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Handbook of

INDUSTRIAL STATISTICS

Handbook of INDUSTRIAL STATISTICS

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Preface

This book is a reprint of the section on industrial statistics in the *Handbook of Industrial Engineering and Management,** edited by W. G. Ireson and E. L. Grant. It has been published separately to serve as a useful and convenient tool for engineers, statisticians, and teachers of courses in industrial statistics.

At present, we are using this handbook, with additional mimeographed notes, as the text for a first course in industrial statistics at Stanford University. Other teachers have expressed interest in this section for classroom use. It can be used as the basis for a course the way we do, or as a handbook to supplement an existing textbook in industrial statistics. We are in the process of expanding this handbook into a complete textbook.

This handbook attempts to present statistics as a science for making decisions, rather than as an art. It is written for engineers and physical scientists and contains the statistical techniques commonly used by them. A detailed explanation of each technique is included with adequate tables for its application. The explanations are strengthened by fully worked out examples. After a brief discussion of descriptive statistics, the book turns to problems in research and production. Article 2 explains the principles underlying the use of control charts; Article 3 is devoted to sampling inspection and contains a discussion of variable inspection and of attributes inspection, both lot-by-lot and continuous. Sampling tables for variables and attributes applicable to both acceptance and in-process inspection also are included. The next four articles describe the significance tests commonly used in engineering research. Among the topics are comparisons of means and variances, curve fitting, analysis of variance, and the analysis of enumeration data. The significance tests are judged from the point of view of the risks involved; for each test presented complete sets of operating characteristics are included to aid the reader in designing experiments.

We are grateful to Professors E. L. Grant and W. G. Ireson, editors of the *Handbook of Industrial Engineering and Management,* for many helpful comments and, of course, for their permission to reprint this section. We are indebted to many members of the staff of the Statistics Department and the Applied Mathematics and Statistics Laboratory at Stanford and to several classes of students who have helped eradicate some of the errors and to clarify some of the explanations. Particular mention should be made of Mrs. Phyllis Winkler who typed the manuscript, of Mrs. Helen Lieberman and Mrs. Judy Berg who helped with the tedious chore of proofreading, of Mr.

* **Prentice-Hall, Inc., 1955.**

Bruce Taft who worked through most of the examples, of Professor L. E. Moses for many helpful comments, of Professor Herbert Solomon without whose aid this book might never have been completed, and of Mrs. Gladys Garabedian for invaluable assistance in the final stages.

Finally, we are indebted to Professor Sir Ronald A. Fisher, Cambridge, and to Messrs. Oliver and Boyd, Ltd., Edinburgh, for permission to reprint Table IV from their book, *Statistical Methods for Research Workers.*

Albert H. Bowker
Gerald J. Lieberman
May 3, 1955
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1. BASIC STATISTICAL CONCEPTS

1.1 INTRODUCTION

The science of statistics deals with drawing conclusions from observed data; the popular conception of statistics is that it involves large masses of data and concerns itself with percentages, averages, or presentation of data in tables or charts. This represents only a small part of the field today and is of less interest to industrial engineers than are other aspects of statistics—e.g., quality control, sampling inspection, and the design and analysis of experiments. These latter topics are the major ones presented in this section.

Most scientific investigations, whether concerned with the effect of a new drug on polio, methods of allaying traffic congestion in a big city, consumer reaction to a new product, or the quality of manufactured products, depend on observations, even if they are of a rudimentary sort. In scientific and industrial experimentation, these observations are taken to study the effect of variation of certain factors or the relation between certain factors. One may wish to study the quality of a raw material from a new supplier, the relation between tensile strength and hardness for a particular alloy, or the optimum combination of conditions in a manufacturing process. Ultimately, these observations are to be used for making decisions; the remainder of this section deals with providing procedures for making decisions with preassigned risks on the basis of the limited information in samples. These procedures are illustrated by examples. Many of these examples contain published data with references to the source.

1.2 EMPIRICAL DISTRIBUTIONS AND HISTOGRAMS

A basic notion of statistics is the notion of variation. For example, there is no single figure for the life of all incandescent lamps produced under certain conditions; some may last many times as long as others. Consider, for example, the data in Table 13.1 on the lifetimes in hours of 417 40-w 110-v internally frosted incandescent lamps taken from forced life tests.★ The life

★ D. J. Davis, "An Analysis of Some Failure Data," *Journal of the American Statistical Association*, Vol. 47, 1953.

varies from a low of 225 hr to a maximum of 1690. Many factors including variations in raw materials, workmanship, function of automatic machinery, and, in fact, test conditions may account for these differences; some of these factors may be controlled carefully but some pattern of variation is inherent in all observational data. Perhaps no one would expect all light bulbs to have exactly the same life, but even the results of the most carefully controlled experiments, such as those designed to measure the velocity of light, exhibit variation due to experimental error. Returning to the light bulbs, the data

TABLE 13.1 ITEM LIFETIMES FOR INCANDESCENT LAMPS

Date	Item lifetimes										Average of sample
1-2-47	1067	919	1196	785	1126	936	918	1156	920	948	997
1-9-47	855	1092	1162	1170	929	950	905	972	1035	1045	1012
1-16-47	1157	1195	1195	1340	1122	938	970	1237	956	1102	1121
1-23-47	1022	978	832	1009	1157	1151	1009	765	958	902	978
1-30-47	923	1333	811	1217	1085	896	958	1311	1037	702	1027
2-6-47	521	933	928	1153	946	858	1071	1069	830	1063	937
2-13-47	930	807	954	1063	1002	909	1077	1021	1062	1157	998
2-20-47	999	932	1035	944	1049	940	1122	1115	833	1320	1029
2-27-47	901	1324	818	1250	1203	1078	890	1303	1011	1102	1088
3-6-47	996	780	900	1106	704	621	854	1178	1138	951	923
3-13-47	1187	1067	1118	1037	958	760	1101	949	992	966	1014
3-20-47	824	653	980	935	878	934	910	1058	730	980	888
3-27-47	844	814	1103	1000	788	1143	935	1069	1170	1067	993
4-3-47	1037	1151	863	990	1035	1112	931	970	932	904	993
4-10-47	1026	1147	883	867	990	1258	1192	922	1150	1091	1053
4-17-47	1039	1083	1040	1289	699	1083	880	1029	658	912	971
4-23-47	1023	984	856	924	801	1122	1292	1116	880	1173	1017
5-1-47	1134	932	938	1078	1180	1106	1184	954	824	529	986
5-8-47	998	996	1133	765	775	1105	1081	1171	705	1425	1015
5-15-47	610	916	1001	895	709	860	1110	1149	972	1002	922
5-22-47	990	1141	1127	1181	856	716	1308	943	1272	917	1045
5-29-47	1069	976	1187	1107	1230	836	1034	1248	1061	1550	1130
6-5-47	1240	932	1165	1303	1085	813	1340	1137	773	787	1058
6-12-47	1438	1009	1002	1061	1277	892	900	1384	1148		1123
6-19-47	1117	1225	1176	709	1485	1225	1011	1028	1227	1277	1148
6-26-47	1222	912	885	1562	1118	1197	976	1080	924	1233	1111
7-3-47	1135	623	983	883	1088	1029	1201	898	970	1058	987
7-10-47	1160	831	1023	1354	1218	1121	1172	1169	1113	1308	1147
7-17-47	1166	1470	1635	1141	1555	1054	1461	1057	1228	1187	1295
8-7-47	1016	744	1197	1122	666	1022	964	1085	612	1003	943
8-14-47	1235	942	1055	893	1235	1056	968	1056	1014	1096	1055
8-21-47	1013	889	1430	926	1297	1033	1024	1103	1385		1122
8-28-47	1077	813	1121	960	1156	1033	1255	225	525	675	884
9-4-47	1211	995	924	732	935	1173	1024	1254	1014		1029
9-11-47	798	1080	862	1220	1024	1170	1120	898	918	1086	1018
9-18-47	1028	1122	872	826	1337	965	1297	1096	1068	943	1055
9-25-47	1490	918	609	985	1233	985	985	1075	1240	985	1051
10-2-47	1105	1243	1204	1203	1310	1262	1234	1104	1303	1185	1215
10-9-47	759	1404	944	1343	932	1055	1381	816	1067	1252	1095
10-16-47	1248	1324	1000	984	1220	972	1022	956	1093	1358	1118
10-23-47	1024	1240	1157	1415	1385	824	1690	1302	1233	1331	1260
10-30-47	1109	827	1209	1202	1229	1079	1176	1173	769	905	1068

in Table 13.1 recorded serially do not present a clear picture of the nature of the variation, and it is more useful to present the data in a frequency table (Table 13.2), grouping adjacent observations into classes, which are usually called class intervals or cells.

TABLE 13.2 FREQUENCY TABLE FOR LENGTH OF LIFE OF INCANDESCENT LAMPS

Class interval (*100 hr*)	*Frequency* (f)	*Mid-point of class interval* (m)	*Deviation from arbitrary origin* (*1050*) *in class intervals* (d)	fd	fd^2
200–299	1	250	−8	−8	64
300–399	...	350	−7	...	...
400–499	...	450	−6	...	...
500–599	3	550	−5	−15	75
600–699	10	650	−4	−40	160
700–799	21	750	−3	−63	189
800–899	43	850	−2	−86	172
900–999	91	950	−1	−91	91
1000–1099	87	1050	0	0	0
1100–1199	79	1150	1	79	79
1200–1299	44	1250	2	88	176
1300–1399	24	1350	3	72	216
1400–1499	9	1450	4	36	144
1500–1599	3	1550	5	15	75
1600–1699	2	1650	6	8	48
Totals	417			−5	1489

Calculations

1. Raw data

(a) Mean: $\bar{x} = \dfrac{\sum_{i=1}^{N} x_i}{N} = \dfrac{435{,}921}{417} = 1{,}045.37$

(b) Standard deviation:

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{N-1} = \frac{\sum x_i^2 - (\sum x_i)^2/N}{N-1}$$

$$= \frac{470{,}808{,}333 - 190{,}027{,}118{,}241/417}{416} = 36{,}316.85$$

$$s = 190.57$$

2. Grouped data

(a) Mean:

$$\bar{x} \text{ in class intervals from arbitrary origin of } 1050 = \frac{\sum fd}{N}$$

$$= -0.012$$

$$\bar{x} \text{ in original units} = \text{arbitrary origin} + \frac{\sum fd}{N}(\text{class interval})$$

$$= 1050 - 0.012(100) = 1049$$

(b) Standard deviation:

$$s \text{ in class interval units} = \sqrt{\frac{\sum fd^2 - (\sum fd)^2/N}{N-1}}$$

$$= \sqrt{\frac{1489 - (5)^2/417}{416}}$$

$$s \text{ in original units} = s \text{ in class interval units (class interval)}$$

$$= 189$$

The most common method of presenting graphically such data as Table 13.1 is in terms of a histogram (Fig. 13.1), which consists of a number of columns with sides equal to the class interval boundaries and height proportional to the frequency.

For comparative or summary purposes, it is often useful to describe an observed frequency distribution by one or two numbers; the most common method is to use the arithmetic mean, which is a measure of central tendency and the sample standard deviation, which is a measure of dispersion around the mean. The calculation of the mean and standard deviation from both the raw data and the grouped data are illustrated in Table 13.2.★ For small

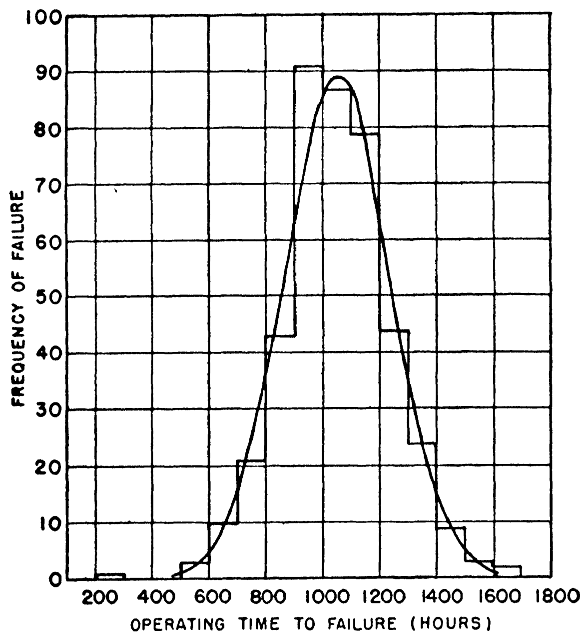


FIG. 13.1 LIFE LENGTH HISTOGRAM FOR INCANDESCENT LAMPS.

★ Throughout this section terms such as $x_1 + x_2 + x_3 + x_4 + x_5$ will be represented as $\sum_{i=1}^{5} x_i$. When no confusion exists, this may instead be written as $\sum x_i$. Terms such as $x_{11} + x_{12} + x_{13} + x_{14} + x_{21} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} + x_{34}$ will be represented as $\sum_{i=1}^{3} \sum_{j=1}^{4} x_{ij}$.

numbers of observations, grouped data computations are not usually advantageous.

1.3 THEORETICAL DISTRIBUTIONS

In the preceding article we described the concept of variables and the presentation of a large volume of data in the form of a histogram. Many decisions from observed data are based on small samples that are assumed to be drawn at random from a larger source, a so-called parent universe or population. To make valid inferences on the basis of small samples it is necessary to make some assumptions about the form of this population.

It is worth while to introduce here an important notion about these models; the constants that characterize these populations are called parameters and should always be clearly distinguished from the quantities we calculate from the observations, i.e., statistics. Thus the statistic, the arithmetic mean of a sample, may differ from the parameter, the true mean value★ of the population.

1.3.1 Normal distribution. The most important distribution in statistics is the normal distribution. This distribution has a symmetric bell-shaped form and tends to infinity in both directions. One of the classical theorems of probability states in essence that if observed quantities can be considered to be the result of large numbers of additive chance effects, the distribution of these quantities should be approximately normal. This theory, plus a mass of empirical evidence, indicates that the normal distribution may be assumed as the underlying population for a large number of industrial problems.

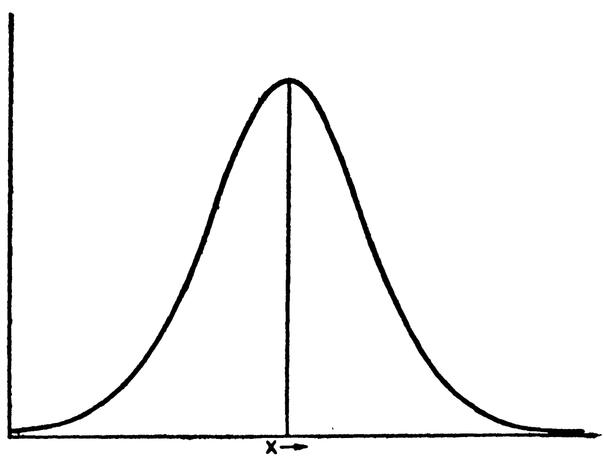


FIG. 13.2 NORMAL DISTRIBUTION.

★ If $f(x)$ represents the equation for the probability distribution, the true mean or population mean given by $\mu = \int_{-\infty}^{+\infty} xf(x)\,dx$ is a parameter.

The equation for the normal curve is

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

The area under the curve is 1. The curve is determined by the two parameters μ, the population mean, and σ, the population standard deviation.★

The normal curve in standard form has area equal to 1, but it is sometimes desirable to construct a normal distribution which has the same area as a histogram. In this case, we write the equation

$$\frac{Nw}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

where N is the total number of observations and w is the width of the class interval.

Areas of the normal distribution may be found in Table 13.3.

Most of the techniques of analysis presented in the remaining articles depend on the distribution of statistics based on random samples from a normal distribution. The sample mean has a normal distribution; in fact, for large samples the distribution of the sample mean from any distribution will be approximately normal. Other distributions, the chi-square, the t distribution, and the F distribution, are the distributions of various functions of means and variances of samples from a normal distribution. The explicit forms of these distributions need not concern us.

1.3.2 **The binomial distribution.** If we have a series of n independent trials and if at each trial p is the probability that the event will occur, the probability that r events occur in the n trials is $\binom{n}{r}p^r q^{n-r}$, where $q = 1 - p$, and $\binom{n}{r}$ is the number of combinations of n things taken r at a time.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{and} \quad n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

The most common application in industrial work is lot-by-lot acceptance inspection, where the lot is large compared with the sample size; p is the fraction defective in the lot, n is the size of a sample drawn at random from the lot, and r is the observed number of defectives.

The observed number of defectives is some number from 0 to n, and hence

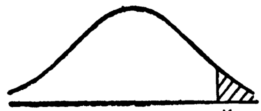
$$\sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} = 1$$

★ $\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx = 1;\quad \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} x e^{-(x-\mu)^2/2\sigma^2}\,dx = \mu;$

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}\,dx = \sigma^2$$

TABLE 13.3 AREAS UNDER THE NORMAL CURVE FROM K_α TO ∞★

$$\int_{K_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \alpha$$



Kα

K_α	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139

K_α	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
3	.00135	$.0^{3}968$	$.0^{3}687$	$.0^{3}483$	$.0^{3}337$	$.0^{3}233$	$.0^{3}159$	$.0^{3}108$	$.0^{4}723$	$.0^{4}481$
4	$.0^{4}317$	$.0^{4}207$	$.0^{4}133$	$.0^{5}854$	$.0^{5}541$	$.0^{5}340$	$.0^{5}211$	$.0^{5}130$	$.0^{6}793$	$.0^{6}479$
5	$.0^{6}287$	$.0^{6}170$	$.0^{7}996$	$.0^{7}579$	$.0^{7}333$	$.0^{7}190$	$.0^{7}107$	$.0^{8}599$	$.0^{8}332$	$.0^{8}182$
6	$.0^{9}987$	$.0^{9}530$	$.0^{9}282$	$.0^{9}149$	$.0^{10}777$	$.0^{10}402$	$.0^{10}206$	$.0^{10}104$	$.0^{11}523$	$.0^{11}260$

★ Reprinted by permission from Frederick E. Croxton, ***Elementary Statistics with Applications in Medicine*** (New York: Prentice-Hall, Inc., 1953), p. 323.

For example, suppose $n = 18$, $p = 0.10$.

r	Probability of r defectives
0	0.150
1	0.300
2	0.284
3	0.168
4	0.070
5	0.022
6	0.005
7	0.001
8	0.000
.	.
.	.
.	.
18	0.000

The distribution function of the above binomial distribution function is plotted in Fig. 13.3.

2. STATISTICAL QUALITY CONTROL: CONTROL CHARTS

2.1 INTRODUCTION

In his book *Statistical Quality Control,* Eugene L. Grant★ states: "Measured quality of manufactured product is always subject to a certain amount of variation as a result of chance. Some stable 'system of chance causes' is inherent in any particular scheme of production and inspection. Variation within this stable pattern is inevitable. The reasons for variation

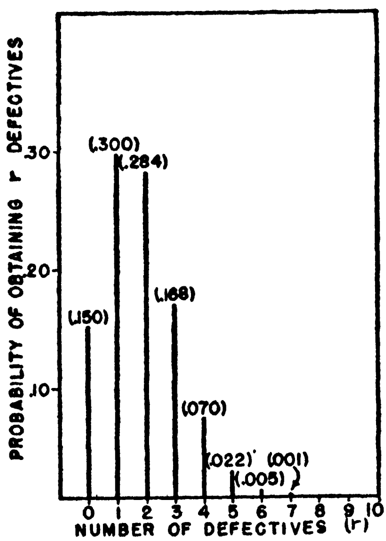


FIG. 13.3 DISTRIBUTION FUNCTION FOR THE BINOMIAL DISTRIBUTION WITH $n = 18$ AND $p = 0.10$.

★ *Statistical Quality Control,* 2nd ed. (New York: McGraw-Hill Book Company Inc., 1952), p. 3.

outside this stable pattern may be discovered and corrected. . . ." These variations outside the stable pattern are known as *assignable causes* of quality variation. A process operating in the absence of any assignable causes of erratic fluctuations is said to be in *statistical control.*

To the manufacturer, the primary purpose of the control chart is to provide a basis for *action.* The introduction of a control chart aids in determining the capabilities of the production process. Action is taken when these estimated capabilities are unsatisfactory in relation to the design specifications. Furthermore, once the process capabilities have been determined, and are satisfactory, action is taken only when the control chart indicates that the process has fallen out of control, e.g., assignable causes of variation have entered.

2.2 OBTAINING DATA FROM RATIONAL SUBGROUPS

The essential feature of the control chart method is the drawing of inferences about the production process on the basis of samples drawn from the production line. The success of the technique depends upon grouping observations under consideration into subgroups or samples, within which a stable system of chance causes is operating, and between which the variations may be due to assignable causes whose presence is suspected or considered possible. Order of production is one of the more commonly used bases for obtaining rational subgroups. If items are coming from more than one source, the source may be a basis for rational subgrouping.

The size of the subgroup or sample usually is not less than 4. In industry 5 seems to be the most common.★ It is preferable that all samples be of equal size.

2.3 CONTROL CHARTS FOR VARIABLES: $\bar{x}$ CHARTS

2.3.1 Statistical concepts. If observations on an item are normally distributed, with mean $\bar{x}'$ and standard deviation σ', it is possible to find the probability that an observation will lie in an interval by referring to Table 13.3. Furthermore, if we denote the variable by x, the probability that x will be within the interval $\bar{x}' \pm 3\sigma'$ is 0.9973. In other words, if a plot such as that in Fig. 13.4 is made, on the average only 27 observations out of 10,000 will be outside the above interval, if the *population mean is $\bar{x}'$ and the population standard deviation is σ'.*† If the underlying population is normal with mean $\bar{x}'$ and standard deviation σ', the population of averages of subgroups of n drawn from the above population is also normal with mean $\bar{x}'$ and stand-

★ When characteristics are classified into two groups, those containing defects and those not containing defects, and no other measurement is recorded, the sample size is usually much larger.

† The notation in the articles on quality control is that recommended by the American Society for Quality Control. In the remaining articles, the usual notation found in texts on industrial statistics will be used—i.e., μ will represent the population mean, and σ will represent the population standard deviation.

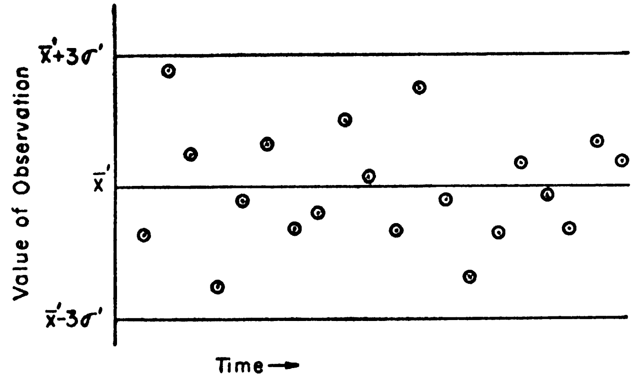


FIG. **13.4** PLOT OF INDIVIDUAL OBSERVATIONS.

ard deviation $\sigma'_{\bar{x}} = \sigma'/\sqrt{n}$. Denoting the n observations by $x_1, x_2, \ldots, x_n$, the average of these n observations, $\bar{x}$, is defined as

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

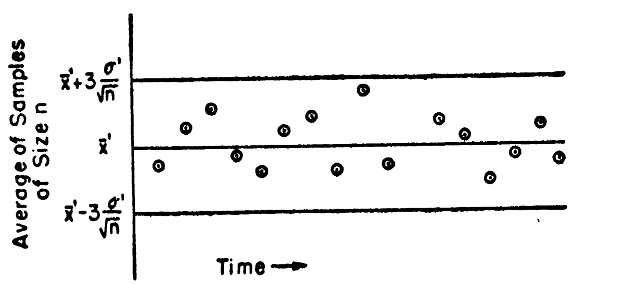
If instead of plotting individual values as in Fig. 13.4, averages of samples of n are plotted, it is expected that on the average only 27 in 10,000 values of these averages will fall outside the interval

$$\bar{x}' \pm 3\sigma'_{\bar{x}} = \bar{x}' \pm \frac{3\sigma'}{\sqrt{n}}$$

The plot in Fig. 13.5 is referred to as a control chart for $\bar{x}$. Here $\bar{x}' + 3\sigma'/\sqrt{n}$ is referred to as the upper control limit (UCL) while $\bar{x}' - 3\sigma'/\sqrt{n}$ is referred to as the lower control limit (LCL).

In all the above discussion it was assumed that the underlying distribution is normal. Although in actual practice many distributions of observations are "nearly" normally distributed, many others do not resemble a normal distribution at all. However, it can be shown that under fairly weak restrictions,

FIG. **13.5** CONTROL CHART FOR $\bar{x}$.



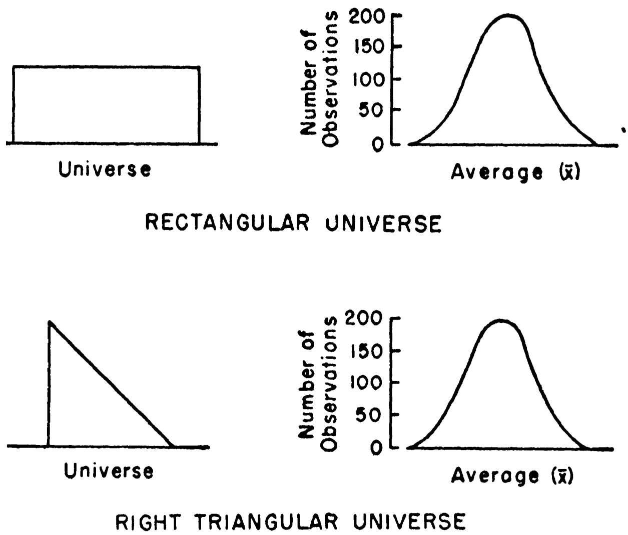


FIG. **13.6** DISTRIBUTION OF THE SAMPLE AVERAGE FROM A RECTANGULAR UNIVERSE AND A RIGHT TRIANGULAR UNIVERSE.

the average of samples of n tend toward normality as n gets large. This theorem is called the central limit theorem, and a proof can be found in most standard texts in mathematical statistics.

From a practical point of view n need not be too large before the results of the theorem begin to apply. In Fig. 13.6, it is demonstrated that even when the underlying distribution is from a rectangular or a triangular distribution, the distribution of $\bar{x}$ values from samples of 4 is approximately normal.

It is evident, then, that the central limit theorem is one of the keys to the success of the control chart for $\bar{x}$. No matter what the underlying distribution may be (there are certain weak conditions that must be satisfied), the above theorem states that the theory about the properties of the normal distribution is applicable, provided, of course, sample averages are considered as the variable plotted on the control chart.

2.3.2 **Estimates of $\bar{x}'$.** In Fig. 13.5, a control chart for $\bar{x}$ is shown, where the control limits are drawn in as functions of $\bar{x}'$ and σ'. In most practical applications $\bar{x}'$ and σ' are not known, and consequently estimates of these parameters must be obtained. It is desirable that these estimates be based on at least 25 subgroups of n observations. Naturally, the larger the number of subgroups, the better the estimates of the population parameters, provided the process is in control. Suppose the estimates are based upon k subgroups of size n. Denoting the averages of the subgroups by $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$, the best estimate of $\bar{x}'$, the population mean, is

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_k}{k}$$

where $\bar{\bar{x}}$ is also the average of all the nk observations.

2.3.3 Estimate of σ' by $\bar{\sigma}$. As was pointed out in the previous article, it is usually necessary to estimate σ', the population standard deviation. Define, for each subgroup:

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}$$

$$= \sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2 - n\bar{x}^2}{n}}$$

Let $\bar{\sigma}$ be the average of the standard deviations of the k subgroups, i.e.,

$$\bar{\sigma} = \frac{\sigma_1 + \sigma_2 + \ldots + \sigma_k}{k}$$

TABLE **13.4** FACTORS FOR COMPUTING CONTROL CHART LINES★

Number of observations in sample, n	Chart for averages			Chart for standard deviations						Chart for ranges						
	Factors for control limits			Factors for central line		Factors for control limits				Factors for central line		Factors for control limits				
	A	A_1	A_2	c_2	$1/c_2$	B_1	B_2	B_3	B_4	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	3.760	1.880	0.5642	1.7725	0	1.843	0	3.267	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.732	2.394	1.023	0.7236	1.3820	0	1.858	0	2.568	1.693	0.5907	0.888	0	4.358	0	2.575
4	1.500	1.880	0.729	0.7979	1.2533	0	1.808	0	2.266	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	1.596	0.577	0.8407	1.1894	0	1.756	0	2.089	2.326	0.4299	0.864	0	4.918	0	2.115
6	1.225	1.410	0.483	0.8686	1.1512	0.026	1.711	0.030	1.970	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	1.277	0.419	0.8882	1.1259	0.105	1.672	0.118	1.882	2.704	0.3698	0.833	0.205	5.203	0.076	1.924
8	1.061	1.175	0.373	0.9027	1.1078	0.167	1.638	0.185	1.815	2.847	0.3512	0.820	0.387	5.307	0.136	1.864
9	1.000	1.094	0.337	0.9139	1.0942	0.219	1.609	0.239	1.761	2.970	0.3367	0.808	0.546	5.394	0.184	1.816
10	0.949	1.028	0.308	0.9227	1.0837	0.262	1.584	0.284	1.716	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.973	0.285	0.9300	1.0753	0.299	1.561	0.321	1.679	3.173	0.3152	0.787	0.812	5.534	0.256	1.744
12	0.866	0.925	0.266	0.9359	1.0684	0.331	1.541	0.354	1.646	3.258	0.3069	0.778	0.924	5.592	0.284	1.716
13	0.832	0.884	0.249	0.9410	1.0627	0.359	1.523	0.382	1.618	3.336	0.2998	0.770	1.026	5.646	0.308	1.692
14	0.802	0.848	0.235	0.9453	1.0579	0.384	1.507	0.406	1.594	3.407	0.2935	0.762	1.121	5.693	0.329	1.671
15	0.775	0.816	0.223	0.9490	1.0537	0.406	1.492	0.428	1.572	3.472	0.2880	0.755	1.207	5.737	0.348	1.652
16	0.750	0.788	0.212	0.9523	1.0501	0.427	1.478	0.448	1.552	3.532	0.2831	0.749	1.285	5.779	0.364	1.636
17	0.728	0.762	0.203	0.9551	1.0470	0.445	1.465	0.466	1.534	3.588	0.2787	0.743	1.359	5.817	0.379	1.621
18	0.707	0.738	0.194	0.9576	1.0442	0.461	1.454	0.482	1.518	3.640	0.2747	0.738	1.426	5.854	0.392	1.608
19	0.688	0.717	0.187	0.9599	1.0418	0.477	1.443	0.497	1.503	3.689	0.2711	0.733	1.490	5.888	0.404	1.596
20	0.671	0.697	0.180	0.9619	1.0396	0.491	1.433	0.510	1.490	3.735	0.2677	0.729	1.548	5.922	0.414	1.586
21	0.655	0.679	0.173	0.9638	1.0376	0.504	1.424	0.523	1.477	3.778	0.2647	0.724	1.606	5.950	0.425	1.575
22	0.640	0.662	0.167	0.9655	1.0358	0.516	1.415	0.534	1.466	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	0.626	0.647	0.162	0.9670	1.0342	0.527	1.407	0.545	1.455	3.858	0.2592	0.716	1.710	6.006	0.443	1.557
24	0.612	0.632	0.157	0.9684	1.0327	0.538	1.399	0.555	1.445	3.895	0.2567	0.712	1.759	6.031	0.452	1.548
25	0.600	0.619	0.153	0.9696	1.0313	0.548	1.392	0.565	1.435	3.931	0.2544	0.709	1.804	6.058	0.459	1.541
Over 25	$\frac{3}{\sqrt{n}}$	$\frac{3}{\sqrt{n}}$	.....	.....	.....	*	**	*	**	.....	.....	.....	....	.....	....	.....

$* \; 1 - \frac{3}{\sqrt{2n}}$ $\quad ** \; 1 + \frac{3}{\sqrt{2n}}$

★ Reproduced by permission from *ASTM Manual on Quality Control of Materials*, American Society for Testing Materials, Philadelphia, Pa., 1951.

An estimate of σ' is then $\bar{\sigma}/c_2$, where values of c_2 for different n can be found in Table 13.4.

To summarize, the estimated central line for the control chart for $\bar{x}$ is $\bar{\bar{x}}$, and the estimated control limits are $\bar{\bar{x}} \pm 3\bar{\sigma}/\sqrt{n}\, c_2$. Values of $3/(\sqrt{n}\, c_2) = A_1$

TABLE 13.5 FORMULAS FOR CENTRAL LINES AND CONTROL LIMITS

Statistic	Standards given		Analysis of past data	
	Central line	Limits	Central line	Limits
Average, using σ'	$\bar{x}'$	$\bar{x}' \pm A\sigma'$	$\bar{\bar{x}}$	$\bar{\bar{x}} \pm A_1\bar{\sigma}$
Average, using R	—	—	$\bar{\bar{x}}$	$\bar{\bar{x}} \pm A_2\bar{R}$
Standard deviation	$c_2\sigma'$	$B_1\sigma'$, $B_2\sigma'$	$\bar{\sigma}$	$B_3\bar{\sigma}$, $B_4\bar{\sigma}$
Range	$d_2\sigma'$	$D_1\sigma'$, $D_2\sigma'$	$\bar{R}$	$D_3\bar{R}$, $D_4\bar{R}$

can be found in Table 13.4 so that the estimated control limits can be written as $\bar{\bar{x}} \pm A_1\bar{\sigma}$.

2.3.4 Estimate of σ' by $\bar{R}$. Define, as the range (R) of a subgroup of n observations, the difference between the largest and smallest value. Let $\bar{R}$ be the average of the ranges of the k subgroups, i.e.,

$$\bar{R} = \frac{R_1 + R_2 + \ldots + R_k}{k}$$

Another estimate of σ' is $\bar{R}/d_2$. Values of d_2 for different values of n can be found in Table 13.4.

In estimating the control limits for $\bar{x}$ with $\bar{R}$ as an estimate of σ' the limits become $\bar{\bar{x}} \pm 3\bar{R}/\sqrt{n}\, d_2$. Values of $3/(\sqrt{n}\, d_2) = A_2$ can be found in Table 13.4, so that the estimated control limits can be written as $\bar{\bar{x}} \pm A_2\bar{R}$.

It must be pointed out that although $\bar{R}$ is simpler to calculate than $\bar{\sigma}$, the estimate of σ' based upon $\bar{\sigma}$ is a better estimate in the sense that on the average $\bar{\sigma}/c_2$ will lie nearer to σ' than $\bar{R}/d_2$ will.

2.3.5 Starting a control chart for $\bar{x}$. It has been pointed out that observations should be classified into rational subgroups, and each subgroup should contain at least 4 observations. The determination of the minimum number of subgroups is a compromise between obtaining the guidance of the control chart as quickly as possible, and the desire for the guidance to be reliable. Usually, at least 25 subgroups should be chosen.

With the data at hand, trial control limits can be calculated. Estimates of $\bar{x}'$ and σ' can be obtained in the manner described previously. The trial control limits are then $\bar{\bar{x}} \pm A_1\bar{\sigma}$ or $\bar{\bar{x}} \pm A_2\bar{R}$, depending upon which estimate of σ' is chosen. If σ' and/or $\bar{x}'$ are known, the control limits should be calculated, using the known values, e.g., if both are known, $\bar{x}' \pm 3\sigma'/\sqrt{n}$. Values of $3/\sqrt{n}$ can be found in Table 13.4, so that the control limits can be written $\bar{x}' \pm A\sigma'$.

Returning to the case where the parameters $\bar{x}'$ and σ' are unknown, and estimates have to be computed, some modifications may have to be made. Lack of control is usually indicated by points falling outside the limits. If all the points fall within the limits, the process is said to be in control. However, there is no assurance that assignable causes of variation are not present, and that this is a constant cause system. It merely means that for practical purposes it pays to act as if no assignable causes of variation are present, although realizing that an error of judgment is quite possible.

The trial control limits serve the purpose of determining whether past operations are in control. To continue using these limits as a basis for action on future production may require a revision of the trial limits, especially if a lack of control is exhibited by points falling outside the trial control limits. If the process is not in control, all the points do not come from a stable population. However, it is evident that future control limits should be based upon data from a controlled process. As a practical rule, then, those points falling outside the trial control limits are eliminated, and new trial control limits are computed using the remaining points. This procedure may be continued until all points fall within the control limits. There is no theoretical justification for this, other than the fact that those points falling outside the trial control limits are more likely to belong to another population, i.e., may be due to some assignable cause.

The control chart when used as a basis for action on future production may be set, using aimed-at values of $\bar{x}'$ and/or σ'. In this case, the control limits are modified by using the aimed-at values as if they were the known values in the computations. If this is done, and the control chart exhibits a lack of control in the future, the trouble may be that the process is not in control at the aimed-at values, but is in control at some other values. For example, suppose that an item is being produced at a level such that the mean $\bar{x}'$ is equal to 20. If the aimed-at value is $\bar{x}' = 25$, the control chart based on $\bar{x}' = 25$ will exhibit a lack of control. Suppose the mean can be shifted by a change in machine setting. The interpretation, then, is that there is an assignable cause present, namely, the machine setting, preventing the production process from operating at the aimed-at value. Yet, with the actual machine setting, the process is in control at $\bar{x}' = 20$. The term "state of control" may have to be interpreted in the above light. Aimed-at values of $\bar{x}'$ or σ' are often used when they can be achieved by a simple adjustment of the machine.

2.3.6 Relation between control limits and specification limits. After ascertaining at what level the process is in control by means of the control chart, it remains to determine whether or not the process can meet the specification limits set for the item. In Arts. 2.3.3 and 2.3.4, it was shown that σ' can be estimated from $\bar{\sigma}$ or $\bar{R}$, i.e., estimate of $\sigma' = \bar{\sigma}/c_2$ or $\bar{R}/d_2$. Furthermore, $\bar{x}'$ can be estimated from $\bar{\bar{x}}$. Consequently, it is known that if $\bar{x}'$ and σ' are really equal to the estimates, almost all the individual items will lie between $\bar{x}' \pm 3\sigma'$, i.e., on the average 9973 out of 10,000.

Specification limits are usually specified by giving an aimed-at value plus an interval surrounding this value. These specification limits *should not* be interpreted as meaning that all the items produced will fall within these limits; but rather *almost* all. If the designer is pressed, he will admit to allowing, say 1 in 1000, to fall outside. Once the number is ascertained, the corresponding probability limits can be obtained from the estimated $\bar{x}'$ and σ'.★ For example, suppose the designer specifies specification limits of 20 ± 2, and he states that only 1 in 1000 should fall outside these limits. If the estimate of $\bar{x}'$ is 20.40 and $\sigma' = 1$, from the normal tables it follows that on that average, 999 out of 1000 items will fall between

$$[\bar{x}' - 3.29\sigma', \bar{x}' + 3.29\sigma'] = [17.11, 23.69]$$

The specification limits allow only [18, 22]. Consequently, the above process cannot meet the specification limits, and either the process must be changed or the specifications revised. To summarize, the specification limits must first be properly interpreted. As a practical rule, they are often interpreted to mean the allowance of 27 in 10,000 to fall outside. This corresponds to $3\sigma'$ limits. Then $\bar{x}' \pm 3\sigma'$★ is calculated and compared with the specification limits. If the specifications fall outside the probability limits, either the process must be altered, or the specifications revised.

2.3.7 Interpretation of control charts for $\bar{x}$. As long as the process is in control, almost all the values of $\bar{x}$ will fall within the control limits. In this case, no action need be taken. A point falling outside these limits is a signal to hunt for trouble. In some instances, production may be stopped until a source of trouble has been discovered. Occasionally, approximately 27 in 10,000 times, an error will be made in that trouble will be sought even though nothing has gone wrong with the process. On the other hand, if something has gone wrong with the process, points will begin to fall outside the control limits. For example, a shift in the mean $\bar{x}'$ (σ' remaining constant) will result in points falling outside the control limits. This is usually indicated by points falling above *or* below the limits (but never both), depending on whether the shift is in the positive or negative direction. On the other hand, an increase in σ' ($\bar{x}'$ remaining constant) will result in points falling above *and* below the control limits. In addition to examining the process, when points fall outside the control limits, the control limits themselves should be re-examined, and perhaps brought up to date.

It has been pointed out that an error may be committed when action is taken after a point falls outside the control limits. There is a small probability that a point will fall outside the control limits even though the process is in control, i.e., if $3\sigma'_{\bar{x}}$ limits are used the probability is 0.0027. This probability of falling outside the control limits when the process is in control is known as the probability of committing an *error of the first type* or *type 1 error*. Similarly, if the process goes out of control, there is a probability greater than 0, that a point will fall within the control limits. This is known as the probability of an *error of the second type* or *type 2 error*. These two errors are related. A decrease in one results in an increase in the other. An increase in one results in

★ Assuming the estimated values of $\bar{x}'$ and σ' are equal to $\bar{x}'$ and σ', respectively.

a decrease in the other. The use of $3\sigma'_{\bar{x}}$ limits implies a probability of a type 1 error of 0.0027. If the process jumps out of control, and the new level is specified, the probability of a type 2 error can be computed. If $2\sigma'_{\bar{x}}$ limits were used instead of $3\sigma'_{\bar{x}}$, the probability of a type 1 error is increased to about 0.0455, but the probability of a type 2 error is decreased.

2.3.8 **Example.** Suppose $\bar{x}'$ is 25, and $\sigma' = 1$, and there are 4 observations in each subgroup. The control limits are then $25 \pm \frac{3}{2}$. The probability of a type 1 error is 0.0027. If the mean shifts to 27, the probability of a point falling inside the control limits of $25 \pm \frac{3}{2}$ is 0.1587. On the other hand, if $2\sigma'_{\bar{x}}$ is used as control limits (25 ± 1) the probability of a type 1 error is increased to 0.0455, whereas the probability of committing a type 2 error is only 0.0228.

It is evident that type 1 errors or type 2 errors (but not both) can be made as small as is desirable, at the expense of increasing the other type error. It is also evident that using $3\sigma'_{\bar{x}}$ limits is conservative from the point of view of considering the type 1 error. A point falling outside the control limits is almost sure evidence that the process is no longer in control. On the other hand, one cannot be assured that a small shift has not taken place even if the points fall within the control limits.

2.4 R CHARTS AND σ CHARTS

2.4.1 **Statistical concepts.** Although both R and σ *do not* have normal distributions, both of these functions are chance variables, each having a distribution. Furthermore, the population mean (of these chance variables) plus and minus three standard deviations (of these variables) contain almost all of the underlying population. Consequently, for control chart purposes, it is necessary to calculate both the population mean value and standard deviation of these chance variables.

The population average of σ is $c_2\sigma'$. The standard deviation of σ is

$$\sigma_\sigma = [2(n-1) - 2nc_2^2]^{1/2}\frac{\sigma'}{\sqrt{2n}}$$

The control limits are then $c_2\sigma' \pm 3\sigma_\sigma$ which can be written

$$\sigma'\left(c_2 \pm \frac{3}{\sqrt{2n}}[2(n-1) - 2nc_2^2]^{1/2}\right)$$

The factors

$$B_2 = c_2 + \frac{3}{\sqrt{2n}}[2(n-1) - 2nc_2^2]^{1/2}$$

$$B_1 = c_2 - \frac{3}{\sqrt{2n}}[2(n-1) - 2nc_2^2]^{1/2}$$

can be obtained from Table 13.4. Thus

$$UCL_\sigma = B_2\sigma'; \qquad LCL_\sigma = B_1\sigma'$$

When σ' is unknown, it can be estimated from $\bar{\sigma}/C_2$, so that the estimated control limits are written

$$\bar{\sigma}\left\{1 \pm \frac{3}{\sqrt{2n}\,c_2}\,[2(n-1) - 2nc_2^2]^{1/2}\right\}$$

The factors

$$B_4 = 1 + \frac{3}{\sqrt{2n}\,c_2}\,[2(n-1) - 2nc_2^2]^{1/2}$$

$$B_3 = 1 - \frac{3}{\sqrt{2n}\,c_2}\,[2(n-1) - 2nc_2^2]^{1/2}$$

can be obtained from Table 13.4. The estimated control limits are then

$$UCL_\sigma = B_4\bar{\sigma}; \qquad LCL_\sigma = B_3\bar{\sigma}$$

Note that the center line is $c_2\sigma'$ if σ' is known, and $\bar{\sigma}$ if σ' is unknown.

The control limits for R are obtained in a similar manner. The population average of R is $d_2\sigma'$. The standard deviation of R is $\sigma_R = d_3\sigma'$. The control limits are then $d_2\sigma' \pm 3\sigma_R$, which can be written $\sigma'(d_2 \pm d_3)$. The factors

$$D_2 = d_2 + d_3; \qquad D_1 = d_2 - d_3$$

can be obtained from Table 13.4. Thus

$$UCL_R = D_2\sigma'; \qquad LCL_R = D_1\sigma'$$

When σ' is unknown, it can be estimated from $\overline{R}/d_2$ so that the estimated control limits are written

$$\overline{R}\left[1 \pm \frac{d_3}{d_2}\right]$$

The factors

$$D_4 = 1 + \frac{d_3}{d_2}; \qquad D_3 = 1 - \frac{d_3}{d_2}$$

can be obtained from Table 13.4. The estimated control limits are then

$$UCL_R = D_4\overline{R}; \qquad LCL_R = D_3\overline{R}$$

The center line is $d_2\sigma'$ if σ' is known, and $\overline{R}$ if σ' is unknown.

2.4.2 **Setting up a control chart for R or σ.** The range of each subgroup should be obtained and $\overline{R}$ calculated from these values. The control limits are obtained from

$$UCL_R = D_4\overline{R}; \qquad LCL_R = D_3\overline{R}$$

If a σ chart is desired, similar calculations are made in accordance with the rules described above. If all the points fall inside the control limits, no modification is made unless it is desired to reduce the process dispersion. In this case, an aimed-at value of σ' should be used in the calculations. When the $\overline{R}$ (or σ) chart indicates a possible lack of control by points falling outside the control limits, it is desirable to estimate the value of σ' that might be attained if the dispersion were brought into control. A method of estimation is to eliminate those points out of control (only those above the UCL_R if points fall both above and below) and recompute the values of the control limits based only on the remaining observations. If more points fall out of control, the procedure is carried out once again.

The final revised values of $\bar{R}$, $\bar{\sigma}$, or σ', whichever is used, may also be used to obtain new control limits for $\bar{x}$. This has the effect of tightening the limits on the $\bar{x}$ chart, making them consistent with a σ' that may be estimated from the revised $\bar{R}$ or $\bar{\sigma}$.

Control limits should be revised from time to time as additional data are accumulated.

2.5 EXAMPLE OF $\bar{x}$ AND R CHART

The following are the $\bar{x}$ and R values for 20 subgroups of five readings.

The specifications for this product characteristic are 0.4037 ± 0.0010. The values given are the last two figures of the dimension reading, i.e., 31.6 should be 0.40316.

Subgroup	$\bar{x}$	R	Subgroup	$\bar{x}$	R	Subgroup	$\bar{x}$	R
1	34.0	4	8	32.6	13	15	33.8	7
2	31.6	4	9	33.8	19	16	31.6	5
3	30.8	2	10	37.8	6	17	33.0	5
4	33.0	3	11	35.8	4	18	28.2	3
5	35.0	5	12	38.4	4	19	31.8	9
6	32.2	2	13	34.0	14	20	35.6	6
7	33.0	5	14	35.0	4			

$\sum \bar{x} = 671.0$, $\bar{\bar{x}} = 33.6$; $\sum R = 124$; $\bar{R} = 6.20$

The trial control limits are computed from

$$\bar{\bar{x}} \pm A_2\bar{R}; \qquad UCL_{\bar{x}} = 37.2; \qquad UCL_R = (2.115)(6.20) = 13.1$$

$$\bar{\bar{x}} \pm (0.577)(6.20); \qquad LCL_{\bar{x}} = 30.0; \qquad LCL_R = 0$$

$$\bar{\bar{x}} \pm 3.58$$

The control charts for $\bar{x}$ and R with trial limits are shown in Fig. 13.7. Since some points fall outside the control limits, the process is assumed to be out of control. Eliminating these points, new control limits are computed.

$$\sum \bar{x} = 5660; \qquad \bar{\bar{x}} = 33.3$$

$$\sum R = 91; \qquad \bar{R} = 5.06$$

$$UCL_{\bar{x}} = 33.3 + (0.577)(5.06) = 33.3 + 2.92 = 36.2$$

$$LCL_{\bar{x}} = 33.3 - (0.577)(5.06) = 33.3 - 2.92 = 30.4$$

$$UCL_R = (2.115)(5.06) = 10.7; \qquad LCL_R = (2.115)(0) = 0$$

None of the remaining points fall outside the limits. If it is now assumed that the process can be brought into control at this level we find $\bar{x}' = 33.3$ and $\sigma' = 2.175$. The value of σ' is obtained from

$$d_2 = \bar{R}/\sigma'; \qquad \sigma' = \bar{R}/d_2 = 5.06/2.326 = 2.175$$

$$[\bar{x}' - 3\sigma', \bar{x}' + 3\sigma'] = [\bar{x}' - 6.525, \bar{x}' + 6.525] = [26.8, 39.8]$$

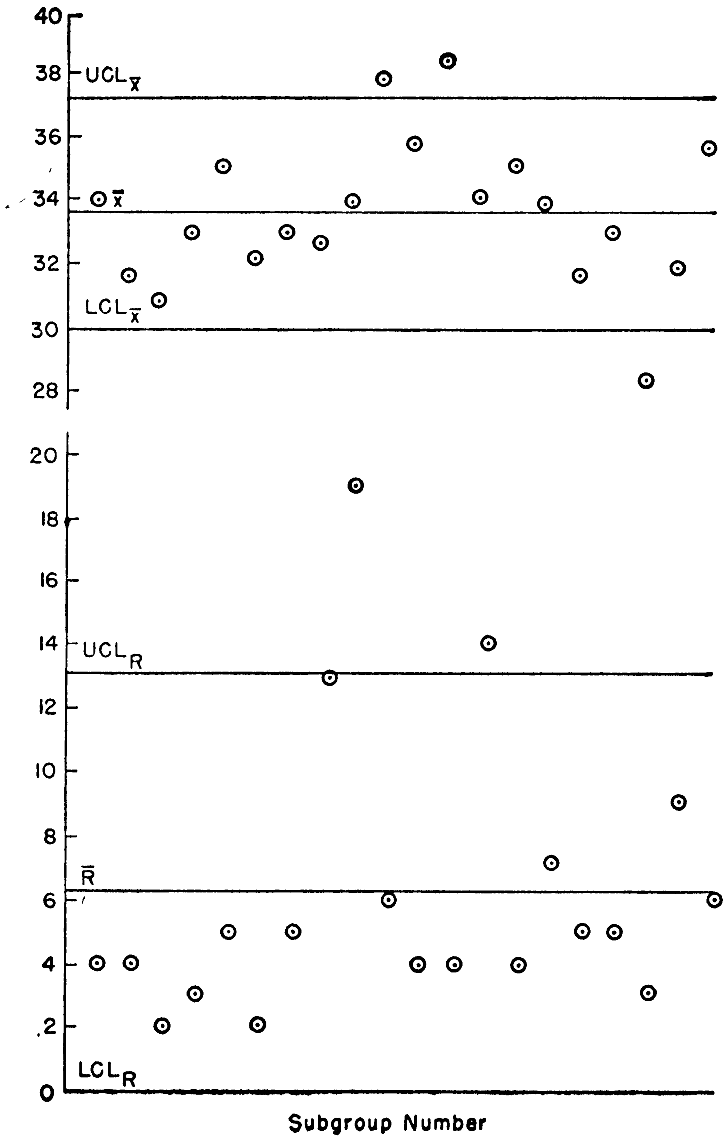


FIG. **13.7** CONTROL CHART FOR $\bar{x}$ AND R.

In terms of the actual data, $[\bar{x}' - 3\sigma', \bar{x}' + 3\sigma'] = [0.4027, 0.4047]$. The specifications are

$$[0.4037 - 0.0010, 0.4037 + 0.0010] = [0.4027, 0.4047]$$

so that this production process is able to meet these specifications even though the process is not centered at the nominal value of 0.4037 provided the process remains in control at the above level.

2.6 CONTROL CHART FOR FRACTION DEFECTIVE

2.6.1 Relation between control charts based on variables data and charts based on attribute data. The $\bar{x}$ and R control charts are charts for variables, i.e., for quality characteristics that can be measured and expressed in numbers. However, many quality characteristics can be observed only as attributes, i.e., by classifying the item into one of two classes, usually defective or nondefective. Furthermore, with the existing techniques, an $\bar{x}$ and R chart can be used for only one measurable characteristic at a time. For example, if an item consists of 10,000 measurable characteristics, each characteristic is a candidate for an $\bar{x}$ and R chart. However, it would be impossible to have 10,000 such charts, and only the most important and troublesome would be charted. As an alternative to $\bar{x}$ and R charts, and as a substitute when characteristics are measured only by attributes, a control chart based on the fraction defective can be used. This is known as a p chart. A p chart can be applied to quality characteristics that are actually observed as attributes even though they may have been measured as variables. The cost of obtaining attribute data is usually less than that for obtaining variables data. The cost of computing and charting may also be less, since one p chart can apply to any number of characteristics.

Basically, the p chart has the same objective as an $\bar{x}$ and R chart. It discloses the presence of assignable causes of variation, although it is not so sensitive as an $\bar{x}$ and R chart.

2.6.2 Statistical theory. The statistical theory of the binomial distribution discussed in Art. 1.3 is applicable here. Let the probability of an item being defective be given by p', so that the probability of obtaining exactly d defectives in a sample of n is $\binom{n}{d}p'^d(1 - p')^{n-d}$. The probability of obtaining a defectives or less in a sample of n is

$$\sum_{d=0}^{a}\binom{n}{d}p'^d(1 - p')^{n-d} = \binom{n}{0}p'^0(1 - p')^n + \binom{n}{1}p'^1(1 - p')^{n-1} + \dots + \binom{n}{a}p'^a(1 - p')^{n-a}$$

Furthermore, the mean value of the total number of defectives in a sample of n is np', and the standard deviation is $\sqrt{np'(1 - p')}$.

If the fraction defective p is defined as the ratio of the number of defectives d to the total number of items in the sample n—i.e., d/n—the mean value of the fraction defective is p' and the standard deviation is $\sqrt{p'(1 - p')/n}$.

As a working rule, the control limits for the fraction defective are defined as

$$UCL_p = p' + 3\sqrt{\frac{p'(1 - p')}{n}}; \qquad LCL_p = p' - 3\sqrt{\frac{p'(1 - p')}{n}}$$

It is important to note that the probability of d/n falling within these limits depends on the value of p', even if the process is in control.★ However, almost

★ This is quite different from the $\bar{x}$ chart, where the probability of $\bar{x}$ falling between $\bar{x}' \pm 3\sigma'/\sqrt{n}$ is 0.9973 for any values of $\bar{x}'$ and σ' provided the process is in control.

all the d/n will fall within the above limits provided the process is in control. Furthermore, the above limits result in a simple empirical rule and for large n, result in an accurate approximation.

If p' is not known, it is usually estimated from past data. The rules are similar to those used in estimating the parameters for the $\bar{x}$ charts. The estimate in this case is d_T/n_T where d_T is the total number of defectives found in the past data of size n_T.

It often happens in control charts for fraction defective that the size of the subgroup varies. In this case, three possible solutions to the problem are as follows: (1) Compute control limits for every subgroup and show these fluctuating limits on the p chart. (2) Estimate the average subgroup size, and compute one set of limits for this average and draw them on the control chart. This method is approximate and is appropriate only when the subgroup sizes are not too variable. Points near the limits may have to be re-examined in accordance with (1). (3) Draw several sets of control limits on the chart corresponding to different subgroup sizes. This method is also approximate and is actually a cross between (1) and (2). Again, points falling near the limits should be re-examined in accordance with (1).

2.6.3 Starting the control chart. The subgroup size is usually large compared with that used for $\bar{x}$ and R charts. The main reason for this is that if p' is very small, and n is small, the expected number of defectives in a subgroup will be very close to 0.

For each subgroup compute p, where

$$p = \frac{\text{number of defectives in the subgroup}}{\text{number inspected in the subgroup}}$$

Whenever practicable, no fewer than 25 subgroups should be used to compute trial control limits. These limits are computed from the data by finding the average fraction defective $\bar{p}$, where

$$\bar{p} = \frac{\text{total number of defectives during period}}{\text{total number inspected during period}}$$

and substituting in the relations

$$UCL_p \doteq \bar{p} + \frac{3\sqrt{\bar{p}(1-\bar{p})}}{\sqrt{n}}; \qquad LCL_p = \bar{p} - \frac{3\sqrt{\bar{p}(1-\bar{p})}}{\sqrt{n}}$$

Inferences about the existence of control or lack of control can be drawn in a manner similar to that described for the $\bar{x}$ and R chart.

2.6.4 Continuing the p chart. The preliminary plot reveals two facts, namely, whether or not the process is in an apparent state of control, and the quality level. If the process appears to be in control, it may be in control at too high a level, i.e., the estimate of p' is higher than can be tolerated. In this case, the production process must be examined, and possible major changes made. On the other hand, the estimate of p' may indicate a good quality level even though the process may appear to be out of control. In this case assignable causes should be sought and eliminated. Of course, control limits can be determined on the basis of an aimed-at value of p', but an indicated lack of control must be interpreted with this in mind.

At first glance, points falling below the lower control limit may appear to be desirable. However, this may be attributed to a poor estimate of p', although it may possibly indicate a change for the better in the quality level; and tracking down the assignable cause may enable an improvement to be made in the production process. In either case, points falling below the lower control limit call for a re-examination of the control chart or the production process or both.

2.6.5 Example. A sample of 50 pieces is drawn from the production of the last two hours from a single spindle automatic screw machine and each item is checked by go and no-go gauges for several possible sources of defectives. The number of defective items found in 25 such successive samples were:

Subgroup	d	p	Subgroup	d	p	Subgroup	d	p
1	1	0.02	9	1	0.02	18	0	0.00
2	2	.04	10	0	.00	19	0	.00
3	5	.10	11	0	.00	20	1	.02
4	6	.12	12	1	.02	21	1	.02
5	3	.06	13	0	.00	22	0	.00
6	5	.10	14	1	.02	23	0	.00
7	2	.04	15	0	.00	24	1	.02
8	1	.02	16	2	.04	25	0	.00
			17	1	.02			

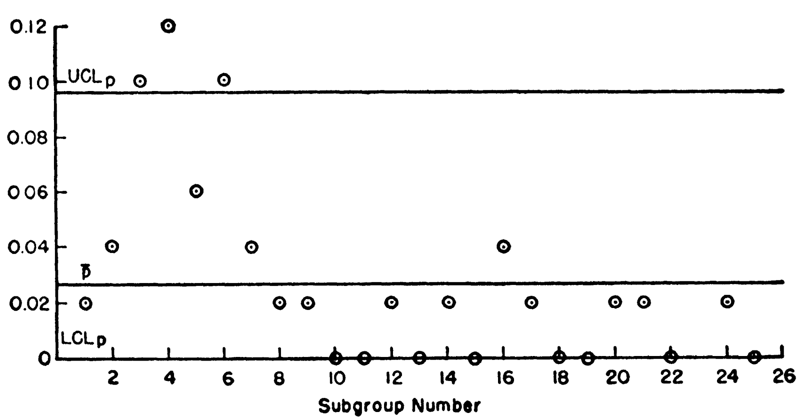
$$\bar{p} = \frac{34}{1250} = 0.0272$$

$$UCL = 0.0272 + \frac{3\sqrt{(0.0272)(0.9728)}}{7.071} = 0.0963$$

$$LCL = 0$$

The control chart for p with trial limits is shown in Fig. 13.8. Many points fall outside the control limits, and it appears that an assignable cause of variation was present when the first samples were taken that was not present when

FIG. **13.8** CONTROL CHART FOR p.



the later samples were taken. Recomputing the control limits starting with the seventh sample we find

$$\bar{p} = \frac{12}{950} = 0.0126$$

$$UCL = 0.0126 + \frac{3\sqrt{(0.0126)(0.9874)}}{7.071} = 0.0599$$

$$LCL = 0$$

No points fall outside these limits and hence we use these as the control limits for future production.

2.7 CONTROL CHARTS FOR DEFECTS

2.7.1 Difference between a defect and a defective. An item is considered to be defective if it fails to conform to the specifications in any of the characteristics. Each characteristic that does not meet the specifications is a defect. An item is defective if it contains at least one defect.

The c chart is a control chart for defects per unit. The unit considered may be a single item, a group of items, part of an item, etc. The unit is examined and the number of defects found is recorded on the c chart. If, for each unit, there are numerous opportunities for defects to occur, and if the probability of a defect occurring in a particular spot is small, the statistical theory for the c chart is based on the Poisson distribution. Some examples where c charts are applicable are in counting the number of defective rivets on an airplane wing, the number of imperfections in a piece of cloth, etc.

2.7.2 Statistical theory. The probability of finding c defects in an item where the number of defects follows the Poisson law is $c'^c e^{-c'}/c!$, where c' is the *average* number of defects per unit, and the standard deviation is $\sqrt{c'}$.

As a working rule, control limits for the c chart are defined as

$$UCL_c = c' + 3\sqrt{c'}; \qquad LCL_c = c' - 3\sqrt{c'}$$

As in the case of the control chart for the fraction defective, these limits do not guarantee that a fixed percentage of the population will lie between them for all values of c' even if the process is in control, i.e., the probability of c falling between these limits depends on the value of c'. However, for practical purposes, it can be assumed that "almost all the values" will fall between the control limits provided the process is in control.

If c' is not known, it is usually estimated from past data. The rules are similar to those used in estimating the parameters from the $\bar{x}$ charts. The estimate in this case is $\bar{c} = c_T/N_T$ where c_T is the total number of defects found in N_T units.

2.7.3 Starting and continuing the control. The discussion on starting and continuing the control chart for fraction defective given in Arts. 2.6.3 and 2.6.4 is pertinent to the c chart.

2.7.4 Example. The following table gives the number of missing rivets noted at final inspection on 25 airplanes. Read column downward and from left to right.

8	15	23	9	10
16	8	16	9	22
14	11	9	14	7
19	21	25	11	28
11	12	15	9	9

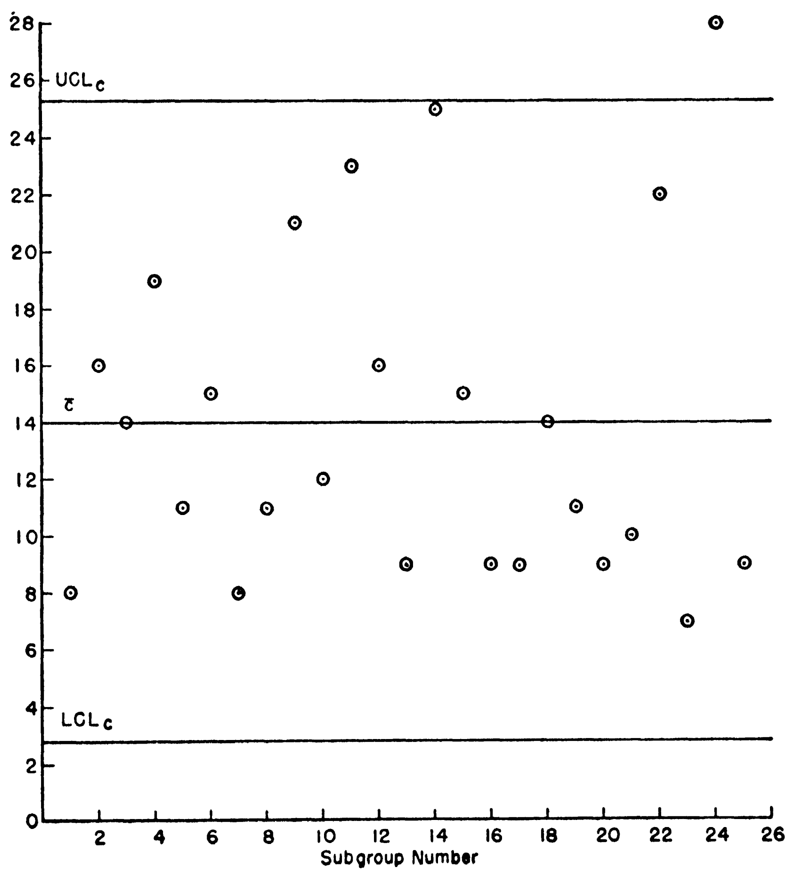
$$\bar{c} = \frac{351}{25} = 14.04$$

$$UCL_c = 14.04 + 3\sqrt{14.04} = 25.26$$

$$LCL_c = 14.04 - 3\sqrt{14.04} = 2.82$$

The control chart for c with trial limits is shown in Fig. 13.9. However, the next to the last plane falls outside the control limits, and it would be best to base future calculations on a $\bar{c}$ which did not involve this airplane.

FIG. **13.9** CONTROL CHART FOR c.



$$\bar{c} = \frac{323}{24} = 13.46$$

$$UCL_c = 13.46 + 3\sqrt{13.46} = 24.47$$

$$LCL_c = 13.46 - 3\sqrt{13.46} = 2.45$$

The fourteenth airplane now falls outside the control limits, so that $\bar{c}$ is again recomputed without the results for this plane.

$$\bar{c} = \frac{298}{23} = 12.96$$

$$UCL_c = 12.96 + 3\sqrt{12.96} = 23.76$$

$$LCL_c = 12.96 - 3\sqrt{12.96} = 2.16$$

3. SAMPLING INSPECTION

3.1 INTRODUCTION

Sampling inspection is of two kinds, namely, lot-by-lot sampling inspection and continuous sampling inspection. In the former, items are formed into lots, a sample is drawn from the lot, and the lot is either accepted or rejected on the basis of the quality of the sample. This is most appropriate for acceptance inspection. In continuous sampling inspection, current inspection results are used to determine whether sampling inspection or screening inspection is to be used for the next articles to be inspected. Sampling plans are further classified depending on whether the quality characteristics are measured and expressed in numbers, i.e., variables inspection; or whether articles are classified only as defective or nondefective, i.e., attributes inspection.

