline{w'l'}.$$

It is seen from § 136 that the work done by the eddy stresses per unit volume is

$$\widehat{xz}\frac{\partial \bar{u}}{\partial z}+\widehat{yz}\frac{\partial \bar{v}}{\partial z} \qquad \text{......(108),}$$

or

$$\rho\left\{K_{mx}\left(\frac{\partial \bar{u}}{\partial z}\right)^2+K_{my}\left(\frac{\partial \bar{v}}{\partial z}\right)^2\right\} \qquad \text{......(109),}$$

where K_{mx}, K_{my} are the appropriate coefficients of eddy diffusivity for momentum along the x and y axes. K_{mx} and K_{my} need not of necessity be equal, as was pointed out by Richardson.

The condition that turbulence should increase may therefore be written

$$K_{mx}\left(\frac{\partial \bar{u}}{\partial z}\right)^2+K_{my}\left(\frac{\partial \bar{v}}{\partial z}\right)^2>\frac{gK_T}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right) \qquad \text{......(110).}$$

* *Proc. Roy. Soc.* A, **97**, 1920, p. 354.

If $K_{mx} = K_{my} = K_m$, the condition becomes

$$\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z}\right)^2 > g\frac{K_T}{K_m}\frac{1}{T}\left(\frac{\partial T}{\partial z} + \Gamma\right) \qquad \text{......(111).}$$

As we have seen above, the factor K_T/K_m is at most unity and may be considerably less than unity. In the atmosphere it does not appear to differ widely from unity, if we judge by the reasonably close agreement of the values of K derived from Eiffel Tower observations of wind and temperature.

Fig. 60 reproduces an application of this criterion to the breakdown of an inversion, taken from a paper by C. S. Durst*. The lower diagram gives $\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z}\right)^2$, here represented as $\left(\frac{\partial v}{\partial h}\right)^2$, and it is seen that the motion becomes turbulent when this quantity exceeds $\frac{g}{T}\left(\frac{\partial T}{\partial z} + \Gamma\right)$.

Richardson's criterion agrees with that given by Prandtl except for a factor $\frac{1}{2}$ which appears on the right-hand side of the inequality as derived by Prandtl. The difference between the treatments of Richardson and Prandtl is that the latter treats the element of air as starting from rest and attaining a velocity w after a vertical distance l, whereas Richardson effectively assumes that it has the velocity w during the whole of the distance l. It is readily seen that this accounts for the factor $\frac{1}{2}$ in Prandtl's inequality. The example quoted above from Durst's paper, and other examples of the application of this criterion given by Richardson in various papers, appear to indicate the validity of Richardson's form of the criterion.

It should be noted that if the atmosphere is vertically stable, so that $(\partial T/\partial z) + \Gamma$ is positive, turbulence can only be maintained if $\partial \bar{u}/\partial z$ exceeds a certain definite limit. If $(\partial T/\partial z) + \Gamma$ is negative, any motion will become turbulent if slightly disturbed. In the atmosphere large gradients of velocity occur mainly in the immediate neighbourhood of the ground, as an effect of surface friction, and it is here that $(\partial T/\partial z) + \Gamma$ is most liable to assume negative values. Thus the layer of air in contact with the ground is the one in which turbulence will most readily occur. Richardson has pointed out (*loc. cit.*) that vertical gradients of wind of very considerable magnitude may occur near the upper boundary of the troposphere, and that turbulence may also arise in that region. In the neighbourhood of a surface of discontinuity between two streams of air moving with different velocities large gradients of velocity occur, and eddies form which tend to smooth out the surface of discontinuity.

A criterion such as is given above affords an explanation of the smoothing out of turbulence at the ground after sunset, when the ground cools by radiation to the sky. The growth of an inversion is accompanied by the suppression of turbulence, though there may be a large increase of wind with height.

G. I. Taylor† has investigated the effect of variation in density on the stability of superposed streams of fluid, and has shown that for three streams,

* *Q.J. Roy. Met. Soc.* **59**, 1933, p. 131.
† *Proc. Roy. Soc.* A, **132**, 1931, p. 499.

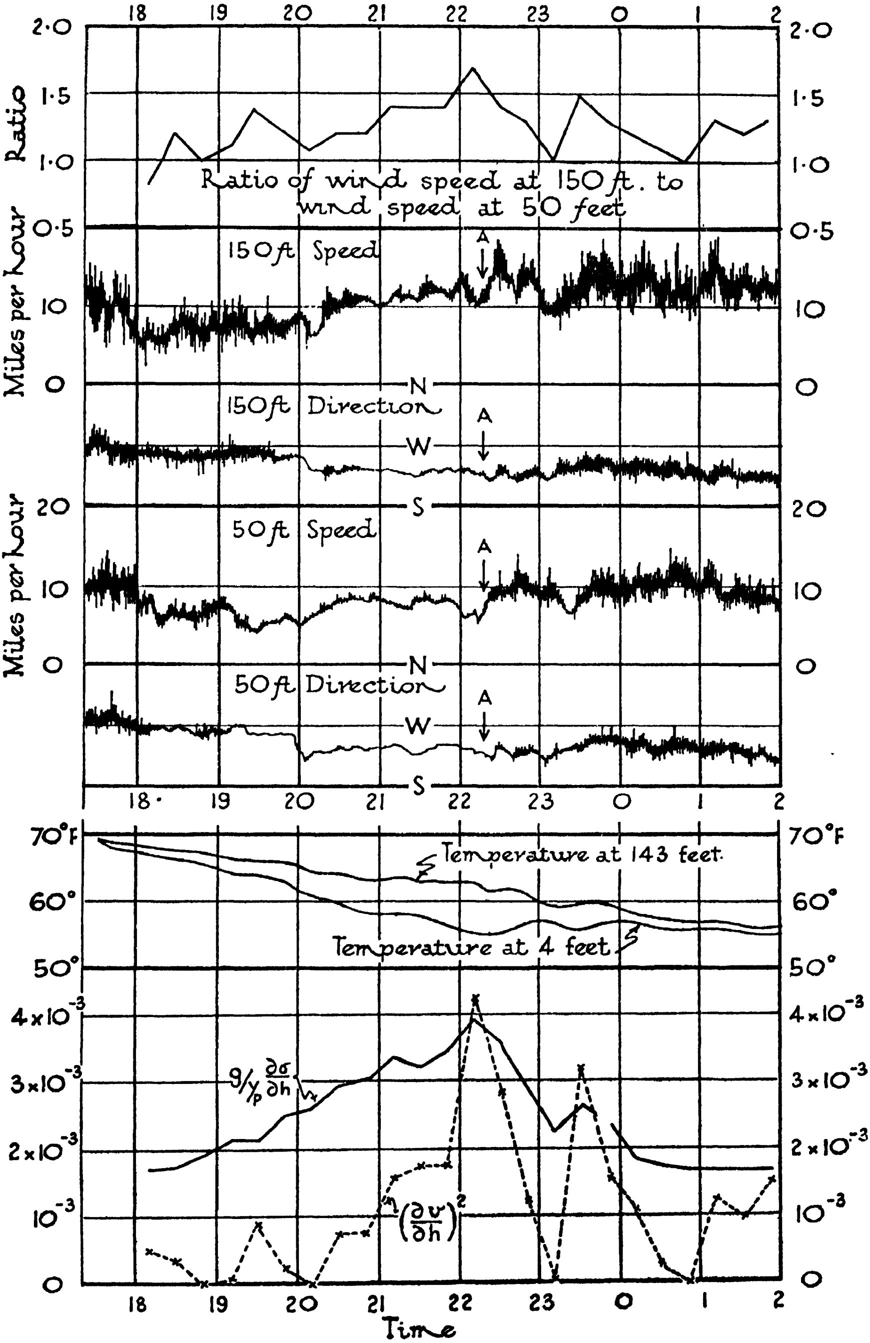


Fig. 60. Durst's application of Richardson's criterion.

of which the intermediate one is of small thickness h, instability of the wave-motion first arises when

$$\left(\frac{\partial \bar{u}}{\partial z}\right)^2 = -2g\frac{1}{\rho}\frac{\partial \rho}{\partial z} \qquad \text{......(112).}$$

Taylor found that the first type of instability which occurs when $\left(\frac{\partial \bar{u}}{\partial z}\right)^2$ reaches the limiting value $-\frac{2g}{\rho}\frac{\partial \rho}{\partial z}$ involves a tendency for the intermediate fluid to collect into lumps which form separate eddies and are projected alternatively upward and downward into the neighbouring fluid.

Taylor further showed that in a fluid in which $\frac{\partial \bar{u}}{\partial z}$ and $\frac{1}{\rho}\frac{\partial \rho}{\partial z}$ are not functions of the height z, all waves are stable if

$$\left(\frac{\partial \bar{u}}{\partial z}\right)^2 < -4g\frac{1}{\rho}\frac{\partial \rho}{\partial z} \qquad \text{......(113),}$$

but that no waves are possible, stable or unstable, if

$$\left(\frac{\partial \bar{u}}{\partial z}\right)^2 > -4g\frac{1}{\rho}\frac{\partial \rho}{\partial z} \qquad \text{......(114).}$$

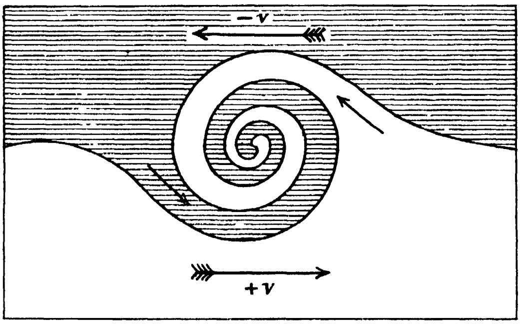


Fig. 61. Mallock's representation of the eddy.

These inequalities apply to continuous distributions of density, and should be contrasted with the equality (112) above which applies to stability with one intermediate layer. In (112) the coefficient 2 becomes 2·11 for two intermediate layers, and Taylor has suggested that it may become 4 when the number of layers becomes infinite. Taylor's discussion applies, however, only to infinitesimal displacements, and is not strictly applicable when the motion becomes finite. The work of Rosenhead* is probably more closely akin to the actual physical phenomena. The idea underlying Rosenhead's work is to be found in a paper by Mallock†. Fig. 61, which shows the nature of an eddy as pictured by Mallock, gives a picture of the process of mixing which is readily understood.

Rosenhead investigates the flow of a stream of density ρ and velocity U in the direction of the axis of x, above a stream of the same density, having

* *Proc. Roy. Soc.* A, **134**, 1932, p. 170. † *Aero. Res. Committee, R. and M.*, No. 314.

velocity U in the opposite direction. The motion is treated as two-dimensional, the axis of x being the undisturbed surface. The flow is continuous and irrotational on both sides of the surface of separation. Initially the surface of separation is of the form of a sine-curve of small amplitude. The solution of the equations of motion by the method of small oscillations contains a first and a second order term. The first order term is a sine-term of increasing amplitude. The second order term, which eventually dominates, introduces a term which is antisymmetrical with respect to a crest, and hence the disturbance does not grow symmetrically.

In Rosenhead's analysis the surface of discontinuity, which is a vortex sheet, is replaced by a distribution of finite elemental vortices along its trace, and the paths of the vortices are determined by a numerical step by step method. It appears that the effect of instability on a surface of discontinuity of sine form is to produce concentrations of vorticity at equal intervals along the surface, and that the surface of discontinuity tends to roll up round these points of concentration. The process of development is shown in fig. 62, which is a rough reproduction of Rosenhead's fig. 4 (*loc. cit.*), and the rolling up of the surface of separation is seen to be similar to that represented by Mallock, as shown in fig. 61 above. An interesting feature of the phenomenon is that while the initial conditions may be represented by two irrotational motions, the conditions represented by the final stage of fig. 62 indicate that in portions of the fluid a finite distribution of vorticity may become observable*.

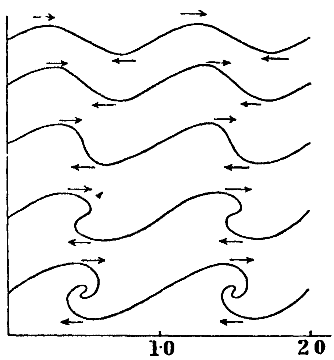


Fig. 62.
Rosenhead's representation of the development of the eddy.

Prandtl† summarises the question of the criterion of turbulence as follows: In large volumes stability depends on the dimensionless number

$$-\frac{g}{\rho}\frac{\partial \rho}{\partial z} \Big/ \left(\frac{\partial u}{\partial z}\right)^2,$$

which in the atmosphere is replaced by

$$\frac{g}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right) \Big/ \left(\frac{\partial u}{\partial z}\right)^2 \qquad \text{......(115).}$$

The critical value of this number, above which turbulence tends to die out, and below which turbulence sets in, is estimated at unity by Richardson, at $\frac{1}{2}$ by Prandtl, and $\frac{1}{4}$ by Taylor and Goldstein (from small oscillations). Prandtl states (*loc. cit.*) that preliminary experiments indicated a limiting value of $\frac{1}{4}$ to $\frac{1}{2}$. The agreement with Richardson's criterion shown in fig. 60 appears, however, to indicate that in the atmosphere the limiting value of expression (115) above is unity.

* See also A. R. Low, *Nature*, **121**, 1928, p. 576.
† *Beitr. Phys. fr. Atmos.* **19**, 1932, p. 188.

The above discussion has no application to purely dynamical stability. Rayleigh* considered the stability of laminated steady motion in which the undisturbed velocity U is parallel to the axis of x, and is a function of z, with the vorticity changing suddenly from one layer to another. He discussed the effect of superposing a small disturbing velocity of a simple harmonic type, and found that the two-dimensional motion should be stable if $\partial^2 U/\partial z^2$ is of the same sign throughout the fluid. Taylor† derived the same result for a perfectly generalised disturbance, and gave a physical explanation of the origin of the instability in cases where $\partial^2 U/\partial z^2$ changes sign, in terms of the eddy transport of momentum.

Prandtl and Tietjens‡ have investigated this question still further, and have studied in detail various velocity profiles. The gradient of velocity appears to be of outstanding importance in questions of stability, and a considerable amount of work along these lines is now in progress.

Douglas§ has used Richardson's criterion to show that there is an upper limit to the slope of a surface of discontinuity beyond which even an inversion or an isothermal patch will become turbulent, and consequently will be smoothed out.

§ 153. *The maintenance of super-adiabatic lapse-rates in the atmosphere*

Reference was made earlier, on p. 214, to the result derived by Rayleigh that a fluid can remain stable with density increasing upward, if

$$\frac{\rho_1 - \rho_0}{\rho_0} < \frac{27\pi^4 \kappa\nu}{4gh^3} \qquad \text{......(116)},$$

or, if $\partial\rho/\partial z$ be the upward gradient of density, if

$$\frac{h}{\rho}\frac{\partial \rho}{\partial z} < \frac{27\pi^4 \kappa\nu}{4gh^3} \qquad \text{......(117)}.$$

In the atmosphere the transfer of heat and momentum is not brought about by the molecular processes of viscosity and conduction, but by eddy diffusion or eddy diffusion and radiative diffusion combined. Jeffreys‖ has shown that it is permissible in a compressible fluid to use the potential temperature in lieu of absolute temperature, or, to a sufficient degree of approximation, to use the deviations of temperature from an adiabatic distribution. We then replace $\frac{1}{\rho}\frac{\partial \rho}{\partial z}$ by $-\frac{1}{T}\left(\frac{\partial T}{\partial z} + \Gamma\right)$ for compressible fluid, and the inequality then becomes

$$-\frac{\partial T}{\partial z} - \Gamma < \frac{27\pi^4 \kappa\nu}{4gh^4 T} \qquad \text{......(118)}.$$

In a turbulent atmosphere the transport of heat and momentum by turbulence greatly exceeds that by molecular diffusion, and κ and ν should be replaced

* *Collected Papers*, **3**, p. 375. † *Phil. Trans. Roy. Soc.* A, **215**, 1915, p. 1.
‡ *Zeit. Angew. Math. u. Mech.* **1**, 1921, p. 431.
§ *Q.J. Roy. Met. Soc.* **50**, 1924, Appendix, p. 359.
‖ *Proc. Camb. Phil. Soc.* **26**, pt 2, 1930.

by K. If K is of the order of 10^5, the maximum lapse-rate in a depth h is given by

$$-\frac{\partial T}{\partial z}-\Gamma=\frac{27\pi^4 10^{10}}{4gh^4 T}=\frac{2\cdot 35 . 10^7}{h^4} \text{ approximately.}$$

If $h=10$ metres $=10^3$ cm,

$$\left(-\frac{\partial T}{\partial z}-\Gamma\right)_{\max}=2\cdot 35\times 10^{-5}\ ^\circ\text{C/cm}=2\cdot 35\ ^\circ\text{C/100 m.}$$

If $h=2$ metres, the maximum lapse-rate becomes about $1\cdot 5^\circ$ C per metre, and for $h=1$ metre, the maximum becomes 12° C per metre. For other values of K the maximum possible value of $-(\partial T/\partial z)-\Gamma$ which can persist will be proportional to K^2.

In quieter conditions when turbulence is less active the transfer of heat by radiation may exceed that by eddy diffusion, and it is then necessary to replace κ in the inequality (118) above by the radiative diffusivity K_R, and to replace ν by K. The resulting maximum lapse-rate is smaller than was derived on the supposition that κ and ν should be replaced by K, whose value was taken as 10^5, but the values so derived are still much in excess of the adiabatic lapse-rate, for very shallow layers. The point which is of interest is that the occurrence of very large lapse-rates in shallow layers of air could have been predicted by theory, and that such lapse-rates actually occur and are readily observed. Further, even when K is assumed to have a value as great as 10^6, the maximum value of $-(\partial T/\partial z)-\Gamma$ which can persist in deep layers is only a very minute fraction of Γ. It follows that, except in very shallow layers, the limiting condition for stability is fixed by the adiabatic lapse-rate as derived in § 21 above.

§ 154. *Total flow along and across the isobars*

By the use of equations (67) or (68) it is readily possible to compute the flow of air along and across the isobars up to any given height, provided the geostrophic wind G does not change with height. The total flow across a vertical surface of height h and unit width, set up at right angles to the isobar, is

$$\int_0^h \rho u\,dz=D_x, \text{ say} \qquad \ldots\ldots(119),$$

and the total flow across a vertical surface of height h and unit width, set up parallel to the isobar, is

$$\int_0^h \rho v\,dz=D_y, \text{ say} \qquad \ldots\ldots(119a).$$

Neglecting variations of ρ,

$$\begin{aligned}
D_x+iD_y&=\rho\int_0^h (u+iv)\,dz\\
&=\rho Gh+\sqrt{2}\rho G\sin\alpha\int_0^h e^{-Bz+i(\alpha+\frac{3}{4}\pi-Bz)}\,dz\\
&=\rho Gh-\frac{\rho G\sin\alpha}{B}\left[e^{-Bz+i(\alpha+\pi/2-Bz)}\right]_0^h\\
&=\rho Gh+\frac{\rho G\sin\alpha}{B}\{-\sin\alpha+i\cos\alpha-e^{-Bh}[\sin(\alpha-Bh)+i\cos(\alpha-Bh)]\},
\end{aligned}$$

and
$$\left.\begin{aligned} D_x &= \rho G h - \frac{\rho G \sin\alpha}{B}\{\sin\alpha + e^{-Bh}\sin(\alpha - Bh)\} \\ D_y &= \frac{\rho G \sin\alpha}{B}\{\cos\alpha - e^{-Bh}\cos(\alpha - Bh)\} \end{aligned}\right\} \quad \text{......(120).}$$

When h is great the factor e^{-Bh} becomes very small and may be neglected. The total flow across the isobar into low pressure is then $\rho G \sin\alpha\cos\alpha/B$ per unit length of the isobar, while the total flow across the surface of unit width at right angles to the isobars is $\rho G h - \rho G \sin^2\alpha/B$. As shown in § 144, e^{-Bh} is in practice negligible when h is 1 km, and the flow across the isobars above this level is negligible.

When $K = 10^5$, $\alpha = 22\frac{1}{2}°$, $B = 2{\cdot}4 \times 10^{-5}$, $h = 1$ km $= 10^5$ cm,
$G = 10$ m/sec $= 1000$ cm/sec, $\rho = 1{\cdot}25 \times 10^{-3}$ gm/cm^3,
the values of D_x and D_y are

$$\begin{aligned} D_x &= 1{\cdot}25 \times 10^5 \text{ gm/sec} = 125 \text{ kg/sec} \\ &= 100 \text{ (m)}^3 \text{ of air per sec,} \\ D_y &= 0{\cdot}2 \times 10^5 \text{ gm/sec} = 20 \text{ kg/sec} \\ &= 16 \text{ (m)}^3 \text{ of air per sec.} \end{aligned}$$

Here D_x and D_y are the flows across surfaces whose width is 1 cm. In the lowest kilometre the flow across the isobars is comparable with the flow along the isobars. The drift of air across the isobars may be of importance in connection with the formation of rain in depressions, since the air carries with it a supply of moisture. Some further consideration is given to this point in a later section in connection with the discussion of rainfall in depressions (see § 190). It is probable that the compensating outflow across the isobars is at higher levels, where the moisture content is small, and the contribution of the inward drift of moisture in the lowest layers to the rainfall in the depression may be considerable, particularly at a front.

The frictional flow of air across a front inclined at an angle β to the isobars is deduced from the component normal to the front of the deviation from the geostrophic wind. The total flow up to great heights is readily deduced from equations (120) above, leaving out the geostrophic term which is independent of α, and neglecting e^{-Bh}. The total flow is

$$\frac{\rho G \sin\alpha}{B}\cos(\alpha - \beta) = D_y \sec\alpha\cos(\alpha - \beta) \qquad \text{......(121),}$$

where D_y is as defined above, and expressed in equations (120). This is greatest when $\alpha = \beta$, and decreases as β increases beyond this value.

§ 155. *The boundary layer*

When a fluid flows over a solid boundary the portions of the fluid in immediate contact with the boundary are at rest, and there is a thin layer of fluid at the boundary across which there is a rapid increase of velocity, and within which the motion is laminar. This "boundary layer" is a time-mean phenomenon, and it is not to be supposed that this layer is always constituted of the same fluid. The turbulence which may prevail at some distance from the boundary will from time to time break through the layer, carrying away portions of the fluid which instantaneously constitute it, but as soon as the individual eddy has removed a portion of the boundary layer normal processes will tend to build it up again.

Reference has already been made in § 145 to Stanton's measurements of flow in pipes, according to which the velocity at the outer limit of the boundary layer is 0·6 times the velocity in the middle. In equation (75) is represented Taylor's estimate of the skin friction at the ground as $0{\cdot}004\rho V^2$ per unit area, where the coefficient 0·004 is a constant deduced from the flow in tubes, and V is the velocity "near" the surface. It is assumed that V may be taken at the outer limit of the boundary layer.

From the above considerations it is possible to obtain an estimate of the thickness of the boundary layer. Call this thickness δ. Then the shearing force per unit area of the surface is $\mu V/\delta$. This is equal to the skin friction, and hence

$$\mu V/\delta = 0{\cdot}004\rho V^2 \qquad \text{......(122)},$$

or

$$\delta = \frac{\mu}{0{\cdot}004\rho V} = \frac{\nu}{0{\cdot}004 V}.$$

Putting $\nu = 0{\cdot}15$, this reduces to

$$\delta = 37/V.$$

If V_0 is the velocity at a great distance from the boundary, $V = 0{\cdot}6V_0$,

$$\delta = \frac{37}{V} = \frac{60}{V_0} \qquad \text{......(123)}.$$

The thickness δ is given in cm and the velocity V in cm/sec. For such velocities as are important in the atmosphere, V is of the order of at least 400 cm/sec, and δ is then of the order of 1 mm. Equation (123) above is in reasonable agreement with such aerodynamical observations as are available. For a discussion of the boundary layer from the aerodynamical standpoint the reader is referred to Prandtl-Tietjens, *Hydro- und Aero-Mechanik* (Berlin), Chapter v. Reference should also be made to the observations of Stanton in the paper cited in the footnote on p. 242.

The existence of the boundary layer is a fact of considerable importance in meteorology, particularly in connection with problems involving evaporation. Within this layer the transport of heat or of water-vapour is brought about entirely by molecular diffusion, while outside it, in the turbulent region, the transfer is brought about by eddies. It has already been emphasised that the boundary layer is a time-mean phenomenon, and that there is a slow exchange of fluid between the layer and the outer region of turbulent motion. Thus the water-vapour produced by evaporation at a liquid surface spreads outward through the laminar layer by molecular diffusion, while the air within that layer is from time to time exchanged by the action of eddies which penetrate it.

The concept of the boundary layer is useful in discussing the action of the wet-bulb thermometer, as well as the phenomena of evaporation from sheets of water. Some brief consideration of these two cases will be found in §§ 156, 157 below.

§ 156. *The derivation of the hygrometer equation*

In deriving the August-Apjohn equation in § 47 above it was assumed that a portion of the normal air of the environment of the wet bulb takes up the required amount of water from the wet bulb to become saturated at the temperature of the wet bulb. The weakness of this theory is that it is not possible in practice to separate out the air into two streams, one of which becomes saturated while the other is unaffected by the presence of the wet bulb.

An alternative theory due to G. I. Taylor was first published by Skinner, in an article in the *Dictionary of Applied Physics*, **3**. It was reproduced by Whipple in a recent paper*, and is summarized below.

Let T and e be the absolute temperature and vapour-pressure in the free air at some distance from the wet bulb, T' and e' the values at the surface of the wet bulb, and T'' and e'' the values at the boundary between the laminar layer and the turbulent layer. Let κ be the molecular diffusivity (for heat) of air, and let D be the coefficient of diffusion of water-vapour into air. Then if δ be the thickness of the boundary layer, the rate at which heat is conducted to the wet bulb is, per unit area,

$$\kappa \rho c_p (T'' - T')/\delta.$$

The rate at which water-vapour diffuses outward from density $\epsilon \rho e'/p$ to the density $\epsilon \rho e''/p$ is

$$D\epsilon\rho (e' - e'')/p\delta.$$

The incoming heat provides the latent heat of evaporation of the outward diffusing water-vapour. Hence

$$\kappa \rho c_p (T'' - T') = LD\epsilon\rho (e' - e'')/p \qquad \text{......(124)}.$$

Let E be the rate at which the air within the boundary layer is being replaced by normal air from the turbulent region. E will be reckoned as volume per unit area of the surface of the outer boundary of the laminar layer, per unit time. The physical significance of E is not very clear, but this is of no great importance if we may assume that E has the same significance in the transfer of heat and of water-vapour.

The rate at which heat is carried to the boundary layer is

$$E (T - T'') \rho c_p.$$

The rate at which water-vapour is carried outward is

$$E\epsilon\rho (e'' - e)/p.$$

Since these two must balance

$$(T - T'') c_p = L\epsilon (e'' - e)/p \qquad \text{......(125)}.$$

* *Proc. Phys. Soc. London*, **45**, pt 2, 1933, p. 307.

The two equations (124) and (125) involve T'' and e'', which are not measured quantities. These two variables can be eliminated together only if $\kappa = D$. For then by addition of the two equations we find

$$\left.\begin{aligned} T-T' &= \frac{L\epsilon}{c_p}(e'-e)/p \\ \text{or} \quad e'-e &= \frac{pc_p}{L\epsilon}(T-T') \end{aligned}\right\} \qquad \text{......(126).}$$

This is the hygrometer equation already derived in § 47. Taylor justified the above derivation of equation (126) by the use of a value of D of 0·198, a value appropriate to a temperature of 0° C, which should be compared with the corresponding value of κ, 0·17. There is some uncertainty as to the most reliable value of D. The estimate of 0·198, due to Winkelmann, was published in 1884. More recent estimates of D are nearer to 0·24. If this value is accepted, the validity of the derivation of the hygrometer equation given above is considerably impaired. (Some measured values of D are given in Table IX, p. 406.)

§ 157. *Evaporation from the surface of water*

(*a*) EVAPORATION INTO STILL AIR*

When the evaporation takes place into still air, the water-vapour is diffused by molecular motion. Let D be the coefficient of diffusion of water-vapour into air. Then if V is the mass of water-vapour per unit mass of air, expressed as a function of three co-ordinates x, y, z, of which z is vertical, the equation of diffusion is

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} = D\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) \qquad \text{......(127).}$$

Let V_0 be the difference between V at $z=0$ and at a great distance. In the steady state $\partial V/\partial t = 0$, and the equation reduces to

$$\nabla^2 V = 0 \qquad \text{......(128).}$$

The rate of transfer of water-vapour upward from the boundary ($z=0$)

$$= -\iint D\rho \frac{\partial V}{\partial z}\, dS \qquad \text{......(129),}$$

the integral being taken over the whole surface of the water. But (128) is the equation which must be satisfied by the potential in electrostatic problems, and in our case the analogous electrostatic problem is that of a charged plate coinciding with the water surface. Let σ be the density of electrostatic charge on the surface. Then

$$\partial V/\partial z = -4\pi\sigma$$

and hence

$$-\iint \frac{\partial V}{\partial z}\, dS = -4\pi \iint \sigma\, dS = 4\pi C V_0 \qquad \text{......(130),}$$

where C is the electrostatic capacity of the plate. Hence the rate of transference outwards is

$$4\pi D\rho C V_0 \qquad \text{......(131).}$$

* This treatment follows a paper by Jeffreys, *Phil. Mag.* **35**, 1918, p. 270.

This determines the total amount of evaporation from the water surface. Since C is proportional to the linear dimension of the conductor for similar plates, it follows that the evaporation is proportional to the linear dimension of the surface. It is also proportional to the difference in V at the water surface, where the air is saturated, and at a great distance, where the value of V corresponds to the state of the air before it has drifted over the water surface.

(b) Evaporation into a steady wind

Jeffreys (*loc. cit.*) discussed the evaporation into a steady wind, taking account of the effects of turbulence. The following is a brief abstract of his treatment. Let u be the velocity of the wind, which will be assumed to blow along the x-axis. Then, since there is no wind along the y- or z-axis, equation (127) may be written

$$\frac{dV}{dt}=\frac{\partial V}{\partial t}+u\frac{\partial V}{\partial x}=K\nabla^2 V \qquad \text{......(132)},$$

where K is the eddy diffusivity, assumed not to vary with height. Outside the boundary layer of rapid shearing the equation may be written

$$u\frac{\partial V}{\partial x}=K\frac{\partial^2 V}{\partial z^2} \qquad \text{......(133)}.$$

Let $h^2=K/u$. Then a solution of (133) is

$$\left.\begin{aligned} &V=V_0\left(1-\operatorname{Erf}\frac{zx^{-\frac{1}{2}}}{2h}\right) \text{ when } x \text{ is positive}\\ &V=0 \text{ when } x \text{ is negative}\end{aligned}\right\} \qquad \text{......(134)}.$$

This makes $\partial V/\partial z=V_0/h\sqrt{\pi x}$ over the wet surface. The rate of evaporation is therefore

$$K\rho\frac{\partial V}{\partial z}=V_0\rho\sqrt{\frac{Ku}{\pi x}} \qquad \text{......(135)}$$

per unit area. The amount evaporated between 0 and x over a strip dy in width is by integration

$$2\rho V_0\,(Kux/\pi)^{\frac{1}{2}}\,dy.$$

If the length of the strip from one margin to the other is l, thus neglecting the diffusion sideways at the edges, the total evaporation is

$$2\rho V_0\,(Ku/\pi)^{\frac{1}{2}}\int l^{\frac{1}{2}}\,dy \qquad \text{......(136)}$$

the integration being taken over the whole area. For areas of the same shape, in which a is a linear dimension, the evaporation is proportional to $a^{\frac{3}{2}}$, and in particular for a circle of radius a the total evaporation is

$$3{\cdot}95\rho V_0\,(Kua^3)^{\frac{1}{2}} \qquad \text{......(137)}.$$

In equations (134) to (137) V_0 is to be taken as the difference of the humidity mixing ratio at the water surface, where the air is saturated, and at a great distance. Jeffreys also considered the limitations of his results, and showed that they apply in the open air to sheets of water for which a is between say 10 cm and 250 metres.

(c) Giblett's investigation

Giblett* solved the diffusion equation, using as his boundary condition an empirical formula for the total evaporation from unit surface.

$$\text{Evaporation} = A\,(e_s - e)\,(1 + cu) \qquad \ldots\ldots(138),$$

where A and c are constants, e_s is the saturation vapour-pressure at the surface temperature, and e is the vapour-pressure in the air "near" the surface (where "near" presumably means just outside the boundary layer), and u is the horizontal velocity of the wind "near" the surface. Giblett regarded the eddy diffusivity K as not varying with height. In view of the assumptions made the results are of limited application.

(d) Sutton's treatment of evaporation

In a recent paper Sutton† has shown that Jeffreys' analysis can be extended to apply to the case where the eddy diffusivity and the wind velocity vary with height, the velocity varying in accordance with the law

$$u = u_1\,(z/z_1)^{n/(2-n)} \qquad \ldots\ldots(139),$$

where n is the parameter defined in § 150.

The analysis is too lengthy to be reproduced here, but the final result can be stated very briefly. The mean rate of evaporation from a surface of unit breadth and of length x_0 downwind is

$$E = Mu_1^{(2-n)/(2+n)}\,a^{2/(2+n)}\,x_0^{2/(2+n)} \qquad \ldots\ldots(140),$$

where M is an absolute constant, u_1 is the velocity at a standard height z_1, as in equation (139), and a is a quantity depending both on n and on the physical constants of the atmosphere, but independent of u_1.

For a strip of length x_0 downwind and breadth y_0 across wind the evaporation is y_0 times that from a strip of unit width.

For an elliptic area with semi-axes r_1 downwind, and r_2 across wind, the total evaporation is readily found by integration of (140) to be

$$E = M'u_1^{(2-n)/(2+n)}\,a^{2/(2+n)}\,r_1^{2/(2+n)}\,r_2 \qquad \ldots\ldots(141).$$

For a circular area of radius r the last expression gives

$$E = M'u_1^{(2-n)/(2+n)}\,a^{2/(2+n)}\,r^{(4+n)/(2+n)} \qquad \ldots\ldots(142).$$

Sutton has compared these results with various experimental results. In such a comparison it is necessary to bear in mind that since a is a function of n it can only be treated as constant in a series of observations if n remains constant during the whole series. For purely molecular diffusion $n = 1$, and the evaporation from a rectangular area whose axis x_0 is downwind is then

$$E = Mu_1^{\frac{1}{3}}\,a^{\frac{2}{3}}\,x_0^{\frac{2}{3}}\,y_0 \qquad \ldots\ldots(143).$$

* *Proc. Roy. Soc.* A, **99**, 1921, p. 472.
† *Proc. Roy. Soc.* A, **146**, 1934.

Gallenkamp* found that his observations fitted the formula

$$E \propto x_0^{0\cdot 6} y_0 \qquad \ldots\ldots(144),$$

which is in excellent agreement with (143).

Molecular diffusion with $n = 1$ gives, from equation (142),

$$E \propto r^{1\cdot 67} \qquad \ldots\ldots(145).$$

Gallenkamp's observations yielded an index 1·6.

In a turbulent medium it is found that, for a wide range of Reynolds number,

$$u = u_1 (z/z_1)^{1/7} \qquad \ldots\ldots(146),$$

corresponding to $n = \frac{1}{4}$. With this value of n the above equations show that the variation of the amount of evaporation with wind velocity is given by

$$E \propto u_1^{7/9}$$

or

$$E \propto u_1^{0\cdot 78} \text{ approximately} \qquad \ldots\ldots(147).$$

Himus†, as the result of a long series of experiments, gave

$$E \propto u_1^{0\cdot 77} \qquad \ldots\ldots(148).$$

Sutton's theory thus leads to results which agree in a remarkable manner with experimental results. It must be borne in mind, however, that the theory only yields a comparison of the evaporation on occasions when n has the same value. But provided n has the same value on all occasions, the theory developed by Sutton will give the ratio of the evaporation from surfaces of different extent and with different wind velocities.

* *Met. Zeit.* **36**, 1919.

† *Inst. of Chem. Eng.*, Conference on Vapour Absorption and Adsorption (1929).

Note on § 150.

In the paper by Sutton referred to in § 157 (*d*), p. 266, it is suggested that R_ξ should be defined by

$$R_\xi = \left(\frac{\nu}{\nu + \overline{w'^2}\xi}\right)^n.$$

The adoption of this form instead of that given in equation (97), p. 250 does not involve any change in the nature of the results derived in §§ 150, 151.

CHAPTER XIV

THE CLASSIFICATION OF WINDS

§ 158. *The terms in the equations of motion*

THE equations of motion of air over the earth's surface contain a statement of the forces acting upon the air, and of the accelerations which the air undergoes. The forces acting upon the air are (*a*) gravity, (*b*) hydrostatic pressure, and (*c*) friction. In addition there is the so-called "deviating force" due to the earth's rotation. The equations of motion of air moving over a rotating earth referred to axes drawn to East, North and vertical respectively, may be written

$$\frac{du}{dt}+2\omega\,(w\cos\phi-v\sin\phi)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+K\frac{\partial^2 u}{\partial z^2} \qquad \text{......(1)},$$

$$\frac{dv}{dt}+2\omega u\sin\phi \qquad =-\frac{1}{\rho}\frac{\partial p}{\partial y}+K\frac{\partial^2 v}{\partial z^2} \qquad \text{......(2)},$$

$$\frac{dw}{dt}-2\omega u\cos\phi \qquad =-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \qquad \text{......(3)}.$$

These equations are sufficiently exact for all practical purposes except for the terms $K\,\partial^2u/\partial z^2$, $K\,\partial^2v/\partial z^2$ in the first and second equations. These terms represent the frictional effects, whose nature is not known with complete certainty, as was shown in the preceding chapter. These equations were used by Jeffreys* as a basis for a classification of winds. Most of the present chapter is based on Jeffreys' paper.

In the first place, the pressure terms in the equations are always important, otherwise each portion of the fluid would pursue its path independently, without being appreciably interfered with by the impacts of surrounding portions. The latter is far from being true in the atmosphere, the mean free path being very small by comparison with the horizontal displacements with which we are concerned in atmospheric motions. In any type of wind the pressure terms are therefore of fundamental importance.

The terms on the left-hand side of equation (3) are usually negligible, and are in practically all circumstances small by comparison with gravity. The easterly wind velocity u is at most 100 metres per second, and ω is 7×10^{-5} cm/sec. Thus $2\omega u\sin\phi$ is at most about 1·5 cm/sec², which is small by comparison with g. The term dw/dt can be estimated roughly. The fall of a hailstone requires that the frictional resistance of the upward current shall be less than the weight of the hailstone. The resistance to the motion of a sphere is of the order of $0{\cdot}1\pi\rho a^2w^2$, and weight is $\frac{4}{3}\pi da^3g$, where a is the radius of the

* *Q.J. Roy. Met. Soc.* **48**, 1922, p. 29.

stone and d its density. Hence the velocity required in an ascending current to maintain the hailstone without acceleration is given by

$$0{\cdot}1\pi\rho a^2 w^2 = \tfrac{4}{3}\pi d a^3 g \quad \text{or} \quad w^2 = 10^7 a.$$

If $a = 3$ cm for an unusually large hailstone, the vertical velocity is about 55 m/sec. If the vertical acceleration is f, and h is the height to which the air has ascended,

$$w^2 = 2fh.$$

Let $h = 3$ km $= 3 \times 10^5$ cm, then f is 50 cm/sec². This value of the vertical acceleration is small by comparison with gravity, and it is therefore legitimate to omit the term dw/dt from the left-hand side of equation (3) which we can now write

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0, \quad \text{or} \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = -g \qquad \text{......(4).}$$

In general vertical motion of any appreciable magnitude only takes place in eddies, and for most purposes it is legitimate to neglect w in equation (1). The equations of mean horizontal motion are then

$$\frac{du}{dt} - 2\omega v \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} + K\frac{\partial^2 u}{\partial z^2} \qquad \text{......(5),}$$

$$\frac{dv}{dt} + 2\omega u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} + K\frac{\partial^2 v}{\partial z^2} \qquad \text{......(6),}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad \text{......(7).}$$

Since the pressure term is always important, it follows that at least one of the other terms is comparable in magnitude with it. Winds may be classified according to the identity of the term or terms in question.

Case 1. Rotational and frictional terms negligible. Here

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{......(8),}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{......(9).}$$

The acceleration is measured by the pressure gradient. Jeffreys calls this class of wind "Eulerian".

Case 2. Rotational terms far in excess of both accelerational and frictional terms. The equations now reduce to

$$-2\omega v \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{......(10),}$$

$$2\omega u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{......(11).}$$

The gradient of pressure is here balanced by the "deviating force". Winds of such a type are called "geostrophic".

Case 3. Frictional terms exceeding the rotational and accelerational terms. The equations of motion reduce to

$$K\frac{\partial^2 u}{\partial z^2}=\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{......(12)},$$

$$K\frac{\partial^2 v}{\partial z^2}=\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{......(13)}.$$

The wind will in general blow along the pressure gradient, friction preventing its velocity from increasing steadily. For such winds Jeffreys suggests the name "antitriptic".

§ 159. *The application of the classification*

It is convenient, following Jeffreys, to classify the winds of the globe according to their horizontal extent, as follows:

(*a*) World-wide phenomena, including the general circulation and its seasonal variation.

(*b*) Phenomena on a continental scale, including monsoon winds and associated changes of pressure.

(*c*) Phenomena on a scale comparable with the British Isles. In view of the fact that even so large an island as Australia only produces modifications of secondary importance in the general circulation, it seems likely that moderate-sized and small islands do not to any great extent modify the general and continental circulations in their neighbourhood.

(*d*) Small scale phenomena, including all atmospheric disturbances whose horizontal dimensions in at least one direction are of the order of kilometres or tens of kilometres. This category includes tropical cyclones, tornadoes, land and sea breezes, mountain and valley winds, and winds of the Föhn type.

It is advantageous to begin by deciding in which of these cases the accelerational term is greater than the rotational term. A strong wind of any of the first three classes has a velocity of about 20 m/sec. The acceleration du/dt is comparable with the ratio u^2/l, where l is the linear dimension of the disturbance. This will be small compared with $u/2\omega \sin \phi$. If u is 20 m/sec, then at the poles $u/2\omega \sin \phi$ is 140 km, and in latitude 20° its value is 800 km. When l has this value, du/dt and the rotational term are about equal. For greater velocities the horizontal extent is increased in proportion. It is clear, however, that in cases a, b and c above the rotational terms exceed the accelerational term, so that none of the winds in these classes can be Eulerian, but are either geostrophic or antitriptic. In the typical tropical revolving storm, the velocities frequently reach 70 m/sec, while the dimensions in typical cases are of the order of 50–100 km. The dimensions are therefore such that the accelerational term exceeds considerably the rotational term. The tropical cyclone is therefore not geostrophic.

Observations show that winds of type (*a*), (*b*) and (*c*) deviate from the direction of the isobars by not more than 2 to 4 points at the surface, and by much less at heights above 0·5 km. These winds are therefore mainly geostrophic in character. In the tropical storm, a particle performs one revolution about the centre in a few hours, while the life of the storm is several days. The frictional term must therefore be much smaller than the accelerational term, and the motion in these storms is therefore Eulerian. The same result has been stated by Sir Napier Shaw, who pointed out that the winds are "cyclostrophic" in the tropical storm and the tornado.

§ 160. *Geostrophic winds*

The result stated above, that large scale phenomena in the atmosphere are essentially geostrophic in character, is in accordance with the results already stated in Chapter IX, where it was shown that at levels removed from the effects of surface friction the wind could be treated as closely approximating to the gradient wind, provided no rapid change of pressure were taking place. Further, the comparison of observed winds with the gradient winds computed from synoptic charts, carried out by Gold (see p. 184 above), also shows that the winds are in the main geostrophic. The only point which is left obscure is the closeness of the geostrophic approximation in general. The comparison of observed and computed winds, being based on pilot balloon observations in clear weather, is thereby restricted to occasions when the pressure distribution is normally not changing rapidly, and does not apply to the central regions of depressions, in which cloud and rain make pilot ballon observations impracticable.

The closeness of the geostrophic approximation can be readily deduced by the use of equation (4) of Chapter IX, if the pressure distribution is not changing. For simplicity take the symmetrical depression, so that the pressure is a function only of distance r from the centre. Let the velocity at distance r be v, and let $v=r\zeta$, so that ζ is the angular velocity about the centre. The equation of motion is

$$\frac{1}{\rho}\frac{dp}{dr}=r\zeta\,(2\omega\sin\phi+\zeta).$$

The gradient wind velocity $r\zeta$ is obtained by solving this equation for ζ, taking the appropriate solution as shown in Chapter IX. The geostrophic wind is

$$\frac{1}{2\omega\sin\phi}\,\frac{1}{\rho}\frac{dp}{dr}=\frac{1}{2\omega\sin\phi}\,r\zeta\,(2\omega\sin\phi+\zeta)=r\zeta\,(1+\zeta/2\omega\sin\phi).$$

Thus the ratio of the geostrophic to the true wind in such a system is $1+\dfrac{\zeta}{2\omega\sin\phi}$. If therefore the angular velocity of the air about the centre is comparable with $2\omega\sin\phi$, the difference between the geostrophic wind and the true wind will be comparable in magnitude with these two winds. In some intense depressions ζ approaches $2\omega\sin\phi$. A rough computation indicated that in the depression of December 16, 1917, ζ was about $1{\cdot}6\omega\sin\phi$. In such a case therefore the geostrophic wind is nearly double the true wind. In an anticyclone of November 16, 1922, ζ was about $\frac{1}{3}\omega\sin\phi$, and the geostrophic wind was therefore about five-sixths of the true wind.

The statement in the preceding paragraph can be extended to any non-symmetrical system, by defining r as the radius of curvature of the path of the air at any point, so that ζ is the angular velocity about the centre of curvature. When the curvature is small, the difference between the geostrophic and the true wind is usually relatively small. But it is clear that it is not justifiable in general to assume the geostrophic wind to be a close approximation to the true wind without further consideration. In particular near the central regions of anticyclones, and still more near the centres of depressions, the approximation breaks down. This adds considerably to the difficulties of theoretical meteorology, in which the equations only become reasonably tractable when the winds are assumed to be geostrophic. Very brief consideration shows that the winds of the globe cannot be accurately geostrophic. For geostrophic winds would blow accurately round the isobars, and transfer of air into or out of the isobar would be impossible. The fundamental problems of meteorology are, however, associated with the transfer of air across the isobars, and it is clear that any satisfactory theory must take into consideration the accelerational and frictional terms in the equations of motion.

The frictional terms are about one-third the magnitude of the rotational terms at the ground, since the air at the ground flows across the isobars at an angle of about $\tan^{-1}\frac{1}{3}$. At greater heights the influence of friction rapidly decreases and at heights above about 1 km it appears to be negligible. It can be said that, in general, friction exerts a modifying rather than a controlling influence in atmospheric phenomena. Friction only enters when the motion has been initiated by other agencies.

There are two aspects of the problem of comparing the actual wind with the geostrophic wind. In the first place we may perhaps require an estimate of the upper wind at some given level for some such practical purpose as air navigation, and it is then legitimate to regard the geostrophic wind as an approximation to the actual wind, though the use of the approximation will demand a reasonable exercise of judgment. In the second place we may require to consider the physical changes taking place in a mass of air, and while the geostrophic wind may again be regarded as an approximation to the actual wind it is the deviation from geostrophic motion which is of importance in determining changes of pressure. Here the accelerational terms become important. Thus while a good instantaneous picture of the winds is derived by the geostrophic approximation, the changes which are taking place are to be measured by deviations from the geostrophic wind, and involve the accelerational terms in the equations of motion.

§ 161. *Antitriptic winds*

Jeffreys has discussed (*loc. cit.*) the nature of the land and sea breeze and of mountain and valley winds, and has shown from mathematical arguments that these winds are mainly antitriptic. The sea breeze at Aberdeen has been found to set in suddenly as a breeze from the sea, and to veer in the course of the day until it blows nearly parallel to the coast. This suggests that the rotational terms are of importance in the motion of these winds. They have, however, the essential characteristics of all antitriptic winds, in that they blow roughly at right angles to the isotherms, and extend to only a small altitude.

No satisfactory detailed theory of land and sea breezes, or of katabatic winds, has been worked out hitherto. Some idea of the volume of papers on mountain and valley winds can be gathered from a paper by Wagner*, but it is clear that a number of different phenomena have been grouped under the name of "Mountain and Valley winds". A discussion of some detailed observations of katabatic winds in a valley of the Cotswold Hills has been given by Heywood†, who showed that on occasions these winds are extremely shallow. The land and sea breezes are also very shallow, extending normally to a height of about 200 feet only. These breezes are produced by the difference of temperature over land and sea.

* *Met. Zeit.* **49**, 1932, p. 329.
† *Q.J. Roy. Met. Soc.* **59**, 1933, p. 47.

CHAPTER XV

THE TRANSFORMATIONS OF ENERGY IN THE ATMOSPHERE

§ 162. *Incoming radiant energy*

THE direct source of energy in the atmosphere is the incoming solar beam, whose amount can be measured by the "solar constant". The phenomena which are directly associated with the disposal of the incoming radiant energy have already been discussed in Chapter VI. For our present purpose we may regard this energy as in four categories, which are:

(*a*) Reflected from clouds, and in varying degrees from the atmosphere itself, and portions of the earth's surface: this radiation is ineffective in producing thermodynamical changes in the atmosphere and may be left out of further consideration.

(*b*) Absorbed in the atmosphere.

(*c*) Used up in evaporation of water from the earth's surface.

(*d*) Used in heating the earth's surface.

The energy in class (*c*) above becomes latent heat of evaporation, and is again liberated when the water-vapour condenses, mainly in middle levels of the atmosphere. The energy absorbed by the earth's surface is radiated again outwards, and, possibly after repeated absorption and re-radiation, passes out again through the atmosphere into space. The energy in all four categories eventually passes out into space as radiation, usually with much longer wavelengths than occur in the incoming solar beam, but a portion at least of this energy goes through varied transformations before it is finally passed out to space.

