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ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them: without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

PHILOSOPHICAL TRANSACTIONS

I. *The Emission of Electrons under the Influence of Chemical Action.*

By O. W. RICHARDSON, F.R.S., *Wheatstone Professor of Physics, University of London, King's College.*

Received October 21,—Read November 18, 1920.

§ 1.—SEVERAL investigators have claimed that electrons are emitted from metals under the influence of chemical action, but the only claim* which seems well substantiated is that of HABER and JUST,† who found that when drops of caesium or of the liquid alloy of sodium and potassium are attacked, at a low pressure, by a number of chemically active gases, the drops lose a negative but not a positive electric charge. The electric currents set up with the drops negatively charged are stopped by the application in a suitable manner of relatively small magnetic fields. This shows that the currents are carried by electrons emitted from the drops.‡

The object of the present investigation has been to obtain quantitative information about this interesting phenomenon, and, more especially, to ascertain the magnitude of the kinetic energy of the emitted electrons and the mode of its distribution among them. The importance of the subject lies in the fact that it is the only way, so far as I am aware, in which any information at all can be made available as to the distribution of energy among the individual products—molecular, atomic, ionic or electronic—of a chemical reaction. The majority of the experiments have been directed towards obtaining the curves showing the relation between the chemical electron current and the applied electromotive force for the case of a small spherical source concentric with a large spherical electrode. If the currents are small and the gas pressure is low, so that the motion of the liberated electrons is determined entirely by the applied electric field and is interfered with neither by the molecules of the gas nor by the fields of force arising from other electrons, we should anticipate that these electron currents would exhibit saturation with zero applied potential difference; subject to the additional proviso

* Possibly some of the cases examined by REBOUL ('C. R.', vol. CXLIX., p. 110 (1909), and vol. CLII., p. 1660 (1911)), may turn out to be an exception to this statement.

† 'Ann. der Physik,' vol. 30, p. 411 (1909); *ibid.*, vol. 36, p. 308 (1911).

‡ A survey of the previous work in this and allied fields, together with an account of the results of some of the earlier experiments of the present research, will be found on pp. 290–298 of my book 'The Emission of Electricity from Hot Bodies' (London, 1916). It will be seen that the earlier experiments gave results which differ in some important particulars from those obtained later under more satisfactory conditions. Cf. also *ibid.*, pp. 49 *et seq.*, and pp. 128 *et seq.*

that the electrons are liberated by the chemical action in such a manner that for practical purposes they can be considered to be clear of the fields of force of the atoms or molecules of origin. As to the validity or otherwise of this last assumption there was no prior evidence, but the experiments show that it is at least approximately satisfied and that small accelerating electric fields have little or no effect in increasing the electron emission. Subject to the validity of the assumptions referred to, the currents in any accelerating field should be constant, whilst their values in any retarding field will be a measure of the number of electrons whose kinetic energy when emitted exceeds the equivalent applied retarding potential difference. It should, in fact, be equal to the number of such electrons divided by the electronic charge. A characteristic curve satisfying such conditions for the case of the concentric spherical electrodes will therefore solve the problem so far as the total kinetic energy is concerned. Stated in this way the matter appears very simple, but it has been found in practice to be fraught with very serious difficulties which have taken a long time to overcome.

In all the experiments the metal acted on was a liquid alloy of sodium and potassium. In the majority of them this had an initial composition corresponding to the formula NaK_2 . The alloy of this formula appears* to have the lowest melting point, and it is a quite mobile liquid at room temperatures. In some of the early experiments an alloy of higher melting point having the initial composition NaK was used. This fact will

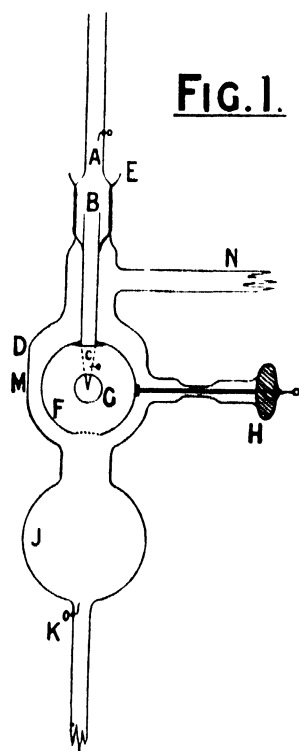


FIG. 1.

be mentioned when it is necessary to refer to those experiments. Where no specific reference to the composition of the alloy is made, it can be assumed that it was close to that given by the formula NaK_2 . There is reason to believe that as the alloy is used the potassium is consumed faster than the sodium, with a consequent increase in the viscosity of the alloy. This effect is not so noticeable with the alloy NaK_2 as with NaK , which after a time becomes almost solid.

After a certain amount of preliminary skirmishing with apparatus of different designs, that finally used in the measurement of the characteristic curves for concentric spheres is shown in a working drawing in fig. 1. The supply of the liquid alloy was kept in a glass reservoir (a separating funnel) vertically above the tube A to which it was sealed. Between A and the reservoir was a glass stop-cock for regulating the flow of the alloy. A fine platinum wire trailed down the extension BC and was sealed through the glass between A and the stop-cock, and so brought to the outside of the apparatus for the purpose of applying any desired potential to the alloy in the tube BC. The dilated portion B of the tube AC was ground to fit air-tight into the neck of the main bulb D, and the joint could be sealed with mercury or wax at the

* H. LE CHATELIER, 'Recueil de Constantes Physiques,' p. 352, fig. 3 (Paris, 1913).

lip E. Below C the glass tube was drawn out to a fine nozzle, and the lower end of the tube about C was silvered, and the silver coating connected through to the alloy inside by a platinum wire sealed through the glass. The object of this was to maintain the outside of the tube at a definite electrical potential and to prevent trouble due to charges developing on the glass surface. Matters were so arranged that the nozzle was about a millimetre above the centre of the copper sphere F. This was attached to a stout wire supported by glass and connected through to the outside by the platinum seal shown at H. Part of the weight of the sphere was taken by a copper pin which butted into a small hump blown in the tube wall at M. The copper sphere, which was 5 cm. in diameter, was provided with openings in front and behind as indicated at G for the purpose of throwing a powerful beam of light of suitable frequency on to the drops which formed at the nozzle. The light was supplied by a Westinghouse Cooper-Hewitt quartz mercury vapour lamp and admitted through a tube sealed into the main bulb. This tube is at right angles to the plane of the figure, which does not show it. The end was ground flat and closed with a thin quartz plate cemented on with sealing-wax. The copper electrode F was provided with holes at the top and at the bottom. The drops fell through the lower hole into the bulb J, which was closed by a stop-cock below K. The alloy in J was connected to earth by the sealed platinum wire K. When a sufficient amount of the alloy had collected in J, dry carbon dioxide or nitrogen could be admitted at N, and the alloy forced back into the reservoir by a system of tubes and stop-cocks not shown. In this way a sample of the alloy could be used a considerable number of times without dismembering the apparatus.

The side tube N led to the phosphorus pentoxide bulb, the MacLeod gauge, the generator of inert gas just referred to, the pumping system (Gaede mercury pump and Geryk backing pump), and the generator of the chemically active gases under consideration. This consisted of a closed vertical tube about 5 mm. in diameter provided with a stop-cock and connected with a point near N through about a metre of similar glass tubing running horizontally. The gases dealt with were carbonyl chloride (COCl_2), chlorine (Cl_2), hydrochloric acid (HCl) and water vapour (H_2O). The three first were condensed into the generating tube by means of liquid air in a thermos vessel placed outside it, and the amount released into the apparatus could be varied by manipulating the height of the liquid air outside the generating tube. The water vapour was supplied from mixtures of water and sulphuric acid or from crystals of $\text{CaCl}_2 \cdot 6\text{H}_2\text{O}$.

Most of the experiments deal with COCl_2 . Except in the most recent experiments this was prepared by boiling a mixture of 20 parts chloroform, 50 parts potassium bichromate and 400 parts sulphuric acid in a flask with a reflex condenser attached. The gas was freed from hydrochloric acid by bubbling through water, from water by bubbling through sulphuric acid, and from chlorine by passing through a U-tube containing small pieces of antimony, and then condensed in a freezing mixture of ice and salt. For the most recent experiments we have been able to secure a pre-war sample

of COCl_2 by KAHLBAUM. This seems to have the same properties as that which was prepared in the laboratory. The chlorine used was prepared by the action of hydrochloric acid on manganese dioxide, washed through sulphuric acid and condensed by liquid air. As a source of HCl gas strong hydrochloric acid was taken and the pressure reduced by cooling it to a low temperature. For H_2O , either water was taken and the vapour pressure reduced by mixing it with an excess of sulphuric acid, or crystals of $\text{CaCl}_2 \cdot 6\text{H}_2\text{O}$ cooled below the room temperature were used.

The electrical arrangements were for the most part of an ordinary character and do not call for detailed description. The currents were measured by a quadrant electrometer having, except when otherwise stated, a sensitiveness of 570 divisions per volt. It is necessary that this instrument should be sensitive, as, although the currents dealt with are of considerable magnitude, the differences of potential to be investigated are small. One quadrant was connected to the point H in fig. 1, and the other to earth. In most of the experiments capacities varying from 0.01 to 1 mfd. were added to the earthed quadrant to reduce the deflections to convenient proportions. Potentials varying continuously by any desired amount between ± 20 volts could be applied to the drop, by means of a sliding contact on a rheostat fed by a battery with one end earthed, and were read by a double-scale Weston Voltmeter.

In some of the preliminary experiments the alloy was forced through the nozzle in a fine stream by admitting inert gas to a pressure of several atmospheres to the space above the main body of the alloy in the reservoir. This method was found to be unsatisfactory, as the stream was apt to get diverted on to the copper ball, owing to some minute particle of solid getting into the nozzle or to some slight deposit forming unsymmetrically at its edge. In the experiments for which data are given the driving pressure was only the atmospheric pressure, and the alloy flowed in a steady succession of uniform spherical drops. These were about 2 mm. in diameter and flowed at the rate of about 6 drops a minute in the experiments on which reliance is placed. Different nozzles have been tried and other conditions varied, so that experiments have been made with drops from about 1 mm. to 1 cm. in diameter and flowing at rates between about 1 per minute and 15 per minute. So far as I have been able to ascertain, the effects recorded are not influenced appreciably by the size and rate of the drops. The size (about 3 mm. diameter) and the rate (about 6 per minute) chiefly aimed at were chosen as being convenient to work with and easy to attain. It should be mentioned that with very slow drops irregular results may be obtained, as the emission is greatest when each drop starts, and falls off as the surface becomes protected by a layer of the reaction products. This effect is always present, but it does not appear to lead to serious trouble if the number of drops is not under four per minute.

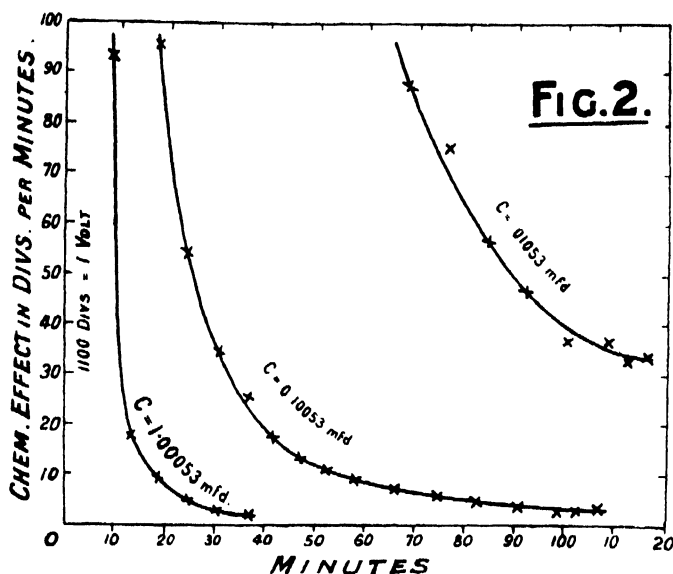
With the viscous alloys containing a high percentage of sodium, exceptionally large and slow drops can be obtained. Some of these formed so slowly that they became covered with a visible white coating of the reaction products. As the drop increased in

size this coating would fracture from time to time, exposing the bright alloy underneath the crack. Simultaneous with each fracture a sudden increase of the rate of electrometer deflection could be observed, owing to the increased emission from the clean alloy. Needless to say, no quantitative data are obtainable from experiments under such conditions. In the experiments on which reliance is placed, the rate of the drops was so rapid and the pressure of the attacking gas so low that no visible change occurred in the appearance of the surface of the drops.

In general it has not been found possible so to regulate the pressure of the attacking gas that the saturation current would stay constant. There is, however, no great difficulty in maintaining conditions so that the saturation current increases or diminishes in a regular manner with lapse of time. For example, a common method of procedure in dealing with COCl_2 has been the following:—Initially the COCl_2 would be strongly cooled by immersion of the generating tube in liquid air. The cock between the generator and the testing apparatus would then be opened and all the gas pumped out of the whole apparatus to a pressure of 0.001 mm. or less. The connection to the pumps would then be shut off and this vacuum would be maintained, provided the COCl_2 was well immersed in the liquid air. The alloy would then be allowed to drop, charged to a suitable negative potential, and the copper ball connected with the electrometer. If the electrical conditions were satisfactory there would be no deflection under these circumstances. The next step would be gradually to reduce the depth of immersion of the COCl_2 in the liquid air. This could be done by turning a screw table which supported the thermos bottle below the generator. Meanwhile continuous observation was kept on the electrometer spot, and at a certain stage a small movement would set in, showing that some active gas was beginning to reach the apparatus. It was generally convenient to raise the thermos bottle a little at this stage, as owing to a lag in the thermal changes the lowering process was generally somewhat overdone before the effects were perceptible. At about this stage transient effects would frequently be observed. These may be attributable either to a more volatile active contaminant present in the chemical used, or to some small trace of the latter which had condensed on the upper walls of the generator. These walls would no doubt warm up quickly when the liquid air was reduced. These transient effects would disappear after a little while, and the electron currents under a given applied voltage would be found to increase very steadily and deliberately as the generating tube gradually warmed up. If the rate of increase became inconveniently large, it could be checked by raising the liquid air and repeating the processes substantially as described until a complete set of observations had been obtained.

In some cases it was found preferable to work in a reverse manner, by allowing an excess of the active gas to flow into the apparatus and to observe the phenomena under examination as this gas was consumed. The consumption takes place rather slowly, probably owing to adsorption of the gas by the walls of the apparatus. A typical example of the diminution of the saturation current with time after admitting COCl_2 ,

at a low pressure, is shown in fig. 2. The three curves correspond to units of current diminishing approximately by successive factors of 10, according to the different capacities, given on the respective curves, added to the electrometer system.



At the beginning of this experiment the pressure was less than 0.001 mm. and the admission of the COCl_2 did not show any ascertainable increase. It was therefore at most 0.001 mm. After 20 minutes the pressure had risen to 0.0015 mm., after 70 minutes it was 0.003 mm., and after 100 minutes 0.004 mm. These data strongly suggested that the admitted COCl_2 was mainly adsorbed and did not make itself felt on the gauge until the chlorine had been fixed and the carbon monoxide, a much less adsorbent gas, liberated. It will be seen that the chemical emission is most vigorous at first and rapidly falls off. The rate of decay, whether considered absolutely or in proportion to the amount of emission, steadily diminishes with lapse of time. This probably means that the active gas initially is that in the immediate neighbourhood of the drops, whereas later on it diffuses from more remote parts of the apparatus. It is probable that the rate of emission is proportional to the chemical action occurring, and that this is likewise proportional to the partial pressure of the active gas at the surface of the drop.

Whilst fig. 2 exhibits the decay of the effect in a typical way under the conditions referred to, it is not typical of the conditions holding during the majority of the measurements, when a much higher degree of constancy of the saturation current was attained. For example, in one set the saturation current only varied between the limits 26.5 and 31.5 during the whole experiment, which lasted over two hours. However, this change with time was always present and its effect had to be eliminated. To accomplish this, determinations of the current under varied conditions, as, for example, under different voltages, were alternated with measurements of the saturation current under some

standard voltage. The results were then expressed as fractions of the standard saturation current ruling at the time the particular measurements were taken. The instantaneous value of the standard current was ascertained in various ways, depending on the rapidity with which it was varying. When the variation was small it was sufficient to take the arithmetic mean of the determinations of the standard current immediately preceding and following the measurement of the current under the given conditions. For somewhat larger degrees of variation the geometric means of the corresponding current values were found to give reliable results. Where the rate of variation was still more rapid, it was necessary to note the times of the various measurements, to plot a curve like fig. 2 showing the value of the standard current as a function of the time, and to ascertain from this curve the instantaneous value of the standard current at the time of the particular experiment. This method is, of course, one which is reliable under any circumstances; but, generally speaking, the rate of change of the standard current was so slow that it was not necessary to record the times at which the different measurements were made. This was an advantage, as each experiment involved a large number of settings and readings. The experimental manipulation was not particularly easy in any event, and each additional item which had to be recorded made a series of operations increasingly tiring and correspondingly liable to involve erroneous records.

This preliminary discussion will, I hope, give a general idea of the method of experimenting adopted. It will probably be more profitable to leave further details until the results of the particular experiments are dealt with.

§ 2.—*The Characteristic (Current — E.M.F.) Curve for Carbonyl Chloride.*

These data all refer to the system in which the source of emission is a spherical drop of the alloy NaK₂ of radius about 1 mm. surrounded by a concentric spherical electrode of copper of radius 2.5 cm. The pressure of the gas in the apparatus increased fairly uniformly with the duration of each experiment, the extremes of pressure recorded lying between the limits 0.001 mm. and 0.087 mm. With the exception of one experiment, the final pressure did not exceed 0.020 mm. It is probable that, in the exceptional case in which the final pressure of 0.087 mm. was recorded, there was a small leakage of air from outside the apparatus. The data afford no evidence that the maximum pressure of the COCl₂ ever exceeded 0.001 mm., and it may have been much less. The pressures recorded are those of the unabsorbed products of the reaction, and are therefore probably due to carbon monoxide. The sensitiveness of the electrometer was 570 divisions per volt, and the capacity of the electrometer and its connections 0.00012 mfd.

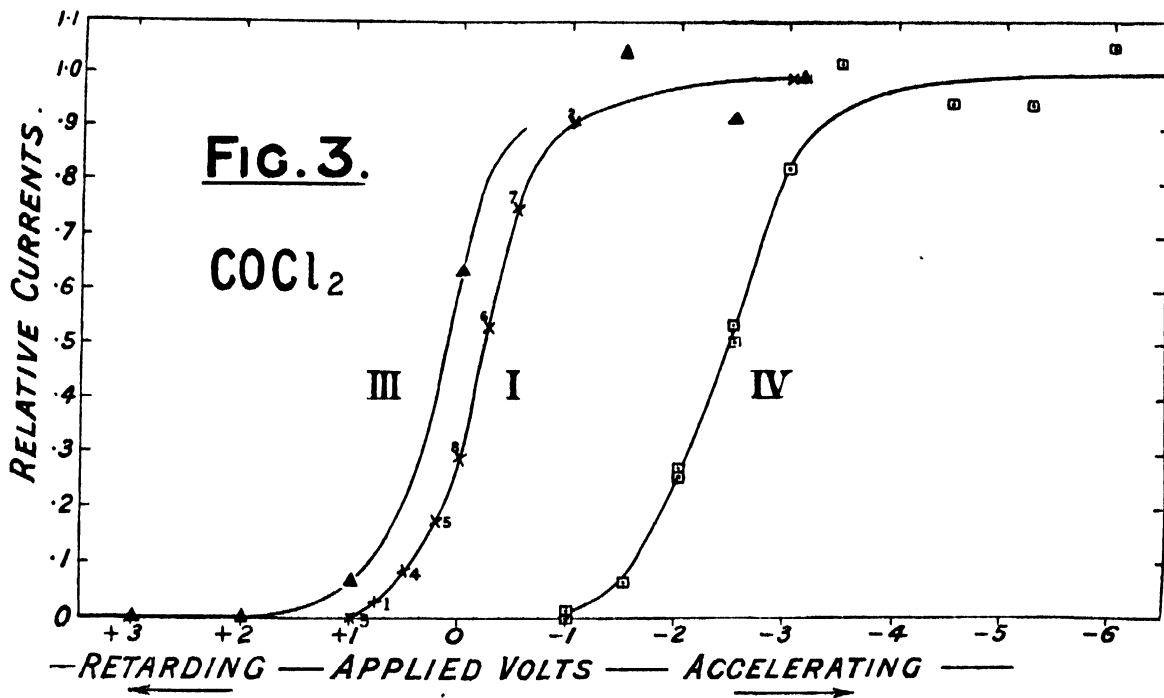
In most of the series of measurements a determination of the photo-electric current under the same voltage, due to the blue light from the mercury vapour lamp transmitted through a Wratten filter No. 50, was made immediately after each determination of

the chemical current. The actual measurement made was that of the combined photo-electric and chemical emission, but the instantaneous value of the chemical current could be obtained from the determination immediately preceding after allowing for its variation with the time, and thus the value of the photo-electric current obtained by subtraction. The light transmitted by the filter is not strictly monochromatic, but for most practical purposes it can be regarded as consisting of the violet group lying between 4347 and 4358. Whilst I have not made a spectrographic examination of the light transmitted by this filter, it probably lets through a certain amount of the group 4046-4077 and a smaller amount of 4916. Consequently no particular quantitative reliance is placed on these photo-electric data, but they afford a valuable indication as to the difference between the chemical and photo-electric characteristic curves and also supply a useful guide as to the state of the surface of the alloy. Later on, when accurate photo-electric information became imperative, a monochromatic illuminator and a set of light filters were obtained. The reliability of these later photo-electric measurements, in so far as it is dependent on the monochromatism of the light used, is to be regarded as of a distinctly higher order than those dealt with in this part of the paper (see p. 23 *et seq.*).

Preliminary tests with this apparatus showed that the chemical-effect currents were not increasing appreciably when the negative (accelerating) potential on the drops was increased from 2 to 3 volts; so that - 3 volts was adopted as the potential for measuring the standard current. The procedure adopted, when photo-electric measurements were included, was first to measure the current with the light cut off at - 3 volts, then with the light still cut off at the voltage under test, say, x volts, then at x volts with the drop illuminated, then at - 3 volts again with the light off, then with the light off at a new voltage, say, y volts; then at y volts with the light on, then at - 3 volts with the light off again, and so on, until a complete set of data had been obtained. From these results the values of the relative chemical currents, *i.e.*, the values of the fraction obtained by dividing the current at the voltage under test by the instantaneous standard current and also the values of the photo-electric currents at the different voltages, were calculated in the manner already explained. When no photo-electric measurements were made the procedure was the same, except that the measurements with the drop illuminated were omitted. In all cases the successive settings and readings were made as rapidly as possible.

At first sight the results got in this way seemed very inconsistent. For example, in different experiments made at intervals perhaps of some weeks, but under conditions which were identical so far as I could ascertain, the relative currents at zero volts (as compared with - 3 volts as standard) would differ by 100 per cent. or more. Later on it appeared that the current at zero volts had vanished altogether, and that the current, instead of being approximately saturated, was increasing rapidly between - 2 and - 3 volts. However, when the precaution was taken of ensuring that the standard current was really on the saturation part of the curve, and when the results

of the whole of the completed experiments were plotted and compared together, it was found that there was a definite method about these variations. This will be recognised from an inspection of fig. 3, which shows three of the characteristics as actually determined. The numbers alongside the crosses on curve I express the order in which



the different points were determined. It will be seen that each characteristic is consistent enough in itself. The variability, normally about 5 per cent., which seems rather pronounced on the flat part of the curves, is due to the cumulative effect of the errors inherent in the determination of the currents. The percentage error should be independent of the magnitude of the currents, and consequently is not so noticeable on the lower parts of the diagram where the relative currents are smaller. It is probable that the main source of error lies in the variation of the rate of emission with the growth of the drops, and it will be noticed that it is most pronounced in curve III., where the drops were formed with exceptional slowness (at the rate of 4 in 90 seconds). However, I believe that the accuracy of the measurements is sufficient to establish the conclusions which it is intended to draw from them.

It will be noticed that the shapes of the curves are all much the same, the most noticeable difference being that they are spaced widely apart on the voltage axis. In fact, it is clear from an inspection of fig. 3, that if any two of the curves are given suitable horizontal displacements, they can be made nearly to coincide with the remaining one. The explanation of such a result seemed obvious. The shapes of the characteristic curves are the same in the different experiments, but they occur at different places along the voltage axis. Now the voltage V plotted along this axis is that given by a

voltmeter connected, in effect, across the gap between the drop of alloy and the copper sphere. The actual difference of potential across this gap is not equal to V , but is equal to $V + K$ where K is the contact potential difference between the surfaces of the alloy and the copper sphere. If now the value of K is changing between one experiment and another, the observed results are completely accounted for. It will be necessary that K should exhibit an extreme variation of as much as 2.5 volts, but this does not seem a surprising requirement when it is remembered that the contact electromotive force between the alkali metals and copper is comparable with this amount. The changes in the contact electromotive force which occur are to be attributed to changes in the surfaces of the alloy and of the copper due to the gases and vapours to which they are exposed, and to changes in the copper surface due to splashing with the alloy and possibly to absorption of the vapours of the alkali metals by the copper.

To test this hypothesis, all the sets of observations with COCl_2 which were sufficiently complete to form a reliable guide were collected together and the relative currents in terms of the standard tabulated. In cases such as curve IV. in fig. 3, where the current was not saturated at -3 volts, the standard voltage was chosen about 3 volts negative to a voltage at which the current was about 50 per cent. of the final saturation value. The precise value of this voltage does not matter much, as the current in this region is not varying with the voltage to an extent ascertainable by these experiments. What is important is that the standard current should be saturated, and it was convenient to employ the least voltage that would make sure of this. It was also felt that until more information was available about the phenomenon, it was desirable to employ as the voltage for the determination of the standard current a voltage which would occupy the same position on each characteristic. For example, it was not, and is still not, known with certainty whether there is a small variation of current with voltage on the flat parts of the curves. If there is any such change, the values of the relative currents would be affected by the value of the voltage at which it was decided to measure the standard saturation current. The method adopted ensures that there are no errors arising from considerations of this character, which would be appreciable in comparison with the unavoidable experimental error. In all the curves but two the original three-volt standard could be retained. In fact, it approximately satisfied the condition just described, the current at zero volts in these cases being on the rapidly rising part of the characteristic.

The relative currents thus obtained were then plotted for each series against the actual volts given by the voltmeter and the amount of displacement along the voltage axis was judged, which would be necessary to bring all the curves as nearly as possible to coincidence. This amount is, of course, in general different for the different curves, but is the same for every experimental point belonging to any one curve. The displacements are not applied to the curves, which constitute a secondary inference from the experimental data, but, to the primary source of evidence, the experimental points themselves. The absolute position of the composite curve in relation to the scale of

voltage is at present an entirely arbitrary matter, in default of any knowledge of the actual contact difference of potential K occurring in any single experiment. The displacements have, however, been so chosen as to bring the point A (fig. 4), formed by the intersection of a horizontal line through the standard saturation value unity and the dotted extension of the straight or slightly concave part FB of the characteristic, over the zero on the volt scale.

That the point A will not be far from that which corresponds to the condition of zero electric field between the two electrodes, *i.e.*, $V + K = 0$, might be anticipated on the following grounds. The data in fig. 3, and still more in fig. 4, indicate that there is no considerable direct effect of the applied field in helping the electron emission. In fact, I have frequently made explicit tests to see if the part DE of the curve slopes upwards, as appears to be suggested by the points in fig. 4, but I have never been able to convince myself that it does. Some experiments have given a small increase, some a small decrease, and others no change with rising voltage. This is a point which deserves further examination, but if there is any effect* of this kind it is certainly small. It is presumed, of course, that the voltages used in such tests are not such as to generate appreciable ionization in the surrounding gas. Inasmuch as the direct effect of the electric field is negligible, the curving of the part BCD of the characteristic is to be attributed to such secondary factors as the mutual repulsion of the electrons, electron reflection at the copper electrode, the return to the drop of some of the emitted electrons owing to collisions with gas molecules, and the effects of the holes in the copper sphere. Inasmuch as under ideal conditions in which the electron-emission or saturation current is very small, the vacuum is so high that gas collisions are negligible, there is no electron reflection at the receiving electrode, and the receiving electrode is a complete sphere, in the absence of electric field across the gap every electron once emitted from the drop will reach the copper sphere, because there is nothing to turn it back or to enable it to go elsewhere. Thus under such ideally perfect conditions the characteristic will consist of a flat saturation part AE intersecting with a part such as AFG, in which the emitted electrons are returned to the source by the opposition of the retarding field. This argument does not establish the coincidence of the point A with the zero potential difference, inasmuch as there is nothing *a priori* which enables us to foretell the shape of the part AFG. I have, however, succeeded in making independent experiments which determine, within limits, the contact difference of potential K and the characteristic curve for the chemical emission simultaneously. These experiments, which will be described later, fix the position of true zero volts somewhat to the left of the point A.

Fig. 4 shows the results of displacing the data along the voltage axis in the manner I have explained. The magnitudes of the displacement, and certain details which may

* For a case in which there is a definite indication of diminishing currents with rising voltage on this part of the curve, see p. 25 *infra* (fig. 12).

be of importance relating to the different series of observations, are set forth in the following table and remarks:—

TABLE I.— COCl_2 .

Number of series.	Mark indicating experimental points of series on curves.	Initial pressure, millimetres of Hg.	Final pressure, millimetres of Hg.	Displacement in volts from volt-meter readings to scale position in fig. 4.	Date.
I.	×	0.0015	0.016	+ 1.00	3/8/1916
II.	⊙	0.004	0.020	+ 1.45	25/7/1916
III.	▲	0.001	0.011	+ 0.55	25/9/1916
IV.	⊠	0.003	0.087	+ 3.30	2/11/1916
V.	⊕	0.002	0.012	+ 2.62	3/11/1916
VI.	•	0.001	0.038		16/9/1916

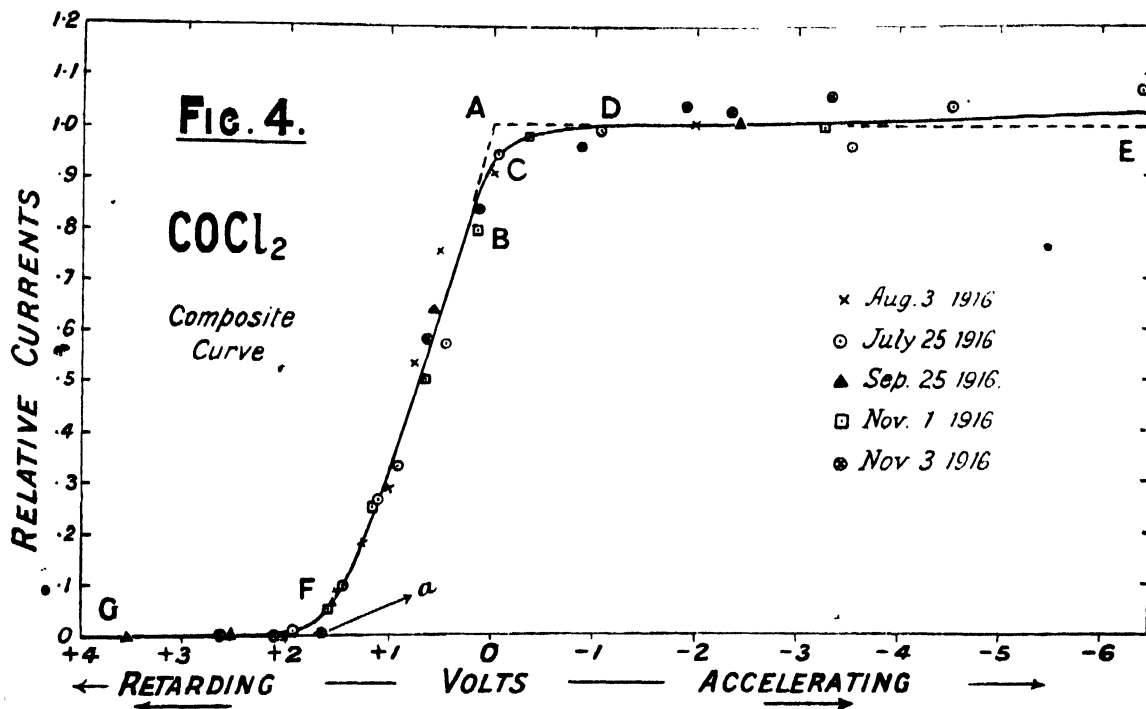
(The numerals I., &c., and marks ×, &c., are relevant to the data in figs. 3 and 5, as well as in fig. 4.)

Remarks on each Series.

- I. ×.—Generator adjusted at the beginning of the experiment and later. The effect rose slowly after each adjustment. Chemical saturation current: minimum value 45 divisions per minute with cap. 0.01 mfd., maximum 200 divisions per minute, cap. 0.04 mfd. Saturation photo-electric current, 64 divisions per minute with 0.04 mfd. Maximum chemical and photo-electric effects thus of same order of magnitude. Gas was pumped out in the middle of these experiments without affecting the results. These observations were very consistent. The only apparent defect was a small insulation leak which had to be allowed for. The slight variability of this makes the point of intersection of the curve (I., fig. 3) and the voltage axis a little doubtful. A test at the end of the series with the mercury line 4355 gave 252 divisions per minute and with the green line 0.7 division per minute, the capacity being 0.01 in each case.
- II. ⊙.—Generator adjusted at the beginning and again at the middle of the series. Effect rose before and after adjustment. Maximum chemical saturation current, 160 divisions per minute, minimum 42 per minute; λ 4355 photo-electric saturation current, 160 divisions per minute; all with capacity 0.02 mfd. Six drops per minute, diameter of drops 3 to 4 mm.
- III. ▲.—Generator not adjusted in these experiments. Chemical effect dropped steadily most of the time, then rose a bit towards the end and fell off again. Chemical saturation current: maximum 28 divisions in 60 seconds, capacity 0.03 mfd.; minimum 37 per minute, capacity 0.01 mfd. λ 4355 photo-electric saturation, 53 per minute with 0.03 mfd. capacity. Four drops in 90 seconds.

- IV. ☐.—Generator adjusted twice. Chemical effect rose after first adjustment. After second it fell, then rose again. It was large most of the time. •Chemical saturation current: maximum 700 divisions per minute with capacity 0.04 mfd., minimum 7 divisions per minute with capacity 0.01 mfd. There was no detectable photo-electric current with $\lambda = 4355$ in this experiment. An effect equal to about 1 per cent. of the chemical emission should have been detected. Considerable gas pressure was present, and there may have been a small air leak into the apparatus. Six drops per minute.
- V. ⊕.—No adjustment of generator in this series. Chemical effect small but very steady, only varying between 26.5 and 31.5 divisions per minute with 0.01 mfd. capacity: photo-electric λ 4355 saturation, 300 divisions per minute with 0.01 mfd.
- VI. •.—No adjustment of generator. Chemical saturation current rose gradually from 18 to 268 divisions per minute with 0.01 mfd. and then fell to 120 per minute. Photo-electric 4355, saturation current 240 divisions per minute with 0.01 mfd.

An examination of the foregoing remarks shows that the series of observations I. to V., which are comprised in fig. 4, embrace a wide variety of conditions. Thus the saturation chemical current is varied over the range from 700 divisions per minute with 0.04



mfd. or 8.2×10^{-10} amperes to 7 divisions per minute with 0.01 mfd., or 2.05×10^{-12} amperes. The photo-electric saturation currents under practically the same illumination in each case varied between the limits 8.8×10^{-11} amperes and something under

3×10^{-13} amperes. The pressure of gas in the apparatus covered the wide range from 0.001 mm. to 0.087 mm. In the different series the chemical saturation currents rose or fell with time or did each of these alternately. In one series it was almost constant. In one case the standard photo-electric current was 10 times as great as the standard chemical current, in another it was less than 1 per cent. of it, whilst in another case these currents were about equal. The number of drops per second varied between 1 in 10 seconds and 1 in $22\frac{1}{2}$ seconds, possibly more. The size of the drops probably varied also.

In spite of these wide changes in the conditions, all the points in fig. 4 fall close to the continuous curve drawn. It is not certain that they are not all coincident with this curve within the limits of likely experimental error, with the single exception of the observation marked *a*. An examination of the data makes it practically certain that the electrometer was not functioning when this observation was taken, as the photo-electric observation immediately following it also gave a zero deflection when it should have given a measurable deflection according to the run of the rest of the photo-electric curve.

The coincidence of the points with the smooth curve in fig. 4 proves that this curve gives, approximately at any rate, the true shape of the COCl_2 characteristic, and confirms the suspicion that the moving about along the volt axis, as in fig. 3 of the experimental curves, is due to changes in the contact potential between the surfaces.

The plot of VI. • did not agree with series I. to V., and is not included in fig. 4, but is shown separately in fig. 5, together with a copy DEF of the composite curve in

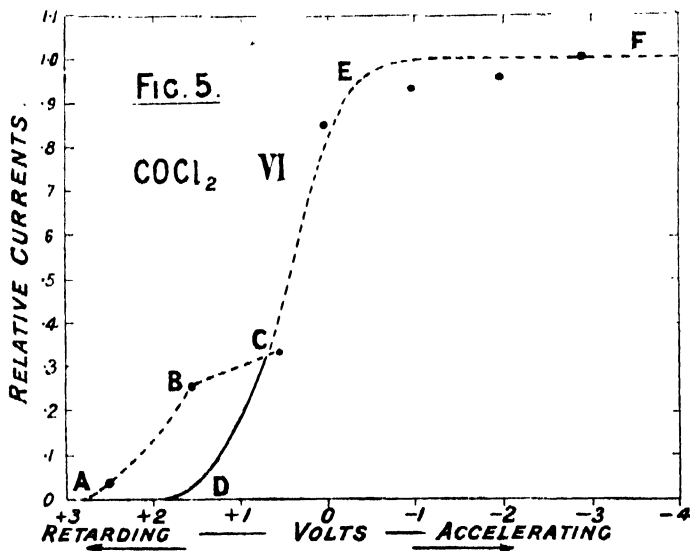
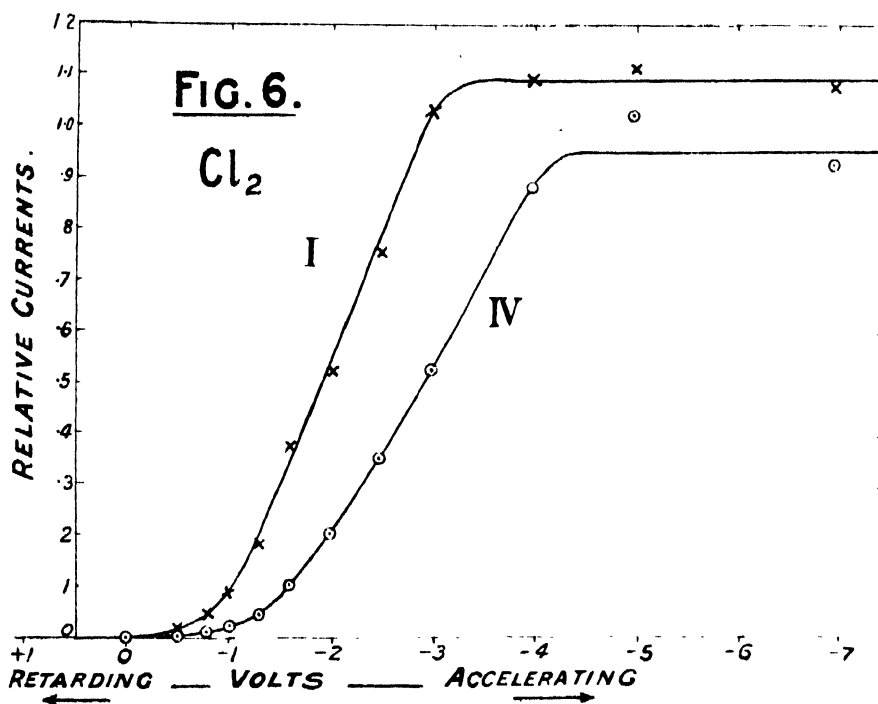


fig. 4 shifted 0.25 volt to the right of its position in that diagram. In this experiment the points to the right of +1 volts were taken first and the others later. It will be seen that the first five points taken agree with fig. 4, but the later points would have to be given a different displacement to bring them on to the common curve. This seems a

clear indication of a change in the contact potential difference between the surfaces taking place during the course of a single set of observations.

§ 3.—*The Characteristic Curves for Chlorine.*

The manipulation in the case of chlorine was very similar to that in the case of COCl_2 and the results are of the same general character. Two typical characteristics as actually determined are shown in fig. 6. The curves show a flat saturation part for



accelerating voltages exceeding about four. There is the same tendency to haphazard displacements of the curves parallel to the voltage axis as in the case of COCl_2 . The magnitude of this displacement seems, however, to tend to be smaller with chlorine, the maximum separation I have observed in six characteristics taken with this gas being 0.95 volt. The fact that the two curves shown in fig. 6 do not appear to be parallel is due to the distortion caused by the different vertical scales. The standard current has the arbitrary value 1.09 in one case and 0.95 in the other.

As in the case of COCl_2 , the various series have been reduced to unit saturation current and displaced by varying amounts along the vertical axis to form a composite curve. The various data referring to the relevant experimental series are given in the following table and remarks :—

TABLE II.—Cl₂.

Number of series.	Mark indicating experimental points of series on curves.	Initial pressure, millimetres of Hg.	Final pressure, millimetres of Hg.	Displacement in volts from volt-meter readings to scale position in fig. 7.	Date.
I.	×	0·003	0·080	0·00	20/4/1917
II.	⊙	0·002	0·052	- 0·10	22/4/1917
III.	▲	0·002	0·002	- 0·25	22/4/1917
IV.	⊠	0·002	0·040	- 0·55	22/4/1917
V.	⊕	0·003	0·100	- 0·70	17/4/1917
VI.	•	0·002	0·080	+ 0·25	25/4/1917

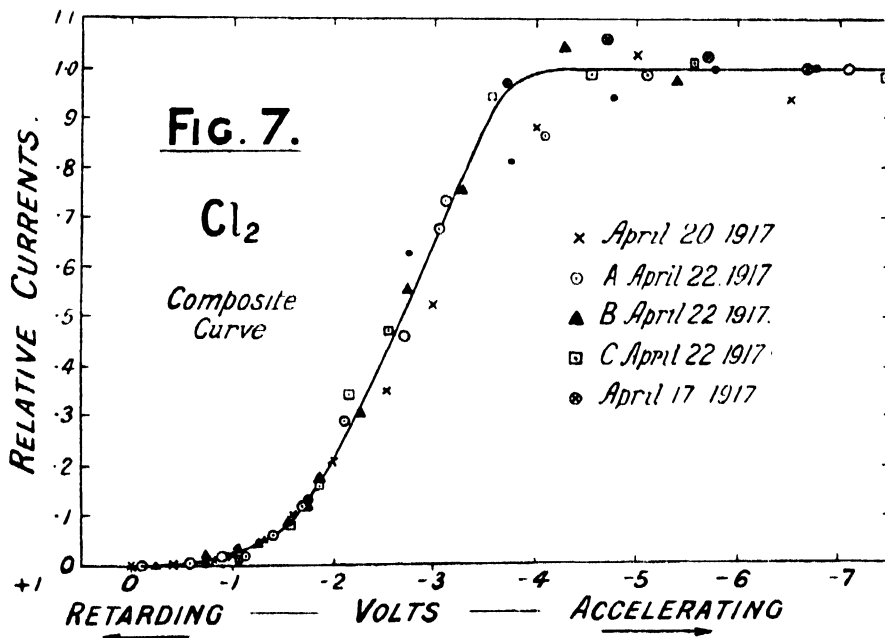
[The numbers I., &c., refer to the curves in fig. 6 as well as fig. 7.]

Remarks on each Series.

- I. ×.—No adjustment of generator. Chemical saturation current increased uniformly and slowly from about 200 divisions per minute to about 800/min. and photo-electric (4355) saturation current equal to about 100/min., all with additional capacity 0·01 mfd. Pump stopped and gas allowed to accumulate.
- II. ⊙.—No adjustment. Chemical saturation current increased slowly from 310/min. to 508/min., photo-electric 60/min., all with 0·01 mfd. Pump stopped and gas accumulating. One drop per minute.
- III. ▲.—No adjustment. Chemical saturation current increased slowly from 10/min. to 15/min., and then towards the end rose rapidly to 300/min.; photo-electric 12/min.; all with 0·01 mfd. In this series the chlorine was kept immersed in liquid air and the pumps running continuously. One drop per minute.
- IV. ⊠.—No adjustment. Chemical saturation current increased slowly from 250/min. with 0·02 mfd. to 500 per minute with 0·04 mfd., photo-electric 18/min. with 0·02 mfd. Pump shut off and gas accumulating. One drop in 110 seconds.
- V. ⊕.—Chlorine generator shut off from apparatus and effects presumably due to chlorine adsorbed in the apparatus. Chemical saturation current fell slowly, the measured limits being 135/min. and 97/min. with 0·000121 mfd., but this difference is partly exaggerated by errors of measurement. Photo-electric 370/min. with 0·000121 mfd. Pump shut off and products accumulating. Six drops per minute.
- VI. •.—Chlorine generator shut off and other arrangements as in V. ⊕. Chemical saturation current rose slowly from 209/min. to 258/min. with 0·000121 mfd. Photo-electric (4355) saturation current 1300/min. with 0·000121 mfd. One drop in 110 seconds.

These data show that the experimental conditions under which chlorine has been tested cover a wide range as in the case of COCl_2 . Thus the chemical saturation current is varied between the limits 500/min. with 0.04 mfd., equal to 5.9×10^{-10} amperes, and about 3.5×10^{-13} amperes. The photo-electric saturation current varies between about 3×10^{-11} amperes and 10^{-12} amperes. The chemical emission increases and decreases with time in the different series; sometimes it is almost constant, at others it is changing rapidly. The pressure of gas in the apparatus ranges from 0.002 mm. to 0.100 mm. The chemical effect is nearly 60 times the photo-electric in one series, whereas it is only about one-sixth of the photo-electric in another, and this ratio has intermediate values in other cases. In five of the series the gaseous products of the reactions occurring were allowed to accumulate, whereas in one of them the pumps were run continuously. The rate at which the drops fell was varied between 1 in 10 seconds and 1 in 110 seconds.

In spite of this wide variation in the conditions, all the points fall on the smooth curve in fig. 7 to the degree of accuracy which it seems reasonable to expect. An exception is perhaps furnished by the 3 points \times of series I. at -2.5 , -3 and -4 volts.



It is unlikely that chance errors, which would have to be rather large in any event, would make these three points, which were determined in succession at the beginning of the series, lie so persistently to the right of the graph required by the rest. It seems likely that we have here another example of a change in the contact potential taking place during the actual course of the experiments, as in the case of series VI. with COCl_2 . Some of the points in I. \times , II. \circ , and VI. \bullet might seem to suggest a slower voltage approach to saturation than the normal, and it was thought that this effect might be

attributable to the rather high gas pressures which developed in these experiments. The fact that III. ▲, in which the pressure was kept down to 0·002 by operating the pumps continuously, does not show this tendency, would seem rather to support such a contention. An examination in detail of the data, however, is against it. For example, in I. × and VI. ● the currents at the high voltages were measured at the lower pressures and in II. ⊙ conversely. On the whole it seems likely that any such appearance is illusory and, except in the case of I. ×, where a displacement caused by change in contact potential during the series seems probable, due to chance errors.

It appears, then, that fig. 7 determines the shape of the chlorine characteristics. No determination of the contact potential with chlorine present has yet been made, so that the true zero on the volt scale in fig. 7 cannot yet be assigned with certainty. All that can be said at present is that an examination of the relative positions of the chemical and of the photo-electric curves which were taken at the same time shows that the true zero of fig. 7 lies to the right of $-2\cdot2$ volts.

§ 4.—*Hydrochloric Acid.*

The effects given by this gas are of a much smaller magnitude than those given by carbonyl chloride and by chlorine. It was impossible to make accurate measurements of them with the apparatus used with the latter gases, and such fragmentary data as were obtained only enable qualitative statements to be made. The characteristics appear to be similar generally to those given by the other gases, and to drop from saturation to zero current within a range comparable with one volt.

§ 5.—*The Characteristic Curves for Water.*

These experiments were made in 1915 with an apparatus which differed in some details from that shown in fig. 1 and used in the experiments described up to this point. Instead of the glass tube ABC of fig. 1, the alloy was fed into the testing vessel through a copper tube fitted with a tapered silver nozzle. This had a fine hole bored vertically down the centre and was screwed on to the copper tube. The other electrode was 7·5 cm. in diameter. The alloy used was also different, having the composition NaK instead of NaK₂. The electrometer sensitiveness was 540 divisions per volt and the capacity of the electrometer and connections 0·00032 mfd. There is an element of doubt about the correctness of the voltmeter set up which was used in some of these early experiments, and I am only prepared to state the applied potential differences as relatively correct. The units are probably volts, but they may be as small as 0·6 volt. The data for the series of observations which were completed are given in fig. 8, and some of the details in Table III. and in the remarks which follow.

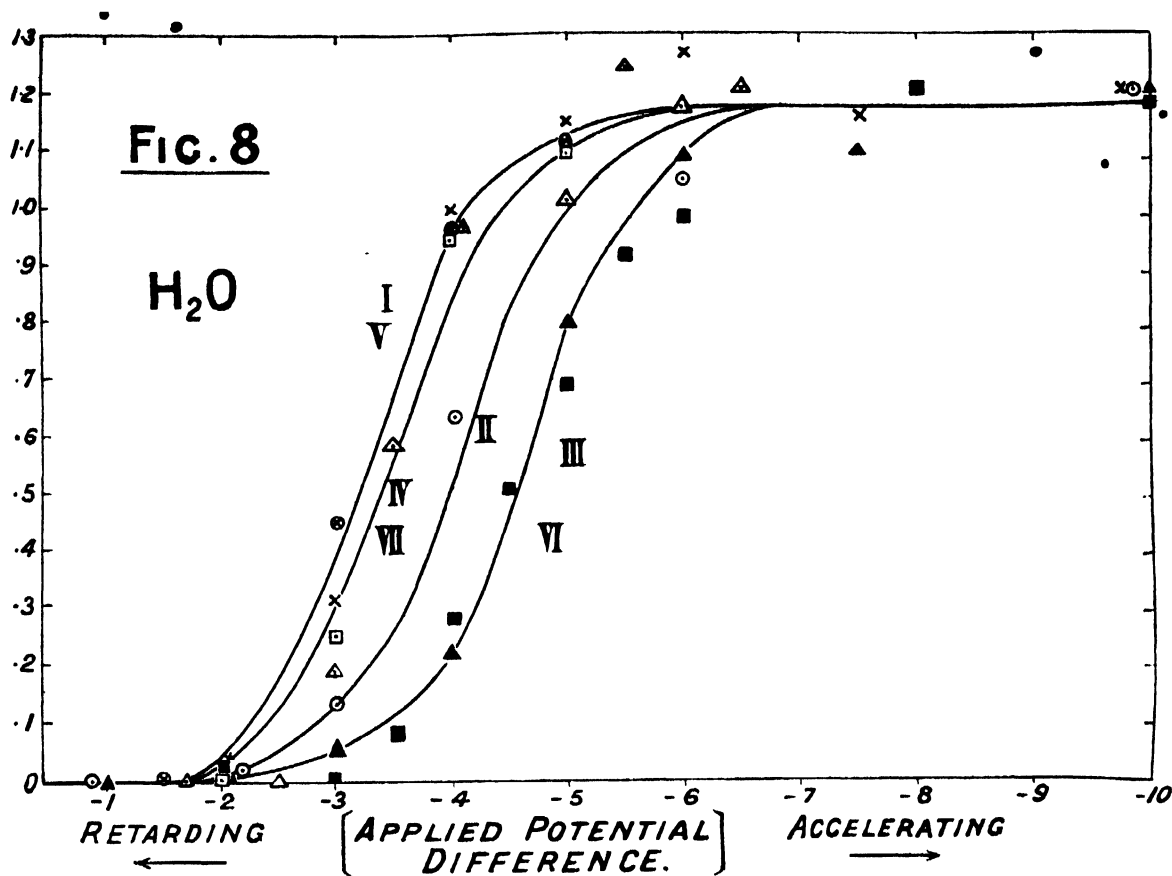


TABLE III.—H₂O.

Number of series.	Mark indicating experimental points of series on curves.	Displacement on volt-meter scale from voltmeter readings to scale position in fig. 9.	Date.	Page of notebook.
I.	×	- 1.15	1915	Vol. 1, p. 77
II.	⊙	- 0.75	1915	" " p. 74
III.	▲	0.00	1915	" " p. 68
IV.	⊠	- 1.00	1915	" " p. 55
V.	⊕	- 1.30	1915	" " p. 54
VI.	■	+ 0.05	6/8/1915	" 2, p. 52
VII.	△	- 1.00	13/8/1915	" " p. 58

[The numbers I., &c., and marks ×, &c., refer to figs. 8 and 9 equally.]

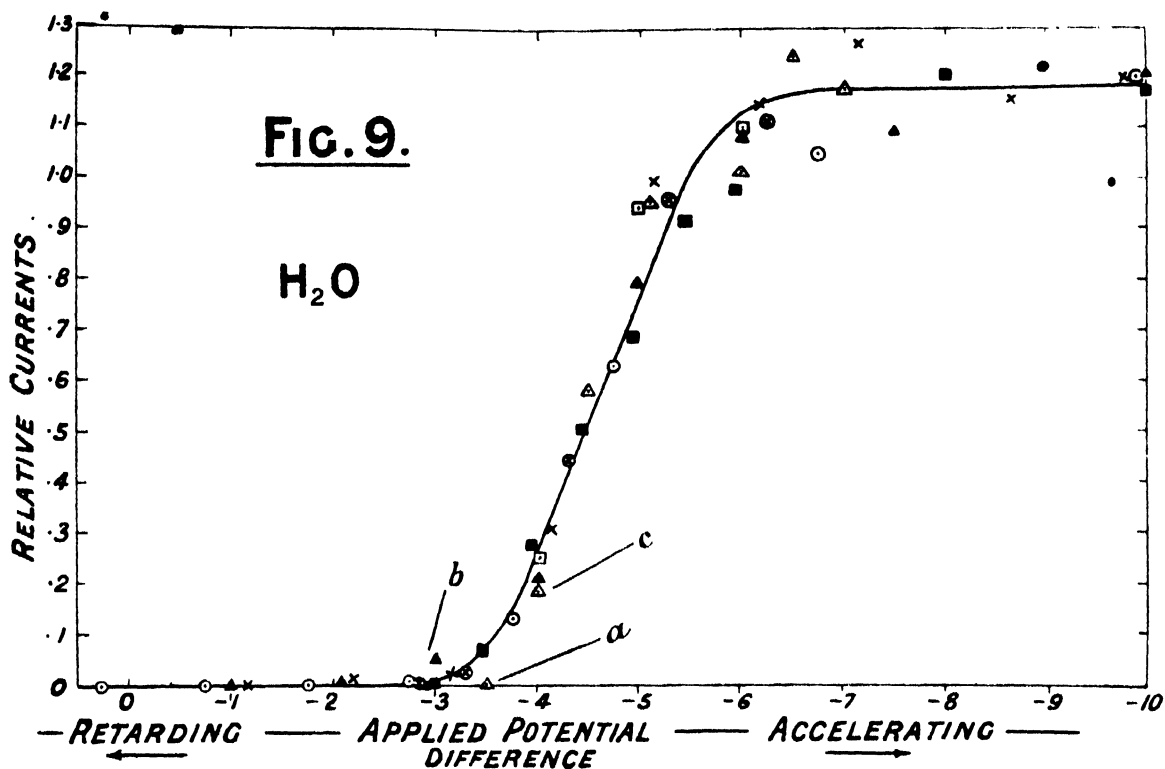
Remarks.

The gas was not allowed to accumulate, but the pumps were kept running continuously in all these experiments. The gas pressure is not recorded, but probably it did not

much exceed 0.002 mm. at any time. In V. \oplus the water vapour was obtained by placing crystals of $\text{CaCl}_2 \cdot 6\text{H}_2\text{O}$ in the generating tube and immersing it in ice. In IV. \square the same crystals were used but the container was immersed in a mixture of ice and salt. In all other cases mixtures of sulphuric acid with different suitable proportions of water were used, and the mixture was kept at the temperature of the room. In VII. \triangle equal parts of the two liquids were taken. The water vapour was given off quietly and the liquid did not boil. In I. to V. the drops fell at the rate of about one per minute, in VI. at the rate of seven per minute. After VI. a large structure with a wart-like appearance had grown on the end of the nozzle, and in VII. the globules came out of this structure. They were very large and looked dirty when attacked by the H_2O . In the previous experiments, in which much larger currents were obtained, the oxidised layer either looked quite white or was invisible. It was noticed that the oxidised metal showed no appreciable photo-electric effect with the quartz mercury lamp except just when the oxidised skin broke and the bright metal shone through. In series I. to V. the electrometer deflections were reduced by adding suitable capacity from an adjustable air condenser. The added capacities are not recorded in these cases. In VI. 0.2 mfd. was added from a standard condenser and the maximum saturation current was about 50 divisions per minute. In VII. the capacity was that of the apparatus alone (0.00032 mfd.) and the saturation current was 50 divisions per minute in this case also. VI. and VII. probably correspond respectively to the largest and smallest effects dealt with.

In fig. 8 the individual data I. to VII. are plotted as they were experimentally determined. The points for I. \times and V. \oplus fall so close together that they appear to fall on a single curve. The same applies also to each of the pairs IV. \square , VII. \triangle and III. \blacktriangle , VI. \blacksquare . Thus the seven sets of data appear to fall on four distinct curves. However, these curves are all nearly parallel to one another, just as was the case with the curves for COCl_2 and Cl_2 . They have therefore been subjected to the same treatment, namely, given the arbitrary displacements which are recorded in Table III. parallel to the volt axis. The result is shown in fig. 9. It will be seen that, with the exception of the points marked *a*, *b* and *c*, all the points then lie on a single smooth curve to within the degree required by the probable experimental errors. Of the excepted points, *b* is doubtful owing to a rapid variation of emission with time when it was taken, and *a* and *c* should not, except for the sake of having a complete record of the observations, have been included in the diagram, as they are determined by minute deflections comparable in magnitude with the variations in the natural leak of the electrometer system in this particular experiment.

Fig. 9 shows that the form of the characteristic curve for H_2O is of the same general nature as those for Cl_2 and COCl_2 . No determinations of the contact potential *K* with water present have been made, so that the position of the true zero of potential difference on fig. 9 is uncertain.

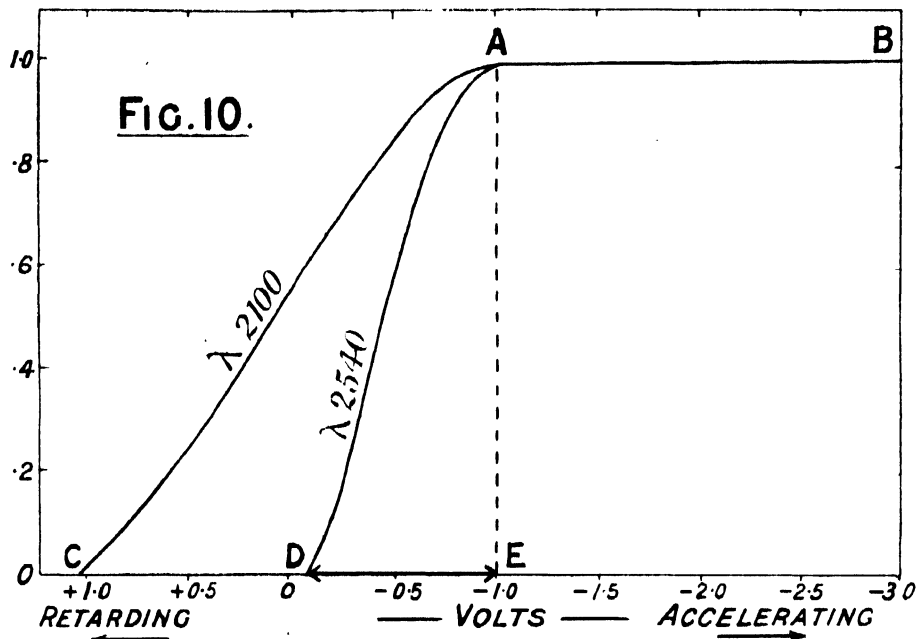


§ 6.—Location of Zero on the Volt Scale for COCl_2 .

A knowledge merely of the shapes of the characteristic curves does not enable us to say much very positively about the kinetic energy of the liberated electrons, owing to the uncertain and changeable contact potential difference. In fact, it is necessary to find the position on the volt scale which corresponds to zero electric field between the electrodes, either by a determination of the contact potential difference for some particular case or by something which is equivalent to this. In principle the position of the zero on the volt axis can be found from a study of photo-electric data. Experiments by the writer and Prof. K. T. COMPTON,* in which a structure similar to that used in the preceding experiments was employed, namely, a small photo-electric source at the centre of a large spherical electrode, have shown that the photo-electric characteristic curves have the form shown in fig. 10. They consist of a flat part AB joined to a sloping part such as AC or AD, whose slope diminishes with diminishing wave-length of the monochromatic exciting light. The sloping portions all converge to a common point A of the saturation part of the characteristic independently of the wave-length used. A is the position of the true zero of potential difference, and, in fact, these small currents are completely saturated in any accelerating electric field, however small. Thus one method of locating the zero would be to find the convergence point A to the saturation value of different monochromatic characteristics such as CA, DA. This would, of

* 'Phil. Mag.,' vol. 24, p. 575 (1912).

course, have to be done simultaneously with the measurements of the chemical emission, and an examination of the preceding data will show that it is hopeless to attempt to determine this point accurately under these conditions. For it is just as the currents approach the maximum value that the absolute errors become largest and the exact



location of the points becomes most difficult. This difficulty can be avoided if we can make an auxiliary determination of ν_0 the threshold frequency of the light which is just high enough to excite any photo-electric emission at all. For if D is the point where the foot of a characteristic such as DA for some particular frequency ν cuts the volt axis and DE is denoted by V , then

$$eV = h(\nu - \nu_0)$$

where e is the charge on an ion, h is PLANCK'S constant and e and V are in the same absolute units. By using very large photo-electric currents, points such as D can be determined with great accuracy and comparative ease, whereas increasing the magnitude of the emission does not diminish the inaccuracies to which determinations of points like A are liable.

Data to which this second method can be applied were obtained in connection with the Series I. \times COCl_2 of 3/8/1916 (p. 12). Observations at the end of this series showed that with the blue filter the photo-electric saturation current was 248 divisions per minute, and that with the green filter it was only 0.6 div./min. At the same time the chemical saturation current was 4.0 divs./min., a capacity of 0.01 mfd. being included in each of these measurements. The pressure of the reaction products had risen to 0.016 mm. at this stage. The fact that the blue light of wave-length 4347-4358 and

frequency 6.89×10^{14} gave a large deflection shows that this frequency was considerably higher than the threshold value. If the small value of the current with $\lambda 5460$, frequency 5.49×10^{14} could be relied on as genuine, it would establish ν_0 as very close to $\lambda 5460$ and just on the low-frequency side of it. On the other hand, if this small deflection is spurious or due to a trace of light of higher frequency mixed with the $\lambda 5460$, the threshold value must be well on the low-frequency side of $\lambda 4350$ on account of the large deflection given by $\lambda 4350$. This experiment shows that ν_0 was somewhere between 5.49×10^{14} and 6.89×10^{14} at this stage. A plot of the photo-electric data got during series I. \times shows that the photo-electric current with $\lambda 4350$ met the voltage axis at -0.20 volt. This was immediately below the point at which the chemical current attained 0.50 of its saturation value. Taking the frequency as 6.89×10^{14} for $\lambda 4350$ and 5.49×10^{14} for $\lambda 5460$, the value of $(\nu - \nu_0)h/e$ is 0.58 volt for $\nu_0 = 5.49 \times 10^{14}$ and zero for $\nu_0 = 6.89 \times 10^{14}$. Thus the true zero must lie between a point which is coincident with the -0.20 volt on the voltage scale and with the 50 per cent. point on the chemical characteristic and a point which is 0.58 volt negative to these points. At this period the contact potential difference between the two electrodes then was between 0.2 volt and 0.78 volt, and the true zero between the points where the chemical current had attained between 0.50 and 0.86 of its saturation value. Comparing with the composite curve in fig. 4, this makes the true zero lie between $+0.12$ and $+0.70$ volt on that diagram. It must be definitely to the right of $+0.70$ on account of the large deflections given by $\lambda 4355$, but how far to the right these data do not determine.

The importance of fixing this zero led me to make a renewed attack on this part of the problem recently. For success in this determination it is necessary to have a series of intense sources of monochromatic radiation not too far apart on the frequency-scale. It is also necessary to carry out the successive measurements with the utmost rapidity. By this time the apparatus shown in fig. 1 had come to grief and a new testing vessel of simpler construction, shown to scale in fig. 11, was employed. Apart from the smaller dimensions, the principal change consists in the substitution of a cylindrical copper electrode, 2 cm. in diameter, instead of a spherical one. The alloy used was NaK_2 and the COCl_2 was the KAHLBAUM specimen already referred to. The monochromatic sources were the lines of the mercury arc spectrum projected on to the drops by a Hilger glass monochromatic illuminator.

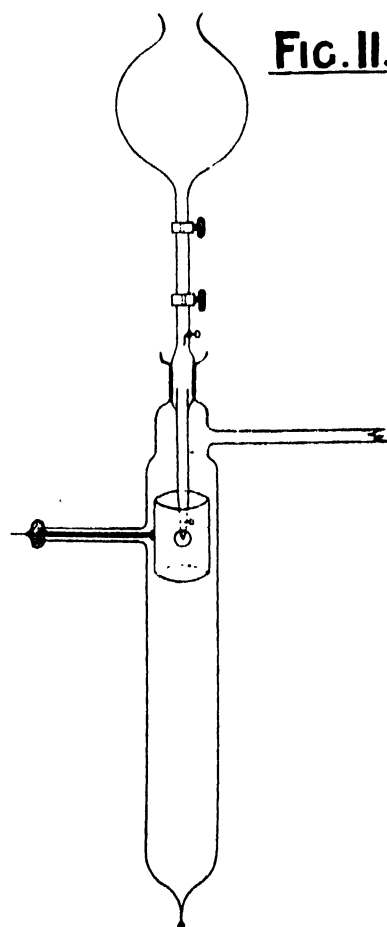


FIG. II.

Some useful data pertaining to the lines used are collected in the following Table:—

TABLE IV.—*Mercury Spectrum (visible)*.

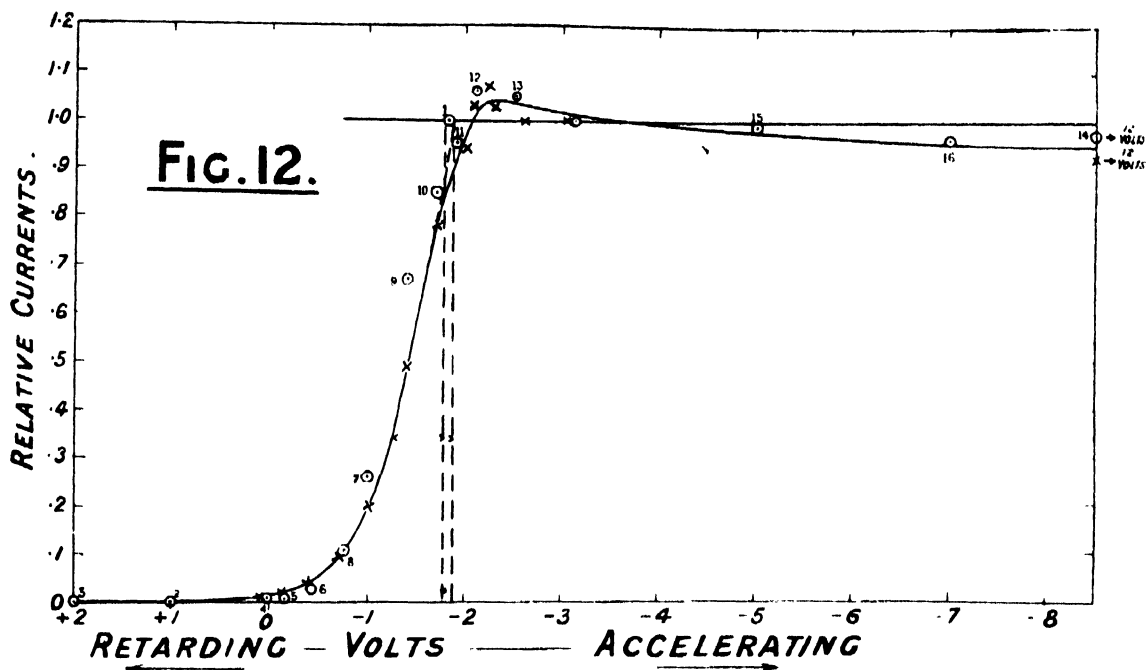
Colour.	Wave-length between (cm. $\times 10^6$).	Frequency (per second).	Relative intensity.*
Yellow	5.769 and 5.790	5.19×10^{14}	23.6
Green	5.460	5.49×10^{14}	18.7
Blue	4.916	6.10×10^{14}	1.0
Violet 1	4.347 and 4.358	6.89×10^{14}	14.7
Violet 2	4.046 and 4.077	7.40×10^{14}	8.2

In addition to the illuminator, I had the use of a large selection of colour filters kindly lent by Messrs. Hilger. Unfortunately, those which might have been useful cut down the working radiation too much. For monochromatism the dispersion of the illuminator has therefore alone been relied on. As a matter of fact, the lines were all sharp and widely separated, and no trouble has been experienced, owing either to overlapping or to stray light of other wave-lengths. The lamp was run at the maximum brightness at which it could be depended on to run smoothly and remain constant. To prevent trouble due to stray light it was necessary to enclose the lamp in a large tin box, and to prevent over-heating a water-cooling system had to be provided. Once the apparatus was adjusted the successive lines could be rapidly thrown on the drops simply by adjusting the screw setting of the illuminator. In every case the satisfactoriness of the mechanical setting was confirmed by visual observation.

Very steady conditions have been obtained with these arrangements. The data for two determinations of the characteristic curves with this apparatus are shown in fig. 12. One of these, \times , was taken the same day as and immediately preceding the determination of the true zero to be described in a moment, and the other, \odot , as early as possible the following day. The smooth curve is drawn to the first set of points marked \times , and will be seen to cover them very closely. The first set marked \times agree with the second, except that (1) they lie about 0.12 volt to the right, indicating a change by this amount in the contact potential difference between the two experiments, and (2) they give higher values for the small currents at the foot of the curve on the left-hand

* These energy measurements are taken from a paper by E. S. JOHANSEN in 'Strahlentherapie,' vol. 6, p. 55 (1915), and are for a different design of lamp which ran on 220 volts, taking 2.6 amperes with 128 volts between the poles of the arc. The quartz mercury vapour lamp used in these experiments ran on 200 volts and took 3.0 amperes with 150 volts between the poles. No doubt the relative intensities of the lines will vary to some extent with different installations, but for the present purpose it is only necessary to know the relative energies approximately.

side. These small currents are important, and the collective data bearing on them will be considered later. An examination of the experimental conditions on which the results plotted in fig. 12 are based shows that the values denoted by \odot are here liable to much more serious errors than those denoted by \times , which therefore are alone relied



on in obtaining this part of the curve. The numbers written alongside the points marked \odot indicate the order in which these points were determined. It should be mentioned that point 1 is unreliable owing to a very rapid change in the standard current when it was determined.

The second series of points indicate a small but definite falling off in the saturation currents with rising potentials. This may be due to some bias of chance errors, but it does not seem likely to be, as the data for the first series also support it. There is no evidence of such an effect in the COCl_2 data for the spherical electrode plotted in fig. 4, but it may be masked by the errors of observation. In fact, the experimental data which look most reliable for this part of the curve point to a rising current with rising negative voltages in those experiments. On the other hand, although the matter was not explicitly tested at the time the experiments were made, the chlorine data which form the basis of fig. 7 show, on re-examination, a distinct drooping tendency towards higher voltages. The evidence is thus not clear that this tendency is peculiar to the cylindrical anode. It may be illusory, but that is unlikely. It may be a general effect present with the spherical electrodes, but masked by errors. There are several physical effects which the electric field might exert on the drops which might give rise to such a phenomenon.

Apart from the hump near -2 volts, the characteristic in fig. 12 is not very different from that in fig. 4. The horizontal stretch from the 50 per cent. point to the 1 per cent. point is for fig. 12, \times , 1.40 volts, as compared with the value 1.20 volts of the stretch from the 50 per cent. to the 1 per cent. point for the curve in fig. 4. On the other hand, the stretch from the 90 per cent. to the 50 per cent. point in fig. 12 is only 0.46 volt, whereas it is 0.65 volt in fig. 4. The total stretch from the 90 per cent. to the 1 per cent. point is practically the same in both cases, the difference between the two curves being that fig. 12 is relatively steeper near the top. The data in fig. 12 are only relied on to give the shape of the characteristic. It is not supposed that the relative currents and voltages had these values when the photo-electric data were being determined. The instantaneous values of the relative chemical currents were determined at the same time as the photo-electric data.

In taking the data in fig. 12 the pressure in each series was less than 0.001 mm. at the beginning and equal to 0.004 mm. at the end.

The determination of the threshold frequency ν_0 was carried out immediately after the series \times of fig. 12 was completed, the pressure in the apparatus being 0.0045 mm. The measurements are shown in the following table, -3 being used for the saturation voltage :—

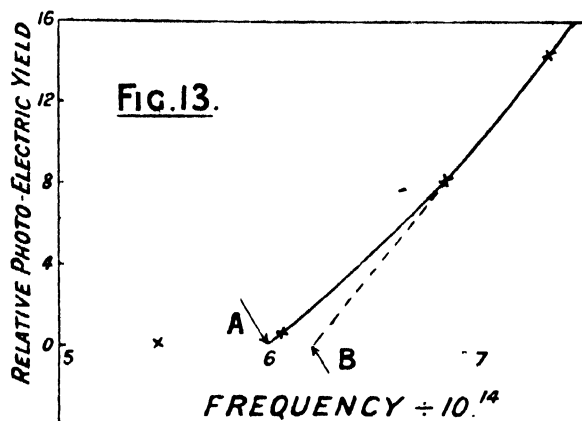
TABLE V.

Nature of light used.	Electrometer deflection in scale divisions per minute.	Additional capacity microfarads.	Deflection due to light.	Deflection per unit light energy.
No light.	47	0.01		
λ 4055	162	0.01	116	14.15
No light.	45	0.01		
λ 4355	160	0.01	117.5	8.00
No light.	40	0.01		
λ 4916	39	0.01	0.5	0.5
No light.	37	0.01		
λ 5460	35	0.01	-0.5	-0.027
No light.	34	0.01		

The deflections in the second column when no light was used are due to the chemical effect which was falling slowly and steadily during these experiments. The values in the fourth column are got by subtracting the instantaneous chemical effect (got by taking the means of the preceding and following measurements) from the combined effect due to light and chemical action when the light is on the drop. The values in the last column are got by dividing those in the fourth column by the relevant energy data given in Table IV.

The deflections per unit-light energy are plotted against the frequency of the exciting

light in fig. 13. According to measurements of the writer and K. T. COMPTON,* curves plotted from such data intersect the frequency axis at the threshold frequency ν_0 . If we assume the half-millimeter deflection with λ 4916 to mean something, this fixes the position of ν_0 at the point A or $6 \cdot 02 \times 10^{14}$. On the other hand, if the reading at λ 4916 is disregarded, it follows from a consideration of the λ 4055 and λ 4355 points



alone that the intersection cannot lie to the right of the point B or $6 \cdot 25 \times 10^{14}$, which is the point where a straight line through these two points meets the axis. Thus this experiment establishes the value of ν_0 as lying between the limits $6 \cdot 02 \times 10^{14}$ and $6 \cdot 25 \times 10^{14}$. It is unfortunate that the line λ 4916 is so weak, otherwise these limits could be narrowed very considerably.

Immediately after determining ν_0 the measurements for determining the intersection of the photo-electric characteristic for the line violet 2 (λ 4046 to 4077) with the voltage axis were taken. The pressure was still 0.0045 mm. The photo-electric currents, in scale divisions in 60 seconds with 0.01 mfd. additional capacity connected, with the electrometer, were as shown in the following table in the order in which the points were taken:—

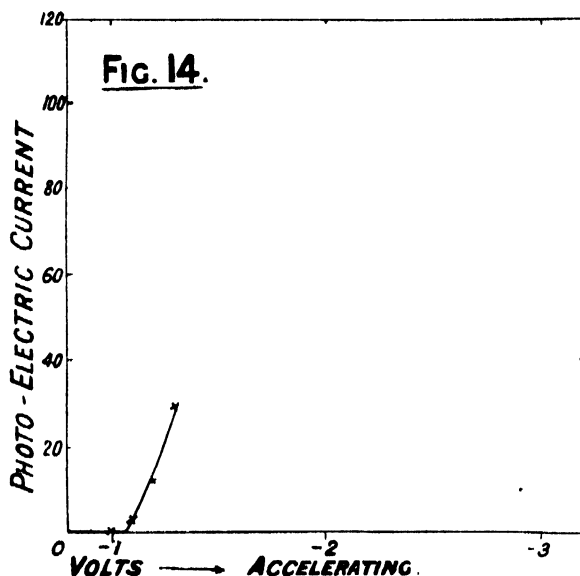
TABLE VI.

Volts	- 3	- 1.8	- 1.0	- 1.2	- 1.1	- 1.3
Current	112	101	0.5	12	3	29.5

These data are plotted in fig. 14, from which it will be seen that the photo-electric characteristic for frequency $7 \cdot 40 \times 10^{14}$ cuts the voltage axis at $- 1 \cdot 07$ volts. Thus the true zero on the volt scale at this time was $-(1 \cdot 07 - h/e(\nu - \nu_0))$, where $\nu = 7 \cdot 40 \times 10^{14}$ and $h/e = 4 \cdot 124 \times 10^{-15}$, is expressed in appropriate units to give volts. Now $h/e(\nu - \nu_0)$ is equal to $\cdot 569$ or $\cdot 474$ volt, according to which of the limiting values of ν_0 determined above we take. Adding $1 \cdot 07$ to these, it follows that

* 'Phil. Mag.', vol. 26, p. 562 (1913).

we have ascertained the position of the true zero at that time as lying between the limits -1.544 and -1.639 volts.



The next point is to ascertain what was the position at that time of the chemical characteristic curve in relation to the volt scale used. Relevant chemical data were taken at the same time as the photo-electric data given in Table VI. An examination of these data shows that with -1.00 volts the chemical current was then 34 per cent. of its saturation value at -3 volts. It follows that for any COCl_2 characteristic curve for this apparatus the true zero will lie between the limits of 0.54 and 0.64 volts to the right of that voltage for which the current is equal to 34 per cent. of the saturation value. Applying this to the curve, points thus:— \times , in fig. 12, the two limiting values are given by the two broken vertical lines at -1.78 and -1.88 volts respectively. The inclined broken line represents the tangent to the characteristic at the point of inflection. It will be seen to intersect the horizontal line through the unit 3-volt saturation current value at the same point as the vertical line through the right-hand zero limit, to the accuracy within which the lines can be drawn. Thus the true zero lies between the intersection of the tangent at the point of inflection and the horizontal saturation current line and a point one-tenth of a volt to the left of this intersection.

These results support the conclusions already drawn from less complete data in the experiments with the spherical anode. In that case the true zero was only located to within 0.58 volt, but it was definitely placed to the left of the corresponding intersection. It therefore seems reasonable to assume that either with spherical or cylindrical anodes the true zero in this potential scale lies a fraction of a volt to the left of the point of intersection of the tangent at the point of inflection with the horizontal saturation line. Considering the nature of the experiments, the location of this zero

to a higher degree of accuracy than one-tenth of a volt would obviously be a matter of some difficulty.

The data now under consideration clearly determine the contact differences of potential ruling between the electrode surfaces during the experiments. If v is the velocity with which the electrons reach the receiving surface, V the retarding potential recorded by the voltmeter, K the contact potential, then for illumination by light of frequency ν of a source whose threshold frequency is ν_0 , in general,

$$\frac{1}{2}mv^2 = h(\nu - \nu_0) - e(V + K).$$

At the intersection of a curve such as that in fig. 14 with the volt axis, V is just sufficiently great to reduce v to zero. Calling this value of V , V_0 , clearly

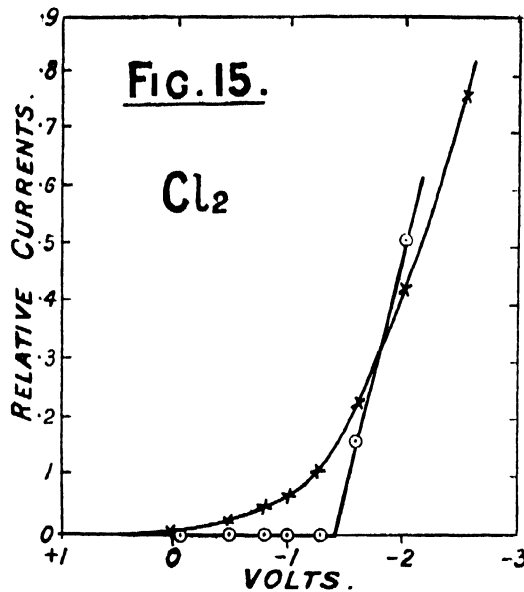
$$K = h/e(\nu - \nu_0) - V_0.$$

For the experiment to which fig. 14 refers, $V_0 = -1.07$, $\nu = 7.40 \times 10^{14}$, and ν_0 is between 6.02×10^{14} and 6.25×10^{14} . Consequently the contact potential K was between the two limits 1.544 and 1.639 volts. For the slightly different conditions ruling when the curves in fig. 12 were taken, the limits for the contact potential would be:—For the full curve with points thus, \odot , 1.62 and 1.72 volts; for the series with points thus, \times , 1.78 and 1.88 volts.

§ 7.—*The Approach to the Voltage Axis.*

It is well known that photo-electric characteristic curves for monochromatic illumination approach the voltage axis at finite angles, indicating a finite limit to the maximum kinetic energy, whereas the characteristic curves for thermionic electron currents approach this axis asymptotically. An examination of figs. 4, 6, 7 and 12 suggests that in this respect the chemical electron curves differentiate themselves from the photo-electric and resemble the thermionic ones. The point is an important one for the interpretation of the results. It is, perhaps, not so easy to be sure about it as might appear from an inspection of the diagrams, inasmuch as in many cases this part of the curves depends on the measurement of small deflections liable to considerable errors. Probably the best way of testing this question is by photo-electric and chemical curves taken simultaneously, since any errors will then be liable to affect both curves in a similar manner. Fig. 15 shows the results of such an experiment with chlorine using the spherical electrode. The points marked \times represent the relative chemical currents, and those marked \odot the relative photo-electric currents taken simultaneously. The values are relative to the -3 volt values, and the currents were not saturated at this voltage. Some trouble arose during the measurements on the saturation part of the curve, and the value of the saturation current could not be got accurately, so that the vertical scale may be a little different from that of most of the other figures in this

paper. This experiment leaves little doubt as to the reality of the gradual approach to the voltage axis in the case of the chemical effect. It is the more convincing because, during the determination of the two points at 0 and -0.5 volts, the standard current



at -3 volts was increased to about 450 divisions per two minutes (cap. 0.01 mfd.); whereas, during the determination of the rest of the points in fig. 15, the standard current varied between 20 and 30 divisions per two minutes, with the same capacity. Notwithstanding this very great change in the absolute value of the standard current, the two end points are seen to fall on the same curve as the rest. The actual numerical values may be of interest and are given in the next table:—

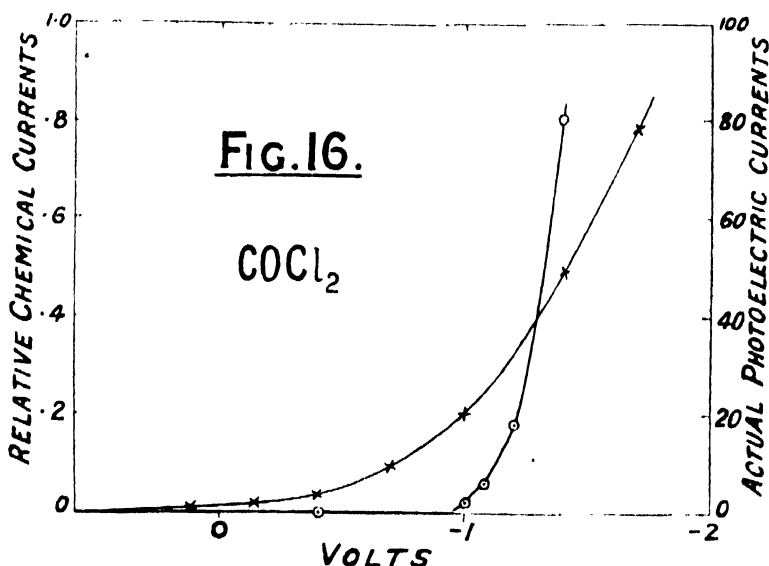
TABLE VII.—Chlorine.