An alternative to sampling inspection is to screen every item. The cost of such a scheme is prohibitive, with perfect quality rarely achieved. Still another advantage of sampling inspection schemes—if properly designed—is that they create more effective pressure for quality improvement, and therefore result in the submission of better quality product for inspection.

Although the above factors were recognized somewhat prior to World War II, and a few sampling inspection schemes were then in use, it wasn't until the outbreak of hostilities that modern sampling inspection received its impetus. Good sampling plans replaced bad ones, sometimes at the expense of an increase in the total amount of inspection but almost always resulting indirectly in better quality being produced.

Any lot-by-lot sampling plan has as its primary purpose the acceptance of good lots and the rejection of bad lots. It is important to define what is meant by a good lot. Naturally, the consumer would like all of his accepted lots to be free of defectives. On the other hand, the manufacturer will usually consider this to be an unreasonable request since some defectives are bound to appear in the manufacturing process. If the manufacturer screens the lot a few times he may get rid of all the defectives, but at the prohibitive cost of screening. This cost will naturally be reflected in his price to the consumer. Ordinarily, the consumer can really tolerate some defectives in his lot, provided the number is not too large. Consequently, the manufacturer and the consumer get together and agree on what constitutes good quality. If lots are submitted at this quality or better, the lot should be accepted, otherwise, rejected. Again this is an imposing task and can be accomplished only at the expense of screening. It is at this point that sampling inspection, with its corresponding advantage of reduced inspection costs, can be instituted. This advantage should not be minimized. Few manufacturers or consumers, whichever has to bear the cost of inspection, can stay in business very long if all lots are screened.

3.2 DRAWING THE SAMPLE

A decision must be made on what shall constitute a lot for acceptance purposes, for each lot must be identified. Each lot should represent, as nearly as possible, the output of one machine or process during one interval of time, so that all parts or products in the lot have been turned out under essentially the same conditions. Wherever practicable, parts from different sources or different conditions should not be mixed into one lot. The power of the sampling plans to distinguish between good and bad lots is dependent on the variation from lot to lot, and provision should be made to maintain the identity of, and prevent the mixing of, the different lots.

A sample from each lot supplies the information on which the decision to accept or reject the lot is based. Therefore it is essential that the sample be drawn from each lot in a random manner. A sample is random if every piece in the lot has an equal chance of being selected. This may be accomplished by using a table of random numbers, a deck of cards, or some such chance method, to insure the equal chance for each piece. Human attempts to randomize a sample without such aids often result in biased samples.

3.3 LOT-BY-LOT SAMPLING INSPECTION BY ATTRIBUTES: SINGLE SAMPLING

A single sampling procedure can be characterized by the following: One sample of n items is drawn from a lot of N items; the lot is accepted if the number of defectives d in the sample does not exceed c. Here c is referred to as the acceptance number.

Certain risks must be taken if sampling inspection is to be used. A graph of these risks plotted as a function of the incoming lot quality (p') is known as an operating characteristic (OC) curve.

If quality is good, it is desirable to have the probability of acceptance high. On the other hand, if quality is bad, it is desirable to have the probability of acceptance small. Note that if p' is 0, the lot contains no defectives and hence the lot will always be accepted, i.e., $L(p') = 1$. If p' is 1 the entire

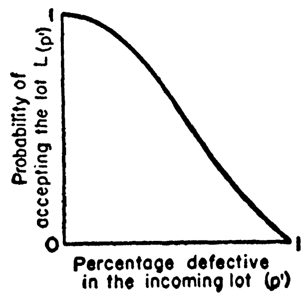


FIG. 13.10 OPERATING CHARACTERISTIC CURVE.

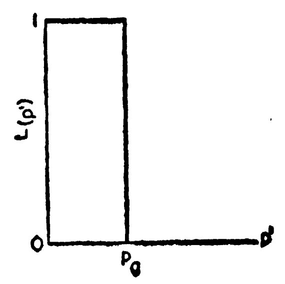
lot is defective and hence the sample will always contain all defectives so that the lot will always be rejected, i.e., $L(p') = 0$.

Assume that a quality standard p_0 is established and that all lots better than this standard are considered to be "good" and all lots worse are considered to be "bad." An "ideal" OC curve would then be of the form shown in Fig. 13.11, with all good lots accepted and all bad ones rejected. Obviously, no sampling plan can have such a curve. The degree of approximation to the ideal curve depends on n and c.

If c is held constant, and n is increased, the slope becomes steeper. On the other hand, holding n constant and changing the acceptance number has the effect of shifting the OC curve to the left or right.

If the lot size N is large compared with the sample size n, the OC curve is essentially independent of the lot size. In other words, if the OC curve of a sampling plan defined by a sample of size n, a lot of size N, and an acceptance number of c is compared with the OC curve of a sampling plan defined by a sample of size n, a lot of size $N_1 (N_1 > N)$, and an acceptance number of c, the difference is negligible, if the sample size is small compared with the lot size. Since this is the situation in most industrial applications, we will make this assumption throughout the rest of this section. As a result, a single sampling plan will now be defined by only two numbers, the sample size n and acceptance number c.★

FIG. 13.11 IDEAL OPERATING CHARACTERISTICS CURVE.



★ From a mathematical viewpoint, this implies that the binomial distribution can be used as an approximation to the hypergeometric distribution.

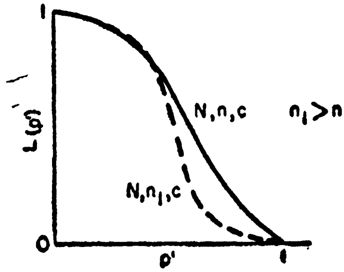


FIG. 13.12 EFFECT ON OC OF VARING n.

The single sampling procedure can be viewed from another side. By drawing a sample of n, and looking at the number of defectives d present, the consumer is essentially estimating the percentage defective, i.e., $\hat{p} = d/n$ is an estimate of the incoming quality p'. A lot is rejected if this estimate is too high, i.e., if $\hat{p} > c/n = p^*$. Therefore, a procedure which says to reject a lot if the number of defectives d is greater than c in a sample of size n is equivalent to rejecting a lot if the estimated percentage defective, $d/n = \hat{p}$, is greater than $c/n = p^*$. This viewpoint will be useful later in the discussion of sampling inspection by variables.

3.3.1 Choosing a sampling plan. The consumer has at his disposal the choice of one of many OC curves. The one chosen should reflect his views as to the cost of making wrong decisions. By varying n and c, he can always find an OC curve which will pass through two preassigned points. In other words, by specifying the points $[p_1', L(p_1')]$ and $[p_2', L(p_2')]$, an n and a c can be found such that the OC curve passes through these two points. Before using sampling inspection, then, the consumer can locate two points p_1' and p_2' such that if quality is submitted better than p_1', he will accept the lot with probability greater than $L(p_1') = 1 - \alpha$; and if quality is submitted worse than p_2' he will accept the lot with probability less than $L(p_2') = \beta$. Here α is known as the producer's risk and β is known as the consumer's risk. The long-run average of submitted quality is known as the process average.

3.3.2 Calculation of OC curves for single sampling plans. It was pointed out previously that a sampling plan is defined by two numbers: n, the sample

FIG. 13.13 EFFECT ON OC OF VARYING c.

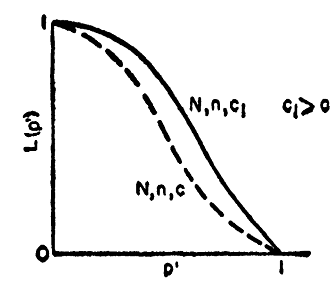


TABLE 13.6 SUMMATION OF TERMS OF POISSON'S EXPONENTIAL BINOMIAL LIMIT★

1,000 × probability of c or less occurrences of event that has average number of occurrences equal to c' or np'

c' or np' \ c	0	1	2	3	4	5	6	7	8	9
0.02	980	1,000								
0.04	961	999	1,000							
0.06	942	998	1,000							
0.08	923	997	1,000							
0.10	905	995	1,000							
0.15	861	990	999	1,000						
0.20	819	982	999	1,000						
0.25	779	974	998	1,000						
0.30	741	963	996	1,000						
0.35	705	951	994	1,000						
0.40	670	938	992	999	1,000					
0.45	638	925	989	999	1,000					
0.50	607	910	986	998	1,000					
0.55	577	894	982	998	1,000					
0.60	549	878	977	997	1,000					
0.65	522	861	972	996	999	1,000				
0.70	497	844	966	994	999	1,000				
0.75	472	827	959	993	999	1,000				
0.80	449	809	953	991	999	1,000				
0.85	427	791	945	989	998	1,000				
0.90	407	772	937	987	998	1,000				
0.95	387	754	929	984	997	1,000				
1.00	368	736	920	981	996	999	1,000			
1.1	333	699	900	974	995	999	1,000			
1.2	301	663	879	966	992	998	1,000			
1.3	273	627	857	957	989	998	1,000			
1.4	247	592	833	946	986	997	999	1,000		
1.5	223	558	809	934	981	996	999	1,000		
1.6	202	525	783	921	976	994	999	1,000		
1.7	183	493	757	907	970	992	998	1,000		
1.8	165	463	731	891	964	990	997	999	1,000	
1.9	150	434	704	875	956	987	997	999	1,000	
2.0	135	406	677	857	947	983	995	999	1,000	

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TABLE 13.6 SUMMATION OF TERMS OF POISSON'S EXPONENTIAL BINOMIAL LIMIT (*Continued*)

c' or np' \ c	0	1	2	3	4	5	6	7	8	9
2.2	111	355	623	819	928	975	993	998	1,000	
2.4	091	308	570	779	904	964	988	997	999	1,000
2.6	074	267	518	736	877	951	983	995	999	1,000
2.8	061	231	469	692	848	935	976	992	998	999
3.0	050	199	423	647	815	916	966	988	996	999
3.2	041	171	380	603	781	895	955	983	994	998
3.4	033	147	340	558	744	871	942	977	992	997
3.6	027	126	303	515	706	844	927	969	988	996
3.8	022	107	269	473	668	816	909	960	984	994
4.0	018	092	238	433	629	785	889	949	979	992
4.2	015	078	210	395	590	753	867	936	972	989
4.4	012	066	185	359	551	720	844	921	964	985
4.6	010	056	163	326	513	686	818	905	955	980
4.8	008	048	143	294	476	651	791	887	944	975
5.0	007	040	125	265	440	616	762	867	932	968
5.2	006	034	109	238	406	581	732	845	918	960
5.4	005	029	095	213	373	546	702	822	903	951
5.6	004	024	082	191	342	512	670	797	886	941
5.8	003	021	072	170	313	478	638	771	867	929
6.0	002	017	062	151	285	446	606	744	847	916

	10	11	12	13	14	15	16
2.8	1,000						
3.0	1,000						
3.2	1,000						
3.4	999	1,000					
3.6	999	1,000					
3.8	998	999	1,000				
4.0	997	999	1,000				
4.2	996	999	1,000				
4.4	994	998	999	1,000			
4.6	992	997	999	1,000			
4.8	990	996	999	1,000			
5.0	986	995	998	999	1,000		
5.2	982	993	997	999	1,000		
5.4	977	990	996	999	1,000		
5.6	972	988	995	998	999	1,000	
5.8	965	984	993	997	999	1,000	
6.0	957	980	991	996	999	999	1,000

TABLE **13.6** SUMMATION OF TERMS OF POISSON'S EXPONENTIAL BINOMIAL LIMIT (*Continued*)

c' or np' \ c	0	1	2	3	4	5	6	7	8	9
6.2	002	015	054	134	259	414	574	716	826	902
6.4	002	012	046	119	235	384	542	687	803	886
6.6	001	010	040	105	213	355	511	658	780	869
6.8	001	009	034	093	192	327	480	628	755	850
7.0	001	007	030	082	173	301	450	599	729	830
7.2	001	006	025	072	156	276	420	569	703	810
7.4	001	005	022	063	140	253	392	539	676	788
7.6	001	004	019	055	125	231	365	510	648	765
7.8	000	004	016	048	112	210	338	481	620	741
8.0	000	003	014	042	100	191	313	453	593	717
8.5	000	002	009	030	074	150	256	386	523	653
9.0	000	001	006	021	055	116	207	324	456	587
9.5	000	001	004	015	040	089	165	269	392	522
10.0	000	000	003	010	029	067	130	220	333	458
	10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
6.2	949	975	989	995	998	999	1,000			
6.4	939	969	986	994	997	999	1,000			
6.6	927	963	982	992	997	999	999	1,000		
6.8	915	955	978	990	996	998	999	1,000		
7.0	901	947	973	987	994	998	999	1,000		
7.2	887	937	967	984	993	997	999	999	1,000	
7.4	871	926	961	980	991	996	998	999	1,000	
7.6	854	915	954	976	989	995	998	999	1,000	
7.8	835	902	945	971	986	993	997	999	1,000	
8.0	816	888	936	966	983	992	996	998	999	1,000
8.5	763	849	909	949	973	986	993	997	999	999
9.0	706	803	876	926	959	978	989	995	998	999
9.5	645	752	836	898	940	967	982	991	996	998
10.0	583	697	792	864	917	951	973	986	993	997
	20	**21**	**22**							
8.5	1,000									
9.0	1,000									
9.5	999	1,000								
10.0	998	999	1,000							

TABLE 13.6 SUMMATION OF TERMS OF POISSON'S EXPONENTIAL BINOMIAL LIMIT (*Continued*)

c' or np' \ c	0	1	2	3	4	5	6	7	8	9
10.5	000	000	002	007	021	050	102	179	279	397
11.0	000	000	001	005	015	038	079	143	232	341
11.5	000	000	001	003	011	028	060	114	191	289
12.0	000	000	001	002	008	020	046	090	155	242
12.5	000	000	000	002	005	015	035	070	125	201
13.0	000	000	000	001	004	011	026	054	100	166
13.5	000	000	000	001	003	008	019	041	079	135
14.0	000	000	000	000	002	006	014	032	062	109
14.5	000	000	000	000	001	004	010	024	048	088
15.0	000	000	000	000	001	003	008	018	037	070
	10	11	12	13	14	15	16	17	18	19
10.5	521	639	742	825	888	932	960	978	988	994
11.0	460	579	689	781	854	907	944	968	982	991
11.5	402	520	633	733	815	878	924	954	974	986
12.0	347	462	576	682	772	844	899	937	963	979
12.5	297	406	519	628	725	806	869	916	948	969
13.0	252	353	463	573	675	764	835	890	930	957
13.5	211	304	409	518	623	718	798	861	908	942
14.0	176	260	358	464	570	669	756	827	883	923
14.5	145	220	311	413	518	619	711	790	853	901
15.0	118	185	268	363	466	568	664	749	819	875
	20	21	22	23	24	25	26	27	28	29
10.5	997	999	999	1,000						
11.0	995	998	999	1,000						
11.5	992	996	998	999	1,000					
12.0	988	994	997	999	999	1,000				
12.5	983	991	995	998	999	999	1,000			
13.0	975	986	992	996	998	999	1,000			
13.5	965	980	989	994	997	998	999	1,000		
14.0	952	971	983	991	995	997	999	999	1,000	
14.5	936	960	976	986	992	996	998	999	999	1,000
15.0	917	947	967	981	989	994	997	998	999	1,000

TABLE 13.6 SUMMATION OF TERMS OF POISSON'S EXPONENTIAL BINOMIAL LIMIT (*Continued*)

c' or np' \ c	4	5	6	7	8	9	10	11	12	13
16	000	001	004	010	022	043	077	127	193	275
17	000	001	002	005	013	026	049	085	135	201
18	000	000	001	003	007	015	030	055	092	143
19	000	000	001	002	004	009	018	035	061	098
20	000	000	000	001	002	005	011	021	039	066
21	000	000	000	000	001	003	006	013	025	043
22	000	000	000	000	001	002	004	008	015	028
23	000	000	000	000	000	001	002	004	009	017
24	000	000	000	000	000	000	001	003	005	011
25	000	000	000	000	000	000	001	001	003	006

c' or np' \ c	14	15	16	17	18	19	20	21	22	23
16	368	467	566	659	742	812	868	911	942	963
17	281	371	468	564	655	736	805	861	905	937
18	208	287	375	469	562	651	731	799	855	899
19	150	215	292	378	469	561	647	725	793	849
20	105	157	221	297	381	470	559	644	721	787
21	072	111	163	227	302	384	471	558	640	716
22	048	077	117	169	232	306	387	472	556	637
23	031	052	082	123	175	238	310	389	472	555
24	020	034	056	087	128	180	243	314	392	473
25	012	022	038	060	092	134	185	247	318	394

c' or np' \ c	24	25	26	27	28	29	30	31	32	33
16	978	987	993	996	998	999	999	1,000		
17	959	975	985	991	995	997	999	999	1,000	
18	932	955	972	983	990	994	997	998	999	1,000
19	893	927	951	969	980	988	993	996	998	999
20	843	888	922	948	966	978	987	992	995	997
21	782	838	883	917	944	963	976	985	991	994
22	712	777	832	877	913	940	959	973	983	989
23	635	708	772	827	873	908	936	956	971	981
24	554	632	704	768	823	868	904	932	953	969
25	473	553	629	700	763	818	863	900	929	950

c' or np' \ c	34	35	36	37	38	39	40	41	42	43
19	999	1,000								
20	999	999	1,000							
21	997	998	999	999	1,000					
22	994	996	998	999	999	1,000				
23	988	993	996	997	999	999	1,000			
24	979	987	992	995	997	998	999	999	1,000	
25	966	978	985	991	994	997	998	999	999	1,000

size and c, the acceptance number. Thus the sampling procedure is to accept a lot if the number of defectives d in a sample of n does not exceed c. The OC curve is defined by this procedure. If lot quality is submitted at p' percent defective, the probability of accepting the lot is $L(p')$.

$$\begin{aligned} L(p') &= \sum_{d=0}^{c} \binom{n}{d}(p')^d(1-p')^{n-d} \\ &= (1-p')^n + \binom{n}{1}(p')(1-p')^{n-1} + \binom{n}{2}(p')^2(1-p')^{n-2} \\ &\quad + \dots + \binom{n}{n}(p')^c \end{aligned}$$

where $\binom{n}{d} = \dfrac{n!}{d!(n-d)!}$, and $0!$ is defined to be equal to 1.

Such calculations as the above are often cumbersome, and an approximation is desirable. Approximate answers may be obtained very rapidly by the use of the Poisson distribution, which is tabulated in Table 13.6.★ The larger the n and smaller the p, the closer the approximate answer is to the true probability.

3.3.3 **Example.** What is the probability of accepting a lot whose incoming quality is 4.0% defective, using a sample of size 30 and an acceptance number $c = 1$?

$$\begin{aligned} L(0.04) &= \sum_{d=0}^{1} \binom{30}{d}(0.04)^d(0.96)^{30-d} \\ &= (0.96)^{30} + \frac{(30)!}{(29)!(1)!}0.04(0.96)^{29} = 0.661 \end{aligned}$$

Using the Poisson approximation,

$$np' = 30 \times 0.04 = 1.20; \qquad c = 1; \qquad L(0.04) = 0.663$$

which, in this case, is in good agreement with the correct value.

3.4 DOUBLE SAMPLING PLANS

A double sampling procedure can be characterized by the following: a sample of n_1 items are drawn from a lot; the lot is accepted if there are no more than c_1 defective items. If there are between $c_1 + 1$ and c_2 defective items, a second sample of size n_2 is drawn; the lot is accepted if there are no more than c_2 defectives in the combined sample of $n_1 + n_2$; the lot is rejected if there are more than c_2 defective items in the combined sample of $n_1 + n_2$.

Thus a lot will be accepted on the first sample if the incoming quality is very good. Similarly, a lot will be rejected on the first sample if the incoming quality is very bad. If the lot is of intermediate quality, a second sample may have to be taken. Double sampling plans have the psychological advantage of giving a second chance to doubtful lots. Furthermore, they can have the additional advantage of requiring fewer total inspections, on the average,

★ Here $L(p')$ is approximately the value read out of Table 13.6 with entries c and np'.

than single sampling plans for any given quality protection. On the other side of the ledger, double sampling plans can have the following disadvantages: (1) more complex administrative duties; (2) inspectors will have variable inspection loads depending upon whether one or two samples are taken; (3) the maximum amount of inspection exceeds that for single sampling plans (which is constant).

3.4.1 OC curves for double sampling plans. A lot will be accepted if and only if (1) the number of defectives d_1 in the first sample of n_1 does not exceed c_1, i.e., $d_1 \leq c_1$; (2) there are more than c_1 defectives in the first sample but no more than c_2 defectives, the total number of defectives $d_1 + d_2$, in the combined sample of $n_1 + n_2$ does not exceed c_2, i.e., if $c_1 < d_1 \leq c_2$, then $d_1 + d_2 \leq c_2$.

Thus for any incoming quality p', the probability of accepting the lot is

$$\begin{aligned} L(p') &= Pr\ (d_1 \leq c_1, \text{ given } p') \\ &\quad + Pr\ (d_1 + d_2 \leq c_2, \text{ given } c_1 < d_1 \leq c_2; p') \\ &= Pr\ (d_1 \leq c_1 | p') + Pr\ (d_1 + d_2 \leq c_2 | c_1 < d_1 \leq c_2; p') \end{aligned}$$

where the symbol | reads "given" and $Pr\ (d_1 \leq c|p')$ reads "the probability that d is less than or equal to c, given p'."

$$\begin{aligned} L(p') &= \sum_{d_1=0}^{c_1} \binom{n_1}{d_1} (p')^{d_1} (1 - p')^{n_1 - d_1} \\ &\quad + \sum_{d_1=c_1+1}^{c_2} \sum_{d_2=0}^{c_2-d_1} \binom{n_1}{d_1} (p')^{d_1} (1 - p')^{n_1 - d_1} \\ &\quad \cdot \binom{n_2}{d_2} (p')^{d_2} (1 - p')^{n_2 - d_2} \\ &= \sum_{d_1=0}^{c_1} \binom{n_1}{d_1} (p')^{d_1} (1 - p')^{n_1 - d_1} \\ &\quad + \sum_{d_1=c_1+1}^{c_2} \left[\binom{n_1}{d_1} (p')^{d_1} (1 - p')^{n_1 - d_1} \right. \\ &\quad \left. \cdot \sum_{d_2=0}^{c_2-d_1} \binom{n_2}{d_2} (p')^{d_2} (1 - p')^{n_2 - d_2}\right] \end{aligned}$$

3.4.2 Example. A large lot of $\frac{3}{4}$-inch screws is submitted for sampling inspection by means of double sampling. If the first sample of 20 contains no defectives, the lot is to be accepted. If it contains more than two defectives, the lot is to be rejected. Otherwise, a second sample of 40 is to be drawn, and the lot is to be accepted if the total number of defectives in both samples does not exceed two. This plan can be characterized by the following table.

Sample	Sample size	Combined samples: Size	Combined samples: Acceptance number	Combined samples: Rejection number
First	20	20	c_1 0	3
Second	40	(n_1+n_2) 60	c_2 2	3

Find the probability of accepting the lot if $p' = 5\%$.

(1) $\sum_{d_1=0}^{0} \binom{n_1}{d_1}(p')^{d}(1 - p')^{n_1-d_1} = (1 - p')^{20} = (0.95)^{20} = 0.358$

(2) $\binom{n_1}{1}(p')(1 - p')^{n_1-1} \sum_{d_2=0}^{1} \binom{n_2}{d_2}(p')^{d_2}(1 - p')^{n_2-d_2}$

$= \binom{n_1}{1}(p')(1 - p')^{n_1-1}[(1 - p')^{n_2} + \binom{n_2}{1}(p')(1 - p')^{n_2-1}]$

$= \binom{20}{1}0.05(0.95)^{19}[(0.95)^{40} + \binom{40}{1}0.05(0.95)^{39}]$

$= 0.377[0.129 + 0.270] = 0.150$

(3) $\binom{n_1}{2}(p')^2(1 - p')^{n_1-2} \sum_{d_2=0}^{0} \binom{n_2}{d_2}(p')^{d_2}(1 - p')^{n_2-d_2}$

$= \binom{n_1}{2}(p')^2(1 - p')^{n_1-2}(1 - p')^{n_2}$

$= \binom{20}{2}(0.05)^2(0.95)^{18}(0.95)^{40} = 0.189 \times 0.129 = 0.02438$

$L(p') = 0.532$

Using the Poisson approximation we have

(1) 0.368

(2)★ $(0.736 - 0.368)(0.406) = 0.368 \times 0.406 = 0.149$

(3)★ $(0.920 - 0.736)(0.135) = 0.184 \times 0.135 = 0.025$

$L(p') = 0.542$

3.5 MULTIPLE SAMPLING PLANS

A multiple sampling procedure can be represented on a table such as the following:

TABLE 13.7 MULTIPLE SAMPLING PLAN

Sample	Sample size	Combined samples		
		Size	Acceptance number	Rejection number
First	n_1	n_1	c_1	r_1
Second	n_2	$n_1 + n_2$	c_2	r_2
Third	n_3	$n_1 + n_2 + n_3$	c_3	r_3
Fourth	n_4	$n_1 + n_2 + n_3 + n_4$	c_4	r_4
Fifth	n_5	$n_1 + n_2 + n_3 + n_4 + n_5$	c_5	r_5
Sixth	n_6	$n_1 + n_2 + n_3 + n_4 + n_5 + n_6$	c_6	r_6
Seventh	n_7	$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7$	c_7	$c_7 + 1$

★ The Poisson table is set up to give $\sum_{d=0}^{c} \binom{n}{d} p^d(1 - p)^{n-d}$. To get an individual term, say $\binom{n}{c} p^c(1 - p)^{n-c}$, we take

$$\sum_{d=0}^{c} \binom{n}{d} p^d(1 - p)^{n-d} - \sum_{d=0}^{c-1} \binom{n}{d} p^d(1 - p)^{n-d} = \binom{n}{c} p^c(1 - p)^{n-c}$$

A first sample of n_1 is drawn, the lot is accepted if there are no more than c_1 defectives, the lot is rejected if there are more than r_1 defectives. Otherwise a second sample of n_2 is drawn. The lot is accepted if there are no more than c_2 defectives in the combined sample of $n_1 + n_2$. The lot is rejected if there are more than r_2 defectives in the combined sample of $n_1 + n_2$. The procedure is continued in accordance with the above table. If, by the end of the sixth sample, the lot is neither accepted or rejected, a sample of n_7 is drawn. The lot is accepted if the number of defectives in the combined sample of $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7$ does not exceed c_7. Otherwise, the lot is rejected. Note that $c_1 < c_2 < \ldots < c_7$ and $c_i < r_i$ for all i. Of course, plans can be devised which permit any number of samples before a decision is reached. A multiple sampling plan will generally involve less inspection, on the average, than the corresponding single or double sampling plan guaranteeing the same protection. The advantage is quite important, since inspection costs are directly related to sample sizes. On the other hand, some of the disadvantages of multiple sampling plans are: (1) they usually require higher administrative costs than single or double sampling plans; (2) the variability of inspection load introduces difficulties in scheduling inspection time; (3) higher caliber inspection personnel may be necessary to guarantee proper use of the plans; (4) adequate storage facilities must be provided for the lot while multiple sampling is being carried on.

3.6 CLASSIFICATION OF SAMPLING PLANS

Published tables of sampling plans have been classified into four categories, namely, by acceptable quality level (AQL), by lot tolerance per cent defective (LTPD), by point of control, and by average outgoing quality limit (AOQL).

3.6.1 Classification by AQL. Acceptance procedures based on the acceptable quality level (AQL) generally make use of the process average to determine the sampling plan to be used. The AQL may be viewed as the highest per cent defective that is acceptable as a process average. In normal sampling, a lot at AQL quality will have a high probability of acceptance. The probability of acceptance, $1 - \alpha$, at the AQL is usually set near the 95% point, and α is known as the producer's risk. Thus a producer has good protection against rejection of submitted lots from a process that is at the AQL or better. On the other hand, this type of classification does not specify anything about the protection the consumer has against the acceptance of a lot worse than the AQL.

3.6.2 Classification by LTPD. The lot tolerance per cent defective (LTPD) usually refers to that incoming quality above which there is a small chance that a lot will be accepted. This probability is usually taken to be near the 10% point. Thus a consumer, inspecting lots submitted from a process that is at the LTPD or worse, has a small probability of accepting such lots. This probability is known as the consumer's risk. On the other hand, this type of classification does not specify anything about the protection the producer has against the rejection of lots better than the LTPD.

3.6.3 Classification by point of control. The point of control is the lot quality for which the probability of acceptance is 0.50. The concept here is one of splitting the risk between producer and consumer. Lots submitted from processes whose quality is better than the point of control have a probability of acceptance that is higher than 50%. Lots submitted from processes whose quality is worse than the point of control have a probability of acceptance smaller than 50%.

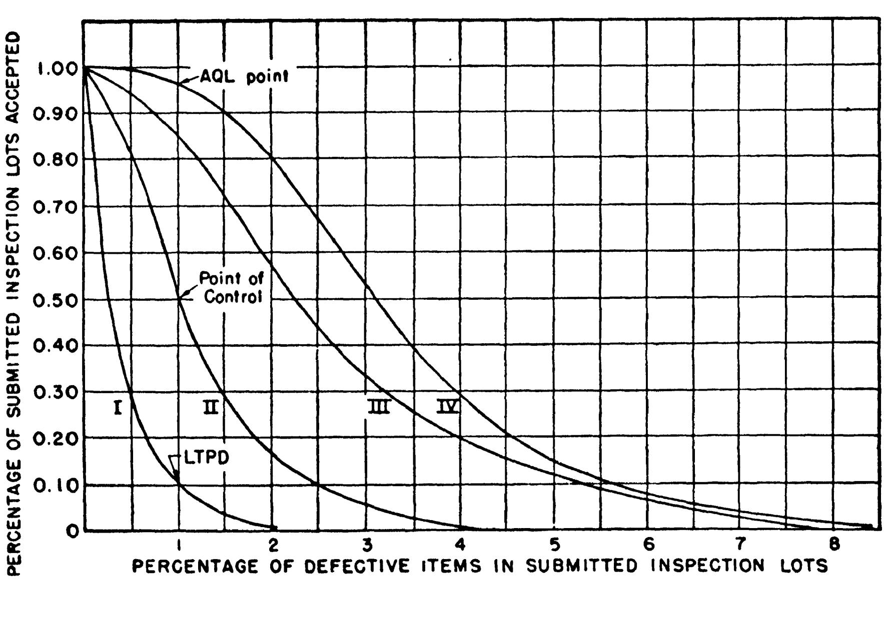
3.6.4 Classification by AOQL. The average outgoing quality limit (AOQL) does not refer to a point on the OC curve, but rather to the upper limit on outgoing quality that may be expected in the long run when all rejected lots are subjected to 100% inspection, with all defective articles removed and replaced by good articles. The average outgoing quality (AOQL) can be computed from the formulas

$$\text{AOQ} = p'L(p'); \qquad \text{AOQL} = \max_{p'} \text{AOQ}$$

where $\max_{p'}$ AOQ is the maximum AOQ over the range $p' = 0$ to $p' = 1$.

Figure 13.14 shows a graphic comparison of these four methods of indexing acceptance plans in relation to a stated quality standard. The four curves

FIG. **13.14** SINGLE SAMPLING PLANS CLASSIFIED ON
I. 1% LTPD: $n = 225$; $c = 50$.
II. POINT OF CONTROL = 1%: $n = 50$; $c = 1$.
III. 1% AOQL: $n = 75$; $c = 1$.
IV. AQL = 1%: $n = 150$; $c = 4$.



shown in this figure are OC curves for single sampling plans that are indexed in their respective tables by the 1.0% defective figure. The most lenient plan (curve IV) is that classified by the AQL; in this plan with an AQL of 1.0%, four defectives are permitted in a sample of 150. The strictest plan (curve I) is the LTPD plan permitting 0 defectives in a sample of 225. The point of control plan (curve II) permits 1 defective in a sample of 150; here the probability of acceptance of a 1.0% defective lot is 0.50. The 1.0% AOQL plan allows 1 defective in a sample of 75; the OC curve (curve III) is intermediate between the AQL 1.0% and the point of control 1.0% plans.

3.7 DODGE-ROMIG TABLES

In 1944, H. F. Dodge and H. G. Romig published a volume of attribute sampling tables called *Sampling Inspection Tables.*★ These tables originated in the Bell Telephone Laboratories and reflect many years of experience with acceptance sampling in the Bell Telephone System. It should be pointed out that these tables were originally prepared for use within the Bell Telephone System. They were designed primarily to minimize the total amount of inspection, considering total sampling inspection and screening inspection of rejected lots. The Dodge-Romig volume contains four sets of tables, as follows:

(1) Single Sampling Lot Tolerance Tables
(2) Double Sampling Lot Tolerance Tables
(3) Single Sampling AOQL Tables
(4) Double Sampling AOQL Tables.

3.7.1 Single sampling lot tolerance tables. This set indexes plans according to the following lot tolerance per cent defectives (LTPD) with the consumer's risk = 0.10.

0.5%	2.0%	4.0%	7.0%
1.0%	3.0%	5.0%	10.0%

Table 13.8 presents a typical table from Dodge-Romig single sampling LTPD. All the sampling plans in this table have the same LTPD. Furthermore, if rejected lots are screened, the table gives the AOQL values for each plan. For any given lot size there is a choice of six plans, each for a different value of the process average. If the plan chosen corresponds to the true process average of the production process, and rejected lots are screened, the plan will guarantee that the average amount of inspection, considering inspection of samples and screening of rejected lots, will be smaller than for any of the remaining five plans.† The process average is usually estimated from past data by taking the ratio of the total number of defectives to the total number inspected. If there are no past data to estimate the process average, the plan

★ H. F. Dodge and H. G. Romig, *Sampling Inspection Tables* (New York: John Wiley and Sons, Inc., 1944).

† The average amount of inspection (AOI) is given by the formula

$$\text{AOI} = nL(p') + N[1 - L(p')].$$

TABLE **13.8** EXAMPLE OF DODGE-ROMIG SINGLE SAMPLING LOT TOLERANCE TABLES

Lot tolerance per cent defective = 5.0%; consumer's risk = 0.10.

Process average %	0–0.05			0.06–0.50			0.51–1.00			1.01–1.50			1.51–2.00			2.01–2.50		
Lot size	n	c	AOQL %	n	c	AOQL %	n	c	AOQL %	n	c	AOQL %	n	c	AOQL %	n	c	AOQL %
1–30	All	0	0	All	0	0	All	0	0	All	0	0	All	0	0	All	0	0
31–50	30	0	0.49	30	0	0.49	30	0	0.49	30	0	0.49	30	0	0.49	30	0	0.49
51–100	37	0	0.63	37	0	0.63	37	0	0.63	37	0	0.63	37	0	0.63	37	0	0.63
101–200	40	0	0.74	40	0	0.74	40	0	0.74	40	0	0.74	40	0	0.74	40	0	0.74
201–300	43	0	0.74	43	0	0.74	70	1	0.92	70	1	0.92	95	2	0.99	95	2	0.99
301–400	44	0	0.74	44	0	0.74	70	1	0.99	100	2	1.0	120	3	1.1	145	4	1.1
401–500	45	0	0.75	75	1	0.95	100	2	1.1	100	2	1.1	125	3	1.2	150	4	1.2
501–600	45	0	0.76	75	1	0.98	100	2	1.1	125	3	1.2	150	4	1.3	175	5	1.3
601–800	45	0	0.77	75	1	1.0	100	2	1.2	130	3	1.2	175	5	1.4	200	6	1.4
801–1,000	45	0	0.78	75	1	1.0	105	2	1.2	155	4	1.4	180	5	1.4	225	7	1.5
1,001–2,000	45	0	0.80	75	1	1.0	130	3	1.4	180	5	1.6	230	7	1.7	280	9	1.8
2,001–3,000	75	1	1.1	105	2	1.3	135	3	1.4	210	6	1.7	280	9	1.9	370	13	2.1
3,001–4,000	75	1	1.1	105	2	1.3	160	4	1.5	210	6	1.7	305	10	2.0	420	15	2.2
4,001–5,000	75	1	1.1	105	2	1.3	160	4	1.5	235	7	1.8	330	11	2.0	440	16	2.2
5,001–7,000	75	1	1.1	105	2	1.3	185	5	1.7	260	8	1.9	350	12	2.2	490	18	2.4
7,001–10,000	75	1	1.1	105	2	1.3	185	5	1.7	260	8	1.9	380	13	2.2	535	20	2.5
10,001–20,000	75	1	1.1	135	3	1.4	210	6	1.8	285	9	2.0	425	15	2.3	610	23	2.6
20,001–50,000	75	1	1.1	135	3	1.4	235	7	1.9	305	10	2.1	470	17	2.4	700	27	2.7
50,001–100,000	75	1	1.1	160	4	1.6	235	7	1.9	355	12	2.2	515	19	2.5	770	30	2.8

should be selected from the right-hand column of the table. This gives good lots a better chance of acceptance.

It must be emphasized that if screening of rejected lots does not take place, these plans still guarantee a stated LTPD.

3.7.2 Double sampling lot tolerance tables. This set indexes plans according to the same LTPD's as for the single sampling LTPD plans. A typical double sampling LTPD table is shown in Table 13.9.

This table has the same features as the single sampling LTPD tables with the exception of using double sampling instead of single sampling. The first sample is always smaller than the corresponding single sampling plan, but if a second sample is required, the combined sample size exceeds the sample size of the single sampling plan. However, the average amount of sampling inspection is generally smaller with the double sampling plan. Since the sec-

TABLE 13.9 EXAMPLE OF DODGE-ROMIG DOUBLE SAMPLING LOT TOLERANCE TABLES

Lot tolerance per cent defective = 5.0%; consumer's risk = 0.10.

Process average %	0–0.05						0.06–0.50						2.01–2.50					
	Trial 1		Trial 2			AOQL in %	Trial 1		Trial 2			AOQL in %	Trial 1		Trial 2			AOQL in %
Lot size	n_1	c_1	n_2	n_1+n_2	c_2		n_1	c_1	n_2	n_1+n_2	c_2		n_1	c_1	n_2	n_1+n_2	c_2	
1–30	All	0	..	..	.	0	All	0	..	..	.	0	All	0	..	..	.	0
31–50	30	0	..	..	.	0.49	30	0	..	..	.	0.49	30	0	..	..	.	0.49
51–75	38	0	..	..	.	0.59	38	0	..	..	.	0.59	38	0	..	..	.	0.59
76–100	44	0	21	65	1	0.64	44	0	21	65	1	0.64	44	0	21	65	1	0.64
101–200	49	0	26	75	1	0.84	49	0	26	75	1	0.84	49	0	51	100	2	0.91
201–300	50	0	30	80	1	0.91	50	0	30	80	1	0.91	50	0	100	150	4	1.1
301–400	55	0	30	85	1	0.92	55	0	55	110	2	1.1	85	0	105	190	6	1.3
401–500	55	0	30	85	1	0.93	55	0	55	110	2	1.1	85	1	140	225	7	1.4
501–600	55	0	30	85	1	0.94	55	0	60	115	2	1.1	85	1	165	250	8	1.5
601–800	55	0	35	90	1	0.95	55	0	65	120	2	1.1	120	2	185	305	10	1.6
801–1,000	55	0	35	90	1	0.96	55	0	65	120	2	1.1	120	2	210	330	11	1.7
1,001–2,000	55	0	35	90	1	0.98	55	0	95	150	3	1.3	175	4	260	435	15	2.0
2,001–3,000	55	0	65	120	2	1.2	55	0	95	150	3	1.3	205	5	375	580	21	2.3
3,001–4,000	55	0	65	120	2	1.2	55	0	95	150	3	1.3	230	6	420	650	24	2.4
4,001–5,000	55	0	65	120	2	1.2	55	0	95	150	3	1.4	255	7	445	700	26	2.5
5,001–7,000	55	0	65	120	2	1.2	55	0	95	150	3	1.4	255	7	495	750	28	2.6
7,001–10,000	55	0	65	120	2	1.2	55	0	120	175	4	1.5	280	8	540	820	31	2.7
10,001–20,000	55	0	65	120	2	1.2	55	0	120	175	4	1.5	280	8	660	940	36	2.8
20,001–50,000	55	0	65	120	2	1.2	55	0	150	205	5	1.7	305	9	745	1050	41	2.9
50,001–100,000	55	0	65	120	2	1.2	55	0	150	205	5	1.7	330	10	810	1140	45	3.0

ond sample is not always taken (depending on the submitted quality), the process average is usually estimated from the first sample only.

3.7.3 Single sampling AOQL tables. This set indexes plans according to the following average outgoing quality limits (AOQL).

0.1%	1.5%	4.0%
0.25%	2.0%	5.0%
0.5%	2.5%	7.0%
0.75%	3.0%	10.0%
1.0%		

A typical single sampling AOQL table is shown in Table 13.10.

TABLE **13.10** EXAMPLE OF DODGE-ROMIG SINGLE SAMPLING AOQL TABLES

Average outgoing quality limit = 2.0%.

Process average %	0–0.04			0.05–0.40			0.41–0.80			0.81–1.20			1.21–1.60			1.61–2.00		
Lot size	n	c	p_t%	n	c	p_t%	n	c	p_t%	n	c	p_t%	n	c	p_t%	n	c	p_t%
1–15	All	0	....	All	0	....	All	0	....	All	0	....	All	0	....	All	0	....
16–50	14	0	13.6	14	0	13.6	14	0	13.6	14	0	13.6	14	0	13.6	14	0	13.6
51–100	16	0	12.4	16	0	12.4	16	0	12.4	16	0	12.4	16	0	12.4	16	0	12.4
101–200	17	0	12.2	17	0	12.2	17	0	12.2	17	0	12.2	35	1	10.5	35	1	10.5
201–300	17	0	12.3	17	0	12.3	17	0	12.3	37	1	10.2	37	1	10.2	37	1	10.2
301–400	18	0	11.8	18	0	11.8	38	1	10.0	38	1	10.0	38	1	10.0	60	2	8.5
401–500	18	0	11.9	18	0	11.9	39	1	9.8	39	1	9.8	60	2	8.6	60	2	8.6
501–600	18	0	11.9	18	0	11.9	39	1	9.8	39	1	9.8	60	2	8.6	60	2	8.6
601–800	18	0	11.9	40	1	9.6	40	1	9.6	65	2	8.0	65	2	8.0	85	3	7.5
801–1,000	18	0	12.0	40	1	9.6	40	1	9.6	65	2	8.1	65	2	8.1	90	3	7.4
1,001–2,000	18	0	12.0	41	1	9.4	65	2	8.2	65	2	8.2	95	3	7.0	120	4	6.5
2,001–3,000	18	0	12.0	41	1	9.4	65	2	8.2	95	3	7.0	120	4	6.5	180	6	5.8
3,001–4,000	18	0	12.0	42	1	9.3	65	2	8.2	95	3	7.0	155	5	6.0	210	7	5.5
4,001–5,000	18	0	12.0	42	1	9.3	70	2	7.5	125	4	6.4	155	5	6.0	245	8	5.3
5,001–7,000	18	0	12.0	42	1	9.3	95	3	7.0	125	4	6.4	185	6	5.6	280	9	5.1
7,001–10,000	42	1	9.3	70	2	7.5	95	3	7.0	155	5	6.0	220	7	5.4	350	11	4.8
10,001–20,000	42	1	9.3	70	2	7.6	95	3	7.0	190	6	5.6	290	9	4.9	460	14	4.4
20,001–50,000	42	1	9.3	70	2	7.6	125	4	6.4	220	7	5.4	395	12	4.5	720	21	3.9
50,001–100,000	42	1	9.3	95	3	7.6	160	5	5.9	290	9	4.9	505	15	4.2	955	27	3.7

All the sampling plans in this table have the same AOQL, assuming all rejected lots are screened. Furthermore, the table gives the LTPD values for each plan. Like all the Dodge-Romig plans these plans have the property that the average amount of inspection is smallest for that plan which belongs to the class under whose heading the true process average falls.

3.7.4 Double sampling AOQL tables. This set indexes plans according to the same AOQL's as for the single sampling AOQL plans. A typical double sampling AOQL table is shown in Table 13.11.

TABLE **13.11** EXAMPLE OF DODGE-ROMIG DOUBLE SAMPLING AOQL TABLES

Average outgoing quality limit = 2.0%.

Process average %	0–0.04						0.05–0.40						1.61–2.00					
	Trial 1		Trial 2				Trial 1		Trial 2				Trial 1		Trial 2			
Lot size	n_1	c_1	n_2	n_1+n_2	c_2	p_t%	n_1	c_1	n_2	n_1+n_2	c_2	p_t%	n_1	c_1	n_2	n_1+n_2	c_2	p_t%
1–15	All	0	..	..	.	...	All	0	..	..	.	...	All	0	..	..	.	...
16–50	14	0	..	..	.	13.6	14	0	..	..	.	13.6	14	0	..	..	.	13.6
51–100	21	0	12	33	1	11.7	21	0	12	33	1	11.7	23	0	23	46	2	10.9
101–200	24	0	13	37	1	11.0	24	0	13	37	1	11.0	27	0	28	55	2	9.6
201–300	26	0	15	41	1	10.4	26	0	15	41	1	10.4	32	0	48	80	3	8.4
301–400	26	0	16	42	1	10.3	26	0	16	42	1	10.3	36	0	69	105	4	7.6
401–500	27	0	16	43	1	10.3	30	0	35	65	2	9.0	60	1	90	150	6	7.0
501–600	27	0	16	43	1	10.3	31	0	34	65	2	8.9	65	1	95	160	6	6.8
601–800	27	0	17	44	1	10.2	31	0	39	70	2	8.8	70	1	120	190	7	6.4
801–1,000	27	0	17	44	1	10.2	32	0	38	70	2	8.7	70	1	145	215	8	6.2
1,001–2,000	33	0	37	70	2	8.5	33	0	37	70	2	8.5	110	2	205	315	11	5.5
2,001–3,000	34	0	41	75	2	8.2	34	0	41	75	2	8.2	160	3	310	470	15	4.7
3,001–4,000	34	0	41	75	2	8.2	38	0	62	100	3	7.3	235	5	415	650	20	4.3
4,001–5,000	34	0	41	75	2	8.2	38	0	62	100	3	7.3	275	6	475	750	23	4.2
5,001–7,000	35	0	40	75	2	8.1	38	0	62	100	3	7.3	280	6	575	855	26	4.1
7,001–10,000	35	0	40	75	2	8.1	38	0	62	100	3	7.3	320	7	645	965	29	4.0
10,001–20,000	35	0	40	75	2	8.1	39	0	66	105	3	7.2	395	9	835	1,230	37	3.9
20,001–50,000	35	0	40	75	2	8.1	43	0	92	135	4	6.6	480	11	1,090	1,570	46	3.7
50,001–100,000	35	0	45	80	2	8.0	43	0	92	135	4	6.6	580	13	1,460	2,040	58	3.5

This table has the same features as the single sampling AOQL tables with the exception of using double sampling instead of single sampling.

3.8 MILITARY STANDARD 105A

There have been various sets of published sampling tables using the AQL as an index. The first set of tables was developed in 1943 for Army Ordnance, and with some changes and extensions, became the Army Service Forces tables, used during the final years of World War II.

An Administration Manual, *Standard Sampling Inspection Procedures*, was issued by the Navy in October, 1945. This manual, including tables and OC (operating characteristic) curves for all the acceptance sampling plans given, had been prepared by the Statistical Research Group of Columbia University during World War II. Although the tables and procedures in this manual were similar in many respects to those of the Army Service Forces, sequential (multiple) sampling plans were included in addition to single and double sampling plans, and there were several other important points of difference. The actual tables from this manual were issued by the Navy Department in April, 1946, as Appendix X to *General Specifications for Inspection of Material.* After the unification of the armed services, Appendix X was adopted by the Department of Defense in February, 1949, as JAN-STD 105.

In 1949, a book entitled *Sampling Inspection*, edited by H. A. Freeman, M. Friedman, F. Mosteller, and W. A. Wallis★ was published. This book was prepared by the Statistical Research Group of Columbia University, and with a few minor differences, the tables in *Sampling Inspection* are identical with JAN-STD 105.

JAN-STD 105 has now been superseded by MILITARY STANDARD 105A, *Sampling Procedures and Tables for Inspection by Attributes*, adopted in September, 1950, and printed for general use in December, 1950. It is evident that the new standard is likely to be used in the acceptance inspection of many billions of dollars' worth of items purchased by Army, Navy, and Air Force. Moreover, like its predecessors, MIL-STD-105A will doubtless have a great influence on acceptance procedures used in industry.

MIL-STD-105A is available for private use and is for sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. Most of the remaining paragraphs in this section on sampling inspection by attributes will be concerned with a discussion of this document.

3.8.1 Classification of defects. Defects are grouped into three classes, i.e., critical, major, and minor. A critical defect "is one that judgment and experience indicate could result in hazardous or unsafe conditions for individuals using or maintaining the product; or for major end item units of product, such as ships, aircraft, or tanks, a defect that could prevent performance of their tactical functions." A major defect "is a defect, other than critical,

★ H. A. Freeman, M. Friedman, F. Mosteller, and W. A. Wallis, *Sampling Inspection* (New York: McGraw-Hill Book Company, Inc., 1946).

that could result in failure, or materially reduce the usability of the unit of product for its intended purpose." A minor defect "is one that does not materially reduce the usability of the unit of product for its intended purpose, or is a departure from established standards having no significant bearing on the effective use or operation of the unit."

3.8.2 Acceptable quality levels. The acceptable quality level (AQL) "is a nominal value expressed in terms of per cent defective or defects per hundred units, whichever is applicable, specified for a given group of defects of a product." The distinction between a defect and a defective is readily seen if we define a defective item as one containing one or more defects. If the percentage defective is relatively small, more than one defect will occur on an item infrequently. Consequently, in this case, the mathematical theory using defects is essentially equivalent to using defectives. The values of the AQL's are

0.015	0.25	2.5	25.0	250.0
0.035	0.40	4.0	40.0	400.0
0.065	0.65	6.5	65.0	650.0
0.10	1.0	10.0	100.0	1000.0
0.15	1.5	15.0	150.0	

Here AQL values of 10.0 or less are expressed either in per cent defective or in defects per hundred units; those over 10.0 are expressed in defects per hundred units only. These points are approximately in geometric progression and multiples of 1, 1.5, 2.5, 4.0, and 6.5. The probabilities of acceptance of submitted lots having these AQL's range from about 0.80 for the smallest sample sizes to about 0.998 for the largest sample sizes. The particular AQL values to be used for a given product "shall be specified by the Government. The Government may, at its option, specify a separate AQL for all the defects of a given class considered collectively or for any particular type of defect considered individually. A single AQL value may be specified for the group of all defects for which a product is inspected, or separate AQL values may be specified for each of two or more subgroups of the defects."

3.8.3 Normal, tightened, and reduced inspection. The government determines whether to use normal, tightened, or reduced inspection at the start of a contract. During the course of a contract, the government determines whether to use normal, tightened, or reduced inspection as follows:

Normal inspection is used when the estimated process average is not outside the applicable upper and lower limits shown in Table 13.12. These limits are obtained from the relation

$$\text{upper limit} = \text{AQL} + 300\sqrt{\frac{\text{AQL}(1 - \text{AQL})}{n}}$$

$$\text{lower limit} = \text{AQL} - 300\sqrt{\frac{\text{AQL}(1 - \text{AQL})}{n}}$$

where AQL is read in per cent.

Tightened inspection is instituted when the estimated process average exceeds the applicable upper limit shown in Table 13.12. Normal inspection is reinstated when the estimated process average is equal to or less than the

TABLE **13.12** LIMITS OF THE PROCESS AVERAGE

(Lower limits for AQL's from 0.015 to 4.0; MIL-STD-105A)

Number of sample units included in estimated process average	Acceptable quality levels											
	0.015	0.035	0.065	0.10	0.15	0.25	0.40	0.65	1.0	1.5	2.5	4.0
25–34	★	★	★	★	★	★	★	★	★	★	★	★
35–49	★	★	★	★	★	★	★	★	★	★	★	★
50–74	★	★	★	★	★	★	★	★	★	★	★	★
75–99	★	★	★	★	★	★	★	★	★	★	★	★
100–124	★	★	★	★	★	★	★	★	★	★	★	★
125–149	★	★	★	★	★	★	★	★	★	★	★	★
150–199	★	★	★	★	★	★	★	★	★	★	★	★
200–249	★	★	★	★	★	★	★	★	★	★	★	★
250–299	★	★	★	★	★	★	★	★	★	★	★	0.38
300–349	★	★	★	★	★	★	★	★	★	★	★	0.67
350–399	★	★	★	★	★	★	★	★	★	★	0.05	0.90
400–449	★	★	★	★	★	★	★	★	★	★	0.20	1.09
450–549	★	★	★	★	★	★	★	★	★	★	0.38	1.32
550–649	★	★	★	★	★	★	★	★	★	★	0.56	1.55
650–749	★	★	★	★	★	★	★	★	★	0.11	0.71	1.73
750–899	★	★	★	★	★	★	★	★	★	0.22	0.85	1.91
900–1,099	★	★	★	★	★	★	★	★	0.05	0.34	1.00	2.10
1,100–1,299	★	★	★	★	★	★	★	★	0.13	0.44	1.13	2.27
1,300–1,499	★	★	★	★	★	★	★	0.004	0.20	0.52	1.23	2.40
1,500–1,699	★	★	★	★	★	★	★	0.045	0.25	0.58	1.31	2.50
1,700–1,899	★	★	★	★	★	★	★	0.080	0.29	0.63	1.41	2.59
1,900–2,249	★	★	★	★	★	★	★	0.119	0.34	0.69	1.46	2.68
2,250–2,749	★	★	★	★	★	★	0.021	0.166	0.40	0.77	1.55	2.80
2,750–3,499	★	★	★	★	★	★	0.061	0.217	0.46	0.84	1.65	2.93
3,500–4,999	★	★	★	★	★	0.020	0.109	0.279	0.54	0.94	1.77	3.08
5,000–6,999	★	★	★	★	★	0.056	0.155	0.338	0.61	1.03	1.89	3.23
7,000–8,999	★	★	★	★	0.020	0.188	0.188	0.380	0.66	1.09	1.97	3.33
9,000–10,999	★	★	★	0.005	0.034	0.100	0.210	0.408	0.70	1.13	2.03	3.40
11,000–13,499	★	★	★	0.015	0.046	0.116	0.230	0.434	0.73	1.17	2.08	3.46
13,500–17,499	★	★	0.003	0.024	0.057	0.129	0.248	0.456	0.76	1.20	2.12	3.52
17,500–22,499	★	★	0.011	0.033	0.068	0.144	0.266	0.479	0.79	1.24	2.16	3.58
22,500 and up	★	0.003	0.021	0.045	0.083	0.163	0.290	0.510	0.83	1.29	2.23	3.65

TABLE **13.12** LIMITS OF THE PROCESS AVERAGE (*Continued*)

(Lower limits for AQL's from 6.5 to 1000.0; MIL-STD-105A)

Number of sample units included in estimated process average	Acceptable quality levels											
	6.5	10.0	15.0	25.0	40.0	65.0	100.0	150.0	250.0	400.0	650.0	1,000.0
25–34	★	★	★	★	5.07	20.47	44.8	82.4	162.7	289.5	509.2	825.3
35–49	★	★	★	1.86	10.72	27.67	53.7	93.3	176.8	307.4	532.0	853.6
50–74	★	★	0.25	5.95	15.90	34.28	61.9	103.3	189.8	323.8	552.9	879.5
75–99	★	★	2.54	8.92	19.66	39.07	67.8	110.6	199.1	335.7	568.0	898.3
100–124	★	1.04	4.02	10.83	22.07	42.14	71.7	115.3	205.2	343.3	577.7	910.4
125–149	★	1.89	5.07	12.18	23.79	44.33	74.4	118.6	209.5	348.7	584.7	918.9
150–199	0.71	2.82	6.20	13.64	25.64	46.69	77.3	122.2	214.1	354.6	592.1	928.2
200–249	1.40	3.67	7.24	14.99	27.34	48.86	80.0	125.5	218.3	360.0	599.0	936.7
250–299	1.88	4.27	7.99	15.95	28.55	50.40	81.9	127.8	221.4	363.8	603.8	942.7
300–349	2.25	4.73	8.55	16.67	29.47	51.57	83.3	129.6	223.7	366.7	607.5	947.3
350–399	2.55	5.10	9.00	17.25	30.17	52.50	84.5	131.0	225.5	369.0	610.5	951.0
400–449	2.79	5.40	9.36	17.72	30.79	53.26	85.4	132.2	227.0	370.9	612.9	954.0
450–549	3.08	5.76	9.80	18.29	31.51	54.18	86.6	133.6	228.8	373.2	615.8	957.6
550–649	3.38	6.13	10.25	18.87	32.25	55.12	87.7	135.0	230.6	375.5	618.8	961.3
650–749	3.61	6.41	10.61	19.33	32.83	55.86	88.7	136.1	232.1	377.3	621.1	964.1
750–899	3.84	6.70	10.95	19.78	33.39	56.58	89.6	137.2	233.5	379.1	623.4	967.0
900–1,099	4.08	7.00	11.32	20.26	34.00	57.35	90.5	138.4	235.0	381.0	625.8	970.0
1,100–1,299	4.29	7.26	11.64	20.67	34.52	58.02	91.3	139.4	236.3	382.7	627.9	972.6
1,300–1,499	4.46	7.46	11.89	20.99	34.93	58.53	92.0	140.2	237.3	384.0	629.6	974.6
1,500–1,699	4.59	7.63	12.09	21.25	35.26	58.95	92.5	140.8	238.1	385.0	630.9	976.3
1,700–1,899	4.70	7.76	12.26	21.46	35.53	59.30	92.9	141.3	238.8	385.9	632.0	†
1,900–2,249	4.82	7.92	12.45	21.71	35.83	59.69	93.4	141.9	239.6	386.8	†	†
2,250–2,749	4.97	8.10	12.68	22.00	36.21	60.16	94.0	142.7	240.5	†	†	†
2,750–3,499	5.13	8.30	12.92	22.32	36.61	60.67	94.6	143.4	†	†	†	†
3,500–4,999	5.33	8.54	13.22	22.70	37.09	61.29	95.4	†	†	†	†	†
5,000–6,999	5.51	8.78	13.50	23.06	37.55	61.88	†	†	†	†	†	†
7,000–8,999	5.64	8.94	13.70	23.32	37.88	†	†	†	†	†	†	†
9,000–10,999	5.73	9.05	13.84	23.50	†	†	†	†	†	†	†	†
11,000–13,499	5.82	9.15	13.96	†	†	†	†	†	†	†	†	†
13,500–17,499	5.89	9.24	†	†	†	†	†	†	†	†	†	†
17,500–22,499	5.96	†	†	†	†	†	†	†	†	†	†	†
22,500 and up	†	†	†	†	†	†	†	†	†	†	†	†

★ Number of sample units included in estimated process average is insufficient for reduced inspection.
† Number of sample units included in estimated process average is too great. Discard older results.

TABLE **13.12** LIMITS OF THE PROCESS AVERAGE (*Continued*)

(Upper limits for AQL's from 0.015 to 4.0; MIL-STD-105A)

Number of sample units included in estimated process average	Acceptable quality levels											
	0.015	0.035	0.065	0.10	0.15	0.25	0.40	0.65	1.0	1.5	2.5	4.0
25–34	‡	‡	‡	‡	‡	‡	‡	5.103	6.52	8.27	11.23	15.05
35–49	‡	‡	‡	‡	‡	‡	3.328	4.383	5.63	7.17	9.82	13.26
50–74	‡	‡	‡	‡	‡	2.155	2.810	3.722	4.81	6.17	8.52	11.62
75–99	‡	‡	‡	‡	1.396	1.858	2.434	3.243	4.22	5.44	7.59	10.43
100–124	‡	‡	‡	0.996	1.248	1.667	2.193	2.935	3.83	4.97	6.98	9.67
125–149	‡	‡	‡	0.911	1.143	1.532	2.021	2.716	3.56	4.64	6.55	9.13
150–199	‡	‡	0.644	0.818	1.030	1.386	1.836	2.481	3.27	4.28	6.09	8.54
200–249	‡	0.410	0.575	0.733	0.926	1.251	1.666	2.264	3.00	3.95	5.67	8.00
250–299	‡	0.374	0.527	0.673	0.851	1.155	1.545	2.110	2.81	3.72	5.36	7.62
300–349	0.219	0.347	0.490	0.627	0.795	1.083	1.453	1.993	2.67	3.54	5.13	7.33
350–399	0.205	0.325	0.460	0.590	0.750	1.025	1.380	1.900	2.55	3.40	4.95	7.10
400–449	0.193	0.307	0.436	0.561	0.714	0.978	1.321	1.824	2.46	3.28	4.80	6.91
450–549	0.179	0.286	0.407	0.525	0.670	0.921	1.249	1.732	2.34	3.14	4.62	6.68
550–649	0.165	0.264	0.377	0.488	0.625	0.863	1.175	1.638	2.23	3.00	4.44	6.45
650–749	0.151	0.247	0.354	0.459	0.589	0.817	1.117	1.564	2.13	2.89	4.29	6.27
750–899	0.143	0.231	0.331	0.430	0.555	0.772	1.061	1.492	2.04	2.78	4.15	6.09
900–1,099	0.131	0.213	0.307	0.400	0.518	0.724	1.000	1.415	1.95	2.66	4.00	5.90
1,100–1,299	0.121	0.197	0.286	0.374	0.485	0.683	0.948	1.348	1.87	2.56	3.87	5.73
1,300–1,499	0.113	0.185	0.270	0.354	0.461	0.651	0.907	1.296	1.80	2.48	3.77	5.60
1,500–1,699	0.107	0.175	0.256	0.337	0.440	0.625	0.874	1.255	1.75	2.42	3.69	5.50
1,700–1,899	0.102	0.167	0.245	0.324	0.424	0.604	0.847	1.220	1.71	2.37	3.59	5.41
1,900–2,249	0.096	0.158	0.233	0.308	0.405	0.579	0.817	1.181	1.66	2.31	3.54	5.32
2,250–2,749	0.089	0.147	0.218	0.290	0.383	0.550	0.779	1.134	1.60	2.23	3.45	5.20
2,750–3,499	0.081	0.136	0.202	0.270	0.358	0.518	0.739	1.083	1.54	2.16	3.35	5.07
3,500–4,999	0.071	0.121	0.182	0.246	0.328	0.480	0.691	1.021	1.46	2.06	3.23	4.92
5,000–6,999	0.062	0.108	0.164	0.222	0.300	0.444	0.645	0.962	1.39	1.97	3.11	4.77
7,000–8,999	0.056	0.098	0.151	0.206	0.280	0.418	0.612	0.920	1.34	1.91	3.03	4.67
9,000–10,999	0.052	0.091	0.142	0.195	0.266	0.400	0.590	0.892	1.30	1.87	2.97	4.60
11,000–13,499	0.048	0.085	0.133	0.185	0.254	0.384	0.570	0.866	1.27	1.83	2.92	4.54
13,500–17,499	0.044	0.080	0.127	0.176	0.243	0.371	0.552	0.844	1.24	1.80	2.88	4.48
17,500–22,499	0.041	0.075	0.119	0.167	0.232	0.356	0.534	0.821	1.21	1.76	2.84	4.42
22,500 and up	0.036	0.067	0.109	0.155	0.217	0.337	0.510	0.790	1.17	1.71	2.77	4.35

TABLE **13.12** LIMITS OF THE PROCESS AVERAGE (*Concluded*)

(Upper limits for AQL's from 6.5 to 1,000.0; MIL-STD-105A)

Number of sample units included in estimated process average	Acceptable quality levels											
	6.5	10.0	15.0	25.0	40.0	65.0	100.0	150.0	250.0	400.0	650.0	1,000.0
25–34	20.58	27.47	36.39	52.62	74.93	109.53	155.2	217.6	337.3	510.5	790.8	1,174.7
35–49	18.30	24.64	32.93	48.14	69.28	102 33	146.3	206.7	323.2	492.6	768.0	1,146.4
50–74	16.21	22.05	29.75	44.05	64.10	95.72	138.1	196.7	310.2	476.2	747.1	1,120.5
75–99	14.70	20.17	27.46	41.08	60.34	90.93	132.2	189.4	300.9	464.3	732.0	1,101.7
100–124	13.73	18.96	25.98	39.17	57.93	87.86	128.3	184.7	294.8	456.7	722.3	1,089.6
125–149	13.03	18.11	24.93	37.82	56.21	85.67	125.6	181.4	290.5	451.3	715.3	1,081.1
150–199	12.29	17.18	23.80	36.36	54.36	83.31	122.7	177.8	285.9	445.4	707.9	1,071.8
200–249	11.60	16.33	22.76	35.01	52.66	81.14	120.0	174.5	281.7	440.0	701.0	1,063.3
250–299	11.12	15.73	22.01	34.05	51.45	79.60	118.1	172.2	278.6	436.2	696.2	1,057.3
300–349	10.75	15.27	21.45	33.33	50.53	78.43	116.7	170.4	276.3	433.3	692.5	1,052.7
350–399	10.45	14.90	21.00	32.75	49.83	77.50	115.5	169.0	274.5	431.0	689.5	1,049.0
400–449	10.21	14.60	20.64	32.28	49.21	76.74	114.6	167.8	273.0	429.1	687.1	1,046.0
450–549	9.92	14.24	20.20	31.71	48.49	75.82	113.4	166.4	271.2	426.8	684.2	1,042.4
550–649	9.62	13.87	19.75	31.13	47.75	74.88	112.3	165.0	269.4	424.5	681.2	1,038.7
650–749	9.39	13.59	19.39	30.67	47.17	74.14	111.3	163.9	267.9	422.7	678.9	1,035.9
750–899	9.16	13.30	19.05	30.22	46.61	73.42	110.4	162.8	266.5	420.9	676.6	1,033.0
900–1,099	8.92	13.00	18.68	29.74	46.00	72.65	109.5	161.6	265.0	419.0	674.2	1,030.0
1,100–1,299	8.71	12.74	18.36	29.33	45.48	71.98	108.7	160.6	263.7	417.3	672.1	1,027.4
1,300–1,499	8.54	12.54	18.11	29.01	45.07	71.47	108.0	159.8	262.7	416.0	670.4	1,025.4
1,500–1,699	8.41	12.37	17.91	28.75	44.74	71.05	107.5	159.2	261.9	415.0	669.1	1,023.7
1,700–1,899	8.30	12.24	17.74	28.54	44.47	70.70	107.1	158.7	261.2	414.1	668.0	†
1,900–2,249	8.18	12.08	17.55	28.29	44.17	70.31	106.6	158.1	260.4	413.2	†	†
2,250–2,749	8.03	11.90	17.32	28.00	43.79	69.84	106.0	157.3	259.5	†	†	†
2,750–3,499	7.87	11.70	17.08	27.68	43.39	69.33	105.4	156.6	†	†	†	†
3,500–4,999	7.67	11.46	16.78	27.30	42.91	68.71	104.6	†	†	†	†	†
5,000–6,999	7.49	11.22	16.50	26.94	42.45	68.12	†	†	†	†	†	†
7,000–8,999	7.36	11.06	16.30	26.68	42.12	†	†	†	†	†	†	†
9,000–10,999	7.27	10.95	16.16	26.50	†	†	†	†	†	†	†	†
11,000–13,499	7.18	10.85	16.04	†	†	†	†	†	†	†	†	†
13,500–17,499	7.11	10.76	†	†	†	†	†	†	†	†	†	†
17,500–22,499	7.04	†	†	†	†	†	†	†	†	†	†	†
22,500 and up	†	†	†	†	†	†	†	†	†	†	†	†

‡ Normal inspection for these AQL's does not provide sample sizes this small.
† Number of sample units included in estimated process average is too great. Discard older results.