One of the effects of the heating of the atmosphere is to produce an increase of gravitational potential energy, which is convertible into kinetic energy, and the main problem to be considered in the present chapter is the maintenance of the kinetic energy of the atmosphere substantially unchanged from year to year.

§ 163. *The classification of energy*

The complete statement of the types of energy in the atmosphere should include electrical and magnetic energy. We restrict our attention, however, to four types, which appear to be those of meteorological importance:

(1) Kinetic energy of the general circulation, and of the currents in depressions and anticyclones.

(2) Turbulent energy, or the energy associated with eddies in the main currents.

(3) Potential energy (gravitational).

(4) Thermal energy.

It will be noted that energy of radiation is not included in the above classification. Meteorologically this energy is ineffective until it has been absorbed by the atmosphere when it is manifested by a change in items 3 and 4 of the classification.

If no condensation or evaporation is taking place, changes in items 3 and 4 are connected by a simple relationship. Any element of mass in the atmosphere which is heated or cooled behaves as though its specific heat were c_p, the specific heat at constant pressure. For a change ΔT in the temperature of unit mass, the loss of heat is $c_p \Delta T$. Of this a portion $c_v \Delta T$ represents the change of thermal energy, and the remainder $(c_p - c_v)\,\Delta T$ the work done against the pressure of the environment.

In a column of unit horizontal cross-section, extending to the top of the atmosphere assumed of uniform constitution, there is a simple relationship between the total gravitational potential energy and the total internal energy $c_v \int_0^\infty \rho T dz$. The total potential energy

$$= \int_0^\infty g\rho z\, dz = -\int_{p_0}^0 z\, dp = -\Big[pz\Big]_0^\infty + \int_0^\infty p\, dz$$

$$= R \int_0^\infty \rho T dz$$

$$= \frac{AR}{c_v} \times \text{total internal energy in heat units.}$$

The coefficient AR/c_v may be written $(c_p - c_v)/c_v$; or $\gamma - 1$. Hence the potential energy is $\gamma - 1$ times the internal energy, and if any change of temperature takes place within the column, the resulting change of potential energy is $\gamma - 1$ times the resulting change of internal energy. To put this in another way, when a given quantity of heat is added to any column of the atmosphere, a fraction $1/\gamma$ of it is used in increasing the internal energy of the air, and the remaining fraction $(\gamma - 1)/\gamma$ is used in increasing the potential energy. Changes in internal and potential energy are thus closely bound together.

Various writers, including Margules, include in their statements of classes of energy the energy associated with distribution of pressure. To add the effect of pressure distribution to the four types named above would be to include some types of energy twice over, since this effect is accounted for under 3 and 4 above.

§ 164. *Transformations of energy in the atmosphere*

Since the great atmospheric motions are without exception turbulent, there is a continual transformation of their kinetic energy into turbulent kinetic energy. This transformation is irreversible, and the energy of the great currents cannot be reinforced by eddy motions. There is a continual degradation of the kinetic energy of eddies into heat, as the effect of viscosity. This process also acts in a lesser degree upon the kinetic energy of the great currents, and this kinetic energy is therefore being continually degraded to heat energy, either directly, or through an intermediate stage as turbulent energy.

There is in addition an interchange between the turbulent energy and the combined potential and thermal energies, in a direction which depends on the degree of stability of the vertical distribution of temperature. Richardson* has shown that when the lapse-rate exceeds the adiabatic, there is a transformation of thermal and potential energy into turbulent energy, and that when the lapse-rate is less than the adiabatic the transformation is in the reverse direction (*vide* § 152 above).

This statement cannot include all the possible transformations in the atmosphere, since it only provides for a steady degradation of the kinetic energy of the great currents of the atmosphere. There must therefore be some other transformation which provides for the reinforcement of the kinetic energy of these currents, and this transformation can only be from the combined potential and thermal energies.

The possible transformations of energy are therefore the following:

(*a*) $1 \begin{cases} \nearrow 2 \rightarrow (4+3) \\ \searrow (4+3) \end{cases}$

(*b*) $2 \rightarrow (4+3)$ by viscosity

(*c*) $2 \rightarrow (4+3)$ stable atmosphere
$(4+3) \rightarrow 2$ unstable atmosphere

(*d*) $(4+3) \rightarrow 1$.

Items 3 and 4 occur together in this list, since changes of internal energy and of gravitational potential energy are always associated as shown in § 163 above.

The first three classes of transformation have been discussed in an earlier chapter. The fourth class of transformation, from the thermal and gravitational potential energy to kinetic energy, which remains to be considered, is the central problem of meteorology. It includes (1) the maintenance of the general circulation of the atmosphere against turbulence and friction, and (2) the origin of depressions and anticyclones of middle latitudes. It is shown above, and is indeed obvious *a priori*, that the kinetic energy of the large scale currents can only be produced and maintained by the consumption of potential and thermal energy. The real problem is to explain the precise mechanism by which the transformation of energy is brought about, and to picture the method by which the kinetic energy is organised, to produce and maintain the currents observed in the atmosphere.

In the first place we shall endeavour to assess the rate at which the kinetic energy of the main currents is destroyed by turbulence, in order to have a working estimate of the rate at which the kinetic energy is being renewed.

* *Proc. Roy. Soc.* A, **97**, 1920, p. 354.

§ 165. *The kinetic energy of the atmosphere, and its dissipation by turbulence*

If we take the parallels of 30° North and South as separating the main easterly and westerly circulations of the atmosphere, the easterly belt, covering a range of 60° in latitude, will contain half of the mass of the atmosphere. Assuming the average pressure to be one atmosphere, we deduce the approximate mass of air in this belt to be $2{\cdot}7 \times 10^{21}$ grammes or $2{\cdot}7 \times 10^{15}$ metric tons. Taking the mean velocity to be 10 m/sec, a value which is in agreement with the distribution of pressure, we find the kinetic energy of the equatorial belt to be $\frac{1}{2} \times 2{\cdot}7 \times 10^{21} \times 10^{6}$ ergs $= 1{\cdot}35 \times 10^{27}$ ergs.

The combined masses of the polar caps will not differ appreciably from the mass of the equatorial belt, and the amount of momentum of the two circulations should balance, since they are produced by internal reactions within the earth's atmosphere. The equality of moments of momentum demands that the westerly velocity should be the greater, and the energy of the combined polar caps should exceed somewhat that of the equatorial belt.

It appears safe to assume that the total kinetic energy of the earth's atmosphere is of the order of 3×10^{27} ergs.

This estimate does not specifically include the energy of the circulations around cyclones and anticyclones. An estimate has been given by Sir Napier Shaw* of the kinetic energy of a cyclone which formed over the lower part of the North Sea between July 27 and August 3, 1917. Its diameter was about 1400 km and the depth at the centre was 10 millibars. The kinetic energy developed was $1{\cdot}5 \times 10^{24}$ ergs.

If the kinetic energy of the earth's atmosphere, roughly computed above, were spread uniformly over the earth's surface, the amount contained in the portion of the atmosphere over a circle of diameter 1400 km would be 10^{24} ergs, so that in the case considered the kinetic energy of the cyclone was about 50 per cent higher than the average kinetic energy of the general circulation.

It has been shown in Chapter XIII, p. 240, that the effect of turbulence may be represented by an internal frictional force R, given in equation (71), p. 240. The rate at which work is done by this force is equal to the product of R by the component of the velocity along the direction of R. The total rate of loss of kinetic energy of the main current from the ground up to height z is readily obtained by integration. Its amount is†

$$2\omega\rho \sin\alpha \sin\phi \frac{G^2}{B}\{\cos\alpha - e^{-Bz}\cos(\alpha - Bz)\} \qquad \ldots\ldots(1).$$

If the same simple law of distribution of wind with height given in equation (68), p. 239, persisted up to the top of the atmosphere, then for a column

* *Dict. App. Phys.* **3**, p. 84.

† See Brunt, *Phil. Mag.* Feb. 1926.

extending from the ground to the top of the atmosphere the rate of dissipation of energy would amount to

$$2\omega\rho \sin\phi \sin\alpha \cos\alpha \frac{G^2}{B} \qquad \text{......(2).}$$

With the assumed law of distribution of wind with height this dissipation is effectively brought about in the lowest kilometre of the atmosphere.

The rate of dissipation of energy per unit volume is

$$K\rho \left(u \frac{\partial^2 u}{\partial z^2} + v \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{......(3).}$$

A rough estimate by Brunt (*loc. cit.*) shows that the rate of dissipation of energy in the column above 1 m^2 of the earth's surface is as follows:

From the ground to 1 km	3×10^{-3} kilowatts
From 1 km to 10 km	2×10^{-3} ,,
Total	5×10^{-3} ,,

This total is the rate at which the kinetic energy of the earth's atmosphere is being dissipated by turbulence. It is probably an overestimate for those regions of the earth's atmosphere in which the winds are normally light at all heights, but so far as order of magnitude is concerned it may be accepted with a fair degree of faith in spite of the numerous approximations made. On the whole it appears likely to be an overestimate rather than an underestimate.

The total energy above 1 m^2 of the earth's surface, on the assumption of uniform velocity of 10 m/sec, is 5×10^{12} ergs. The rate of dissipation by turbulence is 5×10^7 ergs per m^2 per sec. Hence if the same rate of dissipation were maintained for 10^5 seconds, or $1\frac{1}{6}$ days, the whole kinetic energy would be destroyed in that time. If the rate of dissipation is assumed to be proportional at each instant to the total kinetic energy, then the total kinetic energy is reduced to one-hundredth of its original value in six days. As no sensible change takes place in the kinetic energy of the general circulation of the atmosphere, we conclude that the loss by turbulence is being continually made up by the conversion of solar energy into kinetic energy.

§ 166. *Comparison of the eddy dissipation of kinetic energy with the radiation coming in from the sun*

The value of the solar constant being taken as 2 gramme-calories per cm^2 per minute, then the amount of radiation coming from the sun into the earth's atmosphere

$= 2$ gm-cal per cm^2 per min
$= 8{\cdot}36$ joules per cm^2 per min
$= 83600$ joules per m^2 per min
$= 1400$ joules per m^2 per sec
$= 1{\cdot}4$ kilowatts per m^2.

But the radiation intercepted by the earth is the portion of the beam of solar radiation intercepted by an area πr^2, where r is the radius of the earth. This is spread over the whole area of the earth $4\pi r^2$, and hence the effective incoming radiation per m^2 is one-fourth of the figure given above, and amounts to 0·35 kilowatt per m^2.

Also Aldrich estimates that the effect of reflexion from the earth's surface and from clouds is to reduce by 37 per cent the amount of solar radiation available for absorption and subsequent conversion into kinetic energy. The effective incoming solar radiation is therefore 0·22 kilowatt per m^2 when averaged over the whole of the earth's surface.

The rate of dissipation by turbulence was given above as 5×10^{-3} kilowatts per m^2, which is only a little over 2 per cent of the effective incoming solar radiation.

Thus the conversion of a little over 2 per cent of the incoming solar radiation into kinetic energy will suffice to make up for the dissipation by turbulence.

§ 167. *The development of circulation between sources of heat and cold*

Sandström* and Bjerknes† have given a theorem which appears at first sight to be of great importance in meteorology, that permanent circulation is only possible if the source of heat is lower than the source of cold. Wenger‡ gave a slightly different form of the same theorem, that the source of heat must be at higher pressure than the source of cold.

The general aspects of these theorems have been discussed by H. Jeffreys§ in a paper on fluid motions produced by differences of temperature and humidity. Jeffreys shows that if a fluid is in equilibrium, every level surface within it, or in contact with it, must be isothermal, and if any solid is wholly surrounded by fluid, the total rate of inflow of heat from the solid to the fluid is zero. For physical applications he states the theorem in another form as follows: "if a difference of temperature is maintained over any level surface within or in contact with a fluid, or if heat is supplied to, or withdrawn from any region within the fluid, the fluid will move, and will continue to move until such difference of temperatures or supply or removal of heat ceases". Jeffreys' theorem also holds with regard to the supply of material constituents as well as heat, and permanent equilibrium will only be possible if the theorem stated above holds separately for temperature, and for every material constituent.

Sandström's argument is that the supply of energy required in order to maintain a steady motion of fluid against viscosity must arise from the work done on the fluid. If the fluid is in a steady state its centre of mass has no vertical motion, and no work is done by gravity. The energy must therefore arise from work done against fluid pressure in expansion and contraction, and if there is to be a net gain of work, the expansion must be done under greater

* Goteborg, *Vet. Handl.* **17** (4), 1916.
† Leipzig, *Abh. Ges. Wiss.* **35**, 1916, p. 29; also *Physikalische Hydrodynamik*, para. 52.
‡ *Phys. Zeit.* **17**, 1916. § *Q.J. Roy. Met. Soc.* **51**, 1925, p. 347.

pressure than the contraction. The portions of their paths in which elements of fluid become hotter must lie in regions of higher pressure than the places where they become colder. But as Jeffreys states, the correct interpretation of the last statement is that, as the fluid is contracting on its way from the warm to the cold source and expanding on its way from the cold to the warm source, the part of the path from cold to warm source must lie below the part of the path from warm to cold.

It is in any case dangerous to proceed by analogy with the results derived in the experiments of Sandström and Bjerknes in closed tubes to interpret phenomena in the atmosphere. In the latter heat is redistributed to only a very slight extent by molecular conduction, and turbulence and radiation are of far greater importance, as was shown earlier in Chapter XIII. The thermal phenomena in the atmosphere cannot be related to strictly localised sources of heat and cold. Cooling or heating by radiation may take place from a very large volume of air. Further, a given portion of the earth's surface will be a source of heat for polar currents and a source of cold for equatorial currents. Any interpretation of meteorological phenomena as due to the action of simple sources of heat and cold must therefore proceed with caution, taking account of radiation and turbulence. If the lower atmosphere is heated, or a region in the upper atmosphere is cooled, say by radiation, there will be a tendency to set up instability, leading to turbulence and possible large scale convection. Thus the instability which sets in during the afternoon in summer is mainly due to the diurnal heating of the surface levels while the upper levels maintain their initial temperature.

§ 168. *Equations of energy*

If equations (39), (40) and (41) of p. 166 are multiplied by u, v, w respectively and added, the result may be written

$$\frac{1}{2}\frac{d}{dt}(u^2+v^2+w^2) = -gw - \frac{1}{\rho}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)+(uX+vY+wZ) \quad \ldots\ldots(4),$$

or

$$\frac{d}{dt}\{\tfrac{1}{2}(u^2+v^2+w^2)+gz\} = -\frac{1}{\rho}\left(\frac{dp}{dt}-\frac{\partial p}{\partial t}\right)+(uX+vY+wZ) \quad \ldots\ldots(5).$$

The quantity in the curly bracket on the left-hand side of the last equation is the sum of the kinetic and potential energies of unit mass of fluid. The equation states that the rate of increase of the sum of the kinetic and potential energies of unit mass of fluid is equal to the rate at which work is done upon it by the joint effect of the pressure forces and the frictional forces.

The thermal equation of energy as given in equation (23), p. 37, may be written

$$J\frac{dQ}{dt} = Jc_v\frac{dT}{dt}+p\frac{d}{dt}\frac{1}{\rho} \quad \ldots\ldots(6),$$

or

$$J\frac{dQ}{dt} = Jc_p\frac{dT}{dt}-\frac{1}{\rho}\frac{dp}{dt} \quad \ldots\ldots(7),$$

where J is the mechanical equivalent of heat. The second of these equations may be used to eliminate dp/dt from equation (5) above, giving

$$J\frac{dQ}{dt}=Jc_v\frac{dT}{dt}+\frac{d}{dt}\{\tfrac{1}{2}(u^2+v^2+w^2)+gz\}-\frac{1}{\rho}\frac{\partial p}{\partial t}-(uX+vY+wZ) \quad\text{......(8).}$$

If the pressure distribution is not changing $\partial p/\partial t$ is zero, and this equation reduces to

$$J\frac{dQ}{dt}=\frac{d}{dt}\{Jc_vT+\tfrac{1}{2}(u^2+v^2+w^2)+gz\}-(uX+vY+wZ) \quad\text{......(9).}$$

This equation is useful in some applications, but another form of equation is more useful in the first consideration of questions of energy. Equation (6) may be written

$$J\frac{dQ}{dt}=Jc_v\frac{dT}{dt}-\frac{p}{\rho^2}\frac{d\rho}{dt}=Jc_v\frac{dT}{dt}+\frac{p}{\rho}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \quad\text{......(10),}$$

from equation (30) of p. 163. From equations (4) and (10), by addition, we find

$$J\frac{dQ}{dt}=\frac{d}{dt}\{Jc_vT+\tfrac{1}{2}(u^2+v^2+w^2)+gz\}$$
$$-(uX+vY+wZ)+\frac{1}{\rho}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)+\frac{p}{\rho}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \quad\text{......(11).}$$

In this equation Q is the amount of heat added to unit mass of fluid, and the quantity following the operator d/dt on the right-hand side is the sum of the internal, kinetic and potential energies of unit mass of fluid. To obtain the corresponding relation for any mass of fluid, the equation is multiplied by $\rho\,dx\,dy\,dz$ and integrated through the mass under consideration.

Let

$$\left.\begin{aligned}I&=Jc_v\int\rho T\,dx\,dy\,dz\\K&=\tfrac{1}{2}\int\rho(u^2+v^2+w^2)\,dx\,dy\,dz\\P&=\int g\rho z\,dx\,dy\,dz\end{aligned}\right\} \quad\text{......(12),}$$

so that I, K and P are the total internal, molar-kinetic and potential energies of the mass in question. Then from equation (11) above

$$J\int\rho\frac{dQ}{dt}\,dx\,dy\,dz=\text{total rate of addition of heat}$$
$$=\frac{\partial}{\partial t}(I+K+P)-\iiint\rho\,(uX+vY+wZ)\,dx\,dy\,dz$$
$$+\iiint\left\{\left(u\frac{\partial p}{\partial x}+p\frac{\partial u}{\partial x}\right)+\left(v\frac{\partial p}{\partial y}+p\frac{\partial v}{\partial y}\right)+\left(w\frac{\partial p}{\partial z}+p\frac{\partial w}{\partial z}\right)\right\}dx\,dy\,dz \quad\text{......(13).}$$

The last integral in this equation is readily shown to be equal to

$$\int pV_n\,dS$$

taken over the whole boundary of the mass of fluid considered, where dS is an element of surface of the boundary, and V_n is the component of velocity normal to the boundary. If the fluid is limited by solid boundaries, or by a system of stream lines, V_n is everywhere zero, and the integral vanishes. For the moment we retain the term in the equation, which may now be written

Total rate of addition of heat

$$=\frac{\partial}{\partial t}(I+K+P)-\iiint\rho(uX+vY+wZ)\,dx\,dy\,dz+\int pV_n\,dS \quad \text{......(14).}$$

The equation expresses the fact that the energy added to the fluid in the form of heat is used partly in overcoming frictional forces, partly in increasing the sum of the internal, kinetic and potential energies of the fluid, and partly in expansion of the volume occupied by the fluid against the pressure of its environment. This equation is a complete statement of the principle of conservation of energy in a fluid, and if we had initially assumed the principle of conservation of energy, the equation could have been written down immediately.

Margules proceeded somewhat differently. In equation (10), he considered

$$\iiint p\frac{d}{dt}\left(\frac{1}{\rho}\right)dx\,dy\,dz, \quad \text{or} \quad -\iiint\frac{p}{\rho^2}\frac{d\rho}{dt}\,dx\,dy\,dz,$$

which represents the total work of expansion against pressure, and equated it to $\partial A/\partial t$, so that A is in some respects of the nature of a potential. The thermal equation (10) may then be written

$$\text{Total rate of addition of heat}=\frac{\partial}{\partial t}(I-A) \quad \text{......(15).}$$

Also equation (14) may be written

$$\frac{\partial}{\partial t}(K+P+A)=\iiint\rho(uX+vY+wZ)\,dx\,dy\,dz-\int pV_n\,dS \quad \text{......(16).}$$

It is of interest to consider the magnitude of the effect of the frictional term. The rate of dissipation of energy per unit volume of fluid is given by expression (69), of § 106, p. 180 above, where $\mu=\nu\rho$. Let the change of velocity be of the order of 10 m/sec per km, a rate which is large in the free air. Then all the differential coefficients in the equation are of the order of 10^{-2} sec^{-1}, and their squares of the order 10^{-4} sec^{-2}. The rate of dissipation in 1 cm^3 is therefore (assuming $\mu=1{\cdot}5\times10^{-4}$)

$$1{\cdot}5\times10^{-4}\times10^{-4}\times20=3\times10^{-7}.$$

The amount of kinetic energy present in 1 cm^3, assuming a velocity of the order of 10 metres sec^{-1} or 10^3 cm sec^{-1}, is $5\times10^5\rho$. Thus the amount dissipated per second is only about 10^{-12} of the whole, so that in one day 10^{-7} of the kinetic energy would be dissipated by viscous forces in smooth currents having the gradient of velocity of the order of 10 metres sec^{-1} per kilometre. Thus if the flow of air were smooth or laminar flow, the rate of dissipation of the kinetic energy would be negligible.

Atmospheric motion as we observe it is very seldom laminar, being in almost all cases turbulent, and the forces X, Y, Z must contain the effects of turbulence. The energy degraded through the effect of turbulence finally into thermal energy is in fact converted into thermal energy through the action of viscous forces. The eddies produce large gradients of velocity within restricted regions, and these gradients are so great that the dissipation far exceeds that evaluated on the supposition of laminar flow.

The magnitude of the dissipation by turbulence of the kinetic energy of the large scale motions of the atmosphere has been estimated in § 165 above. It is always in the same direction, tending to destroy the large-scale motions. In discussing the generation of the motions of the atmosphere which are associated with depressions or anticyclones we may therefore in the first place leave the effects of viscosity and turbulence out of consideration, with the understanding that their neglect will lead to overestimates of the motions produced by any conversion of potential into kinetic energy. The same result is achieved if we let Q represent the sum of the heat added from external sources, and the heat produced by the viscous degradation of molar-kinetic energy.

It has been shown in § 163 above that in a clear atmosphere, in which there is no evaporation or condensation, there is a simple relationship between the gravitational potential and the heat content of a column of air extending from the ground to the upper limit of the atmosphere. Let P' and I' be the gravitational potential and the total heat content of a column standing on unit cross-section. Then

$$P' = \frac{c_p - c_v}{c_v} I' = (\gamma - 1) I' \qquad \text{......(17).}$$

If the mass of fluid to be considered is bounded laterally by vertical solid boundaries, then it is readily seen from equation (17) that

$$P = \frac{c_p - c_v}{c_v} I,$$

where P and I now represent the total potential and internal energies. It is also seen that in any changes which take place

$$\frac{\partial P}{\partial t} = (\gamma - 1) \frac{\partial I}{\partial t} \qquad \text{......(18),}$$

and that of any heat energy added to the mass a fraction $1/\gamma$ is used in increasing the internal energy, and a fraction $(\gamma - 1)/\gamma$ in increasing the potential energy. Any loss or gain of potential energy is accompanied by a proportionate loss of internal energy.

In these circumstances we may take

$$I + P = \gamma I = J c_p \iiint \rho T \, dx \, dy \, dz \qquad \text{......(19).}$$

§ 169. *Energy liberated when vertical interchange of masses occurs*

The region over which the vertical interchange takes place is regarded as a closed region, having a boundary across which no transfer of mass takes place. It is assumed that friction is negligible and that no heat is given to or extracted from the fluid affected, so that the total

$$K + P + I \quad \text{or} \quad K + \gamma I$$

is unaltered. The air being initially at rest, the final amount of kinetic energy is

$$\gamma \times \text{change in } I \qquad \text{......(20).}$$

For a layer from the ground to height z,

$$I=c_v\int_0^z Tp\,dz=\frac{c_v}{R}\int_0^z p\,dz.$$

Fig. 63 shows a case considered in some detail by Margules. The initial state (*a*) shows a heavy cold mass of air 1, above a potentially warmer mass 2. The two masses are interchanged, and their final state is shown in (*b*).

The pressure at the ground is p_0, at the boundary of the two masses p_1, and at the upper boundary of the mass 1, p_2. In the final state p_0 and p_2 are unchanged, but the pressure at the middle boundary is now p_1'. For the sake of simplification in the computation we assume each mass to have a uniform adiabatic lapse-rate, so that the temperatures are completely specified by the potential temperatures θ_1 and θ_2 in conjunction with the pressures.

Let T_0, T_1 be the initial temperatures at the surface and at the lower limit of the second mass, and let T_0', T_1' be the final temperatures. The temperature at a height z in the lower mass is $T_0-\Gamma z$ and the pressure is

$$p=p_0\left(\frac{T_0-\Gamma z}{T_0}\right)^{\gamma/(\gamma-1)} \qquad \text{......(21)},$$

where Γ is the dry adiabatic lapse-rate. Hence

$$\int_0^z p\,dz=\left[\frac{p_0T_0}{\Gamma}\frac{\gamma-1}{2\gamma-1}\left(\frac{T_0-\Gamma z}{T_0}\right)^{(2\gamma-1)/(\gamma-1)}\right]_z^0,$$

$$\frac{1}{R}\int_0^z p\,dz=\frac{1}{g}\frac{\gamma}{2\gamma-1}(p_0T_0-p_zT_z) \qquad \text{......(22)}$$

since $$R\Gamma=g\frac{AR}{c_p}=g\frac{c_p-c_v}{c_p}=g\frac{\gamma-1}{\gamma}.$$

We now require to evaluate the expression (22) above for the initial and final conditions represented in fig. 63 (*a*) and (*b*). The initial value of $\frac{1}{R}\int_0^z p\,dz$ through the two masses is

$$\frac{1}{g}\frac{\gamma}{2\gamma-1}\{p_0T_0-p_1(T_0-\Gamma h_1)+p_1T_1-p_2(T_2-\Gamma h_2)\} \qquad \text{......(23)}.$$

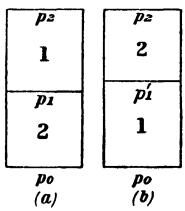


Fig. 63. Margules' diagram showing warm and cold layers inverted.

In the final state p_0 and p_2 are unchanged. The pressure at the surface of separation of the two masses is now $p_1'=p_0-p_1+p_2$. The surface temperature T_0' is given by

$$T_0'p_0^{(\gamma-1)/\gamma}=T_1p_1^{(\gamma-1)/\gamma}$$

and the temperature T_1' at the lower boundary of the upper mass is given by

$$T_1'p_1'^{(\gamma-1)/\gamma}=T_0p_0^{(\gamma-1)/\gamma}.$$

Margules carried out a detailed computation for the case where initially the depth of each layer was 2000 metres, and the drop in temperature at the boundary was 3° C. Evaluating the change in I from equation (23) above, and equating the kinetic energy developed to the loss of potential and internal energy, he found that the velocity which would be acquired by the moving masses would be nearly 15 m/sec.

In a second example Margules took as the initial state two masses lying side by side, each of depth 3000 metres, with a difference of 5° C in temperature, and having equal horizontal extent. In the final state the warmer mass has ascended and lies above the colder, which has now spread laterally so as to cover the whole of the surface area previously covered by the combined masses. The details of the computation are similar to those of the first example, and in the final result it is shown that the amount of potential and internal energy given up is sufficient to produce a mean velocity of 12·2 m/sec, while if the temperature difference is 10° C, the mean velocity is 17·3 m/sec.

Margules' computations are carried through with great accuracy, since the final net gain of kinetic energy is the relatively small difference of two large quantities. We do not here propose to dwell upon the details of the computations. The important fact is that the re-adjustment of such unstable situations as Margules pre-supposes liberates sufficient energy to account for very considerable velocities.

Margules carried out his computations on the basis that all the potential energy given up during the readjustment was converted into kinetic energy. In practice this could never be strictly true. Some of the energy is converted into energy of eddies, and the values for kinetic energy deduced by Margules are upper limits to the kinetic energy developed.

It has been raised as an objection to the work of Margules that large differences of temperature such as 10° C are not observed as discontinuous differences, and that differences of temperature in the atmosphere are in practice found to be continuous. But it is not unusual to find differences of the order of 10° C over a very small horizontal range of distance, and this objection need not be treated too seriously. The fundamental difficulty involved in the adoption of Margules' work as an explanation of the genesis of motions in the atmosphere lies rather in explaining how the energy liberated in the way he discusses is able to organise itself into the horizontal circulations which we observe in the atmosphere.

§ 170. *Single layer in unstable equilibrium*

The analogous problem of the energy liberated when a layer in which the lapse-rate exceeds the adiabatic can be computed by similar methods, though the solution is not capable of simple analytical expression.

Let fig. 64 (*a*) represent the distribution of temperature with height, the lapse-rate β exceeding the dry adiabatic. When the unstable layer is inverted, each portion of the layer retains its original potential temperature, and the potential temperature will now increase with height. Let the initial surface temperature be T_0.

The thin layer initially at P, at pressure p, with difference of pressure dp, will now be at P' (fig. 64 (*b*)) with pressure

$$p_0 + p_1 - p = p', \text{ say} \qquad \text{......(24)}.$$

Its initial temperature was $T = T_0 - \beta z$. This is related to the pressure by equation (19) of p. 35

$$\frac{T}{T_0} = \frac{T_0 - \beta z}{T_0} = \left(\frac{p}{p_0}\right)^{R\beta/g} = \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma \,.\, (\beta/\Gamma)} \qquad \text{......(25)}.$$

In the new position at P' let its temperature be T'. Then

$$\frac{T'}{T} = \left(\frac{p'}{p}\right)^{(\gamma-1)/\gamma} = \left(\frac{p_0 + p_1 - p}{p}\right)^{(\gamma-1)/\gamma} \qquad \text{......(26)}.$$

The total internal and potential energy of the layer in its initial state

$$= \frac{Jc_p}{R}\int_0^h p\,dz = \frac{Jc_p p_0}{R}\int_0^h \left(\frac{T_0 - \beta z}{T_0}\right)^{\gamma/(\gamma-1)\,.\,(\Gamma/\beta)} dz = \frac{Jc_p}{g + R\beta}(p_0 T_0 - p_1 T_h) \qquad \text{......(27)}.$$

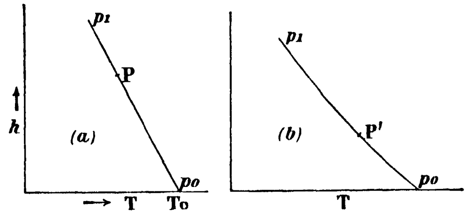


Fig. 64. An unstable layer inverted.

The total internal and potential energy of the layer in its final state

$$= Jc_p \int \rho' T' dz' = -\frac{Jc_p}{g}\int T' dp' = -\frac{Jc_p}{g}\int T' dp \qquad \text{......(28)}.$$

From (25) and (26)

$$\frac{T'}{T_0} = \left(\frac{p_0 + p_1 - p}{p}\right)^{(\gamma-1)/\gamma}\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma\,.\,(\beta/\Gamma)}$$

$$= (p_0 + p_1 - p)^{(\gamma-1)/\gamma}\, p^{(\gamma-1)/\gamma\,.\,(\beta-\Gamma)/\Gamma}\, p_0^{-(\gamma-1)/\gamma\,.\,(\beta/\Gamma)}.$$

Hence the total internal and potential energy of the layer

$$= \frac{Jc_p}{g} T_0 p_0^{-(\gamma-1)/\gamma\,.\,(\beta/\Gamma)} \int_{p_1}^{p_0} (p_0 + p_1 - p)^{(\gamma-1)/\gamma}\, p^{(\gamma-1)/\gamma\,.\,(\beta-\Gamma)/\Gamma}\, dp \text{......(29)}.$$

This integral is not capable of simple evaluation in a finite form. The exponent $(\gamma - 1)/\gamma$ is equal to 0·29, and if $(\beta - \Gamma)/\Gamma$ is much less than unity, as in practice it must be when the layer we are considering is deep, then $(\gamma - 1)/\gamma\,.\,(\beta - \Gamma)/\Gamma$ is very small. If for example the lapse-rate is 10 per cent greater than the dry adiabatic $(\gamma - 1)/\gamma\,.\,(\beta - \Gamma)/\Gamma = 0{\cdot}029$, and the factor $p^{(\gamma-1)/\gamma\,.\,(\beta-\Gamma)/\Gamma}$ will only vary very slowly with p, and we may without serious error put it equal to

$$\left(\frac{p_0 + p_1}{2}\right)^{(\gamma-1)/\gamma\,.\,(\beta-\Gamma)/\Gamma},$$

and take it outside the integral sign. The total internal and potential energy of the layer may then be written

$$\frac{\mathfrak{J}c_p}{g} T_0 \left(\frac{p_0+p_1}{2p_0}\right)^{(\gamma-1)/\gamma \cdot (\beta-\Gamma)/\Gamma} p_0^{-(\gamma-1)\gamma \cdot (\beta/\Gamma)} \int_{p_1}^{p_0} (p_0+p_1-p)^{(\gamma-1)/\gamma} dp$$

$$= \frac{\mathfrak{J}c_p}{g} T_0 \frac{\gamma}{2\gamma-1} \left(\frac{p_0+p_1}{2p_0}\right)^{(\gamma-1)/\gamma \cdot (\beta-\Gamma)/\Gamma} p_0^{-(\gamma-1)\gamma \cdot (\beta/\Gamma)} \left(p_0^{(2\gamma-1)/\gamma} - p_1^{(2\gamma-1)/\gamma}\right) \quad \text{......(30).}$$

The total kinetic energy generated is the difference between expressions (27) and (30). The result cannot be stated in a simple form, even with the approximations made above. The mean velocity generated in the particular case where $T_0 = 300°$, $p_0 = 1000$ mb, $p_1 = 700$ mb has been evaluated by Littwin, and the results are given by Koschmieder in his *Dynamische Meteorologie*, p. 337, from which we extract the results that for $\beta = 1{\cdot}05\Gamma$, $\beta = 1{\cdot}1\Gamma$, $\beta = 1{\cdot}25\Gamma$ the mean velocity generated by the re-adjustment of the unstable layer is 8·7 m/sec, 10·8 m/sec, and 16·6 m/sec, respectively. Thus the amount of kinetic energy generated is sufficient to produce velocities of the order of magnitude of those which are observed in the atmosphere. But here again we are faced with the same difficulty of seeing how these velocities can organise themselves into the horizontal circulations which we observe in the atmosphere.

§ 171. *Effect of the presence of water-vapour*

The examples quoted from Margules refer to dry air. Margules also discussed in detail the effect of condensation upon the liberation of energy, and concluded that the presence of water-vapour had little effect upon the amount of energy liberated. This is undoubtedly true when we consider the liberation of energy by the readjustment of masses of air such as are represented in figs. 63 and 64, when the depth of the disturbed layers is prescribed beforehand. The effect of water-vapour is to modify greatly the depths of the layers which are effective, and in this manner it exerts a predominating influence on the phenomena produced. Moreover its presence leads to instability setting in with lapse-rates which would be stable in dry air.

§ 172. *Maintenance of a difference of pressure by addition of heat*

(*a*) Dry Air

Margules has considered the maintenance of a circulation in which air moves from high pressure to low and tends to annihilate the difference of pressure and consequently the wind circulation associated with it. In fig. 65, the circulation is around $ABCD$ in the sense $A \to B \to C \to D \to A$. The pressure P_1 at A is higher than the pressure P_2 at B, and the pressure p_2 at C is higher than the pressure p_1 at D.

At the surface the air has initially the temperature T_1 at A. In moving to B its temperature falls adiabatically to T_1', but at B heat is added and its temperature raised to T_2. The air ascends from B to C adiabatically so that the temperature T_3 at C is approximately $T_2 - \Gamma h$, h being the height BC, and Γ the dry adiabatic lapse-rate. The air moves adiabatically from C to D, arriving at D with a temperature T_3'. At D heat is abstracted, so that the temperature falls to T_4, and the air descends along DA, arriving at A with the original temperature T_1. Hence $T_4 = T_1 - \Gamma h$.

$$\text{Heat added at } B \quad = c_p\,(T_2 - T_1'),$$
$$\text{Heat subtracted at } D = c_p\,(T_3' - T_4).$$

Since $T_4 = T_1 - \Gamma h$, $T_3 = T_2 - \Gamma h$, the quantity of heat converted into work

$$= c_p\,\{(T_2 - T_1') - (T_3' - T_4)\}$$
$$= c_p\,(T_1 - T_1' + T_3 - T_3')$$
$$= c_p T_1 \left\{1 - \left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma}\right\} + c_p T_3 \left\{1 - \left(\frac{p_1}{p_2}\right)^{(\gamma-1)/\gamma}\right\} \quad \text{......(31)}.$$

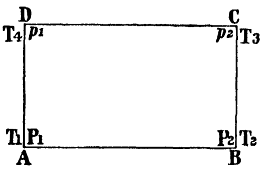


Fig. 65. Margules' diagram for a cycle in a vertical plane.

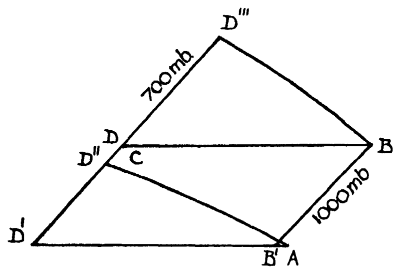


Fig. 66. The same cycle represented in a tephigram.

The evaluation of the amount of energy converted into work in any definite cycle in the atmosphere is most readily carried out by means of the tephigram.

Let $\quad P_1 = 1020$ mb, $\quad P_2 = 1000$ mb,

$\quad p_1 = 700$ mb, $\quad p_2 = 720$ mb,

$\quad T_1 = 273°$ A, $\quad T_2 = 288°$ A.

Any unit mass which goes around the circuit $ABCD$ goes through the cycle of changes $AB'BDD'A$ in fig. 66, where B' and D' represent the condition of the mass before the addition or subtraction of heat at B and D respectively. The amount of heat converted into work is represented by the area $B'BDD'$, which computation in an actual tephigram showed to amount to $1{\cdot}6 \times 10^7$ ergs, or 0·375 calorie, per gramme of air. This energy, if completely converted into kinetic energy, would suffice to give the air a velocity of 60 m/s.

(*b*) Saturated Air

In saturated air the phenomena are complicated by the latent heat associated with condensation and evaporation. If unit mass of air is to perform a true cycle and to reach A with the same water content as it had initially, the changes of state must be represented by some such cycle as $AB'BD'''D''A$, whose area is clearly only very little different from that of $AB'BDD'A$. It is supposed that water is added to the air descending along $D''A$ to maintain it saturated at all points.

§ 173. *A cycle of changes in the atmosphere*

A possible cycle in the atmosphere has been discussed by Sir Napier Shaw*, in which air ascends at the equator and travels at high levels into latitude 60°, where it descends to the sea surface, after which it returns along the surface towards the equator. The cycle can be represented readily in a tephigram. In fig. 67 E represents the state of the air at the equator before it starts its ascent. It ascends at first along the dry adiabatic EA, attaining saturation at A, at a pressure of 900 mb. Its further ascent is along the saturated adiabatic AB,

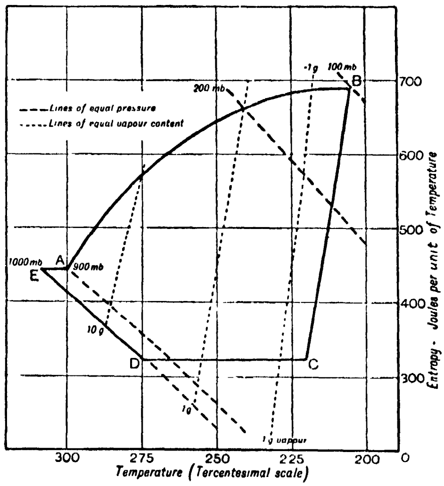


Fig. 67. An atmospheric cycle shown in the tephigram.

B having a pressure of 100 mb. Its journey northward is accompanied by a steady loss of heat by radiation, which enables it to descend to lower levels as it travels poleward, retaining its vapour content practically unchanged.

This part of its path is represented by BC. At C it is in a condition to descend adiabatically to the surface, which it reaches in a condition specified by the point D, at a temperature of 275° A. Its further history, during the return journey to the equator, is represented by DE in the diagram. In this stage of its path it acquires heat and moisture from the surface of the earth.

The amount of heat converted into work is represented by the closed area $EABCDE$, which in the example shown amounts to $2{\cdot}4 \times 10^8$ ergs per gramme. If this amount of energy were used in setting the gramme of air in motion, its velocity would be $\sqrt{4{\cdot}8 \times 10^8}$ cm/sec, or about 220 m/sec. Naturally we do not seriously suggest that all the energy in the cycle is devoted to setting in motion the limited mass of air which performs the cycle. It is actually devoted to setting in motion much larger masses of air which surround the moving mass.

* *Dict. App. Phys.* **3**, p. 82.

CHAPTER XVI

THE FORMATION OF DEPRESSIONS AND ANTI-CYCLONES: EFFECTS OF CONVERGENCE AND DIVERGENCE IN THE ATMOSPHERE

§ 174. *The formation of revolving fluid in the atmosphere*

In Chapter VIII it was shown that the most important factor in the formation of large-scale circulations in the atmosphere is represented by the third term on the right-hand side of equation (48), the term which measures the effect of convergence or divergence of fluid in the atmosphere. In the present chapter we shall consider this factor in further detail, treating the motion as two-dimensional and symmetrical. The results derived are those which are sometimes referred to as "revolving fluid".

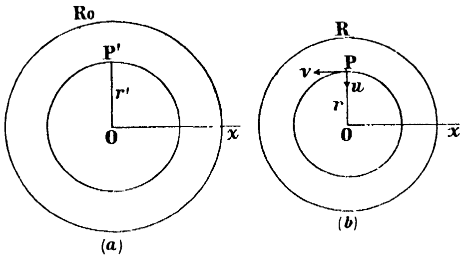


Fig. 68. The development of revolving fluid in the atmosphere.

In the first place we shall consider the type of motion which is produced in a revolving horizontal disc of incompressible fluid, when fluid is drawn off from the centre of the disc. This problem was first discussed by Rayleigh*, and the analysis which he gave was extended by Brunt† to take account of the rotation of the earth.

The motion is assumed to be symmetrical about the centre of the disc, which has unit vertical thickness. Except at the centre the motion will be assumed to be everywhere horizontal. The disc has initially an external radius R_0 and rotates with angular velocity ζ in a counter-clockwise direction, relative to an axis Ox fixed in the earth (fig. 68 (*a*)). Fluid is drawn off at O, and the primary result is to produce a convergence of the fluid in the disc inward towards O. At a later stage let the external radius of the disc be R, and let an element of air which was initially at P', distant r' from O, be now situated at P, distant r from O. Let the velocity at P be v transverse to OP relative to the earth, and u along PO, measured positively inwards. The ring of particles

* *Proc. Roy. Soc.* A, **93**, 1916, p. 148. † *Ibid.* **99**, 1921, p. 397.

of radius r' in fig. 68 (*a*) has become the ring of radius r in fig. 68 (*b*), and since the annulus originally between r' and R_0 is now between r and R,

$$\pi (R_0^2 - r'^2) = \pi (R^2 - r^2)$$

or

$$R_0^2 - r'^2 = R^2 - r^2 \qquad \text{......(1).}$$

The rate of total inward flow across the cylinder of radius r is independent of r since it is equal to the rate of removal of fluid at O. Hence

$$ru = \text{const.} = B \qquad \text{......(2).}$$

The horizontal plane relative to which the motion is measured itself rotates with the earth, with an angular velocity $\omega \sin \phi$ about the vertical. Hence the transverse motion in space of the fluid at P relative to O is $v + r\omega \sin \phi$. The only force in the horizontal plane acting on the air at P is the pressure gradient, which from considerations of symmetry must act along PO. Hence the angular momentum about O remains constant during the motion and the angular momentum about O of the air at P' in fig. 68 (*a*) and at P in fig. 68 (*b*) will be the same; i.e.

$$r'^2 (\zeta + \omega \sin \phi) = rv + r^2 \omega \sin \phi$$

or

$$rv = (r'^2 - r^2)\, \omega \sin \phi + r'^2 \zeta \qquad \text{......(3).}$$

Substituting for r'^2 from equation (1), we find

$$rv = (R_0^2 - R^2)\, \omega \sin \phi + (R_0^2 - R^2 + r^2)\, \zeta,$$

or

$$v = r\zeta + \frac{(R_0^2 - R^2)(\omega \sin \phi + \zeta)}{r} \qquad \text{......(4).}$$

Equation (4) shows that the effect of the removal of fluid from the centre of a disc rotating with uniform angular velocity ζ is to superpose upon the uniform angular velocity a distribution of velocity

$$v = A/r,$$

where A is a constant. This added distribution is equivalent to a simple vortex. Its intensity is proportional to $\pi (R_0^2 - R^2)$, or to the amount of fluid which has been removed from a disc of unit thickness.

When the term $\omega \sin \phi$ is omitted from equation (4), the result is identical with that derived by Rayleigh. Equation (4) is also applicable to the case where $\zeta = 0$, corresponding to an initial state of rest. The distribution of velocity at any subsequent time is then given by

$$v = \frac{(R_0^2 - R^2)\, \omega \sin \phi}{r} \qquad \text{......(5).}$$

There need be no definite outer boundary in this case, and $R_0^2 - R^2$ is to be interpreted as $1/\pi$ times the total amount of fluid removed since the beginning of the motion.

The analysis shows that if fluid be removed from the centre of a disc of incompressible fluid originally having uniform angular velocity, or in other words, rotating as a solid, the consequent convergence towards the centre has the effect of superposing a simple vortex

$$vr = \text{const.}$$

upon the solid rotation

$$v = \zeta r.$$

If the fluid outside the disc be initially at rest, it will converge inwards to fill up the space left otherwise vacant by the shrinkage of the disc, and will take up the velocity distribution given by equation (5) in the region outside the radius R of the disc.

The converse case of addition of fluid at the centre of a disc originally revolving as a solid with angular velocity ζ is equally readily derived by similar considerations. The added fluid spreads out horizontally, causing the disc to increase in size, so that the external radius R at the later stage is greater than the original radius R_0. Suppose at any instant t the added fluid forms a disc of radius R', while the outer radius of the original fluid is R. The fluid which originally constituted the disc of radius R_0 is now contained between $r=R'$ and $r=R$. Hence

$$\pi(R^2-R'^2)=\pi R_0^2$$

or

$$R^2-R_0^2=R'^2 \qquad \text{......(6).}$$

Equation (4) still holds for the original fluid, i.e. for $R'<r<R$. It may be written in the form

$$v=\zeta r-\frac{(\zeta+\omega\sin\phi)R'^2}{r} \qquad \text{......(7).}$$

Thus the effect of the addition of fluid at the centre is to superpose upon the original solid rotation a simple vortex in a clockwise direction. This is true whether ζ is positive or negative, whether the original motion is counter-clockwise or clockwise.

The motion of the added fluid requires separate consideration. If the fluid enters at O with no horizontal velocity, it must spread out horizontally so as to have no rotation in space about the vertical axis through O. It therefore rotates clockwise relative to the earth with an angular velocity $\omega\sin\phi$. The complete motion thus requires three specifications:

$$\left.\begin{array}{lll}\text{For} & r<R' & v=-r\omega\sin\phi\\ & R'<r<R & v=\zeta r-\dfrac{(\zeta+\omega\sin\phi)R'^2}{r}\\ \text{and for} & r>R & v=-\dfrac{(\omega\sin\phi)R'^2}{r}\end{array}\right\} \qquad \text{......(8),}$$

it being assumed that the fluid outside the disc was originally at rest. Here v is positive for counter-clockwise rotation about the centre.

If the initial motion is zero, so that fluid is added at a point to a medium at rest, the equations reduce to two:

$$\left.\begin{array}{lll}\text{For} & r\leqslant R' & v=-r\omega\sin\phi\\ & r\geqslant R' & v=-\dfrac{\omega\sin\phi\,.\,R'^2}{r}\end{array}\right\} \qquad \text{......(9).}$$

Thus the effect of the addition of fluid is to give the fluid originally present the motion of a simple vortex, revolving clockwise, the added fluid forming a central disc having clockwise rotation as a solid.

It is not here suggested that the results derived above afford complete explanations of the depression or anticyclone, but the results will have to be borne in mind in discussing these pressure distributions, in whose formation convergence and divergence have to be considered.

The results given above could also have been derived from equation (64) of Chapter VIII. Let ψ be the stream-line function, then if ζ is the vorticity, since the motion is symmetrical,

$$\frac{\partial^2\psi}{\partial\theta^2}=0 \quad \text{and} \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)=2\zeta,$$

which on integration yields

$$r\frac{\partial\psi}{\partial r}=2\int\zeta r\,dr=\zeta r^2+A, \text{ if } \zeta \text{ is constant,}$$

$$\frac{\partial\psi}{\partial r}=\zeta r+\frac{A}{r}.$$

The effect of the convergence or divergence, therefore, is to impose on the original rotation as a solid a vr-vortex of intensity A. By the adjustment of the appropriate value of the constant A in the last equation, we can as before derive all the results previously obtained. This is left to the reader as an exercise in analysis. Attention is drawn to the method here, in order to emphasise the utility of the equations of § 105.

The same results can also be derived by the use of equation (48) of Chapter VIII, by giving to the area F the appropriate value $\pi r^2 \sin\phi$. The derivation of the above results by this method can also be left as an exercise for the reader. The derivation given in the present chapter perhaps gives a better idea of the physical factors involved in the processes of convergence and divergence, showing that convergence is associated with the growth of cyclonic circulations, and divergence with the growth of anticyclonic circulations.

§ 175. *The tropical cyclone and the tornado as revolving fluid*

Let us consider how we may apply the considerations of revolving fluid developed above to explain the tropical cyclone or the tornado.

The most obvious method of visualising the removal of air is by means of a convection current. If convection on a large scale acts for a sufficiently long period over a restricted area, the effect on the air in the surrounding region is to superpose a simple vortex ($vr=$const.) upon the previously existing motion. In the special case where the air is originally at rest, the final motion consists only of the simple vortex. We need not regard the external boundary R_0 as having any objective existence, and we may interpret $(R_0{}^2-R^2)$ in equation (4) above as $1/\pi$ times the amount of air removed from a layer of unit thickness.