Applied volts (voltmeter readings)	2.5	2.0	1.6	1.3	1.0	0.8	0.5	0
Chemical effect currents (scale divs. per 2 mins.)	18	11	6	3	2	1	8	2
Relative chemical currents	0.78	0.43	0.24	0.116	0.068	0.05	0.025	0.004
Relative photo-electric currents		0.51	0.16	0.00	0.00	0.00	0.00	0.00

The sensitiveness of the electrometer was 570 divisions per volt, the added capacity 0.01 mfd., making the total capacity equal to 0.01012 mfd. ; the pressure was 0.002 mm. throughout and the pump running continuously in this experiment.

Fig. 16 shows some similar data for COCl_2 , except that the photo-electric and chemical curves were not taken simultaneously. The photo-electric curves, with the mercury lines λ 4046 to λ 4077, were taken the day before the chemical data, and no doubt there was some alteration in the contact potential between the electrodes in the interval between the experiments. The chemical data shown by crosses are a replot of those similarly indicated in fig. 12. The photo-electric data, shown thus \odot , are actual currents in scale divisions per minute, with a capacity of 0.01053 mfd. and an electrometer sensitiveness of 1100 divisions per volt. The saturation current was 130 in the same units in these photo-electric experiments. These curves again show quite clearly the reality of the small currents with relatively large opposing voltages for the chemical emission. The actual data for fig. 16 are shown on the next page.

The second, third and fourth rows in this table and the last but one refer to the chemical emission, the fifth, sixth and seventh and the last to the photo-electric. The corrections for the electrometer drift in the fourth and seventh rows have been reduced respectively



to the same units as the corresponding deflections in the second and fifth. Assuming that the determinations are liable to errors of the same magnitude as this drift, an assumption which is certainly conservative, the small chemical current at + 0.12 volt should be reliable to within 30 per cent. and the errors in the others are comparatively trifling. The same test applied to the photo-electric data shows that the value at - 1.0 should be correct to 16 per cent., the value at - 0.4 meaningless and the others reliable. With regard to the small electrometer deflections recorded in some of these experiments, it should be pointed out that these can be measured with considerable confidence when the large capacities employed are added to the instrument. The pressure during these chemical measurements varied between the limits of 0.002 and 0.004 mm., and during the photo-electric between 0.0015 and 0.0025 mm.

TABLE VIII.—COCl₂.

Applied volts (as given by voltmeter)	- 1.7	- 1.4	- 1.2	- 1.08	- 1.0	- 0.7	- 0.4	- 0.14	+ 0.12
Chemical current (scale divs./min.)	79	109			59	11	5	3	31
Capacity (mfd.)	0.02053	0.02053			0.02053	0.02053	0.02053	0.02053	0.00053
Correction for leak (scale divs./min.)	- 0.26	- 0.26			- 0.26	- 0.26	- 0.26	- 0.26	- 10
Photo-electric currents (scale divs./min.)		80	360	114	34		- 12		
Capacity (mfd.)		0.01053	0.00053	0.00053	0.00053		0.00053		
Correction for leak (scale divs./min.)		+ 0.35	+ 7	+ 7	+ 7		+ 7		
Relative chemical currents	0.785	0.490			0.200	0.092	0.034	0.016	0.0067
Photo - electric currents (scale divs./min. with cap. 0.01053)		80	18.5	6.08	2.06		- 0.251		

On referring to fig. 12 it may be noticed that the values taken the following day and shown thus \odot on that diagram, are not in good agreement with the crosses on this part of the curve. However, they are nothing like so reliable. The reading at $+0.12$ volt is only 15 per cent. of the electrometer drift; that at -0.12 , 37 per cent.; that at -0.42 , 80 per cent.; and it is not until the comparatively large relative current at -1.0 volt is reached that the accuracy becomes better than that of the worst point shown in fig. 16.

§ 8.—*The Effect of the Different Gases on the Photo-electric Threshold Frequency.*

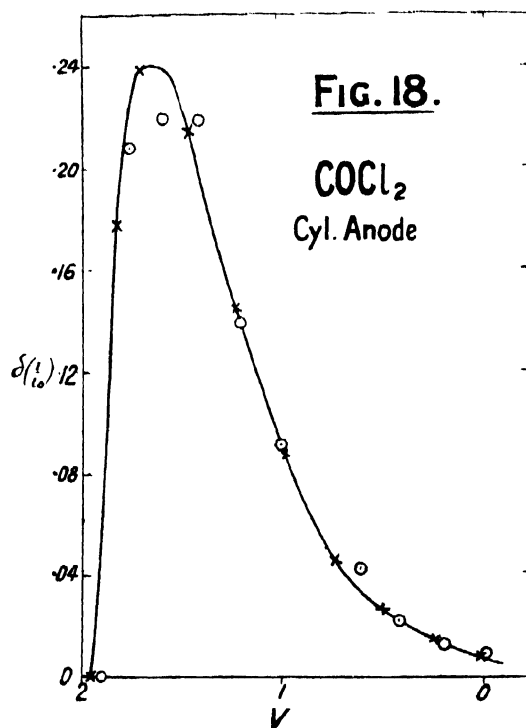
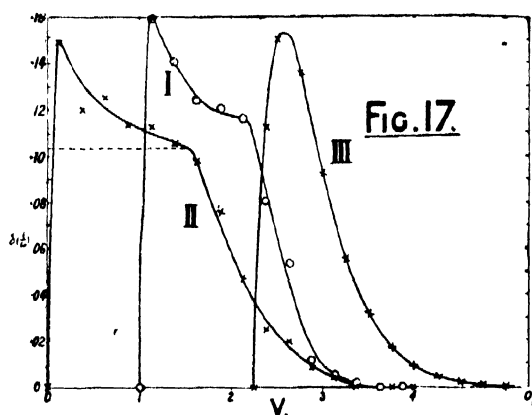
In the one case for which this frequency has been closely estimated (COCl_2) we have seen that it was very near to $\lambda 4900$ ($6.02 \times 10^{14} < \nu_0 < 6.25 \times 10^{14}$). This ν_0 is not to be regarded as a reliable constant, but one which varies with apparently trifling changes in the conditions. Thus in one COCl_2 experiment no photo-electric current could be got with the full light of the mercury lamp through the blue filter (mostly $\lambda 4355$). The chemical effect was working quite well and there was nothing otherwise to differentiate this experiment from others which gave quite large photo-electric currents with $\lambda 4355$. With this exception, so far as I can recollect or tell from the record, measurable photo-electric currents were always obtainable with $\lambda 4355$ both in COCl_2 , chlorine and water vapour. With COCl_2 it is probable that the threshold frequency was always higher than that of the green line $\lambda 5460$ (frequency 5.49×10^{14}). At any rate, tests made from time to time with the green filter never gave any deflections which could be relied on to mean anything. As regards chlorine, all that can be said is that the threshold frequency was less than that of $\lambda 4355$ (6.89×10^{14}) in all the experiments here referred to. In some of the tests with water vapour present it was found that not only $\lambda 4355$, but also the green line $\lambda 5460$ and the yellow line $\lambda 5769$ — $\lambda 5790$ gave photo-electric currents. The same was true of the light from a photographic dark-room lamp, and photo-electric currents were also obtained with the radiation from an incandescent lamp filtered through a solution of iodine in carbon disulphide of such strength as to make the lamp quite invisible through it. Evidently when water vapour is present the threshold frequency can lie in the infra-red part of the spectrum.

§ 9.—*Analysis of the Curves.*

The obvious way of attacking the problem of the distribution of kinetic energy among the emitted electrons is to take the curves, such as figs. 4, 7 and 12, and find the increments in current corresponding to equally spaced intervals dV along the voltage axis. These will be proportional to the number of electrons whose energies lie between eV and $e(V + dV)$. This method will certainly give correctly, to the degree of accuracy within which it can be operated and to the accuracy within which V is known, the distribution of velocity among the electrons as they in fact reach the receiving electrode.

It is not so reliable a guide to the distribution among the electrons as emitted at the source, on account of the difficulty in estimating the effects of the factors enumerated on p. 11 *ante*, which tend to prevent the attainment of saturation at zero field, in modifying that distribution before the electrons reach the receiving electrode. The picture may also be distorted owing to the inexact location of the true zero of potential difference. A study of similar phenomena, as displayed by thermionically emitted electrons, supplies a useful guide towards the interpretation of the present chemical data. In the thermionic case it is found that such factors exert their major effect in distorting the original distribution in the neighbourhood of zero field, and the shapes of the curves in the larger retarding fields are little affected. Thus this method of analysis leads to much more reliable results for the faster than for the slower electrons.

Such an analysis, when applied to fig. 4, is shown in fig. 17, curve I.; when applied to fig. 7, in fig. 17, curve II., and when applied to fig. 12, in fig. 18. In each case an



assumed zero has been taken which may be considerably wrong in the two curves in fig. 17, but which cannot be out by more than 0.10 volt in fig. 18, if, as I believe, the experimental determinations are reliable. Curves I. and II. of fig. 17 are seen to be very much alike and considerably different from fig. 18. The data in fig. 18 which are here referred to are the experimental points marked thus \odot . The meaning of the crosses and the full curve on this diagram will be explained below. However, all three have certain important points in common. They show a distribution, such that for low energies the number within a given range $dE = edV$ varies only slowly with E

(or V), whereas for high energies, *i.e.*, further to the right on each diagram, it falls off very rapidly. Moreover, the shape of the rapidly falling part of the curve is much the same in each case, and in fact on this part of the curves the ordinates are very nearly proportional to $e^{-\alpha v}$, where α is a constant for any one curve. This can be seen if the logarithms of the ordinates are plotted against V when the points fall very nearly on a straight line. On the other hand, the slowly varying part of the curve shows distinct indications of possessing a maximum in fig. 18, whereas in curves I. and II. it falls away continuously from the initial value. This disagreement at low energies suggests trouble from the factors referred to above and points to the desirability of keeping for the present to the high energy part of the curve in trying to find an interpretation of the results. The outstanding feature of this part of the curve is the fact that it falls away very approximately in proportion to the factor $e^{-\alpha v}$, which at once suggests a Maxwell distribution of energy among the electrons, since this distribution is dominated by a factor of this form. I have therefore calculated the currents which would be obtained on the assumption that the energy of the electrons is a Maxwell distribution pertaining to some, as yet undetermined, temperature T , and compared the results of the calculations with the experimental data.

For a small source surrounded by a large electrode the direction of motion is immaterial, and it is only the magnitude of the total kinetic energy which determines whether the emitted electrons will reach the receiving electrode against a given retarding potential difference. If the distribution is Maxwellian, the proportion with energies between u and $u + du$ is equal to

$$\frac{A}{k^2 T^2} u du e^{-\frac{u}{kT}},$$

where A is an undetermined constant and k is BOLTZMANN'S constant. They reach the surrounding electrode if $u \geq eV$. Hence the current against an opposing potential difference V is

$$\begin{aligned} i &= \frac{A}{k^2 T^2} \int_{eV}^{\infty} u du e^{-\frac{u}{kT}} = A \left(1 + \frac{eV}{kT} \right) e^{-\frac{eV}{kT}} \\ &= i_0 \left(1 + \frac{eV}{kT} \right) e^{-\frac{eV}{kT}}, \end{aligned}$$

if i_0 is the value of i when $V = 0$, *i.e.*, the value of the saturation current. If the currents are expressed as fractions of the saturation value, the proportion of the maximum current, or, what is equal to this, the proportion of the emitted electrons, having energies between the limits eV and $e(V + dV)$, is given by

$$d \left(\frac{i}{i_0} \right) = \frac{e^2}{k^2 T^2} V dV e^{-\frac{eV}{kT}}.$$

The following table of values of these various quantities calculated, to the accuracy of the slide rule, for $T = 1500^\circ\text{K}$, taking $e = 4.8 \times 10^{-10}$ and $k = 1.346 \times 10^{-16}$ will

be found useful. These values happened to be computed for $T = 1500$ in connection with another matter. However, since all the functions are homogeneous in V/T , they can at once be applied to any other temperature T^1 by simply multiplying all the voltages by $T^1/1500$. Of course, in dealing with the values of $d\left(\frac{i}{i_0}\right)$ the appropriate value of the factor $\frac{e dV}{kT}$ has to be used. This can readily be ascertained for any values of T and dV from the numbers in the second column of the table.

TABLE IX.

V.	$\frac{eV}{kT}$.	$e^{-\frac{eV}{kT}}$.	$\left(1 + \frac{eV}{kT}\right)e^{-\frac{eV}{kT}}$.	$\frac{eV}{kT}e^{-\frac{eV}{kT}}$.
0	0	1	1	0
0.05	0.398	0.672		0.268
0.10	0.795	0.452	0.809	0.360
0.20	1.59	0.205	0.528	0.326
0.30	2.385	0.0926	0.314	0.222
0.40	3.18	0.0418	0.175	0.133
0.50	3.975	0.0189	0.0940	0.0751
0.60	4.77	0.00855	0.0493	0.0408
0.70	5.565	0.00386	0.0253	0.0214
0.80	6.36	0.00175	0.0129	0.0111
0.90	7.155	0.000787	0.00642	0.00563
1.00	7.95	0.000355	0.00318	0.00282
1.10	8.745	0.000160	0.00156	0.00140
1.20	9.54	0.0000730	0.000768	0.00070
1.30	10.335	0.0000331	0.000374	0.00034
1.40	11.13	0.0000148	0.000179	
1.50	11.925	0.0000068	0.000088	

A set of values of $\frac{eV}{kT}e^{-\frac{eV}{kT}}$ is plotted, on an arbitrary scale, in curve III., fig. 17. The right-hand part of this shows a close resemblance with the rapidly dropping parts-

on the right-hand sides of curves I. and II., but the low-voltage parts of the curves are quite different. Apart from the difference in shape, the low-voltage parts of I. and II. are much broader than in the case of III. relative to the rest of the figure. However, the breadth here is determined entirely by the assumed position of the zero. In curve II. this is at 1 volt in fig. 17 and corresponds to the zero in fig. 4. This zero has, however, been shown to be at least 0.12 volt outside the possible limits for the true zero as determined by the photo-electric data, and it may quite well be out by as much as half a volt. In that case curve I. would have to come up from the volt axis at about 1.5 volts instead of at 1 volt, which would have made the curve show a much stronger resemblance to curve III. The position of the true zero for the chlorine data plotted in curve II. is still more doubtful. These curves illustrate well the difficulties involved in the analysis of the energy distribution unless the position of the zero is correctly known.

In fig. 18, which refers to the data for COCl_2 with the cylindrical electrode, the position of the actual zero is known to within 0.10 volt, and it is obvious that these experimental points (marked thus \odot) show a much closer resemblance to the requirements of the Maxwell distribution. The crosses in this figure do, in fact, represent a Maxwell distribution for the temperature $T = 3600^\circ\text{K}$, and the same range $dV = 0.20$ volt as the experimental points deduced from fig. 12, and the full curve is drawn to meet the theoretical points. It will be seen that the points given by the experiments are very close to the theoretical curve, and, in fact, the agreement is as close as could be expected, except in the immediate neighbourhood of the maximum. As has been explained already, there are disturbing causes which are likely to produce deviations in this region, and these deviations should be in the direction indicated by the discrepancy shown by the diagram. It should be pointed out that the only quantity which can be arbitrarily varied in fitting the theoretical curve to the experimental points is the single parameter T . The only other variable entering is the stretch dV , which is already fixed by the way in which the experimental points are reduced from the observational data. Given T and dV , the value of the ordinate for any particular value of V is determined absolutely.

I believe that the data exhibited in fig. 18 furnish strong evidence that the distribution of energy among the chemically emitted electrons is of a very simple character and is identical with that which would be possessed by the molecules of a gas at a certain definite temperature. The fraction where kinetic energy lies between u and $u + du$ appears, in fact, to be given by

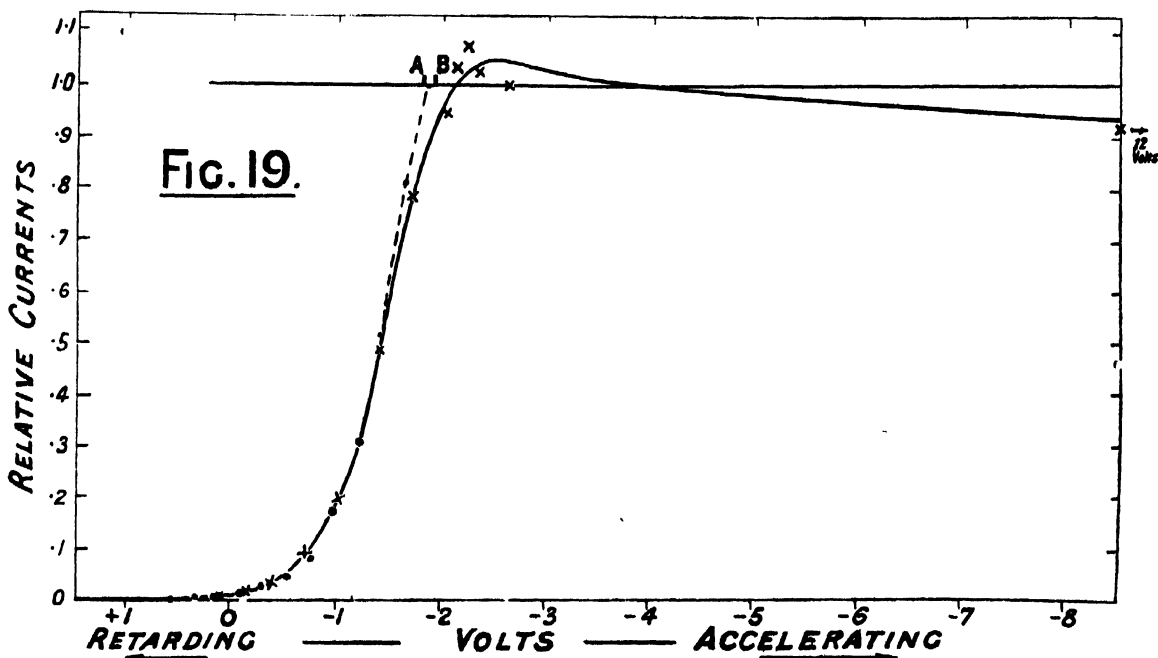
$$\frac{dn}{n} = \frac{du}{kT} \frac{u}{kT} e^{-\frac{u}{kT}}.$$

For COCl_2 acting on NaK_2 the value of T appears to be close to 3600°K . The evidence would, of course, be stronger if it were unequivocally supported by the analysis of figs. 4 and 7. It may be that too much emphasis is being laid on the uncertain data and the disturbing causes which affect curves I. and II. of fig. 17. It may be that

there is something real in the extra width of the low-voltage parts of the curves. Something of this kind might arise on the not improbable contingency that, in reality, more than one temperature T is involved. It is impossible to give a definite answer to such questions until more experimental evidence is forthcoming, but another basis of judgment on the data now available may be got by treating them a little differently.

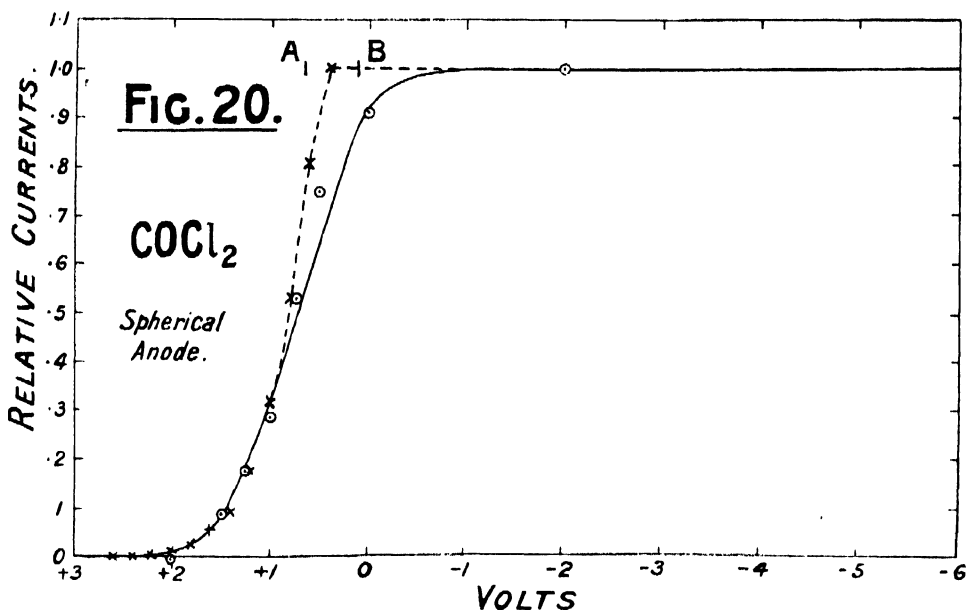
Instead of dealing with the proportion of electrons whose energies lie between eV and $e(V + dV)$, I shall now calculate the fraction of the saturation current which will be able to flow against an opposing potential V on the assumption that the initial velocity distribution is a Maxwell distribution for some temperature T , and compare the result with the experimental characteristic curves. The values of such currents for $T = 1500$ are the values of $\left(1 + \frac{eV}{kT}\right) e^{-\frac{eV}{kT}}$ in the fourth column of Table IX. for the voltages in the same row given by the first column. For any other value T^1 of T the corresponding voltages are obtained by multiplying the values in the first column by the ratio $\frac{T^1}{1500}$.

In determining the appropriate value of T the values of i/i_0 have been made to fit the experimental characteristics at 0.314 and at 0.0493. Unless the position of the zero on the voltage scale is known, it is necessary to fix two points in this way in order to determine the scale of T . Once this is done, all the other points are given by the corresponding numbers in the fourth and first columns of Table IX., subject to the transformation of the voltages in the first column in the ratio of the temperatures. Incidentally it may be remarked that this process, subject to the correctness of the hypothesis, gives the true zero as the point at which the value of i/i_0 becomes unity. This affords a further check on the interpretation of the data. The results of this treatment are shown in figs. 19, 20 and 21, which will now be considered in turn.



The full curve in fig. 19 is a replica of fig. 12, and the crosses denote the series of experimental points on which it is based. The auxiliary series of points in fig. 12 have been left out for clearness. The currents calculated as explained are shown by the points marked thus \bullet , and when necessary the curve joining them is indicated by a broken line. This line, therefore, gives the characteristic as it would be if it were determined entirely by the distribution of velocity of the electrons, and if this were a Maxwell distribution for the temperature $T = 3300^\circ\text{K}$, which is the value given by fitting the points referred to above. It will be seen that the theoretical curve agrees with the experimental characteristic for all retarding voltages exceeding about -0.20 volts. There is a small deviation in the neighbourhood of zero volts which is the counterpart of the discrepancy between the theoretical curve and the experimentally derived points in fig. 18 in the same region. This process places the true zero voltage at -1.84 volts in figs. 12 and 19; the limits assigned by the photo-electric measurements were -1.78 and -1.88 volts, and the value here found lies almost midway between them.

Fig. 20 deals similarly with the earlier data for COCl_2 given by the spherical electrode. The full curve is the composite curve of fig. 4 and the points \odot denote the best single

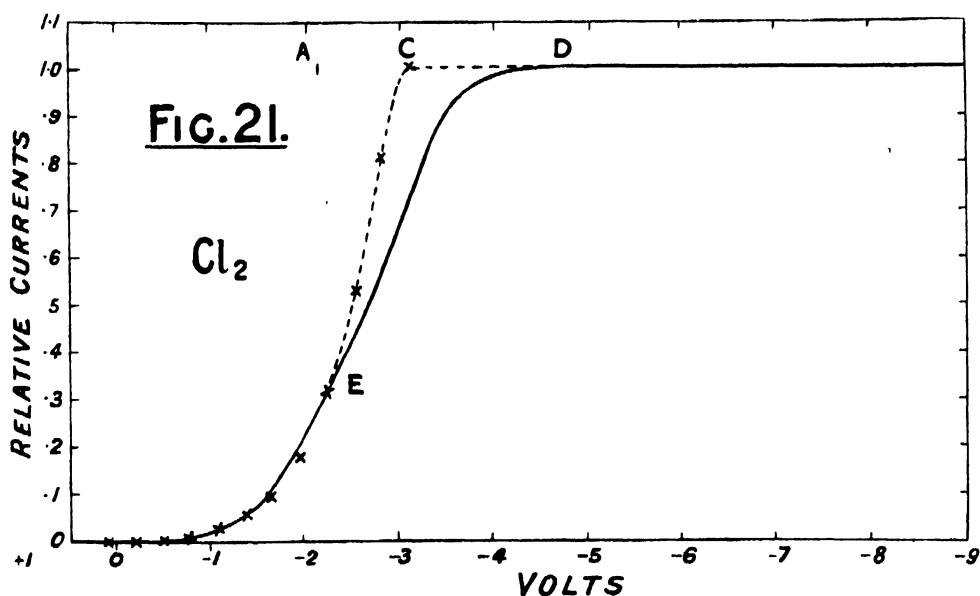


set of data on which this curve is based. The crosses denote the calculated values on the assumption of MAXWELL'S distribution, the fit at the two assigned points requiring a value of $T = 3000$ in this case. The broken lines show the theoretical characteristic as thus calculated. It will be seen that the agreement with the composite curve is satisfactory at the higher retarding voltages (shown on the left of the diagram), but that the deviation in the neighbourhood of zero volts is much more considerable than in fig. 19. However, this deviation is greatly minimised if the circular points are considered rather than the composite curve, and it may be that the rather considerable errors in the determination of this part of the curve have displaced it unduly to the right-hand

side. In any event the deviation is of such character and magnitude that further investigation may assign it to causes outside the energy distribution. The true position of zero volts given by this calculation is $+0.40$ volts on the scale and is well within the limits, marked by vertical lines at A and B on the diagram, set by the photo-electric observations.

It will be seen that figs. 18, 19 and 20 lead to three separate estimates of the value of T for the case of COCl_2 , viz., 3600, 3300 and 3000. Of these the first and second use the same experimental data but treat them in different ways. The third employs different data but treats them in the same way as the second. The mean of these determinations is $T = 3300^\circ\text{K}$. It is probable that an agreement to within 10 per cent. of the value of T is as much as it is reasonable to expect at the present stage of development of the subject.

Turning to fig. 21, the full curve is a copy of the chlorine curve in fig. 7 and the broken curve represents the theoretical characteristic assuming a Maxwell distribution, the



calculated points being shown by the crosses. The value of T for this curve was found to be 4350. The general appearance of fig. 21 is the same as that presented by the COCl_2 data in fig. 20, and, in general, the same remarks apply to both curves. The triangular area CDE, which measures the discrepancy between the calculated curve and the experimental characteristic, is somewhat larger in fig. 21 than in fig. 20, but the increase is only about in proportion to the greater horizontal extension of the chlorine diagram corresponding to the higher value of T . Whatever the difference between the full and broken curves is due to, it is probably caused by similar factors operating in the case of both gases. The value of the true zero given by the calculated curve in fig. 21 is -3.11 volts, which is well to the right of the left-hand limit -2.17 volts

set by the photo-electric currents and shown at A. The right-hand limit could not be determined at this stage.

Another estimate of the value of T for chlorine can be got by assuming that for COCl_2 and comparing the widths of corresponding parts of curves such as I. and II. in fig. 17. Keeping to the rapidly falling parts of these curves, which are comparatively free from errors due to the uncertain zero and to the disturbing causes in its neighbourhood, I find that the width from the $66\frac{2}{3}$ per cent. point to the 4 per cent. point, is for chlorine, 1.387 volts, and for COCl_2 , 0.840 volt. Taking the value of T for COCl_2 to be 3300, this would give the value for chlorine as $3300 \times \frac{1.387}{0.840} = 5450$. This is considerably higher than the value got from a study of fig. 21, but it will be remembered that the corresponding operation on fig. 20 also gave a low value for COCl_2 . Under the circumstances probably the best estimate that we can make for chlorine is the mean of the two values, viz., 4900°K .

In each case the position of the true zero given by assuming the Maxwell distribution falls within the limits set by the direct photo-electric measurements. If we assume that the position of this zero is correctly fixed in this way, we can get another check on the relative values of T. The displacement in volts from the zero to the place where the current is a fixed small fraction of the saturation value, small enough to avoid the disturbances in the neighbourhood of zero volts, should be in the proportion of the respective values of T. The three sets of data give the following results, using 0.05 as the value of the small fraction :—

COCl_2 *Cylindrical anode*—

$$\left. \begin{array}{l} \text{zero} \quad . \quad . \quad . \quad = \quad 1.84 \\ 5 \text{ per cent.} \quad . \quad . \quad = \quad 0.52 \end{array} \right\} \text{Displacement} = 1.32 \text{ volts.}$$

COCl_2 *Spherical anode*—

$$\left. \begin{array}{l} \text{zero} \quad . \quad . \quad . \quad = \quad 0.40 \\ 5 \text{ per cent.} \quad . \quad . \quad = \quad 1.60 \end{array} \right\} \text{Displacement} = 1.20 \text{ volts.}$$

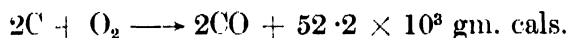
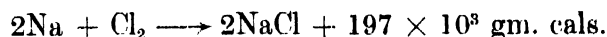
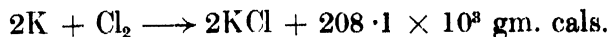
Cl_2 *Spherical anode*—

$$\left. \begin{array}{l} \text{zero} \quad . \quad . \quad . \quad = \quad 3.11 \\ 5 \text{ per cent.} \quad . \quad . \quad = \quad 1.36 \end{array} \right\} \text{Displacement} = 1.75 \text{ volts.}$$

The numbers 1.32, 1.20 and 1.75 are in the proportion 3300, 3000 and 4400, and are therefore in agreement with the values already obtained.

§ 10.—*Thermo-chemical Considerations.*

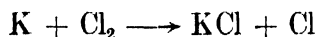
An upper limit to the value of T can be obtained from thermo-chemical data. From 'Recueil de Constantes Physiques,' pp. 333 and 339, it appears that the heats of certain relevant chemical reactions are as follows :—



The data are for solid K, Na, KCl and NaCl, for carbon in the form of diamond, and for gaseous Cl_2 , O_2 , CO and COCl_2 , in each case per gramme molecule, with the equations as written. It is difficult to conceive of any way in which the average energy of the electrons could exceed the equivalent of the heat of formation of two molecules of KCl from two molecules of K and one of Cl_2 ; in fact, it is practically certain to be much less than this. In general, if H is the heat available in gm. cal. per gramme molecule we shall have as a limit for T

$$\frac{3}{2}kT \leq \frac{JH}{N},$$

where k is BOLTZMANN'S constant $1 \cdot 346 \times 10^{-16}$, J is the mechanical equivalent of heat $4 \cdot 184 \times 10^7$, and N is the number of molecules per gramme molecule $6 \cdot 2 \times 10^{23}$. If we put $H = 208 \cdot 1 \times 10^3$ this gives as a limit for T the value $69 \cdot 7 \times 10^3$. This is about 14 times the value deduced from the experiments. However, something has to be subtracted from the value of H for the unknown heat of formation of the liquid alloy from the solid constituents, and the assumption that the whole heat of the reaction is passed on to one electron is only worthy of consideration for the purpose of fixing a limit for T. It is much more likely that the heat available is evenly divided among the different atoms taking part in the reaction. The number of these is doubtful owing to the somewhat uncertain degree of association of the reacting atom with its neighbours in the alloy. It is also possible that the actual reaction in which an electron is expelled is not correctly expressed by the equation of the end products written above, but is some intermediate reaction, such as



or



or the like. In such a case the value of the heat available may be quite different from that given by the end products. It is impossible to make any precise statements where the data are so indefinite, but I feel that when all these factors are taken into account the value of T given by the experiments on chlorine is a reasonable one.

This position is strongly supported when the value given by COCl_2 is compared with that given by Cl_2 . It is most likely that the mechanism of the reaction is much the same in both cases, any difference arising mainly from the fact that the Cl_2 is now loaded up with the CO group. This would have two effects. It would reduce the total amount of energy available by the difference of the heats of formation of CO and COCl_2 , *i.e.*, by 18×10^3 gm. cal. for each gramme molecule of COCl_2 , and it would

increase by two the number of atoms among which the energy is divisible. Let us consider a supposititious case in order to see what the effect of these differences will be. If we assume 6 as a reasonable number of atoms, including electrons as a special kind of atom, to be concerned in the reaction which emits an electron in the case of chlorine, there will be two more, or eight, in the case of COCl_2 . Six atoms in the case of Cl_2 require a value of $H = 88.4 \times 10^8$ to give $T = 4900$. This value of H is about half the total given by the equation of the end products after making a probable allowance for the unknown heat of formation of the liquid alloy, and is thus not unreasonable if the reaction in which the electron is emitted is of an intermediate type. The value of H for the case of COCl_2 now becomes $88.4 - 18 = 70.4$, and one-eighth of this is 8.8 , which on division by $\frac{2}{3} \frac{J}{kN}$ gives $T = 2950$, a value quite near to that given by the experiments. This argument shows that the values of T deduced from the experiments are not in conflict with thermo-chemical data so far as our knowledge of the reactions enables us to apply it.

In conclusion I wish to thank my assistant, Mr. J. W. BURROWS, for his invaluable help. I also gladly acknowledge my indebtedness to the Government Grant Committee of the Royal Society for a grant which has defrayed part of the cost of the apparatus and materials used.

II. *The Problem of Finite Focal Depth revealed by Seismometers.*

By GEORGE W. WALKER, *A.R.C.Sc., M.A., F.R.S., formerly Fellow of Trinity College, Cambridge.*

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THE results discussed in this paper were obtained about five years ago. Except for a brief reference in 'B.A. Reports,' 1917 ('Seismology') they have not been published. There were two reasons for this delay: (1) additional data were desired in a matter of somewhat critical importance in the measured properties of the earth; (2) my official duties left little spare time for the pursuit of a purely scientific branch of seismology which required a good deal of tentative numerical computation.

At the present date it appears that one must abandon all hope that additional results which were expected from Russian observatories can be obtained. Moreover, it appears very doubtful if relevant data from any of the Allied countries can be expected for some years to come. Accordingly, publication of the results so far obtained now seems desirable, and may serve to show how urgent is the need for the equipment of a few seismological observatories capable of obtaining the data that are wanted.

A brief introduction to the problem is necessary, although it covers ground which is fairly familiar to those interested in seismometry.

If we have complete data giving the brachistochronic time for a seismic ray (say the P wave) to travel from a given point on the earth's surface to any other point on the surface, it is possible to calculate the way in which the speed of propagation of the ray varies with the depth. Two methods are open: (1) we may use the differential equation for the path of the ray, or (2) we may use the integral equation obtained from this. Both methods correctly carried out must give the same result, and it is merely a question of convenience which one adopts. By using a comparatively rough graphical method based on (1), WIECHERT and ZOPPRITZ showed, about 14 years ago, from their accumulated data that the speed for the P wave increases from 7·17 km./sec. at the surface to 12·7 km./sec. at a depth of 1500 km., while from 1500 km. to over 3000 km. depth the speed increases but slightly. No data are available for investigating greater depths.

More recently KNOTT ('Roy. Soc. Proc.,' 1918–1919) applied the second method to the same data as was used by ZÖPPRITZ, and his results do not differ materially from those of the earlier and rougher method.

A very important supposition has to be made, however, before either of these methods can be applied. That is, that the true focus is either at the surface or so near the surface that a small correction can be made for it. If, however, the focus is at a considerable

and unknown depth both methods fail, as is obvious from the consideration that, since all the rays we can observe must have passed out from the focus, we have no data for the comparatively large range of distance AB on the surface for which brachistochronic rays would *not* penetrate as deep as does the observed seismic ray. In fact, another unknown element enters into the connexion between the observed time curve and the variation of speed with depth.

A finite depth of focus implies a minimum angle of emergence at some point on the earth's surface and a point of inflexion on the time curve, and *vice versa*. Now, in considering ZÖPPRITZ's accepted time curve for P, we recognize that up to $\Delta = 1000$ km. the curve is probably hypothetical, but from 1000 km. to 13,000 km. there is no indication of a point of inflexion. It is not until we come to consider GALITZIN's direct measurements of the angle of emergence that we are confronted with a most marked minimum angle of emergence near $\Delta = 4000$ km., implying a point of inflexion on the time curve and a very considerable depth of focus. As GALITZIN's observations form the whole basis of this paper, they are reproduced here (although published elsewhere), as the reader may desire to have them convenient for direct reference.

TABLE I.

Epicentral distance. Δ in kilometres.	For P.		
	e from time curve.	\bar{e} computed.	\bar{e} observed at Pulkovo.
	0	0	
0	0	22	
500	11	23	—
1,000	21	27	—
1,500	30	32	—
2,000	37	37	—
2,500	44	42	48
3,000	49	47	44
3,500	53	52	43
4,000	57	54	42
4,500	60	58	43
5,000	63	60	44
5,500	65	62	46
6,000	65	62	48
6,500	65	63	51
7,000	65	63	54
7,500	66	63	58
8,000	66	64	62
8,500	67	64	65
9,000	67	65	67
9,500	68	66	68
10,000	69	67	70
10,500	70	67	71
11,000	70	68	72
11,500	71	69	72
12,000	72	70	73
12,500	73	71	73
13,000	74	72	74

The quantity \bar{e} is called the apparent angle of emergence at the surface, and is defined by

$$\tan \bar{e} = Z/H$$

where

Z is the observed vertical component of displacement

and

H is the observed horizontal component of displacement.

Thus \bar{e} is the direct subject of measurement by vertical and horizontal seismometers.

The angle e is called the true angle of emergence of the brachistochronic ray. It cannot be directly measured, but may be calculated from the time curve for P by the formula

$$\cos e = V_1 \frac{dT}{dA},$$

where V_1 is the speed for longitudinal waves at the surface, and T the time is supposed expressed in terms of the epicentral distance A .

When the conditions of reflexion are examined it can be shown that for a longitudinal ray incident

$$\cos e = \frac{V_1}{V_2} \left\{ \frac{1}{2} (1 + \sin \bar{e}) \right\}^{\frac{1}{2}},$$

where V_2 is the surface speed of transversal waves.

It is seen from the table that the values of \bar{e} calculated from ZÖPPRITZ'S curve do not agree with the values of \bar{e} directly measured at Pulkovo. The discrepancy is so marked that we may set aside the supposition that the Pulkovo values are merely instrumental errors. In a matter so important GALITZIN would hardly have published them if he had not felt assured that they were substantially correct. There remain two alternatives: (1) that the ratio V_1/V_2 for Pulkovo depends on the angle of impingence in such a way as to exactly annul the discrepancy. The probability of such compensation of actual facts to explain a theoretical formula must be regarded as small, and so we are left with alternative (2) that within the limits of possible error in the time curve we can modify it so as to agree with the direct measures of \bar{e} . We shall show that this alternative is quite possible within quite a large range of Δ . But we must at once point out the somewhat startling consequence of accepting the Pulkovo numbers as correct.

It has been shown that a ray which emerges with a minimum angle must have set out from the focus in a direction at right angles to the radius vector from the earth's centre to the focus. Thus for a minimum angle at $\Delta = 4000$ km. we find that even for a uniform earth the depth of focus required is about 0.2 of the earth's radius, or about 1250 km. The actual value may be a little less or a little more, according to the way in which speed varies with the depth. Anyhow, this is a much larger estimate of depth than has formerly been suggested, viz., of order less than 100 km.

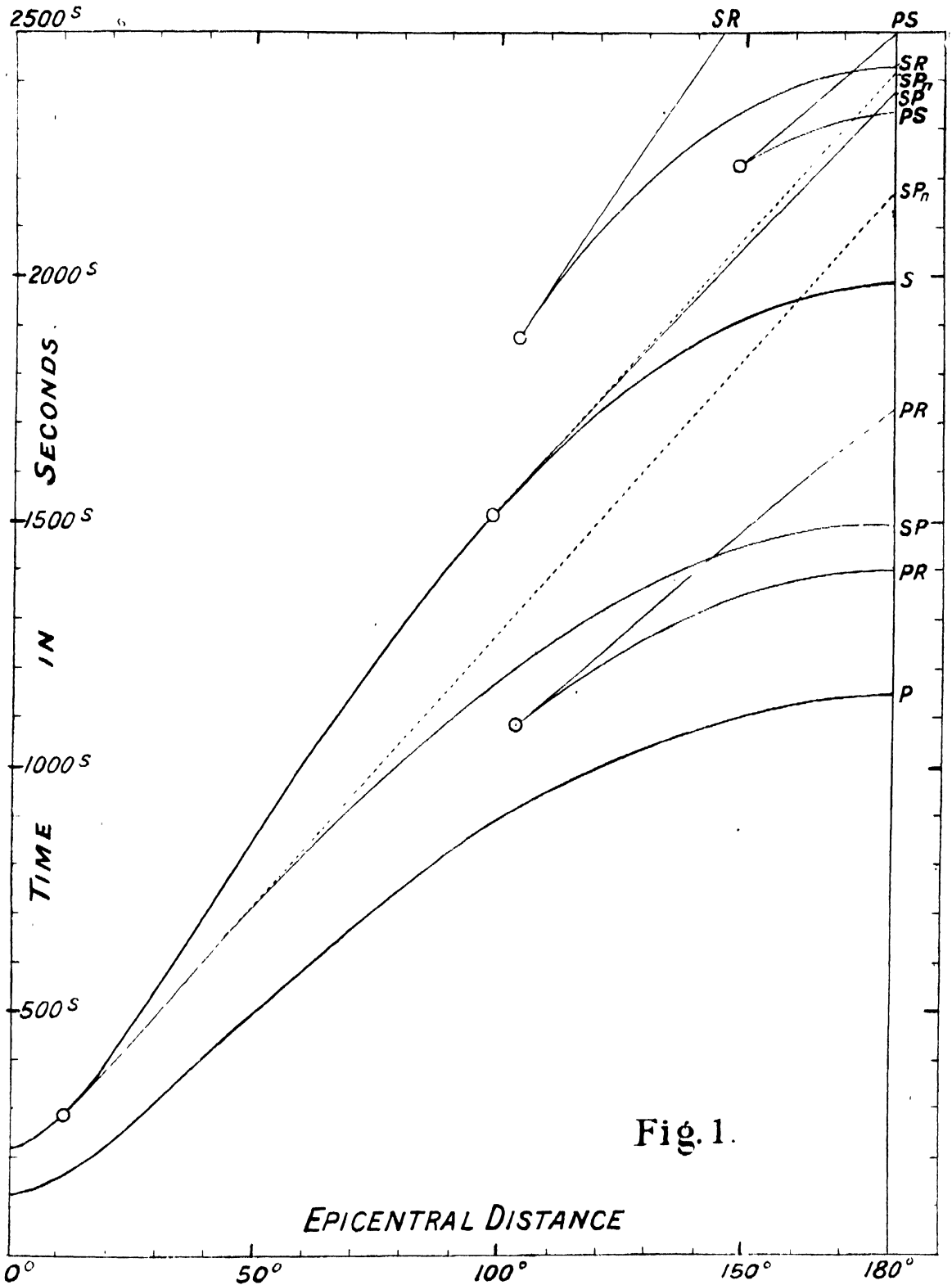


Fig. 1.

Uniform earth $R = 6370$ km. Focal depth $= 0.2 R$. $V_1/V_2 = \sqrt{3}$. $V_1 = 10$ km. per sec.

A number of novel consequences with regard to reflexion follow if we admit such a great depth of focus, so that before showing how a time curve can be deduced from the Pulkovo observed angles \bar{c} , we may within advantage consider what is to be expected in a uniform earth, as thereby we shall be in a better position to deal with what may be inferred from the observed data.

We shall select for discussion a uniform earth $R = 6370$ km., depth of focus $= 0.2 R$, with V_1/V_2 having the theoretical ratio $\sqrt{3}$, while V_1 is taken as 10 km./sec. These numbers are taken partly for convenience of calculation and partly to get as near as possible to the actual case.

The times for P and S may be computed for different epicentral distances from the trigonometrical formula for the paths traversed. The results for epicentral distances from 0 degree up to 180 degrees are shown in fig. 1. We may note that the point of inflexion on the time curve is very ill-defined, and might easily escape detection by direct observations of the time.

The direct measurement of the angle of emergence is, however, fairly precise, and from such measurements we can, in fact, calculate the time curve more accurately than we can determine it by direct observations of the time.

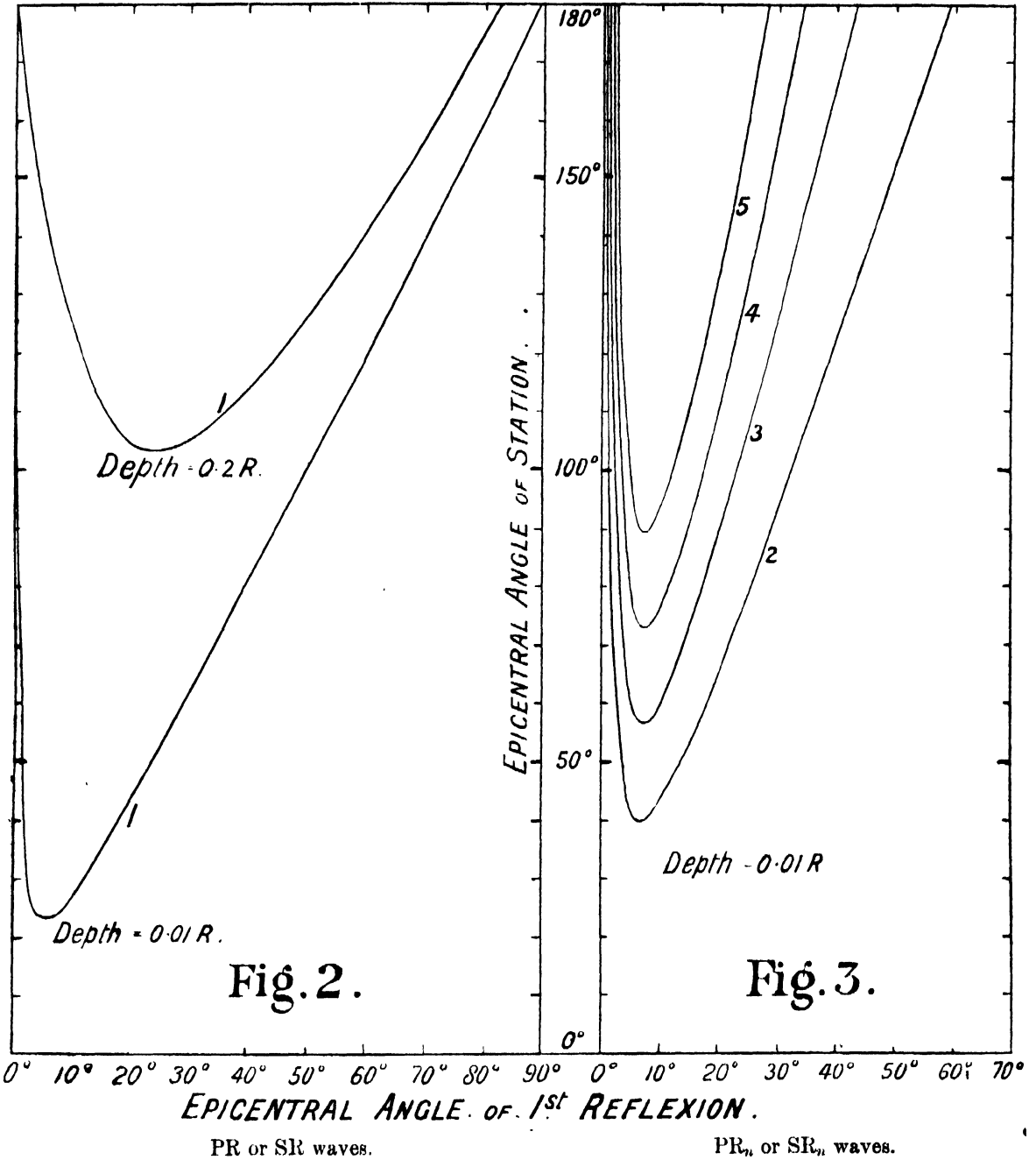
We may further note that the time increases but slowly for the first 1000 km., and since the angle of impingence for this region is not far short of 90 degrees, the true P might escape observation by horizontal seismographs, since the ground motion is almost entirely vertical.

Passing to waves reflected at the surface, we consider first waves which maintain their longitudinal character throughout. We may call them $PR \dots PR_n$ waves. The simplest way of computing is to choose the point at which the first reflexion takes place and then calculate the epicentral distance to the final point of emergence (the station). The results are shown in figs. 2 and 3, where, in order to lead up to the large depth of focus, we have first shown the effect for depth $0.01 R$ (about 64 km.). We find that for this depth we cannot get a reflexion at all until Δ is about 23 degrees, and that for $\Delta > 23$ degrees there are two PR waves, the reflexion taking place at two different points, and they occur at different times. For depth as small as $0.01 R$ we see that we can proceed to PR_n , where n is moderately large.

When we pass to a focal depth $0.2 R$ we find that the smallest epicentral distance for which we can get PR is 103 degrees, and beyond this we have two PR's. But when we try to calculate the PR_2 we find that the least epicentral distance is over 180 degrees, and so we stop. The corresponding times for PR_1 are calculated and shown in fig. 1, and we note that the earlier arrival refers to the PR_1 , which is reflected at the smaller distance from the epicentre. Figs. 2 and 3 are equally applicable to S waves in which the vibration is at right angles to the diametral plane through focus and station. The times for the SR waves are shown in fig. 1.

We consider next waves which undergo change from longitudinal to transversal, or *vice versa*, on reflexion. PS or SP, which for a very shallow focus would arrive together

on the seismogram, are now totally separate phenomena for a deep focus. Fig. 5 shows that PS cannot occur until Δ is 149 degrees, and beyond this there are two PS's for any

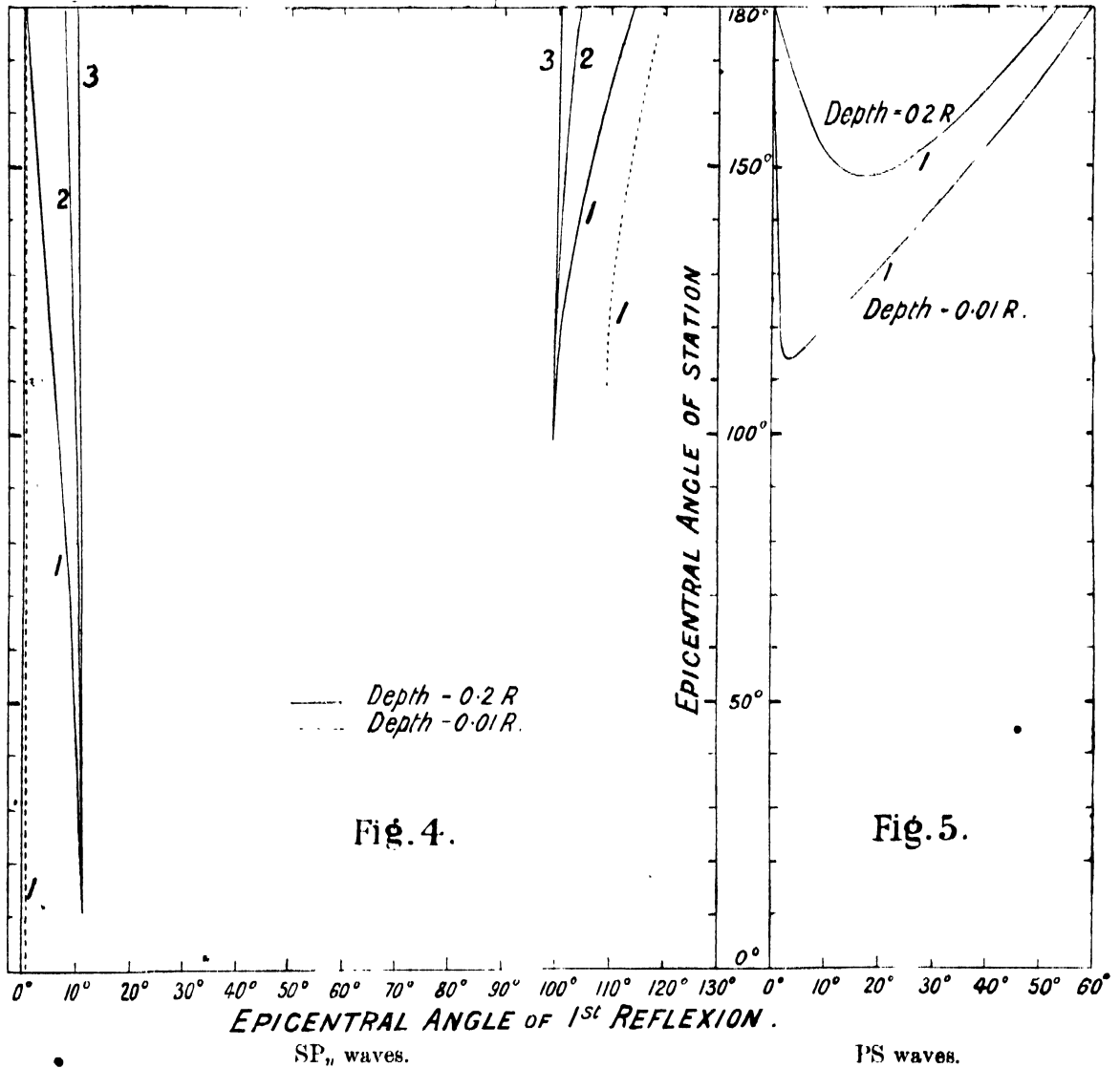


given Δ . There is no PS_2 until Δ exceeds 180 degrees, and we do not pursue it. The times of PS are included in fig. 1. Fig. 4 explains the position as regards $SP \dots SP_\infty$.

We cannot get an SP wave at all until $\Delta = 11$ degrees, and thereafter it may continue to be shown up to 180 degrees, the point at which reflexion takes place approaching the

epicentre as Δ increases. At $\Delta = 99$ degrees a second SP wave makes its appearance and may continue up to 180 degrees. The times for these are included in fig. 1, dotted lines showing the limiting waves SP_x .

From $\Delta = 11$ degrees up to $\Delta = 99$ degrees the reflexion of S waves with vibration in the diametral plane through focus and station is complex, and it is thus natural to suppose that this is the region within which the manufacture of Rayleigh waves goes on.



We should infer that these would not appear until $\Delta = 11$ degrees, but for greater Δ 's we should have a continuous succession of contributions of Rayleigh waves starting immediately after S. We need not confuse this with the long-wave phase, which appears to be a crustal phenomenon.

This now completes the effects to be expected for the hypothetical case selected. If it is desired to look into the question of magnitude of the effects at different points,

the investigation might proceed by aid of the tables and diagrams in 'Phil. Trans. Roy. Soc.,' vol. 218, A, p. 373, etc. We shall only point out that, on account of the large angle of impingence in some regions, some of the P effects would only be shown on vertical component seismographs.

The preceding discussion on elementary lines of the effect of finite depth of focus has led to inferences which differ in a very marked degree from those we are accustomed to draw from actual seismograms.

Naturally our results are qualitative in the first instance, and we must be prepared to find quantitative alterations when data for the earth are available. For example, we may be prepared to find that the epicentral distances at which PR, SR or PS start are less than the values we have calculated. In the case of SR or PS this may be so (we have not yet the required data to decide the matter), but in the case of PR, GALITZIN'S data, which we have taken as our basis, settle this at once. Anticipating the proof which will be given later, we find that PR starts at 11,000 km., which is only slightly less than the 103 degrees calculated. This, however, implies that the effects hitherto interpreted as PR_1 , PR_2 , etc., cannot be so described, but in place we may be able to interpret them as SP_1 , SP_2 , etc. Their capricious occurrence in practice favours this suggestion, and direct test can be made by means of the vertical component seismograph.

There are other possibilities as well as serious difficulties when a deep focus is considered, but we cannot discuss the problem with advantage until we have the requisite data. Hence we now proceed to show what may be deduced from GALITZIN'S data themselves and what additional data are wanted before proper tests can be applied.

Fig. 6 shows in graphical form the Pulkovo observed angle of emergence \bar{e} for various

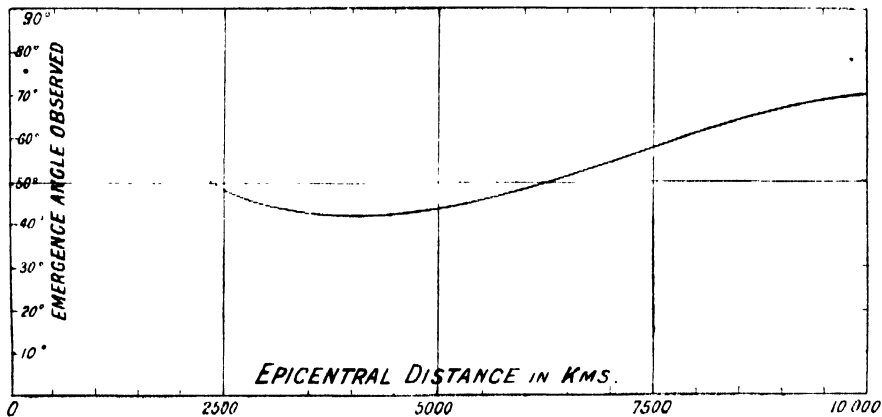


Fig. 6.

epicentral distances Δ . No data are given for $\Delta < 2500$ km., and the dotted part from $\Delta = 0$ to $\Delta = 2500$ km. is hypothetical. It is certain that \bar{e} must be 90 degrees at $\Delta = 0$.

From the formulæ, p. 3, we have

$$V_2 \frac{dT}{d\Delta} = \left\{ \frac{1}{2}(1 - \sin \bar{e}) \right\}^{\frac{1}{2}},$$

so that we can at once calculate $V_2 \frac{dT}{d\Delta}$ as a function of Δ from the numbers in fig. 6.

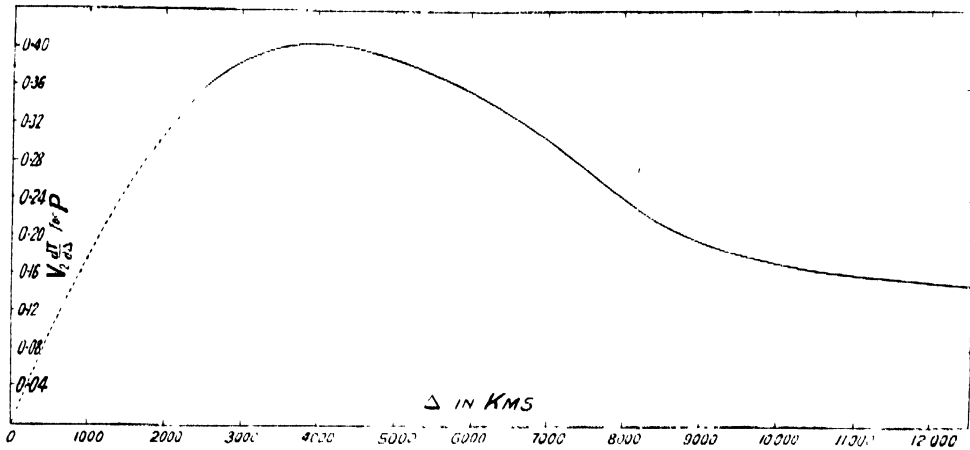


Fig. 7.

The results are shown graphically in fig. 7. Graphical integration of the curve now gives us

$$I = \int_0^{\Delta} \left\{ \frac{1}{2}(1 - \sin \bar{e}) \right\}^{\frac{1}{2}} d\Delta$$

as a function of Δ , whence

$$T = A + \frac{I}{V_2}, \text{ where } A \text{ is a constant,}$$

gives T as a function of Δ .

Table II. gives the value of the integral I for different epicentral distances. Since ZÖPPRITZ's time curve meets with general acceptance for the purpose of determining epicentres, we first seek to see how far we can fit our new time curve with ZÖPPRITZ's. Taking first ZÖPPRITZ's value for V_2 , viz., 4.01 km./sec., we find that the two curves fit over the range 6000 km. to 12,000 km. with a discrepancy ± 11 seconds, but the discrepancy rises to 100 seconds at 3000 km. Taking a larger V_2 one can fit the curves together over various ranges. For example, taking $V_2 = 5.63$ km./sec. we get the values shown in Table II., where over the range 3500 km. to 8000 km. the discrepancy ranges through only ± 5 secs., an error we might quite well admit. But large differences must arise towards the epicentre, for on the present view A must be a substantial number representing the time from focus to epicentre. No special significance is to be attached to the above calculation beyond showing that in the middle range of distances we need not make any large departure from ZÖPPRITZ's time curve.

TABLE II.

Δ kms.	\bar{c} .	I.	$1/5 \cdot 63$ secs.	Z secs.	$Z - 1/5 \cdot 63$
0	90	0	0	0	0
500	79	25	4	69	65
1,000	69	95	17	136	119
1,500	61	203	36	199	163
2,000	54	344	61	257	196
2,500	48	512	91	310	219
3,000	44	699	124	358	234
3,500	43	894	159	402	243
4,000	42	1,096	195	442	247
4,500	43	1,299	231	478	247
5,000	44	1,497	266	512	246
5,500	46	1,688	300	542	242
6,000	48	1,871	332	572	240
6,500	51	2,044	363	601	238
7,000	54	2,205	392	631	239
7,500	58	2,351	418	660	242
8,000	62	2,481	441	688	247
8,500	65	2,596	461	716	255
9,000	67	2,700	480	743	263
9,500	68	2,797	497	769	272
10,000	70	2,888	513	795	282
10,500	71	2,972	528	820	292
11,000	72	3,055	543	844	301
11,500	72	3,133	556	867	311
12,000	73	3,209	570	888	318
12,500	73	3,283	583	909	326
13,000	74	3,355	596	929	333

We now proceed to show how a direct test may be applied to the Pulkovo data, and one which will give a determination of A and V_2 .

If \bar{c} is the emergence angle of a ray, then the Pulkovo data gives us two distances, say, Δ_1 and Δ_2 , for which \bar{c} is the same. We hence infer that a PR wave will be reflected at Δ_1 and pass to epicentral distance $2\Delta_1 + \Delta_2$, and another PR wave will be reflected at Δ_2 and pass to distance $2\Delta_2 + \Delta_1$. *E.g.*, $\bar{c} = 48$ degrees gives $\Delta_1 = 2500$, and $\Delta_2 = 6000$, from which we get distances to station 11,000 km. and 14,500 km.

In this way fig. 8 has been determined directly from the Pulkovo data. It shows that the least distance at which PR occurs is 11,000 km., and for greater distances there are two PR waves for a given epicentral distance. The curve is in very close agreement with the theoretical curve in fig. 2.

The test of the validity of the Pulkovo data is, then, whether for $\Delta > 11,000$ km. we can identify the two PR waves on the seismogram. The Pulkovo *Bulletins* for 1913-1914 show quite a number of records for $\Delta > 11,000$ km., and it would seem desirable that a careful study of the seismograms for such distances should be made. Should the search prove successful the curve fig. 8 will then give two distances, Δ_1 and Δ_2 , for which

$\Delta = 2\Delta_1 + \Delta_2$ and $\Delta = 2\Delta_2 + \Delta_1$, and there is a check on this by means of the measured \bar{c} for the two waves. Further, if I , I_1 and I_2 are known integrals for Δ , Δ_1 and Δ_2 and T_1 and T_2 , the observed time intervals between P and the two PR waves, we have

$$T_1 = 2A + (2I_1 + I_2 - I)/V_2$$

and

$$T_2 = 2A + (2I_2 + I_1 - I)/V_2,$$

which theoretically suffice to determine A and V_2 .

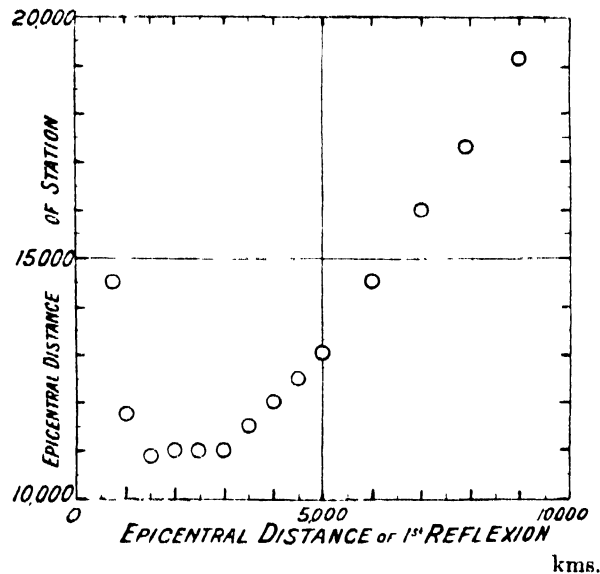


Fig. 8.

It is manifestly a matter of great importance in seismology to settle whether the large depth of focus suggested by the Pulkovo observations can be maintained.

If confirmation is obtained by the research suggested above, the problem of deducing the focal depth and the velocity at any depth must be attacked anew. The two things are related, and it looks as if the process of analysis would be largely tentative. In any case, it is clear that the speed at any depth cannot be uniquely determined until the focal depth is fixed. But one might hazard a guess that a large depth of focus would probably lead to a smaller variation of speed with depth than has been deduced by ZÖPPRITZ.

An investigation on S waves might proceed on similar lines, although the relation between the angle of impingence and the apparent angle of emergence is more complex than for P waves; *cf.* 'Phil. Trans.,' *l.c. ante*, p. 378.

The equations are

$$\cos e = V_2 \frac{dT}{d\Delta},$$

$$\tan \bar{e} = \frac{V_2 \sin e' \tan 2e}{V_1 \sin e}, \quad \text{where} \quad \cos e' = \frac{V_1}{V_2} \cos e.$$

The matter is further complicated by the circumstance that for a certain range of Δ the reflexion of S waves vibrating in the diametral plane is complex. Some attempts to estimate e for S waves were given in the paper referred to, p. 388, but they are isolated results, and what we require is a systematic investigation for a large range of Δ .

In the absence of such data, we may tentatively proceed a little way in the problems arising from finite depth of focus by adding to the time curve obtained from the Pulkovo data the values of $S - P$ given by ZÖPPRITZ, which are known to be not seriously in error over the middle range. Although I have made some calculations in this direction, one cannot proceed very far, and it is an obviously unsatisfactory method.

Summary.

Observations of the emergence angle of P waves at Pulkovo suggest that the depth of focus is of order one-fifth of the earth's radius. It is shown that important modifications would have to be made in the interpretation of seismograms and in the attempt to determine how speed of propagation depends on depth. It is further shown that an important test of the accuracy of the Pulkovo values can be made by a careful scrutiny of seismograms for distances $> 11,000$ km. Further progress cannot be made until this research has been carried out, and until we have corresponding measures of the angle of emergence of S waves by means of three component seismometers.

III. *Eddy-Current Losses in Cylindrical Conductors, with Special Applications to the Alternating Current Resistances of Short Coils.*

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INTRODUCTION.

IT is well-known that a considerable proportion of the effective resistance of inductive coils when used at radio frequencies is caused by the eddy-currents set up in the wires of the coils by the alternating magnetic field in which they are situated, and that in extreme cases the alternating current resistance may amount to more than one hundred times the direct current resistance. It is therefore important to have reliable formulæ for the eddy-current resistance of such coils in order to determine the conditions which will reduce the eddy-current losses to a minimum.

The simplest case, that of a long straight cylindrical wire under the action of its own current, has been treated by KELVIN,* RAYLEIGH,† HEAVISIDE,‡ and others. The general effect is known as the “skin effect,” because the current tends to concentrate more and more upon the skin of the conductor as the frequency increases.

The case of two parallel wires forming a go-and-return circuit has been considered theoretically by NICHOLSON,§ and experimentally examined by KENNELLY.|| KENNELLY found that when the wires are close together, the added resistance due to the proximity of the wires may be of the same order as that due to the simple skin effect.

NICHOLSON’S theoretical treatment includes the possibility that the dimensions of the system may be comparable with the wave-length of the disturbance. His formula is very complicated and difficult to apply numerically. A formula {formula (47)}

* ‘Math. and Phys. Papers,’ vol. 3, 1889.

† ‘Phil Mag.,’ vol. 21, 1886.

‡ ‘Electrical Papers,’ vol. 2, p. 64.

§ ‘Phil. Mag.,’ vol. 18, p. 417, 1909.

|| ‘Trans. A.I.E.E.,’ vol. 35, part 2, p. 1953, 1915. CURTIS (‘Bull. Bureau of Standards,’ 1920) has recently published a formula for this case which gives agreement with KENNELLY’S results.

for this case is obtained in Section 8 of the present paper. This formula is shown (Section 9) to give results in close accordance with KENNELLY'S observations.

In order to reduce the eddy-current losses solid wire is often replaced by stranded wire in which a bundle of thin separately insulated wires are interlaced symmetrically with each other, the notion being that the sum of the eddy losses in the individual wires shall be less than the eddy loss in the corresponding solid wire. LINDEMANN* verified this experimentally at certain frequencies, but also found that if a solid wire coil and stranded wire coil were compared at various frequencies, the stranded wire coil increased in resistance more slowly at the lower frequencies but less slowly at the high frequencies, until above a certain frequency the stranded wire coil had a greater effective resistance than the solid wire coil.

HOWE† has treated the problem of straight stranded wire conductors, assuming the eddy losses to increase as the square of the frequency, and from his formulæ has shown that at high frequencies it is difficult to make the resistance of the stranded wire less than that of solid wire of equal section.

In view of the extensive use of stranded wire in the construction of coils for high-frequency currents it is important that the limitations of stranded wire in reducing effective resistance should be known, so that the present investigation includes the consideration of such coils. From the formulæ obtained, conclusions are drawn in regard to the utility of stranding and in regard to what degree of stranding it is necessary to employ, before any improvement over solid wire coils may be expected.

In formulæ hitherto given for the effective resistance of coils, one or other of the following limitations occur:—

- (1) The coil is very long.
- (2) The frequency is limited to so low a value that the "square of frequency" law holds.
- (3) The coil is wound with wire of square section.

The formulæ deduced in this paper differ from those already established in that—

- (1) The dimensions of the winding sections of the coils are small compared with the coil radii.
- 2) There is no limitation imposed upon the frequency.
- (3) The wire is taken to be circular.

In regard to (1) it is shown that coils of this type have better alternating-current time constants than long coils.

In regard to variation with frequency, the factor governing the upper limit to the application of the square law is the magnitude of f/R_0 where f is the frequency and R_0 is the direct current resistance per unit length of the wire used. If (in C.G.S. units)

* 'Deut. Phys. Gesell.,' 1909, p. 382; 1910, p. 572. 'Jahrbuch der Drahtlosen Telegraphie,' 1911, p. 561.

† 'Roy. Soc. Proc.,' A, vol. 93, p. 468, 1917.

f/R_0 is less than 0.225 the eddy-current losses vary as $(f/R_0)^2$ to an accuracy of one per cent. At higher frequencies the variation is slower, the ultimate rate of variation being as $(f/R_0)^{1/2}$. A knowledge of these limiting rates of variation enables an immediate explanation to be given of LINDEMANN'S results with stranded wire coils.

A solid wire in a given alternating field has eddy losses which are a function of $f/R_0 = \phi(f/R_0)$ say. If the solid wire is replaced by s strands of the same total metallic section, the loss per strand in the same field is $\phi(f/sR_0)$ and the total loss in the s strands is $s\phi(f/sR_0)$.

• Thus, as regards the losses due to the general field of the remainder of the coil, we must replace $\phi(f/R_0)$ by $s\phi(f/sR_0)$ in passing from solid to stranded wire.

At low frequencies $\phi(f/R_0) = C(f/R_0)^2$ where C is a constant independent of the stranding, so that the respective losses are $C(f/R_0)^2$ and $Cs(f/sR_0)^2 = C(f/R_0)^2/s$.

The effect of stranding at low frequencies is thus to reduce these losses in the ratio $1/s$.

At high frequencies $\phi(f/R_0) = C'(f/R_0)^{1/2}$ and the losses are $C'(f/R_0)^{1/2}$ and $C's(f/sR_0)^{1/2} = C's^{1/2}(f/R_0)^{1/2}$, or the effect of stranding at high frequencies is to increase the losses in the ratio $s^{1/2}/1$.

Since in inductive coils the general field produces the main losses, LINDEMANN'S results are explained.

(A). EDDY-CURRENT LOSSES IN A CYLINDER IN AN ALTERNATING MAGNETIC FIELD.

(1) The cylinder is supposed to be non-magnetic and to have electrical conductivity k . Its radius is a . The magnetic field is perpendicular to the axis of the cylinder and does not vary along the axis; otherwise its form is general. The field alternates with frequency $\omega/2\pi$, and the alternations are so slow that the dielectric current can be neglected in comparison with the conductance current. This means that the wave length of the disturbance producing the field is large compared with the dimensions of the cylinder. On the other hand, the cylinder is supposed to be long enough to render its end effects negligible.

The procedure is to represent the electric and magnetic forces by rotors* $Ee^{i\omega t}$, &c. The values of these rotors are found at all points in terms of the (given) undisturbed field. Then by application of POYNTING'S Theorem over unit length of the surface of the cylinder, the energy flow into the cylinder is determined.

This energy flow may be regarded as made up of two portions, one continuous and the other alternating. The former portion is the energy dissipated by eddy-currents set up in the cylinder.

(2) Take the axis of the cylinder as the axis of a right-handed system of cylindrical

* These are the rotating vectors used to represent these quantities on the vector diagram. The term is chosen to distinguish them from the space vectors which are also involved in the problem.

co-ordinates (z, r, θ) . With the assumed conditions the electromagnetic equations are

$$\left. \begin{aligned} -i\omega P &= \frac{1}{r} \frac{\partial E}{\partial \theta}, & i\omega Q &= \frac{\partial E}{\partial r} \\ \frac{1}{r} \frac{\partial (Qr)}{\partial r} - \frac{1}{r} \frac{\partial P}{\partial \theta} &= 4\pi k E \end{aligned} \right\}, \dots \dots \dots (1)$$

in which P, Q represent the components of the magnetic force acting along and perpendicular to r , and E represents the electric force acting parallel to z .

Eliminating P and Q , the equation to be satisfied by E is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E}{\partial \theta^2} = 4\pi k i \omega E, \dots \dots \dots (2)$$

the normal solution of which is

$$E = R_n \cos n\theta + S_n \sin n\theta, \dots \dots \dots (3)$$

in which R_n and S_n are functions of r both satisfying the equation

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR_n}{dr} \right) - \frac{n^2}{r^2} R_n = 4\pi k i \omega R_n, \dots \dots \dots (4)$$

Writing $\lambda^2 = -4\pi k i \omega$ and putting x for λr , (4) may be written

$$\mathfrak{D}^2 R_n + (x^2 - n^2) R_n = 0, \dots \dots \dots (5)$$

in which

$$\mathfrak{D} \equiv x \frac{d}{dx}.$$

This is the general differential equation for the BESSEL functions, so that inside the cylinder the appropriate solution of (4) is

$$R_n = A_n J_n(\lambda r), \dots \dots \dots (6)$$

the second solution being excluded, since the electric force is not infinite at the axis.

Outside the cylinder k is zero, so that the solution of (4) is

$$R_n = B_n r^n + C_n / r^n \dots \dots \dots (6A)$$

except when $n = 0$, in which case

$$R_0 = B_0 \log_e r + C_0 \dots \dots \dots (6B)$$

In order to maintain the continuity of E and $\frac{\partial E}{\partial r}$ at the boundary of the cylinder, A_n, B_n, C_n must satisfy the relations

$$\left. \begin{aligned} A_n J_n(\lambda\alpha) &= B_n \alpha^n + C_n / \alpha^n \\ A_n \lambda \alpha J'_n(\lambda\alpha) &= n (B_n \alpha^n - C_n / \alpha^n) \end{aligned} \right\} n \neq 0$$

$$\left. \begin{aligned} A_0 J_0(\lambda\alpha) &= B_0 \log_e \alpha + C_0 \\ A_0 \lambda \alpha J'_0(\lambda\alpha) &= B_0 \end{aligned} \right\},$$

or, expressing A_n, C_n in terms of B_n and making use of the properties of the BESSEL functions,

$$\left. \begin{aligned} A_n &= 2n B_n \alpha^n / \lambda \alpha J_{n-1}(\lambda\alpha) \\ C_n &= B_n \alpha^{2n} J_{n+1}(\lambda\alpha) / J_{n-1}(\lambda\alpha) \\ A_0 &= B_0 / \lambda \alpha J'_0(\lambda\alpha) \\ C_0 &= B_0 \{ J_0(\lambda\alpha) / \lambda \alpha J'_0(\lambda\alpha) - \log_e \alpha \} \end{aligned} \right\} \dots \dots \dots (7)$$

The general solution of (2) is the sum of the normal solutions of the type (3), so that the electric force may be expressed as a FOURIER series, whose form inside the cylinder is

$$E_1 = B_0 \frac{J_0(\lambda r)}{\lambda \alpha J'_0(\lambda \alpha)} + 2 \sum_1^{\infty} \frac{n B_n \alpha^n J_n(\lambda r)}{\lambda \alpha J_{n-1}(\lambda \alpha)} \cos(n\theta + \alpha_n), \dots \dots \dots (8)$$

and outside the cylinder is

$$E_2 = B_0 \left(\log_e \frac{r}{\alpha} + \frac{J_0(\lambda \alpha)}{\lambda' \alpha J'_0(\lambda \alpha)} \right) + \sum_1^{\infty} B_n r^n \left\{ 1 + \left(\frac{\alpha}{r} \right)^{2n} \frac{J_{n+1}(\lambda \alpha)}{J_{n-1}(\lambda \alpha)} \right\} \cos(n\theta + \alpha_n). \dots (8A)$$

The corresponding series for P and Q follow by differentiation using the relation (1).

The combination of the cosine and sine terms into the form $\cos(n\theta + \alpha_n)$ is permissible, since the ratios of the arbitrary constants are the same for both the sine and cosine series. The values of B_n and α_n may be determined when the form of the undisturbed field is given.

(3) *Energy Dissipation in the Cylinder.*—From (8A) and (1) the values of E and Q at the surface of the cylinder are

$$E = B_0 \chi_0 + \sum_1^{\infty} B_n \alpha^n (1 + \chi_n) \cos(n\theta + \alpha_n). \dots \dots \dots (9)$$

$$Q = -\frac{i}{\omega \alpha} \left\{ B_0 + \sum_1^{\infty} n B_n \alpha^n (1 - \chi_n) \cos(n\theta + \alpha_n) \right\}. \dots \dots \dots (10)$$

in which

$$\left. \begin{aligned} \chi_n &= J_{n+1}(\lambda\alpha) / J_{n-1}(\lambda\alpha) \\ \chi_0 &= J_0(\lambda\alpha) / \lambda \alpha J'_0(\lambda\alpha) = \frac{1}{4} (1 + \chi_2) - \frac{2}{\lambda^2 \alpha^2} \end{aligned} \right\} \dots \dots \dots (11)$$

If e, q represent the instantaneous values of E and Q, the rate at which energy flows

into the cylinder through a small area $d\sigma$ about the point a , θ is $eqd\sigma/4\pi$ by POYNING'S theorem, and the integral rate of flow into unit length of the cylinder is

$$\frac{\alpha}{4\pi} \int_0^{2\pi} \rho q d\sigma. \dots \dots \dots (12)$$

Now, by (9) and (10), e and q have the forms $\sum u_n \cos(n\theta + \alpha_n)$ and $\sum v_n \cos(n\theta + \alpha_n)$ respectively.

Using these forms in (12), and remembering that

$$\int_0^{2\pi} \cos(n\theta + \alpha_n) \cos(m\theta + \alpha_m) d\theta = 0, \quad n \neq m,$$

the integral rate of flow becomes

$$\frac{\alpha}{4} (2u_0v_0 + \sum u_n v_n). \dots \dots \dots (13)$$

To determine the products $u_n v_n$, the obvious method is to determine the real parts of the complex coefficients in (9) and (10). The product $u_n v_n$ expressed as a function of time would then be of the form $w_n \cos \omega t \cos(\omega t + \phi_n)$, so that the rate of energy dissipation in the cylinder would be $\frac{1}{2} w_n \cos \phi_n$.

A better method is to make use of the method of conjugates. If U, V are the conjugates* of two complex quantities U, V ; u, v their real parts, then

$$u = \frac{1}{2} (U + U'), \quad v = \frac{1}{2} (V + V'),$$

so that

$$uv = \frac{1}{4} (U + U')(V + V').$$

Further, if U, V rotate with time in the same sense, U', V' will rotate in the opposite sense, so that UV' and VU' will not rotate. The steady portion of uv is thus

$$\frac{1}{4} (UV' + U'V).$$

Applying this to the present case, the steady flow into unit length of the cylinder is from (9), (10) and (13)

$$W = \frac{i}{8\omega} \left\{ B_0 B'_0 (\chi_0 - \chi'_0) + \sum_1^\infty n \alpha^{2n} B_n B'_n (\chi_n - \chi'_n) \right\},$$

in which as before the accents denote conjugates.

Now χ_n is a function of λa and $\lambda^2 = +4\pi k i \omega$, so that, putting

$$z^2 = 4\pi k i \omega a^2, \dots \dots \dots (14)$$

χ_n may be written

$$\chi_n = \phi_n(z) - i \psi_n(z), \dots \dots \dots (15)$$

from which

$$i(\chi_n - \chi'_n) = 2\psi_n(z),$$

* If $U = A e^{i\psi}$, then $U' = A e^{-i\psi}$ is the conjugate of U .

and when $n = 0$

$$i(\chi_0 - \chi'_0) = \frac{4}{z^2} + \frac{1}{2}\psi_2(z).$$

Using these expressions in the equation for W with (B_n) for the modulus of B_n ,

$$W = \frac{1}{4\omega} \left[(B_0)^2 \left\{ \frac{2}{z^2} + \frac{1}{4}\psi_2(z) \right\} + \sum_1^n n (B_n)^2 a^{2n} \psi_n(z) \right]. \quad (16)$$

The energy dissipation in unit length of the cylinder is thus expressed in terms of coefficients B_n depending on the form of the applied magnetic field and of functions ψ_n having argument $z = 2a(\pi k\omega)^{\frac{1}{2}}$. The functions ϕ_n, ψ_n are discussed in the next section. As regards the coefficients B_n , if the components of the magnetic force in the undisturbed field are P_0, Q_0 these components may be expressed in the forms

$$\left. \begin{aligned} P_0 &= \sum_1^n K_n r^{n-1} \sin(n\theta + \alpha_n) \\ Q_0 &= \frac{K_0}{r} + \sum_1^n K_n r^{n-1} \cos(n\theta + \alpha_n) \end{aligned} \right\} \dots \dots \dots (17)$$

at all points outside the cylinder as these expressions are derivatives of a potential function satisfying LAPLACE'S equation and constant along the axis of the cylinder.

Further, by differentiation of (8A), similar expressions to (17) are obtained, when λ is made zero—that is, when the disturbance due to eddy-currents in the cylinder is removed. These expressions are identical with (17) if we make

$$B_0 = i\omega K_0, \quad B_n = i\omega K_n/n.$$

Hence, using K_n in place of B_n in (16)

$$W = \frac{1}{4\omega} \left[(K_0)^2 \left\{ \frac{2}{z^2} + \frac{1}{4}\psi_2(z) \right\} + \sum_1^n (K_n)^2 a^{2n} \psi_n(z)/n \right]. \quad (18)$$

(4) *The Functions ϕ_n and ψ_n .*—These functions are defined by

$$\phi_n(z) - i\psi_n(z) = J_{n+1}(\sqrt{-iz})/J_{n-1}(\sqrt{-iz}).$$

Series formulæ for these functions have been developed by the author.*

The cases $n = 1$ and $n = 2$ are the most important ones, and in these cases ϕ and ψ may be expressed in terms of ber and bei functions as follows:—

Let

$$\left. \begin{aligned} X(z) &= \text{ber}^2 z + \text{bei}^2 z \\ V(z) &= \text{ber}'^2 z + \text{bei}'^2 z \\ Z(z) &= \text{ber} z \text{ber}' z + \text{bei} z \text{bei}' z \\ W(z) &= \text{ber} z \text{bei}' z - \text{bei} z \text{ber}' z \end{aligned} \right\} \dots \dots \dots (19)$$

* BUTTERWORTH, 'Proc. Phys. Soc. Lond.,' vol. XXV, p. 294, 1913.

Then

$$\left. \begin{aligned} \phi_1(z) &= \frac{2}{z} \frac{W(z)}{X(z)} - 1, & \phi_2(z) &= \frac{4}{z} \frac{Z(z)}{V(z)} - 1 \\ \psi_1(z) &= \frac{2}{z} \frac{Z(z)}{X(z)}, & \psi_2(z) &= \frac{4}{z} \frac{W(z)}{V(z)} - \frac{8}{z^2} \end{aligned} \right\} \dots \dots \dots (20)$$

The combinations W/X , Z/X , W/V and Z/V are tabulated.*

In the limiting cases of z very small or very large, it may be shown from the formulæ already cited that ϕ_n and ψ_n assume the following simple forms:—

$$z \text{ small} \dots \dots \left\{ \begin{aligned} \phi_n &= -2z^4/(2n)^2 (2n+2) (2n+4) \\ \psi_n &= z^2/2n (2n+2) \end{aligned} \right\} \dots \dots \dots (21A)$$

$$z \text{ large} \dots \dots \phi_n = -1, \quad \psi_n = 2n/\sqrt{2z} \dots \dots \dots (21B)$$

In regard to the limitations of (21A), the following table of values of ψ_1 , ψ_2 (the functions most generally used) has been calculated:—

z	ψ_1	ψ_2	ψ_1/z^2	ψ_2/z^2
0.0	0.0000	0.0000	0.1250	0.04167
0.5	0.03119	0.01041	0.1248	0.04164
1.0	0.1215	0.04149	0.1215	0.04149
1.5	0.2458	0.0918	—	—
2.0	0.3448	0.1563	0.0862	0.03908
2.5	0.3770	0.2244	—	—
3.0	0.3600	0.2827	0.0400	0.03141
3.5	0.3257	0.3212	—	—
4.0	0.2920	0.3389	—	—
4.5	0.2643	0.3408	—	—
5.0	0.2416	0.3337	—	—

For ψ_1 , (21A) is a good approximation up to $z = 0.5$ and a fair approximation up to $z = 1$. For higher values of n the range of (21A) increases.