AQL while tightened inspection is in effect. Under tightened inspection, the producer's risk is increased, whereas the consumer's risk is decreased. In other words, the probability of accepting bad lots (as well as good lots) is decreased. This is accomplished by reducing the acceptance numbers while keeping the sample size fixed.

Reduced inspection is instituted, if the government so desires, provided that all the following conditions are satisfied.

"Condition A. The preceding 10 lots have been under normal inspection and none have been rejected.
Condition B. The estimated process average is less than the applicable lower limit shown in Table 13.12.
Condition C. Production is at a steady rate."

Normal inspection is reinstated if any one of the following conditions occur while reduced inspection is in effect.

"Condition A. A lot is rejected.
Condition B. The estimated process average is greater than the AQL.
Condition C. Production becomes irregular or delayed.
Condition D. The Government deems that normal inspection should be reinstated."

Under reduced inspection, the sample size is decreased, thereby increasing the consumer's risk and decreasing (slightly) the producer's risk.

The use of tightened and reduced inspection is an important feature in the success of MIL-STD-105A. If quality is submitted close to but above the AQL, there is still a relatively large probability that it will be accepted. Although this appears to be harmful on the surface, in reality, it is not too damaging. In the first place, rejected lots cost the manufacturer a great deal of money. In fact, too many rejected lots can easily force a manufacturer out of business. For example, few manufacturers can tolerate even a 20% rejection rate for their lots. Second, if lots are constantly being submitted above the AQL, this will be reflected in the process average, which results in tightened inspection. This will cause even more of the manufacturer's lots to be rejected. Consequently, the manufacturer is actually forced to keep his submitted quality better than the agreed-upon level.

On the other hand, if the manufacturer continues to submit quality a great deal better than the AQL, the government is able to use reduced inspection, which results in a saving of inspection costs.

3.8.4 Sampling plans. Sample sizes are designated by code letters from A to Q. The sample size code letter depends on the inspection level and the lot size. There are three inspection levels, I, II, and III. Unless the government specifies otherwise, inspection level II is used. The sample size code letter applicable to the specified inspection level and for lots of given size is obtained from Table 13.13.

TABLE **13.13** SAMPLE SIZE CODE LETTERS (MIL-STD-105A)

Lot size	Inspection levels		
	I	II	III
2 to 8	A	A	C
9 to 15	A	B	D
16 to 25	B	C	E
26 to 40	B	D	F
41 to 65	C	E	G
66 to 110	D	F	H
111 to 180	E	G	I
181 to 300	F	H	J
301 to 500	G	I	K
501 to 800	H	J	L
801 to 1,300	I	K	L
1,301 to 3,200	J	L	M
3,201 to 8,000	L	M	N
8,001 to 22,000	M	N	O
22,001 to 110,000	N	O	P
110,001 to 550,000	O	P	Q
550,001 and over	P	Q	Q

Since it has been pointed out that the OC curves are essentially independent of lot size (for large lots), it may appear incongruous that one of the entries to the table is lot size. However, a moment's reflection will reveal that to the inspection agency the acceptance of a bad lot is much more serious when the lot size is large compared with when it is small. Consequently, better protection in the form of steeper OC curves is desired for large lots.

The appropriate master sampling table is selected as follows:

For normal or tightened inspection

- AQL values of 10.0 or less
  - Single sampling: Table 13.14.
  - Double sampling: MIL-STD-105A, pp. 12-13.
  - Multiple sampling: MIL-STD-105A, pp. 14-15.
- AQL values of greater than 10.0
  - Single sampling: Table 13.14.

For reduced inspection and for all AQL values

- Single sampling: Table 13.15.

The Standard contains the OC curve for each of the plans. Each OC curve corresponds to a single sampling plan, double sampling plan, and multiple sampling plan.

3.9 DESIGNING YOUR OWN ATTRIBUTE PLAN

In many cases, it will not be desirable to select a plan according to the standard procedure of MIL-STD-105A, implying, as it does, a rather arbitrary relation between lot size and sample size. If the characteristics of

TABLE **13.14** MASTER TABLE FOR NORMAL AND TIGHTENED INSPECTION (Single sampling; MIL-STD-105A)

Sample size code letter	Sample size	Acceptable quality levels (normal inspection)																									
		0.015		0.035		0.065		0.10		0.15		0.25		0.40		0.65		1.0		1.5		2.5		4.0		6.5	
		Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re
A	2	↓		↓		↓		↓		↓		↓		↓		↓		↓		↓		↓		↓		↓	
B	3	↓		↓		↓		↓		↓		↓		↓		↓		↓		↓		↓		↓		0	1
C	5	↓		↓		↓		↓		↓		↓		↓		↓		↓		↓		↓		0	1	↑	
D	7	↓		↓		↓		↓		↓		↓		↓		↓		↓		↓		0	1	↑		↓	
E	10	↓		↓		↓		↓		↓		↓		↓		↓		↓		0	1	↑		↓		1	2
F	15	↓		↓		↓		↓		↓		↓		↓		↓		0	1	↑		↓		1	2	2	3
G	25	↓		↓		↓		↓		↓		↓		↓		0	1	↑		↓		1	2	2	3	3	4
H	35	↓		↓		↓		↓		↓		↓		0	1	↑		↓		1	2	2	3	3	4	5	6
I	50	↓		↓		↓		↓		↓		0	1	↑		↓		1	2	2	3	3	4	4	5	6	7
J	75	↓		↓		↓		↓		0	1	↑		↓		1	2	2	3	3	4	4	5	6	7	9	10
K	110	↓		↓		↓		0	1	↑		↓		1	2	2	3	3	4	4	5	6	7	8	9	12	13
L	150	↓		↓		0	1	↑		↓		1	2	2	3	3	4	4	5	5	6	8	9	11	12	17	18
M	225	↓		0	1	↑		↓		1	2	2	3	3	4	4	5	5	6	8	9	11	12	17	18	24	25
N	300	0	1	↑		↓		1	2	2	3	3	4	4	5	5	6	7	8	10	11	14	15	20	21	32	33
O	450	↑		↓		1	2	2	3	3	4	4	5	5	6	7	8	10	11	14	15	20	21	29	30	43	44
P	750	↓		1	2	2	3	3	4	4	5	6	7	8	9	11	12	15	16	20	21	31	32	45	46	68	69
Q	1,500	1	2	2	3	3	4	5	6	7	8	9	10	13	14	18	19	25	26	35	36	56	57	81	82	124	125
		0.035		0.065		0.10		0.15		0.25		0.40		0.65		1.0		1.5		2.5		4.0		6.5		10.0	
		Acceptable quality levels (tightened inspection)																									

Sample size code letter	Sample size	Acceptable quality levels (normal inspection)																					
		10.0		15.0		25.0		40.0		65.0		100.0		150.0		250.0		400.0		650.0		1000.0	
		Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re
A	2	0	1	↓		↓		1	2	2	3	3	4	5	6	8	9	12	13	19	20	28	29
B	3	↑		↓		1	2	2	3	3	4	5	6	8	9	12	13	18	19	28	29	41	42
C	5	↓		1	2	2	3	3	4	5	6	8	9	12	13	19	20	29	30	44	45	65	66
D	7	1	2	2	3	3	4	5	6	7	8	11	12	17	18	26	27	39	40	60	61	89	90
E	10	2	3	3	4	5	6	7	8	10	11	15	16	23	24	36	37	54	55	83	84	123	124
F	15	3	4	4	5	7	8	10	11	15	16	22	23	33	34	51	52	78	79	121	122	178	179
G	25	5	6	7	8	11	12	16	17	24	25	35	36	51	52	80	81	124	125	192	193	↑	
H	35	7	8	10	11	15	16	22	23	33	34	48	49	69	70	110	111	168	169	↑			
I	50	9	10	13	14	20	21	30	31	46	47	67	68	96	97	151	152	↑					
J	75	13	14	19	20	29	30	43	44	66	67	96	97	138	139	↑							
K	110	18	19	26	27	40	41	60	61	93	94	135	136	↑									
L	150	24	25	34	35	53	54	80	81	123	124	↑											
M	225	34	35	48	49	76	77	115	116	↑													
N	300	44	45	63	64	98	99	↑															
O	450	62	63	89	90	↑																	
P	750	98	99	↑																			
Q	1,500	184	185																				
		15.0		25.0		40.0		65.0		100.0		150.0		250.0		400.0		650.0		1000.0			
		Acceptable quality levels (tightened inspection)																					

↓ = Use first sampling plan below arrow. When sample size equals or exceeds lot size, do 100% inspection. ↑ = Use first sampling plan above arrow. Ac = Acceptance no. Re = Rejection no. Tightened sampling plans are not provided for AQL: 0.015.

TABLE 13.15 MASTER TABLE FOR REDUCED INSPECTION (Single sampling only; MIL-STD-105A)

Sample size code letter	Sample size	Acceptable quality levels																							
		0.015		0.035		0.065		0.10		0.15		0.25		0.40		0.65		1.0		1.5		2.5		4.0	
		Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re	Ac	Re
A, B, C, D	2	Reduced inspection not available for AQL: 0.015																		0	1				
E	2																	↓		0	1			↓	
F	3															↓		0	1	↑		↓		1	2
G	5													↓		0	1	↑		↓		1	2	1	2
H	7											↓		0	1	↑		↓		1	2	1	2	2	3
I	10									↓		0	1	↑		↓		1	2	1	2	2	3	2	3
J	15							↓		0	1	↑		↓		1	2	1	2	2	3	2	3	3	4
K	22					↓		0	1	↑		↓		1	2	1	2	2	3	2	3	3	4	4	5
L	30			↓		0	1	↑		↓		1	2	1	2	2	3	2	3	3	4	4	5	5	6
M	45			0	1	↑		↓		1	2	1	2	2	3	2	3	3	4	4	5	5	6	7	8
N	60			↑		↓		1	2	1	2	2	3	2	3	3	4	4	5	5	6	6	7	9	10
O	90			↓		1	2	1	2	2	3	2	3	3	4	4	5	5	6	6	7	9	10	11	12
P	150			1	2	1	2	2	3	2	3	3	4	4	5	5	6	7	8	9	10	11	12	14	15
Q	300			1	2	2	3	3	4	4	5	5	6	6	7	8	9	11	12	13	14	16	17	21	22

Sample size code letter	Sample size	Acceptable quality levels 6.5 Ac	6.5 Re	10.0 Ac	10.0 Re	15.0 Ac	15.0 Re	25.0 Ac	25.0 Re	40.0 Ac	40.0 Re	65.0 Ac	65.0 Re	100.0 Ac	100.0 Re	150.0 Ac	150.0 Re	250.0 Ac	250.0 Re	400.0 Ac	400.0 Re	650.0 Ac	650.0 Re	1000.0 Ac	1000.0 Re
A, B, C, D	1					1	2	1	2	1	2	2	3	4	5	5	6	8	9	12	13	15	16	19	20
E	2	1	2	1	2	2	3	3	4	3	4	4	5	7	8	10	11	13	14	17	18	22	23	29	30
F	3	1	2	2	3	2	3	3	4	5	6	7	8	10	11	13	14	17	18	21	22	29	30	37	38
G	5	2	3	3	4	3	4	5	6	7	8	10	11	13	14	17	18	22	23	29	30	39	40	↑	
H	7	3	4	3	4	5	6	7	8	9	10	13	14	16	17	20	21	27	28	36	37	↑			
I	10	3	4	4	5	6	7	9	10	12	13	16	17	19	20	24	25	33	34	↑					
J	15	4	5	6	7	8	9	12	13	15	16	19	20	24	25	31	32	↑							
K	22	5	6	8	9	11	12	14	15	18	19	23	24	31	32	↑									
L	30	7	8	11	12	12	13	16	17	21	22	29	30	↑											
M	45	10	11	13	14	15	16	20	21	27	28	↑													
N	60	12	13	15	16	18	19	24	25	↑															
O	90	14	15	18	19	23	24	↑																	
P	150	18	19	23	24	↑																			
Q	300	28	29	37	38																				

↓ = Use first sampling plan below arrow. When sample size equals or exceeds lot size, do 100 per cent inspection. ↑ = Use first sampling plan above arrow. Ac = Acceptance number. Re = Rejection number.

TABLE **13.16** VALUES OF np_1 FOR WHICH THE PROBABILITY OF ACCEPTANCE OF c OR FEWER DEFECTIVES IN A SAMPLE OF n IS $P(A)$ *

[To find the fraction defective p, corresponding to a probability of acceptance $P(A)$ in a single sampling plan with sample size n and acceptance number c, divide by n the entry in the row for the given c and the column for the given $P(A)$].

c	$P(A) =$ 0.995	$P(A) =$ 0.990	$P(A) =$ 0.975	$P(A) =$ 0.950	$P(A) =$ 0.900	$P(A) =$ 0.750	$P(A) =$ 0.500	$P(A) =$ 0.250	$P(A) =$ 0.100	$P(A) =$ 0.050	$P(A) =$ 0.025	$P(A) =$ 0.010	$P(A) =$ 0.005
0	0.00501	0.0101	0.0253	0.0513	0.105	0.288	0.693	1.386	2.303	2.996	3.689	4.605	5.298
1	0.103	0.149	0.242	0.355	0.532	0.961	1.678	2.693	3.890	4.744	5.572	6.638	7.430
2	0.338	0.436	0.619	0.818	1.102	1.727	2.674	3.920	5.322	6.296	7.224	8.406	9.274
3	0.672	0.823	1 090	1.366	1.745	2.535	3.672	5.109	6.681	7.754	8.768	10.045	10.978
4	1.078	1.279	1.623	1.970	2.433	3.369	4.671	6.274	7.994	9.154	10.242	11.605	12.594
5	1.537	1.785	2.202	2.613	3.152	4.219	5.670	7.423	9.275	10.513	11.668	13.108	14.150
6	2.037	2.330	2.814	3.286	3.895	5.083	6.670	8.558	10.532	11.842	13.060	14.571	15.660
7	2.571	2.906	3.454	3.981	4.656	5.956	7.669	9.684	11.771	13.148	14.422	16.000	17.134
8	3.132	3.507	4.115	4.695	5.432	6.838	8.669	10.802	12.995	14.434	15.763	17.403	18.578
9	3.717	4.130	4.795	5.426	6.221	7.726	9.669	11.914	14.206	15.705	17.085	18.783	19.998
10	4.321	4.771	5.491	6.169	7.021	8.620	10.668	13.020	15.407	16.962	18.390	20.145	21.398
11	4.943	5.428	6.201	6.924	7.829	9.519	11.668	14.121	16.598	18.208	19.682	21.490	22.779
12	5.580	6.099	6.922	7.690	8.646	10.422	12.668	15.217	17.782	19.442	20.962	22.821	24.145
13	6.231	6.782	7.654	8.464	9.470	11.329	13.668	16.310	18.958	20.668	22.230	24.139	25.496
14	6.893	7.477	8.396	9.246	10.300	12.239	14.668	17.400	20.128	21.886	23.490	25.446	26.836
15	7.566	8.181	9.144	10.035	11.135	13.152	15.668	18.486	21.292	23.098	24.741	26.743	28.166
16	8.249	8.895	9.902	10.831	11.976	14.068	16.668	19.570	22.452	24.302	25.984	28.031	29.484
17	8.942	9.616	10.666	11.633	12.822	14.986	17.668	20.652	23.606	25.500	27.220	29.310	30.792
18	9.644	10.346	11.438	12.442	13.672	15.907	18.668	21.731	24.756	26.692	28.448	30.581	32.092
19	10.353	11.082	12.216	13.254	14.525	16.830	19.668	22.808	25.902	27.879	29.671	31.845	33.383
20	11.069	11.825	12.999	14.072	15.383	17.755	20.668	23.883	27.045	29.062	30.888	33.103	34.668
21	11.791	12.574	13.787	14.894	16.244	18.682	21.668	24.956	28.184	30.241	32.102	34.355	35.947
22	12.520	13.329	14.580	15.719	17.108	19.610	22.668	26.028	29.320	31.416	33.309	35.601	37.219
23	13.255	14.088	15.377	16.548	17.975	20.540	23.668	27.098	30.453	32.586	34.512	36.841	38.485
24	13.995	14.853	16.178	17.382	18.844	21.471	24.668	28.167	31.584	33.752	35.710	38.077	39.745
25	14.740	15.623	16.984	18.218	19.717	22.404	25.667	29.234	32.711	34.916	36.905	39.308	41.000
26	15.490	16.397	17.793	19.058	20.592	23.338	26.667	30.300	33.836	36.077	38.096	40.535	42.252
27	16.245	17.175	18.606	19.900	21.469	24.273	27.667	31.365	34.959	37.234	39.284	41.757	43.497

28	17.004	17.957	19.422	20.746	22.348	25.209	28.667	32.428	36.080	38.389	40.468	42.975	44.738
29	17.767	18.742	20.241	21.594	23.229	26.147	29.667	33.491	37.198	39.541	41.649	44.190	45.976
30	18.534	19.532	21.063	22.444	24.113	27.086	30.667	34.552	38.315	40.690	42.827	45.401	47.210
31	19.305	20.324	21.888	23.298	24.998	28.025	31.667	35.613	39.430	41.838	44.002	46.609	48.440
32	20.079	21.120	22.716	24.152	25.885	28.966	32.667	36.672	40.543	42.982	45.174	47.813	49.666
33	20.856	21.919	23.546	25.010	26.774	29.907	33.667	37.731	41.654	44.125	46.344	49.015	50.888
34	21.638	22.721	24.379	25.870	27.664	30.849	34.667	38.788	42.764	45.266	47.512	50.213	52.108
35	22.422	23.525	25.214	26.731	28.556	31.792	35.667	39.845	43.872	46.404	48.676	51.409	53.324
36	23.208	24.333	26.052	27.594	29.450	32.736	36.667	40.901	44.978	47.540	49.840	52.601	54.538
37	23.998	25.143	26.891	28.460	30.345	33.681	37.667	41.957	46.083	48.676	51.000	53.791	55.748
38	24.791	25.955	27.733	29.327	31.241	34.626	38.667	43.011	47.187	49.808	52.158	54.979	56.956
39	25.586	26.770	28.576	30.196	32.139	35.572	39.667	44.065	48.289	50.940	53.314	56.164	58.160
40	26.384	27.587	29.422	31.066	33.038	36.519	40.667	45.118	49.390	52.069	54.469	57.347	59.363
41	27.184	28.406	30.270	31.938	33.938	37.466	41.667	46.171	50.490	53.197	55.622	58.528	60.563
42	27.986	29.228	31.120	32.812	34.839	38.414	42.667	47.223	51.589	54.324	56.772	59.717	61.761
43	28.791	30.051	31.970	33.686	35.742	39.363	43.667	48.274	52.686	55.449	57.921	60.884	62.956
44	29.598	30.877	32.824	34.563	36.646	40.312	44.667	49.325	53.782	56.572	59.068	62.059	64.150
45	30.408	31.704	33.678	35.441	37.550	41.262	45.667	50.375	54.878	57.695	60.214	63.231	65.340
46	31.219	32.534	34.534	36.320	38.456	42.212	46.667	51.425	55.972	58.816	61.358	64.402	66.529
47	32.032	33.365	35.392	37.200	39.363	43.163	47.667	52.474	57.065	59.936	62.500	65.571	67.716
48	32.848	34.198	36.250	38.082	40.270	44.115	48.667	53.522	58.158	61.054	63.641	66.738	68.901
49	33.664	35.032	37.111	38.965	41.179	45.067	49.667	54.571	59.249	62.171	64.780	67.903	70.084

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TABLE 13.17 VALUES OF np_1 AND c FOR CONSTRUCTING SINGLE SAMPLING PLANS WHOSE OC CURVE IS REQUIRED TO PASS THROUGH THE TWO POINTS $(p_1, 1-\alpha)$ AND (p_2, β) *

(Here p_1 is the fraction defective for which the risk of rejection is to be α, and p_2 is the fraction defective for which the risk of acceptance is to be β. To construct the plan, find the tabular value of p_2/p_1 in the column for the given α and β which is equal to or just greater than the given value of the ratio. The sample size is found by dividing the np_1 corresponding to the selected ratio by p_1. The acceptance number is the value of c corresponding to the selected value of the ratio).

c	Values of p_2/p_1 for: $\alpha = 0.05$, $\beta = 0.10$	$\alpha = 0.05$, $\beta = 0.05$	$\alpha = 0.05$, $\beta = 0.01$	np_1	c	Values of p_2/p_1 for: $\alpha = 0.01$, $\beta = 0.10$	$\alpha = 0.01$, $\beta = 0.05$	$\alpha = 0.01$, $\beta = 0.01$	np_1
0	44.890	58.404	89.781	0.052	0	229.105	298.073	458.210	0.010
1	10.946	13.349	18.681	0.355	1	26.184	31.933	44.686	0.149
2	6.509	7.699	10.280	0.818	2	12.206	14.439	19.278	0.436
3	4.890	5.675	7.352	1.366	3	8.115	9.418	12.202	0.823
4	4.057	4.646	5.890	1.970	4	6.249	7.156	9.072	1.279
5	3.549	4.023	5.017	2.613	5	5.195	5.889	7.343	1.785
6	3.206	3.604	4.435	3.286	6	4.520	5.082	6.253	2.330
7	2.957	3.303	4.019	3.981	7	4.050	4.524	5.506	2.906
8	2.768	3.074	3.707	4.695	8	3.705	4.115	4.962	3.507
9	2.618	2.895	3.462	5.426	9	3.440	3.803	4.548	4.130
10	2.497	2.750	3.265	6.169	10	3.229	3.555	4.222	4.771
11	2.397	2.630	3.104	6.924	11	3.058	3.354	3.959	5.428
12	2.312	2.528	2.968	7.690	12	2.915	3.188	3.742	6.099
13	2.240	2.442	2.852	8.464	13	2.795	3.047	3.559	6.782
14	2.177	2.367	2.752	9.246	14	2.692	2.927	3.403	7.477
15	2.122	2.302	2.665	10.035	15	2.603	2.823	3.269	8.181
16	2.073	2.244	2.588	10.831	16	2.524	2.732	3.151	8.895
17	2.029	2.192	2.520	11.633	17	2.455	2.652	3.048	9.616
18	1.990	2.145	2.458	12.442	18	2.393	2.580	2.956	10.346
19	1.954	2.103	2.403	13.254	19	2.337	2.516	2.874	11.082

20	1.922	2.065	2.352	14.072	20	2.287	2.458	2.799	11.825
21	1.892	2.030	2.307	14.894	21	2.241	2.405	2.733	12.574
22	1.865	1.999	2.265	15.719	22	2.200	2.357	2.671	13.329
23	1.840	1.969	2.226	16.548	23	2.162	2.313	2.615	14.088
24	1.817	1.942	2.191	17.382	24	2.126	2.272	2.564	14.853
25	1.795	1.917	2.158	18.218	25	2.094	2.235	2.516	15.623
26	1.775	1.893	2.127	19.058	26	2.064	2.200	2.472	16.397
27	1.757	1.871	2.098	19.900	27	2.035	2.168	2.431	17.175
28	1.739	1.850	2.071	20.746	28	2.009	2.138	2.393	17.957
29	1.723	1.831	2.046	21.594	29	1.985	2.110	2.358	18.742
30	1.707	1.813	2.023	22.444	30	1.962	2.083	2.324	19.532
31	1.692	1.796	2.001	23.298	31	1.940	2.059	2.293	20.324
32	1.679	1.780	1.980	24.152	32	1.920	2.035	2.264	21.120
33	1.665	1.764	1.960	25.010	33	1.900	2.013	2.236	21.919
34	1.653	1.750	1.941	25.870	34	1.882	1.992	2.210	22.721
35	1.641	1.736	1.923	26.731	35	1.865	1.973	2.185	23.525
36	1.630	1.723	1.906	27.594	36	1.848	1.954	2.162	24.333
37	1.619	1.710	1.890	28.460	37	1.833	1.936	2.139	25.143
38	1.609	1.698	1.875	29.327	38	1.818	1.920	2.118	25.955
39	1.599	1.687	1.860	30.196	39	1.804	1.903	2.098	26.770
40	1.590	1.676	1.846	31.066	40	1.790	1.887	2.079	27.587
41	1.581	1.666	1.833	31.938	41	1.777	1.873	2.060	28.406
42	1.572	1.656	1.820	32.812	42	1.765	1.859	2.043	29.228
43	1.564	1.646	1.807	33.686	43	1.753	1.845	2.026	30.051
44	1.556	1.637	1.796	34.563	44	1.742	1.832	2.010	30.877
45	1.548	1.628	1.784	35.441	45	1.731	1.820	1.994	31.704
46	1.541	1.619	1.773	36.320	46	1.720	1.808	1.980	32.534
47	1.534	1.611	1.763	37.200	47	1.710	1.796	1.965	33.365
48	1.527	1.603	1.752	38.082	48	1.701	1.785	1.952	34.198
49	1.521	1.596	1.743	38.965	49	1.691	1.775	1.938	35.032

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an OC curve are specified in advance, it is possible to determine the sample size and acceptance number accordingly. This article will deal with procedures for determining sampling plans when two points are specified, and with finding the OC curve of an arbitrary sampling plan.

3.9.1 Computing the OC curve of a single sampling plan. In Art. 3.3.2 a method for computing the OC curve of a single sampling plan with sample size n and acceptance number c was presented. The OC curve may also be constructed from Table 13.16 by dividing each entry in the row for the given c by the sample size. The fraction defective p' for which the probability of acceptance is shown in the column heading, is obtained.

Example. For $n = 75$, $c = 4$; find the OC curve.

Dividing the numbers in the row $c = 4$ by 75 we find the following 13 points on the OC curve.

Probability of acceptance	Fraction defective	Probability of acceptance	Fraction defective
0.995	$\frac{1.078}{75} = 0.01437$	0.250	$\frac{6.274}{75} = 0.08365$
0.990	$\frac{1.279}{75} = 0.01705$	0.100	$\frac{7.994}{75} = 0.1066$
0.975	$\frac{1.623}{75} = 0.02164$	0.050	$\frac{9.154}{75} = 0.1221$
0.950	$\frac{1.970}{75} = 0.02627$	0.025	$\frac{10.242}{75} = 0.1372$
0.900	$\frac{2.433}{75} = 0.03244$	0.010	$\frac{11.605}{75} = 0.1547$
0.750	$\frac{3.369}{75} = 0.04492$	0.005	$\frac{12.594}{75} = 0.1679$
0.500	$\frac{4.671}{75} = 0.06228$		

3.9.2 Finding a sampling plan whose OC curve passes through two points. As pointed out in Art. 3.3, it is often useful to specify two points on an OC curve; p_1' and p_2' such that $L(p_1') = 1 - \alpha$ and $L(p_2') = \beta$. Here p_1' is usually denoted as acceptable quality (quality we want to accept) and p_2' usually represents unacceptable quality (quality we want to reject). Hence α and β are usually taken as small numbers, 0.01, 0.05, or 0.10. For these values, sampling plans may be found in Table 13.17. To construct a plan for a given p_1', α and p_2', β, calculate the ratio p_2'/p_1' and find the entry in the appropriate α, β column which is equal to or just greater than the desired ratio. The acceptance number is read off directly and the sample size is determined by dividing $N_{p'_1}$ by p_1'.

Example. $p_1' = 0.02$, $p_2' = 0.04$, $\alpha = 0.05$, $\beta = 0.05$. Find n and c.

$$\frac{p_2'}{p_1'} = \frac{0.04}{0.02} = 2$$

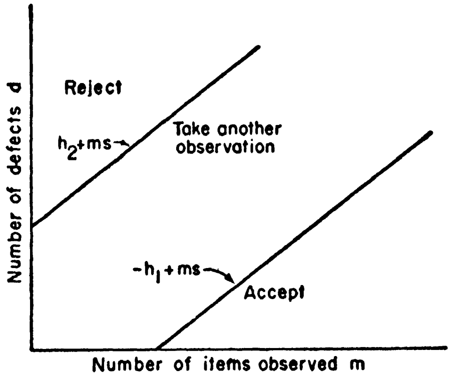


FIG. **13.15** GRAPHIC PROCEDURE FOR ITEM BY ITEM SEQUENTIAL SAMPLING PLAN.

The value in the table just greater than 2 is 2.030. Hence, $c = 21$, $n = 14.894/0.02 = 745$.

3.9.3 Design of item-by-item sequential plans. It is not possible to give simple formulas for the acceptance and rejection numbers in double or multiple sampling plans; however, simple formulas do exist for item-by-item sequential plans, i.e., plans in which the decision to accept, reject, or continue sampling is made after each observation. The plan can be represented graphically as a pair of parallel lines (Fig. 13.15). The plan is determined by the acceptance line $a_m = -h_1 + ms$ and the rejection line $r_m = h_2 + ms$. To calculate h_1, h_2, and s, we introduce the quantities

$$b = \ln\frac{1-\alpha}{\beta}; \quad a = \ln\frac{1-\beta}{\alpha}; \quad g_1 = \ln\frac{p_2'}{p_1'}; \quad g_2 = \ln\frac{1-p_1'}{1-p_2'}$$

Then

$$h_1 = \frac{b}{g_1+g_2}; \quad h_2 = \frac{a}{g_1+g_2}; \quad s = \frac{g_2}{g_1+g_2}$$

Example. Suppose we want $p_1' = 0.01$, $p_2' = 0.10$, $\alpha = 0.05$, $\beta = 0.20$; we calculate

$$b = \ln\frac{0.95}{0.20} = 1.55815; \quad a = \ln\frac{0.80}{0.05} = 2.77259$$

$$g_1 = \ln\frac{0.10}{0.01} = 2.30258; \quad g_2 = \ln\frac{0.99}{0.90} = 0.09531$$

$$h_1 = 0.649796; \qquad h_2 = 1.156407; \qquad s = 0.039747$$

and get the equations

$$a_m = -0.649796 + 0.039747m; \qquad r_m = 1.156407 + 0.039747m$$

Usually the sequential plan is applied in a tabular rather than graphical form; a_m and r_m are computed for every m and listed in Table 13.18. Note that a_m is rounded down to the nearest integer and r_m is rounded up to the nearest integer.

TABLE **13.18** TABULAR PROCEDURE FOR ITEM-BY-ITEM SEQUENTIAL SAMPLING PLAN

Number of observations m	Acceptance number a_m	Rejection number r_m	Observed number of defects d
1	...	...	
2	...	...	
3	...	2	
·	·	·	
·	·	·	
·	·	·	
17	0	2	
·	·	·	
·	·	·	
·	·	·	
22	0	3	
·	·	·	
·	·	·	
·	·	·	
42	0	3	
·	·	·	
·	·	·	
·	·	·	
47	1	4	
·	·	·	
·	·	·	
·	·	·	
67	2	4	
·	·	·	
·	·	·	
·	·	·	
72	2	5	
·	·	·	
·	·	·	
·	·	·	
97	3	6	
·	·	·	
·	·	·	
·	·	·	
117	4	6	
·	·	·	
·	·	·	
·	·	·	
122	4	7	
·	·	·	
·	·	·	
·	·	·	
143	5	7	
·	·	·	
·	·	·	
·	·	·	
148	5	8	

Simple formulas may be found for the probability of acceptance and the expected number of observations from five values of the fraction defective including the two points defining the plan.

$$p' = 0; \qquad L(p') = 1; \qquad \bar{n}_0 = \frac{h_1}{s}$$

$$p' = p_1'; \qquad L(p') = 1 - \alpha; \qquad \bar{n}_{p'_1} = \frac{(1-\alpha)h_1 - \alpha h_2}{s - p_1'}$$

$$p' = s; \qquad L(p') = \frac{h_2}{h_1 + h_2}; \qquad \bar{n}_s = \frac{h_1 h_2}{s(1-s)}$$

$$p' = p_2'; \qquad L(p') = \beta; \qquad \bar{n}_{p'_2} = \frac{(1-\beta)h_2 - \beta h_1}{p_2' - s}$$

$$p' = 1; \qquad L(p') = 0; \qquad \bar{n}_1 = \frac{h_2}{1-s}$$

In the above example,

$$p' = 0; \qquad L(0) = 1; \qquad \bar{n}_0 = \frac{0.649796}{0.039747} = 16.348$$

$$p' = 0.01; \qquad L(0.01) = 0.95;$$

$$\bar{n}_{.01} = \frac{(0.95)(0.649796) - (0.05)(1.156407)}{0.029747} = 18.808$$

$$p' = 0.039747;$$

$$L(0.039747) = \frac{1.156407}{0.649796 + 1.156407} = 0.64024;$$

$$\bar{n}_{(.039747)} = \frac{(0.649796)(1.156407)}{(0.039747)(0.960253)} = 19.688$$

$$p' = 0.10; \qquad L(0.10) = 0.20;$$

$$\bar{n}_{.10} = \frac{0.80(1.156407) - 0.20(0.649796)}{0.10 - 0.039747} = 13.197$$

$$p' = 1.00; \qquad L(1.00) = 0; \qquad \bar{n}_{1.00} = \frac{1.156407}{0.960253} = 1.2043$$

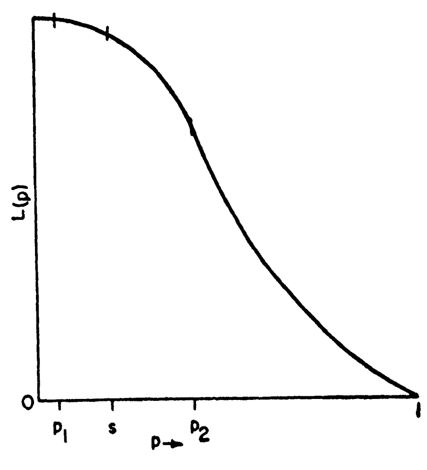


FIG. 13.16 OC CURVE OF SEQUENTIAL PLAN SKETCHED FROM FIVE POINTS.

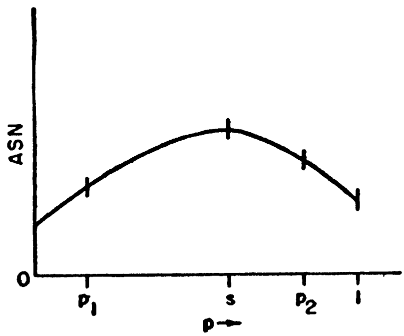


FIG. 13.17 ASN CURVE OF SEQUENTIAL PLAN SKETCHED FROM FIVE POINTS.

Usually these five points may be used to make an adequate sketch of the OC and ASN curves.

If more points are needed, they may be found by substituting values between 1 and ∞ for x in the equations

$$L(p') = \frac{x^{h_1+h_2} - x^{h_1}}{x^{h_1+h_2} - 1}; \qquad p' = \frac{x^s - 1}{x - 1}$$

The point $[p',L(p')]$ has the conjugate $[p'_c,L(p'_c)]$ given by

$$L(p'_c) = \frac{L(p')}{x^{h_1}}; \qquad p'_c = p'(x^{1-s})$$

The expected number of observations is

$$\bar{n}_{p'} = \frac{L(p')(h_1 + h_2) - h_2}{s - p'}$$

3.10 LOT-BY-LOT SAMPLING INSPECTION BY VARIABLES: INTRODUCTION

The principles underlying variables sampling inspection plans have been explained in considerable detail in *Sampling Inspection by Variables*★ by Bowker and Goode, and *Techniques of Statistical Analysis*† by Eisenhart, Hastay, and Wallis. Briefly, it is assumed that the item qualities are normally distributed about the universe mean $\bar{x}'$ with standard deviation σ' either known or unknown.

The criteria for acceptance by variables are put in the following forms.

3.11 ONE-SIDED TESTS—KNOWN σ'

For an upper specification limit U, with known σ', accept the lot if $\hat{p}_U \leq p^*$, where p^* is a preassigned constant (similar to the acceptance number) and

★ A. H. Bowker and H. P. Goode, *Sampling Inspection by Variables* (New York: McGraw-Hill Book Company, Inc., 1952).

† C. Eisenhart, M. W. Hastay, and W. A. Wallis, *Techniques of Statistical Analysis* (New York: McGraw-Hill Book Company, Inc., 1947).

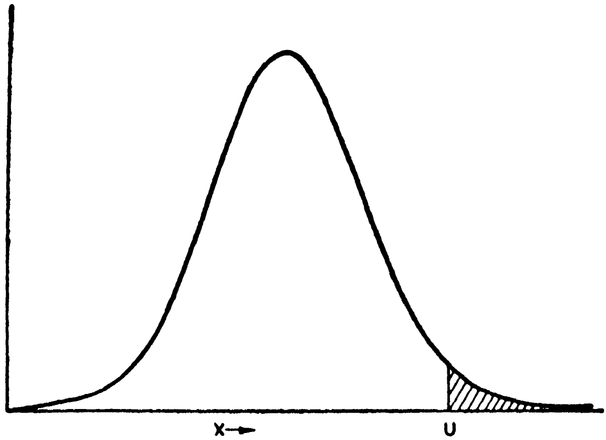
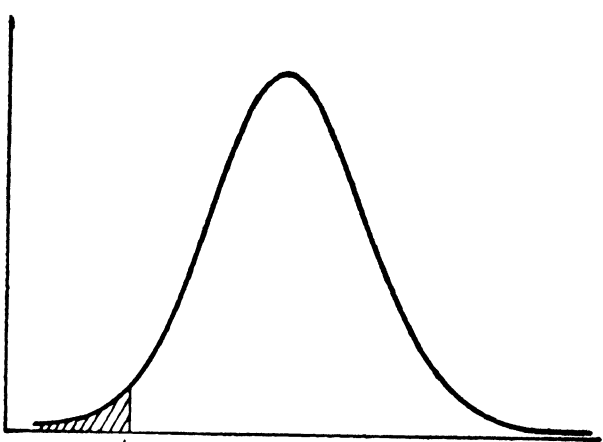


FIG. **13.18** PART OF POPULATION LYING ABOVE THE UPPER SPECIFICATION LIMIT.

$$\hat{p}_U = \int_{\left(\frac{U-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

is an estimate of the percentage of the population, based on a sample of n items, falling above U. Then $\left(\frac{U - \bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}$ is computed and $\hat{p}_U$ is found by entering the table of the normal distribution (Table 13.3). As an alternative procedure, it can be shown that $\hat{p}_U \leq p^*$ if and only if $\bar{x} + k\sigma' \leq U$, where k is a constant specified by the acceptance plan, i.e., for fixed n, different values of k yield different OC curves. Hence, accept the lot if $\bar{x} + k\sigma' \leq U$; otherwise reject the lot.

FIG. **13.19** PART OF POPULATION LYING BELOW A LOWER SPECIFICATION LIMIT.



For a lower specification limit L with known σ', accept the lot if $\hat{p}_L \leq p^*$, where p^* is a preassigned constant and

$$\hat{p}_L = \int_{-\infty}^{\left(\frac{L-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

is an estimate of the percentage of the population, based on a sample of n items, falling below L. Then $\left(\frac{L - \bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}$ is computed and $\hat{p}_L$ is found by entering the table of the normal distribution (Table 13.3).

Alternatively, it can be shown that $\hat{p}_L \leq p^*$ if and only if $\bar{x} - k\sigma' \geq L$. Hence, accept the lot if $\bar{x} - k\sigma' \geq L$; otherwise reject the lot.

3.12 ONE-SIDED TESTS—UNKNOWN σ'

For an upper specification limit U with unknown σ', accept the lot if $\hat{p}_U \leq p^*$, where p^* is a preassigned constant (similar to the acceptance number) and $\hat{p}_U$ is an estimate of the percentage of the population, based on a sample of n items, falling above U. If the sample standard deviation

$$s = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{\sum x_i^2 - n\bar{x}^2}{n-1}}$$

is used as an estimate of σ', then $\hat{p}_U$ is a function of $\bar{x}$, s, and U. This function has been tabulated by the Applied Mathematics and Statistics Laboratory of Stanford University and can be obtained on request. This estimate has optimum statistical properties. Alternatively, it can be shown that $\hat{p}_U \leq p^*$ if and only if $\bar{x} + ks \leq U$. Hence, accept the lot if $\bar{x} + ks \leq U$; otherwise reject the lot.

If the average range of v subgroups of 5, $\bar{R} = \sum_{i=1}^{v} R_i/v$, is used as an estimate of σ', then $\hat{p}_U$, the estimate of the percentage above U in a sample of $n = 5v$, is a function of $\bar{x}$, $\bar{R}$, and U. Again, as an alternate procedure, it can be shown that $\hat{p}_U \leq p^*$ if and only if $\bar{x} + k\bar{R}/2.3259 \leq U$. Hence, accept the lot if $\bar{x} + k\bar{R}/2.3259 \leq U$; otherwise reject the lot.

For a lower specification limit L, with unknown σ', accept the lot if $\hat{p}_L \leq p^*$, where p^* is a preassigned constant and $\hat{p}_L$ is an estimate of the percentage of the population, based on a sample of n items, falling below L. If s is used as an estimate of σ', then $\hat{p}_L$ is a function of $\bar{x}$, s, and L, and has been tabulated by the Applied Mathematics and Statistics Laboratory of Stanford University. The alternate procedure is derived from the relation $\hat{p}_L \leq p^*$ if and only if $\bar{x} - ks \geq L$. Hence, accept the lot if $\bar{x} - ks \geq L$; otherwise reject the lot.

If the average range of v subgroups of 5, $\bar{R} = \sum_{i=1}^{v} R_i/v$, is used as an esti-

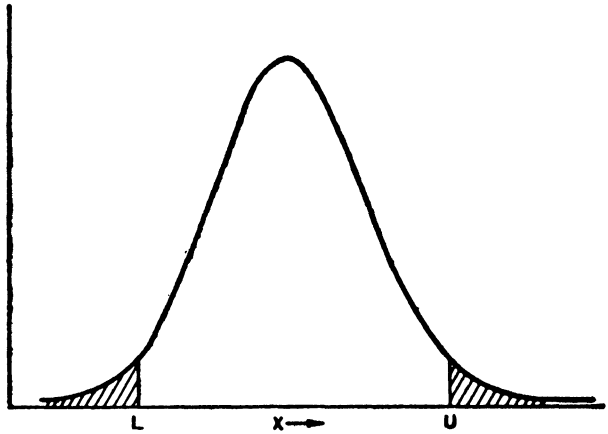


FIG. 13.20 PART OF POPULATION LYING ABOVE UPPER SPECIFICATION LIMIT AND BELOW LOWER SPECIFICATION LIMIT.

mate of σ', then $\hat{p}_L$ is a function of $\bar{x}$, $\bar{R}$, and L. Alternatively, it can be shown that $\hat{p}_L \leq p^*$ if and only if $\bar{x} - k\bar{R}/2.3259 \geq L$.

3.13 TWO-SIDED TESTS—KNOWN σ'

For a two-sided specification test, with known σ', accept the lot if $\hat{p}_L + \hat{p}_U \leq p^*$, where

$$\hat{p}_L = \int_{-\infty}^{\left(\frac{L-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

and

$$\hat{p}_U = \int_{\left(\frac{U-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

Hence, compute $\left(\frac{L-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}$ and $\left(\frac{U-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}$ and enter the tables of the normal distribution to obtain $\hat{p}_L$ and $\hat{p}_U$.

3.14 TWO-SIDED TESTS—UNKNOWN σ'

For a two-sided specification test, with unknown σ', accept the lot if $\hat{p}_L + \hat{p}_U \leq p^*$, where $\hat{p}_L$ and $\hat{p}_U$ are the estimates mentioned in Art. 3.12.

3.15 VARIABLES PLANS THAT MATCH MIL-STD-105A

Table 13.19 presents one-sided unknown standard deviation variables plans based on the sample standard deviation which most closely match

TABLE 13.19 TABLE SHOWING ONE-SIDED UNKNOWN STANDARD DEVIATION VARIABLES PLANS WHICH MOST CLOSELY MATCH ATTRIBUTE PLANS IN MIL-STD-105A★

Sample size code letter	Acceptable quality levels											
	0.065	0.10	0.15	0.25	0.40	0.65	1.0	1.5	2.5	4.0	6.5	10.0
A												$n = 2$ $k = 0.6$
B											$n = 3$ $k = 1.0$	
C										$n = 4$ $k = 1.3$		
D									$n = 5$ $k = 1.5$			$n = 5$ $k = 0.8$
E								$n = 7$ $k = 1.6$			$n = 7$ $k = 1.0$	$n = 7$ $k = 0.6$
F							$n = 10$ $k = 1.8$			$n = 10$ $k = 1.3$	$n = 10$ $k = 1.0$	$n = 10$ $k = 0.8$
G						$n = 15$ $k = 2.0$			$n = 15$ $k = 1.5$	$n = 15$ $k = 1.3$	$n = 15$ $k = 1.1$	$n = 15$ $k = 0.8$
H					$n = 20$ $k = 2.0$			$n = 20$ $k = 1.7$	$n = 20$ $k = 1.5$	$n = 20$ $k = 1.3$	$n = 20$ $k = 1.0$	$n = 20$ $k = 0.8$
I				$n = 25$ $k = 2.25$			$n = 25$ $k = 1.8$	$n = 25$ $k = 1.7$	$n = 25$ $k = 1.5$	$n = 25$ $k = 1.3$	$n = 30$ $k = 1.1$	$n = 30$ $k = 0.8$
J			$n = 30$ $k = 2.5$			$n = 35$ $k = 2.0$	$n = 35$ $k = 1.8$	$n = 35$ $k = 1.7$	$n = 35$ $k = 1.6$	$n = 35$ $k = 1.4$	$n = 40$ $k = 1.2$	$n = 50$ $k = 1.0$
K		$n = 35$ $k = 2.5$			$n = 40$ $k = 2.25$	$n = 40$ $k = 2.0$	$n = 40$ $k = 1.8$	$n = 40$ $k = 1.7$	$n = 50$ $k = 1.6$	$n = 50$ $k = 1.4$	$n = 50$ $k = 1.2$	
L	$n = 40$ $k = 2.5$			$n = 50$ $k = 2.25$								

★ Matches which require a sample size for variables plans greater than 50 are not given.

TABLE **13.20** TABLE SHOWING KNOWN STANDARD DEVIATION VARIABLES PLANS WHICH MOST CLOSELY MATCH ATTRIBUTE PLANS IN MIL-STD-105A

Sample size code letter	Acceptable quality levels											
	0.065	0.10	0.15	0.25	0.40	0.65	1.0	1.5	2.5	4.0	6.5	10.0
A												$n = 2$ $p^* = 17.5$
B											$n = 2$ $p^* = 10.8$	
C										$n = 2$ $p^* = 5.71$		
D									$n = 3$ $p^* = 4.43$			$n = 4$ $p^* = 18.9$
E								$n = 3$ $p^* = 2.91$			$n = 5$ $p^* = 12.7$	$n = 6$ $p^* = 22.8$
F							$n = 4$ $p^* = 1.96$			$n = 6$ $p^* = 8.22$	$n = 7$ $p^* = 15.0$	$n = 8$ $p^* = 22.1$
G						$n = 4$ $p^* = 1.10$			$n = 7$ $p^* = 4.90$	$n = 9$ $p^* = 8.83$	$n = 11$ $p^* = 12.7$	$n = 13$ $p^* = 21.0$
H					$n = 4$ $p^* = 0.788$			$n = 8$ $p^* = 3.37$	$n = 10$ $p^* = 6.21$	$n = 12$ $p^* = 9.13$	$n = 15$ $p^* = 15.0$	$n = 18$ $p^* = 20.6$
I				$n = 5$ $p^* = 0.547$			$n = 9$ $p^* = 2.32$	$n = 11$ $p^* = 4.28$	$n = 14$ $p^* = 6.23$	$n = 16$ $p^* = 8.37$	$n = 20$ $p^* = 12.4$	$n = 24$ $p^* = 18.4$
J			$n = 5$ $p^* = 0.352$			$n = 10$ $p^* = 1.53$	$n = 13$ $p^* = 2.78$	$n = 16$ $p^* = 4.08$	$n = 19$ $p^* = 5.47$	$n = 23$ $p^* = 8.22$	$n = 29$ $p^* = 12.2$	$n = 34$ $p^* = 17.7$
K		$n = 6$ $p^* = 0.238$			$n = 10$ $p^* = 1.00$	$n = 14$ $p^* = 1.37$	$n = 17$ $p^* = 2.80$	$n = 20$ $p^* = 3.73$	$n = 25$ $p^* = 5.58$	$n = 30$ $p^* = 7.41$	$n = 36$ $p^* = 11.1$	$n = 41$ $p^* = 16.7$
L	$n = 6$ $p^* = 0.165$			$n = 11$ $p^* = 0.737$	$n = 15$ $p^* = 1.35$	$n = 18$ $p^* = 2.03$	$n = 22$ $p^* = 2.70$	$n = 26$ $p^* = 3.35$	$n = 33$ $p^* = 5.40$	$n = 40$ $p^* = 7.43$	$n = 49$ $p^* = 11.5$	

TABLE **13.21** TABLE SHOWING ONE-SIDED UNKNOWN STANDARD DEVIATION VARIABLES PLANS BASED ON THE RANGE WHICH MOST CLOSELY MATCH ATTRIBUTE PLANS IN MIL-STD-105A★

Sample size code letter	Acceptable quality levels											
	0.065	0.10	0.15	0.25	0.40	0.65	1.0	1.5	2.5	4.0	6.5	10.0
A												$n = 2^a$ $k = 0.6$
B											$n = 3^b$ $k = 0.8$	
C										$n = 4^c$ $k = 1.2$		
D									$n = 7^d$ $k = 1.4$			$n = 7^d$ $k = 0.8$
E								$n = 10$ $k = 1.6$			$n = 7^d$ $k = 1.0$	$n = 7^d$ $k = 0.6$
F							$n = 10$ $k = 1.7$			$n = 10$ $k = 1.2$	$n = 15$ $k = 1.0$	$n = 15$ $k = 0.8$
G						$n = 15$ $k = 2.0$			$n = 15$ $k = 1.5$	$n = 20$ $k = 1.3$	$n = 20$ $k = 1.1$	$n = 20$ $k = 0.8$
H					$n = 25$ $k = 2.25$			$n = 25$ $k = 1.7$	$n = 25$ $k = 1.5$	$n = 25$ $k = 1.3$	$n = 25$ $k = 1.0$	$n = 25$ $k = 0.8$
I				$n = 20$ $k = 2.25$			$n = 25$ $k = 1.8$	$n = 30$ $k = 1.6$	$n = 30$ $k = 1.4$	$n = 35$ $k = 1.3$	$n = 35$ $k = 1.1$	$n = 35$ $k = 0.8$
J			$n = 35$ $k = 2.5$			$n = 40$ $k = 2.0$	$n = 40$ $k = 1.8$	$n = 40$ $k = 1.7$	$n = 40$ $k = 1.5$	$n = 40$ $k = 1.3$	$n = 50$ $k = 1.1$	$n = 50$ $k = 1.0$
K		$n = 35$ $k = 2.5$			$n = 50$ $k = 2.25$	$n = 50$ $k = 2.0$	$n = 50$ $k = 1.8$	$n = 50$ $k = 1.7$	$n = 50$ $k = 1.5$	$n = 50$ $k = 1.3$	$n = 50$ $k = 1.1$	
L	$n = 40$ $k = 2.5$			$n = 50$ $k = 2.25$								

★ Matches which require a sample size for variables plans greater than 50 are not given.
[a] For $n = 2$, the test criteria is of the form $\bar{x} + kR/1.128$ where R is the range of the sample of 2.
[b] For $n = 3$, the test criteria is of the form $\bar{x} + kR/1.693$ where R is the range of the sample of 3.
[c] For $n = 4$, the test criteria is of the form $\bar{x} + kR/2.059$ where R is the range of the sample of 4.
[d] For $n = 7$, the test criteria is of the form $\bar{x} + kR/2.704$ where R is the range of the sample of 7.

attribute plans in MIL-STD-105A. The criteria for accepting lots is as follows: for an upper specification limit, accept the lot if $\bar{x} + ks \leq U$; for a lower specification limit, accept the lot if $\bar{x} - ks \geq L$.

Table 13.20 presents one- and two-sided known σ' variables plans which most closely match attribute plans in MIL-STD-105A. The criteria for accepting lots is as follows: for an upper specification limit, accept the lot if

$$\hat{p}_U = \int_{\left(\frac{U-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz \leq p^*$$

for a lower specification limit, accept the lot if

$$\hat{p}_L = \int_{-\infty}^{\left(\frac{L-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz \leq p^*$$

for a two-sided specification limit, accept the lot if

$$\hat{p}_L + \hat{p}_U = \int_{-\infty}^{\left(\frac{L-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz + \int_{\left(\frac{U-\bar{x}}{\sigma'}\right)\frac{\sqrt{n}}{\sqrt{n-1}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz \leq p^*$$

Table 13.21 presents one-sided unknown standard deviation variables plans based on the average range which most closely match attribute plans in MIL-STD-105A. The criteria for accepting lots is as follows: for an upper specification limit, accept the lot if $\bar{x} + \dfrac{k\bar{R}}{2.3259} \leq U$; for a lower specification limit, accept the lot if $\bar{x} - \dfrac{k\bar{R}}{2.3259} \geq L$.

3.16 INTRODUCTION: CONTINUOUS SAMPLING INSPECTION

The preceding articles on sampling inspection have dealt with lot-by-lot inspection when items are classified according to attribute data or variables data. In the following articles, inspection procedures will be given for items that are not broken into lots, and that are sampled by attributes. In these continuous sampling procedures, current inspection results are used to determine whether sampling inspection or screening inspection is to be used for the next items to be inspected.

3.17 DODGE CONTINUOUS SAMPLING PLAN (CSP-1)

In 1943, Dodge published a continuous sampling plan in the *Annals of Mathematical Statistics*.★ The procedure, as stated by Dodge, is as follows:

★ H. F. Dodge, "A Sampling Inspection Plan for Continuous Production," *Annals of Mathematical Statistics*, Vol. 14, September 1943.

"(a) At the outset, inspect 100% of the units consecutively as produced and continue such inspection until i units in succession are found clear of defects.

(b) When i units in succession are found clear of defects, discontinue 100% inspection, and inspect only a fraction f of the units, selecting individual sample units one at a time from the flow of product in such a manner as to assure an unbiased sample.

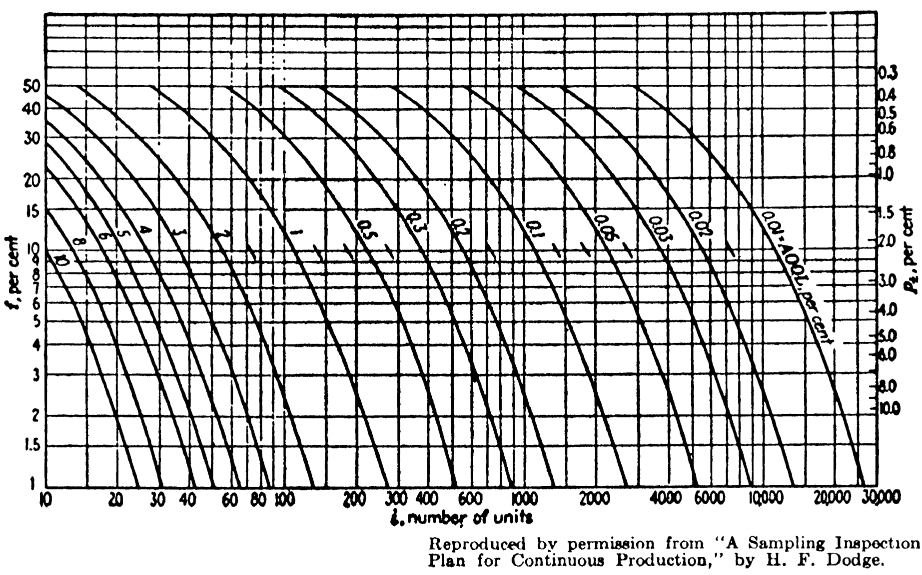
(c) If a sample unit is found defective, revert immediately to a 100% inspection of succeeding units and continue until again i units in succession are found clear of defects, as in paragraph (a).

(d) Correct or replace with good units all defective units found."

In his paper, Dodge studied the properties of this plan, and presented equations and charts for determining the average outgoing quality limit (AOQL) as functions of the parameters f and i, under the assumption that the process is in a state of statistical control. A production process is said to be in statistical control if there is a positive constant $p \leq 1$ such that, for every item produced, the probability that it is defective is p, and is independent of the state (defective or nondefective) of all the other items produced.

Figure 13.21 presents the necessary chart for the selection of the appropriate plan.

3.17.1 Example. Suppose the desired AOQL is 4% and f is chosen to be $\frac{1}{5}$. From Fig. 13.21, i is found to be 17. The plan is as follows:



Reproduced by permission from "A Sampling Inspection Plan for Continuous Production," by H. F. Dodge.

FIG. **13.21** CURVES FOR DETERMINING VALUES OF f AND i FOR A GIVEN VALUE OF AOQL IN DODGE'S PLAN FOR CONTINUOUS PRODUCTION.

Inspect all the units as produced and continue such inspection until 17 units in succession are found clear of defects.

When 17 units in succession are found clear of defects, discontinue 100% inspection, and inspect only one out of five units. As soon as a defective is found, revert to 100% inspection of succeeding units and continue until again 17 units in succession are found clear of defects. Then resume sampling inspection.

3.17.2 **Further results with the Dodge procedure.** Although the Dodge procedure is always applicable, it only guarantees the specified AOQL provided the process is in a state of statistical control. Lieberman★ has shown that the Dodge procedure guarantees an AOQL whether or not the process is in a state of statistical control. In fact, without the assumption of control, and for a given f and i,

$$\text{AOQL} = \frac{1/f - 1}{1/f + i}$$

In the above example, if the assumption of control is *not* made and an AOQL of 4% is desired with $f = \frac{1}{5}$, it is necessary to take $i = 95$.

In 1951, H. F. Dodge and Miss M. N. Torrey† developed two modifications of CSP-1. They are referred to as CSP-2 and CSP-3. These plans remove the feature of reverting to 100% inspection as soon as one defect is found; instead, 100% inspection is reinstated only when one defect falls too closely on the heels of another during sampling inspection, that is, when their separation is smaller than a prescribed minimum spacing. Naturally, the probability of accepting a short run of product of poor quality is higher in these plans than in the original one. These last two plans differ in that one provides for the inspection of four additional units whenever a defect is found under conditions that do not require reinstating 100% inspection, thereby providing protection against short runs of poor product.

3.18 GIRSHICK CONTINUOUS SAMPLING PLAN

In 1948, M. A. Girshick‡ presented a continuous sampling plan. It is defined by the three integers m, N, and K. The plan operates as follows: The units of product in the production sequence are divided into segments of size K. Inspection begins by selecting at random one item from each consecutive segment of K items. The items are inspected in sequence and the number of defectives found, as well as the number of items examined, are cumulated. This procedure is continued until the cumulative number of defectives reaches m. At this point, the size of the sample n is compared with the integer N. If $n \geq N$, the product which has passed through inspection is

★ G. J. Lieberman, "A Note on Dodge's Continuous Inspection Plan," *Annals of Mathematical Statistics*, Vol. 24, September 1953.

† H. F. Dodge and M. N. Torrey, "Additional Continuous Sampling Inspection Plans," *Industrial Quality Control*, Vol. 7, March 1951.

‡ M. A. Girshick, "Sampling Inspection Plans for Continuous Production," unpublished paper delivered at the May 10, 1948, meeting of the Institute of Mathematical Statistics.

considered acceptable and the inspection procedure is repeated on the new incoming product. If, on the other hand, $n < N$, the following actions are taken: (a) the next $N - n$ segments are inspected 100%; and (b) after that, the inspection procedure is repeated. This procedure always guarantees that the AOQL cannot exceed $(K - 1)m/KN$.

3.18.1 Example. If an AOQL of 4% is desired with $K = 5$ and $m = 3$, the value of N for the Girshick plan is found by solving the equation $0.04 = (4 \times 3)/(5N)$ for N, i.e., $N = 60$.

Thus inspection begins by selecting at random one item from each consecutive segment of five items. This procedure is continued until the cumulative number of defectives reaches three. At this point, the size of the sample n is compared with 60. If $n \geq 60$, the product that has passed through inspection is considered acceptable, and the inspection procedure is repeated on the new incoming product. If, on the other hand, $n < 60$, the next $60 - n$ segments are inspected 100%. After that, the initial inspection procedure is repeated.

4. COMMON SIGNIFICANCE TESTS

4.1 INTRODUCTION

Significance tests are important tools in scientific and industrial experimentation; they are used for making decisions on the basis of the limited information available in samples and provide procedures for making these decisions with preassigned risks. For example, an experimenter may wish to determine whether a new method of sealing vacuum tubes will increase their life, whether a new alloy will have an increased breaking strength, whether a new source of raw material has resulted in a change in the quality of output, or whether a special treatment of concrete will have an effect on breaking strength.

Consider the breaking strength example: We are really concerned with two distributions, one the distribution of breaking strength of concrete made by the standard method, and the other the distribution of concrete receiving the special treatment. Our problem is to decide whether these two distributions are the same or different, and in particular to decide if their mean breaking strengths are the same. We make the hypothesis that there is no difference in mean breaking strength, and will accept or reject this hypothesis on the basis of experimental evidence. A number of test specimens of both the special and standard treatment will be made up and the compressive strength of each test specimen determined. A statistic will be computed from these measurements.

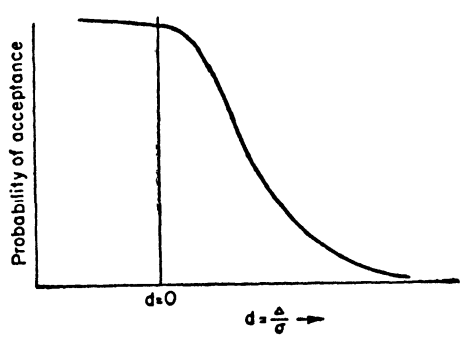
Ideally, before the data are at hand, the experimenter should decide on his rule of rejection and on the number of items to include in the sample. As explained in the detailed instructions for the various cases included in this article, the test for the equality of two population means is based on the difference of the sample means (say h = mean of special treatment minus mean of standard treatment) divided by some measure of their sampling variability. In this case, we will keep the standard method of treatment unless the special method actually increases the breaking strength, and we are interested in rejecting the hypothesis of equality only in this case. Hence, we will reject the hypothesis only if the sample mean of the new material is substantially greater than the sample mean of the old, i.e., h is positive. This kind of test that rejects when differences are in one direction only is called a one-tailed test, since the region of rejection consists of one tail of the distribution of the test statistic. Often we are interested in rejecting the hypothesis of equality of means when the true means differ in either direction, as when we want to see if a new source of raw materials has changed the quality of output. In this case, we reject when the difference of the sample means is either too large or too small; such tests are called two-tailed tests.

4.2 OC CURVES OF THE TEST

Suppose we let Δ equal the true difference in true means, i.e., $\Delta =$ mean of special treatment population minus mean of standard treatment population, and we are interested in the one-tailed test. Now we have indicated that we want a test that will have a high probability of rejecting the hypothesis of equality of means for Δ positive and a high probability of accepting the hypothesis (low probability of rejecting) if $\Delta \leq 0$. Of course, in repeated sampling from populations with a given Δ we will accept some of the time and reject the rest of the time; the percentage of times we accept the hypothesis is the probability of acceptance. A plot of the probability of acceptance as a function of Δ is called the operating characteristic curve of the test; just as the protection of an acceptance sampling plan is characterized by the OC curve (the probability of accepting the lot as a function of presented quality) so the risk of making the wrong decision in any significance test is characterized by the OC curve.

An operating characteristic curve for a one-sided test will look something like Fig. 13.22. Operating characteristic curve for a two-sided test will

FIG. 13.22 OC CURVE OF THE ONE-SIDED TEST FOR THE EQUALITY OF TWO MEANS.



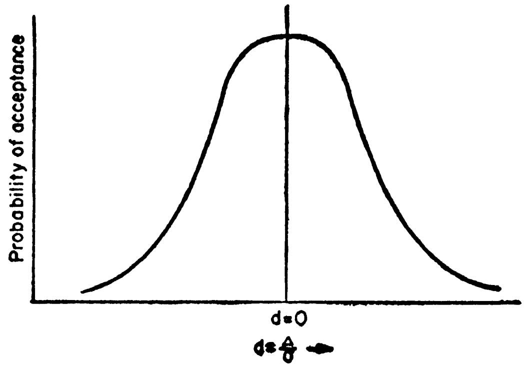


FIG. **13.23** OC CURVE OF THE TWO-SIDED TEST FOR THE EQUALITY OF TWO MEANS.

look something like Fig. 13.23. The probability of rejecting the hypothesis ($\Delta = 0$ in this case) when it is true (1 − probability of acceptance as given by the OC curve) is often called the level of significance or size of the test. The letter α will be used to denote the level of significance. Standard practice is to take α as 0.05 or 0.01 depending on how serious it is from a practical point of view to reject the hypothesis when it is true.