We may think of the convection current as produced by local inequalities of temperature or of water-vapour content, or by the effect of a surface of discontinuity in causing the warmer current to flow up over the colder. If a large mass of air is set in upward motion, the turbulence at the boundary of the rising air will cause a partial mixing of the rising air with the environment, some of the excess of temperature or moisture of the rising air being shared with the environment. The mixture formed will therefore also be lighter than the normal environment, and will also tend to rise. We may therefore regard the ascending current as having a scouring effect upon the environment; the process is known as "eviction of air".

The effect of the condensation of water-vapour is probably an important factor in maintaining convection on a large scale. In any layer in which the lapse-rate is intermediate between the dry adiabatic and the saturated adiabatic any air which attains saturation in the course of its upward motion will thereafter become increasingly warmer than its environment, except in so far as mixing of the boundary restricts the difference. If therefore we can visualise warm damp air at the surface being set in upward motion by its buoyancy, there is no difficulty in understanding the continuance of the upward motion as an effect of the condensation of water-vapour.

The main difficulty which confronts this theory at this stage is that of accounting for a diminution of pressure at the centre. At first sight it appears that the convergence of air towards the centre, combined with ascent in the vertical of air at the centre itself, should cause an increase of pressure at the centre, yielding a pressure gradient opposing the converging motion. The motion cannot continue for any length of time unless there is present some mechanism for removing the air which has ascended. The simplest mechanism which we can postulate is an upper current whose direction differs from that of the current in the lower troposphere. It is possible that the outward motion of cirrus from the centre of cyclonic systems is to be taken as evidence of the existence of such a current. In the absence of some means of removal of the evicted air the development of the system we have described is impossible and a thunderstorm is a more likely occurrence than a cyclonic system.

Observation has shown that convection is not by itself likely to be sufficient to produce a cyclonic system. Tropical cyclones originate in regions where the air is damp and the surface temperature is high, but if these conditions alone were sufficient to produce cyclonic systems, it is difficult to see why they are not far more plentiful than they actually are. Unfortunately little is known of the general wind structure associated with the formation of tropical cyclones, and it is left to future observation to decide the question.

Two main difficulties still confront the theory, and call for further explanation: (*a*) Why is the cyclone not torn to pieces by the variation of wind with height? (*b*) What protects the cyclone from filling in at the top? We shall consider these points briefly in turn:

(*a*) The developing cyclone does not, in its initial stages, extend to great heights, and so is not affected by the marked shear of wind which is frequently observed between 4 and 9 km.

(*b*) The top of the cyclone is protected by the gradual diminution of the horizontal pressure gradients and of the wind velocity, so that no tendency to fill in at the top should occur.

The central portion of the revolving storm should initially be a region of cloud and heavy rain. At a later stage, when the convection has ceased, the inner portion is protected from inflow of the surrounding air by the ring of high velocity outside the core. When this stage has been reached, the cyclone has placed itself on a dynamical rather than a thermal footing, and it continues in existence until its energy is dissipated by friction and turbulence. In this later stage, the central portion of the storm is not of necessity rainy or even cloudy.

If a cyclone could be brought into existence in the manner we have outlined above, it should, in its later stages, possess great stability, and should be capable of acting as a "centre of attraction" for currents of air originally outside its sphere of action. Such currents would be accelerated when approaching the cyclone, and slowed down when receding from it, eventually passing away with approximately the same speed with which they approached the system. Shaw and Lempfert, in their examination of surface air trajectories*, found many examples of such motion.

Before leaving this part of the subject, we would emphasise one point: that a diminution of pressure and a counter-clockwise wind circulation can only be brought about by vertical motion and convergence of the air to take the place of that removed by convection. It is not possible to form such a system by horizontal divergence from a point. For divergence from a centre must produce a clockwise circulation, as the diverging air is deviated round by the effect of the earth's rotation, and a balance between the wind and pressure distribution would then be impossible.

There is a feature of the revolving fluid theory to which no reference has yet been made, and which is usually left out of consideration. The velocity distribution which is produced by the effect of convergence is

$$vr = \text{const.}$$

This requires an infinite velocity at $r = 0$. In practice this is impossible. The air is removed not at a point, but over a finite area, and the revolving field is set up outside this area of convection.

In the formation of the tornado, the instability of the air near the ground is an essential feature of the initial conditions, and it is probable that it is the surface layer which is drawn in to form the core of the vortex. The surface air has its motion checked by friction with the ground, and it thus reaches the inner region with relatively small transverse velocity. Only by this supposition does it appear possible to account for the non-existence of the infinite velocities at the centre demanded by the formal theory.

It is to be noted that if a depression could be formed in still air, then apart from the drift of air across the isobars in the lowest layers, due to friction with the ground, the depression would always contain the same air, since the motion would be circular about the centre of lowest pressure.

* "Life-history of Surface Air Currents." M.O. 174.

If the equations of motion be written in the form

$$\frac{du}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+lv \qquad \text{......(10)},$$

$$\frac{dv}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-lu \qquad \text{......(11)},$$

$$\frac{dw}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \qquad \text{......(12)},$$

where d/dt represents total differentiation "following the fluid" and where $l=2\omega \sin \phi$, ω being the angular velocity of rotation of the earth; then if the system of equations are satisfied by a certain set of values of u, v, w and p, it will also be satisfied by $u+U$, v, w and $p-\rho lUy$, where U is a constant. In other words, the same system of relative velocities can exist, with a uniform velocity of translation U, provided a pressure gradient of the appropriate amount to give U as a geostrophic velocity is superposed upon the original pressure distribution. By means of this result it is possible to pass from the effect of convection at a fixed point in fluid at rest to that of convection in fluid in uniform motion always taking place at a point which has the same uniform motion. The result is to superpose on the uniform motion a vr-vortex, whose centre is carried along with the stream.

§ 176. *The "normal" cyclone or cartwheel depression*

Sir Napier Shaw* assimilated the depression to a horizontal cartwheel spinning about a vertical axis, having a uniform velocity of translation in its own plane. Though the conception is admittedly a considerable simplification of the phenomena we observe in the average depression, it is worth while considering it in some detail, as it helps to fix certain ideas which will be of value in the later development of the subject.

In fig. 69 let O be the centre of the revolving disc of fluid, and let ζ be the angular velocity about a vertical axis through O. Let U be this velocity of translation along Ox. Let O' be the point on the line through O perpendicular to Ox, and such that

$$OO'\zeta=U.$$

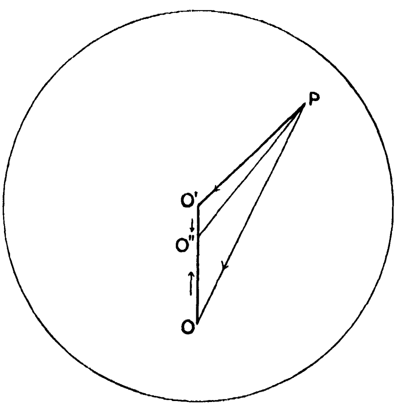


Fig. 69. Shaw's "normal" cyclone.

Then O' is the instantaneous centre of rotation of the disc, and it is a matter of simple geometry to prove that the instantaneous velocity of any other point P' of the disc is $\zeta O'P$, perpendicular to $O'P$, corresponding to instantaneous

* *Manual of Meteorology*, 4, Chapter IX.

rotation ζ about O'. Shaw calls O' the "centre of winds", or the "kinematic" centre, and O, the centre of the revolving fluid, he calls the "tornado" centre.

For the spinning disc, neglecting translation, the distribution of pressure is immediately written down by the use of equations (3) and (4) of Chapter IX:

$$\frac{p}{\rho} = \text{const.} + \tfrac{1}{2}\zeta\,(2\omega \sin\phi + \zeta)\, r^2 \qquad \text{......(13).}$$

It has been shown above that the translation with velocity U adds a term $-2\omega \sin\phi \,.\, Uy$ to the right-hand side of this equation. The pressure distribution for the moving cyclone is therefore

$$\frac{p}{\rho} = \text{const.} + \tfrac{1}{2}\zeta\,(2\omega \sin\phi + \zeta)\left\{x^2 + \left(y - \frac{2\omega \sin\phi \,.\, U}{\zeta\,(2\omega \sin\phi + \zeta)}\right)^2\right\} \qquad \text{......(14).}$$

The isobars are therefore concentric circles whose centre is at O'', where

$$OO'' = \frac{2\omega \sin\phi \,.\, U}{\zeta\,(2\omega \sin\phi + \zeta)} = \frac{2\omega \sin\phi}{2\omega \sin\phi + \zeta}\, OO' \qquad \text{......(15).}$$

Thus O'' is between O and O', in fig. 69.

If for brevity in writing we suppose the cyclone to be moving towards East, then the centre of winds is to the north of the centre of isobars, while the true centre of revolving fluid is to the south of the centre of isobars. The latter, the tornado centre, is the point at which convection should most readily occur, though in the synoptic chart there should be nothing to distinguish this point from any other. Perhaps the most curious feature of Shaw's normal cyclone is that the centre of isobars possesses no special dynamical significance.

There remains the consideration of the relation of the cyclone to its environment. The constancy of the angular velocity involves an increase of linear velocity with distance, and this increase cannot proceed indefinitely. Either there must be a discontinuity of velocity at a boundary, or the spinning disc must be surrounded by a region of decreasing angular velocity in which the cyclone merges slowly into its environment. Shaw states (*loc. cit.*) "the distribution of velocity in the ordinary cyclones of our maps suggests the simple vortex with velocity inversely proportional to distance, for the vortex margin of a cyclone". But there is a striking distinction between the inner and outer regions, in that the inner region has one definite centre of winds, and one definite centre of isobars, while each ring of the outer region has its own centres of winds and of isobars.

For since
$$OO'' = \frac{2\omega \sin\phi \,.\, U}{\zeta\,(2\omega \sin\phi + \zeta)},$$

it follows that OO'' increases as ζ diminishes. As a result of this, the isobars are more crowded to the right-hand side of the path than to the left-hand side.

§ 177. *Gradient wind in the normal cyclone*

It has been seen that the effect of translation of a horizontal disc is to cause the centre of the isobars to move from the centre of the revolving fluid to a point to the left of the path, but without any change in the spacing of the isobars. The gradient winds in the moving cyclone will correspond to a spin with angular velocity ζ about O'', the centre of the isobars. The gradient wind at P is therefore $\zeta O''P$ at right angles to $O''P$, while the true wind is $\zeta O'P$ at right angles to $O'P$. The true wind is equal to the gradient wind, *plus* a wind $\zeta O'O''$ forward. Hence in a normal cyclone the true wind is made up of the gradient wind blowing around the isobars, *plus* a component $\zeta O'O''$ blowing along the direction of translation. The true wind thus blows out of the isobars in the front half of the cyclone and into the isobars in the rear half of the cyclone.

Shaw (*loc. cit.* p. 245) has given a slightly different interpretation of the geometry of the preceding paragraph. He states "The wind calculated from the gradient by the full formula, using the curvature of the isobars, gives the true wind in the free air not at the point at which the gradient is taken, but at a point distant from it along a line at right angles to the path and on the left of it by the amount $\dfrac{U}{2\omega \sin \phi + \zeta}$". The reader can readily satisfy himself of the equivalence of the two views. We consider it more useful at this stage to interpret the geometry as in the preceding paragraph.

It is however easy to misinterpret the result we have stated above, that the true wind blows everywhere across the isobars in a forward direction. For the added component forward is equal to $\zeta O'O''$, which is less than $\zeta OO'$ or U. Thus the isobaric system is moving forward more rapidly than the winds blow across the isobars, and *relative to the moving isobars* the true wind has a backward component, into the depression in the front half, and out of the depression in the rear half. This again is an obvious geometrical deduction from fig. 69. For the fluid is revolving about O, while the isobars are centred about O'', which both move forward with a velocity U. The fluid contained within the isobaric system must therefore have a backward motion $\zeta OO''$ through the isobaric system. Thus the fluid within any selected isobar is continually being changed, the outflow in the rear being renewed at the front. At the same time, in the normal cyclone, the whole disc of revolving fluid retains its identity unchanged.

The results derived above for the disc revolving as a solid can be readily extended to any other symmetrical circular system. The following discussion is based on a note in a paper by C. K. M. Douglas*.

Let O (fig. 70) be the centre of the revolving fluid, in which the velocity is a function of distance from O only, and let the system be travelling with velocity U parallel to Ox. Let the field of pressure be regarded as made up of p', the pressure in the system when not moving ($p' = f(r)$), and $-2\omega \sin \phi . \rho U y$

* *Q.J. Roy. Met. Soc.* **55**, pp. 123–51, 1929, Appendix, p. 146.

which allows for travel. The pressure gradient at any point P is made up of $\partial p'/\partial r$ along OP and $-2\omega \sin \phi . \rho U$ parallel to the axis of y. Let v be the velocity due to the rotation about O alone. Then if PD and PC be respectively perpendicular to the wind direction and the isobars of the combined field, the following relations hold

$$\tan \chi = \frac{v \sin \theta - U}{v \cos \theta} = \tan \theta - \frac{2\omega \sin \phi . \rho U}{2\omega \sin \phi . \rho v \cos \theta} \qquad \text{......(16)},$$

$$\tan \psi = \tan \theta - \frac{2\omega \sin \phi . \rho U}{\frac{\partial p'}{\partial r} \cos \theta} \qquad \text{......(17)},$$

and

$$\frac{\partial p'}{\partial r} = 2\omega \sin \phi . \rho v + \frac{\rho v^2}{r} \qquad \text{......(18)}.$$

Equation (17) expresses the condition that the isobar is perpendicular to PC. In the last of these equations v is always positive. Hence $\frac{\partial p'}{\partial r} > 2\omega \sin \phi . \rho U$, and hence, if $\cos \theta > 0$, $\tan \psi > \tan \chi$ and $\psi > \chi$, while if $\cos \theta < 0$, $\psi < \chi$. Thus the air flows in across the isobars in the rear half of the depression and out

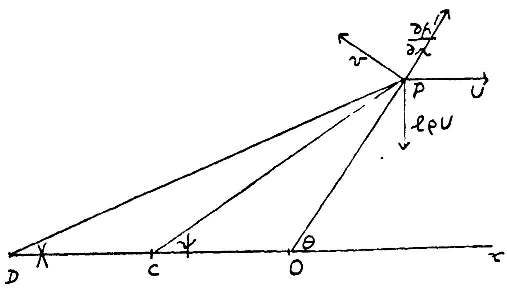


Fig. 70. Revolving fluid in any cyclone.

across the isobars in the front half of the depression. But as for the normal cyclone it is easily seen that this forward flow across the instantaneous position of the isobars is less than the forward velocity of the isobars themselves, so that the actual flow of air is into the depression in the front half, with an outward flow in the rear half.

For let the total velocity at P be made up of V along the isobar at P of the total field (perpendicular to CP) and V' forward. Then

$$V \sin \psi - V' = v \sin \theta - U \qquad \text{......(19)},$$

$$V \cos \psi = v \cos \theta \qquad \text{......(20)},$$

and

$$\frac{1}{\rho} \frac{\partial p'}{\partial r} = 2\omega \sin \phi . v + \frac{v^2}{r}.$$

By division of (19) by (20)

$$\tan\psi - \frac{V'}{V\cos\psi} = \tan\theta - \frac{U}{v\cos\theta} = \tan\psi - \frac{V'}{v\cos\theta} \qquad \text{......(21)},$$

$$= \tan\theta - \frac{2\omega\sin\phi\,.\,\rho U}{\dfrac{\partial p'}{\partial r}\cos\theta} - \frac{V'}{v\cos\theta}, \text{ from (17).}$$

Hence

$$\frac{U - V'}{v\cos\theta} = \frac{2\omega\sin\phi\,.\,\rho U}{\dfrac{\partial p'}{\partial r}\cos\theta} \qquad \text{......(22)},$$

and

$$V' = \frac{U}{\dfrac{1}{\rho}\dfrac{\partial p'}{\partial r}}\left\{\frac{1}{\rho}\frac{\partial p'}{\partial r} - 2\omega\sin\phi\,.\,v\right\} = U\frac{v^2}{r}\bigg/\frac{1}{\rho}\frac{\partial p'}{\partial r} \qquad \text{......(23)}.$$

Substituting for p' from equation (18) above we readily find that this agrees with the results derived for the normal cyclone.

Equations (22) and (23) show that while the winds at any point in a cyclone are forward across the instantaneous position of the isobars, the motion relative to the moving isobars is backward across the isobars. As already pointed out above for the normal or cartwheel depression, this does not involve any real change in the identity of the air within the depression. It is merely a consequence of the use of the isobars to define the depression, instead of the actual flow of the air.

This result can be generalised for any system of isobars, circular or otherwise, moving unchanged with a velocity of translation U. We now define O in fig. 70 as the centre of curvature of the isobar through P due to p' alone, and r as the radius of curvature. Equation (23) is then derived as before. This appears to indicate that the wind has a component across the isobars in their instantaneous position, parallel to the direction of motion, and of magnitude $U\frac{v^2}{r}\Big/\frac{1}{\rho}\frac{\partial p'}{\partial r}$. This component is zero for straight isobars, so that in this case the actual wind blows along the instantaneous position of the isobars.

§ 178. *Fujiwhara's vortical theory*

In two papers published in 1923 Fujiwhara* developed a theory of the growth of cyclones, by the amalgamation of vortices. While the hydrodynamics of a perfect fluid leads to the result that two linear vortices rotating in the same sense will repel each other†, Fujiwhara showed by means of observations of whirls in water that the opposite is the case in real fluids. He found that small whirls tend to approach and amalgamate, giving larger whirls. The phenomenon had been described by earlier writers, notably Mrs Ayrton‡ and Ahlborn§.

* *Q.J. Roy. Met. Soc.* **49**, 1923, p. 75 and p. 105.
† Lamb, *Hydrodynamics*, Chapter VII.
‡ *Proc. Roy. Soc.* A, **96**, 1919, p. 255.
§ *Phys. Zeit.* **23**, p. 59.

Fujiwhara's idea apparently is that the vorticity associated with local variations of wind can be absorbed into one central vortex, and so produce the nascent cyclone. But since the publication of the original papers in 1923, no further work has been done along these lines to the knowledge of the present writer.

Fujiwhara's theory resolves itself into regarding the depression as eventually produced by frictional forces, whose first expression takes the form of local differences of wind. These differences may be represented by three components of vorticity, whose axes are

(*a*) Parallel to the mean wind direction;

(*b*) Horizontal and perpendicular to the mean wind direction;

(*c*) Vertical.

It is the vorticity component with vertical axis which is important in Fujiwhara's theory. But eddies with vertical axes are equally likely to occur with either sign, and there is no reason to suppose that the vorticity integrated over a large area should not approach zero. The theory does not therefore appear to be in satisfactory agreement with the normal behaviour of frictional eddies.

CHAPTER XVII

THE IDEA OF AIR MASSES

§ 179. *The life-history of surface air currents*

THE earlier study of depressions of middle latitudes concentrated attention on the forms of the isobars, and related the weather to the isobars. The first clear effort to consider the physical processes in a depression from the point of view of the differences in the air masses which form the depression was that described by Shaw and Lempfert in the "Life-history of Surface Air Currents"*. In this paper, whose value can scarcely be over-estimated, the authors traced the motions of air currents by means of hourly or two-hourly charts, and trajectories of surface air are reproduced in the memoir for a number of depressions, both fast- and slow-moving. The reader is referred to the original memoir for the fuller details of the results obtained, and a brief résumé only is given here of the main conclusions:

1. In travelling storms, in the front portion the motion of air is from higher to lower pressure, and is associated with falling temperature and the gradual formation of cloud; while in the rear portion, sometimes from quite near the centre, there is motion from low pressure to higher pressure, with rising temperature and improving weather.

2. A fast-travelling storm draws in air into the central region from the front right-hand side of the path, and throws out an equal amount on the same side in the rear. The trajectories which represent this exchange form loops enclosing the centre of the depression. In addition one of the two fast-travelling storms studied in detail showed a well-marked broad westerly current south of the centre, moving with a speed slightly greater than that of the centre of the depression. The heavy rainfall was limited to a narrow band to the left-hand side of the path of the centre.

3. A slow-moving depression takes in air from both sides of its path. In the example studied in detail, that of November 11–13, 1901, there were two distinct classes of trajectories. The front right-hand quadrant of the depression was filled by a warm southerly current which flowed practically in a straight line towards the centre, where it was apparently lost by ascent over the colder air of the front left-hand quadrant. This current yielded a series of practically straight trajectories. The other type of trajectory originated to the left-hand side of the path of the centre, and swept round the rear of the depression, finally taking a generally west-east direction, though some of these trajectories in their later stages approached the centre from a south-westerly direction. The current represented by these trajectories was distinctly cooler than

* M.O. 174.

the southerly current in the front right-hand quadrant, and so it formed a barrier over which the latter could rise on reaching the line of the path of the centre. In the depression of November 11–13, 1901, the rainfall was far more intense than in the fast-moving storms, though the distribution was very similar, the intensest rain occurring in a narrow band to the left of the path of the centre.

4. The distribution of rainfall in all the cases considered could be explained as the effect of (*a*) convergence within a warm damp current, (*b*) the ascent of warm damp air over a colder current, (*c*) the displacement of warm damp air by a colder current. These three factors were found to be of varying importance in the different synoptic situations studied, but in the main they afforded a satisfactory explanation of the distribution of rainfall with the depressions. In particular, the heaviest rainfall, which occurred in a relatively narrow band to the left-hand side of the path of the centre, is readily explained by the

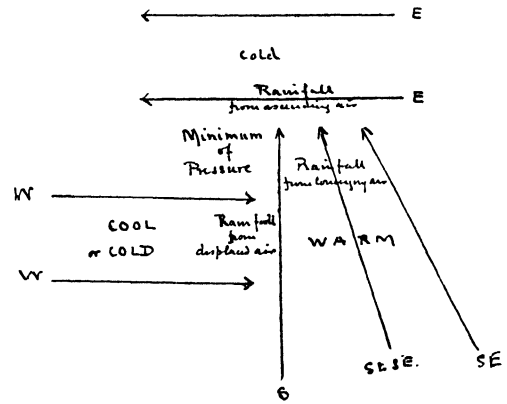


Fig. 71. Shaw's representation of the air currents in a cyclone.

ascent of the warm southerly or south-westerly current of the front right-hand quadrant over the colder easterly current of the front left-hand quadrant. Shaw summarised these results by the diagram of fig. 71. Of this diagram Shaw wrote "The north-west quadrant is not filled up. It is the region where the air sometimes bends round from the east to the north-west and west, but the air supply for the westerly current is not always derived in that way". The representation in fig. 71 is a simplified one which must not be taken as fitting in detail all cases which can occur, but it is perhaps no more exaggerated than any other single diagram which has been put forward to represent the complexities of the depression of middle latitudes.

5. The difference between a fast- and a slow-travelling storm cannot be accounted for by regarding the storm as a vortex travelling in a main current. In fact, the study of the trajectories drawn for different depressions leave the reader with a very pronounced impression that the travel of a depression is not the travel of a definite mass of air, but the travel of a state of disturbance; it gives, however, no hint of the precise nature or cause of the disturbance.

6. The transition from one persistent wind direction to another is relatively sudden.

7. Over the eastern North Atlantic currents from the South soon disappear in the centre of depressions. Only in fast-travelling storms does the air curl round the centre of depressions, forming a loop, but it is possible that the portion of such a trajectory from the central region outward is followed by a different air supply.

(This appears to be rather a dangerous generalisation from an isolated case. One would more naturally expect looped trajectories in a slow-moving rather than in a fast-moving depression. It is found in practice that when trajectories are drawn by the use of gradient winds, loops are frequent in "occluded" depressions, the air in some cases going round the centre two or three times.)

8. Air currents not from the South are of longer duration, and may persist and reach the trade winds, or turn round the rear of a depression and approach the centre from southward.

9. The flow of air along the southern side of the great Atlantic area of low pressure, which is associated with a series of approximately parallel isobars, may consist of a combination or alternation of currents of different direction, force, and temperature, with marked meteorological changes attending the sudden transition from one current to another.

10. When air moves over the sea the temperature of the air is governed by, and rapidly approaches, the temperature of the sea. The passage of warm air over cold sea generally leads to the formation of mist or fog.

11. The centres of well-marked anticyclonic areas could not be identified as regions of origin of surface air currents. These appeared rather to originate in the shoulders or protuberances of anticyclones, in ridges of high pressure, or along the trough lines of V-shaped depressions and parts of the central areas of travelling storms.

12. Well-defined anticyclones which persist for many days are for the most part inert and comparatively isolated masses of air, taking little part in the circulation which goes on around them.

The twelve paragraphs above give only a bare outline of the total results derived by Shaw and Lempfert. They suffice, however, to indicate the development of the conception of air masses as the features to be studied in relation to the phenomena of weather. These ideas have been much more fully developed by the Norwegian school of meteorologists in connection with the "polar front" methods of analysis of synoptic charts, and the discussion of the depression of middle latitudes in greater detail will be taken up in the next chapter in relation to the Norwegian methods.

§ 180. *The classification of air masses*

An examination of synoptic charts shows the existence in the atmosphere of extensive air masses possessing a homogeneous or quasi-homogeneous character. Within such air masses there will not usually be found large local differences of temperature, wet-bulb temperature, or visibility. The boundary separating two air masses of different history is in general recognisable by the abrupt local change in motion and in other properties of the air.

The earlier writers on the polar front methods (see next chapter) distinguished two main classes of air only, polar and tropical, whose properties were sharply defined.

Polar air is defined as air which originates in high latitudes, and travels southward over a surface of sea or land whose temperature increases southward. The heating of the lower layers of the air by contact with the earth's surface produces a tendency to instability in the lower layers, with consequent turbulent mixing. The main features of polar air are therefore low temperature, low absolute humidity, good visibility, and instability in the lower layers.

Tropical air is defined as air which originates in low latitudes and travels poleward over a surface of land or sea whose temperature decreases with increasing latitudes. The cooling of the lower layers by the surface produces stable stratification in the surface layers, and the main features of tropical air are therefore relatively high temperature and humidity, and stable stratification in the lower layers. It is frequently stated that in tropical air the visibility is low, and it is clear that in tropical air which has passed over desert regions there might be present a considerable quantity of dust which in the absence of turbulence would be retained in the lower layers, so giving poor surface visibility. Pick* has shown that the visibility criterion cannot be used with any degree of safety, "exceptional" visibility being sometimes found in tropical air. Pick's data suggest that the recent history of the air mass is more important than its remote history, in determining the visibility within it.

The features described in the last two paragraphs will, in theory, draw a clear distinction between polar and tropical air. The phenomena of condensation also differ in the two classes on account of the differences in relation to stability. In polar air the clouds tend to take the form of cumulus or cumulonimbus, and any rain which occurs is in showers with squally winds. In tropical air condensation tends to take the form of stratus clouds at different heights, and sea fogs are also frequent.

The nomenclature has a special sense which is frequently overlooked. The names "polar air" and "tropical air" are not applied to air masses while in the polar and tropical regions respectively, but to air masses which originate in these regions and move to middle latitudes. It is therefore incorrect to say that tropical air which has spent a long time in polar regions becomes polar air.

Later writers† on polar front analysis have distinguished four classes of air—equatorial, sub-tropical, sub-polar, and arctic. Bergeron further divides each of these into two sub-classes according as they are of maritime or continental origin. Other writers adopt somewhat different criteria, and it is perhaps fair

* *Q.J. Roy. Met. Soc.* **55**, 1929, p. 81; *ibid.* p. 195.

† See Bergeron, *Met. Zeit.* **65**, 1930, p. 246. For a detailed discussion of the problem of air masses see also Bergeron, "Über die drei-dimensionale Wetter-Analyse", *Geof. Publ.* **5**, No. 6.

to say that at present there is a striking absence of uniformity of practice. This is in part due to the difference of emphasis laid upon geographical origin and subsequent history.

The essential fact to be remembered is that air masses tend to retain at high levels their original characteristics, for long periods. The origin of an air mass can therefore usually be detected by means of upper air observations more readily than by any other means. Very strong evidence in support of this view has been given by Douglas*. In the lower layers the characteristics of an air mass may be very markedly affected by its recent history, in particular the nature of the surface over which it has passed, and the time it has spent in the same latitude.

Polar air which has passed over a long stretch of sea is usually described as "maritime polar air". Good examples of this are the outbursts in winter of cold air between the west coast of Greenland and the American continent, the air moving South, then East, and frequently approaching the British Isles as a south-westerly current. In such currents there is frequently a high lapse-rate in the surface layers, associated with the original polar conditions at high levels.

In modern Norwegian practice "tropical air" is not very common, the name being rightly restricted to air originating in really low latitudes, say south of 40° N. Other warm masses may be called "maritime polar air", "returning maritime polar air", and in summer "continental air". There is a tendency to use the name "maritime polar air" too frequently, even for air which has spent a long time in summer between latitudes 50° and 62°, which might more appropriately be called "maritime air".

The number of possible classes of air masses is almost infinite, and in practice the forecaster has to make a compromise between the excessive complexity of the atmosphere and undue simplicity of classification. There is thus an arbitrary element in the classification of air masses and the drawing of fronts separating them which makes it unprofitable in this place to devote much time to a discussion of details. We shall return to the subject in the next chapter, in discussing the drawing of fronts between air masses.

There is no direct evidence that above 1 or 2 km the mean lapse-rate differs appreciably in polar and equatorial air. True instability for dry air rarely extends beyond 1 km, but a lapse-rate greater than the saturated adiabatic, and slightly greater than the average, is often found in polar air which has been subjected to prolonged heating over the sea. This may lead to the development of winter thunderstorms, heavy local rain, and in some cases to the formation of depressions. Various statistical studies of polar air masses have included cases where there had been no prolonged heating, and also cases where subsidence had occurred, and the importance of the effects produced by prolonged surface heating has not always been brought out in such studies.

* *Q.J. Roy. Met. Soc.* **51**, 1925, p. 229.

CHAPTER XVIII

THE POLAR FRONT AND ITS RELATION TO THE DEVELOPMENT OF CYCLONES

§ 181. *The depression as a wave disturbance in a surface of discontinuity*

HELMHOLTZ* many years ago drew attention to the possibility of two currents flowing side by side, having different temperatures and different velocities, and separated by a surface of discontinuity, the currents flowing either in the same or in opposite directions. The dynamical conditions for equilibrium yield a relatively simple equation for the inclination of the surface of separation (*vide* Chapter x).

A number of writers have suggested that cyclones tend to form at such surfaces of separation of cold and warm air. Thus Bigelow† in 1902 pointed out that on the whole cyclones could not be regarded as having warm centres or cold centres, but that the centres of cyclones were found on lines separating warm and cold currents. The evidence adduced by Bigelow in favour of this view related to North America, but similar results were confirmed for the depressions of North-western Europe by Hanzlik, and for Asiatic depressions by von Ficker. Further Shaw and Lempfert, in the "Life-history of Surface Air Currents" (*vide* Chapter XVII) showed that the phenomena of weather in depressions were to be explained by the interaction of air currents of different origin and of different temperatures and humidities.

The relation of the depression to the lines of discontinuity between different air masses has been developed in recent years by the Norwegian school of meteorologists from two different points of view, the first, developed by Professor V. Bjerknes ‡ from a mathematical standpoint, regarding the depression as a wave in the surface of separation, the second developed by J. Bjerknes§, H. Solberg, and others, concerning itself rather with the discussion of the physical processes at fronts, and the evolution of methods of application of physical principles to the analysis of synoptic charts and the practical problems of forecasting. The surface of separation between the warm and cold currents in the depression of middle latitudes has been named the "polar front", and the Norwegian methods of analysis are usually referred to as the polar front methods, or frontal methods.

V. Bjerknes visualises the cyclone as a wave in the *polar front*, the surface of discontinuity between the mild westerly currents of middle latitudes and the

* H. Helmholtz, "Über Atmosphärische Bewegungen", *Ges. Abh.* **2**, p. 289.

† *Monthly Weather Review*, 1902, p. 251.

‡ "On the dynamics of the circular vortex", *Geof. Publ.* **3**; No. 4.

§ J. Bjerknes and H. Solberg, *Geof. Publ.* **2**, No. 3, **3**, No. 1; also J. Bjerknes, *ibid.* **1**, No. 2.

cold easterly currents of high latitudes. It has not yet been possible to give a complete mathematical development of the problem of waves in an inclined surface of discontinuity. The problem is of extreme complexity, and the arguments used by V. Bjerknes are of a general character, and present numerous gaps. The waves are gravitational waves, in which initially the vertical displacements are much greater than the horizontal displacements. But according to Bjerknes the effect of the deviating force due to the earth's rotation will be to cause the horizontal displacements to increase enormously in proportion to those which would exist in the absence of the deviating force. No satisfactory mathematical proof of this step in the argument has yet been given. The wave increases in amplitude, and a depression develops at its northern crest. Thus the theory is strongly reminiscent of Emden's* theory of sunspots.

The computation of the wave-length which should correspond to any particular distribution of temperature on the two sides of the surface of discontinuity leads to an estimate of the wave-length which is of the order of 500 metres. Thus Wegener† estimates that in order to obtain a wave-length of the order of magnitude of the diameter of an average depression we should require to make the difference of temperature at the two sides of the surface of discontinuity vanishingly small.

V. Bjerknes‡ has given an expression for the velocity of propagation of a wave at a horizontal surface of discontinuity of temperature, which he finds to be $0{\cdot}53\sqrt{(T'-T)(1000-p_1)}$ metres per second, where $T'-T$ is the discontinuity of temperature, p_1 is the pressure in millibars at the level of the surface of discontinuity, and 1000 mb is adopted as the mean pressure at M.S.L. If 700 mb is adopted for p_1, corresponding roughly to the average pressure at a surface of discontinuity extending from the surface of the earth through the height of a depression, the velocity is $9\sqrt{T'-T}$ metres per second. This is of the right order of magnitude, but the difficulty associated with the smallness of the wave-length remains unresolved. The extreme difficulty of the mathematical discussion of such a problem as the association of fronts with depressions makes progress difficult. Much remains to be done, but a vast amount of the preliminary work has already been done by V. Bjerknes and his associates§.

§ 182. *The polar front methods of analysis of charts*

The inter-relation of the polar front and the depression, as depicted by J. Bjerknes and Solberg (*loc. cit.*), is most readily explained by reference to fig. 72*a–d*. In this figure, diagram *a* represents a portion of the undisturbed polar front, diagram *b* represents the distortion of the front by the intrusion northward of the warm air, and diagram *c* represents a later stage in the growth

* Emden, *Gaskugeln*, 1st edn., 1907, p. 441.
† *Met. Zeit.* 1921, p. 300.
‡ *Loc. cit. ante*, p. 28.
§ *Vide Physikalische Hydrodynamik* by V. Bjerknes and others.

of the distortion. At stage *c* there is a well-marked depression centred at the most northerly point of the tongue of warm air. The bulge in the surface of separation, together with the newly formed cyclone, moves eastwards with the warm currents. The part of the polar front along the eastern edge of the warm tongue is called the *warm front*, and the part along the western edge the *cold front*. (This refers to a depression moving in an easterly direction. In a depression moving in any other direction the positions of the fronts relative to the path of the centre remain unaltered.) At the warm front the warm air climbs up over the cold air, giving precipitation over a wide area, the rain falling through the cold air.

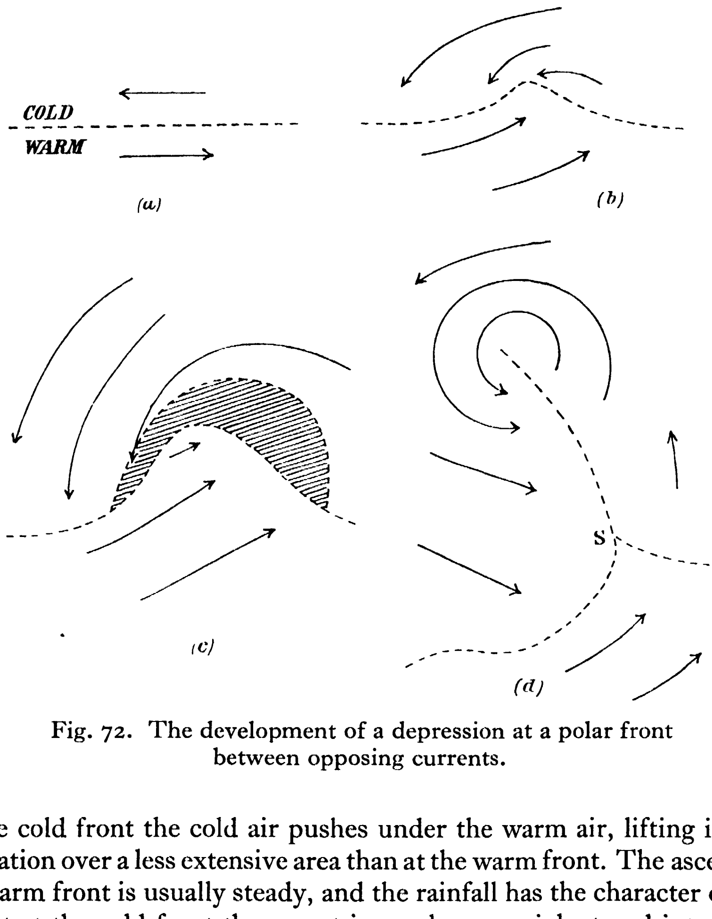


Fig. 72. The development of a depression at a polar front between opposing currents.

At the cold front the cold air pushes under the warm air, lifting it, giving precipitation over a less extensive area than at the warm front. The ascent of air at the warm front is usually steady, and the rainfall has the character of steady rain; but at the cold front the ascent is much more violent and intermittent, accompanied by squally winds, and the clouds are of the cumulus or cumulonimbus type, with frequently a long roll of cloud over the front. This distribution of weather phenomena and of the types of cloud in different parts of the depression is represented in fig. 72*e*, in which the central diagram represents the typical depression in active development, while the upper and lower diagrams represent vertical West-East sections taken north and south of the path of the centre respectively. It is to be noted that when the warm and cold fronts are strongly marked the isobars are refracted at the fronts, as shown in fig. 73.

A vertical section through the right-hand side of the lower diagram in fig. 72*e* from the cold air up through the front into the warm air will show the

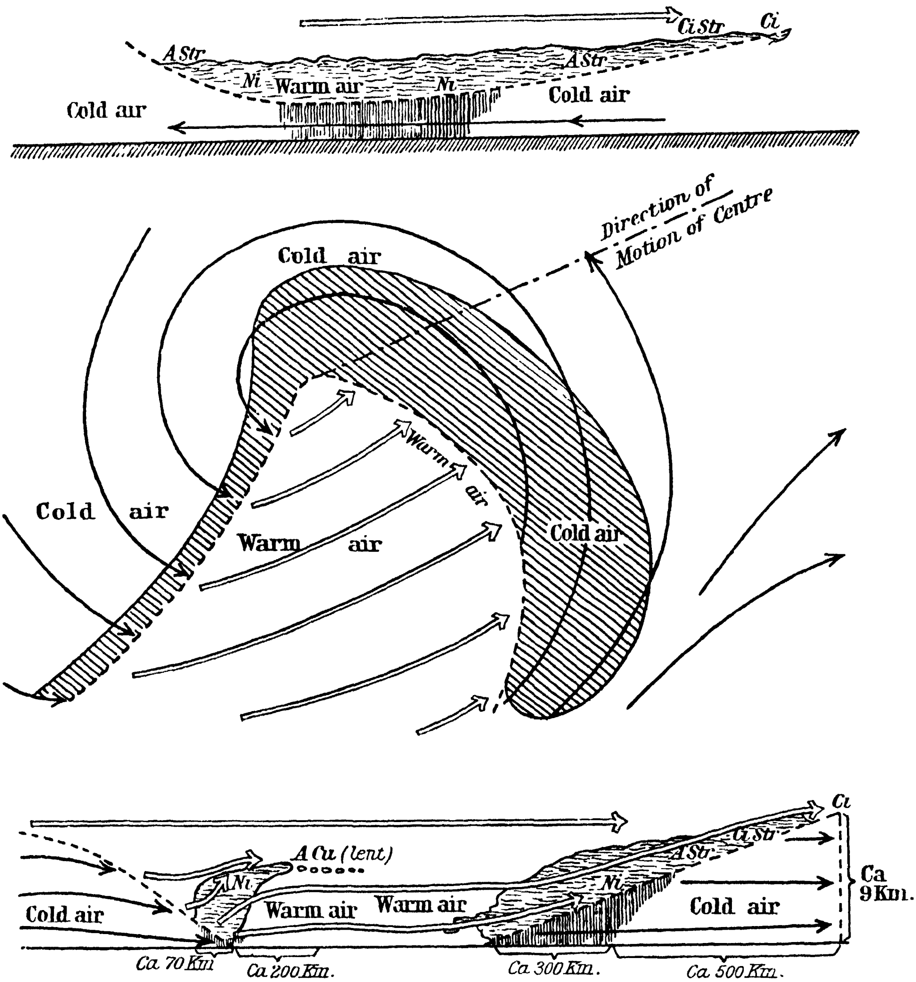


Fig. 72*e*. The typical polar front depression.

following characteristics. Near the ground the lapse-rate in the polar air will be high, possibly approaching the dry adiabatic, except over land in winter, when, in continental air, or in air cooled at the surface by radiation, a considerable inversion may occur. At the frontal surface the temperature in the warm air is higher than that in the cold air at the same level. There will therefore be a check in the fall of temperature, possibly amounting to a pronounced inversion, the change taking place in a layer whose relative humidity is high, approaching saturation.

On account of the polar current being heated by the earth's surface and the tropical current being cooled by the earth's surface, it follows that in the lower layers the lapse-rate is greater in polar than in tropical air. The difference in temperature is accordingly more accentuated at moderate heights than it is at the actual surface. This is very clearly borne out by observations. Douglas* has produced direct evidence showing that in disturbed westerly conditions round the British Isles the difference of temperature between cold and warm air masses in juxtaposition is more than twice as great at 4 km as at the surface. There is however no direct evidence that at heights above 1 or 2 km the lapse-rate in genuine polar air is appreciably higher than that in genuine tropical air.

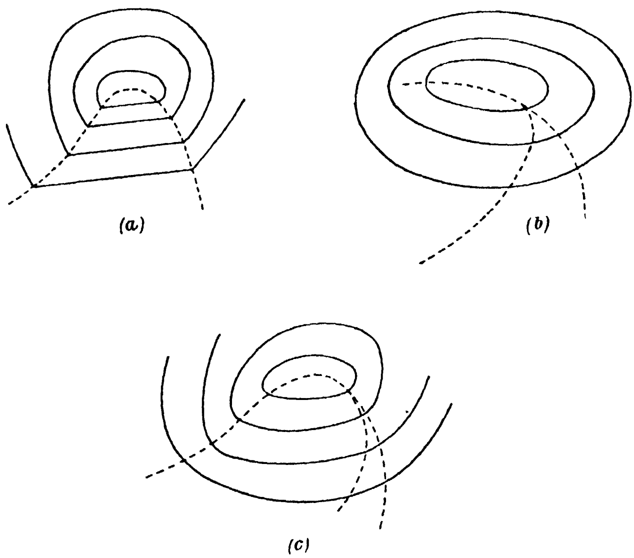


Fig. 73. The development of a back-bent occlusion.

In the course of time the cold air in the rear pushes forward, raising the warm air and narrowing the tongue of warm air at the ground. Eventually the cold front overtakes the warm front, lifting the warm air up from the ground, and the depression is then said to be *occluded*. The occlusion begins at the centre, where the cold front has a shorter path to cover before overtaking the warm front, and works progressively outwards (see fig. 72*d*). So far as the surface layers are concerned the depression then consists entirely of cold air, though there will still be a warm sector in the upper air for some time after occlusion has taken place at the surface. The occlusion progresses upwards as the warm air is displaced still higher, until eventually the depression consists entirely of cold air, and has approached to a nearly symmetrical vortex.

* *Q.J. Roy. Met. Soc.* **47**, 1921, p. 23.

Continuous rain may fall for 12 to 24 hours after the surface occlusion, on account of the progressive rise of the air in the warm sector at high levels. With the cessation of the continuous rain the depression begins to decay, since it no longer has a store of potential energy to draw upon, and friction and turbulence produce a steady diminution of kinetic energy.

In general there will be a difference of temperature between the cold air in advance of the warm front and the cold air in the rear of the cold front, so that when occlusion takes place there will still be some difference of temperature at the line of occlusion. Two kinds of occlusion may therefore arise, according as the cold air in the rear is colder or warmer than the cold air in front of the line of occlusion. These are shown in fig. 74*a* and *b*. The first is essentially a

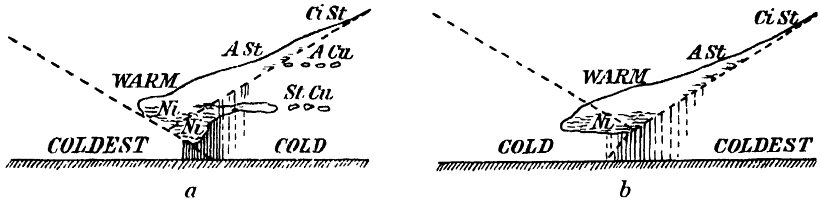


Fig. 74. Types of occlusion.

cold front with a narrow rain belt, while the second is a warm front with a broader belt of rain.

The somewhat simple picture of fig. 72*e* may be considerably complicated by the appearance of secondary cold fronts within the cold air in the rear of the cyclone. If the temperature contrasts are slight at these secondary fronts, the only result is the production of narrow rain belts, but if one of the secondary fronts shows a very marked contrast of temperature it will have the effect of enlarging the effective warm sector, so that all the air between this front and the warm front acts as a warm sector, giving correspondingly greater supplies of energy to the cyclone.

Some of the earlier papers on the polar front picture the development of the occlusion as shown in fig. 75, in which an island of warm air is isolated at the centre. This development is not typical of the depression as we know it, and is limited to regions where orographic causes hold up the travel of the warm front in such places as off the coast of Norway. This type of development is now known as "seclusion".

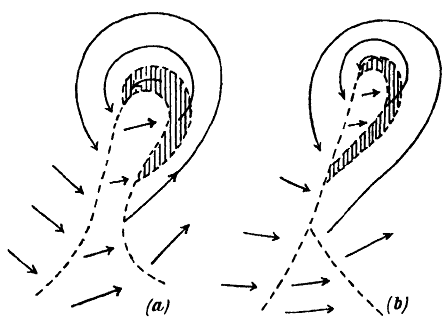


Fig. 75. The seclusion of depressions.

The cyclone moves with the warm current or in the direction of the isobars in the warm sector, with a speed which never exceeds, and is usually a little less than, that of the warm air. There is no evidence that the speed of travel ever exceeds the wind speed at 10,000 feet, though in some depressions, more particularly in summer, it may exceed the wind speed at 2000 feet. After occlusion the depression usually becomes nearly stationary, and if it is isolated from other depressions it remains stationary in its dying stage. Frequently, however, the dying depression gets caught up in the circulation around other and more vigorous systems.

The fact that the depression moves in the direction of the isobars in the warm sector is confirmed by an examination of the data of observation, even in cases where the motion has differed widely from the mean tracks as usually laid down in the older textbooks. The growth of the typical Bjerknes depression is in the direction from marked dissymmetry toward symmetry. In its initial stage the depression shows a marked warm sector, but as the depression deepens the warm sector shrinks, in most cases with great rapidity. In its final occluded state the depression is seen to be of the nature of a whirl, surrounded by a close approach to a solid current, which protects the centre from the encroachment of any new currents. Only in very special circumstances can a new current approach the centre of an occluded depression, and any new development in an occluded depression is likely to be of the nature of the development of a new centre.

The brief account which has been given above of the frontal structure of a depression must be regarded as only an outline. The phenomena can be further complicated when the air masses involved are not homogeneous, or when their motion is affected by mountain ranges, which may hold up the portion of a warm front in the lower layers for a considerable time, giving rise to heavy rainfall.

The description given in the preceding paragraphs follows in the main the lines of the description given by Bjerknes and Solberg (*loc. cit.*) in 1922. Since that time no very considerable advance has been made along these lines, though J. Bjerknes and others have shown in a series of papers that the phenomena are in many cases more complicated than was assumed in the original description. When unexpected developments occur they can as a rule be explained by the nature of the development of the fronts, though it is rarely possible in such cases to forecast the developments in detail.

Fig. 72 above visualises the depression as originating at a straight front separating easterly and westerly currents. There is, however, no *a priori* reason why a front should be straight unless the isobars in the two currents are straight; but strictly straight isobars are exceptional rather than the rule. Most developments over the Central and Eastern Atlantic and round the British Isles occur at discontinuities in a westerly current, the course of the development being as shown in fig. 76*a* to *d*. Initially there is no east wind, and east winds only develop as the depression develops, as part of the circulation round the centre of low pressure. A good example of this is shown in the development of the depression discussed in detail by Bergeron and Swoboda*. This depression originated at a front separating two westerly currents, the

* *Veröff. Geoph. Inst. Leipzig*, Ser. III, **3**, p. 63.

more northerly of which was maritime polar air, while the more southerly was genuine tropical air which had come round the Azores anticyclone.