In regard to (21B), its region of application has not been reached at $z = 5$, but if we take a second approximation we find, when z is large,

$$z^2\psi_1 = \sqrt{2z} - 1, \quad z^2\psi_2 = 2\sqrt{2z} - 6. \dots \dots \dots (21c)$$

These formulæ give the following values for ψ_1 , ψ_2 :—

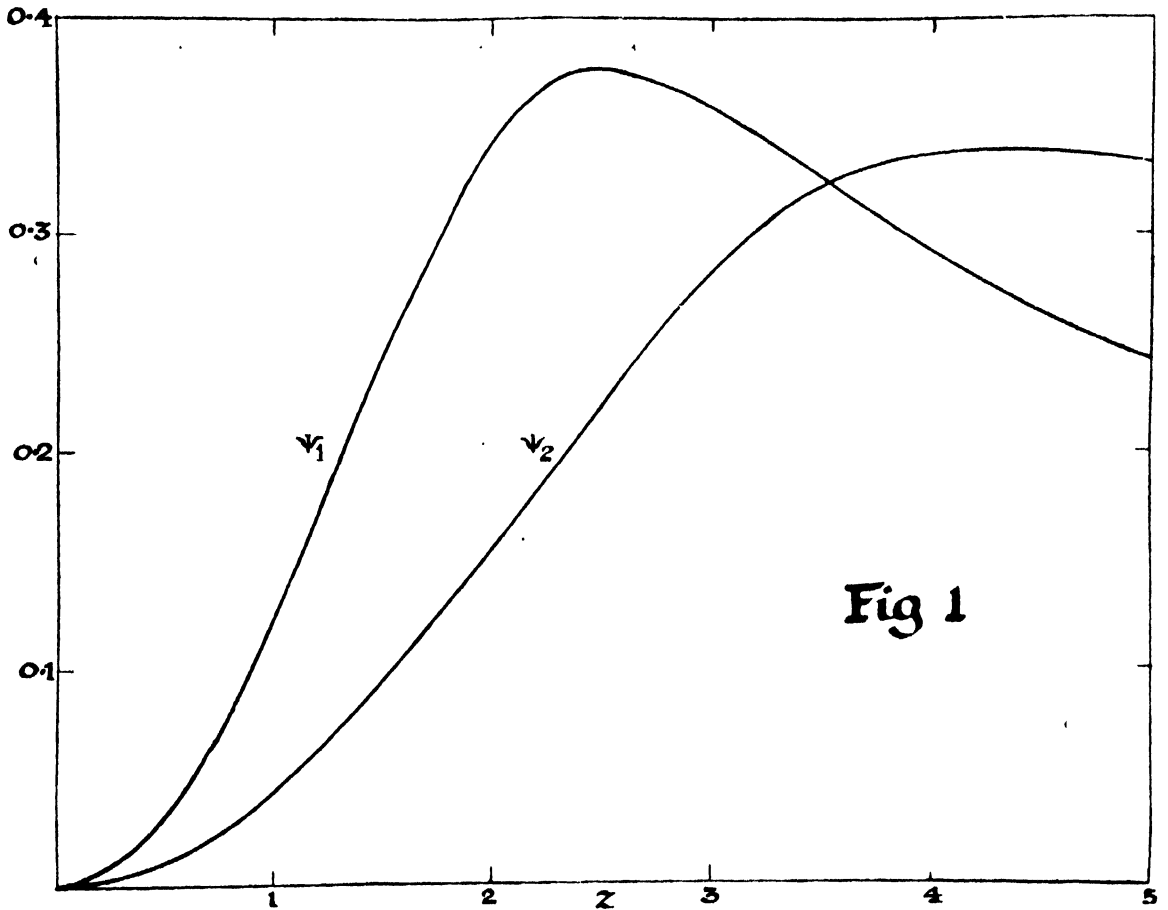
$z =$	2	3	4	5
$\psi_1 =$	0.457	0.360	0.291	0.243
$\psi_2 =$	-0.086	0.276	0.332	0.326

* SAVIDGE, 'Phil. Mag.', 6, 19, p. 49, 1910. ROSA and GROVER, 'Bull. Bureau of Stands.', p. 226, 1912.

while the actual values are

$\psi_1 =$	0.345	0.360	0.292	0.242
$\psi_2 =$	0.156	0.287	0.339	0.334

so that if z is greater than 3 the values given by (21c) are fair approximations to the true values. The values of ψ_1 and ψ_2 from $z = 0$ to $z = 5$ are plotted in fig. 1.



(5) *Eddy-Current Losses at Low Frequencies.*—The argument z is related to the radius and conductance of the cylinder and the frequency of alternation of the field by the formula

$$z^2 = 4\pi k\omega a^2.$$

If R_0 is the electrical resistance per unit length of the cylinder, the formula may be written

$$z^2 = 4\omega/R_0. \dots \dots \dots (22)$$

The frequency $\omega/2\pi$ will be defined to be *low* when z is less than unity, so that the

condition of low frequency is $\omega < R_0/4$. When this holds, formula (21A) applies in regard to ψ , so that by (18) the rate of dissipation of energy is

$$W = \frac{1}{8} R_0 (K_0)^2 + \frac{\omega^2}{R_0} \left\{ \frac{1}{8} (K_0)^2 + \sum_1^{\infty} (K_n)^2 \alpha^{2n} / 2n^2 (2n+2) \right\} \dots \dots \dots (23)$$

Now by (17) the terms involving K_0 are due to a field whose components outside the cylinder are $Q_0 = K_0/r$, $P_0 = 0$. This field can only be due to a current of magnitude $I = \frac{1}{2} K_0$ distributed symmetrically round the axis and flowing parallel to the axis.

Hence the energy dissipation due to such a current is

$$W_1 = \frac{1}{2} R_0 \left(1 + \frac{1}{2} \frac{\omega^2}{R_0^2} \right) I^2 \dots \dots \dots (24)$$

This is the usual formula for the skin effect at low frequencies.

If a uniform field H is acting on the cylinder, then $H = K_1$, $K_2 = K_3 = \dots = 0$, so that the energy dissipation due to a uniform field H is

$$W_2 = \frac{1}{8} \omega^2 H^2 \alpha^2 / R_0 \dots \dots \dots (25)$$

The remaining terms are due to non-uniformity of the field.

If the external field is expressed in a FOURIER series of the form (17), and if the coefficient of the term $\cos(n\theta + \alpha_n)$ in the series for Q_0 has the value L_n at the surface of the cylinder, then this portion of the field contributes an amount

$$\omega^2 \alpha^2 L_n^2 / 2n^2 (2n+2) R_0$$

to the energy dissipation.

The way in which n occurs in this expression shows how unimportant are the higher terms of the FOURIER series in producing eddy losses at low frequencies.

The assumption that the external field is uniform and has its central value will, therefore, in most cases give a good approximation to the actual loss when the frequency is low. In illustration, suppose the external field to be due to a thin wire carrying current I , and stretched parallel to the cylinder at a distance D from the axis. The value of Q_0 in the plane common to the axis and the wire is $2I/(D-r)$, or, in ascending powers of r ,

$$\frac{2I}{D} \left(1 + \frac{r}{D} + \frac{r^2}{D^2} + \dots \right),$$

so that

$$L_n = \frac{2I}{D} \frac{\alpha^n}{D^n},$$

and therefore the energy dissipation is

$$\frac{\omega^2 \alpha^2 I^2}{R_0 D^2} \left(\frac{1}{1^2 \cdot 2} + \frac{1}{2^2 \cdot 3} \frac{\alpha^2}{D^2} + \frac{1}{3^2 \cdot 4} \frac{\alpha^4}{D^4} + \dots \right) \dots \dots \dots (26)$$

In the extreme case in which the wire touches the cylinder, the sum of the series becomes

$$\frac{\pi^2}{6} - 1 = 0.64493.$$

Upon the assumption of a uniform field, the first term of the series would be the only one employed; and as this is 1/2, the correction due to non-uniformity of the field is a multiplying factor ranging from 1.00 to 1.29.

If there are two thin parallel wires and the cylinder is situated symmetrically between them, the axes of wires and cylinder being coplanar, the alternate terms of the series (26) vanish, and the losses become

$$\frac{4\omega^2 a^2 I^2}{R_0 D^2} \left(\frac{1}{1^2 2} + \frac{1}{3^2 4} \frac{a^4}{D^4} + \frac{1}{5^2 6} \frac{a^8}{D^8} + \dots \right) \dots \dots \dots (27)$$

if the currents flow in opposite directions in the two wires, and

$$\frac{4\omega^2 a^2 I^2}{R_0 D^2} \left(\frac{1}{2^2 3} \frac{a^2}{D^2} + \frac{1}{4^2 5} \frac{a^6}{D^6} + \dots \right) \dots \dots \dots (28)$$

when the currents flow in the same direction.

When the wires touch the cylinder, (27) reduces to

$$\frac{4\omega^2 I^2}{R_0} \left(\frac{\pi^2}{8} - \log_e 2 \right) = \frac{4\omega^2 I^2}{R_0} \times 0.54055$$

and (28) to

$$\frac{4\omega^2 I^2}{R_0} \left(\frac{\pi^2}{24} + \log_e 2 - 1 \right) = \frac{4\omega^2 I^2}{R_0} \times 0.10438.$$

The uniform field theory would give $\frac{4\omega^2 I^2}{R_0} \times 0.5000$ and zero respectively for these cases.

(6) *Eddy-Current Losses at High Frequencies.*—At very high frequencies

$$\nu_n = 2n/\sqrt{2z} = n\sqrt{R_0/2\omega},$$

so that by (18) the energy dissipation is

$$W = \frac{1}{4}\sqrt{R_0\omega/2} \left\{ \frac{1}{2} (K_0)^2 + \sum_1^\infty (K_n)^2 a^{2n} \right\} \dots \dots \dots (29)$$

The first term is due to a current $I = \frac{1}{2}K_0$ distributed symmetrically round the axis of the cylinder and flowing in a direction parallel to the axis, and when this is the only factor producing the field the energy dissipation is

$$W_1 = \frac{1}{2}\sqrt{R_0\omega/2} I^2 \dots \dots \dots (30)$$

This is the formula for the skin effect at high frequencies.

The energy dissipation due to a uniform field H is got by putting $K_1 = H$, $K_2 = K_3 = \dots = 0$, giving

$$W_2 = \frac{1}{4} H^2 \alpha^2 \sqrt{R_0 \omega / 2} \dots \dots \dots (31)$$

If the field is non-uniform, a comparison of (29) with (17) shows that the various harmonic terms in the field produce terms of equal importance in the expression for the eddy-current losses.

The examples of the last section give, for the single thin wire,

$$W = \sqrt{R_0 \omega / 2} \frac{\alpha^2 I^2}{D^2} \left(1 + \frac{\alpha^2}{D^2} + \frac{\alpha^4}{D^4} + \dots \right) = \frac{\alpha^2 I^2}{D^2 - \alpha^2} \times \sqrt{R_0 \omega / 2}, \dots \dots (32)$$

and for the pair of wires,

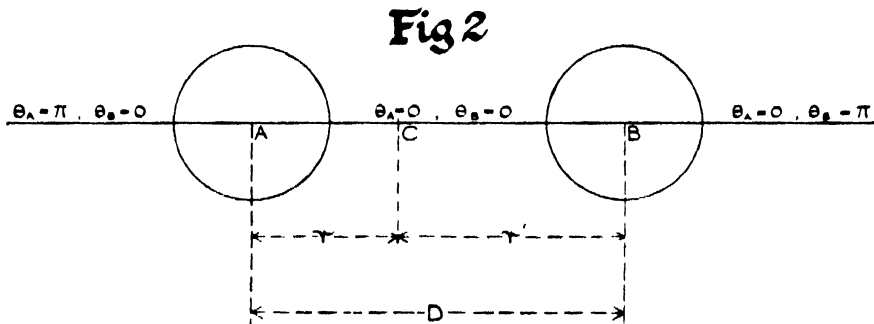
$$W = \frac{\alpha^2 I^2}{D^4 - \alpha^4} I^2 \sqrt{R_0 \omega / 2} \quad \text{or} \quad \frac{\alpha^4}{D^4 - \alpha^4} I^2 \sqrt{R_0 \omega / 2}, \dots \dots (33)$$

according as the currents flow in opposite or the same direction in the two wires.

As regards (32), it is seen that the uniform field theory may be applied if we take as the uniform field the field at the point where the tangent plane through the wire touches the surface of the cylinder.

(B). EDDY-CURRENT LOSSES IN TWO PARALLEL CYLINDERS CARRYING EQUAL CURRENTS.

(7) If the field acting on the cylinder is due to currents in neighbouring cylinders, then, because of the distortion of the current distribution in these cylinders, the external field acting on the cylinder under consideration is itself variable with frequency, and the assumption that this field is that which would occur if *all* eddy-currents are absent will lead to wrong results. The case of two similar parallel cylinders carrying equal currents may be solved by considerations of symmetry.



(8) Let the cylinders each have radius a , and let the distance of their centres be D . Take two systems of cylindrical co-ordinates (fig. 2), the first system (r, θ_A) having A as origin and AB as the line of zero θ , and the second system (r', θ_B) having B as

origin and BA as the line of zero θ . Consider the field at the point C external to the cylinders situated on AB and due to two current systems flowing in the two cylinders parallel to the axes and symmetrical on either side of AB. By symmetry α_n of equation (17) is zero; and since θ is zero, we have from (10), using K_n instead of B_n , for the first system of co-ordinates,

$$Q = \frac{K_0}{r'} + \sum_1^{\infty} K_n r'^{n-1} \left\{ 1 - \left(\frac{\alpha}{r'} \right)^{2n} \chi_n \right\} \dots \dots \dots (34)$$

for the second system

$$Q = \frac{K'_0}{r'} + \sum_1^{\infty} K'_n r'^{n-1} \left\{ 1 - \left(\frac{\alpha}{r'} \right)^{2n} \chi_n \right\} \dots \dots \dots (35)$$

Also, if the currents are equal and *similarly* directed in the two cylinders,

$$K'_n = -K_n :$$

if *oppositely* directed.

$$K'_n = +K_n.$$

Further, (34) may be divided into two portions,

$$Q_1 = \frac{K_0}{r'} - \sum_1^{\infty} K_n \frac{\alpha^{2n}}{r'^{n+1}} \chi_n \dots \dots \dots (36)$$

and

$$Q_2 = \sum_1^{\infty} K_n r'^{n-1} \dots \dots \dots (37)$$

the former arising from causes *inside* the cylinder A, and the latter from causes *outside* A, which in this case are located *inside* B, and therefore Q_2 has also the value

$$Q_2 = \frac{K'_0}{r'} - \sum_1^{\infty} K'_n \frac{\alpha^{2n}}{r'^{n+1}} \chi_n \dots \dots \dots (38)$$

Equating (37) and (38),

$$\sum_1^{\infty} K_n r'^{n-1} = \frac{K'_0}{r'} - \sum_1^{\infty} K'_n \frac{\alpha^{2n}}{r'^{n+1}} \chi_n \dots \dots \dots (39)$$

Putting $K'_n = \pm K_n$, $r' = D - r$, expanding the right-hand side of (39) in ascending powers of r , and equating coefficients of r^{n-1} , a series of equations are obtained to determine K_n in terms of K_0 . Now if I is the total current in either wire, $K_0 = 2I$, so that the method yields the values of K_n completely. Thus, in the case where the currents are similarly directed, $K'_n = -K_n$; and on equating coefficients we find

$$\begin{aligned} K_1 \alpha &= -2I\mu + (K_1 \alpha) \mu^2 \chi_1 + (K_2 \alpha^2) \mu^3 \chi_2 + \dots \\ K_2 \alpha^2 &= -2I\mu^2 + 2(K_1 \alpha) \mu^3 \chi_1 + 3(K_2 \alpha^2) \mu^4 \chi_2 + \dots \\ K_3 \alpha^3 &= -2I\mu^3 + 3(K_1 \alpha) \mu^4 \chi_1 + 6(K_2 \alpha^2) \mu^5 \chi_2 + \dots, \end{aligned}$$

in which $\mu \equiv a/D$ and is less than $1/2$.

Solving by successive approximations to the order μ^4 ,

$$\left. \begin{aligned} K_1 a &= -2I\mu \{ 1 + \mu^2 \chi_1 + \mu^4 (\chi_1^2 + \chi_2) + \dots \} \\ K_2 a^2 &= -2I\mu^2 (1 + 2\mu^2 \chi_1 + \dots) \\ K_3 a^3 &= -2I\mu^3 (1 + \dots) \end{aligned} \right\} \dots \dots \dots (40)$$

In the expression for the eddy-current losses (18) the moduli of the complex quantities K_1, K_2, K_3 are required, which, from (40) with $\chi_n = \phi_n - i\psi_n$, are

$$\left. \begin{aligned} (K_1)^2 &= \frac{4I^2}{D^2} \{ 1 + 2\mu^2 \phi_1 + \mu^4 (2\phi_2 + 3\phi_1^2 - \psi_1^2) + \dots \} \\ (K_2)^2 &= \frac{4I^2}{D^4} (1 + 4\mu^2 \phi_1 + \dots) \\ (K_3)^2 &= \frac{4I^2}{D^6} (1 + \dots) \end{aligned} \right\} \dots \dots \dots (41)$$

Substituting in (18), the energy dissipation per unit length in either cylinder is given by

$$W = \omega I^2 \left[\frac{2}{z^2} + \frac{1}{4}\psi_2 + \mu^2 \psi_1 \left\{ 1 + \mu^2 \left(2\phi_1 + \frac{1}{2} \frac{\psi_2}{\psi_1} \right) + \mu^4 \left(2\phi_2 + \frac{1}{3} \frac{\psi_3}{\psi_1} + 3\phi_1^2 - \psi_1^2 + \frac{2\phi_1 \psi_2}{\psi_1} \right) \right\} \right] \dots (42)$$

When the currents are in opposite directions a similar treatment gives

$$W = \omega I^2 \left[\frac{2}{z^2} + \frac{1}{4}\psi_2 + \mu^2 \psi_1 \left\{ 1 + \mu^2 \left(\frac{1}{2} \frac{\psi_2}{\psi_1} - 2\phi_1 \right) + \mu^4 \left(3\phi_1^2 - \psi_1^2 - 2\phi_2 - \frac{2\phi_1 \psi_2}{\psi_1} + \frac{1}{3} \frac{\psi_3}{\psi_1} \right) \right\} \right] \dots (43)$$

In (42) and (43) the term

$$\omega I^2 \left(\frac{2}{z^2} + \frac{1}{4}\psi_2 \right)$$

is due to the ordinary skin effect. Since $2\omega/z^2 = \frac{1}{2}R_0$, this term may be written $\frac{1}{2}R_0 I^2 \{ 1 + F(z) \}$ in which $F(z) \{ \equiv \frac{1}{8}z^2 \psi_2 \}$ is plotted in fig. 3 up to $z = 5$. When z is greater than 5, then, by (21c),

$$F(z) \doteq (\sqrt{2z} - 3)/4. \dots \dots \dots (44)$$

This is shown in fig. 3 by the broken line A.

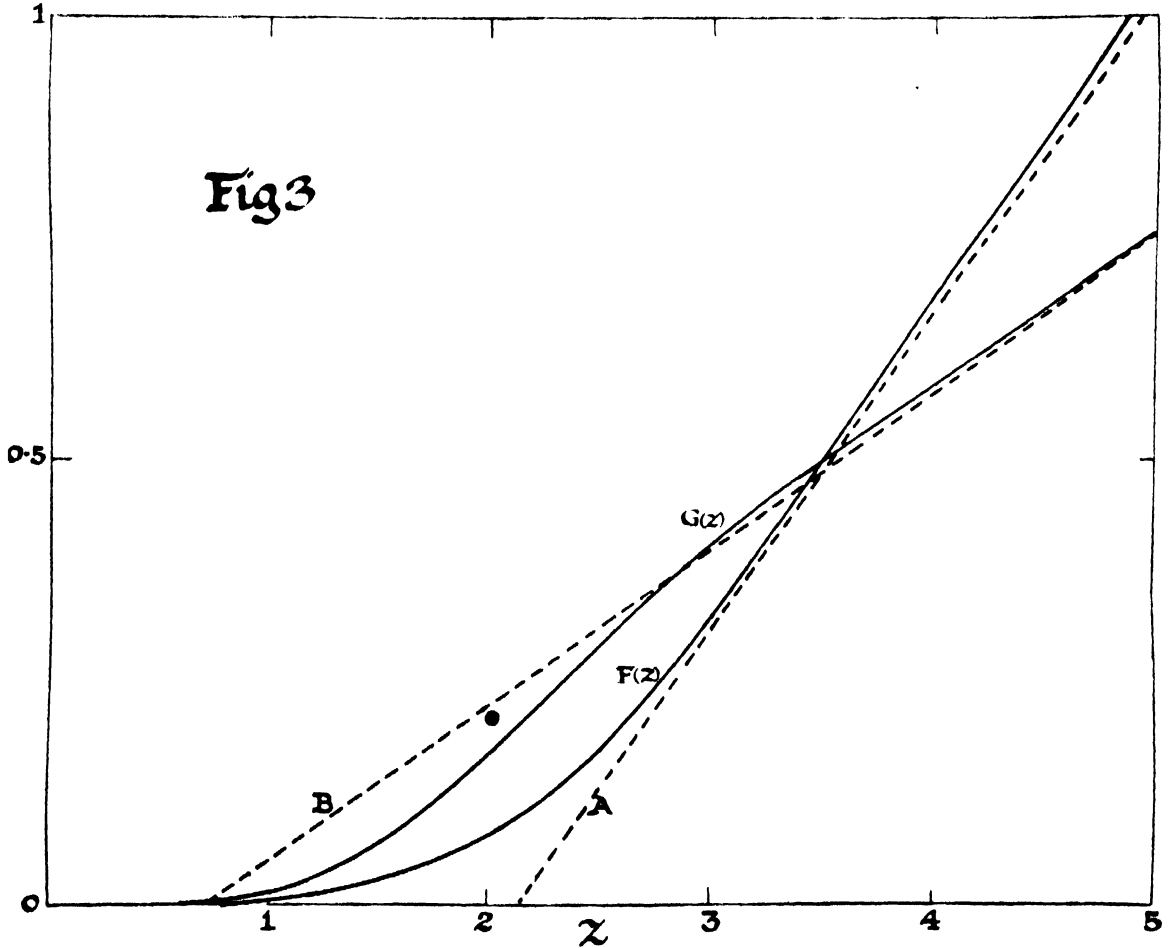
The next term, $\mu^2 \psi_1$, is due to the proximity of the two cylinders when they are so far apart that the quantity in $\{ \}$ may be regarded as practically unity. It would have been the term obtained by assuming the current in the second cylinder as concentrated on the axis and producing a uniform field of strength $2 I/D$ on the first cylinder. Including the proximity effect to this order we may write

$$W = \frac{1}{2} R_0 I^2 \left\{ 1 + F(z) + \frac{d^2}{D^2} G(z) \right\}, \dots \dots \dots (45)$$

in which $d (\equiv 2a)$ is the diameter of either cylinder, and $G(z) \equiv \frac{1}{8}z^2\sqrt{z}$ is plotted in fig. 3, its limiting value

$$G(z) \doteq (\sqrt{2z-1})/8 \dots \dots \dots (46)$$

being indicated by the broken line (B).



Finally the expression included in { } includes the effect both of disturbance of current distribution due to proximity and of non-uniformity of the field.

At low frequencies, by (21A), ϕ_n is negligible and both (42) and (43) give for { }

$$1 + \frac{1}{6}\mu^2 + \frac{1}{18}\mu^4 + \dots$$

This is identical with the result obtained for the thin wire (Equation 26), so that at low frequencies the distortion of distribution due to reaction of eddy-currents is quite negligible, and the effect of non-uniformity of the field will give in the extreme case where the cylinders are touching a correcting factor of amount 1.0456 to be applied to the result obtained by the uniform field theory.

At frequencies so high that (21B) holds, the factor { } becomes

$$1 - \mu^2 - 2\mu^4$$

when the currents are similarly directed, and

$$1 + 3\mu^2 + 10\mu^4$$

when the currents are oppositely directed ; while, if distortion is neglected, its value is

$$1 + \mu^2 + \mu^4$$

by (32). Thus the effects of distortion and of non-uniformity of field are equally important. When the currents are similarly directed the distortion is such as to tend to reduce the losses, and when the currents are oppositely directed the losses are increased.

The term involving μ^4 will contribute less than 1 per cent. at high frequencies if $D > 2.5d$ with the currents oppositely directed, and if $D > 2d$ with the currents similarly directed.

To this accuracy we may write for any frequency

$$W = \frac{1}{2}R_0 \left\{ 1 + F(z) + G(z) \frac{d^2}{D^2} \left(1 + H(z) \frac{d^2}{D^2} \right) \right\} I^2, \dots \dots \dots (46)$$

in which

$$H(z) = \frac{1}{4} \left(\frac{1}{2} \frac{\psi_2}{\psi_1} \pm 2\phi_1 \right).$$

If we assume the remaining terms to be in geometrical progression (46) may be written

$$W = \frac{1}{2}R_0 \left\{ 1 + F(z) + \frac{G(z)}{1 - \frac{d^2}{D^2} H(z)} \frac{d^2}{D^2} \right\} I^2. \dots \dots \dots (47)$$

At low frequencies this formula gives 1.043 as the correction for non-uniformity when the cylinders touch, and will certainly hold to 1 per cent. up to $D = 2d$ at extremely high frequencies.

$H(z)$ is plotted in fig. 4, up to $z = 5$, the curve I holding when the currents are oppositely directed, and II when the currents are similarly directed.

Since the effective resistance R' of a coil system is such that

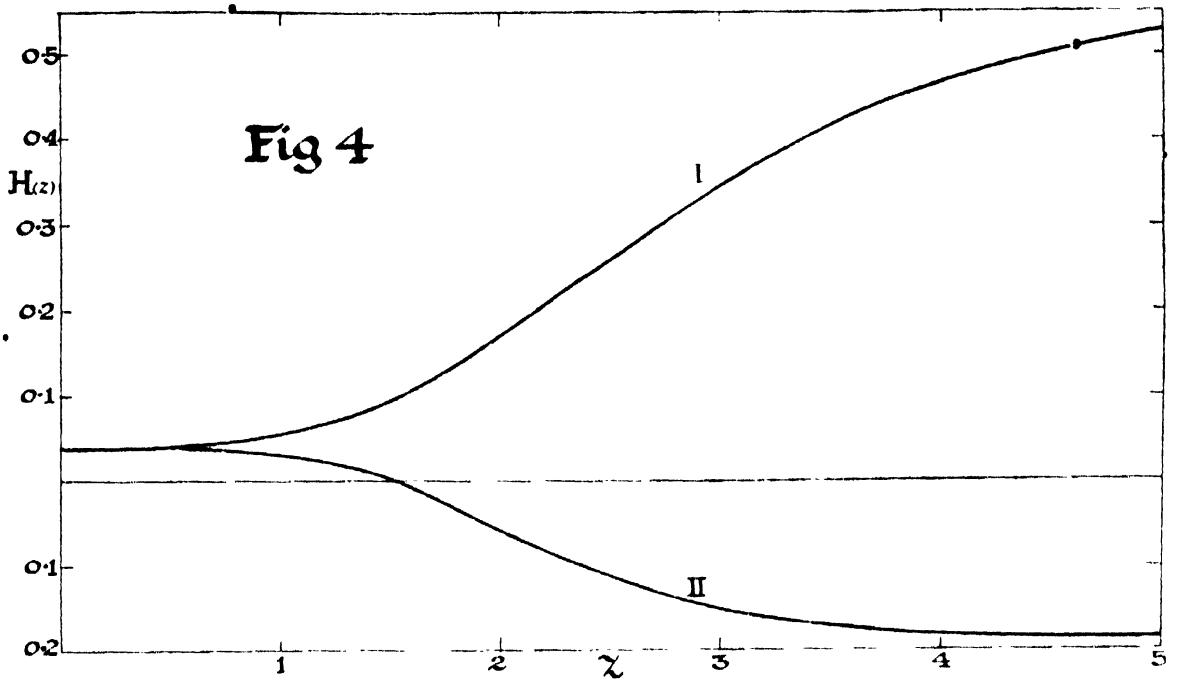
$$W = \frac{1}{2}R'I^2,$$

the effective resistance (apart from electrostatic capacities) per unit length of a pair of parallel wires is given by the formula*

$$R' = R_0 \left\{ 1 + F(z) + \frac{G(z)}{1 - \frac{d^2}{D^2} H(z)} \frac{d^2}{D^2} \right\}, \dots \dots \dots (48)$$

in which d is the diameter of either wire D the distance of their centres, and F, G, H are functions of $z \{ \equiv 2 \sqrt{\omega/R_0} \}$ drawn in fig. 3 and 4, and tabulated below.

* At extremely high frequencies it may be shown that ratio of the resistance of a go-and-return system to the skin resistance is given by $D/\sqrt{D^2-d^2}$. Formula (48) is then 3 per cent. in error when $d = 0.8D$.



VALUES of F, G, H in Formula (48).

$$z = 2\sqrt{\omega/R_0} \begin{cases} \omega/2\pi = \text{frequency.} \\ R_0 = \text{D.C. resistance of cylinder per centimetre.} \end{cases}$$

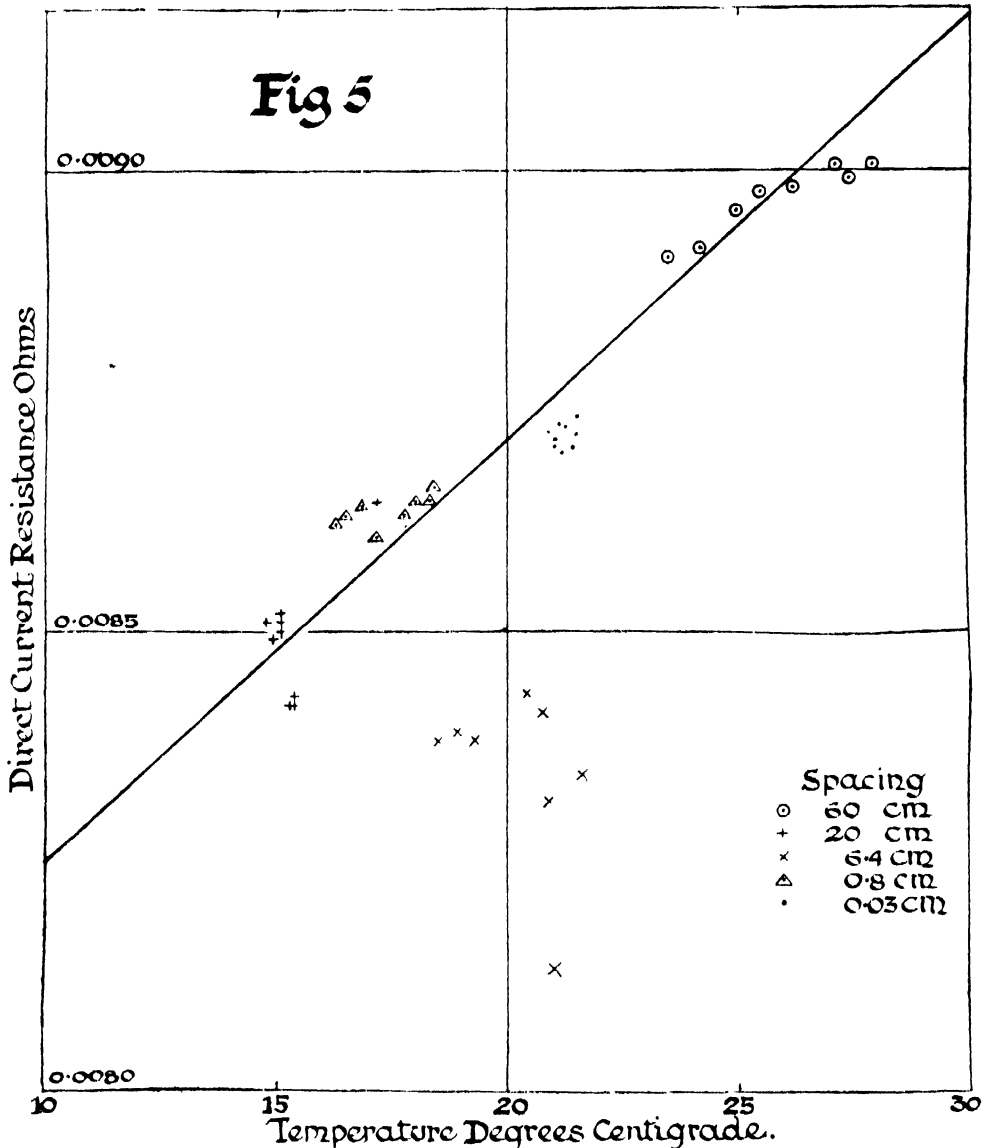
z.	F (z).	G (z).	H (z).	
			I. Currents Opposite.	II. Currents Similar.
0.0	0.000	0.000	0.0417	+ 0.0417
0.5	0.000326	0.000975	0.042	+ 0.041
1.0	0.00519	0.01519	0.053	+ 0.033
1.5	0.0258	0.0691	0.092	+ 0.001
2.0	0.0782	0.1724	0.169	- 0.056
2.5	0.1756	0.295	0.263	- 0.114
3.0	0.318	0.405	0.348	- 0.152
3.5	0.492	0.499	0.416	- 0.170
4.0	0.678	0.584	0.466	- 0.176
4.5	0.862	0.669	0.503	- 0.181
5.0	1.042	0.755	0.530	- 0.185
Large	$(\sqrt{2z}-3)/4$.	$(\sqrt{2z}-1)/8$	0.750	- 0.250

(9) *Test of Formula (48) by Comparison with Experimental Results.*—An extensive series of measurements of the resistance of a go-and-return system of parallel conductors has been made by KENNELLY, LAWS and PIERCE.*

* 'Trans. American Inst. El. Eng.,' Vol 35, Part 2, 1953, 1915.

The wire used was copper of diameter 1.168 cm., and the frequencies employed ranged from 60 to 5,000 cycles per second, so that the range of z in formula (48) is from 0.84 to 7.7. The spacing ($D - d$) varied from 0.03 cm. to 60 cm., so that the observations afford a very complete check on the adequacy of the formula.

The resistance of the loop, which for direct currents was of the order 0.01 ohm, was



measured by an alternating current bridge. In this method variation of contact resistance would probably be the chief source of trouble. As to whether this effect is appreciable, may be judged by a comparison of the tabulated direct current resistances after allowing for temperature variations. In fig. 5 this is done by plotting the resistances on a resistance-temperature diagram. Of the five groups of observations at spacings 60 cm., 20 cm., 6.4 cm., 0.8 cm., 0.03 cm., four show to within half a per

cent. a linear increase of resistance with temperature, this increase agreeing with the temperature coefficient of copper. The fifth group, that corresponding to the spacing 6.4 cm., is low. The discrepancy is removed if we suppose the measured resistance in error by 4 per cent. (0.0004 ohm).

In the following tables the ratio of alternating current resistance R' to the direct current resistance R has been calculated from formula (48) and compared with the observed value. This ratio is not independent of the temperature as the eddy-current resistance varies with temperature in a different way to that of the direct current resistance. The value of z has been calculated using that value of R which corresponds to the temperature of the observation as deduced from fig. 5.

Tables comparing observed values of the effective resistance of two parallel wires with the calculated values :—

θ = temperature of observation.

f = frequency in cycles per second.

R = direct current resistance.

$R_s - R$ = increase in resistance due to skin effect.

R_p = increase in resistance due to proximity of wires.

$R' = R_s + R_p$ = total alternating current resistance.

TABLE I.—Spacing = 60 cm.

θ° C.	23.5	24.2	25.0	25.6	26.2	27.1	27.4	27.9
f	60	306	888	1600	2040	3065	3950	5000
Calculated	$\left\{ \begin{array}{l} R_s/R \\ R_p/R \\ R'/R \end{array} \right.$	1.0047	1.108	1.560	2.045	2.270	2.708	3.030	3.372
		Negligible	—	—	—	—	—	—	—
		1.0047	1.108	1.560	2.045	2.270	2.708	3.030	3.372
Observed	R'/R	1.0038	1.111	1.587	2.042	2.279	2.694	3.034	3.361
Difference per cent.	+0.1	-0.2	-1.4	+0.2	-0.4	+0.5	-0.1	+0.3

TABLE II.—Spacing = 20 cm.

θ° C.	17.2	15.2	15.2	15.0	14.9	15.2	15.4	15.3	15.4
f	60	288	868	1663	2061	3063	3112	3860	5040
Calculated	$\left\{ \begin{array}{l} R_s/R \\ R_p/R \\ R'/R \end{array} \right.$	1.0048	1.106	1.578	2.120	2.328	2.775	2.790	3.075	3.472
		0.00004	0.0006	0.002	0.002	0.003	0.003	0.004	0.004	0.004
		1.0048	1.107	1.580	2.122	2.331	2.778	2.799	3.079	3.476
Observed	R'/R	1.0058	1.106	1.584	2.120	2.313	2.755	2.781	3.067	3.446
Difference per cent.	-0.1	+0.1	-0.3	+0.1	+0.8	+0.8	+0.5	+0.4	+0.9

TABLE III.—Spacing = 6.4 cm.

θ° C. ...	18.5	18.9	19.3	20.4	20.7	20.9	21.0	21.0	21.6	—
f ...	60	266	582	923	1465	2019	1992	3028	3960	5320
Calculated										
R_s/R ...	1.0048	1.088	1.340	1.602	1.984	2.280	2.268	2.728	3.075	3.470
R_p/R ...	0.0003	0.0045	0.0099	0.0132	0.017	0.021	0.020	0.026	0.031	0.036
R'/R ...	1.0051	1.093	1.350	1.615	2.001	2.301	2.288	2.754	3.106	3.506
Observed R'/R	1.0087	1.100	1.354	1.640	2.037	2.344	2.322	2.851	3.145	3.558
Difference per cent. ...	-0.4	-0.6	-0.3	-1.6	-1.8	-1.9	-1.5	-3.6	-1.2	-1.5

This group is abnormal on the resistance-temperature diagram. If we assume R' and R as measured are *both* too small by a constant amount = 0.04 R , the group becomes normal on the resistance-temperature diagram and gives the following values replacing the observed R'/R :—

Corrd. Obsd. R'/R ...	1.008	1.096	1.343	1.617	2.000	2.296	2.275	2.784	3.062	3.466
Difference per cent. ...	-0.3	0.3	+0.5	-0.2	+0.0	+0.2	+0.6	-1.1	+1.5	+1.5

TABLE IV.—Spacing = 0.8 cm.

θ° C.	—	—		16.3	16.5	16.9	17.2	17.8	18.0	18.3	18.4
f ...	60	239	671	1068	1509	1991	1988	2486	3028	3880	4900
Calculated											
R_s/R ...	1.0050	1.073	1.424	1.732	2.028	2.283	2.280	2.517	2.744	3.067	3.409
R_p/R ...	0.0052	0.061	0.187	0.254	0.314	0.382	0.380	0.424	0.477	0.550	0.620
R'/R ...	1.0102	1.134	1.611	1.986	2.342	2.665	2.660	2.941	3.221	3.617	4.029
Observed R'/R ...	1.0124	1.132	1.604	1.981	2.330	2.643	2.638	2.912	3.179	3.587	3.955
Difference per cent. ...	-0.2	+0.2	+0.4	+0.3	+0.5	+0.8	+0.8	+1.0	+1.3	+0.8	+1.8

The calculated values are in general too high. A spacing 0.85 cm. would give the following calculated values and differences :—

R'/R ...	1.0100	1.131	1.602	1.973	2.326	2.646	2.641	2.920	3.197	3.589	3.998
Difference per cent. ...	-0.3	-0.1	-0.1	-0.4	-0.2	+0.1	+0.1	+0.3	+0.6	+0.1	+0.4

TABLE V.—Spacing = 0.03 cm.

θ° C.	21.1	21.4	21.5	21.5	21.2	21.0	20.9	21.0	21.1
f	60	236	740	1000	1473	2038	3058	3918	5170
Calculated $\left\{ \begin{array}{l} R_s/R \dots \dots \dots \\ R_p/R \dots \dots \dots \\ R'/R \dots \dots \dots \end{array} \right.$	1.005	1.068	1.464	1.658	1.995	2.31	2.74	3.06	3.46
	0.015	0.165	0.759	1.00	1.37	1.83	2.56	3.09	3.72
	1.020	1.243	2.223	2.66	3.37	4.11	5.30	6.15	7.18
Observed R'/R	1.017	1.244	2.231	2.688	3.460	4.272	5.522	6.449	7.512
Difference per cent.	+0.3	-0.1	-0.4	-1.1	3.2	3.5	4.3	5.0	-4.5

The calculated values are in general too low, but with so small a spacing R' is varying rapidly. The formula gives the following values of R'/R when the wires touch :

R'/R	1.021	1.253	2.288	2.75	3.52	4.35	5.61	6.57	7.71
Difference per cent.	+0.4	+0.7	+2.6	+2.2	+1.7	+1.4	+1.8	+1.6	+2.7

In Tables I. and II. the skin effect is the only one of importance and very good agreement is obtained. These tables really check the experimental observations as the skin effect formula is well established. Tables IV. and V. form the real test of the proximity effect. It is seen that the small discrepancies are sufficiently accounted for by a slight adjustment (0.5 mm. at most) in the spacing. In Table III. the skin effect is predominant, but there is a rather large discrepancy. It is noteworthy that this group also shows a discrepancy on the resistance-temperature diagram, and that both the discrepancies are removed if we assume the measured values of R' and R to be both in error by 0.04 R .

(C) LOSSES IN PARALLEL WIRE SYSTEMS AND IN SHORT COILS.

(10) When the field acting upon the cylinder is uniform and has magnitude H , then by (18) the eddy-current losses per unit of length are given by

$$W = \frac{1}{4} \omega \alpha^2 H^2 \sqrt{\rho_1} (z),$$

or eliminating ω by $\omega = z^2 R_0 / 4$, and putting $\frac{1}{8} z^2 \sqrt{\rho_1} = G(z)$, $2a = d$,

$$W = \frac{1}{8} R_0 d^2 G^2(z) H^2. \dots \dots \dots (49)$$

Consider a system of parallel wires each of diameter d and occupying a square space of side D . Let these wires carry equal currents I in the same direction. Then, if the spacing is not too close, the currents may be supposed to be concentrated on the axes and producing uniform fields acting on the other wires. The field acting upon any wire s may be written $H_s = 2Ik_s/D$ where k_s is a numerical quantity depending upon the distribution of the wires and the position of the wire s in the system. By (49) the eddy-current loss in the wire s due to the field of the neighbouring wires is

$$W = \frac{1}{2} R_0 k_s^2 \frac{d^2}{D^2} G^2(z) I^2 \dots \dots \dots (50)$$

per unit of length.

If there are n wires each of length l , the total loss due to the proximity of the wires will therefore be

$$\frac{1}{2} R_0 l \frac{d^2}{D^2} G(z) I^2 \sum_1^n k_s^2;$$

or if the wires are connected in series to give a total direct current resistance $R = R_0 n l$, the added resistance due to the proximity of the wires is

$$R_p = u_n R \frac{d^2}{D^2} G(z), \dots \dots \dots (51)$$

in which

$$u_n \equiv \frac{1}{n} \sum_1^n k_s^2, \dots \dots \dots (52)$$

and depends only on the geometry of the axes of the wires.

The total resistance (apart from capacity effects) is got by adding the skin effect to (52) so that the formula for the alternating current resistance R' of the parallel wire system is

$$R' = R \left\{ 1 + F(z) + u_n \frac{d^2}{D^2} G(z) \right\} \dots \dots \dots (53)$$

This formula may also be applied to circular coils if the winding section is small in comparison with the radius. It remains to determine the values of u_n .

(11) *Single Layer Systems*.—If all the axes lie in one plane (fig. 6), then, numbering the wires 1, 2, 3, s n from left to right,

$$k_s = \frac{1}{s} + \frac{1}{s+1} + \dots + \frac{1}{n-s}.$$

From this formula the following values of k_s^2 are calculated:—

s	$n = 4$	$n = 8$	$n = 16$	$n = 24$
1	3.35	6.70	11.00	14.00
2	0.25	2.10	5.06	7.30
3	0.25	0.61	2.82	4.63
4	3.35	0.06	1.61	3.14
5		0.06	0.89	2.16
6		0.61	0.42	1.49
7		2.10	0.14	0.98
8		6.70	0.02	0.62
9			&c.	0.36
10				0.18
11				0.06
12				0.01

These and similarly deduced results give for u_n ,

$n =$	2	4	6	8	10
$u_n =$	1.00	1.80	2.16	2.37	2.51
$n =$	12	16	24	32	inf.
$u_n =$	2.61	2.74	2.91	3.00	3.29

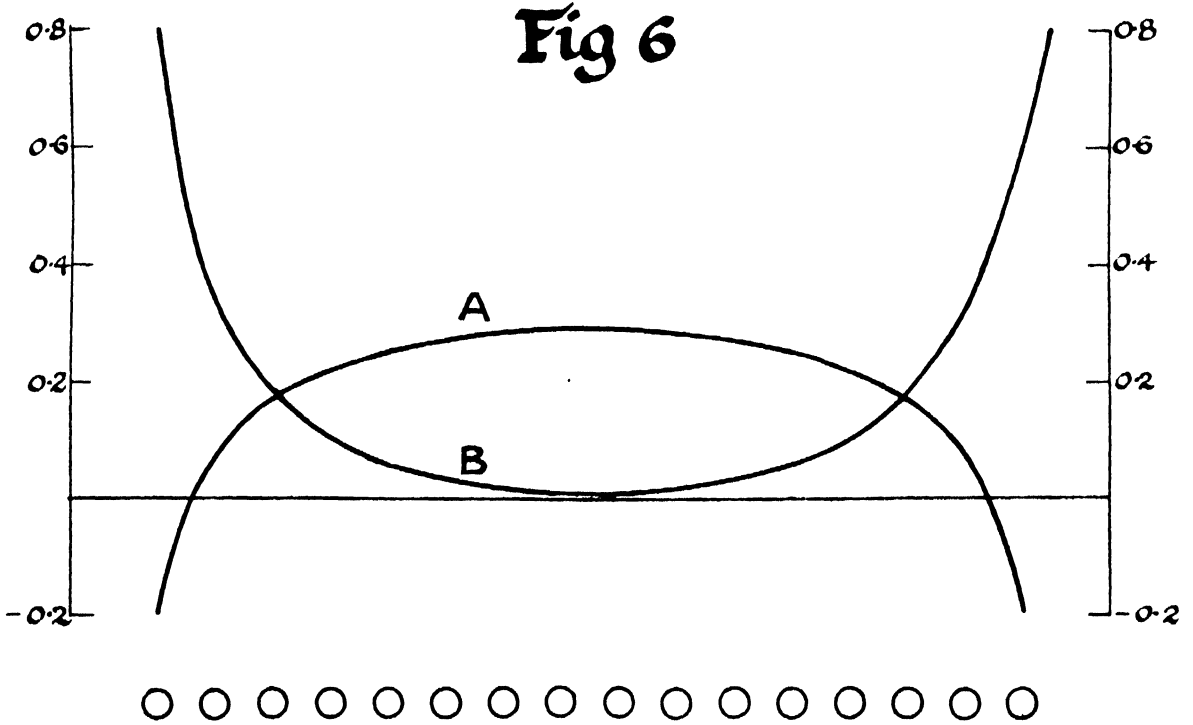


Fig. 6. Single layer 16-wire system.

Curve A shows the distribution of effective resistance produced by eddy-current losses due to the proximity of the wires.

Curve B shows the distribution of the losses throughout the system.

The value of u_n when $n = \infty$ is obtained as follows. Consider a long strip of width l composed of Nl parallel wires each carrying a current I . The field at a distance x from the edge in this strip is $2NI \log (l - x)/x$, and the mean square field is

$$\begin{aligned}
 H_m^2 &= \frac{4N^2I^2}{l} \int_0^l \log^2 \frac{l-x}{x} dx \\
 &= 16N^2I^2 \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right) \\
 &= \frac{1}{3} \pi^2 N^2 I^2.
 \end{aligned}$$

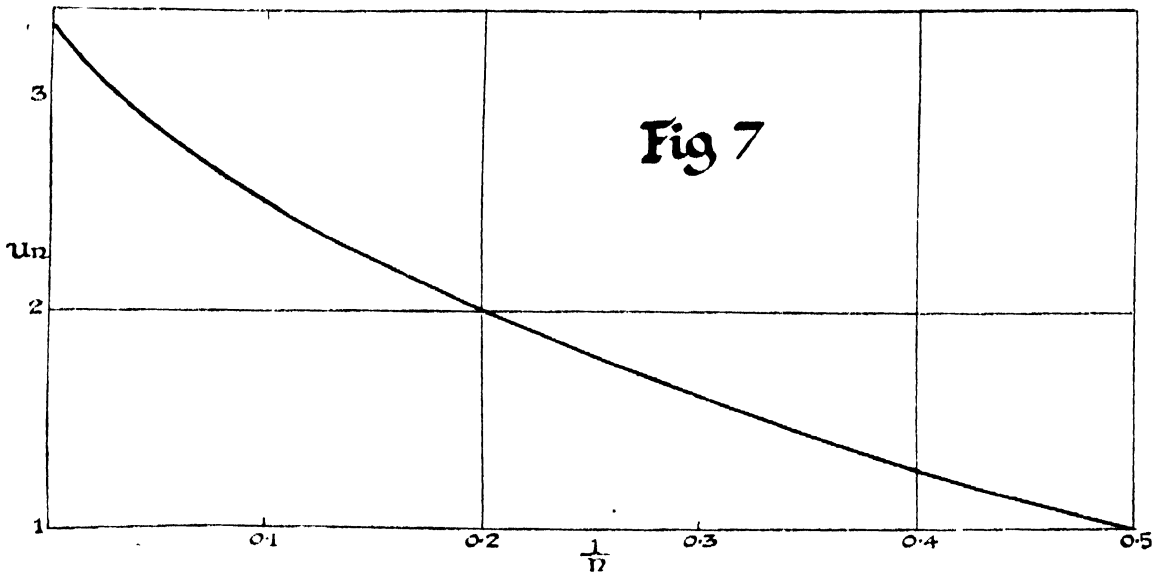
But since $N = 1/D$,

$$H_m^2 = 4N^2I^2u_n,$$

so that

$$\text{Lt } n \rightarrow \infty u_n = \frac{1}{8} \pi^2 = 3.2899.$$

The tabulated values of u_n when plotted against $1/n$ (fig. 7) enable the value of u_n to be determined for any number of wires.



(12) *Distribution of Losses in Single Layer System.*—The energy loss in the wire s , due to the fields of the other wires, is proportional to k_s^2 . From the table for k_s^2 it is seen that this is greatest in the end wires. Thus, in the case of four wires, 93 per cent. of this loss takes place in the two outer wires and only 7 per cent. in the two inner wires. For a greater number of wires, if the system is divided into four equal sections the distribution of loss is still such that approximately 93 per cent. of the loss occurs in the two outer sections. This may be of importance in measurements of effective resistance based on the determination of the increase in temperature of the system, and account should be taken of the possibility of a temperature distribution for alternating currents different from that for direct currents.

(13) *Distribution of Resistance in Single Layer System.*—The distribution of eddy-current losses throughout the system does not represent the distribution of effective resistance. In fact the energy required to produce the losses in any wire is supplied by the currents flowing in the other wires, and therefore the other wires behave as if certain resistances were added to them.

In order to determine these equivalent resistances, consider two coils (I) and (II) carrying currents I_1, I_2 which produce fields acting on a cylinder carrying no current. Ultimately these currents will be assumed equal, so that for simplicity they will be considered in phase. Let the fields in the neighbourhood of the cylinder due to these currents have intensities

$$H_1 = \alpha I_1, \quad H_2 = \beta I_2 \quad \dots \dots \dots (54)$$

and let them be inclined at an angle ϕ .

By (49) the eddy-current loss in the cylinder is

$$\begin{aligned}
 W &= \gamma (H_1^2 + H_2^2 + 2H_1H_2 \cos \phi) \\
 &= \gamma (\alpha^2 I_1^2 + \beta^2 I_2^2 + 2\alpha\beta I_1 I_2 \cos \phi) \dots \dots \dots (55)
 \end{aligned}$$

in which

$$\gamma \equiv \frac{1}{8} R_v d^2 G(z).$$

Now, instead of the cylinder being present, suppose the two circuits carrying the

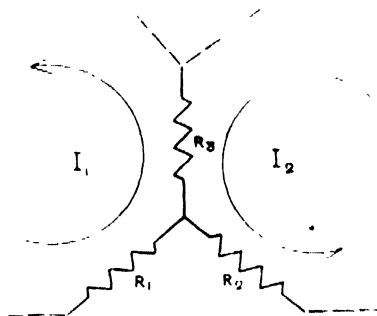


Fig 8

currents I_1, I_2 to be linked by the resistance system shown in fig. 8. The rate of dissipation of energy by this system is

$$\begin{aligned}
 W' &= \frac{1}{2} \{ R_1 I_1^2 + R_2 I_2^2 + R_3 (I_1 - I_2)^2 \} \\
 &= \frac{1}{2} (R_1 + R_3) I_1^2 + \frac{1}{2} (R_2 + R_3) I_2^2 - R_3 I_1 I_2, \dots \dots \dots (56)
 \end{aligned}$$

W' is identical with W if the resistances have the values

$$\left. \begin{aligned}
 R_1 &= 2\alpha (\alpha + \beta \cos \phi) \gamma, & R_2 &= 2\beta (\beta + \alpha \cos \phi) \gamma \\
 R_3 &= -2 (\alpha\beta \cos \phi) \gamma
 \end{aligned} \right\} \dots \dots \dots (57)$$

In the case $I_1 = I_2$, no current flows through R_3 , so that the potential differences produced by the eddy losses in the cylinder are then such that they may be represented by the resistances R_1, R_2 in series with the respective coils.

Applying to a single layer system, suppose we require the resistance to be added to a wire s which would represent the contribution of s to the eddy losses in the whole system. Let the wire s be the coil (I), another wire r be the cylinder, and the remaining wires be the coil (II). αI is the field acting on r due to the current in s , βI is the field acting on r due to the currents in the remaining wires, so that $(\alpha + \beta \cos \phi) I$ is the nett field acting on r resolved in the direction of αI . Since in the single layer system the fields due to individual wires are collinear, $(\alpha + \beta \cos \phi) I$ is the total field in which the wire r is situated. It is to be regarded as positive when the field due to s is the same sense as the total field.

Hence if H_r is the whole field acting on r , and H_{rs} that portion of the field contributed by the wire s , the resistance to be added to the wire s to imitate the eddy losses in the remainder of the system is

$$R_{ps} = \frac{2\gamma}{I^2} \left\{ \sum_{r=1}^{r=s-1} H_{rs} \cdot H_r + \sum_{r=s+1}^{r=n} H_{rs} \cdot H_r \right\} \dots \dots \dots (58)$$

Remembering that

$$H_{rs} = \frac{2I}{(r-s)D}, \quad H_r = \frac{2I}{D} \left(\frac{1}{r} + \dots + \frac{1}{n+r} \right),$$

(58) may be written

$$R_{ps} = u_{ns} R \frac{d^2}{D^2} G(z), \quad \dots \dots \dots (59)$$

where u_{ns} is a numerical quantity depending on the number of wires and the position of s .

The general distribution of resistance is sufficiently illustrated by the case of a 16-wire system. The values of u_{ns} for this system are found by the above method to be

$$s = \begin{cases} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 \end{cases}$$

$$u_{ns} = \quad -0.19 \quad +0.08 \quad 0.17 \quad 0.22 \quad 0.25 \quad 0.27 \quad 0.28 \quad 0.29$$

The values of u_{ns} when added should give the value of u_n for the whole system. Thus $\sum_{s=1}^{s=16} u_{ns} = 2.74$, and this agrees with the tabulated value of u_n for the 16-wire system.

As regards the negative value of u_{ns} for the extreme wires, it must be remembered that the equivalent resistances for each wire are such as to imitate the *potential differences* produced by the eddy losses in the wires, and it is quite possible that the phase relations may produce a rise in potential in phase with the current in *part* of a system. The only condition that is essential is that the value of u_n for the *whole* system shall be positive. Curves A and B of fig. 6 (p. 79) show the distribution of proximity resistance and of loss respectively for a 16-wire system.

(D). SINGLE LAYER SOLID WIRE COILS.

(14) *Single Layer Coils. Effect of radius of curvature of Coils.*—A single layer circular coil, whose width of winding is small compared with the radius of the coil, differs only slightly from a straight parallel wire system, so that formula (53) will hold for a coil of this kind as a first approximation. The slight differences which occur, due to helicity of winding and owing to the fact that the wire (regarded as a cylinder under the action of a transverse field) is curved, are probably too small to be measurable and the mathematical difficulties too great to make a theoretical treatment possible.

A more important difference is the modification of the transverse field acting on the

individual wires. In a straight single layer system the field acting on one wire is perpendicular to the surface of the layer, but in a single layer coil not only is this normal field modified but there also exists a component of the field acting along the surface of the layer. The case where the number of wires in the coil is large will alone be considered.

Solenoidal Coil.—Let the mutual inductance (M) between two equal parallel coaxial circles of radius a and separation b be written

$$M = 4\pi\alpha f(2a/b). \quad (60)$$

Then*

$$f(\mu) = \log 4\mu - 2 + \frac{3}{4}\frac{1}{\mu^2} (\log 4\mu + \frac{1}{3}) - \frac{1}{6}\frac{5}{4}\frac{1}{\mu^4} (\log 4\mu - \frac{2}{3}\frac{1}{3}) + \frac{3}{2}\frac{5}{6}\frac{1}{\mu^6} (\log 4\mu - \frac{2}{3}\frac{1}{3}) + \dots \quad (61)$$

so long as $\mu < 1$.

From this expression it is readily shown by differentiation and integration that the radial and axial components of the field at the point on the prolongation of the surface of a cylindrical coil distant ξ from the edge are given by

$$H_N = 2nI \left\{ f\left(\frac{2a}{\xi}\right) - f\left(\frac{2a}{b+\xi}\right) \right\}$$

$$H_T = 2nI \int_{2a/b+\xi}^{2a/\xi} \frac{d(\mu f)}{d\mu} \frac{d\mu}{\mu^2}, \quad (62)$$

n being the number of turns per unit of length and I the current.

When ξ is very small, H_N tends to the value $H_0 - h$, in which

$$H_0 = 2nI \log \frac{b+\xi}{\xi}, \quad (63)$$

and is the field due to a straight strip of width b , while

$$h = 2nI \left\{ \frac{3}{4}\frac{1}{\mu^2} (\log 4\mu - \frac{1}{3}) - \frac{1}{6}\frac{5}{4}\frac{1}{\mu^4} (\log 4\mu - \frac{2}{3}\frac{1}{3}) + \frac{3}{2}\frac{5}{6}\frac{1}{\mu^6} (\log 4\mu - \frac{2}{3}\frac{1}{3}) \dots \right\}$$

$$\mu = 2a/b.$$

H_T tends to the value

$$H_T = 2nI \left\{ \frac{\log 4\mu}{\mu} - \frac{1}{4}\frac{1}{\mu^3} (\log 4\mu - 1) + \frac{3}{8}\frac{1}{\mu^5} (\log 4\mu - \frac{7}{6}) - \frac{2}{2}\frac{5}{6}\frac{1}{\mu^7} (\log 4\mu - \frac{2}{3}\frac{1}{3}) + \dots \right\}$$

$$\mu = 2a/b.$$

To find the normal and axial components at any point on the surface of the coil,

* BUTTERWORTH, 'Phil. Mag.', vol. 31, p. 216, 1916.

divide the coil into two portions A and B to the left and right of the point in question. Then the two components of the field are

$$(H_0 - H'_0) - (h - h'), \quad H_T + H'_T$$

where the accented letters refer to the field due to the portion B, and the unaccented letters to that due to A. $H_0 - H'_0$ is the normal field for a straight strip, and $h - h'$ the correction on the normal field due to curvature, while $H_T - H'_T$ is the field tangential to the layer due to curvature.

Formulae (64) and (65) give the following values for h and H_T :—

$b/2a$	$h/2nI$	$H_T/2nI$
0.1	0.025	0.369
0.2	0.079	0.595
0.3	0.149	0.767
0.4	0.229	0.902
0.5	0.313	1.010
0.6	0.400	1.096
0.7	0.486	1.167
0.8	0.569	1.225
0.9	0.651	1.273
1.0	0.728	1.313

from which the values of $h - h'$, $H_T - H'_T$ may be calculated for any point on the surface of a coil if $b/2a < 1$.

For the eddy loss formula we require the mean square field acting on the coils; that is, denoting $H_0 - H'_0$ by H , $h - h'$ by H_1 , $H_T + H'_T$ by H_2 , we require the mean value of

$$(H - H_1)^2 + H_2^2 = H^2 - 2HH_1 + H_1^2 + H_2^2$$

throughout the surface of the coil.

As regards the integrations required in determining this mean value, the integral of H^2 leads to the straight system formula; that of H_1^2 and H_2^2 may be carried out by approximate methods, since H_1^2 and H_2^2 are finite throughout the range of integration. The integral of $H \cdot H_1$ is obtained as follows. Choose the length of the coil as twice the unit of length so that $H/2nI = \log \frac{1+x}{1-x}$ at a point on the surface distant x from the centre. Suppose H_1 may be expressed in the form

$$H_1 = \alpha + \beta x + \gamma x^2 + \delta x^3 + \dots$$

The integrals required are then of the form

$$\int_{-1}^1 x^n \log \frac{1+x}{1-x} dx.$$

When s is even these integrals are zero, and when s is odd have the values

$$\frac{4}{s+1} \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{s} \right).$$

Neglecting terms beyond x^4 in the series for H_1 ,

$$\int_{-1}^1 HH_1 dx = 4nI (\beta + \frac{2}{3}\delta).$$

There is no need to evaluate β and δ for $H_1(x_1) - H_1(-x_1) = 2x_1(\beta + \delta x_1^2)$, which immediately gives $\beta + \frac{2}{3}\delta$ if we put $x_1^2 = 2/3$. Greater accuracy may be obtained by suitably choosing a series of values of x , &c., to take into account the higher terms of the series, but the above expression is sufficient for the present purpose.

The evaluation of the integrals by the above methods leads to the following table of values for u_n in applying formula (53) to solenoidal coils of length b and radius a .

SINGLE Layer Solenoidal Coils. Radius = a . Length = b . u_n in formula (53).

$b/2a$	=	0.0	0.2	0.4	0.6	0.8	1.0
u_n	=	3.29	3.63	4.06	4.50	4.93	5.28

The assumption $u_n = 3.29 + b/a$ will give results which do not differ by more than 2 per cent. from the above values.

Flat Coils.—By methods similar in principle to those used in determining the values of u_n for solenoidal coils, the following values of u_n have been found:—

SINGLE Layer Flat Coils. r = inner radius. R = outer radius.

r/R	=	1.0	0.9	0.8	0.7	0.6	0.5
u_n	=	3.29	3.36	3.58	3.84	4.24	4.78

(15) *Single Layer Coils at High Frequencies.*—When z is greater than 3, $F(z)$ and $G(z)$ assume the simplified forms

$$F(z) = (\sqrt{2z}-3)/4, \quad G(z) = (\sqrt{2z}-1)/8,$$

so that formula (53) becomes

$$R' = \alpha + \beta z, \quad \dots \dots \dots (66)$$

in which

$$\alpha = \frac{1}{8}R(2 - u_n d^2/D^2), \quad \beta = \frac{\sqrt{2}R}{8}(2 + u_n d^2/D^2).$$

Now $u_n d^2/D^2$ seldom exceeds 6, so that α/β will usually lie between + 0.7 and - 0.4.

Even when z is as small as 3, α is therefore less than $\frac{1}{4}\beta z$. For most purposes it is sufficiently accurate to take

$$R' = \beta z. \quad \dots \dots \dots (67)$$

From the relation $z^2 = 4\omega/R_0$ we may write

$$z = 0.0479\sqrt{f/R'_0} = 830/\sqrt{R'_0\lambda}, \quad \dots \dots \dots (68)$$

in which f is the frequency in cycles per second, λ the wave-length in metres, R'_0 the resistance of the wire of the coil in ohms per 1,000 yards.

Hence (67) becomes

$$R' = A/\sqrt{\lambda} = A'\sqrt{f}, \quad \dots \dots \dots (69)$$

where A and A' are given by

$$\left. \begin{aligned} A &= 146.7 (2 + u_n d^2/D^2) R/\sqrt{R'_0} \\ A' &= 8.47 \times 10^{-3} (2 + u_n d^2/D^2) R/\sqrt{R'_0} \end{aligned} \right\} \dots \dots \dots (70)$$

(16) *Comparison of Formula (69) with Experimental Observations.*—LINDEMANN and HÜTER* have measured the effective resistances of a series of single layer coils over a range of wave-lengths to which formula (69) is applicable.

Their method was to bring the coil into resonance with an air condenser at the required wave-length and to measure the effective resistance in this condition by adding a known non-inductive resistance and observing the reduction in current. The method measures the resistance of the whole circuit of which the coil is a part, so that the resistance of the coil may be deduced if the resistances of the non-inductive portions of the circuit are known, and the condenser is assumed to be free from loss.

Their results included four solenoidal coils wound with thick solid wire, and for these coils they found that the effective resistance could be expressed in the form

$$R' = A/\sqrt{\lambda} + B/\lambda^2. \quad \dots \dots \dots (71)$$

The data given by LINDEMANN and HÜTER for these coils enable the value of A in formula (70) to be calculated. The results are given in the following Table, which includes also three other coils measured by a similar method at the National Physical Laboratory.

It is seen that the value of A as calculated from formula (70) is in good agreement with the value of A determined experimentally. In the calculation, the value of u_n taken has been the value given in the short table for solenoidal coils in Section 14. These values have been deduced upon the assumption of a large number of turns. Calculations based on the value of u_n , as deduced from a straight system containing the same number of wires as there were turns in the coil, were found to give a value of A which in every case was lower than the value deduced from observations.

* 'Verh. Deutsch. Phys. Gesellschaft,' vol. 15, 1913, p. 219.

TABLE.—Effective Resistance of Single Layer Solenoidal Coils.

Coil.	Radius <i>a</i> cm.	Length <i>b</i> cm.	Turns <i>n</i> .	Dia- meter of wire <i>d</i> mm.	Direct Current R ohms.	Induct- ance L micro- henries.	Experimental.		Calcula- ted A.	Obsr.	Range of Wave- length metres.
							A.	B.			
1	15.5	1.8	7	2.2	—	36	9.5	2.14×10^4	10.4	L.H.	—
2	9.7 ₅	2.6	11	2	0.0382	43	11.8	2.15×10^4	11.2	L.H.	100 3500
3	11.3	2.0	13	0.71	0.425	75	25.1	7.4×10^4	25.5	N.P.L.	150 600
4	10.0	6.5	18	3	—	80	14.3	7.0×10^4	13.2	L.H.	—
5	9.8	2.1 ₅	15	1.1	—	84	25.5	7.0×10^4	25.6	L.H.	—
6	11.7 ₅	1.0	13	0.71	0.444	100	49.0	6.8×10^4	50	N.P.L.	200 600
7	8.7 ₅	7.3	40	1.62	0.193	320	56	1.7×10^6	55	N.P.L.	600 1500

The direct current resistances were not known in the cases of the coils 1, 4, 5. For the calculation of A, their values were deduced from the dimensions of the coils using the resistivity of copper as given by the known direct current resistance of Coil 3.

It was to throw some light upon the properties of the second term (B/λ^2) in LINDEMANN'S equation that the three coils marked N.P.L. were measured. Coil No. 6, which was wound with D.S.C. wire and held together by wax and silk without any other frame, was measured first to confirm LINDEMANN'S results. The observations are given below for this coil and are typical. In the Table, R_1 is the measured resistance of the coil. R' is obtained from R_1 by dividing by $(1 - \omega^2 LC)^2$, where C is the measured self capacity of the coil. The values of A and B are deduced by plotting $R'\sqrt{\lambda}$ against $\lambda^{-1/2}$ (formula (71)).

COIL No. 6. $L = 100.0$ microhenries. $C = 20 \mu\mu F$.

λ metres.	R_1 ohms.	R' ohms.	$A/\sqrt{\lambda}$.	B/λ^2 .	R' (formula (71)).
206	7.2	5.0 ₀	3.4 ₂	1.6 ₁	5.0 ₃
229	6.2	4.6 ₄	3.2 ₃	1.3 ₁	4.5 ₅
260	5.2	4.1 ₆	3.0 ₄	1.0 ₁	4.0 ₅
294	4.3	3.6 ₁	2.8 ₅	0.7 ₉	3.6 ₄
350	3.5	3.1 ₀	2.6 ₄	0.5 ₆	3.2 ₀
463	2.8	2.6 ₂	2.2 ₈	0.3 ₂	2.6 ₀
500	2.6	2.4 ₆	2.1 ₉	0.2 ₇	2.4 ₆
543	2.5	2.3 ₈	2.1 ₀	0.2 ₃	2.3 ₃
584	2.3	2.3 ₀	2.0 ₂	0.2 ₀	2.2 ₂

Coil No. 3 was of bare wire and supported by eight pieces of ebonite upon which equidistant grooves had been cut to keep the wires in position, these ebonite pieces being spaced equally round an octagonal wooden frame. The arrangement involved no metal except the wire of the coil. Coil 3 approximately imitates Coil 6, except that the insulation had been removed and the spacing increased. Coil No. 7 was wound on a wooden frame with D.S.C. wire and no wax. It is included in the table to increase the range of inductance and to put a severe test on the curvature correction for u_n .

In regard to the second term in formula (71), it is interesting to notice that if B is divided by L^2 , the result is of the same order of magnitude for all the coils tested although the inductance increases nine fold. Thus :—

Coil No. :	1	2	3	4	5	6	7
B/L ² =	16.3	11.6	8.5	10.9	9.9	7.2	16.7

A leakage of conductance G would contribute a term $\omega^2 L^2 G$ to the expression for the effective resistance. In terms of wave-length this becomes $3.56 L^2 G / \lambda^2$ if λ is in metres, L in microhenries, and G in micromhos. In order to imitate the resistance B/λ^2 by such a leakage the value of $1/G$ must range from 0.2 to 0.4 megohm to give the values observed for B.

As to whether leakage is the cause of the second term in LINDEMANN'S equation, and as to whether it lies in the coil or the remainder of the circuit is a matter which requires further investigation. There is no doubt, however, that the first term of LINDEMANN'S equations may be closely predicted by formula (69).

(17) *Conditions for Minimum Eddy-Current Losses in Single Layer Coils.*—The inductance of a single layer solenoidal coil of radius a and length b may be written

$$L = 4\pi ab^2 X / D^2, \dots \dots \dots (72)$$

in which D is the distance apart of two consecutive turns and X is a function of a/b . The effective resistance of the coil is by (53)

$$R' = R (1 + F + u_n G d^2 / D^2) \dots \dots \dots (73)$$

where F, G depend on the frequency and diameter of wire only, while u_n is a function of a/b . The values of X and u_n for the range of a/b 1.0 to 2.4 are given below, the latter being obtained by interpolation from the table of Section 14, and the former from RAYLEIGH'S formula

$$X = \log_e 8a/b - 1/2 + b^2/32a^2 (\log_e 8a/b - 1/4). \dots \dots \dots (74)$$

a/b =	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4
u_n =	4.29	4.10	3.98	3.87	3.80	3.73	3.69	3.65
X =	1.651	1.816	1.958	2.084	2.195	2.296	2.388	2.466

With wire of a given diameter, the value of R'/L depends upon the values of D, a, b . If the length of wire (l) is also fixed, a may be expressed in terms of l, D and b , since

$$l = 2\pi ab/D. \quad \dots \dots \dots (75)$$

Writing $D = \xi d, b = \eta a$ we have from (72), (73), (75)

$$R'/L = R \left(\frac{\pi d}{2l^3} \right)^{\frac{1}{2}} \frac{\xi^{\frac{1}{2}} (1 + F + u_n G / \xi^2)}{\eta^{\frac{1}{2}} X} \dots \dots \dots (76)$$

The minimum value of R'/L is required for variations of ξ and η ; the former gives the best spacing of the wires with a coil of given shape, and the latter gives the best shape.

Best Spacing.—(76) is minimum with regard to variation of ξ when

$$\xi^2 = 3u_n G / (1 + F), \dots \dots \dots (77)$$

and then

$$R'/L = \frac{1}{3} R \left(\frac{\pi d}{2l^3} \right)^{\frac{1}{2}} \{ 3G(1 + F) \}^{\frac{1}{2}} \frac{u_n^{\frac{1}{2}}}{\eta^{\frac{1}{2}} X} \dots \dots \dots (78)$$

Condition (77) shows that at the best spacing the proximity losses are one-third the skin losses. If the best spacing is not employed, then, writing $R'/L = \tau$, and letting τ_0, ξ_0 be the values of τ, ξ when the spacing is best,

$$\tau/\tau_0 = \frac{1}{4} (\xi/\xi_0)^{\frac{1}{2}} \{ 3 + (\xi_0/\xi)^2 \} \dots \dots \dots (79)$$

from which the following values are found:—

$\xi/\xi_0 (\equiv D/D_0) =$	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3
$\tau/\tau_0 =$	1.120	1.053	1.019	1.004	1.000	1.003	1.012	1.023
$\xi/\xi_0 =$	1.4	1.5						
$\tau/\tau_0 =$	1.037	1.055						

Best Shape.—Keeping the best spacing, the best shape is that value of a/b which will make $(a/b)^{\frac{1}{2}} u_n^{\frac{1}{2}} / X$ a minimum. The following values are obtained from the table given above for u_n and X :—

$a/b =$	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4
$X/u_n^{\frac{1}{2}} (a/b)^{\frac{1}{2}} =$	1.147	1.165	1.172	1.174	1.172	1.167	1.161	1.152

so that if condition (77) is possible the best shape is $a/b = 1.6$, and then

$$R'/L = 1.872R (d/l^3)^{\frac{1}{2}} \{ G(1 + F) \}^{\frac{1}{2}} \dots \dots \dots (80)$$

When z is very large, $F = 2, G = \sqrt{2z}/4$, so that at very high frequencies

$$R'/L = 0.557Rz (d/l^3)^{\frac{1}{2}},$$

or with

$$R = 4\rho l/\pi d^2, \quad z^2 = 2\pi^2 f^2 d^2/\rho,$$

$$R'/L = 3.15 (f\rho/d)^{\frac{1}{2}} \dots \dots \dots (81)$$

If the frequency f is not sufficiently high for the approximation to hold, (81) must be replaced by

$$R'/L = \gamma(z) (f\rho/ld)^{\frac{1}{2}}, \dots \dots \dots (81A)$$

in which $\gamma(z) \equiv 3.15 \{G(1+F)^{\frac{1}{2}}\}^{\frac{1}{2}}/z$ and has the following values:—

$z =$	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	∞
$\gamma(z) =$	3.81	3.61	3.53	3.48	3.43	3.41	3.39	3.37	3.15

It is seen that $\gamma(z)$ does not vary much throughout a large range of frequency, and if the formula

$$R'/L \approx 3.4 \{f\rho/ld\}^{\frac{1}{2}} \dots \dots \dots (81B)$$

is used this will represent the best value of alternating current time constant attainable so long as z is greater than 2. Taking for copper $\rho = 1,600$ C.G.S., expressing f in terms of wave-length λ (in metres), and supposing L to be in microhenries, l and d in centimetres,

$$R'/L = 2.35/\sqrt{ld\lambda}.$$

Thus A in LINDEMANN'S formula cannot be less than

$$A_{min} = 2.35 L/\sqrt{ld}.$$

In illustration we have for coils No. 1, 2, 3 of the table of Section 16,

$$A_{min.} = 7.1, \quad 8.6, \quad 22$$

while

$$A_{min.}/A_{actual} = 0.75, \quad 0.73, \quad 0.86.$$

The ratio $2.35 L/A\sqrt{ld}$ may be taken as a measure of the efficiency of any coil.

Condition that Equation (77) may be satisfied.—Since, if the best spacing is used, $a/b = 1.6$ is always the best shape, we have from (77), with $u_n = 3.87$,

$$(D/d)^2 = 11.61G/1+F. \dots \dots \dots (82)$$

This gives the following values for D/d :—

$z =$	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$D/d =$	0.425	0.88 ₅	1.36	1.70 ₅	1.89	1.97	2.01	2.04	2.07 ₅
$z =$	6.0	7.0	8.0	9.0	10.0	inf.			
$D/d =$	2.14	2.17	2.20	2.22 ₅	2.24 ₅	2.41			

Now D/d must always be greater than unity in practice, so that if z is less than 1.6, close winding is the best. When z exceeds 1.61, spacing rapidly becomes advantageous, the best spacing at very high frequencies being $D = 2.4d$. As regards departure from the best spacing, the table of τ/τ_0 shows that the time constant will vary by less than 1 per cent. from the best value if D/D_0 lies between 0.85 and 1.18, by less than 5 per cent. if D/D_0 lies between 0.79 and 1.28.

If close winding is employed (keeping $a/b = 1.6$), the losses at high frequencies will be 42 per cent. greater than when the proper spacing is employed. Fig. 9 summarizes the

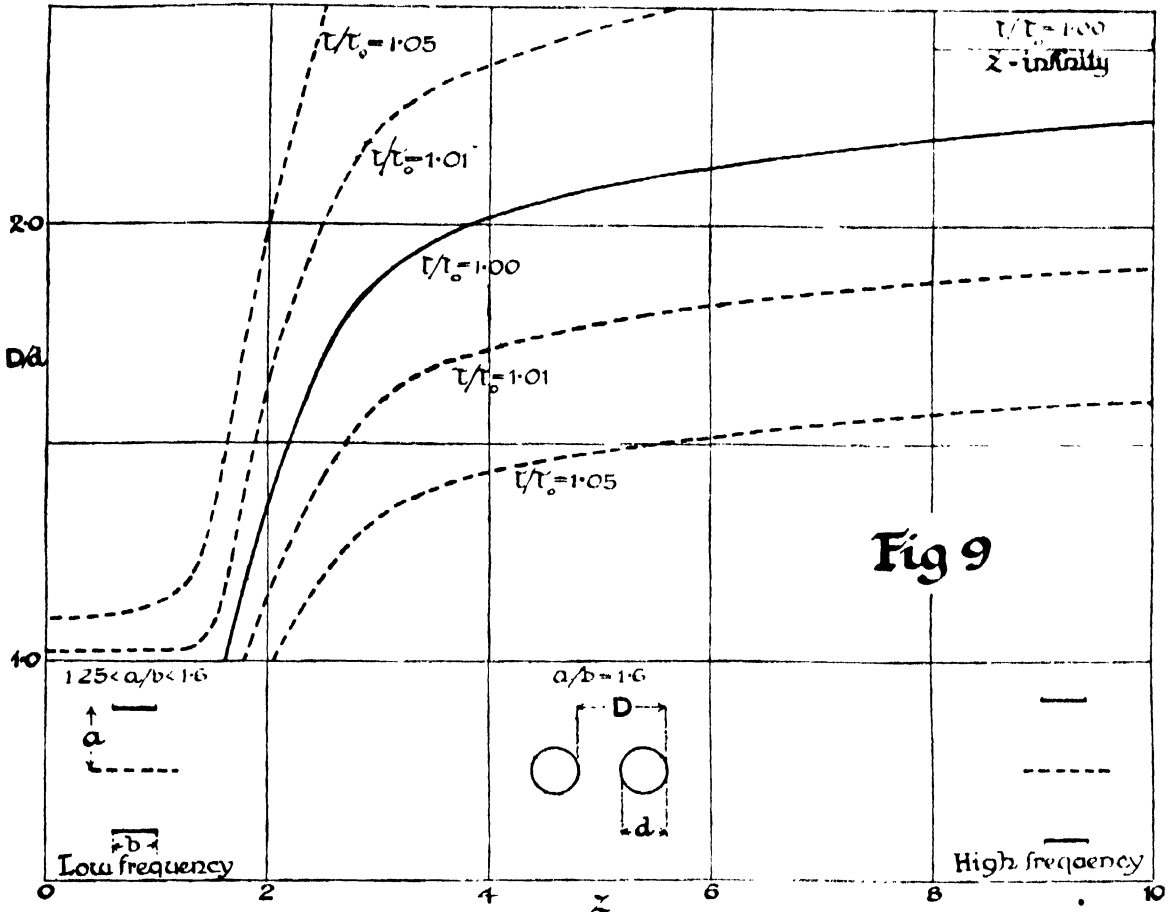


Fig. 9. Single layer solenoidal coils—Solid wire.

- R' = Alternating current resistance, L = Inductance.
- τ = R'/L , τ_0 = Minimum R'/L .
- z = $830/\sqrt{R'_0\lambda}$, R'_0 = D.C. resistance in ohms per 1,000 yards
- λ = Wave-length in metres.

Frequency high if $z > 2$.

At high frequencies best shape is $a/b = 1.6$, best spacing (average) is $D/d = 2$.

At low frequencies best shape (average) is $a/b = 1.4$ and close winding is the best.

results of this section. The full curve gives the best spacing, and the broken curves show the limits allowable if losses 1 per cent. and 5 per cent. greater than the minimum losses are permissible. The figure may be used to grade coils for different ranges of frequency.

Thus the 5 per cent. limit will be attained if below $z = 2$ we make $a/b = 1.4^*$ and employ close winding, and above $z = 2$ we make $a/b = 1.6$ and space the wire so that

* Between $z = 0$ and $z = 1.6$ the best value of a/b rises from 1.25 (the steady current value) to 1.6. The value $a/b = 1.4$ will never produce a loss exceeding by 0.2 per cent. the minimum loss.

$D = 2d$. In estimating, for a given diameter of wire, the wave-length at which spacing should be employed, the formula

$$R'_0 \lambda = 172,000 \dots \dots \dots (83)$$

(deduced from (68) with $z = 2$) is useful. Thus, with No. 28 S.W.G. wire, R'_0 in ohms per 1,000 yards is 140; so that if λ is less than 1,230 metres, spaced winding should be used.

Flat Coils.—A similar treatment for flat coils has yielded the following results:—

Spacing becomes advantageous at practically the same value of z (viz., $z = 1.6$) as that for solenoidal coils. At any value of z greater than 1.6, the best ratio of inner to outer radius is $r/R = 0.60$, and the best spacing is 1.04 times that for solenoidal coils. When both these conditions are satisfied the eddy-current losses in the flat coil are 5 per cent. greater than those in the solenoidal coil wound in the best way with the same wire. When z is less than 1.6, close winding is the best, and the best ratio of radii varies from 0.4 to 0.6. The value $r/R = 0.5$ will therefore be the best ratio to take for a range of working from $z = 0$ to $z = 1.6$.

(E). SINGLE LAYER STRANDED WIRE COILS.

18. *Single Layer Stranded Wire Coils.*—If we replace the solid wire in a coil by a number of insulated parallel filaments of the same copper section and connected in parallel, the direct current resistance of the coil will be unaltered, but for alternating currents the distribution of current between the filaments will not be uniform unless the filaments are interwoven in such a way that they traverse similar paths.

As a non-uniform current distribution will cause increased losses, and twisting will produce increased length in the filaments, it is a matter for investigation as to what gain may be expected by using stranded wire coils.

It will be assumed that each strand traces out a helix about the axis of stranding, the angle of which is α . Actually the radius of the helix will vary along a particular strand; but this need not be considered in getting the average result, as we may pass from one strand to its replacing strand, and thus keep at the same distance from the axis throughout the length of the stranded wire. The further assumption that α is constant throughout the section will assure that there are the same number of strands in the same axial length of any layer whatever its distance from the axis.

19. If r is the direct current resistance per unit length of one strand, the skin resistance of each strand in unit axial length of the wire is

$$r_0 \sec \alpha \{1 + F(z)\}$$

where $z^2 = \pi k \omega \delta^2$, δ being the diameter of one strand. Upon this must be superposed the resistances representing the losses due to two fields:—

(a) A field H_1 due to the strands in the same turn of the coil as that in which the

strand in question is situated. This field is normal to the axis of stranding and tangential to the cylinder on which the strand is wound.

(b) A field H_2 due to the remaining turns in the coil. This field will also be assumed to be normal to the axis of stranding, and will have two components $H_2 \cos \theta$ tangential to the cylinder and $H_2 \sin \theta$ normal to the cylinder, θ being the angular position of the element under consideration.

Thus an element of one strand is situated in a nett field whose components tangential and normal to the winding cylinder are $H_1 + H_2 \cos \theta$, $H_2 \sin \theta$. These may further be resolved into components along and at right angles to the direction of the element, viz. :—

$$\begin{aligned} H_A &= (H_1 + H_2 \cos \theta) \sin \alpha, \\ H_B &= (H_1 + H_2 \cos \theta) \cos \alpha, H_2 \sin \theta. \end{aligned}$$

Now, as regards the axial component of the field, it may be shown that if a cylinder is placed with its axis along the direction of an alternating field, the losses in the cylinder are one-half the losses which would occur if the cylinder were placed at right angles to the field.