A family of OC curves of the one-tailed test for various values of the sample size is given in Fig. 13.24. As in the case of sampling inspection, increasing the number of observations increases the slope of the OC curve; the probability of accepting the hypothesis $\Delta = 0$ if Δ is really 0 remains the same, i.e., the significance level can be achieved for any sample size. What changes is the probability of deciding that $\Delta = 0$ if the true situation is such that the means actually differ; for any given $\Delta > 0$ the probability of accepting will decrease as n increases. To pick a sample size, the experimentalist must pick a difference Δ that is important from the practical point of view and then choose the number of observations which gives him small chance of accepting the hypothesis if the true difference is as great as the practical difference. Very often he must weigh the cost of taking additional observations

FIG. **13.24** FAMILY OF OC CURVES OF THE ONE-TAILED TEST FOR THE EQUALITY OF MEANS.

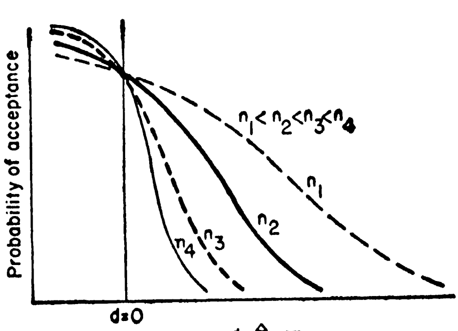


TABLE **13.22** SUMMARY OF SIGNIFICANCE TESTS: TESTING FOR THE VALUE OF A SPECIFIED PARAMETER

Hypothesis	Notation for hypothesis	Qualifying conditions	Statistic used in test	Reference to detailed explanation	Rule of rejection	Tables	Choice of sample size
The mean of a normal distribution has a specified value	$\mu = \mu_0$	known σ	$U = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma}$	Normal dist. Art. 4.4	$U \geq K_\alpha$ if we wish to reject when the true mean exceeds μ_0.	13.3	Figs. 13.27 and 13.28
					$U \leq -K_\alpha$ if we wish to reject when the true mean is less than μ_0.	13.3	Figs. 13.27 and 13.28
					$\|U\| \geq K_{\alpha/2}$ if we wish to reject when the true mean departs in either direction from μ_0.	13.3	Figs. 13.25 and 13.26
	$\mu = \mu_0$	unknown σ	$t = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s}$	t Art. 4.5	$t \geq t_{\alpha;n-1}$ if we wish to reject when the true mean exceeds μ_0.	13.25	Figs. 13.31 and 13.32
					$t \leq -t_{\alpha;n-1}$ if we wish to reject when the true mean is less than μ_0.	13.25	Figs. 13.31 and 13.32
					$\|t\| \geq t_{\alpha/2;n-1}$ if we wish to reject when the true mean departs in either direction from μ_0.	13.25	Figs. 13.29 and 13.30
The standard deviation (or variance) of a normal distribution has a specified value	$\sigma = \sigma_0$		$\chi^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{\sigma_0^2}$	χ^2 Art. 4.6	$\chi^2 \geq \chi^2_{\alpha;n-1}$ if we wish to reject when the true standard deviation exceeds σ_0.	13.26	Figs. 13.35 and 13.36
					$\chi^2 \leq \chi^2_{1-\alpha;n-1}$ if we wish to reject when the true standard deviation is less than σ_0.	13.26	Figs. 13.37 and 13.38
					$\chi^2 \leq \chi^2_{1-\alpha/2;n-1}$ or $\chi^2 \geq \chi^2_{\alpha/2;n-1}$ if we wish to reject when the standard deviation departs in either direction from σ_0.	13.26	Figs. 13.33 and 13.34

Hypothesis	Notation for hypothesis	Qualifying conditions	Statistic used in test
The means of two normal distributions are equal	$\mu_x = \mu_y$	Known standard deviations σ_x, σ_y	$U = \dfrac{\bar{x} - \bar{y}}{\sqrt{\sigma_x^2/n_x + \sigma_y^2/n_y}}$
The means of two normal distributions are equal	$\mu_x = \mu_y$	Unknown standard deviations with $\sigma_x = \sigma_y$	$t = \dfrac{\bar{x} - \bar{y}}{\sqrt{\dfrac{1}{n_x} + \dfrac{1}{n_y}}\sqrt{\dfrac{\sum_{i=1}^{n_x} (x_i - \bar{x})^2 + \sum_{i=1}^{n_y} (y_i - \bar{y})^2}{n_x + n_y - 2}}}$
The means of two normal distributions are equal	$\mu_x = \mu_y$	Unknown standard deviations with σ_x and σ_y not necessarily equal	$t' = \dfrac{\bar{x} - \bar{y}}{\sqrt{s_x^2/n_x + s_y^2/n_y}}$
The variance of two normal distributions are equal	$\sigma_x^2 = \sigma_y^2$		$F = \dfrac{s_x^2}{s_y^2} = \dfrac{\sum_{i=1}^{n_x} (x_i - \bar{x})^2/(n_x - 1)}{\sum_{i=1}^{n_y} (y_i - \bar{y})^2/(n_y - 1)}$ In a two-sided test, put the larger mean square in the numerator

against the advantage in decreasing the probability of accepting the null hypothesis when it is false.

In practice, there are two major limitations to the use of OC curves. Very often, the number of observations is fixed in advance, by custom, limitation of testing equipment, or indeed the statistical analysis may be of secondary importance based on data taken for another purpose. Even in this case, a look at the OC curve is important, since it gives an idea of the type of differences you are likely to detect and hence an indication of the sensitivity of the analysis. In the second place, it often happens that the OC curve depends on parameters in which you are not interested, since the OC curve for the test of equality of two means depends on the ratio of the difference of the means to the standard deviation. However, even if only the general magnitude of the standard deviation is known, the OC curves are useful in designing experiments. The experimentalist must realize that whenever he picks a sample size he is implicitly picking an OC curve, and the more information he has available in making this decision the better. The OC curves for most of the standard significance tests are presented in this chapter.

Reference to detailed explanation	Rule of rejection	Table	Choice of sample size
Art. 4.7	$U \geq K_\alpha$ if we wish to reject when $\mu_x > \mu_y$.	13.3	Fig. 13.27 and 13.28
	$U \leq -K_\alpha$ if we wish to reject when $\mu_x < \mu_y$.	13.3	Fig. 13.27 and 13.28
	$\|U\| \geq K_{\alpha/2}$ if we wish to reject whenever the means differ.	13.3	Fig. 13.25 and 13.26
Art. 4.8	$t \geq t_{\alpha;n_x+n_y-2}$ if we wish to reject when $\mu_x > \mu_y$.	13.25	Fig. 13.31 and 13.32
	$t \leq -t_{\alpha;n_x+n_y-2}$ if we wish to reject when $\mu_x < \mu_y$.	13.25	Fig. 13.31 and 13.32
	$\|t\| \geq t_{\alpha/2;n_x+n_y-2}$ if we wish to reject whenever $\mu_x \neq \mu_y$.	13.25	Fig. 13.29 and 13.30
Art. 4.9	$t' \geq t_{\alpha;v}$ if we wish to reject when $\mu_x > \mu_y$. $t' \leq -t_{\alpha;v}$ if we wish to reject when $\mu_x < \mu_y$. $\|t'\| \geq t_{\alpha/2;v}$ when we wish to reject when $\mu_x \neq \mu_y$. where the degrees of freedom v is given by the closest integer to $v = -2 + \left(\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}\right)^2/[(s_x^2/n_x)^2/(n_x+1) + (s_y^2/n_y)^2/(n_y+1)]$.	13.25 13.25 13.25	There are no results available
Art. 4.10	$F \geq F_{\alpha;n_x-1,n_y-1}$ if we wish to reject when $\sigma_x > \sigma_y$.	13.27	Fig. 13.41 and 13.42
	$F \geq F_{\alpha/2;n_x-1,n_y-1}$ if $s_x^2 > s_y^2$ and we wish to reject whenever $\sigma_x \neq \sigma_y$.	13.27	Fig. 13.39 and 13.40
	$F \geq F_{\alpha/2;n_y-1,n_x-1}$ if $s_x^2 < s_y^2$ and we wish to reject whenever $\sigma_x \neq \sigma_y$.	13.27	Fig. 13.39 and 13.40

All the tests described in this article are based on the assumption that the populations from which we draw samples are normal distributions. This assumption, which underlies a good deal of statistical technique today, is, of course, never exactly true, but has been shown to be approximately true by both empirical and theoretical studies, and the techniques derived on this assumption are adequate for most practical problems. Care should be taken, however, not to apply one-sided tests to very skew distributions.

A summary of the tests discussed in this section can be found in Tables 13.22 and 13.23.

4.3 NOTATION

The notation used in the following discussion will be the standard statistical notation found in most statistics texts. The observations $x_1, x_2, x_3, \ldots, x_n$ will be assumed to be drawn from a normal distribution with mean μ and variance σ^2.

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

is an estimate of μ, and

$$s^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1} = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n-1}$$

is an estimate of σ^2. The best computational form for s^2 is given by

$$s^2 = \frac{x_1^2 + x_2^2 + \ldots + x_n^2 - n\bar{x}^2}{n-1}$$

Here α is the level of significance, and K_α, $t_{\alpha;\nu}$, $\chi^2_{\alpha;\nu}$, and $F_{\alpha;\nu_1,\nu_2}$ are respectively the α percentage points of the normal distribution, the α percentage point of Student's t distribution with ν degrees of freedom, the α percentage point of the chi-square distribution with ν degrees of freedom, and the α percentage point of the F distribution with ν_1 and ν_2 degrees of freedom.

4.4 TEST OF THE HYPOTHESIS THAT THE MEAN OF A NORMAL DISTRIBUTION HAS A SPECIFIED VALUE WHEN THE STANDARD DEVIATION IS KNOWN

Notation for hypothesis	*Test statistic*
$\mu = \mu_0$	$U = \sqrt{n}\,(\bar{x} - \mu_0)/\sigma$
Rule of rejection	*Rule for choosing sample size*
$U \geq K_\alpha$ if we wish to reject when the true mean exceeds μ_0.	Select a value $\mu_1 > \mu_0$ for which we want to reject the null hypothesis with high probability, and calculate $d = (\mu_1 - \mu_0)/\sigma$, and select n from the OC curve of Fig. 13.27 or 13.28.
$U \leq -K_\alpha$ if we wish to reject when the true mean is less than μ_0.	Choose a value $\mu_1 < \mu_0$ which we want to reject and enter Fig. 13.27 or 13.28 with $d = (\mu_0 - \mu_1)/\sigma$.
$\lvert U\rvert \geq K_{\alpha/2}$ if we wish to reject when the true mean departs from μ_0 in either direction.	Select a value Δ that is a departure from μ_0 we want to reject with high probability and enter Fig. 13.25 or 13.26 with $d = \lvert\Delta\rvert/\sigma$.

TABLE **13.24** PERCENTAGE POINTS OF THE NORMAL DISTRIBUTION

Level of significance	*For one-sided tests*	*For two-sided tests*
$\alpha = 0.05$	$K_{0.05} = 1.645$	$K_{0.025} = 1.960$
$\alpha = 0.01$	$K_{0.01} = 2.326$	$K_{0.005} = 2.576$

4.4.1 Example. A manufacturer produces a special alloy steel with an average tensile strength of 125,800 psi. A change in the composition of the alloy is said to increase the breaking strength. A sample of 20 items of the new material is tested, and the average tensile strength is found to be 127,000.

The standard deviation of the tensile strength is known to be 300 psi. To decide whether to adopt the new alloy, we test the hypothesis that the breaking strength is unchanged, i.e., that the new value could have arisen by chance from a population with true mean of 125,800. The statistic U is calculated:

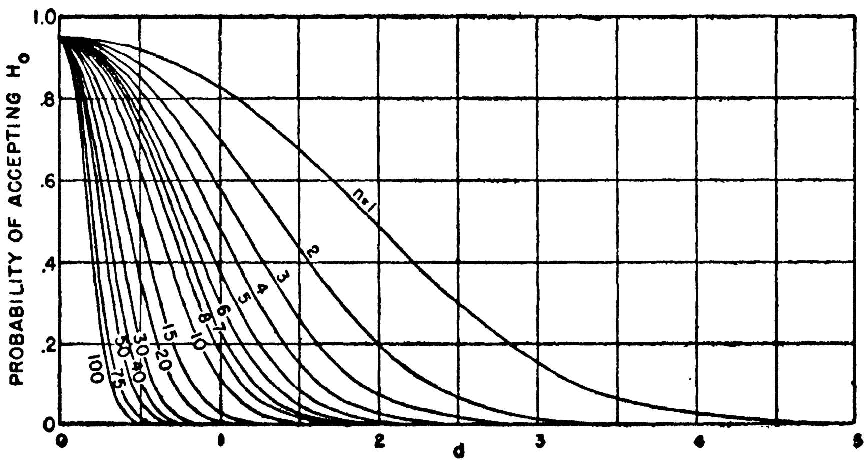
$$U = \frac{\sqrt{n}\,(\bar{x} - \mu_0)}{\sigma} = \frac{\sqrt{20}\,(127{,}000 - 125{,}800)}{300} = 17.888$$

Since we are interested in rejecting the hypothesis that the breaking strength is unchanged only if it is increased, we use a one-tailed test. As our observed value of U exceeds the critical value for the 1% level $K_{.01} = 2.326$, we reject the hypothesis and conclude that the new alloy produces greater tensile strength than the old.

In this case, we based our decision on a sample of 20 observations; the sample size was picked arbitrarily. However, the operating characteristic curves of the U test in Fig. 13.28 (for 0.01 level of significance) provide an objective basis for the selection of a sample size. These curves give the probability of accepting the hypothesis as a function of the difference between the true mean and the hypothesized one, in σ units, e.g., for a sample size of 20 the probability of accepting is about 0.10 for a difference of $0.80\sigma = 240$. Ordinarily, we would use this graph before running the experiment and select the sample size which will have a high probability of detecting a difference that is important from a practical point of view.

FIG. 13.25 OPERATING CHARACTERISTICS OF THE TWO-SIDED NORMAL TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.05.

Reproduced by permission from "Operating Characteristics for the Common Statistical Tests of Significance" by Charles D. Ferris, Frank E. Grubbs, Chalmers L. Weaver, *Annals of Mathematical Statistics*, June, 1946.



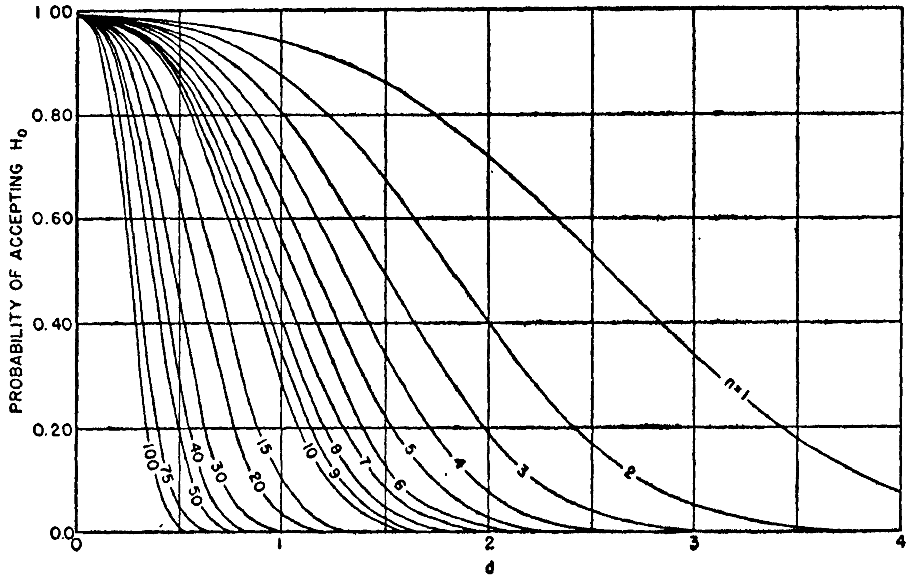
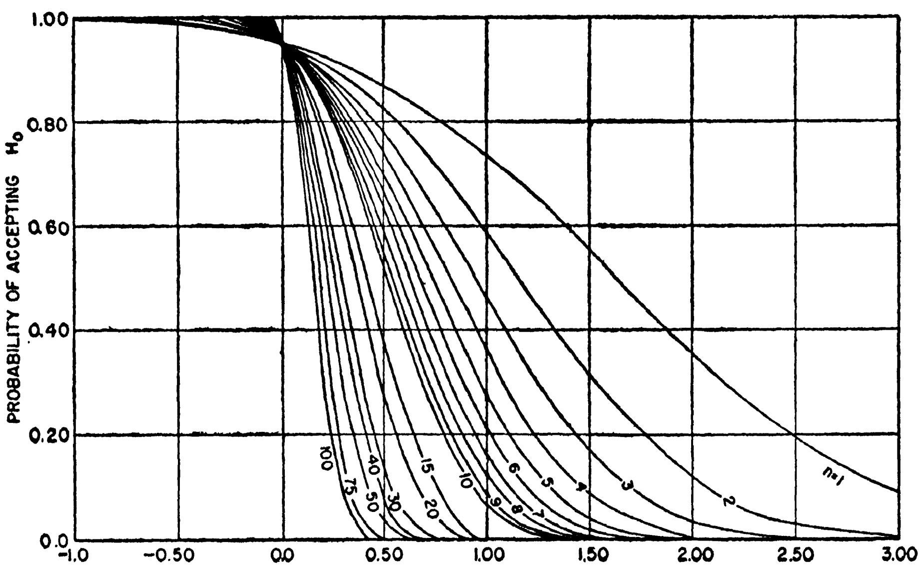


FIG. **13.26** OPERATING CHARACTERISTICS OF THE TWO-SIDED NORMAL TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.01.

FIG. **13.27** OPERATING CHARACTERISTICS OF THE ONE-SIDED NORMAL TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.05.



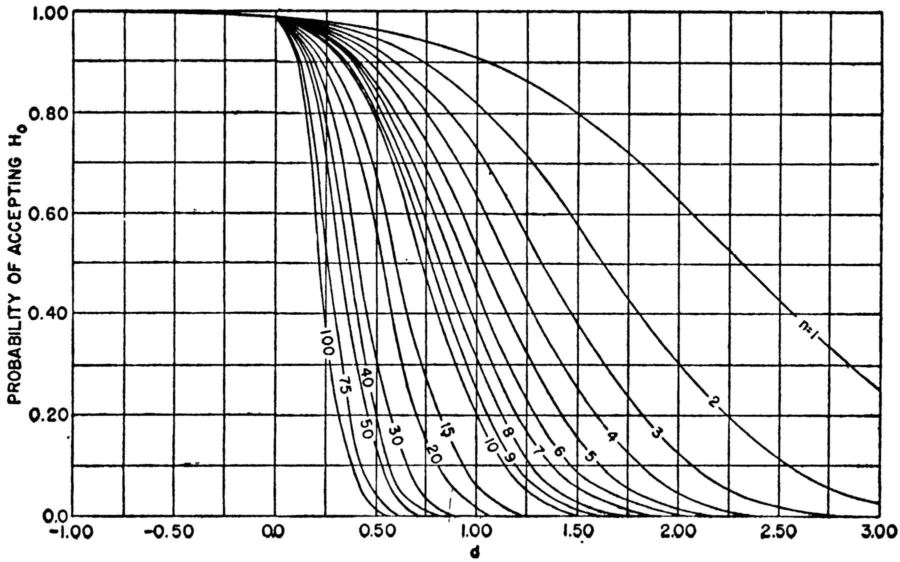


FIG. **13.28** OPERATING CHARACTERISTICS OF THE ONE-SIDED NORMAL TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.01.

4.5 TEST OF THE HYPOTHESIS THAT THE MEAN OF A NORMAL DISTRIBUTION HAS A SPECIFIED VALUE WHEN THE STANDARD DEVIATION IS UNKNOWN

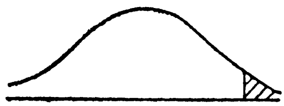
Notation for hypothesis	*Test statistic*
$\mu = \mu_0$	$t = \sqrt{n}\,(\bar{x} - \mu_0)/s$
Rule of rejection	*Rule for choosing sample sizes*
$t \geq t_{\alpha;n-1}$ if we wish to reject when the true mean exceeds μ_0.	Choose a value of $d = (\mu_1 - \mu_0)/\sigma$★ for which we wish to reject the null hypothesis with high probability and enter Fig. 13.31 or 13.32 to find the necessary sample size.
$t \leq -t_{\alpha;n-1}$ if we wish to reject when the true mean is less than μ_0.	Choose a value of $d = (\mu_0 - \mu_1)/\sigma$ for which we wish to reject the null hypothesis with high probability and enter Fig. 13.31 or 13.32 to find the necessary sample size.
$\lvert t\rvert \geq t_{\alpha/2;n-1}$ if we wish to reject when the true mean differs from μ_0 in either direction.	Choose a value of $d = \left\lvert\dfrac{\mu_1 - \mu_0}{\sigma}\right\rvert$ for which we wish to reject the null hypothesis with high probability and enter Fig. 13.29 or 13.30 to find the necessary sample size.

★ The operating characteristic curve of the t test, unlike the U test, depends on the ratio of $\mu_0 - \mu_1$ to σ, although the test procedure is independent of σ, as is the level of significance. Usually a rough idea of the magnitude of σ is sufficient for choosing the necessary sample size.

4.5.1 Example. In the manufacture of a food product, the label states that the box contains 10 lb. The boxes are filled by machine, and it is of interest to determine whether or not the machine is set properly. Previous experience has indicated that the standard deviation is approximately 0.05 lb, although this is not known precisely. The company feels that the present setting is un-

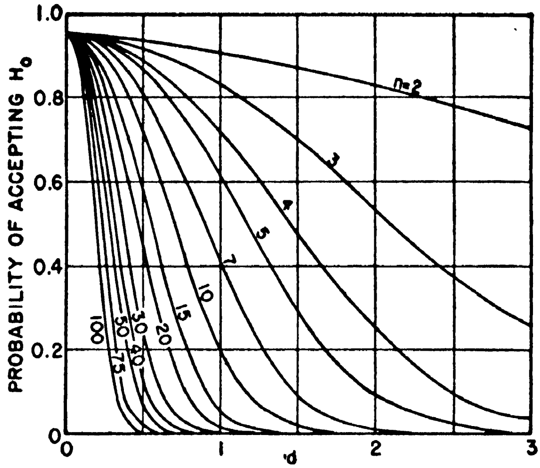
TABLE **13.25** PERCENTAGE POINTS OF THE t DISTRIBUTION★

Table of $t\alpha$; ν—the 100 α percentage point of the t distribution for ν degrees of freedom



ν \ α	.10	.050	.025	.010	.005	α / ν
1	3.078	6.314	12.706	31.821	63.657	1
2	1.886	2.920	4.303	6.965	9.925	2
3	1.638	2.353	3.182	4.541	5.841	3
4	1.533	2.132	2.776	3.747	4.604	4
5	1.476	2.015	2.571	3.365	4.032	5
6	1.440	1.943	2.447	3.143	3.707	6
7	1.415	1.895	2.365	2.998	3.499	7
8	1.397	1.860	2.306	2.896	3.355	8
9	1.383	1.833	2.262	2.821	3.250	9
10	1.372	1.812	2.228	2.764	3.169	10
11	1.363	1.796	2.201	2.718	3.106	11
12	1.356	1.782	2.179	2.681	3.055	12
13	1.350	1.771	2.160	2.650	3.012	13
14	1.345	1.761	2.145	2.624	2.977	14
15	1.341	1.753	2.131	2.602	2.947	15
16	1.337	1.746	2.120	2.583	2.921	16
17	1.333	1.740	2.110	2.567	2.898	17
18	1.330	1.734	2.101	2.552	2.878	18
19	1.328	1.729	2.093	2.539	2.861	19
20	1.325	1.725	2.086	2.528	2.845	20
21	1.323	1.721	2.080	2.518	2.831	21
22	1.321	1.717	2.074	2.508	2.819	22
23	1.319	1.714	2.069	2.500	2.807	23
24	1.318	1.711	2.064	2.492	2.797	24
25	1.316	1.708	2.060	2.485	2.787	25
26	1.315	1.706	2.056	2.479	2.779	26
27	1.314	1.703	2.052	2.473	2.771	27
28	1.313	1.701	2.048	2.467	2.763	28
29	1.311	1.699	2.045	2.462	2.756	29
Inf.	1.282	1.645	1.960	2.326	2.576	Inf.

★ Reproduced from J. E. Freund, *Modern Elementary Statistics* (New York: Prentice-Hall, Inc., 1952), Table II, p. 390, Control Values of t. This table is abridged from Table IV of R. A. Fisher, *Statistical Methods for Research Workers*, published by Oliver & Boyd Ltd., Edinburgh, by permission of the author and publishers.



Reproduced by permission from "Operating Characteristics for the Common Statistical Tests of Significance" by Charles D. Ferris, Frank E. Grubbs, Chalmers L. Weaver, *Annals of Mathematical Statistics*, June, 1946.

FIG. 13.29 OPERATING CHARACTERISTICS OF THE TWO-SIDED t TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.05.

satisfactory if the machine fills the boxes so that the mean weight differs from 10 lb by more than 0.1 lb. From Fig. 13.29 and $d = 0.1/0.05 = 2$, a sample size of 6 is sufficient to insure rejection with probability 0.95 if the mean weight differs from 10 lb by more than 0.1 lb. The data are: 9.99 lb, 9.99 lb, 10.00 lb, 10.11 lb, 10.09 lb, 9.95 lb.

$$\bar{x} = 10.021667\star$$

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} = \frac{\sum x_i^2 - n\bar{x}^2}{n-1} = \frac{602.6229 - 602.6029}{5} = 0.0040$$

$$s = 0.0632$$

$$\frac{\sqrt{n}(\bar{x} - 10)}{s} = \frac{\sqrt{6}(10.0217 - 10)}{0.0632} = 0.84$$

$$t_{.025;5} = 2.571 \qquad \text{for 5 degrees of freedom}$$

Thus $|t| \leq t_{.025;5}$ so that it is unnecessary to change the machine setting.

★ It is important to compute $\bar{x}$ to more places than appears necessary for the significance of the final result. The reason becomes clear upon looking at the calculation of s^2 where a subtraction operation is involved.

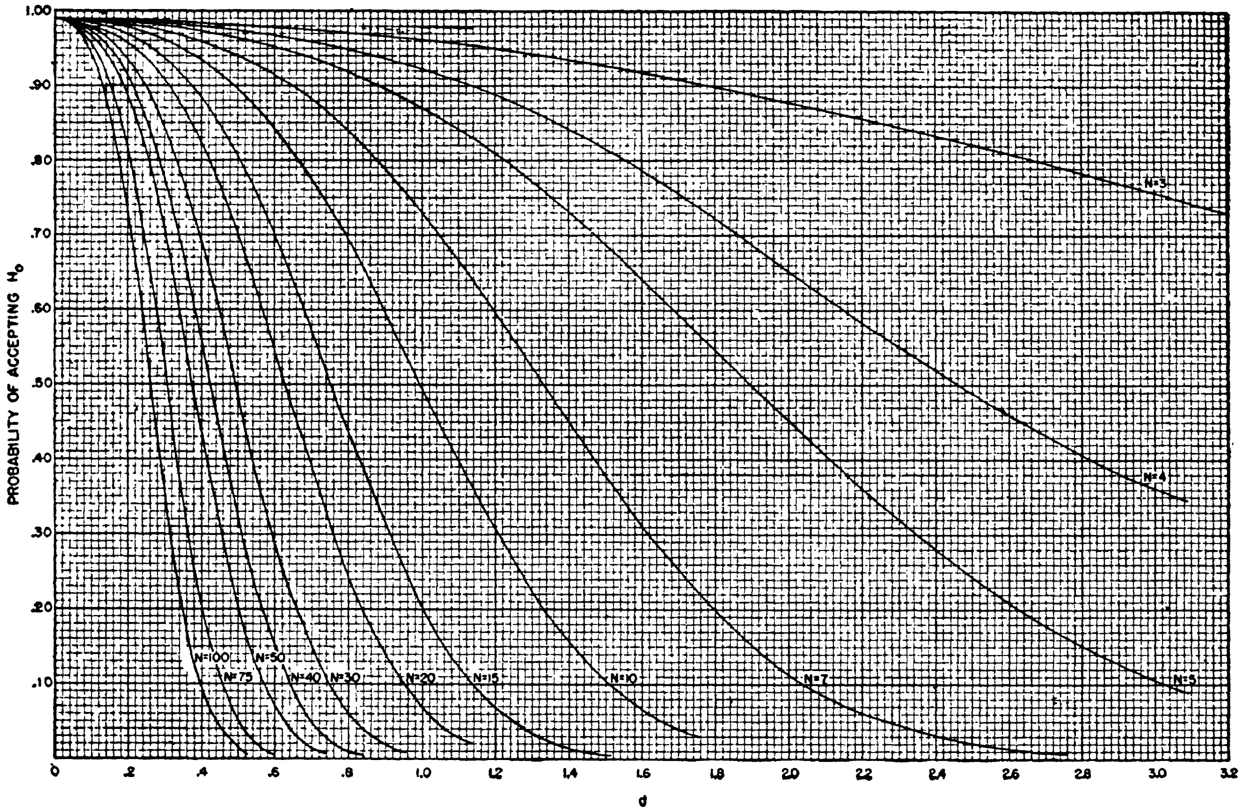


FIG. **13.30** OPERATING CHARACTERISTICS OF THE TWO-SIDED t TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.01.

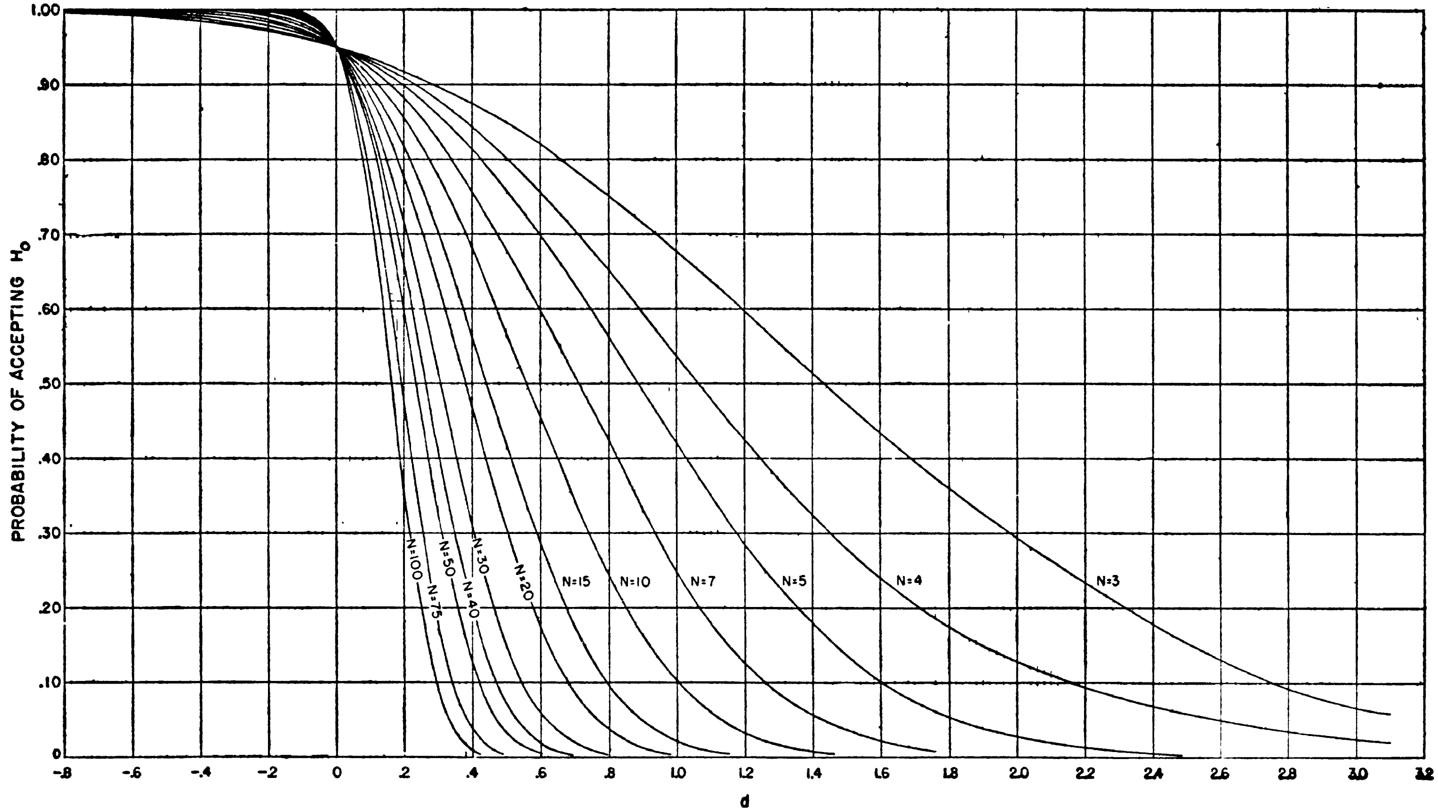


FIG. **13.31** OPERATING CHARACTERISTICS OF THE ONE-SIDED t TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.05.

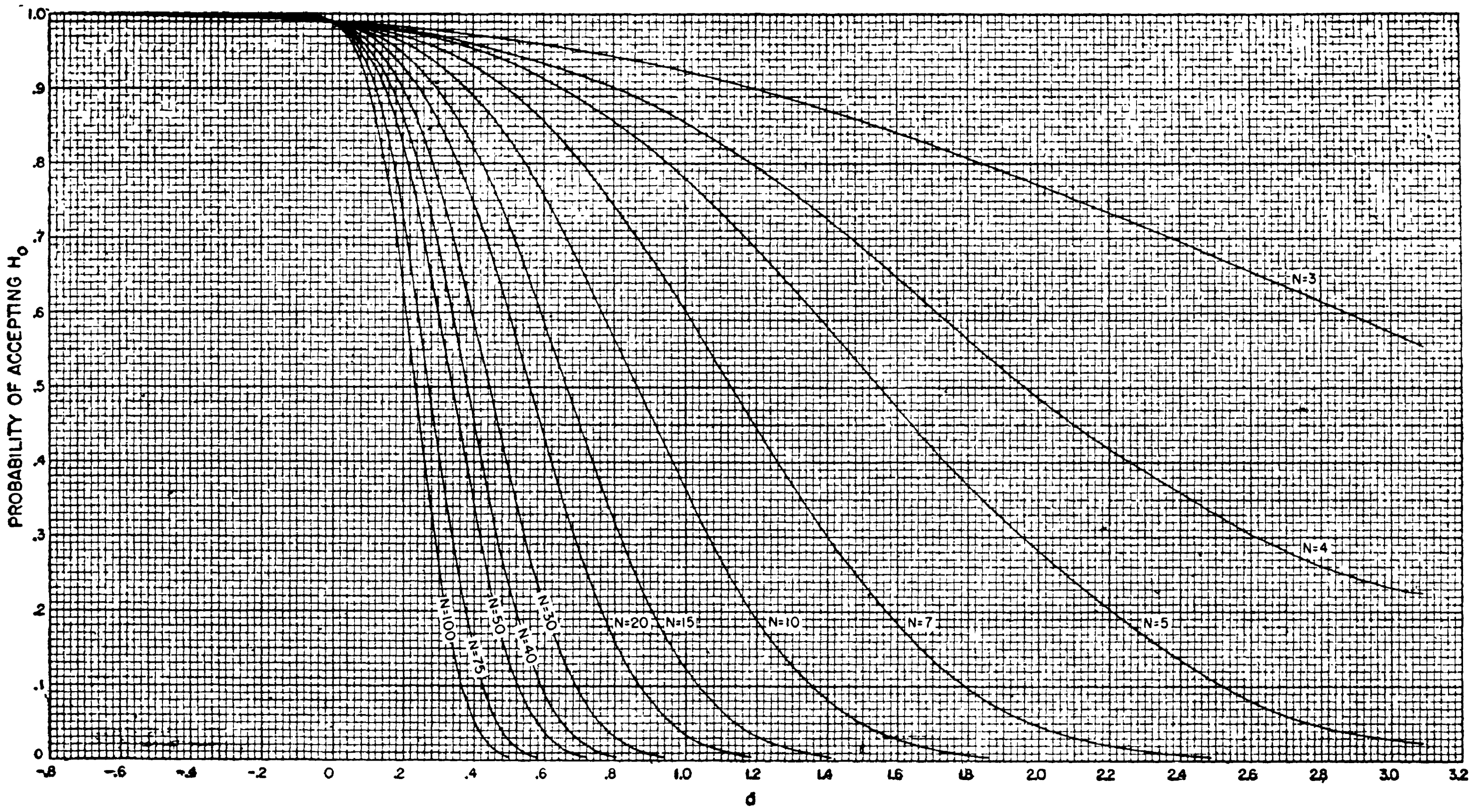


FIG. **13.32** OPERATING CHARACTERISTICS OF THE ONE-SIDED t TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.01.

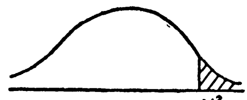
4.6 TEST OF HYPOTHESIS THAT THE VARIANCE OF A NORMAL DISTRIBUTION HAS A SPECIFIED VALUE

Notation for hypothesis	*Test statistic*
$\sigma = \sigma_0$	$\chi^2 = (n - 1)s^2/\sigma_0^2$
Rule of rejection	*Rule for choosing sample sizes*
$\chi^2 \geq \chi^2_{\alpha;n-1}$ if we wish to reject when the true standard deviation exceeds σ_0.	Select a value $\sigma_1 > \sigma_0$ for which we wish to reject the null hypothesis with high probability. Calculate $\lambda = \sigma_1/\sigma_0$ and enter Fig. 13.35 or 13.36 to find the required sample size.
$\chi^2 \leq \chi^2_{1-\alpha;n-1}$ if we wish to reject when the true standard deviation is less than σ_0.	Select a value $\sigma_1 < \sigma_0$ for which we wish to reject the null hypothesis with high probability. Calculate $\lambda = \sigma_1/\sigma_0$ and enter Fig. 13.37 or 13.38 to find the required sample size.
$\chi^2 \geq \chi^2_{\alpha/2;n-1}$ or $\chi^2 \leq \chi^2_{1-\alpha/2;n-1}$ if we wish to reject when the true standard deviation differs from σ_0 in either direction.	Select a value of the relative error σ_1/σ_0 for which we wish to reject the null hypothesis with high probability. Enter Fig. 13.33 or 13.34 to find the required sample size.

Percentage points of the chi-square distribution can be found in Table 13.26.

4.6.1 Example. The standard deviation of a dimension of standard product is $\sigma = 0.1225$ in. A new product is under consideration and will be adopted if the variability of this dimension is not larger. The decision about variability will be based on a sample of items of the new product. If σ is as large as 0.2450 in., we want to be quite sure to reject. From Fig. 13.36 it appears that a sample size of 25 will give a probability of acceptance of 0.01 for $\lambda = 0.2450/0.1225 = 2$ if we make the test at the 1% level. A sample of 25 items is drawn and s^2 is found to be 0.0384. Hence $\chi^2 = (n - 1)s^2/\sigma_0^2 = 61.4$, which exceeds $\chi^2_{.01;24} = 43.0$. The new product is not adopted.

Table of $\chi^2_{\alpha;\nu}$—the 100 α percentage point of



$\chi^2_{\alpha;\nu}$

ν \ α	.995	.99	.98	.975	.95	.90	.80	.75	.70	.50
1	$.0^4393$	$.0^3157$	$.0^3628$	$.0^3982$	.00393	.0158	.0642	.102	.148	.455
2	.0100	.0201	.0404	.0506	.103	.211	.446	.575	.713	1.386
3	.0717	.115	.185	.216	.352	.584	1.005	1.213	1.424	2.366
4	.207	.297	.429	.484	.711	1.064	1.649	1.923	2.195	3.357
5	.412	.554	.752	.831	1.145	1.610	2.343	2.675	3.000	4.351
6	.676	.872	1.134	1.237	1.635	2.204	3.070	3.455	3.828	5.348
7	.989	1.239	1.564	1.690	2.167	2.833	3.822	4.255	4.671	6.346
8	1.344	1.646	2.032	2.180	2.733	3.490	4.594	5.071	5.527	7.344
9	1.735	2.088	2.532	2.700	3.325	4.168	5.380	5.899	6.393	8.343
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	6.737	7.267	9.342
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	7.584	8.148	10.341
12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	8.438	9.034	11.340
13	3.565	4.107	4.765	5.009	5.892	7.042	8.634	9.299	9.926	12.340
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	10.165	10.821	13.339
15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	11.036	11.721	14.339
16	5.142	5.812	6.614	6.908	7.962	9.312	11.152	11.912	12.624	15.338
17	5.697	6.408	7.255	7.564	8.672	10.085	12.002	12.792	13.531	16.338
18	6.265	7.015	7.906	8.231	9.390	10.865	12.857	13.675	14.440	17.338
19	6.844	7.633	8.567	8.907	10.117	11.651	13.716	14.562	15.352	18.338
20	7.434	8.260	9.237	9.591	10.851	12.443	14.578	15.452	16.266	19.337
21	8.034	8.897	9.915	10.283	11.591	13.240	15.445	16.344	17.182	20.337
22	8.643	9.542	10.600	10.982	12.338	14.041	16.314	17.240	18.101	21.337
23	9.260	10.196	11.293	11.688	13.091	14.848	17.187	18.137	19.021	22.337
24	9.886	10.856	11.992	12.401	13.848	15.659	18.062	19.037	19.943	23.337
25	10.520	11.524	12.697	13.120	14.611	16.473	18.940	19.939	20.867	24.337
26	11.160	12.198	13.409	13.844	15.379	17.292	19.820	20.843	21.792	25.336
27	11.808	12.879	14.125	14.573	16.151	18.114	20.703	21.749	22.719	26.336
28	12.461	13.565	14.847	15.308	16.928	18.939	21.588	22.657	23.647	27.336
29	13.121	14.256	15.574	16.047	17.708	19.768	22.475	23.567	24.577	28.336
30	13.787	14.953	16.306	16.791	18.493	20.599	23.364	24.478	25.508	29.336

For values of $\nu > 30$, approximate values for χ^2 may be obtained from the expression $\nu\left[1 - \frac{2}{9\nu} \pm \frac{x}{\sigma}\sqrt{\frac{2}{9\nu}}\right]^3$, where $\frac{x}{\sigma}$ is the normal deviate cutting off the corresponding tails of a normal distribution. If $\frac{x}{\sigma}$ is taken at the 0.02 level, so that 0.01 of the normal distribution is in each tail, the expression yields χ^2 at the 0.99 and 0.01

OF THE χ^2 DISTRIBUTION

the χ^2 distribution for ν degrees of freedom

.30	.25	.20	.10	.05	.025	.02	.01	.005	.001	α / ν
1.074	1.323	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827	1
2.408	2.773	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815	2
3.665	4.108	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.268	3
4.878	5.385	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.465	4
6.064	6.626	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.517	5
7.231	7.841	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457	6
8.383	9.037	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.322	7
9.524	10.219	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.125	8
10.656	11.389	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877	9
11.781	12.549	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588	10
12.899	13.701	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264	11
14.011	14.845	15.812	18.549	21.026	23.337	24.054	26.217	28.300	32.909	12
15.119	15.984	16.985	19.812	22.362	24.736	25.472	27.688	29.819	34.528	13
16.222	17.117	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.123	14
17.322	18.245	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.697	15
18.418	19.369	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252	16
19.511	20.489	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.790	17
20.601	21.605	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312	18
21.689	22.718	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.820	19
22.775	23.828	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.315	20
23.858	24.935	26.171	29.615	32.671	35.479	36.343	38.932	41.401	46.797	21
24.939	26.039	27.301	30.813	33.924	36.781	37.659	40.289	42.796	48.268	22
26.018	27.141	28.429	32.007	35.172	38.076	38.968	41.638	44.181	49.728	23
27.096	28.241	29.553	33.196	36.415	39.364	40.270	42.980	45.558	51.179	24
28.172	29.339	30.675	34.382	37.652	40.646	41.566	44.314	46.928	52.620	25
29.246	30.434	31.795	35.563	38.885	41.923	42.856	45.642	48.290	54.052	26
30.319	31.528	32.912	36.741	40.113	43.194	44.140	46.963	49.645	55.476	27
31.391	32.620	34.027	37.916	41.337	44.461	45.419	48.278	50.993	56.893	28
32.461	33.711	35.139	39.087	42.557	45.722	46.693	49.588	52.336	58.302	29
33.530	34.800	36.250	40.256	43.773	46.979	47.962	50.892	53.672	59.703	30

points. For very large values of ν, it is sufficiently accurate to compute $\sqrt{2\chi^2}$, the distribution of which is approximately normal around a mean of $\sqrt{2\nu - 1}$ and with a standard deviation of 1.

This table is taken by consent from *Statistical Tables for Biological, Agricultural, and Medical Research*, by R. A. Fisher and F. Yates, published by Oliver and Boyd, Edinburgh, and from *Biometrika*, Vol. XXXII, Part II, October 1941, pp. 187-191, "Table of Percentage Points of the χ^2 Distribution," by Catherine M. Thompson. Reproduced in Croxton *Elementary Statistics with Applications in Medicine*, Appendix VI, pp. 328-329, Values of χ^2.

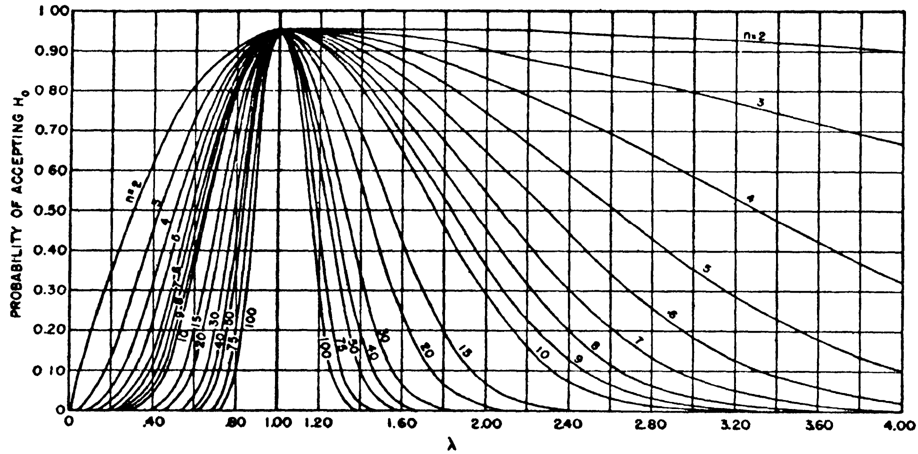
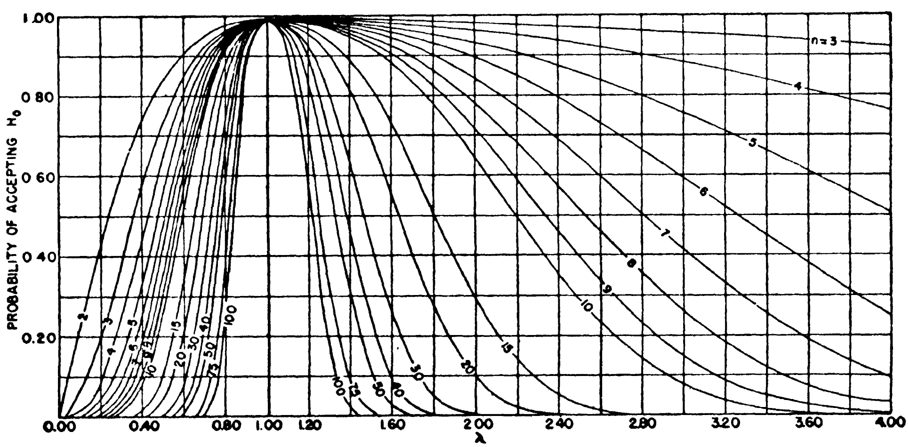


FIG. **13.33** OPERATING CHARACTERISTICS OF THE TWO-SIDED CHI-SQUARE TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.05.

FIG. **13.34** OPERATING CHARACTERISTICS OF THE TWO-SIDED CHI-SQUARE TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.01.



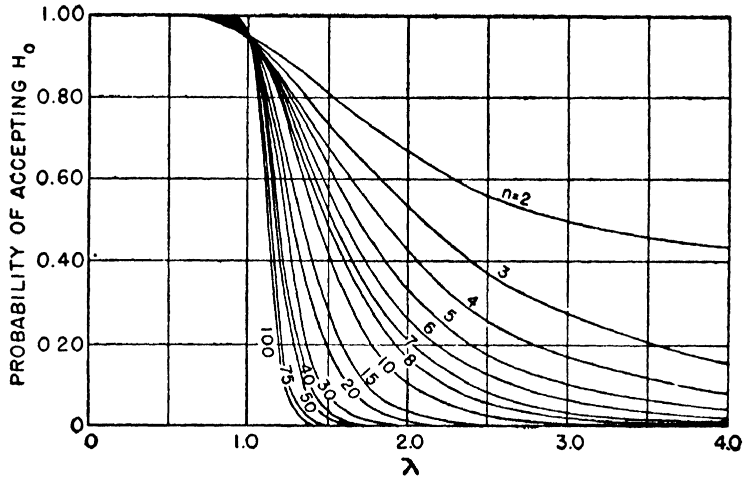


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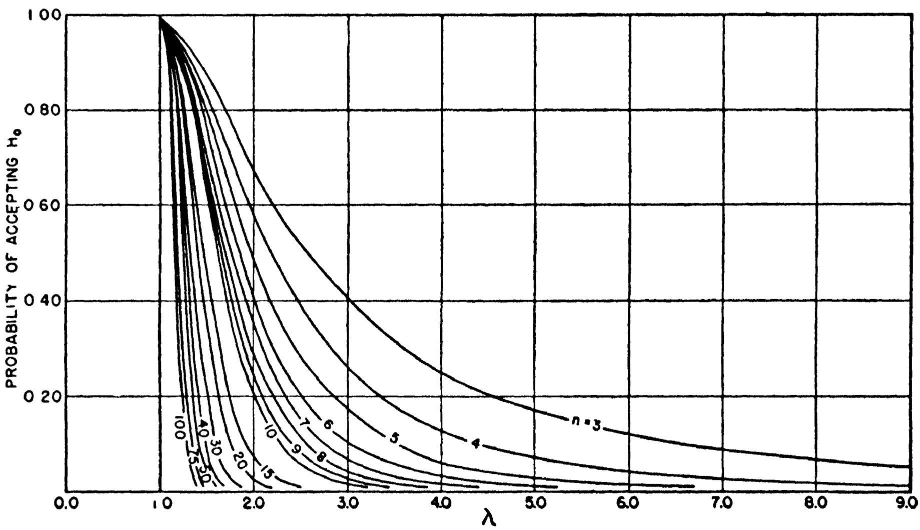
FIG. **13.35** OPERATING CHARACTERISTICS OF THE ONE-SIDED (UPPER TAIL) CHI-SQUARE TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.05.

FIG. **13.36** OPERATING CHARACTERISTICS OF THE ONE-SIDED (UPPER TAIL) CHI-SQUARE TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.01.



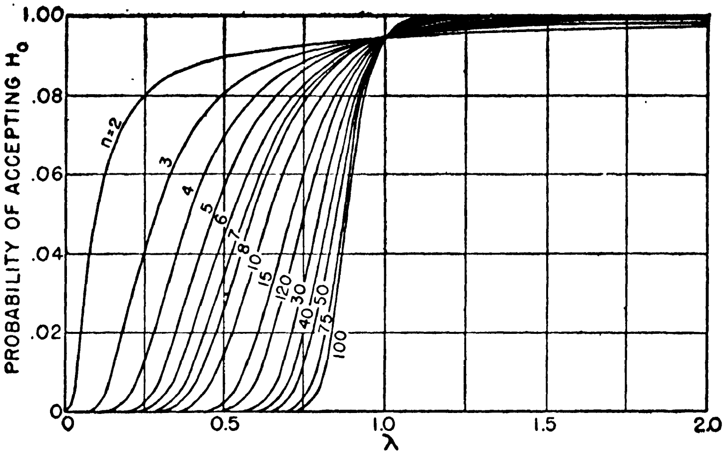


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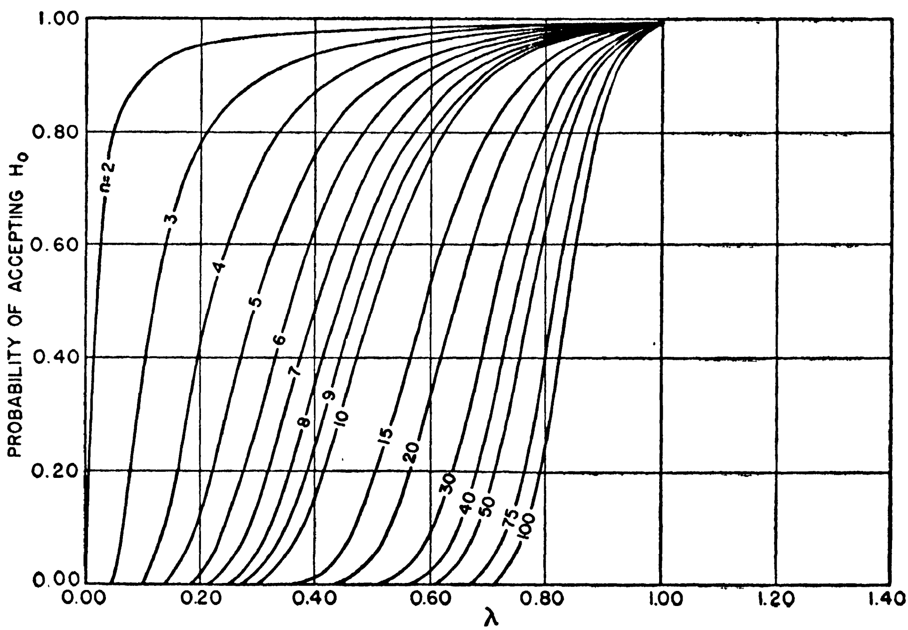
FIG. **13.37** OPERATING CHARACTERISTICS OF THE ONE-SIDED (LOWER TAIL) CHI-SQUARE TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.05.

FIG. **13.38** OPERATING CHARACTERISTICS OF THE ONE-SIDED (LOWER TAIL) CHI-SQUARE TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.01.



4.7 TEST OF THE HYPOTHESIS THAT THE MEANS OF TWO NORMAL DISTRIBUTIONS ARE EQUAL WHEN BOTH STANDARD DEVIATIONS ARE KNOWN

Notation for hypothesis	*Test statistic*
$\mu_x = \mu_y$	$U = (\bar{x} - \bar{y})/\sqrt{\sigma_x^2/n_x + \sigma_y^2/n_y}$
Rule of rejection	*Rule for choosing sample sizes, assuming* $n_x = n_y = n$
$U \geq K_\alpha$ if we wish to reject when $\mu_x > \mu_y$.	Select a value $\mu_x - \mu_y > 0$ for which we want to reject the null hypothesis with high probability. Calculate★ $d = (\mu_x - \mu_y)/\sqrt{\sigma_x^2 + \sigma_y^2}$ and select n from the OC curve of Fig. 13.27 or 13.28.
$U \leq -K_\alpha$ if we wish to reject when $\mu_x < \mu_y$.	Select a value $\mu_x - \mu_y < 0$ for which we want to reject the null hypothesis with high probability. Calculate★ $d = (\mu_y - \mu_x)/\sqrt{\sigma_x^2 + \sigma_y^2}$ and select n from the OC curve of Fig. 13.27 or 13.28.
$\lvert U \rvert \geq K_{\alpha/2}$ if we wish to reject when the true difference in means is either positive or negative.	Select a value $\lvert\mu_x - \mu_y\rvert > 0$ for which we want to reject the null hypothesis with high probability. Calculate★ $d = \left\lvert \dfrac{\mu_x - \mu_y}{\sqrt{\sigma_x^2 + \sigma_y^2}} \right\rvert$ and select n from the OC curve of Fig. 13.25 or 13.26.

★ In order to find the required sample size for the OC curves given, it is necessary to choose $n = n_x = n_y$. However, if n_x and n_y are fixed in advance, $n_x \neq n_y$, the resulting protection using the above rule can be obtained by entering the curves using d and $n = (\sigma_x^2 + \sigma_y^2)/[(\sigma_x^2/n_x) + (\sigma_y^2/n_y)]$.

TABLE **13.24** PERCENTAGE POINTS OF THE NORMAL DISTRIBUTION

Level of significance	*For one-sided tests*	*For two-sided tests*
$\alpha = 0.05$	$K_{.05} = 1.645$	$K_{.025} = 1.960$
$\alpha = 0.01$	$K_{.01} = 2.326$	$K_{.005} = 2.576$

Example. There is evidence that surface finish has an effect on the endurance limit (reversed bending) of steel. An experiment is performed on 0.4% carbon steel, using both unpolished and polished smooth-turned specimens. The finish on the smooth-turned polished specimens was obtained by polishing with No. 0 and 00 emery cloth.

For 0.4% carbon steel, it has been estimated that an average increase in the endurance limit of approximately 7500 psi should be detected on the polished specimens. Furthermore, polishing should not have any effect on the standard deviation of the endurance limit, which is known from the performance of numerous endurance limit experiments to be 4000 psi. From Fig. 13.28 (1% level of significance) with $d = 7500/\sqrt{(4000)^2 + (4000)^2} = 1.33$, we find that about eight observations on each group of specimens are required to have a probability of 0.9 of detecting a difference as large as 7500 psi. The data is as follows:

Endurance limit for polished 0.4% carbon steel (x)	Endurance limit for unpolished 0.4% carbon steel (y)
86,500	82,600
91,900	82,400
89,400	81,700
84,000	79,500
89,900	79,400
78,700	69,800
87,500	79,900
83,100	83,400
$\bar{x}$ = 86,375	$\bar{y}$ = 79,838

$$U = \frac{6537}{\sqrt{(4000)^2/8 + (4000)^2/8}} = 3.27; \qquad K_{.01} = 2.326$$

Hence we reject the hypothesis at the 1% significance level that surface polish has no effect on the endurance limit, and conclude that polished 0.4% carbon steel specimens have a higher mean endurance limit than unpolished ones.

4.8 TEST OF THE HYPOTHESIS THAT THE MEANS OF TWO NORMAL DISTRIBUTIONS ARE EQUAL ASSUMING THAT THE STANDARD DEVIATIONS ARE UNKNOWN BUT EQUAL*; TWO-SAMPLE t TEST

Notation for the hypothesis	*Test statistic*
$\mu_x = \mu_y$	$t = \dfrac{\bar{x} - \bar{y}}{\sqrt{\dfrac{1}{n_x} + \dfrac{1}{n_y}}\sqrt{\dfrac{\sum (x_i - \bar{x})^2 + \sum (y_i - \bar{y})^2}{(n_x + n_y - 2)}}}$
Rule of rejection	***Rule for choosing sample size assuming*** $n_x = n_y = n$.
$t \geq t_{\alpha;n_x+n_y-2}$ if we wish to reject when $\mu_x > \mu_y$.	The OC curve depends on $d = (\mu_x - \mu_y)/2\sigma$. Estimate that value of d for which we want to be sure to reject the hypothesis and pick n' from Fig. 13.31 or 13.32. The required sample size is $(n' + 1)/2$.
$t \leq -t_{\alpha;n_x+n_y-2}$ if we wish to reject when $\mu_x < \mu_y$.	Estimate the $d = (\mu_x - \mu_y)/2\sigma$ for which we want to be sure to reject, and pick n' from Fig. 13.31 or 13.32. The required sample size is $(n' + 1)/2$.
$\|t\| \geq t_{\alpha/2;n_x+n_y-2}$ if we wish to reject whenever μ_x is not equal to μ_y.	Estimate the $d = \|(\mu_x - \mu_y)/2\sigma\|$ for which we want to be sure to reject, and pick n' from Fig. 13.29 or 13.30. The required sample size is $(n' + 1)/2$.

Example. A manufacturer of electric irons produces these items in two plants. Both plants have the same suppliers of small parts. A saving can be made by purchasing thermostats for plant B from a local supplier. A single lot was purchased from the local supplier and it was desired to test whether

* A test for the equality of standard deviations is given in Art. 4.10.

or not these new thermostats were as accurate as the old. It was decided to test the irons on the 550°F setting, and the actual temperatures were to be read to the nearest 0.1° with a thermocouple. With the old supplier, very few complaints were received, and the manufacturer feels that the switch should not be made if the average temperature changes by more than 10.5°. The order of magnitude of the standard deviation is roughly 10° for the old supplier, and there is no reason to suspect that it will be different for the new supplier. For $d = 10.5/(2 \times 10) = 0.525$, and from Fig. 13.29, $n' = 45$, corresponding to a probability of 0.9 of detecting a change of 10.5° or more. Hence $n = 23$. The data are:

New supplier x (degrees F)		Old supplier y (degrees F)	
530.3	559.1	559.7	564.6
559.3	555.0	534.7	554.5
549.4	538.6	554.8	553.0
544.0	551.1	545.0	538.4
551.7	565.4	544.6	548.3
566.3	554.9	538.0	552.9
549.9	550.0	550.7	535.1
556.9	554.9	563.1	555.0
536.7	544.7	551.1	544.8
558.8	536.1	553.8	558.4
538.8	569.1	538.8	548.7
	543.3		560.3

$$\sum x = 12{,}684.3 \qquad \sum y = 12{,}648.3$$
$$\bar{x} = 551.491304 \qquad \bar{y} = 549.926086$$
$$\sum x^2 = 6{,}997{,}436.07 \qquad \sum y^2 = 6{,}957{,}333.23$$
$$\sum (x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2 = 2{,}154.930$$
$$\sum (y_i - \bar{y})^2 = \sum y_i^2 - n\bar{y}^2 = 1{,}703.130$$
$$\frac{\sum (x_i - \bar{x})^2 + \sum (y_i - \bar{y})^2}{n_1 + n_2 - 2} = \frac{3858.06}{44} = 87.68318$$
$$|t| = \frac{551.491304 - 549.926086}{\sqrt{2/23}\sqrt{87.68318}} = 0.567 \leq t_{\alpha/2} = 2.02$$

Therefore we accept the hypothesis at the 5% level of significance that there is no difference in the mean temperatures of the two suppliers.

4.8.1 Procedure based on correlated samples. An important special case of the t test arises when the observations are "paired"; each pair being taken under the same experimental conditions, with the conditions varying from pair to pair. In this case, the test is made on the differences of the observations, and is carried out as a one sample problem, testing the hypothesis that the mean of the difference is zero according to the procedure of Art. 4.5, where the observations now consist of the differences.

Example. A research laboratory is embarking on an experiment to determine whether or not outside temperature has any effect on the accuracy of a Bourdon gage. The experiment consists of placing the gage in a steam line which is completely insulated from room temperature. The line passes through two adjacent rooms. One room always remains at 32°F, whereas the temperature in the other can be varied. Since there is a time lag in changing the temperature of the controlled room, it is likely that the steam pressure will fluctuate during this time interval. Consequently, no effort is made to maintain the same pressure for a period of time. It can be assumed that the pressure is the same at the stations in each room and will be kept at approximately 60 psi. The experiment should be designed such that a difference as great as 0.75 psi should be picked up with high probability, say 0.9. The order of magnitude of the standard deviation is 0.4 psi. This is obtained as follows: The manufacturer usually specifies that these gages are good to $\pm 2\%$. If we interpret this to mean that most of the observations will lie between $\pm 2\%$ of the true value, and in particular, if "most" means 0.9987, we have true value $\pm 2\%$ = true value $\pm 3\sigma$. If the true value is near 60 psi, we have $3\sigma = 1.2$ or $\sigma = 0.4$. Hence the standard deviation of the differences is $(\sqrt{2})(0.4) = 0.57$. It must be emphasized that this is just an estimate of the order of magnitude of the standard deviation and should be used only for determining sample size. Referring to Fig. 13.29, $d = (0.75)/(\sqrt{2})(0.4) = 1.3$, so that eight observations are required. The data are:

Pressure read at room temp. = 32° x	Pressure read at other room temp. y	$x - y$	$(x - y)^2$
60.3	61.0	−0.7	0.49
59.9	60.0	−0.1	0.01
59.5	60.9	−1.4	1.96
58.5	58.2	0.3	0.09
59.6	59.9	−0.3	0.09
59.0	60.4	−1.4	1.96
60.1	59.6	0.5	0.25
59.4	59.9	−0.5	0.25

$$s^2 = \frac{5.10 - 8(0.450)^2}{7} = 0.497; \quad \sum (x - y) = -3.6; \quad \sum (x - y)^2 = 5.10$$

$$s = 0.705; \qquad |t| = \left| \frac{-0.45\sqrt{8}}{0.705} \right| = \frac{1.406}{0.705} = 1.99 \leq t_{\alpha/2} = 2.365$$

Therefore the data do not reveal any significant difference between gauge readings when the temperature changes.

4.9 TEST OF THE HYPOTHESIS THAT THE MEANS OF TWO NORMAL DISTRIBUTIONS ARE EQUAL WHEN THE STANDARD DEVIATIONS ARE UNKNOWN AND NOT NECESSARILY EQUAL

Notation for the hypothesis	*Test statistic*
$\mu_x = \mu_y$	$t' = \dfrac{\bar{x} - \bar{y}}{\sqrt{s_x^2/n_x + s_y^2/n_y}}$
Rule of rejection	*Formula for obtaining the degrees of freedom v*
$t' \geq t_{\alpha;v}$ if we wish to reject when $\mu_x > \mu_y$.	$v = \dfrac{(s_x^2/n_x + s_y^2/n_y)^2}{(s_x^2/n_x)^2/(n_x+1) + (s_y^2/n_y)^2/(n_y+1)} - 2$
$t' \leq -t_{\alpha;v}$ if we wish to reject when $\mu_x < \mu_y$.	
$\lvert t' \rvert \geq t_{\alpha/2;v}$ if we wish to reject whenever μ_x is not equal to μ_y.	