Shaw has suggested* that the initial condition for the development of a depression is not the occurrence of two currents in opposite directions, but rather of two currents at right angles to one another, associated with the occurrence of a right-angled kink in the isobars. In a large number of cases it is undoubtedly possible to trace such a scheme as Shaw suggests, which accords with Exner's view that the depression of middle latitudes is due to the invasion of the cold air of polar regions into the zone of westerly winds (see

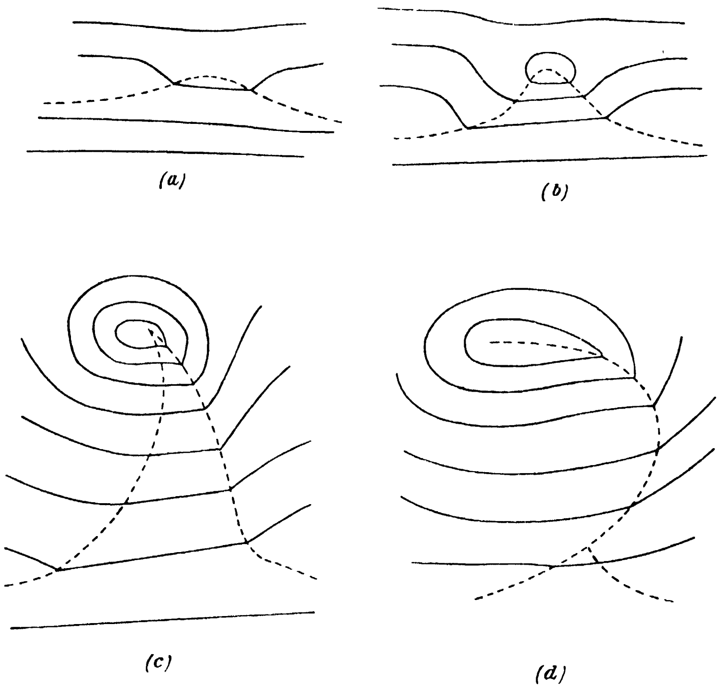


Fig. 76. The development of a depression at a boundary between two westerly currents.

§ 195 below), but in the two cases illustrated by Shaw the bulge in the isobars is not the initial stage, the depression being then definitely in existence.

Depressions sometimes form entirely within a polar current, but the air in the southern portion of these depressions will have moved over a longer trajectory in middle latitudes than the air in the northerly portion. Thus even within these depressions there may occur fairly marked contrasts of temperature. An example was provided by a depression which on November 3, 1910, was centred over England. Upper air soundings at Pyrton Hill, Berlin and Vienna showed a marked rise of temperature with the approach of the centre of the depression. It is of some interest to note that on November 5 when this

* *Manual of Meteorology*, 4, p. 289, fig. 75.

depression had passed away eastward, a very deep depression developed rapidly off North-west Ireland, entirely in polar air. This rather suggests that when conditions in the upper air have become favourable for the formation of depressions in polar air, those conditions tend to persist. Douglas cites several other cases, notably the depressions of March 3 and 5, 1909, which also formed entirely in polar air. In the first of these, the temperatures in the central part of the depression were below the normal, but in the outer regions temperatures were still lower.

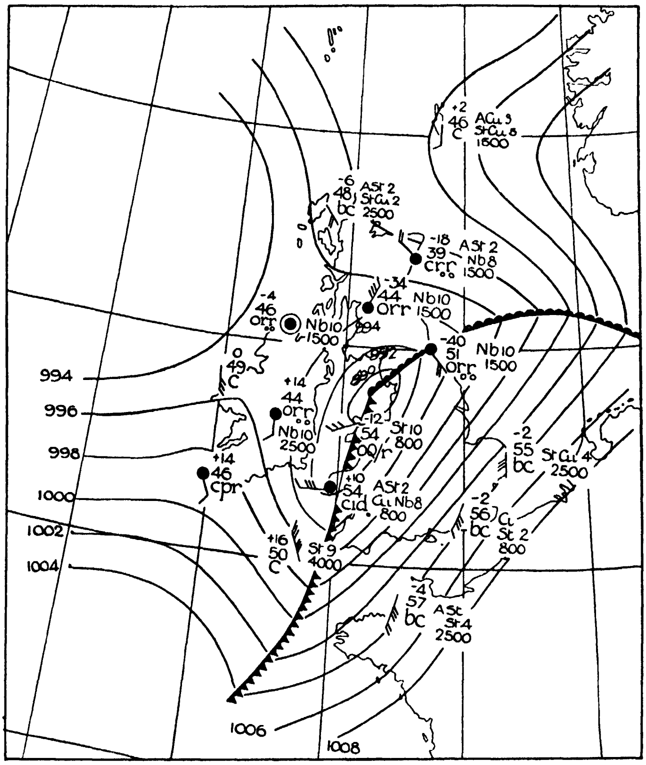


Fig. 77. The depression of October 22, 1932.

In a later paper J. Bjerknes* has shown that the phenomena of occlusion may lead to a subsidiary cold front. As the occlusion first develops near the centre of the depression, there is, in some depressions at least, a tendency for the centre of the cyclone to move along the occlusion to the angle of the warm sector. The occluded front then tends to become a subsidiary cold front as shown in fig. 73*c* above. Such a front is usually referred to as a "back-bent occlusion". In addition there is in such cases a tendency for the depression to be strengthened by reason of its re-acquiring a warm sector, as well as by reason of the part of the cold air between the main and the subsidiary cold

* *Geophys. Mem.* No. 50.

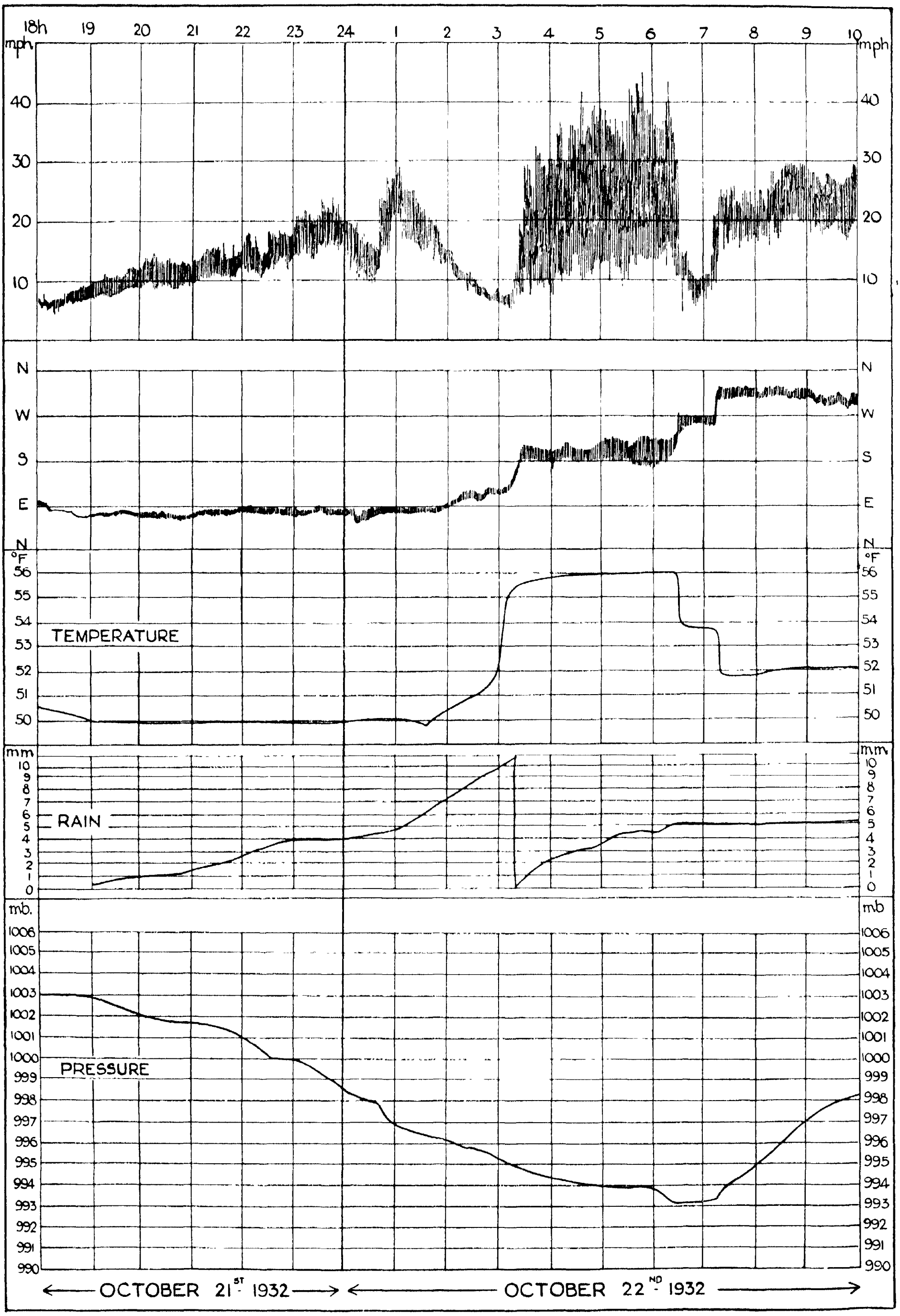


Fig. 78. Autographic records at Holyhead, October 21–22, 1932.

front acting as a warm sector and helping to supply energy to the depression.

It is not practicable in the space here available to illustrate all the points brought out in the above description by means of synoptic charts selected for the purpose, and only certain typical situations can be illustrated. Fig. 77 shows a typical polar front depression, having well-marked warm and cold fronts. With the observations available on the chart it is not possible to place with complete certainty the accurate position of the warm front. Fig. 78 reproduces the autographic records for Holyhead. The thermograph shows that up to about 3h Holyhead was in the cold air in advance of the warm front. In passing through the warm front the temperature rose about 2° F in about 1½ hours, then rose 3° F in the course of a few minutes, after which there was a further rise of 1° F in 3 hours, the highest temperature occurring just before the advent of the cold front. The fall of temperature at the passage of the cold front was in two very distinct stages, separated by an interval of about 50 minutes, indicating that the front was double at Holyhead, as in fact it was at several stations. The lowest pressure was recorded from about 6h to 7h, corresponding to the time of highest temperature. On the anemogram is shown the arrival of the warm front at about 3h 20 m, with an increase of wind and a change of direction from ESE to S. This occurred at the same time as the sudden rise of temperature. The first cold front arrived at about 6h 30 m, with a drop of wind, and a veer to W by S, and the second cold front arrived at 7h 20 m, with a further veer to NW and an increase of wind. The times of the arrival of the two cold fronts as fixed by the anemogram agrees with the two separate falls of temperature shown by the thermogram. The hyetograph indicates that rain fell continuously from 19h on the 21st to 7h on the 22nd, being heaviest from 2h to 4h on the 22nd, when the warm front was practically over the station. The total rainfall during the whole period from 19h on the 21st to 7h on the 22nd was 15 mm, of which 5 mm fell between 2h and 4h.

The further motion of the depression was parallel to the isobars in the warm sector; at 13h it was centred slightly east of the Firth of Forth, and by 7h on the 23rd was centred over the coast of Norway in latitude 64°, the pressure at the centre being 984 mb.

Fig. 79 shows the upper air observations made at Duxford at 7h 30m on the 22nd. It will be seen that the lapse-rate was stable up to the level of about 700 mb, even for saturated air. At this time Duxford was not more than 140 miles from the warm front.

The reader is recommended to examine the charts in *Geophysical Memoir*, No. 50, in which J. Bjerknes has illustrated a number of interesting phenomena. One depression selected from the three discussed in this memoir is represented in figs. 80 and 81, which reproduce the synoptic charts for 7h and 13h on January 23, 1926.

The 13h chart shows that the depression has moved roughly parallel to the isobars in the warm sector. These two charts present some typical features of polar front depressions. The association of the heavy rain with the fronts is very clearly marked. On the 7h chart the cloud in the part of the warm sector over Wales is St. and St.-Cu., but a subsidiary attenuated warm front running from about Plymouth to Spurn Head is marked by Nimbus, as is the whole of the principal front. The 13h chart shows, in addition to the principal warm and cold fronts, a back-bent occlusion running from the centre in a SSW direction across the East of Ireland, a subsidiary cold front from Anglesey to Plymouth, and a subsidiary warm front crossing the warm sector, shown on the 7h chart.

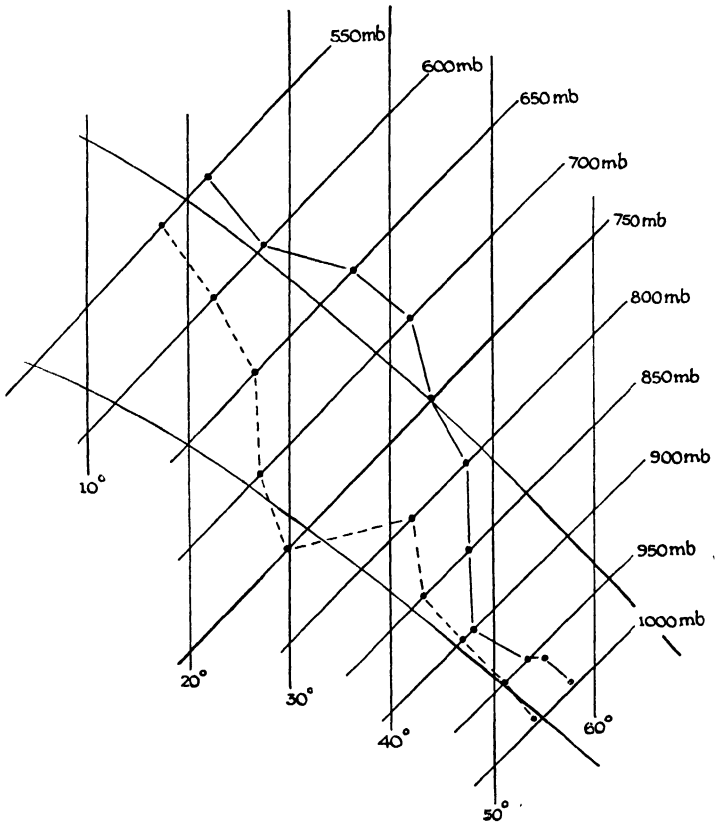


Fig. 79. Upper air observations at Duxford on October 22, 1932.
Wet bulb temperatures are shown by the broken line.

The association of different features of the weather with the fronts is best studied by means of autographic records. Those for Valentia, Holyhead, and Eskdalemuir are shown in figs. 82, 83 and 84. Consider first the records at Valentia (fig. 82). The cold front passed Valentia at 4h on the 23rd, and the back-bent occlusion passed about 10h. These times are best determined from the barograph. At 4h the wind veered and decreased in strength, and a steady fall of temperature started and continued afterwards for several hours. The rain preceded the arrival of the front, commencing about 3h. The passage of the back-bent occlusion was marked by no pronounced change of surface

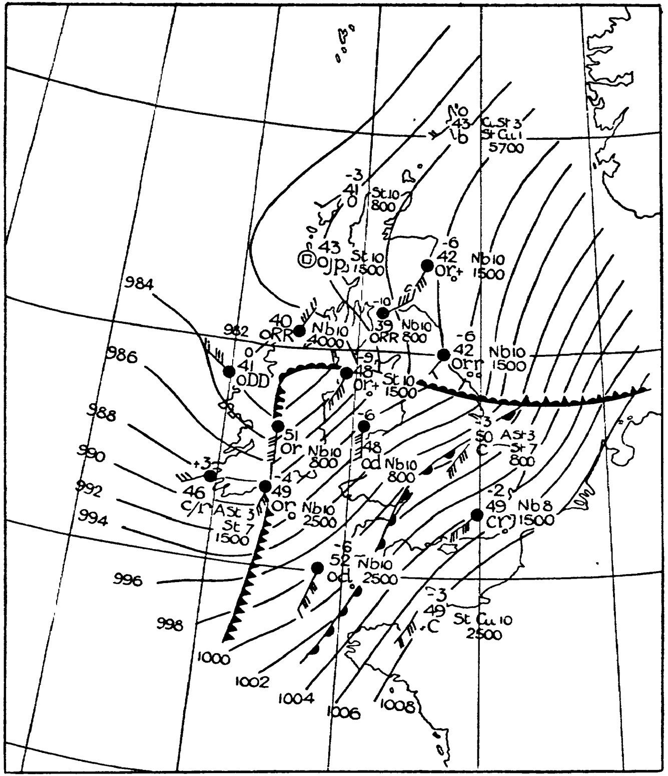


Fig. 80.

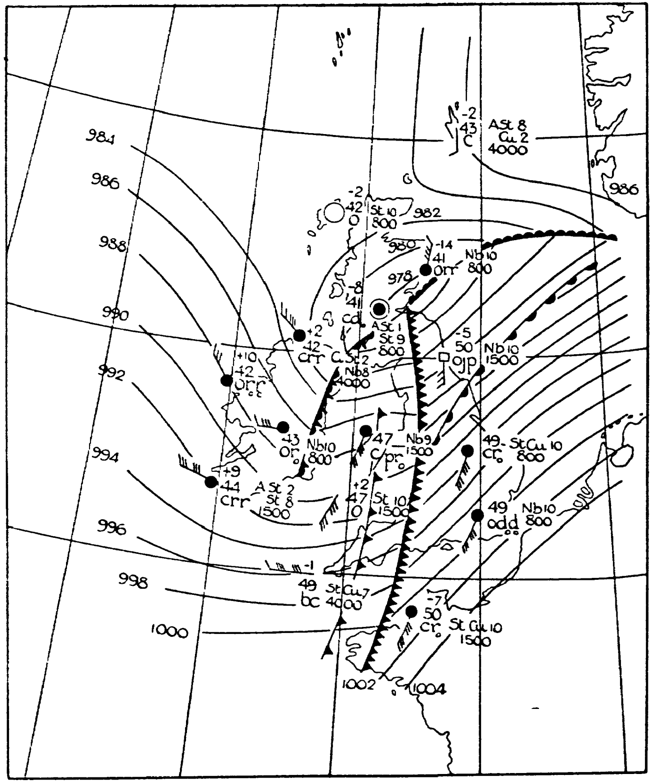


Fig. 81.

Figs. 80, 81. The depression of January 23, 1926. Charts for 7 h and 13 h. In charts showing fronts, the warm front is indicated by rounded teeth, the cold front by pointed teeth, an occlusion by a combination of rounded and pointed teeth, and secondary or attenuated fronts by the appropriate teeth spaced out.

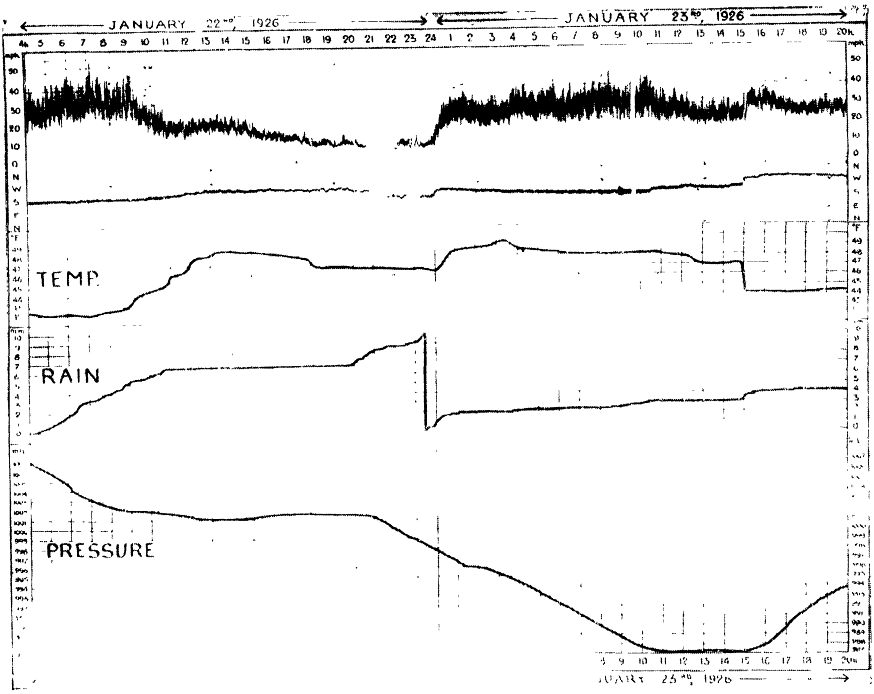


Fig. 82. Autographic records for January 22–23, 1926, at Valentia.

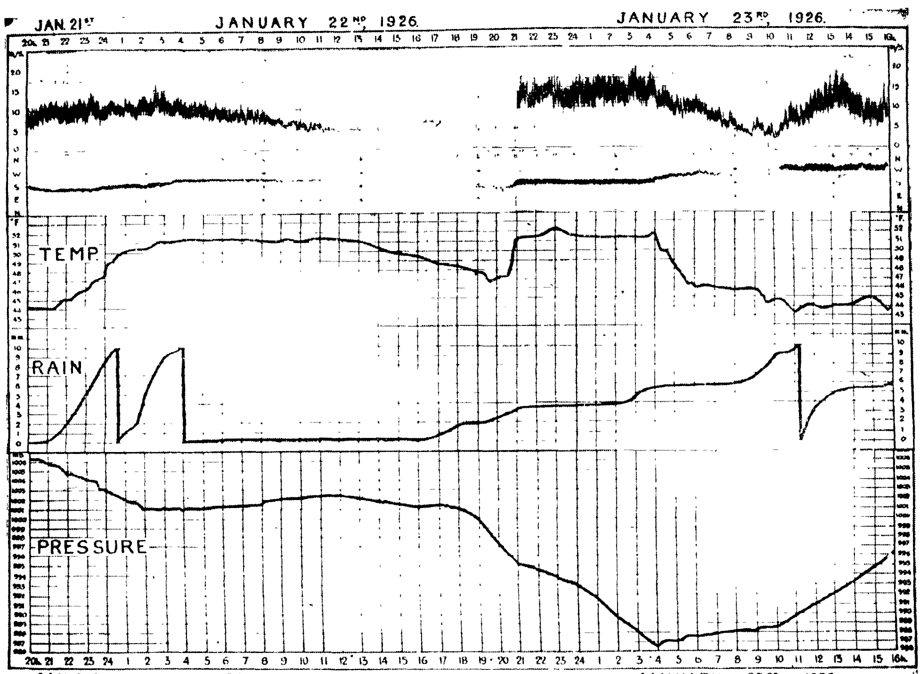


Fig. 83. Autographic records for January 22–23, 1926, at Holyhead.

temperature, but there was a well-defined increase in the strength of the wind, and heavy rain fell for several hours.

The "principal" cold front passed Holyhead at about 11h on the 23rd; the secondary cold front, which in fig. 81 runs down the West coast of Wales, passed Holyhead soon after 13h; and the back-bent occlusion passed about 15h. An examination of fig. 83 shows that the fall of surface temperature at 11h, when the principal cold front passed, was very slight, as was the fall at the passage of the subsidiary cold front. The greatest fall of temperature

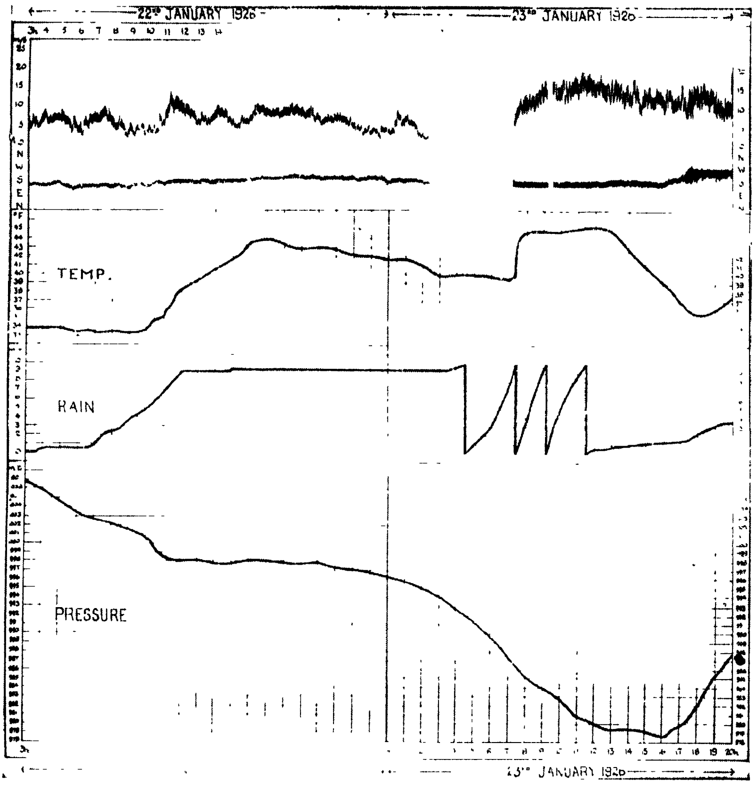


Fig. 84. Autographic records for January 22–23, 1926, at Eskdalemuir.

occurred at the passage of the back-bent occlusion at 15h, when a drop of 3° F occurred. The rain which fell in the warm sector in advance of the cold front fell at Holyhead between 8h and 11h, after which no further rain fell until 15h, when for a short time there was heavy rain, beginning at the same time as the sharp fall of temperature, while the wind veered from SW to W, and increased in strength. The fall of pressure was checked at 11h, the rise beginning at 15h.

Eskdalemuir (fig. 84) was in the cold air north of the warm front at 7h, and by 13h was in the rear of the cold front, having in the interval passed through the warm sector. Fig. 84 shows that the warm front passed at about 7h 20m, the temperature rising rapidly 5° F, while the wind increased rapidly in strength from the West. The cold front passed about 12h 30m, but the fall of temperature was distributed over more than 5 hours. Very heavy rain started some three hours before the passage of the warm front, and continued during the passage of the warm sector. Much lighter rain fell after the passage of the cold front.

The importance of the back-bent occlusion in the scheme of fronts is borne out by the changes which occurred at Valentia at 10h on the 23rd, at

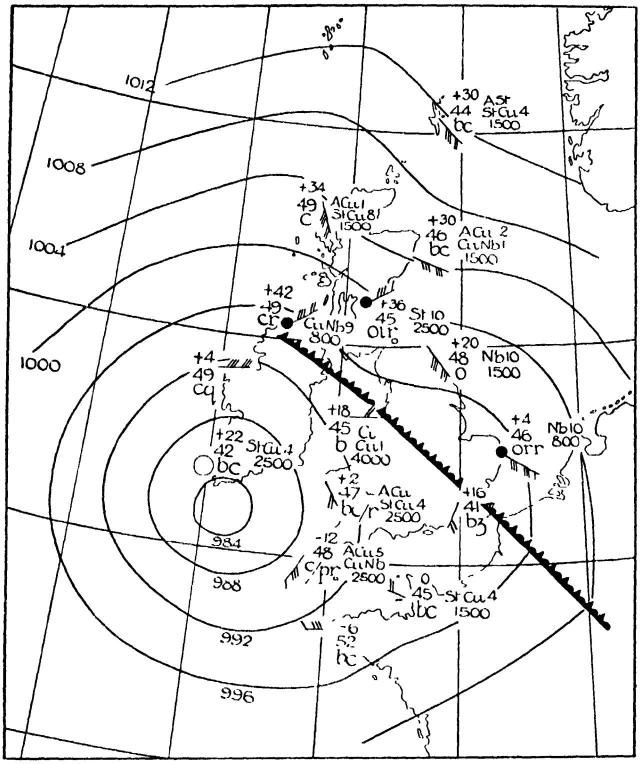


Fig. 85. An occluded depression, November 15, 1933.

Holyhead at 15h, and at Eskdalemuir at about 16h. At Valentia heavy and continuous rain occurred at the passage of the back-bent occlusion; the rain was heavy for a short time at Holyhead, and less heavy but lasting for a longer period at Eskdalemuir. At Eskdalemuir heavy rain lasted from 4h 30m till 11h 30m on the 23rd, when the station was near the centre of the depression. Such rain must have required the bodily ascent of enormous quantities of damp air.

In fig. 85 is shown an occluded depression of November 15, 1933, the chart reproducing the conditions at 18h. The line of occlusion is shown running from the north of Ireland across central England down to about Dijon. There is a clearly marked line of rain running parallel with the surface occlusion, while south of the line of occlusion conditions were generally fair. There remains some discontinuity of temperature at the front, the cold air which has curved round in the rear of the depression being warmer than the cold air in advance. The discontinuity of surface wind at the line of occlusion is also clearly marked.

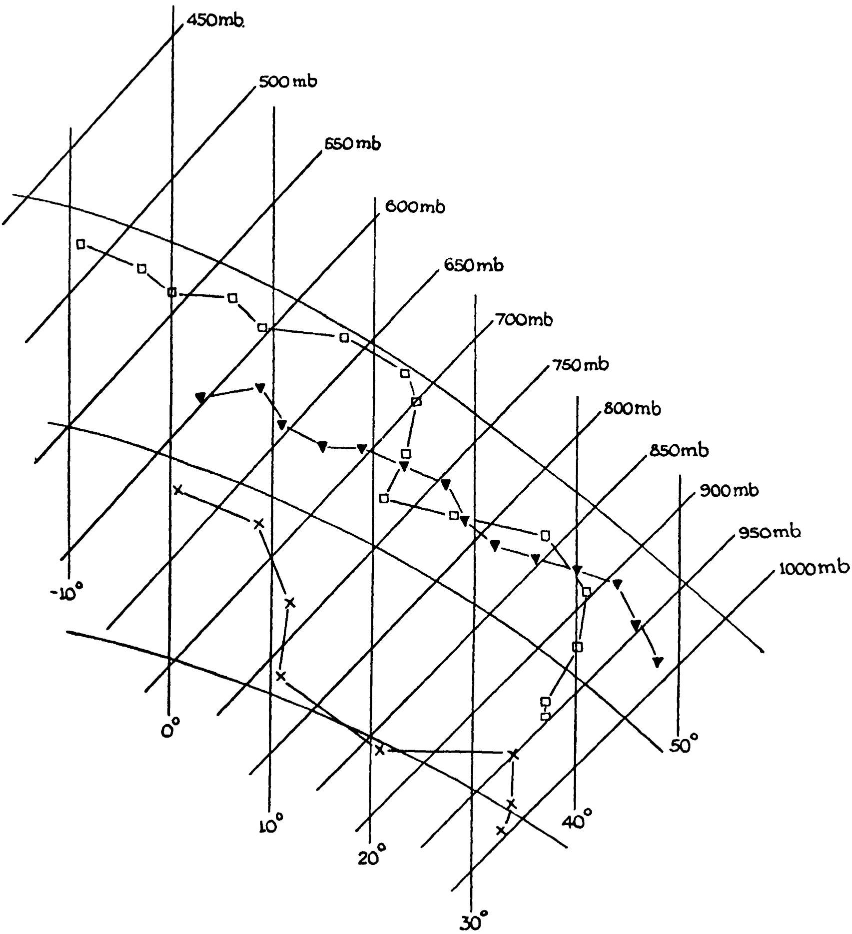
The line of occlusion had been clearly marked on the synoptic chart since 13h on the 14th, when it ran roughly North-South some 100 miles west of Ireland. The depression had moved slowly southward, the pressure at its centre increasing very slowly. Subsequently to the time represented in fig. 85 the depression continued to move southward, and to fill up very slowly. The motion is in the direction of the strongest winds, which are the winds of force 9 shown in the western edge of the depression, over Ireland. By 18h on the 17th it was centred over Portugal, the pressure at the centre being about 996 mb. Its later history was complicated by the formation of fresh centres over southern Spain.

The motion of the line of occlusion is not without interest. As the depression moved southward the line of occlusion swung round the depression counter-clockwise, and by the morning of the 16th had ceased to be a clearly marked feature of the system.

In fig. 86 are shown the upper air ascents at South Farnborough at 11h 50m, at Lindenberg at 6h, and at Munich at 9h on the 15th. The South Farnborough observations were made entirely in the southerly current. The Lindenberg ascent apparently penetrated into this current at about $2\frac{1}{2}$ km, and since Lindenberg was some 500 km from the surface line of occlusion, the front was sloped forward about 1 in 200, the form of the occlusion being that shown in fig. 74 (*b*). Munich was in the cold surface current, but from about 1 km to $2\frac{1}{2}$ km was in the current which had come around in the rear of the occlusion, while above that height there appeared to be warm tropical air. It is doubtful whether this air is to be interpreted as the remnant of the warm sector. It is more probably tropical air which has recently encroached into the depression.

Such differences between the two air masses on the two sides of an occlusion are not always found. J. Bjerknes and Palmén* have given upper air data for the depression of March 28–31, 1928, which show that except in the lowest kilometre the temperature conditions were practically identical in the two masses on the two sides of the warm sector, the tropopause being at 7 km above each. Palmén† suggests that above occluded depressions the tropopause is normally very low for the latitude, say at 5–8 km over occluded depressions over NW Europe.

* *Beitr. Phys. fr. Atmos.* **21**, 1933, p. 53.
† *Mitteilungen Met. Inst. Helsingfors*, No. 25, 1933.



▼——▼ South Farnborough, November 15, 1933, 11 h 50 m.
×——× Lindenberg, November 15, 1933, 6 h.
□——□ Munich, November 15, 1933, 9 h.

Fig. 86. Upper air observations on November 15, 1933.

§ 183. *The formation of secondaries*

When a depression has become occluded, the cold front trails behind the depression as in fig. 72*d*. Secondaries tend mainly to form at a bend in the trailing front. Every trailing front does not give rise to secondaries, nor do secondaries invariably form only at trailing fronts. They can on occasion form within the polar current itself, and then usually at a secondary front in the cold air.

There are some regions in which secondaries tend to form at the angle of the residual warm sector in an occluded depression, as at *S* in fig. 72*d*. The phenomenon is in part orographic, due to the warm front being held up by mountain ranges. This is particularly liable to happen over the Skagerrak, when a depression centre passes to north of Scandinavia, and such secondaries are known as "Skagerrak cyclones".

It is not uncommon for a secondary depression and a primary depression to amalgamate. The charts *a* and *b* of fig. 87 illustrate this. Chart *a* for 7h on October 8, 1932, shows a well-marked secondary depression over southern England, while chart *b* shows that by 18h on the 8th the two centres had amalgamated. Later during the 9th a further secondary developed, as shown in chart *c*, which represents conditions at 7h on the 9th. The two centres rotated counter-clockwise round each other, as seen from chart *d*, which represents conditions at 18h on the 9th. This rotation was continued, and by 7h on the 10th one centre was over the North Sea, and the other over Brest. Observations at Duxford on the 8th, in the southerly current, showed an isothermal layer from 950 to 900 mb, and above this, up to at least the height of 500 mb, a lapse-rate almost exactly equal to the dry adiabatic.

§ 184. *Families of depressions*

A series of depressions may form along the same surface of discontinuity. The easternmost depression is the oldest and most fully developed, and the other depressions form in turn, each on the trailing front behind its predecessor. Each successive depression passes a little farther south than the preceding one, until a stage is attained when a broad northerly current behind a depression sweeps on into the trade wind zone. This ends the series or family of depressions, and the next depression will form usually on a more northern track, and on a new front.

J. Bjerknes and Solberg state that when a cyclone family passes, on the average four depressions are observed at any particular place which is situated centrally in the depression belt. The number is, however, rather variable, and single depressions may occur.

The family of depressions described by Bjerknes and Solberg occurs at the south-eastern edge of a cold current which bursts out in a south-westerly direction from the polar regions. The outbursts are roughly periodic, with a mean period of about 24 days.

§ 185. *The regeneration of depressions*

In general the occlusion of a depression is followed, after an interval of 12 to 24 hours, by gradual decline in its intensity. But it is not an infrequent occurrence for a depression to be revived after having reached the stationary or dying stage. In fig. 73*c* it was suggested that the centre of the depression moved along the occluded front until it reached the angle of the warm sector. When this occurs, the depression has again acquired a warm sector in its

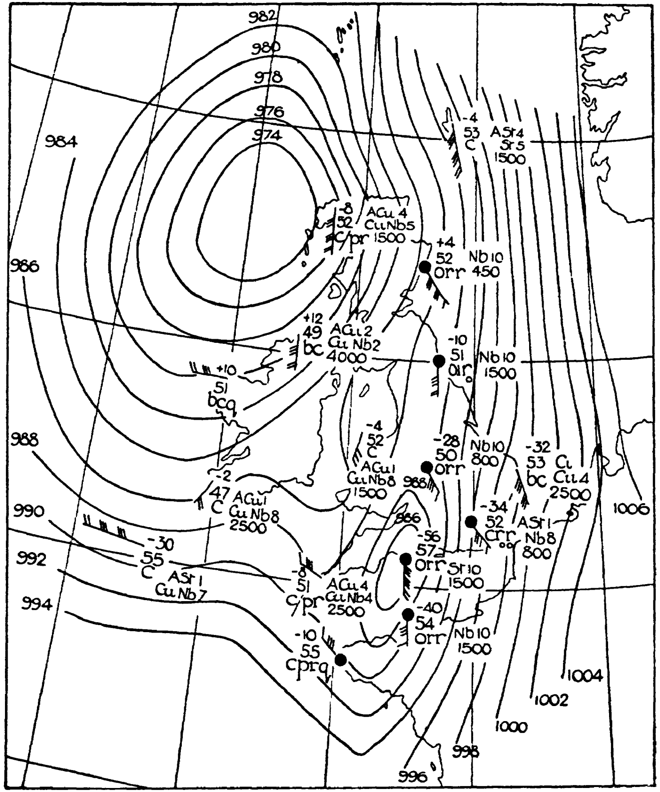


Fig. 87*a*. 7 h., October 8.

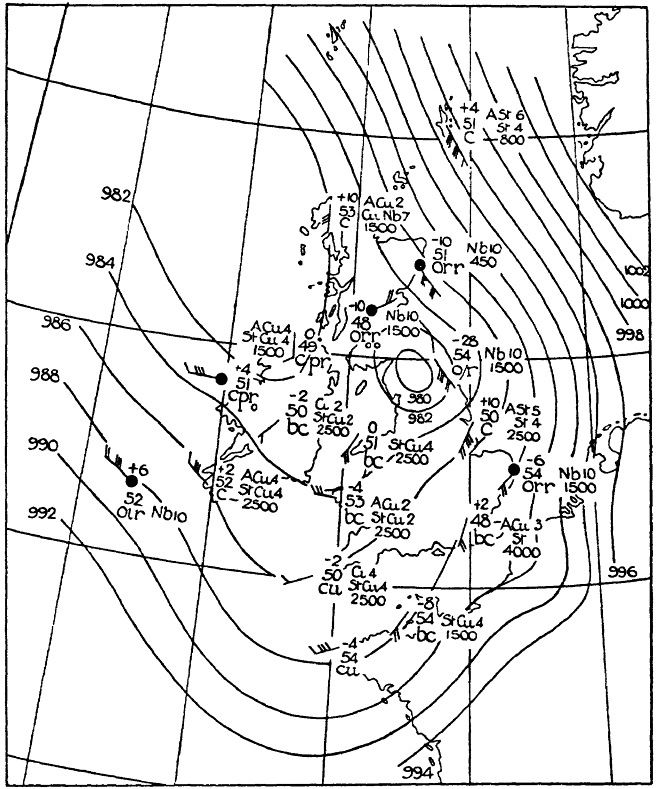


Fig. 87*b*. 18 h., October 8.

Fig. 87. Charts for October 8–9, 1932, showing the amalgamation of a depression and a secondary, and the rotation of two centres counter-clockwise.

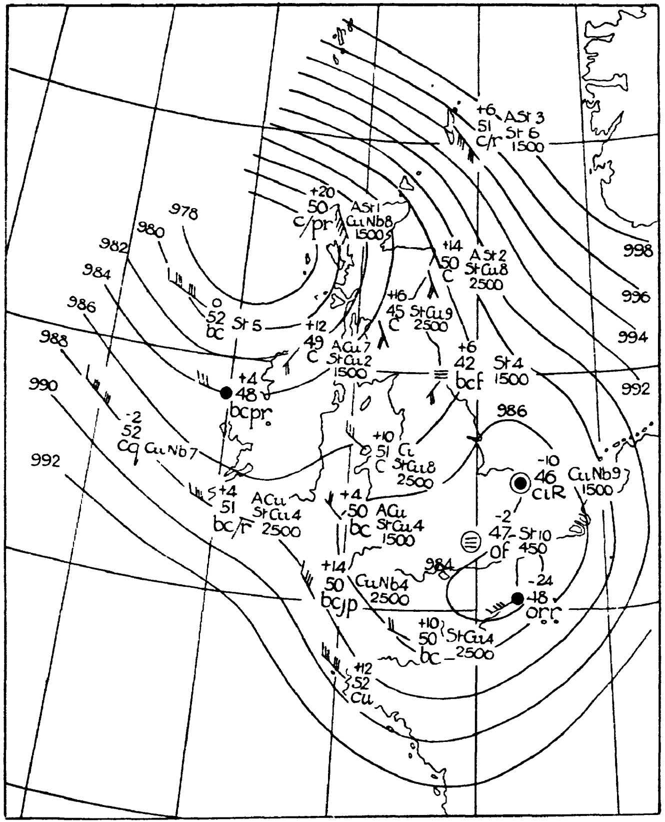


Fig. 87*c*. 7 h., October 9.

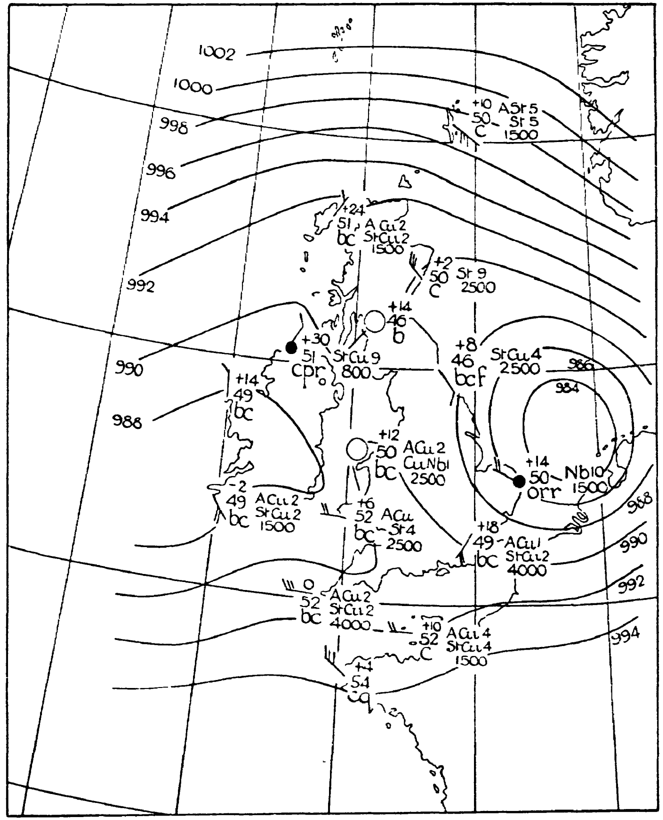


Fig. 87*d*. 18 h., October 9.

central region, and it may then go on growing in intensity until it has again become occluded.

Another method by which a depression may be revived is by the approach of a new stream of cold air from a distance. It has already been suggested in § 183 above that this can only be brought about with difficulty. The occluded depression being surrounded by a ring of solid current, it cannot in normal circumstances be penetrated by a fresh current. Any new development implies the breakdown of the solid rotation. This is likeliest to happen where the cyclic circulation is weakest, in the outer region of the depression. It has been suggested that when the front which marks the advance of cold air reaches the centre of the depression or near it, this front, together with the line of occlusion, can mark out a new warm sector which will furnish the necessary supply of potential energy to regenerate the depression. Examples have been worked out in detail by Schröder*. A particularly interesting example is furnished by case 6 in the "Life-history of Surface Air Currents", in which a depression moved up the west coasts of Ireland and Scotland, becoming occluded and stationary over the northern coast of Scotland. Later a cold current flowing across southern Scandinavia and the Baltic caused the depression to intensify and to move across the North Sea in a south-easterly direction.

There is some uncertainty as to what really happens in such cases. Maps at very short intervals might show the development of a new centre, which absorbs the old one, so giving the appearance of regeneration of the original centre. The original working charts drawn by Shaw and Lempfert for their case 6 mentioned above rather suggest the formation of a new centre over southern Scandinavia and the subsequent rotation of the two centres round one another, after the manner of the depression shown in fig. 87*c* and *d*. It is not suggested that regeneration of depressions never occurs, but special care is required in any effort to make practical use of this idea.

§ 186. *Some general aspects of polar front depressions*

The picture of the life-history of the typical depression, as presented by J. Bjerknes and Solberg, is of great value in that it unifies into one single picture a number of facts which had previously remained isolated. It is not to be questioned that cyclones can form at boundaries between warm and cold currents and that their motion follows the direction of motion of the warm current. The phenomena of occlusion are readily found on the synoptic chart, and the value of the polar front method of analysis of charts is of the greatest practical value in forecasting, particularly in detailed forecasting for such periods as 6 hours.

Difficulties begin to arise, however, when we try to draw a mental picture of the processes involved in the birth of the depression. The surface of discontinuity as depicted by Helmholtz is stable so long as the slope of the surface is adjusted to the velocities. It is true that, as Helmholtz pointed

* *Veroff. Leipzig*, Ser. II, 4, h. 2.

out, the mathematical surface of separation does not correspond precisely to the physical reality, which usually presents not a mathematical surface but a layer of transition from warm to cold air. Helmholtz* added without formal proof the suggestion that the layer of transition might be unstable, and might therefore be forced to ascend. But the action which Helmholtz visualised is only a very feeble one. It is clear that the ascent of the layer of transition is not in itself sufficient to produce the observed fall of pressure, and moreover there is not in general a tendency for the layer of transition to be attenuated as the depression deepens.

The mathematical treatment of the problem becomes extremely complicated, once the hypothesis of steady motion is abandoned, and vertical components are introduced into the equations of motion. The most plausible way of stating the case is perhaps to suppose that the originally straight polar front surface is buckled or corrugated through the action of some irregularities in the meteorological conditions, and that the corrugation forms a channel along which the warm air is conducted upwards. The horizontal circulation is generated as a result of the convergence of air to take the place of that which ascends, as described in Chapter XVI. Some evidence has been adduced by C. K. M. Douglas† to show that the centre of an incipient depression is a thermal centre. If this is once accepted, the further development of the depression will be facilitated by the distortion of the front by the circulation around the newly formed centre.

The Norwegian writers associate the development of a depression at the polar front with the evolution and breaking down of a wave in the front itself, though there is no necessary connection between the work of J. Bjerknes and others on the frontal methods of analysis and the theoretical work of V. Bjerknes. The evolution of a depression does not appear to be very closely analogous to the formation of a wave. In a wave there is an alternation of potential and kinetic energy. But the depression visualised by J. Bjerknes changes potential energy into kinetic energy, which is finally dissipated by turbulence, and reappears as thermal energy. The process is therefore irreversible.

The polar front depression shows an evolution from the non-symmetrical to the symmetrical, from lack of spin to something approaching pure rotation. It is likely that an approach to purely rotational motion occurs at a fairly early stage in the development shown in fig. 72, and this should be taken into account in drawing the fronts. For example, the back-bent occlusion of fig. 73 *c* will rotate round the depression and will suffer distortion on account of this motion. Recent discussions of depressions have tended to emphasise the frontal aspect and to neglect the rotational aspect. It should be remembered, however, that the ascent of air along a restricted portion of the front must produce some convergence towards the centre, leading to rotation about the centre, as was suggested in Chapter XVI. Any complete view of the depression

* *Ges. Abh.* **2**, p. 302.
† *Q.J. Roy. Met. Soc.* **50**, 1924, pp. 339–63; esp. p. 357.

must take account of the rotational as well as of the frontal aspect, and must take into consideration the tendency of the fronts to rotate about the centre.

No satisfactory explanation of the fact that depressions tend to become occluded has yet been put forward. If the whole system of fronts moved with the geostrophic wind velocity, there should be little tendency for the depression to occlude. In practice it is found that while the cold front advances at about the speed of the gradient wind (and occasionally faster than the gradient wind), the warm front moves slower than the gradient wind by an amount which on the average is 10 m.p.h., and which may be as much as 20 m.p.h. The advance of the cold front is not to any appreciable extent retarded by friction at the ground. It is true that the layers of cold air in immediate contact with the ground may be retarded more than the air at some distance above the ground, so that the cold front overhangs in the lowest layers. But the cold air at say 2000 feet is not affected by friction, and moves forward so as to overhang the warm air, producing an unstable arrangement which breaks down from time to time, giving squally winds. After such a breakdown the overhanging front begins to build up again, and the process may be repeated again and again, without holding up appreciably the advance of the effective cold front.

The warm front cannot be propagated in quite the same way, and the surface layers of air both in front of and behind the warm front may be retarded by surface friction. At higher levels surface friction can have no appreciable effect, and the front at say 2000 feet should advance freely so far as surface friction is concerned. The only way in which the retardation of the warm front can be explained kinematically is by the assumption that the cold air has an acceleration down the slope of the front, sufficient to produce the difference between the rate of advance observed and the gradient wind.

In an effort to discuss the occlusion of the warm sector along the lines suggested above, by considering separately the conditions at the warm and the cold front, we are at a disadvantage in not discussing the depression as a whole. The fact that a depression forms at a front, and continues to deepen, one of the features of the deepening being the occlusion, indicates some deep-seated instability in the whole system. No complete explanation of the occlusion of the depression is likely to be evolved until the nature of this instability has been found. Some further evidence bearing on this point is adduced later in connection with the discussion of rainfall in depressions (see § 190).

§ 187. *The drawing of fronts*

A front as drawn on a synoptic chart should separate masses of air which have some definite differences in physical characteristics of temperature, humidity, transparency, or motion. Some or all of these characteristics may differ for the two masses. In actual synoptic practice only fronts defined with reference to differences of temperature have been formally defined and are in general use, the other factors mentioned being treated as subsidiary aids in defining the position of the fronts. It was pointed out in Chapter x above that a surface of discontinuity between two air masses in steady motion must intersect the surface either at a trough of low pressure, or in a line having stronger gradients of pressure on the high-pressure side than on the low-pressure side. When the surface of discontinuity is not parallel to the isobars but cuts across them at a finite angle, it is clear that there must be a sharp bend in the isobars where they cross a surface of real discontinuity. If the surface of discontinuity is replaced by a zone of transition, the sharp refraction of the isobars is smoothed out; and if the zone of transition is wide, the isobars at the zone may be rounded as were the isobars on the charts drawn in "pre-frontal" times. But granted the presence of a clear-cut surface of discontinuity, the element which should most clearly define its position is obviously pressure, provided observations are available from a close network of stations. The isobars will distinguish the sharp discontinuity from the wide frontal zone by the sharp bend in the former, as compared with the rounded form appropriate to the latter.