Therefore the loss in an element $d\lambda$ of one strand due to the fields H_1 , H_2 is by (49)

$$dW = \frac{1}{8} r_0 d\lambda \delta^2 G(z) (H_B^2 + \frac{1}{2} H_A^2).$$

Now, as we pass along one strand, the value of θ increases uniformly as we are rotating relative to the field H_2 . Hence the average loss in one strand is got by replacing H_A^2 , H_B^2 by their mean values throughout a complete cycle of θ ; that is, H_A^2 , H_B^2 are replaced by

$$\sin^2 \alpha (H_1^2 + \frac{1}{2} H_2^2), \quad \cos^2 \alpha (H_1^2 + \frac{1}{2} H_2^2) + \frac{1}{2} H_2^2$$

The loss per unit axial length of the stranded wire is thus

$$W = \frac{1}{8} r_0 \sec \alpha G(z) \delta^2 \{ H_1^2 (1 - \frac{1}{2} \sin^2 \alpha) + H_2^2 (1 - \frac{1}{4} \sin^2 \alpha) \}.$$

(20) Since each strand carries the same current, the value of H_1 at a distance r from the axis is $2Ir/a_0^2$, where I is the whole current and a_0 the over-all radius of the stranded wire, the number of strands being assumed large.

Again the number of strands crossing an annular belt of width dr in the cross-section of the wire is $2rs dr/a_0^2$, s being the whole number of strands. The mean value of H_1^2 throughout the section is therefore

$$8I^2 \int_0^{a_0} r^3 dr/a_0^6 = 2I^2/a_0^2 = 8I^2/d_0^2,$$

in which $d_0 = 2a_0$.

The field H_2 is the same as that for the corresponding solid wire coil, and the mean value of H_2^2 throughout the coil is $4\mu_0 I^2/D^2$, D being the distance apart of consecutive

turns. Using these values in W and adding the skin losses, the following formula is obtained for the effective resistance of a stranded wire coil:—

$$R' = R_0 \sec \alpha \left[1 + F(z) + s^2 \delta^2 G(z) \left\{ \frac{2}{d_0^2} (1 - \frac{1}{2} \sin^2 \alpha) + \frac{u_n}{D^2} (1 - \frac{1}{4} \sin^2 \alpha) \right\} \right] \quad (84)$$

in which

- R_0 ($\equiv r_0/s$) is the direct current resistance per unit length of the equivalent solid wire,
- s = No. of strands,
- δ = diameter of each strand,
- α = angle of twist,
- d_0 = over-all diameter of stranded wire,
- D = separation of turns in coil;

while in the calculation of F and G , $z^2 = \pi k \omega \delta^2$.

(21) If the twist is so small that $\sin \alpha = \alpha$, (84) becomes

$$R' = R_0 \left[1 + F + s^2 \delta^2 G \left(\frac{2}{d_0^2} + \frac{u_n}{D^2} \right) + \frac{1}{2} \alpha^2 \left\{ 1 + F + \frac{1}{2} u_n G \frac{s^2 \delta^2}{D^2} \right\} \right] \quad (85)$$

The correction due to the twist will therefore be less than 1 per cent. if $\alpha < 0.14$ radian (8°). In determining the most efficient coil only the main term in (85) need be considered.

(22) The quantities fixed will be taken to be the length of wire, the number of strands, and the diameter of each strand. Under these conditions it is clear from (85) that the best value of d_0 is $d_0 = D$, as adjustment of d_0 will have a negligible effect on the inductance. Then

$$R' = R_0 \left[1 + F + \frac{s^2 \delta^2}{D^2} G(2 + u_n) \right] \quad (86)$$

The best value of D and shape of coil then follow by a method identical with that for the solid wire coil, except that $s\delta$ replaces d , and $2 + u_n$ replaces u_n .

The method gives as the conditions for the best time-constant

$$a/b = 1.5 \quad (87)$$

$$(D/s\delta)^2 = 17.76G/(1 + F) \quad (88)$$

and the value of the time-constant is then

$$R'/L = 1.111\gamma(z) \sqrt{f\rho/l s \delta} \quad (89)$$

$\gamma(z)$ being calculated from the diameter of a single strand using (81A).

(23) *Limits of Application.*—The quantity $s\delta^2$ is the diameter of solid copper wire having the same section as the copper section of the stranded wire, so that if it were possible to pack the circle D entirely with copper, $D^2/s\delta^2$ could never fall below unity. Actually the limit is greater than unity, partly because the wire is circular, but also

because of twisting and the need for symmetrical distribution. Thus if three wires each of diameter δ are arranged so that their centres form an equilateral triangle, the diameter of the cylinder in which these wires can be twisted is

$$(1 + 2\sqrt{3}) n\delta = 2.155n\delta,$$

in which n is a factor greater than unity, introduced to allow for insulation between contiguous wires. Denoting $D^2/s\delta^2$ by μ^2 the value of μ for the three-wire system is $1.244n$. If three three-wire systems are twisted, then, assuming rigidity, the over-all diameter of the resulting nine-wire system is $(2.155)^2 n\delta$, and the value of μ is $(1.244)^2 n$.

Generally, if the operation is repeated p times, the result is a 3^p system for which

$$D = (2.155)^p n\delta, \quad \mu = (1.244)^p n.$$

Applying this result to (88), it is seen that, as $G/(1 + F)$ increases with frequency, there is a lower limit of frequency below which the conditions may not be satisfied. If we depart from the condition of best internal space the resulting increase in R/L follows a law similar to that for solid wire. If we allow a 10 per cent. variation, the actual value of μ may range between $0.63 \mu_0$ and $1.75 \mu_0$ where μ_0 is the ideal value of μ , and this may be used to extend the lower limit of the range of application. At the higher frequencies, although $G/1 + F$ tends to the finite value $1/2$, the spacing required is so large as to give unpractical coils.

If we set as practical limits to n the values 1.1 and 3.3, and allow a 10 per cent. variation, the wave-length limits for copper wire of the usual gauges used in stranding are given in the following table:—

TABLE giving Limits of Application of Formulæ (87), (88), (89).

λ = Wave-length in metres.

Wire No. S.W.G.	=	42	40	38	36
No. of strands :					
3	$\left\{ \begin{array}{l} \lambda = \\ \lambda = \end{array} \right.$	0	0	0	0
		430	630	960	1570
9	$\left\{ \begin{array}{l} \lambda = \\ \lambda = \end{array} \right.$	10	10	20	30
		600	850	1330	2150
27	$\left\{ \begin{array}{l} \lambda = \\ \lambda = \end{array} \right.$	80	120	180	290
		900	1290	2030	3200
81	$\left\{ \begin{array}{l} \lambda = \\ \lambda = \end{array} \right.$	140	200	300	500
		1270	1800	2800	4500

The shorter wave-length assumes $n = 3.3$ and the longer wave-length assumes $n = 1.1$. If we introduce a third system with $n = 2.2$ for the mid-regions, a choice of one or other of these systems will enable the time-constant (89) to be secured to within 10 per cent. throughout the range of the table. The table shows clearly the transference of the applicability of the results to the regions of lower frequency as the stranding becomes finer. The observed inferiority of stranded wire coils at short wave-lengths is thus due to lack of internal spacing at these wave-lengths.

(24) *Comparison of Stranded Wire Coils with Solid Wire Coils.*—Assuming both coils to have the same length of wire, the same total copper section and wound to give the best *time constants*, the ratio of the time-constants is by (81A) and (89),

$$\tau'/\tau = 1.111\gamma(z)/s^4\gamma(s^2z), \dots \dots \dots (90)$$

since $s\delta^2 = d^2$ and z is proportional to d .

In (90) $\tau' = R'/L$ for the stranded wire coil and τ that for the solid wire. Now, the ratio $\gamma(z)/\gamma(s^2z)$ lies between 1 and 2² for all possible values of z and s , so that the formula

$$\tau'/\tau \doteq 1.2/s^4 \dots \dots \dots (91)$$

may be taken as comparing the two cases.

For the 3-system we have therefore

s	—	3	9	27	81
τ'/τ	—	0.91	0.69	0.53	0.40

when the same length of wire is used in both coils.

If coils of equal *inductance* are compared, the conditions are different, as the spacing for stranded wire coils is not the same as that for solid wire coils. In fact, throughout the range for which spacing is advantageous,

$$D \doteq 2d$$

for the solid wire coils, and for stranded wire coils on the 3-system having the same copper section,

$$D = (1.244)^n nd.$$

For coils of the same shape and of radius a , the inductance L is proportional to a^3/D^2 and the length of wire l is proportional to a^2/D . Hence, to keep the *inductance* constant, l^3/D must be constant. We have then

$$l \propto D^{1/3}, \quad a \propto D^{2/3},$$

and from (89)

$$R'/L \propto D^{-1/3}.$$

Thus, if accented letters refer to the stranded wire coils, the relative values for equal inductance are

$$(\ell'/\ell)^3 = (a'/a)^{3/2} = (1.244)^p n/2$$

and

$$\tau'/\tau \doteq 1.2/s^{1/2} \{(1.244)^p n/2\}^{1/6},$$

in the extreme case, when $s = 81$, $p = 4$, $n = 3 \cdot 3$, $\ell' = 1.58\ell$, $a' = 2.5a$, $\tau'/\tau = 0.32$.

(25) *Modification of Formulæ when Strands are few in number.*—The value of the field H_1 was calculated on the assumption of a large number of strands, giving $H_1^2 = 8I^2/d_0^2$. If there are two touching strands each of diameter δ , the field acting on one strand due to a current $I/2$ in the other is I/δ , and d_0 (the diameter of the circumscribing circle) is 2δ . Hence, for this case, $H_1^2 = 4I^2/d_0^2$. The factor $2/d_0^2$ in the formula (84) must therefore be replaced by $1/d_0^2$. With three strands whose centres form an equilateral triangle of side δ , $H_1^2 = 4I^2/3\delta^2$ and $d_0 = 2.155\delta$, so that $2/d_0^2$ is replaced by $1.55/d_0^2$.

With four strands in square order, the centres form a square of side δ , $H_1^2 = 9I^2/8\delta^2$ and $d_0 = 2.414\delta$; $2/d_0^2$ is replaced by $1.65/d_0^2$. These new values react on the conditions (87), (88), (89); the "shape" condition (87) is practically unaltered. The "spacing" condition (88) gives slightly too high a value for D , viz. :—

$S = 2$	3	4
$D/D_0 = 0.90$	0.96	0.97

where D_0 is the value calculated by (88) and D the true value.

The factor 1.111 in (89) must be replaced by 1.054, 1.078, 1.084 when $s = 2, 3, 4$. These differences are small, so that the theory may be safely applied even when the strands are few.

(F). MANY-LAYERED COILS.

(26) *Coils of Many Layers.*—Let the winding section of the coil be $b \times c$. Let c/b be small, and let b be small compared with the radius of the coil. Let there be m layers in the depth c and n turns per layer.

The field at any point in the section will have two components H_b and H_c parallel to b and c respectively, which will act independently in producing eddy-current losses.

As regards H_b , the field acting on a single wire is the same as that for a single layer coil for which $D = b/mn$. Thus the added resistance due to the action of H_c is

$$\frac{1}{3}\pi^2 RG(z) (mnd/b)^2,$$

d being the diameter of the wire and R the direct current resistance of the coil.

As regards H_b , each layer behaves as a current sheet having current density nI/b .

In the immediate neighbourhood of the sheet the component of the field parallel to the sheet is

$$h = 2\pi nI/b$$

and reverses its direction as we pass through the sheet.

Assuming h to be constant throughout the depth c , the field acting on the first layer is $(m - 1)h$, on the second layer $(m - 3)h$, and generally on the r^{th} layer $(m - 2r - 1)h$.

The mean square value for all the layers is therefore

$$h^2 \{(m-1)^2 + (m-3)^2 + (m-5)^2 + \dots\} / m = (m^2 - 1) h^2 / 3 = \frac{4}{3} \pi^2 (m^2 - 1) (nI/b)^2.$$

Upon applying this result to (49) it follows that the added resistance due to the action of H_c is

$$\frac{1}{3} \pi^2 (m^2 - 1) R G (z) (nd/b)^2.$$

Adding these resistances to the skin resistance, the formula for the resistance of a many-layered coil is

$$R' = R \{1 + F + \frac{1}{3} \pi^2 (2m^2 - 1) (nd/b)^2 G\}. \dots \dots \dots (92)$$

The corresponding formula for a stranded wire coil is obtained by replacing F by $F + 2s^2 \delta^2 G / d_0^2$ and d by $s\delta$. F and G in this case are calculated, using the diameter of a single strand.

Assuming that the correction for curvature for the many-layered coil is of the same form as that for the single layer coil, the following formula includes all the previous formulæ—

$$R' = R (1 + F + MG) \dots \dots \dots (93)$$

in which for solid wire coils

$$M = u_n (2m^2 - 1) (nd/b)^2 \dots \dots \dots (94)$$

and for stranded wire coils

$$M = 2 (s \delta / d_0)^2 + u_n (2m^2 - 1) (ns \delta / b^2)^2. \dots \dots \dots (95)$$

(27) *Best Conditions for Many-layered Coils.*—If different coils are wound with the same length and diameter of wire on the same shape of frame, and with the same spacing between the wires but with different radii, then the inductance will vary as m^3 while the resistance will be of the form

$$\alpha + \beta m^2,$$

in which

$$\alpha = R \{1 + F - u_n (nd/b)^2 G\}, \quad \beta = 2u_n (nd/b)^2 G.$$

At low frequencies F and G are negligibly small, so that increasing the number of layers will always improve the time-constant. At high frequencies the best time-constant is obtained when

$$\alpha = 3\beta m^2.$$

which gives

$$6m^2 + 1 = (1 + F)/Gu_n (nd/b)^2. \dots \dots \dots (96)$$

Assuming the condition such as to make m large,

$$6 (mnd/b)^2 \doteq (1 + F)/u_n G, \dots \dots \dots (97)$$

an expression determining the total number of turns ($N = m \times n$). With this value of N we obtain from

$$L = 4\pi N^2 \alpha X \left(\frac{\alpha}{b}\right), \quad l = 2\pi \alpha N,$$

the relation

$$L^2 = \frac{2l^3}{\pi d} \frac{b}{\alpha} X^2 \left(\frac{1 + F}{6u_n G}\right)^{\frac{1}{2}}$$

for the inductance of the coil, and this leads to the same value for a/b as for single-layer coils, viz. :—

$$a/b = 1.6, \quad u_n = 3.87. \dots \dots \dots (98)$$

When both conditions are satisfied

$$R'/L = 1.187\gamma(z) \sqrt{f\rho/l d}. \dots \dots \dots (99)$$

When $z > 1$, condition (97) with $u_n = 3.87$ shows that $(mnd/b)^2 < 3$; and since nd/b is of the order unity, m will not exceed 2. Many-layered coils are therefore only of advantage when $z < 1$. When this is the case, $G = z^4/64$ and F is negligible.

Condition (97) may then be written, when $a/b = 1.6$,

$$Nd/b = 1.66/z^2;$$

or, expressing z in terms of wave-length and diameter, and assuming the wire to be of copper of resistivity 1600 C.G.S. units,

$$N = 2.8 \times 10^{-4} \lambda \alpha / d^3, \dots \dots \dots (100)$$

λ being the wave-length in metres, a the coil radius in centimetres, and d the diameter of the wire in millimetres.

For stranded wire coils of the same total copper section the conditions are

$$a/b = 1.6$$

while

$$N = 2.8 \times 10^{-4} \lambda \alpha \sqrt{s} / d^3, \dots \dots \dots (101)$$

$$R'/L = 1.187\gamma(z) \sqrt{f\rho/l s \delta}, \dots \dots \dots (102)$$

d being the diameter of the equivalent solid copper, δ the diameter of one strand, s the number of strands, and z is calculated from the diameter of a single strand.

Thus the gain in time-constant by using stranded wire is $1/s^4$; but, in addition, a larger number of turns, and therefore an increased inductance, may be obtained with stranded wire, while maintaining the best conditions.

(28) *Design of Coils of Large Inductance.*—If a coil of large inductance is required to have minimum effective resistance at a specified wave-length, the conditions

$$a/b = 1.6,$$

$$N = 2.8 \times 10^{-4} \lambda \alpha \sqrt{s/d^3}, \quad \dots \dots \dots (A)$$

together with the formula for the inductance

$$L = 25.5 N^2 \alpha, \quad \dots \dots \dots (B)$$

determine the radius, shape and number of turns for a given diameter of wire.

Usually these coils are required to resonate with a condenser of given capacity. In this case, if C is the resonating capacity in micro-microfarads,

$$\lambda^2 = 3.55 \times 10^{-3} LC. \quad \dots \dots \dots (C)$$

Eliminating L and λ^2 between (A), (B), (C), we find

$$\alpha^3 = 1.4 \times 10^8 d^6 s C,$$

a relation independent of the number of turns. Thus, whatever inductance is used, the coils must all have the same radius if wound with the same type of wire. In illustration, let the wire consist of nine strands, each of diameter 0.2 mm., and let the resonating capacity be 1,000 $\mu\mu$ F.

Then

$$s = 9, \quad d = \sqrt{s} \delta = 0.6,$$

from which

$$a \doteq 9 \text{ cm.}$$

Thus, if $L = 20mh$, $N = 297$. As the winding length $b = 5.6$ cm., this could be arranged by having 6 layers of 50 turns each. To avoid large self-capacities the winding should be "sliced."



IV. *The Influence of Satellites upon the Form of Saturn's Ring.*

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§ 1. *Introduction.*

IN his "Adams' Prize Essay" * for the year 1856, MAXWELL showed that the rings of the planet Saturn could only be stable for small disturbances on the theory that they were composed of meteorites sufficiently small. This has been confirmed since by spectroscopic evidence and is now generally accepted. In continuance of the same idea, the various divisions of the rings have been accounted for by presuming that, in those positions where a single particle moving in a circular orbit about the planet would have a period simply commensurate with that of one of the nearer satellites of Saturn, instability would result. This idea has been fully emphasized recently by LOWELL. † His observations at Flagstaff have disclosed a large number of additional divisions in the rings (see Appendix to this paper). They have the appearance of fine lines traced on the surface of the rings. In each case LOWELL is able to show that the divisions occur at intervals of periods commensurable with that of satellite Minus. The periods have the ratios such as $\frac{3}{5}$, $\frac{1}{11}$, $\frac{1}{2}$, $\frac{1}{11}$, &c. LOWELL has stated the argument for this view in 'Bulletin,' 32, p. 189. If the action of one body upon another revolving about a third be examined by the method of the variation of arbitrary constants, in the expressions for the periodic inequalities in the radius vector and the longitude, there appear terms of the type $[C/(pn - qn')] \cos \{(pn - qn')t + Q\}$, where n and n' are the mean motions of the perturbing and perturbed bodies, p and q are integers, and the remaining quantities are constants. It is clear that when the ratio n/n' is approximately equal to q/p , then the inequality will become very large.

We may take a satellite of Saturn as one of the bodies and one of the particles forming the ring as the other; if $n/n' = q/p$, approximately, then the particle will

* MAXWELL'S 'Collected Works,' I., p. 288.

† LOWELL, 'Observatory Bulletin,' No. 66.

depart considerably from its unperturbed path and collision with other particles will result. In this way the divisions in the ring have been explained.

Some doubt has been cast upon this theory, and it has been shown* that even when n and n' are commensurable, a closer examination of the motion leads to the conclusion that the denominator will not vanish.

It is also noticeable that this explanation takes no account of the attraction of the numerous particles upon one another, which may be considerable.

A re-examination of the matter is made in the present paper. As the satellites of Saturn are all approximately in the same plane as the ring, the problem is formulated in two dimensions only. The satellite is assumed to follow an unperturbed circular orbit, and the problem reduces to a slight variation of the "restricted problem" of three bodies. We shall consider the effect of this satellite upon a number of particles forming a single ring round the planet, subject to their mutual attraction as well as that of the satellite and of Saturn. The actual Saturnian rings are supposed to be composed of a number of such rings arranged concentrically. These will have some effect one upon the other, but, for the present, this effect is disregarded.

In his paper, MAXWELL considered the single ring of particles only. He found that the equations of motion could be satisfied by assuming that the particles rotated round the primary in a circle with suitable angular motion. He then examined the effect of a small arbitrary disturbance upon them, and his results show that the disturbances would remain small if the masses of the particles were sufficiently small. That is, the ring would be "ordinarily" stable.

In the present paper the plan is different. The disturbance of the ring of particles by the satellite is examined, with a view to determining under what conditions the departure from a certain fixed circle will be large. It is clear that if the departures do become large, collisions with adjacent rings of particles will result, and the particles will leave the vicinity of the original circle irrevocably. In this case a division in the ring will result. It is with this meaning that the terms stability and instability have been used in the paper. But, as will be pointed out again in its proper place, the orbits in which the departure from the circular form does not become great with increase of time may yet become "ordinarily" unstable if further small arbitrary displacements are imposed upon them.

The results of this paper will therefore indicate some, but not necessarily all, of the positions of divisions in the rings due to instability of whatever kind.

In §§ 2 to 4 an analytical theory is fully worked out on the supposition of equal particles in each ring. In § 5 it is shown how amendments may be introduced to cover the case of unequal particles. The application to the Saturnian system is given in § 6, and the last paragraph summarises the results obtained.

* TISSERAND, 'Méc. Céleste,' vol. iv., p. 420.

§ 2. *Formation of the Equations.*

Let M be the mass of the primary and m' the mass of the principal satellite which is assumed to describe an unperturbed circle round the primary. Take the origin at M . Let there be n particles forming a ring round the primary, subject to attraction from M , m' , and one another, and let the mass and co-ordinates at time t of particle λ be $m_\lambda, r_\lambda, \theta_\lambda$. If the co-ordinates of m' at the same time are r', θ' , then the motion of particle λ will be produced by forces which are the derivatives of the function

$$\mathbf{F} \equiv \frac{M + m_\lambda}{r_\lambda} + \frac{m'}{\Delta_\lambda} - \frac{m'r_\lambda}{r'^2} \cos(\theta' - \theta_\lambda) + \sum_\mu \frac{m_\mu}{D_{\lambda\mu}} - \sum_\mu \frac{m_\mu r'_\lambda}{r'_\mu{}^2} \cos(\theta_\mu - \theta_\lambda);$$

where

$$\Delta_\lambda{}^2 = r'^2 + r_\lambda{}^2 - 2r'r_\lambda \cos(\theta' - \theta_\lambda).$$

and

$$D_{\lambda\mu}{}^2 = r_\mu{}^2 + r_\lambda{}^2 - 2r_\mu r_\lambda \cos(\theta_\mu - \theta_\lambda).$$

The equations of motion of m_λ are then

$$\left. \begin{aligned} \frac{d^2 r_\lambda}{dt^2} - r_\lambda \left(\frac{d\theta_\lambda}{dt} \right)^2 &= \frac{\partial \mathbf{F}}{\partial r_\lambda}, \\ \frac{1}{r_\lambda} \frac{d}{dt} (r_\lambda{}^2 \dot{\theta}_\lambda) &= \frac{1}{r_\lambda} \frac{\partial \mathbf{F}}{\partial \theta_\lambda}. \end{aligned} \right\} \dots \dots \dots (1)$$

As we are assuming that m' describes an unperturbed circle,

$$r' = a' \quad \text{and} \quad \theta' = \omega't + \epsilon',$$

where $\omega'^2 a'^3 = M + m' = M$, with sufficient approximation.

Let us assume now that the remaining particles are moving in the vicinity of the vertices of a regular polygon of radius a . Then we may put

$$\begin{aligned} r_\lambda &= a + \rho_\lambda, \\ \theta_\lambda &= \omega t + \epsilon + \lambda \cdot 2\pi/n + \sigma_\lambda, \end{aligned}$$

for all values of λ from 1 to n , where ρ and σ are assumed small, so that squares, products, and higher powers of them and their first derivatives with regard to the time may be neglected.

The equations (1) now reduce to

$$\left. \begin{aligned} \frac{d^2}{dt^2} \rho_\lambda - 2\alpha\omega \frac{d}{dt} \sigma_\lambda - \omega^2 \rho_\lambda - \alpha\omega^2 &= \left(\frac{\partial \mathbf{F}}{\partial r_\lambda} \right)_0 + \sum_\mu \rho_\mu \left(\frac{\partial^2 \mathbf{F}}{\partial r_\mu \partial r_\lambda} \right)_0 \\ &+ \sum_\mu \sigma_\mu \left(\frac{\partial^2 \mathbf{F}}{\partial \theta_\mu \partial r_\lambda} \right)_0, \\ \alpha \frac{d^2 \sigma_\lambda}{dt^2} + 2\omega \frac{d\rho_\lambda}{dt} &= \left(\frac{1}{r_\lambda} \frac{\partial \mathbf{F}}{\partial \theta_\lambda} \right)_0 + \sum_\mu \rho_\mu \left(\frac{\partial^2 \mathbf{F}}{r_\lambda \partial r_\mu \partial \theta_\lambda} \right)_0 \\ &+ \sum_\mu \sigma_\mu \left(\frac{\partial^2 \mathbf{F}}{r_\lambda \partial \theta_\mu \partial \theta_\lambda} \right)_0. \end{aligned} \right\} \dots \dots (2)$$

In order to determine the derivatives, we write in the formula for Δ_λ , $\phi_\lambda = \theta' - \theta_\lambda$, and $\alpha = r_\lambda/r'$. Then

$$\begin{aligned} \Delta_\lambda^{-1} &= \{1 + \alpha^2 - 2\alpha \cos \phi_\lambda\}^{-\frac{1}{2}} \div r' \\ &= \left\{ \frac{1}{2} b_0 + b_1 \cos \phi + \dots + b_i \cos i\phi + \dots \right\} \div r' \end{aligned}$$

by FOURIER'S series.

This series will be taken as absolutely and uniformly convergent.

We find then

$$\begin{aligned} \frac{\partial \mathbf{F}}{\partial r_\lambda} &= -\frac{\mathbf{M} + m_\lambda}{r_\lambda^2} + \frac{m'}{r'^2} \left\{ \frac{1}{2} \frac{\partial b_0}{\partial \alpha} + \dots + i b_i \cos i\phi + \dots \right\} - \frac{m'}{r'^2} \cos \phi \\ &\quad - \sum_\mu m_\mu \left\{ \frac{r_\lambda - r_\mu \cos(\theta_\mu - \theta_\lambda)}{D_{\lambda\mu}^3} + \frac{\cos(\theta_\mu - \theta_\lambda)}{r_\mu^2} \right\}, \\ \sum_\mu \rho_\mu \frac{\partial^2 \mathbf{F}}{\partial r_\lambda \partial r_\mu} &= \left[2 \frac{\mathbf{M} + m_\lambda}{r_\lambda^3} + \frac{m'}{r'^3} \frac{\partial^2}{\partial \alpha^2} \left(\frac{1}{2} b_0 + \dots + b_i \cos i\phi \dots \right) \right. \\ &\quad \left. - \sum_\mu m_\mu \left\{ \frac{1}{D_{\lambda\mu}^3} - \frac{3(r_\lambda - r_\mu \cos(\theta_\mu - \theta_\lambda))^2}{D_{\lambda\mu}^5} \right\} \right] \rho_\lambda \\ &\quad + \sum_\mu m_\mu \left[\frac{\cos(\theta_\mu - \theta_\lambda)}{D_{\lambda\mu}^3} + \frac{3\{r_\lambda - r_\mu \cos(\theta_\mu - \theta_\lambda)\} \{r_\mu - r_\lambda \cos(\theta_\mu - \theta_\lambda)\}}{D_{\lambda\mu}^5} \right. \\ &\quad \left. + \frac{2 \cos(\theta_\mu - \theta_\lambda)}{r_\mu^3} \right] \rho_\mu, \\ \sum_\mu \sigma_\mu \frac{\partial^2 \mathbf{F}}{\partial \theta_\mu \partial r_\lambda} &= \left[\frac{m'}{r'^2} \frac{\partial}{\partial \alpha} (b_1 \sin \phi + \dots + i b_i \sin i\phi + \dots) - \frac{m'}{r'^2} \sin \phi \right] \sigma_\lambda \\ &\quad - \sum_\mu m_\mu \left[\frac{r_\mu \sin(\theta_\mu - \theta_\lambda)}{D_{\lambda\mu}^3} - \frac{3\{r_\lambda - r_\mu \cos(\theta_\mu - \theta_\lambda)\} r_\lambda r_\mu \sin(\theta_\mu - \theta_\lambda)}{D_{\lambda\mu}^5} \right. \\ &\quad \left. - \frac{\sin(\theta_\mu - \theta_\lambda)}{r_\mu^2} \right] (\sigma_\mu - \sigma_\lambda), \\ \frac{1}{r_\lambda} \frac{\partial \mathbf{F}}{\partial \theta_\lambda} &= \frac{m'}{r' r_\lambda} (\dots + i b_i \sin i\phi + \dots) - \frac{m'}{r'^2} \sin \phi + \sum_\mu m_\mu \left[\frac{r_\mu \sin(\theta_\mu - \theta_\lambda)}{D_{\lambda\mu}^3} - \frac{1}{r_\mu^2} \sin(\theta_\mu - \theta_\lambda) \right], \end{aligned}$$

$$\begin{aligned} \sum_{\mu} \rho_{\mu} \frac{\partial^2 \mathbf{F}}{r_{\lambda} \partial r_{\mu} \partial \theta_{\lambda}} &= \left[\frac{m'}{r'^2 r_{\lambda}} \frac{\partial}{\partial \alpha} (b_1 \sin \phi + \dots + i b_i \sin i\phi + \dots) \right. \\ &\quad - \frac{m'}{r' r_{\lambda}^2} (\dots + i b_i \sin i\phi \dots) \\ &\quad - \sum_{\mu} m_{\mu} \left\{ \frac{3 r_{\mu} \sin (\theta_{\mu} - \theta_{\lambda}) \{ r_{\lambda} - r_{\mu} \cos (\theta_{\mu} - \theta_{\lambda}) \}}{D_{\mu\lambda}^6} \right\} \Big] \rho_{\lambda} \\ &\quad + \sum_{\mu} m_{\mu} \left[\frac{\sin (\theta_{\mu} - \theta_{\lambda})}{D_{\lambda\mu}^3} - \frac{3 r_{\mu} \sin (\theta_{\mu} - \theta_{\lambda}) \{ r_{\mu} - r_{\lambda} \cos (\theta_{\mu} - \theta_{\lambda}) \}}{D_{\lambda\mu}^6} \right. \\ &\quad \left. + \frac{2}{r_{\mu}^3} \sin (\theta_{\mu} - \theta_{\lambda}) \right] \rho_{\mu}, \end{aligned}$$

$$\begin{aligned} \sum_{\mu} \sigma_{\mu} \frac{\partial^2 \mathbf{F}}{r_{\lambda} \partial \theta_{\mu} \partial \theta_{\lambda}} &= \left[- \frac{m'}{r' r_{\lambda}} (\dots i^2 b_i \cos i\phi \dots) + \frac{m'}{r'^2} \cos \phi \right] \sigma_{\lambda} \\ &\quad + \sum_{\mu} m_{\mu} \left[\frac{r_{\mu} \cos (\theta_{\mu} - \theta_{\lambda})}{D_{\mu\lambda}^3} - \frac{3 r_{\mu}^2 r_{\lambda} \sin^2 (\theta_{\mu} - \theta_{\lambda})}{D_{\lambda\mu}^6} \right. \\ &\quad \left. - \frac{1}{r_{\mu}^2} \cos (\theta_{\mu} - \theta_{\lambda}) \right] (\sigma_{\mu} - \sigma_{\lambda}). \end{aligned}$$

In the summations of the right-hand members, μ takes all integral values from 1 to n , except $\mu = \lambda$.

The zero values of these derivatives are obtained by putting

$$\begin{aligned} r' &= \alpha', & r_{\lambda} &= \alpha, \\ \theta' &= \omega' t + \epsilon', & \theta_{\lambda} &= \omega t + \epsilon + \lambda 2\pi/n. \end{aligned}$$

Whence

$$\Delta_{\lambda}^2 = \alpha'^2 + \alpha^2 - 2\alpha\alpha' \cos \phi$$

where ϕ now is

$$(\omega' - \omega) t + \epsilon' - \epsilon - \lambda 2\pi/n;$$

and

$$D_{\lambda\mu} = 2\alpha \sin (\mu - \lambda) \pi/n.$$

Then

$$\begin{aligned} (\partial \mathbf{F} / \partial r_{\lambda})_0 &= - \frac{M + m_{\lambda}}{\alpha^2} + \frac{m'}{\alpha'^2} \left(\frac{1}{2} \frac{db_0}{d\alpha} + \dots + \frac{db_i}{d\alpha} \cos i\phi + \dots \right) - \frac{m'}{\alpha'^2} \cos \phi \\ &\quad - \sum_{\mu} m_{\mu} \left\{ \frac{1}{4\alpha^2 \sin (\mu - \lambda) \pi/n} + \frac{1}{\alpha^2} \cos (\mu - \lambda) 2\pi/n \right\}, \end{aligned}$$

$$\begin{aligned} \sum_{\mu} \rho_{\mu} \left(\frac{\partial^2 \mathbf{F}}{\partial r_{\lambda} \partial r_{\mu}} \right)_0 &= \left[\frac{2(M + m_{\lambda})}{\alpha^3} + \frac{m'}{\alpha'^3} \frac{\partial^2}{\partial \alpha^2} \left(\frac{1}{2} b_0 + \dots + b_i \cos i\phi + \dots \right) \right. \\ &\quad \left. - \sum_{\mu} m_{\mu} \left\{ \frac{1}{8\alpha^3 \sin^3 (\mu - \lambda) \pi/n} - \frac{3}{8\alpha^3 \sin (\mu - \lambda) \pi/n} \right\} \right] \rho_{\lambda} \\ &\quad + \sum_{\mu} m_{\mu} \left[\frac{\cos (\mu - \lambda) 2\pi/n}{8\alpha^3 \sin^3 (\mu - \lambda) \pi/n} + \frac{3}{8\alpha^3 \sin (\mu - \lambda) \pi/n} + \frac{2 \cos (\mu - \lambda) 2\pi/n}{\alpha^3} \right] \rho_{\mu}, \end{aligned}$$

$$\begin{aligned} \sum_{\mu} \sigma_{\mu} \left(\frac{\partial^2 \mathbf{F}}{\partial \theta_{\mu} \partial r_{\lambda}} \right) &= \left[\frac{m'}{a'^2} \frac{\partial}{\partial \alpha} (b_1 \sin \phi + \dots + ib_i \sin i\phi \dots) - \frac{m'}{a'^2} \sin \phi \right] \sigma_{\lambda} \\ &\quad + \sum_{\mu} m_{\mu} \left[\frac{\delta \cos (\mu - \lambda) \pi / \mu}{8a'^2 \sin^2 (\mu - \lambda) \pi / \mu} + \frac{\sin (\mu - \lambda) 2\pi / \mu}{a'^2} \right] (\sigma_{\mu} - \sigma_{\lambda}), \\ \left(\frac{1}{r_{\lambda}} \frac{\partial \mathbf{F}}{\partial \theta_{\lambda}} \right) &= \frac{m'}{a'a} (\dots + ib_i \sin i\phi \dots) - \frac{m'}{a'^2} \sin \phi \\ &\quad + \sum_{\mu} m_{\mu} \left[\frac{\cos (\mu - \lambda) \pi' / \mu}{4a'^2 \sin^2 (\mu - \lambda) \pi / \mu} - \frac{1}{a'^2} \sin (\mu - \lambda) 2\pi / \mu \right], \\ \sum_{\nu} \rho_{\nu} \left(\frac{\partial^2 \mathbf{F}}{r_{\lambda} \partial r_{\nu} \partial \theta_{\nu}} \right) &= \left[\frac{m'}{a'^2 a} \frac{\partial}{\partial \alpha} (b_1 \sin \phi + \dots + ib_i \sin i\phi \dots) \right. \\ &\quad \left. - \frac{m'}{a'a^2} (b_1 \sin \phi + \dots + ib_i \sin i\phi \dots) - \sum_{\mu} m_{\mu} \left\{ \frac{3 \cos (\mu - \lambda) \pi / \mu}{8a'^3 \sin^2 (\mu - \lambda) \pi / \mu} \right\} \right] \rho_{\lambda} \\ &\quad + \sum_{\mu} m_{\mu} \left[-\frac{\delta \cos (\mu - \lambda) \pi / \mu}{8a'^3 \sin^2 (\mu - \lambda) \pi / \mu} + \frac{2}{a'^3} \sin (\mu - \lambda) 2\pi / \mu \right] \rho_{\mu}, \\ \sum_{\mu} \sigma_{\mu} \left(\frac{\partial^2 \mathbf{F}}{r_{\lambda} \partial \theta_{\mu} \partial \theta_{\lambda}} \right) &= \left[-\frac{m'}{a'a} (b_1 \cos \phi + \dots + i^2 b_i \cos i\phi \dots) + \frac{m'}{a'^2} \cos \phi \right] \sigma_{\lambda} \\ &\quad + \sum_{\mu} m_{\mu} \left[\frac{\cos (\mu - \lambda) 2\pi / \mu}{8a'^2 \sin^3 (\mu - \lambda) \pi / \mu} - \frac{3 \sin^2 (\mu - \lambda) 2\pi / \mu}{32a'^2 \sin^5 (\mu - \lambda) \pi / \mu} \right. \\ &\quad \left. - \frac{1}{a'^2} \cos (\mu - \lambda) 2\pi / \mu \right] (\sigma_{\mu} - \sigma_{\lambda}). \end{aligned}$$

In the summations of the right-hand members, μ takes all integral values from 1 to n , except $\mu = \lambda$.

Next assume that all the small particles forming the ring are equal to one another. That is $m_{\lambda} = m$.

Further, let $\rho_{\lambda+1} = \beta \rho_{\lambda}$, for all values of λ .

Then

$$\rho_{\lambda} = \rho_{\lambda+n} = \beta^n \rho_{\lambda},$$

whence

$$\beta^n = 1;$$

or

$$\beta = \cos \frac{2s\pi}{n} + i \sin \frac{2s\pi}{n}, \text{ where } i = \sqrt{-1},$$

and s takes all integral values from 0 to $n-1$.

The quantities appearing under the signs of summation are then :

$$\sum_{\mu} \frac{m}{a'^2} \cos (\mu - \lambda) 2\pi / \mu = -\frac{m}{a'^2};$$

$$\sum_{\mu} \frac{m}{a'^2} \sin (\mu - \lambda) 2\pi / \mu = 0;$$

$$\sum_{\mu} \frac{m}{4a'^2 \sin (\mu - \lambda) \pi / \mu} = \frac{m}{a'^2} \mathbf{K};$$

$$\begin{aligned} & \sum_{\mu} \frac{m}{8\alpha^3} \left[\frac{\cos (\mu-\lambda) 2\pi/n \{ \cos s(\mu-\lambda) 2\pi/n + i \sin s(\mu-\lambda) 2\pi/n \} - 1}{\sin^3 (\mu-\lambda) \pi/n} \right] \\ & \quad + \sum_{\mu} \frac{3m}{8\alpha^3} \left[\frac{\cos s(\mu-\lambda) 2\pi/n + i \sin s(\mu-\lambda) 2\pi/n + 1}{\sin (\mu-\lambda) \pi/n} \right] \\ & = \sum_{\mu} \frac{m}{\alpha^3} \left\{ \frac{1}{2} \frac{\cos^2 s(\mu-\lambda) \pi/n}{\sin (\mu-\lambda) \pi/n} - \frac{1}{4} \frac{\sin^2 s(\mu-\lambda) \pi/n \cos^2 (\mu-\lambda) \pi/n}{\sin^3 (\mu-\lambda) \pi/n} \right\} = -\frac{m}{\alpha^3} L_s; \\ & \sum_{\mu} \frac{m}{8\alpha^2} \frac{\cos (\mu-\lambda) \pi/n}{\sin^2 (\mu-\lambda) \pi/n} \{ \cos s(\mu-\lambda) \pi/n + i \sin s(\mu-\lambda) \pi/n - 1 \} \\ & = i \frac{m}{8\alpha^2} \sum_{\mu} \frac{\cos (\mu-\lambda) \pi/n \sin s(\mu-\lambda) \pi/n}{\sin^2 (\mu-\lambda) \pi/n} = i \frac{m}{\alpha^2} M_s; \\ & \sum_{\mu} \frac{m}{8\alpha^3} \frac{\cos (\mu-\lambda) \pi/n}{\sin^2 (\mu-\lambda) \pi/n} \{ \cos s(\mu-\lambda) \pi/n + i \sin s(\mu-\lambda) \pi/n + 3 \} = i \frac{m}{\alpha^3} M_s; \\ & \sum_{\mu} \frac{m}{8\alpha^2} \left[\frac{\cos (\mu-\lambda) 2\pi/n}{\sin^3 (\mu-\lambda) \pi/n} - \frac{3}{4} \frac{\sin^2 (\mu-\lambda) 2\pi/n}{\sin^3 (\mu-\lambda) \pi/n} \right] \{ \cos s(\mu-\lambda) 2\pi/n + i \sin s(\mu-\lambda) 2\pi/n - 1 \} \\ & = \frac{m}{\alpha^2} \sum_{\mu} \left\{ \frac{1}{2} \frac{\sin^2 s(\mu-\lambda) \pi/n \cos^2 (\mu-\lambda) \pi/n}{\sin^3 (\mu-\lambda) \pi/n} + \frac{1}{4} \frac{\sin^2 s(\mu-\lambda) \pi/n}{\sin (\mu-\lambda) \pi/n} \right\} = \frac{m}{\alpha^2} N_s. \end{aligned}$$

The quantities K , L_s , M_s , N_s can readily be found by direct summation when n , the number of particles, and s are known.

In re-writing the differential equations (2), we may now omit the suffixes of ρ and σ . Change the independent variable from t to $\phi = (\omega' - \omega)t + \epsilon' - \epsilon - \lambda \cdot 2\pi/n$. Also put $m/M = v$, $m'/M = v'$, $\omega'/\omega = \kappa'$, $(\kappa' - 1)^{-1} = \kappa$, and, to secure homogeneity, replace ρ by $\alpha\rho$. Let us further assume that $\omega^2 a^3 = M$, and $\omega'^2 a'^3 = M$ (the latter holds very approximately when m' describes a circle), so that we have $a/a' = (\omega'/\omega)^3$.

The differential equations then become

$$\begin{aligned} \frac{d^2\rho}{d\phi^2} - 2\kappa \frac{d\sigma}{d\phi} &= \kappa^2 \kappa'^3 v' \frac{\partial}{\partial \alpha} \left(\frac{1}{2} b_0 + \dots + b_1 \cos i\phi + \dots \right) - K \kappa^2 v - \kappa^2 \kappa'^3 v' \cos \phi \\ & \quad + \left[3\kappa^2 + \kappa^2 \kappa'^2 v' \frac{\partial^2}{\partial \alpha^2} \left(\frac{1}{2} b_0 + \dots + b_1 \cos i\phi + \dots \right) - \kappa^2 v L_s \right] \rho \\ & \quad + \left[\kappa^2 \kappa'^3 v' \frac{\partial}{\partial \alpha} \left(b_1 \sin \phi + \dots + i b_1 \sin i\phi + \dots - \sin \phi \right) + \kappa^2 v M_s \right] \sigma, \end{aligned}$$

and

$$\begin{aligned} \frac{d^2\sigma}{d\phi^2} + 2\kappa \frac{d\rho}{d\phi} &= v' \kappa^2 \kappa'^3 \left(b_1 \sin \phi + \dots + i b_1 \sin i\phi + \dots \right) - v' \kappa^2 \kappa'^3 \sin \phi \\ & \quad + \left[v' \kappa^2 \kappa'^3 \left(\dots + i \frac{\partial b_1}{\partial \alpha} \sin i\phi + \dots \right) - v' \kappa^2 \kappa'^3 \left(\dots + i b_1 \sin i\phi + \dots \right) \right. \\ & \quad \left. - v \kappa^2 L_s \right] \rho \\ & \quad + \left[-v' \kappa^2 \kappa'^3 \left(\dots + i^2 b_1 \cos i\phi + \dots \right) + v' \kappa^2 \kappa'^3 \cos \phi + v \kappa^2 N_s \right] \sigma. \end{aligned} \tag{3}$$

The equations (2) may be replaced, under the suppositions made, by equations (3). Equations (2) form a system of n pairs of linear equations of the second order. The complete integral will therefore involve $4n$ arbitrary constants. The system of equations (3) will give the same result, for the solution of (3) will be a function of s involving four arbitrary constants. By giving s its n values, $0, 1, 2, \dots, (n-1)$, we arrive at the complete integral involving $4n$ arbitrary constants.

Now it has been shown by TISSERAND* that for large values of n , whatever the value of s may be, the limiting value of L_s is $0.0194n^3$, $N_s = 2L_s$ and $M_s = 0$.

These values largely simplify the discussion of the stability of the system.

Lastly the equations (3) may be written, for convenience, in the form

$$\left. \begin{aligned} \rho'' - 2\kappa\sigma' + (\Theta_{1,0} + \Theta_{1,1} \cos \phi + \dots + \Theta_{1,r} \cos r\phi + \dots) \rho \\ + (\Theta_{2,1} \sin \phi + \Theta_{2,2} \sin 2\phi + \dots + \Theta_{2,r} \sin r\phi + \dots) \sigma \\ = \Theta_{3,0} + \Theta_{3,1} \cos \phi + \dots + \Theta_{3,r} \cos r\phi + \dots \\ \sigma'' + 2\kappa\rho' + (\Theta_{1,1} \sin \phi + \Theta_{1,2} \sin 2\phi + \dots + \Theta_{1,r} \sin r\phi + \dots) \rho \\ + (\Theta_{2,0} + \Theta_{2,1} \cos \phi + \dots + \Theta_{2,r} \cos r\phi + \dots) \sigma \\ = \Theta_{3,1} \sin \phi + \dots + \Theta_{3,r} \sin r\phi + \dots \end{aligned} \right\} \dots \dots \dots (4)$$

The values of the quantities Θ are :

$$\left. \begin{aligned} \Theta_{1,0} &= -3\kappa^2 - \frac{1}{2} \nu' \kappa^2 \kappa'^2 \frac{\partial^2 b_0}{\partial \alpha^2} + \nu \kappa^2 L_s & ; \\ \Theta_{1,r} &= -\nu' \kappa^2 \kappa'^2 \frac{\partial^2 b_r}{\partial \alpha^2} & ; \quad (r \neq 0) \\ \Theta_{2,r} &= -\nu' \kappa^2 \kappa'^{1/2} \nu' \frac{\partial b_r}{\partial \alpha} & ; \\ \Theta_{3,0} &= \frac{1}{2} \nu' \kappa^2 \kappa'^{1/2} \frac{\partial b_0}{\partial \alpha} - \nu \kappa^2 K & ; \\ \Theta_{3,1} &= \nu' \kappa^2 \kappa'^{1/2} \frac{\partial b_1}{\partial \alpha} - \nu' \kappa^2 \kappa'^{1/2} & ; \\ \Theta_{3,r} &= \nu' \kappa^2 \kappa'^{1/2} \frac{\partial b_r}{\partial \alpha} & ; \quad (r \neq 0, 1) \\ \Theta_{4,r} &= -\nu' \kappa^2 \kappa'^{1/2} \nu' \frac{\partial b_r}{\partial \alpha} + \nu' \kappa^2 \kappa'^{1/2} \nu' b_r & ; \\ \Theta_{5,0} &= -\nu \kappa^2 N_s & ; \\ \Theta_{5,r} &= \nu' \kappa^2 \kappa'^{1/2} \nu'^2 b_r & ; \quad (r \neq 0) \\ \Theta_{6,r} &= \nu' \kappa^2 \kappa'^{1/2} \nu' b_r & ; \quad (r \neq 1) \\ \Theta_{6,1} &= \nu' \kappa^2 \kappa'^{1/2} b_1 - \nu' \kappa^2 \kappa'^{1/2} & . \end{aligned} \right\} \dots \dots \dots (5)$$

* 'Méc. Céleste,' vol. ii., p. 184.

The best methods of determining the values of b_r , and its first and second derivatives for known values of α , or a/a' , are given by TISSERAND.* The complete evaluation of a number of these quantities for various ratios, applicable to the solar system, is given by PONTÉCOULANT.† For the purpose of estimating the order of the numerical values of the quantities $\Theta_{r,s}$, we may take the highest ratio α likely to occur as that of the outer edge of the ring to the mean distance of Mimas. This ratio is 0.7461 (see Appendix for data). PONTÉCOULANT gives the values for $\alpha = 0.72333$, which we may use to avoid laborious calculation. If we take $r' = 7 \cdot 10^{-8}$, the value for Mimas, we find

$$\begin{aligned} \Theta_{1,0} &= -20.2590 + 6.7530rL_s; \\ \Theta_{1,1} &= -1.35 \cdot 10^{-6}, & \Theta_{1,2} &= -1.53 \cdot 10^{-6}, & \Theta_{1,3} &= -1.63 \cdot 10^{-6} \dots; \\ \Theta_{2,1} &= -4.07 \cdot 10^{-7}, & \Theta_{2,2} &= -7.39 \cdot 10^{-7}, & \Theta_{2,3} &= -9.34 \cdot 10^{-6} \dots; \\ \Theta_{3,0} &= -1.46 \cdot 10^{-7} - 1.79 \cdot 10^{-7}K, & \Theta_{3,1} &= 4.07 \cdot 10^{-7}, & \Theta_{3,2} &= 3.69 \cdot 10^{-7} \dots, \\ \Theta_{5,0} &= -13.5060rL_s. \end{aligned}$$

It is clear that, compared with $\Theta_{1,0}$, all products and squares of the remaining Θ 's may be neglected.

§ 3. *Solution of the Equations.*

(a) The complementary function.

The equations

$$\left. \begin{aligned} \rho'' - 2\kappa\rho' + \rho\Sigma\Theta_{1,r} \cos r\phi + \sigma\Sigma\Theta_{2,r} \sin r\phi &= 0, \\ \sigma'' + 2\kappa\sigma' + \rho\Sigma\Theta_{3,r} \sin r\phi + \sigma\Sigma\Theta_{5,r} \cos r\phi &= 0. \end{aligned} \right\} \dots \dots \dots (6)$$

belong to the class of homogeneous linear differential equations with periodic coefficients. The integral is known to be the sum of the forms $e^{i\phi}f(\phi)$, where $f(\phi)$ is a periodic function of ϕ with the same period as the coefficients in the equations (6). Equations of this form in one dependent variable have been discussed by WHITTAKER,‡ YOUNG,§ INCE,|| and BAKER.¶ The present solution is a simple extension of the work of these writers.

Let

$$\begin{aligned} \rho &= e^{i\phi}A, \\ \sigma &= e^{i\phi}X, \end{aligned}$$

* 'Méc. Céleste,' vol. i., p. 270, *et seq.*
 † 'Système du Monde,' vol. 3, pp. 353-376.
 ‡ 'Proc. Inter. Congress Math.' vol. 1, 1912; 'Proc. Edin. Math. Soc.' xxxii., p. 76.
 § 'Proc. Edin. Math. Soc.' xxxii., p. 81.
 || 'Monthly Notices R.A.S.' lxxv., 5, p. 436.
 ¶ H. F. BAKER, 'Phil. Trans.' A., vol. 216, p. 129.

where A and X are, as has been said, purely periodic functions of period 2π . On substituting in equations (6) we find

$$\left. \begin{aligned} c^2 A + 2cA' + A'' - 2\kappa(cX + X') + A \sum \Theta_{1,r} \cos r\phi + X \sum \Theta_{2,r} \sin r\phi &= 0 \\ c^2 X + 2cX' + X'' + 2\kappa(cA + A') + X \sum \Theta_{4,r} \sin r\phi + A \sum \Theta_{3,r} \sin r\phi &= 0 \end{aligned} \right\} \quad (7)$$

Let us now assume that A and X can be represented in the most general way by a series of terms in Θ with suitable coefficients, the coefficients being periodic functions of ϕ with period 2π . That is, let

$$\begin{aligned} A &= A_0 \sin(n\phi - \tau) + \sum \sum A_{r,s} \Theta_{r,s} + \sum \sum \sum B_{r,s,p,q} \Theta_{r,s} \Theta_{p,q} + \dots, \\ X &= X_0 \cos(n\phi - \tau) + \sum \sum X_{r,s} \Theta_{r,s} + \sum \sum \sum Y_{r,s,p,q} \Theta_{r,s} \Theta_{p,q} + \dots \end{aligned}$$

In these expressions A_0 and X_0 will be arbitrary constants, n is an arbitrary integer* and τ a parameter which will be defined presently.

We shall assume that the index c is of the form

$$c = \sum \sum c_{r,s} \Theta_{r,s} + \sum \sum \sum d_{r,s,p,q} \Theta_{r,s} \Theta_{p,q} + \dots$$

Then, if we substitute these values in equations (7) and equate to zero those terms which do not involve any Θ except $\Theta_{1,0}$ and $\Theta_{5,0}$, which are large compared with the others, we find

$$\left\{ \begin{aligned} (\Theta_{1,0} - n^2) A_0 + 2\kappa n X_0 &= 0, \\ 2\kappa n A_0 + (\Theta_{5,0} - n^2) X_0 &= 0. \end{aligned} \right\} \quad \dots \dots \dots (8)$$

On eliminating A_0 and X_0 we find

$$(\Theta_{1,0} - n^2)(\Theta_{5,0} - n^2) - 4\kappa^2 n^2 = 0. \quad \dots \dots \dots (9)$$

In general, the given values of $\Theta_{1,0}$ and $\Theta_{5,0}$ will not satisfy the identity (9) for any integral value of n . Let us replace $\Theta_{1,0}$ by $\alpha_{1,0}$, where $\alpha_{1,0}$ is a quantity which satisfies the relation

$$(\alpha_{1,0} - n^2)(\Theta_{5,0} - n^2) - 4\kappa^2 n^2 = 0. \quad \dots \dots \dots (10)$$

For some suitable value of n , it will usually be found that $\alpha_{1,0}$ approximates closely to $\Theta_{1,0}$.

Following the method of WHITTAKER previously referred to, let us now assume that

$$(\Theta_{1,0} - n^2)(\Theta_{5,0} - n^2) - (\alpha_{1,0} - n^2)(\Theta_{5,0} - n^2) = \sum \sum a_{r,s} \Theta_{r,s} + \sum \sum \sum r_{r,s,p,q} \Theta_{r,s} \Theta_{p,q} + \dots,$$

or

$$\Theta_{1,0} = n^2 + \frac{4\kappa^2 n^2}{\Theta_{5,0} - n^2} + \sum \sum a_{r,s} \Theta_{r,s} + \sum \sum \sum b_{r,s,p,q} \Theta_{r,s} \Theta_{p,q} + \dots \quad (11)$$

* The use of n is to be distinguished from a former use where it referred to the number of particles in the ring.

We now substitute the assumed values for A , X , c , $\Theta_{1,0}$ in equations (7) and equate to zero the coefficients of each term in $\Theta_{r,0}$, $\Theta_{r,1}$, $\Theta_{p,q}$, &c. It will be found that the relations (8) are satisfied identically. Two conditions further must be imposed in order that all the unknown coefficients may be determined. These are :

- (i) The term $\cos (u\phi - \tau)$ must not appear in the series for A ;
- (ii) The solutions for A and X must be purely periodic with period 2π .

The condition (i) amounts to a definition of τ , and condition (ii) secures that no part of the exponent shall appear in the periodic series. Further, these conditions determine uniquely the undetermined coefficients in the series for $\Theta_{1,0}$ and c . The work from this point is purely mechanical though long. The following sample sufficiently indicates its character.

On equating to zero the terms involving $\Theta_{1,r}$ we find

$$\left. \begin{aligned} 2c_{1,r}uA_0 \cos (u\phi - \tau) + A''_{1,r} - 2\kappa c_{1,r}X_0 \cos (u\phi - \tau) \\ - 2\kappa X'_{1,r} + \alpha_{1,0}A_{1,r} + \alpha_{1,r}A_0 \sin (u\phi - \tau) + A_0 \cos r\phi \sin (u\phi - \tau) = 0 \\ - 2c_{1,r}uX_0 \sin (u\phi - \tau) + X''_{1,r} + 2\kappa c_{1,r}A_0 \sin (u\phi - \tau) + 2\kappa A'_{1,r} + \Theta_{5,0}X_{1,r} = 0 \end{aligned} \right\} \quad (12)$$

In the case when r is not $2u$ or u , it is clear that

$$c_{1,r} = 0 \quad \text{and} \quad \alpha_{1,r} = 0.$$

Equation (12) then reduces to

$$\left. \begin{aligned} A''_{1,r} - 2\kappa X'_{1,r} + \alpha_{1,0}A_{1,r} + \frac{1}{2}A_0 [\sin \{(u+r)\phi - \tau\} + \sin \{(u-r)\phi - \tau\}] = 0, \\ X''_{1,r} + 2\kappa A'_{1,r} + \Theta_{5,0}X_{1,r} = 0. \end{aligned} \right\} \quad (13)$$

Solving in the usual way we find

$$A_{1,r} = -\frac{A_0u^2 \{ (u+r)^2 - \Theta_{5,0} \} \sin \{ (u+r)\phi - \tau \}}{2r(2u+r) \{ \alpha_{1,0}\Theta_{5,0} - u^2(u+r)^2 \}} + \frac{A_0u^2 \{ (u-r)^2 - \Theta_{5,0} \} \sin \{ (u-r)\phi - \tau \}}{2r(2u-r) \{ \alpha_{1,0}\Theta_{5,0} - u^2(u-r)^2 \}},$$

$$X_{1,r} = -\frac{A_0u^2\kappa(u+r) \cos \{ (u+r)\phi - \tau \}}{r(2u+r) \{ \alpha_{1,0}\Theta_{5,0} - u^2(u+r)^2 \}} + \frac{A_0u^2\kappa(u-r) \cos \{ (u-r)\phi - \tau \}}{r(2u-r) \{ \alpha_{1,0}\Theta_{5,0} - u^2(u-r)^2 \}}.$$

In the special case where $r = u$, we have

$$c_{1,u} = 0, \quad \alpha_{1,u} = 0,$$

and

$$\left. \begin{aligned} A''_{1,u} - 2\kappa X'_{1,u} + \alpha_{1,0}A_{1,u} + \frac{1}{2}A_0 [\sin (2u\phi - \tau) - \sin \tau] = 0, \\ X''_{1,u} + 2\kappa A'_{1,u} + \Theta_{5,0}X_{1,u} = 0. \end{aligned} \right\} \quad (14)$$

From which

$$A_{1,u} = \frac{A_0(4u^2 - \Theta_{5,0}) \sin (2u\phi - \tau)}{6(4u^2 - \alpha_{1,0}\Theta_{5,0})} + \frac{A_0}{2\alpha_{1,0}} \sin \tau,$$

and

$$X_{1,u} = \frac{2}{3} \frac{\kappa u A_0 \cos (2u\phi - \tau)}{4u^2 - \alpha_{1,0}\Theta_{5,0}}.$$

Again, in the special case where $r = 2n$, we find in place of equations (12), the following

$$\left. \begin{aligned} 2c_{1,2n}nA_0 \cos(n\phi - \tau) + A''_{1,2n} - 2\kappa c_{1,2n}X_0 \cos(n\phi - \tau) - 2\kappa X'_{1,2n} \\ + \alpha_{1,0}A_{1,2n} + \alpha_{1,2n}A_0 \sin(n\phi - \tau) + \frac{1}{2}A_0 \{ \sin(3n\phi - \tau) - \sin(n\phi - \tau) \cos 2\tau \\ - \cos(n\phi - \tau) \sin 2\tau \} = 0, \\ -2c_{1,2n}nX_0 \sin(n\phi - \tau) + X''_{1,2n} + 2\kappa c_{1,2n}A_0 \sin(n\phi - \tau) + 2\kappa A'_{1,2n} + \Theta_{5,0}X_{1,2n} = 0. \end{aligned} \right\} \quad (15)$$

In order to avoid the explicit appearance of ϕ , we must have

$$\alpha_{1,2n} = \frac{1}{2} \cos 2\tau.$$

Since we have already stipulated that A must not contain any term in $\cos(n\phi - \tau)$, $c_{1,2n}$ must be so chosen as to make quantities involving $\cos(n\phi - \tau)$ annul. Hence we must have

$$\left\{ \begin{aligned} 2c_{1,2n}nA_0 - 2\kappa c_{1,2n}X_0 - \frac{1}{2}A_0 \sin 2\tau \} \cos(n\phi - \tau) - 2\kappa X'_{1,2n} = 0, \\ \{ -2c_{1,2n}nX_0 + 2\kappa c_{1,2n}A_0 \} \sin(n\phi - \tau) + X''_{1,2n} + \Theta_{5,0}X_{1,2n} = 0. \end{aligned} \right\} \quad (16)$$

Whence

$$c_{1,2n} = \frac{n(\Theta_{5,0} - n^2) \sin 2\tau}{4(\alpha_{1,0}\Theta_{5,0} - n^4)},$$

$$X_{1,2n} = \frac{1}{8} \frac{(n^2 - \alpha_{1,0})(\Theta_{5,0} + n^2) \sin 2\tau A_0 \sin(n\phi - \tau)}{\kappa n(\alpha_{1,0}\Theta_{5,0} - n^4)}.$$

To the value for $X_{1,2n}$ must be added the further particular solution arising from the term $\frac{1}{2}A_0 \sin(3n\phi - \tau)$ in (15). It is

$$A_{1,2n} = \frac{(\Theta_{5,0} - 9n^2) A_0 \sin(3n\phi - \tau)}{16(\alpha_{1,0}\Theta_{5,0} - 9n^4)},$$

$$X_{1,2n} = \frac{3n\kappa A_0 \cos(3n\phi - \tau)}{8(\alpha_{1,0}\Theta_{5,0} - 9n^4)}.$$

Proceeding in this way, we have the following results:—

Terms not involving argument Θ :

In A

$$A_0 \sin(n\phi - \tau).$$

In X

$$X_0 \cos(n\phi - \tau).$$

In c

None.

Also

$$2\kappa n X_0 = -(\alpha_{1,0} - n^2) A_0,$$

when

$$\alpha_{1,0} = n^2 + 4\kappa^2 n^2 / (\Theta_{5,0} - n^2).$$

Terms involving argument $\Theta_{1,r}$, where r is not u nor $2u$:

$$a_{1,r} = 0, \quad c_{1,r} = 0,$$

$$A_{1,r} = -\frac{A_0 n^2 \{ (u+r)^2 - \Theta_{5,0} \} \sin \{ (u+r) \phi - \tau \}}{2r (2u+r) \{ a_{1,0} \Theta_{5,0} - n^2 (u+r)^2 \}} + \frac{A_0 n^2 \{ (u-r)^2 - \Theta_{5,0} \} \sin \{ (u-r) \phi - \tau \}}{2r (2u-r) \{ a_{1,0} \Theta_{5,0} - n^2 (u-r)^2 \}},$$

$$X_{1,r} = -\frac{A_0 n^2 \kappa (u+r) \cos \{ (u+r) \phi - \tau \}}{r (2u+r) \{ a_{1,0} \Theta_{5,0} - n^2 (u+r)^2 \}} + \frac{A_0 n^2 \kappa (u-r) \cos \{ (u-r) \phi - \tau \}}{r (2u-r) \{ a_{1,0} \Theta_{5,0} - n^2 (u-r)^2 \}}.$$

Terms involving argument $\Theta_{1,u}$:

$$a_{1,u} = 0, \quad c_{1,u} = 0,$$

$$A_{1,u} = \frac{A_0 (4u^2 - \Theta_{5,0}) \sin (2u\phi - \tau)}{6 (4u^2 - a_{1,0} \Theta_{5,0})} + \frac{A_0 \sin \tau}{2a_{1,0}},$$

$$X_{1,u} = \frac{2\kappa u A_0 \cos (2u\phi - \tau)}{3 (4u^2 - a_{1,0} \Theta_{5,0})}.$$

Terms involving argument $\Theta_{1,2u}$:

$$a_{1,2u} = \frac{1}{2} \cos 2\tau, \quad c_{1,2u} = \frac{u (\Theta_{5,0} - u^2) \sin 2\tau}{4 (a_{1,0} \Theta_{5,0} - u^4)}.$$

$$A_{1,2u} = \frac{(\Theta_{5,0} - 9u^2) A_0 \sin (3u\phi - \tau)}{16 (a_{1,0} \Theta_{5,0} - 9u^4)},$$

$$X_{1,2u} = -\frac{3\kappa A_0 \cos (3u\phi - \tau)}{8 (a_{1,0} \Theta_{5,0} - 9u^4)} + \frac{(u^2 - a_{1,0}) (\Theta_{5,0} - u^2) \sin 2\tau A_0 \sin (u\phi - \tau)}{8\kappa u (a_{1,0} \Theta_{5,0} - 9u^4)}$$

Terms involving argument $\Theta_{2,r}$, where r is not u nor $2u$:

$$a_{2,r} = 0, \quad c_{2,r} = 0,$$

$$A_{2,r} = \frac{n^2 \{ \Theta_{5,0} - (u+r)^2 \} X_0 \sin \{ (u+r) \phi - \tau \}}{4r (u+r) \{ a_{1,0} \Theta_{5,0} - n^2 (u+r)^2 \}} + \frac{n^2 \{ \Theta_{5,0} - (u-r)^2 \} X_0 \sin \{ (u-r) \phi - \tau \}}{4r (u-r) \{ a_{1,0} \Theta_{5,0} - n^2 (u-r)^2 \}},$$

$$X_{2,r} = \frac{X_0 n^2 \kappa (u+r) \cos \{ (u+r) \phi - \tau \}}{2r (u+r) \{ a_{1,0} \Theta_{5,0} - n^2 (u+r)^2 \}} + \frac{X_0 n^2 \kappa (u-r) \cos \{ (u-r) \phi - \tau \}}{2r (u-r) \{ a_{1,0} \Theta_{5,0} - n^2 (u-r)^2 \}}.$$

Terms involving argument $\Theta_{2,u}$:

$$a_{2,u} = 0, \quad c_{2,u} = 0,$$

$$A_{2,u} = \frac{X_0 (\Theta_{5,0} - 4u^2) \sin (2u\phi - \tau)}{6 (a_{1,0} \Theta_{5,0} - 4u^4)} - \frac{X_0 \sin 2\tau}{2a_{1,0}},$$

$$X_{2,u} = -\frac{2X_0 \kappa u \cos (2u\phi - \tau)}{3 (a_{1,0} \Theta_{5,0} - 4u^4)}.$$

Terms involving argument $\Theta_{2,2n}$:

$$a_{2,2n} = -\frac{\kappa n \cos 2\tau}{\Theta_{5,0} - n^2}, \quad c_{2,2n} = \frac{\kappa n^2 \sin 2\tau}{2(n^4 - a_{1,0}\Theta_{5,0})},$$

$$A_{2,2n} = \frac{(\Theta_{5,0} - 9n^2) X_0 \sin(3n\phi - \tau)}{9n^4 - a_{1,0}\Theta_{5,0}},$$

$$X_{2,2n} = \frac{3\kappa n X_0 \cos(3n\phi - \tau)}{8(9n^4 - a_{1,0}\Theta_{5,0})} + \frac{(\Theta_{5,0} + n^2) \kappa n X_0 \sin 2\tau \sin(n\phi - \tau)}{2(\Theta_{5,0} - n^2)(n^4 - a_{1,0}\Theta_{5,0})}.$$

Terms involving argument $\Theta_{4,r}$, where r is not n nor $2n$:

$$c_{4,r} = 0, \quad a_{4,r} = 0,$$

$$A_{4,r} = \frac{A_0 n^2 \kappa (n+r) \sin\{(n+r)\phi - \tau\}}{r(2n+r)\{a_{1,0}\Theta_{5,0} - n^2(n+r)^2\}} + \frac{A_0 n^2 \kappa (n-r) \sin\{(n-r)\phi - \tau\}}{r(2n-r)\{a_{1,0}\Theta_{5,0} - n^2(n-r)^2\}},$$

$$X_{4,r} = \frac{A_0 n^2 \{(n+r)^2 - a_{1,0}\} \cos\{(n+r)\phi - \tau\}}{2r(2n+r)\{a_{1,0}\Theta_{5,0} - n^2(n+r)^2\}} + \frac{A_0 n^2 \{(n-r)^2 - a_{1,0}\} \cos\{(n-r)\phi - \tau\}}{2r(2n-r)\{a_{1,0}\Theta_{5,0} - n^2(n-r)^2\}}.$$

Terms involving argument $\Theta_{4,n}$:

$$c_{4,n} = 0, \quad a_{4,n} = 0,$$

$$A_{4,n} = \frac{2n\kappa A_0 \sin(2n\phi - \tau)}{3(a_{1,0}\Theta_{5,0} - 4n^4)},$$

$$X_{4,n} = \frac{A_0(4n^2 - a_{1,0}) \cos(2n\phi - \tau)}{6(a_{1,0}\Theta_{5,0} - 4n^4)} - \frac{A_0 \cos \tau}{2\Theta_{5,0}}.$$

Terms involving argument $\Theta_{4,2n}$:

$$a_{4,2n} = \frac{n\kappa \cos 2\tau}{\Theta_{5,0} - n^2}, \quad c_{4,2n} = \frac{n^2 \kappa \sin 2\tau}{2(a_{1,0}\Theta_{5,0} - n^4)},$$

$$A_{4,2n} = \frac{3\kappa n A_0 \sin(3n\phi - \tau)}{8(a_{1,0}\Theta_{5,0} - 9n^4)},$$

$$X_{4,2n} = \frac{(9n^2 - a_{1,0}) A_0 \cos(3n\phi - \tau)}{16(a_{1,0}\Theta_{5,0} - 9n^4)} + \frac{A_0 \cos 2\tau \cos(n\phi - \tau)}{2(\Theta_{5,0} - n^2)} \\ + \frac{(a_{1,0} + n^2) \sin 2\tau A_0 \sin(n\phi - \tau)}{4(a_{1,0}\Theta_{5,0} - n^4)}$$

Terms involving argument $\Theta_{5,r}$, when r is not n nor $2n$:

$$c_{5,r} = 0, \quad a_{5,r} = 0,$$

$$A_{5,r} = \frac{X_0 n^2 \kappa (n+r) \sin\{(n+r)\phi - \tau\}}{r(2n+r)\{a_{1,0}\Theta_{5,0} - n^2(n+r)^2\}} - \frac{X_0 n^2 \kappa (n-r) \sin\{(n-r)\phi - \tau\}}{r(2n-r)\{a_{1,0}\Theta_{5,0} - n^2(n-r)^2\}},$$

$$X_{5,r} = \frac{X_0 n^2 \{(n+r)^2 - a_{1,0}\} \cos\{(n+r)\phi - \tau\}}{2r(2n+r)\{a_{1,0}\Theta_{5,0} - n^2(n+r)^2\}} - \frac{X_0 n^2 \{(n-r)^2 - a_{1,0}\} \cos\{(n-r)\phi - \tau\}}{2r(2n-r)\{a_{1,0}\Theta_{5,0} - n^2(n-r)^2\}}.$$

Terms involving argument $\Theta_{s,n}$:

$$a_{s,n} = 0, \quad c_{s,n} = 0,$$

$$A_{s,n} = \frac{2X_0 \kappa 2n \sin(2n\phi - \tau)}{3(\alpha_{1,0}\Theta_{s,0} - 4n^4)} - \frac{X_0 \cos \tau}{2\Theta_{s,0}},$$

$$X_{s,n} = \frac{X_0(4n^2 - \alpha_{1,0}) \cos(2n\phi - \tau)}{6(\alpha_{1,0}\Theta_{s,0} - 4n^4)}.$$

Terms involving argument $\Theta_{s,2n}$:

$$a_{s,2n} = -\frac{(\alpha_{1,0} - n^2) \cos 2\tau}{2(\Theta_{s,0} - n^2)}, \quad c_{s,2n} = \frac{n(\alpha_{1,0} - n^2) \sin 2\tau}{4(\alpha_{1,0}\Theta_{s,0} - n^4)},$$

$$A_{s,2n} = \frac{3n\kappa X_0 \sin(3n\phi - \tau)}{8(9n^4 - \alpha_{1,0}\Theta_{s,0})},$$

$$X_{s,2n} = \frac{9(n^2 - \alpha_{1,0}) X_0 \cos(3n\phi - \tau)}{16(9n^4 - \alpha_{1,0}\Theta_{s,0})} + \frac{(\alpha_{1,0} - n^2) \cos 2\tau \Lambda_0 \cos(n\phi - \tau)}{4\kappa n(\Theta_{s,0} - n^2)} - \frac{(\alpha_{1,0} + n^2)(\alpha_{1,0} - n^2) \sin 2\tau \Lambda_0 \sin(n\phi - \tau)}{8\kappa n(\alpha_{1,0}\Theta_{s,0} - n^4)}.$$

Terms involving powers products of the Θ 's follow in similar fashion.

If we summarize the parts specially required, we find

$$(\Theta_{1,0} - n^2)(\Theta_{s,0} - n^2) = 4\kappa^2 n^2 + \frac{1}{2}(\Theta_{s,0} - n^2) \cos 2\tau \Theta_{1,2n} - \kappa n \cos 2\tau \Theta_{2,2n} + \kappa n \cos 2\tau \Theta_{4,2n} - \frac{1}{2}(\alpha_{1,0} - n^2) \cos 2\tau \Theta_{s,2n} + \dots ; \dots \quad (17)$$

and

$$2c(\alpha_{1,0}\Theta_{s,0} - n^4) = \frac{1}{2}n(\Theta_{s,0} - n^2) \sin 2\tau \Theta_{1,2n} - \kappa n^2 \sin 2\tau \Theta_{2,2n} + \kappa n^2 \sin 2\tau \Theta_{4,2n} - \frac{1}{2}n(\alpha_{1,0} - n^2) \sin 2\tau \Theta_{s,2n} + \dots ; \dots \quad (18)$$

where, as already stated,

$$(\alpha_{1,0} - n^2)(\Theta_{s,0} - n^2) = 4\kappa^2 n^2.$$

It is necessary to examine the expressions just obtained in order to see whether the complete integral of equations (6) has been found.

The integer n is determined so as most nearly to satisfy the relation

$$(\Theta_{1,0} - n^2)(\Theta_{s,0} - n^2) = 4\kappa^2 n^2,$$

when

$$\Theta_{1,0}, \Theta_{s,0} \text{ and } \kappa \text{ are known.}$$

The negative value of n will also satisfy this relation.

On solving equation (17), for each value of n there will be, in general, two values of 2τ , equal and opposite in sign. So that altogether there are four distinct values of 2τ obtainable. Each of these with the corresponding value of n will give a

different value of c on substituting in equation (18), and different values for Λ and X . Hence there are four distinct solutions and these when multiplied by arbitrary constants will give the complete primitive of equations (6).

(b) The Particular Integral.

We have now to determine the particular integral of equations (4). We shall assume only one general term on the right-hand side and take the complete solution as the sum of a series of the corresponding solutions. The equations may therefore be written

$$\left. \begin{aligned} \rho'' - 2\kappa\rho' + \rho\Sigma\Theta_{1,r} \cos r\phi + \sigma\Sigma\Theta_{2,r} \sin r\phi &= \frac{1}{2}\Theta_{3,m}e^{+im\phi}, \\ \sigma'' + 2\kappa\sigma' + \rho\Sigma\Theta_{4,r} \sin r\phi + \sigma\Sigma\Theta_{5,r} \cos r\phi &= 0. \end{aligned} \right\} \dots \dots (19)$$

Assume $\rho = e^{im\phi}\Lambda$,*
and $\sigma = e^{im\phi}X$,

where X and Λ as before are functions of ϕ . On substituting in equations (19) and reducing, we find

$$\left. \begin{aligned} -m^2\Lambda + 2im\Lambda' + \Lambda'' - 2\kappa(imX + X') + \Lambda\Sigma\Theta_{2,r} \cos r\phi + X\Sigma\Theta_{2,r} \sin r\phi &= \frac{1}{2}\Theta_{3,m}, \\ -m^2X + 2imX' + X'' + 2\kappa(im\Lambda + \Lambda') + \Lambda\Sigma\Theta_{4,r} \sin r\phi + X\Sigma\Theta_{5,r} \cos r\phi &= 0. \end{aligned} \right\} \dots (20)$$

As a solution we now take

$$\begin{aligned} \Lambda &= \Lambda_0 + \Sigma\Sigma A_{r,s}\Theta_{r,s} + \Sigma\Sigma\Sigma\Sigma B_{r,s,p,q}\Theta_{r,s}\Theta_{p,q} + \dots, \\ X &= X_0 + \Sigma\Sigma X_{r,s}\Theta_{r,s} + \Sigma\Sigma\Sigma\Sigma Y_{r,s,p,q}\Theta_{r,s}\Theta_{p,q} + \dots \end{aligned}$$

In these summations all the Θ 's in the coefficients of ρ and σ are to be included except $\Theta_{1,0}$ and $\Theta_{5,0}$. Λ_0 and X_0 are constants, and the other coefficients functions of ϕ .

Now substitute these expansions for Λ and X in (20), and equate to zero the terms involving no Θ except $\Theta_{1,0}$ and $\Theta_{5,0}$. We then have

$$\begin{aligned} -m^2\Lambda_0 - 2\kappa imX_0 + \Theta_{1,0}\Lambda_0 &= \frac{1}{2}\Theta_{3,m}, \\ -m^2X_0 + 2\kappa im\Lambda_0 + \Theta_{5,0}X_0 &= 0. \end{aligned}$$

Whence

$$\begin{aligned} \Lambda_0 &= \frac{1}{2}\Theta_{3,m}(\Theta_{5,0} - m^2) \div \{(\Theta_{1,0} - m^2)(\Theta_{5,0} - m^2) - 4\kappa^2 m^2\}, \\ X_0 &= -\kappa im\Theta_{3m} \div \{(\Theta_{1,0} - m^2)(\Theta_{5,0} - m^2) - 4\kappa^2 m^2\}. \end{aligned}$$

Next, taking the coefficient of $\Theta_{1,r}$, we have the equations

$$\left. \begin{aligned} -m^2\Lambda_{1,r} + 2im\Lambda'_{1,r} + \Lambda''_{1,r} - 2\kappa(imX_{1,r} + X'_{1,r}) + \Theta_{1,0}\Lambda_{1,0} + \Lambda_0 \cos r\phi &= 0 \\ -m^2X_{1,r} + 2imX'_{1,r} + X''_{1,r} + 2\kappa(im\Lambda_{1,r} + \Lambda'_{1,r}) + \Theta_{5,0}X_{1,r} &= 0 \end{aligned} \right\} \dots (21)$$

* The use of m here to represent an integer is to be carefully distinguished from its previous use to represent mass.

We shall form the solution by taking only $e^{i\phi}$ in the term $\cos r\phi$. Changing the sign of r will then give the other part. Assuming that $A_{1,r}$ and $X_{1,r}$ vary as $e^{i\phi}$, we have

$$\left. \begin{aligned} A_{1,r}(-m^2 - 2mr - r^2 + \Theta_{1,0}) - 2\kappa l(m+r)X_{1,r} &= -\frac{1}{2}A_0, \\ A_{1,r}(m+r)2\kappa l + X_{1,r}(-m^2 - 2mr - r^2 + \Theta_{5,0}) &= 0. \end{aligned} \right\} \dots \dots \dots (22)$$

From these

$$A_{1,r} = -\frac{1}{2}A_0 \{ \Theta_{5,0} - (m+r)^2 \} \div [\{ \Theta_{1,0} - (m+r)^2 \} \{ \Theta_{5,0} - (m+r)^2 \} - 4\kappa^2(m+r)^2],$$

and

$$X_{1,r} = +\frac{1}{2}A_0 \cdot 2\kappa l(m+r) \div [\{ \Theta_{1,0} - (m+r)^2 \} \{ \Theta_{5,0} - (m+r)^2 \} - 4\kappa^2(m+r)^2].$$

On determining the corresponding values for the term $e^{-i\phi}$ and combining the two, we have

$$\left. \begin{aligned} A_{1,r} &= -\frac{1}{2}A_0 e^{i\phi} \{ \Theta_{5,0} - (m+r)^2 \} \div [\{ \Theta_{1,0} - (m+r)^2 \} \{ \Theta_{5,0} - (m+r)^2 \} - 4\kappa^2(m+r)^2] \\ &\quad -\frac{1}{2}A_0 e^{-i\phi} \{ \Theta_{5,0} - (m-r)^2 \} \div [\{ \Theta_{1,0} - (m-r)^2 \} \{ \Theta_{5,0} - (m-r)^2 \} - 4\kappa^2(m-r)^2], \\ X_{1,r} &= A_0 e^{i\phi} \kappa l(m+r) \div [\{ \Theta_{1,0} - (m+r)^2 \} \{ \Theta_{5,0} - (m+r)^2 \} - 4\kappa^2(m+r)^2] \\ &\quad + A_0 e^{-i\phi} \kappa l(m-r) \div [\{ \Theta_{1,0} - (m-r)^2 \} \{ \Theta_{5,0} - (m-r)^2 \} - 4\kappa^2(m-r)^2]. \end{aligned} \right\} (23)$$

Expression (23) shows that $A_{1,r}$ and $X_{1,r}$ are factored by A_0 , which is a multiple of $\Theta_{3,m}$. Now the terms in the expansions of A and X that we are seeking are $A_{1,r}\Theta_{1,r}$ and $X_{1,r}\Theta_{1,r}$. Since both of these involve the product $\Theta_{3,m}\Theta_{1,r}$, it is clear that they may be neglected in comparison with the values of A_0 and X_0 .

We have further to determine the parts of A and X arising from a term $\frac{1}{2l}\Theta_{6,m}e^{i\mu\phi}$ in the right-hand member of the second equations (4). These can be written down from the results already given, and are

$$X_0 = \frac{1}{2l}\Theta_{6,m}(\Theta_{5,0} - m^2) \div \{ (\Theta_{1,0} - m^2)(\Theta_{5,0} - m^2) - 4\kappa^2 m^2 \},$$

$$A_0 = \kappa m \Theta_{6,m} \div \{ (\Theta_{1,0} - m^2)(\Theta_{5,0} - m^2) - 4\kappa^2 m^2 \}.$$

Hence to the degree of accuracy we are using, we may summarise the results as:

$$\left. \begin{aligned} \rho &= \sum_m [\Theta_{3,m}(\Theta_{5,0} - m^2) \cos m\phi + 2\kappa m \Theta_{6,m} \cos m\phi] \div [(\Theta_{1,0} - m^2)(\Theta_{5,0} - m^2) - 4\kappa^2 m^2], \\ \sigma &= \sum_m [2\kappa m \Theta_{3,m} \sin m\phi + \Theta_{6,m}(\Theta_{5,0} - m^2) \sin m\phi] \div [(\Theta_{1,0} - m^2)(\Theta_{5,0} - m^2) - 4\kappa^2 m^2]. \end{aligned} \right\} (24)$$

Except when the denominators are small, it is seen that, owing to the very small factors $\Theta_{3,m}$ and $\Theta_{6,m}$, the values of ρ and σ derived from the above equations are very small.

§ 4. Discussion of the Solutions of the Equations for the Case of Equal Particles.

(i.) The complementary function.

• Equations (17) and (18) which determine the value of the exponent c , may be re written here,

$$(\Theta_{1,0} - n^2)(\Theta_{5,0} - n^2) = 4\kappa^2 n^2 + \left\{ \frac{1}{2}(\Theta_{5,0} - n^2)\Theta_{1,2n} - \kappa n \Theta_{2,2n} + \kappa n \Theta_{4,2n} - \frac{1}{2}(\alpha_{1,0} - n^2)\Theta_{5,2n} \right\} \cos 2\tau \quad (25)$$

$$2c(\alpha_{1,0}\Theta_{5,0} - n^4) = \left\{ \frac{1}{2}n(\Theta_{5,0} - n^2)\Theta_{1,2n} - \kappa n^2 \Theta_{2,2n} + \kappa n^2 \Theta_{4,2n} - \frac{1}{2}n(\alpha_{1,0} - n^2)\Theta_{5,2n} \right\} \sin 2\tau \quad (26)$$

In these $\alpha_{1,0}$ is determined by the relation

$$(\alpha_{1,0} - n^2)(\Theta_{5,0} - n^2) = 4\kappa^2 n^2.$$

It is noticeable that the coefficient of $\sin 2\tau$ in (26) is n times that of $\cos 2\tau$ in (25). Owing to the smallness of the quantities Θ (excepting $\Theta_{1,0}$ and $\Theta_{5,0}$), it is clear that the coefficients of $\cos 2\tau$ and $\sin 2\tau$ are both very small quantities. Now real values of c are only given by real values of τ , and conversely. Hence in order that (25) may give real values of τ it is necessary that the expression

$$(\Theta_{1,0} - n^2)(\Theta_{5,0} - n^2) - 4\kappa^2 n^2 \quad (27)$$

should be less than, or at most equal to, the coefficient of $\cos 2\tau$. That is, the real values of c will be in the vicinity of these values of κ that make (27) vanish. The actual limits of the zone in which real values of c are found will be given by

$$(\Theta_{1,0} - n^2)(\Theta_{5,0} - n^2) = 4\kappa^2 n^2 \pm \left\{ \frac{1}{2}(\Theta_{5,0} - n^2)\Theta_{1,2n} \mp \kappa n \Theta_{2,2n} \pm \kappa n \Theta_{4,2n} - \frac{1}{2}(\alpha_{1,0} - n^2)\Theta_{5,2n} \right\} \quad (28)$$

There are four groups of signs possible in this expression, and there will result four values of κ . The outermost and innermost of these will define the zone in which some real value of c appears, and this zone will be the zone of instability. Owing, however, to the extreme smallness of the coefficient of $\sin 2\tau$ in (26), it is clear that c will be extremely small, in general; that is, the modulus of instability will be small and departure from the zone will be slow. In one case, however, c may be quite large. That is, when the coefficient of c , $\alpha_{1,0}\Theta_{5,0} - n^4$, is exceedingly small.