Example. A manufacturer of automobile crankshafts was troubled with the problem of bend in the final shaft. Bend may be caused by the length and weight of the shaft, by improper setup of machine tools, by the heat treating process, or by some combination of these causes. It is suspected that nitriding the shaft, i.e., the process that hardens the surface of the shaft by a heat and nitrous oxide reaction with the steel, is the main cause for bend. Twenty-five shafts were measured before nitriding at the front main center journal by a dial indicator gauge accurate to the 0.0001 in. Similarly, another 25 shafts were measured at the same spot after the nitriding operation. It cannot be assumed that the variability in the bend is the same before and after nitriding. We test the hypothesis that the mean value of the bend is the same before and after nitriding, wishing to reject if the average bend after nitriding is larger. Denoting by x the values before nitriding and by y the values after nitriding, the following results were obtained:

$$\sum_{i=1}^{25} x = 203 \times 10^{-4} \qquad \sum_{i=1}^{25} y = 453 \times 10^{-4}$$

$$\bar{x} = 0.0008120 \qquad \bar{y} = 0.001812$$

$$\sum_{i=1}^{25} x^2 = 2{,}933 \times 10^{-8} \qquad \sum_{i=1}^{25} y^2 = 14{,}362 \times 10^{-8}$$

$$s_x^2 = \frac{2{,}933 \times 10^{-8} - 25(0.0008120)^2}{24} \qquad s_y^2 = \frac{14{,}362 \times 10^{-8} - 25(0.001812)^2}{24}$$

$$= 53.5 \times 10^{-8} \qquad = 256 \times 10^{-8}$$

$$\frac{s_x^2}{n_x} = 2.14 \times 10^{-8} \qquad \frac{s_y^2}{n_y} = 10.24 \times 10^{-8}$$

$$v = \frac{[(2.14)10^{-8} + (10.24)10^{-8}]^2}{(2.14)^2(10^{-16})/26 + (10.24)^2(10^{-16})/26} - 2 = 34$$

$$t' = \frac{0.0008120 - 0.001812}{\sqrt{(2.14)10^{-8} + (10.24)10^{-8}}} = -\frac{0.001}{(3.52)10^{-4}} = -2.84$$

$$t_{.05;34} = 1.69$$

Since $t' \leq t_{\alpha;v}$ we reject the hypothesis that at the 5% significance level the means are the same before and after nitriding, and we conclude that the average bend is larger after nitriding.

4.10 TEST OF THE HYPOTHESIS THAT THE VARIANCES OF TWO NORMAL DISTRIBUTIONS ARE EQUAL

Notation for the hypothesis	*Test statistic*
$\sigma_x^2 = \sigma_y^2$ or $\sigma_x = \sigma_y$	$F = \dfrac{[\sum (x_i - \bar{x})^2]/(n_x - 1)}{[\sum (y_i - \bar{y})^2]/(n_y - 1)} = \dfrac{s_x^2}{s_y^2}$
Rule of rejection	*Rule for choosing sample sizes*
$F \geq F_{\alpha;n_x-1,n_y-1}$ if we wish to reject when $\sigma_x > \sigma_y$. The notation x and y is arbitrary in a one-sided test; let x be the symbol for the variable with possible larger variance.	The OC curve depends on $\lambda = \sigma_x/\sigma_y$, and n_x and n_y. The OC curve for $n_x = n_y = n$ is given in Fig. 13.41 and 13.42. Pick a λ for which we want to reject and choose n such that the probability of accepting for that λ is sufficiently small.
In a two-sided test, let s_x^2 be the larger sample variance. Reject if $F \geq F_{\alpha/2;n_x-1,n_y-1}$.	The OC curve depends on $\lambda = \sigma_x/\sigma_y$ and is given in Fig. 13.39 and 13.40 for the case $n_x = n_y = n$. Pick the n for a specified λ such that the probability of accepting is sufficiently small.

Percentage points of the F distribution may be found in Table 13.27.

Example. The standard deviation of a particular dimension of a metal component is small enough so that it is satisfactory in subsequent assembly; a new supplier of metal plate is under consideration and will be preferred if the standard deviation of his product is not larger as the cost of his product is lower than that of the present supplier. From Fig. 13.41, it is decided to base this decision on a sample of 100 items from each supplier, since it is desired to insure that the probability of switching to the new product is less than 0.02 if $\lambda \geq 1.5$; data on dimensions are relatively easy to obtain, and small numbers of observations give relatively little protection against erroneous decisions. The following data are obtained:

$$\text{New supplier:} \quad s^2 = 0.00041$$

$$\text{Current supplier:} \quad s^2 = 0.00057$$

$$F = \frac{(0.00041)}{(0.00057)} < F_{.05;99,99}$$

Since the value of F does not exceed $F_{.05;99,99}$, the hypothesis of equality of variances is adopted.

TABLE 13.27 10 PERCENTAGE POINT OF THE F DISTRIBUTION*

Table of $F_{.10;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1); Degrees of freedom for the denominator (ν_2)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	15	20	30	50	100	200	500	∞
1	39.9	49.5	53.6	55.8	57.2	58.2	58.9	59.4	59.9	60.2	61.2	61.7	62.3	62.7	63.0	63.2	63.3	63.3
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.42	9.44	9.46	9.47	9.48	9.49	9.49	9.49
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.20	5.18	5.17	5.15	5.14	5.14	5.14	5.13
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.87	3.84	3.82	3.80	3.78	3.77	3.76	3.76
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.24	3.21	3.17	3.15	3.13	3.12	3.11	3.10
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.87	2.84	2.80	2.77	2.75	2.73	2.73	2.72
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.63	2.59	2.56	2.52	2.50	2.48	2.48	2.47
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.46	2.42	2.38	2.35	2.32	2.31	2.30	2.29
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.34	2.30	2.25	2.22	2.19	2.17	2.17	2.16
10	3.28	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.24	2.20	2.16	2.12	2.09	2.07	2.06	2.06
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.17	2.12	2.08	2.04	2.00	1.99	1.98	1.97
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.10	2.06	2.01	1.97	1.94	1.92	1.91	1.90
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.05	2.01	1.96	1.92	1.88	1.86	1.85	1.85
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.01	1.96	1.91	1.87	1.83	1.82	1.80	1.80
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	1.97	1.92	1.87	1.83	1.79	1.77	1.76	1.76
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.94	1.89	1.84	1.79	1.76	1.74	1.73	1.72
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.91	1.86	1.81	1.76	1.73	1.71	1.69	1.69
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.89	1.84	1.78	1.74	1.70	1.68	1.67	1.66
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.86	1.81	1.76	1.71	1.67	1.65	1.64	1.63
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.84	1.79	1.74	1.69	1.65	1.63	1.62	1.61
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.81	1.76	1.70	1.65	1.61	1.59	1.58	1.57
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.78	1.73	1.67	1.62	1.58	1.56	1.54	1.53
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.76	1.71	1.65	1.59	1.55	1.53	1.51	1.50
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.74	1.69	1.63	1.57	1.53	1.50	1.49	1.48
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.72	1.67	1.61	1.55	1.51	1.48	1.47	1.46
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.66	1.61	1.54	1.48	1.43	1.41	1.39	1.38
50	2.81	2.41	2.20	2.06	1.97	1.90	1.84	1.80	1.76	1.73	1.63	1.57	1.50	1.44	1.39	1.36	1.34	1.33
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.60	1.54	1.48	1.41	1.36	1.33	1.31	1.29
80	2.77	2.37	2.15	2.02	1.92	1.85	1.79	1.75	1.71	1.68	1.57	1.51	1.44	1.38	1.32	1.28	1.26	1.24
100	2.76	2.36	2.14	2.00	1.91	1.83	1.78	1.73	1.70	1.66	1.56	1.49	1.42	1.35	1.29	1.26	1.23	1.21
200	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66	1.63	1.52	1.46	1.38	1.31	1.24	1.20	1.17	1.14
500	2.72	2.31	2.10	1.96	1.86	1.79	1.73	1.68	1.64	1.61	1.50	1.44	1.36	1.28	1.21	1.16	1.12	1.09
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.49	1.42	1.34	1.26	1.18	1.13	1.08	1.00

Example: $P\{F_{.10,8,20} < 2.00\} = 90\%$.

$F_{.90;\nu_1,\nu_2} = 1/F_{.10;\nu_2,\nu_1}$. Example: $F_{.90;8,20} = 1/F_{.10;20,8} = 1/2.42 = 0.413$.

Approximate formula for ν_1 and ν_2 larger than 30: $\log_{10} F_{.10;\nu_1,\nu_2} \approx \dfrac{1.1131}{\sqrt{h - 0.77}} - 0.527\left(\dfrac{1}{\nu_1} - \dfrac{1}{\nu_2}\right)$,

where $\dfrac{1}{h} = \dfrac{1}{2}\left(\dfrac{1}{\nu_1} + \dfrac{1}{\nu_2}\right)$.

* This table is abridged from *Statistical Tables and Formulas*, by A. Hald, published by John Wiley & Sons, Inc., a major part of which has been abridged from "Tables of Percentage Points of the Inverted Beta (F) Distribution," computed by M. Merrington and C. M. Thompson, *Biometrika* Vol. 33, 1943, 73-88, by permission of the proprietors, or reproduced from Table V of R. A. Fisher and F. Yates: *Statistical Tables*, Oliver and Boyd, Edinburgh, by permission of the authors and the publishers.

Degrees of freedom for the denominator (ν_2)	Degrees of freedom for the numerator (ν_1) 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	161	200	216	225	230	234	237	239	241	242	243	244	245	245	246	246	247	247
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68	8.67
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91	3.90
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48	3.47
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58	2.57
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27	2.26
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23	2.22
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17	2.15
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11	2.10
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.23	2.20	2.18	2.15	2.13	2.11	2.09	2.07
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.21	2.18	2.15	2.13	2.11	2.09	2.07	2.05
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16	2.14	2.11	2.09	2.07	2.05	2.04
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.02
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.17	2.13	2.10	2.08	2.06	2.04	2.02	2.00
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00	1.99
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.14	2.10	2.08	2.05	2.03	2.01	1.99	1.97
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.96
32	4.15	3.29	2.90	2.67	2.51	2.40	2.31	2.24	2.19	2.14	2.10	2.07	2.04	2.01	1.99	1.97	1.95	1.94
34	4.13	3.28	2.88	2.65	2.49	2.38	2.29	2.23	2.17	2.12	2.08	2.05	2.02	1.99	1.97	1.95	1.93	1.92
36	4.11	3.26	2.87	2.63	2.48	2.36	2.28	2.21	2.15	2.11	2.07	2.03	2.00	1.98	1.95	1.93	1.92	1.90
38	4.10	3.24	2.85	2.62	2.46	2.35	2.26	2.19	2.14	2.09	2.05	2.02	1.99	1.96	1.94	1.92	1.90	1.88
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00	1.97	1.95	1.92	1.90	1.89	1.87
42	4.07	3.22	2.83	2.59	2.44	2.32	2.24	2.17	2.11	2.06	2.03	1.99	1.96	1.93	1.91	1.89	1.87	1.86
44	4.06	3.21	2.82	2.58	2.43	2.31	2.23	2.16	2.10	2.05	2.01	1.98	1.95	1.92	1.90	1.88	1.86	1.84
46	4.05	3.20	2.81	2.57	2.42	2.30	2.22	2.15	2.09	2.04	2.00	1.97	1.94	1.91	1.89	1.87	1.85	1.83
48	4.04	3.19	2.80	2.57	2.41	2.29	2.21	2.14	2.08	2.03	1.99	1.96	1.93	1.90	1.88	1.86	1.84	1.82
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.99	1.95	1.92	1.89	1.87	1.85	1.83	1.81
55	4.02	3.16	2.77	2.54	2.38	2.27	2.18	2.11	2.06	2.01	1.97	1.93	1.90	1.88	1.85	1.83	1.81	1.79
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92	1.89	1.86	1.84	1.82	1.80	1.78
65	3.99	3.14	2.75	2.51	2.36	2.24	2.15	2.08	2.03	1.98	1.94	1.90	1.87	1.85	1.82	1.80	1.78	1.76
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.93	1.89	1.86	1.84	1.81	1.79	1.77	1.75
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.91	1.88	1.84	1.82	1.79	1.77	1.75	1.73
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.90	1.86	1.83	1.80	1.78	1.76	1.74	1.72
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.89	1.85	1.82	1.79	1.77	1.75	1.73	1.71
125	3.92	3.07	2.68	2.44	2.29	2.17	2.08	2.01	1.96	1.91	1.87	1.83	1.80	1.77	1.75	1.72	1.70	1.69
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.85	1.82	1.79	1.76	1.73	1.71	1.69	1.67
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80	1.77	1.74	1.72	1.69	1.67	1.66
300	3.87	3.03	2.63	2.40	2.24	2.13	2.04	1.97	1.91	1.86	1.82	1.78	1.75	1.72	1.70	1.68	1.66	1.64
500	3.86	3.01	2.62	2.39	2.23	2.12	2.03	1.96	1.90	1.85	1.81	1.77	1.74	1.71	1.69	1.66	1.64	1.62
1000	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89	1.84	1.80	1.76	1.73	1.70	1.68	1.65	1.63	1.61
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75	1.72	1.69	1.67	1.64	1.62	1.60

Example: $P\{F_{.05;8,20} < 2.45\} = 95\%$.

$F_{.95;\nu_1,\nu_2} = 1/F_{.05;\nu_2,\nu_1}$. Example: $F_{.95;8,20} = 1/F_{.05;20,8} = 1/3.15 = 0.317$.

$F_{.05;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1)

19	20	22	24	26	28	30	35	40	45	50	60	80	100	200	500	∞	Degrees of freedom for the denominator (ν_2)
248	248	249	249	249	250	250	251	251	251	252	252	252	253	254	254	254	1
19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	2
8.67	8.66	8.65	8.64	8.63	8.62	8.62	8.60	8.59	8.59	8.58	8.57	8.56	8.55	8.54	8.53	8.53	3
5.81	5.80	5.79	5.77	5.76	5.75	5.75	5.73	5.72	5.71	5.70	5.69	5.67	5.66	5.65	5.64	5.63	4
4.57	4.56	4.54	4.53	4.52	4.50	4.50	4.48	4.46	4.45	4.44	4.43	4.41	4.41	4.39	4.37	4.37	5
3.88	3.87	3.86	3.84	3.83	3.82	3.81	3.79	3.77	3.76	3.75	3.74	3.72	3.71	3.69	3.68	3.67	6
3.46	3.44	3.43	3.41	3.40	3.39	3.38	3.36	3.34	3.33	3.32	3.30	3.29	3.27	3.25	3.24	3.23	7
3.16	3.15	3.13	3.12	3.10	3.09	3.08	3.06	3.04	3.03	3.02	3.01	2.99	2.97	2.95	2.94	2.93	8
2.95	2.94	2.92	2.90	2.89	2.87	2.86	2.84	2.83	2.81	2.80	2.79	2.77	2.76	2.73	2.72	2.71	9
2.78	2.77	2.75	2.74	2.72	2.71	2.70	2.68	2.66	2.65	2.64	2.62	2.60	2.59	2.56	2.55	2.54	10
2.66	2.65	2.63	2.61	2.59	2.58	2.57	2.55	2.53	2.52	2.51	2.49	2.47	2.46	2.43	2.42	2.40	11
2.56	2.54	2.52	2.51	2.49	2.48	2.47	2.44	2.43	2.41	2.40	2.38	2.36	2.35	2.32	2.31	2.30	12
2.47	2.46	2.44	2.42	2.41	2.39	2.38	2.36	2.34	2.33	2.31	2.30	2.27	2.26	2.23	2.22	2.21	13
2.40	2.39	2.37	2.35	2.33	2.32	2.31	2.28	2.27	2.25	2.24	2.22	2.20	2.19	2.16	2.14	2.13	14
2.34	2.33	2.31	2.29	2.27	2.26	2.25	2.22	2.20	2.19	2.18	2.16	2.14	2.12	2.10	2.08	2.07	15
2.29	2.28	2.25	2.24	2.22	2.21	2.19	2.17	2.15	2.14	2.12	2.11	2.08	2.07	2.04	2.02	2.01	16
2.24	2.23	2.21	2.19	2.17	2.16	2.15	2.12	2.10	2.09	2.08	2.06	2.03	2.02	1.99	1.97	1.96	17
2.20	2.19	2.17	2.15	2.13	2.12	2.11	2.08	2.06	2.05	2.04	2.02	1.99	1.98	1.95	1.93	1.92	18
2.17	2.16	2.13	2.11	2.10	2.08	2.07	2.05	2.03	2.01	2.00	1.98	1.96	1.94	1.91	1.89	1.88	19
2.14	2.12	2.10	2.08	2.07	2.05	2.04	2.01	1.99	1.98	1.97	1.95	1.92	1.91	1.88	1.86	1.84	20
2.11	2.10	2.07	2.05	2.04	2.02	2.01	1.98	1.96	1.95	1.94	1.92	1.89	1.88	1.84	1.82	1.81	21
2.08	2.07	2.05	2.03	2.01	2.00	1.98	1.96	1.94	1.92	1.91	1.89	1.86	1.85	1.82	1.80	1.78	22
2.06	2.05	2.02	2.00	1.99	1.97	1.96	1.93	1.91	1.90	1.88	1.86	1.84	1.82	1.79	1.77	1.76	23
2.04	2.03	2.00	1.98	1.97	1.95	1.94	1.91	1.89	1.88	1.86	1.84	1.82	1.80	1.77	1.75	1.73	24
2.02	2.01	1.98	1.96	1.95	1.93	1.92	1.89	1.87	1.86	1.84	1.82	1.80	1.78	1.75	1.73	1.71	25
2.00	1.99	1.97	1.95	1.93	1.91	1.90	1.87	1.85	1.84	1.82	1.80	1.78	1.76	1.73	1.71	1.69	26
1.99	1.97	1.95	1.93	1.91	1.90	1.88	1.86	1.84	1.82	1.81	1.79	1.76	1.74	1.71	1.69	1.67	27
1.97	1.96	1.93	1.91	1.90	1.88	1.87	1.84	1.82	1.80	1.79	1.77	1.74	1.73	1.69	1.67	1.65	28
1.96	1.94	1.92	1.90	1.88	1.87	1.85	1.83	1.81	1.79	1.77	1.75	1.73	1.71	1.67	1.65	1.64	29
1.95	1.93	1.91	1.89	1.87	1.85	1.84	1.81	1.79	1.77	1.76	1.74	1.71	1.70	1.66	1.64	1.62	30
1.92	1.91	1.88	1.86	1.85	1.83	1.82	1.79	1.77	1.75	1.74	1.71	1.69	1.67	1.63	1.61	1.59	32
1.90	1.89	1.86	1.84	1.82	1.80	1.80	1.77	1.75	1.73	1.71	1.69	1.66	1.65	1.61	1.59	1.57	34
1.88	1.87	1.85	1.82	1.81	1.79	1.78	1.75	1.73	1.71	1.69	1.67	1.64	1.62	1.59	1.56	1.55	36
1.87	1.85	1.83	1.81	1.79	1.77	1.76	1.73	1.71	1.69	1.68	1.65	1.62	1.61	1.57	1.54	1.53	38
1.85	1.84	1.81	1.79	1.77	1.76	1.74	1.72	1.69	1.67	1.66	1.64	1.61	1.59	1.55	1.53	1.51	40
1.84	1.83	1.80	1.78	1.76	1.74	1.73	1.70	1.68	1.66	1.65	1.62	1.59	1.57	1.53	1.51	1.49	42
1.83	1.81	1.79	1.77	1.75	1.73	1.72	1.69	1.67	1.65	1.63	1.61	1.58	1.56	1.52	1.49	1.48	44
1.82	1.80	1.78	1.76	1.74	1.72	1.71	1.68	1.65	1.64	1.62	1.60	1.57	1.55	1.51	1.48	1.46	46
1.81	1.79	1.77	1.75	1.73	1.71	1.70	1.67	1.64	1.62	1.61	1.59	1.56	1.54	1.49	1.47	1.45	48
1.80	1.78	1.76	1.74	1.72	1.70	1.69	1.66	1.63	1.61	1.60	1.58	1.54	1.52	1.48	1.46	1.44	50
1.78	1.76	1.74	1.72	1.70	1.68	1.67	1.64	1.61	1.59	1.58	1.55	1.52	1.50	1.46	1.43	1.41	55
1.76	1.75	1.72	1.70	1.68	1.66	1.65	1.62	1.59	1.57	1.56	1.53	1.50	1.48	1.44	1.41	1.39	60
1.75	1.73	1.71	1.69	1.67	1.65	1.63	1.60	1.58	1.56	1.54	1.52	1.49	1.46	1.42	1.39	1.37	65
1.74	1.72	1.70	1.67	1.65	1.64	1.62	1.59	1.57	1.55	1.53	1.50	1.47	1.45	1.40	1.37	1.35	70
1.72	1.70	1.68	1.65	1.63	1.62	1.60	1.57	1.54	1.52	1.51	1.48	1.45	1.43	1.38	1.35	1.32	80
1.70	1.69	1.66	1.64	1.62	1.60	1.59	1.55	1.53	1.51	1.49	1.46	1.43	1.41	1.36	1.32	1.30	90
1.69	1.68	1.65	1.63	1.61	1.59	1.57	1.54	1.52	1.49	1.48	1.45	1.41	1.39	1.34	1.31	1.28	100
1.67	1.65	1.63	1.60	1.58	1.57	1.55	1.52	1.49	1.47	1.45	1.42	1.39	1.36	1.31	1.27	1.25	125
1.66	1.64	1.61	1.59	1.57	1.55	1.53	1.50	1.48	1.45	1.44	1.41	1.37	1.34	1.29	1.25	1.22	150
1.64	1.62	1.60	1.57	1.55	1.53	1.52	1.48	1.46	1.43	1.41	1.39	1.35	1.32	1.26	1.22	1.19	200
1.62	1.61	1.58	1.55	1.53	1.51	1.50	1.46	1.43	1.41	1.39	1.36	1.32	1.30	1.23	1.19	1.15	300
1.61	1.59	1.56	1.54	1.52	1.50	1.48	1.45	1.42	1.40	1.38	1.34	1.30	1.28	1.21	1.16	1.11	500
1.60	1.58	1.55	1.53	1.51	1.49	1.47	1.44	1.41	1.38	1.36	1.33	1.29	1.26	1.19	1.13	1.08	1000
1.59	1.57	1.54	1.52	1.50	1.48	1.46	1.42	1.39	1.37	1.35	1.32	1.27	1.24	1.17	1.11	1.00	∞

Approximate formula for ν_1 and ν_2 larger than 30: $\log_{10} F_{.05;\nu_1,\nu_2} \simeq \dfrac{1.4287}{\sqrt{h} - 0.95} - 0.681\left(\dfrac{1}{\nu_1} - \dfrac{1}{\nu_2}\right)$, where $\dfrac{1}{h} = \dfrac{1}{2}\left(\dfrac{1}{\nu_1} + \dfrac{1}{\nu_2}\right)$.

Degrees of freedom for the denominator (ν_2)	Degrees of freedom for the numerator (ν_1)																	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	648	800	864	900	922	937	948	957	963	969	973	977	980	983	985	987	989	990
2	38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4
3	17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.4	14.3	14.3	14.3	14.3	14.2	14.2	14.2
4	12.2	10.6	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.79	8.75	8.72	8.69	8.66	8.64	8.62	8.60
5	10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.57	6.52	6.49	6.46	6.43	6.41	6.39	6.37
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.41	5.37	5.33	5.30	5.27	5.25	5.23	5.21
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.71	4.67	4.63	4.60	4.57	4.54	4.52	4.50
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.24	4.20	4.16	4.13	4.10	4.08	4.05	4.03
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.91	3.87	3.83	3.80	3.77	3.74	3.72	3.70
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.66	3.62	3.58	3.55	3.52	3.50	3.47	3.45
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.47	3.43	3.39	3.36	3.33	3.30	3.28	3.26
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.32	3.28	3.24	3.21	3.18	3.15	3.13	3.11
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.20	3.15	3.12	3.08	3.05	3.03	3.00	2.98
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.09	3.05	3.01	2.98	2.95	2.92	2.90	2.88
15	6.20	4.76	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	3.01	2.96	2.92	2.89	2.86	2.84	2.81	2.79
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.93	2.89	2.85	2.82	2.79	2.76	2.74	2.72
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.87	2.82	2.79	2.75	2.72	2.70	2.67	2.65
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.81	2.77	2.73	2.70	2.67	2.64	2.62	2.60
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.76	2.72	2.68	2.65	2.62	2.59	2.57	2.55
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.72	2.68	2.64	2.60	2.57	2.55	2.52	2.50
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.68	2.64	2.60	2.56	2.53	2.51	2.48	2.46
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.65	2.60	2.56	2.53	2.50	2.47	2.45	2.43
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.62	2.57	2.53	2.50	2.47	2.44	2.42	2.39
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.59	2.54	2.50	2.47	2.44	2.41	2.39	2.36
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.56	2.51	2.48	2.44	2.41	2.38	2.36	2.34
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.54	2.49	2.45	2.42	2.39	2.36	2.34	2.31
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.51	2.47	2.43	2.39	2.36	2.34	2.31	2.29
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.49	2.45	2.41	2.37	2.34	2.32	2.29	2.27
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.48	2.43	2.39	2.36	2.32	2.30	2.27	2.25
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.46	2.41	2.37	2.34	2.31	2.28	2.26	2.23
32	5.53	4.15	3.56	3.22	3.00	2.84	2.72	2.62	2.54	2.48	2.43	2.38	2.34	2.31	2.28	2.25	2.22	2.20
34	5.50	4.12	3.53	3.19	2.97	2.81	2.69	2.59	2.52	2.45	2.40	2.35	2.31	2.28	2.25	2.22	2.19	2.17
36	5.47	4.09	3.51	3.17	2.94	2.79	2.66	2.57	2.49	2.43	2.37	2.33	2.29	2.25	2.22	2.20	2.17	2.15
38	5.45	4.07	3.48	3.15	2.92	2.76	2.64	2.55	2.47	2.41	2.35	2.31	2.27	2.23	2.20	2.17	2.15	2.13
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.33	2.29	2.25	2.21	2.18	2.15	2.13	2.11
42	5.40	4.03	3.45	3.11	2.89	2.73	2.61	2.51	2.44	2.37	2.32	2.27	2.23	2.20	2.16	2.14	2.11	2.09
44	5.39	4.02	3.43	3.09	2.87	2.71	2.59	2.50	2.42	2.36	2.30	2.26	2.21	2.18	2.15	2.12	2.10	2.07
46	5.37	4.00	3.42	3.08	2.86	2.70	2.58	2.48	2.41	2.34	2.29	2.24	2.20	2.17	2.13	2.11	2.08	2.06
48	5.35	3.99	3.40	3.07	2.84	2.69	2.57	2.47	2.39	2.33	2.27	2.23	2.19	2.15	2.12	2.09	2.07	2.05
50	5.34	3.98	3.39	3.06	2.83	2.67	2.55	2.46	2.38	2.32	2.26	2.22	2.18	2.14	2.11	2.08	2.06	2.03
55	5.31	3.95	3.36	3.03	2.81	2.65	2.53	2.43	2.36	2.29	2.24	2.19	2.15	2.11	2.08	2.05	2.03	2.01
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.01	1.98
65	5.27	3.91	3.32	2.99	2.77	2.61	2.49	2.39	2.32	2.25	2.20	2.15	2.11	2.07	2.04	2.01	1.99	1.97
70	5.25	3.89	3.31	2.98	2.75	2.60	2.48	2.38	2.30	2.24	2.18	2.14	2.10	2.06	2.03	2.00	1.97	1.95
80	5.22	3.86	3.28	2.95	2.73	2.57	2.45	2.36	2.28	2.21	2.16	2.11	2.07	2.03	2.00	1.97	1.95	1.93
90	5.20	3.84	3.27	2.93	2.71	2.55	2.43	2.34	2.26	2.19	2.14	2.09	2.05	2.02	1.98	1.95	1.93	1.91
100	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24	2.18	2.12	2.08	2.04	2.00	1.97	1.94	1.91	1.89
125	5.15	3.80	3.22	2.89	2.67	2.51	2.39	2.30	2.22	2.15	2.10	2.05	2.01	1.97	1.94	1.91	1.89	1.86
150	5.13	3.78	3.20	2.87	2.65	2.49	2.37	2.28	2.20	2.13	2.08	2.03	1.99	1.95	1.92	1.89	1.87	1.84
200	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18	2.11	2.06	2.01	1.97	1.93	1.90	1.87	1.84	1.82
300	5.08	3.74	3.16	2.83	2.61	2.45	2.33	2.23	2.16	2.09	2.04	1.99	1.95	1.91	1.88	1.85	1.82	1.80
500	5.05	3.72	3.14	2.81	2.59	2.43	2.31	2.22	2.14	2.07	2.02	1.97	1.93	1.89	1.86	1.83	1.80	1.78
1000	5.04	3.70	3.13	2.80	2.58	2.42	2.30	2.20	2.13	2.06	2.01	1.96	1.92	1.88	1.85	1.82	1.79	1.77
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.99	1.94	1.90	1.87	1.83	1.80	1.78	1.75

Example: $P\{F_{.025,8,20} < 2.91\} = 97.5\%$.

$F_{.975;\nu_1,\nu_2} = 1/F_{.025;\nu_2,\nu_1}$. Example: $F_{.975,8,20} = 1/F_{.025;20,8} = 1/4.00 = 0.250$.

POINT OF THE F DISTRIBUTION

$F_{.025;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1)

19	20	22	24	26	28	30	35	40	45	50	60	80	100	200	500	∞	Degrees of freedom for the denominator (ν_2)
992	993	995	997	999	1000	1001	1004	1006	1007	1008	1010	1012	1013	1016	1017	1018	1
39.4	39.4	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	2
14.2	14.2	14.1	14.1	14.1	14.1	14.1	14.1	14.0	14.0	14.0	14.0	14.0	14.0	13.9	13.9	13.9	3
8.58	8.56	8.53	8.51	8.49	8.48	8.46	8.44	8.41	8.39	8.38	8.36	8.33	8.32	8.29	8.27	8.26	4
6.35	6.33	6.30	6.28	6.26	6.24	6.23	6.20	6.18	6.16	6.14	6.12	6.10	6.08	6.05	6.03	6.02	5
5.19	5.17	5.14	5.12	5.10	5.08	5.07	5.04	5.01	4.99	4.98	4.96	4.93	4.92	4.88	4.86	4.85	6
4.48	4.47	4.44	4.42	4.39	4.38	4.36	4.33	4.31	4.29	4.28	4.25	4.23	4.21	4.18	4.16	4.14	7
4.02	4.00	3.97	3.95	3.93	3.91	3.89	3.86	3.84	3.82	3.81	3.78	3.76	3.74	3.70	3.68	3.67	8
3.68	3.67	3.64	3.61	3.59	3.58	3.56	3.53	3.51	3.49	3.47	3.45	3.42	3.40	3.37	3.35	3.33	9
3.44	3.42	3.39	3.37	3.34	3.33	3.31	3.28	3.26	3.24	3.22	3.20	3.17	3.15	3.12	3.09	3.08	10
3.24	3.23	3.20	3.17	3.15	3.13	3.12	3.09	3.06	3.04	3.03	3.00	2.97	2.96	2.92	2.90	2.88	11
3.09	3.07	3.04	3.02	3.00	2.98	2.96	2.93	2.91	2.89	2.87	2.85	2.82	2.80	2.76	2.74	2.72	12
2.96	2.95	2.92	2.89	2.87	2.85	2.84	2.80	2.78	2.76	2.74	2.72	2.69	2.67	2.63	2.61	2.60	13
2.86	2.84	2.81	2.79	2.77	2.75	2.73	2.70	2.67	2.65	2.64	2.61	2.58	2.56	2.53	2.50	2.49	14
2.77	2.76	2.73	2.70	2.68	2.66	2.64	2.61	2.58	2.56	2.55	2.52	2.49	2.47	2.44	2.41	2.40	15
2.70	2.68	2.65	2.63	2.60	2.58	2.57	2.53	2.51	2.49	2.47	2.45	2.42	2.40	2.36	2.33	2.32	16
2.63	2.62	2.59	2.56	2.54	2.52	2.50	2.47	2.44	2.42	2.41	2.38	2.35	2.33	2.29	2.26	2.25	17
2.58	2.56	2.53	2.50	2.48	2.46	2.44	2.41	2.38	2.36	2.35	2.32	2.29	2.27	2.23	2.20	2.19	18
2.53	2.51	2.48	2.45	2.43	2.41	2.39	2.36	2.33	2.31	2.30	2.27	2.24	2.22	2.18	2.15	2.13	19
2.48	2.46	2.43	2.41	2.39	2.37	2.35	2.31	2.29	2.27	2.25	2.22	2.19	2.17	2.13	2.10	2.09	20
2.44	2.42	2.39	2.37	2.34	2.33	2.31	2.27	2.25	2.23	2.21	2.18	2.15	2.13	2.09	2.06	2.04	21
2.41	2.39	2.36	2.33	2.31	2.29	2.27	2.24	2.21	2.19	2.17	2.14	2.11	2.09	2.05	2.02	2.00	22
2.37	2.36	2.33	2.30	2.28	2.26	2.24	2.20	2.18	2.15	2.14	2.11	2.08	2.06	2.01	1.99	1.97	23
2.35	2.33	2.30	2.27	2.25	2.23	2.21	2.17	2.15	2.12	2.11	2.08	2.05	2.02	1.98	1.95	1.94	24
2.32	2.30	2.27	2.24	2.22	2.20	2.18	2.15	2.12	2.10	2.08	2.05	2.02	2.00	1.95	1.92	1.91	25
2.29	2.28	2.24	2.22	2.19	2.17	2.16	2.12	2.09	2.07	2.05	2.03	1.99	1.97	1.92	1.90	1.88	26
2.27	2.25	2.22	2.19	2.17	2.15	2.13	2.10	2.07	2.05	2.03	2.00	1.97	1.94	1.90	1.87	1.85	27
2.25	2.23	2.20	2.17	2.15	2.13	2.11	2.08	2.05	2.03	2.01	1.98	1.94	1.92	1.88	1.85	1.83	28
2.23	2.21	2.18	2.15	2.13	2.11	2.09	2.06	2.03	2.01	1.99	1.96	1.92	1.90	1.86	1.83	1.81	29
2.21	2.20	2.16	2.14	2.11	2.09	2.07	2.04	2.01	1.99	1.97	1.94	1.90	1.88	1.84	1.81	1.79	30
2.18	2.16	2.13	2.10	2.08	2.06	2.04	2.00	1.98	1.95	1.93	1.91	1.87	1.85	1.80	1.77	1.75	32
2.15	2.13	2.10	2.07	2.05	2.03	2.01	1.97	1.95	1.92	1.90	1.88	1.84	1.82	1.77	1.74	1.72	34
2.13	2.11	2.08	2.05	2.03	2.00	1.99	1.95	1.92	1.90	1.88	1.85	1.81	1.79	1.74	1.71	1.69	36
2.11	2.09	2.05	2.03	2.00	1.98	1.96	1.93	1.90	1.87	1.85	1.82	1.79	1.76	1.71	1.68	1.66	38
2.09	2.07	2.03	2.01	1.98	1.96	1.94	1.90	1.88	1.85	1.83	1.80	1.76	1.74	1.69	1.66	1.64	40
2.07	2.05	2.02	1.99	1.96	1.94	1.92	1.89	1.86	1.83	1.81	1.78	1.74	1.72	1.67	1.64	1.62	42
2.05	2.03	2.00	1.97	1.95	1.93	1.91	1.87	1.84	1.82	1.80	1.77	1.73	1.70	1.65	1.62	1.60	44
2.04	2.02	1.99	1.96	1.93	1.91	1.89	1.85	1.82	1.80	1.78	1.75	1.71	1.69	1.63	1.60	1.58	46
2.02	2.01	1.97	1.94	1.92	1.90	1.88	1.84	1.81	1.79	1.77	1.73	1.69	1.67	1.62	1.58	1.56	48
2.01	1.99	1.96	1.93	1.91	1.88	1.87	1.83	1.80	1.77	1.75	1.72	1.68	1.66	1.60	1.57	1.55	50
1.99	1.97	1.93	1.90	1.88	1.86	1.84	1.80	1.77	1.74	1.72	1.69	1.65	1.62	1.57	1.54	1.51	55
1.96	1.94	1.91	1.88	1.86	1.83	1.82	1.78	1.74	1.72	1.70	1.67	1.62	1.60	1.54	1.51	1.48	60
1.95	1.93	1.89	1.86	1.84	1.82	1.80	1.76	1.72	1.70	1.68	1.65	1.60	1.58	1.52	1.48	1.46	65
1.93	1.91	1.88	1.85	1.82	1.80	1.78	1.74	1.71	1.68	1.66	1.63	1.58	1.56	1.50	1.46	1.44	70
1.90	1.88	1.85	1.82	1.79	1.77	1.75	1.71	1.68	1.65	1.63	1.60	1.55	1.53	1.47	1.43	1.40	80
1.88	1.86	1.83	1.80	1.77	1.75	1.73	1.69	1.66	1.63	1.61	1.58	1.53	1.50	1.44	1.40	1.37	90
1.87	1.85	1.81	1.78	1.76	1.74	1.71	1.67	1.64	1.61	1.59	1.56	1.51	1.48	1.42	1.38	1.35	100
1.84	1.82	1.79	1.75	1.73	1.71	1.68	1.64	1.61	1.58	1.56	1.52	1.48	1.45	1.38	1.34	1.30	125
1.82	1.80	1.77	1.74	1.71	1.69	1.67	1.62	1.59	1.56	1.54	1.50	1.45	1.42	1.35	1.31	1.27	150
1.80	1.78	1.74	1.71	1.68	1.66	1.64	1.60	1.56	1.53	1.51	1.47	1.42	1.39	1.32	1.27	1.23	200
1.77	1.75	1.72	1.69	1.66	1.64	1.62	1.57	1.54	1.51	1.48	1.45	1.39	1.36	1.28	1.23	1.18	300
1.76	1.74	1.70	1.67	1.64	1.62	1.60	1.55	1.51	1.49	1.46	1.42	1.37	1.34	1.25	1.19	1.14	500
1.74	1.72	1.69	1.65	1.63	1.60	1.58	1.54	1.50	1.47	1.44	1.41	1.35	1.32	1.23	1.16	1.09	1000
1.73	1.71	1.67	1.64	1.61	1.59	1.57	1.52	1.48	1.45	1.43	1.39	1.33	1.30	1.21	1.13	1.00	∞

Approximate formula for ν_1 and ν_2 larger than 30: $\log_{10} F_{.025;\nu_1,\nu_2} \simeq \dfrac{1.7023}{\sqrt{h - 1.14}} - 0.846\left(\dfrac{1}{\nu_1} - \dfrac{1}{\nu_2}\right)$, where $\dfrac{1}{h} = \dfrac{1}{2}\left(\dfrac{1}{\nu_1} + \dfrac{1}{\nu_2}\right)$.

Degrees of freedom for the denominator (ν_2)	Degrees of freedom for the numerator (ν_1) 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	Multiply the numbers of the first row (ν_2 = 1) by 10.																	
1	*405*	*500*	*540*	*563*	*576*	*586*	*593*	*598*	*602*	*606*	*608*	*611*	*613*	*614*	*616*	*617*	*618*	*619*
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	27.1	27.0	26.9	26.9	26.8	26.8	26.8
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.4	14.3	14.2	14.2	14.2	14.1	14.1
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.96	9.89	9.82	9.77	9.72	9.68	9.64	9.61
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72	7.66	7.60	7.56	7.52	7.48	7.45
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47	6.41	6.36	6.31	6.27	6.24	6.21
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67	5.61	5.56	5.52	5.48	5.44	5.41
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11	5.05	5.00	4.96	4.92	4.89	4.86
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71	4.65	4.60	4.56	4.52	4.49	4.46
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40	4.34	4.29	4.25	4.21	4.18	4.15
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16	4.10	4.05	4.01	3.97	3.94	3.91
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96	3.91	3.86	3.82	3.78	3.75	3.72
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.86	3.80	3.75	3.70	3.66	3.62	3.59	3.56
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73	3.67	3.61	3.56	3.52	3.49	3.45	3.42
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55	3.50	3.45	3.41	3.37	3.34	3.31
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52	3.46	3.40	3.35	3.31	3.27	3.24	3.21
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37	3.32	3.27	3.23	3.19	3.16	3.13
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36	3.30	3.24	3.19	3.15	3.12	3.08	3.05
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23	3.18	3.13	3.09	3.05	3.02	2.99
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.24	3.17	3.12	3.07	3.03	2.99	2.96	2.93
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12	3.07	3.02	2.98	2.94	2.91	2.88
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.14	3.07	3.02	2.97	2.93	2.89	2.86	2.83
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03	2.98	2.93	2.89	2.85	2.82	2.79
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	3.06	2.99	2.94	2.89	2.85	2.81	2.78	2.75
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96	2.90	2.86	2.82	2.78	2.74	2.72
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.99	2.93	2.87	2.82	2.78	2.75	2.71	2.68
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90	2.84	2.79	2.75	2.72	2.68	2.65
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.93	2.87	2.81	2.77	2.73	2.69	2.66	2.63
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84	2.79	2.74	2.70	2.66	2.63	2.60
32	7.50	5.34	4.46	3.97	3.65	3.43	3.26	3.13	3.02	2.93	2.86	2.80	2.74	2.70	2.66	2.62	2.58	2.55
34	7.44	5.29	4.42	3.93	3.61	3.39	3.22	3.09	2.98	2.89	2.82	2.76	2.70	2.66	2.62	2.58	2.55	2.51
36	7.40	5.25	4.38	3.89	3.57	3.35	3.18	3.05	2.95	2.86	2.79	2.72	2.67	2.62	2.58	2.54	2.51	2.48
38	7.35	5.21	4.34	3.86	3.54	3.32	3.15	3.02	2.92	2.83	2.75	2.69	2.64	2.59	2.55	2.51	2.48	2.45
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66	2.61	2.56	2.52	2.48	2.45	2.42
42	7.28	5.15	4.29	3.80	3.49	3.27	3.10	2.97	2.86	2.78	2.70	2.64	2.59	2.54	2.50	2.46	2.43	2.40
44	7.25	5.12	4.26	3.78	3.47	3.24	3.08	2.95	2.84	2.75	2.68	2.62	2.56	2.52	2.47	2.44	2.40	2.37
46	7.22	5.10	4.24	3.76	3.44	3.22	3.06	2.93	2.82	2.73	2.66	2.60	2.54	2.50	2.45	2.42	2.38	2.35
48	7.19	5.08	4.22	3.74	3.43	3.20	3.04	2.91	2.80	2.72	2.64	2.58	2.53	2.48	2.44	2.40	2.37	2.33
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.79	2.70	2.63	2.56	2.51	2.46	2.42	2.38	2.35	2.32
55	7.12	5.01	4.16	3.68	3.37	3.15	2.98	2.85	2.75	2.66	2.59	2.53	2.47	2.42	2.38	2.34	2.31	2.28
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50	2.44	2.39	2.35	2.31	2.28	2.25
65	7.04	4.95	4.10	3.62	3.31	3.09	2.93	2.80	2.69	2.61	2.53	2.47	2.42	2.37	2.33	2.29	2.26	2.23
70	7.01	4.92	4.08	3.60	3.29	3.07	2.91	2.78	2.67	2.59	2.51	2.45	2.40	2.35	2.31	2.27	2.23	2.20
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.48	2.42	2.36	2.31	2.27	2.23	2.20	2.17
90	6.93	4.85	4.01	3.54	3.23	3.01	2.84	2.72	2.61	2.52	2.45	2.39	2.33	2.29	2.24	2.21	2.17	2.14
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.43	2.37	2.31	2.26	2.22	2.19	2.15	2.12
125	6.84	4.78	3.94	3.47	3.17	2.95	2.79	2.66	2.55	2.47	2.39	2.33	2.28	2.23	2.19	2.15	2.11	2.08
150	6.81	4.75	3.92	3.45	3.14	2.92	2.76	2.63	2.53	2.44	2.37	2.31	2.25	2.20	2.16	2.12	2.09	2.06
200	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.34	2.27	2.22	2.17	2.13	2.09	2.06	2.02
300	6.72	4.68	3.85	3.38	3.08	2.86	2.70	2.57	2.47	2.38	2.31	2.24	2.19	2.14	2.10	2.06	2.03	1.99
500	6.69	4.65	3.82	3.36	3.05	2.84	2.68	2.55	2.44	2.36	2.28	2.22	2.17	2.12	2.07	2.04	2.00	1.97
1000	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43	2.34	2.27	2.20	2.15	2.10	2.06	2.02	1.98	1.95
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.18	2.13	2.08	2.04	2.00	1.97	1.93

Example: $P\{F_{.01;8,20} < 3.56\} = 99\%$.

$F_{.99;\nu_1,\nu_2} = 1/F_{.01;\nu_2,\nu_1}$. Example: $F_{.99;8,20} = 1/F_{.01;20,8} = 1/5.36 = 0.187$.

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$F_{.01;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1)

Multiply the numbers of the first row ($\nu_2 = 1$) by 10.

19	20	22	24	26	28	30	35	40	45	50	60	80	100	200	500	∞	Degrees of freedom for the denominator (ν_2)
620	*621*	*622*	*623*	*624*	*625*	*626*	*628*	*629*	*630*	*630*	*631*	*633*	*633*	*635*	*636*	*637*	1
99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	2
26.7	26.7	26.6	26.6	26.6	26.5	26.5	26.5	26.4	26.4	26.4	26.3	26.3	26.2	26.2	26.1	26.1	3
14.0	14.0	14.0	13.9	13.9	13.9	13.8	13.8	13.7	13.7	13.7	13.7	13.6	13.6	13.5	13.5	13.5	4
9.58	9.55	9.51	9.47	9.43	9.40	9.38	9.33	9.29	9.26	9.24	9.20	9.16	9.13	9.08	9.04	9.02	5
7.42	7.40	7.35	7.31	7.28	7.25	7.23	7.18	7.14	7.11	7.09	7.06	7.01	6.99	6.93	6.90	6.88	6
6.18	6.16	6.11	6.07	6.04	6.02	5.99	5.94	5.91	5.88	5.86	5.82	5.78	5.75	5.70	5.67	5.65	7
5.38	5.36	5.32	5.28	5.25	5.22	5.20	5.15	5.12	5.09	5.07	5.03	4.99	4.96	4.91	4.88	4.86	8
4.83	4.81	4.77	4.73	4.70	4.67	4.65	4.60	4.57	4.54	4.52	4.48	4.44	4.42	4.36	4.33	4.31	9
4.43	4.41	4.36	4.33	4.30	4.27	4.25	4.20	4.17	4.14	4.12	4.08	4.04	4.01	3.96	3.93	3.91	10
4.12	4.10	4.06	4.02	3.99	3.96	3.94	3.89	3.86	3.83	3.81	3.78	3.73	3.71	3.66	3.62	3.60	11
3.88	3.86	3.82	3.78	3.75	3.72	3.70	3.65	3.62	3.59	3.57	3.54	3.49	3.47	3.41	3.38	3.36	12
3.69	3.66	3.62	3.59	3.56	3.53	3.51	3.46	3.43	3.40	3.38	3.34	3.30	3.27	3.22	3.19	3.17	13
3.53	3.51	3.46	3.43	3.40	3.37	3.35	3.30	3.27	3.24	3.22	3.18	3.14	3.11	3.06	3.03	3.00	14
3.40	3.37	3.33	3.29	3.26	3.24	3.21	3.17	3.13	3.10	3.08	3.05	3.00	2.98	2.92	2.89	2.87	15
3.28	3.26	3.22	3.18	3.15	3.12	3.10	3.05	3.02	2.99	2.97	2.93	2.89	2.86	2.81	2.78	2.75	16
3.18	3.16	3.12	3.08	3.05	3.03	3.00	2.96	2.92	2.89	2.87	2.83	2.79	2.76	2.71	2.68	2.65	17
3.10	3.08	3.03	3.00	2.97	2.94	2.92	2.87	2.84	2.81	2.78	2.75	2.70	2.68	2.62	2.59	2.57	18
3.03	3.00	2.96	2.92	2.89	2.87	2.84	2.80	2.76	2.73	2.71	2.67	2.63	2.60	2.55	2.51	2.49	19
2.96	2.94	2.90	2.86	2.83	2.80	2.78	2.73	2.69	2.67	2.64	2.61	2.56	2.54	2.48	2.44	2.42	20
2.90	2.88	2.84	2.80	2.77	2.74	2.72	2.67	2.64	2.61	2.58	2.55	2.50	2.48	2.42	2.38	2.36	21
2.85	2.83	2.78	2.75	2.72	2.69	2.67	2.62	2.58	2.55	2.53	2.50	2.45	2.42	2.36	2.33	2.31	22
2.80	2.78	2.74	2.70	2.67	2.64	2.62	2.57	2.54	2.51	2.48	2.45	2.40	2.37	2.32	2.28	2.26	23
2.76	2.74	2.70	2.66	2.63	2.60	2.58	2.53	2.49	2.46	2.44	2.40	2.36	2.33	2.27	2.24	2.21	24
2.72	2.70	2.66	2.62	2.59	2.56	2.54	2.49	2.45	2.42	2.40	2.36	2.32	2.29	2.23	2.19	2.17	25
2.69	2.66	2.62	2.58	2.55	2.53	2.50	2.45	2.42	2.39	2.36	2.33	2.28	2.25	2.19	2.16	2.13	26
2.66	2.63	2.59	2.55	2.52	2.49	2.47	2.42	2.38	2.35	2.33	2.29	2.25	2.22	2.16	2.12	2.10	27
2.63	2.60	2.56	2.52	2.49	2.46	2.44	2.39	2.35	2.32	2.30	2.26	2.22	2.19	2.13	2.09	2.06	28
2.60	2.57	2.53	2.49	2.46	2.44	2.41	2.36	2.33	2.30	2.27	2.23	2.19	2.16	2.10	2.06	2.03	29
2.57	2.55	2.51	2.47	2.44	2.41	2.39	2.34	2.30	2.27	2.25	2.21	2.16	2.13	2.07	2.03	2.01	30
2.53	2.50	2.46	2.42	2.39	2.36	2.34	2.29	2.25	2.22	2.20	2.16	2.11	2.08	2.02	1.98	1.96	32
2.49	2.46	2.42	2.38	2.35	2.32	2.30	2.25	2.21	2.18	2.16	2.12	2.07	2.04	1.98	1.94	1.91	34
2.45	2.43	2.38	2.35	2.32	2.29	2.26	2.21	2.17	2.14	2.12	2.08	2.03	2.00	1.94	1.90	1.87	36
2.42	2.40	2.35	2.32	2.28	2.26	2.23	2.18	2.14	2.11	2.09	2.05	2.00	1.97	1.90	1.86	1.84	38
2.39	2.37	2.33	2.29	2.26	2.23	2.20	2.15	2.11	2.08	2.06	2.02	1.97	1.94	1.87	1.83	1.80	40
2.37	2.34	2.30	2.26	2.23	2.20	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.91	1.85	1.80	1.78	42
2.35	2.32	2.28	2.24	2.21	2.18	2.15	2.10	2.06	2.03	2.01	1.97	1.92	1.89	1.82	1.78	1.75	44
2.33	2.30	2.26	2.22	2.19	2.16	2.13	2.08	2.04	2.01	1.99	1.95	1.90	1.86	1.80	1.75	1.73	46
2.31	2.28	2.24	2.20	2.17	2.14	2.12	2.06	2.02	1.99	1.97	1.93	1.88	1.84	1.78	1.73	1.70	48
2.29	2.27	2.22	2.18	2.15	2.12	2.10	2.05	2.01	1.97	1.95	1.91	1.86	1.82	1.76	1.71	1.68	50
2.25	2.23	2.18	2.15	2.11	2.08	2.06	2.01	1.97	1.93	1.91	1.87	1.81	1.78	1.71	1.67	1.64	55
2.22	2.20	2.15	2.12	2.08	2.05	2.03	1.98	1.94	1.90	1.88	1.84	1.78	1.75	1.68	1.63	1.60	60
2.20	2.17	2.13	2.09	2.06	2.03	2.00	1.95	1.91	1.88	1.85	1.81	1.75	1.72	1.65	1.60	1.57	65
2.18	2.15	2.11	2.07	2.03	2.01	1.98	1.93	1.89	1.85	1.83	1.78	1.73	1.70	1.62	1.57	1.54	70
2.14	2.12	2.07	2.03	2.00	1.97	1.94	1.89	1.85	1.81	1.79	1.75	1.69	1.66	1.58	1.53	1.49	80
2.11	2.09	2.04	2.00	1.97	1.94	1.92	1.86	1.82	1.79	1.76	1.72	1.66	1.62	1.54	1.49	1.46	90
2.09	2.07	2.02	1.98	1.94	1.92	1.89	1.84	1.80	1.76	1.73	1.69	1.63	1.60	1.52	1.47	1.43	100
2.05	2.03	1.98	1.94	1.91	1.88	1.85	1.80	1.76	1.72	1.69	1.65	1.59	1.55	1.47	1.41	1.37	125
2.03	2.00	1.96	1.92	1.88	1.85	1.83	1.77	1.73	1.69	1.66	1.62	1.56	1.52	1.43	1.38	1.33	150
2.00	1.97	1.93	1.89	1.85	1.82	1.79	1.74	1.69	1.66	1.63	1.58	1.52	1.48	1.39	1.33	1.28	200
1.97	1.94	1.89	1.85	1.82	1.79	1.76	1.71	1.66	1.62	1.59	1.55	1.48	1.44	1.35	1.28	1.22	300
1.94	1.92	1.87	1.83	1.79	1.76	1.74	1.68	1.63	1.60	1.56	1.52	1.45	1.41	1.31	1.23	1.16	500
1.92	1.90	1.85	1.81	1.77	1.74	1.72	1.66	1.61	1.57	1.54	1.50	1.43	1.38	1.28	1.19	1.11	1000
1.90	1.88	1.83	1.79	1.76	1.72	1.70	1.64	1.59	1.55	1.52	1.47	1.40	1.36	1.25	1.15	1.00	∞

Approximate formula for ν_1 and ν_2 larger than 30: $\log_{10} F_{.01;\nu_1,\nu_2} \simeq \dfrac{2.0206}{\sqrt{h} - 1.40} - 1.073\left(\dfrac{1}{\nu_1} - \dfrac{1}{\nu_2}\right)$, where $\dfrac{1}{h} = \dfrac{1}{2}\left(\dfrac{1}{\nu_1} + \dfrac{1}{\nu_2}\right)$.

Degrees of freedom for the numerator (ν_1)

Multiply the numbers of the first row ($\nu_1 = 1$) by 100.

Degrees of freedom for the denominator (ν_2)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	*162*	*200*	*216*	*225*	*231*	*234*	*237*	*239*	*241*	*242*	*243*	*244*	*245*	*246*	*246*	*247*	*247*	*248*
2	198	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199
3	55.6	49.8	47.5	46.2	45.4	44.8	44.4	44.1	43.9	43.7	43.5	43.4	43.3	43.2	43.1	43.0	42.9	42.9
4	31.3	26.3	24.3	23.2	22.5	22.0	21.6	21.4	21.1	21.0	20.8	20.7	20.6	20.5	20.4	20.4	20.3	20.3
5	22.8	18.3	16.5	15.6	14.9	14.5	14.2	14.0	13.8	13.6	13.5	13.4	13.3	13.2	13.1	13.1	13.0	13.0
6	18.6	14.5	12.9	12.0	11.5	11.1	10.8	10.6	10.4	10.2	10.1	10.0	9.95	9.88	9.81	9.76	9.71	9.66
7	16.2	12.4	10.9	10.0	9.52	9.16	8.89	8.68	8.51	8.38	8.27	8.18	8.10	8.03	7.97	7.93	7.87	7.83
8	14.7	11.0	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.10	7.01	6.94	6.87	6.81	6.76	6.72	6.68
9	13.6	10.1	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.31	6.23	6.15	6.09	6.03	5.98	5.94	5.90
10	12.8	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.75	5.66	5.59	5.53	5.47	5.42	5.38	5.34
11	12.2	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.32	5.24	5.16	5.10	5.05	5.00	4.96	4.92
12	11.8	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.99	4.91	4.84	4.77	4.72	4.67	4.63	4.59
13	11.4	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.72	4.64	4.57	4.51	4.46	4.41	4.37	4.33
14	11.1	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.51	4.43	4.36	4.30	4.25	4.20	4.16	4.12
15	10.8	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.33	4.25	4.18	4.12	4.07	4.02	3.98	3.95
16	10.6	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.18	4.10	4.03	3.97	3.92	3.87	3.83	3.80
17	10.4	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	4.05	3.97	3.90	3.84	3.79	3.75	3.71	3.67
18	10.2	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.94	3.86	3.79	3.73	3.68	3.64	3.60	3.56
19	10.1	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.84	3.76	3.70	3.64	3.59	3.54	3.50	3.46
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.76	3.68	3.61	3.55	3.50	3.46	3.42	3.38
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.68	3.60	3.54	3.48	3.43	3.38	3.34	3.31
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.61	3.54	3.47	3.41	3.36	3.31	3.27	3.24
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.55	3.47	3.41	3.35	3.30	3.25	3.21	3.18
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.50	3.42	3.35	3.30	3.25	3.20	3.16	3.12
25	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64	3.54	3.45	3.37	3.30	3.25	3.20	3.15	3.11	3.08
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.40	3.33	3.26	3.20	3.15	3.11	3.07	3.03
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.36	3.28	3.22	3.16	3.11	3.07	3.03	2.99
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.32	3.25	3.18	3.12	3.07	3.03	2.99	2.95
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.29	3.21	3.15	3.09	3.04	2.99	2.95	2.92
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.25	3.18	3.11	3.06	3.01	2.96	2.92	2.89
32	9.09	6.28	5.17	4.56	4.17	3.89	3.68	3.52	3.39	3.29	3.20	3.12	3.06	3.00	2.95	2.90	2.86	2.83
34	9.01	6.22	5.11	4.50	4.11	3.84	3.63	3.47	3.34	3.24	3.15	3.07	3.01	2.95	2.90	2.85	2.81	2.78
36	8.94	6.16	5.06	4.46	4.06	3.79	3.58	3.42	3.30	3.19	3.10	3.03	2.96	2.90	2.85	2.81	2.77	2.73
38	8.88	6.11	5.02	4.41	4.02	3.75	3.54	3.39	3.25	3.15	3.06	2.99	2.92	2.87	2.82	2.77	2.73	2.70
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	3.03	2.95	2.89	2.83	2.78	2.74	2.70	2.66
42	8.78	6.03	4.94	4.34	3.95	3.68	3.48	3.32	3.19	3.09	3.00	2.92	2.86	2.80	2.75	2.71	2.67	2.63
44	8.74	5.99	4.91	4.31	3.92	3.65	3.45	3.29	3.16	3.06	2.97	2.89	2.83	2.77	2.72	2.68	2.64	2.60
46	8.70	5.96	4.88	4.28	3.90	3.62	3.42	3.26	3.14	3.03	2.94	2.87	2.80	2.75	2.70	2.65	2.61	2.58
48	8.66	5.93	4.85	4.25	3.87	3.60	3.40	3.24	3.11	3.01	2.92	2.85	2.78	2.72	2.67	2.63	2.59	2.55
50	8.63	5.90	4.83	4.23	3.85	3.58	3.38	3.22	3.09	2.99	2.90	2.82	2.76	2.70	2.65	2.61	2.57	2.53
55	8.55	5.84	4.77	4.18	3.80	3.53	3.33	3.17	3.05	2.94	2.85	2.78	2.71	2.66	2.61	2.56	2.52	2.49
60	8.49	5.80	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.82	2.74	2.68	2.62	2.57	2.53	2.49	2.45
65	8.44	5.75	4.68	4.11	3.73	3.46	3.26	3.10	2.98	2.87	2.79	2.71	2.65	2.59	2.54	2.49	2.45	2.42
70	8.40	5.72	4.65	4.08	3.70	3.43	3.23	3.08	2.95	2.85	2.76	2.68	2.62	2.56	2.51	2.47	2.43	2.39
80	8.33	5.67	4.61	4.03	3.65	3.39	3.19	3.03	2.91	2.80	2.72	2.64	2.58	2.52	2.47	2.43	2.39	2.35
90	8.28	5.62	4.57	3.99	3.62	3.35	3.15	3.00	2.87	2.77	2.68	2.61	2.54	2.49	2.44	2.39	2.35	2.32
100	8.24	5.59	4.54	3.96	3.59	3.33	3.13	2.97	2.85	2.74	2.66	2.58	2.52	2.46	2.41	2.37	2.33	2.29
125	8.17	5.53	4.49	3.91	3.54	3.28	3.08	2.93	2.80	2.70	2.61	2.54	2.47	2.42	2.37	2.32	2.28	2.24
150	8.12	5.49	4.45	3.88	3.51	3.25	3.05	2.89	2.77	2.67	2.58	2.51	2.44	2.38	2.33	2.29	2.25	2.21
200	8.06	5.44	4.41	3.84	3.47	3.21	3.01	2.85	2.73	2.63	2.54	2.47	2.40	2.35	2.30	2.25	2.21	2.18
300	8.00	5.39	4.37	3.80	3.43	3.17	2.97	2.81	2.69	2.59	2.51	2.43	2.37	2.31	2.26	2.21	2.17	2.14
500	7.95	5.36	4.33	3.76	3.40	3.14	2.94	2.79	2.66	2.56	2.48	2.40	2.34	2.28	2.23	2.19	2.14	2.11
1000	7.92	5.33	4.31	3.74	3.37	3.11	2.92	2.77	2.64	2.54	2.45	2.38	2.32	2.26	2.21	2.16	2.12	2.09
∞	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.43	2.36	2.29	2.24	2.19	2.14	2.10	2.06

Example: $P\{F_{.005;8,20} < 4.09\} = 99.5\%$.

$F_{.995;\nu_1,\nu_2} = 1/F_{.005;\nu_2,\nu_1}$. Example: $F_{.995;8,20} = 1/F_{.005,20,8} = 1/6.61 = 0.151$.

POINT OF THE F DISTRIBUTION

$F_{.005;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1)

Multiply the numbers of the first row ($\nu_2 = 1$) by 100.