Surface temperatures are notoriously unreliable for tracing fronts. Temperatures at say 2000 feet would afford a much more satisfactory criterion, but such temperatures are not usually available. In actual practice the forecaster has to rely very largely on indirect evidence. Over the Atlantic observations of wind are of value in fixing fronts, and over the western coasts of the British Isles the barometer tendency is also frequently useful. The tendency usually increases with the approach of a cold front, and after the passage of the front may either be steady, or show a rise or a slower fall. The pressure usually rises after the passage of a cold front, and a positive tendency may indicate that a front has passed the station, though this is by no means universally true, and a rise of pressure may occur in tropical air, being due to the advance of an anticyclone or a general surge of pressure. Dew-points are also of use in fixing fronts, though they are apt to mislead if deduced from observations when rain is falling. Even when rain is not falling, water-vapour content is not a very conservative property. Willett* has stated that when northerly currents in winter pass over the Gulf of Mexico the water content at the surface may rise from 1 to 15 grammes per kilogramme of dry air in 36 to 48 hours, the corresponding dew-points being 0° F and 68° F. Other useful factors are the types and heights of clouds, and the presence of precipitation. The evidence in favour of the association of rain and fronts is so overwhelming that if it is known from other evidence that warm and cold air masses are in juxtaposition in a certain region, it is legitimate to connect with this fact any rain which falls in that region. Considerable caution is necessary however in making such use of precipitation alone to define a front, since the rain often runs ahead of the front as shown on a mean sea level chart, and also since a few casual showers may appear in line on the chart without being due to an occlusion.

It will be gathered then that the drawing of fronts on a synoptic chart cannot be regarded as a purely impersonal scientific operation, unless a close net of pressure observations is available. The use of indirect methods cannot

* "American Air-Mass Properties", *Mass. Inst. Tech.*, *Met. Papers*, **1**, No. 4 (esp. p. 4).

be wholly impersonal, since each individual will have his own system of weighting the indirect factors he uses. As a result of this, the drawing of fronts is full of pitfalls for the unwary. It is all too easy to draw fronts over the Atlantic representing depressions as having warm sectors with sharply bent isobars in regions where no observations are available, and where rounded isobars are equally likely. Next to this in attractiveness is the drawing of occlusions in regions where no evidence of the previous existence of a warm sector is available. A genuine occlusion as defined by J. Bjerknes and H. Solberg originates in the passing away of a warm sector, and though an occlusion may have steady rain associated with it for some time after the occlusion appears it does not of necessity follow that any three rain-dots in line on the synoptic chart should be joined by an occlusion or other form of front. The criticisms which can be brought against the methods based on fronts and air masses are mainly due to the complexity of the atmosphere. J. Bjerknes and others have shown that cases occur in which large masses of air may be treated as homogeneous, but one finds, not infrequently, that the air over a limited region cannot be treated as a single mass, but can more appropriately be described as a patchwork of masses of air of different life-history. In such a case the practical meteorologist has to effect a compromise between the complexity of the actual atmosphere and the simplicity which is desirable in any representation which shall be of practical use in the analysis of the weather.

The difficulties in the way of drawing fronts are confirmed by an examination of the charts showing fronts issued by different European meteorological services. The differences are astonishing, not merely in the details of the fronts, but in their main outline, and the forms of isobars based on the same observations frequently show wide variations.

Frontal analysis is closely related to the movements of air as shown by trajectories. A skilful use of fronts and trajectories together cannot fail to assist the forecaster in understanding the processes which are taking place. The difficulty in the way of the general use of these two methods jointly is that there is no absolutely certain method of drawing a trajectory of air at 2000 feet, the gradient wind being often unreliable. Such a trajectory, when it can be drawn, represents the motion of a far greater mass of air than does the surface trajectory. If the trajectory of the warm air apparently crosses a well-defined front, it can be concluded that the warm air has certainly gone up. It is therefore advisable to check the trajectory at different stages with the fronts.

§ 188. *Sharp and diffuse fronts*

The records of temperature reproduced above in figs. 78, 82, 83 and 84 indicate that very wide differences in the sharpness of fronts are to be seen, and it is of importance to obtain some indication of the causes which underlie this variability. We shall consider some of these causes in turn.

(*a*) Subsidence

When a mass of air subsides, i.e. descends and spreads out laterally, it is dynamically heated and its lapse-rate is also modified. In § 24, p. 46 above, we showed that if a mass of air at a mean pressure p covers a horizontal area S, then the difference between its lapse-rate and the dry adiabatic is proportional to Sp. Thus when air subsides the difference between its lapse-rate and the dry adiabatic increases, its initial stability or instability being increased. Further, since in the absence of rain its absolute humidity cannot change, the relative humidity decreases with increasing temperature. The occurrence of very low relative humidity may usually be taken as an indication that the air has subsided. The reverse of this is not true, and all air which has recently subsided will not of necessity have low relative humidity. Rain falling from a higher level and partially or wholly evaporating in the subsiding air will maintain high relative humidity and will lower the temperature of the subsiding air both by evaporation and by direct conduction. In such cases the wet-bulb potential temperature will be a more useful criterion of the origin of the air (see § 50), though this is not entirely satisfactory, being vitiated by the effect of cooling by direct conduction from the air to the raindrops. Subsidence normally occurs in cold air, and may occur either at a warm or a cold front. If the cold air adjacent to the front subsides, it has at the surface a higher temperature than the original cold air. J. Bjerknes* suggests that this results in the front originally separating the warm and cold masses being replaced by two fronts, one separating the undescended cold air from the cold air which has subsided and another separating the cold air which has subsided from the warm air. The contrast of temperature may in such a case be concentrated at the first of these two surfaces. The air in the transitional zone will show marked dryness if no rain has fallen through it. The double front may at a later stage develop into a single front through the ascent of the air warmed by subsidence, the ascent involving convergence and precipitation.

Bergeron has suggested the name *frontolysis* to denote the smoothing out of a front from discontinuity to continuity, and the name *frontogenesis* to denote the reverse process.

The subsidence of air behind a cold front was very clearly described by M. A. Giblett in a note on "Upper Air conditions after a line squall" in *Nature*, **112**, 1923, p. 863. In this case, two aeroplane ascents were made, one immediately before the arrival of the cold front, and one some hours after the cold front had passed, and the results are plotted in fig. 88. Between the two ascents, the temperature fell at all heights below 3 km. In the second ascent, the cold air showed an adiabatic lapse-rate up to about 1·3 km, while between this level and 3 km conditions were approximately isothermal, with extreme dryness. The air between 1·3 km and 3 km can only be explained as air descended from the upper part of the cold wedge.

The phenomenon of descent of cold air at the cold front is of very frequent

* M.O., *Geophys. Mem.* No. 50.

occurrence. In the autographic records it should, according to J. Bjerknes, show itself as a duplication of the cold front, the transitional air being usually drier than either the warm or cold currents which have not descended. But as noted above extreme dryness will not always occur as the evaporation of falling raindrops may lead to high humidity. A good example of a double cold

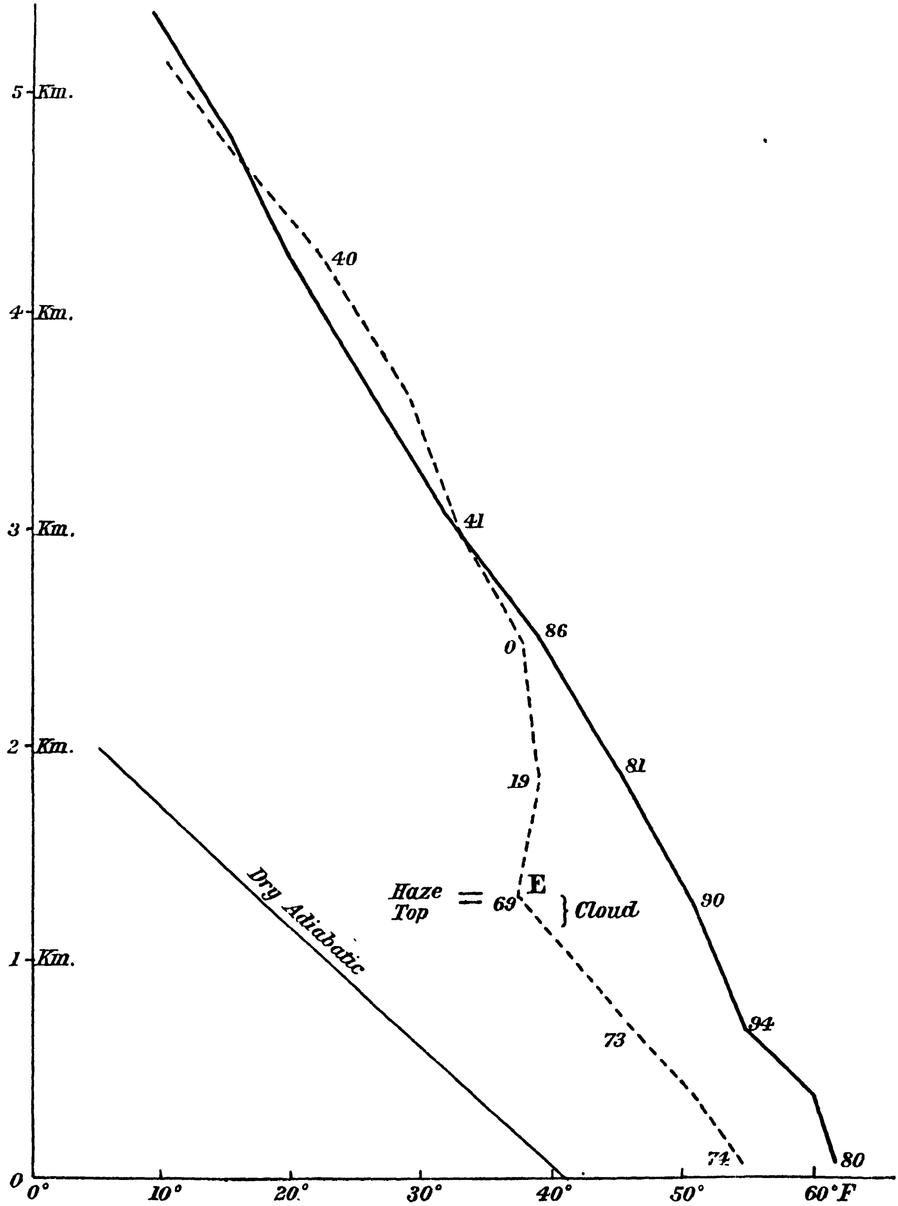


Fig. 88. The effects of subsidence; upper air conditions behind a line squall. The numbers shown against the curves indicate relative humidity.

front over England on September 1, 1925, has been described by R. S. Read in the *Q.J. Roy. Met. Soc.* for October 1925. In this case, as in that cited by Bjerknes, the rain fell mainly at the first cold front, and the humidities in the air behind this were still high, though rather lower than in the warm air.

It is by no means established that subsidence is the cause, or even the main cause, of double fronts. Some of the factors enumerated below, notably (*e*) and (*f*), may be of considerable importance. J. Bjerknes (*loc. cit.*) states that air which has subsided near the cold front "may have been heated by descending, but kept wet by the rain". This is not consistent with the known effects of the evaporation of water into air. If the rain falls continuously through the subsiding air, the latter will descend along a saturated adiabatic, retaining its wet-bulb temperature unchanged. The lapse-rate within an appreciable mass will then vary much less than when the air is dry. If the air first subsides, and afterwards is cooled and wetted by rain falling through it, then, apart from effects due to turbulence, the result will be precisely the same as before, since the wet-bulb potential temperature of each element of air will remain unchanged, and the whole mass should approach saturation.

(*b*) Inclination of the front to the isobars

It was shown in Chapter XIII, p. 261, equation (121), that friction at the ground causes a drift of air across the isobars, and that if a front is inclined at an angle β to the isobars, the frictional drift of air across the front, which is assumed to move with the geostrophic velocity, is proportional to $\cos(\alpha-\beta)$, α being the angle between the surface wind and the isobars. The frictional drift will help to keep the warm air up close to the front, and will tend to keep the front sharply defined, particularly if the angle between the front and the isobars is greater in the cold than in the warm air. The effect is greatest when $\beta=\alpha$, and diminishes as β increases. When the angle between the front and the isobars is large, the effect of surface friction in checking the motion of the warm air may cause the slope of the front to become extremely small, and turbulence will then tend to smooth out the front more effectively. Thus fast-moving warm fronts which lie across the isobars at a considerable angle will usually be more diffuse than those which lie nearly along the isobars. This is illustrated in figs. 81 and 84, which show that the warm front was sharp at Eskdalemuir, where the angle between it and the isobars was small; while at Holyhead, where the angle between the front and the isobars was large, the front was less sharp.

In the case of a cold front it is less important that the surface air should keep up with the front, since if the cold air comes in first at the 600 metres level it will normally descend owing to thermal instability. Thus a cold front should normally be sharper than a warm front, except when subsidence from considerable heights occurs, in which case the cold front may become smoothed out as discussed in (*a*) above.

(c) Precipitation

Reference has already been made to the effect of precipitation on subsiding air. The cooling effect of rain and snow, both by evaporation and direct conductive cooling, may be of very considerable significance in the formation or accentuation of fronts. In an air-mass which is reasonably homogeneous initially it is possible for precipitation to produce large local differences of temperature. An example was shown over the British Isles on May 9, 1932. Croydon in the rain area had at 13h a temperature of 40° F, while Cranwell, in the same air-mass but outside the rain area, had a temperature of 52° F. Since at the warm front the rain falls through the cold air, which is usually relatively dry, the effect of cooling by the rain is to enhance the difference of temperature. At cold fronts the rain falls to a greater extent through the warm air, and the cooling by rain extends to the warm air as well as the cold air. As a result, the cold front may become considerably more diffuse when judged by temperature records than when judged by wind records. Cases in which the warm air before a cold front is cooled by rain occur mainly in summer, when the lapse-rate is steep, and the air dry before the rain starts. It will be recalled that the limit to which the temperature can be lowered by evaporation is set by the initial wet-bulb temperature (see § 47).

The contrasts of temperature at the front in fast-moving line squalls is sometimes diminished in this manner. In such systems there is frequently some rain before the front, a brief heavy shower at the arrival of the cold air, and then a clearance. A wider belt of rain behind the cold front occurs more frequently with slow-moving fronts, though fronts show considerable individual variation. Fronts which show contrasts of temperature as great at the ground as in the free air can probably be explained in part by the accentuating effect of rainfall, and some fronts in polar air may possibly be explained as the effects of rainfall.

(d) Molecular and eddy diffusion

If initially a perfectly sharp mathematical surface of discontinuity existed between warm and cold masses, the effect of molecular diffusion would be to cause the front to become less clear cut, though molecular diffusion is so ineffective that the result would be inappreciable. In the atmosphere the effects of wave motion in the front, combined with the diffusing power of eddies formed in the warm and cold currents, will lead to the formation of a zone of transition. The computations made in Chapter XIII, on the height to which turbulence can extend its influence in a given time, show (see p. 223) that within a time of 6 hours the zone of transition may become as much as $\frac{1}{2}$ km in depth, and therefore some 50 km in horizontal extent. Reference was made in § 152 above to the demonstration by Douglas that there is an upper limit to the angle of slope of the front, beyond which turbulence increases rapidly, and tends to produce a wide zone of turbulent mixing instead of a sharp discontinuity. The growth of turbulent mixing is likely to be rapid if the air-masses are not homogeneous. There is usually a component of velocity parallel to the front, which results in variations of acceleration normal to the front if the masses are not homogeneous.

(*e*) Variation of wind with height

One effect of the variation of wind with height is to cause the warm front at say 2000 feet to be carried ahead of the front defined by surface temperatures, which is fixed by the air which lags behind on account of surface friction. The slope of the front near the surface is then extremely small, and the rain is associated with the steeper part of the front. In such a system it will be the front at some such height as 2000 feet which will be associated with the normal frontal changes in tendencies, with the cessation of rain, and with the trough indicated by the barograph. The passage of the front indicated by the surface observations may also produce a feeble trough on the barograph. The conditions near the surface will favour turbulent mixing, and the sharp front will tend to be replaced by a zone of transition. Variations of wind with height above the level of say 2000 feet may introduce additional complexity, and will promote turbulent mixing. It is thus clear that at a warm front the conditions may be extremely complex. There is no accepted method of fixing the front, though most meteorologists appear to accept the rain as the best criterion for its position. As already pointed out in § 187, the use of the rain for this purpose may involve arguing in a circle, unless there is independent evidence for the existence of a front.

The normal increase of wind with height from the surface up to say 500 metres causes the cold air at a cold front to overrun the front at the surface, giving the cold wedge an overhanging nose. In one or two cases the backward slope between the warm surface air and the overhanging cold air has been directly observed. The system so postulated is unstable, and the cold air must tend to fall to the surface, leading to some turbulent mixing, and giving two fronts separated by an intermediate zone. The form of cold wedge with a nose at say 500–600 metres above the ground might account for the rain frequently observed in advance of a cold front, but it cannot be said that any adequate explanation of the latter has yet been found.

(*f*) Variations of temperature within the warm or cold air

When there are conditions favourable to the warming or cooling of the warm or cold air or both, frontal phenomena may be complicated. The case which suggests itself is that of the warming of a shallow mass of cold air by passage over warm land or sea. The difference of temperature between adjacent warm and cold masses may then be practically annihilated, so that the existing front becomes so changed as to be undetectable from surface temperature records alone. The night cooling of air by ground cooled by radiation, depending as it does on the cloud amount, leads to distributions of temperature at the surface which may simulate fronts, though the phenomena may be purely superficial.

The absence of large variations of absolute humidity is probably the best check in such cases. But it is often difficult, on the basis of observations at 7h, to analyse with certainty the air-masses shown on the chart.

The factors enumerated above possibly do not exhaust the list of factors which may affect the form and sharpness of fronts, but they suffice to show that each front has its own special features, and that general principles are not easily reached. So far no dynamical discussion of the factors which affect the sharpness of fronts is available, and the arguments given above are of necessity in very general terms.

§ 189. *Upper air conditions above depressions*

Sharply defined surfaces of subsidence are often observed in the polar air within depressions, but the inversions at these surfaces are usually small by comparison with the differences between air-masses.

Observations show that in the upper air over depressions sharp surfaces of discontinuity between air masses of different origin are rare, and that the nearest approach to sharpness is a discontinuity smoothed through a range of 0·5 to 2 km of height, which can frequently be traced to heights of 6 km. Probably the commonest occurrence is a wide frontal zone of transition, which is not of necessity to be explained as a degenerate front, but is in all probability the normal form of transition from warm to cold air at higher levels. When therefore we refer to fronts in the upper air, it is to be understood that these fronts are not of necessity visualised as sharp discontinuities. Such fronts may have a very important bearing on rainfall, and on the development of depressions. An occlusion with dissimilar polar currents on the two sides comes into the category of upper air fronts.

In discussing the very important question of the three-dimensional structure of depressions, we must be on our guard against certain vagaries which in practice complicate the subject. A well-marked front will cut the surface at a trough of low pressure, or, less frequently, parallel to the isobars, with higher gradient of pressure on the side of high pressure than on the side of low pressure. Some meteorologists appear to regard the converse proposition as true, that every trough is to be interpreted as associated with a front, even when there is no observational evidence of any difference between the currents on the two sides. Such cases could equally well be interpreted as rotary systems carried round the main centre in the circulation of the main depression. It is by no means uncommon for a dying primary depression to swing round a newer and more active centre (cf. fig. 87*c*, *d*). The phenomena then observed at all heights indicate approximate symmetry about the old centre, though on the isobaric chart the dying depression may appear as a trough of low pressure. Cases also occur where a vortex in the polar current causes distortion of a front, with subsequent accentuation of the depression at the centre. There are also occasions when a development on the front itself superposes some vorticity on the polar current near it, the result appearing as a rounded trough. There is always some tendency for rain to occur at a trough, on account of frictional convergence, and some fronts drawn across the isobars are troughs of this type, in which the effects of rainfall may simulate closely the phenomena at a true front.

Douglas* showed that in well-marked currents, leaving anticyclones out of account, the temperature conditions in the upper air depended very largely on the eventual origin of the currents, going backwards not just a few hours but for 3 days or even a week. The variables which showed a close correlation with the southward displacement in the previous three days included the pressure at 9 km, the height of the base of the stratosphere, the temperature at the base of the stratosphere, as well as the temperature within the troposphere. Douglas's results suggest that polar currents extend to well within the stratosphere, and that they therefore bring low temperatures at all heights within the troposphere, and carry with them the low and warm stratosphere of high latitudes, while genuine tropical currents bring high temperatures in the troposphere, and carry with them the high and cold stratosphere of low latitudes. This accords well with the results derived by Gold†, who found that the stratosphere was high and cold over the southerly current in front of a depression, and low and warm over the northerly current in the rear of a depression. These investigations show that when currents of air of widely different origin are brought into juxtaposition, steep temperature gradients will occur in the upper air at all heights within the troposphere.

Above the surface of discontinuity a large increase of wind is a regular feature. Cirrus velocities range up to 150 m.p.h., and frequently exceed 100 m.p.h. in winter, and occasionally do so in summer. This accords with the effects to be anticipated from the temperature gradients in the neighbourhood of the front (see Chapter XI).

The trough of lowest pressure is farther west in the upper air than at the surface. When a depression moves eastward the cirrus may continue to move rapidly from a southerly direction for some time after the wind at 15,000 feet has veered and fallen off in velocity. Also, after the passage of a cold front the tops of the showers attain heights of 20,000 feet or more. These observational facts, together with the correlation coefficients derived by Douglas (*loc. cit.*), indicate that the effects of cold air extend beyond 20,000 feet.

The motion at the cirrus level above a depression has been given in diagrammatic form by several writers. Fig. 89 reproduces the diagrams of Douglas‡. Fig. 89*a* represents the conditions in a depression with a large warm sector, and fig. 89*b* represents a depression well advanced towards the dying stage. The first diagram indicates that in a young depression it is only in the warm sector and the central region that the cirrus motion is in the direction of motion of

* *Q.J. Roy. Met. Soc.* **51**, 1925, p. 229. † M.O., *Geophys. Mem.* No. 5.
‡ *Q.J. Roy. Met. Soc.* **48**, 1922, p. 342.

the depression. At cirrus levels the thermal wind defined in Chapter XI is so large that the cirrus motion over the polar air near the front tends to be parallel to the front. This tendency is perhaps the most marked feature of fig. 89. Some depressions have in their initial stages a warm sector extending through 180°, and having the front running as an approximately straight line through the centre. In such cases there is across the central region of the depression a current of cirrus parallel to the front, and in the direction of motion of the centre.

Douglas's diagrams show that the structure of the depression changes even at the cirrus level as the depression progresses towards occlusion, and that no single diagram can be drawn to represent the whole circulation in a depression. The detailed structure probably varies from one depression to another, even at comparable stages of development. Douglas indeed hints in one of his papers that depressions only have one completely common feature, a centre of low pressure with a counter-clockwise circulation of winds.

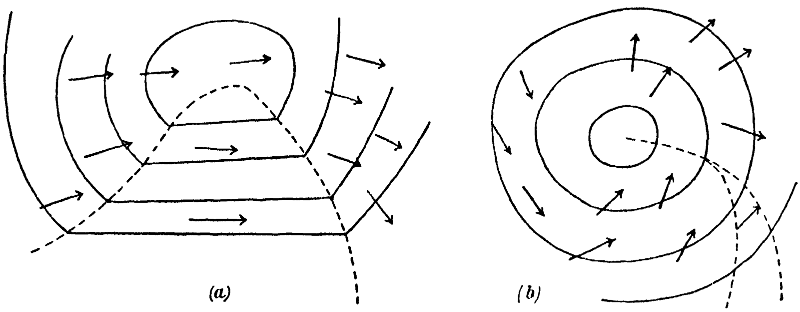


Fig. 89. Motion at cirrus levels above depressions.

The diagrams of fig. 89*a* and *b* above, together with the diagrams of the surface circulation already given in figs. 72 and 76, yield a picture of the three-dimensional structure of the typical depression. The motion at the level of the alto-clouds is intermediate between the surface and the cirrus motions, but nearer to the latter as a rule. In the early stages of the development of the depression there may be no closed cyclonic isobars at the cirrus level, but closed isobars show up in the later stages, the lowest pressure occurring to North-west of the surface centre. As a result, it is only far behind the surface centre of low pressure in a depression having a warm sector that the wind at cirrus levels veers to NW. In fig. 89 no NW current is shown in the rear of the depression at cirrus levels, since its inclusion would give a misleading impression of a net flow toward South. The displacement to West and North-west of upper air features, relative to the surface features, leads to one of the most troublesome features of forecasting. It has the result that when a depression moves eastward, an observer at a fixed point observes changes in the lower troposphere before those in the upper troposphere or the stratosphere. Thus upper air observations do not in general give an earlier indication than surface observations of changes which are taking place; in fact the surface indications come first. The lag between changes at the ground and at the cirrus level is very clearly shown when a wedge of high pressure moving eastward is followed by a new depression. The upper wind will then in general continue from North-west for some time after the surface wind has backed to South-west. Thus apart from the information which they yield concerning the stability or instability of air masses, and the consequent chance of rain or thunderstorms, observations of upper winds and temperatures have been far less useful to the forecaster than was anticipated some 15 years ago. Such observations are more valuable for the subsequent investigation of depressions and other phenomena than for use in day to day forecasting of changes over periods of a day or more.

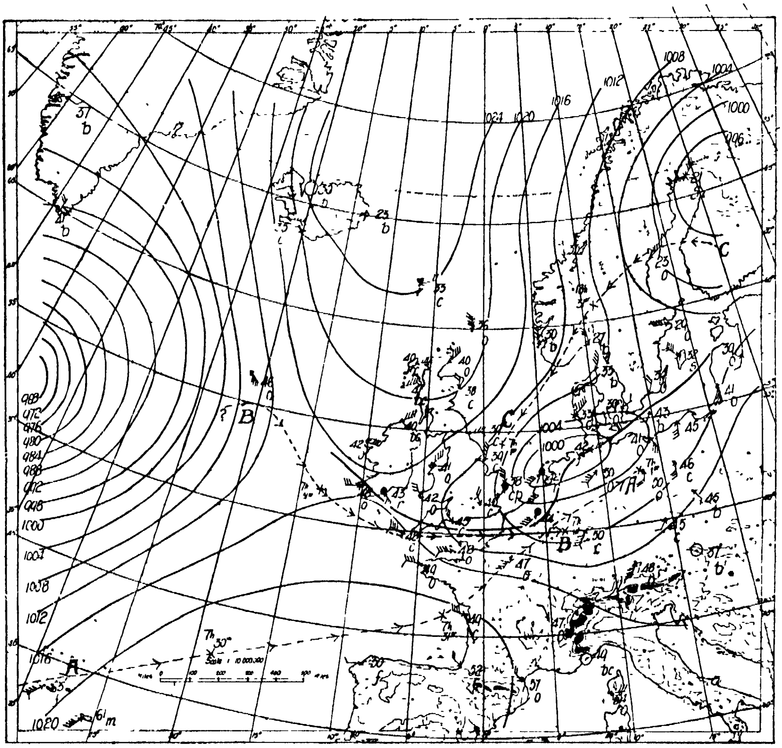


Fig. 90. A depression with a marked equatorial sector, April 1, 1909.

It is usually difficult to obtain observations of wind or temperature in the central region of a depression, and on account of the relatively short duration of the phase in which a depression has a well-marked warm sector it is not surprising that so far little information relating to warm sectors has been obtained by sounding balloons. The best method available for studying warm sectors is the examination of continuous records from high mountain observatories. The observatory on the Pic du Midi in the Pyrenees, at a height of 2859 metres, lies near the Atlantic, but south of the main cyclone tracks, and only relatively few depressions affect its pressure records, though its temperature records show fairly frequently large fluctuations associated with the passage of depressions across the British Isles. Douglas* collected data for 18 shallow winter depressions during the years 1895–1914 which had well-defined warm sectors and whose centres passed within 300 miles of the Pic du Midi. When the centre was nearest to the Pic du Midi the temperature was on the average 1° C above normal (after allowing for diurnal and seasonal variations), and apparently showed remarkable uniformity in the different cases. At the extreme boundary of the system, the temperature at the Pic du Midi averaged 5° C below normal, though occasionally reaching 10° C below normal; the highest temperature was a little in advance of the centre; and after the passage of the centre a pronounced fall of temperature, averaging 7° C, was shown. These figures give a picture of the conditions at 3 km in a shallow depression having a warm sector. Douglas quotes in the same paper another interesting case. On April 1, 1909, a small depression centred over the northern coast of Germany (fig. 90) showed a marked equatorial sector. Gradient wind trajectories showed that the air in the warm sector had come from south of latitude 40°, passing across the southern part of the Bay of Biscay. The air in the northerly current in the rear of the depression had come from the Northern Baltic, while the air in the region just in advance of the cold sector had come from the Atlantic, from about latitude 55° N, longitude 20° W, with doubtful earlier origin, and had swept over the English Channel. Upper air soundings were made in these three currents at Berlin, Pyrton Hill and Paris, respectively. These are shown in fig. 91. At Berlin, in the equatorial current, the temperature was above the April normal, and was much higher than at Pyrton Hill at all levels within the troposphere, the difference amounting to 18° C at 6–8 km, but at Berlin the tropopause was higher and the stratosphere colder than at Pyrton Hill. On April 2 a kite ascent at Berlin showed that the temperature at 3 km, now behind the centre, had fallen 11° C, but no observations could be made to appreciable heights in the intervening

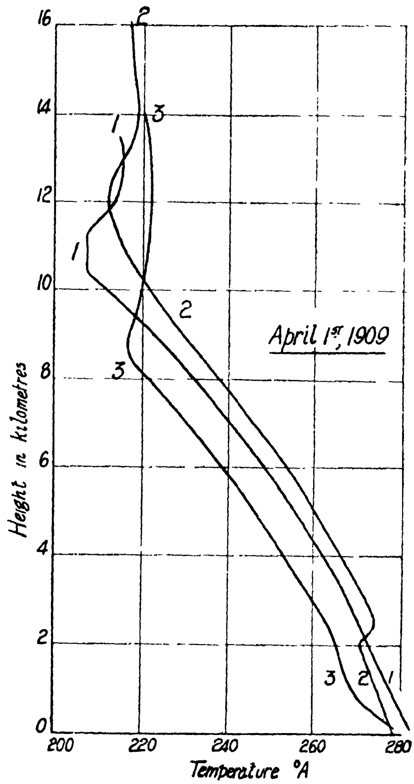


Fig. 91. Upper air observations on April 1, 1909. 1. Berlin, 8 h. 2. Paris, 4 h. 3. Pyrton Hill, 17 h.

* *Q.J. Roy. Met. Soc.* **50**, 1924, p. 339.

period. The records reproduced in fig. 91 show how great the difference can be between air masses of different origin brought into juxtaposition, though the surface temperatures may differ only slightly. They also demonstrate in a striking manner the fact that the warm sector is warmer than the surrounding polar air at all levels within the troposphere. Confirmation of this will be found scattered through very many papers by Douglas and others.

§ 190. *Rainfall in depressions*

The fact that the presence of a front is not by itself sufficient to produce rainfall was emphasised by Brunt and Douglas*, who also pointed out that in a depression in which the winds were everywhere geostrophic there could be no appreciable rainfall, since no convergence of winds is possible so long as the geostrophic condition is satisfied.

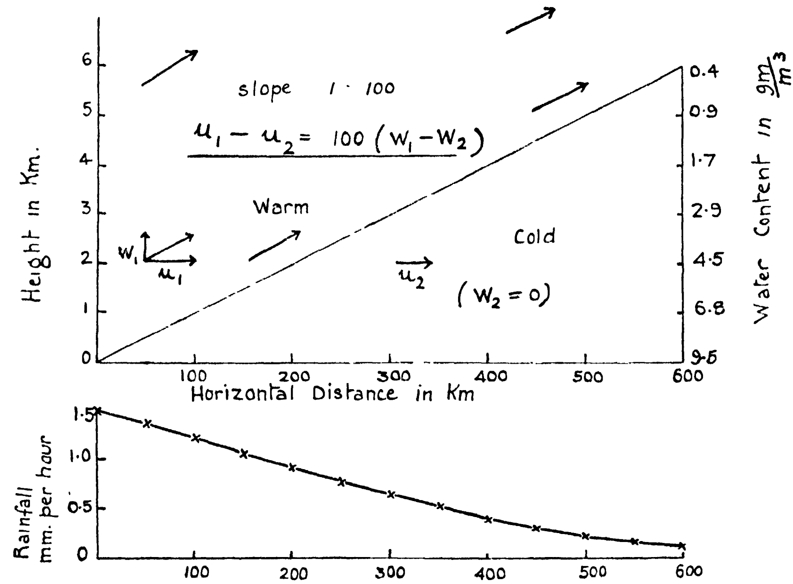


Fig. 92. Rainfall at a front.

At a sharp front where the warm air rises up over the cold air, the rainfall can be computed for any assumed values of the slope, of the differences in horizontal velocities on the two sides of the surface, and of the temperature and humidity of the warm air. Let the components of horizontal velocity at right angles to the front be u_1 and u_2 in the warm and cold air respectively, and let w_1 and w_2 be the corresponding vertical velocities. We assume $w_2 = 0$. If ψ is the angle of the slope, then from equation (18), Chapter X, p. 197, taking $u_1 - u_2 = 4$ m/sec, $\tan\psi = 0{\cdot}01$, and accordingly $w_1 = 0{\cdot}04$ m/sec, the rate of rainfall at different distances from the front is shown in the lower part of fig. 92 for conditions shown in the upper part.

It is assumed that all the condensed water falls as rain. The warm air is

* *Mem. R. Met. Soc.* **3**, No. 22.

assumed to be saturated and in convective equilibrium at all heights, so that the maximum rainfall must occur at the front itself. If the warm air fell short of saturation at all heights, there would be a narrow rainless band in advance of the front. Both cases are to be found in practice, as well as intermediate cases in which the air is saturated at some levels and unsaturated at others. A layer of about 6 km is assumed to take part in the upward motion.

It was pointed out in Chapter x that the winds in the warm and cold masses would have slightly differing components at right angles to the front. Let A and B be two points on a front, and let p_A, p_B be the corresponding pressures. The component of pressure gradient along the front is $(p_A - p_B)/AB$. For a discontinuity of 3° C the density is 1 per cent less in the warm air than in the cold air, and if the winds are geostrophic $u_1 - u_2 = 0{\cdot}01 u_2$. Let u_2 be 20 m/sec, and take the other variables as in the preceding computation. The resulting rainfall is now less than 0·1 mm per hour, a value considerably less than is observed at actual fronts. The mere presence of a front, acting through the density effect, is thus not capable of producing heavy rain, though it may produce cloud and slight rain in advance of a warm front. At a cold front it can only produce slight subsidence.

When a system of V-shaped isobars moves across without change of form, there is an inflow of warm air across the front as a result of surface friction. It has been shown in § 154 above that when the geostrophic wind velocity is 10 m/sec, $\alpha = 22\frac{1}{2}°$, and $K = 10^5$, the total frictional inflow across a front inclined at an angle β to the isobars is about 15 cos $(\alpha - \beta)$ m^3 per second per cm of the front. If the slope of the front is 1 in 100, and the initial state of the air is as shown in fig. 92 above, then it is seen from that figure that the air in rising through $\frac{1}{2}$ km loses 1·5 grammes of water per cubic metre, so that 15 m^3 lose 22 grammes of water in so ascending. The air moves 50 km in a horizontal direction, and the 22 grammes of water per cm of front is distributed over a strip $\frac{1}{2}$ cm wide and 50 km long. The amount of rain per cm^2 per sec is thus $4 \,.\, 10^{-6}$ cc, which is equivalent to 0·14 mm per hour. Such rain would be very light drizzle, and even in the most favourable circumstances could not yield rainfall in amounts comparable with that due to isallobaric convergence, discussed below. The rainfall which has been computed above is moreover the maximum which corresponds to $\alpha - \beta = 0$, or corresponding to a front parallel to the surface wind. When the angle between the front and the isobars is large, the rainfall is diminished in proportion to cos $(\alpha - \beta)$.

The above computation assumes that the warm air ascends a uniform slope. There is definite observational evidence that the form of cold fronts is as shown in fig. 93, the trough line being below the nose of the cold air. If the warm air ascends rapidly from the ground to the height of the nose, which may possibly be as high as 500 metres above the ground, the amount of rainfall which we have assumed to be distributed over a belt 50 km long will be con-

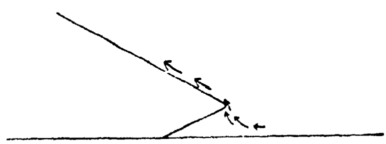


Fig. 93. Form of the cold front and its effect on rainfall.

centrated, giving a narrow belt of heavy rain along the trough, with a wider belt of light drizzle behind it.

At an earlier stage it was pointed out that in a system of geostrophic winds no rain can occur. We can now supplement this by saying that the deviations from geostrophic winds in the surface layers, due to surface friction, cannot produce more than light rain at a front. Heavy rain in depressions must therefore be produced by deviations from geostrophic winds other than those due to surface friction.

The work of Brunt and Douglas quoted in § 112 indicates that in an isallobaric low converging winds having a component of 5 m/sec across the isallobars may occur. The same writers gave an estimate of the amount of rainfall produced by convergence of this magnitude over a circular area of radius 200 km, the convergence being assumed to extend up to 3 km. The total inflow of air across the cylinder, assuming an inward velocity of 5 m/sec, or 0·005 km/sec, is

$$2\pi \times 3 \times 200 \times 0{\cdot}005 \text{ km}^3 \quad \text{or} \quad 6\pi \text{ km}^3.$$

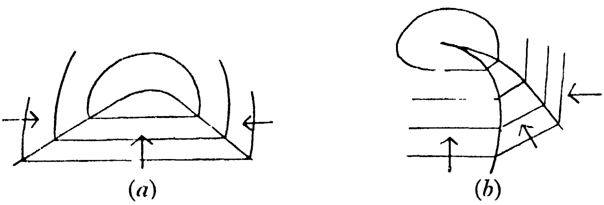


Fig. 94. Isallobars in a deepening depression.

Assuming a mean water content of 6 grammes per m^3 (see fig. 92), we find that the air brings in $6\pi \times 6 \times 10^9$ grammes of water per second; and if it is assumed that this is all deposited in the form of rain over an area of $\pi\ (200)^2\ \text{km}^2$ which forms the base of the cylinder, the rate of rainfall is 10^{-4} cc per cm^2 per second, or 10^{-3} mm per second, or 4 mm of rain per hour. This is the right order of magnitude for fairly heavy rain, and since the amount of inflow into an area is proportional to its circumference, while the resulting rain is distributed over the whole area, it follows that with a given rate of convergence the rainfall is inversely proportional to the radius of the area over which the convergence takes place.

The computations which have been briefly summarised above show that heavy rainfall can only be produced by deviations from geostrophic winds leading to convergence. Convergence of the isallobaric components of winds (see § 112 above) at a trough is produced when the gradient is steepening or when a trough is becoming more elongated, and it is well known that in such cases the rainfall is heavy. The argument used above must be applied with caution to such cases, since the angle of slope of the surface of discontinuity is likely to vary.

When a depression is growing deeper, with the gradient of pressure everywhere increasing, the isallobaric component is everywhere at right angles to the isobars, and in fig. 94 (*a*) and (*b*) are shown the isallobaric components in different parts of a deepening depression at two stages of its existence. In stage (*a*) the convergence is great at both warm and cold fronts, causing the warm sector to shrink, with heavy rainfall at both fronts. In stage (*b*) there is little convergence at the cold front, and there is little rain at the cold front at this stage, the frictional rain discussed in an earlier paragraph being then also small, on account of the large angle between the front and the isobars. At the warm front the convergence may still be considerable. The isallobaric component frequently exceeds 5 m/sec in the warm sector and in the cold air in advance of the warm front. The isallobaric effect at the latter (see fig. 94) manifests itself in the fact that the rate of advance of the warm front is usually less than the geostrophic wind; the advance of the cold front is accelerated by the isallobaric component in the early stages of development, as shown in fig. 94 (*a*), but this effect diminishes with time, and by the time the stage of development shown in fig. 94 (*b*) is attained this effect has practically disappeared.

The use of isallobars requires a word of caution. In so far as the isallobars are due to development and not to the drift in a general current, the argument used above is sound; but it may require emphasising that a depression always formed of the same mass of revolving fluid and carried in a general current would produce a system of isallobars, though there would be no convergence or divergence within it. Unfortunately it has not been possible to separate out the parts of the isallobaric distribution due to motion from that due to development.

It is assumed in most of the arguments used above that the warm air is damp at all heights. If the warm air is dry, convergence may fail to produce heavy rainfall. After a spell of fine weather it is frequently noted that rain does not readily occur in the subsequent depression, and this is to be explained by the depression being fed with warm dry air at heights usually about 5000 feet.

Another feature of depressions in relation to rain must be mentioned at this stage. The lapse-rate in the warm sector, even in depressions which produce very heavy rainfall, may be less than the saturated adiabatic up to considerable heights. The occurrence of rain in such cases cannot therefore be attributed to thermal instability of damp air at low levels. Examples of this stability are shown in figs. 79 and 91. Douglas* has collected details of a number of cases in which heavy rain has occurred while the lapse-rate was stable for saturated air. This is of course not universally the case, and other cases could be cited in which heavy rain was associated with a lapse-rate unstable for saturated air. But as rain which is continuous for any considerable period must be due to the bodily ascent of large air masses, and not to local movements within the mass, there is in reality no special reason for expecting that the lapse-rate should be unstable for damp air at times when rain is falling, though instability extending to great heights must have important effects.

J. Bjerknes and Palmén† have also shown that in the depression of March 29–31, 1928, the mean lapse-rate up to 6 km was stable for saturated air, and

* *Q.J. Roy. Met. Soc.* **55**, 1929, p. 127; also *ibid.* **60**, 1934, p. 143.
† *Beitr. Phys. fr. Atmos.* **21**, 1933, p. 53.

while in the polar air the lapse-rate was almost exactly equal to the saturated adiabatic up to 6 km.

The conclusion appears to be inevitable that rainfall in a depression is associated with convergence on a large scale, the convergence being brought about by large scale convection, which in turn is due essentially, not to thermal instability but to dynamical instability. The assumption of some essential dynamical instability would account for the forced ascent of damp air even when it is not thermally unstable, and would at the same time afford a reasonable method of explaining the fact that depressions become occluded. It is perhaps all too easy to think of a depression as consisting of a warm front and a cold front, which can be treated separately. This is obviously unsound, and the depression should be looked upon as one entity.

§ 191. *The vertical structure and extent of depressions*

While the large depressions which occur in middle latitudes appear to extend well into the stratosphere, small depressions frequently occur whose circulation is restricted to a height of 3 km or less. Figs. 95 and 96 reproduced from a paper by Douglas* give an interesting case of a depression on November 17, 1910, centred over the Bay of Biscay and moving rapidly eastward to the Alps. The centre of this depression passed near Bordeaux, where the barograph indicated a fall of pressure of 23 mb (fig. 96), with a pronounced minimum of pressure, while the thermograph indicated a narrow warm sector. The records at Pic du Midi (2859 m), 140 miles farther south, showed practically no change of pressure, but a broad warm sector with no pronounced discontinuity was indicated by the temperature records. This depression could scarcely be said to exist at all at a height of 3 km.

It is probable that all depressions with large warm sectors are superficial structures of this type, having no cyclonic circulation over them in the upper troposphere. Douglas† showed that the depression of November 2–4, 1925, which had a well-marked warm sector and produced a brisk fall of surface pressure of 12 mb, left the pressure at 4 km unaltered, while above 5 km pressure over the centre of the depression was higher than the pressure in the polar air at the same level the next day.

The large depressions of middle latitudes normally develop from smaller ones, which in the initial stages do not extend beyond a few kilometres of height. It is difficult to conceive of the development which then occurs as a mere progressive extension upwards through the troposphere. This question is discussed in the next section in connection with the advection of low pressure at high levels.

* *Q.J. Roy. Met. Soc.* **50**, 1924, p. 339.
† *Ibid.* **55**, 1929, p. 123 (esp. para. 3. 1).

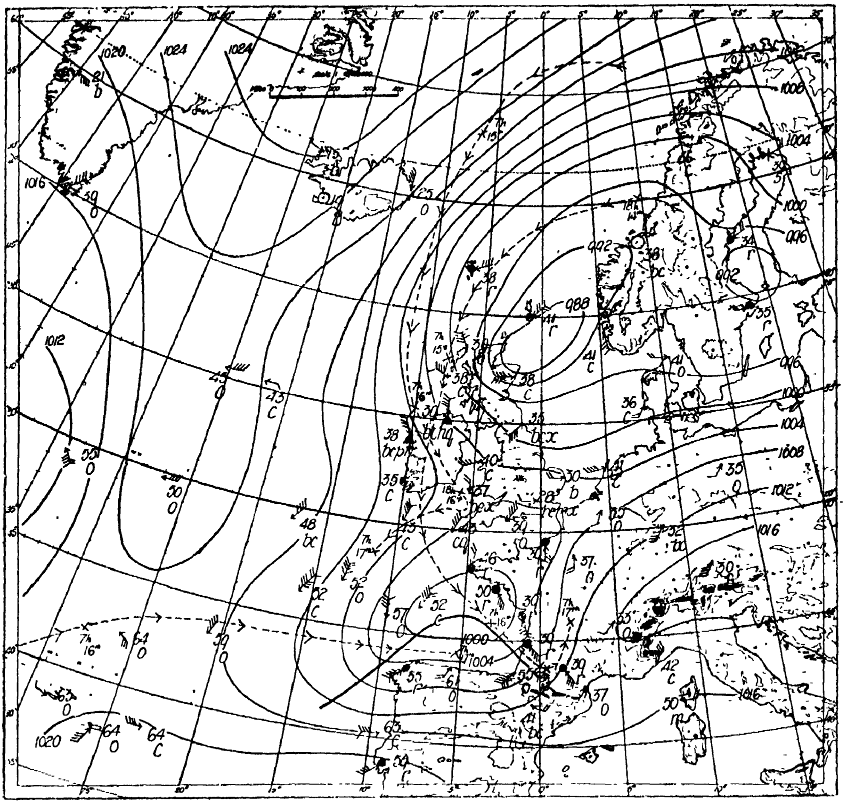


Fig. 95. The depression of November 17, 1910.

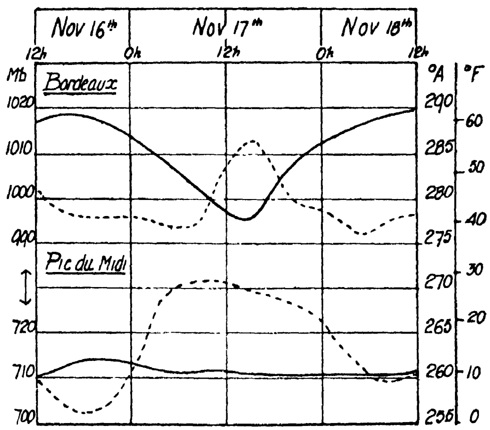


Fig. 96. Pressure (full line) and temperature (broken line) at Bordeaux and Pic du Midi, November 16–18, 1910.

§ 192. *The causes of formation of depressions*

In § 104 above it was suggested that the only effective cause of production of horizontal circulations on a large scale in the atmosphere is convergence. Further, in Chapter XVI it was suggested that a cyclonic circulation could be set up in a disc of air by the removal upwards of air from its centre. No attempt was then made to suggest any cause of removal of air, but it is probable that when depressions form in polar air, the procedure is similar to that visualised in Chapter XVI, and that the air which is removed upward ascends partly on account of thermal instability. In a polar current the temperature is initially low at all heights in the troposphere. As the current attains lower latitudes it is continually warmed from below by contact with the surface of the earth, until instability is set up through a considerable range of height. The ascent of the lower unstable air is presumably brought about by some kind of trigger action, and the wind circulation is in the main produced by convergence towards the region of ascent.

If the depressions in polar air are to be explained as suggested above, the tendency for more than one such depression to occur is readily explained. For the conditions in the main polar current are likely to persist so long as the current itself persists, and the instability should therefore also persist. But it cannot be claimed that the details of the process of formation of such depressions are understood. The amount of upper air information relating to the early stages of their development is extremely meagre, and we can only guess at the nature of the trigger action involved. Moreover, an analysis of relevant synoptic charts shows that in the polar air there is usually a secondary cold front between different portions of the polar current which have travelled over varying distances over a sea surface. In a current of maritime polar air sweeping first southward and then eastward the southern portion has had a longer trajectory over a usually warmer sea surface. The precise effect of the horizontal temperature gradients thus set up is not fully understood.

The formation of depressions at polar fronts is still less understood. The study of synoptic charts shows that while depressions frequently form at polar fronts depressions can also form in polar air (possibly through action at secondary fronts), and that fronts can exist without formation of depressions.

Thus depressions and fronts appear to be capable of separate existence, as well as of existence associated together. The main difficulty is to explain the beginnings of a depression at a front. Once a depression has formed, its interaction with the front will lead to distortion of the surface of discontinuity in a manner which forms a corrugation in the front, and the corrugation will form a channel up which the warm air can readily ascend. Mathematical discussion of the conditions at a surface of discontinuity does not lead to any useful result, as these conditions can only be specified at the ground. The magnitudes of the discontinuities of wind and temperature probably vary with height, so that the surface, even when it is sharply defined, need not be a plane surface.

One of the most difficult aspects of the problem concerns the lateral removal of air in sufficient quantity to produce the observed fall of pressure. Shaw suggested that the removal could be brought about by a strong current in the upper air, where velocity was much in excess of that of the currents in the lower troposphere. Such a current must exist when deep layers of warm and cold air are in juxtaposition. In a note in *Q.J. Roy. Met. Soc.* Jan. 1931, Douglas has given some consideration to this particular question. He cites December 14, 1929, as an example of the occurrence of strong cirrus motion above a front at which no depression developed. The gradient wind over South Scotland was 50 m.p.h., while cirrus velocities were estimated at 190 m.p.h. at Renfrew, and 150 m.p.h. at Leuchars (assuming a height of 5 miles). No depression developed, and the rainfall at the front was small. Somewhat similar conditions prevailed on October 24, 1929, above a front which gave heavy rain, exceeding 20 mm in 12 hours in places. This appears to indicate that the combination of convection and a strong upper current does not suffice to produce a depression.