Each of the quantities Θ is a function of $\frac{a}{a'}$ or of κ . Further $\Theta_{1,0}$ and $\Theta_{5,0}$ involve both the mass of the particles and the number of them. Both of these are entirely unknown. All that can be said is that MAXWELL'S criterion,* that is,

$$\nu < \frac{2}{p^3},$$

* TISSERAND, 'Méc. Céleste,' vol. ii., p. 184.

where ν is the ratio of the mass of a particle to the mass of Saturn and p is the number of particles in the ring, must be fulfilled.

ν appears in the expressions for $\Theta_{1,0}$ and $\Theta_{5,0}$ in the form νL_s . It has been mentioned that $L_s < 0.0194\mu^3$ for all values of s . Hence $\nu L_s < 0.0194\nu\mu^3$. For the present we shall regard νL_s as a variable parameter and discuss the solutions relative to this parameter.

In order to locate the zone of instability, we equate expression (27) to zero. Writing it in full, but omitting the term involving ν' , which will be exceedingly small and will hardly affect the result, we find

$$\{(3 - \nu L_s)\kappa^2 + n^2\} \{2\nu L_s \kappa^2 + n^2\} - 4\kappa^2 n^2 = 0. \dots \dots \dots (29)$$

This equation, regarded as involving an unknown quantity n^2/κ^2 , is precisely the equation used by MAXWELL to determine the condition of stability of the ring of particles when unperturbed by any satellite. The condition of the reality of n^2/κ^2 leads to the upper limit for ν just quoted. In our problem we may take the unknown quantity as κ^2/n^2 , and then assuming a value for νL_s , solve the equation. The values of κ (for differing values of n) will give the position of the zones of instability of a ring of particles of mass and number assumed. Or, conversely, taking a position of instability, as shown by telescopic observations of the ring, we may determine the corresponding value of νL_s , which establishes the order of value of the mass and number of particles at that point.

I have found that the latter process leads to no satisfactory result, and hence I do not record the work.

It is interesting to examine the meaning of the condition previously referred to, that the maximum instability is found when $(\Theta_{1,0}\Theta_{5,0} - n^4)$ is approximately zero. On referring again to equation (28), it is clear that the broadest zone of instability will be found, owing to the extreme smallness of the last member, when $(\Theta_{1,0} - n^2)(\Theta_{5,0} - n^2) - 4\kappa^2 n^2$ changes most slowly with κ . This will occur when the equation (29) has equal roots. Equal roots appear when, by the variation of the parameter νL_s , κ/n passes from real to imaginary values, or when*

$$\nu L_s = 0.039.$$

This is the upper limit of the criterion previously quoted from MAXWELL, and would imply that all the particles were of such mass and number as to be on the border-line of instability.

*When νL_s has this value, we find that

$$2\nu L_s(3 - \nu L_s)\kappa^4 - n^4 = 0;$$

or

$$\Theta_{5,0}\Theta_{1,0} - n^4 = 0.$$

* TISSERAND, *loc. cit.*, p. 183.

Whence, by (11),

$$\alpha_{1,0}\Theta_{5,0}-n^4 = -\Theta_{5,0} \{ \Sigma \Sigma \alpha_{r,s} \Theta_{r,s} + \dots \}.$$

This right-hand member is of the same order of value as the factor of $\sin 2\tau$ in (26). In this case, then, c may assume a high value. But it is noticeable that only at the limit of MAXWELL'S relation is great instability to be found.

When expression (27) has a value far from zero, either by virtue of the value of κ/n or the value of νL_s , it is clear from (25) that 2τ is imaginary and hence c is imaginary, the solution being stable.

It might be inferred from this that if values of νL_s were chosen such that MAXWELL'S relation were not fulfilled the effect of the satellite would be to stabilise what would otherwise be an unstable system. As pointed out already, however, the original equations and their solutions, as given here, simply give the motion of the particles in the vicinity of certain circles. In some cases the motion may be such that the particles depart rapidly from this zero circle; this we have termed instability. In other cases the solutions may indicate that the particles never move far from the zero circle; and this type of motion we have termed stable. But it is clear that if a small arbitrary displacement were given to each of the particles in the latter case, nothing in this paper precludes the possibility of their departure finally from the zero circle. That is, they may be again unstable. What we have found here is a series of orbits for the particles when subject to the attractions of Saturn, a satellite, and one another. Those in which the particles have large inequalities result in collisions with the neighbouring rings of particles and hence a complete departure from their former positions. Those which have no large inequalities and hence avoid collisions with neighbouring rings of particles may yet prove unstable when an arbitrary disturbance is further imposed upon them.

(ii) The particular integral.

In the expression (24) there appears a denominator of the form

$$(\Theta_{1,0}-m^2)(\Theta_{5,0}-m^2)-4\kappa^2m^2. (30)$$

Here m takes all positive integral values including zero. When the conditions are such, therefore, that expression (30) is approximately zero, the term in the particular integral will become very great and departure from the orbit will result. This expression is the same as (29), which, it has been pointed out, gives the positions of the unstable solutions of the complementary function. It may therefore be said that all the unstable positions are in the vicinity of the zero values of (30), and the following remarks apply equally to both parts of the solutions.

Referring to the form (29) it is seen that there are two variables, κ/n and νL_s . For a given value of νL_s there are in general two values of κ/n , and for a given value of κ/n there are two values of νL_s . In the figure (p. 125), the relation between κ/n and νL_s is shown graphically, only those values of νL_s which satisfy MAXWELL'S criterion being

chosen. It will be seen that κ/n increases slowly from unity as νL_e increases from zero, until νL_e reaches the value 0.039. At this point the curve turns back and rises rapidly to an asymptote at $\nu L_e = 0$.

In the case when expression (30) is exactly zero, it is seen from elementary principles that the independent variable ϕ would appear explicitly. With passage of time, therefore, ρ and σ would increase linearly in magnitude and there would be complete departure of the particles from the vicinity of $r = a$.

§ 5. *Case where the Particles forming the Ring are of Unequal Masses.*

The previous equations (2) were reduced to the form (3) on the supposition that all the masses m_λ were of the same value $m_\lambda = \nu M$. We now proceed to the modifications introduced when these masses are all distinct in value.

Equations (2) with the same reductions as before, but maintaining the separate values m_λ , become :

$$\begin{aligned}
 \rho''_\lambda - 2\kappa\rho'_\lambda &= \kappa^2 \kappa''/\nu' \left\{ \frac{d}{d\alpha} \left(\frac{1}{2}b_0 + \dots + b_i \cos i\phi + \dots \right) - \cos \phi \right\} - \kappa^2 E_\lambda \\
 &+ \left[3\kappa^2 + \kappa^2 \kappa''/\nu' \frac{d^2}{d\alpha^2} \left(\frac{1}{2}b_0 + \dots + b_i \cos i\phi + \dots \right) - \kappa^2 F_\lambda \right] \rho_\lambda + \sum_\mu \kappa^2 G_{\mu,\lambda} \rho_\mu \\
 &+ \left[\kappa^2 \kappa''/\nu' \frac{d}{d\alpha} \left(b_1 \sin \phi + \dots + i b_i \sin i\phi + \dots \right) - \kappa^2 \kappa''/\nu' \sin \phi - \kappa^2 H_\lambda \right] \sigma_\lambda \\
 &+ \sum_\mu \kappa^2 J_{\mu,\lambda} \sigma_\mu ; \\
 \sigma''_\lambda + 2\kappa\sigma'_\lambda &= \nu' \kappa^2 \kappa''/\nu^3 \left(b_1 \sin \phi + \dots + i b_i \sin i\phi + \dots \right) - \nu' \kappa^2 \kappa''/\nu^3 \sin \phi + \nu \kappa^2 E'_\lambda \\
 &+ \left[\nu' \kappa^2 \kappa''/\nu^3 \left(\dots + i \frac{db_i}{d\alpha} \sin i\phi + \dots \right) + \nu' \kappa^2 \kappa''/\nu^3 \left(\dots + i b_i \sin i\phi \dots \right) \right. \\
 &\quad \left. - \kappa^2 F'_\lambda \right] \sigma_\lambda + \sum_\mu \kappa^2 G_{\mu,\lambda} \rho_\mu \\
 &+ \left[-\nu' \kappa^2 \kappa''/\nu^3 \left(\dots + i^2 b_i \cos i\phi + \dots \right) + \nu' \kappa^2 \kappa''/\nu^3 \cos \phi - \kappa^2 H'_\lambda \right] \sigma_\lambda \\
 &+ \sum_\mu \kappa^2 J'_{\mu,\lambda} \sigma_\mu .
 \end{aligned} \tag{31}$$

In these equations

$$\begin{aligned}
 E_\lambda &= \sum_\mu \frac{m_\mu}{M} \left\{ \frac{1}{4 \sin(\mu-\lambda) \pi/n} + \cos(\mu-\lambda) 2\pi/n \right\}, \\
 F_\lambda &= \sum_\mu \frac{m_\mu}{M} \left\{ \frac{1}{8 \sin^3(\mu-\lambda) \pi/n} - \frac{3}{\sin(\mu-\lambda) \pi/n} \right\}, \\
 G_{\mu,\lambda} &= \frac{m_\mu}{M} \left\{ \frac{\cos(\mu-\lambda) 2\pi/n}{8 \sin^3(\mu-\lambda) \pi/n} + \frac{\frac{3}{8}}{\sin(\mu-\lambda) \pi/n} + 2 \cos(\mu-\lambda) 2\pi/n \right\},
 \end{aligned}$$

$$\begin{aligned}
 \Pi_\lambda &= \sum_\mu \frac{m_\mu}{M} \left\{ \frac{\cos(\mu-\lambda)\pi/n}{8\sin^2(\mu-\lambda)\pi/n} + \sin(\mu-\lambda)2\pi/n \right\}, \\
 J_{\mu,\lambda} &= \frac{m_\mu}{M} \left\{ \frac{\cos(\mu-\lambda)\pi/n}{8\sin^2(\mu-\lambda)\pi/n} + \sin(\mu-\lambda)2\pi/n \right\}, \\
 E'_\lambda &= \sum_\mu \frac{m_\mu}{M} \left\{ \frac{\cos(\mu-\lambda)\pi/n}{4\sin^2(\mu-\lambda)\pi/n} - \sin(\mu-\lambda)2\pi/n \right\}, \\
 F'_\lambda &= \sum_\mu \frac{m_\mu}{M} \left\{ \frac{3\cos(\mu-\lambda)\pi/n}{8\sin^2(\mu-\lambda)\pi/n} \right\}, \\
 G'_{\mu,\lambda} &= \frac{m_\mu}{M} \left\{ -\frac{\cos(\mu-\lambda)\pi/n}{8\sin^2(\mu-\lambda)\pi/n} + 2\sin(\mu-\lambda)2\pi/n \right\}, \\
 H'_\lambda &= \sum_\mu \frac{m_\mu}{M} \left\{ \frac{\cos(\mu-\lambda)2\pi/n}{8\sin^3(\mu-\lambda)\pi/n} - \frac{3\sin^2(\mu-\lambda)2\pi/n}{\sin^5(\mu-\lambda)\pi/n} - \cos(\mu-\lambda)2\pi/n \right\}, \\
 J'_{\mu,\lambda} &= \frac{m_\mu}{M} \left\{ \frac{\cos(\mu-\lambda)2\pi/n}{8\sin^3(\mu-\lambda)\pi/n} - \frac{3\sin^2(\mu-\lambda)2\pi/n}{\sin^5(\mu-\lambda)\pi/n} - \cos(\mu-\lambda)2\pi/n \right\}. \quad (32)
 \end{aligned}$$

These equations may be written, with a slightly different meaning of the quantities Θ from those formerly obtaining, in the form,

$$\left. \begin{aligned}
 \rho''_\lambda - 2\kappa\sigma'_\lambda + \rho_\lambda \sum_r \{ \Theta_{1,r} \cos r\phi \} + \kappa^2 \sum_\mu G_{\mu,\lambda} \rho_\mu + \sigma_\lambda \sum_r \{ \Theta_{2,r} \sin r\phi \} \\
 \qquad \qquad \qquad + \kappa^2 \sum_\mu J_{\mu,\lambda} \sigma_\mu = \sum_r \Theta_{3,r} \cos r\phi, \\
 \sigma''_\lambda + 2\kappa\rho'_\lambda + \rho_\lambda \sum_r \{ \Theta_{4,r} \sin r\phi \} + \kappa^2 \sum_\mu G'_{\mu,\lambda} \rho_\mu + \sigma_\lambda \sum_r \{ \Theta_{5,r} \cos r\phi \} \\
 \qquad \qquad \qquad + \kappa^2 \sum_\mu J'_{\mu,\lambda} \sigma_\mu = \sum_r \Theta_{6,r} \sin r\phi.
 \end{aligned} \right\} \quad (33)$$

We shall determine the particular integral arising from one term of the right-hand member of the first equation, writing it typically $\frac{1}{2}\Theta_{3,m}e^{im\phi}$.

Assume that

$$\rho_\lambda = A_\lambda e^{im\phi},$$

and

$$\sigma_\lambda = X_\lambda e^{im\phi}$$

for all values of λ .

Equations (33) then become

$$\left. \begin{aligned}
 -m^2 A_\lambda + 2imA'_\lambda + A''_\lambda - 2\kappa(imX_\lambda + X'_\lambda) + A_\lambda \sum_r \{ \Theta_{1,r} \cos r\phi \} \\
 \qquad \qquad \qquad + \kappa^2 \sum_\mu A_\mu G_{\mu,\lambda} + X_\lambda \sum_r \{ \Theta_{2,r} \sin r\phi \} + \kappa^2 \sum_\mu X_\mu J_{\mu,\lambda} = \frac{1}{2}\Theta_{3,m}, \\
 -m^2 X_\lambda + 2imX'_\lambda + X''_\lambda + 2\kappa(mA_\lambda + A'_\lambda) + A_\lambda \sum_r \{ \Theta_{4,r} \sin r\phi \} \\
 \qquad \qquad \qquad + \kappa^2 \sum_\mu A_\mu G'_{\mu,\lambda} + X_\lambda \sum_r \{ \Theta_{5,r} \cos r\phi \} + \kappa^2 \sum_\mu X_\mu J'_{\mu,\lambda} = 0.
 \end{aligned} \right\} \quad (34)$$

There are n pairs of equations in this form corresponding to the n values of λ .

As a solution we now take

$$A_\lambda = A_0^\lambda + \sum \sum A_{r,\lambda}^\lambda \Theta_{r,\lambda} + \dots$$

$$X_\lambda = X_0^\lambda + \sum \sum X_{r,\lambda}^\lambda \Theta_{r,\lambda} + \dots$$

with the same restrictions as before.

Substitute in (34) and equate to zero the terms involving no Θ but $\Theta_{1,0}^\lambda$ and $\Theta_{3,0}^\lambda$. Then, for all values of λ from 1 to n ,

$$\left. \begin{aligned} -m^2 A_0^\lambda - 2\kappa m X_0^\lambda + \Theta_{1,0}^\lambda A_0^\lambda + \kappa^2 \sum A_0^\mu G_{\mu,\lambda} + \kappa^2 \sum X_0^\mu J_{\mu,\lambda} &= \frac{1}{2} \Theta_{3,0}^\lambda \\ -m^2 X_0^\lambda + 2\kappa m A_0^\lambda + \Theta_{3,0}^\lambda X_0^\lambda + \kappa^2 \sum A_0^\mu G'_{\mu,\lambda} + \kappa^2 \sum X_0^\mu J'_{\mu,\lambda} &= 0. \end{aligned} \right\} \dots (35)$$

These $2n$ equations can be solved by the usual processes to give the values of the constants A_0^λ and X_0^λ . It is not necessary for us to work out the results in detail, it is sufficient to note that the determinant of the left-hand members will appear as the denominator in each case. The determinant is the following :

$$\left| \begin{array}{cccccc} -m^2 + \Theta_{1,0}^1 & \kappa^2 G_{2,1} & \kappa^2 G_{3,1} & \dots & \kappa^2 G_{n,1} & ; & -2\kappa m & \kappa^2 J_{2,1} & \kappa^2 J_{3,1} & \dots & \kappa^2 J_{n,1} \\ 2\kappa m & \kappa^2 G'_{2,1} & \kappa^2 G'_{3,1} & \dots & \kappa^2 G'_{n,1} & ; & -m^2 + \Theta_{3,0}^1 & \kappa^2 J'_{2,1} & \kappa^2 J'_{3,1} & \dots & \kappa^2 J'_{n,1} \\ \kappa^2 G_{1,2} & -m^2 + \Theta_{1,0}^2 & \kappa^2 G_{3,2} & \dots & \kappa^2 G_{n,2} & ; & \kappa^2 J_{1,2} & -2\kappa m & \kappa^2 J_{3,2} & \dots & \kappa^2 J_{n,2} \\ \kappa^2 G'_{1,2} & 2\kappa m & \kappa^2 G'_{3,2} & \dots & \kappa^2 G'_{n,2} & ; & \kappa^2 J'_{1,2} & -m^2 + \Theta_{3,0}^2 & \kappa^2 J'_{3,2} & \dots & \kappa^2 J'_{n,2} \\ \dots & \dots & \dots & \dots & \dots & & \dots & \dots & \dots & \dots & \dots \\ \kappa^2 G_{1,n} & \kappa^2 G_{2,n} & \kappa^2 G_{3,n} & \dots & -m^2 + \Theta_{1,0}^n & ; & \kappa^2 J_{1,n} & \kappa^2 J_{2,n} & \kappa^2 J_{3,n} & \dots & -2\kappa m \\ \kappa^2 G'_{1,n} & \kappa^2 G'_{2,n} & \kappa^2 G'_{3,n} & \dots & 2\kappa m & ; & \kappa^2 J'_{1,n} & \kappa^2 J'_{2,n} & \kappa^2 J'_{3,n} & \dots & -m^2 + \Theta_{3,0}^n \end{array} \right| \dots (36)$$

This determinant corresponds to the denominators in expressions (24). When it vanishes or becomes small, it is clear, as before, that the terms of the solution tend to become large, and instability follows.

In estimating the values of F, G, H and J, which appear in the above determinant, it is to be noted that m_μ/M is exceedingly small for all values of μ . But the quantities in which it appears may be large by virtue of the small denominators which are involved. In the expression for F_λ , the term $3/\sin(\mu-\lambda)\pi/n$ may be neglected in comparison with the first term for large values of n . Also, $\sum \frac{m_\mu}{M} \frac{1}{8 \sin^3(\mu-\lambda)\pi/n}$ will lie between zero and $\frac{\bar{m}}{M} \simeq \frac{1}{8 \sin^3(\mu-\lambda)\pi/n}$ since all the signs are positive, if \bar{m} is the greatest value of m_μ appearing in the ring. Hence F_λ lies between zero and $0.0096 n^3 \bar{m}/M$ in value.

In the same way the value of $G_{\mu,\lambda}$ will arise almost wholly from the first term. The largest value it may have will be $\bar{m}n^3/8\pi^3 M$ or $0.004n^3 \bar{m}/M$. $E_\lambda, H_\lambda, J_{\mu,\lambda}, E'_\lambda, F'_\lambda$ and $G'_{\mu,\lambda}$ are seen to be one order lower in the reciprocal of $\sin(\mu-\lambda)\pi/n$ and therefore may be neglected. H'_λ has the limit $-0.0192n^3 \bar{m}/M$, and $J'_{\mu,\lambda}$ the limit $-0.008n^3 \bar{m}/M$.

We shall assume that the number of particles in any ring is large. It is probable that they vary in magnitude from the infinitesimally small up to the limit given by MAXWELL. Hence the values of the expressions F_λ , $G_{\mu,\lambda}$, H'_λ and J'_λ will vary over a range of values, between the given limits, as λ takes its successive values.

Reverting to determinant (36), we see that it may now be written

$$\left. \begin{array}{cccccccc}
 -m^2 + \Theta_{1,0}^1, & \kappa^2 G_{2,1}, & \kappa^2 G_{3,1}, \dots, & \kappa^2 G_{n,1} & ; & -2\kappa m & , & 0 & , & 0 & , \dots, & 0 \\
 2\kappa m & , & 0 & , & 0 & , \dots, & 0 & ; & -m^2 + \Theta_{5,0}^1, & \kappa^2 J'_{2,1} & , & \kappa^2 J'_{3,1}, \dots, & \kappa^2 J'_{n,1} \\
 \kappa^2 G_{1,2} & , & -m^2 + \Theta_{1,0}^2, & \kappa^2 G_{2,2}, \dots, & \kappa^2 G_{n,2} & ; & 0 & , & -2\kappa m & , & 0 & , \dots, & 0 \\
 0 & , & 2\kappa m & , & 0 & , \dots, & 0 & ; & \kappa^2 J'_{1,2} & , & -m^2 + \Theta_{5,0}^2, & \kappa^2 J'_{3,2}, \dots, & \kappa^2 J'_{n,2} \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 \kappa^2 G_{1,n} & , & \kappa^2 G_{2,n} & , & \kappa^2 G_{3,n}, \dots, & -m^2 + \Theta_{1,0}^n & ; & 0 & , & 0 & , & 0 & , \dots, & -2\kappa m \\
 0 & , & 0 & , & 0 & , \dots, & 2\kappa m & ; & \kappa^2 J'_{1,n} & , & \kappa^2 J'_{2,n} & , & \kappa^2 J'_{3,n}, \dots, & -m^2 + \Theta_{5,0}^n
 \end{array} \right\} (37)$$

For all conditions satisfying MAXWELL's criterion, the quantities $G_{\lambda,\mu}$, $J'_{\lambda,\mu}$ will be small. So that, provided κ^2 is not too great, the value of the determinant (37) will be small for those values of κ that satisfy the relation

$$\left. \begin{array}{cccccccc}
 -m^2 + \Theta_{1,0}^1, & 0 & , \dots, & 0 & , & -2\kappa m & , & 0 & , \dots, & 0 \\
 2\kappa m, & , & 0 & , \dots, & 0 & , & -m^2 + \Theta_{5,0}^1, & 0 & , \dots, & 0 \\
 0 & , & -m^2 + \Theta_{1,0}^2, & \dots, & 0 & , & 0 & , & -2\kappa m & , \dots, & 0 \\
 0 & , & 2\kappa m & , \dots, & 0 & , & 0 & , & -m^2 + \Theta_{5,0}^2, & \dots, & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & , & 0 & , \dots, & -m^2 + \Theta_{1,0}^n, & 0 & , & 0 & , \dots, & 2\kappa m \\
 0 & , & 0 & , \dots, & 2\kappa m & , & 0 & , & 0 & , \dots, & -m^2 + \Theta_{5,0}^n
 \end{array} \right\} = 0 \quad (38)$$

This relation is satisfied by those values of κ which satisfy the equation

$$(-m^2 + \Theta_{1,0}^\lambda)(-m^2 + \Theta_{5,0}^\lambda) - 4\kappa^2 m^2 = 0, \dots \dots \dots (39)$$

where λ takes all its integral values in turn. Further it is easily shown that, on any distribution with n large, $-F_\lambda = \frac{1}{2}H'_\lambda$. Hence we fall back upon the same type of equation as we had in the case of equal particles (equation (29)) where we replace nI_λ by F_λ .

Instead of treating the equation (39) separately for the various integral values of λ , since n is large, we may imagine a single equation with the assumption that F_λ is an arbitrary variable parameter. The determinant (37) will then be small, and instability result for all the values of κ given by (39), for all values of the parameter F_λ that exist. With a wide range of values of F_λ corresponding to a wide range in

the magnitudes of the masses of the particles, we may expect to find a broad region of instability.

It can readily be shown that the condition (39) would also be produced if the general case of unequal particles were solved for the complementary function in the same way as has been done for the case of equal particles, which produced (29). This work is not reproduced owing to the length and complexity of the expressions, and also because the results are wholly contained in the condition (39) produced from the particular integral.

§ 6. *Application of the Results to the Saturnian System.*

Equation (39) written out in full is

$$\{\kappa^2(3 - F_\lambda) + m^2\} \{2F_\lambda \kappa^2 + m^2\} - 4\kappa^2 m^2 = 0. \dots \dots \dots (40)$$

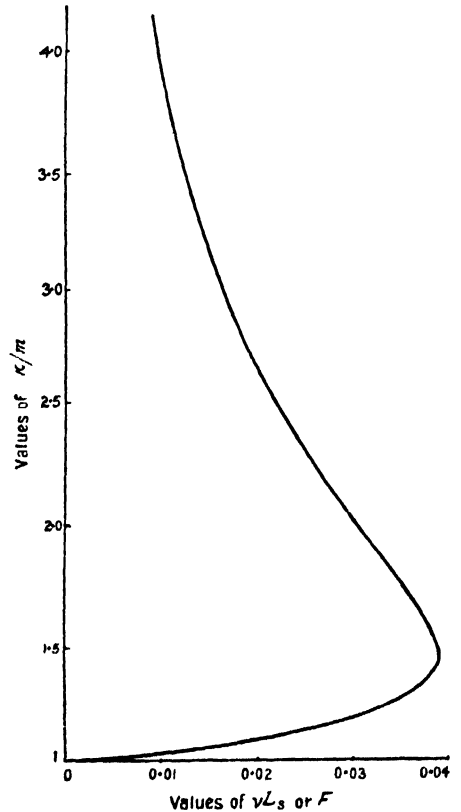
In this equation m is any integer and F_λ may vary between zero and $0.0096n^3m/M$. As the distinctions indicated by the suffix λ are now of no importance, it may be dropped. The solutions of (40) will give approximately the positions where divisions in the Ring of Saturn may be expected.

For any given value of F , there are four values of κ/m , two pairs equal with opposite signs. For any given value of κ/m there are two values of F ; one, however, being greater than the MAXWELL limit, is excluded. The limiting value of F for real values of κ/m is 0.039. This is, of course, the same result as that found by MAXWELL.

The relation between κ/m and F is shown in the figure, and the table shows actual numerical values.

We may readily assume that in the existing Rings of Saturn there are particles of all masses from the infinitesimal to MAXWELL'S upper limit. These will give rise to varying values of F , depending upon the masses of the particles adjacent to the particle under consideration. The

maximum value of F is itself small compared with unity; we shall then arrive at a limit of κ by taking $F = 0$ in equation (40). We find thus that the boundary of a division should occur at $\kappa/m = 1$, for each integral value of m .



νI_s or F.	κ/m .		$m = 1.$		$m = 2.$
			a/a' .		a/a' .
0	1	∞	0	1	0.6299
0.01	1.0395	3.9336	0.0243	0.8223	0.6456
0.015	1.0639	3.1412	0.1534	0.7745	0.6549
0.020	1.0940	2.6509	0.2102	0.7349	0.6656
0.030	1.1750	2.0156	0.2810	0.6333
0.038	1.3353	1.5772	0.3981	0.5117	
0.039	1.4424	—	0.4549		

Remembering that $\kappa = -\omega(\omega' - \omega)^{-1}$, $\omega^2 a^3 = \omega'^2 a'^3$, we find:

$$\begin{aligned} \text{For } m = 1, \quad \omega = \infty, \quad a/a' = 0, \\ m = 2, \quad \omega/\omega' = 2, \quad a/a' = 0.62996, \\ m = 3, \quad \omega/\omega' = \frac{3}{2}, \quad a/a' = 0.76289, \\ m = 4, \quad \omega/\omega' = \frac{4}{3}, \quad a/a' = 0.82524. \end{aligned}$$

[It should be remarked that a positive value of κ gives positions without the satellite orbit, and a negative value of κ gives positions within. As κ appears in (40) in the form of a square, both positive and negative forms result. We should therefore have the same phenomena in a ring of particles beyond the satellite orbit as we find within].

The result $a = 0$ implies a division of the ring at the origin. This would fall within the planet itself. But if the zone consequent upon the variation of F is extensive, it may extend beyond the surface of the planet and show a clearance of particles there.

For $m = 2$, $a/a' = 0.62996$. In the case of satellite Mimas this should indicate the commencement of a division in the ring at distance 16.9". CASSINI'S Division begins at 16.87" and ends at 17.64". This agreement is very remarkable.

Reference to the figure shows that in the vicinity of $\kappa/m = 1$, κ increases with F. But as κ increases so does a/a' . Hence the instability caused by the larger values of F should be in positions corresponding to larger values of κ , that is, to larger values of a/a' . In other words, the division should extend outwards. This agrees with the observational data just quoted. We may then attribute the production of CASSINI'S Division to Mimas.

For $n = 3$, $a/a' = 0.76289$.

For satellite Mimas, this should cause a division at distance 20.46". This is just beyond the outer edge of Ring A, which terminates at 20.01".

Considering next the satellite Enceladus, we should find a division at the origin

for $n = 1$, and at distance $21\cdot69''$ for $n = 2$. The last is again just beyond the limits of Ring A.

The remaining satellites all produce instability at the origin, but the other points at which this occurs are outside the existing ring.

We may use the observations of the dimensions of CASSINI'S Division to determine the maximum value of F appearing. As we have already found, the inner edge corresponds closely to $F = 0$. The radius of the outer edge is $a = 17\cdot64''$. Hence for satellite Mimas $a/a' = 0\cdot65781$, giving $\kappa/m = 1\cdot0720$.

If now equation (40) be solved for F , taking this value for κ/m , the result is $F = 0\cdot0173$. Hence we may conclude that F ranges from zero to $0\cdot0173$.

Using this value of F , we now proceed to the study of the roots of equation (40). Solving, we find

$$\kappa/m = \pm 1\cdot0720 \quad \text{and} \quad \kappa/m = \pm 2\cdot8917.$$

Take $m = 1$. Then

$$a/a' = 0\cdot1712 \quad \text{and} \quad a/a' = 0\cdot7535.$$

We may expect to find a clearance of particles from $a/a' = 0$ to $0\cdot1712$; and from $a/a' = 0\cdot7535$ to unity.

The first gives the extent of the clearance near the origin.

For the various satellites its dimensions are :

Mimas	$a = 4\cdot59''$,
Enceladus	$a = 6\cdot16''$,
Tethys	$a = 7\cdot30''$,
Dioné	$a = 9\cdot34''$,
Rhea	$a = 13\cdot07''$,
Titan	$a = 29\cdot94''$.

The radius $a = 9\cdot34''$ indicates approximately the inner radius of the Crêpe Ring, while $a = 13\cdot07''$ indicates more closely the inner radius of Ring B.

Applying the second ratio, $a/a' = 0\cdot7535$, to Mimas, we find radius $a = 20\cdot2''$. There should be a clearance of particles from $20\cdot2''$ up to the satellite itself. This indicates with considerable precision the termination of the whole ring, which has a radius $20\cdot01''$.

These results are subject to modification owing to the effect of the oblateness of the planet Saturn and the influence of one ring upon another. But the agreement of theory and observation in this first approximation is sufficiently remarkable.

The interpretation of the effect of Dioné and Rhea on the inner parts of the ring is not clear. From the theory one would expect that any satellite could affect a clearance of particles from the origin up to a radius given by $a/a' = 0\cdot1712$. In that case Titan, the largest of the satellites, should dissipate the whole of the existing rings, for this ratio carries us far beyond the outer radius.

There are therefore two facts to explain. First, the existence of the Crêpe Ring within the dissipative area of Rhea, and second, the existence of the bright rings within the dissipative area of Titan. In connection with the first, LOWELL has noted a definite black band within Ring B, so that there is a clearance of particles between the Crêpe Ring and the bright rings. It would appear as though the dissipative power of the satellites was only effective near the outer boundary of the unstable area about the origin. To discuss this, let us examine the analytical results.

It has already been pointed out how very small the exponent c is, as given by (18), indicating a very slow rate of dispersion. Consider, instead, the numerators of the expressions (24), the vanishing of the denominators of which causes the instability. The numerators are small because of the quantities $\Theta_{3,m}$ and $\Theta_{6,m}$. In the case under discussion, $m = 1$. From (5)

$$\Theta_{3,1} = \nu' \kappa^2 \kappa'^{1/2} \left\{ \frac{db_1}{d\alpha} - 1 \right\},$$

$$\Theta_{6,1} = \nu' \kappa^2 \{ \kappa'^{2/3} b_1 - \kappa'^{1/3} \}.$$

Using the well-known expression for b_1 ,* we find

$$\Theta_{3,1} = \nu' \kappa^2 \kappa'^{1/2} \left\{ \frac{9}{8} \alpha^2 + \frac{7}{2} \frac{5}{8} \alpha^4 + \dots \right\}$$

$$= \nu' \{ \omega'/\omega - 1 \}^{-2} \left\{ \frac{9}{8} \alpha^4 + \frac{7}{2} \frac{5}{8} \alpha^6 + \dots \right\}$$

$$\Theta_{6,1} = \nu' \kappa^2 \{ \alpha (\alpha + \frac{3}{8} \alpha^3 + \frac{1}{1} \frac{5}{8} \alpha^5 \dots) - \alpha^2 \}$$

$$= \nu' \{ \omega'/\omega - 1 \}^{-2} \left\{ \frac{9}{8} \alpha^4 + \frac{1}{1} \frac{5}{8} \alpha^6 \dots \right\}.$$

For small values of α , ω'/ω is small, and the value of $(\omega'/\omega - 1)^{-2}$ will be greater than, but not far from, unity. Hence the values of $\Theta_{3,1}$ and $\Theta_{6,1}$ depend approximately upon the fourth power of α or α/α' . It is clear then that the numerators in (24) will be vanishingly small except for the larger values of α/α' .

The physical meaning is that, while instability will always take place when the denominators vanish, the rate of dissipation will be small except for the largest values of α which are permissible. There will also be a uniform grading in the rate of dissipation as α increases.

Applying this result to the case of Saturn's satellites, we may expect to find actually a clearance only near the outer limits of the areas under consideration. The areas of clearance of the first three satellites fall within the body of the planet. Dioné causes the clearance between the surface of the planet at 8'65" and 9'34", which is approximately the commencement of the Crêpe Ring. The limit of the area of clearance of Rhea is 13'07", and only near that boundary is the action effective, the Crêpe Ring being undispersed in the weaker part of the field. The bright rings are clearly in the weak part of Titan's field of clearance, and so continue to exist. It is obvious, however, that with passage of time the Crêpe Ring will be dispersed by Rhea and the whole by Titan.

* TISSERAND, vol. i., p. 272.

We have found the maximum value of F appearing as $0\cdot0173$. It was previously shown that the limiting value of F was $0\cdot0096n^3\bar{m}/M$. Hence

$$0\cdot0096n^3\bar{m}/M = 0\cdot0173,$$

which gives

$$\begin{aligned}\bar{m}/M &= \frac{0\cdot0173}{0\cdot0096n^3} \\ &= 1\cdot8/n^3.\end{aligned}$$

That is, the size of the largest particles is just below that given by MAXWELL'S criterion.

§ 7. *Summary and Conclusion.*

(1) Assuming that a planet is surrounded by concentric rings of particles performing approximately circular orbits when unperturbed, and that the influence of one ring upon another may be neglected to a first approximation, the effect upon these rings of a satellite performing also an unperturbed circular orbit is discussed.

If the particles in the rings are all equal, it is shown that we should expect, in certain places, large perturbations to take place, such that the particles in a particular ring would leave that ring and mingle with those of other rings, and so leave a "division."

(2) As there is no reason to believe that the particles in any ring are all equal, the analysis is extended to cover the case of unequal particles.

We assume that in any ring the number of particles is large, and that therefore we shall probably have one specimen at least of all particles from the smallest to the largest.

It is then shown that the divisions would become more extended, and therefore more readily visible.

(3) On the supposition that some of the particles at any rate are indefinitely small, we obtain CASSINI'S Division at once. On making use of the dimensions of this division to estimate the greatest magnitude of the particles in any ring, we find the following results:—

Satellite Mimas should produce a clearance of particles from radius $20\cdot2''$ up to itself. The ring should therefore terminate at $20\cdot2''$. Observation shows that it terminates at $20\cdot01''$.

Satellite Mimas should produce a division from radius $16\cdot9''$ to $17\cdot64''$. (This last measurement was used as a datum for estimating the magnitude of the greatest particles.) Observation gives the limits of CASSINI'S Division as $16\cdot87''$ and $17\cdot64''$.

Satellite Dioné should produce a clearance of particles from the region of the surface of the planet up to radius $9\cdot34''$. The Crêpe Ring is observed to begin with a diffused edge at $10\cdot83''$.

Satellite Rhea should also produce a clearance of particles up to radius 13'07". The inner edge of Ring B is observed to commence at 13'21".

The existence of the Crêpe Ring in a dissipative area is also discussed.

(4) By the inclusion of the effect of the oblateness of Saturn and the influence of one ring of particles upon another these results might be still further improved.

The theory presented therefore gives a closely quantitative account of the salient features of Saturn's Ring. The numerous smaller divisions observed by LOWELL and others are not accounted for; but, for the reasons given in § 4, their existence is not excluded.

(5) The dimensions of CASSINI'S Division show that particles of all sizes up to a limit just short of that imposed by MAXWELL for stability exist in the rings.

Appendix on the Data of the Problem.

I. Dimensions and divisions of the ring in seconds of arc at mean distance* :—

Distance from centre of planet to—	
Inner edge of Crêpe Ring	10'83"
Inner edge of Ring B	13'00"
Divisions of Ring B {	B1 13'39"
	B2 14'04"
	B3 14'74"
	B4 15'32"
	B5 15'69"
	B6 15'95"
Outer edge of Ring B	16'87"
Inner edge of Ring A	17'64"
Division in Ring A	19'00"
Outer edge of Ring A	20'01"

II. Equatorial diameter of Saturn 17'30"

III. Elements of satellites :—

	Mean distance.	Mass as fraction of Saturn.
Mimas	26'82"	$7 \cdot 10^{-8}$
Enceladus	34'43"	$25 \cdot 10^{-8}$
Tethys	42'66"	$11 \cdot 10^{-7}$
Dioné	54'59"	$18 \cdot 7 \cdot 10^{-7}$
Rhea	76'38"	$4 \cdot 10^{-6}$
Titan	174'8"	$2 \cdot 1 \cdot 10^{-4}$

* LOWELL, 'Observatory Bulletin,' No. 68, and "Lecture" on April 26, 1916, in 'Journal of Royal Astron. Soc. of Canada.'

*The Analysis of Sound.**

PART I.—*The Experimental Analysis of Sound in Air and Water : Some Experiments towards a Sound Spectrum.* By GUY BARLOW, D.Sc. (Wales, Lond., Birm.), F.Inst.P., Lecturer in Physics in the University of Birmingham, and H. B. KEENE, D.Sc. (Birm.), F.Inst.P., Lecturer in Physics in the University of Birmingham.

PART II.—*The Theory of Analysis of an Electric Current by Periodic Interruption.* By G. BARLOW, D.Sc.

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PART I.—THE EXPERIMENTAL ANALYSIS OF SOUND IN AIR AND WATER : SOME EXPERIMENTS TOWARDS A SOUND SPECTRUM.

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* The experiments described in this paper are published with the permission of the Admiralty. They form part of an investigation carried out by the authors in the Physics Department of the University of Birmingham during the years 1916 and 1917, at the suggestion of Sir O. LODGE on behalf of the Board of Invention and Research.

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INTRODUCTION.

A METHOD of analysing an alternating current termed "Analysis by Periodic Interruption" was worked out by G. BARLOW in May, 1916, and is described by him in some detail in Part II. The present paper, Part I., gives an account of certain experiments in which this method has been applied to the analysis of sound vibrations in air and water with the object of obtaining "sound spectra." The principle of the method may be stated as follows. The alternating current circuit contains a direct current galvanometer and also an interrupter of which the speed can be varied over the whole range of frequency to be investigated. Generally the type of interrupter used is such

that the intervals during which the circuit is open and closed are equal. When the interruptions synchronize with any component $A \sin 2\pi n t$ of the current, fig. 1 (a), the galvanometer responds by giving a steady deflection of magnitude depending on the phase difference. Fig. 1 (b) shows interruptions and current in the same phase. Practically it is better to allow a slight difference in frequency; the galvanometer then oscillates slowly to and fro as the phase alters. The maximum amplitude of the galvanometer swings is then proportional to the amplitude A of the component current—actually

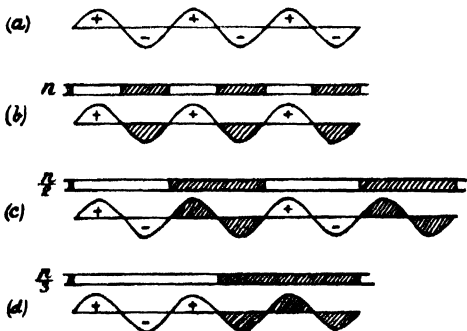


Fig. 1. Interruption of a simple harmonic current at frequencies $n, n/2, n/3$.

it measures A/π . In making the analysis the frequency of interruption is slowly increased over the whole range. The approach to the condition of synchronism is indicated by

a rapid oscillation of small amplitude followed by slower oscillations of greater amplitude until the maximum is reached. Afterwards the oscillations die down in the reverse order. This characteristic motion exhibited by the galvanometer will be referred to by the term "response." In this way the amplitude of each component may be determined. At the same time the corresponding frequency is obtained by observing, at the moment of maximum, the frequency at which the interrupter is driven. For a component of given amplitude the range of frequency over which the response is greater than half its maximum value, and which may be called the "width of response," is the *same* at all frequencies. For example, if a response at 10/sec. falls to half value for frequencies of interruption of 9 and 11/sec., then one at 1000/sec. will fall to half value at 999 and 1001/sec. It will therefore be seen that it is necessary to have perfect control over the speed of interruption, especially in the higher frequency region, and the same time must be spent in sweeping over a range such as 1000–1100/sec. as over 10–110/sec. In measuring a response the rate at which the speed of interruption may be changed is conditioned by the period of the galvanometer. It is necessary that the speed should not change sensibly during an interval of time of the order of the galvanometer period. The galvanometer may be of any type, but its vibrations should be well damped so as to be nearly dead-beat. A suitable period is 3 seconds. Under these conditions the width of response is 0.7/sec.

It is a peculiarity of this method of analysis that a *single* simple harmonic component of frequency, n , gives rise to responses not only when the frequency of interruption is n , but also when it is $\frac{1}{3}n$, $\frac{1}{5}n$, $\frac{1}{7}n$, &c., and these responses have amplitudes $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$ of the fundamental response. These responses will be called "Subharmonics." Their origin is made clear in fig. 1, which also shows why the even-order subharmonics $\frac{1}{2}n$, $\frac{1}{4}n$, $\frac{1}{6}n$, &c., are non-existent. When the alternating current represented by (a) is interrupted at its frequency n , all the negative elements are suppressed as shown in (b), giving a unidirectional current in the galvanometer. When interrupted at $\frac{1}{2}n$, as in (c), an equal number of positive and negative elements are passed through giving no resultant current in the galvanometer. But when interrupted at $\frac{1}{3}n$ as represented in (d), there is a resultant current due to the odd positive elements. This is the third order subharmonic, and it will be seen by comparing (b) and (d) that it has one-third the magnitude of the fundamental.

The presence of these subharmonics is not so objectionable in practice as one might expect, in fact their frequencies and relative magnitudes have on certain occasions assisted in the identification of the fundamental with which they are associated. There is a close analogy with grating spectra, inasmuch as each subharmonic corresponds to a spectrum of a different order. The even orders are absent just as in a grating where the opaque and transparent parts of the grating-element are equal in width. If the intervals of make and break are unequal, then the even-order subharmonics are introduced.

A type of interrupter has been constructed in which by repeating the sequence of

intervals shown in fig. 2 it has been found possible to eliminate the subharmonics $\frac{1}{2}n$, $\frac{1}{3}n$, $\frac{1}{4}n$, &c., in addition to the even orders. The fundamental response is reduced to three-fourths of its usual value, and there are certain other disadvantages suggested by the few experiments



Fig. 2.

which have so far been made.

In order to analyse by this method of periodic interruption mechanical vibration of a solid body or of sound waves in air or water, the vibration must be converted into an electrical current which in wave-form faithfully represents the original motion. Some distortion of the wave-form would not be a serious objection, provided it followed a simple relation allowing correction to be made. Actually very few methods of converting vibration into current are available, and none of these is free from objection. Among the most practicable are :—

- (1) Variation of electrical resistance by pressure, *e.g.* carbon microphone.
- (2) Variation of electrical resistance by change in thermal conditions, *e.g.* Tucker Hot Wire Microphone.
- (3) Electromotive force generated by induction, magnetophones, &c.

It may be pointed out that all these methods depend on induction (assuming a transformer is used in (1) and (2)), and the final current therefore represents the *velocity* of the vibration under investigation, but this is not an objection from the point of view of analysis.

For, suppose the original vibration is resolved into simple harmonic components—

$$a_1 \sin (2\pi n_1 t + \alpha_1) + a_2 \sin (2\pi n_2 t + \alpha_2) + \&c. ;$$

then, assuming no other form of distortion, the current will be proportional to

$$2\pi a_1 n_1 \cos (2\pi n_1 t + \alpha_1) + 2\pi a_2 n_2 \cos (2\pi n_2 t + \alpha_2) + \&c.$$

The analysis of this current will then give correctly the frequencies of all the component vibrations, but in each case the amplitude is magnified in proportion to the corresponding frequency. The product $2\pi an$, representing the maximum velocity, is itself an appropriate measure of the importance of the component, as the relative energies for different components are proportional to $(an)^2$.

In the present experiments the determination of the frequencies of the components has been effected with all the accuracy desired, but as it has not yet been found possible to avoid selective action due to resonance of diaphragms, the amplitudes of the components are not faithfully represented. No attempt has been made to deduce the absolute amplitudes of motion of the original vibration.

When the components of a vibration have strictly commensurable frequencies, as in a harmonic series, the phase relations of the components are quite definite, and the determination of the relative phases might be of value—in fact it would be necessary

if it were required to reconstruct the wave-form of the vibration. It would not be difficult to adapt the method of analysis by interruption for the determination of phase-difference, but no experiments in this direction have yet been made. Hence at present the method is incomplete in that it fails to take account of wave-form. If the wave-form were required it would appear simpler to deal with it directly by means of an oscillograph method than to build it up from a complete analysis.

APPARATUS.

The interrupter (fig. 3) consisted of a brass cylinder made up of five discs ; the first was complete and served as a slip-ring, the other four contained ebonite segments giving respectively 1, 4, 16 and 64 interruptions per revolution of the cylinder. Contact was made by means of two small brushes cut from $\frac{1}{16}$ mm. sheet brass, each brush possessing four or five separate fingers. One brush pressed lightly on the slip-ring, the other on whichever disc was the most convenient for the frequency under examination. The electric contact was found to be satisfactory when the surfaces were kept clean and well lubricated with machine-oil. The cylinder, insulated with ebonite, was mounted directly on the shaft of an electric motor the

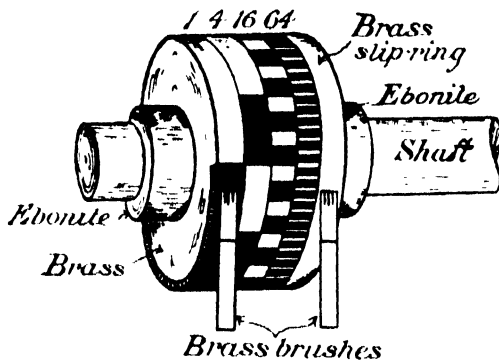


Fig. 3. Interrupter.

speed of which could be regulated over the range 3-30 revs./sec. The four discs gave overlapping ranges of interruption frequencies with a total range of 3 to over 2000/sec.

It was required that the rotation of the interrupter should be extremely uniform and perfectly under control. This is especially important for analysis at high frequencies ; thus for 1000/sec. an irregularity of rotation of 1 in 2000 would in one second completely reverse the phase of the response, and with a galvanometer of 3 sec. period the full value of the response would not be obtained.

Much preliminary work was done in examining the conditions necessary for steadiness and smooth running of small motors. Two forms of apparatus have been constructed :—

(1) A Siemens-Schücker 12-volt $\frac{1}{8}$ h.p. motor was directly coupled to a flywheel (radius 11 cm., mass $7\frac{1}{2}$ kgm.) to prevent sudden changes of speed. This apparatus, which was used in nearly all the laboratory experiments, was suitable for exact measurements, as the motor could be made to run very slowly over any required small range of speed, and this range could be repeated by using the finger as a brake on the flywheel. Since plain lined bearings were used, there was the disadvantage that the ultimate speed attained was limited only by the work done in friction, and this varied with the state of the lubrication. The great weight of the flywheel made this frictional

work so considerable compared with the power of the motor that it was not feasible to use an eddy-current brake to give further stability.

(2) This apparatus (fig. 4) was made as light as possible so as to be easily portable.

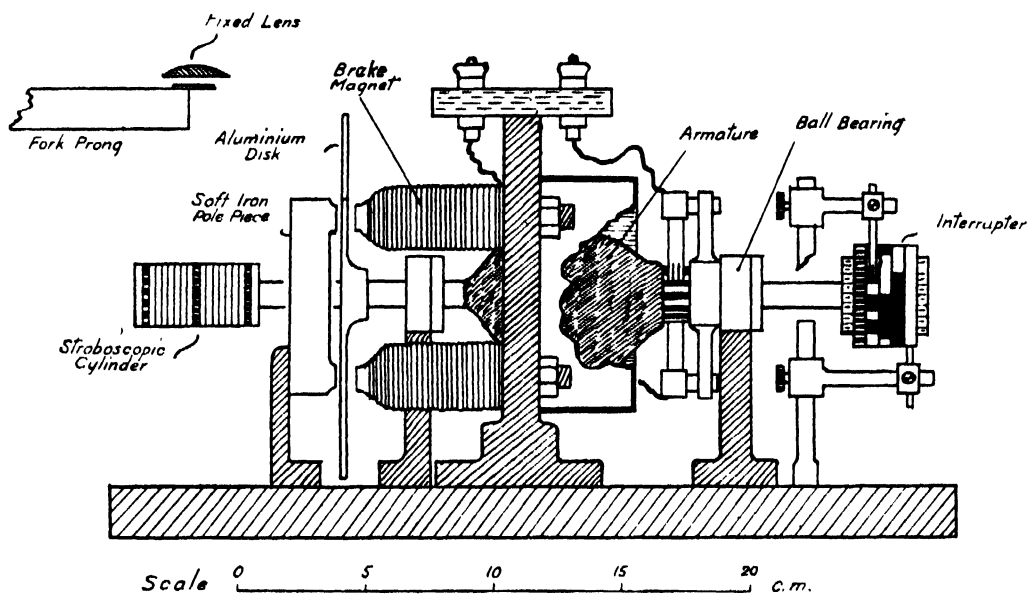


Fig. 4. Portable motor-driven interrupter (elevation).

The shaft of a Siemens-Schückert 12-volt, $\frac{1}{40}$ h.p. motor was lengthened and fitted with a couple of ball-bearings carried by brackets from a rigid base-plate. No fly-wheel was used, the required steadiness being given by an eddy-current brake consisting of an aluminium disc (radius = 7 cm.) attached to the shaft and spinning between the four poles (distant $5\frac{1}{2}$ cm. from the axis) of an electromagnet excited by a constant current. In this case the friction of the bearings only formed a small fraction of the total work done.

The different circuits were excited as follows :—

Armature	6 volts, 0-3 amperes.
Field	6 „ 3 „
Brake	2 „ 1 ampere.

Except in special cases the whole range of frequency could be covered by variation of the armature current alone.

In addition to portability (the total weight was 8 kgm.) this apparatus possessed, on account of its small inertia, the advantage of extreme rapidity in attaining a steady speed. For a given adjustment the final speed was reached in about 3 seconds from rest, whereas the former apparatus required 10 minutes.

The chief source of trouble in obtaining a constant speed was found to be due to irregular variations in brush-contact on the commutator of the motor. Both carbon and solid copper brushes were found to be unsatisfactory on this account. These were

eventually replaced by springy brushes built up of long thin strips of phosphor-bronze, each strip being slit longitudinally into alternately two and three fingers. In this way a number of independent contacts was obtained, and much steadier conditions of running of the motor were secured. Even with the improved brush design occasional irregularities in speed still occur, and are usually traceable to the contact conditions on the commutator. This appears to be the outstanding difficulty in obtaining a constant speed of revolution with an electric motor.

The speed of the motor was measured by a stroboscopic method.* For this purpose the shaft carries a cylinder the surface of which is divided into 21 rings. Each ring is marked out into 20, 21, 22 . . . 41 equally spaced black squares with white intervals, and for ease in identification every fifth ring is tinted red. The stroboscopic cylinder is viewed through double slits mounted on the prongs of a maintained fork (64/sec.) giving 128 views per second of the rotating patterns. The speed is determined by observing the number of the ring which appears to be stationary. When, as is generally the case, no ring is exactly stationary, then two consecutive rings are seen to rotate slowly in opposite directions with different speeds. By measuring the rate of progression of one of these the required frequency may be obtained with a degree of accuracy limited only by the constancy of the motor speed. In practice it is sufficient to interpolate by estimation, as this can be done without giving an error in the frequency of more than $\frac{1}{3}$ per cent.

The laboratory experiments were made with a Broca galvanometer (10 ohms), and, when required, a transformer having a primary resistance of 4 ohms and secondary of 90 ohms. At the reservoir a Broca galvanometer (100 ohms) was used, and also a transformer with resistances 60 and 110 ohms. The period of the galvanometer was in both cases adjusted to be 3 sec., and then made almost dead-beat. For the purpose of dealing with vibrations of great complexity, it would appear quite practicable to modify the present apparatus to give a photographic record of the "spectrum."

ANALYSIS OF A CURRENT.

Before proceeding to analyse sound vibrations the following experiments were made to test the reliability of the method by applying it to analyse alternating currents of known characteristics.

(1) *Simple Harmonic Current.*

The current was generated in a small coil, wound in the form of a figure 8, by the motion through it of a U-shaped magnet (made from a piece of knitting-needle 4 cm. long) attached either to the prong of an electrically maintained fork, or in the case of the lower frequencies to an electrically maintained steel strip.

* RAYLEIGH, 'Phil. Mag.,' 1907.

The amplitude of the current i_0 in c.g.s. units was calculated from the formula

$$i_0 = \frac{JN \cdot 2\pi na}{\sqrt{R^2 + 4\pi^2 n^2 L^2}},$$

where,

J = magnetic flux cut by one turn of the coil per cm. displacement, directly determined by means of a ballistic galvanometer.

N — number of turns on the coil, usually 1-10 turns.

n — frequency.

a = amplitude of motion of the magnet, determined by a microscope with eyepiece scale.

R = resistance of the circuit (9 ohms) in c.g.s. units.

L = self-induction of the circuit. The only appreciable self-induction was due to the Broca galvanometer (·0073 henry).

On interruption the current measured by the galvanometer response is i_0/π , and from the known sensitiveness of the galvanometer (35 div. per microampere) the magnitude of the response can be calculated and compared with the observed value.

The results for experiments made with frequencies ranging from 1-2000/sec. are given in the following table:—

Frequency. n .	Amplitude of motion. a .	Amplitude of current, i_0 .	Galvanometer deflection (observed).	Galvanometer deflection (calculated).
	cm.	amp.		
1·07	0·200	$0·67 \times 10^{-6}$	70*	73
6·4	0·170	3·4	35	38
11·5	0·210	3·1	38	34
14·9	0·200	3·8	47	43
27·5	0·210	7·4	70	83
95	0·200	11·0	120	120
500	0·025	14·3	190	160
990	0·002	1·26	20	14
2040	0·003	3·8	42	40

The experimental error is likely to be greatest in the case of the two highest frequencies owing to the small amplitudes to be measured. Moreover, the 2040 fork had to be sustained by bowing. The agreement between the observed and calculated values for all frequencies is as good as can be expected, but it only holds if account is taken of self-induction, since at the higher frequencies the correction for impedance is very large.

* Galvanometer with 9·4 sec. period, giving 340 div. per microampere.

The currents were very nearly pure, but contained traces of the even harmonics. Thus the analysis for the 95 fork gave the current-amplitudes of the components n , $2n$, $4n$, in the ratio $1 : \frac{1}{2} : \frac{1}{6}$. These harmonics may have been present in the fork vibration, but it is more likely that they were produced by want of uniformity in the magnetic flux of the U-magnet. The so-called subharmonic responses described above were also observed.

(2) *Close Pair of Simple Harmonic Components – Resolving-Power.*

The current was produced by putting in series two U-magnet generators on separate forks of frequency 64/sec., which could be adjusted to have a slight difference of frequency by means of sliding loads on one of them. The responses were examined for each alone, and then for both together. When the difference in frequency was reduced to 0.6/sec. the two components could still be resolved, owing to a distinct drop between the two maxima. Separation was also effected when the components were very unequal in magnitude, e.g. a ratio 5 : 1. In all cases the double nature of the response was at once evident from the characteristic beating of the galvanometer oscillations, whereby it was readily distinguished from that due to a single frequency. When the components are unequal the beating is more distinct on the side of the smaller. The difference in frequency of the components can be determined directly by the frequency of the beats without observing the positions of the two maxima.

Experiments on resolving-power were not made at higher frequencies, but theory shows that two components with the above limit of frequency-difference should be resolved whatever their absolute values, e.g. 1000 and 1000.6/sec.

(3) *Current Containing Harmonic Series.*

Two types of current were produced simultaneously from the same electrically maintained fork with mercury contact, 32/sec.—the first by induction in a couple of turns of wire round the electromagnet of the fork, the second by using a small air-transformer consisting of a few turns of wire (giving negligible self-induction) in which the current in the primary was interrupted by a separate platinum contact attached to the fork-prong and dipping into mercury. The level of the mercury was adjusted to give equal time intervals of make and break. By means of a "throw-over" key the amplitude of corresponding harmonics in the two currents could be compared.

Harmonic	Frequency.	Current-amplitude with iron-transformer.	Current-amplitude with air-transformer.
1	32	280	215
2	64	150	10 erratic
3	96	240	210
4	128	210	20 erratic
5	160	180	205
6	192	175	30 erratic
7	224	135	200
8	256	150	20 erratic
9	288	100	190
10	320	130	40 erratic
11	352	85	180
12	384	80	20 erratic
13	416	85	160
14	448	65	20 erratic
15	480	150	160
16	512	135	120 fairly good
17	544	140	170
18	576	140	140
19	608	115	145
20	640	120	130
21	672	110	100
22	704	120	120
23	736	95	100
24	768	100	80
25	800	90	130
29	928	85	90
32	1024	90	100
33	1056	70	100
35	1120	60	130
37	1184	70	130
40	1280	95	120
41	1312	80	110
48	1536	60	90
49	1568	50	60

After the twenty-fifth the examination of several harmonics was omitted. The table shows that the current produced by the air-transformer consists of practically odd harmonics only, until the higher frequencies are reached, and their amplitudes are nearly equal. This is what would be expected from the nature of the make and break.

(4) *Complex Current.*

In this experiment six maintained forks were used with their U-magnet generators in series. The current-amplitude and frequency were first measured for each generator taken separately, then all were excited and a complete analysis made. This "spectrum" is given in fig. 5. It will be seen that all six primary constituents were found, together with their various subharmonics, and also a weak octave of the fork A.

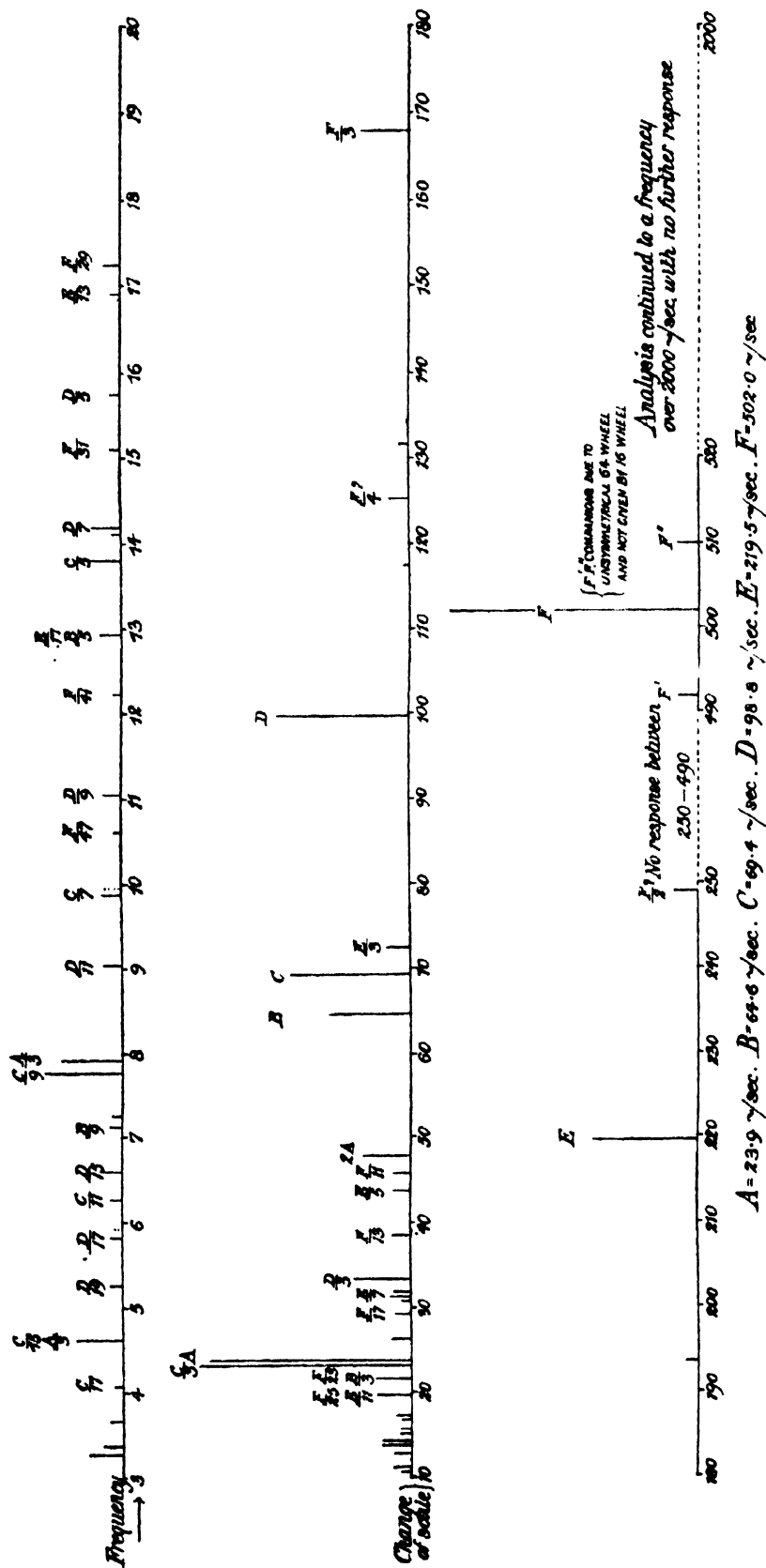


Fig. 5. Analysis of complex current. Range of analysis 3 ~ sec. to 2000 ~ sec.

The frequencies and amplitudes of the primaries were as follows :—

	Frequency.	Amplitude (Galvanometer response).
A	23·9	90
B	64·6	42
C	69·4	62
D	98·8	68
E	219·5	55
F	502	130

These values are sensibly the same as were obtained in the separate examination except for some small differences in amplitude easily accounted for by unavoidable variations in the amplitudes of the forks during the experiment. As the whole range explored extended from 3–2000/sec. and every response found was critically examined, this analysis occupied a long time—about four hours.

In the diagram the response F is shown accompanied by companions F' and F'', forming a triplet. These companions are due to irregularities in the spacing of the segments of the 64 interrupter-disc. They are always given with this disc, but not with the others in which, presumably, the spacing is more uniform.

When two responses occur close together the amplitude of each is reinforced by the other. Several such examples occur in the diagram, notably in the case of A and C/3.

ANALYSIS OF SOUND IN AIR.

(a) *Magnetophone Receiver.*

A Graham's Patent "Loud-talking Apparatus" (iron diaphragm 10 cm. diameter, 0·061 cm. thick) was placed in series with the Broca galvanometer and interrupter. This instrument, although not extremely sensitive, proved to be satisfactory for analysing sounds. Examples of some of the experiments made are here briefly indicated :—

- (1) Bowed fork, $n = 512$, at 8 metres gave a good response at this frequency.
- (2) Voice, singing a note, $n = 256$, with moderate intensity gave a deflection off the scale.
- (3) Organ pipe, blown at a distance of a few feet. Analysis gave the fundamental $n = 531$ and also the third harmonic. The change to the octave on over-blowing was readily shown.
- (4) In determining the frequency of a fork it is not necessary that the vibration should be sustained. By tapping a 640 fork with a rubber hammer the frequency was determined by the interrupter to $\frac{1}{2}$ per cent.

- (5) In a similar way the fundamental frequencies were found for three metal diaphragms (3 in. diameter). The values obtained were 345, 890, 1520/sec.
- (6) Highly damped vibrations. The resonator box belonging to a standard 384 fork was tapped in front of the trumpet of the magnetophone. The maximum deflection was obtained at 344/sec., and the fork itself gave a response at 382/sec.

In those cases in which the vibrations are not sustained there is no regular response, but merely a kick on the galvanometer for each excitation. At any one speed the kicks may be of either sign, and they vary in magnitude over a certain range according to the phase-difference. The frequency is determined by the interrupter speed for which the range is a maximum, but, as would be expected, this maximum becomes ill-defined when the vibrations are highly damped.

In the above experiments electrically maintained forks were not used owing to their direct magnetic action on the magnetophone.

(b) Carbon Granule Microphone Receiver.

An ordinary commercial instrument (G.E. Co.) of the "solid-back" type with conical mouthpiece was suspended by thin rubber cords. Some preliminary experiments in which the receiver was enclosed in an exhausted vessel showed that no appreciable vibrations were communicated through the supports to the microphone when suspended in this way.

The electrical connections are shown in fig. 6. When required an additional resistance was placed in the galvanometer circuit to reduce the sensitiveness. As compared with the magnetophone the microphone has the following advantages:—

- (i.) Greater sensitiveness.
- (ii.) It is non-inductive, and therefore not affected by stray alternating fields from electric forks, &c.

The disadvantages are:—

- (i.) Resonance: The above receiver has a natural frequency about 1030/sec.
- (ii.) The *mean* resistance during vibration differs from the normal resistance (it is actually increased), and therefore the starting and stopping of a sustained sound gives kicks (in opposite directions) on the galvanometer.
- (iii.) The resistance, and consequently the sensitiveness, is therefore subject to uncontrollable variation.

(1) *Pure Tones.*—A fork 512, mounted on its resonator, when bowed or struck near the receiver gave, on analysis, deflections off the scale.

When required an additional resistance

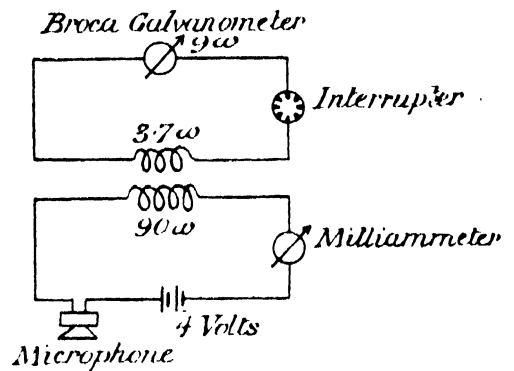


Fig. 6.

When bowed 8 metres away a response 150 div. was obtained. (The magnetophone gave 20 div. under the same conditions.) When a 512 resonance box was placed in front of the microphone the response was increased to over 200 div.

Fork 2048 bowed at 2 metres gave 200 div.

It was noticed that the overtone of a certain pipe blown gently gave abnormal disturbances of the microphone without the interrupter. This disturbing frequency was found to be about 1030/sec. Starting and stopping a 1024 fork gave violent kicks on the galvanometer, indicating a large increase of microphone resistance during the vibration. This change of resistance was shown directly on the milliammeter in the microphone circuit. This effect suggested that the microphone would be abnormally sensitive to a note of about this frequency. This was actually the case; thus it was shown that the microphone as a detector was as sensitive as the ear for this particular frequency. *E.g.*, fork 1024 when bowed in adjoining room with doors shut was detected by the galvanometer; to the ear the note was only just audible.

Also this instrument was extremely sensitive to taps on a fork resonator-box, and also in a less degree to almost any sharp taps given by wooden objects—*e.g.*, putting down tool on table or walking about on wooden floor; locomotive whistles from the railway near by were especially effective.

(2) *Note from Harmonical (ELLIS).*—The note selected had a fundamental frequency of 66/sec. All harmonics up to the twentieth (the analysis was not carried further) were observed and their amplitudes measured. The results are given in the following table:—

Harmonic.	Frequency.	Response.	Harmonic.	Frequency	Response.
1	66	100	11	726	70
2	132	100	12	792	10
3	198	50	13	858	15
4	264	40	14	924	5
5	330	60	15	990	80
6	396	20	16	1056	60
7	462	50	17	1122	60
8	528	14	18	1188	10
9	594	25	19	1254	5
10	660	50	20	1320	10

It will be seen that some of the harmonics are as important as the fundamental. The fifteenth, sixteenth and seventeenth harmonics are probably enhanced on account of their frequencies being nearly in resonance with the microphone. The subharmonics are not recorded in the table.

In illustration of the complexity arising from the subharmonics of strong harmonics, the following example from another experiment is given:—

The harmonical note 263/sec. contained a strong fourth harmonic of frequency $263 \times 4 = 1052$. This was greatly enhanced on account of its being so nearly in

resonance with the microphone. The subharmonic series of 1052 was in consequence pronounced. Thus in the experiment a marked response was observed at 151, which is $1052 \div 7$ nearly, *i.e.* the seventh subharmonic of the fourth harmonic of the fundamental vibration. In practice the presence of the subharmonics is generally of less inconvenience than this example might suggest, but it will be seen that analysis is particularly affected if the receiver possesses strong resonance-points very much *above* the region of frequency under analysis.

(3) *Complex Sound.*—In order to test the reliability of the analysis, it was thought desirable to produce a complex sound containing known constituents each of which could be examined separately. Four electrically maintained forks were placed on the same table, and grouped in front of the microphone receiver suspended on rubber cords independently of the table. The four forks selected had the following frequencies:—A 71.3, B 89, C 100, D 261/sec. With all forks sounding together a careful analysis was first made, in which each response was examined separately in order to determine accurately its magnitude and frequency. The result of this analysis is shown in fig. 7. It will be observed that the following frequencies were present:—

- (i.) The four fundamental vibrations A, B, C and D.
- (ii.) Weak octave of C. Those of A and B were not observed, while the octave of D was out of range.
- (iii.) All the important subharmonics of A, B, C and D, except in those cases where they were masked by other responses, *e.g.*, $\frac{1}{3}D$ was missed, as it was nearly coincident with the fundamental B.

With the exception of uncertain responses at 158 and 161/sec., all frequencies found in the analysis are accounted for. The frequency 161 may be due to a combination-tone, either $D - C = 161$, or $A + B = 160.3$. A separate investigation showed that this response only occurred when the forks A and B were sounding together, and that it was not always present even under those conditions.

The analysis was now repeated by making a rapid sweep of the whole range by continuously increasing the speed of interruption *without stopping*

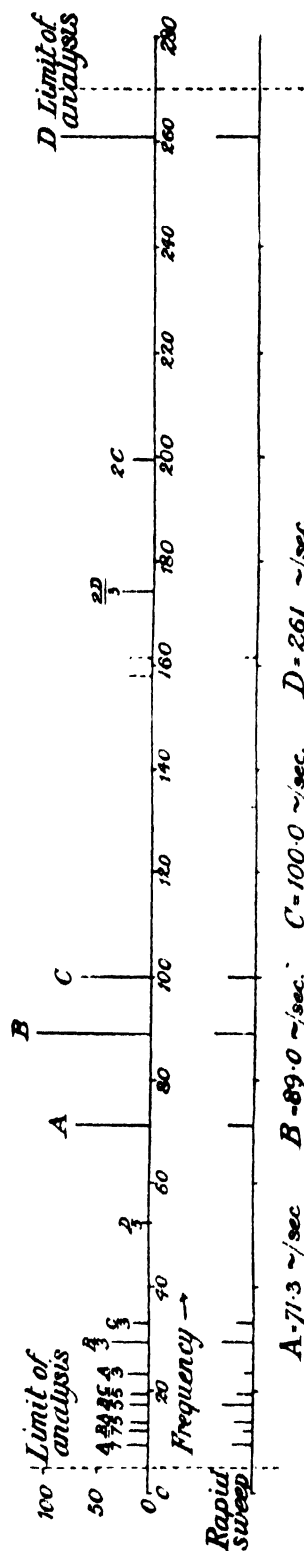


Fig. 7. Analysis of complex sound vibration. Range of analysis 5 ~ sec. to 270 ~ /sec.

to examine the separate responses. The results of this sweep are given in fig. 7, where it will be seen that no important vibrations were missed, and that their frequencies were obtained with almost the same degree of accuracy as in the more careful analysis. As would be expected, the amplitudes as determined in this rapid sweep were generally less than before, and could only be determined roughly.

Other experiments which cannot be described here have shown that all responses, however small or irregular, which occur in an analysis can always be accounted for. In fact, analysis has sometimes indicated the presence in the source of quite unsuspected vibrations which have been afterwards shown to exist by other means.

Replacing the galvanometer by a telephone, a weak sound of machine-like quality was heard, the frequency $D = 261$ being most pronounced, and also a marked throbbing which was probably associated with the heating of B and C (89 and 100).

(4) *Analysis as affected by a General Background of Sound.*—The object of these experiments was to determine to what extent a definite vibration could be masked by the presence of a general “background” of sound. A suitable background was found to be produced by placing a roaring bunsen near the suspended microphone. As heard in the telephone this background completely masked the note from a 256 maintained fork, so that one could not detect when the fork started or stopped. On analysis it was found that the background alone gave a disturbance on the galvanometer at all frequencies of interruption, while the fork response was 15 times that due to the background, showing that in this case the analysis is vastly superior to the ear using the telephone. Similar experiments were made with a 512 fork, but it was found much more difficult to mask this note, two roaring bunsens close to the microphone being required. The analysis appeared less efficient in distinguishing the fork note from the background, the ratio of fork response to background disturbance being of the order 8 : 1. The galvanometer disturbance due to the background showed a maximum when the speed of interruption was of the order of 1000/sec. Previous experiments have shown that this is the resonance region of the microphone.

The masking effects of backgrounds appear to be of considerable interest and importance, and require fuller investigation.

ANALYSIS OF SOUND IN WATER.

It was decided to give special attention to low frequencies ranging from about 5–150/sec. A number of low-frequency sounders and receivers of different types were used, and will be described before the experiments.

(a) *Sources of Sound.*

(1) *Cylindrical Sounder.* (A type used by Lord RAYLEIGH for experiments in air, ‘Phil Mag.’ 1907.)—This was a metal can maintained in bell-like vibration by means

of an electromagnet excited by a current of about 1 ampere from a spring-interrupter using 4 volts. Two sizes of this type of sounder were constructed. These were generally driven at their natural frequencies.

	No. 1.	No. 2.
Length	12·8 cm.	16·1 cm.
Diameter	7·8 cm.	9·4 cm.
Thickness	0·68 mm.	0·30 mm.
Natural frequency in water	111/sec.	21/sec.

With these sounders the intensity fell off exceedingly rapidly with the distance, probably owing to consecutive segments moving in opposite phases causing short-circuiting of the vibrations in the water. They have also directional properties, the sound being sent out radially with four symmetrically placed maxima. The rapid falling off of intensity with distance is an advantage in tank experiments, where multiple reflections are a source of disturbance.

(2) *Double Diaphragm Sounder*.—This is non-directional, and on account of its great range has been used in most of the experiments carried out in the reservoir.

It consisted of a shallow cylindrical cavity closed by two vertical metal diaphragms, one of which carried at its centre an electromagnet, and the other an equal mass of soft iron (fig. 8). The electromagnet was excited by an intermittent current from a spring-interrupter. The space between the diaphragms communicated with the water by a small hole at the bottom. The apparatus could be used either completely filled with air or filled with water, with the exception of a small chamber of about 30 c.c. near the top of the cavity, in which the air remained trapped. The sounder was held by a 12-ft. iron rod, by means of which it could be lowered over the side of a boat to the desired depth. When used air-filled, a long piece of pressure tubing was attached to the cavity and air pumped in until the air-bubbles escaping from the hole at the bottom were seen rising in the water. In the boat from which the sounder was hung were the cells (4 volts) and the spring-interrupter, these being connected by long leads to the electromagnet within the sounder (current 2–2½ amperes). The boat could be anchored in any desired position, and the apparatus when once adjusted would run for hours at a time without attention.

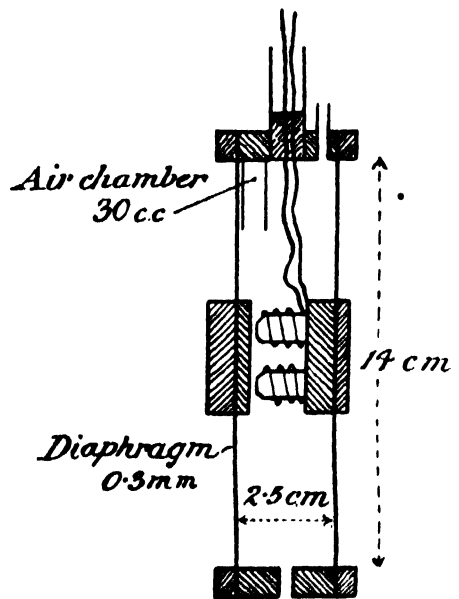


Fig. 8. Double diaphragm sounder (section).

restoring force on the diaphragm was mainly due to the elasticity of the air in the cavity (the restoring force due to the elasticity of the diaphragm being relatively unimportant) giving when air-filled a natural frequency of about 67/sec., the exact value increasing with the depth of the sounder below the surface, the variation being roughly $\frac{1}{2}$ per cent. per foot. Although the effect of introducing water into the cavity is to increase the effective inertia of the diaphragms, the frequency is raised on account of the greatly increased restoring force given by the residual air. With the 30 c.c. of air generally used the natural frequency was about 123/sec. In practice the sounder was generally driven near the natural frequency corresponding to the working conditions; the maximum energy is then emitted, and as a nearly pure tone. If driven at much lower frequencies the intensity was diminished, and, as would be expected, any harmonic near resonance with the natural frequency of the diaphragm became prominent.

(3) *Single Diaphragm Sounder*.—A thin iron diaphragm was bolted on to the end of a massive cylindrical iron pot (fig. 9). A mass (380 gm.) of soft iron was attached

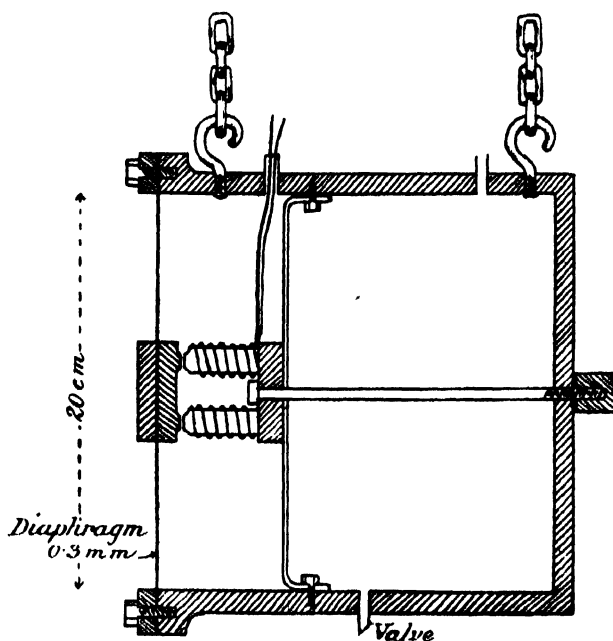


Fig. 9. Single diaphragm sounder (section).

to the centre of the diaphragm, behind which the exciting magnet was rigidly held. The cavity was always air-filled at the pressure of the surrounding water in order to avoid distortion of the diaphragm. The sounder was hung from a punt by a double chain at a depth of $4\frac{1}{2}$ feet.

The instrument was found to be unsatisfactory in practice owing to the extreme sensitiveness of the diaphragm to changes of pressure due to small variations in depth. It had a natural frequency in water of 47/sec., a value much higher than was expected. The effect of the restoring force due to the air-cavity can be avoided by using a single diaphragm with both sides in contact with the water, but such an instrument would

not be an efficient radiator for low frequencies in consequence of short-circuiting action referred to above, and also it would be bi-directional.

(4) *Evinrude Row-boat Motor*.—To obtain a more powerful and complex source an Evinrude row-boat motor was used.

This was a 2–3 h.p. single-cylinder, two-cycle petrol motor fitted with a two-blade propeller. In most of the experiments it was attached to the stern of a 13-ft. centre-board sailing dinghy, which either circled round the receiving instrument or was kept in a fixed position, about 100 feet out, by steel wires made fast to posts on the bank of the reservoir. The frequency of the piston, which must be regarded as the fundamental frequency, could be varied from about 10–11/sec. when the boat was fixed, while a frequency of 14·5/sec. could be obtained when the boat was in motion. During a single experiment the motor was found to run at a very constant frequency, and proved to be a very convenient source of vibration for the purpose of analysis.

(b) *Receivers*.

In all the water experiments described below attention was usually confined to the range of frequency 5–150/sec.

In order to obtain a faithful analysis the receiver should be either non-resonant or its resonance frequency should be well above the range under investigation. In the latter case the receiver will be insensitive in this range, and the results of the analysis will be complicated by the disturbance of subharmonics of the diaphragm frequency (see p. 144). We did not succeed in devising an ideal receiver for low-frequency vibrations, although fairly satisfactory results were obtained with the Rubber Diaphragm Receiver. Owing to damping the latter did not give unduly sharp resonance, and its resonance frequency could be varied to eliminate selective action.

(1) *Metal Diaphragm Receiver*.—The construction of this instrument is shown in fig. 10. Owing to the thinness of the diaphragm, it is necessary to compensate the external water-pressure. This was done by putting the air cavity in communication with a cylindrical reservoir the lower end of which was provided with a small hole to admit the water. If the cross-section of the cylinder is sufficiently great, then it is easily seen that the level of the water in it will not alter much with change in depth, and hence the air-pressure in the instrument will remain approximately at the pressure of the water outside. By adjusting the level of the reservoir with respect to the diaphragm, the compensation can be made exact for any particular depth.

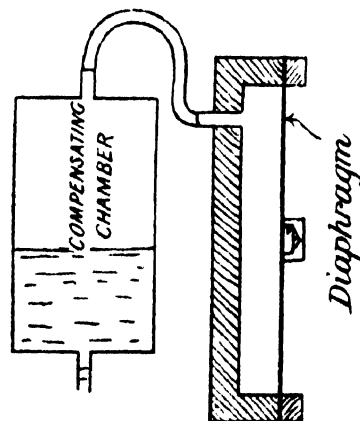


Fig. 10. Metal diaphragm receiver (section).

The microphone was attached by its base to the centre of the diaphragm, and enclosed

in a small sealed box. The diaphragm was of tinned iron, 10 cm. diameter and 0.3 mm. thick, and with a natural frequency in water of 120/sec.

(2) *Rubber Diaphragm Receiver with Adjustable Natural Frequency.*—With this instrument the object was to obtain a much lower natural frequency. The effect of the air-cavity in giving an additional restoring force to the diaphragm, as already referred to under “Sounders,” has now to be taken into account. In this receiver the diaphragm was made of very thin rubber, so that the restoring force was almost entirely due to the enclosed air. Advantage was taken of this fact to make the instrument of adjustable frequency by changing the volume of the air-cavity behind the diaphragm. The principle of compensation for hydrostatic pressure described above was again applied.

The diaphragm, with the microphone attached at its centre, formed one end of a brass tube 30 cm. long and 5 cm. in diameter, the opposite end being closed by a screwed cap (fig. 11). Inside the brass tube was a solid brass plunger the position of which

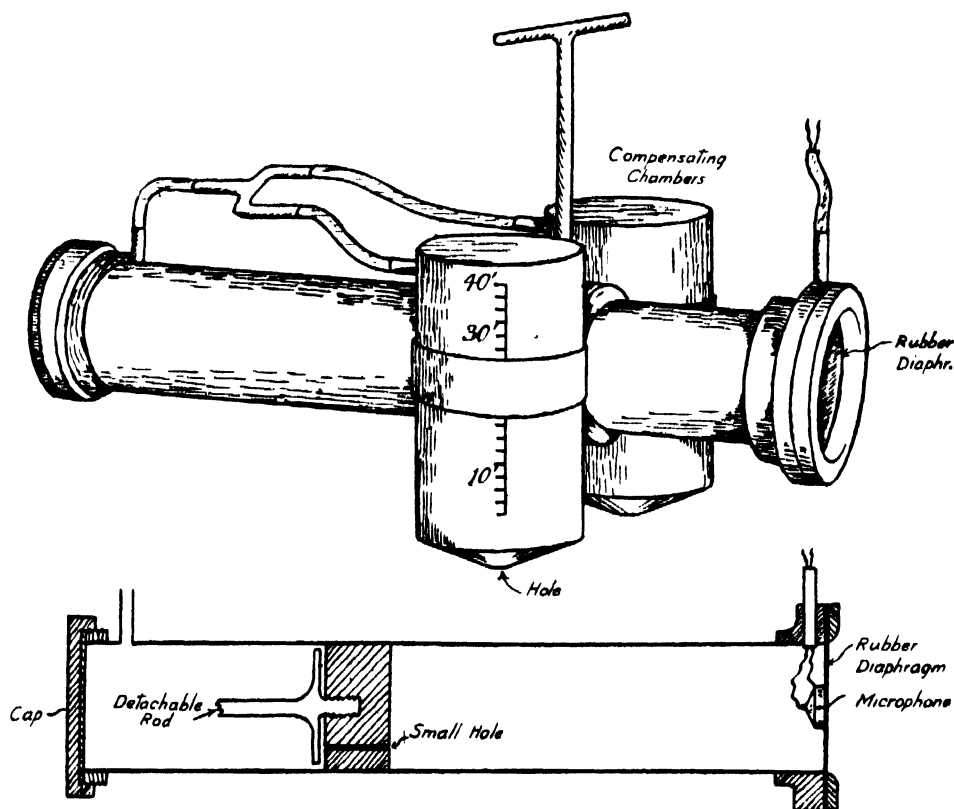


Fig. 11. Rubber diaphragm receiver.

could be varied by means of a detachable rod. A very small hole in the plunger allowed equalisation of pressure on either side, while at the far end of the cylinder was a side tube communicating with two symmetrically placed compensating chambers. The total volume of the cylinder was 600 c.c., and the chambers were 500 c.c. each. This allowed a compensation to be made for depths down to 40 feet. For convenience in setting the chambers, a scale of depths was marked on the sides.