19	20	22	24	26	28	30	35	40	45	50	60	80	100	200	500	∞	Degrees of freedom for the denominator (ν_2)
248	*248*	*249*	*249*	*250*	*250*	*250*	*251*	*251*	*252*	*252*	*253*	*253*	*253*	*254*	*254*	*255*	1
199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	200	200	2
42.8	42.8	42.7	42.6	42.6	42.5	42.5	42.4	42.3	42.3	42.2	42.1	42.1	42.0	41.9	41.9	41.8	3
20.2	20.2	20.1	20.0	20.0	19.9	19.9	19.8	19.8	19.7	19.7	19.6	19.5	19.5	19.4	19.4	19.3	4
12.9	12.9	12.8	12.8	12.7	12.7	12.7	12.6	12.5	12.5	12.5	12.4	12.3	12.3	12.2	12.2	12.1	5
9.62	9.59	9.53	9.47	9.43	9.39	9.36	9.29	9.24	9.20	9.17	9.12	9.06	9.03	8.95	8.91	8.88	6
7.79	7.75	7.69	7.64	7.60	7.57	7.53	7.47	7.42	7.38	7.35	7.31	7.25	7.22	7.15	7.10	7.08	7
6.64	6.61	6.55	6.50	6.46	6.43	6.40	6.33	6.29	6.25	6.22	6.18	6.12	6.09	6.02	5.98	5.95	8
5.86	5.83	5.78	5.73	5.69	5.65	5.62	5.56	5.52	5.48	5.45	5.41	5.36	5.32	5.26	5.21	5.19	9
5.30	5.27	5.22	5.17	5.13	5.10	5.07	5.01	4.97	4.93	4.90	4.86	4.80	4.77	4.71	4.67	4.64	10
4.89	4.86	4.80	4.76	4.72	4.68	4.65	4.60	4.55	4.52	4.49	4.44	4.39	4.36	4.29	4.25	4.23	11
4.56	4.53	4.48	4.43	4.39	4.36	4.33	4.27	4.23	4.19	4.17	4.12	4.07	4.04	3.97	3.93	3.90	12
4.30	4.27	4.22	4.17	4.13	4.10	4.07	4.01	3.97	3.94	3.91	3.87	3.81	3.78	3.71	3.67	3.65	13
4.09	4.06	4.01	3.96	3.92	3.89	3.86	3.80	3.76	3.73	3.70	3.66	3.60	3.57	3.50	3.46	3.44	14
3.91	3.88	3.83	3.79	3.75	3.72	3.69	3.63	3.58	3.55	3.52	3.48	3.43	3.39	3.33	3.29	3.26	15
3.76	3.73	3.68	3.64	3.60	3.57	3.54	3.48	3.44	3.40	3.37	3.33	3.28	3.25	3.18	3.14	3.11	16
3.64	3.61	3.56	3.51	3.47	3.44	3.41	3.35	3.31	3.28	3.25	3.21	3.15	3.12	3.05	3.01	2.98	17
3.53	3.50	3.45	3.40	3.36	3.33	3.30	3.25	3.20	3.17	3.14	3.10	3.04	3.01	2.94	2.90	2.87	18
3.43	3.40	3.35	3.31	3.27	3.24	3.21	3.15	3.11	3.07	3.04	3.00	2.95	2.91	2.85	2.80	2.78	19
3.35	3.32	3.27	3.22	3.18	3.15	3.12	3.07	3.02	2.99	2.96	2.92	2.86	2.83	2.76	2.72	2.69	20
3.27	3.24	3.19	3.15	3.11	3.08	3.05	2.99	2.95	2.91	2.88	2.84	2.78	2.75	2.68	2.64	2.61	21
3.20	3.18	3.12	3.08	3.04	3.01	2.98	2.92	2.88	2.84	2.82	2.77	2.72	2.69	2.62	2.57	2.55	22
3.15	3.12	3.06	3.02	2.98	2.95	2.92	2.86	2.82	2.78	2.76	2.71	2.66	2.62	2.56	2.51	2.48	23
3.09	3.06	3.01	2.97	2.93	2.90	2.87	2.81	2.77	2.73	2.70	2.66	2.60	2.57	2.50	2.46	2.43	24
3.04	3.01	2.96	2.92	2.88	2.85	2.82	2.76	2.72	2.68	2.65	2.61	2.55	2.52	2.45	2.41	2.38	25
3.00	2.97	2.92	2.87	2.83	2.80	2.77	2.72	2.67	2.64	2.61	2.56	2.51	2.47	2.40	2.36	2.33	26
2.96	2.93	2.88	2.83	2.79	2.76	2.73	2.67	2.63	2.59	2.57	2.52	2.47	2.43	2.36	2.32	2.29	27
2.92	2.89	2.84	2.79	2.76	2.72	2.69	2.64	2.59	2.56	2.53	2.48	2.43	2.39	2.32	2.28	2.25	28
2.88	2.86	2.80	2.76	2.72	2.69	2.66	2.60	2.56	2.52	2.49	2.45	2.39	2.36	2.28	2.24	2.21	29
2.85	2.82	2.77	2.73	2.69	2.66	2.63	2.57	2.52	2.49	2.46	2.42	2.36	2.32	2.25	2.21	2.18	30
2.80	2.77	2.71	2.67	2.63	2.60	2.57	2.51	2.47	2.43	2.40	2.36	2.30	2.26	2.19	2.15	2.11	32
2.75	2.72	2.66	2.62	2.58	2.55	2.52	2.46	2.42	2.38	2.35	2.30	2.25	2.21	2.14	2.09	2.06	34
2.70	2.67	2.62	2.58	2.54	2.50	2.48	2.42	2.37	2.33	2.30	2.26	2.20	2.17	2.09	2.04	2.01	36
2.66	2.63	2.58	2.54	2.50	2.47	2.44	2.38	2.33	2.29	2.27	2.22	2.16	2.12	2.05	2.00	1.97	38
2.63	2.60	2.55	2.50	2.46	2.43	2.40	2.34	2.30	2.26	2.23	2.18	2.12	2.09	2.01	1.96	1.93	40
2.60	2.57	2.52	2.47	2.43	2.40	2.37	2.31	2.26	2.23	2.20	2.15	2.09	2.06	1.98	1.93	1.90	42
2.57	2.54	2.49	2.44	2.40	2.37	2.34	2.28	2.24	2.20	2.17	2.12	2.06	2.03	1.95	1.90	1.87	44
2.54	2.51	2.46	2.42	2.38	2.34	2.32	2.26	2.21	2.17	2.14	2.10	2.04	2.00	1.92	1.87	1.84	46
2.52	2.49	2.44	2.39	2.36	2.32	2.29	2.23	2.19	2.15	2.12	2.07	2.01	1.97	1.89	1.84	1.81	48
2.50	2.47	2.42	2.37	2.33	2.30	2.27	2.21	2.16	2.13	2.10	2.05	1.99	1.95	1.87	1.82	1.79	50
2.45	2.42	2.37	2.33	2.29	2.26	2.23	2.16	2.12	2.08	2.05	2.00	1.94	1.90	1.82	1.77	1.73	55
2.42	2.39	2.33	2.29	2.25	2.22	2.19	2.13	2.08	2.04	2.01	1.96	1.90	1.86	1.78	1.73	1.69	60
2.39	2.36	2.30	2.26	2.22	2.19	2.16	2.09	2.05	2.01	1.98	1.93	1.87	1.83	1.74	1.69	1.65	65
2.36	2.33	2.28	2.23	2.19	2.16	2.13	2.07	2.02	1.98	1.95	1.90	1.84	1.80	1.71	1.66	1.62	70
2.32	2.29	2.23	2.19	2.15	2.11	2.08	2.02	1.97	1.93	1.90	1.85	1.79	1.75	1.66	1.60	1.56	80
2.28	2.25	2.20	2.15	2.12	2.08	2.05	1.99	1.94	1.90	1.87	1.82	1.75	1.71	1.62	1.56	1.52	90
2.26	2.23	2.17	2.13	2.09	2.05	2.02	1.96	1.91	1.87	1.84	1.79	1.72	1.68	1.59	1.53	1.49	100
2.21	2.18	2.13	2.08	2.04	2.01	1.98	1.91	1.86	1.82	1.79	1.74	1.67	1.63	1.53	1.47	1.42	125
2.18	2.15	2.10	2.05	2.01	1.98	1.94	1.88	1.83	1.79	1.76	1.70	1.63	1.59	1.49	1.42	1.37	150
2.14	2.11	2.06	2.01	1.97	1.94	1.91	1.84	1.79	1.75	1.71	1.66	1.59	1.54	1.44	1.37	1.31	200
2.10	2.07	2.02	1.97	1.93	1.90	1.87	1.80	1.75	1.71	1.67	1.61	1.54	1.50	1.39	1.31	1.25	300
2.07	2.04	1.99	1.94	1.90	1.87	1.84	1.77	1.72	1.67	1.64	1.58	1.51	1.46	1.35	1.26	1.18	500
2.05	2.02	1.97	1.92	1.88	1.84	1.81	1.75	1.69	1.65	1.61	1.56	1.48	1.43	1.31	1.22	1.13	1000
2.03	2.00	1.95	1.90	1.86	1.82	1.79	1.72	1.67	1.63	1.59	1.53	1.45	1.40	1.28	1.17	1.00	∞

Approximate formula for ν_1 and ν_2 larger than 30: $\log_{10} F_{.005;\nu_1,\nu_2} \simeq \dfrac{2.2373}{\sqrt{h} - 1.61} - 1.250\left(\dfrac{1}{\nu_1} - \dfrac{1}{\nu_2}\right)$, where $\dfrac{1}{h} = \dfrac{1}{2}\left(\dfrac{1}{\nu_1} + \dfrac{1}{\nu_2}\right)$.

TABLE 13.27 (*Continued*) 0.1 PERCENTAGE POINT OF THE F DISTRIBUTION

Table of $F_{.001;\nu_1,\nu_2}$

Degrees of freedom for the numerator (ν_1)

Multiply the numbers of the first row ($\nu_2 = 1$) by 1000.

Degrees of freedom for the denominator (ν_2)	1	2	3	4	5	6	7	8	9	10	15	20	30	50	100	200	500	∞
1	*405*	*500*	*540*	*562*	*576*	*586*	*593*	*598*	*602*	*606*	*616*	*621*	*626*	*630*	*633*	*635*	*636*	*637*
2	998	999	999	999	999	999	999	999	999	999	999	999	999	999	999	999	999	999
3	168	148	141	137	135	133	132	131	130	129	127	126	125	125	124	124	124	124
4	74.1	61.2	56.2	53.4	51.7	50.5	49.7	49.0	48.5	48.0	46.8	46.1	45.4	44.9	44.5	44.3	44.1	44.0
5	47.0	36.6	33.2	31.1	29.8	28.8	28.2	27.6	27.2	26.9	25.9	25.4	24.9	24.4	24.1	23.9	23.8	23.8
6	35.5	27.0	23.7	21.9	20.8	20.0	19.5	19.0	18.7	18.4	17.6	17.1	16.7	16.3	16.0	15.9	15.8	15.8
7	29.2	21.7	18.8	17.2	16.2	15.5	15.0	14.6	14.3	14.1	13.3	12.9	12.5	12.2	11.9	11.8	11.7	11.7
8	25.4	18.5	15.8	14.4	13.5	12.9	12.4	12.0	11.8	11.5	10.8	10.5	10.1	9.80	9.57	9.46	9.39	9.34
9	22.9	16.4	13.9	12.6	11.7	11.1	10.7	10.4	10.1	9.89	9.24	8.90	8.55	8.26	8.04	7.93	7.86	7.81
10	21.0	14.9	12.6	11.3	10.5	9.92	9.52	9.20	8.96	8.75	8.13	7.80	7.47	7.19	6.98	6.87	6.81	6.76
11	19.7	13.8	11.6	10.4	9.58	9.05	8.66	8.35	8.12	7.92	7.32	7.01	6.68	6.41	6.21	6.10	6.04	6.00
12	18.6	13.0	10.8	9.63	8.89	8.38	8.00	7.71	7.48	7.29	6.71	6.40	6.09	5.83	5.63	5.52	5.46	5.42
13	17.8	12.3	10.2	9.07	8.35	7.86	7.49	7.21	6.98	6.80	6.23	5.93	5.62	5.37	5.17	5.07	5.01	4.97
14	17.1	11.8	9.73	8.62	7.92	7.43	7.08	6.80	6.58	6.40	5.85	5.56	5.25	5.00	4.80	4.70	4.64	4.60
15	16.6	11.3	9.34	8.25	7.57	7.09	6.74	6.47	6.26	6.08	5.53	5.25	4.95	4.70	4.51	4.41	4.35	4.31
16	16.1	11.0	9.00	7.94	7.27	6.81	6.46	6.19	5.98	5.81	5.27	4.99	4.70	4.45	4.26	4.16	4.10	4.06
17	15.7	10.7	8.73	7.68	7.02	6.56	6.22	5.96	5.75	5.58	5.05	4.78	4.48	4.24	4.05	3.95	3.89	3.85
18	15.4	10.4	8.49	7.46	6.81	6.35	6.02	5.76	5.56	5.39	4.87	4.59	4.30	4.06	3.87	3.77	3.71	3.67
19	15.1	10.2	8.28	7.26	6.61	6.18	5.84	5.59	5.39	5.22	4.70	4.43	4.14	3.90	3.71	3.61	3.55	3.51
20	14.8	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24	5.08	4.56	4.29	4.01	3.77	3.58	3.48	3.42	3.38
22	14.4	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99	4.83	4.32	4.06	3.77	3.53	3.34	3.25	3.19	3.15
24	14.0	9.34	7.55	6.59	5.98	5.55	5.23	4.99	4.80	4.64	4.14	3.87	3.59	3.35	3.16	3.07	3.01	2.97
26	13.7	9.12	7.36	6.41	5.80	5.38	5.07	4.83	4.64	4.48	3.99	3.72	3.45	3.20	3.01	2.92	2.86	2.82
28	13.5	8.93	7.19	6.25	5.66	5.24	4.93	4.69	4.50	4.35	3.86	3.60	3.32	3.08	2.89	2.79	2.73	2.70
30	13.3	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39	4.24	3.75	3.49	3.22	2.98	2.79	2.69	2.63	2.59
40	12.6	8.25	6.60	5.70	5.13	4.73	4.43	4.21	4.02	3.87	3.40	3.15	2.87	2.64	2.44	2.34	2.28	2.23
50	12.2	7.95	6.34	5.46	4.90	4.51	4.22	4.00	3.82	3.67	3.20	2.95	2.68	2.44	2.24	2.14	2.07	2.03
60	12.0	7.76	6.17	5.31	4.76	4.37	4.09	3.87	3.69	3.54	3.08	2.83	2.56	2.31	2.11	2.01	1.93	1.89
80	11.7	7.54	5.97	5.13	4.58	4.21	3.92	3.70	3.53	3.39	2.93	2.68	2.40	2.16	1.95	1.84	1.77	1.72
100	11.5	7.41	5.85	5.01	4.48	4.11	3.83	3.61	3.44	3.30	2.84	2.59	2.32	2.07	1.87	1.75	1.68	1.62
200	11.2	7.15	5.64	4.81	4.29	3.92	3.65	3.43	3.26	3.12	2.67	2.42	2.15	1.90	1.68	1.55	1.46	1.39
500	11.0	7.01	5.51	4.69	4.18	3.82	3.54	3.33	3.16	3.02	2.58	2.33	2.05	1.80	1.57	1.43	1.32	1.23
∞	10.8	6.91	5.42	4.62	4.10	3.74	3.47	3.27	3.10	2.96	2.51	2.27	1.99	1.73	1.49	1.34	1.21	1.00

Example: $P\{F_{.001;8,20} < 5.44\} = 99.9\%$.

$F_{.999;\nu_1,\nu_2} = 1/F_{.001;\nu_2,\nu_1}$. Example: $F_{.999,8,20} = 1/F_{.001;20,8} = 1/10.5 = 0.095$.

Approximate formula for ν_1 and ν_2 larger than 30: $\log_{10} F_{.001;\nu_1,\nu_2} \simeq \dfrac{2.6841}{\sqrt{h - 2.09}} - 1.672\left(\dfrac{1}{\nu_1} - \dfrac{1}{\nu_2}\right)$, where $\dfrac{1}{h} = \dfrac{1}{2}\left(\dfrac{1}{\nu_1} + \dfrac{1}{\nu_2}\right)$.

4.11 PROBLEMS OF ESTIMATION: CONFIDENCE INTERVALS

A common statistical problem is to estimate parameters on the basis of a sample. A single number (e.g., a sample mean) is occasionally adequate but usually it is desirable to give an indication of the reliability of these estimates and present an interval that may be said to be "likely" to contain the true value. More precisely, an interval is said to be a confidence interval of size $(1 - \alpha)$ if in the long run $100(1 - \alpha)\%$ of intervals constructed in this way contain the true value.

For example, suppose we are interested in estimating the mean breaking strength of 1040 carbon steel; ten specimens are tested. One possibility would

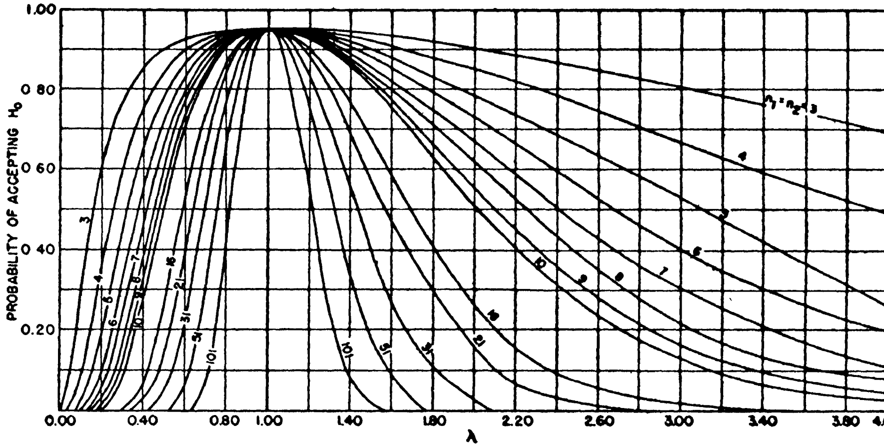
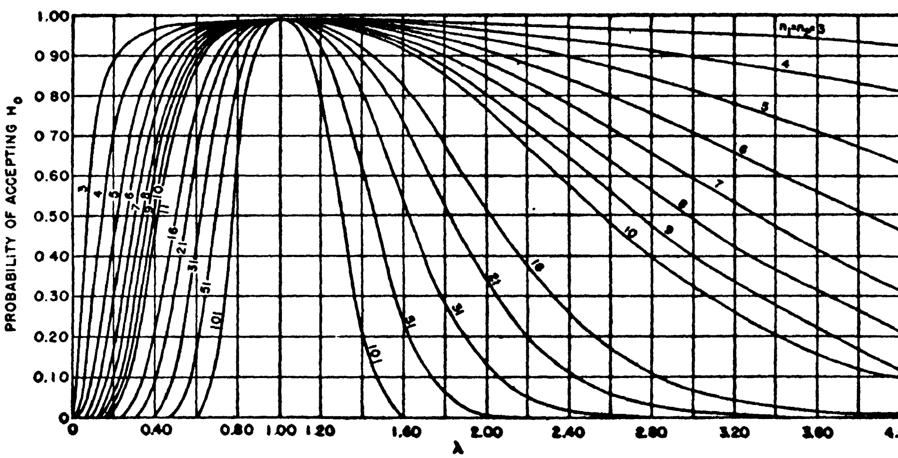
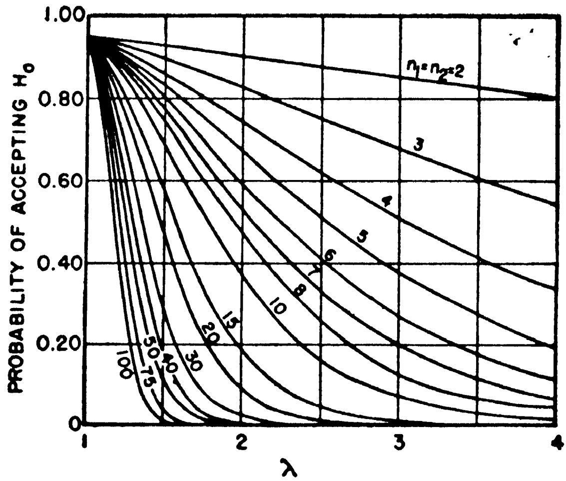


FIG. **13.39** OPERATING CHARACTERISTICS OF THE TWO-SIDED F TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.05.

FIG. **13.40** OPERATING CHARACTERISTICS OF THE TWO-SIDED F TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.01.





Reproduced by permission from "Operating Characteristics for the Common Statistical Tests of Significance" by Charles D. Ferris, Frank E. Grubbs, Chalmers L. Weaver, *Annals of Mathematical Statistics*, June, 1946.

FIG. **13.41** OPERATING CHARACTERISTICS OF THE ONE-SIDED F TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.05.

FIG. **13.42** OPERATING CHARACTERISTICS OF THE ONE-SIDED F TEST FOR A LEVEL OF SIGNIFICANCE EQUAL TO 0.01.

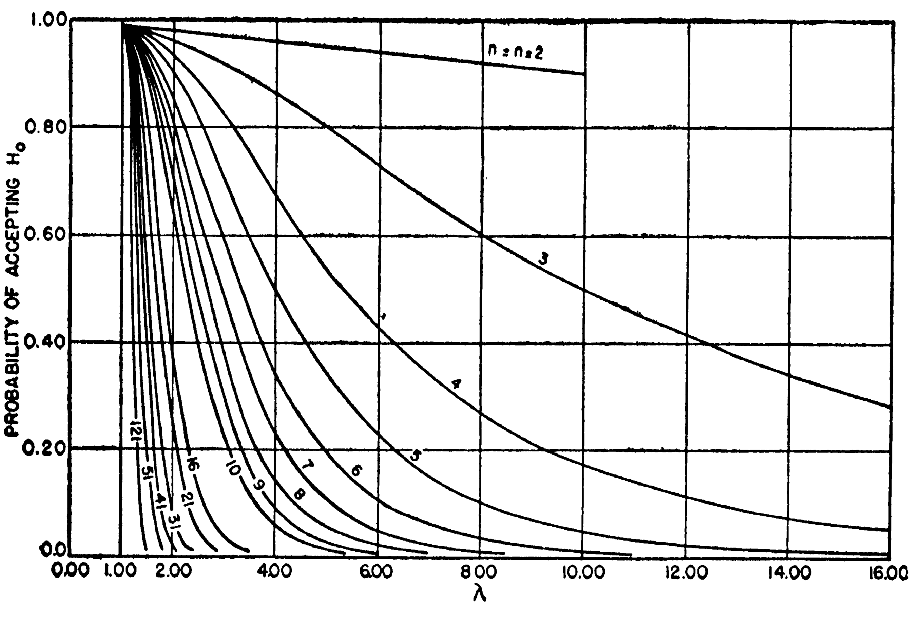


TABLE **13.28** SUMMARY OF ESTIMATES BY CONFIDENCE INTERVALS AND BY SINGLE POINTS

Parameters	Notation	Qualifying conditions	Formula	Tables to find percentage points	Point estimates
Mean of a normal distribution	μ	known σ	$\bar{x} - K_{\alpha/2}(\sigma/\sqrt{n}) \leq \mu \leq \bar{x} + K_{\alpha/2}(\sigma/\sqrt{n})$	Normal table, Table 13.3	$\bar{x}$
Standard deviation of a normal distribution	σ		$s\sqrt{\dfrac{n-1}{\chi^2{}_{\alpha/2;n-1}}} \leq \sigma \leq s\sqrt{\dfrac{n-1}{\chi^2{}_{1-\alpha/2;n-1}}}$	Chi-square table, Table 13.26	s
Mean of a normal distribution	μ	unknown σ	$\bar{x} - (t_{\alpha/2;n-1})\dfrac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + (t_{\alpha/2;n-1})\dfrac{s}{\sqrt{n}}$	t tables, Table 13.25	$\bar{x}$
Difference between the means of two normal distributions	$\Delta = \mu_x - \mu_y$	$x_1, \ldots, x_{n_x}$ are observations in first sample; $y_1, \ldots, y_{n_y}$ are observations in second sample.	$\bar{x} - \bar{y} - K_{\alpha/2}\sqrt{\left(\dfrac{\sigma_x{}^2}{n_x} + \dfrac{\sigma_y{}^2}{n_y}\right)} \leq \Delta \leq \bar{x} - \bar{y} + K_{\alpha/2}\sqrt{\left(\dfrac{\sigma_x{}^2}{n_x} + \dfrac{\sigma_y{}^2}{n_y}\right)}$		$\bar{x} - \bar{y}$
Difference between the means of two normal distributions	$\Delta = \mu_x - \mu_y$	$\sigma_x = \sigma_y = \sigma$ unknown σ	$(\bar{x} - \bar{y}) - t_{\alpha/2;n_x+n_y-2}\sqrt{\dfrac{1}{n_x} + \dfrac{1}{n_y}}\sqrt{\dfrac{\sum(x_i - \bar{x})^2 + \sum(y_i - \bar{y})^2}{n_x + n_y - 2}}$ $\leq \Delta \leq (\bar{x} + \bar{y}) + [t_{\alpha/2;n_x+n_y-2}]$ $\left[\sqrt{\dfrac{1}{n_x} + \dfrac{1}{n_y}}\sqrt{\dfrac{\sum(x_i - \bar{x})^2 + \sum(y_i - \bar{y})^2}{n_x + n_y - 2}}\right]$		$\bar{x} - \bar{y}$

be to present the average of these numbers as an estimate of the mean breaking strength, and this is the best estimate if a single number is desired. However, this single number may by chance depart substantially from the true mean value, and no indication of the magnitude of the departure is available. For many purposes, a more useful estimate is an interval which contains the true value with high probability.

Procedures for calculating confidence intervals most likely to arise in practical work are given in Table 13.28; these intervals usually involve percentage points of the standard distributions that are presented in Tables 13.3, 13.25, and 13.26. The notation for the percentage points is explained in Table 13.28.

5. CURVE FITTING

5.1 INTRODUCTION

It is common practice for engineers to represent relationships by means of graphs. Although these graphs are depicted as smooth curves, rarely do the experimental points lie on the fitted curves. This is usually explained by the fact that the experimental points are chance variables. The usual procedures are to plot the points and (1) connect all the points as in Fig. 13.43; (2) fair the best line in by eye as in Fig. 13.44; (3) fair the best curve in by eye as in Fig. 13.45.

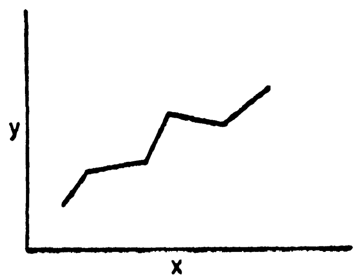


FIG. 13.43 RELATION BETWEEN x AND y OBTAINED BY CONNECTING POINTS.

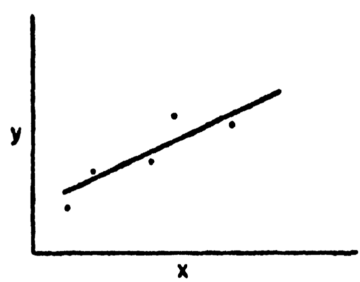


FIG. 13.44 RELATION BETWEEN x AND y OBTAINED BY DRAWING IN BEST LINE BY EYE.

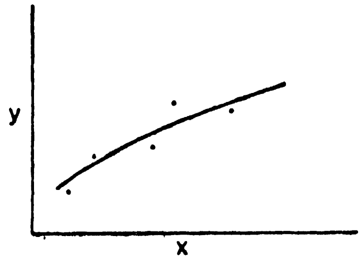


FIG. 13.45 RELATION BETWEEN x AND y OBTAINED BY DRAWING IN BEST CURVE BY EYE.

It is useful to discuss these three methods when there is a priori knowledge that the plotted relation is linear and when the plotted relation is curvilinear.

If there is an underlying linear relationship, procedure (1) has serious implications. In the first place, the problem is really to obtain the best estimate of the true underlying relationship (the word estimate is used because on the basis of a sample the true relationship is never obtained). The procedure pictured in Fig. 13.43 does not represent a linear relationship, and it is intuitively evident that the best estimate of a straight line should be a straight line. Furthermore, the performance of another experiment with the same variables would lead to a completely different graph. Very little confidence can be placed in curves that will differ radically with the performance of another experiment using the same variables. Frequently, graphs are used for prediction purposes, and no statements about the chances of being right or wrong can be made if procedure (1) is used.

The second procedure depicted in Fig. 13.44 is probably most commonly used. This has two serious disadvantages. Two people confronted with the same data will draw different lines through the points, so that there is not a single interpretation of the data. If these graphs are used for prediction purposes, the best point estimates based on the faired line will differ for different people. Furthermore, since fairing in a line is not an objective procedure, probability statements about the slope and intercept cannot be made.

The third procedure depicted in Fig. 13.45 can be dismissed for essentially the same reasons as those for procedures (1) and (2). In addition, it is again intuitively evident that the best estimate of a true linear relationship should be a straight line.

If the underlying relationship is curvilinear, the three procedures can be dismissed for essentially the same reasons as discussed above.

The method commonly used for fitting linear and curvilinear relations is the method of least squares. Besides the advantage of giving a unique solution, i.e., a single line, this procedure has several optimum statistical properties. Furthermore, with some knowledge about the underlying distribution, confidence statements about the parameters can be made.

5.2 SIMPLE LINEAR CURVE FITTING

Two cases will be distinguished, namely, where there exists an underlying physical linear relationship, and where there exists a degree of association between the variables.

5.2.1 An underlying physical linear relationship. In this case, there is a mathematical relationship of the form $y = B_0 + B_1x$, which relates the variables. Examples of this case are the equations of physics:

$$\text{velocity} = \text{acceleration} \times \text{time} + \text{initial velocity}$$

$$\text{force} = \text{mass} \times \text{acceleration}$$

where the intercept, in the second example, is 0. Upon accumulating data for such relations, all the observed points $y_1, y_2, \ldots, y_n$ do not lie on a straight line because of experimental errors. However, it is reasonable to assume that the distribution of errors is such that for a fixed value of x, the population mean value of the y's is $B_0 + B_1x$, where B_0 and B_1 are the intercept and slope to be estimated from a sample $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Here x is the independent variable and is *assumed* to be known without error. From a practical point of view, it is sufficient to have the error in x small compared with the variability in y. In this section it will be assumed that $y_1, y_2, \ldots, y_n$ are independent chance variables having a normal distribution★ with mean $B_0 + B_1x$ and standard deviation σ (constant for all x_i).

5.2.2 A degree of association between chance variables x and y. Measurements on one variable, say x, may be relatively inexpensive compared with measurements on y. Consequently, one would like to predict y from a knowledge of x. For example, determining abrasion loss is difficult, whereas measuring hardness by means of a Rockwell hardness machine is relatively simple. There exists a degree of association between abrasion loss and hardness. Similarly, rainfall in one area is associated with rainfall in a surrounding area. From measurements of the amount of rainfall in one area, one would like to predict rainfall in the other area.

A linear relationship exists between x and y if for a fixed x, the population mean value of y, given x, is of the form $B_0 + B_1x$. A sufficient condition for the existence of a linear relationship is that both x and y have a bivariate normal distribution.† The model implies that the actual observations $y_1, y_2, \ldots, y_n$ should not be on a straight line, but for fixed x, the population mean value of y should fall on the line $B_0 + B_1x$, where B_0 and B_1 are the intercept and slope, respectively, to be estimated from a sample $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. In this section it will be assumed that the y's are at least independ-

★ The equations for the least square estimates of B_0 and B_1 do not depend on the assumption of normality. However, tests of hypothesis and confidence statements presented later do depend on this assumption.

† If x and y are related such that x is normally distributed, the mean of y, given x, is a linear function of x, and the variance of y, given x, is constant, x and y are said to have a bivariate normal distribution.

ently normally distributed★ with mean $B_0 + B_1x$ and standard deviation σ (for all x_i).†

Thus, returning to the abrasion loss-hardness example, if it is assumed that abrasion loss y and hardness x have a bivariate normal distribution, the average value of abrasion loss, given hardness, is $B_0 + B_1$ (hardness).

5.3 LEAST SQUARE ESTIMATES OF B_0 AND B_1

It should be noted that the cases presented in Arts. 5.2.1 and 5.2.2 fall into the general model of the following:

For fixed x, the $y_1, y_2, \ldots, y_n$ are independently normally distributed with mean value $B_0 + B_1x$ and standard deviation σ (for all x_i).

The problem is to estimate B_0 and B_1 by the method of least squares. Denote by b_0 and b_1 the least squares estimates of B_0 and B_1, respectively. The line is of the form $\tilde{y} = b_0 + b_1x$ and is usually referred to in statistics as a regression line. The least squares estimates of B_0 and B_1 are obtained by minimizing the sum of squares of the deviations about the estimated line with respect to b_0 and b_1, i.e., minimizing $\sum_{i=1}^{n} (y_i - \tilde{y}_i)^2$. The results are

$$b_1 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{\sum_{i=1}^{n} x_iy_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

$$b_0 = \bar{y} - b_1\bar{x}$$

5.4 SIGNIFICANCE TEST FOR B_0 AND B_1

In Art. 4 various significance tests for the mean and standard deviation of a normal distribution are given. In a similar manner, significance tests can be made for B_0 and B_1, and these are outlined in Table 13.29. These tests depend on the assumptions mentioned above. The reader is referred to the discussion in Art. 4 regarding significance tests.

5.5 POINT ESTIMATES AND CONFIDENCE INTERVALS FOR THE LINEAR MODEL

The point estimates and interval estimates for the various parameters are given in Table 13.30.

5.6 PREDICTING AN INTERVAL WITHIN WHICH A FUTURE OBSERVATION y^* CORRESPONDING TO x^* WILL LIE

Table 13.30 presents a confidence interval for the parameter, the mean value of y^* corresponding to x^*. It is often the case that a confi-

★ The equations for the least square estimates of B_0 and B_1 do not depend on the assumption of normality. However, tests of hypothesis and confidence statements presented later do depend on this assumption.

† That x and y have a bivariate normal distribution implies this condition.

TABLE **13.29** SIGNIFICANCE TESTS FOR COEFFICIENTS OF STRAIGHT LINE

Hypothesis	Test statistic[a]	Rule of rejection[b]
$B_0 = B'_0$	$t = \dfrac{b_0 - B'_0}{\sqrt{s^2_{y\|x}\left(\dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum (x_i - \bar{x})^2}\right)}}$	$\|t\| \geq t_{\alpha/2;n-2}$
$B_1 = B'_1$	$t = \dfrac{b_1 - B'_1}{\sqrt{s^2_{y\|x}\,(1/\sum (x_i - \bar{x})^2)}}$	$\|t\| \geq t_{\alpha/2;n-2}$
$B_0^{xy} = B_0^{uv}$ where mean $y = B_0^{xy} + B_1^{xy}\,x$; mean $v = B_0^{uv} + B_1^{uv}\,u$	$t = \dfrac{b_0^{xy} - b_0^{uv}}{\sqrt{\dfrac{(n_{xy} - 2)s^2_{y\|x} + (n_{uv} - 2)s^2_{v\|u}}{n_{xy} + n_{uv} - 4}}\sqrt{\dfrac{1}{n_{xy}} + \dfrac{1}{n_{uv}} + \dfrac{\bar{x}^2}{\sum (x_i - \bar{x})^2} + \dfrac{\bar{u}^2}{\sum (u_i - \bar{u})^2}}}$	$\|t\| \geq t_{\alpha/2;n_{xy}+n_{uv}-4}$
$B_1^{xy} = B_1^{uv}$ where mean $y = B_0^{xy} + B_1^{xy}\,x$; mean $v = B_0^{uv} + B_1^{uv}\,u$	$t = \dfrac{b_1^{xy} - b_1^{uv}}{\sqrt{\dfrac{(n_{xy} - 2)s^2_{y\|x} + (n_{uv} - 2)s^2_{v\|u}}{n_{xy} + n_{uv} - 4}}\sqrt{\dfrac{1}{\sum (x_i - \bar{x})^2} + \dfrac{1}{\sum (u_i - \bar{u})^2}}}$	$\|t\| \geq t_{\alpha/2;n_{xy}+n_{uv}-4}$

TABLE **13.29** SIGNIFICANCE TESTS FOR COEFFICIENTS OF STRAIGHT LINE (*Continued*)

OC curve	Notation
Compute $$d = \frac{\lvert B'_0 - B^*_0 \rvert}{\sigma\sqrt{[1/n + \bar{x}^2/\sum (x_i - \bar{x})^2]\,(n-1)}}$$ and refer to Fig. 13.29 or 13.30, using the curve for $n - 1$.	Here σ is the standard deviation of y about the mean $B_0 + B_1x$, $s^2_{y\lvert x}$ is the estimate of σ^2, and is equal to $$s^2_{y\lvert x} = \frac{\sum_{i=1}^{n} [y_i - (b_0 + b_1x_i)]^2}{n-2} = \frac{\sum (y_i - \bar{y})^2 - \dfrac{[\sum (x_i - \bar{x})(y_i - \bar{y})]^2}{\sum (x_i - \bar{x})^2}}{n-2}$$ where B_0^* is the value of the true intercept for use in referring to the OC curve.
Compute $$d = \frac{\lvert B'_1 - B^*_1 \rvert}{\sigma\sqrt{(n-1)/\sum (x_i - \bar{x})^2}}$$ and refer to Fig. 13.29 or 13.30, using the curve for $n - 1$.	Here B^*_1 is the value of the true slope for use in referring to the OC curve.
Compute $$d = \frac{\lvert \delta_0 \rvert}{\sqrt{n_{xy} + n_{uv} - 3}\;\sigma_{y\lvert x}\sqrt{\dfrac{1}{n_{xy}} + \dfrac{1}{n_{uv}} + \dfrac{\bar{x}^2}{\sum (x_i - \bar{x})^2} + \dfrac{\bar{u}^2}{\sum (u_i - \bar{u})^2}}}$$ and refer to Fig. 13.29 or 13.30, using the curve for $n_{xy} + n_{uv} - 3$.	This is the problem of comparing coefficients from two straight lines. The subscripts x and y refer to one line, whereas u and v refer to the second line. Then $\sigma_{y\lvert x}$ is the unknown standard deviation of y about the mean $B_0^{xy} + B_1^{xy}x$, and $\sigma_{v\lvert u}$ is the unknown standard deviation of v about the mean $B_0^{uv} + B_1^{uv}u$. It is assumed that $\sigma_{y\lvert x} = \sigma_{v\lvert u}$; δ_0 is the difference in intercept of the two lines for use in referring to the OC curve. Here δ_1 is the difference in slopes of the two lines for use in referring to the OC curve.
Compute $$d = \frac{\lvert \delta_1 \rvert}{\sigma_{y\lvert x}\sqrt{\left(\dfrac{1}{\sum (x_i - \bar{x})^2} + \dfrac{1}{\sum (u_i - \bar{u})^2}\right)(n_{xy} + n_{uv} - 3)}}$$ and refer to Fig. 13.29 or 13.30, using the curve for $n_{xy} + n_{uv} - 3$.	

[a] If the population standard deviation is known, these values should be used in place of the sample estimates, and K_α should be used instead of t_α.
[b] These values are for two-sided tests, and the values of t_α are given in Table 13.25.

TABLE **13.30** ESTIMATION OF PARAMETERS FOR LINEAR MODEL

Parameter	Symbol for estimate	Estimation computation formula	$100\,(1-\alpha)\,\%$ confidence interval	Notes
B_1	b_1	$\dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$	$b_1 \pm \dfrac{t_{(\alpha/2;n-2)}\, s_{y\vert x}}{\sqrt{\sum (x_i - \bar{x})^2}}$	
B_0	b_0	$\bar{y} - b_1\bar{x}$	$b_0 \pm t_{(\alpha/2;n-2)}\, s_{y\vert x} \sqrt{\dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum (x_i - \bar{x})^2}}$	
Average value of y corresponding to x^*, i.e., $B_0 + B_1x^*$	$\bar{y}^*$	$b_0 + b_1x^*$	$b_0 + b_1x^* \pm t_{(\alpha/2;n-2)}\, s_{y\vert x} \sqrt{\dfrac{1}{n} + \dfrac{(x^* - \bar{x})^2}{\sum (x_i - \bar{x})^2}}$	
Value of x corresponding to observed value of y' for the case when there is an underlying physical relationship.	x'	$\dfrac{y' - b_0}{b_1} = x'$	$x' + b_1 \dfrac{(y' - \bar{y})}{c} \pm t_{(\alpha/2;n-2)} \dfrac{s_{y\vert x}}{c} \sqrt{\left(1 + \dfrac{1}{n}\right)c + A(y' - \bar{y})^2}$	$c = b_1^2 - t^2_{\alpha/2}\,(s^2_{y\vert x})(A)$ $A = 1/\sum (x_i - \bar{x})^2$ If $(x' - \bar{x})^2/\sum (x_i - \bar{x})^2$ is small, then we have for the confidence interval, $(y' - b_0)/b_1 \pm t_{\alpha/2}\, s_{y\vert x}\sqrt{(1 + 1/n)}/b_1$
$\sigma^2_{y\vert x}$	$s^2_{y\vert x}$	$\dfrac{\sum [y_i - (b_0 + b_1x_i)]^2}{n-2} = \dfrac{\sum (y_i - \bar{y})^2 - [\sum (x_i - \bar{x})(y_i - \bar{y})]^2/\sum (x_i - \bar{x})^2}{n-2}$	$\dfrac{(n-2)s^2_{y\vert x}}{\chi^2_{\alpha/2;n-2}};\ \dfrac{(n-2)s^2_{y\vert x}}{\chi^2_{1-\alpha/2;n-2}}$	

dence statement about the mean value is unimportant, whereas a probability statement about a future observation is relevant. For example, tensile strength of cement is related to the curing time (t) by: strength $= \alpha e^{-\beta/t}$. This function is transformed by taking logarithms:

$$\ln \text{strength} = \ln \alpha - \frac{\beta}{t} = \ln \alpha - \beta z$$

where $z = 1/t$, which is now linear in z and can be estimated by the method of least squares. The cement manufacturer is interested in the average tensile strength of his cement after a particular period of time, i.e., a confidence statement, whereas a builder is interested in the tensile strength of his particular batch of cement to determine whether it will carry the required load. After a specified period of time, say 28 days, the builder would like to have such a statement as the probability is $1 - \alpha$ that the tensile strength of his batch of cement will lie in a specified interval.

The following statement can be made. The probability is $1 - \alpha$ that the value of a future observation y^* corresponding to x^* will lie in the interval

$$b_0 + b_1 x^* \pm t_{(\alpha/2;n-2)}\, s_{y|x} \sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}}$$

5.7 CORRELATION

A measure of the degree of association between two variables is the correlation coefficient ρ. An estimate of ρ is given by the sample correlation coefficient r, which is defined

$$r = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}$$

In engineering applications, the correlation coefficient does not play a very important role. The correlation coefficient can be derived from the slope of the fitted least squares line, e.g.,

$$r = b_1 \sqrt{\frac{\sum (x_i - \bar{x})^2}{\sum (y_i - \bar{y})^2}}$$

Consequently it is clear that the correlation coefficient does not contain any additional information. However, a significance test for $\rho = 0$ is equivalent to testing whether $B_1 = 0$, i.e., whether or not a linear relation is present. The test for $B_1 = 0$ is given in Table 13.29.

To test the hypothesis that $\rho = 0$, reject when

$$|t| = \left| \frac{r}{\sqrt{1 - r^2}} \right| \sqrt{n - 2} \geq t_{\alpha/2}$$

where values of t_α can be found in Table 13.25.

TABLE **13.31** WORKSHEET FOR THE FITTING OF STRAIGHT LINES BY THE METHOD OF LEAST SQUARES

General data

x represents $\underline{\qquad\qquad}$

$$\sum_{i=1}^{n} x_i = \underline{\qquad};\ \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} = \underline{\qquad};\ \left(\sum_{i=1}^{n} x_i\right)^2 \Big/ n = \left(\sum_{i=1}^{n} x_i\right)(\bar{x}) = \underline{\qquad}$$

$$\sum_{i=1}^{n} x_i^2 = \underline{\qquad};\ \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 \Big/ n = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \underline{\qquad}$$

y represents $\underline{\qquad\qquad}$

$$\sum_{i=1}^{n} y_i = \underline{\qquad};\ \bar{y} = \sum_{i=1}^{n} \frac{y_i}{n} = \underline{\qquad};\ \left(\sum_{i=1}^{n} y_i\right)^2 \Big/ n = \left(\sum_{i=1}^{n} y_i\right)(\bar{y}) = \underline{\qquad}$$

$$\sum_{i=1}^{n} y_i^2 = \underline{\qquad};\ \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2 \Big/ n = \sum_{i=1}^{n} (y_i - \bar{y})^2 = \underline{\qquad}$$

$$n = \underline{\qquad};\ \frac{1}{n} = \underline{\qquad};\ \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \Big/ n = (\bar{x}) \sum_{i=1}^{n} y_i = \underline{\qquad}$$

$$\sum_{i=1}^{n} x_i y_i = \underline{\qquad};\ \sum_{i=1}^{n} x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)/n = \sum (x_i - \bar{x})(y_i - \bar{y}) = \underline{\qquad}$$

$$\frac{\left[\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})\right]^2}{\sum (x_i - \bar{x})^2} = \underline{\qquad}$$

Data for confidence intervals and predictions for $x = x^*$

Data for the equation of line

$$b_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \underline{\qquad}$$

$$b_0 = \bar{y} - b_1\bar{x} = \underline{\qquad}$$

$$\text{equation of line} = \hat{y} = b_0 + b_1 x$$
$$= \underline{\qquad} + \underline{\qquad}\, x$$

Estimate of standard deviation

$$(n - 2)(s^2_{y|x}) = \sum_{i=1}^{n} (y_i - \bar{y})^2$$
$$- \frac{\left[\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})\right]^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \underline{\qquad}$$

$$s^2_{y|x} = \underline{\qquad};\ s_{y|x} = \sqrt{s^2_{y|x}} = \underline{\qquad}$$

TABLE **13.31** WORKSHEET FOR THE FITTING OF STRAIGHT LINES BY THE METHOD OF LEAST SQUARES (*Continued*)

Data for significance tests	Data for confidence intervals and predictions for $x = x^*$
$\dfrac{s^2_{y\|x}}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \underline{\qquad};\quad \sqrt{\dfrac{s^2_{y\|x}}{\sum_{i=1}^{n}(x_i-\bar{x})^2}} = \underline{\qquad};\ \bar{x}^2 = \underline{\qquad}$	$(x^*-\bar{x})^2 = \underline{\qquad};\quad \dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \underline{\qquad}$
$\dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \underline{\qquad};\quad \dfrac{1}{n}+\dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \underline{\qquad}$	$\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \underline{\qquad}$
$(s^2_{y\|x})\left[\dfrac{1}{n}+\dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right] = \underline{\qquad}$	$(s^2_{y\|x})\left[\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right] = \underline{\qquad}$
$\sqrt{(s^2_{y\|x})\left[\dfrac{1}{n}+\dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]} = \underline{\qquad}$	$\sqrt{(s^2_{y\|x})\left[\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]} = \underline{\qquad}$
	$1+\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \underline{\qquad}$
	$(s^2_{y\|x})\left[1+\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right] = \underline{\qquad}$
	$\sqrt{(s^2_{y\|x})\left[1+\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]} = \underline{\qquad}$

TABLE **13.31a** WORKSHEET FOR THE FITTING OF STRAIGHT LINES BY THE METHOD OF LEAST SQUARES

General data

x represents nominal value

$$\sum_{i=1}^{n} x_i = 27.0; \quad \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} = 1.125; \quad \left(\sum_{i=1}^{n} x_i\right)^2 / n = \left(\sum_{i=1}^{n} x_i\right)(\bar{x}) = 30.375$$

$$\sum_{i=1}^{n} x_i^2 = 38.2500; \quad \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 / n = \sum_{i=1}^{n} (x_i - \bar{x})^2 = 7.875$$

y represents actual value

$$\sum_{i=1}^{n} y_i = 27.022; \quad \bar{y} = \sum_{i=1}^{n} \frac{y_i}{n} = 1.1259; \quad \left(\sum_{i=1}^{n} y_i\right)^2 / n = \left(\sum_{i=1}^{n} y_i\right)(\bar{y}) = 30.424552$$

$$\sum_{i=1}^{n} y_i^2 = 38.279818; \quad \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2 / n = \sum_{i=1}^{n} (y_i - \bar{y})^2 = 7.855266$$

$$n = 24; \quad \frac{1}{n} = 0.041667; \quad \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) / n = (\bar{x}) \sum_{i=1}^{n} y_i = 30.39975$$

$$\sum_{i=1}^{n} x_i y_i = 38.26375; \quad \sum_{i=1}^{n} x_i y_i - (\sum x_i)(\sum y_i)/n = \sum (x_i - \bar{x})(y_i - \bar{y}) = 7.86400$$

$$\frac{\left[\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})\right]^2}{\sum (x_i - \bar{x})^2} = \underline{\qquad}$$

Data for confidence intervals and predictions for $x = x^*$

Data for the equation of line

$$b_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = 0.998603$$

$$b_0 = \bar{y} - b_1 \bar{x} = -0.000672882$$

$$\text{equation of line} = \hat{y} = b_0 + b_1 x$$
$$= 0.000672882 + 0.998603x$$

Estimate of standard deviation

$$(n-2)(s^2_{y|x}) = \sum_{i=1}^{n} (y_i - \bar{y})^2 - \frac{\left(\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})\right)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = 0.0023$$

$$s^2_{y|x} = 0.00010455; \quad s_{y|x} = \sqrt{s^2_{y|x}} = 0.010225$$

TABLE 13.31a WORKSHEET FOR THE FITTING OF STRAIGHT LINES BY THE METHOD OF LEAST SQUARES (*Continued*)

Data for significance tests	Data for confidence intervals and predictions for $x = x^*$
$\dfrac{s^2_{y\|x}}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \underline{\qquad}$; $\sqrt{\dfrac{s^2_{y\|x}}{\sum_{i=1}^{n}(x_i - \bar{x})^2}} = \underline{\qquad}$; $\bar{x}^2 = \underline{\qquad}$	$(x^* - \bar{x})^2 = 0.015625$; $\dfrac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = 0.001984$
$\dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \underline{\qquad}$; $\dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \underline{\qquad}$	$\dfrac{1}{n} + \dfrac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = 0.043651$
$(s^2_{y\|x})\left[\dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right] = \underline{\qquad}$	$(s^2_{y\|x})\left[\dfrac{1}{n} + \dfrac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right] = 0.000004564$
$\sqrt{(s^2_{y\|x})\left[\dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]} = \underline{\qquad}$	$\sqrt{(s^2_{y\|x})\left[\dfrac{1}{n} + \dfrac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]} = 0.002136$
	$1 + \dfrac{1}{n} + \dfrac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = 1.043651$
	$(s^2_{y\|x})\left[1 + \dfrac{1}{n} + \dfrac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right] = 0.0001091$
	$\sqrt{(s^2_{y\|x})\left[1 + \dfrac{1}{n} + \dfrac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]} = 0.01045$

5.7.1 Example. A screw manufacturer is interested in giving out data to his customers on the relation between nominal and actual lengths. The following results were observed:

Nominal x	Actual y		
$\frac{1}{4}$ in.	0.262	0.262	0.245
$\frac{1}{2}$	0.496	0.512	0.490
$\frac{3}{4}$	0.743	0.744	0.751
1	0.976	1.010	1.004
$1\frac{1}{4}$	1.265	1.254	1.252
$1\frac{1}{2}$	1.498	1.518	1.504
$1\frac{3}{4}$	1.738	1.759	1.750
2	2.005	1.992	1.992

In this problem, there is a definite physical relation

$$\text{actual value} = B_0 + B_1 \text{ (nominal value)}$$

If the manufacturing process were perfect, it would be expected that $B_0 = 0$ and $B_1 = 1$. It should be noted that without using regression, one could make a confidence statement about the average of length of "1-inch" screws on the basis of the three numbers in the table; but a much more precise statement can be made by using all the data, together with the fact that linear regression is applicable. (1) Estimate the above relation. (2) For nominal 1-in. screws, find a confidence interval for the average actual value of screw length. (3) For a 1-in. screw find an interval such that the actual value will lie in this interval with probability 0.95.

(1) Estimated line: $\tilde{y} = 0.000672882 + 0.998603x$

(2) Confidence interval for the average actual value of 1-in. screws:

$$b_0 + b_1x^* \pm t_{(\alpha/2;n-2)}\, s_{y|x}\sqrt{\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum (x_i - \bar{x})^2}}$$

$$0.000672882 + 0.998603 \pm 2.074 \times 0.002136$$

$$0.99485 \leq \text{average actual value} \leq 1.00371$$

(3) Interval such that the probability is 0.95 that the true length of a 1-in. screw will be in this interval:

$$0.999275 \pm 0.01045 \times 2.074$$

$$0.97760 \leq \text{true length} \leq 1.02095$$

5.7.2 Example. The following results on the tensile strength of specimens of cold drawn copper have been recorded in a laboratory.

These variables fall into the second category, namely, a degree of association between x and y. Furthermore, it is reasonable to assume that x and y have a bivariate normal distribution, so that the population mean value of $y = B_0 + B_1x$. Hence, it remains to find the estimated line $\tilde{y} = b_0 + b_1x$ by the method of least squares. (1) Test the hypothesis that $B_0 = -100{,}000$. (2) Test the hypothesis that $B_1 = 1900$. (3) Get 95% confidence intervals for B_1; B_0; the mean value of tensile strength corresponding to a Brinell hardness of 105.0; and $\sigma_{y|x}$. (4) Find an interval such that the probability is 0.95 that the value of tensile strength for a Brinell hardness of 105.0 will lie in this interval.

y Tensile strength (psi)	x Brinell hardness number
38,871	104.2
40,407	106.1
39,887	105.6
40,821	106.3
33,701	101.7
39,481	104.4
33,003	102.0
36,999	103.8
37,632	104.0
33,213	101.5
33,911	101.9
29,861	100.6
39,451	104.9
40,647	106.2
35,131	103.1

(1) Hypothesis: $B_0 = -100{,}000$; reject if

$$|t| = \left| \frac{b_0 + 100{,}000}{\sqrt{s_{y|x}^2[1/n + \bar{x}^2/\sum (x_i - \bar{x})^2]}} \right| \geq t_{.025,13} = 2.160$$

$$|t| = \left| \frac{-149{,}786.5 + 100{,}000}{11{,}879.8} \right| = 4.19 \geq 2.160$$

Therefore reject the hypothesis that $B_0 = -100{,}000$.

(2) Hypothesis: $B_1 = 1900$; reject if

$$|t| = \left| \frac{b_1 - 1900}{\sqrt{s_{y|x}^2(1/\sum (x_i - \bar{x})^2)}} \right| \geq t_{.025,13} = 2.160$$

$$|t| = \left| \frac{1799.025 - 1900}{114.5} \right| = 0.881 < 2.160$$

Therefore accept the hypothesis that $B_1 = 1900$.

(3) Confidence intervals:

(a) for B_1

$$b_1 \pm \frac{t_{\alpha/2;n-2}\, s_{y|x}}{\sqrt{\sum (x_i - \bar{x})^2}}$$

$$1799.025 \pm 2.160(114.5)$$

$$1551.7 \leq B_1 \leq 2046.3$$

(b) for B_0

$$b_0 \pm t_{\alpha/2;n-2}\, s_{y|x}\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum (x_i - \bar{x})^2}}$$

$$-149{,}786.5 \pm 2.160(11{,}879.8)$$

$$-175{,}446.9 \leq B_0 \leq -124{,}126.1$$

(c) for mean value of tensile strength corresponding to a Brinell hardness of 105.0.

$$b_0 + b_1x^* \pm t_{\alpha/2;n-2}\, s_{y|x}\sqrt{\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum (x_i - \bar{x})^2}}$$

$$-149{,}786.5 + 1{,}799.025(105.0) \pm 2.160(252.56) = 39{,}111.1 \pm 545.53$$

$$38{,}565.6 \leq \text{mean value of tensile strength} \leq 39{,}656.6$$

TABLE **13.31b** WORKSHEET FOR THE FITTING OF STRAIGHT LINES BY THE METHOD OF LEAST SQUARES

General data

x represents Brinell hardness number

$$\sum_{i=1}^{n} x_i = 1556.3; \quad \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} = 103.753; \quad \left(\sum_{i=1}^{n} x_i\right)^2 / n = \left(\sum_{i=1}^{n} x_i\right)(\bar{x}) = 161{,}471.3126$$

$$\sum_{i=1}^{n} x_i^2 = 161{,}520.87; \quad \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 / n = \sum_{i=1}^{n} (x_i - \bar{x})^2 = 49.56$$

y represents tensile strength

$$\sum_{i=1}^{n} y_i = 553{,}016; \quad \bar{y} = \sum_{i=1}^{n} \frac{y_i}{n} = 36{,}867.7; \quad \left(\sum_{i=1}^{n} y_i\right)^2 / n = \left(\sum_{i=1}^{n} y_i\right)(\bar{y}) = 20{,}388{,}446{,}420$$

$$\sum_{i=1}^{n} y_i^2 = 20{,}557{,}291{,}078; \quad \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2 / n = \sum_{i=1}^{n} (y_i - \bar{y})^2 = 168{,}844{,}660$$

$$n = 15; \quad \frac{1}{n} = 0.0666667; \quad \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) / n = (\bar{x}) \sum_{i=1}^{n} y_i = 57{,}377{,}253.38$$

$$\sum_{i=1}^{n} x_i y_i = 57{,}466{,}413.1; \quad \sum_{i=1}^{n} x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)/n = \sum (x_i - \bar{x})(y_i - \bar{y}) = 89{,}159.7$$

$$\frac{\left[\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})\right]^2}{\sum (x_i - \bar{x})^2} = 160{,}400{,}567$$

Data for confidence intervals and predictions for $x = x^*$

Data for the equation of line

$$b_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = 1{,}799.025$$

$$b_0 = \bar{y} - b_1\bar{x} = -149{,}786.5$$

equation of line is $\hat{y} = b_0 + b_1 x$
$= -149{,}786.5 + 1{,}799.025x$

Estimate of standard deviation

$$(n - 2)(s^2_{y|x}) = \sum_{i=1}^{n} (y_i - \bar{y})^2 - \frac{\left[\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})\right]^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = 8{,}444{,}093$$

$$s^2_{y|x} = 649{,}546; \quad s_{y|x} = \sqrt{s^2_{y|x}} = 805.945$$

TABLE **13.31b** WORKSHEET FOR THE FITTING OF STRAIGHT LINES BY THE METHOD OF LEAST SQUARES (*Continued*)

Data for significance tests	Data for confidence intervals and predictions for $x = x^*$
$\dfrac{s^2_{y\|x}}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = 13{,}106$; $\sqrt{\dfrac{s^2_{y\|x}}{\sum_{i=1}^{n}(x_i-\bar{x})^2}} = 114.5$; $\bar{x}^2 = 10{,}764.7$	$(x^*-\bar{x})^2 = 1.555$; $\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = 0.03135$
$\dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = 217.2051$; $\dfrac{1}{n}+\dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = 217.2718$	$\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = 0.0982$
$(s^2_{y\|x})\left[\dfrac{1}{n}+\dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right] = 141{,}128{,}029$	$(s^2_{y\|x})\left[\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right] = 63{,}785.4$
$\sqrt{(s^2_{y\|x})\left[\dfrac{1}{n}+\dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]} = 11{,}879.8$	$\sqrt{(s^2_{y\|x})\left[\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]} = 252.56$
	$1+\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = 1.0982$
	$(s^2_{y\|x})\left[1+\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right] = 713{,}331.4$
	$\sqrt{(s^2_{y\|x})\left[1+\dfrac{1}{n}+\dfrac{(x^*-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]} = 844.59$

(d) for $\sigma^2_{y|x}$

$$\frac{(n-2)s^2_{y|x}}{\chi^2_{(\alpha/2;n-2)}}; \qquad \frac{(n-2)s^2_{y|x}}{\chi^2_{(1-\alpha/2;n-2)}}$$

$$\frac{13(649{,}546)}{5.01}; \qquad \frac{13(649{,}546)}{24.7}$$

$$1{,}685{,}449 \geq \sigma^2_{y|x} \geq 341{,}866$$

(4) Interval such that the probability is 0.95 that the value of tensile strength for a Brinell hardness of 105.0 will be in this interval.

$$b_0 + b_1x^* \pm t_{(\alpha/2;n-2)}s_{y|x}\sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum (x_i - \bar{x})^2}}$$

$$39{,}111.1 \pm 2.160(844.59)$$

$$37{,}286.8 \leq \text{tensile strength} \leq 40{,}935.4$$

If we wished to estimate hardness from tensile strength, we would have simply reversed the roles of x and y, since we are working in Case 2.

5.8 RELATIONS BETWEEN SEVERAL VARIABLES

In Arts. 5.1 to 5.7 the problem considered was that of finding a relation which would enable one to predict a variable from a knowledge of some one other variable. However, in many cases several factors are relevant to the prediction of just one variable. The relations between height and weight of individuals is different at different ages and different for men and women; an estimate of weight based on height, age, and sex will be better than one based on height alone. The relation between tensile strength and Rockwell hardness may depend on the density of the specimens. For each hardness-density combination the tensile strength will vary about a certain mean; we are interested in a formula giving the mean for each combination. As in the case of a single variable, it is assumed that the variance will be the same for each combination.

5.8.1 Least-squares equations. Suppose, for example, we are interested in predicting y, and let x_1 and x_2 be the independent variables. Assume that the true underlying relationship is of the form

$$\text{average value of } y = B_0 + B_1x_1 + B_2x_2$$

If we denote the estimates of the B_i by b_i, the least squares estimate is $\tilde{y} = b_0 + b_1x_1 + b_2x_2$, where b_1 and b_2 are found from the following set of equations:

$$\sum_{i=1}^{n} (y_i - \bar{y})(x_{1i} - \bar{x}_1) = b_1 \sum_{i=1}^{n} (x_{1i} - \bar{x}_1)^2 + b_2 \sum_{i=1}^{n} (x_{1i} - \bar{x}_1)(x_{2i} - \bar{x}_2)$$

$$\sum_{i=1}^{n} (y_i - \bar{y})(x_{2i} - \bar{x}_2) = b_1 \sum_{i=1}^{n} (x_{1i} - \bar{x}_1)(x_{2i} - \bar{x}_2) + b_2 \sum_{i=1}^{n} (x_{2i} - \bar{x}_2)^2$$

$$b_0 = \bar{y} - b_1\bar{x}_1 - b_2\bar{x}_2$$

These equations are known as normal equations and can be solved by solving the first for b_1 in terms of b_2 and substituting this expression for b_1 in the second, or by any other method. In general, we might consider a regression equation of the sort

$$\text{average value of } y = B_0 + B_1x_1 + B_2x_2 + \ldots + B_kx_k$$

with estimates

$$\tilde{y} = b_0 + b_1x_1 + b_2x_2 + \ldots + b_kx_k$$

where $b_1, b_2, \ldots, b_k$ are found by the solution of a set of k simultaneous equations in k unknowns. For $k = 2$ and 3, detailed instructions are given in Tables 13.34 and 13.35.

5.9 TEST OF SIGNIFICANCE

There are many hypotheses which we might be interested in testing, but the most important question is whether the independent variables taken as a whole lead to an improvement in our ability to forecast y.

For $k = 2$. This is equivalent to testing the hypothesis that $(B_1,B_2) = (0,0)$, i.e., mean value of $y = B_0$. The results may be arranged into Table 13.32,

TABLE 13.32 REGRESSION TABLE FOR $k = 2$

Variation	Sum of squares	Degrees of freedom	Mean square
Due to regression	$b_1c_1 + b_2c_2$	2	$(b_1c_1 + b_2c_2)/2$
About regression	$\sum_{i=1}^{n} (y_i - \tilde{y}_i)^2$	$n - 3$	$\sum_{i=1}^{n} (y_i - \tilde{y}_i)^2/(n - 3)$
Total	$\sum_{i=1}^{n} (y_i - \bar{y})^2$	$n - 1$	

where c_1 and c_2 are defined in the computational Table 13.34 and $\sum_{i=1}^{n}(y_i - \tilde{y}_i)^2$ is obtained by subtraction.* Reject the hypothesis that $(B_1,B_2) = (0,0)$ if

$$F = \frac{(b_1c_1 + b_2c_2)/2}{\sum_{i=1}^{n} (y_i - \tilde{y}_i)^2/(n - 3)} \geq F_{\alpha;2,n-3}$$

where $F_{\alpha;2,n-3}$ is the α percentage point of the F distribution found in Table 13.27.

For $k = 3$. To test H_0: $(B_1,B_2,B_3) = (0,0,0)$, arrange the results into Table 13.33,

* The following is an identity:

$$\sum_{i=1}^{n} (y_i - \bar{y})^2 = b_1c_1 + b_2c_2 + \sum_{i=1}^{n} (y_i - \tilde{y}_i)^2$$

TABLE **13.33** REGRESSION TABLE FOR $k = 3$

Variation	Sum of squares	Degrees of freedom	Mean square
Due to regression	$b_1c_1 + b_2c_2 + b_3c_3$	3	$(b_1c_1 + b_2c_2 + b_3c_3)/3$
About regression	$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2$	$n - 4$	$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2/(n - 4)$
Total	$\sum_{i=1}^{n} (y_i - \bar{y})^2$	$n - 1$	

where c_1, c_2, and c_3 are defined in the computational Table 13.35 and $\sum_{i=1}^{n} (y_i - \hat{y}_i)^2$ is obtained by subtraction.★

Reject the hypothesis that $(B_1,B_2,B_3) = (0,0,0)$ if

$$F = \frac{(b_1c_1 + b_2c_2 + b_3c_3)/3}{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2/(n - 4)} \geq F_{\alpha;3,n-4}$$

where $F_{\alpha;3,n-4}$ is the α percentage point of the F distribution found in Table 13.27.

5.10 NONLINEAR REGRESSION

Nonlinear regression falls into the above model. For example,

$$\text{average value of } y = B_0 + B_1Z + B_2Z^2$$

Since the Z's are fixed variables, we can write the quadratic equation in the form

$$\text{average value of } y = B_0 + B_1x_1 + B_2x_2$$

where $x_1 = Z$ and $x_2 = Z^2$. Hence the least squares estimate of the B's can be obtained using the procedure presented above. (See Tables 13.34 and 13.35.)

5.11 EXAMPLE

During the years 1951 and 1952, the County of Santa Clara, in California, contracted for a cloud-seeding operation.† In order to evaluate the effectiveness of this operation, the county and the surrounding area were di-

★ The following is an identity:

$$\sum_{i=1}^{n} (y_i - \bar{y})^2 = b_1c_1 + b_2c_2 + b_3c_3 + \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

† G. J. Lieberman, *An Evaluation of the 1951-1952 Cloud Seeding Experiment in Santa Clara County*, Stanford University, Stanford, California.

vided geographically into regions. The rainfall in the Los Gatos, Palo Alto, and Santa Clara (y) region was to be predicted from the rainfall in the Berkeley, Crockett, Oakland, San Francisco area (x_1), the Merced, Modesto, and Stockton area (x_2), and the Big Sur, Coalinga, King City, and Paso Robles area (x_3). These areas were determined in such a manner that they contained long-record stations, and all the stations within a group were geographically located close to one another. In making this rainfall study, storm totals, summed

TABLE **13.34** COMPUTATIONAL SCHEME FOR SOLVING THE NORMAL EQUATIONS

$$\hat{y} = b_0 + b_1x_1 + b_2x_2$$

The normal equations presented in Art. 5.8 are of the form

$$a_{11}b_1 + a_{12}b_2 = c_1; \qquad a_{21}b_1 + a_{22}b_2 = c_2$$

where

$$a_{11} = \sum_{i=1}^{n} (x_{1_i} - \bar{x}_1)^2 = \sum_{i=1}^{n} x_{1_i}^2 - n\bar{x}_1^2 = \text{———}$$

$$a_{21} = a_{12} = \sum_{i=1}^{n} (x_{1_i} - \bar{x}_1)(x_{2_i} - \bar{x}_2) = \sum_{i=1}^{n} x_{1_i}x_{2_i} - n\bar{x}_1\bar{x}_2 = \text{———}$$

$$a_{22} = \sum_{i=1}^{n} (x_{2_i} - \bar{x}_2)^2 = \sum_{i=1}^{n} x_{2_i}^2 - n\bar{x}_2^2 = \text{———}$$

$$c_1 = \sum_{i=1}^{n} (x_{1_i} - \bar{x}_1)(y_i - \bar{y}) = \sum_{i=1}^{n} x_{1_i}y_i - n\bar{x}_1\bar{y} = \text{———}$$

$$c_2 = \sum_{i=1}^{n} (x_{2_i} - \bar{x}_2)(y_i - \bar{y}) = \sum_{i=1}^{n} x_{2_i}y_i - n\bar{x}_2\bar{y} = \text{———}$$

operation	b_1	b_2	c	check
(1)	a_{11}	a_{12}	c_1	$a_{11} + a_{12} + c_1$
(2)	a_{21}	a_{22}	c_2	$a_{21} + a_{22} + c_2$
(3) (1) repeated	μ_{11}	μ_{12}	μ_{13}	
(4) (3) divided by a_{11}	1	V_{12}	V_{13}	
(5) (2) $- V_{12} \times$ (3)	0	μ_{22}	μ_{23}	
(6) (5) divided by μ_{22}	0	1	V_{23}	

Solution

(7) $b_2 = V_{23}$

(8) $b_1 = V_{13} - V_{12}b_2$

(9) $b_0 = \bar{y} - b_1\bar{x}_1 - b_2\bar{x}_2$

The check is started with line 3. All the operations from line 3 on are performed on the values in the check column. For each line, then, the sum of the entries in all the columns should check with the entry in the check column.