There remains a possibility that the nascent cyclone has a purely thermal centre, a vortex formed by convergence towards a centre of convection, and that the further development is due to the interaction of the vortex and the front. A centre of convection is equivalent to the sink of hydrodynamics, and tends to set up a circulation represented by a simple vortex ($vr = \text{constant}$) over the region surrounding it. The interaction of the vortex and its surroundings can be readily seen from fig. 97, which comprises charts taken from a paper by Kaye and Durst*. In the first of these charts there is shown a depression formed in polar air, centred over Lake Superior, while a front sweeps from the Gulf of Mexico across the Atlantic to a depression south of Greenland. In the second chart, which shows conditions 24 hours later, the depression has moved to Newfoundland, but 12 hours later it has moved rather more slowly to the north-east of Newfoundland. The interest of these charts is in the motion of the front, which is drawn in towards the centre of the depression so that the depression has in the third chart a well-marked warm sector, while pressure has fallen very considerably. In the next 12 hours the centre moved to latitude 53° N, longitude 35° W, so that the motion had become much more rapid, and was now in the direction of the isobars in the warm sector. The depression had deepened very considerably.

In this case we have a weak cyclonic circulation strengthened by the interaction with a front, and it is possible that this represents the initial stages of all depressions which form at polar fronts. There is a further possibility that the spin of the depression is not produced in the lower atmosphere as in the case cited, but is present in the upper atmosphere, being brought to the neighbourhood of the front by advection. This view has found favour with many writers. Exner first put forward this view, and Douglas, as the result of the examination of many individual cyclones, appears to have reached a somewhat similar conclusion.

* *Q.J. Roy. Met. Soc.* April 1932.

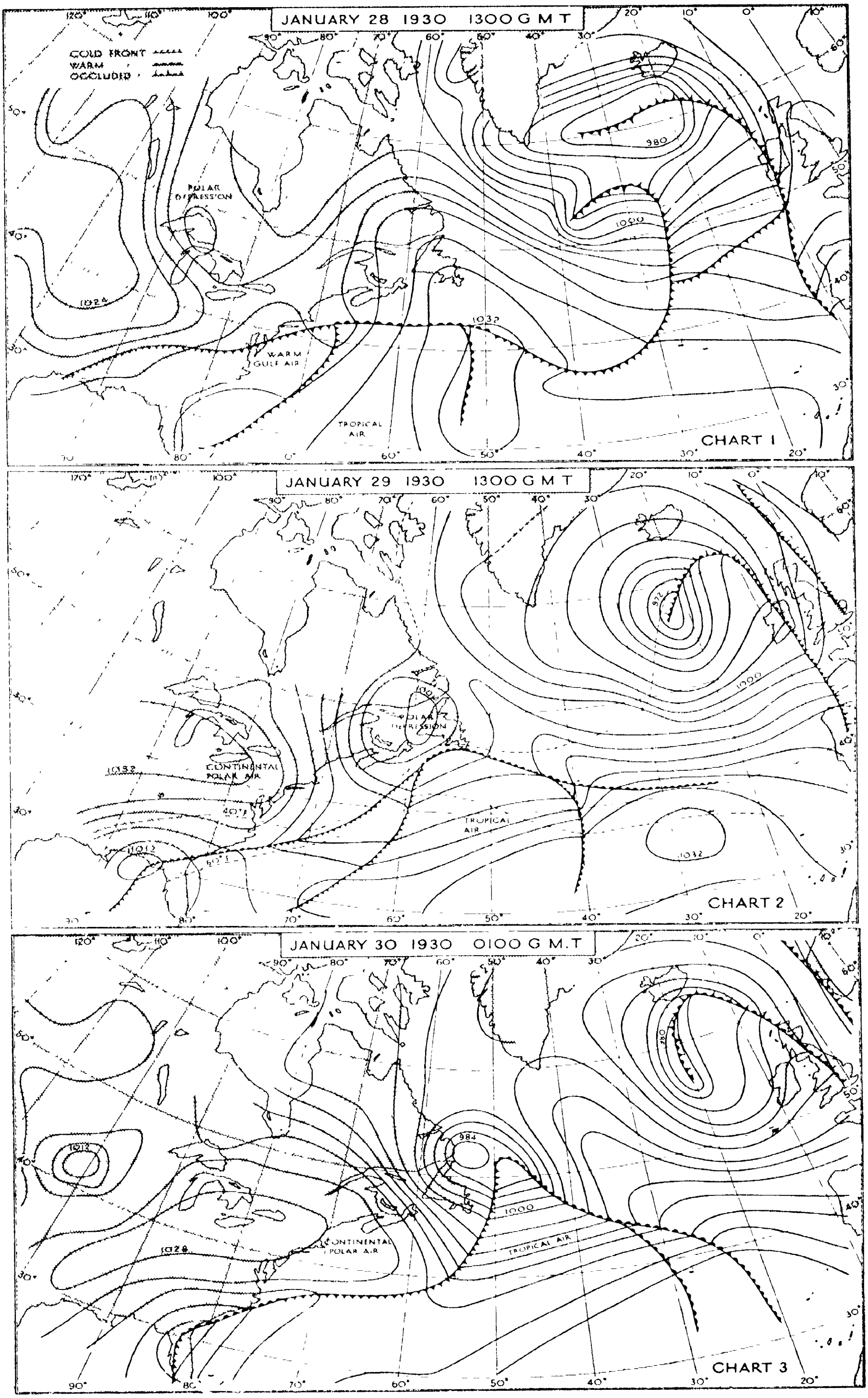


Fig. 37.—Formation of depressions over the Western Atlantic

The argument put forward by Douglas is as follows. Since in the warm sector the horizontal temperature gradient is not generally steep, the possibility of producing a fall of pressure by drawing in still warmer air is distinctly limited. Further, as shown in Chapter XVI increasing cyclonic circulation implies convergence, so that when a depression deepens the required withdrawal of air takes place right above it. Over the developing cyclone observation shows the existence of strong cirrus motion on the polar side of the front, so that there appears to be no possibility of a supply of cyclonic angular momentum being available in the upper currents. Douglas in fact regards the circulation at cirrus level above a Bjerknes depression as anticyclonic. He therefore falls back on the only obvious alternative, an advectional effect with air spreading over at high levels, bringing with it lower temperature and pressure in the troposphere, and a low and warm stratosphere. The rapid deepening of a large depression is then to be regarded as associated with the spreading of polar air right over the system, simultaneously with the diminution and final disappearance of the warm sector. It is well known by observation that a pronounced polar current behind a depression favours its deepening. In the earlier stages of development of the depression the "high depression" is not of necessity closed at the centre, though in the later stages closed isobars may develop. This accords with the ideas put forward in § 189 above in connection with the cirrus motion at different stages of the development of the depression. The "low depression" formed at a polar front is not in itself capable of spreading up through the whole troposphere, and the large depressions which extend up to the base of the stratosphere are on this theory regarded as due to the amalgamation of a high level depression with a low depression, the former shearing across the latter. The rapid fall of pressure in big depressions is largely due to the advection of low pressure in the upper air, while the circulation at low levels is built up mainly through convergence towards the centre produced by the displacement of the warm sector by polar air. It has not yet been possible to treat this complicated problem mathematically. It appears however that when the advection of low pressure in the upper air superposes lower pressures on the lower air, some convergence into the area of diminished pressure must take place, with a growth of cyclonic circulation.

Douglas points out that in no single case has observation indicated the existence of a symmetrical vortex above a depression with a genuine warm sector. He suggests that the upper cold depression may possibly be a vortex travelling in the general current, but in the early stages of the development the high level depression lies altogether outside the system shown on the surface chart. It is possible that the lower system influences the upper one, by some mechanism not yet understood, but the whole problem is at the present time unresolved, and cannot be resolved until more upper air data over nascent and young depressions become available.

J. Bjerknes in a recent memoir* has suggested that the changes in height and

* *Geof. Publ.* **9**, No. 9. See also *Physikalische Hydrodynamik*, §§ 182–4.

temperature of the tropopause above a given place can be accounted for as due to the lateral movements of the tropopause, to North and South, produced by the effect of the surface circulation. The warm air flowing upwards over a calotte of cold air is forced to diverge on account of the restriction of the range of height in which it can flow, and an anticyclonic circulation is produced in the upper troposphere, while at the other side of the calotte of cold air descent of the warm air is associated with convergence and the growth of cyclonic circulation. Here again the phenomena visualised are too complex for mathematical treatment, and no complete theory has yet been evolved.

It is probable that the type of motion visualised by J. Bjerknes sometimes occurs. If, however, the same mass of air goes up and then comes down again, the motion will involve the growth of spin during ascent, and the loss of this spin during the subsequent descent. A better view of the phenomenon is obtained when we postulate that the ascent and descent are in different air masses, the ascent being of warm air masses, and the descent being of cold air masses. The tendency for cold air masses extending originally through the whole troposphere to subside and diverge is well established by observation; and the growth of circulation in this manner is in accordance with the Margules scheme of growth of kinetic energy. If circulations can be thus generated at high levels, they should be capable of continued existence for a relatively long time, since the turbulent dissipation should be small in such systems. It is only in the lowest kilometre of the atmosphere that frictional effects are considerable, and at high levels these effects are slight (see § 165 above). In J. Bjerknes' scheme the upper air circulations would not be long-lived, but it is certain that upper air cyclic circulations which are set up in the atmosphere are capable of continued existence for fairly long periods. Dines found that the standard deviation of pressure was 10·4 mb at 9 km, and 11·1 mb at the surface. This evidence tends to confirm the existence of upper-air circulations. Systems already in existence high up would complicate the phenomena, and it is possible that the mechanism suggested above is unduly simplified.

It is not suggested that the depressions and anticyclones shown on the surface charts normally *originate* at high levels. All the researches of the last 15 years conflict with this view, and suggest a low-level origin for these systems; the high-level system amalgamates at a later stage of development with the low-level system.

In some of his earlier papers on revolving fluid Shaw suggested that an ascending current of warm air would undergo turbulent mixing with its environment, so that an increased amount of air could be removed from the core of the nascent depression by turbulence, the process being known as "eviction". The cyclonic circulation is in this theory a direct product of convergence. The evicted air was visualised by Shaw as carried away in a current whose motion was more rapid than that of the depression. The convection current cannot *generate* angular momentum; it can only re-distribute it; and if over the area of the depression the angular momentum is to increase in the cyclonic sense and the pressure is to diminish, the air which rises in the convection current must be carried away in the upper air, either as the effect of a shearing current, or of diverging currents. In either case there must be a growth of anticyclonic motions at high levels, but either cause would suffice to distribute this anticyclonic motion, and the excess of pressure associated with it, over such a wide area as to show no appreciable effect at the surface. R. A. Watson* has quoted evidence of anticyclonic motion at cirrus levels above tropical cyclones, and it is possible that such motion represents the high-level counterpart of the tropical cyclone.

§ 193. *The energy of depressions*

A theory of the origin of depressions must take account of the necessity of explaining the development of colossal amounts of kinetic energy, and the removal of colossal amounts of air to provide for the fall of pressure. All theories which have been put forward to explain the genesis of depressions are eventually based on the existence of horizontal differences of temperature. It is because the horizontal gradients of temperature from equator to pole are greater in winter than in summer that depressions are more frequent and more intense in winter than in summer. It should be added that the air contains more water-vapour in summer than in winter, so that any theory which would explain depressions purely on the basis of water-vapour should demand greater frequency and greater intensity in summer than in winter.

Margules ascribed the development of kinetic energy in the depression to the readjustment of masses in unstable equilibrium, which liberates large amounts of energy. Some of Margules' computations are summarised in Chapter XV above, where it is shown that differences of temperature of 10° C, which are relatively common in the atmosphere in winter, involves the existence of sufficient potential energy, to supply the required kinetic energy by redistribution of the masses. The computations of Margules afford the only available criterion for the amount of energy which can be developed in the atmosphere, and the agreement found between the maximum possible development of kinetic energy and the observed kinetic energy in depressions, with differences of temperature of say 10° C, is, to say the least, very striking. The actual processes which Margules considers are more nearly analogous to squall phenomena than the development of depressions, but if there are sloping surfaces of separation between warm and cold masses of air it is not possible even to form a mental picture of the horizontal and vertical motions associated with the readjustment of potential energy, much less to represent these motions analytically.

There can however be little doubt that the formation of the depressions is to be explained by the displacement of warm air by cold air. In the growth of the circulation of the depression convergence probably is a factor of prime importance. The growth of an intense depression is accompanied by heavy rainfall over a wide area, in itself evidence of the displacement of large masses

* *Q.J. Roy. Met. Soc.* **53**, 1927, p. 446.

of warm damp air by heavier air, and of the liberation of large quantities of potential and thermal energies.

There is a difficulty in the way of applying the ideas of Margules to a Bjerknes depression to which the writer's attention has been drawn by C. K. M. Douglas. It was seen in equation (9) of Chapter x that the slope of a surface of discontinuity for a given difference of temperature is proportional to the differences of velocity. The liberation of potential energy would require a decrease in the slope of the surface, whereas a development of kinetic energy requires an increase in the slope.

It should be noted that there is no essential contradiction between the work of Margules and the idea that a depression may originate by pure convection. The latter involves a diminution of potential energy, and in the end the kinetic energy developed by convergence must amount to the potential energy released by the convection.

§ 194. *Kobayasi's theory of the formation of fronts in a vortex as a result of horizontal temperature gradients*

Kobayasi* considered the effect of horizontal variation of temperature in the general field upon the distribution in an originally symmetrical cyclone. In order to discuss the question mathematically he assumed a distribution of pressure in the cyclone giving everywhere a constant gradient of pressure in the central or principal part of the cyclone, whose outer boundary is a circle of radius R, while outside this region the transverse velocity is given by

$$vr = \text{constant}.$$

He further assumed that at a given level the air drifts across the isobars at an angle α, which decreases from the ground upwards, and that the cyclone moves through a field of uniform horizontal temperature gradient. The convergence of air to the centre involves the ascent of air at the centre, and the result is that air within a region which is of the form of a parabola (*EF* of fig. 98) with its focus at the centre of the cyclone is eventually removed upwards, while air currents from the two opposite sides outside this region are brought into juxtaposition along a line which trails backward from the centre to the right of the path of the cyclone. This line is accordingly a line of discontinuity of temperature. Kobayasi further showed that the line of discontinuity is advanced in the direction of motion of the cyclone as α, the angle of drift across the isobars, is diminished. Hence the surface which is traced out by the discontinuity at different levels slopes forward and produces a system in which the colder air at say 500 metres is brought in above warmer air at the surface. This provides a mechanism at the cold front capable of giving the squally winds which are actually observed, and provides a simple explanation for the existence of a front in the rear of a cyclone.

Kobayasi further suggests that the temperature discontinuity left behind by

* *Q.J. Roy. Met. Soc.* **49**, 1923, p. 177.

one cyclone may act as steering line for the next cyclone. This portion of his argument is not as readily acceptable as the earlier portion. Subsequent researches have shown that cyclones do not move along the direction of the front in advance of the centre, but along the direction of the isobars in the warm current. Further, Kobayasi does not prove that what he calls the "steering line", the line of discontinuity produced by one cyclone and drawn into the ambit of the next, will have any steering properties.

The ideas developed by Kobayasi have not been applied to any Atlantic depressions. It may probably be assumed with safety that they are not applicable to the ordinary depression with a well-marked warm sector, but the

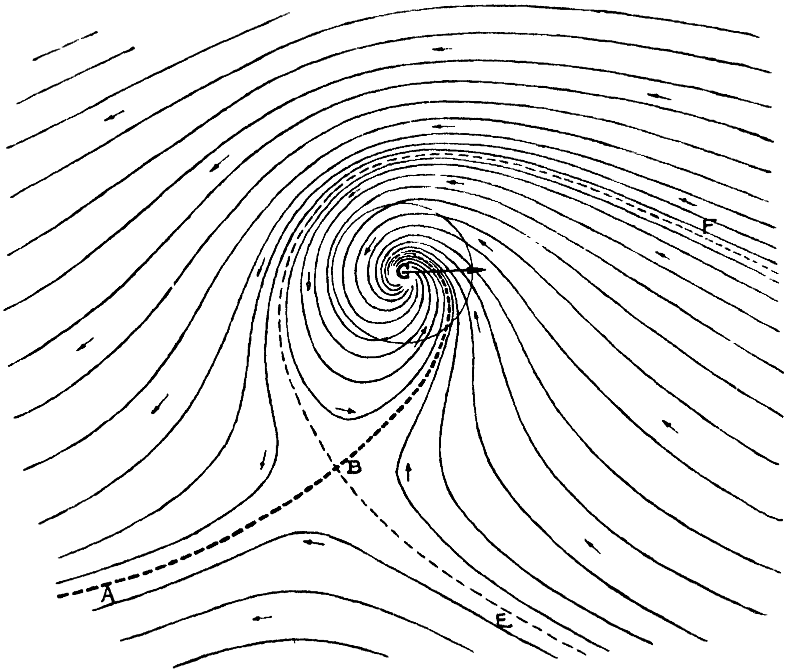


Fig. 98. Kobayasi's diagram to illustrate the formation of fronts.

possibility of these phenomena occurring in the depressions which form in polar air deserves further study.

It is further possible that Kobayasi's ideas would find closer application in tropical cyclones. Horiguti* points out the absence of lines of discontinuity in the typhoon area. Father Gherzi† expresses the same view. Mal and Desai‡ have however traced fronts to very great distances from the centre of a tropical cyclone, but Father Gherzi suggests (*loc. cit.*) that these fronts are phenomena of the surrounding region rather than of the true typhoon central region. A recent paper by Desai and Basur§ points to the inner structure of tropical cyclones not being symmetrical.

* Y. Horiguti, *Mems. Imp. Marine Obs. Kobe, Japan*, **2**, No. 3, **3**, Nos. 2, 3.
† *Q.J. Roy. Met. Soc.* **58**, 1932, p. 303.
‡ *Ind. Met. Dept., Sci. Notes*, **4**, No. 39, 1931. § *Beitr. Geoph.* **40**, h. 1, 1933.

§ 195. *Exner's barrier theory of depressions*

Exner has outlined in his *Dynamische Meteorologie* (2nd edn.), Chapter XII, a view of the origin of depressions which is in some respects similar to the Norwegian view, in other respects contradictory to it. The cyclone is regarded as analogous to the whirl which forms in the lee of a rock jutting out into a stream, or in the lee of the pillars of a bridge. Exner suggests that the place of the rock or bridge pillar may be taken by high land, particularly near the ocean, and cites in particular the massif of Greenland, which, in his view, interferes with the westerly current of those latitudes, producing a drop in pressure east of the southern projection of Greenland. This gives the westerly current a component of motion to north, and produces the Iceland minimum.

Exner points out, however, that most depressions do not form at such projections, but at the limits of currents of air. The initial stage of development is the outbreak of a cold mass of air from polar regions into the westerly current of middle latitudes, the cold air sweeping southward in the form of a tongue, as shown in fig. 99(*a*) below.

The tongue of cold air *C* cuts off the supply of warm air from the corner *A*, causing a fall of pressure at *A*. The pressure gradients set up produce the

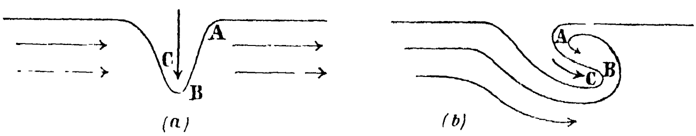


Fig. 99. Exner's barrier scheme of formation of depressions.

circulation shown in fig. 99(*b*), the cold tongue *C* being drawn round the centre of low pressure as shown. The bursts of cold air over North America, which are frequently accompanied by depressions on their eastern edges, are regarded by Exner as essentially similar to the scheme outlined above, and a map showing such an outburst in January 1895 is reproduced by Exner (*loc. cit.* p. 340).

Exner devised an experiment to illustrate his theory. A circular vessel which can be rotated about its centre has a cylinder of ice at its centre and is heated at its periphery. Colouring matter is placed at the centre, so that the motion of the cold water can be traced. A photograph reproduced by Exner (*loc. cit.* p. 341) shows intermittent outbursts of cold water into the warm outer regions, showing considerable resemblance in their subsequent motion to the cyclonic and anticyclonic circulations of the atmosphere.

Exner's scheme for the formation of a cyclone resembles the Norwegian scheme in that it demands the interaction of warm and cold currents, but in other ways it is in direct contrast with the Norwegian views. It replaces a continuous polar front by discontinuous outbursts of cold polar air, and, if the present writer has completely understood it, demands that the actual centre of the cyclone should be filled with warm air, so that the "warm sector" should enclose the centre of the cyclone. This is not in agreement with the facts of observation. The location of the outburst of cold air should show a marked tendency to occur east of Greenland, over Nova Zemblya, east of the western mountain ranges of North America, and over the eastern portion of the northern coasts of Asia.

The production of the cyclone is regarded as due to dynamical interaction of the warm and cold air, but the original outburst of cold air is regarded as produced by a purely thermal cause—the difference of density due to differences of temperature in the two air masses; and the energy of the cyclone is eventually derived from the thermal energy of the meridional circulation of the atmosphere. Exner made no claim that his theory was completely developed. He could give no estimate of the extent of the fall of pressure in the lee of the cold tongue, and it has been objected that the fall of pressure so produced must be insignificant. Until some arithmetical estimate has been given of this fall of pressure, the situation as between the adherents and opponents of the theory remains unresolved.

There is however a marked contrast between this theory and that developed by J. Bjerknes and Solberg. The latter also regard the first stage in the development of a cyclone family as the outburst of a mass of cold air from the polar basin; but a *family* of cyclones forms along the front which separates the cold mass from the warm air to the east of it. Exner however would have *one* cyclone form at the northern corner of the warm air east of the tongue of cold air. On either scheme the cyclone becomes a definite feature of the exchange of air between different latitudes.

Exner's theory is a logical outcome of the study of the outbursts of cold air over Asia and Western Europe which formed the subject of a series of papers by Ficker. In a paper in the *Meteorologische Zeitschrift* for March 1923 Ficker gives a very complete bibliography of his own papers and of those of other authors on the subject. Ficker explicitly states that outbursts of cold or warm air could only explain the shallow cyclones and anticyclones which do not extend beyond the lower troposphere. He regards the depression which extends to the stratosphere as the combination of a depression in the lower atmosphere with a depression at high levels, the latter being formed by high level outbursts of warm air from the North.

It would be an error to regard the theory of Exner as an alternative to the Norwegian polar front theory. The depressions which form over the Atlantic and over the region of the British Isles generally form at the boundary between two portions of the westerly current which have originated in different latitudes, and there is definitely no trace of an initial outburst of cold air into the warm westerly current. Most of the depressions which affect the British Isles are therefore Norwegian rather than Austrian depressions. But there is not so marked a contrast of temperatures in the cold waves which move south over Western Europe as in the cold waves over America or Asia, and it is not clear whether Exner's views may not be closely applicable to other regions of the earth. In any case we have to guard against thinking of the depression as something which always has a typical form, and arises in a typical way. There is only one fundamentally typical feature common to all depressions, a centre of low pressure. Other features can show almost every conceivable variation, and it is possible that different depressions arise in different ways, some as the result of large-scale convection in unstable air, some at polar fronts as pictured by Bjerknes, and some in the lee of cold outbursts as pictured by Exner. The meteorologist has to bear these different possibilities in mind, as alternatives which shall apply in individual cases.

CHAPTER XIX

ANTICYCLONES

§ 196. *Types of anticyclones*

VE have already seen that an anticyclone is a centre of high pressure in /hich the wind circulates clockwise about the centre. Consideration of the ppropriate solution of the gradient wind equation showed that the winds in 1e anticyclone spin round the centre at a slower rate than the rotation of the ırth beneath them, so that, regarded as a circulation in space, the anticyclone ; a slow counter-clockwise circulation. The angular velocity about the centre 1 an anticyclone has thus an upper limit, equal to the angular velocity of the orizon, $\omega \sin \phi$. There is a corresponding limit to the pressure gradient. We ave also seen (p. 184) that the geostrophic wind is an underestimate of the radient wind in an anticyclone, and an overestimate in a cyclone. Thus for given gradient of pressure, the gradient wind is greater in an anticyclone ıan in a cyclone, or, conversely, a given gradient wind velocity requires a ›wer gradient of pressure in an anticyclone than in a cyclone.

The argument used in Chapter XVI that convergence is necessary in order to roduce a cyclonic circulation can be applied to show that divergence is :quired in order to produce anticyclonic circulation in the lower atmosphere. 'his divergence is most readily visualised as due to the accumulation of air high levels causing a general settling down at low levels, with a gradual ›read outwards from the centre. This would give the required clockwise rculation relative to the earth in the levels in which divergence occurs. But ıe transport of air to a restricted region at high levels is not readily visualised. ' it took place by a general convergence towards a point or a small area, we ıould expect to find a cyclonic circulation at the level of convergence. But is is not found by observation over the anticyclones of middle latitudes, ough it is present above the anticyclone which forms over Asia in winter.

The earlier writers on the subject regarded an anticyclone as a region of cold r, the excess of pressure being due to the excess of density in the lower oposphere. But Hann, and later other writers, showed that on the average e anticyclone is warmer than the cyclone at levels of 3–8 km. The strato- ›here is higher above the anticyclone than above the cyclone, and the fall of mperature with height is continued farther over the anticyclone, yielding a ›lder stratosphere.

Hanzlik* made a detailed study of European anticyclones, and found two stinct types, the cold and the warm anticyclone. The cold anticyclone is a latively shallow vertical structure, whose excess of pressure is due to excess

* *Denkschr. Wien. Akad.* **48**, 1909, pp. 163–256.

of density in the lower troposphere. It usually moves fairly rapidly. If it slows down or becomes stationary it tends, according to Hanzlik, to be transformed into the second type, the warm anticyclone. The warm anticyclone extends into the stratosphere, in which region it is colder than the normal for latitude and time of year. The region of lowest stratospheric temperature is not in general directly above the highest pressure at mean sea level, but is usually displaced to west or northwest of the latter. The "axis" of a warm anticyclone is therefore inclined to west or north-west in the same way as the "axis" of a cyclone, though occasionally it is inclined to south*.

Hesselberg† suggested that the high temperature in the anticyclone can only be explained by the descent of air. Compression alone can only give a relatively small increase of temperature. Radiation is also insufficient to produce the rapid changes of temperature which accompany changes of pressure and there remains only descent of air as a possible explanation of the high temperatures which occur. An anticyclone which is warmer than the normal in the lower troposphere must have a slower diminution of pressure with height than the normal in that region, so that the excess of pressure above normal increases with height, and the high pressure can only be explained by the addition of mass in the upper layers of the atmosphere.

The following extract from Table XII of the memoir by W. H. Dines‡ on the characteristics of the free atmosphere will give an idea of the extent of the differences between cyclonic and anticyclonic temperatures at different levels, the mean surface pressure being respectively 989 mb and 1026 mb.

Height in km	1	2	3	4	5	6	7	8	10	12	14
Mean temp. °A	277	273	268	262	256	249	242	235	233	220	220
Cyclone °A	276	270	263	256	249	242	234	228	225	225	224
Anticyclone °A	279	276	271	265	259	253	246	238	225	217	215
Anticyclone−mean °C	2	3	3	3	3	4	4	3	2	−3	−5
Anticyclone−cyclone °C	3	6	8	9	10	11	12	10	0	−8	−9

The difference between cyclone and anticyclone, and the difference between anticyclone and the mean conditions, are both greatest round about 6–8 km.

Dines gives (*loc. cit.* pp. 62–3) tables which indicate uniformity of pressure at 20 km in all latitudes and at all times, which would appear to indicate that any addition of mass must be below the level of 20 km, though above the level of 8 km. But the result given by Dines does not agree with that of Dobson§ on the variation of wind with height. Dobson found that while on the average the winds over the British Isles fall off to zero at 20 km the winds at Lindenberg are still of considerable strength at that height.

§ 197. *The cold anticyclone*

The cold anticyclones of Europe appear to be mainly generated by cold currents from polar regions brought southward in the rear of depressions. There was an anticyclone over Scandinavia on March 5–6, 1931, with very high

* Runge, Leipzig Dissertation, 1931.
† *Met. Zeit.* **32**, 1915, p. 311.
‡ M.O., *Geophys. Mem.* No. 13.
§ *Q.J. Roy. Met. Soc.* **46**, 1920, p. 54.

surface pressure, at the centre exceeding 1040 mb (see fig. 100). Upper air ascents at Kjeller, in the NE current on the eastern side of the centre, at 18h, and at Duxford, in the SE current on the southern side of the centre, at 10h, are shown in fig. 101. Temperatures were much higher in the SE current than in the NE current, being about 34° F higher at 15,000 feet, and over

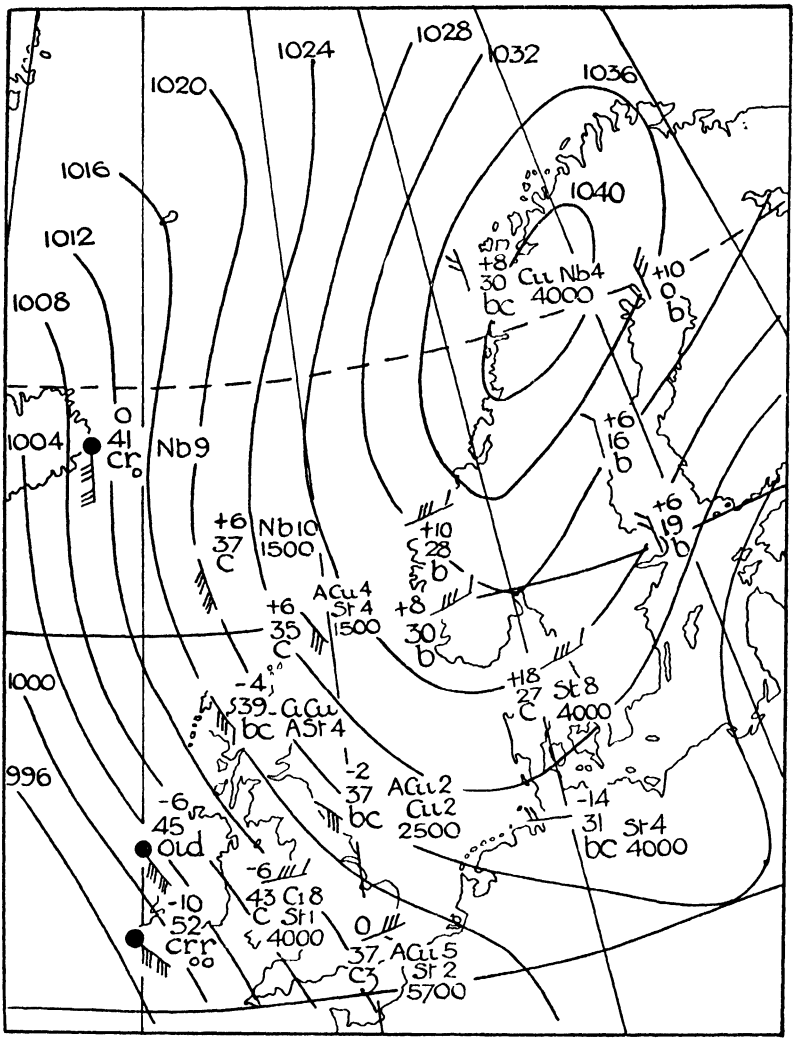
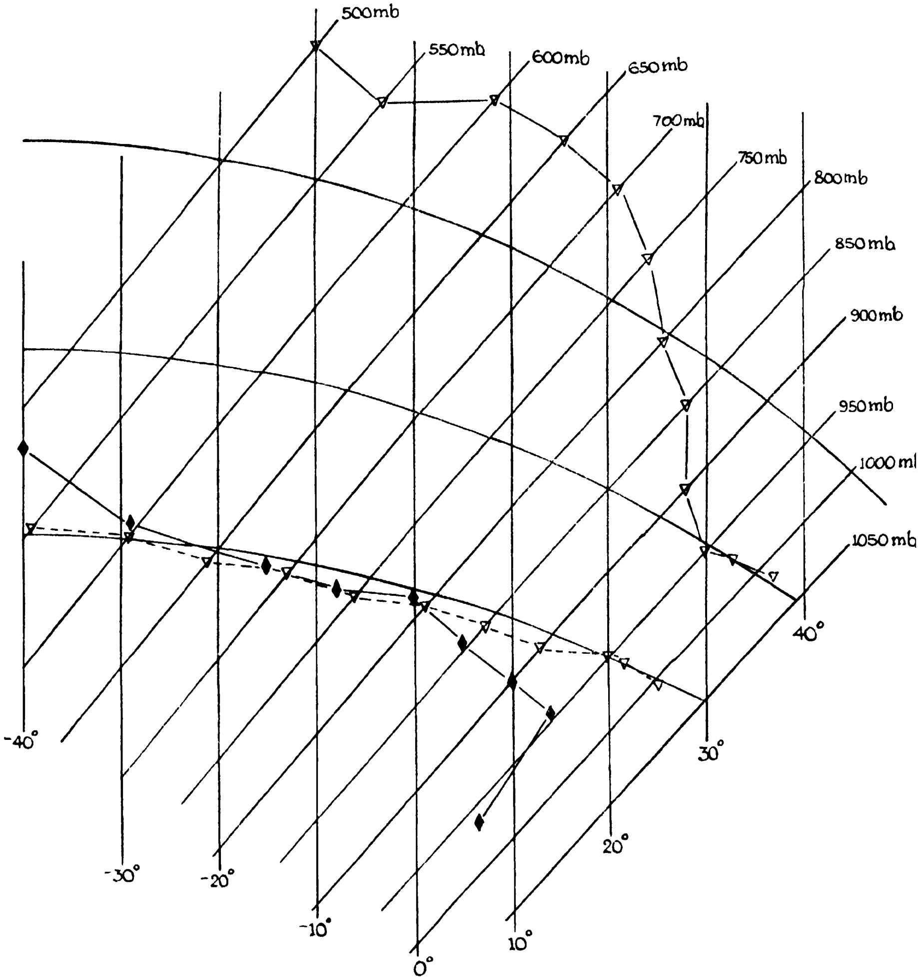


Fig. 100. The anticyclone of March 5, 1931.

20° F higher at the ground. The surface inversion shown at Kjeller was due o radiational cooling of the ground. Above the first 2000 feet, the Duxford ascent showed a layer of practically steady temperature up to 8000 feet. In at east the central part of this isothermal layer the relative humidity was about 70 per cent, while at 8000 feet, and also at 1000–2000 feet it approached 100 per cent. At Kjeller the relative humidity was reported as 35 per cent through the whole of the ascent, a value which indicates that the air had probably subsided. It is readily seen from a Hertz diagram or tephigram that air which had relative humidity 35 per cent at the ground might have started saturated from a level of about 2 km and descended adiabatically.



1. ▽——▽ Duxford, March 5, 1931, 11 h 50 m.
2. ▽----▽ Duxford, March 9, 1931, 9 h.
3. ♦——♦ Kjeller, March 5, 1931, 8 h.

Fig. 101. Upper air observations, March 5, 1931.

By 18h on the 6th pressure at the centre of the anticyclone was about 1044 mb, but after that it diminished slowly. The anticyclone was however maintained for some time, and drifted slowly westward to Iceland. At 7h on the 9th it was centred over the north-eastern end of Iceland, and the pressure at the centre was a little above 1032 mb. Duxford was now in the NE current, and the conditions in the upper air resembled closely those at Kjeller on the 5th. These conditions are represented by curve 2 in fig. 101. The lapse-rate was very slightly in excess of the saturated adiabatic, and a little less than that observed at Kjeller on the 5th, except in the surface layers, which no longer showed an inversion, but a marked lapse-rate, on account of the heating of the surface layers in passing over the North Sea. The surface layers developed considerable instability, which on the 8th gave rise to heavy falls of snow over the British Isles*.

The changes in the lapse-rate in the NE current represented by the differences between curves (2) and (3) in fig. 101 are in the direction of decreasing stability, and are such as would be produced by surface heating. The resemblance of curves (2) and (3) indicate however a remarkable conservatism of the state of the air in the NE current. At Duxford on the 5th there was a pronounced inversion of 2° F between 900 mb and 860 mb (3200 to 4400 feet), with drier air (R.H. 70 per cent) above damp air (R.H. 95 per cent).

§ 198. *The warm anticyclone*

The warm anticyclones which extend into the stratosphere are different in their relation to their environment. They are warmer than their environment in the troposphere, and colder in the stratosphere, which is higher and colder than the normal for the latitude and time of the year. The excess of pressure above normal is thus to be explained by the excess of density at high levels. On account of the high temperature in the troposphere it follows that the excess of pressure above normal increases with height in the troposphere. The correlation coefficients between different variables in the upper air which were evaluated by W. H. Dines (see p. 21 *ante*) bear out the statements made above. These coefficients show that high pressure at 9 km is associated with a high and cold stratosphere, and a warm and cold troposphere. The results derived by W. H. Dines are confirmed by Schedler†, who evaluated similar correlation coefficients for data accumulated in the course of a series of daily observations at Lindenberg. Schedler's coefficients were all slightly lower than those of Dines, but the confirmation of the general results is striking. Schedler investigated the contribution of different layers of the atmosphere to the total changes of pressure, and found that, on the whole, the first three kilometres above the ground contribute slightly to the net surface change of pressure, the next 5 or 6 kilometres act weakly against it, so that the changes of pressure at 8 or 9 km are at least as great as those at the ground. Thus effectively the whole pressure variation must be accounted for by the changes above 9 km. Schedler has estimated that 40 per cent of the variation was due to the layer from 9 to 14 km, and the remaining 40 per cent to the layers above 14 km.

The problem of explaining the warm anticyclone thus reduces in the main to that of explaining the occurrence of low temperatures at high levels. Two

* *Vide* note by W. R. Morgans, *Met. Mag.* April 1928, p. 53.

† *Beiträge Phys. fr. Atmos.* **7**, p. 88.

alternatives are possible. The air at high levels may be cooled *in situ* by radiation, or the cold air at high levels may be brought in from lower latitudes. The first alternative is difficult to discuss on account of our lack of knowledge of the time scale of radiation phenomena in the highest layers. The active agent which effects the changes we are considering may be the water-vapour in the lowest layers, or the occurrence of sheets of cloud. From the point of view of radiational phenomena the fundamental difference between low and high latitudes consists in their respective high and low water-vapour content of the lower troposphere. The effect of the water-vapour at these levels is to warm the lowest layers at the expense of the highest. But the water-vapour content of the atmosphere is not in general as great in anticyclones as in cyclones, and this line of approach to the problem is not promising. Indeed the main problem appears to be far removed from the possibility of explanation by radiation. This is borne out by the phenomena in cyclones, in which the occurrence of cloud sheets must lead to the reflexion back into space of a considerable part of the incoming solar beam, as a result of which the amount of long-wave radiation passing out into the stratosphere is considerably reduced. Yet the temperature of the stratosphere above a cyclone is higher than the normal, showing that the cyclone is not to be explained as a radiational phenomenon. The validity of this argument is borne out by the observations of temperature of the stratosphere over India. During the monsoon the stratosphere is colder than in winter, though the surface temperatures are higher, the reason being that solar radiation is sent back into space by the cloud sheet associated with the monsoon rainfall. The surface temperatures are high on account of the current of warm air from lower latitudes.

The reader cannot fail to notice that there is a close apparent analogy between the distributions above cyclones and anticyclones, and those above polar and equatorial regions. Over cyclones and in polar regions the troposphere is cold, and the stratosphere low and warm. Over anticyclones and in equatorial regions the troposphere is high and the stratosphere cold. It is thus an obvious step to ascribe the formation of anticyclones to the transfer of air from equatorial regions to high latitudes, by means of solid currents which extend high into the stratosphere. This suggestion is so obvious that it is perhaps advisable to add a word of caution that its obviousness may be in no sense physically justifiable.

The curves of fig. 102 are based on the figures given by Wagner* of the distribution of pressure over the globe in winter and summer. The differences between the mean surface pressures in different latitudes are so slight that it is obvious that no large differences of pressure in middle latitudes could be produced by the bodily transfer from low to high latitudes of sections of the atmosphere. Round about 20 to 22 km the pressure attains approximate uniformity over the whole globe, and the increase of pressure which could be produced in latitude 60° by the transfer from latitude 10° to latitude 60° of a section of the upper air reaching from the top of the atmosphere down to

* *Handbuch der Klimatologie*, **1**, Teil F, "Klimatologie der freien Atmosphäre", p. F 67.

some chosen level in the troposphere can therefore be readily evaluated from the diagram by taking the differences of pressure in latitudes 10° and 60°, at that level. These differences are approximately as follows:

Height in km		0	2	4	6	8	10	12	14	16	18	20	22
Pressure Difference	Summer	2	5	11	17	23	23	21	16	11	6	3	2
	Winter	4	11	19	24	27	27	22	15	8	3	1	0

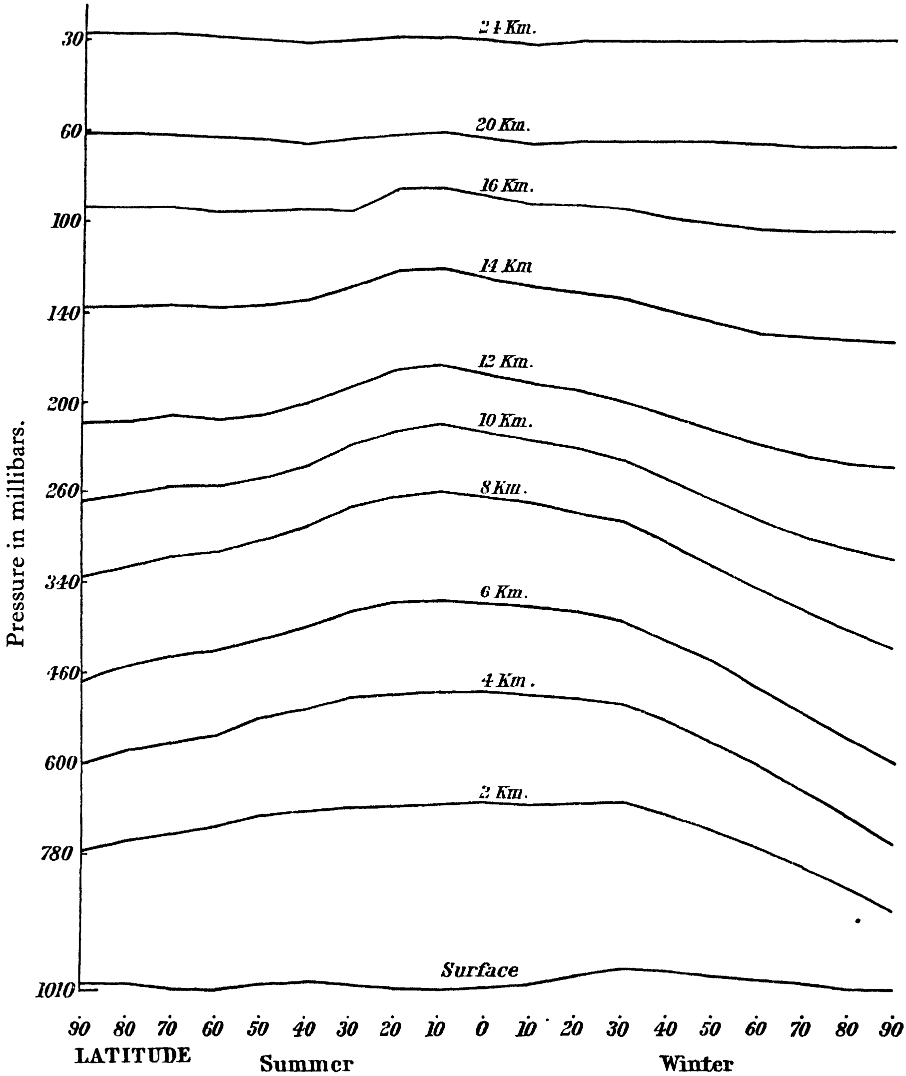


Fig. 102. The distribution of pressure at different heights. On the vertical scale of variation of the curves the interval between the successive numbers from 780 upwards shown on the vertical axis is 40 mb.

Thus in latitude 60° if the portion of the atmosphere above about 9 km could be replaced by the corresponding portion of the atmosphere from latitude 10° a change of pressure of from 23 to 27 mb would be produced.

It was suggested by Gold* that some anticyclones may be due to the bulging northward of the cold equatorial current in the stratosphere, and subsequent writers have advocated somewhat similar ideas. Such outbursts to northward of cold air in the upper troposphere and stratosphere would provide an increase of pressure of the right order of magnitude, but it must be emphasised that if the outburst to northward includes the whole of the troposphere, the increment of pressure due to cold air at high levels is destroyed by the diminution due to the high temperature at lower levels.

Exner† suggested that a warm southerly current of great breadth might extend into the stratosphere and bring with it the cold high stratosphere of more southerly latitudes. The arguments adduced above make it difficult to explain by this means the increment of pressure which produces the anticyclone as we observe it. The suggestion of an outburst of cold air at high levels is more plausible. The phenomena must become more complicated in the lower troposphere. For if an anticyclonic circulation is set up in lower levels, it will draw into its ambit warm southerly air on its western side and cold northerly air on its eastern side. The contribution of the lowest layers to the pressure distribution will thus be unsymmetrical relative to the centre of high pressure in the upper troposphere, the maximum pressure at the surface being displaced to the east. This point was brought out by Exner‡ in a discussion of some of Schedler's results.

The explanation of the cold anticyclone does not present any insuperable difficulty. The accumulation of a subsiding mass of cold air explains most of the phenomena, and the development is facilitated when the surface of the earth cools rapidly by radiation, as when the ground is covered with snow. The crucial difficulty is the explanation of the growth of the cold anticyclone into a warm one. Recent writers tend to the view that the apparent growth up into the stratosphere of an originally shallow anticyclone is to be explained as an amalgamation of a low-level anticyclone with a high-level anticyclone. The occurrence of closed isobaric systems at high levels is indicated by the far from negligible frequency of cirrus motion from the east. Such systems correspond to islands of warm or cold air which observations have shown to exist. Islands of cold air at high levels are to be explained as high-level outbursts of air from low latitudes. Douglas§ distinguishes three main phases in the life-history of the typical anticyclone or depression. "Firstly, the systems are confined to low levels; secondly, the (still unclosed) high-level systems are two or three hundred miles to west of the (often closed) low-level systems; finally, the upper systems are roughly coincident with the lower systems." These words present a picture which is in its main outlines similar to that of many other writers. The high-level system, whether depression or anticyclone, is in the main regarded as the effect of advection from high or low latitudes. Perhaps

* "International Kite and Balloon Ascents", M.O., *Geoph. Mem.* No. 5.

† *Dynamische Meteorologie*, p. 358.

‡ *Met. Zeit.* 1921, p. 21.

§ *Q.J. Roy. Met. Soc.* **59**, 1933, p. 62.

the strongest evidence in favour of advection as a fundamental cause of anticyclones is to be found in a letter to *Nature* from L. H. G. Dines* in which are given the correlation coefficients between the potential temperature at the tropopause and certain other fundamental variables. The variables correlated, and the suffixes used to denote them, are shown below:

	Suffix
Pressure at 9 km	3
Height of the tropopause	4
Temperature at the tropopause	5
Pressure at the tropopause	6
Potential temperature at the tropopause	7

The correlation coefficients obtained are shown below, the column headed "smoothed" giving values derived by using departures from smoothed monthly means instead of the observed values:

		Smoothed
r_{34}	0·82	0·82
r_{36}	−0·70	—
r_{37}	0·89	0·82
r_{47}	0·82	0·81
r_{56}	0·79	—

The standard deviation of the potential temperature at the tropopause was nearly 10° C, and since departures may range up to twice the standard deviation or more, it is implied in these results that the potential temperature at the tropopause may alter by 20–25° in a week or so. So big a change in potential temperature (or entropy) is not readily explained by any known process, if it is assumed that the change referred to is a change in the condition of a fixed mass of air, and we are forced to the conclusion that the change is in reality the replacement of the original air by a fresh supply from some other latitude. If we consider the potential temperature at the tropopause in an anticyclone, to find air of the same potential temperature in a cyclone we should require to penetrate well into the stratosphere, and to find elsewhere air of the same potential temperature at approximately the same level, we should require to go to much lower latitude. The conclusion that the air at the tropopause in an anticyclone has recently come from lower latitudes appears inevitable, particularly as the extremes of potential temperature in depressions and anticyclones are well within the range provided by the poles and the equator.

One serious problem is the growth of anticyclonic rotation at high levels. One plausible explanation is that of J. Bjerknes (see § 192, p. 353 above) that it is produced where warm air diverges in the upper troposphere after flowing up a warm front surface.

In the present state of our knowledge of what occurs when anticyclones develop or die away, it is not possible to give any complete theoretical treatment, and future advance will only be possible by the discussion of actual observations. Much of the argument hitherto used in discussing anticyclones is rather of the nature of guesswork.

* *Nature*, **127**, 1931, p. 815.

It was suggested by Hanzlik that the warm anticyclone is a stable system liable to persist for considerable periods, and this suggestion has frequently been used as a basis in forecasting, though it is probably much exaggerated. An exceptionally warm and apparently well-established anticyclone on September 15, 1932, collapsed completely within 48 hours, and other examples could be quoted to show that Hanzlik's suggestion cannot with safety be adopted as a basis of practical forecasting.

§ 199. *Some observational data*

A number of individual warm anticyclones have been investigated by different writers. Mügge*, using data accumulated on two international days—May 7, 1909, and May 19, 1910—found it possible to draw streamlines of vectorial mean winds at three levels, from 1 to 4 km, from 4 to 8 km, and from 8 km to the tropopause.