Using the full volume of the tube, the natural frequency was 40/sec., and when the plunger was pushed up within a few centimetres of the diaphragm the natural frequency became about 100/sec. There was considerable damping, as a result of which the natural frequency was not pronounced, and was difficult to determine.

Listening Arrangements.—In most of the experiments it was found useful to put a telephone in the receiver circuit in place of the galvanometer and interrupter in order to listen to the sound. Such observations are recorded below as "sound in telephone."

In the case of low-frequency vibrations, *e.g.* 50/sec., the sound in the telephone does not give a reliable indication of the character of the vibration in the water, since for such a frequency the telephone responds only to the harmonics, and if they are not present then no sound is heard.

A Broca tube also served as a useful listening device. The diaphragms were flat sheets of tinned iron, 15 cm. diameter. The natural frequency in water was approximately 100/sec., and showed marked tuning with depth due to change in pressure. The sounder 123/sec. could be heard at a distance of 400 yards.

It was found that a short length of rubber tube, 1 cm. external diameter, walls 2 mm. thick, formed a simple type of non-resonant Broca tube. The lower end was sealed and weighted with a lead sinker, the upper end was connected to a stethoscope.

Except the cylindrical sounder first described, both sounders and receivers were non-directional.

(c) *Reservoir Experiments with Sounders.*

The Barnt Green Fishing and Boating Club kindly gave permission to make use of Great Bittell Reservoir (near Birmingham), together with a number of boats, &c., for experimental purposes. The reservoir has an area of 100 acres, with a maximum depth of 40 feet. The receivers were connected to the shore by cable, and the analysis was made in the boathouse where the interrupter and 100-ohm Broca galvanometer were set up. With few exceptions it was found impossible even in the calmest weather to use the receivers when hung from a moored boat, since a very small motion at the point of support produced a large disturbance on the galvanometer. Satisfactory results were only obtained when the receiver was either hung from a tripod or placed directly on the bottom of the reservoir. The tripod was constructed from bamboos 11 feet long and provided with a sinker on each foot. When in the required position, the lowering rope was sunk with a lead weight in order to prevent surface disturbances being directly communicated to the apparatus. In some experiments a small cork marker was used to indicate the position of the tripod.

Determination of Natural Frequency of Sounders and Receivers in Water.—This presented considerable difficulty, and it may be of interest to indicate the different methods which have been used.

Sounders :—

- (i.) A platinum contact was attached to the diaphragm, and the sounder made self-driving after the manner of an electrically maintained fork. The frequency was then determined either by analysis of the current induced in a single turn of wire acted on inductively by the circuit, or by analysis of the current generated by a U-magnet attached to the sounder diaphragm. The platinum contact worked quite well under water, but this system of driving the sounder was not suitable for general use.
- (ii.) With the sounder immersed, the diaphragm was gently tapped with a rubber hammer at intervals of 1 second, and an analysis made of the current induced in the windings of the electromagnet, as in earlier experiments on the analysis of damped vibrations.

Receivers :—

- (i.) By experiments in which a sounder was driven at various frequencies and the receiver used to obtain a resonance curve.
- (ii.) The natural frequency of the diaphragm was excited by impulses such as an oar-splash or a tap on a neighbouring boat. In the latter case the analysis showed in addition the natural vibration of the boat.
- (iii.) A method more satisfactory than the last was to analyse the general water disturbance on a rough day. (Compare experiments on Analysis of Background, p. 146.)

Metal Diaphragm Receiver (Natural frequency, 120/sec.).—This instrument in moderately calm weather could be used, when hung from an anchored boat with a galvanometer disturbance of no more than about 1 div., corresponding to a faint background in the telephone. Under rougher conditions a strong diaphragm noise was produced in which on certain occasions the separate impulses could be identified with the lapping of the water against the boat. The sounder could be distinctly heard at a distance of 60 feet.

With this instrument remarkable variations of the intensity of the sound with depth were observed, showing the existence of an almost silent layer on the bottom. Slowly raising the receiver from the bottom to the surface showed a rapid increase in the first 6 feet, followed by a slower falling off towards the surface, the maximum occurring at 6–9 feet from the bottom. The observed amplitude at 6 feet was in some cases 25 times that at 1 foot from the bottom. The sound in the telephone varied in a similar way. These effects as tested by the telephone appeared to be the same in different parts of the reservoir, where the depths were 12, 18 and 32 feet, and independent of the frequency which ranged from 60–130/sec. Sounds due to taps on the boat, splashing of oars, &c., and also natural disturbances, were modified in the same way. No definite effect of this kind was observed when using the other receivers of lower frequency.

The amplitude of the vibration fell off with horizontal distance from the source with

remarkable rapidity. Putting amplitude $\propto \frac{1}{(\text{distance})^p}$, the value of the index p (see below) is generally 3 to 3.5, which makes the energy fall off with the sixth or seventh power of the distance. This gives rise to considerable difficulty in making exact measurements, as a large error is introduced by a small change in distance.

The variation in amplitude with distance and depth is shown in the following examples:—

Depth of water.	Frequency of sounder.	Depth of sounder.	Depth of receiver.	Horizontal distance.	Response.	Index p .
ft. 12	105	ft. 6	ft. On bottom.	ft. 4	Off scale > 250 18 3	3.5
				45		
				75		
15	124	10	9	40	270	3.5
			9	80		
			9	130	< 5	> 2.6
			On bottom	80	3	
			6	80	16	
17	66.5*	3	11	135	7	3.0
			11	60	80	
			On bottom	60	3	
			9	60	150	
			3	60	80	

Rubber Diaphragm Receiver.—When hung from a boat in very calm weather the galvanometer showed oscillations of 20–40 div. with an occasional 200 divs., but on the bottom the disturbance was reduced to 1–2 div. A residual disturbance of this magnitude was always present under the quietest conditions, even when out of the water, and this may represent the natural limit of steadiness of the microphone. The instrument was used either on the bottom or on the tripod, and even then quiet weather was essential. On a windy day there were large disturbances always closely associated with the gusts. Their magnitude was not changed by altering the natural frequency of the receiver.†

With this apparatus a number of experiments were made in which the sounder was

* In this case the weak octave 133/sec. present in the sounder was reinforced by resonance with the receiver. This response varied with depth in the same way as the fundamental.

† Some experiments made in the laboratory at a later date furnished the explanation of these disturbances. It will be seen from the construction of the apparatus that slow pressure changes can be communicated to the air cavity through the compensating reservoirs. Such pressure changes would not produce any motion of the diaphragm. The effects on the microphone are due to the direct action of the air pressure on the lid of the button. This was confirmed by blowing air into the cavity of the instrument. It was then found that when the microphone was subjected to a sustained additional pressure the microphone current showed a rapid increase followed by a slow exponential recovery, the transformed current causing

driven successively at different frequencies. Since the driving current contains harmonics the forced motion of the sounder will also contain these harmonics, and if the frequency of one of them is near the natural frequency, then on account of resonance that harmonic will be very prominent, and may be much greater than the fundamental. But the receiver also has a selective action, and will enhance by resonance any harmonic which is near its natural frequency. These effects were observed in the analysis but will not be further described.

The variation of amplitude with distance was determined using the two sounders near their natural frequencies, and the results obtained are given in the following table :—

Depth of water.	Frequency of sounder.	Depth of sounder.	Depth of receiver.	Horizontal distance.	Response.	Index p .
ft. 18	44.5	ft. 4.5	ft. 12	ft. 57 150	190 9	} 3.1
17	58	9	On bottom.	60 165	180 <5	

The high value of the index is in agreement with the results obtained with the Metal Diaphragm Receiver.

A particular source of disturbance with the Rubber Diaphragm Receiver was found to be due to the vibrations caused by trains passing at a distance of rather more than a quarter of a mile. The effects were noticed when analysing in the region 30–40/sec., where on occasions definite responses of over 100 div. were obtained. These disturbances greatly added to the difficulty of the experiments, especially when examining vibrations of very small amplitude.

(d) *Sound Spectrum of Evinrude Motor* (Reservoir experiments).

A description of the motor together with the general arrangements has already been given (p. 149). Analysis was made with both receivers. They were either placed on the bottom or hung from the tripod 6 feet above the bottom. In each case a spectrum was obtained consisting of a harmonic series, n , $2n$, $3n$, $4n$, &c., having for its fundamental the frequency n of the motor. In general the higher harmonics have smaller intensities, and only in a few special cases could measurements be extended beyond the first ten harmonics.

the galvanometer to give a complete oscillation, as would be expected. The exponential recovery, which extended over some 15 sec., was evidently due to the air leaking into the partially air-tight button. *The whole effect completely disappeared when a small hole was drilled through the side of the button.*

It was thought that the instrument might now be less sensitive to low frequency vibrations. An experiment in a tank showed that for a frequency of 100/sec. the sensitiveness was reduced to about one-half.

It would thus appear that in the reservoir experiments the wind pressure was transmitted through the water to the receiver. The metal diaphragm receiver did not show these effects since its button is enclosed in a sealed cavity.

The Rubber Diaphragm Receiver was found to be much more satisfactory than any other type tried, and was used in all the experiments described below.

Variation of Spectrum with Motor Speed.—It was soon found that even under closely identical conditions the character of the spectrum often differed greatly on repeating the experiments. It has not been found possible to explain this variation completely, but it is partly due to change in motor frequency. Experiment showed that quite small variations in the motor frequency—variations originally regarded as of little consequence—may greatly affect the relative intensities of the “lines” in the spectrum. There are several ways in which this may take place :—

- (i.) In the experiments the motor speed was always changed by altering the phase of the ignition ; this may result in a change in the character of the vibration of the motor itself.
- (ii.) Mere change of speed may bring certain harmonics into, or out of, resonance with natural frequencies of the boat.
- (iii.) Change of speed may bring certain harmonics into resonance with the receiver.

Experiments show that (iii.) at least is an important factor. Unfortunately, the frequency of the Evinrude motor (with boat fixed) can only be varied by about 10 per cent., *i.e.* from 10–11/sec.

Three examples of the variations in the spectrum are given in fig. 12, where the

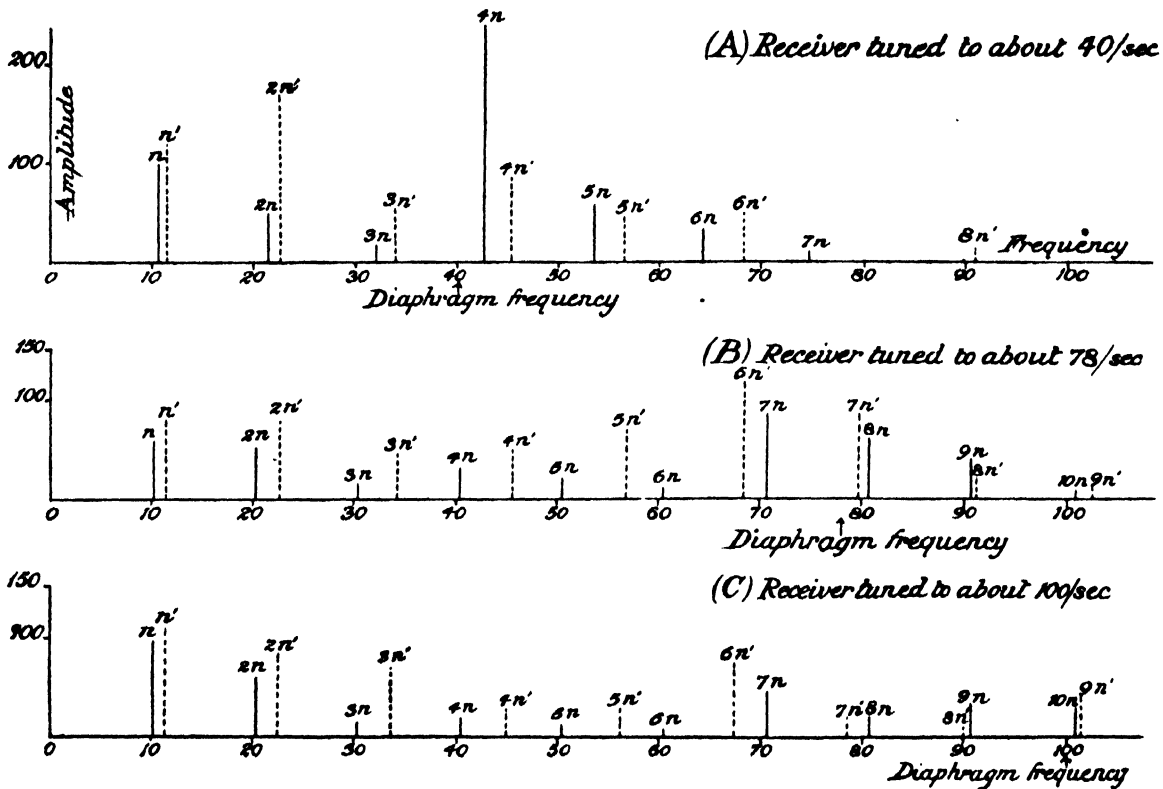


Fig. 12. Variation of spectrum of Evinrude with motor speed and natural frequency of receiver. Depth, 15 feet. Receiver on bottom. Distance 13 feet ; broadside on.

harmonic series for low and high speeds are represented by continuous and broken lines respectively. All three experiments were carried out under identical conditions as regards the position of the receiver, which was placed on the bottom at a depth of 15 feet. The only difference made in the conditions was in the natural frequency of the receiver: in the first experiment this was made as low as possible, about 40/sec., by using the full volume of the air-cavity; in the second it was raised to about 80/sec., and in the last to about 100/sec. The last two natural frequencies are only known approximately.

It will be seen that in all three cases the responses n , $2n$, $3n$, are increased with increased speed. In the first experiment $4n$ is very prominent, and showed a decrease with speed, evidently due to its being put out of resonance with the diaphragm. The effect of diaphragm resonance is also shown in the second and third experiments, which give strong responses near 80 and 100 respectively, frequencies for which no responses were detected in the first experiment. On the other hand, there seem to be certain permanent features in the spectrum—for example, the third harmonic for the lower speed always being very weak, and the harmonic near 68 being always prominent. These characteristics may be associated with either the boat or the motor.

Variation of Spectrum with Direction.—The receiver was placed in various directions with respect to the axis of the boat, the distance from the propeller being constant. Variations in the spectrum were observed, but as consistent results were not obtained on repeating the measurements for any one position, no definite conclusions can be drawn. In general the experiments showed that direction has no *marked* influence on either the intensity or distribution in the spectrum.

Variation of Spectrum with Depth.—It was not possible to moor the motor-boat in the deeper parts of the reservoir. Some experiments were therefore made with the motor circling round a cork marker indicating the position of the receiver, which was—(a) on bottom, (b) on tripod 6 feet above bottom. The depth of water was 26 feet, and the radius of the circle either 20 feet or 60 feet. The spectrum of the *boat in motion* did not differ essentially from results obtained with the boat moored in shallow water. The motor frequency was 14·2/sec. instead of 10–11/sec., so that no exact comparison is justifiable. The intensities observed were less than those for equal horizontal distances in shallow water, but not in greater proportion than is accounted for by the greater actual (oblique) distance between receiver and boat.

Variation of Amplitude with Distance.—Experiments were made to determine the variation of amplitude with distance in the case of the most prominent vibrations up to the fifth harmonic. The most useful results would be obtained at distances great compared with the dimensions of the boat, but on account of the rapid decrease in intensity it was quite impossible to make observations at distances greater than 70 feet. At these distances disturbances were comparable in magnitude with the effects to be observed.

Since the source is on the surface and receiver on or near the bottom, any change in

distance also involves a change in direction with respect to the vertical, and if nodal planes exist in the water one would expect a more complex relation than that given by the former experiments using non-directional sounders. There is the additional complication due to reflection from top and bottom, which will be different in the two sets of experiments owing to the sounder always being placed below, instead of on, the surface.

Putting, as formerly (p. 153), amplitude $\propto \frac{1}{(\text{distance})^p}$, the values of the index p for the different harmonics are given in the following table :—

Depth of water.*	Frequency, n .	Responses at horizontal distance.				Index p .	
		(14 ft.)		(24 ft.)			
ft. 13	11·1	n	90	25		2·4	
		$2n$	210	100		1·4	
		$3n$	130	37		2·3	
		$4n$	230	60		2·5	
		$5n$	40	7		3·2	
16	11·2		(13½ ft.)	(21 ft.)	(43 ft.)	(69 ft.)	
		n	80	25	5	0	2·4†
		$2n$	45	25	10	0	1·3†
		$3n$	45	10	10	5	—
		$4n$	140	70	23	10	1·6†
26 Boat circling.	14·2		(21 ft.)		(42 ft.)		
		n	28	15			0·91 (1·1)‡
		$2n$	14	5			1·5 (1·6)‡
		$3n$	25	3			3·1 (4·9)‡

The distances given in the table are horizontal distances, and these differ from the true distances more in deep than in shallow water. This correction has not been made, as it does not appear to lead to a simpler law.

In fig. 13 the values of the log (amplitude) are plotted against log (distance) for the third experiment in the table, in which the greatest range of distance was covered. It will be seen that the graphs for n , $2n$ and $4n$ are practically straight lines giving a constant value for p as determined by the slope of the line in each case. The results

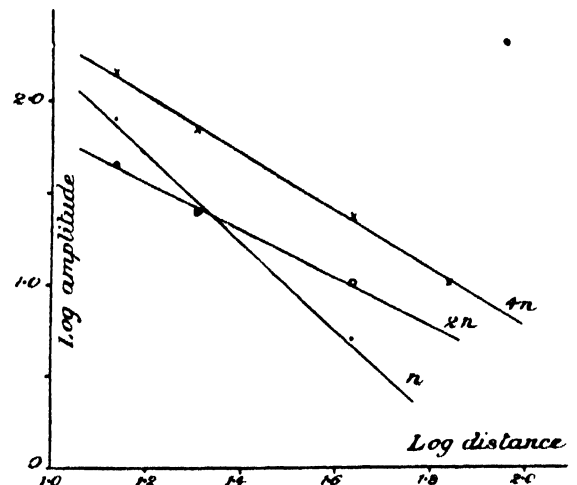


Fig. 13. Variation of amplitude with distance—
Index graph.

* The receiver was 6 feet above the bottom.

† From graph, fig. 13.

‡ These values of p are calculated for the oblique distances.

for $3n$ are quite abnormal, as no falling off in amplitude was observed for the distances 21 feet and 43 feet. This is the only occasion on which such an effect has been recorded, and may be accounted for by accidental disturbance.

The table shows that in any one experiment the various harmonics have different values of the index p , indicating that the character of the spectrum really changes with the distance, but the values of p depend on the experimental conditions. The range of p is from 1 to 5, and in general the higher harmonics fall off most rapidly with the distance (*i.e.*, have larger values of p) and the harmonic $2n$ least rapidly of all. It is not known whether this peculiarity of $2n$ is associated with the absolute frequency or with the propeller action in which this vibration may have its origin.

The distances were always measured from the propeller, but it appeared that the propeller only acts as an additional source. One can *see* the sides of the boat vibrating, and in calm weather the ripples radiating from the stationary boat are quite evident.

On several occasions the boat-vibration was examined by placing in the boat a frequency-meter of the vibrating reed type. Within the range of this instrument (25-50/sec.) responses were obtained corresponding to the harmonics which were simultaneously detected in the water. With the rubber diaphragm receiver used in these experiments the sound in the telephone was distinct up to 50 feet. Listening directly with the Broca tube and rubber tube, the sound was poor even at distances of about 20 feet.

Subharmonics.—In the diagrams representing the spectrum of the Evinrude motor the subharmonics have, for the sake of clearness, been omitted. As the higher harmonics are generally small, and not observable beyond the tenth, there were very few subharmonics to confuse the analysis. Most of the subharmonics were below the fundamental, and this region was disregarded in the analysis. The fundamental response often showed some irregularity due to the superimposed subharmonic $\frac{1}{3}(3n)$.

The measurement of the amplitudes of the vibrations up to the tenth harmonic occupies about 10 minutes.

Analysis by Telephone.—It is interesting to note that by putting a telephone (in place of the galvanometer) in series with the interrupter the fundamental frequency (10/sec.) of the Evinrude motor can be determined with great precision, as the beats are very marked when the interrupter is running near that frequency. Also the frequency of the harmonics can be determined in a similar way, although with them the effect is not so evident.

PART II.—THE THEORY OF ANALYSIS OF AN ELECTRIC CURRENT BY PERIODIC INTERRUPTION.

The present paper contains an account of the theory of the analysis of an alternating current by the method of periodic interruption. The experiments in which this method has been applied for the analysis of sounds in air and water are described in Part I.

Let the current to be analysed be a function of the time expressed by

$$y = f(t)$$

and represented by (a) in fig. 14. We may regard the effects of interruption as merely

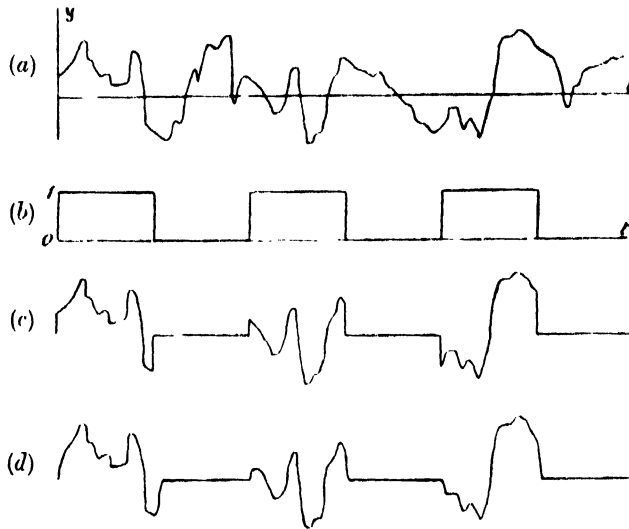


Fig. 14. Interruptions of current.

the result of multiplying the instantaneous value of the current y by a factor which has periodically the values 1 and 0. For equally spaced interruptions of frequency n^* and giving equal time intervals for open and closed circuit, this factor is represented graphically by the periodic form (b) (fig. 14), and has for its expression the Fourier series :

$$\frac{1}{2} + \frac{2}{\pi} \left(\sin nt + \frac{1}{3} \sin 3nt + \frac{1}{5} \sin 5nt + \&c. \right).$$

The resulting intermittent current, represented by (c) (fig. 14), is therefore given by

$$y' = \left\{ \frac{1}{2} + \frac{2}{\pi} \left(\sin nt + \frac{1}{3} \sin 3nt + \frac{1}{5} \sin 5nt + \&c. \right) \right\} f(t).$$

This expression must now be resolved into S.H. components. These may then be regarded as forces of S.H. type acting on the galvanometer system—or on the telephone diaphragm—and giving rise to a forced motion which is readily calculated. If the galvanometer possesses very little damping, only those components which have a frequency near that of the galvanometer will produce any appreciable motion. But we shall suppose, in accordance with experimental requirements, that the galvanometer is heavily damped, and for convenience we shall here take it as being exactly “dead-beat.” Under these conditions there will be no resonance, and a component is only

* The quantities $n, n_1, n_0, p, \&c.$, really angular velocities, are, for the sake of brevity, here referred to as frequencies, the 2π being everywhere omitted. They are strictly the radian frequency of phase change.

effective in producing appreciable motion when its frequency is *less* than that of the (undamped) galvanometer. The full amplitude of the component is not exhibited by the galvanometer unless the component has zero frequency; if it has half that of the galvanometer the amplitude of the motion produced is 80 per cent. of the full value (see fig. 15, which shows how the forced amplitude varies with the frequency p of the force).

On the other hand, a telephone in the circuit will render audible only those components which are of sufficiently high frequency to excite the diaphragm and the ear.

If the circuit contains appreciable self-induction the effects of interruption are greatly complicated—at least theoretically—and special assumptions as to the conditions which hold at “break” must be made in order to proceed with the investigation. The general effect will be to round off the sharp corners at “make” and “break” in curve (c) (fig. 14) as represented in curve (d). Provided the “time-constant” of the circuit is small, only the components of high frequency will be seriously modified by self-induction. The order of frequency affected is $1 \div (\text{time-constant})$, or R/L if the circuit has resistance R and induction L . (For a Broca galvanometer alone R/L is about 1200/sec.)

Actually we are not likely to be greatly concerned with the modification introduced, for this would imply that we are using the analysis to determine a component of frequency so high that, on account of ordinary impedance, it already must have undergone considerable distortion. We shall, therefore, neglect self-induction and proceed to consider certain special types of current.

(1) *Steady Direct Current.*

$$y = A, \text{ a constant.}$$

Then

$$y' = \frac{A}{2} + \frac{2A}{\pi} (\sin nt + \frac{1}{3} \sin 3nt + \&c.).$$

The galvanometer is deflected by $A/2$. The sound in the telephone is represented by the uneven harmonic series of tones with the interruption frequency n as fundamental. This is the “interrupter note”; it is characterised by the wave-form (b) (fig. 14). When n is very low only the numerous high harmonics in the audible region are effective, and the sound is well described as a purr. As n is increased the note becomes more musical in quality, and its pitch is recognisable.

(2) *S.H. Current.*

$$y = A \sin n_1 t.$$

Putting $A \sin n_1 t$ in place of $f(t)$ in the general equation for y' , this may be written :

$$y' = \frac{A}{2} \sin n_1 t + \frac{A}{\pi} \left\{ \cos (n - n_1) t - \cos (n + n_1) t + \frac{1}{3} \cos (3n - n_1) t - \frac{1}{3} \cos (3n + n_1) t \right. \\ \left. + \frac{1}{5} \cos (5n - n_1) t - \frac{1}{5} \cos (5n + n_1) t + \&c. \dots \right\}.$$

The tones present are therefore

$$n_1, \quad n \pm n_1, \quad 3n \pm n_1, \quad 5n \pm n_1, \quad \&c.$$

The difference tones $n - n_1, 3n - n_1, \&c.$, have zero value of frequency when

$$n = n_1, \quad \frac{1}{3}n_1, \quad \frac{1}{5}n_1, \quad \&c.,$$

with the amplitudes

$$\frac{A}{\pi}, \quad \frac{1}{3} \frac{A}{\pi}, \quad \frac{1}{5} \frac{A}{\pi}, \quad \&c.$$

These represent the fundamental and "subharmonic" responses.

We shall now suppose we are running through the fundamental response so that $n = n_1$ nearly. Put $n - n_1 = p$ where p is the small difference in frequency between interrupter and current. In going up, *i.e.* with increasing speed, through the response p changes from negative to positive.

The general expression may now be written

$$y' = \frac{A}{2} \sin n_1 t + \frac{A}{\pi} \left\{ \cos pt - \cos (2n - p)t + \frac{1}{3} \cos (2n + p)t - \frac{1}{5} \cos (4n - p)t \right. \\ \left. + \frac{1}{7} \cos (4n + p)t - \frac{1}{9} \cos (6n - p)t + \&c. \right\}.$$

The frequencies of the components are

$$p, \quad n_1, \quad 2n \pm p, \quad 4n \pm p, \quad 6n \pm p, \quad \&c.$$

The galvanometer responds to p . It is seen that there is a series of beating tones, $2n, 4n, 6n, \&c.$, and all these beat with the *same* frequency $2p$, *i.e.* twice the frequency of the oscillations of the galvanometer at the same instant (confirmed by experiment). The amplitudes of the beating tones are never equal, but they tend to equality for the higher harmonics of the series. This explains the beating which is heard in the telephone even when n_1 is well below the limits of audibility.

For synchronism ($p = 0$) there is produced the single note of fundamental n_1 containing the *even* harmonics $2n, 4n, \&c.$ The original *tone* n_1 may be inaudible, and in any case it is weakened by interruption, but the addition of the harmonics will in general render audible the resulting *note* n_1 .

When n is near $2n_1$ put $n - 2n_1 = p$. The component frequencies are then

$$n_1, \quad n_1 + p, \quad 3n_1 + p, \quad 5n_1 + 3p, \quad 7n_1 + 3p, \quad \&c.$$

In this case there is no galvanometer response. There is only one beating pair, $n_1, n_1 + p$, the amplitudes $A/2, A/\pi$ of which are sufficiently near equality to give a marked beating, but, of course, this will not be heard unless n_1 is within the audible range. It will be noted that the beats have the frequency p in this case, instead of $2p$ as above.

Similarly, when n is near $3n_1$ it is easily shown that there is no response and no beating. [Confirmed by experiment.]

Now take n near $\frac{1}{3}n_1$ and put $3n - n_1 = p$. Then

$$y' = \frac{A}{2} \sin n_1 t + \frac{A}{\pi} \left\{ \cos (2n-p)t - \cos (4n-p)t + \frac{1}{3} \cos pt - \frac{1}{3} \cos (6n-p)t + \frac{1}{5} \cos (2n+p)t - \frac{1}{5} \cos (8n-p)t + \frac{1}{7} \cos (4n+p)t - \&c. \right\}.$$

Hence the frequencies are

$$p, \quad n_1, \quad 2n \pm p, \quad 4n \pm p, \quad 6n \pm p, \quad \&c.$$

The galvanometer responds to p ; this is the third order subharmonic of amplitude $A/3\pi$. The beating pairs all beat with the same frequency $2p$, but their amplitudes are more unequal than in the case of the fundamental response.

(3) *Harmonic Series.*

Let the original current be

$$y = A_1 \sin n_1 t + A_2 \sin 2n_1 t + A_3 \sin 3n_1 t + \&c.$$

without regard to phase differences between the constituents.

The interrupted current is now made up of groups of components, there being one group associated with each harmonic. It is sufficient to consider only the case of synchronism with the fundamental of the series, *i.e.* when $n = n_1$ nearly. Putting $n - n_1 = p$ as before, the resulting tones may be tabulated thus:—

Amp. ratio.	$A_1.$	$A_2.$	$A_3.$	$A_4.$	$A_5.$
	$n_1.$	$2n_1.$	$3n_1.$	$4n_1.$	$5n_1.$
1	p $2n-p$	$n-2p$ $3n-2p$	$2n-3p$ $4n-3p$	$3n-4p$ $5n-4p$	$4n-5p$ $6n-5p$
$\frac{1}{3}$	$2n+p$ $4n-p$	$n+2p$ $5n-2p$	$3p$ $6n-3p$	$n-4p$ $7n-4p$	$2n-5p$ $8n-5p$
$\frac{1}{5}$	$4n+p$ $6n-p$	$3n+2p$ $7n-2p$	$2n+3p$ $8n-3p$	$n+4p$ $9n-4p$	$5p$ $10n-5p$
$\frac{1}{7}$	$6n+p$ $8n-p$	$5n+2p$ $9n-2p$	$4n+3p$ $10n-3p$	$3n+4p$ $11n-4p$	$2n+5p$ $12n-5p$

The interrupted current may be regarded as made up of

(a) $\frac{1}{2} (A_1 \sin n_1 t + A_2 \sin 2n_1 t + \&c.).$

This is the original note with half the amplitude.

(b) $\frac{1}{\pi} (A_1 \cos pt + \frac{1}{3} A_3 \cos 3pt + \frac{1}{5} A_5 \cos 5pt + \&c.).$

This is the galvanometer response, which now has a complex character due to the superimposed subharmonics of $3n_1, 5n_1, \&c.$

(c) The beating tones

$$\begin{aligned} & \frac{A_1}{\pi} [(1, \frac{1}{3}) 2n \pm p, \quad (\frac{1}{3}, \frac{1}{3}) 4n \pm p, \quad (\frac{1}{3}, \frac{1}{7}) 6n \pm p, \text{ \&c.}] \text{ beating at } 2p, \\ & \frac{A_2}{\pi} [(1, \frac{1}{3}) n \pm 2p, \quad (1, \frac{1}{5}) 3n \pm 2p, \quad (\frac{1}{3}, \frac{1}{7}) 5n \pm 2p, \text{ \&c.}] \text{ beating at } 4p, \\ & \frac{A_3}{\pi} [(1, \frac{1}{5}) 2n \pm 3p, \quad (1, \frac{1}{7}) 4n \pm 3p, \quad (\frac{1}{3}, \frac{1}{9}) 6n \pm 3p, \text{ \&c.}] \text{ beating at } 6p, \\ & \text{\&c.}, \end{aligned}$$

where the numbers in () brackets indicate the relative amplitudes of the beating pairs.

The sound in the telephone is, therefore, remarkably complex. It will be noticed that the constituents of the original note (*a*) have frequencies which may be written

$$n + p, \quad 2n + 2p, \quad 3n + 3p, \quad \text{\&c.}$$

and these can be associated with components in (c) to give beating at frequency *p*. This appears to be the explanation of the curious fact that in an experiment with a current rich in harmonics the beating heard agrees in frequency with the galvanometer oscillations *p*, although with a S.H. current the beating frequency has the double value *2p*. Examination of the above table shows that this abnormal beating at *p* depends essentially on the co-operation of *consecutive* harmonics of the original current. In all cases the ear appears to appreciate only the slowest beats which are present.

The more general case in which the interruptions are of unequal intervals and unequally spaced may be treated in a similar way to the above. It is only necessary that the interruptions shall be strictly periodic so that they may be represented by a Fourier series.

Simple Response.—If the galvanometer system has a natural undamped period $\frac{2\pi}{n_0}$ and is made exactly dead-beat, the equation of motion due to a S.H. force is

$$\ddot{x} + 2n_0\dot{x} + n_0^2x = F \cos pt.$$

The solution of this for the steady state gives the amplitude of motion

$$a = \frac{F}{n_0^2 + p^2}.$$

When the frequency *p* of the force becomes very small compared with that of the galvanometer *n*₀, the amplitude has the maximum value $a_m = \frac{F}{n_0^2}$ corresponding with the centre of the response.

Hence

$$\frac{a}{a_m} = \frac{n_0^2}{n_0^2 + p^2}.$$

Plotting a/a_m against p/n_0 we obtain the response curve (equation $y = \frac{1}{1+x^2}$) shown by the full line in fig. 15. The amplitude falls to half value when $p = \pm n_0$,

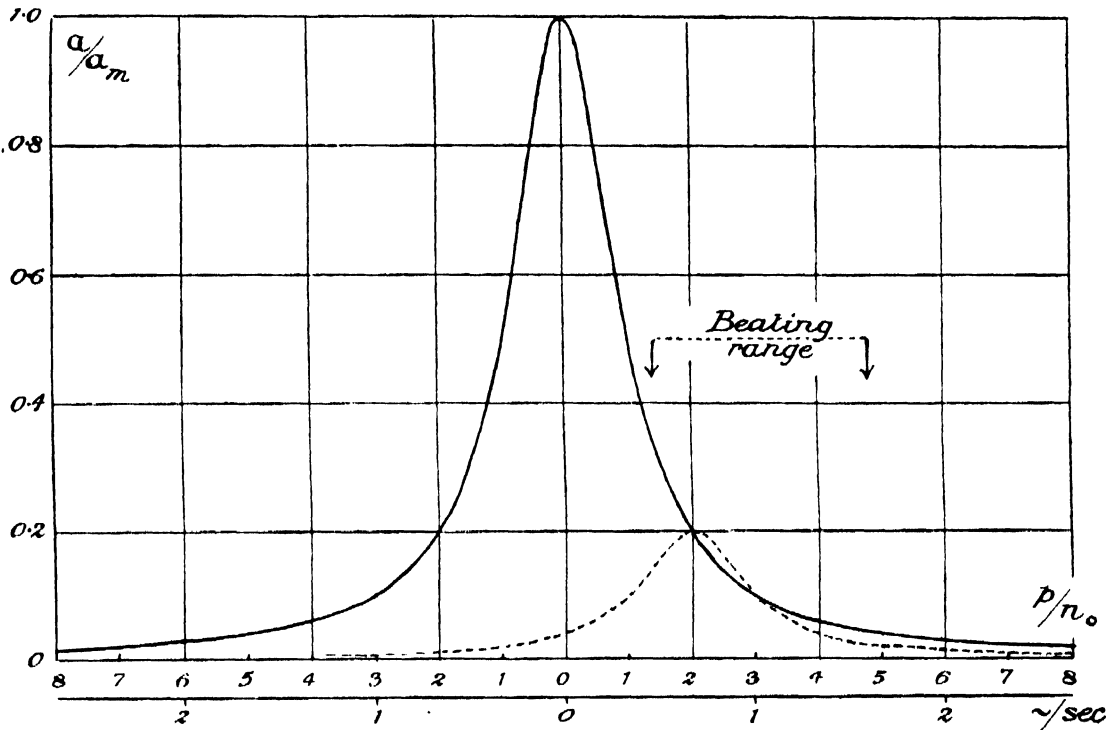


Fig. 15. Response curve.

i.e. when the difference in frequency between interrupter and current is equal to the natural frequency of the galvanometer.

[For the galvanometer with period 3 sec. used in the experiments the half-value amplitude is given for a true frequency of $\frac{1}{3}$ /sec., or the "width of response" is $\frac{2}{3}$ /sec. The true frequency of the forced motion of this galvanometer over the range of the response is indicated on the lower scale in fig. 15.]

Resolving Power.—Consider first the case of two equal components. Let

a_m = maximum amplitude for each separately,

q = frequency difference,

a_1, a_2 = response amplitude, for one component alone, at frequencies $\frac{1}{2}q$ and q respectively from the maximum.

The superposition of the two response curves is shown in fig. 16. The degree of resolution is determined by the ratio

$$\frac{\text{amplitude at central dip}}{\text{amplitude at summit on either side}}$$

Approximately this ratio is $\frac{2a_1}{a_m + a_2}$, neglecting a slight displacement of the maxima.

If we take for the limit of resolution a ratio $5/6$, *i.e.* a drop of 16 per cent. at the centre, this corresponds to the condition $q = 2n_0$. Hence the least difference of frequency resolvable is twice the frequency of the galvanometer. For the experiments this limit is $\frac{2}{3}$ /sec., but practically a rather higher degree of resolution was obtained. Undoubtedly this is due to the fact that the above ratio is that of the maximum amplitudes obtained by taking the two vibrations always in the same phase. If they are taken in opposite phases the two peaks are completely separated, as indicated by the dotted curve in fig. 16. The motion of the galvanometer is in general compounded of

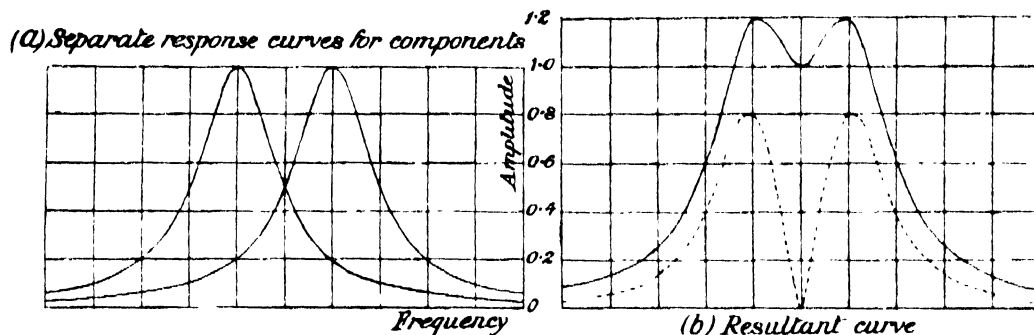


Fig. 16. Resolving power.

two S.H. motions of different amplitude and different frequency, but the maximum excursion is the easiest observed. Midway between, the two components become equal in amplitude and frequency, giving rise to pronounced beating. The beats have a frequency which is variable between the two peaks, but on either side it has the constant value q . For extremely close components (q/n_0 very small) practically the whole response is characterised by beating with frequency q , so that from this point of view the resolving power is unlimited. With components very unequal in magnitude the presence of the companion is generally detected by beating *on one side only* of the response. This will be understood by examination of fig. 15, where the response curve due to a companion of amplitude $\frac{1}{5}$ of the primary is represented by the broken line. The region of most distinct beating is where the two amplitudes approach equality.

Time Required for Analysis.—We shall suppose that the interrupter frequency n is increasing at a uniform rate, $\frac{dn}{dt} = c$, in going through a response due to an isolated component of frequency n_1 . It is required that the value of the maximum deflection shall not possibly be less than a certain fraction, say 90 per cent., of the full response obtained by an infinitely slow rate. Measuring the time t from the instant of synchronism, we may put $n = n_1 + ct$.

If at the instant $t = 0$ the phase difference between current and interrupter happens to be 0 , or π , the conditions will be as favourable as possible. It is only necessary for this phase condition to persist practically unchanged for a time interval sufficient for the galvanometer to deflect. For a galvanometer in the dead-beat condition this interval

is approximately $\frac{1}{2}T$, where T is the period when undamped. A phase change of $\pm \frac{\pi}{8}$ may be allowed at either end of the interval, which we may take as extending from $t = -\frac{1}{4}T$ to $t = +\frac{1}{4}T$. Then since $n - n_1$ is the phase change per sec. at t , the required condition is expressed by

$$\frac{\pi}{8} = \int_0^{1/4 T} (n - n_1) dt = \int_0^{1/4 T} ct dt = \frac{1}{2}c\left(\frac{T}{4}\right)^2,$$

therefore

$$c = \frac{4\pi}{T^2} \text{radian/sec.}$$

Hence the limiting rate is $\frac{2}{T^2}$ /sec. Thus in the experiments, with $T = 3$ sec., unit range 1/sec. can be covered in $4\frac{1}{2}$ sec. But since the phase at synchronism may be unfavourable, the rate must be much slower than the above limit—probably about $\frac{1}{16}$ the rate.

The investigation shows that the limiting rate is the same at all frequencies, and it is inversely as the square of the galvanometer period. The result is of importance in considering the possibilities of making a photographic record of a spectrum by a continuous sweep through the whole range of frequency.

VI. LAGRANGE'S *Ballistic Problem.*

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PART I.

1. *Introduction.*—The first part of this paper, written by LOVE, contains a theoretical solution of the problem of rational hydrodynamics which has been named by writers on ballistics, “LAGRANGE’S problem”; the second part, written by PIDDUCK, gives the application to ballistics.

In the problem* it is supposed that a given mass of gas, which is initially in a uniform state, is contained in a segment of a tube of uniform section. At one end the segment of the tube is bounded by a fixed transverse section, and at the other end the tube is closed by a piston of given mass, which is initially at rest and is free to move along the tube without resistance. It is required to find the subsequent states of the gas and the motion of the piston.

Under the pressure exerted by the gas the piston begins to move, and wave-motion of finite amplitude is set up in the gas. The waves are plane. The theory of plane waves of expansion of finite amplitude has been the subject of much study,† chiefly in connection with the question of the initiation and maintenance of surfaces of discontinuity. The difficulties associated with this question do not arise in Lagrange's problem, because the waves that are generated are always waves of rarefaction, and there is no tendency to discontinuity in waves of this type. Among the results that have been obtained in the theory of plane waves of finite amplitude, two are specially important for our present purpose. The first of these is that there exist waves of the type known as "progressive waves," and that they are the only ones that can advance without discontinuity into gas at rest. They are sometimes described as "motions compatible with rest."‡ The second important result is that the equations governing the propagation of waves which are not compatible with rest can be integrated.§ Such waves will be described in the sequel as "compound waves."

The most important writings in which LAGRANGE'S problem is dealt with are the memoir of HUGONOT cited above, H. HADAMARD'S 'Leçons sur la Propagation des Ondes,' Paris, 1903, and a memoir by F. GOSSOT and R. LIOUVILLE in 'Mémorial des Poudres et Salpêtres,' vol. 17, 1914, p. 1.

The problem is not rendered essentially more difficult if it is supposed that the segment of the tube occupied by gas is bounded by two movable pistons of given masses. Provision can be made for the case of a fixed end by taking the masses of the two pistons to be equal, for then there is never any velocity at the section midway between them.

The tube will be thought of as running from left to right. When the pistons begin to move progressive waves set out, one from the left-hand piston with a front proceeding towards the right, the other from the right-hand piston with a front proceeding towards the left. These waves meet at the middle section, and from that section there then sets out a compound wave, which has an advancing front, proceeding towards the right, and a receding front proceeding towards the left. This wave will be described as the

* S. D. POISSON, "Formules relatives au Mouvement du Boulet . . . extraites des Manuscrits de Lagrange," Paris, 'J. Éc. Pol.,' cah. 21 (1832).

† Reference may be made to LAMB'S 'Hydrodynamics,' ch. 10.

‡ H. HUGONOT, Paris, 'J. Éc. Pol.,' cah. 57 (1887) and cah. 58 (1889).

§ B. RIEMANN, 'Göttingen Abh.,' vol. 8 (1859-60); also 'Ges. math. Werke,' Leipzig, 1876, p. 145.

“first middle wave.” When the advancing and receding fronts of the first middle wave reach the pistons the original progressive waves are obliterated, reflexions take place at the pistons, and new compound waves are generated at the pistons and encroach upon the first middle wave. These waves will be described as the “first reflected wave from the left” (or “from the right” as the case may be). The reflected waves meet at or near the middle section, from which there then sets out a new compound wave called the “second middle wave.” This wave again has two fronts, one advancing and encroaching upon the first reflected wave from the right, and the other receding and encroaching upon the first reflected wave from the left. The two fronts eventually reach the pistons, and then the second middle wave will have obliterated the first reflected waves, and will itself be reflected so as to give rise to new compound waves setting out from the pistons. These will be called the “second reflected wave from the left” (or “from the right” as the case may be). The motion goes on in this way until a piston reaches an end of the tube if the tube is of finite length.

In what follows Articles 2–9 are devoted to giving such an account of the theory of plane waves of finite amplitude as seems to be necessary for the discussion of the problem. Although so much has been written about the subject, it appears to be impossible to find what is wanted in a suitable form. Articles 10, 11 contain the formulæ relating to the two progressive waves. These are already known from the work of GOSSOT and LIOUVILLE, but it seemed to be desirable, for the sake of completeness, to obtain them anew. Articles 12–17 deal with the first middle wave. Sufficient indications of the method of determining this wave have been given by the same writers for the case of equal pistons. The really formidable difficulties of the problem begin to present themselves when an attempt is made to discuss the waves reflected from the moving pistons. In Articles 18–25 an approximate method of solution is found. It seems to be capable of giving results for the first reflected waves correct to any desired order of accuracy. In Articles 26–31 the second middle wave is determined. However far the approximation to the first reflected waves is carried, the second middle wave answering to them can be found by the method here given. Articles 32–40 are devoted to the determination of the second reflected waves. The method used for the first reflected waves does not give a sufficiently close approximation, and a new method is applied. Numerical calculation of a particular example showed that all information that can be of practical importance may be obtained from a solution which does not go beyond the determination of these waves. The results of this calculation belong properly to the second part of the paper.

THEORY OF PLANE WAVES OF FINITE AMPLITUDE.

2. *General Equations.*—The motion is supposed to take place in an unlimited straight tube of uniform cross-section ω . Let x be a co-ordinate measured along the tube, and specifying the position at time t of a plane of particles, which, when $t = 0$, is in

the position specified by x_0 . At the time $t = 0$ the gas is supposed to be at rest. Let p_0 and ρ_0 denote the undisturbed pressure and density, supposed uniform, and let p , ρ , u denote the pressure, density and velocity at time t for the particles specified by x_0 . The equation of continuity is

$$\rho \frac{\partial x}{\partial x_0} = \rho_0,$$

and the equation of motion is

$$\rho \frac{\partial^2 x}{\partial t^2} \frac{\partial x}{\partial x_0} = - \frac{\partial p}{\partial x_0}.$$

On introducing u , which is $\partial x / \partial t$, these equations become

$$\frac{\partial u}{\partial x_0} = - \frac{\rho_0}{\rho^2} \frac{\partial \rho}{\partial t}, \quad \frac{\partial u}{\partial t} = - \frac{1}{\rho_0} \frac{\partial p}{\partial x_0}.$$

It is supposed that p is a uniform function of ρ , and it is convenient to introduce, after RIEMANN, a quantity σ by the defining equation

$$d\sigma = \frac{1}{\rho} \sqrt{\left(\frac{dp}{d\rho}\right)} d\rho$$

and the condition that $\sigma = 0$ when $\rho = 0$. Then the equations become

$$\frac{\partial \sigma}{\partial t} + \Pi \frac{\partial u}{\partial x_0} = 0, \quad \frac{\partial u}{\partial t} + \Pi \frac{\partial \sigma}{\partial x_0} = 0,$$

where Π is a function of ρ defined by the equation

$$\Pi = \frac{\rho}{\rho_0} \sqrt{\left(\frac{dp}{d\rho}\right)}.$$

The quantity Π , which is of the dimensions of a velocity, may be regarded as a known function of σ . The value of Π when $\rho = \rho_0$ is the velocity of sound waves of small amplitude in the undisturbed state of the gas. This will be denoted by a . The equations are of Lagrangian type, and x_0 and t are the independent variables. The quantities ρ and p , like Π , can be regarded as known functions of σ . The value of σ when $\rho = \rho_0$ will be denoted by σ_0 .

3. *Progressive Waves*.—Two quantities r and s may be introduced, after RIEMANN, by the equations

$$\sigma + u = 2r, \quad \sigma - u = 2s,$$

or,

$$\sigma = r + s, \quad u = r - s.$$

The equations of continuity and motion then give the two equations

$$\frac{\partial r}{\partial t} + \Pi \frac{\partial r}{\partial x_0} = 0, \quad \frac{\partial s}{\partial t} - \Pi \frac{\partial s}{\partial x_0} = 0.$$

If s is constant, the second of these equations becomes an identity, and the first can be integrated in the form

$$r = F(x_0 - \Pi t),$$

where F denotes an arbitrary function. This can be proved easily, and the equation can be written

$$x_0 - \Pi t = f(\sigma).$$

In like manner, when r is constant, the first of the two equations becomes an identity, and the second can be integrated in the form

$$x_0 + \Pi t = f(\sigma).$$

A motion with constant r or constant s is described as a "progressive wave." A wave with constant s is propagated in the direction of increase of x , with velocity Π , which depends upon the constant value of s and the local value of r . This is the velocity relative to the *medium*, not the velocity relative to the tube. Similar statements hold for a wave of constant r .

4. *Motion of a Junction.*—When a wave is transmitted into gas at rest, or into a region where there is some other state of motion, there may be discontinuity in the values of the pressure, &c., in the two regions separated by the front of the wave. We consider here the case where there is no such discontinuity, but, while the pressure, &c., have the same values on the two sides of any plane $x = \text{const.}$, the laws of variation of these quantities on the two sides of a wave-front are different. We describe such a moving wave-front as a "junction." Our immediate object is to determine the velocity of a junction relative to the medium. We shall attain this object by supposing that there are very slight differences between the values of any of the quantities on the two sides of the wave-front.

Let w denote the velocity of the junction relative to the medium. In a very short time δt a mass equal to $\rho_0 \omega w \delta t$ has its motion and state changed from those specified by u, p, ρ , to those specified by $u + \Delta u, p + \Delta p, \rho + \Delta \rho$. The increment of momentum must be equal to the impulse of the difference of pressure, and therefore we have the equation

$$\rho_0 \omega w \delta t \Delta u = \omega \Delta p \delta t.$$

Further, the work done during the interval δt by the external pressures on the ends of this element of mass must be equal to the sum of the increments of the kinetic and intrinsic energies of the element. Now the changes of state being adiabatic and very slight, the increment of the intrinsic energy per unit of mass may be put equal to

$$-p \Delta (1/\rho),$$

and therefore we have the equation

$$\omega (p + \Delta p) (u + \Delta u) \delta t - \omega p u \delta t = \frac{1}{2} \rho_0 \omega w \delta t \{ (u + \Delta u)^2 - u^2 \} - \rho_0 \omega w \delta t p \Delta (1/\rho).$$

The two equations containing Δp and Δu give

$$\Delta p = \rho_0 w \Delta u$$

and

$$p \Delta u + u \Delta p = \rho_0 w u \Delta u + \rho_0 w (p/\rho^2) \Delta \rho.$$

The terms $u \Delta p$ and $\rho_0 w u \Delta u$ in the second of these equations cancel, and then, by eliminating Δu between the two equations, we find

$$w^2 = \frac{\rho^2}{\rho_0^2} \frac{\Delta p}{\Delta \rho}.$$

Since there is no actual discontinuity and p is a uniform function of ρ , we may replace $\Delta p/\Delta \rho$ by $dp/d\rho$, and thus obtain the equation

$$w^2 = \frac{\rho^2}{\rho_0^2} \frac{dp}{d\rho},$$

which shows that the velocity of the junction relative to the medium is that which was previously denoted by Π .

If motion is set up in one part of the gas, and advances into previously undisturbed gas, the value of ρ at the junction is ρ_0 , and therefore the velocity of the front of the wave, relative to the medium or to the tube, is that which has been denoted by a .

5. *Nature of the Motion in a Compound Wave.*—Important results can be obtained by regarding x_0 and t as functions of r and s . On interchanging the dependent and independent variables in the equations

$$\frac{\partial r}{\partial t} + \Pi \frac{\partial r}{\partial x_0} = 0, \quad \frac{\partial s}{\partial t} - \Pi \frac{\partial s}{\partial x_0} = 0,$$

we obtain the equations

$$\frac{\partial x_0}{\partial s} - \Pi \frac{\partial t}{\partial s} = 0, \quad \frac{\partial x_0}{\partial r} + \Pi \frac{\partial t}{\partial r} = 0.$$

Now the differentials of x_0 and t are always connected with those of r and s by the formulæ

$$dx_0 = \frac{\partial x_0}{\partial r} dr + \frac{\partial x_0}{\partial s} ds, \quad dt = \frac{\partial t}{\partial r} dr + \frac{\partial t}{\partial s} ds.$$

Hence the places in the medium, and the times, at which any particular value of r is found, vary according to the formulæ

$$dx_0 = \frac{\partial x_0}{\partial s} ds = \Pi \frac{\partial t}{\partial s} ds, \quad dt = \frac{\partial t}{\partial s} ds = \frac{dx_0}{\Pi},$$

and thus it appears that any value of r is transmitted through the medium, in the direction of increase of x , with the velocity Π . In like manner it can be shown that any value of s is transmitted in the opposite direction with the same local velocity.

We have seen that the velocity of a junction relative to the medium is the value of Π at the junction, and it follows that the value of r remains constant along a junction which travels in the direction of increase of x . If the junction travels in the opposite direction, the value of s at the junction remains constant.

The motion consequent upon any initial conditions consists in the transfer of the existing values of r and s through the medium with the variable velocity already described. New values of r and s can be generated at boundaries and transferred through the medium.

6. *General Analysis of Compound Waves.*—When the dependent and independent variables are interchanged in the equations

$$\frac{\partial \sigma}{\partial t} + \Pi \frac{\partial u}{\partial x_0} = 0, \quad \frac{\partial u}{\partial t} + \Pi \frac{\partial \sigma}{\partial x_0} = 0,$$

there result the equations

$$\frac{\partial x_0}{\partial u} + \Pi \frac{\partial t}{\partial \sigma} = 0, \quad \frac{\partial x_0}{\partial \sigma} + \Pi \frac{\partial t}{\partial u} = 0.$$

The first of these shows that there exists a function Z of σ and u which has the properties expressed by the equations

$$x_0 = -\Pi \frac{\partial Z}{\partial \sigma}, \quad t = \frac{\partial Z}{\partial u},$$

and then the second shows that Z satisfies the differential equation

$$\frac{\partial}{\partial \sigma} \left(\Pi \frac{\partial Z}{\partial \sigma} \right) - \Pi \frac{\partial^2 Z}{\partial u^2} = 0.$$

If Z can be found in accordance with this equation, the values of x_0 and t answering to any simultaneous value of σ and u can be deduced.

There is a relation between Z and x , which can be obtained very simply by introducing for a moment a quantity ϖ by the equation

$$\varpi = \rho_0 / \rho,$$

for then we have

$$-\frac{d\sigma}{\Pi} = -\frac{\rho_0}{\rho^2} d\rho = d\varpi,$$

and it follows that we have at once

$$x_0 = \frac{\partial Z}{\partial \varpi}, \quad t = \frac{\partial Z}{\partial u},$$

and

$$\varpi = \frac{\partial x}{\partial x_0}, \quad u = \frac{\partial x}{\partial t}.$$

These are the relations of duality familiar in discussions of partial differential equations,* and we may put

$$Z = x_0 \frac{\partial x}{\partial x_0} + t \frac{\partial x}{\partial t} - x.$$

Actually Z could differ from the right-hand member of this equation by a constant, but as such a constant would be irrelevant, the above will be taken as the relation between Z and x .

The equation satisfied by Z can be written in either of the forms

$$\frac{\partial^2 Z}{\partial \sigma^2} + \left(\frac{1}{\Pi} \frac{d\Pi}{d\sigma} \right) \frac{\partial Z}{\partial \sigma} - \frac{\partial^2 Z}{\partial n^2} = 0$$

or

$$\frac{\partial^2 Z}{\partial r \partial s} + \left(\frac{1}{2\Pi} \frac{d\Pi}{d\sigma} \right) \left(\frac{\partial Z}{\partial r} + \frac{\partial Z}{\partial s} \right) = 0.$$

7. *Relation between Pressure and Density.* The analysis of the problem is not rendered more difficult if the adiabatic relation between pressure p and volume v is taken in the form $p(v - b)^\gamma = \text{const.}$ instead of the more ordinary form $p v^\gamma = \text{const.}$, and the former is more suitable for the applications which we have in view. We shall accordingly take the relation between pressure and density to be

$$p \left(\frac{1}{\rho} - \frac{1}{\beta} \right)^\gamma = p_0 \left(\frac{1}{\rho_0} - \frac{1}{\beta} \right)^\gamma.$$

where β and γ are constants. Then the following results can be obtained without difficulty:—

$$\sigma = \frac{2}{\gamma - 1} \left[\frac{\rho_0 \gamma (\beta - \rho_0)}{\beta \rho_0} \right]^{\frac{1}{2}} \left(\frac{\beta - \rho_0}{\rho_0} - \frac{\rho}{\beta - \rho} \right)^{(\gamma - 1)/2},$$

$$\sigma_0 = \frac{2}{\gamma - 1} \left[\frac{\rho_0 \gamma (\beta - \rho_0)}{\beta \rho_0} \right]^{\frac{1}{2}},$$

$$\alpha = \left[\frac{\rho_0 \gamma \beta}{\rho_0 (\beta - \rho_0)} \right]^{\frac{1}{2}},$$

$$p = p_0 (\sigma / \sigma_0)^{2n+1},$$

$$\Pi = \alpha (\sigma / \sigma_0)^{2n},$$

where $2n$ has been written for $(\gamma + 1)/(\gamma - 1)$.

The equation for Z can now be written

$$\frac{\partial^2 Z}{\partial \sigma^2} + \frac{2n}{\sigma} \frac{\partial Z}{\partial \sigma} - \frac{\partial^2 Z}{\partial n^2} = 0,$$

or

$$\frac{\partial^2 Z}{\partial r \partial s} + \frac{n}{r + s} \left(\frac{\partial Z}{\partial r} + \frac{\partial Z}{\partial s} \right) = 0.$$

* The reduction of the equations governing the propagation of plane waves of finite amplitude to a single partial differential equation of the second order was effected by RIEMANN, who worked with "Eulerian" equations. The use of the principle of duality to connect Z and x was noted by HADAMARD.

8. *Integration of the Equation in Special Cases.*—When n is a positive integer, the equation can be integrated. We write for a moment D for $\partial/\partial u$, and observe that, if D were a constant, the equation

$$\frac{\partial^2 Z}{\partial \sigma^2} + \frac{2n}{\sigma} \frac{\partial Z}{\partial \sigma} - D^2 Z = 0$$

would be a form of RICCATI'S equation, and could be integrated in the form

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^{n-1} \frac{(e^{\sigma D} A + e^{-\sigma D} B)}{\sigma},$$

where A and B are independent of σ . Treating them as functions of u , we obtain the general primitive of the equation for Z in the form

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^{n-1} \left\{ \frac{F(\sigma + u) + f(\sigma - u)}{\sigma} \right\}.$$

9. *More General Integration.*—Interpreting the variables r and s as the co-ordinates of a point in a plane, RIEMANN showed how to integrate the equation for Z when the values of this function and its first differential coefficients are given along an arc of a curve in the plane. If V satisfies the "adjoint" equation

$$\frac{\partial^2 V}{\partial r \partial s} - n \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) \left(\frac{V}{r+s} \right) = 0,$$

the integral

$$\iint \left[\frac{\partial}{\partial r} \left\{ V \left(\frac{\partial Z}{\partial s} + \frac{nZ}{r+s} \right) \right\} - \frac{\partial}{\partial s} \left\{ Z \left(\frac{\partial V}{\partial r} - \frac{nV}{r+s} \right) \right\} \right] dr ds,$$

taken over any area in the plane, is equal to

$$\iint \left[V \left\{ \frac{\partial^2 Z}{\partial r \partial s} + \frac{n}{r+s} \left(\frac{\partial Z}{\partial r} + \frac{\partial Z}{\partial s} \right) \right\} - Z \left\{ \frac{\partial^2 V}{\partial r \partial s} - n \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) \left(\frac{V}{r+s} \right) \right\} \right] dr ds,$$

and therefore vanishes. It follows that the line-integral

$$\int V \left(\frac{\partial Z}{\partial s} + \frac{nZ}{r+s} \right) ds + Z \left(\frac{\partial V}{\partial r} - \frac{nV}{r+s} \right) dr$$

taken round the boundary of the area vanishes.

Let the values of Z and its first differential coefficients be given along an arc AC of a curve, and let P be a point which is not on the arc. Through P let lines PA and PC be drawn parallel to the axes of r and s , and let the area of integration be that bounded

by the arc AC and the lines CP and PA. The contribution of CP to the line-integral is

$$\int_{CP} V \left(\frac{\partial Z}{\partial s} + \frac{nZ}{r+s} \right) ds,$$

which may be written

$$[VZ]_P - [VZ]_C - \int_{CP} Z \left(\frac{\partial V}{\partial s} - \frac{nV}{r+s} \right) ds.$$

The contribution of PA to the line-integral is

$$\int_{PA} Z \left(\frac{\partial V}{\partial r} - \frac{nV}{r+s} \right) dr.$$

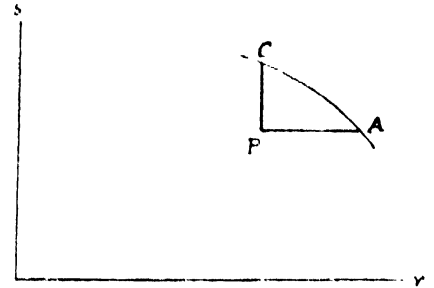


Fig. 1.

Now we can find a function to satisfy the equation for V, to make V = 1 at P, and so that, along CP, where r has the same value as at P, $\partial V/\partial s = nV/(r + s)$, and along PA, where s has the same value as at P, $\partial V/\partial r = nV/(r + s)$. Then the value of Z at P is

$$[VZ]_C - \int_{AC} \left\{ V \left(\frac{\partial Z}{\partial s} + \frac{nZ}{r+s} \right) ds + Z \left(\frac{\partial V}{\partial r} - \frac{nV}{r+s} \right) dr \right\}.$$

The required function V can be shown, after RIEMANN, to be given by the equation

$$V = \left(\frac{r+s}{r'+s'} \right)^n F(n, 1-n, 1, \xi),$$

where F is the symbol for the hypergeometric series,

$$\xi = - \frac{(r-r')(s-s')}{(r+s)(r'+s')},$$

and r', s' are the co-ordinates of P.

It may be observed that if n is an integer the series terminates, and V like Z is expressible in a finite form.

It will be useful hereafter to note the formulæ

$$\frac{\partial V}{\partial r} - \frac{nV}{r+s} = - \frac{(r+s)^{n-2} (s+r')(s-s')}{(r'+s')^{n+1}} \frac{d}{d\xi} F(n, 1-n, 1, \xi),$$

$$\frac{\partial V}{\partial s} - \frac{nV}{r+s} = - \frac{(r+s)^{n-2} (r+s')(r-r')}{(r'+s')^{n+1}} \frac{d}{d\xi} F(n, 1-n, 1, \xi).$$

THE PROGRESSIVE WAVES IN LAGRANGE'S PROBLEM.

10. *The Progressive Wave from the Left.*—Let the positive sense of the axis of x be from left to right, and let the initial positions of the two pistons be given by $x_0 = 0$ and $x_0 = c$, where c is positive. We shall denote the mass of the piston at $x_0 = 0$ by M, and that of the other piston by m.

The progressive wave generated at the piston M is determined by the equation of motion of this piston. This equation is

$$M \frac{\partial u}{\partial t} = -\omega p,$$

and it must hold at $x_0 = 0$ for all positive values of t . It may be written

$$M \frac{\partial u}{\partial t} = -\omega p_0 \left(\frac{\sigma}{\sigma_0} \right)^{2n+1},$$

and, since in the progressive wave s is constant and equal to $\frac{1}{2}\sigma_0$, or $\sigma - u = \sigma_0$, it gives

$$dt = -\frac{M}{\omega p_0} \left(\frac{\sigma_0}{\sigma} \right)^{2n+1} d\sigma,$$

from which, since $\sigma = \sigma_0$ when $t = 0$, we have at $x_0 = 0$

$$t = \frac{M\sigma_0}{2n\omega p_0} \left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n} - 1 \right\}.$$

Put for brevity

$$H = M\sigma_0 a / 2n\omega p_0,$$

then we have the values of σ and t at $x_0 = 0$ connected by the equation

$$t = \frac{H}{a} \left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n} - 1 \right\}.$$

Now in the progressive wave we have

$$x_0 - Ht = f(\sigma),$$

where the function f is to be found from the condition that at $x_0 = 0$ the above relation holds between σ and t . Hence we find

$$f(\sigma) = -H \left\{ 1 - \left(\frac{\sigma}{\sigma_0} \right)^{2n} \right\},$$

and the progressive wave formula can be written

$$\frac{at + H}{x_0 + H} = \left(\frac{\sigma_0}{\sigma} \right)^{2n}.$$

In the motion described by these formulæ any plane of particles, specified by a value of x_0 in the interval $\frac{1}{2}c > x_0 > 0$, remains at rest until $t = x_0/a$, and then moves with a velocity u , which is equal to $\sigma - \sigma_0$. Therefore the value of x answering to these particles at any subsequent time is given by the equation

$$x = x_0 + \int_{x_0/a}^t u \, dt = x_0 + \int_{\sigma_0}^{\sigma} (\sigma_0 - \sigma) 2n \frac{x_0 + H}{a} \frac{\sigma_0^{2n}}{\sigma^{2n+1}} d\sigma,$$

or

$$x = x_0 + \frac{\sigma_0}{a} (x_0 + H) \left[\frac{2n}{2n-1} \left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n-1} - 1 \right\} - \left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n} - 1 \right\} \right].$$

This equation holds so long as the plane of particles is in the region occupied by the progressive wave. In particular, the displacement of the piston M is given by the equation

$$x = -\frac{2n}{2n-1} (\sigma_0 - \sigma) \frac{H}{a} + \left(\frac{2n}{2n-1} \sigma - \sigma_0 \right) t,$$

in which

$$t = \frac{H}{a} \left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n} - 1 \right\}.$$

The corresponding values of Z are found from the formula

$$Z = x_0 \frac{\partial x}{\partial x_0} + ut - x,$$

in which

$$\frac{\partial x}{\partial x_0} = \frac{\rho_0}{\rho}, \quad \frac{\beta - \rho_0}{\rho_0} \frac{\rho}{\beta - \rho} = \left(\frac{\sigma}{\sigma_0} \right)^{2n-1},$$

to be

$$\frac{H}{a} \left\{ \frac{2n}{2n-1} \sigma_0 - \sigma - \frac{\sigma_0}{2n-1} \left(\frac{\sigma_0}{\sigma} \right)^{2n-1} \right\}.$$

11. *The Progressive Wave from the Right.*—The equation of motion of the piston m is

$$m \frac{\partial u}{\partial t} = \omega p,$$

and we put

$$h = m\sigma_0 a / 2n\omega p_0.$$

Since in the progressive wave r is constant and equal to $\frac{1}{2}\sigma_0$, or $\sigma = \frac{1}{2}u = \sigma_0$, the values of σ and t at $x_0 = c$ are found to be connected by the equation

$$t = \frac{h}{a} \left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n} - 1 \right\},$$

and then the progressive wave formula is found to be

$$\frac{at + h}{c + h - x_0} = \left(\frac{\sigma_0}{\sigma} \right)^{2n}.$$

The value of x for any plane of particles specified by a value of x_0 in the interval $\frac{1}{2}c < x_0 < c$, and for any time later than that given by $t = (c - x_0)/a$, is found to be given by the equation

$$x = x_0 + \frac{\sigma_0}{a} (c+h-x_0) \left[\left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n} - 1 \right\} - \frac{2n}{2n-1} \left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n-1} - 1 \right\} \right],$$

which holds so long as the plane of particles is in the region occupied by the progressive wave. In particular, the displacement of the piston m is given by the equation

$$x = c + \frac{2n}{2n-1} (\sigma_0 - \sigma) \frac{h}{a} - \left(\frac{2n}{2n-1} \sigma - \sigma_0 \right) t,$$

in which

$$t = \frac{h}{a} \left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n} - 1 \right\}.$$

The formula for Z is found to be

$$Z = \frac{c+h}{(2n-1)a} \sigma_0 \left\{ \left(\frac{\sigma_0}{\sigma} \right)^{2n-1} - 1 \right\} - \frac{h}{a} (\sigma_0 - \sigma).$$

THE FIRST MIDDLE WAVE.

12. *Conditions satisfied at the Receding Front.*—In the progressive wave from the left s is constant and r variable. The greatest value of r , which is the undisturbed value $\frac{1}{2}\sigma_0$, travels at the front of the wave, and continually diminishing values of r , generated at the piston M , travel after it. Similar statements, with r and s interchanged, hold for the progressive wave from the right. The fronts of both waves travel along the tube with velocity a . When they reach the middle section, a compound wave begins to be generated there, and transmitted in both directions, encroaching upon the original progressive waves. This wave has a receding front, along which s is constant, travelling towards the left, and an advancing front, along which r is constant, travelling towards the right. The constant values of r and s at the two fronts are equal, and each of them is $\frac{1}{2}\sigma_0$.

At the receding front the variations of x_0 and t are connected by the equation

$$dx_0 + \Pi dt = 0,$$

while the values of x_0 , t and σ are connected by the progressive wave formula, which can be written

$$x_0 - \Pi t + H \left(1 - \frac{\Pi}{a} \right) = 0,$$

so that the variations of x_0 , t and Π are connected by the equation

$$dx_0 - (\Pi dt + t d\Pi) - \frac{H}{a} d\Pi = 0.$$

On elimination of dx_0 there results the equation

$$2H dt + \left(t + \frac{H}{a}\right) dH = 0,$$

which can be integrated in the form

$$\left(t + \frac{H}{a}\right)^2 H = \text{const.}$$

To determine the constant there is the condition that when $x_0 = \frac{1}{2}c$ and $t = \frac{1}{2}c/a$, the value of H is a . Hence at the receding front we have

$$(at + H)^2 H = \left(\frac{1}{2}c + H\right)^2 a,$$

or

$$(x_0 + H)(at + H) = \left(\frac{1}{2}c + H\right)^2.$$

13. *Conditions satisfied at the Advancing Front.*—At the advancing front we have in like manner

$$dx_0 - H dt = 0 \quad \text{and} \quad c + h - x_0 - H \left(t + \frac{h}{a}\right) = 0,$$

leading to

$$(at + h)^2 H = \left(\frac{1}{2}c + h\right)^2 a$$

and

$$(c + h - x_0)(at + h) = \left(\frac{1}{2}c + h\right)^2.$$

14. *Conditions determining the First Middle Wave.* It will now be convenient to restrict the value of n to be an integer. This happens when γ has one of the values 3, 5/3, 7/5, 9/7, 11/9, With a view to applications, in which the value of γ is 1.2 nearly, we shall take the value 11/9 for γ , or 5 for n . Then in any compound wave Z has the form

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{F(\sigma + u) + f(\sigma - u)}{\sigma} \right\}.$$

or

$$105\sigma^{-9} F(\sigma + u) - 105\sigma^{-8} F^{(1)}(\sigma + u) + 45\sigma^{-7} F^{(2)}(\sigma + u) - 10\sigma^{-6} F^{(3)}(\sigma + u) + \sigma^{-5} F^{(4)}(\sigma + u) \\ + 105\sigma^{-9} f(\sigma - u) - 105\sigma^{-8} f^{(1)}(\sigma - u) + 45\sigma^{-7} f^{(2)}(\sigma - u) - 10\sigma^{-6} f^{(3)}(\sigma - u) + \sigma^{-5} f^{(4)}(\sigma - u),$$

where $F^{(1)}$, $F^{(2)}$, and so on stand for the first, second, &c., differential coefficients of the function F with respect to its argument. We have to determine the unknown functions from the values of Z at the advancing and receding fronts.

At the advancing front, where $r = \frac{1}{2}\sigma_0$ and $\sigma = \frac{1}{2}\sigma_0 + s$, we have

$$Z = \frac{1}{5}(c + h) \frac{\sigma_0}{a} \left\{ \left(\frac{\sigma_0}{\frac{1}{2}\sigma_0 + s}\right)^5 - 1 \right\} - \frac{h}{a} \left(\frac{1}{2}\sigma_0 - s\right).$$

and at the receding front, where $s = \frac{1}{2}\sigma_0$ and $\sigma = \frac{1}{2}\sigma_0 + r$, we have

$$Z = -\frac{1}{6}H \frac{\sigma_0}{\alpha} \left\{ \left(\frac{\sigma_0}{\frac{1}{2}\sigma_0 + r} \right)^6 - 1 \right\} + \frac{H}{\alpha} (\frac{1}{2}\sigma_0 - r).$$

15. *Determination of the First Middle Wave.*—To determine Z from these conditions we may have recourse to RIEMANN'S method, taking the curve AC to consist of segments of two lines AB and BC , which meet at the point B , where $r = s = \frac{1}{2}\sigma_0$, and are parallel respectively to the axes of s and r .

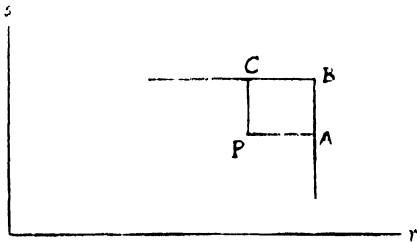


Fig. 2.

We have then

$$\int_{CP} V \left(\frac{\partial Z}{\partial s} + \frac{5Z}{r+s} \right) ds + \int_{PA} Z \left(\frac{\partial V}{\partial r} - \frac{5V}{r+s} \right) dr \\ + \int_{AB} V \left(\frac{\partial Z}{\partial s} + \frac{5Z}{r+s} \right) ds + \int_{BC} Z \left(\frac{\partial V}{\partial r} - \frac{5V}{r+s} \right) dr = 0.$$

This equation is

$$[VZ]_P - [VZ]_C - \int_{CP} Z \left(\frac{\partial V}{\partial s} - \frac{5V}{r+s} \right) ds + \int_{PA} Z \left(\frac{\partial V}{\partial r} - \frac{5V}{r+s} \right) dr \\ + [VZ]_B - [VZ]_A - \int_{AB} Z \left(\frac{\partial V}{\partial s} - \frac{5V}{r+s} \right) ds + \int_{BC} Z \left(\frac{\partial V}{\partial r} - \frac{5V}{r+s} \right) dr = 0,$$

or

$$Z(r', s') = [VZ]_A - [VZ]_B + [VZ]_C + \int_{AB} Z \left(\frac{\partial V}{\partial s} - \frac{5V}{r+s} \right) ds - \int_{BC} Z \left(\frac{\partial V}{\partial r} - \frac{5V}{r+s} \right) dr,$$

where r', s' are the co-ordinates of P . Also at A we have

$$r = \frac{1}{2}\sigma_0, \quad s = s', \quad \zeta = 0, \quad V = \left(\frac{\frac{1}{2}\sigma_0 + s'}{r' + s'} \right)^6, \quad Z = \frac{1}{6}(c+h) \frac{\sigma_0}{\alpha} \left\{ \left(\frac{\sigma_0}{\frac{1}{2}\sigma_0 + s'} \right)^6 - 1 \right\} - \frac{h}{\alpha} (\frac{1}{2}\sigma_0 - s'),$$

so that

$$[VZ]_A = \frac{1}{6}(c+h) \frac{\sigma_0}{\alpha} \left\{ \frac{\sigma_0^6}{(r' + s')^6 (\frac{1}{2}\sigma_0 + s')^6} - \left(\frac{\frac{1}{2}\sigma_0 + s'}{r' + s'} \right)^6 \right\} - \frac{h}{\alpha} (\frac{1}{2}\sigma_0 - s') \left(\frac{\frac{1}{2}\sigma_0 + s'}{r' + s'} \right)^6.$$

At B we have

$$r = s = \frac{1}{2}\sigma_0, \quad Z = 0,$$

so that

$$[VZ]_B = 0.$$

At C we have

$$r = r', \quad s = \frac{1}{2}\sigma_0, \quad \zeta = 0, \quad V = \left(\frac{\frac{1}{2}\sigma_0 + r'}{r' + s'} \right)^6, \quad Z = -\frac{1}{6}H \frac{\sigma_0}{\alpha} \left\{ \left(\frac{\sigma_0}{\frac{1}{2}\sigma_0 + r'} \right)^6 - 1 \right\} + \frac{H}{\alpha} (\frac{1}{2}\sigma_0 - r'),$$

so that

$$[VZ]_C = -\frac{1}{6}H \frac{\sigma_0}{\alpha} \left\{ \frac{\sigma_0^6}{(r' + s')^6 (\frac{1}{2}\sigma_0 + r')^6} - \left(\frac{\frac{1}{2}\sigma_0 + r'}{r' + s'} \right)^6 \right\} + \frac{H}{\alpha} (\frac{1}{2}\sigma_0 - r') \left(\frac{\frac{1}{2}\sigma_0 + r'}{r' + s'} \right)^6.$$

Along AB we have

$$Z = \frac{1}{9} (c+h) \frac{\sigma_0}{a} \left\{ \left(\frac{\sigma_0}{\frac{1}{2}\sigma_0 + s} \right)^9 - 1 \right\} - \frac{h}{a} \left(\frac{1}{2}\sigma_0 - s \right),$$

$$\frac{\partial V}{\partial s} - \frac{5V}{r+s} = \frac{\left(\frac{1}{2}\sigma_0 + s' \right) \left(\frac{1}{2}\sigma_0 - r' \right) \left(\frac{1}{2}\sigma_0 + s \right)^3}{(r'+s')^6} (20 - 180\zeta + 420\zeta^2 - 280\zeta^3),$$

where

$$\zeta = - \frac{\left(\frac{1}{2}\sigma_0 - r' \right) (s-s')}{(r'+s') \left(\frac{1}{2}\sigma_0 + s \right)},$$

so that

$$\begin{aligned} \int_{AB} Z \left(\frac{\partial V}{\partial s} - \frac{5V}{r+s} \right) ds &= \int_s^{\frac{1}{2}\sigma_0} \left[\frac{(c+h)\sigma_0}{9a} \left\{ \left(\frac{\sigma_0}{\frac{1}{2}\sigma_0 + s} \right)^9 - 1 \right\} - \frac{h}{a} \left(\frac{1}{2}\sigma_0 - s \right) \right] \frac{\left(\frac{1}{2}\sigma_0 + s' \right) \left(\frac{1}{2}\sigma_0 - r' \right)}{(r'+s')^6} \\ &\times \left[20 \left(\frac{1}{2}\sigma_0 + s \right)^3 + 180 \frac{\left(\frac{1}{2}\sigma_0 - r' \right) (s-s') \left(\frac{1}{2}\sigma_0 + s \right)^2}{r'+s'} \right. \\ &\left. + 420 \frac{\left(\frac{1}{2}\sigma_0 - r' \right)^2 (s-s')^2 \left(\frac{1}{2}\sigma_0 + s \right)}{(r'+s')^2} + 280 \frac{\left(\frac{1}{2}\sigma_0 - r' \right)^3 (s-s')^3}{(r'+s')^3} \right] ds. \end{aligned}$$

Along BC we have

$$Z = -\frac{1}{9} H \frac{\sigma_0}{a} \left\{ \left(\frac{\sigma_0}{\frac{1}{2}\sigma_0 + r} \right)^9 - 1 \right\} + \frac{H}{a} \left(\frac{1}{2}\sigma_0 - r \right),$$

$$\frac{\partial V}{\partial r} - \frac{5V}{r+s} = \frac{\left(\frac{1}{2}\sigma_0 + r' \right) \left(\frac{1}{2}\sigma_0 - s' \right) \left(\frac{1}{2}\sigma_0 + r \right)^3}{(r'+s')^6} (20 - 180\zeta + 420\zeta^2 - 280\zeta^3),$$

where

$$\zeta = - \frac{\left(\frac{1}{2}\sigma_0 - s' \right) (r-r')}{(r'+s') \left(\frac{1}{2}\sigma_0 + r \right)},$$

so that

$$\begin{aligned} \int_{BC} Z \left(\frac{\partial V}{\partial r} - \frac{5V}{r+s} \right) dr &= \int_{\frac{1}{2}\sigma_0}^r \left[-\frac{H\sigma_0}{9a} \left\{ \left(\frac{\sigma_0}{\frac{1}{2}\sigma_0 + r} \right)^9 - 1 \right\} + \frac{H}{a} \left(\frac{1}{2}\sigma_0 - r \right) \right] \frac{\left(\frac{1}{2}\sigma_0 + r' \right) \left(\frac{1}{2}\sigma_0 - s' \right)}{(r'+s')^6} \\ &\times \left[20 \left(\frac{1}{2}\sigma_0 + r \right)^3 + 180 \frac{\left(\frac{1}{2}\sigma_0 - s' \right) (r-r') \left(\frac{1}{2}\sigma_0 + r \right)^2}{r'+s'} \right. \\ &\left. + 420 \frac{\left(\frac{1}{2}\sigma_0 - s' \right)^2 (r-r')^2 \left(\frac{1}{2}\sigma_0 + r \right)}{(r'+s')^2} + 280 \frac{\left(\frac{1}{2}\sigma_0 - s' \right)^3 (r-r')^3}{(r'+s')^3} \right] dr. \end{aligned}$$

There is no difficulty in the integrations. After they have been performed, we may suppress the accents on r' and s' , and so obtain the formula for Z in the first middle wave expressed as a function of r and s . We find, for example, as the coefficient of $(c+h)\sigma_0/9a$ the expression

$$\begin{aligned} &\left(\frac{\sigma_0}{r+s} \right)^9 - \left(\frac{\frac{1}{2}\sigma_0 + s}{r+s} \right)^9 - \frac{\left(\frac{1}{2}\sigma_0 + s \right) \left(\frac{1}{2}\sigma_0 - r \right)}{(r+s)^6} \{ 9\sigma_0^4 - 5 \left(\frac{1}{2}\sigma_0 + s \right)^4 \} \\ &- \frac{\left(\frac{1}{2}\sigma_0 + s \right) \left(\frac{1}{2}\sigma_0 - r \right)^2}{(r+s)^7} \{ 81\sigma_0^4 - 90\sigma_0^3 \left(\frac{1}{2}\sigma_0 + s \right) + 15 \left(\frac{1}{2}\sigma_0 + s \right)^4 \} \\ &- \frac{\left(\frac{1}{2}\sigma_0 + s \right) \left(\frac{1}{2}\sigma_0 - r \right)^3}{(r+s)^8} \{ 189\sigma_0^4 - 420\sigma_0^3 \left(\frac{1}{2}\sigma_0 + s \right) + 270\sigma_0^2 \left(\frac{1}{2}\sigma_0 + s \right)^2 - 35 \left(\frac{1}{2}\sigma_0 + s \right)^4 \} \\ &- \frac{\left(\frac{1}{2}\sigma_0 + s \right) \left(\frac{1}{2}\sigma_0 - r \right)^4}{(r+s)^9} \{ 126\sigma_0^4 - 420\sigma_0^3 \left(\frac{1}{2}\sigma_0 + s \right) + 540\sigma_0^2 \left(\frac{1}{2}\sigma_0 + s \right)^2 - 315\sigma_0 \left(\frac{1}{2}\sigma_0 + s \right)^3 \\ &\qquad\qquad\qquad + 70 \left(\frac{1}{2}\sigma_0 + s \right)^4 \}. \end{aligned}$$

In this expression we put

$$\frac{1}{2}\sigma_0 - r = \left(\frac{1}{2}\sigma_0 + s\right) - (r + s) = \frac{1}{2}(\sigma_0 + \sigma - u) - \sigma,$$

and find that the expression is the same as

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{\phi_1(\sigma - u)}{\sigma} \right\},$$

where

$$\phi_1(\sigma - u) = 105 \left[\sigma_0^9 - \frac{1}{2^5}(\sigma_0 + \sigma - u)^6 + 126\sigma_0^4 - 210\sigma_0^3(\sigma_0 + \sigma - u) + 135\sigma_0^2(\sigma_0 + \sigma - u)^2 - \frac{315}{8}\sigma_0(\sigma_0 + \sigma - u)^3 + \frac{35}{8}(\sigma_0 + \sigma - u)^4 \right].$$

The remaining terms in the expression for Z may be treated in the same way, and we obtain finally, as the expression for Z in the first middle wave,

$$\begin{aligned} Z = & \frac{(c+h)\sigma_0}{9a} \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{\phi_1(\sigma - u)}{\sigma} \right\} + \frac{h}{a} \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{\psi_1(\sigma - u)}{\sigma} \right\} \\ & - \frac{H\sigma_0}{9a} \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{\phi_1(\sigma + u)}{\sigma} \right\} - \frac{H}{a} \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{\psi_1(\sigma + u)}{\sigma} \right\}, \end{aligned}$$

in which the expression denoted by ϕ_1 has been written down, and ψ_1 is given by the formula

$$\psi_1(\sigma - u) = -\frac{1}{15 \times 2^6} (\sigma_0 + \sigma - u)^5 (\sigma_0 - \sigma + u)^5.$$

It may be observed that the differential coefficient of the function ϕ_1 is given by the equation

$$\phi_1^{(1)}(\sigma - u) = -\frac{3}{2^8} (\sigma_0 + \sigma - u)^4 (\sigma_0 - \sigma + u)^4.$$

Although the actual calculation of Z is rather long, it is comparatively easy to verify that the form obtained satisfies the conditions by which Z was determined.