TABLE **13.35** COMPUTATIONAL SCHEME FOR SOLVING THE NORMAL EQUATIONS

$$\hat{y} = b_0 + b_1x_1 + b_2x_2 + b_3x_3$$

The normal equations are of the form:

$$\begin{aligned} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 &= c_1 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 &= c_2 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 &= c_3 \end{aligned}$$

where

$$a_{11} = \sum_{i=1}^{n} (x_{1_i} - \bar{x}_1)^2 = \sum_{i=1}^{n} x_{1_i}^2 - n\bar{x}_1^2 = \text{———}$$

$$a_{21} = a_{12} = \sum_{i=1}^{n} (x_{1_i} - \bar{x}_1)(x_{2_i} - \bar{x}_2) = \sum_{i=1}^{n} x_{1_i}x_{2_i} - n\bar{x}_1\bar{x}_2 = \text{———}$$

$$a_{22} = \sum_{i=1}^{n} (x_{2_i} - \bar{x}_2)^2 = \sum_{i=1}^{n} x_{2_i}^2 - n\bar{x}_2^2 = \text{———}$$

$$a_{13} = a_{31} = \sum_{i=1}^{n} (x_{1_i} - \bar{x}_1)(x_{3_i} - \bar{x}_3) = \sum_{i=1}^{n} x_{1_i}x_{3_i} - n\bar{x}_1\bar{x}_3 = \text{———}$$

$$a_{23} = \sum_{i=1}^{n} (x_{2_i} - \bar{x}_2)(x_{3_i} - \bar{x}_3) = \sum_{i=1}^{n} x_{2_i}x_{3_i} - n\bar{x}_2\bar{x}_3 = \text{———}$$

$$a_{33} = \sum_{i=1}^{n} (x_{3_i} - \bar{x}_3)^2 = \sum_{i=1}^{n} x_{3_i}^2 - n\bar{x}_3^2 = \text{———}$$

$$c_1 = \sum_{i=1}^{n} (x_{1_i} - \bar{x}_1)(y_i - \bar{y}) = \sum_{i=1}^{n} x_{1_i}y_i - n\bar{x}_1\bar{y} = \text{———}$$

$$c_2 = \sum_{i=1}^{n} (x_{2_i} - \bar{x}_2)(y_i - \bar{y}) = \sum_{i=1}^{n} x_{2_i}y_i - n\bar{x}_2\bar{y} = \text{———}$$

$$c_3 = \sum_{i=1}^{n} (x_{3_i} - \bar{x}_3)(y_i - \bar{y}) = \sum_{i=1}^{n} x_{3_i}y_i - n\bar{x}_3\bar{y} = \text{———}$$

Operation	b_1	b_2	b_3	c	Check
(1)	a_{11}	a_{12}	a_{13}	c_1	$a_{11} + a_{12} + a_{13} + c_1$
(2)	a_{21}	a_{22}	a_{23}	c_2	$a_{21} + a_{22} + a_{23} + c_2$
(3)	a_{31}	a_{32}	a_{33}	c_3	$a_{31} + a_{32} + a_{33} + c_3$
(4) (1) repeated	μ_{11}	μ_{12}	μ_{13}	μ_{14}	
(5) (4) divided by a_{11}	1	V_{12}	V_{13}	V_{14}	
(6) (2) $- V_{12} \times$ (4)	0	μ_{22}	μ_{23}	μ_{24}	
(7) (6) divided by μ_{22}	0	1	V_{23}	V_{24}	
(8) (3) $- V_{13} \times$ (4) $- V_{23} \times$ (6)	0	0	μ_{33}	μ_{34}	
(9) (8) divided by μ_{33}	0	0	1	V_{34}	

solution

(10) $b_3 = V_{34}$

(11) $b_2 = V_{24} - V_{23}b_3$

(12) $b_1 = V_{14} - b_3V_{13} - b_2V_{12}$

(13) $b_0 = \bar{y} - b_1\bar{x}_1 - b_2\bar{x}_2 - b_3\bar{x}_3$

The check is started with line 4. All the operations from line 4 on are performed on the values in the check column. For each line, then, the sum of the entries in all the columns should check with the entry in the check column.

over all rainfall stations within an area, were chosen as the index to be considered. From preliminary analysis it was determined that the cube root of the total rainfall over all the stations within a region in the county, per storm, can be predicted from a linear combination of the cube roots of the total rainfall over all the stations within a region in the control area, per storm, i.e.,

$$\text{average value of } y = B_0 + B_1x_1 + B_2x_2 + B_3x_3$$

From past data, these coefficients were estimated so that

$$\tilde{y} = b_0 + b_1x_1 + b_2x_2 + b_3x_3$$

Assuming no seeding effect in the areas outside the county, the rainfall within the county can be predicted from the above relationship, during the seeding period, and compared with the actual values. If seeding is effective, we should expect the actual rainfall for seeded storms to exceed the predicted values rather consistently.

The data is as follows:

$n = 168;\quad \sum_{i=1}^{168} x_{1_i}y_i = 332.176315;\quad \sum_{i=1}^{168} x_{3_i}^2 = 349.784479$

$\bar{x}_1 = 1.37761;\quad \sum_{i=1}^{168} x_{2_i}y_i = 251.280625;\quad \sum_{i=1}^{168} x_{1_i}x_{2_i} = 284.67716$

$\bar{x}_2 = 1.05103;\quad \sum_{i=1}^{168} x_{3_i}y_i = 315.567472;\quad \sum_{i=1}^{168} x_{1_i}x_{3_i} = 352.654387$

$\bar{x}_3 = 1.27261;\quad \sum_{i=1}^{168} x_{1_i}^2 = 384.327981;\quad \sum_{i=1}^{168} x_{2_i}x_{3_i} = 270.377485$

$\bar{y} = 1.20063;\quad \sum_{i=1}^{168} x_{2_i}^2 = 221.509607;\quad \sum_{i=1}^{168} y_i^2 = 296.885778$

$\sum_{i=1}^{168} x_{1_i}^2 - n\bar{x}_1^2 = 65.494584;\quad \sum_{i=1}^{168} x_{1_i}x_{2_i} - n\bar{x}_1\bar{x}_2 = 41.428439$

$\sum_{i=1}^{168} x_{1_i}x_{3_i} - n\bar{x}_1\bar{x}_3 = 58.122085;\quad \sum_{i=1}^{168} x_{2_i}^2 - n\bar{x}_2^2 = 35.926129$

$\sum_{i=1}^{168} x_{2_i}x_{3_i} - n\bar{x}_2\bar{x}_3 = 45.668373;\quad \sum_{i=1}^{168} x_{3_i}^2 - n\bar{x}_3^2 = 77.701072$

$\sum_{i=1}^{168} x_{1_i}y_i - n\bar{x}_1\bar{y} = 54.303488;\quad \sum_{i=1}^{168} x_{2_i}y_i - n\bar{x}_2\bar{y} = 39.281616$

$\sum_{i=1}^{168} x_{3_i}y_i - n\bar{x}_3\bar{y} = 58.873775$

Operation	b_1	b_2	b_3	c	Check
(1)	65.494584	41.428439	58.122085	54.303488	219.348596
(2)	41.428439	35.926129	45.668373	39.281616	162.304551
(3)	58.122085	45.668373	77.701072	58.873775	240.365305
(4) (1) repeated	65.494584	41.428439	58.122085	54.303488	219.348596
(5) (4) divided by 65.494584	1	0.63254756	0.88743346	0.82912944	3.34911046
(6) (2) −0.63254756 × (4)	0	9.720671	8.903390	4.932078	23.556139
(7) (6) divided by 9.720671	0	1	0.9159234	0.5073804	2.4233038
(8) (3) −0.88743346 (4) −0.9159234 × (6)	0	0	17.966766	6.165637	24.132403

(9) (8) divided by 17.966766 0 0 1 0.3431690 1.3431690

(10) $b_3 = 0.3431690$

(11) $b_2 = 0.5073804 - (0.9159234)(0.3431690)$
$= 0.1930639$

(12) $b_1 = 0.82912944 - (0.88743346)(0.3431690)$
$- (0.63254756)(0.1930639) = 0.4024677$

(13) $b_0 = 1.20063 - (0.4024677)(1.37761) - (0.1930639)(1.05103)$
$- (0.3431690)(1.37761) = 0.60655$

Equation:

$$\hat{y} = 0.0066 + 0.4025x_1 + 0.1931x_2 + 0.3432x_3$$

6. ANALYSIS OF VARIANCE

6.1 INTRODUCTION

In Art. 4, the problem of testing whether two means are equal was considered. The t test was used for making such a comparison. This problem is actually a special case of the more general problem of comparing several means. A possible solution to this general problem is to enumerate every possible pair, and test whether each pair differs by means of the t test. The disadvantage of this procedure is clear upon a little reflection. Even if the population mean is the same for every sample mean, we must expect that if there are enough sample means some will be extremely large and some extremely small. This will lead to one or more fictitious significant differences —with probability much higher than α. This distortion in the significance level introduces another difficulty. Suppose that the 45 possible comparisons among ten means have all been made, and four are significant by an 0.05 level t test; the experimenter must now wonder *which* (if any) of these 4 significant differences are really indicative of true differences.

The appropriate solution to the problem is the analysis of variance. The analysis of variance, however, has a much wider application than this simple generalization, and is probably the most powerful procedure in the field of experimental statistics. In this article, three problems will be treated: (1) A test for homogeneity of means will be given. (2) If the hypothesis of homogeneity is rejected, a procedure for comparing the means will be given. (3) Estimates of the variability due to a particular effect will be found.

6.2 ONE-WAY CLASSIFICATION

Suppose four different quantities of carbon are to be added to a batch of raw material in the manufacture of steel. Six specimens are taken for each of the four quantities, and it is desired to determine the effect of these percentages on the tensile stress of the resultant steel. The data are given in Table 13.36.

TABLE 13.36 TENSILE STRENGTH OF HOT ROLLED STEEL FOR DIFFERENT PERCENTAGE OF CARBON

								$\bar{x}$
Percentage 1	0.10%	23050	36000	31100	32650	30900	31400	30850
Percentage 2	0.20%	41850	25650	46700	34500	36650	31450	36133
Percentage 3	0.40%	47050	43450	43000	38650	41850	35450	41575
Percentage 4	0.60%	49650	73900	66450	74550	62400	63750	65117

It is reasonable to assume that the tensile strength of a specimen consists of the sum of two components; namely, the population mean effect attributed to the addition of the particular percentage of carbon, and a random effect. The random effects are assumed to be independently normally distributed with mean 0 and variance σ^2. Thus, the third specimen having 0.20% of carbon has a tensile strength:

46700 = population mean effect due to 0.20% carbon + random effect

All the specimens having 0.20% carbon have the same mean effect and differ only because of the random effect. More generally, these assumptions can be summarized as follows:

$$X_{ij} = \zeta_i + \epsilon_{ij}$$

where

X_{ij} is the outcome of the jth specimen corresponding to the ith treatment; $i = 1, 2, \ldots, r; j = 1, 2, \ldots, s$;
ζ_i is the population mean effect attributed to the ith treatment;
ϵ_{ij} is the random effect assumed to be independently normally distributed with mean 0 and variance σ^2.

A few words about the selection of the specimens of carbon steel to be used in the experiment are in order. The tensile strength of steel depends on many factors besides the percentage of carbon—the temperature, presence of other alloys, and so forth. It is important in running experiments to determine the effect of carbon on a particular kind of steel to make sure that differences which are asserted to be due to carbon do not arise from some other factors, say all the 0.10% carbon bars were treated at a higher temperature than the others. In some cases other factors which affect the steel may be isolated as factors in the experiment and their effect removed by the techniques of two-way, three-way, and other analysis of variance techniques discussed in the subsequent articles. If this is not done, the specimens to be used with each percentage of carbon should be chosen at random with respect to the other factors which are not specified; this random choice will insure that the conclusions are not vitiated by a coincidence of effects. A table of random numbers is useful in assigning specimens to the specific treatments.

6.2.1 Test for homogeneity. A major problem arising in the analysis of variance is to test the hypothesis that r means are equal, i.e., to test whether all the carbon effects are the same in the example.

This hypothesis can be written as H_0: $\zeta_1 = \zeta_2 = \zeta_3 = \ldots = \zeta_r$, and may

TABLE **13.37** ANALYSIS OF VARIANCE TABLES, ONE-WAY CLASSIFICATION

Source	Sum of squares	Degrees of freedom	Mean square	Average mean square	Test
Between treatments	$SS_1 = s\sum_{i=1}^{r}(\overline{X}_{i.} - \overline{X}_{..})^2$	$r - 1$	$SS^*_1 = SS_1/(r - 1)$	$\sigma^2 + \dfrac{s\sum_{i=1}^{r}(\zeta_i - \bar{\zeta})^2}{r - 1}$	$F = \dfrac{SS^*_1}{SS^*_2}$
Within treatments	$SS_2 = \sum_{i=1}^{r}\sum_{j=1}^{s}(X_{ij} - \overline{X}_{i.})^2$	$r(s - 1)$	$SS^*_2 = SS_2/r(s - 1)$	σ^2	
Total	$SS = \sum_{i=1}^{r}\sum_{j=1}^{s}(X_{ij} - \overline{X}_{..})^2$	$rs - 1$	$SS^* = SS/(rs - 1)$	$\sigma^2 + \dfrac{s\sum_{i=1}^{r}(\zeta_i - \bar{\zeta})^2}{rs - 1}$	

In this table $\overline{X}_{i.} = \sum_{j=1}^{s}\frac{X_{ij}}{s}$; $\overline{X}_{..} = \sum_{i=1}^{r}\sum_{j=1}^{s}\frac{X_{ij}}{rs}$; $\bar{\zeta} = \sum_{i=1}^{r}\frac{\zeta_i}{r}$.

be answered by making complete the analysis of variance table, Table 13.37.

Computational procedure

(1) Calculate totals for each treatment.

$$R_1, R_2, \ldots, R_r$$

(2) Calculate over-all total.

$$T = R_1 + R_2 + \ldots + R_r$$

(3) Compute crude total sum of squares.

$$\sum_{i=1}^{r} \sum_{j=1}^{s} X_{ij}^2 = X_{11}^2 + X_{12}^2 + \ldots + X_{rs}^2$$

(4) Calculate crude sum of squares between treatments.

$$\frac{\sum_{i=1}^{r} R_i^2}{s} = (R_1^2 + R_2^2 + \ldots + R_r^2)/s$$

(5) Calculate correction factor due to mean $= T^2/rs$.

From the above quantities compute

$$(6)\ SS_3 = (4) - (5) = \sum_{i=1}^{r} R_i^2/s - \frac{T^2}{rs}$$

$$(7)\ SS = (3) - (5) = \sum_{i=1}^{r} \sum_{j=1}^{s} X_{ij}^2 - \frac{T^2}{rs}$$

$$(8)\ SS_2 = (7) - (6) = SS - SS_3$$

If

$$F = \frac{SS_3/(r-1)}{SS_2/r(s-1)} = \frac{SS_3^*}{SS_2^*} \geq F_{\alpha;r-1,r(s-1)}$$

we reject the hypothesis

$$\zeta_1 = \zeta_2 = \zeta_3 = \ldots = \zeta_r$$

where $F_{\alpha;r-1,r(s-1)}$ is the upper α percentage point of the F distribution given in Table 13.27.

6.2.2 Test for comparisons of mean.

If the hypothesis

$$H_0\colon \zeta_1 = \zeta_2 = \zeta_3 = \ldots = \zeta_r$$

is rejected, it is possible to make statements about which means differ. For a long time there was no satisfactory solution to this problem. However, Tukey★ and Scheffé† have presented solutions whereby it becomes possible to make contrasts of the form $\zeta_k - \zeta_l$. Tukey's procedure leads to probability statements in the form of confidence intervals for the contrasts

★ J. Tukey, *Allowances for Various Types of Error Rates*, unpublished invited address presented before a joint meeting of the Institute of Mathematical Statistics and the Eastern North American Region of the Biometric Society on March 19, 1952, at Blacksburg, Va.

† H. Scheffé, "A Method for Judging All Contrasts in the Analysis of Variance," *Biometrika*, Vol. 40, June 1953.

TABLE **13.38** TABLE OF FACTORS c^* (5% Significance Level)★

ν \ n*	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	18.0	26.7	32.8	37.2	40.5	43.1	45.4	47.3	49.1	50.6	51.9	53.2	54.3	55.4	56.3	57.2	58.0	58.8	59.6
2	6.09	8.28	9.80	10.89	11.73	12.43	13.03	13.54	13.99	14.39	14.75	15.08	15.38	15.65	15.91	16.14	16.36	16.57	16.77
3	4.50	5.88	6.83	7.51	8.04	8.47	8.85	9.18	9.46	9.72	9.95	10.16	10.35	10.52	10.69	10.84	10.98	11.12	11.24
4	3.93	5.00	5.76	6.31	6.73	7.06	7.35	7.60	7.83	8.03	8.21	8.37	8.52	8.67	8.80	8.92	9.03	9.14	9.24
5	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32	7.47	7.60	7.72	7.83	7.93	8.03	8.12	8.21
6	3.46	4.34	4.90	5.31	5.63	5.89	6.12	6.32	6.49	6.65	6.79	6.92	7.04	7.14	7.24	7.34	7.43	7.51	7.59
7	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43	6.55	6.66	6.76	6.85	6.94	7.02	7.09	7.17
8	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18	6.29	6.39	6.48	6.57	6.65	6.73	6.80	6.87
9	3.20	3.95	4.42	4.76	5.02	5.24	5.43	5.60	5.74	5.87	5.98	6.09	6.19	6.28	6.36	6.44	6.51	6.58	6.65
10	3.15	3.88	4.33	4.66	4.91	5.12	5.30	5.46	5.60	5.72	5.83	5.93	6.03	6.12	6.20	6.27	6.34	6.41	6.47
11	3.11	3.82	4.26	4.58	4.82	5.03	5.20	5.35	5.49	5.61	5.71	5.81	5.90	5.98	6.06	6.14	6.20	6.27	6.33
12	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.40	5.51	5.61	5.71	5.80	5.88	5.95	6.02	6.09	6.15	6.21
13	3.06	3.73	4.15	4.46	4.69	4.88	5.05	5.19	5.32	5.43	5.53	5.63	5.71	5.79	5.86	5.93	6.00	6.06	6.11
14	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46	5.56	5.64	5.72	5.79	5.86	5.92	5.98	6.03
15	3.01	3.67	4.08	4.37	4.59	4.78	4.94	5.08	5.20	5.31	5.40	5.49	5.57	5.65	5.72	5.79	5.85	5.91	5.96
16	3.00	3.65	4.05	4.34	4.56	4.74	4.90	5.03	5.15	5.26	5.35	5.44	5.52	5.59	5.66	5.73	5.79	5.84	5.90
17	2.98	3.62	4.02	4.31	4.52	4.70	4.86	4.99	5.11	5.21	5.31	5.39	5.47	5.55	5.61	5.68	5.74	5.79	4.84
18	2.97	3.61	4.00	4.28	4.49	4.67	4.83	4.96	5.07	5.17	5.27	5.35	5.43	5.50	5.57	5.63	5.69	5.74	5.79
19	2.96	3.59	3.98	4.26	4.47	4.64	4.79	4.92	5.04	5.14	5.23	5.32	5.39	5.46	5.53	5.59	5.65	5.70	5.75
20	2.95	3.58	3.96	4.24	4.45	4.62	4.77	4.90	5.01	5.11	5.20	5.28	5.36	5.43	5.50	5.56	5.61	5.66	5.71
24	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92	5.01	5.10	5.18	5.25	5.32	5.38	5.44	5.50	5.55	5.59
30	2.89	3.48	3.84	4.11	4.30	4.46	4.60	4.72	4.83	4.92	5.00	5.08	5.15	5.21	5.27	5.33	5.38	5.43	5.48
40	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.74	4.82	4.90	4.98	5.05	5.11	5.17	5.22	5.27	5.32	5.36
60	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65	4.73	4.81	4.88	4.94	5.00	5.06	5.11	5.15	5.20	5.24
120	2.80	3.36	3.69	3.92	4.10	4.24	4.36	4.47	4.56	4.64	4.71	4.78	4.84	4.90	4.95	5.00	5.04	5.09	5.13
∞	2.77	3.32	3.63	3.86	4.03	4.17	4.29	4.39	4.47	4.55	4.62	4.68	4.74	4.80	4.84	4.89	4.93	4.97	5.01

* Here n is the number of effects being studied.

★ This table is taken by permission from Joyce M. May, "Extended and Corrected Tables of the Upper Percentage Points of the Studentized Range," *Biometrika*, Vol. 39, Parts 1 and 2, May 1953, p. 192; and from H. O. Hartley, "Corrigenda to Tables of Percentage Points of the Studentized Range," *Biometrika*, Vol. 40, Parts 1 and 2, June 1953, p. 236.

TABLE **13.39** TABLE OF FACTORS c^* (1% Significance Level)★

ν \ n^a	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	90.0	134	164	186	202	216	227	237	246	253	260	266	272	277	282	286	291	295	298
2	14.0	18.9	22.3	24.7	26.6	28.2	29.5	30.7	31.7	32.6	33.4	34.2	34.8	35.5	36.0	36.5	37.0	37.5	38.0
3	8.26	10.56	12.17	13.34	14.25	15.00	15.65	16.20	16.69	17.13	17.53	17.89	18.23	18.54	18.83	19.09	19.33	19.56	19.79
4	6.51	8.08	9.17	9.97	10.58	11.10	11.55	11.93	12.26	12.56	12.84	13.09	13.32	13.53	13.73	13.92	14.09	14.25	14.40
5	5.70	6.97	7.80	8.42	8.91	9.32	9.67	9.97	10.24	10.48	10.70	10.89	11.08	11.24	11.40	11.55	11.68	11.81	11.93
6	5.24	6.32	7.03	7.56	7.97	8.31	8.61	8.87	9.10	9.30	9.49	9.65	9.81	9.95	10.08	10.21	10.32	10.43	10.54
7	4.95	5.92	6.54	7.01	7.37	7.68	7.94	8.17	8.37	8.55	8.71	8.86	9.00	9.12	9.24	9.35	9.46	9.55	9.65
8	4.74	5.63	6.20	6.63	6.96	7.24	7.47	7.68	7.86	8.03	8.18	8.31	8.44	8.55	8.66	8.76	8.86	8.95	9.03
9	4.60	5.42	5.96	6.35	6.66	6.91	7.13	7.33	7.50	7.65	7.79	7.91	8.02	8.13	8.23	8.33	8.41	8.50	8.58
10	4.48	5.26	5.77	6.14	6.43	6.67	6.88	7.06	7.22	7.36	7.49	7.60	7.71	7.81	7.91	7.99	8.08	8.15	8.23
11	4.39	5.14	5.62	5.98	6.25	6.47	6.67	6.84	6.99	7.13	7.25	7.36	7.46	7.56	7.65	7.73	7.81	7.88	7.95
12	4.32	5.04	5.50	5.84	6.10	6.32	6.51	6.67	6.81	6.94	7.06	7.17	7.26	7.36	7.44	7.52	7.60	7.67	7.73
13	4.26	4.96	5.40	5.73	5.98	6.19	6.37	6.53	6.67	6.79	6.90	7.01	7.10	7.19	7.27	7.35	7.42	7.49	7.55
14	4.21	4.89	5.32	5.64	5.88	6.08	6.26	6.41	6.54	6.66	6.77	6.87	6.96	7.05	7.13	7.20	7.27	7.34	7.40
15	4.17	4.83	5.25	5.56	5.80	5.99	6.16	6.31	6.44	6.55	6.66	6.76	6.85	6.93	7.00	7.07	7.14	7.20	7.26
16	4.13	4.78	5.19	5.49	5.72	5.91	6.08	6.22	6.35	6.46	6.56	6.66	6.74	6.82	6.90	6.97	7.03	7.09	7.15
17	4.10	4.73	5.14	5.43	5.66	5.85	6.01	6.15	6.27	6.38	6.48	6.57	6.66	6.73	6.81	6.87	6.94	7.00	7.05
18	4.07	4.70	5.09	5.38	5.60	5.79	5.95	6.08	6.20	6.31	6.41	6.50	6.58	6.65	6.73	6.79	6.85	6.91	6.97
19	4.05	4.66	5.05	5.34	5.55	5.73	5.89	6.02	6.14	6.25	6.34	6.43	6.51	6.58	6.65	6.72	6.78	6.84	6.89
20	4.02	4.63	5.02	5.30	5.51	5.69	5.84	5.97	6.09	6.19	6.28	6.37	6.45	6.52	6.59	6.66	6.71	6.77	6.82
24	3.96	4.54	4.91	5.17	5.37	5.54	5.69	5.81	5.92	6.02	6.11	6.19	6.26	6.33	6.39	6.45	6.51	6.57	6.61
30	3.89	4.45	4.80	5.05	5.24	5.40	5.53	5.65	5.76	5.85	5.93	6.01	6.08	6.14	6.20	6.26	6.31	6.36	6.41
40	3.82	4.36	4.70	4.93	5.11	5.26	5.39	5.50	5.60	5.69	5.77	5.84	5.90	5.96	6.02	6.07	6.12	6.17	6.21
60	3.76	4.28	4.60	4.82	4.99	5.13	5.25	5.36	5.45	5.53	5.60	5.67	5.73	5.78	5.83	5.88	5.93	5.98	6.01
120	3.70	4.20	4.50	4.71	4.87	5.00	5.12	5.21	5.30	5.38	5.44	5.50	5.56	5.61	5.66	5.71	5.75	5.79	5.83
∞	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16	5.23	5.29	5.35	5.40	5.45	5.49	5.53	5.57	5.61	5.64

[a] Here n is the number of effects being studied.

★ This table is taken by permission from May, "Extended and Corrected Tables of the Upper Percentage Points of the Studentized Range"; and from Hartley, "Corrigenda to Tables of Percentage Points of the Studentized Range."

$\zeta_k - \zeta_l$; i.e., the probability is $1 - \alpha$ that the values $\zeta_k - \zeta_l$ for *all* such contrasts ($k = 1, 2, \ldots, r$; $l = 1, 2, \ldots, r$) simultaneously satisfy

$$\bar{X}_{k.} - \bar{X}_{l.} - c \leq \zeta_k - \zeta_l \leq \bar{X}_{k.} - \bar{X}_{l.} + c$$

where c is a factor obtained from Table 13.38 or 13.39 depending on whether $\alpha = 0.05$ or 0.01 and from the analysis of variance table. The means are said to differ significantly when the appropriate confidence interval fails to include 0. Furthermore, if the interval includes only positive values, the difference is said to differ significantly from zero, and to be positive. Similarly, if the confidence interval includes only negative values, the difference is said to differ significantly from zero, and to be negative. If the interval includes 0, the means are said not to differ significantly. Summarizing then, the F test is made for the homogeneity of the means. If this hypothesis is rejected, Tukey's procedure is applied to determine which of the means differ.

A possible short-cut method for determining the homogeneity of all the treatment effects is to check whether $\bar{X}_{\max.} - \bar{X}_{\min.} \pm c$ includes 0. If this contrast includes 0, every other contrast of this type includes 0, and hence all the treatment effects are called equal. If the above contrast fails to include zero, the corresponding treatment effects, and perhaps others, differ significantly. This short-cut method has a level of significance exactly equal to α. However, the data in the analysis of variance table are necessary for determining the factor c, and for other results, so that this table should be completed regardless.

The factor c can be obtained from the data in the analysis of variance table and Table 13.38 or 13.39 depending on the level of significance. Table 13.38 or 13.39 is entered with the indices degrees of freedom and the number of treatments studied (r). The proper degrees of freedom are those corresponding to the degrees of freedom of the within treatments source in the analysis of variance table, i.e., $r(s - 1)$. The factor c^* is read from the table; c is obtained from $c = c^*\sqrt{SS_2^*/s}$.

Scheffé's procedure also leads to confidence statements and if we apply his test when the F test of homogeneity is rejected, the level of significance is preserved. On the other hand, if we apply Tukey's test when the F test of homogeneity is rejected, we increase the significance level *slightly*. However, if the only type of comparison of interest has the form $\zeta_k - \zeta_l$, Tukey's procedure will result in shorter confidence intervals for $\zeta_k - \zeta_l$ than would Scheffé's procedure.★ Consequently, we are going to use the Tukey method, and proceed as if the significance level suffered a negligible increase.

6.2.3 Estimating the variability due to the sources. In the analysis of variance table there is a column headed "Average mean square." Estimates of these quantities are given in the column headed "Mean square." If all the treatment means are equal, the entries in the "Average mean square" column would be σ^2. Consequently, it is expected, if all the means are equal, that the estimates given in the "Mean square" column will be close together. On the

★ If other type of contrasts are also desired, e.g., $(\zeta_1 + \zeta_2 + \zeta_3)/3 - (\zeta_4 + \zeta_5 + \zeta_6)/3$, Scheffé's procedure should be used.

other hand, if the treatments are in fact different, then the average value of the mean square for treatments will be increased by $[s/(r-1)] \sum (\zeta_i - \bar{\zeta})^2$, which will tend to increase the value of F, leading to a significant result and the detection of the differences.

6.2.4 Example. Returning to the data on the tensile strength of carbon steel, the complete analysis of variance table is as follows:

ANALYSIS OF VARIANCE TABLE FOR TENSILE STRENGTH

Source	Sum of squares	Degrees of freedom	Mean square	Average mean square	Test
Between treatments	4,111,498,600	3	1,370,499,533		$F = 31.576$
Within treatments	868,060,500	20	43,403,025		
Total	4,979,559,100	23	216,502,670		

The completed computational procedure is given below:

(1) $R_1, R_2, \ldots, R_r = 185{,}100;\ 216{,}800;\ 249{,}450;\ 390{,}700$

(2) $T = \sum_{i=1}^{4} R_i = 185{,}100 + 216{,}800 + 249{,}450 + 390{,}700 = 1{,}042{,}050$

(3) $$\sum_{i=1}^{4} \sum_{j=1}^{6} x_{ij}^2 = (23{,}050)^2 + (36{,}000)^2 + \ldots + (63{,}750)^2$$
$$= 50{,}224{,}067{,}500$$

(4) $$\frac{\sum_{i=1}^{4} R_i^2}{6} = \frac{(185{,}100)^2 + (216{,}800)^2 + (249{,}450)^2 + (390{,}700)^2}{6}$$
$$= 49{,}356{,}007{,}000$$

(5) $\frac{T^2}{24} = \frac{1{,}085{,}868{,}202{,}500}{24} = 45{,}244{,}508{,}400$

(6) $SS_3 = 49{,}356{,}007{,}000 - 45{,}244{,}508{,}400 = 4{,}111{,}498{,}600$

(7) $SS = 50{,}224{,}067{,}500 - 45{,}244{,}508{,}400 = 4{,}979{,}559{,}100$

(8) $SS_2 = 4{,}979{,}559{,}100 - 4{,}111{,}498{,}600 = 868{,}060{,}500$

Since $F_{.05;3,20} = 3.10$ and $F = 31.576 > 3.10$, the hypothesis that $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4$ is rejected and we conclude that the tensile strengths differ. In order to determine the magnitude of these differences, we compute c for $r = 4$, $r(s-1) = 20$. Here $c^* = 3.96$, where the factor 3.96 is obtained from Table 13.38.

Therefore, $c = 3.96(2689) = 10{,}648$, and any two means differing by this much differ significantly at $\alpha = 0.05$. Thus the following differences are significant.

0.60 from 0.40
0.60 from 0.20
0.60 from 0.10
0.40 from 0.10

Confidence intervals of size 0.95 for the differences can be given, e.g., the confidence interval for difference $\zeta_{.60} - \zeta_{.10}$ is:

$$\bar{X}_{.60} - \bar{X}_{.10} \pm c = 34{,}267 \pm 10{,}648$$

so that

$$23{,}619 \leq \zeta_{.60} - \zeta_{.10} \leq 44{,}915$$

Similarly,

$$\bar{X}_{.60} - \bar{X}_{.40} \pm c = 23{,}542 \pm 10{,}648$$

so that

$$12{,}894 \leq \zeta_{.60} - \zeta_{.40} \leq 34{,}190$$

6.2.5 Randomization tests in the analysis of variance. It has been emphasized in the discussion of the t test in the analysis of variance that the random effects are assumed to be normally and independently distributed. These tests actually have a wider justification if they are considered special cases of what are called permutation tests. Suppose, for example, we are interested in testing the difference between two effects, say percentage of carbon, and suppose that the experimenter takes eight samples (let them be named a, b, c, etc.) and chooses four with a table of random numbers to receive treatment A and the other four treatment B. The results of this experiment are:

	A		B
a	2	f	17
h	15	e	19
b	10	c	18
d	9	g	12

He could test the hypothesis that there is no difference between treatments by calculating a t test or an F test, or consider it a one-way analysis of variance, calculating an F with 1 and 6 degrees of freedom; the value of F is 6 in this problem. Suppose he is interested in a level of significance of 0.05. Even if he is unwilling to assume normality, he can obtain a test of the hypothesis as follows: Consider all possible pairs of samples that can be formed with the eight numbers above; there are $\binom{8}{4} = 70$ different ways for the experiment to turn out (one for each way of selecting 4 of the eight numbers for treatment A). If there is no difference between treatments, each of these outcomes is equally likely. Suppose we adopt the procedure that we will reject the hypothesis of equality of treatments if we have one of the samples with the four largest differences in the mean; the probability of rejection when the hypothesis is true is $4/70 = 0.057$, since all 70 possibilities are equally likely. If the hypothesis is false, large differences are more likely.

A partial list of the 70 outcomes (starting with most extreme ones) is:

A				B				difference in means $\bar{x}_A - \bar{x}_B$	F
2	9	10	12	15	17	18	19	-9	14.84
15	17	18	19	2	9	10	12	$+9$	
2	9	10	15	12	17	18	19	-7.5	6.00
12	17	18	19	2	9	10	15	$+7.5$	

A				B				difference in means $\bar{x}_A - \bar{x}_B$	F
2	9	10	17	12	15	18	19	−6.5	3.61
12	15	18	19	2	9	10	17	+6.5	
2	9	12	15	10	17	18	19	−6.5	
10	17	18	19	2	9	12	15	+6.5	

Observe that only 4 of the 70 possibilities give values of F as large as that in the particular sample obtained.

The significance level of the test is 0.057. If $F = 6$ is looked up in the table it is seen to be significant at the 0.05 level (using the assumption of normality).

This kind of test is called a randomization test. It has the drawback that for medium or large-sized samples the computational burden is enormous. For example, if there are 20 objects divided at random into two groups of 10, then the number of possibilities is $\binom{20}{10} = 184{,}756$, and listing the 5% most extreme cases would be prohibitive.

Both numerical and analytical investigations★ have shown, however, that for medium and large samples, the distribution of F obtained through enumeration of the possibilities is closely approximated by the tabulated F distribution. So without assuming normality we *can* test the hypothesis of no treatment effects by making a randomization test, but instead of exactly evaluating the significance level by actual counting we content ourselves with an approximate evaluation, as yielded by the F tables.

6.3 TWO-WAY CLASSIFICATION

The reader is advised to read the section on the one-way classification before going on to this section. Experiments can be conducted in such a way as to study the effects of several variables in the same experiment. For each variable a number of levels may be chosen for study. When observations are made for all possible combinations of levels the experiment is called a factorial experiment. This section is concerned with a factorial experiment where the effects of two variables at several levels are to be studied. For example, W. J. Youden† reports the following experiment. Counts are made on four samples of radium. If the four specimens are run consecutively, the four counts may be called experiment 1. The experiment may be repeated as often as desired, varying the order of the specimens at random in each experiment. The results of this investigation are given in Table 13.40 where the data are given as the net counting rates less 25.00. The experimenter is interested in knowing whether these samples of radium are different. However, it is possible that the line voltage or other conditions that influence the counting rate varies from experiment to experiment.

★ The agreement between the two values 0.05 and 0.057 above is not merely by coincidence.

† W. J. Youden, "The Interpretation of Chemical Data," *Industrial Quality Control*, May 1952.

TABLE **13.40** NEW COUNTING RATES LESS 25.00

Specimen number	Experiment number				Specimen average
	1	2	3	4	
1	1.46	2.00	1.48	1.51	1.61
2	2.58	3.03	2.82	2.90	2.83
3	4.15	4.61	4.31	4.13	4.30
4	4.54	4.52	4.53	4.15	4.44
Experiment average	3.18	3.54	3.28	3.17	

Thus the model can be written

$$X_{ij} = \zeta_{ij} + \epsilon_{ij}; \qquad i = 1, 2, 3, 4; \;\; j = 1, 2, 3, 4$$

where

X_{ij} is the counting rate of the ith specimen in the jth experiment;
ζ_{ij} is the population mean counting rate of the ith specimen in the jth experiment;
ϵ_{ij} is the random effect assumed to be normally independently distributed with mean 0 and variance σ^2.

6.3.1 **Interaction.** Before proceeding to make inferences about the structure of the ζ_{ij}, it is useful to refer to Table 13.41, which is a table of population means.

TABLE **13.41** TABLE OF POPULATION MEANS FOR COUNTING RATES

Specimen number	Experiment number				Specimen mean	Differences
	1	2	3	4		
1	ζ_{11}	ζ_{12}	ζ_{13}	ζ_{14}	$\zeta_{1.}$	$\varphi_1 = \zeta_{1.} - \zeta_{..}$
2	ζ_{21}	ζ_{22}	ζ_{23}	ζ_{24}	$\zeta_{2.}$	$\varphi_2 = \zeta_{2.} - \zeta_{..}$
3	ζ_{31}	ζ_{32}	ζ_{33}	ζ_{34}	$\zeta_{3.}$	$\varphi_3 = \zeta_{3.} - \zeta_{..}$
4	ζ_{41}	ζ_{42}	ζ_{43}	ζ_{44}	$\zeta_{4.}$	$\varphi_4 = \zeta_{4.} - \zeta_{..}$
Experiment mean	$\zeta_{.1}$	$\zeta_{.2}$	$\zeta_{.3}$	$\zeta_{.4}$	$\zeta_{..}$	
Differences	$\gamma_1 = \zeta_{.1} - \zeta_{..}$	$\gamma_2 = \zeta_{.2} - \zeta_{..}$	$\gamma_3 = \zeta_{.3} - \zeta_{..}$	$\gamma_4 = \zeta_{.4} - \zeta_{..}$		

Here $\zeta_{i.} = \frac{1}{4}\sum_{j=1}^{4} \zeta_{ij}$; $\zeta_{.j} = \frac{1}{4}\sum_{i=1}^{4} \zeta_{ij}$; $\zeta_{..} = \frac{1}{16}\sum_{i=1}^{4}\sum_{j=1}^{4} \zeta_{ij}$. Thus a mean ζ_{ij} can be written

$$\begin{aligned} \zeta_{ij} &= \zeta_{..} + (\zeta_{i.} - \zeta_{..}) + (\zeta_{.j} - \zeta_{..}) + (\zeta_{ij} - \zeta_{i.} - \zeta_{.j} + \zeta_{..}) \\ &= \zeta_{..} + \varphi_i + \gamma_j + \eta_{ij} \end{aligned}$$

where $\eta_{ij} = \zeta_{ij} - \zeta_{i.} - \zeta_{.j} + \zeta_{..}$ and is known as the interaction term. We say that an interaction between two effects, φ_i and γ_j, exists if the joint effect of the two taken simultaneously is different from the sum of their separate effects. Thus either lye or muriatic acid may be an effective cleanser, but taken together they are ineffective, and would be said to interact. Or again, suppose there are five machines together with four workmen and it is desired to test whether machines differ in the number of units produced per day. It is quite possible that the second man may work better on the third machine than any other machine because he has worked for 20 years on such a machine. The resultant increased production cannot be assumed to be a characteristic of this man or of this particular machine, but a combination of only this man with only this machine, and we would say that interaction was present. In the counting rate problem we expect any cause raising the reading for one specimen in an experiment to raise all readings similarly, and so we may confidently assume the interactions between experiments and specimens to be zero, i.e., $\eta_{ij} = 0$ for all i and j.

Returning to the table of means, we find the following linear relationships:

$$\sum_{i=1}^{4} \varphi_i = 0; \quad \sum_{j=1}^{4} \gamma_j = 0; \quad \sum_{i=1}^{4} \eta_{ij} = 0; \quad \sum_{j=1}^{4} \eta_{ij} = 0$$

Furthermore, instead of the original 16 means, we have introduced new parameters which will supply the results. By writing $\zeta_{ij} = \zeta_{..} + \varphi_i + \gamma_j + \eta_{ij}$ we can write the mean for the ith specimen and jth experiment as a constant, $\zeta_{..}$, plus an effect due to the ith specimen which is constant over all columns, φ_i, plus an effect due to the jth experiment which is constant over all rows, γ_j, plus the interaction between the ith specimen and jth experiment. Testing whether (1) the specimens are homogeneous, (2) the experiments are homogeneous, and (3) whether there is any interaction is equivalent to testing:

(1) $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0$;
(2) $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$;
(3) $\eta_{ij} = 0$

for all i and j.

6.4 TWO-WAY CLASSIFICATION—ONE OBSERVATION PER CELL AND NO INTERACTION ASSUMED

In order to make an analysis with only one observation per cell it is necessary to assume that the interaction is 0. The model is as follows:

$$X_{ij} = \zeta_{..} + \varphi_i + \gamma_j + \epsilon_{ij}$$

where

X_{ij} is the outcome of the experiment using the ith row effect and jth column effect, $i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, s$;

$\zeta_{..}$ is a general mean;

φ_i is the effect of adding the ith row treatment; $\sum_{i=1}^{r} \varphi_i = 0$;

γ_j is the effect of adding the jth column treatment; $\sum_{j=1}^{s} \gamma_j = 0$;

ϵ_{ij} is the random effect, which is independently normally distributed with mean 0 and variance σ^2.

6.4.1 Tests for homogeneity. Complete the following analysis of variance table (Table 13.42):

Computational procedure for analysis of variance table is:

(1) Calculate row totals:

$R_1, R_2, \ldots, R_r$

(2) Calculate column totals:

$C_1, C_2, \ldots, C_s$

(3) Calculate over-all total:

$T = R_1 + R_2 + \ldots + R_r$

(4) Calculate crude total sum of squares:

$$\sum_{i=1}^{r} \sum_{j=1}^{s} X_{ij}^2 = X_{11}^2 + X_{12}^2 + \ldots + X_{rs}^2$$

(5) Calculate crude sum of squares between rows:

$$\sum_{i=1}^{r} \frac{R_i^2}{s} = (R_1^2 + R_2^2 + \ldots + R_r^2)/s$$

(6) Calculate crude sum of squares between columns:

$$\sum_{j=1}^{s} \frac{C_j^2}{r} = (C_1^2 + C_2^2 + \ldots + C_s^2)/r$$

(7) Calculate correction factor due to mean $= T^2/rs$.

From the above quantities compute

(8) $SS_4 = (6) - (7) = \sum_{j=1}^{s} \frac{C_j^2}{r} - T^2/rs$

(9) $SS_3 = (5) - (7) = \sum_{i=1}^{r} \frac{R_i^2}{s} - T^2/rs$

(10) $SS = (4) - (7) = \sum_{i=1}^{r} \sum_{j=1}^{s} X_{ij}^2 - T^2/rs$

(11) $SS_2 = (10) - (9) - (8) = SS - SS_3 - SS_4$

If
$$F_4 = \frac{SS_4/(s-1)}{SS_2/(r-1)(s-1)} = \frac{SS_4^*}{SS_2^*} \geq F_{\alpha;s-1,(r-1)(s-1)}$$

we reject the hypothesis that there is no column effect, i.e.,

$$\gamma_1 = \gamma_2 = \gamma_3 = \ldots = \gamma_s = 0$$

TABLE **13.42** ANALYSIS OF VARIANCE TABLE, TWO-WAY CLASSIFICATION, ONE OBSERVATION PER CELL

Source	Sum of squares	Degrees of freedom	Mean square	Average mean square	Test
Between columns	$SS_4 = r \sum_{j=1}^{s} (\overline{X}_{.j} - \overline{X}_{..})^2$	$s - 1$	$SS^*_4 = SS_4/(s-1)$	$\sigma^2 + \frac{r}{s-1} \sum_{j=1}^{s} \gamma_j^2$	$F_4 = SS^*_4/SS^*_2$
Between rows	$SS_3 = s \sum_{i=1}^{r} (\overline{X}_{i.} - \overline{X}_{..})^2$	$r - 1$	$SS^*_3 = SS_3/(r-1)$	$\sigma^2 + \frac{s}{r-1} \sum_{i=1}^{r} \varphi_i^2$	$F_3 = SS^*_3/SS^*_2$
Residue	$SS_2 = \sum_{i=1}^{r} \sum_{j=1}^{s} (X_{ij} - \overline{X}_{i.} - \overline{X}_{.j} + \overline{X}_{..})^2$	$(r-1)(s-1)$	$SS^*_2 = SS_2/(r-1)(s-1)$	σ^2	
Total	$SS = \sum_{i=1}^{r} \sum_{j=1}^{s} (X_{ij} - \overline{X}_{..})^2$	$rs - 1$	$SS^* = SS/(rs-1)$	$\sigma^2 + \frac{r}{rs-1} \sum_{j=1}^{s} \gamma_j^2 + \frac{s}{rs-1} \sum_{i=1}^{r} \varphi_i^2$	

In this table, $\overline{X}_{.j} = \sum_{i=1}^{r} X_{ij}/r$; $\overline{X}_{i.} = \sum_{j=1}^{s} X_{ij}/s$; $\overline{X}_{..} = \sum_{i=1}^{r} \sum_{j=1}^{s} X_{ij}/rs$

where $F_{\alpha;s-1,(r-1)(s-1)}$ is the upper α percentage point of the F distribution given in Table 13.27.

If
$$F_3 = \frac{SS_3/(r-1)}{SS_2/(r-1)(s-1)} = \frac{SS_3^*}{SS_2^*} \geq F_{\alpha;r-1,(r-1)(s-1)}$$

we reject the hypothesis that there is no row effect, i.e.,

$$\varphi_1 = \varphi_2 = \varphi_3 = \ldots = \varphi_r = 0$$

where $F_{\alpha;r-1,(r-1)(s-1)}$ is the upper α percentage point of the F distribution given in Table 13.27.

6.4.2 Tests for comparison of mean effect. If the hypothesis that there is no row effects or no column effects, or both, is rejected, Tukey's method can again be applied to get confidence intervals on the difference between two row effects or two column effects. The probability is $1 - \alpha$ that the value $(\varphi_k - \varphi_l)$ for *all* such contrasts (for $k = 1, 2, \ldots, r$; $l = 1, 2, \ldots, r$) simultaneously satisfy

$$\bar{X}_{k.} - \bar{X}_{l.} - c \leq \varphi_k - \varphi_l \leq \bar{X}_{k.} - \bar{X}_{l.} + c$$

and the probability is $1 - \alpha$ that the values $(\gamma_m - \gamma_n)$ for *all* such contrasts (for $m = 1, 2, \ldots, s$; $n = 1, 2, \ldots, s$) simultaneously satisfy

$$\bar{X}_{.m} - \bar{X}_{.n} - c \leq \gamma_m - \gamma_n \leq \bar{X}_{.m} - \bar{X}_{.n} + c$$

Again two mean effects are judged significantly different when 0 is not included in the confidence interval.

The factor c for row effects and for column effects is obtained from the analysis of variance table and from Table 13.38 or Table 13.39 depending on the level of significance. Table 13.38 or 13.39 is entered with the indices degrees of freedom, and the number of row effects being studied (r) if row effects are of interest. The proper degrees of freedom are those corresponding to the degrees of freedom of the residual source in the analysis of variance table, i.e., $(r - 1)(s - 1)$. The factor c^* is read from the table, and

$$c \text{ (for rows)} = c^*\sqrt{SS_2^*/s}$$

Similarly, if column effects are of interest, the table is entered with indices degrees of freedom and the number of column effects being studied (s). The factor c^* is read from the table and

$$c \text{ (for columns)} = c^*\sqrt{SS_2^*/r}$$

6.4.3 Estimating the variability due to the sources. If there were no row effects and no column effects, the column headed "Average mean square" in the analysis of variance table would contain σ^2 for the row source entry and σ^2 for the column source entry. The residual entry is always σ^2. The entry for the residual source, SS_2^*, in the "Mean square" column estimates σ^2. The values in the "Mean square" column for row effects, SS_3^*, and column effects, SS_4^*, estimate the corresponding value in the "Average mean square" column. The value of SS^* is an estimate of the total variability of the process (corresponding to the "Average mean square" entry). Since

$[(r-1)/s](SS_3^* - SS_2^*)$ is an estimate of $\sum_{i=1}^{r} \varphi_i^2$, and $[(s-1)/r](SS_4^* - SS_2^*)$ is an estimate of $\sum_{j=1}^{s} \gamma_j^2$, the contribution of these main effects to the total variability of the process can be estimated.

In some investigations, concern with reducing the total variability around the general mean is paramount. In such a case, this kind of analysis can be of value in determining the relative sizes of various contributions to the variability. In many other experiments, differences around the general mean are more to be identified than reduced, and this kind of analysis is irrelevant.

Many experiments which can be handled as a two-way analysis of variance might alternatively be attacked as a simpler one-way analysis of variance. For example, in the experiment on radium counts, rows correspond to specimens (difference among which we wish to estimate) and columns to experiments, where each specimen is handled in each experiment; this is a two-way plan. Alternatively, we *could* simply take 16 readings in random order, with each specimen being measured four times. The analysis of this second plan is arithmetically easier but it would seem to be a less precise experiment. Using the data and information given in the "Average mean square" column we can estimate the increase in precision resulting from using the two-way plan. This is done in the following way.

Calculate $SS_2^* + [r/(rs-1)] \sum \gamma_j^2$. This is an estimate of what the residual mean square would be had the experiment been done as a one-way analysis of variance. Clearly, if $\sum \gamma_j^2$ is large this means that the residual mean square for the one-way plan would be inflated from the value obtained by using the two-way plan. The ratio

$$e = \frac{SS_2^* + [r/(rs-1)] \sum \gamma_j^2}{SS_2^*}$$

is called the efficiency of the two-way plan relative to the one-way plan. And we can say approximately that for the same precision as that given by n observations in the two-way plan, $e \times n$ would be needed for a one-way plan.

6.4.4 Example. Let us return to the example of the counting rates for different specimens of radium using different experiments. The completed analysis of variance table is obtained by following the computational procedure.

(1) $R_1, R_2, R_3, R_4 = 6.45, 11.33, 17.20, 17.74$

(2) $C_1, C_2, C_3, C_4 = 12.73, 14.16, 13.14, 12.69$

(3) $T = 6.45 + 11.33 + 17.20 + 17.74 = 52.72$

(4) $\sum_{i=1}^{4} \sum_{j=1}^{4} X_{ij}^2 = (1.46)^2 + (2.00)^2 + \ldots + (4.15)^2$
$= 195.6948$

(5) $\sum_{i=1}^{4} \frac{R_i^2}{4} = \frac{(6.45)^2 + (11.33)^2 + (17.20)^2 + (17.74)^2}{4}$
$= 195.1298$

(6) $\sum_{i=1}^{4} \frac{C_j^2}{4} = \frac{(12.73)^2 + (14.16)^2 + (13.14)^2 + (12.69)^2}{4}$

$= 174.06355$

(7) $T^2/16 = (52.72)^2/16 = 173.7124$

(8) $SS_4 = 174.06355 - 173.7124 = 0.3512$

(9) $SS_3 = 195.1298 - 173.7124 = 21.4174$

(10) $SS = 195.6948 - 173.7124 = 21.9824$

(11) $SS_2 = 21.9824 - 21.4174 - 0.3512 = 0.2138.$

TABLE **13.43** COMPLETED ANALYSIS OF VARIANCE TABLE FOR COUNTING RATE PROBLEM

Source	Sum of squares	Degrees of freedom	Mean square	Test
Between experiments	0.3512	3	0.1171	$F_4 = 4.93$
Between specimens	21.4174	3	7.1391	$F_3 = 300.5$
Residual	0.2138	9	0.0238	
Total	21.9824	15	1.4655	

$F_{.05;3,9} = 3.86$

Consequently, we conclude that the specimen effects differ significantly, and the experiment effects also differ significantly. Hence, determining the largest effects is in order.

For experiments and from Table 13.38 for $\alpha = 0.05$ and for $r = 4$, $s = 4$.

$$c^* = 4.42$$

so that

$$c = 4.42\sqrt{\frac{SS_2^*}{4}} = 0.3408$$

Making the comparison between the second and fourth experiment effect,

$$(3.54 - 3.17) - 0.34 \leq \gamma_2 - \gamma_4 \leq (3.54 - 3.17) + 0.34$$
$$0.03 \leq \gamma_2 - \gamma_4 \leq 0.71$$

It is evident that they differ significantly. Furthermore, the difference $\gamma_2 - \gamma_4$ can be said to lie between 0.03 and 0.71.

It is evident by inspection that the only other significant difference is between γ_2 and γ_1. Hence we can conclude that experiment 2 produced an effect on the counting rate which was significantly different from the others.

For specimens, c^* also equals 4.42, so that c is again 0.3408.

Making the comparison between the third and fourth specimen effects,

$$(4.44 - 4.30) - 0.34 \leq \varphi_4 - \varphi_3 \leq (4.44 - 4.30) + 0.34$$
$$-0.20 \leq \varphi_4 - \varphi_3 \leq 0.48$$

It appears that these two specimens do *not* differ. From inspection of the data, it is evident that these are the only two specimens that do not differ significantly. The difference between specimen effect 4 and specimen effect 1 can be said to lie between 2.49 and 3.17, i.e.,

$$2.49 \leq \varphi_4 - \varphi_1 \leq 3.17$$

6.4.5 Estimates of variability. The estimate of

$$\sigma^2 + \frac{r}{s-1}\sum_{j=1}^{s} \gamma_j^2 = \sigma^2 + \frac{4}{3}\sum_{j=1}^{4} \gamma_j^2$$

is 0.1171. The estimate of σ^2 is 0.0238. Hence, the estimate of $\sum_{j=1}^{4} \gamma_j^2$ is

$$\tfrac{3}{4}(0.1171 - 0.0238) = 0.0699$$

Similarly, the estimate of

$$\sigma^2 + \frac{s}{r-1}\sum_{i=1}^{r} \varphi_i^2 = \sigma^2 + \frac{4}{3}\sum_{i=1}^{4} \varphi_i^2$$

is 7.1391. Hence the estimate of $\sum_{i=1}^{4} \varphi_i^2$ is

$$\tfrac{3}{4}(7.1391 - 0.0238) = 5.3364$$

We may now compute the value of

$$SS_2^* + \frac{r}{rs-1}\sum \gamma_j^2 = 0.0238 + \frac{4}{(4)(4)-1}(0.0699) = 0.0424$$

and the efficiency of the two-way plan used, relative to a one-way plan (where the 16 readings would not be taken in sets of four) is: $0.0424/0.0238 = 1.78$, which means that about $16 \times 1.78 = 29$ observations taken in random order would be needed to give as precise results as those obtained by using the two-way plan.

6.5 TWO-WAY CLASSIFICATION—MORE THAN ONE OBSERVATION PER CELL—INTERACTION

The model is as follows:

$$X_{ijk} = \zeta_{..} + \varphi_i + \gamma_j + \eta_{ij} + \epsilon_{ijk}$$

where

X_{ijk} is the outcome of the kth experiment using the ith row effect and jth column effect, $i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, s$; $k = 1, 2, \ldots, v$;

$\zeta_{..}$ is a general mean;

φ_i is the effect of adding the ith row treatment, $\sum_{i=1}^{r} \varphi_i = 0$;

γ_j is the effect of adding the jth column treatment, $\sum_{j=1}^{s} \gamma_j = 0$;

η_{ij} is the interaction of the ith row treatment with the jth column treatment, $\sum_{i=1}^{r} \eta_{ij} = 0, \sum_{j=1}^{s} \eta_{ij} = 0$;

ϵ_{ijk} is the random effect which is independently normally distributed with mean 0 and variance σ^2.

6.5.1 Tests for homogeneity. Complete Table 13.44. Computational procedure for analysis of variance table:

(1) Calculate row totals:

$R_1, R_2, \ldots, R_r$

(2) Calculate column totals:

$C_1, C_2, \ldots, C_s$

(3) Calculate within cell totals:

$w_{11}, w_{12}, \ldots, w_{rs}$

(4) Calculate over-all total:

$T = R_1 + R_2 + \ldots + R_r$

(5) Calculate crude sum of squares:

$$\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{v} X_{ijk}^2 = X_{111}^2 + X_{112}^2 + \ldots + X_{rsv}^2$$

(6) Calculate crude sum of squares between columns:

$$\sum_{j=1}^{s} \frac{C_j^2}{rv} = \frac{(C_1^2 + C_2^2 + \ldots + C_s^2)}{rv}$$

(7) Calculate crude sum of squares between rows:

$$\sum_{i=1}^{r} \frac{R_i^2}{sv} = \frac{(R_1^2 + R_2^2 + \ldots + R_r^2)}{sv}$$

(8) Calculate crude sum of squares between cells:

$$\sum_{i=1}^{r} \sum_{j=1}^{s} \frac{w_{ij}^2}{v} = \frac{(w_{11}^2 + w_{12}^2 + \ldots + w_{rs}^2)}{v}$$

(9) Calculate correct factor due to mean T^2/rsv.

From the above quantities compute:

(10) $SS_4 = (6) - (9) = \sum_{j=1}^{r} \frac{C_j^2}{rv} - \frac{T^2}{rsv}$

(11) $SS_3 = (7) - (9) = \sum_{i=1}^{r} \frac{R_i^2}{sv} - \frac{T^2}{rsv}$

(12) $SS_1 = (5) - (8)$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{v} X_{ijk}^2 - \sum_{i=1}^{r} \sum_{j=1}^{s} w_{ij}^2/v$$

(13) $SS = (5) - (9)$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{v} X_{ijk}^2 - \frac{T^2}{rsv}$$

TABLE **13.44** ANALYSIS OF VARIANCE TABLE—TWO-WAY CLASSIFICATION—INTERACTION

Source	Sum of squares	Degrees of freedom	Mean square	Average mean square	Test
Between columns	$SS_4 = rv\sum_{j=1}^{s}(\overline{X}_{.j.} - \overline{X}_{...})^2$	$s - 1$	$SS^*_4 = SS_4/(s-1)$	$\sigma^2 + \frac{rv}{s-1}\sum_{j=1}^{s}\gamma_j^2$	$F_4 = SS^*_4/SS^*_1$
Between rows	$SS_3 = sv\sum_{i=1}^{r}(\overline{X}_{i..} - \overline{X}_{...})^2$	$r - 1$	$SS^*_3 = SS_3/(r-1)$	$\sigma^2 + \frac{sv}{r-1}\sum_{i=1}^{r}\varphi_i^2$	$F_3 = SS^*_3/SS^*_1$
Interaction	$SS_2 = v\sum_{i=1}^{r}\sum_{j=1}^{s}(\overline{X}_{ij.} - \overline{X}_{i..} - \overline{X}_{.j.} + \overline{X}_{...})^2$	$(r-1)(s-1)$	$SS^*_2 = SS_2/(r-1)(s-1)$	$\sigma^2 + \frac{v}{(r-1)(s-1)}\sum_{i=1}^{r}\sum_{j=1}^{s}\eta_{ij}^2$	$F_2 = SS^*_2/SS^*_1$
Within cells	$SS_1 = \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{v}(X_{ijk} - \overline{X}_{ij.})^2$	$rs(v-1)$	$SS^*_1 = SS_1/rs(v-1)$	σ^2	
Total	$SS = \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{v}(X_{ijk} - \overline{X}_{...})^2$	$rsv - 1$	$SS^* = SS/(rsv-1)$	$\sigma^2 + \frac{rv}{rsv-1}\sum_{j=1}^{s}\gamma_j^2 + \frac{sv}{rsv-1}\sum_{i=1}^{r}\varphi_i^2 + \frac{v}{rsv-1}\sum_{i=1}^{r}\sum_{j=1}^{s}\eta_{ij}^2$	

In this table

$$\overline{X}_{.j.} = \sum_{i=1}^{r}\sum_{k=1}^{v}X_{ijk}/rv; \qquad \overline{X}_{i..} = \sum_{j=1}^{s}\sum_{k=1}^{v}X_{ijk}/sv$$

$$\overline{X}_{ij} = \sum_{k=1}^{v}X_{ijk}/v; \qquad \overline{X}_{...} = \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{v}X_{ijk}/rsv$$

(14) $SS_2 = (13) - (12) - (11) - (10)$
$= SS - SS_1 - SS_3 - SS_4$

If there is more than one observation per cell and it is assumed *before the experiment is performed* that the interaction is zero, the interaction and within cells sums of squares are combined into a new row, where $SS_1' = SS_1 + SS_2$.

Source	Sum of squares	Degrees of freedom	Mean square	Average mean square
Residual	$SS'_1 = SS_1 + SS_2$	$(r-1)(s-1) + rs(v-1) = rsv - r - s + 1$	$SS'^*_1 = SS'_1/(rsv - r - s + 1)$	σ^2

In all formulas, SS_1 and SS_1^* are replaced by SS_1' and $SS_1'^*$, respectively.

In making tests of significance, when interaction may be present, it is important to first test for interaction.

If $F_2 = SS_2^*/SS_1^* \geq F_{\alpha;(r-1)(s-1),rs(v-1)}$ we reject the hypothesis that there is no interaction present, i.e., that $\eta_{ij} = 0$ for all i, j, where $F_{\alpha;(r-1)(s-1),rs(v-1)}$ is the upper α percentage point of the F distribution given in Table 13.27.

If this hypothesis is rejected, the tests for significance of row effects and column effects are meaningless under the present formulation of the problem. For example, suppose there is interaction present between workmen and machines, and suppose that there are also significant machine effects. If a particular machine effect is positive, and interaction is present, there is no guarantee that this machine will produce more units consistently. Yet, this is what is implied by saying that this machine has a positive effect. Consequently, if interaction is present, tests for row effects and column effects are irrelevant.

If the hypothesis that there is no interaction is accepted, we reject the hypothesis that there are no column effects, i.e.,

$$\gamma_1 = \gamma_2 = \gamma_3 = \ldots = \gamma_s = 0$$

if $F_4 = SS_4^*/SS_1^* \geq F_{\alpha;s-1,rs(v-1)}$.

We reject the hypothesis that there are no row effects, i.e.,

$$\varphi_1 = \varphi_2 = \varphi_3 = \ldots = \varphi_r = 0$$

if $F_3 = SS_3^*/SS_1^* \geq F_{\alpha;r-1,rs(v-1)}$.

6.5.2 Tests for comparison of mean effects. If the interactions are not significant, the procedure for comparing row or column effects is exactly the same as for the two-way classification with one observation per cell except for the following changes.

The factor c for row effects and for column effects is obtained from the analysis of variance table and from Table 13.38 or Table 13.39, depending on the level of significance. Table 13.38 or 13.39 is entered with the indices degrees of freedom, and the number of row effects being studied (r) if row effects are of interest. The proper degrees of freedom are those corresponding to the degrees of freedom of the within cell source in the analysis of variance table, i.e., $rs(v-1)$. The factor c^* is read from the table and

$$c \text{ (for rows)} = c^* \sqrt{SS_1^*/vs}$$

Similarly, if column effects are of interest, the table is entered with indices degrees of freedom, and the number of column effects being studied (s). The factor c^* is read from the table and

$$c \text{ (for columns)} = c^* \sqrt{SS_1^*/vr}$$

Of course, for finding confidence intervals, the calculated terms that are of interest are $\overline{X}_{i..}$ and $\overline{X}_{.j.}$.

6.5.3 Estimating the variability due to the sources. In many problems to which this experimental plan is applicable the main concern is with estimating the size of row effect differences, column effect differences, and magnitudes of interactions. However, if the investigation is principally concerned with estimating the relative sizes of contributions to the total variance around the general mean, these can be estimated.

The column of the analysis of variance table headed "Mean square" is an estimate of the corresponding entry in the column headed "Average mean square." Consequently, we can calculate estimates of $\sum_{i=1}^{r} \varphi_i^2$, $\sum_{j=1}^{s} \gamma_j^2$, and $\sum_{i=1}^{r} \sum_{j=1}^{s} \eta_{ij}^2$.

$$\text{estimate of } \sum_{j=1}^{s} \gamma_j^2 = (SS_4^* - SS_1^*) \frac{s-1}{rv}$$

$$\text{estimate of } \sum_{i=1}^{r} \varphi_i^2 = (SS_3^* - SS_1^*) \frac{r-1}{sv}$$

$$\text{estimate of } \sum_{i=1}^{r} \sum_{j=1}^{s} \eta_{ij}^2 = (SS_2^* - SS_1^*) \frac{(r-1)(s-1)}{v}$$

We cannot appropriately calculate the efficiency of this design to a one-way plan, since the two plans are not really comparable; the information on interactions cannot be obtained from the one-way plan.

6.5.4 Example. Youden★ reports the following experiment to determine the effect of time of aging in the strength of cement. Two mixes of cement were prepared and 6 specimens were made from each mix. Three specimens from each mix were tested after three days and after seven days. Interaction cannot be ignored since it is quite possible that the two mixes might not differ after a short period of time, but would after a longer period. This is equivalent to saying that the effect of an additional period of time is different for the two mixes, i.e., interaction is present. The test specimens are 2-in. cubes that yielded under the indicated loads, which are in units of 10 lb.