On May 7, 1909, an intense anticyclone covered the North Sea and Northern Europe, giving clear weather over a wide area, so making upper air observations possible over most of Western and Central Europe. The lines of flow at the three chosen levels showed a point of divergence of air-flow, which for the level 1 to 4 km was slightly north of Bergen on the West coast of Norway: for the level 8 km to the tropopause the point of divergence was over the North Sea in latitude 55°; and for the intermediate layer the point of divergence was rather indeterminate, but probably situated somewhere between the points found for the upper and lower levels. The distribution of temperature at the tropopause was represented by isotherms which marked out a clearly defined centre of low temperature over Holland, with temperature increasing outwards on all sides of this centre. The marked limitation of the cold centre, particularly to the south, appears to exclude the possibility of explaining the anticyclone as the effect of a continuous flow of cold air from the south, and the divergence of air-flow at all levels in the troposphere indicates that the inflow of cold air into the region took place well within the stratosphere. The general run of the isotherms suggest that the outburst of cold air from lower latitudes came from south-west of the observed centre of lowest temperature. The observed winds over the central region of the anticyclone were however generally from a northerly or north-easterly direction, and the theory suggested previously does not appear to fit the facts at all closely. But it should be borne in mind that the details of the lines of flow given by Mügge are by no means definitely fixed by the observations. A good deal of latitude is possible, using these observations, which are barely sufficient for the purpose. Cave gives (*Structure of the Atmosphere*, etc. p. 105) the details of two balloon ascents on May 7, 1909. At 6.30 p.m. the wind at the ground was 12 m.p.h. from ENE, and there was no marked change of velocity or direction up to 5 km. Beyond this height the velocity diminished, and from 12 to 15 km it was light and very variable in direction.

* *Veröff. Geoph. Inst. Leipzig*, **3**, h. 4.

In Mügge's second case, May 19, 1910, there was an anticyclone centred over Northern Europe, and a second over Italy and the Adriatic. The temperature distribution at the tropopause indicated a region of low temperature stretching over Denmark and Germany to the Mediterranean. The lines of flow in the upper troposphere above 8 km indicated a southerly wind at the south-western edge of the cold tongue, sweeping eastwards over Central Europe. There was no indication (possibly on account of lack of observations) of the limitation to the south of the cold area, and the observations of May 19, 1910, appeared to fit reasonably closely the idea of an outburst of cold air in the stratosphere.

Khanewsky* studied in detail the anticyclone of September 30–October 1, 1908, which was centred over Germany. He found that on the eastern side of the anticyclone the wind was northerly or north-easterly up to heights of 19 km, while at Petersfield on the western side of the anticyclone the wind was south-westerly up to 16 km. The surface temperatures were lowest in the south-eastern section of the anticyclone, but in the lower stratosphere the lowest temperatures occurred in a closed region just east of Denmark, with, apparently, an increase of temperature to the south of this cold island. The stratosphere was very high and cold over Lindenberg, the temperature being $-73{\cdot}9^\circ$ C at the tropopause (13·6 km).

The table of pressures at different heights evaluated by Khanewsky showed that the highest pressure at 15 km occurred at Petersfield in the southerly current, though the temperature at that level was lower at Lindenberg. Cave† has given diagrams showing the variation with height of the wind velocity and direction as observed by sounding balloons on September 30 and October 1. On the second of these days the wind direction was round about southerly up to nearly 18 km; the velocity increased from 4 m/sec at the surface to about 27 m/sec at 12 km, and then diminished to about 8 m/sec at 17 km. On September 30, the wind was southerly at the ground, and backed towards NE and fell to a calm at 2·5 km, beyond which the direction was about SSW, and the velocity 7 m.p.h. up to 5 km, after which there was a rapid increase up to 17 m/sec at 6 km. At Lindenberg, near the centre of the island of lowest temperatures in the stratosphere, the wind was between N and ENE up to 15 km on both days. The north-easterly current was colder than the south-westerly current practically throughout the whole range of height.

The points which appear to be of greatest importance in Khanewsky's analysis are (1) the "island" of low temperature sharply limited to the south, (2) the coldness and depth of the north-easterly current, and (3) the fact that the highest pressure at 15 km occurred in the south-westerly current. The last of these suggests that an outbreak of cold air from the south might have been the primary cause of the anticyclone, and that the subsequent drawing of the cold northerly current into the circulation distorted the form of the anticyclone at the surface, and displaced the highest pressure to the east. Khanewsky ascribes the development of this anticyclone to the thrust of the cold north-

* *Met. Zeit.* **46**, 1929, p. 81. † *Loc. cit.* pp. 130–3.

easterly current towards low latitudes against the general westerly drift of those latitudes, and suggests that this is probably the explanation of all anticyclones.

Runge* studied an anticyclone which was somewhat similar to Khanewsky's, that of December 5, 1912. The anticyclone covered Czecho-Slovakia, Austria and Hungary, while there were depressions to NE and SW of Iceland. In the period December 2 to 5, the anticyclone in the far North had brought in its rear a cold northerly current. The lowest surface temperatures were in the central region of highest pressure, but at 3 km and 5 km the centre of the anticyclone was warmer than its surroundings. At 3 km the highest pressure occurred to north-west of the surface centre, and at 5 km the highest pressure was about WSW of the centre at 3 km. Here as in the anticyclone investigated by Khanewsky there were two currents side by side, a relatively cold north-easterly current, and a warm south-westerly or westerly current. The difference of temperature between Hamburg and Lindenberg, in the warm and cold streams respectively, was about 4° C at the ground and over 15° C at 8 km. The isotherms of the stratosphere were roughly parallel, with lowest temperature to the south, while the height of the tropopause exceeded 14 km in places.

The details quoted above for individual anticyclones do not directly confirm the theoretical views advanced earlier in this chapter, nor can they be said definitely to contradict those views. Rather do they emphasise a point which arose also in connection with the study of depressions, that the detailed structure in the upper atmosphere has a life-history, and that one instantaneous picture of the anticyclone can help but little in determining the course of that life-history.

One further piece of observational evidence remains to be presented. In a recent paper C. S. Durst† has collected values of the rotational velocity about the centre, in anticyclonic areas, showing that with increasing height there is a definite increase in this rotational velocity. In the typical anticyclone in which such a distribution of velocity occurs, subsidence brings to a given level air moving faster than is required by the gradient at that level, and accordingly this air tends to flow in towards the centre. This procedure will tend to maintain the anticyclone against attrition by friction. Durst points out that the vertical distribution of velocity described above will in general be that appropriate to a warm anticyclone, and he explains in this way the persistence of the warm anticyclone. This conclusion is open to doubt to the extent that the observations used do not of necessity represent the distribution of winds all round the anticyclone.

* *Met. Zeit.* **49**, p. 131. † *Q.J. Roy. Met. Soc.* **59**, 1933, p. 231.

§ 200. *Subsidence and divergence in anticyclones*

When cold air subsides it spreads out laterally, and develops an anticyclonic circulation. It was shown by Brunt and Douglas* that there is divergence from a region of high positive isallobars. It is a fact of observation that when an anticyclone is developing the rise of pressure extends over a very considerable area, so that the gradient of the isallobars is very weak, usually corresponding to a diverging velocity of rather less than 1 m/sec, if averaged over periods of the order of 24 hours. The velocities involved in the divergence within a growing anticyclone are therefore not more than about 1 m/sec on the average. The pressure in the upper air rises with the surface pressure, and we may assume as a first approximation that neither the velocity of divergence nor the velocity around the isobars changes with height up to say 3 km. On some occasions both may increase with height.

Now take a cylinder of height h and radius r, and consider what happens to the air within it when the whole mass is given a mean downward velocity w, and a mean velocity of divergence outward v. Let ρ_1 be the mean density from the ground up to height h, and ρ_2 the density at height h. The variations of ρ_1 and ρ_2 with time will be neglected. The equation of continuity gives

$$2\pi r h \rho_1 v = \pi r^2 \rho_2 w,$$

$$w = \frac{2hv}{r} \frac{\rho_1}{\rho_2}.$$

Let $v = 1$ m/sec, $r = 400$ km, $h = 3$ km, $\rho_1/\rho_2 = 1{\cdot}2$; then

$$w = 0{\cdot}018 \text{ m/sec} = 65 \text{ metre/hour} = 1{\cdot}6 \text{ km/day}.$$

This is probably an overestimate of the average value over developing anticyclones, but a value of 1 km per day is probably of the right order of magnitude at the level of 3 km.

In a stationary unchanging anticyclone the only divergence and subsidence is that due to surface friction. Sir Napier Shaw† has estimated that in a large anticyclone with light winds the subsidence is only about 80 metres per day, so slight that the horizontal history of any element of air must far outweigh the vertical history in importance. Such subsidence can have little or no effect in producing increased stability. The relative humidity of the air above an anticyclonic inversion, say at 1·5 km, is usually below 30 per cent, and sometimes below 10 per cent. If the air were saturated when it started to descend, a relative humidity of 25 per cent at 1·5 km would imply a descent of more than 2 km, and a relative humidity of 10 per cent at 1·5 km would imply a descent of about 3·5 km. The adiabatic descent of air through 3·5 km would imply, with normal conditions as to lapse-rate, an increase of temperature of about 25° F at the level finally reached.

The inversions found above anticyclones at relatively small heights are frequently referred to as "surfaces of subsidence". The name is misleading

* *Memoirs R. Met. Soc.* **3**, No. 22. † *The Air and its Ways*, 1923.

in that it implies a surface of separation between air which is descending and air which is not descending. The inversion frequently found occurs at a surface which is horizontal or nearly so, and which does not in general reach the ground, the inversion simply dying away before the surface is reached. It is much more probable that the discontinuity is developed *within* a mass of descending air. Many factors could be suggested to account for its formation. The effects of turbulence, variations in water content, and radiation, combined with subsidence, would in many circumstances suffice to produce inversions at approximately horizontal surfaces, of the magnitude of those observed.

A case of some interest was discussed in detail by Giblett*, using observations at Cranwell on October 19, 1923. These observations showed an inversion within the polar air behind a cold front, the equatorial air being found at higher levels, with no inversion at the lower boundary, which was distinguished by an increase in humidity (see fig. 88). Giblett explained the form of the lower part of the continuous curve in polar air as the effect of the upward diffusion of heat from the surface of the ocean during the passage of the air over the North Atlantic. The stabilising effect of the subsidence would oppose the diffusion upward of turbulence, but in spite of this, turbulence would extend to an increasing height as the surface temperature rose. That the point E represented the limiting height reached by turbulence was confirmed by the haze-top discovered at that height. The air below E had a lapse-rate which approached the dry adiabatic, it was relatively humid, and it had a sheet of cloud near its top, all indicative of thorough mixing. While the inversion discussed by Giblett was associated with a depression, the phenomena were in many respects similar to those observed in anticyclones.

When a mass of air containing a sheet of cloud subsides, the lapse-rate just above the cloud sheet becomes more stable, if it is initially stable, while the lapse-rate within the lower part of the cloud, if initially greater than the saturated adiabatic, becomes greater. The latter effect tends to maintain convection, and the cloud sheet will be maintained if enough moisture is carried up to counteract the effect of the slow subsidence. An examination of fig. 22 shows that if the descent were from 700 mb (say 3 km), at temperature 20° F, with no inversion initially, to 875 mb (say 1·5 km), an inversion of about 14° F would be formed. If later the clouds were dissipated, the inversion could persist, though probably in a weakened form, for some time. The mechanism thus suggested, associating the formation with the presence of clouds, is perhaps the easiest to understand physically, and is in all probability the commonest cause of inversions.

Other factors can also operate to form inversions, provided a rapid change of water-vapour content with height exists, as for example at the top of the layer affected by surface turbulence. Radiational effects have already been mentioned. If rain falls through a mass of air in which there is such a variation of water-vapour content, and is evaporated in falling, the cooling due to evaporation will vary according to the variations of wet-bulb temperature, and

* *Vide*, p. 334 above.

if all the air is brought to saturation thereby, each element of air will be cooled to its initial wet-bulb temperature. The course of events suggested would actually lead to instability.

Sufficient has been said to show that the formation of inversions does not require a sloping surface of subsidence, and that inversions can be formed in other ways than by simple subsidence, as for example by subsidence combined with some inequality of distribution of water-vapour or condensed water in the form of drops, or by turbulence. In view of the number of the factors which may be operative, it is necessary in any particular case to proceed with considerable caution in interpreting the existence of an inversion.

CHAPTER XX

THE GENERAL CIRCULATION OF THE ATMOSPHERE

§ 201. *The surface conditions over the earth*

THE general circulation of the atmosphere has been mentioned briefly in an earlier chapter, and we shall now endeavour to sum up the main facts of observation, with a view to considering the physical causes underlying the phenomena. Some idea of the facts of observation can be derived by a study of figures which represent the mean conditions in the months of January and July. The marked feature which appears from a casual study of these charts is the tendency for high pressure to occur over the continents and low pressure over the oceans in the winter, while in summer this tendency is reversed. In both winter and summer the Southern hemisphere has a well-marked belt of high pressure centred about latitude 30° S, and more marked over the oceans than over the continents (see figs. 5–8). The January chart shows a similar distribution in the Northern hemisphere, but there are well-marked centres of low pressure in the Northern Atlantic (the Icelandic low) and the North Pacific (the Aleutian low), a very extensive anticyclone centred over Asia, and another but less extensive anticyclone over the north-west of North America. In July the sub-tropical anticyclones in the North Atlantic and North Pacific show marked increase in intensity and in extent, while the winter anticyclone over Asia is replaced by a depression centred over North-western India.

The greater wind systems of the atmosphere can be inferred from the pressure distribution by the use of Buys Ballot's law (see also figs. 10, 11). On each side of the equator are winds which have an easterly component, the North-east and South-east trades, extending from the high-pressure belts nearly to the equator. They are conventionally regarded as separated by a belt of calms known as the *doldrums*, but the doldrums are not clearly marked all round the equator, or at all times of the year. Brooks and Braby* showed that in the equatorial Pacific the phenomena appeared to be most easily explained by the presence of a sharp surface of discontinuity between the North-east and South-east trades, particularly in the region east of longitude 180°. In this region, where the South-east trade meets the North-east trade, the former rises over the latter, giving heavy rainfall. Beals† has shown that the doldrum zone of calms in the equatorial Pacific is restricted to the shore ends of the tropical belt over that ocean. Durst‡ has shown that in the region between 25° W and 40° W in the Atlantic, the doldrums fluctuate in width and position, and that

* *Q.J. Roy. Met. Soc.* **47**, 1921, p. 1.
† *Monthly Weather Review*, **55**, 1927, p. 215.
‡ M.O., *Geoph. Mem.* No. 28.

the heaviest rainfall is in the zone of light winds. At times the belt of doldrums over the Atlantic diminishes to vanishing point giving practically a surface of discontinuity separating the North-east and South-east trades.

The phenomena over the continents are somewhat different. The trade winds do not exist over the continents on account of the absence of the tropical high-pressure belts, which cannot form over heated land. The diurnal variation of temperature of the earth's surface gives rise to thermal instability over large areas, giving heavy rainfall over much more extensive areas of the continents than the rainfall zones over the oceans.

The central belts of the sub-tropical anticyclones are regions of calms, and on the poleward sides of these calms the winds are westerly, corresponding with the fall of pressure towards the poles. The westerly winds of middle latitudes extend to about latitude 65° to 70°. They are not steady winds, but are disturbed by the occurrence of travelling depressions and anticyclones. The long tract of the North Atlantic covered by the mean isobar of 1005 mb (figs. 7, 8) indicates the position of the mean track of depressions across the North Atlantic. Nearer to the North Pole pressure again increases, though in the inner polar regions conditions are not symmetrical on account of the irregular distribution of land and sea. Indeed over neither polar region are observations sufficient in number to enable us to say with certainty what the mean conditions are.

In winter the winds over almost the whole of Asia are explained by a clockwise circulation around the Asiatic anticyclone, while in summer conditions over Southern Asia are entirely dominated by the depression which forms to North-west of India, and gives rise to the winds of the South-west monsoon. The South-west monsoon sets in during June, and lasts until about the end of September.

The above brief description gives the salient features of the mean surface conditions over the earth. The conditions from day to day vary considerably, particularly in those regions in which travelling depressions and anticyclones occur, and the mean condition to which we give the name "general circulation" may never occur *in toto*.

One marked feature is to be noted in the charts for both January and July. The flow of air is not everywhere symmetrical along the parallels of latitude. The flow is partly poleward, partly equatorward, as demanded by notions of continuity of air mass, and the substantial constancy of the pressure over the earth's surface. For though the pressure is everywhere of the order of 1000 mb, the limits of pressure found on the earth (at mean sea level) are everywhere within about 30 mb of the mean value. There is therefore no factor in the atmosphere capable of heaping up air in one region at the expense of the rest of the earth.

§ 202. *The circulation in the upper air*

We might begin the consideration of conditions in the upper air by the study of charts of pressure for different heights, such as are given by Shaw, *Manual of Meteorology*, **2**, but before doing so we shall first consider the mean pressure for different latitudes and for a range of heights from 0 to 24 km, as represented in fig. 102. This diagram is based on the estimated mean pressures given by Wagner in the *Handbuch der Klimatologie*, **1**, Teil F, "Klimatologie der freien Atmosphäre", p. F 67. The pressures were computed on the basis of the temperature distribution given by Ramanathan*.

The curves in fig. 102 represent conditions in the Northern hemisphere in summer and winter. The winter conditions are represented on the right-hand side of the diagram, and continuous lines have been drawn across the whole diagram. It is not possible to say that the winter conditions represent the winter conditions of the Southern hemisphere, so that the diagram must not be interpreted as giving the pressure distribution over the whole earth. The observations for the Southern hemisphere are not sufficient to enable a mean curve to be drawn, and it is in any case certain that in the Antarctic the mean pressure distribution does not agree with that in the Arctic.

The curves in the diagram represent the variations with latitude of the pressures at 0, 2, 4, 6, 8, 10, 12, 14, 16, 20 and 24 km. The curves for the uppermost levels are not reliable in detail, as the number of observations of temperature available for these heights is very small.

At the surface the maximum pressure in summer is in latitude 40°, the tropical minimum occurring at about 15° and not at the equator. These correspond respectively to the sub-tropical high pressure, and the doldrum minimum. A second minimum in latitude 65° corresponds to the zone of occurrence of the depressions of middle latitudes. There is a maximum at 90° corresponding to an anticyclone over the North Pole. At 2 km the sub-tropical high-pressure belt has disappeared, as also has the maximum at the North Pole, and the form of the curve at 4 km is almost identical with that at 2 km, indicating a steady rise of mean pressure from the pole to the equator. Beyond 4 km the position of the maximum pressure is shifted to about latitude 13°, and this feature persists to about 20 km. The zone of the depressions in middle latitudes, which is indicated by a minimum in the curve of surface pressures, is represented at heights of 2 to 8 km by a check in the general rise of pressure from the pole to the equator, and a secondary minimum in latitude 65° at 10 km. This secondary minimum shifts nearer to the equator with increasing height, and appears in latitude 40° at 20 km.

Winter conditions differ considerably from the summer conditions described above. At the surface the sub-tropical high-pressure belt is centred about latitude 30°, and from this point to the pole there is a steady decrease of pressure. There is a flat minimum of pressure at the equator. The sub-tropical high-pressure zone is only feebly indicated at 2 km, and at greater

* *Nature*, **123**, 1929, p. 834. See also fig. 12, p. 18 above.

heights, up to 12 km only, is represented by a slight check in the fall of pressure from the equator towards the pole. Above 12 km, there is a definite flattening out of the pressure curve between latitudes 25° and 10°.

The pressure distribution can be readily interpreted in terms of the corresponding winds. Let us consider first the summer conditions. At the surface the winds should be easterly from the doldrum region to latitude 40°, beyond which westerly winds should predominate up to latitude 65°; beyond this to the pole, winds should be predominatingly easterly. At 2 km the winds should be predominatingly westerly from the pole to the sub-tropics, the winds in the tropics being very light. At greater heights the westerly winds should extend from the pole to nearly 10° N, beyond which there should be only light winds, with a tendency for easterly wind directions. The curves indicate that at 12 km the westerlies should be interrupted by a zone of easterly winds in latitudes 60° to 70° and that at still greater heights this zone should extend to the North Pole.

In winter the circulation around the sub-tropical anticyclones disappears before 4 km is reached. Westerly winds should predominate at this level in all latitudes, and this condition should persist up to between 12 and 16 km. At the 16 km level zones of calm appear to be possible in latitudes 10° to 25°, and near the poles.

It is necessary to reiterate the qualification that the curves for upper levels are based on few or no observations at many points, and that the deductions drawn from them above are perhaps not to be relied upon in detail beyond 8 or 10 km at the highest.

We shall next consider the forms of the pressure distribution at the ground, at 2 km, 4 km, and at 8 km, with a view to amplifying or correcting the impressions deduced from fig. 102. The distribution of pressure at 2 km in July for the Northern hemisphere, shown in fig. 103, indicates that the anticyclones over the North Atlantic and the North Pacific, and the depression over North-west India, still persist, though much flattened out, while over North-east Africa there is also a weak anticyclone. At 4 km (fig. 104) the North-west African anticyclone is more marked, the North Atlantic belt of high pressure and the depression over North-west India are both much weakened, and the North Pacific belt of high pressure has disappeared. A new feature at this level is a small anticyclonic belt just north of the Himalayas.

At 8 km (fig. 105) the African anticyclone is still further strengthened, while the North Atlantic anticyclone is displaced farther west, and extends from longitude 50° to 110°, the highest pressure being in latitude 30° over the southern part of the North American continent. The anticyclone which appeared north of the Himalayas at 4 km now extends across from the Persian Gulf to Western China. At all heights from 2 to 8 km the North Pole appears to be a centre of low pressure, and west winds should prevail from the pole down to about 30° N. At 8 km easterly winds should predominate over tropical Africa, North America, and India, while over the Pacific and Atlantic the winds in the tropics cannot be deduced from the charts on account of the fewness of observations, and possibly very light and uncertain gradient of

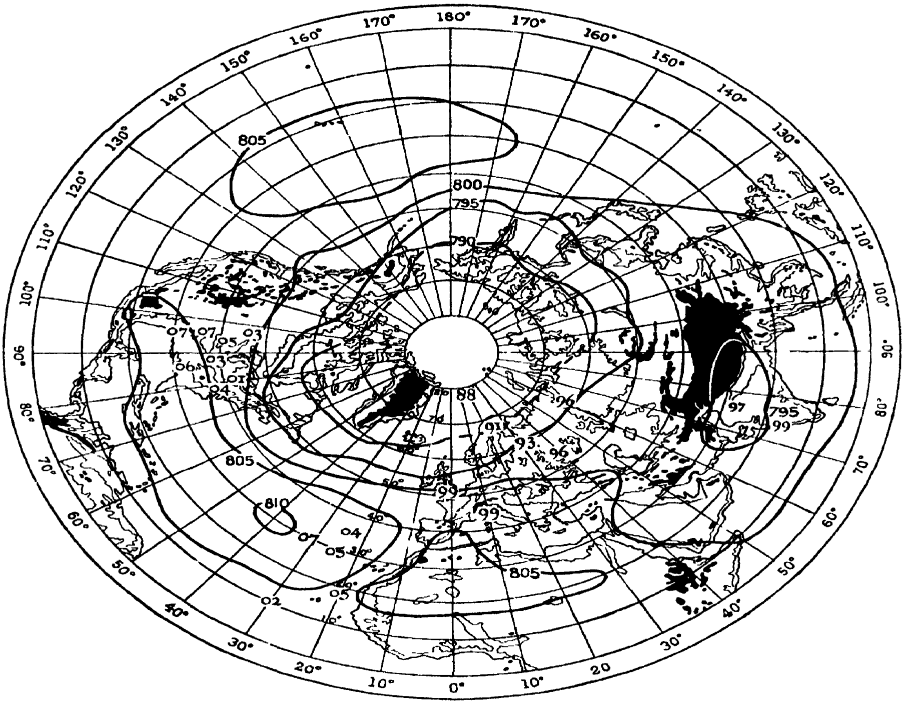


Fig. 103. Normal pressure at 2 km, Northern hemisphere, July.

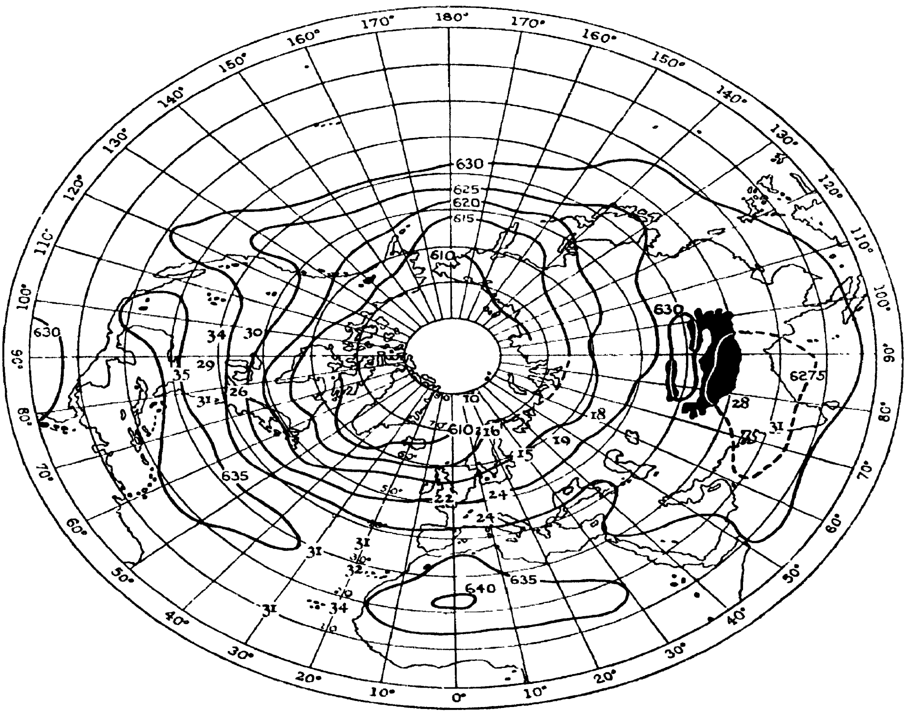


Fig. 104. Normal pressure at 4 km, Northern hemisphere, July.

pressure. The deductions from the charts of pressure distribution for July conflict with the deductions from fig. 102 to the extent that whereas fig. 102 indicates that the sub-tropical anticyclones do not appear at 2 km or any

Fig. 105. Normal pressure at 8 km, Northe·n hemisphere, July.

higher level, figs. 103, 104 and 105 indicate that in summer the sub-tropical anticyclones are clearly marked up to 8 km, with a shift of position from the oceans to the land at a level of 2–4 km. The summer depression over South-west Asia is very weak at 4 km, and is replaced by an anticyclone at 8 km.

The contrast between the pressure distribution over the Northern hemisphere in July at the surface and at 8 km, represented in figs. 8 and 105 respectively, is very remarkable. At the surface the pressure is highest over the oceans, while at 8 km it is highest over the continents.

It has been noted in fig. 7 that in January there is an intense anticyclone centred over Asia, this being one of the most marked features of the surface pressure distribution. Fig. 106 reproduces the pressure distribution over the Northern hemisphere in January at 2 km, indicating that at this height the Asiatic anticyclone has already disappeared, and that the only deviations from a regime of North-South pressure gradients are to be found in a weak remnant

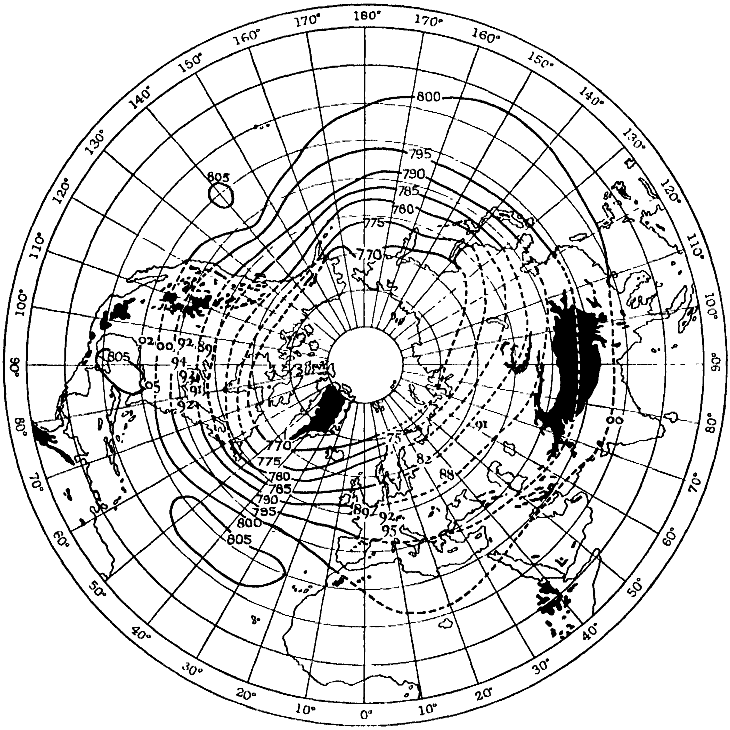


Fig. 106. Normal pressure at 2 km, Northern hemisphere, January.

of the sub-tropical anticyclones, which shows secondary centres over the eastern North Atlantic, the Gulf of Mexico, and the eastern North Pacific. Everywhere from the pole down to latitude 30° N the mean gradients for January are for West winds. At 4 km, fig. 107, the gradient is everywhere favourable for West winds. Thus the deductions drawn from fig. 102 are substantially correct for January (winter).

Over the Southern hemisphere, the sub-tropical belt of high pressure shows far more uniformity in all longitudes at the surface than is the case in the Northern hemisphere (see figs. 5, 6). At 4 km the sub-tropical high-pressure belt is barely appreciable in summer (fig. 108), and apparently has quite disappeared in winter (fig. 109), indicating the prevalence of westerly winds everywhere above this height, and possibly at much lower levels, except in the tropics, concerning which no reliable estimate can be made from the mean distribution of pressure.

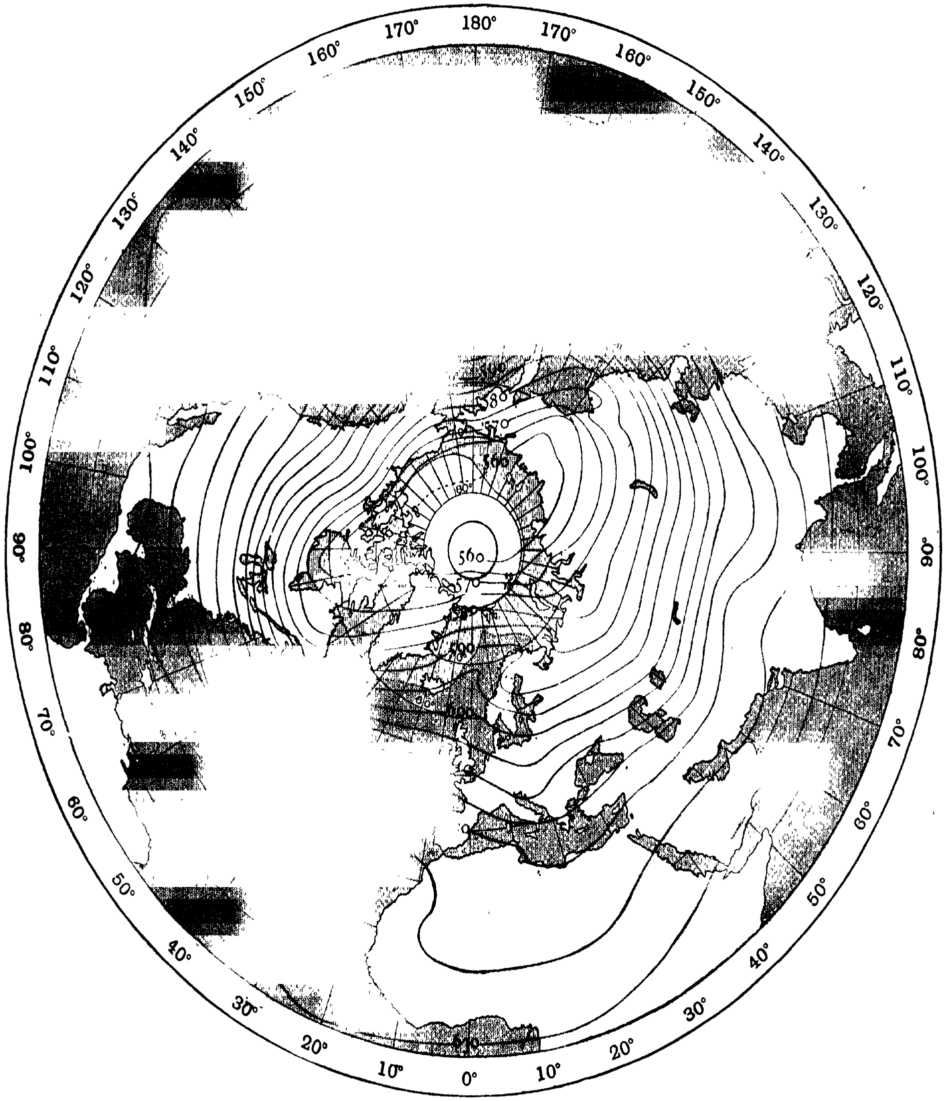


Fig. 107. Normal pressure at 4 km, Northern hemisphere, January.

§ 203. *The observed distribution of winds in the upper air*

The observations of winds in the upper air are still insufficient to enable us to give a complete picture of the wind circulation in detail; but there are sufficient observations available to enable us to check the general conclusions which we have drawn from figs. 103 to 109. In the first place we shall use the monthly mean values of winds given by Wagner in his *Klimatologie der freien Atmosphäre.*

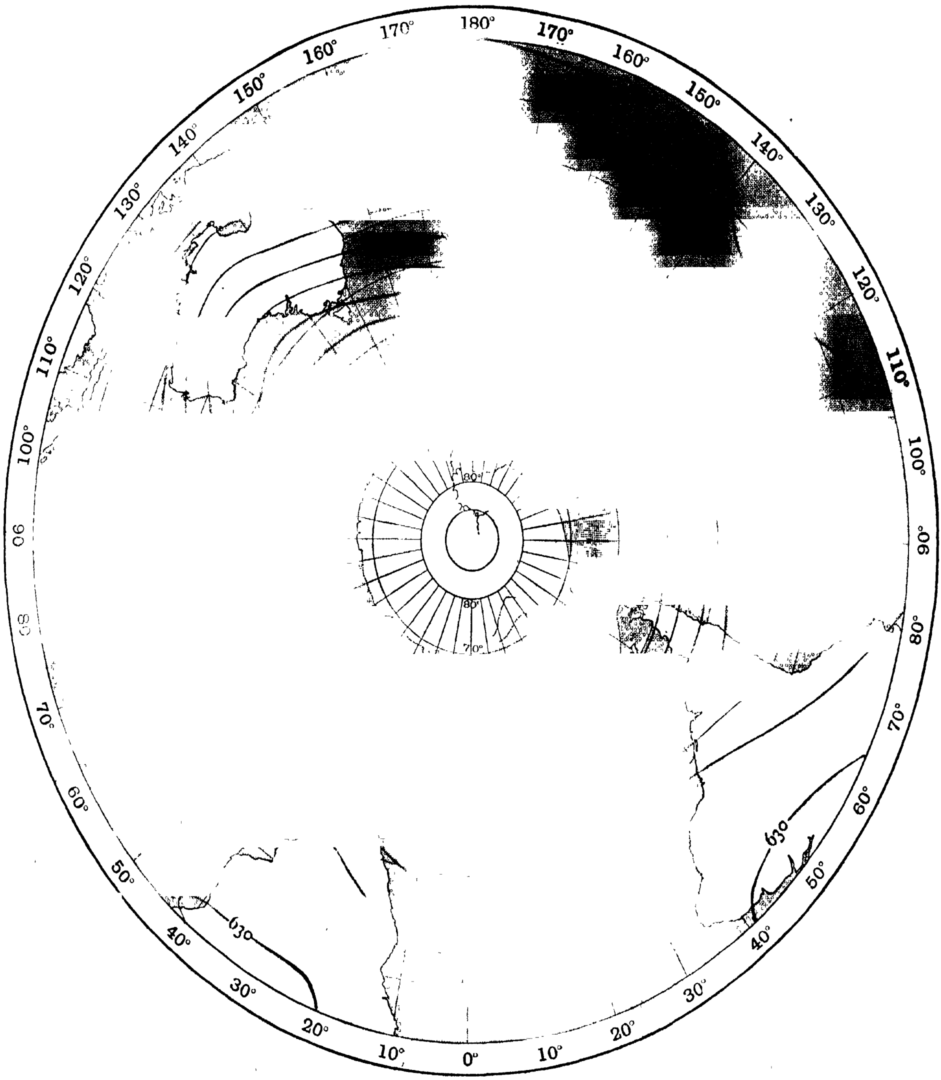


Fig. 108. Normal pressure at 4 km, Southern hemisphere, July.

(*a*) The tropics

At Batavia, in latitude 6° 11′ S, longitude 106° 50′ E, the wind in the lower levels is westerly in the months November to April. The westerly wind extends only to between 1 and 2 km in November, to 4 km in December, and to 5 or

6 km in March and April. In the months March to September there is a westerly or south-westerly wind at levels of about 18 to 22 km, with easterly winds at still greater heights. At the times and heights other than those men-

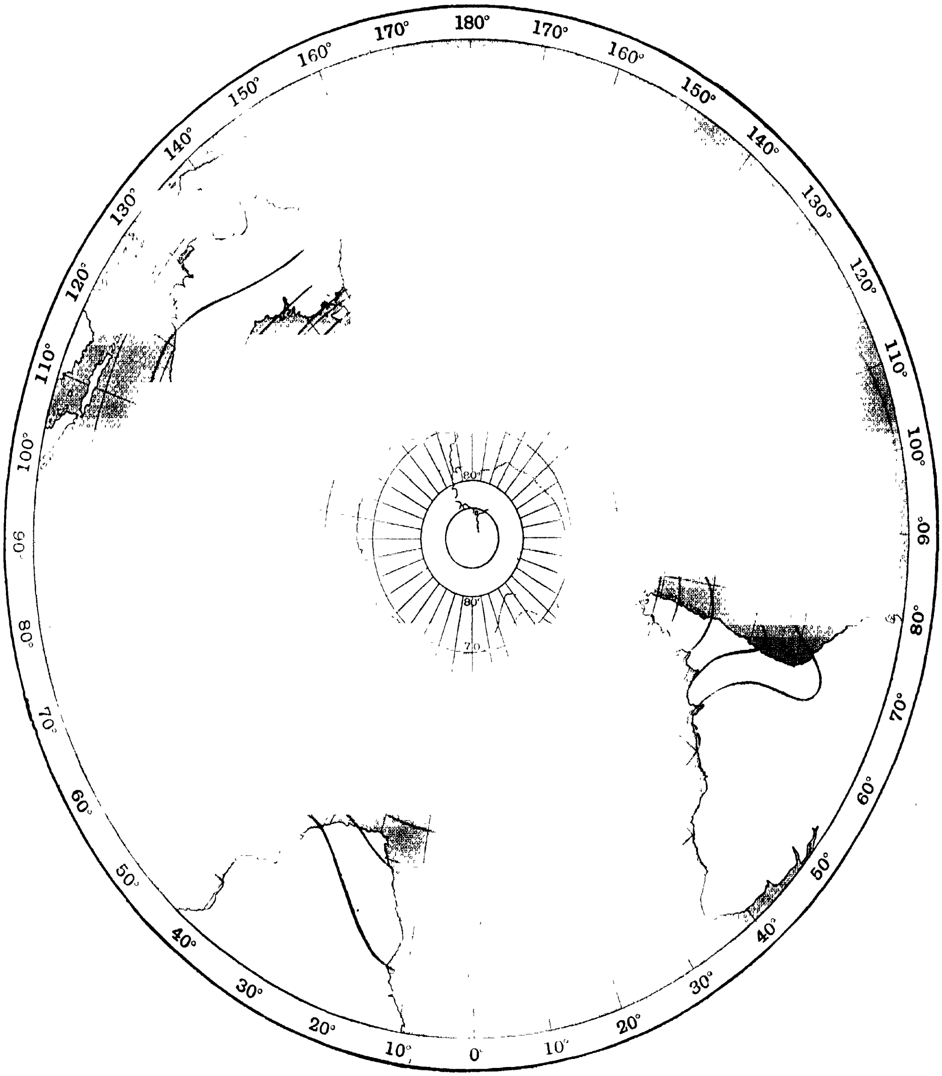


Fig. 109. Normal pressure at 4 km, Southern hemisphere, January.

tioned, the winds are from an easterly direction. Thus in the southern summer the westerly monsoon extends at its maximum to a height of 6 km and has an easterly wind above it. In the southern winter the winds are predominatingly easterly at all heights up to about 18 km, usually with a slight northerly component of velocity. Van Bemmelen's* studies of cirrus motion in the tropics

* *Proc. K. Akad. Wetenschap. Amsterdam*, **20**, 1918, pp. 1313–27.

indicate in general easterly drift everywhere, and confirm the existence of a northerly component in the motion at cirrus levels above Batavia (see figs. 110, 111). Berson and Elias at Lake Victoria Nyanza in tropical Africa (just south of the equator) found the South-east trade from the Southern Indian Ocean and above it a counter-trade wind with a northerly component, most frequently from NW but often from NE. With increasing height the wind took a more predominating southerly component, and at 16 km the few observations obtained indicated a definite tendency to SE winds.

At Honolulu (21° 22′ N, 157° 57′ W) the surface winds are round about NE, veering with height, but remain generally easterly up to heights of 3 to 5 km, and in May and June remain easterly up to 9 km. Beyond these heights the mean wind is westerly or north-westerly.

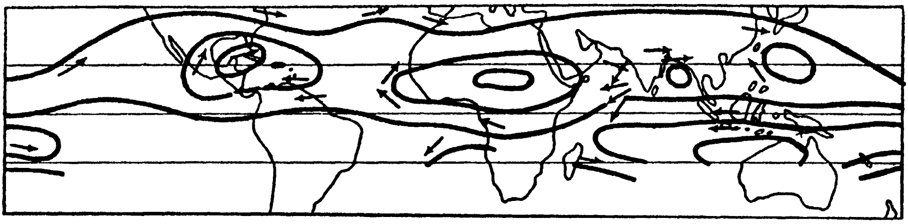


Fig. 110. Lines of flow of cirrus, December–February.

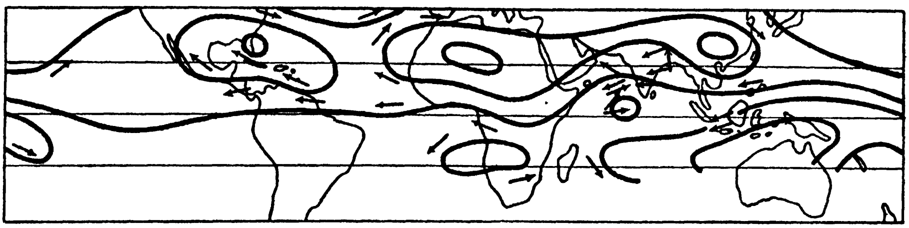


Fig. 111. Lines of flow of cirrus, June–August.

The winds at Samoa (13° 48′ S, 171° 46′ W) are predominatingly easterly in the lower layers, with a westerly wind from 7 km upwards in the southern winter, and from 3 km upwards in the southern summer.

At Mauritius (20° 5′ S, 57·5° E) the mean winds are generally from ESE to E in the lower layers, changing to SW or W at $2\frac{1}{2}$–3 km in the southern winter, and at 4–5 km in the southern summer.

The trade wind region in the North Atlantic was investigated in great detail by Sverdrup* who found that the North-east trade extended only to a very limited height, which varied with both latitude and longitude. The upper boundary slopes downward from latitude 40° to latitude 20°, and from longitude 40° W to longitude 20° W. Above this limit is a zone of mixing

* *Veröff. Geoph. Inst. Leipzig*, **2**, 1917.

or irregular winds, of thickness about 2 km, while above this is the counter-trade wind from a westerly direction.*

To summarise the above data very roughly, we may say that while the winds are usually easterly in the lower layers, there is a definite tendency to find westerly winds in the upper air, above heights which vary with locality from a few kilometres to 9 km.

(*b*) The sub-tropics

Wagner gives mean wind directions for India for the three months of December, April and August, which may be taken as representative of the cool season, the hot season, and the monsoon season, respectively. In December the mean isobars run roughly across from West to East, with high pressure to the North, but the gradient is very slight on the chart of mean isobars for the month. In low latitudes, south of about 20° N, there is at low levels an easterly wind, which is replaced by a south-westerly wind at 8 km at Bangalore in latitude 12° 58′ N, but at heights of from 2 to 3 km farther north. At stations in latitudes 20° to 30°, the surface wind is northerly, backing to west within a range of 2 or 3 km. This agrees with the results derived by Harwood† for the cool season. Harwood's Plate 1 shows that over a strip along the north-eastern frontier of India, covering an area of nearly a half of India, the surface winds are north-westerly, becoming northerly over Burma, while over the rest of India the surface winds are about ENE. The direction of the motion of clouds at middle levels is westerly over the northern half of the peninsula and easterly over the southern portion. High clouds and pilot balloon observations at 9 km gave substantially the same drift as clouds at middle levels. In the hot season the easterly wind of the lowest layers is restricted to the extreme south of India, and the regime of northerly winds in the lowest layers, changing to westerly within a few kilometres, extends farther south. The gradients of pressure shown by the monthly mean pressure charts are very slight and irregular in April, though the Asiatic anticyclone still persists. Harwood's Plate 2 (*loc. cit.*) confirms these results.

During the monsoon season the surface winds are southerly over Burma, east-south-easterly over the strip of country north of the plain of the Ganges, and west or south-west over the remainder of India. At the 2 km level the motion is substantially the same as at the ground. At the level of high clouds, the winds are easterly everywhere east of a line joining Bombay and Agra, and westerly to the west of this line. The change from east to west wind occurs at varying levels, being at about 1 km at Agra, 3 km at Bombay, 5 km at Bangalore in the south. The final result is striking, in that at heights of 10–14 km there is a marked line of discontinuity across the peninsula from south-west to north-east.

* Recent observations over the whole of the Atlantic Ocean, obtained on the research ship "Meteor", have been published in *Wissenschaftliche Ergebnisse*, **15**. A discussion of the observations will be published later.

† *Indian Met. Memoirs*, **24**, pt 8.

Thus the winds of the Indian cold season (the north-east monsoon) are of the same nature as the trade winds of the North Atlantic, having a south-westerly current above them. The upper winds of the south-west monsoon season do not appear capable of such summary description. The east winds in the upper air over a large tract of the country are presumably part of the circulation around the anticyclone which Teisserenc de Bort found over the region of the Himalayas at 4 km (see fig. 104). Harwood suggests (*loc. cit.*) that most of the air of the south-west monsoon is carried off westward by this upper current, and that the greater part of this air passes northward over North-west India, and joins the circulation of middle latitudes, a portion however possibly descending over the Arabian Sea to join the westerly surface monsoon current.

(*c*) Middle latitudes

In the zone of prevailing westerly winds of middle latitudes, conditions are difficult to summarise. The trade winds of low latitudes blow in the same direction with substantially the same velocity for considerable periods of time, and can therefore be described in relatively few words. In middle latitudes however the conditions are continually being disturbed by the passage of depressions and anticyclones, which affect the winds up to levels well in the stratosphere. Wagner* quotes numerous data for the mean wind-transport for places in middle latitudes. At Lindenberg the general direction of air transport is westerly in practically all months, at all levels up to 2 km, but with a component from North in summer and from South in winter. At Vienna the direction is between West and North in all seasons up to a height of 4 km. Mean values for Central Europe indicate westerly winds with a northerly component in summer, up to 18 km, and in winter westerly winds with a southerly component up to 10 km, and a northerly component at greater heights. Over Italy the winds are NE to E in the lowest kilometre, above which there is a rapid change to NW. At 3 km the mean air transfer is between W and NW over the whole of the Mediterranean, agreeing with the circulation around the anticyclone over North Africa at this level.

(*d*) United States

Wagner quotes data for a number of stations in the United States. The results are similar in their main features to those for Europe. At 3 km a broad westerly current sweeps across the country from its northern limit down to latitude 30° N. A portion of this current is deflected southward over the Gulf of California, while a south-westerly current over the Gulf of Mexico joins the main stream moving across the Eastern Atlantic.

(*e*) Northern high latitudes

In the Arctic observations are far from sufficient to give a complete picture of the circulation of the winds. The mean of 252 pilot balloon ascents at Ebeltofthafen in Spitzbergen in 1912–13 gave winds between N and NE in the

* *Klimatologie der freien Atmosphäre*, pp. F 33, 34.

lowest kilometre, with northerly winds above, backing towards west at 7 km. Over Eastern Greenland the winds observed by A. Wenger were predominatingly NW. Over the western coast of Greenland the winds were easterly at the surface veering towards south with increasing height, and reaching south at 8 km. At 10 km the wind direction is backed one point to S by E. The mean of 225 pilot balloon observations at Akureyri and Adalvik in Iceland at various times indicate SSE winds at the surface veering steadily to west at 8 km. Georgii* describes some pilot balloon observations in North-west Iceland (66° 22′ N, 23° 8′ W) which indicated for the periods 25th to 28th June 1927, and 17th to 21st July 1927, outbursts of cold air from about NNW, reaching to a height of at least 15 km, with velocities of 70 m/sec and 60 m/sec respectively at 15 km, where the outflow had its maximum. The winds in the lowest 2 km were intermittently northerly or southerly, but at higher levels the north-westerly currents were clearly developed with strength increasing with height. Southerly winds at great heights were never so strong as the northerly outbursts referred to.