16. *Transformation of the Formula.*—The form taken by Z in the first middle wave is

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{\Phi_1(\sigma + u) + \Psi_1(\sigma - u)}{\sigma} \right\},$$

where

$$\Phi_1(\sigma + u) = -\frac{H\sigma_0}{9a} \phi_1(\sigma + u) - \frac{H}{a} \psi_1(\sigma + u),$$

$$\Psi_1(\sigma - u) = \frac{(c+h)\sigma_0}{9a} \phi_1(\sigma - u) + \frac{h}{a} \psi_1(\sigma - u),$$

so that Φ_1 and Ψ_1 are rational integral functions of the tenth degree. Now it is important to observe that when $1 < \nu < 8$,

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(\sigma + u)^\nu}{\sigma} \right\} = (-1)^\nu \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(\sigma - u)^\nu}{\sigma} \right\},$$

while for $\nu = 9$ we have

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(\sigma + u)^9}{\sigma} \right\} = 2(2 \cdot 4 \cdot 6 \cdot 8) - \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(\sigma - u)^9}{\sigma} \right\},$$

and for $\nu = 10$ we have

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(\sigma + u)^{10}}{\sigma} \right\} = 2u(2 \cdot 4 \cdot 6 \cdot 8 \cdot 10) + \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(\sigma - u)^{10}}{\sigma} \right\}.$$

It follows that the expression for Z can be written either in the form

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{Q_1(\sigma + u)}{\sigma} \right\} + K_1 + L_1 u,$$

or in the form

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{q_1(\sigma - u)}{\sigma} \right\} + k_1 + l_1 u,$$

where

$$K_1 = 2 \cdot (2 \cdot 4 \cdot 6 \cdot 8) \times \text{the coefficient of } (\sigma - u)^9 \text{ in } \Psi_1(\sigma - u),$$

$$L_1 = -2 \cdot (2 \cdot 4 \cdot 6 \cdot 8 \cdot 10) \times \text{the coefficient of } (\sigma - u)^{10} \text{ in } \Psi_1(\sigma - u),$$

$$k_1 = 2 \cdot (2 \cdot 4 \cdot 6 \cdot 8) \times \text{the coefficient of } (\sigma + u)^9 \text{ in } \Phi_1(\sigma + u),$$

$$l_1 = 2 \cdot (2 \cdot 4 \cdot 6 \cdot 8 \cdot 10) \times \text{the coefficient of } (\sigma + u)^{10} \text{ in } \Phi_1(\sigma + u),$$

and Q_1 and q_1 are certain rational integral functions of the 9th degree. The explicit expressions are

$$K_1 = -\frac{1}{3}(c+h)(\sigma_0/a), \quad L_1 = -h/a, \quad k_1 = \frac{1}{3}H\sigma_0/a, \quad l_1 = -H/a,$$

$$Q_1(\sigma + u) = \frac{\sigma_0^{10}}{1890a}(c+h-H) + \frac{h-H}{15 \times 2^9 a} \{ \sigma_0^2 - (\sigma + u)^2 \}$$

$$+ \frac{\sigma_0(c+h+H)}{945 \times 2^8 a} \{ 315\sigma_0^8(\sigma + u) - 420\sigma_0^6(\sigma + u)^3 + 378\sigma_0^4(\sigma + u)^5 - 180\sigma_0^2(\sigma + u)^7 + 35(\sigma + u)^9 \},$$

and $q_1(\sigma - u)$ is obtained from $Q_1(\sigma + u)$ by writing $-(\sigma - u)$ for $(\sigma + u)$.

17. *Incidence of the First Middle Wave on the Pistons.*—The values of all the quantities at the piston M, at the instant when the first middle wave reaches it, are to be found from the formulæ, connected with the receding front of the wave, by putting $x_0 = 0$. We see that the receding front reaches the piston M at the time T_1 , where

$$T_1 = \frac{(\frac{1}{2}c + H)^2}{aH} - \frac{H}{a},$$

that the corresponding value of σ is Σ_1 , where

$$\Sigma_1 = \sigma_0 \left(\frac{H}{\frac{1}{2}c + H} \right)^{1/2},$$

that the corresponding values of r , s , u are R_1 , S_1 , U_1 , where

$$R_1 = \Sigma_1 - \frac{1}{2}\sigma_0, \quad S_1 = \frac{1}{2}\sigma_0, \quad U_1 = -(\sigma_0 - \Sigma_1),$$

and that the corresponding value of Z is Z_1 , where

$$Z_1 = \frac{\Pi}{\alpha} \left\{ \frac{1}{9}\sigma_0 - \Sigma_1 - \frac{\sigma_0}{9} \left(\frac{\sigma_0}{\Sigma_1} \right)^u \right\}.$$

In like manner we see that the advancing front reaches the piston m at the time t_1 , where

$$t_1 = \frac{(\frac{1}{2}c+h)^2}{\alpha h} - \frac{h}{\alpha},$$

and that the corresponding values of σ , r , s , u , Z , are σ_1 , r_1 , s_1 , u_1 , z_1 , where

$$\begin{aligned} \sigma_1 &= \sigma_0 \left(\frac{h}{\frac{1}{2}c+h} \right)^{1/h}, & r_1 &= \frac{1}{2}\sigma_0, & s_1 &= (\sigma_1 - \frac{1}{2}\sigma_0), & u_1 &= \sigma_0 - \sigma_1, \\ z_1 &= \frac{(c+h)\sigma_0}{9\alpha} \left\{ \left(\frac{\sigma_0}{\sigma_1} \right)^u - 1 \right\} - \frac{h}{\alpha} (\sigma_0 - \sigma_1). \end{aligned}$$

THE FIRST REFLECTED WAVES.

18. *Conditions determining the First Reflected Wave from the Left.*—After the instant $t = T_1$ the formulæ belonging to the progressive wave from the left cease to hold in the neighbourhood of $x_0 = 0$, and a new compound wave, the first reflected wave from the left, is generated there and encroaches upon the first middle wave. The junction is characterized by the value R_1 of r . The conditions determining the reflected wave are the condition which holds at the junction, where $r = R_1$, and the condition which holds at the piston, where $x_0 = 0$. It is further necessary that x_0 should vanish when $r = R_1$ and $s = S_1$.

The condition which holds at the junction is that the value of Z , calculated from the formulæ belonging to the reflected wave, should be equal to that calculated from the formulæ belonging to the first middle wave when $r = R_1$. The condition which holds at the piston is the equation of motion of the piston, viz., that at $x_0 = 0$

$$M \frac{\partial u}{\partial t} = -\omega p.$$

The condition that x_0 should vanish when $r = R_1$ and $s = S_1$ is the condition that $\partial Z / \partial \sigma$ should vanish for these values of r and s .

To express the condition which holds at the piston in terms of Z we substitute $p_0(\sigma/\sigma_0)^{11}$ for p , and

$$\frac{\partial x_0}{\partial \sigma} \left/ \left(\frac{\partial x_0}{\partial \sigma} \frac{\partial t}{\partial u} - \frac{\partial x_0}{\partial u} \frac{\partial t}{\partial \sigma} \right) \right.$$

for $\partial u/\partial t$. Then we have

$$\frac{M}{\omega p_0} \frac{\partial x_0}{\partial \sigma} = - \left(\frac{\sigma}{\sigma_0} \right)^{11} \left(\frac{\partial x_0}{\partial \sigma} \frac{\partial t}{\partial u} - \frac{\partial x_0}{\partial u} \frac{\partial t}{\partial \sigma} \right).$$

Again we substitute $-11 \partial Z/\partial \sigma$ for x_0 and $\partial Z/\partial u$ for t , and put $\partial Z/\partial \sigma = 0$, obtaining the equation

$$\frac{\partial^2 Z}{\partial \sigma^2} \frac{\partial^2 Z}{\partial u^2} - \left(\frac{\partial^2 Z}{\partial \sigma \partial u} \right)^2 = - \frac{1011 \sigma_0^{10}}{\alpha \sigma^{11}} \frac{\partial^2 Z}{\partial \sigma^2}.$$

The condition which holds at the junction is that

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{Q_1(\sigma + u)}{\sigma} \right\} + K_1 + L_1 u$$

for all values of σ and u for which $\sigma + u = 2R_1$.

19. *Determination of the First Reflected Wave from the Left.*—These conditions can be satisfied by assuming for Z the form

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{F_1(\sigma + u)}{\sigma} \right\} + K_1 + L_1 u.$$

expanding the unknown function F_1 in the series

$$F_1(\sigma + u) = A_0 + A_1(\sigma + u - 2R_1) + A_2(\sigma + u - 2R_1)^2 + \dots,$$

and finding the coefficients of this series.

The condition which holds at the junction determines the coefficients A_0, A_1, \dots, A_4 . The condition that $\partial Z/\partial \sigma$ vanishes when $r = R_1$ and $s = S_1$ determines the coefficient A_0 . The remaining coefficients are to be determined by the condition which holds at the piston.

We have

$$\begin{aligned} Z = 105 \frac{F_1(\sigma + u)}{\sigma^9} - 105 \frac{F_1^{(1)}(\sigma + u)}{\sigma^8} + 45 \frac{F_1^{(2)}(\sigma + u)}{\sigma^7} \\ - 10 \frac{F_1^{(3)}(\sigma + u)}{\sigma^6} + \frac{F_1^{(4)}(\sigma + u)}{\sigma^5} + K_1 + L_1 u, \end{aligned}$$

from which we find

$$A_0 = Q_1(2R_1), \quad A_1 = Q_1^{(1)}(2R_1), \quad 2! A_2 = Q_1^{(2)}(2R_1), \quad 3! A_3 = Q_1^{(3)}(2R_1), \quad 4! A_4 = Q_1^{(4)}(2R_1).$$

We have also

$$\begin{aligned} \frac{\partial Z}{\partial \sigma} = -945 \frac{F_1(\sigma + u)}{\sigma^{10}} + 945 \frac{F_1^{(1)}(\sigma + u)}{\sigma^9} - 420 \frac{F_1^{(2)}(\sigma + u)}{\sigma^8} \\ + 105 \frac{F_1^{(3)}(\sigma + u)}{\sigma^7} - 15 \frac{F_1^{(4)}(\sigma + u)}{\sigma^6} + \frac{F_1^{(5)}(\sigma + u)}{\sigma^5}, \end{aligned}$$

and the condition that this vanishes when $r = R_1$ and $s = S_1$ gives

$$-945 \frac{A_{10}}{\Sigma_1^{10}} + 945 \frac{A_1}{\Sigma_1^9} - 420 \frac{2! A_2}{\Sigma_1^8} + 105 \frac{3! A_3}{\Sigma_1^7} - 15 \frac{4! A_4}{\Sigma_1^6} + \frac{5! A_5}{\Sigma_1^5} = 0,$$

thus determining the coefficient A_6 . It is seen easily that $5! A_5 = Q_1^{(6)}(2R_1)$.

Now when $\partial Z / \partial \sigma = 0$ the differential equation for Z shows that

$$\frac{\partial^2 Z}{\partial \sigma^2} = \frac{\partial^2 Z}{\partial u^2}.$$

Also we have in general

$$\begin{aligned} \frac{\partial^2 Z}{\partial u^2} &= 105 \frac{F_1^{(2)}(\sigma+u)}{\sigma^9} - 105 \frac{F_1^{(3)}(\sigma+u)}{\sigma^8} + 45 \frac{F_1^{(4)}(\sigma+u)}{\sigma^7} - 10 \frac{F_1^{(5)}(\sigma+u)}{\sigma^6} + \frac{F_1^{(6)}(\sigma+u)}{\sigma^5}, \\ \frac{\partial^2 Z}{\partial \sigma \partial u} &= -945 \frac{F_1^{(1)}(\sigma+u)}{\sigma^{10}} + 945 \frac{F_1^{(2)}(\sigma+u)}{\sigma^9} - 420 \frac{F_1^{(3)}(\sigma+u)}{\sigma^8} \\ &\quad + 105 \frac{F_1^{(4)}(\sigma+u)}{\sigma^7} - 15 \frac{F_1^{(5)}(\sigma+u)}{\sigma^6} + \frac{F_1^{(6)}(\sigma+u)}{\sigma^5}, \end{aligned}$$

and therefore when $\partial Z / \partial \sigma = 0$ we have the equation

$$\begin{aligned} &\sigma^2 \{ 105 F_1^{(2)} - 105 \sigma F_1^{(3)} + 45 \sigma^2 F_1^{(4)} - 10 \sigma^3 F_1^{(5)} + \sigma^4 F_1^{(6)} \}^2 \\ &- \{ 945 F_1^{(1)} - 945 \sigma F_1^{(2)} + 420 \sigma^2 F_1^{(3)} - 105 \sigma^3 F_1^{(4)} + 15 \sigma^4 F_1^{(5)} - \sigma^5 F_1^{(6)} \}^2 \\ &+ 10 H(\sigma_0^{10}/u) \{ 105 F_1^{(2)} - 105 \sigma F_1^{(3)} + 45 \sigma^2 F_1^{(4)} - 10 \sigma^3 F_1^{(5)} + \sigma^4 F_1^{(6)} \} = 0 \end{aligned}$$

as well as the equation

$$945 F_1 - 945 \sigma F_1^{(1)} + 420 \sigma^2 F_1^{(2)} - 105 \sigma^3 F_1^{(3)} + 15 \sigma^4 F_1^{(4)} - \sigma^5 F_1^{(5)} = 0.$$

The equation expressing the condition which holds at the piston is linear in $F_1^{(6)}$, and therefore can be solved for $F_1^{(6)}$ without ambiguity. As it holds for $r = R_1$ and $s = S_1$, it determines the coefficient A_6 . The equation in question holds for all values of σ and u for which the equation expressing the vanishing of $\partial Z / \partial \sigma$ holds, and it can therefore be differentiated totally with respect to σ , u being treated as a function of σ in accordance with the equation $\partial Z / \partial \sigma = 0$. This process yields an equation which determines the coefficient A_7 without ambiguity. A second differentiation yields an equation from which the value of the coefficient A_8 may be found. By proceeding in this way we may obtain as many of the coefficients A as we wish.

This method of determining the coefficients A_6, A_7, \dots is not very well adapted to numerical computation, and other methods will be explained presently.

20. *Determination of the First Reflected Wave from the Right.*—The junction of the reflected wave and the first middle wave is characterised by the value s_1 of s . The conditions determining the reflected wave are the condition which holds at the junction, where $s = s_1$, and the condition which holds at the piston, where $x_0 = c$. Further, x_0 must be equal to c when $r = r_1$ and $s = s_1$.

At the junction, where $s = s_1$, the value of Z must be the same whether it is found from the formulæ belonging to the reflected wave or from those belonging to the first middle wave. At $x_0 = c$ the equation of motion of the piston, viz., the equation

$$m \frac{\partial u}{\partial t} = \omega p$$

must hold. The equation $-\Pi \partial Z / \partial \sigma = c$ must hold at $r = r_1$ and $s = s_1$.

To express these conditions it is convenient to write

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left(\frac{c \sigma_0^{10}}{945 a \sigma} \right) + Z';$$

then Z' satisfies the same differential equation as Z , and $\partial Z' / \partial \sigma$ vanishes when $x_0 = c$.

The condition which holds at the junction is that

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{q_1 (\sigma - u)}{\sigma} \right\} + k_1 + l_1 u$$

for all values of σ and u for which $\sigma - u = 2s_1$.

The condition which holds at the piston is that

$$\frac{\partial^2 Z'}{\partial \sigma^2} \frac{\partial^2 Z'}{\partial u^2} - \left(\frac{\partial^2 Z'}{\partial \sigma \partial u} \right)^2 = \frac{10 h \sigma_0^{10}}{a \sigma^{11}} \frac{\partial^2 Z'}{\partial \sigma^2}$$

when $\partial Z' / \partial \sigma = 0$.

These conditions can be satisfied by assuming for Z the form

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left[\frac{1}{\sigma} \left\{ \frac{c \sigma_0^{10}}{945 a} + f_1 (\sigma - u) \right\} \right] + k_1 + l_1 u,$$

expanding the unknown function f_1 in the series

$$f_1 = a_0 + a_1 (\sigma - u - 2s_1) + a_2 (\sigma - u - 2s_1)^2 + \dots,$$

and finding the coefficients of this series.

For the coefficients a_0, a_1, \dots, a_4 we find

$$a_0 + \frac{c \sigma_0^{10}}{945 a} = q_1 (2s_1), \quad a_1 = q_1^{(1)} (2s_1), \quad 2! a_2 = q_1^{(2)} (2s_1), \quad 3! a_3 = q_1^{(3)} (2s_1), \quad 4! a_4 = q_1^{(4)} (2s_1).$$

The coefficient a_4 is given by the equation

$$-945 \frac{a_0}{\sigma_1^{10}} + 945 \frac{a_1}{\sigma_1^9} - 420 \frac{2! a_2}{\sigma_1^8} + 105 \frac{3! a_3}{\sigma_1^7} - 15 \frac{4! a_4}{\sigma_1^6} + \frac{5! a_5}{\sigma_1^5} = 0.$$

The remaining coefficients can be determined from the condition which holds at the piston in the same way as the corresponding coefficients in the formula belonging to the first reflected wave from the left could be determined.

21. *Relation between Pressure and Velocity at a Piston.*—The equation in terms of σ and u , which holds for the first reflected wave from the left at $x_0 = 0$, is the relation between the pressure on the piston M and its velocity during the time that the wave is being generated. It may also be interpreted as the equation of a certain locus in the plane of r and s . This locus passes through the point (R_1, S_1) , and we may take its equation to be of the form

$$r - R_1 = B_1(s - S_1) + (B_2/\Sigma_1)(s - S_1)^2 + (B_3/\Sigma_1^2)(s - S_1)^3 + \dots$$

Now if the coefficients B were known, we could determine x_0 as a function of r and s , from the known value, zero, of the function along the locus and the values of its differential coefficients along the same curve. These differential coefficients also are known along the locus. To prove this and obtain formulæ for these differential coefficients, we write X for x_0 and observe that the equations of Article 6 show that X satisfies the differential equation

$$\frac{\partial}{\partial u} \left(\frac{1}{\Pi} \frac{\partial X}{\partial u} \right) = \frac{\partial}{\partial \sigma} \left(\frac{1}{\Pi} \frac{\partial X}{\partial \sigma} \right),$$

which can be written either in the form

$$\frac{\partial^2 X}{\partial \sigma^2} - \frac{10}{\sigma} \frac{\partial X}{\partial \sigma} - \frac{\partial^2 X}{\partial u^2} = 0,$$

or in the form

$$\frac{\partial^2 X}{\partial r \partial s} - \frac{5}{r+s} \left(\frac{\partial X}{\partial r} + \frac{\partial X}{\partial s} \right) = 0.$$

Further at $x_0 = 0$ we have

$$\frac{\partial u}{\partial t} = -\frac{\omega p}{M} = -\frac{\omega p_0}{M} \left(\frac{\sigma}{\sigma_0} \right)^{11} = -\frac{\alpha \sigma_0}{10H} \left(\frac{\sigma}{\sigma_0} \right)^{11},$$

and

$$\frac{\partial u}{\partial t} = \frac{\frac{\partial x_0}{\partial \sigma}}{\frac{\partial x_0}{\partial \sigma} \frac{\partial t}{\partial u} - \frac{\partial x_0}{\partial u} \frac{\partial t}{\partial \sigma}} = \frac{\Pi \frac{\partial x_0}{\partial \sigma}}{-\left(\frac{\partial x_0}{\partial \sigma}\right)^2 + \left(\frac{\partial x_0}{\partial u}\right)^2} = \frac{\alpha \left(\frac{\sigma}{\sigma_0}\right)^{10} \frac{\partial x_0}{\partial \sigma}}{-\left(\frac{\partial x_0}{\partial \sigma}\right)^2 \left\{1 - \left(\frac{d\sigma}{du}\right)^2\right\}},$$

where $d\sigma/du$ is to be found from the equation connecting r and s . Thus we have along this locus

$$\frac{\partial x_0}{\partial \sigma} = \frac{10H}{\sigma \left\{1 - \left(\frac{d\sigma}{du}\right)^2\right\}}, \quad \frac{\partial x_0}{\partial u} = -\frac{10H \frac{d\sigma}{du}}{\sigma \left\{1 - \left(\frac{d\sigma}{du}\right)^2\right\}}.$$

The equation for X is similar in form to that for Z, and may be solved by RIEMANN'S method. When this is done the coefficients B in the equation of the locus may be determined by identifying the values of X at $r = R_1$ with those given for x_0 by the formulæ for the first middle wave.

22. *Integration of the Equation for X.*—We write the equation

$$\frac{\partial^2 X}{\partial r \partial s} - \frac{5}{r+s} \left(\frac{\partial X}{\partial r} + \frac{\partial X}{\partial s} \right) = 0,$$

and consider also a function Y which satisfies the adjoint equation

$$\frac{\partial^2 Y}{\partial r \partial s} + 5 \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) \left(\frac{Y}{r+s} \right) = 0.$$

Then the integral

$$\iint \left[\frac{\partial}{\partial r} \left\{ Y \left(\frac{\partial X}{\partial s} - \frac{5X}{r+s} \right) \right\} - \frac{\partial}{\partial s} \left\{ X \left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s} \right) \right\} \right] dr ds$$

taken over any area in the plane of (r, s) vanishes, and therefore the integral

$$\int Y \left(\frac{\partial X}{\partial s} - \frac{5X}{r+s} \right) ds + X \left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s} \right) dr$$

taken round the boundary of the area also vanishes.

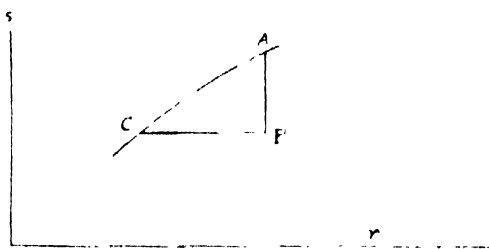


Fig. 3.

We take the area of integration to be bounded by an arc of the locus along which $X = 0$, and two lines parallel to the axes of s and r and meeting at the point P, where $r = r'$ and $s = s'$. Let these be the lines PA and PC. Then we have

$$\begin{aligned} [YX]_A - [YX]_P - \int_{PA} X \left(\frac{\partial Y}{\partial s} + \frac{5Y}{r+s} \right) ds + \int_{AC} Y \left(\frac{\partial X}{\partial s} - \frac{5X}{r+s} \right) ds + X \left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s} \right) dr \\ + \int_{CP} X \left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s} \right) dr = 0, \end{aligned}$$

or, since $X = 0$ on the arc AC,

$$[YX]_P = \int_{AC} Y \frac{\partial X}{\partial s} ds - \int_{PA} X \left(\frac{\partial Y}{\partial s} + \frac{5Y}{r+s} \right) ds + \int_{CP} X \left(\frac{\partial Y}{\partial r} + \frac{5Y}{r+s} \right) dr.$$

We choose Y so that, at P, $Y = 1$, along PA, where $r = r'$, $\partial Y / \partial s = -5Y / (r + s)$, and along CP, where $s = s'$, $\partial Y / \partial r = -5Y / (r + s)$. Then the value of X at P is

$$\int_{AC} Y \frac{\partial X}{\partial s} ds.$$

The requisite form of Y is

$$Y = \left(\frac{r'+s'}{r+s} \right)^6 F(6, -5, 1, \xi),$$

where

$$\xi = -\frac{(r-r')(s-s')}{(r'+s')(r+s)},$$

and we have

$$\begin{aligned} X(r', s') &= \int_{AC} Y \left(\frac{\partial X}{\partial \sigma} - \frac{\partial X}{\partial u} \right) \frac{1}{2} (d\sigma - du) \\ &= \int_{AC} Y \frac{10H}{\sigma} \frac{1 + \frac{d\sigma}{du}}{1 - \left(\frac{d\sigma}{du}\right)^2} \frac{1}{2} \left(\frac{d\sigma}{du} - 1 \right) du \\ &= -5H \int_{AC} \frac{Y}{\sigma} du. \end{aligned}$$

23. *Determination of the Coefficients B.*—The integral $\int_{AC} (Y/\sigma) du$ may be evaluated approximately by assuming, as in Article 21, that the equation of the locus, of which AC is an arc, is of the form

$$(r - R_1)/\Sigma_1 = B_1\delta + B_2\delta^2 + B_3\delta^3 + \dots,$$

where δ stands for $(s - S_1)/\Sigma_1$. Then along AC we have

$$\begin{aligned} u - U_1 &= \Sigma_1 \{ (B_1 - 1)\delta + B_2\delta^2 + B_3\delta^3 + \dots \}, \\ du &= \Sigma_1 \{ (B_1 - 1) + 2B_2\delta + 3B_3\delta^2 + \dots \} d\delta, \\ \frac{Y}{\sigma} &= \frac{(r'+s')^5}{\sigma^6} \left\{ 1 + 30 \frac{(r-r')(s-s')}{(r'+s')\sigma} + 210 \frac{(r-r')^2(s-s')^2}{(r'+s')^2\sigma^2} + \dots \right\}. \end{aligned}$$

Also any inverse power of σ can be expanded in powers of δ by means of the equations

$$\begin{aligned} \sigma &= \Sigma_1 \{ 1 + (B_1 + 1)\delta + B_2\delta^2 + B_3\delta^3 + \dots \}, \\ \sigma^{-\kappa} &= \Sigma_1^{-\kappa} \left\{ 1 - \kappa \frac{\sigma - \Sigma_1}{\Sigma_1} + \frac{\kappa(\kappa + 1)}{2!} \left(\frac{\sigma - \Sigma_1}{\Sigma_1} \right)^2 - \dots \right\}, \end{aligned}$$

which give

$$\begin{aligned} \sigma^{-\kappa} &= \Sigma_1^{-\kappa} \left[1 - \kappa (B_1 + 1)\delta - \left\{ \kappa B_2 - \frac{\kappa(\kappa + 1)}{2!} (B_1 + 1)^2 \right\} \delta^2 \right. \\ &\quad \left. - \left\{ \kappa B_3 - \kappa(\kappa + 1)(B_1 + 1)B_2 + \frac{\kappa(\kappa + 1)(\kappa + 2)}{3!} (B_1 + 1)^3 \right\} \delta^3 \right. \\ &\quad \left. - \left\{ \kappa B_4 - \frac{\kappa(\kappa + 1)}{2!} B_2^2 - \kappa(\kappa + 1)(B_1 + 1)B_3 + \frac{\kappa(\kappa + 1)(\kappa + 2)}{2!} (B_1 + 1)^2 B_2 \right. \right. \\ &\quad \left. \left. - \frac{\kappa(\kappa + 1)(\kappa + 2)(\kappa + 3)}{4!} (B_1 + 1)^4 \right\} \delta^4 - \dots \right]. \end{aligned}$$

To obtain the expression for X on $r = R_1$ we have to put R_1 for r' , so that

$$r - r' = \Sigma_1 (B_1\delta + B_2\delta^2 + B_3\delta^3 + \dots),$$

and for $s-s'$ we have to put $\Sigma_1(\delta-\delta')$, where δ' stands for $(s'-S_1)/\Sigma_1$. If the expansions are carried as far as the fourth order, the result is that, to the fifth order in δ' ,

$$\int_{AC} (Y/\sigma) du = (B_1-1) \delta' + \{ B_2 - (3B_1-2)(B_1-1) \} \delta'^2 + \{ B_3 - 3B_2(2B_1-1) + (7B_1^2-6B_1+2)(B_1-1) \} \delta'^3 + \{ B_4 - 2B_3(3B_1-1) - 3B_2^2 + \frac{1}{2}B_2(42B_1^2-33B_1+5) - \frac{1}{2}(28B_1^3-21B_1^2+9B_1-2)(B_1-1) \} \delta'^4 + \{ B_5 - \frac{1}{5}B_4(30B_1-7) - 6B_2B_3 + \frac{1}{10}B_3(210B_1^2-108B_1+7) + \frac{1}{5}B_2^2(105B_1-27) - \frac{1}{10}B_2(560B_1^3-462B_1^2+81B_1-7) + \frac{1}{5}(126B_1^4-56B_1^3+21B_1^2-6B_1+1)(B_1-1) \} \delta'^5.$$

Now at any point (R_1, s') on $r = R_1$ the formula for the first middle wave gives

$$x_0 = -a \left(\frac{\sigma}{\sigma_0} \right)^{10} \frac{\partial}{\partial \sigma} \left[\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{Q_1(\sigma+u)}{\sigma} \right\} + K_1 + L_1 u \right] = -\frac{a}{\sigma_0^{10}} \{ -945Q_1^{(1)}(2R_1) + 945\sigma Q_1^{(1)}(2R_1) - 420\sigma^2 Q_1^{(2)}(2R_1) + 105\sigma^3 Q_1^{(3)}(2R_1) - 15\sigma^4 Q_1^{(4)}(2R_1) + \sigma^5 Q_1^{(5)}(2R_1) \},$$

in which we have to put $\sigma = \Sigma_1(1+\delta')$. Then, since x_0 vanishes with δ' , we have without any approximation

$$x_0 = -\frac{a}{\sigma_0^{10}} [\{ 945\Sigma_1 Q_1^{(1)}(2R_1) - 840\Sigma_1^2 Q_1^{(2)}(2R_1) + 315\Sigma_1^3 Q_1^{(3)}(2R_1) - 60\Sigma_1^4 Q_1^{(4)}(2R_1) + 5\Sigma_1^5 Q_1^{(5)}(2R_1) \} \delta' + \{ -420\Sigma_1^2 Q_1^{(2)}(2R_1) + 315\Sigma_1^3 Q_1^{(3)}(2R_1) - 90\Sigma_1^4 Q_1^{(4)}(2R_1) + 10\Sigma_1^5 Q_1^{(5)}(2R_1) \} \delta'^2 + \{ 105\Sigma_1^3 Q_1^{(3)}(2R_1) - 60\Sigma_1^4 Q_1^{(4)}(2R_1) + 10\Sigma_1^5 Q_1^{(5)}(2R_1) \} \delta'^3 + \{ -15\Sigma_1^4 Q_1^{(4)}(2R_1) + 5\Sigma_1^5 Q_1^{(5)}(2R_1) \} \delta'^4 + \Sigma_1^5 Q_1^{(5)}(2R_1) \delta'^5].$$

The coefficients B_1, \dots, B_5 can be determined successively by equating the coefficients of powers of δ' in the expressions for $\int_{AC} (Y/\sigma) du$ and $-x_0/5H$. If additional coefficients B_6, \dots , are desired, they may be found by equating to zero the coefficients of powers of δ' higher than the fifth in the expansion of $\int_{AC} (Y/\sigma) du$.

The expansion of $x_0(R_1, s')$ in powers of δ' may, of course, be found from the expression for Z in the first middle wave without transformation to the Q form. In particular it may be proved that $B_1 = 6 - 4\Sigma_1/\sigma_0$.

24. Second Method of determining the Coefficients A .—When the coefficients B are known, the coefficients A_n, \dots may be found in the following way.

Since x_0 , which is $-a(\sigma/\sigma_0)^{10} \partial Z/\partial \sigma$, vanishes at all points of the locus

$$r - R_1 = B_1(s - S_1) + (B_2/\Sigma_1)(s - S_1)^2 + (B_3/\Sigma_1^2)(s - S_1)^3 + \dots,$$

the expression

$$945F_1(2r) - 945\sigma F_1^{(1)}(2r) + 420\sigma^2 F_1^{(2)}(2r) - 105\sigma^3 F_1^{(3)}(2r) + 15\sigma^4 F_1^{(4)}(2r) - \sigma^5 F_1^{(5)}(2r),$$

in which

$$F_1(2r) = A_0 + A_1(2r - 2R_1) + A_2(2r - 2R_1)^2 + \dots,$$

must become identically zero on substitution of

$$\Sigma_1(B_1\delta + B_2\delta^2 + B_3\delta^3 + \dots)$$

for $r - R_1$ and of

$$\Sigma_1\{1 + (B_1 + 1)\delta + B_2\delta^2 + B_3\delta^3 + \dots\}$$

for σ . Now the powers of σ/Σ_1 and $(r - R_1)/\Sigma_1$ can all be expanded in powers of δ , and then the coefficients of the powers of δ in the expansion of

$$945 \frac{F_1(2r)}{\Sigma_1^{10}} - 945 \frac{\sigma}{\Sigma_1} \frac{F_1^{(1)}(2r)}{\Sigma_1^9} + 420 \left(\frac{\sigma}{\Sigma_1}\right)^2 \frac{F_1^{(2)}(2r)}{\Sigma_1^8} - 105 \left(\frac{\sigma}{\Sigma_1}\right)^3 \frac{F_1^{(3)}(2r)}{\Sigma_1^7} \\ + 15 \left(\frac{\sigma}{\Sigma_1}\right)^4 \frac{F_1^{(4)}(2r)}{\Sigma_1^6} - \left(\frac{\sigma}{\Sigma_1}\right)^5 \frac{F_1^{(5)}(2r)}{\Sigma_1^5}$$

can be equated severally to zero. The equations thus arising give the values of A_6, A_7, \dots , successively. Suppressing the algebra, which is rather long, we may write down the results in the following form:—

The equation for A_6 is

$$2B_1 \frac{6! A_6}{\Sigma_1^4} = 945(B_1 - 1) \frac{A_1}{\Sigma_1^9} - 210(5B_1 - 4) \frac{2! A_2}{\Sigma_1^8} + 105(5B_1 - 3) \frac{3! A_3}{\Sigma_1^7} \\ - 30(5B_1 - 2) \frac{4! A_4}{\Sigma_1^6} + 5(5B_1 - 1) \frac{5! A_5}{\Sigma_1^5}.$$

The equation for A_7 is

$$2B_1^2 \frac{7! A_7}{\Sigma_1^3} = 2B_1^2 \left(945 \frac{2! A_2}{\Sigma_1^8} - 945 \frac{3! A_3}{\Sigma_1^7} + 420 \frac{4! A_4}{\Sigma_1^6} - 105 \frac{5! A_5}{\Sigma_1^5} + 15 \frac{6! A_6}{\Sigma_1^4} \right) \\ + 2B_1(B_1 + 1) \left(-945 \frac{2! A_2}{\Sigma_1^8} + 840 \frac{3! A_3}{\Sigma_1^7} - 315 \frac{4! A_4}{\Sigma_1^6} + 60 \frac{5! A_5}{\Sigma_1^5} - 5 \frac{6! A_6}{\Sigma_1^4} \right) \\ + (B_1 + 1)^2 \left(420 \frac{2! A_2}{\Sigma_1^8} - 315 \frac{3! A_3}{\Sigma_1^7} + 90 \frac{4! A_4}{\Sigma_1^6} - 10 \frac{5! A_5}{\Sigma_1^5} \right) \\ + B_2 \left(945 \frac{A_1}{\Sigma_1^9} - 1050 \frac{2! A_2}{\Sigma_1^8} + 525 \frac{3! A_3}{\Sigma_1^7} - 150 \frac{4! A_4}{\Sigma_1^6} + 25 \frac{5! A_5}{\Sigma_1^5} - 2 \frac{6! A_6}{\Sigma_1^4} \right).$$

The equation for A_8 is

$$\begin{aligned}
 4B_1^3 \frac{8! A_8}{\Sigma_1^2} &= 4B_1^3 \left(945 \frac{3! A_3}{\Sigma_1^7} - 945 \frac{4! A_4}{\Sigma_1^6} + 420 \frac{5! A_5}{\Sigma_1^5} - 105 \frac{6! A_6}{\Sigma_1^4} + 15 \frac{7! A_7}{\Sigma_1^3} \right) \\
 &+ 6B_1^2 (B_1 + 1) \left(-945 \frac{3! A_3}{\Sigma_1^7} + 840 \frac{4! A_4}{\Sigma_1^6} - 315 \frac{5! A_5}{\Sigma_1^5} + 60 \frac{6! A_6}{\Sigma_1^4} - 5 \frac{7! A_7}{\Sigma_1^3} \right) \\
 &+ 6B_1 (B_1 + 1)^2 \left(420 \frac{3! A_3}{\Sigma_1^7} - 315 \frac{4! A_4}{\Sigma_1^6} + 90 \frac{5! A_5}{\Sigma_1^5} - 10 \frac{6! A_6}{\Sigma_1^4} \right) \\
 &+ 3 (B_1 + 1)^3 \left(-105 \frac{3! A_3}{\Sigma_1^7} + 60 \frac{4! A_4}{\Sigma_1^6} - 10 \frac{5! A_5}{\Sigma_1^5} \right) \\
 &+ 6B_1 B_2 \left(945 \frac{2! A_2}{\Sigma_1^8} - 1050 \frac{3! A_3}{\Sigma_1^7} + 525 \frac{4! A_4}{\Sigma_1^6} - 150 \frac{5! A_5}{\Sigma_1^5} + 25 \frac{6! A_6}{\Sigma_1^4} - 2 \frac{7! A_7}{\Sigma_1^3} \right) \\
 &+ 3 (B_1 + 1) B_2 \left(-1050 \frac{2! A_2}{\Sigma_1^8} + 1050 \frac{3! A_3}{\Sigma_1^7} - 450 \frac{4! A_4}{\Sigma_1^6} + 100 \frac{5! A_5}{\Sigma_1^5} - 10 \frac{6! A_6}{\Sigma_1^4} \right) \\
 &+ 3B_2 \left(945 \frac{A_1}{\Sigma_1^9} - 1050 \frac{2! A_2}{\Sigma_1^8} + 525 \frac{3! A_3}{\Sigma_1^7} - 150 \frac{4! A_4}{\Sigma_1^6} + 25 \frac{5! A_5}{\Sigma_1^5} - 2 \frac{6! A_6}{\Sigma_1^4} \right).
 \end{aligned}$$

The equation for A_9 is

$$\begin{aligned}
 2B_1^4 \frac{9! A_9}{\Sigma_1} &= 2B_1^4 \left(945 \frac{4! A_4}{\Sigma_1^6} - 945 \frac{5! A_5}{\Sigma_1^5} + 420 \frac{6! A_6}{\Sigma_1^4} - 105 \frac{7! A_7}{\Sigma_1^3} + 15 \frac{8! A_8}{\Sigma_1^2} \right) \\
 &+ 4B_1^3 (B_1 + 1) \left(-945 \frac{4! A_4}{\Sigma_1^6} + 840 \frac{5! A_5}{\Sigma_1^5} - 315 \frac{6! A_6}{\Sigma_1^4} + 60 \frac{7! A_7}{\Sigma_1^3} - 5 \frac{8! A_8}{\Sigma_1^2} \right) \\
 &+ 6B_1^2 (B_1 + 1)^2 \left(420 \frac{4! A_4}{\Sigma_1^6} - 315 \frac{5! A_5}{\Sigma_1^5} + 90 \frac{6! A_6}{\Sigma_1^4} - 10 \frac{7! A_7}{\Sigma_1^3} \right) \\
 &+ 6B_1 (B_1 + 1)^3 \left(-105 \frac{4! A_4}{\Sigma_1^6} + 60 \frac{5! A_5}{\Sigma_1^5} - 10 \frac{6! A_6}{\Sigma_1^4} \right) \\
 &+ 6 (B_1 + 1)^4 \left(15 \frac{4! A_4}{\Sigma_1^6} - 5 \frac{5! A_5}{\Sigma_1^5} \right) \\
 &+ 12B_1^2 B_2 \left(945 \frac{3! A_3}{\Sigma_1^7} - 945 \frac{4! A_4}{\Sigma_1^6} + 420 \frac{5! A_5}{\Sigma_1^5} - 105 \frac{6! A_6}{\Sigma_1^4} + 15 \frac{7! A_7}{\Sigma_1^3} - \frac{8! A_8}{\Sigma_1^2} \right) \\
 &+ 6B_1 (3B_1 + 2) B_2 \left(-945 \frac{3! A_3}{\Sigma_1^7} + 840 \frac{4! A_4}{\Sigma_1^6} - 315 \frac{5! A_5}{\Sigma_1^5} + 60 \frac{6! A_6}{\Sigma_1^4} - 5 \frac{7! A_7}{\Sigma_1^3} \right) \\
 &+ 6 (B_1 + 1) (3B_1 + 1) B_2 \left(420 \frac{3! A_3}{\Sigma_1^7} - 315 \frac{4! A_4}{\Sigma_1^6} + 90 \frac{5! A_5}{\Sigma_1^5} - 10 \frac{6! A_6}{\Sigma_1^4} \right) \\
 &+ 3 (B_1 + 1)^2 B_2 \left(-315 \frac{3! A_3}{\Sigma_1^7} + 180 \frac{4! A_4}{\Sigma_1^6} - 30 \frac{5! A_5}{\Sigma_1^5} \right) \\
 &+ 3B_2^2 \left(420 \frac{2! A_2}{\Sigma_1^8} - 525 \frac{3! A_3}{\Sigma_1^7} + 300 \frac{4! A_4}{\Sigma_1^6} - 100 \frac{5! A_5}{\Sigma_1^5} + 20 \frac{6! A_6}{\Sigma_1^4} - 2 \frac{7! A_7}{\Sigma_1^3} \right) \\
 &+ 6B_1 B_3 \left(945 \frac{2! A_2}{\Sigma_1^8} - 1050 \frac{3! A_3}{\Sigma_1^7} + 525 \frac{4! A_4}{\Sigma_1^6} - 150 \frac{5! A_5}{\Sigma_1^5} + 25 \frac{6! A_6}{\Sigma_1^4} - 2 \frac{7! A_7}{\Sigma_1^3} \right) \\
 &+ 3 (B_1 + 1) B_3 \left(-1050 \frac{2! A_2}{\Sigma_1^8} + 1050 \frac{3! A_3}{\Sigma_1^7} - 450 \frac{4! A_4}{\Sigma_1^6} + 100 \frac{5! A_5}{\Sigma_1^5} - 10 \frac{6! A_6}{\Sigma_1^4} \right) \\
 &+ 3B_3 \left(945 \frac{A_1}{\Sigma_1^9} - 1050 \frac{2! A_2}{\Sigma_1^8} + 525 \frac{3! A_3}{\Sigma_1^7} - 150 \frac{4! A_4}{\Sigma_1^6} + 25 \frac{5! A_5}{\Sigma_1^5} - 2 \frac{6! A_6}{\Sigma_1^4} \right).
 \end{aligned}$$

The method avails for the calculation of as many coefficients as may be desired.

25. *Third Method of determining the Coefficients A.*—Another nearly equally accurate process for finding the coefficients A_0, A_1, \dots , is founded upon an expression for t , valid at the piston M.

The equation of motion of the piston shows that at $x_0 = 0$,

$$\frac{\partial u}{\partial t} = -\frac{\alpha \sigma^{11}}{10H\sigma_0^{10}},$$

and the differential of u is always

$$\frac{\partial u}{\partial x_0} dx_0 + \frac{\partial u}{\partial t} dt,$$

so that, at $x_0 = 0$, t can be expressed as a function of s by the equation

$$t - T_1 = -\frac{10H\sigma_0^{10}}{\alpha\Sigma_1^{11}} \int_{S_1}^s \left(\frac{\sigma}{\Sigma_1}\right)^{-11} \{(B_1 - 1) + 2B_2\delta + 3B_3\delta^2 + \dots\} ds,$$

and thus $t - T_1$ can be expanded in powers of δ or $(s - S_1)/\Sigma_1$. Also, since $t = \partial Z/\partial u$, and T_1 is the value of t given by putting $r = R_1$ and $s = S_1$ in the formula for the first reflected wave from the left, we have

$$\begin{aligned} t - T_1 = & 105 \left\{ \frac{F_1^{(1)}(2r)}{\sigma^9} - \frac{F_1^{(1)}(2R_1)}{\Sigma_1^9} \right\} - 105 \left\{ \frac{F_1^{(2)}(2r)}{\sigma^8} - \frac{F_1^{(2)}(2R_1)}{\Sigma_1^8} \right\} \\ & + 45 \left\{ \frac{F_1^{(3)}(2r)}{\sigma^7} - \frac{F_1^{(3)}(2R_1)}{\Sigma_1^7} \right\} - 10 \left\{ \frac{F_1^{(4)}(2r)}{\sigma^6} - \frac{F_1^{(4)}(2R_1)}{\Sigma_1^6} \right\} + \left\{ \frac{F_1^{(5)}(2r)}{\sigma^5} - \frac{F_1^{(5)}(2R_1)}{\Sigma_1^5} \right\}, \end{aligned}$$

so that a different form of expansion can be obtained for $t - T_1$. By equating coefficients of different powers of δ in the two forms of expansion we obtain again a series of equations giving the values of A_0, A_1, \dots , successively. The results may be recorded as follows:—

The equation for A_0 is

$$\begin{aligned} 105 \left\{ -9(B_1 + 1) \frac{A_1}{\Sigma_1^9} + 2B_1 \frac{2! A_2}{\Sigma_1^8} \right\} - 105 \left\{ -8(B_1 + 1) \frac{2! A_2}{\Sigma_1^8} + 2B_1 \frac{3! A_3}{\Sigma_1^7} \right\} \\ + 45 \left\{ -7(B_1 + 1) \frac{3! A_3}{\Sigma_1^7} + 2B_1 \frac{4! A_4}{\Sigma_1^6} \right\} - 10 \left\{ -6(B_1 + 1) \frac{4! A_4}{\Sigma_1^6} + 2B_1 \frac{5! A_5}{\Sigma_1^5} \right\} \\ + \left\{ -5(B_1 + 1) \frac{5! A_5}{\Sigma_1^5} + 2B_1 \frac{6! A_6}{\Sigma_1^4} \right\} = -10 \frac{H}{\alpha} \left(\frac{\sigma}{\Sigma_1}\right)^{10} (B_1 - 1). \end{aligned}$$

The equation for A_7 is

$$\begin{aligned}
 105 & \left[\{-9B_2 + 45(B_1 + 1)^2\} \frac{A_1}{\Sigma_1^9} + 2 \{B_2 - 9B_1(B_1 + 1)\} \frac{2! A_2}{\Sigma_1^8} + \frac{2^2}{2!} B_1^2 \frac{3! A_3}{\Sigma_1^7} \right] \\
 & - 105 \left[\{-8B_2 + 36(B_1 + 1)^2\} \frac{2! A_2}{\Sigma_1^8} + 2 \{B_2 - 8B_1(B_1 + 1)\} \frac{3! A_3}{\Sigma_1^7} + \frac{2^2}{2!} B_1^2 \frac{4! A_4}{\Sigma_1^6} \right] \\
 & + 45 \left[\{-7B_2 + 28(B_1 + 1)^2\} \frac{3! A_3}{\Sigma_1^7} + 2 \{B_2 - 7B_1(B_1 + 1)\} \frac{4! A_4}{\Sigma_1^6} + \frac{2^2}{2!} B_1^2 \frac{5! A_5}{\Sigma_1^5} \right] \\
 & - 10 \left[\{-6B_2 + 21(B_1 + 1)^2\} \frac{4! A_4}{\Sigma_1^6} + 2 \{B_2 - 6B_1(B_1 + 1)\} \frac{5! A_5}{\Sigma_1^5} + \frac{2^2}{2!} B_1^2 \frac{6! A_6}{\Sigma_1^4} \right] \\
 & + \left[\{-5B_2 + 15(B_1 + 1)^2\} \frac{5! A_5}{\Sigma_1^5} + 2 \{B_2 - 5B_1(B_1 + 1)\} \frac{6! A_6}{\Sigma_1^4} + \frac{2^2}{2!} B_1^2 \frac{7! A_7}{\Sigma_1^3} \right] \\
 & = -10 \frac{H}{\alpha} \left(\frac{\sigma_0}{\Sigma_1} \right)^{10} \{B_2 - \frac{1}{2} (B_1 + 1)(B_1 - 1)\}.
 \end{aligned}$$

The equation for A_8 is

$$\begin{aligned}
 105 & \left[\{-9B_3 + 90(B_1 + 1)B_2 - 165(B_1 + 1)^3\} \frac{A_1}{\Sigma_1^9} + 2 \{B_3 - 9(2B_1 + 1)B_2 + 45B_1(B_1 + 1)^2\} \frac{2! A_2}{\Sigma_1^8} \right. \\
 & \quad \left. + \frac{2^2}{2!} \{2B_1B_2 - 9B_1^2(B_1 + 1)\} \frac{3! A_3}{\Sigma_1^7} + \frac{2^3}{3!} B_1^3 \frac{4! A_4}{\Sigma_1^6} \right] \\
 & - 105 \left[\{-8B_3 + 72(B_1 + 1)B_2 - 120(B_1 + 1)^3\} \frac{2! A_2}{\Sigma_1^8} \right. \\
 & \quad + 2 \{B_3 - 8(2B_1 + 1)B_2 + 36B_1(B_1 + 1)^2\} \frac{3! A_3}{\Sigma_1^7} \\
 & \quad \left. + \frac{2^2}{2!} \{2B_1B_2 - 8B_1^2(B_1 + 1)\} \frac{4! A_4}{\Sigma_1^6} + \frac{2^3}{3!} B_1^3 \frac{5! A_5}{\Sigma_1^5} \right] \\
 & + 45 \left[\{-7B_3 + 56(B_1 + 1)B_2 - 84(B_1 + 1)^3\} \frac{3! A_3}{\Sigma_1^7} \right. \\
 & \quad + 2 \{B_3 - 7(2B_1 + 1)B_2 + 28B_1(B_1 + 1)^2\} \frac{4! A_4}{\Sigma_1^6} \\
 & \quad \left. + \frac{2^2}{2!} \{2B_1B_2 - 7B_1^2(B_1 + 1)\} \frac{5! A_5}{\Sigma_1^5} + \frac{2^3}{3!} B_1^3 \frac{6! A_6}{\Sigma_1^4} \right] \\
 & - 10 \left[\{-6B_3 + 42(B_1 + 1)B_2 - 56(B_1 + 1)^3\} \frac{4! A_4}{\Sigma_1^6} \right. \\
 & \quad + 2 \{B_3 - 6(2B_1 + 1)B_2 + 21B_1(B_1 + 1)^2\} \frac{5! A_5}{\Sigma_1^5} \\
 & \quad \left. + \frac{2^2}{2!} \{2B_1B_2 - 6B_1^2(B_1 + 1)\} \frac{6! A_6}{\Sigma_1^4} + \frac{2^3}{3!} B_1^3 \frac{7! A_7}{\Sigma_1^3} \right] \\
 & + \left[\{-5B_3 + 30(B_1 + 1)B_2 - 35(B_1 + 1)^3\} \frac{5! A_5}{\Sigma_1^5} \right. \\
 & \quad + 2 \{B_3 - 5(2B_1 + 1)B_2 + 15B_1(B_1 + 1)^2\} \frac{6! A_6}{\Sigma_1^4} \\
 & \quad \left. + \frac{2^2}{2!} \{2B_1B_2 - 5B_1^2(B_1 + 1)\} \frac{7! A_7}{\Sigma_1^3} + \frac{2^3}{3!} B_1^3 \frac{8! A_8}{\Sigma_1^2} \right] \\
 & = -10 \frac{H}{\alpha} \left(\frac{\sigma_0}{\Sigma_1} \right)^{10} \{B_3 - \frac{1}{3} (3B_1 + 1)B_2 + 22(B_1 + 1)^2(B_1 - 1)\}.
 \end{aligned}$$

The equation for A_9 is

$$\begin{aligned}
 105 \left[\left\{ -9B_4 + 9 \cdot 10 (B_1 + 1) B_3 + \frac{9 \cdot 10}{2!} B_2^2 - 3 \frac{9 \cdot 10 \cdot 11}{3!} (B_1 + 1)^2 B_2 \right. \right. \\
 \left. \left. + \frac{9 \cdot 10 \cdot 11 \cdot 12}{4!} (B_1 + 1)^4 \right\} \frac{A_1}{\Sigma_1^9} \right. \\
 + 2 \left\{ B_4 - 9 (2B_1 + 1) B_3 - 9B_2^2 + \frac{9 \cdot 10}{2!} (3B_1 + 1) (B_1 + 1) B_2 - \frac{9 \cdot 10 \cdot 11}{3!} B_1 (B_1 + 1)^3 \right\} \frac{2! A_2}{\Sigma_1^8} \\
 + \frac{2^2}{2!} \left\{ 2B_1 B_3 + B_2^2 - 9B_1 (3B_1 + 2) B_2 + \frac{9 \cdot 10}{2!} B_1^2 (B_1 + 1)^2 \right\} \frac{3! A_3}{\Sigma_1^7} \\
 + \frac{2^3}{3!} \left\{ 3B_1^2 B_2 - 9B_1^3 (B_1 + 1) \right\} \frac{4! A_4}{\Sigma_1^6} + \frac{2^4}{4!} B_1^4 \frac{5! A_5}{\Sigma_1^5} \\
 - 105 [\dots] + 45 [\dots] - 10 [\dots] \\
 + \left[\left\{ -5B_4 + 5 \cdot 6 (B_1 + 1) B_3 + \frac{5 \cdot 6}{2!} B_2^2 - 3 \frac{5 \cdot 6 \cdot 7}{3!} (B_1 + 1)^2 B_2 + \frac{5 \cdot 6 \cdot 7 \cdot 8}{4!} (B_1 + 1)^4 \right\} \frac{5! A_5}{\Sigma_1^5} \right. \\
 + 2 \left\{ B_4 - 5 (2B_1 + 1) B_3 - 5B_2^2 + \frac{5 \cdot 6}{2!} (3B_1 + 1) (B_1 + 1) B_2 - \frac{5 \cdot 6 \cdot 7}{3!} B_1 (B_1 + 1)^3 \right\} \frac{6! A_6}{\Sigma_1^4} \\
 + \frac{2^2}{2!} \left\{ 2B_1 B_3 + B_2^2 - 5B_1 (3B_1 + 2) B_2 + \frac{5 \cdot 6}{2!} B_1^2 (B_1 + 1)^2 \right\} \frac{7! A_7}{\Sigma_1^3} \\
 + \frac{2^3}{3!} \left\{ 3B_1^2 B_2 - 5B_1^3 (B_1 + 1) \right\} \frac{8! A_8}{\Sigma_1^2} + \frac{2^4}{4!} B_1^4 \frac{9! A_9}{\Sigma_1} \\
 = -10 \frac{H}{a} \left(\frac{\sigma_0}{\Sigma_1} \right)^{10} \left[B_4 - \frac{11}{2} (2B_1 + 1) B_3 - \frac{11}{2} B_2^2 + \frac{11 \cdot 12}{2!} B_1 (B_1 + 1) B_2 \right. \\
 \left. - \frac{11 \cdot 12 \cdot 13}{3!} \frac{(B_1 + 1)^3}{4} (B_1 - 1) \right],
 \end{aligned}$$

where the law of formation of the terms that are not written down is sufficiently obvious.

The formulæ of this article may, of course, be transformed into those of the previous article by means of the relations by which the coefficients B were expressed in terms of the differential coefficients of Q_1 , and the relations by which the coefficients A_1, \dots, A_9 were expressed in terms of the same differential coefficients. They are useful in numerical work as affording a verification of the values obtained for the coefficients A_6, A_7, \dots , from the previous formulæ.

Formulæ similar to those of the present and preceding articles may be obtained for the coefficients in the expression for Z belonging to the first reflected wave from the right, but it seems hardly worth while to write them down.

THE SECOND MIDDLE WAVE.

26. *Method of determining the Second Middle Wave.*—The first reflected wave from the left meets that from the right at the place and time determined by substituting

R_1 for r and s_1 for s in the formulæ belonging to the first middle wave. When they meet, the first middle wave becomes obliterated, and the second middle wave begins to be generated at the time and place in question, and encroaches upon the two first reflected waves.

To determine the second middle wave we have the conditions that at its advancing front, where $r = R_1$, the Z belonging to it is equal to that belonging to the first reflected wave from the right, and at its receding front, where $s = s_1$, the Z belonging to it is equal to that belonging to the first reflected wave from the left. RIEMANN'S method may be applied in exactly the same way as in Article 15. If P is the point (r', s'), A the point (R_1, s'), B the point (R_1, s_1) and C the point (r', s_1), we have

$$Z(r', s') = [VZ]_A - [VZ]_B + [VZ]_C + \int_{AB} Z \left(\frac{\partial V}{\partial s} - \frac{5V}{r+s} \right) ds - \int_{BC} Z \left(\frac{\partial V}{\partial r} - \frac{5V}{r+s} \right) dr.$$

At A we have

$$r = R_1, \quad s = s', \quad \xi = 0, \quad V = \left(\frac{R_1 + s'}{r' + s'} \right)^5,$$

and Z is the result of substituting R_1 for r and s' for s in the formula

$$Z = k_1 + l_1 u + \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left[\frac{1}{\sigma} \left\{ \frac{c\sigma_0^{10}}{945\alpha} + f_1(\sigma - u) \right\} \right].$$

At B we have

$$r = R_1, \quad s = s_1, \quad \xi = - \frac{(R_1 - r')(s_1 - s')}{(r' + s')(R_1 + s_1)}, \quad V = \left(\frac{R_1 + s_1}{r' + s'} \right)^5 (1 - 20\xi + 90\xi^2 - 140\xi^3 + 70\xi^4),$$

and Z is the result of substituting R_1 for r and s_1 for s in the formulæ for the first middle wave, or in those for either of the first reflected waves. For the present we shall denote it by Z_B , and observe that it is independent of r' and s' .

At C we have

$$r = r', \quad s = s_1, \quad \xi = 0, \quad V = \left(\frac{r' + s_1}{r' + s'} \right)^5,$$

and Z is the result of substituting r' for r and s_1 for s in the formula

$$Z = K_1 + L_1 u + \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{F_1(\sigma + u)}{\sigma} \right\}.$$

Along AB , where $r = R_1$ and s increases from s' to s_1 , we have

$$Z = k_1 + l_1 (R_1 - s) + \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left[\frac{1}{\sigma} \left\{ \frac{c\sigma_0^{10}}{945\alpha} + f_1(\sigma - u) \right\} \right],$$

$$\frac{\partial V}{\partial s} - \frac{5V}{r+s} = \frac{(R_1 + s')(R_1 - r')(R_1 + s)^3}{(r' + s')^6} (20 - 180\xi + 420\xi^2 - 280\xi^3),$$

$$\xi = - \frac{(R_1 - r')(s - s')}{(r' + s')(R_1 + s)};$$

and along BC, where $s = s_1$ and r decreases from R_1 to r' , we have

$$Z = K_1 + L_1 (r - s_1) + \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{F_1(\sigma + u)}{\sigma} \right\},$$

$$\frac{\partial V}{\partial r} - \frac{5V}{r + s} = \frac{(s_1 + r')(s_1 - s')(s_1 + r')^3}{(r' + s')^6} (20 - 180\xi + 420\xi^2 - 280\xi^3),$$

$$\xi = -\frac{(r - r')(s_1 - s')}{(r' + s')(r + s_1)}.$$

The value of Z at (r', s') can be regarded as a sum of terms with the coefficients

$$Z_B, k_1, l_1, c\sigma_0^{10}/945a, \alpha_0, \alpha_1, \dots, K_1, L_1, A_0, A_1, \dots,$$

and each of these terms may be found from the formula for $Z(r', s')$ by performing the integrations where necessary. The result will be to exhibit $Z(r', s')$ as a sum of terms with these coefficients.

27. *Determination of the Second Middle Wave.*—No integration is needed in order to obtain the term which has Z_B as a factor, but it is important to observe that V_B , as a function of r' and s' , can be expressed either in the form

$$V_B = \frac{2}{3} (R_1 + s_1) \left(\frac{1}{\sigma'} \frac{\partial}{\partial \sigma'} \right)^4 \left\{ \frac{(s' - s_1)^4 (s' + R_1)^4}{\sigma'} \right\},$$

or in the form

$$V_B = \frac{2}{3} (R_1 + s_1) \left(\frac{1}{\sigma'} \frac{\partial}{\partial \sigma'} \right)^4 \left\{ \frac{(r' - R_1)^4 (r' + s_1)^4}{\sigma'} \right\}.$$

We shall suppress the accents on r' and s' so as to express the value of Z at (r, s) . The term with coefficient Z_B is

$$-\frac{2}{3} (R_1 + s_1) Z_B \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{(s - s_1)^4 (s + R_1)^4}{\sigma} \right\}.$$

The term with coefficient k_1 is

$$\frac{2}{3} k_1 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{(s - s_1)^4 (s + R_1)^4}{\sigma} \right\}.$$

The term with coefficient l_1 is

$$1\frac{2}{3} l_1 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{(s - s_1)^4 (5R_1 - 4s_1 - s)(s + R_1)^4}{\sigma} \right\}.$$

The terms with coefficients $c\sigma_0^{10}/945a$ and α_0 are

$$\left(c \frac{\sigma_0^{10}}{945a} + \alpha_0 \right) \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left[\frac{1}{\sigma} \left(\frac{s - s_1}{R_1 + s_1} \right)^4 \left\{ 1 + 4 \left(\frac{s + R_1}{R_1 + s_1} \right) + 10 \left(\frac{s + R_1}{R_1 + s_1} \right)^2 \right. \right. \\ \left. \left. + 20 \left(\frac{s + R_1}{R_1 + s_1} \right)^3 + 35 \left(\frac{s + R_1}{R_1 + s_1} \right)^4 \right\} \right].$$

The terms with coefficients $\alpha_1, \alpha_2, \alpha_3$ are

$$\begin{aligned}
 & -2\alpha_1 (R_1 + s_1) \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left[\frac{1}{\sigma} \left(\frac{s-s_1}{R_1+s_1}\right)^4 \left\{ 1 + 3\left(\frac{s+R_1}{R_1+s_1}\right) + 6\left(\frac{s+R_1}{R_1+s_1}\right)^2 \right. \right. \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. \left. + 10\left(\frac{s+R_1}{R_1+s_1}\right)^3 + 15\left(\frac{s+R_1}{R_1+s_1}\right)^4 \right\} \right], \\
 & 2^2\alpha_2 (R_1 + s_1)^2 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left[\frac{1}{\sigma} \left(\frac{s-s_1}{R_1+s_1}\right)^4 \left\{ 1 + 2\left(\frac{s+R_1}{R_1+s_1}\right) + 3\left(\frac{s+R_1}{R_1+s_1}\right)^2 + 4\left(\frac{s+R_1}{R_1+s_1}\right)^3 + 5\left(\frac{s+R_1}{R_1+s_1}\right)^4 \right\} \right], \\
 & -2^3\alpha_3 (R_1 + s_1)^3 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left[\frac{1}{\sigma} \left(\frac{s-s_1}{R_1+s_1}\right)^4 \left\{ 1 + \left(\frac{s+R_1}{R_1+s_1}\right) + \left(\frac{s+R_1}{R_1+s_1}\right)^2 + \left(\frac{s+R_1}{R_1+s_1}\right)^3 + \left(\frac{s+R_1}{R_1+s_1}\right)^4 \right\} \right].
 \end{aligned}$$

The terms with coefficients $\alpha_4, \alpha_5, \dots$, are

$$2^4\alpha_4 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(s-s_1)^4}{\sigma} \right\}, \quad 2^5\alpha_5 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(s-s_1)^5}{\sigma} \right\},$$

and so on.

The term with coefficient K_1 is

$$\frac{2}{3}K_1 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(r-R_1)^4 (r+s_1)^5}{\sigma} \right\}.$$

The term with coefficient L_1 is

$$-\frac{2}{15}L_1 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(r-R_1)^4 (5s_1 - 4R_1 - r) (r+s_1)^6}{\sigma} \right\}.$$

The terms with coefficients A_0, A_1, A_2, A_3 are

$$\begin{aligned}
 & A_0 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left[\frac{1}{\sigma} \left(\frac{r-R_1}{R_1+s_1}\right)^4 \left\{ 1 + 4\left(\frac{r+s_1}{R_1+s_1}\right) + 10\left(\frac{r+s_1}{R_1+s_1}\right)^2 + 20\left(\frac{r+s_1}{R_1+s_1}\right)^3 + 35\left(\frac{r+s_1}{R_1+s_1}\right)^4 \right\} \right], \\
 & -2A_1 (R_1 + s_1) \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left[\frac{1}{\sigma} \left(\frac{r-R_1}{R_1+s_1}\right)^4 \left\{ 1 + 3\left(\frac{r+s_1}{R_1+s_1}\right) + 6\left(\frac{r+s_1}{R_1+s_1}\right)^2 \right. \right. \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. \left. + 10\left(\frac{r+s_1}{R_1+s_1}\right)^3 + 15\left(\frac{r+s_1}{R_1+s_1}\right)^4 \right\} \right], \\
 & 2^2A_2 (R_1 + s_1)^2 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left[\frac{1}{\sigma} \left(\frac{r-R_1}{R_1+s_1}\right)^4 \left\{ 1 + 2\left(\frac{r+s_1}{R_1+s_1}\right) + 3\left(\frac{r+s_1}{R_1+s_1}\right)^2 + 4\left(\frac{r+s_1}{R_1+s_1}\right)^3 + 5\left(\frac{r+s_1}{R_1+s_1}\right)^4 \right\} \right], \\
 & -2^3A_3 (R_1 + s_1)^3 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left[\frac{1}{\sigma} \left(\frac{r-R_1}{R_1+s_1}\right)^4 \left\{ 1 + \left(\frac{r+s_1}{R_1+s_1}\right) + \left(\frac{r+s_1}{R_1+s_1}\right)^2 + \left(\frac{r+s_1}{R_1+s_1}\right)^3 + \left(\frac{r+s_1}{R_1+s_1}\right)^4 \right\} \right].
 \end{aligned}$$

The terms with coefficients A_4, A_5, \dots , are

$$2^4A_4 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(r-R_1)^4}{\sigma} \right\}, \quad 2^5A_5 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(r-R_1)^5}{\sigma} \right\},$$

and so on.

28. *Indication of a General Method.*—If the coefficients a or A with suffixes exceeding 10 were all zero, the expression for Z in the second middle wave could be transformed at once to the form

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{Q_2(\sigma+u)}{\sigma} \right\} + K_2 + L_2 u,$$

or to the form

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{q_2(\sigma-u)}{\sigma} \right\} + k_2 + l_2 u.$$

But if these coefficients do not vanish a transformation of the same kind is still possible. We have, for example,

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(\sigma+u)^{11} + (\sigma-u)^{11}}{\sigma} \right\} = 2^6 5! (\sigma^2 + 11u^2),$$

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(\sigma+u)^{12} - (\sigma-u)^{12}}{\sigma} \right\} = 2^7 5! u (3\sigma^2 + 11u^2),$$

and thus Z can be expressed either in the form

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{Q_2(\sigma+u)}{\sigma} \right\} + K_2 + L_2 u + M_2 (\sigma^2 + 11u^2) + N_2 u (3\sigma^2 + 11u^2) + \dots,$$

or in the form

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{q_2(\sigma-u)}{\sigma} \right\} + k_2 + l_2 u + m_2 (\sigma^2 + 11u^2) + n_2 u (3\sigma^2 + 11u^2) + \dots,$$

where the factors whose coefficients are written K_2, L_2, \dots , or k_2, l_2, \dots , are the homogeneous rational integral functions of σ and u of degrees 0, 1, 2, 3, \dots , which satisfy the differential equation for Z .

When this transformation is effected we may proceed to determine the second reflected waves. The first step is to find sets of coefficients analogous to A_0, \dots, A_5 and a_0, \dots, a_5 . The next is to find sets of coefficients analogous to B_1, B_2, \dots , determining the loci in the plane of (r, s) along which $x_0 = 0$ and $x_0 = c$ during the time that these reflected waves are being generated. By means of the coefficients analogous to B_1, B_2, \dots , sets of coefficients analogous to A_0, A_7, \dots , and a_0, a_7, \dots , may be found, and thus the second reflected waves may be determined.

From the formulæ for Z in the second reflected waves that in the third middle wave may be found, in the same way as the formula for Z in the second middle wave was found from those in the first reflected waves.

The method of solution can be continued, and gives a theoretically complete solution of the problem; but when arithmetical computation is attempted, failure may arise through approximate equality of groups of terms with opposite signs, so that some quantity, which ought to be calculable to five figures, for example, may only be calcu-

lable to three. This difficulty was found to present itself in the calculation of the second reflected waves by this method, and another method had to be sought. An account of this will be given in the theory of the second reflected waves.

29. *Pistons of Equal Mass.*—A considerable reduction in the number of coefficients to be calculated is effected by supposing the two pistons to have the same mass. When this is so $H = h$, and hereafter we shall write everywhere h for H . The calculation of A_0, A_1, \dots, A_3 is then simplified a good deal. Further, it appears that the coefficients a differ only in sign from the coefficients A , or we have

$$a_0 = -A_0, \quad a_1 = -A_1, \dots$$

It is now unnecessary to calculate separately the pressures, velocities, displacements, and times at the two pistons. We shall speak of the piston specified by $x_0 = c$ as the "shot," and of the piston specified by $x_0 = 0$ as the "image of the shot." We shall generally calculate the pressures, &c., for the image of the shot, because a slight simplification is effected by putting x_0 equal to zero.

30. *Incidence of the Second Middle Wave upon the Pistons.*—The value of s at the receding front of the second middle wave is that which has been denoted by s_1 , and in the case of equal pistons it is the same as R_1 or $\Sigma_1 - \frac{1}{2}\sigma_0$. This is therefore the value of s at the image of the shot at the instant when the receding front of the second middle wave reaches it. It will be denoted by S_2 . The corresponding value of r may be found from the formula

$$r - R_1 = B_1(s - S_1) + (B_2/\Sigma_1)(s - S_1)^2 + \dots$$

by putting S_2 for s . It will be denoted by R_2 . From this the corresponding value of σ may be found. It will be denoted by Σ_2 . The corresponding value of u , which is $R_2 - S_2$, will be denoted by U_2 . The corresponding value of Z , denoted by Z_2 , can be found most simply from the formula for the first reflected wave from the left. We have

$$Z_2 = K_1 + L_1 U_2 + 105 \frac{F_1(2R_2)}{\Sigma_2^{10}} - 105 \frac{F_1^{(1)}(2R_2)}{\Sigma_2^8} + 45 \frac{F_1^{(2)}(2R_2)}{\Sigma_2^7} - 10 \frac{F_1^{(3)}(2R_2)}{\Sigma_2^6} + \frac{F_1^{(4)}(2R_2)}{\Sigma_2^5},$$

where

$$\begin{aligned} \frac{F_1(2R_2)}{\Sigma_2^{10}} &= \left(\frac{\Sigma_1}{\Sigma_2}\right)^{10} \left\{ \frac{A_0}{\Sigma_1^{10}} - \frac{A_1}{\Sigma_1^9} \left(\frac{2R_1 - 2R_2}{\Sigma_1}\right) \right. \\ &\quad \left. + \frac{1}{2!} \frac{2! A_2}{\Sigma_1^8} \left(\frac{2R_1 - 2R_2}{\Sigma_1}\right)^2 - \frac{1}{3!} \frac{3! A_3}{\Sigma_1^7} \left(\frac{2R_1 - 2R_2}{\Sigma_1}\right)^3 + \dots \right\}, \\ \frac{F_1^{(1)}(2R_2)}{\Sigma_2^8} &= \left(\frac{\Sigma_1}{\Sigma_2}\right)^8 \left\{ \frac{A_1}{\Sigma_1^9} - \frac{2! A_2}{\Sigma_1^8} \left(\frac{2R_1 - 2R_2}{\Sigma_1}\right) \right. \\ &\quad \left. + \frac{1}{2!} \frac{3! A_3}{\Sigma_1^7} \left(\frac{2R_1 - 2R_2}{\Sigma_1}\right)^2 - \frac{1}{3!} \frac{4! A_4}{\Sigma_1^6} \left(\frac{2R_1 - 2R_2}{\Sigma_1}\right)^3 + \dots \right\}, \end{aligned}$$

$$\begin{aligned} \frac{F_1^{(2)}(2R_2)}{\Sigma_2^8} &= \left(\frac{\Sigma_1}{\Sigma_2}\right)^8 \left\{ \frac{2! A_2}{\Sigma_1^8} - \frac{3! A_3}{\Sigma_1^7} \left(\frac{2R_1-2R_2}{\Sigma_1}\right) \right. \\ &\quad \left. + \frac{1}{2!} \frac{4! A_4}{\Sigma_1^6} \left(\frac{2R_1-2R_2}{\Sigma_1}\right)^2 - \frac{1}{3!} \frac{5! A_5}{\Sigma_1^5} \left(\frac{2R_1-2R_2}{\Sigma_1}\right)^3 + \dots \right\}, \\ \frac{F_1^{(3)}(2R_2)}{\Sigma_2^7} &= \left(\frac{\Sigma_1}{\Sigma_2}\right)^7 \left\{ \frac{3! A_3}{\Sigma_1^7} - \frac{4! A_4}{\Sigma_1^6} \left(\frac{2R_1-2R_2}{\Sigma_1}\right) \right. \\ &\quad \left. + \frac{1}{2!} \frac{5! A_5}{\Sigma_1^5} \left(\frac{2R_1-2R_2}{\Sigma_1}\right)^2 - \frac{1}{3!} \frac{6! A_6}{\Sigma_1^4} \left(\frac{2R_1-2R_2}{\Sigma_1}\right)^3 + \dots \right\}, \\ \frac{F_1^{(4)}(2R_2)}{\Sigma_2^6} &= \left(\frac{\Sigma_1}{\Sigma_2}\right)^6 \left\{ \frac{4! A_4}{\Sigma_1^6} - \frac{5! A_5}{\Sigma_1^5} \left(\frac{2R_1-2R_2}{\Sigma_1}\right) \right. \\ &\quad \left. + \frac{1}{2!} \frac{6! A_6}{\Sigma_1^4} \left(\frac{2R_1-2R_2}{\Sigma_1}\right)^2 - \frac{1}{3!} \frac{7! A_7}{\Sigma_1^3} \left(\frac{2R_1-2R_2}{\Sigma_1}\right)^3 + \dots \right\}. \end{aligned}$$

31. *Transformation of the Formula for the Second Middle Wave.*—In what follows we shall disregard coefficients A beyond A_6 ; if it were desired to include further coefficients A some of the formulæ would require modification, but there is no difficulty arising from the convention to stop at A_6 . The most effective transformation of the formula for Z in the second middle wave is found by putting for Z_B the value derived from the first reflected wave from the right, viz. :—

$$Z_B = k_1 + l_1 (R_1 - s_1) + \left[\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{c\sigma_0^{10}}{945\alpha\sigma} + \frac{\alpha_0 + \alpha_1(2s - 2s_1) + \alpha_2(2s - 2s_1)^2 + \dots}{\sigma} \right\} \right]_{r=R_1, s=s_1},$$

so that the terms contributed to Z by Z_B come to

$$\begin{aligned} -\frac{2}{3} \left\{ k_1 (R_1 + s_1)^6 + 105 \left(\frac{c\sigma_0^{10}}{945\alpha} + \alpha_0 \right) - 105 (R_1 + s_1) \alpha_1 \right. \\ \left. + 45 (R_1 + s_1)^2 2! \alpha_2 - 10 (R_1 + s_1)^3 3! \alpha_3 + (R_1 + s_1)^4 4! \alpha_4 \right\} \\ \times \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left\{ \frac{(s - s_1)^4 (s + R_1)^4}{\sigma (R_1 + s_1)^8} \right\}, \end{aligned}$$

and then, before putting R_1 for s_1 , or $-A_0, -A_1, \dots$, for $\alpha_0, \alpha_1, \dots$, transforming the terms contributed by A_0, A_1, A_2, A_3 to the form

$$\begin{aligned} &\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left[\frac{1}{\sigma} \left\{ A_0 + 2A_1 (r - R_1) + 2^2 A_2 (r - R_1)^2 + 2^3 A_3 (r - R_1)^3 \right\} \right] \\ &+ \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma} \right)^4 \left[\frac{A_0}{\sigma} \left\{ -56 \left(\frac{s_1 + r}{R_1 + s_1} \right)^5 + 140 \left(\frac{s_1 + r}{R_1 + s_1} \right)^6 - 120 \left(\frac{s_1 + r}{R_1 + s_1} \right)^7 + 35 \left(\frac{s_1 + r}{R_1 + s_1} \right)^8 \right\} \right. \\ &\quad - \frac{2A_1 (R_1 + s_1)}{\sigma} \left\{ -21 \left(\frac{s_1 + r}{R_1 + s_1} \right)^5 + 56 \left(\frac{s_1 + r}{R_1 + s_1} \right)^6 - 50 \left(\frac{s_1 + r}{R_1 + s_1} \right)^7 + 15 \left(\frac{s_1 + r}{R_1 + s_1} \right)^8 \right\} \\ &\quad + \frac{2^2 A_2 (R_1 + s_1)^2}{\sigma} \left\{ -6 \left(\frac{s_1 + r}{R_1 + s_1} \right)^5 + 17 \left(\frac{s_1 + r}{R_1 + s_1} \right)^6 - 16 \left(\frac{s_1 + r}{R_1 + s_1} \right)^7 + 5 \left(\frac{s_1 + r}{R_1 + s_1} \right)^8 \right\} \\ &\quad \left. - \frac{2^3 A_3 (R_1 + s_1)^3}{\sigma} \left\{ - \left(\frac{s_1 + r}{R_1 + s_1} \right)^5 + 3 \left(\frac{s_1 + r}{R_1 + s_1} \right)^6 - 3 \left(\frac{s_1 + r}{R_1 + s_1} \right)^7 + \left(\frac{s_1 + r}{R_1 + s_1} \right)^8 \right\} \right]. \end{aligned}$$

The first line of this expression, with the terms contributed by A_1, A_2, \dots , makes up

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{F_1(\sigma+u)}{\sigma} \right\},$$

and the remaining lines are unaltered when $-s$ is written for r .

The terms contributed by K_1 and L_1 are the same as

$$K_1 + L_1 u - \frac{2}{3} K_1 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(s-s_1)^5 (R_1+s)^4}{\sigma} \right\} + \frac{1}{15} L_1 \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{(s-s_1)^5 (R_1+s)^5}{\sigma} \right\},$$

and thus the Z of the second middle wave is expressed entirely as the sum of the Z of the first reflected wave from the left and a function of the form

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{\phi_2(\sigma-u)}{\sigma} \right\}.$$

Further, noting that with equal pistons $l_1 = L_1$, we see that $\phi_2(\sigma-u)$ contains no terms of degree higher than the ninth in s or $\frac{1}{2}(\sigma-u)$. Also we see that it can be expressed as a rational integral function of $(s-s_1)/(R_1+s_1)$ of the ninth degree, and that it contains no terms of degree lower than the fourth. Since Z and $\partial Z/\partial \sigma$ are continuous at $s = s_1$ with the Z and $\partial Z/\partial \sigma$ belonging to the first reflected wave from the left, the function ϕ_2 can contain no terms of the fourth or fifth degree in $(s-s_1)/(R_1+s_1)$. The vanishing of the coefficients of these terms does not introduce any new condition. On replacing a_0, \dots by $-A_0, \dots$, we have the result that in the second middle wave

$$Z = K_1 + L_1 u + \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{F_1(\sigma+u)}{\sigma} \right\} + \Sigma_2^{10} \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{1}{\sigma} (\eta_6 \xi^6 - \eta_7 \xi^7 + \eta_8 \xi^8 - \eta_9 \xi^9) \right\},$$

where ξ is written for $(s-s_1)/(R_1+s_1)$, and $s_1 = R_1 = S_2$, while $\eta_6, \eta_7, \eta_8, \eta_9$ are given by the equations

$$\eta_6 = -\frac{8}{3} (K_1 - k_1) \frac{(2R_1)^9}{\Sigma_2^{10}} - \frac{4c}{27\alpha} \left(\frac{\sigma_0}{\Sigma_2}\right)^{10} + 280\xi_0 + 140\xi_1 + 62\xi_2 + 23\xi_3 + 6\xi_4 - \xi_5,$$

$$\eta_7 = 4 (K_1 - k_1) \frac{(2R_1)^9}{\Sigma_2^{10}} - \frac{8c}{63\alpha} \left(\frac{\sigma_0}{\Sigma_2}\right)^{10} - 240\xi_0 - 120\xi_1 - 52\xi_2 - 18\xi_3 - 4\xi_4 - \xi_7,$$

$$\eta_8 = -\frac{8}{3} (K_1 - k_1) \frac{(2R_1)^9}{\Sigma_2^{10}} - \frac{c}{27\alpha} \left(\frac{\sigma_0}{\Sigma_2}\right)^{10} + 70\xi_0 + 35\xi_1 + 15\xi_2 + 5\xi_3 + \xi_4 - \xi_8,$$

$$\eta_9 = \frac{8}{3} (K_1 - k_1) \frac{(2R_1)^9}{\Sigma_2^{10}} - \xi_9.$$

in which

$$\begin{aligned} \xi_0 &= \frac{A_0}{\Sigma_1^{10}} \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}, & \xi_2 &= -2 \frac{A_1}{\Sigma_1^9} \frac{2R_1}{\Sigma_1} \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}, & \xi_3 &= 2 \frac{2! A_2}{\Sigma_1^8} \left(\frac{2R_1}{\Sigma_1} \right)^2 \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}, \\ \xi_3 &= -4 \frac{3! A_3}{\Sigma_1^7} \left(\frac{2R_1}{\Sigma_1} \right)^3 \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}, & \xi_4 &= 2 \frac{4! A_4}{\Sigma_1^6} \left(\frac{2R_1}{\Sigma_1} \right)^4 \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}, & \xi_5 &= -1 \frac{5! A_5}{\Sigma_1^5} \left(\frac{2R_1}{\Sigma_1} \right)^5 \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}, \\ \xi_6 &= 4 \frac{6! A_6}{\Sigma_1^4} \left(\frac{2R_1}{\Sigma_1} \right)^6 \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}, & \xi_7 &= -3 \frac{7! A_7}{\Sigma_1^3} \left(\frac{2R_1}{\Sigma_1} \right)^7 \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}, & \xi_8 &= 3 \frac{8! A_8}{\Sigma_1^2} \left(\frac{2R_1}{\Sigma_1} \right)^8 \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}, \\ \xi_9 &= -2 \frac{9! A_9}{\Sigma_1} \left(\frac{2R_1}{\Sigma_1} \right)^9 \left(\frac{\Sigma_1}{\Sigma_2} \right)^{10}. \end{aligned}$$

THE SECOND REFLECTED WAVES.

32. *Relation between Pressure and Velocity at a Piston.*—The relation between pressure and velocity at the image of the shot is an equation connecting r and s , which holds at $x_0 = 0$, and can be interpreted as the equation of a certain locus in the plane of r and s . This equation can be written in the form

$$(r - R_2)/\Sigma_2 = B'_1 \delta + B'_2 \delta^2 + B'_3 \delta^3 + \dots,$$

where δ stands for $(s - S_2)/\Sigma_2$, and the coefficients B' are at present undetermined.