★ W. J. Youden, *Statistical Methods for Chemists* (New York: John Wiley & Sons, Inc., 1951), pp. 64-65.

TABLE 13.45 YIELD LOADS FOR CEMENT SPECIMENS

	3-day test	7-day test	$\bar{X}_{i.}$
Mix 1	660 674 648	979 1038 1051	841.7
Mix 2	661 624 652	1070 1066 1053	854.3
$\bar{X}_{.j}$	653.2	1042.8	

The following computations lead to the Analysis of Variance Table.

(1) $R_1, R_2 = 5050, 5126$

(2) $C_1, C_2 = 3919, 6257$

(3) $w_{11}, w_{21}, w_{12}, w_{22} = 1982, 1937, 3068, 3189$

(4) $T = 5050 + 5126 = 10{,}176$

(5) $$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{3} X_{ijk}^2 = (660)^2 + (674)^2 + \ldots + (1053)^2 = 9{,}091{,}732$$

(6) $$\sum_{j=1}^{2}\frac{C_j^2}{6} = \frac{(3919)^2 + (6257)^2}{6} = 9{,}084{,}768$$

(7) $$\sum_{i=1}^{2}\frac{R_i^2}{6} = \frac{(5050)^2 + (5126)^2}{6} = 8{,}629{,}729$$

(8) $$\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{w_{ij}^2}{3} = \frac{(1982)^2 + (1937)^2 + (3068)^2 + (3189)^2}{3} = 9{,}087{,}546$$

(9) $$\frac{T^2}{12} = \frac{(10{,}176)^2}{12} = 8{,}629{,}248$$

(10) $SS_4 = 9{,}084{,}768 - 8{,}629{,}248 = 455{,}520$

(11) $SS_3 = 8{,}629{,}729 - 8{,}629{,}248 = 481$

(12) $SS_1 = 9{,}091{,}732 - 9{,}087{,}546 = 4{,}186$

(13) $SS = 9{,}091{,}732 - 8{,}629{,}248 = 462{,}484$

(14) $SS_2 = 462{,}484 - 4{,}186 - 481 - 455{,}520 = 2{,}297$

ANALYSIS OF VARIANCE TABLE FOR YIELD LOADS ON CEMENT SPECIMENS

Source	Sum of squares	Degrees of freedom	Mean square	Test
Between times	455,520	1	455,520	$F = 870.54$
Between mixes	481	1	481	$F = 0.919$
Interaction	2,297	1	2,297	$F = 4.39$
Within cells	4,186	8	523.25	
Total	462,484	11	42,044	

The F values in the table for interaction and mixes is well below the critical 5% value for 1 and 8 degrees of freedom, i.e., $F_{.05;1,8} = 5.32$. Hence it is concluded that the data do not give sufficient evidence of the presence of interaction or any mix effect. On the other hand, the F test for times is significant, so that it is concluded that there is an effect from the additional four days of aging. Furthermore, a comparison of the column means indicates that

$$\bar{X}_{.2} - \bar{X}_{.1} - c \leq \gamma_2 - \gamma_1 \leq \bar{X}_{.2} - \bar{X}_{.1} + c$$

or

$$(1042.8 - 653.2) - 30.4 = 359.2 \leq \text{mean effect for 7 days} - \text{mean effect for 3 days}$$
$$\leq (1042.8 - 653.2) + 30.4 = 420.0$$

where c is obtained from Table 13.38, using as indices: $s = 2$; degrees of freedom $= rs(v - 1) = 8$; $c^* = 3.26$; so that

$$c = 30.4$$

In this problem we are not interested in finding the contributions of the main effects to the total variability about the mean. However, for illustrative purposes, we calculate the estimates of the magnitude of the main effects and interactions.

estimate of time effects

$$\sum \gamma_j^2 = (SS_4^* - SS_1^*) \frac{s - 1}{rv}$$
$$= (455{,}520 - 523.25) \frac{1}{6} = \frac{454{,}996.75}{6} = 75{,}832.79$$

estimate of mix effects

$$\sum \varphi_i^2 = (SS_3^* - SS_1^*) \frac{r - 1}{sv} = (481 - 523.25) \frac{1}{6}$$
$$= 0 \quad \text{since} \quad \sum \varphi_i^2 \geq 0$$

estimate of interaction effects

$$\sum \sum \eta_{ij}^2 = (SS_2^* - SS_1^*) \frac{1}{3}$$
$$= \frac{(2297 - 523.25)}{3} = \frac{1773.75}{3} = 591.25$$

6.6 THREE-WAY CLASSIFICATION

The reader is cautioned to read the section on two-way classification, since the present article is merely a simple extension. The model is as follows:

$$X_{ijkl} = \zeta_{...} + \varphi_i + \gamma_j + \lambda_k + \eta_{ij} + \psi_{ik} + \mu_{jk} + \rho_{ijk} + \epsilon_{ijkl}$$

where

X_{ijkl} is the outcome of the lth experiment using the ith treatment of factor A, the jth treatment of factor B, and the kth treatment of factor C;

TABLE **13.46** ANALYSIS OF VARIANCE TABLE,

Source	Sum of squares	Degrees of freedom
Between treatments of factor C	$SS_5 = rsv \sum_{k=1}^{t} (\bar{X}_{..k.} - \bar{X}_{....})^2$	$t - 1$
Between treatments of factor B	$SS_4 = rtv \sum_{j=1}^{s} (\bar{X}_{.j..} - \bar{X}_{....})^2$	$s - 1$
Between treatments of factor A	$SS_3 = stv \sum_{i=1}^{r} (\bar{X}_{i...} - \bar{X}_{....})^2$	$r - 1$
Interaction AB	$SS_2^{AB} = vt \sum_{i=1}^{r} \sum_{j=1}^{s} (\bar{X}_{ij..} - \bar{X}_{i...} - \bar{X}_{.j..} + \bar{X}_{....})^2$	$(r - 1)(s - 1)$
Interaction AC	$SS_2^{AC} = vs \sum_{i=1}^{r} \sum_{k=1}^{t} (\bar{X}_{i.k.} - \bar{X}_{i...} - \bar{X}_{..k.} + \bar{X}_{....})^2$	$(r - 1)(t - 1)$
Interaction BC	$SS_2^{BC} = vr \sum_{j=1}^{s} \sum_{k=1}^{t} (\bar{X}_{.jk.} - \bar{X}_{.j..} - \bar{X}_{..k.} + \bar{X}_{....})^2$	$(s - 1)(t - 1)$
Interaction ABC	$SS_2^{ABC} = v \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{t} (\bar{X}_{ijk.} - \bar{X}_{ij..} - \bar{X}_{i.k.} - \bar{X}_{.jk.} + \bar{X}_{i...} + \bar{X}_{.j..} + \bar{X}_{..k.} - \bar{X}_{....})^2$	$(r - 1)(s - 1)(t - 1)$
Within cells	$SS_1 = \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{t} \sum_{l=1}^{v} (X_{ijkl} - \bar{X}_{ijk.})^2$	$rst(v - 1)$
Total	$SS = \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{t} \sum_{l=1}^{v} (X_{ijkl} - \bar{X}_{....})^2$	$rstv - 1$

In this table

$$\bar{X}_{..k.} = \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{l=1}^{v} X_{ijkl}/rsv; \qquad \bar{X}_{.j..} = \sum_{i=1}^{r} \sum_{k=1}^{t} \sum_{l=1}^{v} X_{ijkl}/rtv$$

$$\bar{X}_{i...} = \sum_{j=1}^{s} \sum_{k=1}^{t} \sum_{l=1}^{v} X_{ijkl}/stv; \qquad \bar{X}_{ij..} = \sum_{k=1}^{t} \sum_{l=1}^{v} X_{ijkl}/tv$$

$$\bar{X}_{i.k.} = \sum_{j=1}^{s} \sum_{l=1}^{v} X_{ijkl}/sv; \qquad \bar{X}_{.jk.} = \sum_{i=1}^{r} \sum_{l=1}^{v} X_{ijkl}/rv$$

$$\bar{X}_{ijk.} = \sum_{l=1}^{v} X_{ijkl}/v; \qquad \bar{X}_{....} = \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{t} \sum_{l=1}^{v} X_{ijkl}/rstv$$

THREE-WAY CLASSIFICATION, INTERACTION

Mean square	Average mean square	Test
$SS_5^* = \dfrac{SS_5}{t-1}$	$\sigma^2 + \dfrac{rsv}{t-1}\sum_{k=1}^{t}\lambda_k^2$	$F = \dfrac{SS_5^*}{SS_1^*}$
$SS_4^* = \dfrac{SS_4}{s-1}$	$\sigma^2 + \dfrac{rtv}{s-1}\sum_{j=1}^{s}\gamma_j^2$	$F = \dfrac{SS_4^*}{SS_1^*}$
$SS_3^* = \dfrac{SS_3}{r-1}$	$\sigma^2 + \dfrac{stv}{r-1}\sum_{i=1}^{r}\varphi_i^2$	$F = \dfrac{SS_3^*}{SS_1^*}$
$SS_2^{AB*} = \dfrac{SS_2^{AB}}{(r-1)(s-1)}$	$\sigma^2 + \dfrac{vt}{(r-1)(s-1)}\sum_{i=1}^{r}\sum_{j=1}^{s}\eta_{ij}^2$	$F = \dfrac{SS_2^{AB*}}{SS_1^*}$
$SS_2^{AC*} = \dfrac{SS_2^{AC}}{(r-1)(t-1)}$	$\sigma^2 = \dfrac{vs}{(r-1)(t-1)}\sum_{i=1}^{r}\sum_{k=1}^{t}\psi_{ik}^2$	$F = \dfrac{SS_2^{AC*}}{SS_1^*}$
$SS_2^{BC*} = \dfrac{SS_2^{BC}}{(s-1)(t-1)}$	$\sigma^2 + \dfrac{vr}{(s-1)(t-1)}\sum_{j=1}^{s}\sum_{k=1}^{t}\mu_{jk}^2$	$F = \dfrac{SS_2^{BC*}}{SS_1^*}$
$SS^{ABC*} = \dfrac{SS_2^{ABC}}{(r-1)(s-1)(t-1)}$	$\sigma^2 + \dfrac{v}{(r-1)(s-1)(t-1)}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}\rho_{ijk}^2$	$F = \dfrac{SS_2^{ABC*}}{SS_1^*}$
$SS_1^* = \dfrac{SS_1}{rst(v-1)}$	σ^2	
$SS^* = \dfrac{SS}{rstv-1}$	$\sigma^2 + \dfrac{rsv}{rstv-1}\sum_{k=1}^{t}\lambda_k^2 + \dfrac{rtv}{rstv-1}\sum_{j=1}^{s}\gamma_j^2 + \dfrac{stv}{rstv-1}\sum_{i=1}^{r}\varphi_i^2 + \dfrac{vt}{rstv-1}\sum_{i=1}^{r}\sum_{j=1}^{s}\eta_{ij}^2 + \dfrac{vs}{rstv-1}\sum_{i=1}^{r}\sum_{k=1}^{t}\psi_{ik}^2 + \dfrac{vr}{rstv-1}\sum_{j=1}^{s}\sum_{k=1}^{t}\mu_{jk}^2 + \dfrac{v}{rstv-1}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}\rho_{ijk}^2$	

$i = 1, 2, \ldots, r; j = 1, 2, \ldots, s; k = 1, 2, \ldots, t; l = 1, 2, \ldots, v;$
$\zeta_{\ldots}$ is a general mean;

φ_i is the effect of adding the ith treatment of factor A, $\sum_{i=1}^{r}\varphi_i = 0$;

γ_j is the effect of adding the jth treatment of factor B, $\sum_{i=1}^{s}\gamma_j = 0$;

λ_k is the effect of adding the kth treatment of factor C, $\sum_{k=1}^{t}\lambda_k = 0$;

η_{ij} is the interaction of the ith treatment of factor A with the jth treatment of factor B, $\sum_{i=1}^{r} \eta_{ij} = \sum_{j=1}^{s} \eta_{ij} = 0$;

ψ_{ik} is the interaction of the ith treatment of factor A with the kth treatment of factor C, $\sum_{i=1}^{r} \psi_{ik} = \sum_{k=1}^{t} \psi_{ik} = 0$;

μ_{jk} is the interaction of the jth treatment of factor B with the kth treatment of factor C, $\sum_{j=1}^{s} \mu_{jk} = \sum_{k=1}^{t} \mu_{jk} = 0$;

ρ_{ijk} is the interaction of the ith treatment of factor A, the jth treatment of factor B, and the kth treatment of factor C, $\sum_{i=1}^{r} \rho_{ijk} = \sum_{j=1}^{s} \rho_{ijk} = \sum_{k=1}^{t} \rho_{ijk} = 0$;

ϵ_{ijkl} is the random effect, which is independently normally distributed with mean 0 and variance σ^2.

6.6.1 Tests for homogeneity. Complete the analysis of variance table (Table 13.46).

Computational procedure for the Analysis of Variance Table. For simplicity let $r = 2$, $s = 4$, $t = 3$, $v = 2$, and the data will appear in a table such as Table 13.47.

TABLE **13.47** EXAMPLE OF DATA FOR THREE-WAY CLASSIFICATION

		B_1	B_2	B_3	B_4
A_1	C_1	X_{1111} X_{1112}	X_{1211} X_{1212}	X_{1311} X_{1312}	X_{1411} X_{1412}
	C_2	X_{1121} X_{1122}	X_{1221} X_{1222}	X_{1321} X_{1322}	X_{1421} X_{1422}
	C_3	X_{1131} X_{1132}	X_{1231} X_{1232}	X_{1331} X_{1332}	X_{1431} X_{1432}
A_2	C_1	X_{2111} X_{2112}	X_{2211} X_{2212}	X_{2311} X_{2312}	X_{2411} X_{2412}
	C_2	X_{2121} X_{2122}	X_{2221} X_{2222}	X_{2321} X_{2322}	X_{2421} X_{2422}
	C_3	X_{2131} X_{2132}	X_{2231} X_{2232}	X_{2331} X_{2332}	X_{2431} X_{2432}

Form Tables 13.48, 13.49, and 13.50.

For Table 13.48:

(1) Calculate total $= X_{1+++} + X_{2+++}$

$$= \sum_{i=1}^{2} \sum_{j=1}^{4} \sum_{k=1}^{3} \sum_{l=1}^{2} X_{ijkl}$$

(2) Calculate crude sum of squares and divide by the number of original individuals that have been summed to give the individuals in this table:

$$\sum_{i=1}^{2} \sum_{j=1}^{4} \frac{X_{ij++}^2}{6} = \frac{X_{11++}^2 + X_{12++}^2 + \ldots + X_{24++}^2}{6}$$

TABLE **13.48** TABLE FOR FACTORS A AND B

	B_1	B_2	B_3	B_4	Totals
A_1	X_{11++} = total of all X's whose first two subscripts are 1,1	X_{12++} = total of all X's whose first two subscripts are 1,2	X_{13++} = total of all X's whose first two subscripts are 1,3	X_{14++} = total of all X's whose first two subscripts are 1,4	X_{1+++}
A_2	X_{21++} = total of all X's whose first two subscripts are 2,1	X_{22++} = total of all X's whose first two subscripts are 2,2	X_{23++} = total of all X's whose first two subscripts are 2,3	X_{24++} = total of all X's whose first two subscripts are 2,4	X_{2+++}
Totals:	X_{+1++}	X_{+2++}	X_{+3++}	X_{+4++}	

TABLE **13.49** TABLE FOR FACTORS B AND C

	B_1	B_2	B_3	B_4	Totals
C_1	X_{+11+} = total of all X's whose second subscript is 1 and third is 1	X_{+21+} = total of all X's whose second subscript is 2 and third is 1	X_{+31+} = total of all X's whose second subscript is 3 and third is 1	X_{+41+} = total of all X's whose second subscript is 4 and third is 1	X_{++1+}
C_2	X_{+12+} = total of all X's whose second subscript is 1 and third is 2	X_{+22+} = total of all X's whose second subscript is 2 and third is 2	X_{+32+} = total of all X's whose second subscript is 3 and third is 2	X_{+42+} = total of all X's whose second subscript is 4 and third is 2	X_{++2+}
C_3	X_{+13+} = total of all X's whose second subscript is 1 and third is 3	X_{+23+} = total of all X's whose second subscript is 2 and third is 3	X_{+33+} = total of all X's whose second subscript is 3 and third is 3	X_{+43+} = total of all X's whose second subscript is 4 and third is 3	X_{++3+}
Totals:	X_{+1++}	X_{+2++}	X_{+3++}	X_{+4++}	

TABLE **13.50** TABLE FOR FACTORS A AND C

	C_1	C_2	C_3	Totals
A_1	X_{1+1+} = total of all X's whose first subscript is 1 and third is 1	X_{1+2+} = total of all X's whose first subscript is 1 and third is 2	X_{1+3+} = total of all X's whose first subscript is 1 and third is 3	X_{1+++}
A_2	X_{2+1+} = total of all X's whose first subscript is 2 and third is 1	X_{2+2+} = total of all X's whose first subscript is 2 and third is 2	X_{2+3+} = total of all X's whose first subscript is 2 and third is 3	X_{2+++}
Totals:	X_{++1+}	X_{++2+}	X_{++3+}	

(3) Calculate crude sum of squares between rows and divide by the number of original individuals that have been summed to give the row totals:

$$\sum_{i=1}^{2} \frac{X_{i+++}^2}{24} = \frac{X_{1+++}^2 + X_{2+++}^2}{24}$$

(4) Calculate crude sum of squares between columns and divide by number of original individuals that have been summed to give the column total:

$$\sum_{j=1}^{4} \frac{X_{+j++}^2}{12} = \frac{X_{+1++}^2 + X_{+2++}^2 + X_{+3++}^2 + X_{+4++}^2}{12}$$

(5) Calculate correction factor due to mean which is total squared divided by total number of observations:

$$\frac{(1)^2}{48} = \frac{(\sum\sum\sum\sum X_{ijkl})^2}{48}$$

From the above factors compute:

(6) (2) − (5)

(7) $SS_3 = (3) - (5) = \sum_{i=1}^{2} \frac{X_{i+++}^2}{24} - \frac{(\sum\sum\sum\sum X_{ijkl})^2}{48}$

(8) $SS_4 = (4) - (5) = \sum_{j=1}^{4} \frac{X_{+j++}^2}{12} - \frac{(\sum\sum\sum\sum X_{ijkl})^2}{48}$

(9) $SS_2^{AB} = (6) - (7) - (8)$

In a similar manner, using the other two tables, the following can be computed:

(10) SS_5

(11) SS_2^{BC}

(12) SS_2^{AC}

To get the remaining factors:

(13) Calculate within cell totals from original data:

$$w_{111}, w_{112}, \ldots, w_{423}$$

(14) Calculate crude sum of squares between cells and divide by number of observations per cell:

$$\sum_{i=1}^{2} \sum_{j=1}^{4} \sum_{k=1}^{3} \frac{w_{ijk}^2}{2} = \frac{w_{111}^2 + w_{112}^2 + \ldots + w_{243}^2}{2}$$

(15) Calculate crude sum of squares:

$$\sum_{i=1}^{2} \sum_{j=1}^{4} \sum_{k=1}^{3} \sum_{l=1}^{2} X_{ijkl}^2 = X_{1111}^2 + X_{1112}^2 + \ldots + X_{2432}^2$$

From the above quantities compute:

(16) $SS_1 = (15) - (14) = \sum\sum\sum\sum X_{ijkl}^2 - \frac{\sum\sum\sum w_{ijk}^2}{2}$

(17) $SS = (15) - (5) = \sum\sum\sum\sum X_{ijkl}^2 - \dfrac{(\sum\sum\sum\sum X_{ijkl})^2}{48}$

(18) $SS_2^{ABC} = SS - SS_5 - SS_4 - SS_3 - SS_2^{AB} - SS_2^{AC} - SS_2^{BC} - SS_1$
$= (17) - (10) - (8) - (7) - (9) - (11) - (12) - (16)$

In making tests of significance when interaction may be present, it is important to test first for interaction between ABC.

If $F_2^{ABC} = SS_2^{ABC*}/SS_1^* \geq F_{\alpha;(r-1)(s-1)(t-1),rst(v-1)}$ we reject the hypothesis that there is no interaction between ABC present, i.e., that $\rho_{ijk} = 0$ for all ijk.

If this hypothesis is rejected, the tests for significance of effects A, effects B, and effects C are again meaningless under the present formulation of the problem (see Art. 6.5.1 for a discussion).

If F_2^{ABC} is not rejected, we make tests for interaction between AB, BC, and AC.

If $F_2^{AB} = SS_2^{AB*}/SS_1^* \geq F_{\alpha;(r-1)(s-1),rst(v-1)}$ we reject the hypothesis that there is no interaction between AB present, i.e., that $\eta_{ij} = 0$ for all i, j.

If $F_2^{AC} = SS_2^{AC*}/SS_1^* \geq F_{\alpha;(r-1)(t-1),rst(v-1)}$ we reject the hypothesis that there is no interaction between AC present, i.e., that $\psi_{ik} = 0$ for all i, k.

If $F_2^{BC} = SS_2^{BC*}/SS_1^* \geq F_{\alpha;(s-1)(t-1),rst(v-1)}$ we reject the hypothesis that there is no interaction between BC present, i.e., that $\mu_{jk} = 0$ for all j, k.

The presence of interaction between any two effects eliminates the need for tests of significance for these effects, since such a test is meaningless under the present formulation.

If all four types of interaction may be present, it is necessary that there be more than one observation per cell to make the necessary tests. However, if it is assumed that the interaction between ABC is 0, i.e., $\rho_{ijk} = 0$ for all ijk, a test can be formulated when there is only one observation in each cell, i.e., $v = 1$. The sources "within cells" and "interaction ABC" are eliminated in the analysis of variance table. It is replaced by the source "residual."

Source	Sum of squares	Degrees of freedom	Mean square	Average mean square
Residual	$SS'_1 = \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t} (\bar{X}_{ijk.} - \bar{X}_{ij..} - \bar{X}_{i.k.} - \bar{X}_{.jk.} + \bar{X}_{i...} + \bar{X}_{.j..} + \bar{X}_{..k.} - \bar{X}_{....})^2$	$(r-1)(s-1)(t-1)$	$SS'^*_1 = \dfrac{SS'_1}{(r-1)(s-1)(t-1)}$	σ^2

All the tests are then made with the new source residual ($SS_1'^*$). For ease of computation SS_1' is obtained by subtraction.

Returning to tests of significance, if the interaction terms are nonsignificant, the tests for effects A, B, and C are made in the usual manner.

6.6.2 Tests for comparison of mean effects. If the interactions are nonsignificant, the procedure for comparing effects for factor A, effects for factor B, and effects for factor C is exactly the same as that given for the two-way classification except for the following changes.

The value of c for factor A, for factor B, and for factor C is obtained from the analysis of variance table and from Table 13.38 or Table 13.39 depending on the level of significance. Table 13.38 or 13.39 is entered, using as indices the degrees of freedom, and the number of A effects being studied (r) if A effects are of interest. The proper degrees of freedom are those corresponding to the degrees of freedom of the within cells source in the analysis of variance table, i.e., $rst(v - 1)$. The factor c^* is read from the table, and

$$c \text{ (for } A \text{ effects)} = c^*\sqrt{SS_1^*/vst}$$

Similarly, if B effects are of interest, the table is entered with indices degrees of freedom and the number of B effects being studied (s). The factor c^* is read from the table, and

$$c \text{ (for } B \text{ effects)} = c^*\sqrt{SS_1^*/vrt}$$

If C effects are of interest, the table is entered with indices degrees of freedom and the number of C effects being studied (t). The factor c^* is read from the table, and

$$c \text{ (for } C \text{ effects)} = c^*\sqrt{SS_1^*/vrs}$$

Of course, in considering confidence intervals, the calculated terms that are of interest are $\bar{X}_{i\cdot\cdot\cdot}$, $\bar{X}_{\cdot j\cdot\cdot}$, and $\bar{X}_{\cdot\cdot k\cdot}$.

6.6.3 Estimating the variability due to the sources. Estimates of the entries in the "Average mean square" column are obtained from the "Mean square" column. Hence estimates of

$$\sum_{i=1}^{r} \varphi_i^2, \quad \sum_{j=1}^{s} \gamma_j^2, \quad \sum_{k=1}^{t} \lambda_k^2, \quad \sum_{i=1}^{r}\sum_{j=1}^{s} \eta_{ij}^2, \quad \sum_{i=1}^{r}\sum_{k=1}^{t} \psi_{ik}^2, \quad \sum_{j=1}^{r}\sum_{k=1}^{t} \mu_{jk}^2$$

and $\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t} \rho_{ijk}^2$ can be calculated.

$$\text{estimate of } \sum_{i=1}^{r} \varphi_i^2 = (SS_3^* - SS_1^*)\frac{r-1}{stv}$$

$$\text{estimate of } \sum_{j=1}^{s} \gamma_j^2 = (SS_4^* - SS_1^*)\frac{s-1}{rtv}$$

$$\text{estimate of } \sum_{k=1}^{t} \lambda_k^2 = (SS_5^* - SS_1^*)\frac{t-1}{rsv}$$

$$\text{estimate of } \sum_{i=1}^{r}\sum_{j=1}^{s} \eta_{ij}^2 = (SS_2^{AB*} - SS_1^*)\frac{(r-1)(s-1)}{tv}$$

$$\text{estimate of } \sum_{i=1}^{r}\sum_{k=1}^{t} \psi_{ik}^2 = (SS_2^{AC*} - SS_1^*)\frac{(r-1)(t-1)}{sv}$$

$$\text{estimate of } \sum_{j=1}^{s} \sum_{k=1}^{t} \mu_{jk}^2 = (SS_2^{BC^*} - SS_1^*) \frac{(s-1)(t-1)}{rv}$$

$$\text{estimate of } \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{t} \rho_{ijk}^2 = (SS_2^{ABC^*} - SS_1^*) \frac{(r-1)(s-1)(t-1)}{v}$$

6.6.4 Example. An electronic manufacturer was having difficulty with a particular type of tube, where the mutual conductance was running below the bogie value. Actually this tube is interchangeable in radio sets with an alternate. An experiment was performed to determine the effect of the exhaust variables, plate temperature, and filament lighting on the electrical characteristics of the tubes. Two levels of plate temperature and four levels of filament lighting current were used.

TRANSCONDUCTANCE

		Filament lighting conditions L_1	L_2	L_3	L_4
Plate temperature T_1	Tube	3774	4710	4176	4540
		4364	4180	4140	4530
		4374	4514	4398	3964
	Alt.	4138	4276	4228	4292
		4503	4168	4570	4386
		4352	4244	4280	4666
Plate temperature T_2	Tube	4216	3828	4122	4484
		4524	4170	4280	4332
		4136	4180	4226	4390
	Alt.	4074	4224	4108	4326
		4030	3922	4070	4312
		4350	4146	3940	4426

The analysis of variance table is completed by following the prescribed procedure.

TABLE FOR FACTORS: TEMPERATURE AND LIGHTING CONDITIONS

	L_1	L_2	L_3	L_4	X_{i+++}
T_1	25,505	26,092	25,792	26,378	103,767
T_2	25,330	24,470	24,746	26,270	100,816
X_{+j++}	50,835	50,562	50,538	52,648	

(1) $X_{1+++} + X_{2+++} = 103{,}767 + 100{,}816 = 204{,}583$

(2) $\displaystyle\sum_{i=1}^{2} \sum_{j=1}^{4} \frac{X_{ij++}^2}{6} = \frac{(25{,}505)^2 + (26{,}092)^2 + \ldots + (26{,}270)^2}{6}$

$= 872{,}531{,}808$

(3) $\sum_{i=1}^{2} \frac{X_{i++}^2}{24} = \frac{(103{,}767)^2 + (100{,}816)^2}{24} = 872{,}144{,}006$

(4) $\sum_{j=1}^{4} \frac{X_{+j++}^2}{12} = \frac{(50{,}835)^2 + (50{,}562)^2 + (50{,}538)^2 + (52{,}648)^2}{12}$

$= 872{,}217{,}868$

(5) $\frac{(1)^2}{48} = \frac{(204{,}583)^2}{48} = 871{,}962{,}581$

(6) (2) − (5) = 872,531,808 − 871,962,581 = 569,227

(7) $SS_3 = 872{,}144{,}006 - 871{,}962{,}581 = 181{,}425$

(8) $SS_4 = 872{,}217{,}868 - 871{,}962{,}581 = 255{,}287$

(9) $SS_2^{AB} = 569{,}227 - 255{,}287 - 181{,}425 = 132{,}515$

TABLE FOR FACTORS: TUBE AND LIGHTING CONDITIONS

	L_1	L_2	L_3	L_4	X_{++k+}
Tube	25,388	25,582	25,342	26,240	102,552
Alternate	25,447	24,980	25,796	26,408	102,031
X_{+j++}	50,835	50,562	50,538	52,648	

(1) $X_{++1+} + X_{++2+} = 102{,}552 + 102{,}031 = 204{,}583$

(2) $\sum_{k=1}^{2} \sum_{j=1}^{4} \frac{X_{+jk+}^2}{6} = \frac{(25{,}388)^2 + (25{,}582)^2 + \ldots + (26{,}408)^2}{6}$

$= 872{,}252{,}486$

(3) $\sum_{k=1}^{2} \frac{X_{++k+}^2}{24} = \frac{(102{,}552)^2 + (102{,}031)^2}{24} = 871{,}968{,}236$

(4) $\sum_{j=1}^{4} \frac{X_{+j++}^2}{12} = \frac{(50{,}835)^2 + (50{,}562)^2 + (50{,}538)^2 + (52{,}648)^2}{12}$

$= 872{,}217{,}868$

(5) $\frac{(1)^2}{48} = \frac{(204{,}853)^2}{48} = 871{,}962{,}581$

(6) (2) − (5) = 872,252,486 − 871,962,581 = 289,905

(7) $SS_5 = 871{,}968{,}236 - 871{,}962{,}581 = 5655$★

(8) $SS_4 = 872{,}217{,}868 - 871{,}962{,}581 = 255{,}287$

(9) $SS_2^{BC} = 289{,}905 - 5655 - 255{,}287 = 28{,}963$†

★ This result is denoted by (10) in the computational procedure.
† This result is denoted by (11) in the computational procedure.

TABLE FOR FACTORS, TUBES, AND TEMPERATURES

	Tube	Alternate	X_{i+++}
T_1	51,664	52,103	103,767
T_2	50,888	49,928	100,816
X_{++k+}	102,552	102,031	

(1) $X_{1+++} + X_{2+++} = 103{,}767 + 100{,}816 = 204{,}583$

(2) $$\sum_{i=1}^{2}\sum_{k=1}^{2}\frac{X_{i+k+}^2}{12} = \frac{(51{,}664)^2 + (52{,}103)^2 + (50{,}888)^2 + (49{,}928)^2}{12}$$
$$= 872{,}190{,}435$$

(3) $$\sum_{i=1}^{2}\frac{X_{i+++}^2}{24} = \frac{(103{,}767)^2 + (100{,}816)^2}{24} = 872{,}144{,}006$$

(4) $$\sum_{k=1}^{2}\frac{X_{++k+}^2}{24} = \frac{(102{,}552)^2 + (102{,}031)^2}{24} = 871{,}968{,}236$$

(5) $$\frac{(1)^2}{48} = \frac{(204{,}583)^2}{48} = 871{,}962{,}581$$

(6) $(2) - (5) = 872{,}190{,}435 - 871{,}962{,}581 = 227{,}854$

(7) $SS_3 = 872{,}144{,}006 - 871{,}962{,}581 = 181{,}425$

(8) $SS_5 = 871{,}968{,}236 - 871{,}962{,}581 = 5655$

(9) $SS_2^{AC} = 227{,}854 - 181{,}425 - 5655 = 40{,}774$★

(13) $w_{111}, w_{112}, \ldots, w_{242} = 12{,}512; 13{,}404; \ldots ; 13{,}064$

(14) $$\sum_{i=1}^{2}\sum_{j=1}^{4}\sum_{k=1}^{2}\frac{w_{ijk}^2}{3} = \frac{(12{,}512)^2 + (13{,}404)^2 + \ldots + (13{,}064)^2}{3}$$
$$= 872{,}772{,}468$$

(15) $$\sum_{i=1}^{2}\sum_{j=1}^{4}\sum_{k=1}^{2}\sum_{l=1}^{3} X_{ijkl}^2 = (3774)^2 + (4364)^2 + \ldots + (4426)^2$$
$$= 873{,}947{,}177$$

(16) $SS_1 = 873{,}947{,}177 - 872{,}772{,}468 = 1{,}179{,}709$

(17) $SS = 873{,}947{,}177 - 871{,}962{,}581 = 1{,}984{,}596$

(18) $$SS_2^{ABC} = 1{,}984{,}596 - 5{,}655 - 255{,}287 - 181{,}425$$
$$- 132{,}515 - 40{,}774 - 28{,}963 - 1{,}174{,}709 = 165{,}268$$

The analysis of variance table reveals that only plate conditions give significant results. For plate temperatures,

$$\bar{X}_{1\ldots} = 4323.6 \quad \text{and} \quad \bar{X}_{2\ldots} = 4200.7$$

From Table 13.38, for 32 degrees of freedom and $r = 2$, $c^* = 2.88$, so that $c = 112.8$. Hence,

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★ This result is denoted by (12) in the computational procedure.

ANALYSIS OF VARIANCE TABLE FOR ELECTRONIC TUBE EXPERIMENT

Source	Sum of squares	Degrees of freedom	Mean square	Average mean square	Test
Between tube or alternate	5,655	1	5,655		$F = 0.154 \leq F_{.05;1,32}$
Between plate temperatures	181,425	1	181,425		$F = 4.94 \geq F_{.05;1,32}$
Between lighting conditions	255,287	3	85,095.7		$F = 2.32 \leq F_{.05;3,32}$
Interaction lighting conditions and plate temperatures	132,515	3	44,171.7		$F = 1.20 \leq F_{.05;3,32}$
Interaction lighting conditions and tube or alternate	28,963	3	9,654.3		$F = 0.260 \leq F_{.05;3,32}$
Interaction tube or alternate and plate temperatures	40,774	1	40,774		$F = 1.11 \leq F_{.05;1,32}$
Interaction plate temperatures, lighting conditions and tube or alternate	165,268	3	55,089.3		$F = 1.50 \leq F_{.05;3,32}$
Within cells	1,174,709	32	36,710		
Total	1,984,596	47			

$$(4323.6 - 4200.7) - 112.8 = 10.1 \leq T_1 \text{ effect} - T_2 \text{ effect}$$
$$\leq (4323.6 - 4200.7) + 112.8 = 235.7$$

Thus to increase the mutual conductance, the tubes can be manufactured under plate temperature T_1.

6.7 LATIN SQUARE

In a three-way classification with four levels for each factor, there must be $4^3 = 64$ observations taken in order to analyze the experiment. Furthermore, it may be impossible to obtain observations for all combinations of all levels of the factors. For example, suppose that in the counting rate problem discussed in Arts. 6.3 and 6.4.4 there was an additional factor, order, besides the factors, specimens and experiments, i.e., a short-time effect may exist. It is impossible to obtain observations for all combinations of all levels, since only one specimen can be measured first in the first experiment. (It must be remembered that experiments were actually long-time differences.) The difficulty is met by setting up the procedure so that all factors appear at the same number of levels. This procedure is an arrangement of the levels into a Latin square, which is simply a square array of letters such that every letter appears once and only once in every row and column.

TABLE **13.51** LATIN SQUARE TABLE

		Column factor 1	2	3	4
Row factor	1	*A*	*C*	*B*	*D*
	2	*B*	*D*	*A*	*C*
	3	*D*	*A*	*C*	*B*
	4	*C*	*B*	*D*	*A*

We can identify the letters with the four specimens. The column factor is the experiment and the row factor is the order. Thus the third specimen (C) is used second during the fourth experiment. Note that only 16 observations are required compared with the 64 if the factorial design were used (if it could be used). This saving is at the expense of a loss in degrees of freedom for estimating σ^2. Furthermore, the analysis of such a Latin square design requires that there be no interactions present. In spite of these two drawbacks, the Latin square, when applicable, is of great importance from the point of view of industrial experimentation.

The general model is

$$X_{ij(k)} = \zeta_{...} + \varphi_i + \gamma_j + \lambda_{(k)} + \epsilon_{ij(k)}$$

where the subscript i refers to rows, j to columns, and k to letters in the square. The k is enclosed in parentheses to indicate that it is not independent of i and j. Then

$X_{ij(k)}$ is the outcome of the experiment using the ith row effect, jth column effect, and (k)th letter effect, $i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; k = 1, 2, \ldots, n$;

$\zeta_{...}$ is a general mean;

φ_i is the effect of adding the ith row treatment, $\sum_{i=1}^{n} \varphi_i = 0$;

γ_j is the effect of adding the jth column treatment, $\sum_{j=1}^{n} \gamma_j = 0$;

$\lambda_{(k)}$ is the effect of adding the (k)th letter treatment, $\sum_{(k)=1}^{n} \lambda_{(k)} = 0$;

$\epsilon_{ij(k)}$ is the random effect which is independently normally distributed with mean 0 and variance σ^2.

6.7.1 Tests for homogeneity. Compute the following analysis of variance table (Table 13.52). Computational procedure is as follows:

(1) Calculate row totals:

$R_1, R_2, \ldots, R_n$

(2) Calculate column totals:

$C_1, C_2, \ldots, C_n$

(3) Calculate letter totals:

$L_1, L_2, \ldots, L_n$

(4) Calculate over-all total:

$T = R_1 + R_2 + \ldots + R_n$

(5) Calculate crude sum of squares:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij(k)}^2 = X_{11}^2 + X_{12}^2 + \ldots + X_{nn}^2$$

(6) Calculate crude sum of squares between rows:

$$\sum_{i=1}^{n} \frac{R_i^2}{n} = \frac{R_1^2 + R_2^2 + \ldots + R_n^2}{n}$$

(7) Calculate crude sum of squares between columns:

$$\sum_{j=1}^{n} \frac{C_j^2}{n} = \frac{C_1^2 + C_2^2 + \ldots + C_n^2}{n}$$

(8) Calculate crude sum of squares between letters:

$$\sum_{(k)=1}^{n} \frac{L_{(k)}^2}{n} = \frac{L_1^2 + L_2^2 + \ldots + L_n^2}{n}$$

(9) Calculate correction factor due to mean $= T^2/n^2$.

From the quantities above compute:

(10) $SS_5 = (8) - (9) = \sum_{(k)=1}^{n} \frac{L_{(k)}^2}{n} - \frac{T^2}{n^2}$

(11) $SS_4 = (7) - (9) = \sum_{j=1}^{n} \frac{C_j^2}{n} - \frac{T^2}{n^2}$

(12) $SS_3 = (6) - (9) = \sum_{i=1}^{n} \frac{R_i^2}{n} - \frac{T^2}{n^2}$

(13) $SS = (5) - (9) = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij(k)}^2 - \frac{T^2}{n^2}$

(14) $SS_2 = (13) - (12) - (11) - (10) = SS - SS_3 - SS_4 - SS_5$

TABLE **13.52** ANALYSIS OF VARIANCE TABLE, LATIN SQUARE

Source	Sum of squares	Degrees of freedom	Mean square	Average mean square	Test
Between letters	$SS_5 = n\sum_{(k)=1}^{n}(\overline{X}_{(k)} - \overline{X}_{..})^2$	$n-1$	$SS_5^* = \frac{SS_5}{n-1}$	$\sigma^2 + \frac{n}{n-1}\sum_{(k)=1}^{n}\lambda_{(k)}^2$	$F_5 = \frac{SS_5^*}{SS_2^*}$
Between columns	$SS_4 = n\sum_{j=1}^{n}(\overline{X}_{.j} - \overline{X}_{..})^2$	$n-1$	$SS_4^* = \frac{SS_4}{n-1}$	$\sigma^2 + \frac{n}{n-1}\sum_{j=1}^{n}\gamma_j^2$	$F_4 = \frac{SS_4^*}{SS_2^*}$
Between rows	$SS_3 = n\sum_{i=1}^{n}(\overline{X}_{i.} - \overline{X}_{..})^2$	$n-1$	$SS_3^* = \frac{SS_3}{n-1}$	$\sigma^2 + \frac{n}{n-1}\sum_{i=1}^{n}\varphi_i^2$	$F_3 = \frac{SS_3^*}{SS_2^*}$
Residual	$SS_2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(X_{ij(k)} - \overline{X}_{i.} - \overline{X}_{.j} - \overline{X}_{(k)} + 2\overline{X}_{..})^2$	$(n-1)(n-2)$	$SS_2^* = \frac{SS_2}{(n-1)(n-2)}$	σ^2	
Totals	$SS = \sum_{i=1}^{n}\sum_{j=1}^{n}(X_{ij(k)} - \overline{X}_{..})^2$	n^2-1	$SS^* = \frac{SS}{n^2-1}$	$\sigma^2 + \frac{n}{n^2-1}\left(\sum_{(k)=1}^{n}\lambda_{(k)}^2 + \sum_{j=1}^{n}\gamma_j^2 + \sum_{i=1}^{n}\varphi_i^2\right)$	

In this table $\overline{X}_{(k)}$ = average over-all observations with the kth letter;

$$\overline{X}_{.j} = \sum_{i=1}^{n} X_{ij(k)}/n; \quad \overline{X}_{i.} = \sum_{j=1}^{n} X_{ij(k)}/n; \quad \overline{X}_{..} = \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij(k)}/n^2.$$

If $F_5 = SS_5^*/SS_2^* \geq F_{\alpha;n-1,(n-1)(n-2)}$ we reject the hypothesis that there are no letter effects, i.e., that

$$\lambda_{(1)} = \lambda_{(2)} = \ldots = \lambda_{(n)} = 0$$

If $F_4 = SS_4^*/SS_2^* \geq F_{\alpha;n-1,(n-1)(n-2)}$ we reject the hypothesis that there are no column effects, i.e., that

$$\gamma_1 = \gamma_2 = \ldots = \gamma_n = 0$$

If $F_3 = SS_3^*/SS_2^* \geq F_{\alpha;n-1,(n-1)(n-2)}$ we reject the hypothesis that there are no row effects, i.e., that

$$\varphi_1 = \varphi_2 = \ldots = \varphi_n = 0$$

6.7.2 Tests for comparison of mean effects. The procedure for comparing letter effects, row effects, and column effects is exactly the same as for the two-way classification except for the following changes.

The factor c for all three effects is obtained from the analysis of variance table and from Table 13.38 or 13.39, depending on the level of significance. Table 13.38 or 13.39 is entered, using as indices the degrees of freedom, and the number of levels of each factor (n). The proper degrees of freedom are those corresponding to the degrees of freedom of the residual source in the analysis of variance table, i.e., $(n - 1)(n - 2)$.

The factor c^* is read from the table for all three effects, and c is obtained from $c = c^*\sqrt{SS_2^*/n}$. Of course, for finding confidence intervals, the calculated terms that are of interest are $\bar{X}_{i.}$, $\bar{X}_{.j}$, and $\bar{X}_{(k)}$.

6.7.3 Estimating the variability due to the sources. Estimates of the entries in the "Average mean square" column are obtained from the "Mean square" column. Hence estimates of $\sum_{i=1}^{n} \varphi_i^2$, $\sum_{j=1}^{n} \gamma_j^2$, and $\sum_{(k)=1}^{n} \lambda_{(k)}^2$ can be calculated.

$$\text{estimate of } \sum_{i=1}^{n} \varphi_i^2 = (SS_3^* - SS_2^*)\frac{n-1}{n}$$

$$\text{estimate of } \sum_{j=1}^{n} \gamma_j^2 = (SS_4^* - SS_2^*)\frac{n-1}{n}$$

$$\text{estimate of } \sum_{(k)=1}^{n} \lambda_{(k)}^2 = (SS_5^* - SS_2^*)\frac{n-1}{n}$$

Often an experiment which can be treated as a Latin square might alternatively be attacked as a two-way plan, neglecting either the row or column grouping. For example, the counting rate problem has already been presented where the specimens were measured in random order in each experiment. We might hope to increase the precision by requiring that each specimen be measured first in one experiment, second in another, third in another, and fourth in another. This experimental plan is a more complicated one to carry out, and it will be worth while to assess how much the precision is increased as a guide in deciding how future investigations of a similar sort should be conducted.

The efficiency is calculated as:

$$e = \frac{SS_2^* + \dfrac{1}{n-1}\sum_{i=1}^{n} \varphi_i^2}{SS_2^*}$$

when the row grouping is neglected and

$$e = \frac{SS_2^* + \dfrac{1}{n-1}\sum_{j=1}^{n} \gamma_j^2}{SS_2^*}$$

when the column grouping is neglected.

We can say approximately that for the same precision as that given by the n^2 observations in the Latin square, $e \times n^2$ would be needed for a two-way plan.

6.7.4 Example. Consider the problem of the net counting rates described at the beginning of the article, where the factor "order" has been included. The letters represent specimens. The data are:

TABLE **13.53** NET COUNTING RATES LESS 25.00

Order	Experiment number 1	2	3	4	Order average
1	$A_{1.46}$	$C_{4.61}$	$B_{2.82}$	$D_{4.15}$	3.26
2	$B_{2.58}$	$D_{4.52}$	$A_{1.48}$	$C_{4.13}$	3.18
3	$D_{4.54}$	$A_{2.00}$	$C_{4.31}$	$B_{2.90}$	3.44
4	$C_{4.15}$	$B_{3.03}$	$D_{4.53}$	$A_{1.51}$	3.30
Experiment average	3.18	3.54	3.28	3.17	
Specimen average	$A_{1.61}$	$B_{2.83}$	$C_{4.30}$	$D_{4.44}$	

The analysis of variance table is obtained by following the computational procedure.

(1) $R_1, R_2, R_3, R_4 = 13.04, 12.71, 13.75, 13.22$

(2) $C_1, C_2, C_3, C_4 = 12.73, 14.16, 13.14, 12.69$

(3) $L_{(1)}, L_{(2)}, L_{(3)}, L_{(4)} = 6.45, 11.33, 17.20, 17.74$

(4) $T = 13.04 + 12.71 + 13.75 + 13.22 = 52.72$

(5) $$\sum_{i=1}^{4}\sum_{j=1}^{4} X_{ij(k)}^2 = (1.46)^2 + (4.61)^2 + \ldots + (1.51)^2 = 195.6948$$

(6) $$\sum_{i=1}^{4} \frac{R_i^2}{4} = \frac{(13.04)^2 + (12.71)^2 + (13.75)^2 + (13.22)^2}{4} = 173.8542$$

(7) $$\sum_{j=1}^{n} \frac{C_j^2}{4} = \frac{(12.73)^2 + (14.16)^2 + (13.14)^2 + (12.69)^2}{4} = 174.06355$$

(8) $\sum_{(k)=1}^{4} \frac{L_{(k)}^2}{4} = \frac{(6.45)^2 + (11.33)^2 + (17.20)^2 + (17.74)^2}{4} = 195.1298$

(9) $T^2/n^2 = \frac{(52.72)^2}{16} = 173.7124$

(10) $SS_5 = 195.1298 - 173.7124 = 21.4174$

(11) $SS_4 = 174.06355 - 173.7124 = 0.3512$

(12) $SS_3 = 173.8542 - 173.7124 = 0.1418$

(13) $SS = 195.6948 - 173.7124 = 21.9824$

(14) $SS_1 = 21.9824 - 21.4174 - 0.3512 - 0.1418 = 0.0720$

ANALYSIS OF VARIANCE TABLE FOR LATIN SQUARE DESIGN OF THE COUNTING RATE EXPERIMENT

	Sum of squares	Degrees of freedom	Mean square	Average mean square	Test
Between letters (specimens)	21.4174	3	7.1391		$F = 594.9$
Between columns (experiments)	0.3512	3	0.11707		$F = 9.76$
Between rows (order)	0.1418	3	0.0473		$F = 3.94$
Residual	0.0720	6	0.0120		
Totals	21.9824	15	1.4655		

For a critical value $F_{.05;3,6} = 4.76$, we see that the data indicate that both the specimens and experiments reveal significant differences. To judge which specimens differ, we enter Table 13.38 with $n = 4$ and degrees of freedom $= 6$, and $c^* = 4.90$, so that

$$c = 4.90 \times \tfrac{1}{2}\sqrt{0.0120} = 0.2695$$

and we can make such statements as: specimen 1 differs from specimen 2, 3, and 4; specimen 2 also differs from specimen 3 and 4; and furthermore with a confidence level of 0.95 we can make such statements as

$$(4.44 - 1.61) - 0.27 = 2.56 \le \text{mean effect of specimen 4} - \text{mean effect of specimen 1}$$
$$\le (4.44 - 1.61) + 0.27 = 3.10$$

Similarly experiment 2 differs from experiment 1 and from experiment 4. To estimate the magnitude of the main effects we find

for specimens, $\sum \lambda_{(k)}^2 = (7.1391 - 0.0120)\tfrac{3}{4} = 5.3453$

for experiments, $\sum \gamma_j^2 = (0.11707 - 0.0120)\tfrac{3}{4} = 0.08690$

for order, $\sum \varphi_i^2 = (0.0473 - 0.0120)\tfrac{3}{4} = 0.0265$

We can compute now the efficiency of the Latin square relative to the two-way plan for these data where order is ignored.

$$e = \frac{0.0120 + \frac{1}{4-1}(0.0265)}{0.0120} = 1.74$$

This indicates that we have increased our information per observation by more than half, by taking account of order (in addition to experiments) in the design.

In the earlier treatment of these data, using only a two-way analysis, the value of c for getting confidence intervals was 0.34; in the present analysis the value of c is 0.27, which exhibits tangibly the gain resulting from the use of the Latin square design.

6.8 COMPONENTS OF VARIANCE MODEL

Throughout this article we have treated the problem where each observation is composed of two components, namely, the unknown mean of the population from which the observation is drawn and which is common to all observations within a set, and the deviation from this mean. Depending on the number of factors involved, each set mean could be further subdivided, and tests made on the component parts.

For example, in a small laboratory four thermometers are used interchangeably to make all temperature measurements. A natural question to ask is whether these thermometers differ. The model can be described as an observation consisting of a mean value common to a particular thermometer, and a deviation from this mean. Observations on a thermometer at the same temperature differ only because of these random deviations. A difference in thermometers is equivalent to speaking of a difference in the thermometer means. The techniques described above handle such problems.

However, it is possible that the problem at hand does not fit this model. For example, suppose that there are hundreds of thermometers used interchangeably to make temperature measurements. To perform an experiment with all the thermometers is too costly, and a random sample of 4 thermometers is drawn to determine whether thermometers differ. Naturally, the experimental results will depend heavily on which thermometers are chosen. Yet inferences are to be made about all the thermometers in the laboratory. The measurement may be considered as consisting of the two random components: (1) the deviation of the thermometer measurement from the mean value of the particular thermometer used, and (2) the deviation of the thermometer mean from the over-all mean of all the thermometers in the laboratory.

We are interested in determining not whether these particular thermometers chosen at random differ, but whether all the thermometers in the laboratory, of which these particular four thermometers are samples, differ.

Another way of expressing this is to determine whether the deviation of the population means of the different thermometers from the over-all mean are zero, i.e., the thermometer variance is zero. This type of problem can be characterized in that each observation is regarded as the sum of three components, namely, an unknown constant, the mean, which is common to all observations, and two random components which give rise to the total variability; one component producing the variation within the sets of observations, and the other producing the variation between the sets of observations.

Although it is important to distinguish between the two models formally, the analysis of variance table, computations, and procedures for significance tests are identical in both cases, with the following exceptions: (1) The column in the analysis of variance table headed "Average mean square" is replaced by a column headed "Components of variance." For example, Table 13.42 for the two-way classification would have the following entry

Sources	Components of variance
Between rows:	$\sigma^2 + s\sigma^2_{(\text{rows})}$
Between columns:	$\sigma^2 + r\sigma^2_{(\text{columns})}$
Residual:	σ^2

The entries in the "Mean square" column are now estimates of the entries in the column headed "Components of variance," so that $\sigma^2_{(\text{rows})}$ and $\sigma^2_{(\text{columns})}$ can be estimated. (2) If the possibility of interactions are allowed in the model, the sum of squares for main effects are usually compared with the proper interaction sum of squares in tests of significance. For example, for the two-way classification, with interaction, the test for interaction is still $F_2 = SS_2^*/SS_1^*$. However, the test for column effects involves SS_4^*/SS_2^*, and the test for row effects involves SS_3^*/SS_2^*.

6.9 HARTLEY'S TEST FOR HOMOGENEITY OF VARIANCES

Since a primary assumption in the analysis of variance is that the within-group variability be constant for all groups, we may wish to test the hypothesis that $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2$.

The following test, known as Hartley's $F_{\text{max.}}$ test, will be suitable for the purpose.

Reject the hypothesis that the variances are equal when

$$H = \frac{\text{largest } s_i^2}{\text{smallest } s_i^2} \geq H_\alpha$$

where

$$s_i^2 = \sum_{j=1}^{n} \frac{(X_{ij} - \bar{X}_i)^2}{n-1} = \sum_{j=1}^{n} \frac{X_{ij}^2}{n-1} - \frac{n\bar{X}_i^2}{n-1}$$

and n = number of observations within each group (assumed the same for each group). Values of H_α are found in Table 13.54 by entering with n and k (the number of variances being considered).

TABLE **13.54** UPPER PERCENTAGE POINTS OF THE RATIO $S^2_{\text{max.}}/S^2_{\text{min.}}$ IN A SET OF k MEAN SQUARES, EACH BASED ON n OBSERVATIONS (Normal variation assumed)*

(a) 5% points

n \ k	2	3	4	5	6	7	8	9	10	11	12
3	39.0	87.5	142	202	266	333	403	475	550	626	704
4	15.4	27.8	39.2	50.7	62.0	72.9	83.5	93.9	104	114	124
5	9.60	15.5	20.6	25.2	29.5	33.6	37.5	41.1	44.6	48.0	51.4
6	7.15	10.8	13.7	16.3	18.7	20.8	22.9	24.7	26.5	28.2	29.9
7	5.82	8.38	10.4	12.1	13.7	15.0	16.3	17.5	18.6	19.7	20.7
8	4.99	6.94	8.44	9.70	10.8	11.8	12.7	13.5	14.3	15.1	15.8
9	4.43	6.00	7.18	8.12	9.03	9.78	10.5	11.1	11.7	12.2	12.7
10	4.03	5.34	6.31	7.11	7.80	8.41	8.95	9.45	9.91	10.3	10.7
11	3.72	4.85	5.67	6.34	6.92	7.42	7.87	8.28	8.66	9.01	9.34
13	3.28	4.16	4.79	5.30	5.72	6.09	6.42	6.72	7.00	7.25	7.48
16	2.86	3.54	4.01	4.37	4.68	4.95	5.19	5.40	5.59	5.77	5.93
21	2.46	2.95	3.29	3.54	3.76	3.94	4.10	4.24	4.37	4.49	4.59
31	2.07	2.40	2.61	2.78	2.91	3.02	3.12	3.21	3.29	3.36	3.39
61	1.67	1.85	1.96	2.04	2.11	2.17	2.22	2.26	2.30	2.33	2.36
∞	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

(b) 1% points

n \ k	2	3	4	5	6	7	8	9	10	11	12
3	199	448	729	1036	1362	1705	2063	2432	2813	3204	3605
4	47.5	85	120	151	184	216	249	281	310	337	361
5	23.2	37	49	59	69	79	89	97	106	113	120
6	14.9	22	28	33	38	42	46	50	54	57	60
7	11.1	15.5	19.1	22	25	27	30	32	34	36	37
8	8.89	12.1	14.5	16.5	18.4	20	22	23	24	26	27
9	7.50	9.9	11.7	13.2	14.5	15.8	16.9	17.9	18.9	19.8	21
10	6.54	8.5	9.9	11.1	12.1	13.1	13.9	14.7	15.3	16.0	16.6
11	5.85	7.4	8.6	9.6	10.4	11.1	11.8	12.4	12.9	13.4	13.9
13	4.91	6.1	6.9	7.6	8.2	8.7	9.1	9.5	9.9	10.2	10.6
16	4.07	4.9	5.5	6.0	6.4	6.7	7.1	7.3	7.5	7.8	8.0
21	3.32	3.8	4.3	4.6	4.9	5.1	5.3	5.5	5.6	5.8	5.9
31	2.63	3.0	3.3	3.4	3.6	3.7	3.8	3.9	4.0	4.1	4.2
61	1.96	2.2	2.3	2.4	2.4	2.5	2.5	2.6	2.6	2.7	2.7
∞	1.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

* This table is taken by permission from H. A. David, "Upper 5 and 1% Points of the Maximum F-Ratio," *Biometrika*, Vol. 39, Parts 3 and 4, December 1952, p. 424.

7. ANALYSIS OF ENUMERATION DATA—CHI-SQUARE TESTS

7.1 INTRODUCTION

The problems considered in Art. 4 dealt with the measurements on certain variables; there are many experiments in which we count the number of cases which fall into specified categories. For example, we may record the number of defective items produced by various machines, the number of errors made by several operators, the number of hits and misses made by several firing control devices, the number of accidents as a function of the shift.

We will assume that each sample observation must fall into one and only one of k categories; let $0_1, 0_2, \ldots, 0_k$ be the observed frequencies for each category. We are interested in testing hypotheses about the true relative frequencies.

Category	Observed frequency	Theoretical frequency
1	0_1	E_1
2	0_2	E_2
.	.	.
.	.	.
.	.	.
k	0_k	E_k

The test statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{(0_i - E_i)^2}{E_i}$$

which for large enough samples has an approximate χ^2 distribution with a number of degrees of freedom which depends on how the data are used in computing the E_i.★ Several problems of this sort are described in the following articles.

7.2 THE HYPOTHESIS COMPLETELY SPECIFIES THE RELATIVE FREQUENCIES OF THE CATEGORIES

Consider $p_1, p_2, \ldots, p_k$, where p_i is the true relative frequency of the ith category. In this type of problem, the theoretical frequencies are calculated from the formula $E_i = Np_i$ and the χ^2 statistic has $k - 1$ degrees of freedom.

7.2.1 Example. The total number of defective units in a day's production was tabulated by shifts.

★ The following rules of thumb can be used to assess adequacy of sample size:

1. If χ^2 has from 2 to 15 degrees of freedom, all E_i should be at least 2.5.
2. If χ^2 is computed from a 2×2 table (which will be discussed below), all E_i should be at least 5; however, if all but one of them is at least 5, the remaining one may be as small as 1 with little distortion in significance level.

	Observed frequency	Theoretical frequency
Shift 1:	20	26.67
Shift 2:	36	26.67
Shift 3:	24	26.67

It is of interest to see whether the variation from shift is due to chance or whether there is a real difference in the occurrence of defectives. That is, we test the hypothesis that the true relative frequencies of defectives are the same on all shifts, i.e., $p_1 = p_2 = p_3 = \frac{1}{3}$. Our test statistic is

$$\chi^2 = \frac{(20 - 26.67)^2}{26.67} + \frac{(36 - 26.67)^2}{26.67} + \frac{(24 - 26.67)^2}{26.67}$$
$$= \frac{44.4889}{26.67} + \frac{87.0489}{26.67} + \frac{7.1289}{26.67}$$
$$= 1.6681 + 3.2639 + 0.26730 = 5.1993 < \chi^2_{.05;2}$$

Therefore we accept the hypothesis and conclude that the data do not indicate that the frequencies differ.

7.3 TEST OF INDEPENDENCE IN A TWO-WAY CLASSIFICATION

Often frequency data are tabulated according to two criteria, with a view toward testing whether the criteria are associated. Consider the following analysis of the 157 machine breakdowns during a given quarter.

NUMBER OF BREAKDOWNS

	Machine				
	A	B	C	D	Total per shift
Shift 1	10	6	12	13	41
Shift 2	10	12	19	21	62
Shift 3	13	10	13	18	54
Total per machine	33	28	44	52	157

We are interested in whether the same percentage of breakdowns occur on each machine during each shift or whether there is some difference due perhaps to untrained operators or other factors peculiar to a given shift.

If the number of breakdowns are independent of shifts and machines, the probability of a breakdown occurring in the first shift and in the first machine can be estimated as

$$p_{11} = \frac{41}{157} \times \frac{33}{157} = 0.05489$$

If there are 157 breakdowns, the expected number of breakdowns on this shift and machines is estimated as

$$E_{11} = 157 \times p_{11} = 8.6177$$

Similarly for the third shift and second machine

$$p_{32} = \frac{54}{157} \times \frac{28}{157} = 0.06134$$

$$E_{32} = p_{32} \times 157 = 9.6306$$

This is done for all categories, and

$$\chi^2 = \sum_{i=1}^{3} \sum_{j=1}^{4} \frac{(0_{ij} - E_{ij})^2}{E_{ij}}$$
$$= \frac{(10 - 8.6177)^2}{8.6177} + \frac{(6 - 7.3120)^2}{7.3120} + \dots + \frac{(18 - 17.885)^2}{17.885} = 2.02$$

This is to be compared with the χ^2 statistic given in Table 13.26 for $(3 - 1)(4 - 1) = 6$ degrees of freedom, i.e., $\chi^2_{.05,6} = 12.6$. Hence we accept the hypothesis and conclude that the data do not indicate different percentages of breakdowns on each machine during each shift. In general, for an r by s table, we have

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{(0_{ij} - E_{ij})^2}{E_{ij}}$$

which has an approximate χ^2 distribution with $(r - 1)(s - 1)$ degrees of freedom.

7.4 COMPUTING FORM FOR TEST OF INDEPENDENCE IN A 2 BY 2 TABLE

An important special case of the test of independence arises when both criteria of classification have two categories; the data may be represented in a 2 by 2 table. The data may be represented in the following way:

First Criteria

Second criteria	a	b	$a + b$
	c	d	$c + d$
	$a + c$	$b + d$	

In this case the chi-square statistic which has one degree of freedom may be written

$$\chi^2 = \frac{(ad - bc)^2(a + b + c + d)}{(a + b)(a + c)(b + d)(c + d)}$$

7.4.1 Example. From the following data, determine whether the same percentage of breakdowns occur on each machine using material from each supplier.

NUMBER OF MACHINE BREAKDOWNS

Machine	Supplier A	Supplier B	
I	4	9	13
II	15	3	18
	19	12	31

$$\chi^2 = \frac{(12 - 135)^2(31)}{(13)(19)(12)(18)} = 8.8 \geq \chi^2_{.05;1} = 3.84$$

Therefore, we reject the hypothesis that the percentage of breakdowns is the same for each machine using material from each supplier.

7.5 COMPARISON OF TWO PERCENTAGES

A problem which in many cases is equivalent to the test of independence in a 2 by 2 table is the problem of testing equality of two percentages. Suppose, following Wallis,★ we have the following data on two fire control devices.

	Hits	Misses	
Old	3	197	200
New	4	196	200
			400

Suppose we want to test the hypothesis that the percentage of hits is the same and want to reject when the new is superior, i.e., has a larger percentage of hits. If we let P_E be the observed proportion for the new experimental method, P_S be the observed proportion for the standard, and N be the number of trials with each method, then

$$U = \frac{\sqrt{N}(P_E - P_S)}{\sqrt{(P_E + P_S)[1 - (P_E + P_S)/2]}}$$

has approximately the normal distribution, and we would reject when $U \geq K_\alpha$. For the above data $P_S = 0.015$, $P_E = 0.020$, and

$$U = \frac{\sqrt{200}(0.005)}{\sqrt{0.020 + 0.015(1 - 0.035/2)}} = 0.3795 < U_{.05} = 1.645$$

and we accept the hypothesis and conclude that the data does not reveal that the new device is better than the old one.

In terms of the notation of the last section,

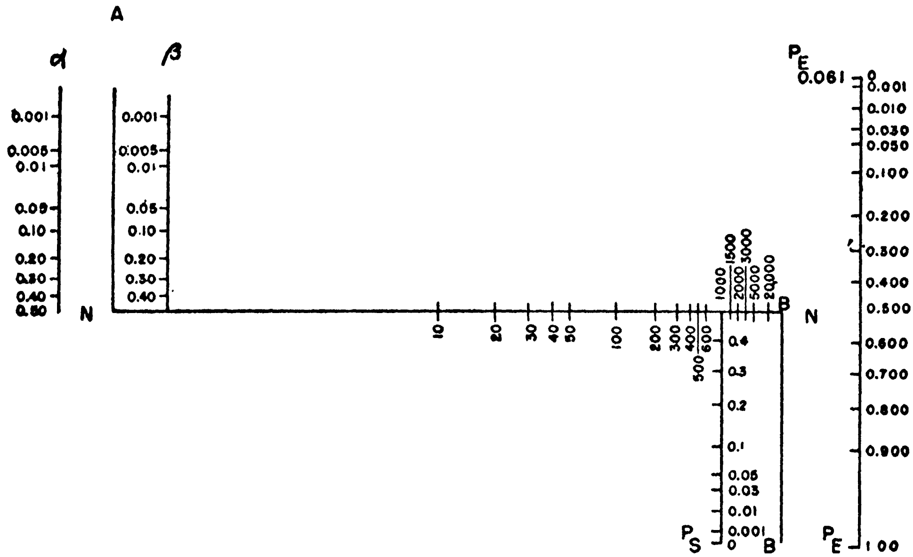
$$U = \frac{\sqrt{a + b + c + d}\,(ad - bc)}{\sqrt{(a + b)(a + c)(b + d)(c + d)}}$$

If we are interested in rejecting whenever the proportions differ, we could reject when $|U| \geq K_{\alpha/2}$ or when $U^2 \geq K^2_{\alpha/2}$. This is exactly the same as analyzing the data by the method in the last article.

★ Eisenhart, Hastay, and Wallis, *Techniques of Statistical Analysis.*

A nomogram for determining sample size in problems of this sort is attached. It requires a guess at the true values of the percentage of hits with the standard and experimental method. Detailed instructions are on the nomogram (Fig. 13.46).

FIG. **13.46** NUMBER OF CASES REQUIRED FOR COMPARING TWO PERCENTAGES.



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