The aerological results of the Greenland Expedition of the University of Michigan† in 1926 and 1927–29 indicate that in a shallow layer near the surface the winds at Mount Evans on the west coast are east to south-east at all times. In the months of November, December, January and February the wind veers to SW or W at 7 km, and to about W at 10 km. In this season the winds show remarkable regularity in the variation of the monthly means with height. In all months the change to between south and south-west is reached in about 2 km. The change with height of the wind direction is generally similar to that in winter in the months of March to July, up to about 7 km, but at greater heights wide variation appears. In July and August 1927 the winds at 7 km were N to NW, but in the same months of 1928 the winds at 7 km were S to SW. In September and October 1927 the mean winds remained in the SE quadrant up to 7 km, and then backed towards north, but in September 1928 the mean wind was SSW at 6 km, NNW at 7 km, and remained in the NW quadrant to considerable heights above 7 km; while in October 1928 the mean wind remained southerly up to 8 km, and then backed through to N at 13 km.

Thus except in the winter months there are considerable variations in the mean winds above Greenland, though there is a predominance of winds in the south-west quadrant at 7 km. An examination of the individual pilot balloon ascents shows marked variations from day to day, or even from hour to hour. Thus on February 3 the wind at 7 km was 17 m/sec from south at 11.38 a.m., but was 35 m/sec from S by W three hours later.

(*f*) Southern hemisphere

For the Southern hemisphere the available information is extremely scanty. Such observations as are available indicate that the winds in the inner tropics are in a generally easterly direction, and that in middle latitudes the mean

* *Arktis*, **1**, h. 3/4. † *Rep. Greenland Exped. Univ. Michigan* (1926–31), pt 1, Aerology.

winds are generally westerly. We have already referred above to the observations at Batavia and elsewhere in the southern Tropics, which indicate predominatingly easterly winds. Wagner's table (*loc. cit.* p. F 63) shows that in Northern Australia, Willis Island (16° 18′ S, 149° 59′ E) the mean winds are ESE up to a height of 3 km, while at Melbourne, Adelaide, Sydney and Christchurch in New Zealand the mean wind direction is about NW up to a height of 3 km.

The observations obtained during the drift of the German Antarctic Expedition in 1911/12 gave mean velocities of 5–7 m/sec at the ground, increasing to 12–15 m/sec at 2 km. The mean directions, according to Wagner's brief summary, were not easterly at any time of the year, and the most frequent wind direction was south-south-westerly at the surface, backing about one point in the first 3 km, then veering through about 17° in the next 5 km; beyond this height the veer became more rapid, so that at 10 km the wind was veered on an average 52° from the surface wind. On an average the wind was round about SSW in the first 6 km, increasing steadily with height, thereafter veering to west.

Observations made on the Scott Antarctic Expedition 1911 are not readily interpreted, on account of the marked topography of the continent, which forces the wind to follow certain directions independently of the free air conditions. The smoke of Erebus (4000 metres) was predominatingly from a westerly direction when the surface wind was less than 20 m.p.h., but with surface winds of 20 m.p.h. or more there occurred a group of south-easterly winds at the summit of Erebus. Medium clouds showed precisely the same features as the smoke of Erebus, northerly and southerly winds being approximately equally frequent. When the surface winds were 20 m.p.h. or more, the motion of high cloud showed three directions of marked maximum frequency, NNE, ESE, and NNW, while with light surface winds the most frequent direction of motion of high cloud were N (21 per cent), NNW (11 per cent) and E (11 per cent). On the south polar plateau the observations of the Scott and Amundsen expeditions combined indicate a group of wind directions of maximum frequency centred at about S by E, and extending three points to each side of this. This result is obtained by taking South to be parallel to the meridian of 160° E. On the Western Plateau the most frequent direction for light winds (up to force 4) was WSW and for strong winds (force 5 or more) S or SSW. The diagrams reproduced by Simpson in the report of the expedition (*Meteorology*, **1**) indicate that the blizzard winds usually blew from a southerly point, and that the effect of the blizzard conditions was superposed upon the other conditions, which in themselves gave winds which might blow out of or into the Antarctic continent. The meteorological conditions over the plateau were therefore not steady, and did not correspond to a steady cyclonic or a steady anticyclonic condition at the South Pole.

We are now in a position to sum up our knowledge of the mean winds over the globe. Around the equator is a boundary between the circulations of the Northern and Southern hemispheres. The boundary is not coincident with the geographical equator itself, but is somewhat north of it at all seasons, but farther north in the northern summer than in the northern winter. The boundary is in places closely defined as a line of demarcation, as for example in the Pacific, as shown by Brooks and Braby; in places it is clearly defined as a zone of definite width and known as the doldrums; elsewhere it is ill-defined and possibly non-existent. Above this boundary is a zone in which the winds are easterly up to considerable heights. Above these east winds there may occur a belt of westerly winds above which there again occur east winds. Between this zone and the central lines of the sub-tropical high pressures are the North-east and South-east trade winds in the Northern and Southern hemispheres respectively. The trade winds blowing towards the equator are relatively shallow, with heights varying with time and with longitude. Above the trade winds are westerly winds, usually with a component from the equator towards the pole, known as the anti-trades, and above the anti-trades are at times, and in some places, upper-trade winds from an easterly direction, possibly an incursion of the equatorial belt of east winds. In middle latitudes the mean winds over most of the earth are westerly at all heights with a tendency for a component from the pole in summer, and towards the pole in winter. Embedded in these winds are regions of easterly winds which may form part of the circulation around such systems as the anticyclone which appears over North Africa at 4 km; also moving cyclones and anticyclones have at certain stages complete circulations around their centres extending up to great heights, and these may therefore introduce temporarily winds from any point of the compass at any heights within the troposphere.

In higher latitudes we can only guess at the nature of the phenomena, but such observations as are available indicate that easterly and westerly winds may alternate, particularly in the lowest kilometre or two; while at greater heights westerly winds predominate with a poleward or equator-ward component alternating irregularly, and from time to time there occur outbursts of cold air from within the Arctic circle, extending to great heights, and so bringing down to middle latitudes vast masses of very cold air.

§ 204. *The problem to be solved*

Having in the last few paragraphs summarised, admittedly in a rough-and ready manner, the observed winds of the globe, we are now in a position to state what is the nature of the problem which has to be faced if the general circulation of the atmosphere is to be explained physically. We require a thread which will connect the motions observed in the atmosphere with the amount of incoming solar radiation reaching different parts of the earth, and with the nature of the atmosphere itself and of the earth's surface. This is, in broad terms, the whole problem. But several subsidiary aspects of the problem deserve special mention. We have already seen (§ 164) that the kinetic energy of the earth's atmosphere is continually being degraded into heat as a result of turbulence and friction. Since on the average there is no observable change in its kinetic energy from year to year, the loss is continually being made good, through the agency of the only possible source of energy—solar radiation. We require to know the precise manner in which some of the energy of incoming solar radiation is used to maintain the kinetic energy of the atmosphere. Again we shall find that the flow of air in the trade winds towards the equator demands the ascent of large masses of air in the inner tropics, the ascended air flowing polewards at high levels. This air must eventually descend again to the earth's surface, and part of our problem is to determine first the agency which fits this air to come down to a lower level, and secondly the part of the earth's surface at which the descent takes place. Another aspect of the larger problem is the explanation of the origin and position of the sub-tropical high-pressure belts. Yet another is the relation of the moving cyclones and anticyclones to the general circulation. To what extent are these lesser circulations necessary for the maintenance of the general circulation; or, to what extent are they brakes in the general circulation which keep the latter from growing indefinitely?

We shall not be able to claim that the problem is completely solved until we have found the explanation of the origin, maintenance, travel and death of the cyclone and anticyclone, and have explained the mechanism which produces variations of pressure over large areas of the earth's surface. Nor at this stage can we say definitely that the cyclone, anticyclone and general circulation are not merely three aspects of one problem.

Unfortunately it must be admitted that no solution of the complete problem as stated above has yet been found. The most we can hope to do at the present stage of the development of meteorology is to enumerate more clearly the aspects of the problem mentioned above, and to connect up some of the observed phenomena with the results of some of the earlier chapters in this book. The present writer is not aware of any "theory" which fits the whole of the phenomena. Theories of the general circulation were rife in the latter half of the last century, but increasing recognition of the tremendous complexity of the problem has led to an increasing disinclination to attempt a general theory.

§ 205. *Some theoretical aspects of the general circulation*

The first point which we have to consider is the distribution of incoming radiation over the earth's surface. Apart from the geometrical considerations which affect this distribution, the most important controlling factor is the cloud amount, since (*vide* § 64) a sheet of cloud sends back into space nearly the whole of the incoming radiation. The figures and diagrams given by Brooks* show that over both land and sea the mean cloudiness is rather greater over the equator than it is within zones to either side of it, these zones showing an annual variation in position following the sun in its motion north or south of the equator. Simpson†, using Brooks' figures for mean cloudiness, estimated the total effective solar radiation in different months of the year, over

* *Memoirs R. Met. Soc.* **1**, No. 10. † *Ibid.* **3**, No. 23.

the whole earth, and found that the maximum occurred in a zone which was approximately beneath the sun, and so was 20°–30° N in July, and 20°–30° S in January. These results are in a general way confirmed by Wüst's estimates of evaporation in different zones of the earth, according to which the evaporation is less over the equator than over zones some little distance to each side of the equator; but Wüst's figures for evaporation are *estimates* and not observations.

The curious way in which the highest temperatures avoid the equator, except at the equinoxes, makes it difficult to accept in all its simplicity the usual statement that on account of the maximum of insolation at the equator the air at the equator rises, its place being taken by air drifting in from both sides. Yet this is what appears to happen. In the Northern hemisphere the air which moves towards the equator to take the place of the ascended air should start as a northerly current, swinging round through NE and eventually to E, on account of the deviating force due to the earth's rotation. This current is the North-east trade-wind of the Northern hemisphere, and its counterpart in the Southern hemisphere is called the South-east trade-wind.

Since there is a continual transport of air towards the equator by the trade-winds, there must be an ascent of air in a region about the equator, and the air which ascends drains away poleward in the upper air; in the Northern hemisphere it will start as a southerly current swinging round through SW eventually to W, on account of the deviating force of the earth's rotation. Some of this air descends again to the earth's surface in quite low latitudes, as is shown by Sverdrup (*loc. cit.*) in his researches on the conditions in the North-east trade of the Eastern Atlantic. Sverdrup found definite evidence of the descent of air in a region 20–25° N, in the poleward moving current of the anti-trade, showing that in the time required to drift to this distance from the equator some of the air is cooled sufficiently to enable it to descend to low levels. On the poleward side of the centre of the sub-tropical high pressure belt the poleward drift of air across the isobars gives generally south-westerly winds at the surface, with westerly winds above. The diminution of temperature poleward produces a gradient of pressure directed towards the poles, at all heights in middle latitudes, if we consider only the mean conditions. But we cannot take this result too literally, since it would involve a net flow of air towards the poles, which is naturally impossible if the mean pressure at the surface is to remain reasonably constant over the earth, and a vast accumulation of the air in high latitudes is to be avoided.

It was pointed out by Exner that it is impossible to have the same distribution of winds all the way round a circle of latitude, since this demands that the gradient of pressure should be directed the same way round the whole circle, which is a physical absurdity. We therefore come to the conclusion that the zone of mean west winds must be broken by regions in which the flow is towards the equator. The synoptic charts drawn from day to day indicate that these regions are associated with travelling cyclones and anticyclones. In the rear of cyclones and in the front of anticyclones broad deep currents of air move equator-ward, carrying a compensating mass of air away from the polar regions. Thus the cyclones and anticyclones of middle latitudes are dynamically necessary for the maintenance of the general circulation.

It is frequently stated in meteorological treatises that if air moves from one latitude to another, retaining its original angular momentum (in space) about the earth's axis, then in its new latitude it will have enormous velocities along the circle of latitude. This statement is highly misleading. If there were no pressure gradients or other forces, a mass of air set in motion would be constantly deviated to the right in the Northern hemisphere and would move with constant velocity; if it were started in a northerly direction it would rapidly swing round until it had a motion to East, after which it would continue to swing round until eventually it had completed a circle. The last statement requires some slight qualification, since the path is only a closed circle if the variation of latitude is neglected, but as we shall see the radius of the path is so small that the argument is in no way affected by this approximation. Let V be the velocity of projection, and r the radius of the path. Then the equation of motion is

$$\frac{V^2}{r} = 2\omega \sin \phi . V.$$

Assume $$V = 20 \text{ m/sec} = 2000 \text{ cm/sec}.$$

In latitude 60°, $$r = \frac{V}{2 \times 7{\cdot}3 \times 10^{-5} \times 0{\cdot}866},$$

or $$r = \frac{2000 \times 10^5}{12{\cdot}6} = 1{\cdot}6 \times 10^7 \text{ cm} = 160 \text{ km}.$$

Thus on a smooth earth with no forces, the maximum displacement could not exceed a few hundred kilometres in latitude 60°. In lower latitudes the displacement would increase in inverse proportion to sin ϕ, but even air projected in latitude 10°, with a velocity of 20 m/sec to North, would not travel to North more than a distance of the order of 1000 km, which is equivalent to about 9° of latitude.

The point of the argument given above is that air leaving low latitudes cannot travel to high latitudes unless it is guided by a suitably arranged pressure gradient.

If there is a pressure gradient in existence the motion of air projected in any direction over the earth's surface will oscillate about the velocity and direction corresponding to the pressure gradient, and the enormous velocities referred to above cannot come into existence. In practice the motion of a mass of air through a large range of latitude, while retaining its original angular momentum about the axis of the earth, can never arise.

It has been pointed out by many writers that it is impossible to derive a theory of the general circulation based only on the known value of the solar constant, the constitution of the atmosphere, and the distribution of land and sea. Not only are the laws which determine the transfer of energy by radiation too complicated to permit this, but the transport of heat by advection through the medium of the general circulation, and the inter-relationships of cloud amount, radiative transfer and the general circulation, whose precise nature are unknown, make it impossible to derive any simple theory. It is only possible to begin by assuming the known temperature distribution, then deriving the corresponding pressure distribution, and finally the corresponding wind circulation.

Jeffreys* in a mathematical treatment of the problem started off with the assumption that in a frictionless atmosphere the temperature variations can substantially be represented by a decrease of the annual mean temperature from the equator poleward. This involves a slower decrease of the pressure with height at the equator than elsewhere, so that there should be an outflow of air poleward in the upper air, and an inflow of air towards the equator everywhere along the surface. This evidently requires easterly winds everywhere at the surface, with westerly winds everywhere in the upper air. This is in contradiction with observations, which indicate that easterly winds occur only within a zone covering about half of the earth's surface, centred at the equator, the prevailing winds elsewhere, except possibly in the polar caps, being westerly.

Exner showed (*Dynamische Meteorologie*, 2te Aufl., p. 216) that purely zonal distributions of winds are not possible on the earth, from considerations of pressure distribution. Jeffreys discussed the same question from a different angle. The existence of a mean velocity which is the same around a parallel of latitude implies that the frictional effect at the surface is everywhere increasing or diminishing the angular momentum about the earth's axis of rotation, and if the mean conditions are to persist, the loss by friction must be compensated by some other agency. The only effective agency is the interchange of air with other latitudes. Consider air at a height z, above a point in latitude ϕ, whose distance from the earth's axis is ϖ. The angular momentum about the earth's axis of unit mass at height z is

$$(\varpi + z\cos\phi)^2(\omega + \dot{\lambda}),$$

where $\dot{\lambda}$ is the rate of increase of longitude of the air, measured positive for eastward motion. Let the eastward velocity be u, and the northward velocity v. Then the total northward flux of angular momentum across the whole parallel is

$$2\pi\int_0^\infty \rho\,(\varpi + z\cos\phi)^3(\omega + \dot{\lambda})\,v\,dz.$$

But since the total mass northward of latitude ϕ is constant,

$$\int_0^\infty \rho\,(\varpi + z\cos\phi)\,v\,dz = 0.$$

Now write $$\int_0^z \rho\,(\varpi + z\cos\phi)\,v\,dz = q,$$

* *Q.J. Roy. Met. Soc.* **52**, 1926, p. 85; a slight correction to this paper, by C. A. Coulson, is given in same Journal, **57**, 1931, p. 161.

so that q is proportional to the total flow northward at heights less than z. The total northward flow of angular momentum is

$$M = 2\pi \int_0^\infty (\varpi + z\cos\phi)^2 (\omega + \dot{\lambda})\, dq,$$

taken through the atmosphere.

Integrating by parts we find, since $q = 0$ at $z = 0$ and at $z = \infty$,

$$M = -2\pi \int_0^\infty q \frac{d}{dz}\{(\varpi + z\cos\phi)^2 (\omega + \dot{\lambda})\}\, dz$$

$$= -2\pi \int_0^\infty q \left\{2\omega(\varpi + z\cos\phi)\cos\phi + u\cos\phi + (\varpi + z\cos\phi)\frac{du}{dz}\right\} dz,$$

since $$(\varpi + z\cos\phi)\,\dot{\lambda} = u.$$

Now put $\varpi = a\cos\phi$ and neglect z by comparison with ϖ. Then

$$M = -2\pi \int_0^\infty q\cos\phi \left(2\omega a\cos\phi + u + a\frac{du}{dz}\right) dz.$$

The terms inside the brackets are of order 4×10^4, 2×10^2, and 4×10^5 cm/sec, so that the last is by far the greatest. Neglecting all but the last, we find

$$M = -2\pi a \int_0^\infty q\cos\phi \frac{du}{dz}\, dz = 2\pi a\cos\phi \int_0^\infty \rho u v a\cos\phi\, dz = 2\pi a^2\cos^2\phi \int_0^\infty \rho u v\, dz.$$

The eastward frictional force per unit area is $-\kappa\rho_s u_s (u_s^2 + v_s^2)^{\frac{1}{2}}$, where κ is the coefficient of skin friction, about 0·003. The eastward friction will produce forces on the air north of latitude ϕ whose moment about the axis will be

$$-\iint \kappa\rho_s u_s (u_s^2 + v_s^2)^{\frac{1}{2}}\, a\cos\phi\, dS,$$

where dS = element of surface,

$$= -\int_0^{2\pi}\int_\phi^{\pi/2} \kappa\rho_s a^3 u_s (u_s^2 + v_s^2)^{\frac{1}{2}} \cos^2\phi\, d\phi\, d\lambda$$

$$= -2\pi a^3 \int_\phi^{\pi/2} \kappa\rho_s u_s (u_s^2 + v_s^2)^{\frac{1}{2}} \cos^2\phi\, d\phi.$$

If conditions are steady, the net loss by friction must equal the gain by advection. Hence

$$-\kappa a \int_\phi^{\pi/2} \rho_s u_s (u_s^2 + v_s^2)^{\frac{1}{2}} \cos^2\phi\, d\phi = \cos^2\phi \int_0^\infty \rho u v\, dz.$$

If the motion is symmetrical about the axis and the winds geostrophic, this equation cannot be satisfied, even allowing for the usual modification near the surface to allow for friction. For in these conditions v is negligible except in the lowest kilometre, and there it is only about $\frac{1}{4}u$. Thus the ratio of the two sides is practically $\kappa a : \frac{1}{4}$ kilometre or 50 to 1. Even allowance for the variation of ϕ cannot reduce this discrepancy to less than 20:1. Thus such winds as are observed are incompatible with a steady symmetrical distribution of pressure, and the maintenance of the polar circulation against friction requires a greater supply of angular momentum from without than can be provided merely by the drift of air across the isobars produced by surface friction.

Exner showed that a pressure distribution in which the isobars cut across a circle of latitude at the same angle all round the earth is impossible. That does not exclude the possibility of the isobars coinciding with the circles of latitude, with the surface air drifting across the isobars as the result of surface friction. Jeffreys' discussion summarised above shows that this is also impossible, since friction would destroy angular momentum far more rapidly than it would be replaced by advection from other latitudes.

Jeffreys used an extension of his argument to consider the phenomena when the restriction to symmetry is abandoned. But it must be noted that in an unsymmetrical distribution the equation for poleward transfer of angular momentum is no longer true. The pressure forces now have a moment about the earth's axis, so that the assumption that a given mass of air carries its angular momentum with it is no longer justifiable, and the argument becomes invalid.

Jeffreys' argument is based on applying the last equation to portions only of a circle of latitude. If v is about equal to u instead of being $\frac{1}{4}u$, and if this condition applies to 6 or 7 kilometres of height, the equation can be satisfied. This is equivalent to assuming, in high latitudes, that the stream of air extends through most of the troposphere. The exchange of air between high and low latitudes is then to be explained as due to deep currents in opposite directions, over different parts of the same circle of latitude. And since the isobars must be closed systems, this demands the cyclones and anticyclones of middle latitudes as an integral part of the machinery which maintains the general circulation in being.

Jeffreys' theory was in the first place developed for a frictionless atmosphere, and as its author pointed out, the introduction of friction demands that in any approximately steady state there must be a balance between East winds and West winds at the surface.

This argument does not bear out the suggestion that the depressions and anticyclones of middle latitudes represent either an instability of the general circulation, or an oscillation about a steady state. Jeffreys' theory, on the contrary, makes it clear that a steady general circulation is impossible *ab initio*, and regards depressions and anticyclones as the irregularities which are inevitable in any circulation when friction is taken into account.

It is not necessary to assume that the westerly circulation of middle latitudes should extend to the poles. Such an extension would be difficult to realise dynamically, as it would demand that the frictional drift of air poleward across the isobars should be compensated by the ascent of air at the poles. In view of the intense cooling of the earth's surface in the polar regions by radiation, the ascent of air there could not be expected. The general result stated by Jeffreys, that skin friction will destroy any circulation in a period of the order of one week unless fresh angular momentum of the same sense is brought in, does not preclude the possibility of anticyclonic conditions at the poles.

It has been shown earlier (§ 110) that both cyclones and anticyclones have a rotation in space in the same sense as the rotation of the earth; and it is therefore possible to have an anticyclonic circulation maintained at the poles by the inflow at high levels of air whose subsequent subsidence provides angular momentum of the right sense to compensate for the losses by surface friction. There appears therefore to be every probability that the average conditions at the poles are anticyclonic.

In a note on Jeffreys' paper Whipple* suggested that if the trade winds represented the drift of air from the cooler parts of the earth to the warmer parts they would be developed in middle latitudes rather than in the tropics. There must therefore be some other cause than convection to force air into the anticyclonic belt. Whipple added that the most probable explanation is that in middle latitudes the strong West upper winds speed up the air at lower levels, dragging it onwards by turbulence on a large scale, and that this air is flung outwards and piled up in the anticyclonic belts. Whipple's suggestion is equivalent to the assumption of departures of the winds from geostrophic, giving a mean flow of air in the lower troposphere of temperate latitudes towards the sub-tropical anticyclones, more than sufficient to compensate for the surface frictional flow in the opposite direction.

The problem was taken up at this stage by C. K. M. Douglas†, who started from the mean pressure distribution over the North Atlantic in January, and estimated the mean frictional inflow to be of the order of 1 m/sec through a layer 600 metres thick. A flow of 0·15 m/sec up to 4 km would suffice to compensate for this loss of air, but something much greater would be needed in order to supply the necessary angular momentum.

In the ordinary theory of turbulence, if u is the component in the West-East direction, there is a term $K\dfrac{\partial^2 u}{\partial z^2}$ on the right-hand side of the equation for du/dt (see equation (37), p. 229), and if $\partial^2 u/\partial z^2$ is positive, the air is speeded up, and in consequence suffers a subsequent deviation to the right, or towards South. Douglas estimates that even with limiting values of 10^5 for K, and 5×10^{-9} cm/sec for $\partial^2 u/\partial z^2$, the southward velocity is only 0·04 m/sec, and is therefore much too slight to provide compensation for the frictional losses.

The deviations from geostrophic winds associated with changing pressure distribution, which are discussed in § 112 above, may range up to 5 m/sec, but are usually less than this. There is no *a priori* reason why such deviations should not be equally distributed in all directions, giving no tendency for a net flow in any one direction.

Douglas suggested that there might be a valve-like action at fronts, letting through the southward moving air, but forcing the northward moving air to ascend. Much of the displacement of tropical air is carried out at fronts running North and South, and these cannot contribute anything to deviations from geostrophic winds which shall have components to North or South. Using the Danish charts of the North Atlantic for December 1909 to February

* *Q.J. Roy. Met. Soc.* **52**, 1926, p. 332. † *Ibid.* **57**, 1931, p. 423.

1910, Douglas found that the maximum effect produced by fronts extending up to 4 km was to give a southward drift of at most 0·2 m/sec, which again is scarcely sufficient to compensate the frictional outflow. This agrees with what might have been anticipated from the theory of Brunt and Douglas. For, since the mean theoretical deviations from geostrophic winds are only of the order of 1–2 m/sec, with no apparent tendency to systematic direction, the mean flow in one direction, due to the influence of fronts, should be much less. We therefore come to the conclusion that in the lowest 4 km of the troposphere there is no net flow of polar air towards the sub-tropical anticyclone, and that the only possible conclusion is that of Jeffreys, that the exchange of air between different latitudes, which is required to make up the losses of momentum due to surface friction, is carried out by currents side by side, and not one above the other.

§ 206. *The present position regarding the theory of the general circulation*

It is not possible at present to put forward any satisfactory theory of the general circulation, in part because the details of the circulation have only been observed very incompletely. There are still very considerable gaps in our knowledge of these details, particularly over the oceans. The observations over the Atlantic Ocean accumulated during the expeditions of the "Meteor" should place our knowledge of the circulation over the Atlantic on a surer footing. The actual observations have been published*, and a volume of discussion is promised for an early date.

The older textbook explanation of why the air carried towards the equator in the trade-winds rises in the doldrums and spreads poleward in the upper air, is not readily acceptable to the modern meteorologist. The zone of highest temperature is centred in latitude 30° N in July, and in latitude 10° S in January. In July this zone is well within the sub-tropical high-pressure belt of the Northern hemisphere, as shown on the chart of mean pressure in fig. 8.

Georgii† has suggested that the source of energy of the tropical circulation, and possibly of the entire atmospheric circulation, is to be found in the sub-tropics. This is in accordance with the distribution of maximum temperature mentioned in the preceding section, which is to be explained by the greater cloudiness in the tropics than in the sub-tropics, that are estimated at 2–3 tenths and 7 tenths respectively. An appreciable fraction, probably one-tenth, of the incoming solar radiation in the trade-wind zones is used up in evaporation, and the ascent of the humid air brought equator-ward in the trade-winds is shown by the cloudiness of the inner tropics. The showery nature of the rain shows that the ascent is intermittent, not continuous.

Fig. 112 shows the distribution of potential temperature in the atmosphere, deduced from the distributions of temperature and pressure shown in figs. 12 and 102. The minor details of this diagram are not reliable, on account of the

* *Wissenschaftliche Ergebnisse*, **15**.

† *Arktis* **1**, h. 3/4.

fact that the mean temperature and pressure in the upper air, particularly in the polar regions, are based on relatively few observations. But the general trend of the lines of equal potential temperature (isentropics) is probably fairly reliable, except at great heights, and in high latitudes. The most striking feature of fig. 112 is the very great increase of potential temperature with height through the whole atmosphere. If, as suggested by Shaw, air tends to move along the isentropic surfaces, it is not difficult to visualise the air which ascends in latitude 10° N as spreading slowly poleward, at first slowly ascending, while air in the polar regions could spread equator-ward down the appropriate isentropic surfaces.

Some of the air which ascends in the inner tropics appears to descend again to low levels even in latitude 23° N. Sverdrup* found in latitude 23° N that from 1–3 km the air is warmer than the air to north or south of it, and is very dry, a condition which can only be explained by its descent from high levels. An examination of fig. 102 shows that at all levels above 2 km there is a fall of pressure from equator to pole. At such levels any additional air interposed into this pressure field must tend to move down the gradient of pressure, and if it moves adiabatically it will slowly ascend at first, along the isentropic surfaces shown in fig. 112. But it is not legitimate to assume the motion to be adiabatic. Radiative processes will be active, and Sverdrup's results referred to above appear to indicate that radiation must produce rapid cooling, and enable some of the air so cooled to penetrate downward through the successive isentropic surfaces, and eventually to reach the surface in quite low latitudes. This type of motion was referred to earlier, in § 173, but the assumption of adiabatic motion there made is obviously unjustifiable. If the motion were truly adiabatic, then air which had succeeded in penetrating upward to the level of even 2 km in latitude 10° N could never again come down to the surface. The conclusion appears to be inevitable that of the air which ascends in the tropics and flows poleward a part cools and descends in the sub-tropical anticyclones, sufficient at least to compensate the outward drift from these anticyclones towards the equator and towards the poles; the remainder of the upper current from the tropics continues beyond the sub-tropics into the zone of the westerlies. The current from the inner tropics starts as a southerly wind and veers through SW to W, when it forms part of the circulation around the upper air cyclone centred at the poles, and replaces the slow drift across the isobars into the polar centres of low pressure. There will be a corresponding sinking of air at the poles, produced by the intense radiative cooling at the earth's surface. The occurrence of periodical outbursts of cold air from the polar regions becomes a dynamical necessity if the pressure at the poles is to be kept from increasing indefinitely. We thus come to the view that the zone of westerly winds of middle latitudes will be filled partly by air which has come from the sub-tropical anticyclones, and partly by air which has come from the polar regions; and it is the clash of these two types of air which produces the travelling depressions and anticyclones of middle latitudes. The

* *Veröff. Geoph. Inst. Leipzig*, **2**, 1917.

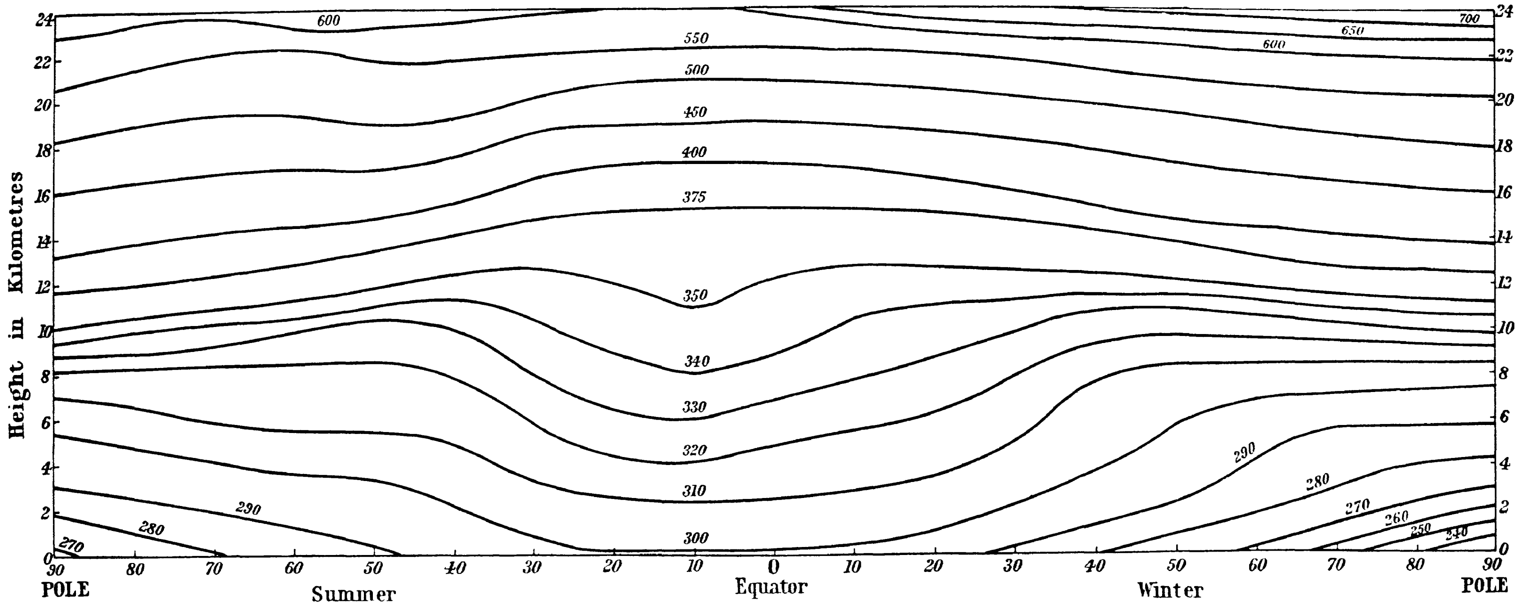


Fig. 112. The distribution of potential temperature in the Northern hemisphere in summer and winter.

polar currents appear to penetrate considerably farther equator-ward in the Southern than in the Northern hemisphere, and the westerly winds of the Southern hemisphere are relatively stronger and more extensive. It is a well-known fact that in middle and high latitudes of the Northern hemisphere tropical air is a comparatively rare phenomenon, it being forced to ascend over the southward penetrating polar air. Much of this polar air curves around again, returning poleward as a south-westerly current, and penetrating back to the polar regions; and, as stated above on p. 398, in the lowest 4 km there appears to be no net flow of polar air towards the sub-tropical anticyclone of the Northern hemisphere.

The greater symmetry of conditions in the Southern hemisphere produces greater regularity in the eastward motion of anticyclones in sub-tropical and low temperate latitudes, and leads to a series of phenomena which can best be described as a series of waves rotating round the globe, each wave consisting of alternate bands of north-westerly and south-westerly winds. Where the eastern edge of the cold south-westerly wind meets the warmer north-westerly, the low pressure trough is marked by cold front phenomena*. Simpson† concluded that the weather of the Antarctic continent was controlled by a series of pressure waves issuing from the South pole or near it, but the possible connection between these waves and those of low temperate regions has not yet been elucidated.

The machinery of the atmospheric circulation visualised above demands a source of heat in the tropics or sub-tropics, to account for the upward motion observed in the tropics, and a source of cold at the poles to account for the descent of air which has penetrated from lower latitudes to the poles. It is not possible to specify with definiteness whether the source of cold is entirely at the surface, or is spread through a considerable range of height. These two sources of heat and cold cannot of themselves account for more than a small part of the observed complexity of the atmospheric circulation. It has been suggested that the cooling of air in the poleward currents occurs at high levels, and this could only be attributed to radiation. Further, the surface of the earth itself is a powerful source of heating for the polar currents which penetrate into middle latitudes. It is in fact impossible to describe the atmospheric engine by comparison with the ordinary heat engine, which has clearly definable sources of heat and cold. In the atmosphere the sources of heat and cold cannot as yet be completely specified, but considerable progress would be possible if a time-scale could be assigned to radiational effects in the upper atmosphere.

From time to time it has been suggested that the katabatic flow of air down cold slopes such as those of the Antarctic continent must contribute very considerably to the kinetic energy of the motion of the atmosphere. Such katabatic flows are usually extremely shallow, and are destroyed by friction and the heating effect of the earth's surface in a very short distance. Thus

* See Kidson, *Q.J. Roy. Met. Soc.* **58**, 1932, p. 219; also **59**, 1933, p. 372.

† *British Antarctic Exped.* 1910–13, Meteorology, **1**.

the British Arctic Air-Route expedition to Greenland under the late G. H. Watkins observed at the base camp on the east coast of Greenland a wind of 129 m.p.h., while at Angmagsalik, a few miles away, only light or moderate winds were observed. From this it may be inferred that these winds are very shallow, and are rapidly destroyed by friction and the effects of turbulent mixing. Very strong katabatic winds also occur along the edge of the Antarctic continent, but it is probable that they do not penetrate far from the coast. There are also other great masses of land which can produce katabatic flow on a large horizontal scale, such as the Himalayas, and the mountains along the western coast of the continent of America, but it is difficult to visualise them as producing any marked contribution to the general circulation of the atmosphere.

§ 207. *Epilogue*

In the last three chapters an attempt has been made to summarise the known facts concerning depressions, anticyclones, and the general circulation. The summary is of necessity incomplete, on account of the paucity of certain types of observations, particularly in the upper air. We are not yet in a position to specify the necessary facts regarding the circulation of the upper winds, or the distribution of temperature, and until we are in a position to do so it will not be possible to specify why the atmosphere behaves as it does. This is particularly the case as regards the general circulation, but it is also to a great extent true of the depression and anticyclone. In each of the last three chapters of this book the difficulties have been largely due to lack of observations of the right type.

But although the three atmospheric features mentioned are still incompletely understood, it would be erroneous to suppose that no advance has been made. From time to time the introduction of new ideas has led to definite progress. During the last 15 years the greatest progress has been made by the frontal methods introduced by the Norwegian school of meteorologists. These methods have thrown new light on the phenomena in depressions, and though they have not yet produced an *explanation* of the depression, they have clarified and vivified the *description* of the depression. The latest writings on the frontal analysis of depressions, on anticyclones, and on the general circulation, all indicate a recognition of the complexity of the phenomena, and less readiness to attempt general theories. This is a step in the right direction, since it encourages the collection of data, and the discussion of facts.

Further progress must depend largely on the introduction of new ideas, as well as on the application of results derived in such fields as radiation and turbulence to the discussion of the general circulation and the local circulations. That meteorology to-day offers a plentiful supply of problems for research is very clearly shown in a series of papers on "Problems of Modern Meteorology" published in the *Quarterly Journal of the Royal Meteorological Society* during the years 1930–34. The reader is referred to this series of articles for summaries of some of the outstanding problems of modern meteorology. Such a series naturally does not exhaust the problems, many of which are not yet sufficiently advanced to be capable of specific statement.

§ 208. *Some additional references*

In addition to the references in the text, the reader is referred to the following books or papers, which amplify the treatment given in the text, or, in some cases, refer to matters not discussed in the text.

Chapter IV

Rossby, *Thermodynamics applied to Air-mass Analysis*, Massachusetts Inst. Tech., Met. Papers, No. 3.

Chapters IV–VII

Sir N. Shaw, *Manual of Meteorology*, **3**.

Chapter VIII

V. Bjerknes and others, *Physikalische Hydrodynamik*.

Chapters XII, XIII

L. F. Richardson, *Weather Prediction by Numerical Process*.

Chapter XVIII

The following papers in *Geofysiske Publikasjoner, Oslo*: **1**, No. 2; **2**, Nos. 3, 4; **3**, Nos. 1, 6; **5**, No. 6; **9**, No. 9.

Bergeron, *Met. Zeit.* 1930, h. 7, p. 246.

C. K. M. Douglas, "Some Aspects of Surfaces of Discontinuity", *Q.J. Roy. Met. Soc.* **55**, 1929, p. 123. (This paper has been largely quoted in the text, but is specially recommended as a very valuable summary.)

Chapter XX

Five papers on "World Weather" by Sir G. T. Walker, in *Indian Met. Memoirs*, **4**, 1923, pt 4, and pt 9. *Memoirs R. Met. Soc.* **2**, No. 17; **3**, No. 24; and **4**, No. 36.

Oscillations of the atmosphere as a whole have been discussed by

Rayleigh, *Collected Papers*, **3**, p. 335.

Margules, *Sitzber. Wiener Akad.* **101**, pt II*a*, p. 597; **102**, pt II*a*, pp. 11 and 1369. These papers were summarized by Trabert, *Met. Zeit.* **20**, 1903, p. 481.

G. I. Taylor, *Proc. Roy. Soc.* A, **126**, 1929, p. 169; *Memoirs R. Met. Soc.* **4**, No. 35.

Chapman, Pramanik and Topping, *Beitr. Geoph. Leipzig*, **33**, 1931, p. 246.

Chapman, *Memoirs R. Met. Soc.* **4**, No. 34.

APPENDIX

Table I

Corrections to be subtracted from actual heights in metres to give heights in dynamic metres

Latitude / Height	0	10	20	30	40	50	60	70	80
1000	21·9	21·8	21·3	20·7	19·8	18·9	18·1	17·4	17·0
2000	43·8	43·5	42·6	41·3	39·6	37·9	36·2	34·9	34·0
3000	65·7	65·3	63·9	62·0	59·4	56·8	54·3	52·3	51·0
4000	90·1	89·5	87·7	85·1	81·7	78·3	74·9	72·3	70·5
5000	114	113	111	107	103	99	95	91	89
6000	137	137	134	130	125	120	115	111	108
7000	161	161	157	153	147	140	135	130	127
8000	185	184	180	175	168	162	155	150	146
9000	209	208	204	198	190	182	175	169	165
10000	234	233	228	222	213	204	196	189	185

Table II

Values of $\left(\frac{1000}{p}\right)^{0.288}$ *for values of p from* 10 *to* 1090 *mb*

p	00	10	20	30	40	50	60	70	80	90
00	∞	3·767	3·085	2·745	2·527	2·370	2·249	2·151	2·070	2·001
100	1·941	1·888	1·842	1·800	1·762	1·727	1·695	1·666	1·639	1·613
200	1·590	1·568	1·547	1·527	1·508	1·491	1·474	1·458	1·443	1·428
300	1·414	1·401	1·388	1·376	1·364	1·353	1·342	1·332	1·321	1·312
400	1·302	1·293	1·284	1·275	1·267	1·259	1·251	1·243	1·235	1·228
500	1·221	1·214	1·207	1·201	1·194	1·188	1·182	1·176	1·170	1·164
600	1·159	1·153	1·148	1·142	1·137	1·132	1·127	1·122	1·118	1·113
700	1·108	1·104	1·099	1·095	1·091	1·086	1·082	1·078	1·074	1·070
800	1·066	1·063	1·059	1·055	1·052	1·048	1·044	1·041	1·038	1·034
900	1·031	1·028	1·024	1·021	1·018	1·015	1·012	1·009	1·006	1·003
1000	1·000	0·997	0·994	0·992	0·989	0·986	0·983	0·981	0·978	0·976

Table III

Table for conversion of temperatures from the Fahrenheit scale to the Absolute scale, from °F to °A

Degrees absolute

°F	0	1	2	3	4	5	6	7	8	9
− 60	221·9	221·3	220·8	220·2	219·7	219·1	218·6	218·0	217·4	216·9
− 50	227·4	226·9	226·3	225·8	225·2	224·7	224·1	223·6	223·0	222·4
− 40	233·0	232·4	231·9	231·3	230·8	230·2	229·7	229·1	228·6	228·0
− 30	238·6	238·0	237·4	236·9	236·3	235·8	235·2	234·7	234·1	233·6
− 20	244·1	243·6	243·0	242·4	241·9	241·3	240·8	240·2	239·7	239·1
− 10	249·7	249·1	248·6	248·0	247·4	246·9	246·3	245·8	245·2	244·7
− 0	255·2	254·7	254·1	253·6	253·0	252·4	251·9	251·3	250·8	250·2
0	255·2	255·8	256·3	256·9	257·4	258·0	258·6	259·1	259·7	260·2
10	260·8	261·3	261·9	262·4	263·0	263·6	264·1	264·7	265·2	265·8
20	266·3	266·9	267·4	268·0	268·6	269·1	269·7	270·2	270·8	271·3
30	271·9	272·4	273·0	273·6	274·1	274·7	275·2	275·8	276·3	276·9
40	277·4	278·0	278·6	279·1	279·7	280·2	280·8	281·3	281·9	282·4
50	283·0	283·6	284·1	284·7	285·2	285·8	286·3	286·9	287·4	288·0
60	288·6	289·1	289·7	290·2	290·8	291·3	291·9	292·4	293·0	293·6
70	294·1	294·7	295·2	295·8	296·3	296·9	297·4	298·0	298·6	299·1
80	299·7	300·2	300·8	301·3	301·9	302·4	303·0	303·6	304·1	304·7
90	305·2	305·8	306·3	306·9	307·4	308·0	308·6	309·1	309·7	310·2
100	310·8	311·3	311·9	312·4	313·0	313·6	314·1	314·7	315·2	315·8
110	316·3	316·9	317·4	318·0	318·6	319·1	319·7	320·2	320·8	321·3
120	321·9	322·4	323·0	323·6	324·1	324·7	325·2	325·8	326·3	326·9
130	327·4	328·0	328·6	329·1	329·7	330·2	330·8	331·3	331·9	332·4
140	333·0	333·6	334·1	334·7	335·2	335·8	336·3	336·9	337·4	338·0

Table IV

Mass in grammes of one cubic metre of dry air

Temperature °A

	200	210	220	230	240	250	260	270	280	290	300
100	174	166	158	152	145	139	134	129	124	120	116
200	349	332	317	303	290	279	268	258	249	240	232
300	523	498	475	455	436	418	402	387	373	360	348
400	697	664	634	606	581	558	536	516	498	480	464
500	872	830	792	758	726	697	670	645	622	601	581
600	1046	996	951	909	871	836	804	774	747	721	697
700	1220	1162	1109	1061	1016	976	938	903	871	841	813
800	1395	1328	1267	1212	1162	1115	1072	1032	995	951	929
900	1569	1494	1426	1364	1307	1254	1206	1161	1120	1081	1045
1000	1743	1660	1584	1515	1452	1394	1340	1290	1244	1201	1161
1040	1813	1725	1648	1576	1510	1450	1394	1342	1294	1249	1208

For saturated air subtract the figures below from the table:

0	0	0	0	0	1	1	2	5	9	16

Table V

Saturation vapour-pressure over water (in millibars)

(After Holborn, Scheel u. Henning, *Wärmetabellen*, *Phys. Tech. Reichsanstalt*)

°A	0	1	2	3	4	5	6	7	8
250	—	—	—	—	—	—	—	1·75	1·91
260	2·25	2·44	2·64	2·86	3·10	3·35	3·62	3·90	4·21
270	4·89	5·27	5·68	6·11	6·57	7·06	7·58	8·13	8·72
280	10·02	10·73	11·48	12·28	13·13	14·02	14·98	15·98	17·05
290	19·38	20·64	21·97	23·38	24·87	26·44	28·09	29·84	31·68
300	35·65	37·80	40·06	42·45	44·93	47·55	50·31	53·20	56·23
310	62·76	67·26	69·92	74·10	77·79	82·00	86·40	91·01	95·84
320	106·1	111·6	117·4	123·4	129·6	135·8	142·9	150·0	157·4
330	173·1	181·4	190·1	199·1	208·6	218·4	228·5	239·1	250·1
340	273·3	285·6	298·3	311·6	325·2	339·5	354·3	369·6	385·5

Saturation vapour-pressure over ice

°A	0	1	2	3	4	5	6	7	8
240	0·27	0·30	0·34	0·39	0·42	0·46	0·51	0·57	0·63
250	0·77	0·85	0·94	1·03	1·11	1·25	1·37	1·50	1·65
260	1·98	2·17	2·37	2·60	2·83	3·10	3·38	3·68	4·01
270	4·76	5·17	5·62	6·12	—	—	—	—	—

At 230° A, 0·09; at 220° A, 0·025; at 210° A, 0·005.

Table VI

Density of saturated water-vapour

	Over water									Over ice		
°A	340	330	320	310	300	290	280	270	260	270	260	250
gm/m³	174·2	113·7	72·0	43·9	25·8	14·5	7·8	3·9	1·87	3·82	1·65	0·66

Table VII

Latent heat of water

(After Holborn, Scheel u. Henning, *Wärmetabellen*)

°C	0	5	10	15	20	25	30	35	40
L	594·9	592·4	590·0	587·5	585·0	582·4	579·8	577·2	574·5

°C	45	50	55	60	70	80	90	100
L	571·8	569·0	566·2	563·4	557·6	551·6	545·5	539·1

Up to 40° C, L may be represented with great accuracy by

$$L = 594{\cdot}9 - 0{\cdot}51t,$$

where t is the temperature in degrees C.

Latent heat of fusion of ice = 79·7 g-cal per gramme.

Table VIII

Temperature radiation of a black body at absolute temperature T

$E = \sigma T^4$ (where σ is Stefan's constant) in g-cal per cm² per min

°A	0	1	2	3	4	5	6	7	8	9
210	0·161	0·164	0·167	0·170	0·173	0·176	0·180	0·183	0·187	0·190
220	0·193	0·197	0·201	0·204	0·208	0·212	0·215	0·219	0·223	0·227
230	0·231	0·235	0·239	0·244	0·248	0·252	0·256	0·261	0·265	0·270
240	0·274	0·279	0·283	0·288	0·293	0·298	0·303	0·308	0·313	0·318
250	0·323	0·328	0·333	0·339	0·344	0·349	0·355	0·360	0·366	0·372
260	0·378	0·384	0·389	0·395	0·401	0·407	0·414	0·420	0·426	0·432
270	0·439	0·446	0·452	0·459	0·466	0·472	0·479	0·486	0·493	0·500
280	0·508	0·515	0·522	0·530	0·537	0·545	0·553	0·560	0·568	0·576
290	0·584	0·592	0·601	0·609	0·617	0·626	0·634	0·643	0·651	0·660
300	0·669	0·678	0·687	0·696	0·706	0·715	0·724	0·734	0·743	0·753
310	0·762	0·772	0·782	0·793	0·803	0·813	0·823	0·834	0·845	0·855
320	0·866	0·877	0·888	0·899	0·910	0·921	0·933	0·944	0·956	0·968

Table IX

Coefficients of conductivity of heat, k, of viscosity, μ, and of diffusion of water-vapour in air, D

Temp. °C	40	30	20	10	0	−10	−20	−30	−40	−50
$10^7 k$	642	625	608	591	574	556	539	522	505	488
$10^6 \mu$	190	186	181	176	171	161	156	151	148	146
$10^5 \rho$, at 1000 mb	111	115	119	123	128	133	138	143	150	156
κ (approx.)	0·24	0·23	0·21	0·20	0·19	0·17	0·16	0·15	0·14	0·13
ν (approx.)	0·17	0·16	0·15	0·14	0·13	0·13	0·11	0·11	0·10	0·09

The coefficient of diffusion of water-vapour in air, represented by D, is given by

$$D = D_0 \, (T/T_0)^{1{\cdot}75},$$

where $D_0 = 0{\cdot}22$ (*Inter. Crit. Tables*, **5**, p. 62), and $T_0 = 273°$ A, and $p_0 = 1$ atmosphere (1013 mb).

Individual values of D_0 vary rather widely. Winkelmann (1884) gave its value as 0·198; Guglielmo (1884) as 0·231; Houdaille (1896) as 0·203; Le Blanc and Wupperman (1916) as 0·224; and Summerhayes (*Proc. Phys. Soc. London*, **42**, 1930, p. 218) as 0·252. The last of these is probably the most reliable value.

The values of $10^7 k$ and of κ have been computed from the values given by Hercus and Sutherland (*Proc. Roy. Soc.* A, **145**, 1934, p. 599) and by Kannuluik and Martin (*ibid.* **144**, 1934, p. 496).

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