To determine these coefficients we have recourse to the method of Articles 21–23. During the progress of the second reflected wave from the left, the value of x_0 at any point in the region occupied by it can be expressed in terms of the values r' and s' of r and s , which occur simultaneously at the point, by the formula

$$x_0 = -5h \int_{AC} (Y/\sigma) du,$$

wherein the integral is taken along the locus from the point A, where $r = r'$, to the point C, where $s = s'$. In this integral

$$u - U_2 = \Sigma_2 \{ (B'_1 - 1) \delta + B'_2 \delta^2 + B'_3 \delta^3 + \dots \},$$

$$du = \Sigma_2 \{ (B'_1 - 1) + 2B'_2 \delta + 3B'_3 \delta^2 + \dots \} d\delta,$$

$$\sigma = \Sigma_2 \{ 1 + (B'_1 + 1) \delta + B'_2 \delta^2 + B'_3 \delta^3 + \dots \},$$

$$\frac{Y}{\sigma} = \frac{(r' + s')^5}{\sigma^6} \left\{ 1 + 30 \frac{(r - r')(s - s')}{(r' + s') \sigma} + 210 \frac{(r - r')^2 (s - s')^2}{(r' + s')^2 \sigma^2} + \dots \right\}.$$

On putting R_2 for r' , we have the value of x_0 along the junction of the second middle wave and the second reflected wave from the left expanded in a series of powers of δ' , or $(s'-S_2)/\Sigma_2$, in the form

$$\begin{aligned} x_0 = & -5h [(B'_1-1)\delta' + \{B'_2-(3B'_1-2)(B'_1-1)\}\delta'^2 \\ & + \{B'_3-3B'_2(2B'_1-1)+(7B'_1{}^2-6B'_1+2)(B'_1-1)\}\delta'^3 \\ & + \{B'_4-2B'_3(3B'_1-1)-3B'_2{}^2-\frac{1}{2}B'_2(42B'_1{}^2-33B'_1+5) \\ & \quad -\frac{1}{2}(28B'_1{}^3-21B'_1{}^2+9B'_1-2)(B'_1-1)\}\delta'^4 \\ & + \{B'_5-\frac{1}{3}B'_4(30B'_1-7)-6B'_2B'_3+{}_{10}B'_2(210B'_1{}^2-108B'_1+7) \\ & \quad +\frac{1}{3}B'_2{}^2(105B'_1-27)-{}_{10}B'_2(560B'_1{}^3-462B'_1{}^2+81B'_1-7) \\ & \quad +\frac{1}{3}(126B'_1{}^4-56B'_1{}^3+21B'_1{}^2-6B'_1+1)(B'_1-1)\}\delta'^5 + \dots]. \end{aligned}$$

Now at any point (R_2, s') on the same junction the value of x_0 can be obtained by forming $-\Pi \partial Z/\partial \sigma$, where

$$Z = K_1 + L_1 u + \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{F_1(\sigma + \eta)}{\sigma} \right\} + \Sigma_2^{10} \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)^4 \left\{ \frac{1}{\sigma} (\eta_6 \xi^n - \eta_7 \xi^7 + \eta_8 \xi^8 - \eta_9 \xi^9) \right\}$$

and

$$\Pi = u(\sigma/\sigma_0)^{10}, \quad \xi = (s-S_2)/2R_1,$$

and putting therein

$$\sigma = R_2 + s', \quad u = R_2 - s', \quad s = s', \quad s' - S_2 = \Sigma_2 \delta',$$

and the result can be expressed in terms of δ' in the form

$$\begin{aligned} \frac{x_0}{a} = & \left(\frac{\Sigma_2}{\sigma_0}\right)^{10} \left[\left\{ 945 \frac{F_1(2R_2)}{\Sigma_2^{10}} - 945(1+\delta') \frac{F_1^{(1)}(2R_2)}{\Sigma_2^9} + 420(1+\delta')^2 \frac{F_1^{(2)}(2R_2)}{\Sigma_2^8} \right. \right. \\ & - 105(1+\delta')^3 \frac{F_1^{(3)}(2R_2)}{\Sigma_2^7} + 15(1+\delta')^4 \frac{F_1^{(4)}(2R_2)}{\Sigma_2^6} - (1+\delta')^5 \frac{F_1^{(5)}(2R_2)}{\Sigma_2^5} \left. \right\} \\ & + \eta_6 \left(\frac{\Sigma_2}{2R_1}\right)^6 \left\{ 945\delta'^6 - 945 \times 3(1+\delta')\delta'^5 + 420 \times \frac{1}{2}\delta'^5(1+\delta')^2\delta'^4 \right. \\ & \quad \left. - 105 \times 15(1+\delta')^3\delta'^3 + 15 \times \frac{1}{2}\delta'^5(1+\delta')^4\delta'^2 - \frac{1}{2}\delta'^5(1+\delta')^6\delta' \right\} \\ & - \eta_7 \left(\frac{\Sigma_2}{2R_1}\right)^7 \left\{ 945\delta'^7 - 945 \times \frac{1}{2}(1+\delta')\delta'^6 + 420 \times \frac{1}{2}\delta'^5(1+\delta')^2\delta'^5 \right. \\ & \quad \left. - 105 \times \frac{1}{2}\delta'^5(1+\delta')^3\delta'^4 + 15 \times \frac{1}{2}\delta'^5(1+\delta')^4\delta'^3 - \frac{3}{4}\delta'^5(1+\delta')^5\delta'^2 \right\} \\ & + \eta_8 \left(\frac{\Sigma_2}{2R_1}\right)^8 \left\{ 945\delta'^8 - 945 \times 4(1+\delta')\delta'^7 + 420 \times 14(1+\delta')^2\delta'^6 \right. \\ & \quad \left. - 105 \times 42(1+\delta')^3\delta'^5 + 15 \times 105(1+\delta')^4\delta'^4 - 210(1+\delta')^5\delta'^3 \right\} \\ & - \eta_9 \left(\frac{\Sigma_2}{2R_1}\right)^9 \left\{ 945\delta'^9 - 945 \times \frac{1}{2}(1+\delta')\delta'^8 + 420 \times 18(1+\delta')^2\delta'^7 \right. \\ & \quad \left. - 105 \times 63(1+\delta')^3\delta'^6 + 15 \times 189(1+\delta')^4\delta'^5 - \frac{1}{2}\delta'^5(1+\delta')^5\delta'^4 \right\} \left. \right]. \end{aligned}$$

The terms independent of δ' in the right-hand member of this equation add up to zero, for x_0 vanishes at (R_2, S_2) ; and, by equating the coefficients of powers of δ' in the two expressions for x_0 , equations are obtained from which the values of the coefficients B' can be found successively.

33. *Relation between Velocity and Time at a Piston.*—The time at which any particular simultaneous values of r and s occur at $x_0 = 0$, can be found by the method of Article 25, and thus the relation between velocity and time at the image of the shot may be traced. We can write down the equation

$$t - T_2 = -10 \frac{h}{a} \frac{\sigma_0^{10}}{\Sigma_2^{11}} \int_{s_2}^s \left(\frac{\sigma}{\Sigma_2} \right)^{-11} \{ (B'_1 - 1) + 2B'_2\delta + 3B'_3\delta^2 + \dots \} ds,$$

in which

$$\delta = (s - S_2) / \Sigma_2, \quad \sigma = \Sigma_2 \{ 1 + (B'_1 + 1)\delta + B'_2\delta^2 + B'_3\delta^3 + \dots \},$$

and thus $t - T_2$ can be expanded in powers of δ in the form

$$t - T_2 = c_1\delta + c_2\delta^2 + c_3\delta^3 + \dots,$$

where

$$c_1 = -\frac{10h}{a} \left(\frac{\sigma_0}{\Sigma_2} \right)^{10} (B'_1 - 1),$$

$$c_2 = -\frac{10h}{a} \left(\frac{\sigma_0}{\Sigma_2} \right)^{10} \{ B'_2 - \frac{1}{2} (B'_1 + 1) (B'_1 - 1) \},$$

$$c_3 = -\frac{10h}{a} \left(\frac{\sigma_0}{\Sigma_2} \right)^{10} \{ B'_3 - \frac{1}{3} (3B'_1 + 1) B'_2 + 22 (B'_1 + 1)^2 (B'_1 - 1) \},$$

$$c_4 = -\frac{10h}{a} \left(\frac{\sigma_0}{\Sigma_2} \right)^{10} \{ B'_4 - \frac{1}{2} (2B'_1 + 1) B'_3 - \frac{1}{2} B'_2^2 + 66B'_1 (B'_1 + 1) B'_2 - \frac{1}{2} (B'_1 + 1)^3 (B'_1 - 1) \},$$

$$c_5 = -\frac{10h}{a} \left(\frac{\sigma_0}{\Sigma_2} \right)^{10} \{ B'_5 - \frac{1}{6} (5B'_1 + 3) B'_4 - 11B'_2 B'_3 + \frac{1}{6} (5B'_1 + 1) (B'_1 + 1) B'_3 + \frac{1}{6} (5B'_1 + 3) B'_2^2 - \frac{5}{6} (5B'_1 - 1) (B'_1 + 1)^2 B'_2 + \frac{2}{3} (B'_1 + 1)^4 (B'_1 - 1) \},$$

...

34. *Displacement of a Piston.*—To obtain the displacement of the image of the shot, we have to find the value of x at $x_0 = 0$ in terms of simultaneous values of r and s occurring on the locus

$$r - R_2 = B'_1 (s - S_2) + (B'_2 / \Sigma_2) (s - S_2)^2 + (B'_3 / \Sigma_2^2) (s - S_2)^3 + \dots$$

Now, when $x_0 = 0$, we have $x = ut - Z$, and for the value of x at (R_2, S_2) , denoted by X_2 , we have $X_2 = U_2 T_2 - Z_2$, so that when $x_0 = 0$, we have

$$x - X_2 = ut - U_2 T_2 - (Z - Z_2).$$

Also we have

$$Z - Z_2 = \int_{(u_2, s_2)}^{(r, s)} \frac{\partial Z}{\partial u} du + \frac{\partial Z}{\partial \sigma} d\sigma,$$

where the integral is taken along the locus, and, since $\partial Z/\partial \sigma$ vanishes along the locus, we get

$$x - x_2 = ut - U_2 T_2 - \int_{(u_2, s_2)}^{(r, s)} t du = \int_{s_2}^s u \frac{dt}{ds} ds,$$

in which

$$u = U_2 + \Sigma_2 \{ (B'_1 - 1) \delta + B'_2 \delta^2 + B'_3 \delta^3 + \dots \},$$

$$t = T_2 + c_1 \delta + c_2 \delta^2 + c_3 \delta^3 + \dots,$$

$$s = S_2 + \Sigma_2 \delta.$$

It follows that, at $x_0 = 0$, x is given by the equation

$$\begin{aligned} x - X_2 = & U_2 c_1 \delta + \{ U_2 c_2 + \frac{1}{2} \Sigma_2 c_1 (B'_1 - 1) \} \delta^2 + \{ U_2 c_3 + \frac{3}{2} \Sigma_2 c_2 (B'_1 - 1) + \frac{1}{2} \Sigma_2 c_1 B'_2 \} \delta^3 \\ & + \{ U_2 c_4 + \frac{3}{2} \Sigma_2 c_3 (B'_1 - 1) + \frac{1}{2} \Sigma_2 c_2 B'_2 + \frac{1}{4} \Sigma_2 c_1 B'_3 \} \delta^4 \\ & + \{ U_2 c_5 + \frac{3}{2} \Sigma_2 c_4 (B'_1 - 1) + \frac{3}{2} \Sigma_2 c_3 B'_2 + \frac{3}{2} \Sigma_2 c_2 B'_3 + \frac{1}{2} \Sigma_2 c_1 B'_4 \} \delta^5 + \dots \end{aligned}$$

which may be written

$$x - X_2 = \xi_1 \delta + \xi_2 \delta^2 + \xi_3 \delta^3 + \dots$$

The formulæ which have been obtained avail to determine the displacement, velocity and pressure at the shot or its image at any time during the progress of the second reflected waves.

35. *General Method for the Second Reflected Waves.*—We shall need to be able to calculate Z , t and x_0 for any simultaneous values of r and s that can occur in the second reflected wave from the left. It is best to obtain formulæ for t and x_0 separately, and not to deduce them from the formula for Z by differentiation, because the formulæ will be approximate, and to obtain the terms of any particular order in t , for example, by differentiation of Z it would be necessary to obtain the terms of order higher by one in the formula for Z . The method of determining the formula for x_0 has been indicated already in Article 32, and the work will be completed presently. The formulæ for t and Z will be found by similar applications of the method of RIEMANN. We begin with the formula for t . After finding formulæ for x_0 , t , Z , we can calculate x from the equation

$$x = x_0 \left\{ \frac{\rho_0}{\beta} + \left(\frac{\sigma_0}{\sigma} \right)^n \left(1 - \frac{\rho_0}{\beta} \right) \right\} + ut - Z.$$

36. *Method of Determining t .*—The value of t along the locus $x_0 = 0$ has been found in Article 33. To obtain the differential coefficients of t along the same locus we have the equations

$$\frac{\partial x_0}{\partial u} + \Pi \frac{\partial t}{\partial \sigma} = 0, \quad \frac{\partial x_0}{\partial \sigma} + \Pi \frac{\partial t}{\partial u} = 0,$$

and

$$\frac{\partial x_0}{\partial \sigma} = \frac{10h}{\sigma \left\{ 1 - \left(\frac{d\sigma}{du} \right)^2 \right\}}, \quad \frac{\partial x_0}{\partial u} = -\frac{10h \frac{d\sigma}{du}}{\sigma \left\{ 1 - \left(\frac{d\sigma}{du} \right)^2 \right\}},$$

where $d\sigma/du$ is to be found from the equation of the locus. Thus we may write

$$\frac{\partial t}{\partial \sigma} = 10 \frac{h\sigma_0^{10}}{\alpha\sigma^{11}} \frac{\frac{d\sigma}{du}}{1 - \left(\frac{d\sigma}{du} \right)^2}, \quad \frac{\partial t}{\partial u} = -10 \frac{h\sigma_0^{10}}{\alpha\sigma^{11}} \frac{1}{1 - \left(\frac{d\sigma}{du} \right)^2}.$$

Now let (r', s') be any point P, and let lines PA, PC parallel to the axes of s and r meet the locus in A, C, as in fig. 3 in Article 22. Then, since t satisfies the same differential equation as Z, the integral

$$\int V \left(\frac{\partial t}{\partial s} + 5 \frac{t}{\sigma} \right) ds + t \left(\frac{\partial V}{\partial r} - 5 \frac{V}{\sigma} \right) dr$$

taken round the contour formed by the arc AC and the lines CP, PA vanishes, and therefore we have the equation

$$t(r', s') = [Vt]_A + \int_{AC} \left(\frac{\partial V}{\partial r} - 5 \frac{V}{\sigma} \right) t dr + \left(\frac{\partial t}{\partial s} + 5 \frac{t}{\sigma} \right) V ds,$$

or, on putting

$$t' = t - T_2,$$

the equation

$$t'(r', s') = [Vt'] + \int_{AC} \left(\frac{\partial V}{\partial r} - 5 \frac{V}{\sigma} \right) t' dr + \left(\frac{\partial t'}{\partial s} + 5 \frac{t'}{\sigma} \right) V ds.$$

But we have, along the locus,

$$\frac{\partial t'}{\partial s} = \frac{\partial t'}{\partial \sigma} - \frac{\partial t'}{\partial u} = -10 \frac{h\sigma_0^{10}}{\alpha\sigma^{11}} \frac{1}{\frac{d\sigma}{du} - 1} = -\frac{5h\sigma_0^{10}}{\alpha\sigma^{11}} \frac{du}{ds},$$

and, by the theory of Article 25, the expression last written is the same as the value of $\frac{1}{2}dt/ds$ along the locus, or we have

$$\frac{\partial t'}{\partial s} = \frac{1}{2} \Sigma_2^{-1} (c_1 + 2c_2\delta + 3c_3\delta^2 + \dots).$$

Also along the locus we have

$$t' = c_1\delta + c_2\delta^2 + c_3\delta^3 + \dots$$

Let s_A denote the value of s at A. Then at A we have

$$r = r', \quad s = s_A, \quad V = \left(\frac{r' + s_A}{r' + s'} \right)^5,$$

$$t' = c_1 \delta_A + c_2 \delta_A^2 + c_3 \delta_A^3 + \dots,$$

where δ_A stands for $(s_A - S_2)/\Sigma_2$.

Along AC we have the formulæ already written for t' and $\partial t'/\partial s$, and we have further

$$V = \left(\frac{r+s}{r'+s'} \right)^5 (1 - 20\xi + 90\xi^2 - 140\xi^3 + 70\xi^4), \quad \xi = - \frac{(r-r')(s-s')}{(r'+s')(r+s)},$$

$$\frac{\partial V}{\partial r} - 5 \frac{V}{\sigma} = \frac{(s+r')(s-s')(r+s)^3}{(r'+s')^6} (20 - 180\xi + 420\xi^2 - 280\xi^3),$$

and we have to put

$$ds = \Sigma_2 d\delta, \quad dr = \Sigma_2 (B'_1 + 2B'_2\delta + 3B'_3\delta^2 + \dots) d\delta.$$

The limits of integration are δ_A and δ' , which is $(s' - S_2)/\Sigma_2$.

The value of δ_A is to be found by reversing the series

$$r - R_2 = (s - S_2) \{ B'_1 + (B'_2/\Sigma_2)(s - S_2) + (B'_3/\Sigma_2^2)(s - S_2)^2 + \dots \},$$

and putting r' for r and s_A for s . If we write ϵ' for $(r' - R_2)/\Sigma_2$ the result is

$$\begin{aligned} \delta_A = & \frac{\epsilon'}{B'_1} - \frac{B'_2}{B'_1{}^2} \epsilon'^2 + \left\{ \frac{2B'_2{}^2}{B'_1{}^5} - \frac{B'_3}{B'_1{}^4} \right\} \epsilon'^3 - \left(\frac{5B'_2{}^3}{B'_1{}^7} - \frac{5B'_2 B'_3}{B'_1{}^6} + \frac{B'_4}{B'_1{}^5} \right) \epsilon'^4 \\ & + \left(14 \frac{B'_2{}^4}{B'_1{}^9} - 21 \frac{B'_2{}^2 B'_3}{B'_1{}^8} + 3 \frac{B'_3{}^2}{B'_1{}^7} + 6 \frac{B'_2 B'_4}{B'_1{}^7} - \frac{B'_5}{B'_1{}^6} \right) \epsilon'^5 + \dots \end{aligned}$$

Thus δ_A is known in terms of r' .

37. *Formula for the Time.*—We work out the formula for t' or $t - T_2$ in terms of δ , or $(s - S_2)/\Sigma_2$, and ϵ , or $(r - R_2)/\Sigma_2$, at any point answering to simultaneous values of r and s which can occur in the second reflected wave from the left. For this we first perform the integrations with respect to r and s and then suppress the accents on δ' and ϵ' . We record the results as far as terms of the fourth order.

The terms of the first order present themselves in the form

$$\left(\frac{1 + \epsilon + \delta_A}{1 + \epsilon + \delta} \right)^5 c_1 \delta_A + \frac{1}{(1 + \epsilon + \delta)^5} \frac{1}{2} c_1 (\delta - \delta_A),$$

and it is simpler to leave the factors

$$\left(\frac{1 + \epsilon + \delta_A}{1 + \epsilon + \delta} \right)^5, \quad \frac{1}{(1 + \epsilon + \delta)^5}$$

as they are, rather than to expand them in powers of ϵ , δ_Λ and δ . In like manner the terms of the second order are

$$\left(\frac{1+\epsilon+\delta_\Lambda}{1+\epsilon+\delta}\right)^6 c_2 \delta_\Lambda^2 + \frac{1}{(1+\epsilon+\delta)^6} \left\{ \frac{1}{2} c_2 + \frac{5}{4} c_1 (B'_1 + 3) \right\} (\delta^2 - \delta_\Lambda^2),$$

the terms of the third order are

$$\begin{aligned} \left(\frac{1+\epsilon+\delta_\Lambda}{1+\epsilon+\delta}\right)^6 c_3 \delta_\Lambda^3 + \frac{1}{(1+\epsilon+\delta)^6} \left\{ \frac{1}{2} c_3 + \frac{5}{3} c_2 (B'_1 + 2) + \frac{5}{3} c_1 (B'_1 + 1) (B'_1 + 5) + \frac{5}{8} c_1 B'_2 \right\} (\delta^3 - \delta_\Lambda^3) \\ + \frac{5}{(1+\epsilon+\delta)^6} c_1 \epsilon (\delta - \delta_\Lambda)^2 - \frac{5}{(1+\epsilon+\delta)^6} c_1 B'_1 (\delta + 2\delta_\Lambda) (\delta - \delta_\Lambda)^2, \end{aligned}$$

and the terms of the fourth order are

$$\begin{aligned} \left(\frac{1+\epsilon+\delta_\Lambda}{1+\epsilon+\delta}\right)^6 c_4 \delta_\Lambda^4 + \frac{1}{(1+\epsilon+\delta)^6} \left\{ \frac{1}{2} c_4 + \frac{5}{8} c_3 (3B'_1 + 5) + \frac{5}{2} c_2 (B'_1 + 1) (B'_1 + 3) + \frac{5}{4} c_2 B'_2 \right. \\ \left. + \frac{5}{4} c_1 (B'_1 + 1)^2 (B'_1 + 7) + \frac{5}{2} c_1 (B'_1 + 3) B'_2 + \frac{5}{8} c_1 B'_3 \right\} (\delta^4 - \delta_\Lambda^4) \\ + \frac{1}{(1+\epsilon+\delta)^6} \left\{ \frac{1}{3} c_2 + \frac{1}{3} c_1 (B'_1 + 7) \right\} \epsilon (\delta + 2\delta_\Lambda) (\delta - \delta_\Lambda)^2 \\ - \frac{1}{(1+\epsilon+\delta)^6} \left\{ \frac{1}{3} c_2 B'_1 + \frac{5}{3} c_1 B'_1 (5B'_1 + 11) + \frac{2}{6} c_1 B'_2 \right\} (\delta^2 + 2\delta_\Lambda \delta + 3\delta_\Lambda^2) (\delta - \delta_\Lambda)^2. \end{aligned}$$

38. *Formula for Z.*—We write Z' for $Z - Z_2$, and seek first a formula for Z' along the locus $x_0 = 0$. The value of Z' along this locus is given by the equation

$$Z' = \int_{(u_0, s_2)}^{(r, s)} \frac{\partial Z}{\partial \sigma} d\sigma + \frac{\partial Z}{\partial u} du,$$

where the integral is taken along the locus $\partial Z / \partial \sigma = 0$, so that

$$Z' = \int_{s_1}^s t \frac{du}{ds} ds = \Sigma_2 \int_0^6 (T_2 + c_1 \delta + c_2 \delta^2 + \dots) (B'_1 - 1 + 2B'_2 \delta + 3B'_3 \delta^2 + \dots) d\delta.$$

Thus the value of Z' along the locus can be expanded in powers of δ in the form

$$Z' = d_1 \delta + d_2 \delta^2 + d_3 \delta^3 + \dots,$$

where

$$\begin{aligned} d_1 &= \Sigma_2 T_2 (B'_1 - 1), \\ d_2 &= \Sigma_2 \left\{ T_2 B'_2 + \frac{1}{2} c_1 (B'_1 - 1) \right\}, \\ d_3 &= \Sigma_2 \left\{ T_2 B'_3 + \frac{2}{3} c_1 B'_2 + \frac{1}{3} c_2 (B'_1 - 1) \right\}, \\ d_4 &= \Sigma_2 \left\{ T_2 B'_4 + \frac{3}{4} c_1 B'_3 + \frac{1}{2} c_2 B'_2 + \frac{1}{4} c_3 (B'_1 - 1) \right\}, \\ d_5 &= \Sigma_2 \left\{ T_2 B'_5 + \frac{4}{5} c_1 B'_4 + \frac{2}{5} c_2 B'_3 + \frac{2}{5} c_3 B'_2 + \frac{1}{5} c_4 (B'_1 - 1) \right\}, \\ &\dots \end{aligned}$$

We require also the differential coefficient $\partial Z'/\partial s$ along the locus, and this is given by the equation

$$\frac{\partial Z'}{\partial s} = -t = - (T_2 + c_1\delta + c_2\delta^2 + \dots).$$

The value of Z' at any point (r', s') is then given by the equation

$$Z'(r', s') = [VZ']_A + \int_{AC} \left(\frac{\partial V}{\partial r} - \frac{5V}{\sigma} \right) Z' dr + \left(\frac{\partial Z'}{\partial s} + \frac{5Z'}{\sigma} \right) V ds,$$

in which V has the same form as in Article 36, the value of Z' at A is

$$d_1\delta_A + d_2\delta_A^2 + d_3\delta_A^3 + \dots,$$

and the integration is taken along the locus.

The result may be recorded in a similar form to that for t' in Article 37. The terms of the first order in the formula for Z' are

$$\left(\frac{1 + \epsilon + \delta_A}{1 + \epsilon + \delta} \right)^5 d_1\delta_A - \frac{1}{(1 + \epsilon + \delta)^5} \Sigma_2 T_2 (\delta - \delta_A),$$

the terms of the second order are

$$\left(\frac{1 + \epsilon + \delta_A}{1 + \epsilon + \delta} \right)^5 d_2\delta_A^2 - \frac{1}{(1 + \epsilon + \delta)^5} \left\{ \frac{1}{2} \Sigma_2 \{ c_1 + 5T_2 (B'_1 + 1) \} - \frac{1}{2} d_1 \right\} (\delta^2 - \delta_A^2),$$

the terms of the third order are

$$\begin{aligned} & \left(\frac{1 + \epsilon + \delta_A}{1 + \epsilon + \delta} \right)^5 d_3\delta_A^3 - \frac{1}{(1 + \epsilon + \delta)^5} \left[\frac{1}{3} \Sigma_2 \{ c_2 + 5c_1 (B'_1 + 1) + 10T_2 (B'_1 + 1)^2 + 5T_2 B'_2 \} \right. \\ & \qquad \qquad \qquad \left. - \frac{1}{3} \{ d_2 + 4d_1 (B'_1 + 1) \} \right] (\delta^3 - \delta_A^3) \\ & + \frac{1}{(1 + \epsilon + \delta)^5} \frac{1}{3} \Sigma_2 (\Sigma_2 T_2 - d_1) B'_1 (\delta + 2\delta_A) (\delta - \delta_A)^2 - \frac{10}{(1 + \epsilon + \delta)^5} \Sigma_2 T_2 \epsilon (\delta - \delta_A)^2, \end{aligned}$$

and the terms of the fourth order are

$$\begin{aligned} & \left(\frac{1 + \epsilon + \delta_A}{1 + \epsilon + \delta} \right)^5 d_4\delta_A^4 - \frac{1}{(1 + \epsilon + \delta)^5} \left[\frac{1}{4} \Sigma_2 \{ c_3 + 5c_2 (B'_1 + 1) + 10c_1 (B'_1 + 1)^2 + 5c_1 B'_2 \} \right. \\ & \qquad \qquad \qquad + 10T_2 (B'_1 + 1)^3 + 20T_2 (B'_1 + 1) B'_2 + 5T_2 B'_3 \} \\ & \qquad \qquad \qquad \left. - \frac{1}{4} \{ d_3 + 4d_2 (B'_1 + 1) + 6d_1 (B'_1 + 1)^2 + 4d_1 B'_2 \} \right] (\delta^4 - \delta_A^4) \\ & - \frac{1}{(1 + \epsilon + \delta)^5} \left[\frac{1}{3} \Sigma_2 \{ c_1 + 4T_2 (B'_1 + 1) \} + \frac{1}{3} d_1 (B'_1 - 5) \right] \epsilon (\delta + 2\delta_A) (\delta - \delta_A)^2 \\ & + \frac{1}{(1 + \epsilon + \delta)^5} \left[\frac{1}{3} \Sigma_2 \{ c_1 B'_1 + 4T_2 B'_1 (B'_1 + 1) + T_2 B'_2 \} - \frac{1}{3} \{ d_2 B'_1 + d_1 B'_1 (3B'_1 + 9) + 2d_1 B'_2 \} \right] \\ & \qquad \qquad \qquad \times (\delta^2 + 2\delta_A\delta + 3\delta_A^2) (\delta - \delta_A)^2. \end{aligned}$$

39. *Formula for x_0 .*—A formula has been obtained in Article 32, and can be written in the form

$$x_0(r', s') = -5h \int_{\delta_A}^{\delta'} \left(\frac{r'+s'}{\sigma} \right)^5 \left\{ 1 + 30 \frac{(r-r')(s-s')}{(r'+s')\sigma} + \dots \right\} \frac{\Sigma_2}{\sigma} (B'_1 - 1 + 2B'_2\delta + 3B'_3\delta^2 + \dots) d\delta,$$

in which we have to put

$$\begin{aligned} r'+s' &= \Sigma_2 (1 + \epsilon' + \delta'), \\ \sigma &= \Sigma_2 \{ 1 + (B'_1 + 1) \delta + B'_2\delta^2 + B'_3\delta^3 + \dots \}, \\ r-r' &= \Sigma_2 \{ B'_1\delta - \epsilon' + B'_2\delta^2 + B'_3\delta^3 + \dots \}, \\ s-s' &= \Sigma_2 (\delta - \delta'). \end{aligned}$$

After the integrations are performed the accents on δ' and ϵ' are to be suppressed. The result may be recorded in the form:—The terms of the first order in the expression for x_0 are

$$-5h (1 + \epsilon + \delta)^5 (B'_1 - 1) (\delta - \delta_A),$$

the terms of the second order are

$$-5h (1 + \epsilon + \delta)^5 \{ B'_2 - 3 (B'_1 + 1) (B'_1 - 1) \} (\delta^2 - \delta_A^2),$$

the terms of the third order are

$$\begin{aligned} &-5h (1 + \epsilon + \delta)^5 \{ B'_3 - 2 (3B'_1 + 1) B'_2 + 7 (B'_1 + 1)^2 (B'_1 - 1) \} (\delta^3 - \delta_A^3) \\ &-150h (1 + \epsilon + \delta)^4 \{ \frac{1}{2} (B'_1 - 1) \epsilon - \frac{1}{6} B'_1 (B'_1 - 1) (\delta + 2\delta_A) \} (\delta - \delta_A)^2, \end{aligned}$$

and the terms of the fourth order are

$$\begin{aligned} &-5h (1 + \epsilon + \delta)^5 \{ B'_4 - 3 (2B'_1 + 1) B'_3 - 3B'_2^2 + 21 B'_1 (B'_1 + 1) B'_2 \\ &\hspace{15em} - 14 (B'_1 + 1)^3 (B'_1 - 1) \} (\delta^4 - \delta_A^4) \\ &-25h (1 + \epsilon + \delta)^4 \{ [2B'_2 - 7 (B'_1 + 1) (B'_1 - 1) \} \epsilon (\delta + 2\delta_A) \\ &\hspace{10em} - \frac{1}{2} \{ (3B'_1 - 1) B'_2 - 7 B'_1 (B'_1 + 1) (B'_1 - 1) \} (\delta^2 + 2\delta_A\delta + 3\delta_A^2) \} (\delta - \delta_A)^2. \end{aligned}$$

40. *State of the Gas at any Time.*—With a view to applications it is important to indicate how the state of the gas may be determined at any time, or when the shot and its image have both travelled an assigned distance. We shall suppose that the time in question is an instant during the generation of the second reflected waves, before the second middle wave is obliterated. Then the central part of the tube is occupied by the second middle wave, and beyond the junctions the rest of the tube, up to the shot and its image, are occupied by the second reflected waves.

An assigned position of the shot and its image answer to a given value of x , and the corresponding value of δ is to be found by solving the equation for x given in Article 34.

This is the value of δ at the image of the shot, and the corresponding values of r and s at the image of the shot are given by formulæ in the same article. Also, δ being known for the image of the shot in this position, the value of t is given by the formula of Article 33. Let this particular value of t be denoted by T_3 , and in like manner let the values of the various quantities at the image of the shot at this time be denoted by attaching a suffix 3 to the letters, thus: $-R_3, \delta_3$.

In the second reflected wave from the left the values of r that occur lie between R_2 and R_3 . To each such value, when $t = T_3$, there answers a value of s and therefore of δ . If in the formula of Article 37 we put T_3 for t and the chosen value for r , the formula becomes an equation giving δ . The chosen value of r determines the corresponding values of ϵ and δ_A , and the deduced value of δ determines the corresponding value of s . Then, simultaneous values of r and s being known, all the quantities can be determined. It seems to be most appropriate to assume a series of suitable values of r and calculate the corresponding values of s . The process of finding δ , by trial, may be simplified by means of a theorem to the effect that the loci, in the plane of (r, s) , which answer to constant values of t and x_0 , are equally inclined to the axis of r . To prove this we have

$$\left(\frac{du}{d\sigma}\right)_{t = \text{const.}} = -\frac{\partial t / \partial \sigma}{\partial u / \partial \sigma} = -\frac{\partial x_0 / \partial \sigma}{\partial u / \partial \sigma} = \left(\frac{d\sigma}{du}\right)_{x_0 = \text{const.}}$$

or

$$\left(\frac{dr - ds}{dr + ds}\right)_{t = \text{const.}} = \left(\frac{dr + ds}{dr - ds}\right)_{x_0 = \text{const.}}$$

or

$$\left(\frac{ds}{dr}\right)_{t = \text{const.}} + \left(\frac{ds}{dr}\right)_{x_0 = \text{const.}} = 0.$$

This theorem shows that a point of given r on the locus $t = T_3$ is not far from the image in $r = R_3$ of the tangent at (R_3, S_3) to the locus along which $x_0 = 0$. Hence a first approximation to the s answering to a given r is $2S_3 - s_A$, where s_A depends upon r in the known way, and therefore a first approximation to the required value of δ is $2\delta_3 - \delta_A$.

The junction of the second reflected wave from the left and the second middle wave is characterized by the value R_2 of r . If, then, the process indicated above is carried out for the value R_2 of r , the result is to give a pair of simultaneous values of r and s , which can occur in the second middle wave at the time when $t = T_3$. Another pair of simultaneous values can be found by finding the common value of r and s which occurs at the central section at the same time. This is to be done by putting $r = s$ and $t = T_3$ in the formula giving t in the second middle wave, and solving the resulting equation for r by trial. When this is done we shall have two pairs of simultaneous values of r and s which occur in the left-hand half of the central part of the tube at time T_3 , and they are the extreme values of r and s which can occur in that part at that time. To obtain other pairs, we may choose an intermediate value of r , substitute in the equation giving t the value T_3 of t and this value of r , and find s by trial. For a first approximation

we may assume that the required point (r, s) is on the straight line joining the two extreme points whose co-ordinates have been determined previously.

After these preliminaries the way is prepared for the numerical computation of any special case.

PART II.

41. *Numerical Constants.*—Prof. LOVE's investigation was undertaken in order to throw light on a vexed question of internal ballistics, namely, how the mass of the propellant should be taken into account in calculating the velocity and pressure in a gun. Its completion has been delayed not only by the analytical complexity of the problem, but also by the time required for the numerical computations. In his original paper LAGRANGE set out from a certain state of the gas assumed as a first approximation, namely, one in which the velocity, at a given epoch, changed uniformly from one end of the gas to the other. Restricting attention to the case of a very heavy gun, the total momentum of gas and projectile is then $(M + \frac{1}{2}C)V$ and the total kinetic energy $\frac{1}{2}(M + \frac{1}{2}C)V^2$, where V is the velocity of the projectile, M its mass, and C that of the propelling charge. LAGRANGE recognized that this state of motion is dynamically possible only in the limiting case of small charges, but made no real progress towards the theory for finite charges, the development of analysis being then inadequate to the problem. Since the ratio C/M in modern guns, though less than with gunpowder, is still of the order $\frac{1}{4}$, the importance of a full numerical discussion of LAGRANGE's problem is evident. The calculations which follow were begun by Prof. LOVE, who determined all the fixed coefficients and the position and velocity of the projectile at various epochs. After verifying these figures I undertook the calculation of the distribution of pressure in the gas, at the times when a new type of wave was either being generated or extinguished, and at the half intervals. Instantaneous combustion is assumed, as it appears hopeless to attempt to allow for the gradual burning of the propellant which occurs in actual guns.

It is assumed that the propellant is cordite M.D., for which the maximum pressure for different densities of the gas, after explosion in a closed vessel, has been measured by NOBLE.* The results at medium pressure are represented approximately by the formula

$$p_0 \left(\frac{1}{\rho_0} - 1 \right) = 9500,$$

giving the pressure p_0 in kilograms per square centimetre when ρ_0 is in absolute measure. This is the formula used in calculating initial pressures. The subsequent expansion of the gas is adiabatic, and will be represented by an equation of the form

$$p \left(\frac{1}{\rho} - 1 \right)^{\gamma} = \text{const.}$$

* Sir A. NOBLE, 'Phil. Trans.,' A, vol. 205, p. 201, 1906.

It appears probable, for various reasons, that the mean adiabatic index γ is in the neighbourhood of 1.2. As we are restricted to a special set of values the value $11/9 = 1.22$ is selected.

The problem discussed in detail is that of a gun of 15 cm. calibre, mass of projectile 50 kg., charge of propellant 12 kg., distance travelled by the projectile from its initial position of rest to the muzzle 6 metres, initial volume of gas behind the projectile (chamber capacity) 30 litres. It is not, of course, possible with instantaneous combustion to keep the maximum pressure the same as it would be in a gun, though the muzzle velocity is much the same. The maximum pressure in this case is 6333 kg./cm.². Had the pressure been kept down to 3000 kg./cm.² by taking a smaller charge, the problem would have been less representative as regards muzzle velocity, and as regards the ratio of the masses of propellant and projectile.

In order to exhibit both the pressure in the gun and the degree in which the back particles partake of the motion of the projectile, eleven planes are taken at equal distances apart in the undisturbed gas, the end planes coinciding with the breech and the base of the projectile respectively. The horizontal line at the top of Plate 1 shows their initial positions. These eleven planes of particles are traced throughout their motion. The particles originally half-way between the breech and the base of the projectile may be called the middle particles,* and we shall choose, as epochs for the curves of pressure (Plate 1), the times at which a "junction" is either at the breech or the base of the projectile, or has just reached the middle particles. A junction is marked with a black circle on the figure.

42. *Details of the Calculation* (Plate 1, curve 1) (Article 10).—The early stages of the calculation call for no comment. We have $\sigma_0 = 960,536.7$ cm./sec., $\rho_0 = 0.4$, $p_0 = 9500\rho_0/(1-\rho_0) = 6333.3$ kg./cm.², $c = 339.5305$ cm. (the initial distance from the breech to the base of the projectile is $\frac{1}{2}c = 169.76525$ cm.), $a = 177,877.1$ cm./sec., $h = 778.0909$ cm. The progressive wave which starts out from the base of the projectile reaches the middle particles ($x_0 = \frac{1}{4}c$) at time $t = 0.0004772$ sec. Particles between there and the breech are still at rest: from these particles to the base of the projectile the velocity of the gas increases almost uniformly to the value 99.6 m./sec., and the pressure falls to 5651.3 kg./cm.². The fall of pressure is remarkable considering that the projectile has only moved a distance of 2.4 cm. from its initial position; and we observe a finite discontinuity in the pressure gradient on the two sides of the junction.

(Plate 1, curve 2).—The progressive wave reaches the breech at time $t = 0.0009544$ sec., when the projectile has moved a distance of 9.28 cm. from its seat and has a velocity of 187.7 m./sec. The pressure falls from 6333.3 kg./cm.² at the breech to 5097.2 kg./cm.² at the base of the projectile.

(Plate 1, curve 3) (Articles 12, 16–17).—The first middle wave begins at the epoch just mentioned, by reflexion of the progressive wave at the breech. To find when it

* These particles must be distinguished from those of the "middle section" of the theory, which here correspond to the breech of the gun.

reaches the middle particles, *i.e.*, when the progressive wave has receded to $x_0 = \frac{1}{2}c$, we solve the equation $(x_0 + h)(ct + h) = (h + \frac{1}{2}c)^2$, giving $t = 0.0014785$ sec. The velocity of the projectile at this time is 275.4 m./sec., its displacement 21.4 cm. The pressure falls from 5151.6 kg./cm.² at the junction to 4598.7 kg./cm.² behind the projectile. In the first middle wave trial and error begins. At the breech $u = 0$ and σ_0/σ is found by trial to give the correct value of t . For intermediate points we have theoretically to find both u and σ by trial to make x_0 and t correct. Actually the smallness of u allows us to neglect powers of u/σ above the second, so that the pressure follows an approximately parabolic law. The difference of pressure in the first middle wave is quite small. At the breech we have 5170.9 kg./cm.², an increase of only 19.3 kg./cm.² over that at the junction, as against a drop of 552.9 kg./cm.² from the junction to the projectile.

(Plate 1, curve 4).—The first middle wave reaches the projectile at time $t = T_1 = 0.0021170$ sec.; when the displacement of the projectile is $-X_1 = 42.191$ cm. and its velocity $-U_1 = 37175.64$ cm./sec. = 371.8 m./sec. The remaining constants at this epoch are $R_1 = 443,092.1$, $S_1 = 480,268.8$, $\Sigma_1 = 923,360.9$. The pressure falls slightly from 4169.1 kg./cm.² at the breech to 4102.5 kg./cm.² at the base of the projectile.

(Plate 1, curve 5) (Articles 18–25).—The first reflected wave begins at $t = T_1$. For the constants we find

$$\begin{aligned} \log(\Lambda_0/\Sigma_1^{10}) &= 6.99416, & \log(\Lambda_1/\Sigma_1^9) &= \bar{9}.98722, & \log(-2! \Lambda_2/\Sigma_1^8) &= \bar{7}.66552, \\ \log(3! \Lambda_3/\Sigma_1^7) &= 5.20603, & \log(-4! \Lambda_4/\Sigma_1^6) &= \bar{4}.53510, & \log(5! \Lambda_5/\Sigma_1^5) &= \bar{3}.35986, \\ \log(6! \Lambda_6/\Sigma_1^4) &= 2.69991, & \log(7! \Lambda_7/\Sigma_1^3) &= \bar{1}.49726, & \log(8! \Lambda_8/\Sigma_1^2) &= 0.02177, \\ \log(9! \Lambda_9/\Sigma_1) &= 0.29583, \\ \log B_1 &= 0.33341, & \log B_2 &= 0.84347, & \log B_3 &= 1.66011, & \log B_4 &= 2.49429, \\ \log B_5 &= 3.33303, & \log B_6 &= 4.18096. \\ K_1 &= -670.58, & \log(-L_1) &= 3.64091. \end{aligned}$$

To find when the first reflected wave reaches the middle particles, we know that $r = R_1$ along a junction with the first middle wave, and s is found by trial, from the formulæ of the first middle wave, to give $x_0 = \frac{1}{2}c$. Knowing r and s , t is known: we find $t = 0.002898$ sec. The part of the first middle wave which still remains is treated as before. The pressure falls from 3316.0 kg./cm.² at the breech to 3304.3 kg./cm.² at the junction. A long process is required to find the pressure in the first reflected wave. Writing $\phi = (r - R_1)/\Sigma_1$ and $\delta = (s - S_1)/\Sigma_1$, at the base of the projectile $\phi = B_1\delta + B_2\delta^2 + \dots + B_n\delta^n$ is a known function of δ . We expand the formulæ of Article 20 to give $t + h/a$, x_0 and $(Z - K_1 - L_1u)/\Sigma_1$ explicitly in terms of ϕ and δ , and try different values of δ until t has its required value 0.002898. Then the pressure at

the base of the projectile is known, and also its velocity and final position. For other points ϕ and δ have to be found to make both x_0 and t correct : the adjustment is facilitated by the fact that uniform division of x_0 corresponds approximately to uniform division of ϕ . At the junction ($x_0 = \frac{1}{4}c$) $\phi = 0$, and at the base of the projectile ($x_0 = 0$) $\phi = -0.019578$. Taking four values of ϕ equally spaced between these, and finding δ to give the correct t , we have four points which correspond nearly to 10, 20, 30 and 40 per cent. division of the initial gas, and are easily adjusted to exact value by interpolation. The pressure falls from 3304.3 kg./cm.² at the junction to 2970.3 kg./cm.² at the base of the projectile. The projectile is displaced 75.4 cm. from its seat, and has velocity 466.2 m./sec.

(Plate 1, curve 6).--The first reflected wave reaches the breech at time $t = 0.003859$ sec., where the pressure is 2610.5 kg./cm.² Other points are found as in the last paragraph. The pressure at the base of the projectile is 2161.6 kg./cm.², the displacement of the projectile 124.3 cm., and its velocity 550.4 m./sec.

(Plate 1, curve 7) (Articles 30-31).--The second middle wave begins at the above epoch $t = 0.003859$, pushing back the first reflected wave along a junction $s = R_1$. This junction reaches the middle particles at time $t = 0.005154$. In the part of the first reflected wave that still remains the pressure falls from 1708.2 kg./cm.² at the junction to 1535.2 kg./cm.² behind the projectile. The displacement of the projectile is 202.1 cm. and its velocity 632.5 m./sec. The second middle wave differs from the first reflected wave by the presence of four additional terms with coefficients given by

$$\begin{aligned} \log \{ -\eta_6 (\Sigma_2/2R_1)^6 \} &= \bar{3}.04397, & \log \{ \eta_7 (\Sigma_2/2R_1)^7 \} &= 3.52835, \\ \log \{ -\eta_8 (\Sigma_2/2R_1)^8 \} &= 3.60452, & \log \{ \eta_9 (\Sigma_2/2R_1)^9 \} &= \bar{3}.34841, \end{aligned}$$

where $\Sigma_2 = 814,358.3$ cm./sec.; also $k_1 = 466.85$. At the breech $u = 0$ or $\phi - \delta = (S_1 - R_1)/\Sigma_1$, leaving ϕ to be found by trial. The pressure at the breech is 1728.0 kg./cm.². For intermediate points we take a number of values of ϕ , find δ by trial to give the correct t , then calculate x_0 and interpolate.

(Plate 1, curve 8).--The second middle wave reaches the base of the projectile where $s = S_2 = R_1$, $r = R_2 = 371,266.2$, giving $\sigma = \Sigma_2$, $t = T_2 = 0.007137$ sec. This point is found without trial. The pressure at the base of the projectile is 1030.2 kg./cm.², its displacement $-X_2 = 335.6$ cm., and its velocity $-U_2 = 71,827$ cm./sec. = 718.3 m./sec. The value of Z is $Z_2 = -177.0$. At the breech we have a pressure of 1085.7 kg./cm.². Other points are calculated as in the last paragraph. We have

$$\begin{aligned} \frac{F_1(2R_2)}{\Sigma_2^{10}} &= 0.00034563, & \frac{F_1^{(1)}(2R_2)}{\Sigma_2^9} &= 0.0000015953, & \frac{F_1^{(2)}(2R_2)}{\Sigma_2^8} &= -0.000020614, \\ \frac{F_1^{(3)}(2R_2)}{\Sigma_2^7} &= 0.00017489, & \frac{F_1^{(4)}(2R_2)}{\Sigma_2^6} &= -0.0005648, & \frac{F_1^{(5)}(2R_2)}{\Sigma_2^5} &= -0.004338. \end{aligned}$$

(Plate 1, curve 9) (Articles 32-40).—The second reflected wave begins at the base of the projectile at time T_2 and pushes back the second middle wave along a junction $r = R_2$. Thus ϕ is known, and the value of δ corresponding to an assigned x_0 is found by trial. We find that the junction reaches the middle particles at time $t = 0.01023$ sec. The pressure at the breech is 650.0 kg./cm.², at the junction 641.0 kg./cm.². For the constants of the second reflected wave we have

$$\begin{aligned} \log B'_1 &= 0.18668, & \log B'_2 &= 0.27390, & \log B'_3 &= 0.82262, & \log B'_4 &= 1.40104, \\ \log B'_5 &= 1.99012, & \log (-c_1) &= 1.08789, & \log c_2 &= 0.10720, & \log (-c_3) &= 1.00167, \\ \log c_4 &= 1.81519, & \log (-c_5) &= 2.56985, & \log \xi_1 &= 3.94418, & \log (-\xi_2) &= 5.07447, \\ \log \xi_3 &= 6.01358, & \log (-\xi_4) &= 6.85566, & \log \xi_5 &= 7.62931. \end{aligned}$$

The method of calculation of the pressures in the second reflected wave has been described in Article 40. The pressure at the base of the projectile is 581.6 kg./cm.², where the displacement is 571.9 cm. and the velocity 801.3 m./sec.

The projectile is so near the muzzle at time $t = 0.01023$ that a fresh chart for the muzzle epoch (displacement 600 cm.) is unnecessary. We find for the time to the muzzle $t = 0.01058$ sec., for the muzzle velocity 807.7 m./sec., and for the pressure at the base of the projectile at this instant 552.6 kg./cm.².

43. *Results.*—The pressure results are collected in Table I., from which Plate 1 is constructed. Plate 2 shows the pressures at the breech and at the base of the projectile, their ratio, the mean pressure, the displacement and velocity of the projectile, and a certain "energy factor" as functions of the time. The mean pressure (P in Table I.) is that which the cordite gases would have after adiabatic expansion, at *uniform* density, to the volume which they actually occupy at time t . The work of expansion in these circumstances will be equal, not to the kinetic energy of the projectile, but to a greater kinetic energy corresponding to a fictitious mass $M + \alpha C$, where

$$\frac{1}{2} (M + \alpha C) V^2 = \frac{3}{2} C p_0 \left(\frac{1}{\rho_0} - 1 \right) \left[1 - \left(\frac{P}{p_0} \right)^{\frac{1}{\gamma_0}} \right].$$

The "energy factor" α may be expected to vary with the distance travelled by the projectile: the lower values given are only approximate.

It is difficult, after a glance at Plate 2, to resist the conclusion that the motion is tending to a limiting form, in which the pressure is approximately represented by $f(y_0) \phi(t)$, with suitable functions f, ϕ . The energy factor α oscillates about a mean value of approximately $1/3$, and the range of oscillation diminishes in time: similarly the pressure ratio oscillates about a value of approximately 0.9 . Moreover, the latter value, like the former, can be obtained from LAGRANGE'S approximation by suitable treatment.* If p' is the pressure at the breech and p that at the base of the projectile,

* F. Gossot and R. Liouville, 'Mémorial des Poudres et Salpêtres,' vol. 13, p. 51, 1905.

TABLE I.

y_0	(1)		(2)		(3)		(4)		(5)		(6)		(7)		(8)		(9)	
	y	p	y	p	y	p	y	p	y	p	y	p	y	p	y	p	y	p
0 (breach) ...	0	6333	0	6333	0	5171	0	4169	0	3316	0	2610	0	1728	0	1086	0	650.0
16.98 ...	16.98	6333	17.06	6208	18.81	5170	21.20	4168	24.30	3316	28.08	2568	36.11	1727	49.93	1085	71.84	649.7
33.95 ...	33.95	6333	34.41	6074	37.62	5168	42.38	4166	48.26	3314	56.38	2532	72.24	1725	99.89	1083	143.8	648.6
50.93 ...	50.93	6333	51.69	5958	56.45	5164	65.60	4163	72.39	3312	84.84	2491	108.4	1721	150.1	1080	216.0	646.8
67.91 ...	67.91	6333	69.28	5836	75.28	5159	84.78	4158	96.52	3309	113.6	2448	144.5	1715	200.2	1076	288.7	644.3
84.88 ...	84.88	6333	87.06	5712	94.12	5152	106.0	4152	120.6	3304	142.8	2404	180.6	1708	250.5	1071	361.8	641.0
101.9 ...	101.9	6196	105.0	5589	113.1	5040	127.2	4145	144.4	3241	172.4	2358	218.1	1676	301.0	1065	436.0	632.1
118.8 ...	119.2	6059	123.3	5465	132.3	4929	148.4	4136	169.0	3174	202.3	2310	255.9	1643	351.7	1058	511.2	620.1
135.8 ...	136.6	5923	141.6	5342	151.7	4818	169.6	4126	193.0	3109	232.6	2262	293.7	1609	402.6	1050	587.0	607.4
152.8 ...	154.3	5787	160.2	5220	171.3	4707	190.7	4115	218.4	3041	263.2	2212	332.0	1574	453.9	1041	663.9	594.9
169.8 (proj) ...	172.2	5651	179.0	5097	191.2	4599	212.0	4102	245.2	2970	294.1	2162	371.9	1535	505.4	1030	741.7	581.6

t = time from beginning of motion in seconds.
 P = pressure in kg cm.² of cordite gas filling the space behind the projectile with uniform density.
 V = velocity of projectile in m. sec.
 z = coefficient necessary to make $\frac{1}{2}(M+zC)V^2$ equal to work of uniform adiabatic expansion.
 y_0 = initial distance of a plane of particles from the breech in cm.
 y = distance of same particles at time t .
 p = pressure in kg. cm.²
 Black letters represent junctions.

p'/p is the ratio of the momenta of gun and projectile, that is $(M + \frac{1}{2}C)/M$, so that the pressure ratio is approximately $\frac{1}{1 + C/2M} = 0.893$. The agreement is to be expected; for Table I. shows how little, relatively, y/y_0 varies with y_0 , so that LAGRANGE'S approximation leads to little error in the total energy and momentum.

44. *Calculation of Recoil.*—Prof. LOVE'S theory also enables us to calculate the distance recoiled by a very heavy gun while the projectile is travelling to the muzzle: this is important since the distance can also be found experimentally. We take from Plate 2 the values of p' and p at intervals of 0.0005 sec. to the muzzle, and calculate $\int p' dt$ and $\int p dt$ by approximate integration. These quantities are proportional to $M'V'$ and MV , where M' , M are the masses of gun and projectile and V' , V their velocities. A second integration gives $M'S'$ and MS , where S' and S are the distances travelled by gun and projectile. For the muzzle epoch we find, in the present problem, $M'S'/MS = 1/0.879$. The recoil distance S' of the gun is therefore the same as for a massless propellant and a projectile of mass $M/0.879 = 56.9$ kg., an addition of 0.57 times the mass of the propellant to that of the projectile. LAGRANGE'S approximation gives 0.5. CRANZ* measured the recoil distance of a rifle, with comparatively slow combustion of the propellant, and obtained factors 0.496, 0.497, 0.477, mean 0.493. The theory of limiting motion would seem to apply with almost equal force to the case of slow combustion; and thus we may regard CRANZ'S experiment as confirming the recoil factor $\frac{1}{2}$ and therefore (indirectly) the energy factor $1/3$. Prof. LOVE has worked out the energy factor for a light projectile of mass 25 kg., and 12 kg. propellant, at epochs corresponding to (4) and (8) in Table I. The values are 0.335 and 0.333.

45. *A Special Solution of the Hydrodynamical Equations.*—Prof. LOVE'S theory having suggested the possibility of the motion tending to a limiting form, it remains to show that the hydrodynamical equations admit of a particular solution in which the pressure is of the form $f(y_0) \phi(t)$. We shall see that the pressure ratio and energy factor corresponding to this exact solution agree closely with those already calculated, and thus support is lent to the view that the limiting motion would be developed sooner or later with other initial conditions, *e.g.*, with gradual introduction of gas from a burning propellant. If y_0 is, as above, the initial distance of a particle from the breech and y its distance at time t , the general hydrodynamical equation is

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = \rho_0 \left(\frac{1}{\rho_0} - 1 \right)^\gamma \gamma \left(\frac{1}{\rho_0} \frac{\partial y}{\partial t} - 1 \right)^{-\gamma-1} \frac{1}{\rho_0} \frac{\partial^2 y}{\partial y_0^2}.$$

Write temporarily $x = y_0$, $z = y - \rho_0 y_0$. Then

$$\frac{\partial^2 z}{\partial t^2} = \frac{\gamma \rho_0}{\rho_0} (1 - \rho_0)^\gamma \frac{\partial^2 z}{\partial x^2} \left(\frac{\partial z}{\partial x} \right)^{-\gamma-1}.$$

* C. CRANZ, 'Zeitschr. f. d. ges. Schiess-u. Sprengstoffwesen,' vol. 2, p. 345, 1907.

A solution of the type $z = f(x) \phi(t)$ is possible if and only if

$$f''(x) = A f(x) \{f'(x)\}^{\gamma+1}, \quad \{\phi(t)\}^\gamma \phi''(t) = B,$$

where A and B are constants connected by the equation

$$B = \frac{\gamma D_0}{\rho_0} (1 - \rho_0)^\gamma A.$$

If S is the area of the cross-section, the equation of motion of the projectile, which is supposed to be at $x = b$, is $pS = Mf(b) \phi''(t)$. Now in general

$$p = \rho_0 (1 - \rho_0)^\gamma \{f'(x) \phi(t)\}^{-\gamma}.$$

Hence the equation of motion of the projectile is satisfied if

$$MBf(b) = Sp_0 (1 - \rho_0)^\gamma \{f'(b)\}^{-\gamma},$$

or

$$A = \frac{\epsilon}{\gamma b f'(b) \{f'(b)\}^\gamma},$$

where $\epsilon = C/M = Sb\rho_0/M$ is the ratio of the mass of the propellant to that of the projectile. Writing $w = f(x)$ and $q = dw/dx$, the first integral of the differential equation for w is

$$q^{\gamma-1} = \frac{2}{A(\gamma-1)} \frac{1}{a^2 - w^2},$$

where a is a constant. Since $f(x)$ vanishes with x , the final integral is

$$\int_0^w (a^2 - w^2)^{\frac{1}{\gamma-1}} dw = \left\{ \frac{2}{A(\gamma-1)} \right\}^{\frac{1}{\gamma-1}} x.$$

Writing $c = f(b)$ for the length of the column of gas at the instant considered, we have therefore

$$\int_0^c (a^2 - w^2)^{\frac{1}{\gamma-1}} dw = \left\{ \frac{2}{A(\gamma-1)} \right\}^{\frac{1}{\gamma-1}} b, \quad \{f'(b)\}^{\gamma-1} = \frac{2}{A(\gamma-1)} \frac{1}{a^2 - c^2}.$$

Substituting for A we have

$$f'(b) = \frac{(\gamma-1)(a^2 - c^2)\epsilon}{2\gamma bc}, \quad \frac{2}{A(\gamma-1)} = \left\{ \frac{(\gamma-1)\epsilon}{2\gamma bc} \right\}^{\gamma-1} (a^2 - c^2)^\gamma,$$

so that

$$\int_0^w (a^2 - w^2)^{\frac{1}{\gamma-1}} dw = \frac{(\gamma-1)\epsilon}{2\gamma c} (a^2 - c^2)^{\frac{\gamma}{\gamma-1}}.$$

This equation determines c/a , and when it is known w is given by

$$\int_0^w (a^2 - w^2)^{\frac{1}{\gamma-1}} dw = \frac{(\gamma-1)\epsilon}{2\gamma c} (a^2 - c^2)^{\frac{\gamma}{\gamma-1}} \frac{x}{b}.$$

The pressure ratio between the two ends of the gas is $R = \{f'(0)/f'(b)\}^\gamma$, where $\{f'(b)/f'(0)\}^{\gamma-1} = a^2/(a^2-c^2)$. Hence

$$R = \left(\frac{a^2-c^2}{a^2}\right)^{\frac{\gamma}{\gamma-1}}.$$

Writing $c = a \sin \theta$ we find $R = \cos^m \theta$, where $m = 2\gamma/(\gamma-1)$ and θ is found by trial from the equation

$$\frac{\sin \theta}{\cos^m \theta} \int_0^\theta \cos^{m-1} \theta d\theta = \frac{\epsilon}{m}.$$

In the case of $\gamma = 11/9$, $m = 11$, we find, after some analytical reduction, the expansion

$$R = 1 - \frac{1}{2}\epsilon + \frac{7}{64}\epsilon^2 - \frac{149}{1280}\epsilon^3 + \dots,$$

valid for small values of ϵ . Either method gives $R = 0.894$ for $\epsilon = 12/50$, the corresponding value of θ being $8^\circ 9' 6''$. It will be noticed that although R is not equal to $(1 + \frac{1}{2}\epsilon)^{-1}$ to the second order, the approximation is still a remarkably good one. The present theory will appear more satisfactory, as it is based on an exact solution valid for all values of ϵ .

As regards the energy factor, the previous definition in terms of the work done from an initial state of uniform density is not convenient, as this state is not one of the previous states of the gas. We may, however, define the energy factor in such a way that the kinetic energy of the gas is $\alpha\epsilon$ times that of the projectile.*

Corresponding to the initial distance x from the breach we have in general $w = a \sin \phi$, where

$$\int_0^\phi \cos^{m-1} \phi d\phi = K \frac{x}{b},$$

and

$$K = \int_0^\theta \cos^{m-1} \theta d\theta = \frac{\epsilon \cos^m \theta}{m \sin \theta}.$$

The corresponding velocity is $V \sin \phi / \sin \theta$. If $x + dx$ corresponds to $\phi + d\phi$, $K dx/b = \cos^{m-1} \phi d\phi$. The kinetic energy of the gas is

$$\frac{1}{2} \frac{CV^2}{K \sin^2 \theta} \int_0^\theta \cos^{m-1} \phi \sin^2 \phi d\phi,$$

and that of the projectile $\frac{1}{2}MV^2$. Hence by definition

$$\alpha = \frac{m}{\epsilon \cos^m \theta \sin \theta} \int_0^\theta \cos^{m-1} \phi \sin^2 \phi d\phi,$$

* This was not done above because the problem would naturally present itself in the other form in practical calculations, where we should seek a factor which will make the kinetic energy of the projectile equal to the work of an assumed massless propellant.

where θ is given by the equation already written down. Using the reduction formulæ we find

$$\alpha = \frac{\epsilon - m \sin^2 \theta}{(m+1) \epsilon \sin^2 \theta},$$

giving $\alpha = 0.325$ when $m = 11$ and $\epsilon = 12/50$. The expansion formula, to the first power in ϵ , is

$$\alpha = \frac{1}{3} - \frac{1}{18} \epsilon.$$

46. *Application to Ballistics.*—To resume, Prof. LOVE's theory supports the factors $\frac{1}{2}$ and $\frac{1}{3}$ up to considerable values of C/M , and shows further that the ratio of the pressures on projectile and breech (Plate 2) begins at once to oscillate about its mean value, reaching its first minimum when the projectile has travelled a distance of only two-thirds of a calibre. We may remark that no support is lent to the theory which appears to be favoured by CHARBONNIER* of more or less violent impulses of pressure on the base of the projectile: the discontinuity is at most one of pressure gradient, which becomes less and less as the motion proceeds. What would happen with gradual introduction of gas from a burning propellant is more conjectural, but nevertheless it seems of interest to examine the consequences of the assumption that the limiting state of motion, contemplated above, is developed almost at once, and maintained ever after. The considerations which we shall advance have no pretence to rigour.†

It is usual to measure maximum pressures in guns by crusher gauges placed at or near the breech. Let P be the pressure at the breech, $P(1 - C/2M)$ that at the base of the projectile at any time, powers of C/M above the first being neglected. Compare the actual motion with that for a massless gas of the same thermodynamical properties, and a projectile of mass m . Then for identical motion of the two projectiles, with $m/M = 1 + C/3M$,

$$\frac{p}{P(1 - C/2M)} = \frac{m}{M},$$

or $p/P = 1 - C/6M$. In order to keep up the parallelism of motion we have to ensure that equal quantities of propellant are burnt in equal times. The rate of regression of the surface of colloidal propellants at different pressures has been measured by VIEILLE in a famous research.‡ MANSELL, who examined cordite M.D. by VIEILLE's method,§ found a rate of regression in a closed vessel approximately proportional to the pressure. If D and d are the diameters of cordite in the two cases (or more generally numbers proportional to the linear dimensions of the grain), equal generation of gas corresponds approximately to the condition

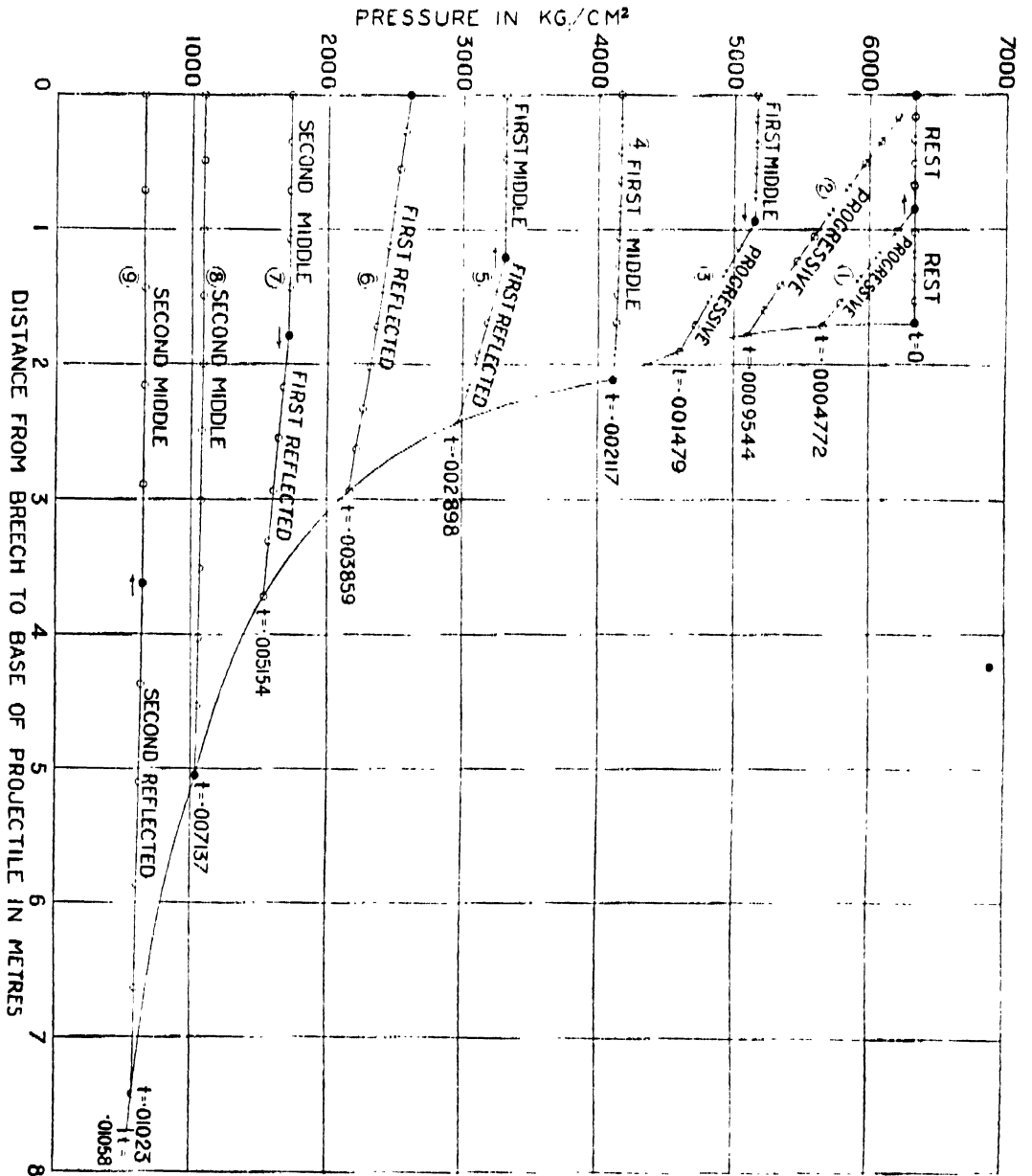
$$\frac{d}{D} = \frac{p}{P(1 - C/4M)},$$

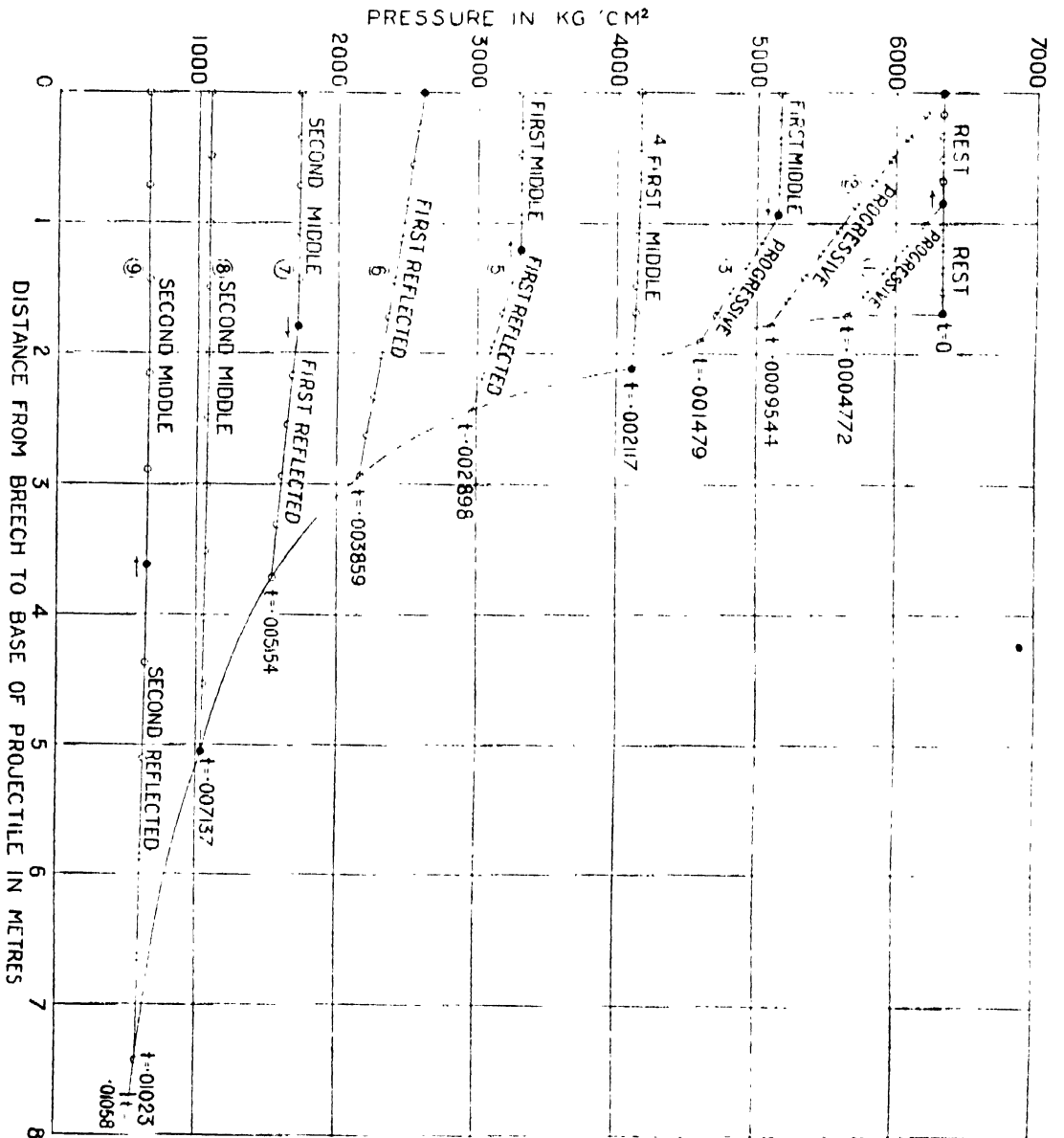
* P. CHARBONNIER, 'Traité de Balistique Intérieure,' Paris, O. Doin, p. 91.

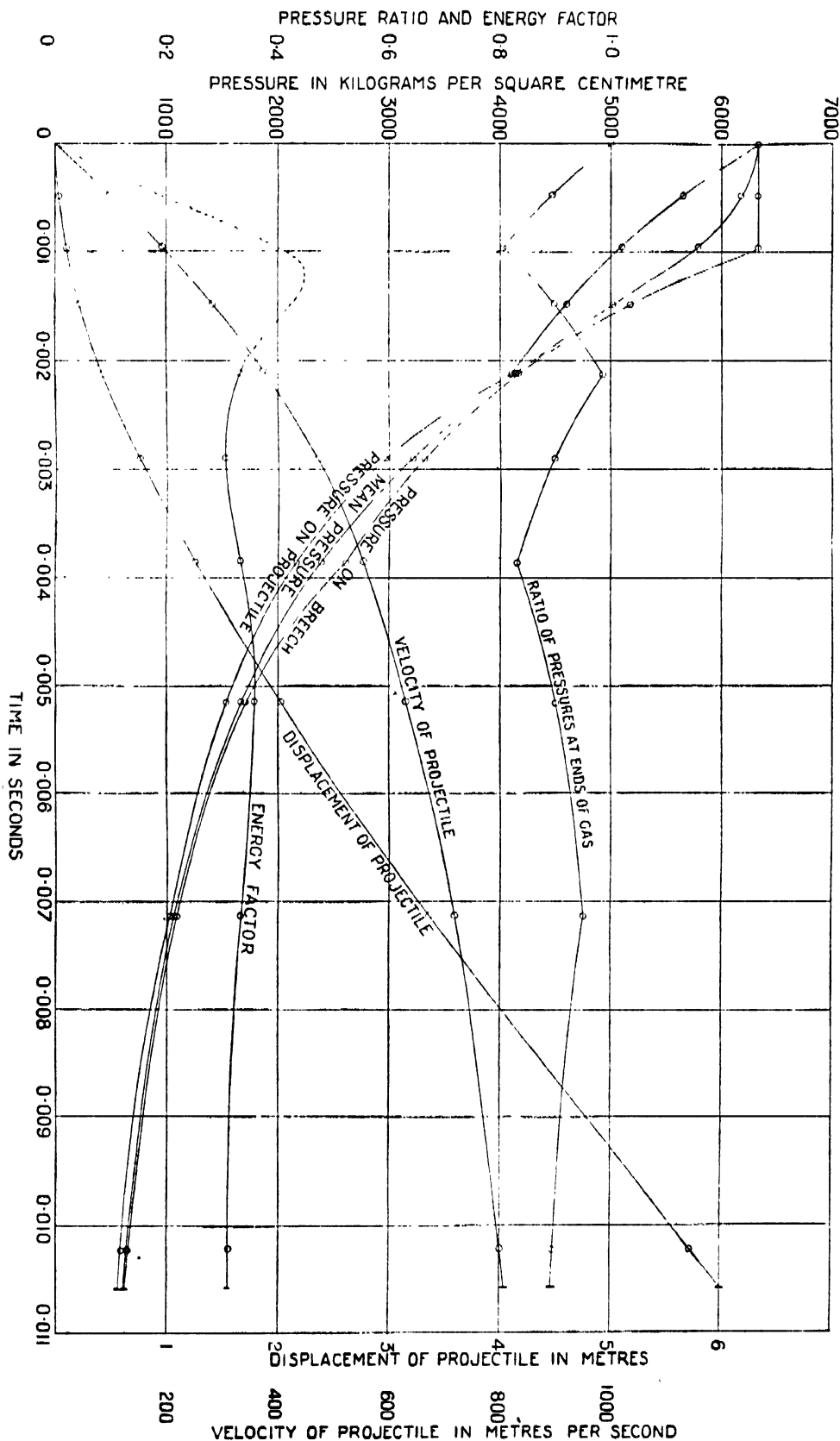
† See also F. GOSSOT and R. LIOUVILLE, *loc. cit.*, pp. 50-52; vol. 17, pp. 61-66, 1914.

‡ P. VIEILLE, 'Mémorial des Poudres et Salpêtres,' vol. 6, p. 256, 1893.

§ J. H. MANSELL, 'Phil. Trans.,' A, vol. 207, p. 243, 1908.







VII. *The Aerodynamics of a Spinning Shell.—Part II.*

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§ 1. *Introduction.*

IN a previous paper* the authors, with others, have described observations of the angular oscillations of the axis of a 3-inch shell over the first 600 feet from the muzzle of the gun, and from an analysis of the observations have obtained information about the forces due to the air. In the experiments, shells were fired from two guns giving different degrees of axial spin to the shell. While the shells fired from the gun giving the more rapid spin were all stable, most of the shells from the other gun were slightly unstable, this condition being shown by the much larger maximum yaw† developed. These unstable rounds were not analysed in (A) as no suitable method of doing so had then been devised.‡ The analysis of these rounds, about one-third of the number fired, forms the subject of the present paper: the results confirm those of (A) and provide some additional information.

The information as to the force system obtained from the stable rounds' was confined to yaws up to 7 degrees or perhaps 10 degrees; by analysis of the unstable rounds this information is extended, though in a fragmentary manner, over the region up to 35 degrees of yaw. On the other hand, no information has been derived from the observed damping of the unstable rounds. The observations are, in respect of the damping, clearly in qualitative agreement with the theory and results of (A), but no method has been devised of making a quantitative analysis of the damping.

The force system on a model shell was also determined at low velocity in the wind channels of the National Physical Laboratory. The results are shown in fig. 2 of (A)

* "The Aerodynamics of a Spinning Shell," *Phil. Trans.*, A, vol. 221, p. 295 (1920). This paper will be cited as (A). The experiments analysed here and in (A) were carried out for the Ordnance Committee, and the results are published with their sanction.

† The "yaw" is the angle between the axis of the shell and the direction of motion of its centre of gravity.

‡ As will appear later, the ordinary solution in elliptic functions of the equations of motion of a top is not adequate for this purpose in the case of large yaws.

and have been used in both papers for extending the results down to low velocities, as in figs. 1 and 2, here.

Shells of four different types, I.-IV. were used. Types I.-III. were of the same external shape (form A), with three different positions of the centre of gravity. Type IV. was of a different external shape (form B). The details are given in (A).*

The experimental data, which have already been discussed in (A), consist of the mass, principal moments of inertia, and position of the centre of gravity for each type of shell; rough values of the forward velocity over the whole range of 600 feet from the muzzle of the gun; the spin of the shell, deduced from the rifling of the bore; the yaw and orientation of the shell's axis at a number of points along the range, deduced from the shape and orientation of the holes punched by the shell in cardboard targets. These cards were set up at intervals of about 60 feet for all the unstable rounds, and it appears from figs. 12 of (A) and figs. 3 and 4 of this paper that they were close enough together for satisfactory curves to be drawn through the observed points representing the variation of the yaw δ and its azimuth ϕ over the whole range.

§ 2. *The Equations of Motion.*

It is convenient to recapitulate the notation of (A). Suppose that OA denotes the direction of the axis, OP the direction of motion of the centre of gravity of the shell; then $\text{AOP} = \delta$, and ϕ is the angle that the plane AOP makes with a fixed plane through OP. $M (= \mu \sin \delta)$ is the couple in the plane AOP which tends to increase δ , A, B and N are the principal moments of inertia and the axial spin of the shell, and we write $\Omega = \text{AN/B}$. Then the equations of motion will be taken in the form†

$$\delta'' + \phi'^2 \sin^2 \delta + \int \frac{2\mu}{B} d \cos \delta = E. \dots \dots \dots (1)$$

$$\phi' \sin^2 \delta + \Omega \cos \delta = F, \dots \dots \dots (2)$$

where E and F are constants. The various assumptions underlying equations (1) and (2) are discussed in detail in (A). If μ is constant the equations are, of course, of the same form as the ordinary integrals of energy and angular momentum for a spinning top, and the complete solution in elliptic functions is standard.

When M is an arbitrary (odd) function of δ the top solution no longer applies, but a solution in elliptic functions is still possible if M has the form $X \sin \delta \{1 - Y (1 - \cos \delta)\}$, where X and Y are constants. This more general form allows the first two terms in the expansion of an arbitrary M to be catered for and can represent M adequately

* *Loc. cit.*, p. 316 and fig. 6. See also fig. 1, here.

† (A), *loc. cit.*, p. 334, equations 3.404, 3.405. For the underlying assumptions see (A) Part I., pp. 301 *sqq.*, 311 *sqq.* These equations are, strictly speaking, not referred to fixed axes, but are approximate equations referred to axes changing direction with OP. Dashes denote differentiations with respect to the time t .

over a wider range of values of δ . By suitably adjusting X and Y, which define the couple, and the initial conditions, a curve showing the variation of δ with the time can be obtained which agrees closely with observations over a complete half-period, so that the above expression for M appears to be adequate up to values of δ of 35 degrees.* Our original approximation with $Y = 0$ fails in general when $\delta > 10$ degrees. The observed curves suffice to determine X and Y for each round, and as observations were taken for a number of different values of the muzzle velocity, M is determined by the experiments over a limited range, as a function of the two variables, v , the velocity of the shell, and δ .

In solving the equations of motion it is convenient to express the couple by means of the non-dimensional coefficients s and q defined by the equation

$$M = \frac{B\Omega^2 \sin \delta}{4s} \{1 - 4qs(1 - \cos \delta)\}. \quad (3)$$

It will appear that the motion with δ permanently zero is stable or unstable according as $s > 1$ or $s < 1$. For the rounds here analysed, s lies between 1.06 and 0.83.

In expressing the results in a standard form it is convenient to use a different non-dimensional coefficient f_M , which is independent of the mass, moments of inertia, size and velocity of the shell, and depends only on the shape of the shell and the non-dimensional variables v/a and δ , where a is the velocity of sound. This is defined by the equation†

$$M = \rho v^2 r^3 \sin \delta f_M(v/a, \delta), \quad (4)$$

where ρ is the air-density, r the radius of the shell, and the quantities involved are expressed in consistent units.

According to (3), f_M is practically constant so long as $\delta < 7$ degrees, and the value of $f_M(v/a, 0)$ is strictly comparable with similar values obtained in (A) by analysis of the stable rounds on the assumption that f_M is independent of δ .

§ 3. Final Results of the Experiment.

Fig. 1 shows curves of $f_M(v/a, 0)$ as a function of v/a for the four types of shell corrected for the effect of the cards‡. They are reproduced without alteration from figs. 4 and 5 of (A) and represent the results for the stable rounds.§ The values derived from the present analysis of the unstable rounds are plotted for comparison ;

* When the yaw exceeds 30 degrees the fit is less satisfactory (e.g., III., 11-13).

† *Loc. cit.*, p. 302, equation 1.103.

‡ § 10 below.

§ The curve for type II. is not actually given, but the data for drawing it can be found in (A) (fig. 13, p. 352).

they show remarkably good agreement between the new results and the old. This confirms the substantial truth of the whole theory; in particular, the agreement of the results for the two twists of rifling verifies that the couple M is unaffected by a change in axial spin in the ratio 3 to 4.

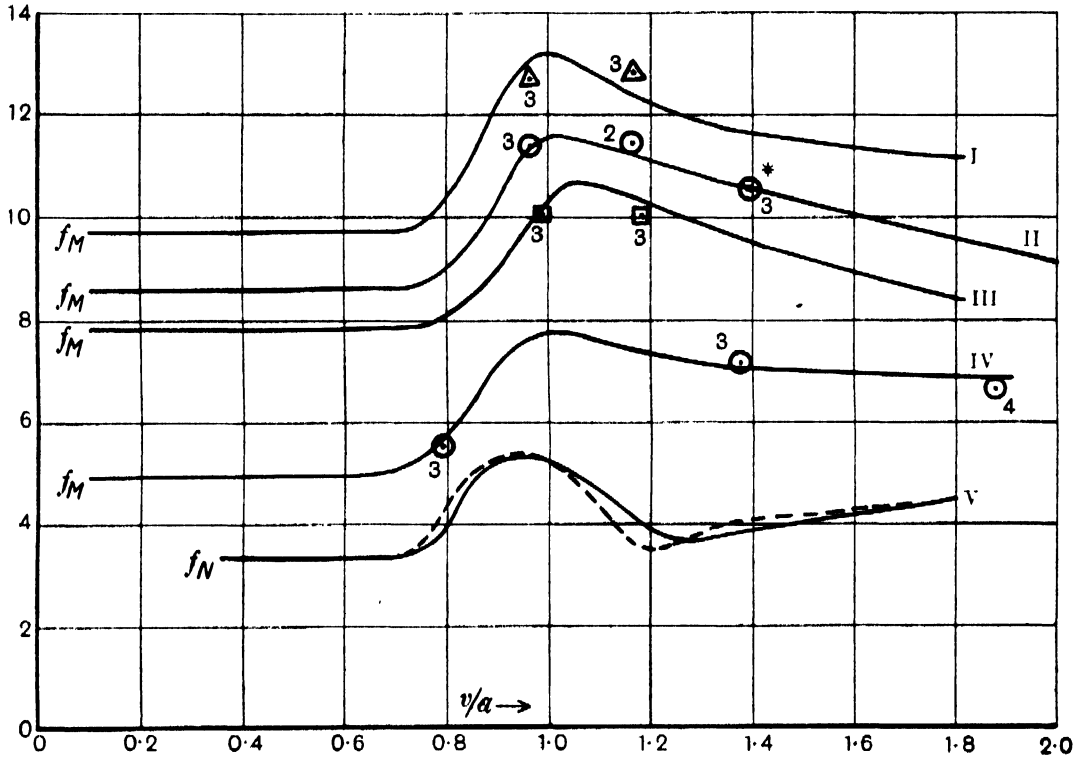


Fig. 1. Values of the couple coefficient $f_M(v/a, 0)$ and the normal force coefficient $f_N(v/a, 0)$ plotted against v/a , corrected for the effect of the cards. The full curves for f_M for types I.-IV. are reproduced without alteration from (A). The plotted points show the means of observations here analysed for the first and second half-periods, and the numbers against them show the number of rounds contributing to each mean. The group marked * was analysed in (A) by a different method with identical results. The origin of f_M for type IV. is displaced downwards 4 units.

The full curve (V) for f_N (shells of form A) represents the complete results of the experiment. The dotted curve reproduces the partial results of (A).

- Curve I.— f_M for 3-inch shells of form A with centre of gravity 4.20 inches from the base. (Type III.)
- Curve II.—The same with centre of gravity 4.73 inches from the base. (Type I.)
- Curve III.—The same with centre of gravity 5.08 inches from the base. (Type II.)
- Curve IV.—The same for form B with centre of gravity 4.965 inches from the base. (Type IV.)

Curves of $f_M \sin \delta$, the complete moment coefficient, considered as a function of both variables v/a and δ corrected for the effect of the cards, are plotted against δ in fig. 2. The information is somewhat fragmentary: in addition to wind channel results, values of f_M , when $\delta > 10$ degrees, are available for shells of type I.-III. for two velocities near $v = a$. For type IV. (pointed shells) values of f_M for large δ are wanting in this region, but exist for two high velocities, and one less than a .

Some general information can be deduced from these curves. At high velocities ($v/a = 2.0$) and large values of δ the couple coefficient actually falls below its low

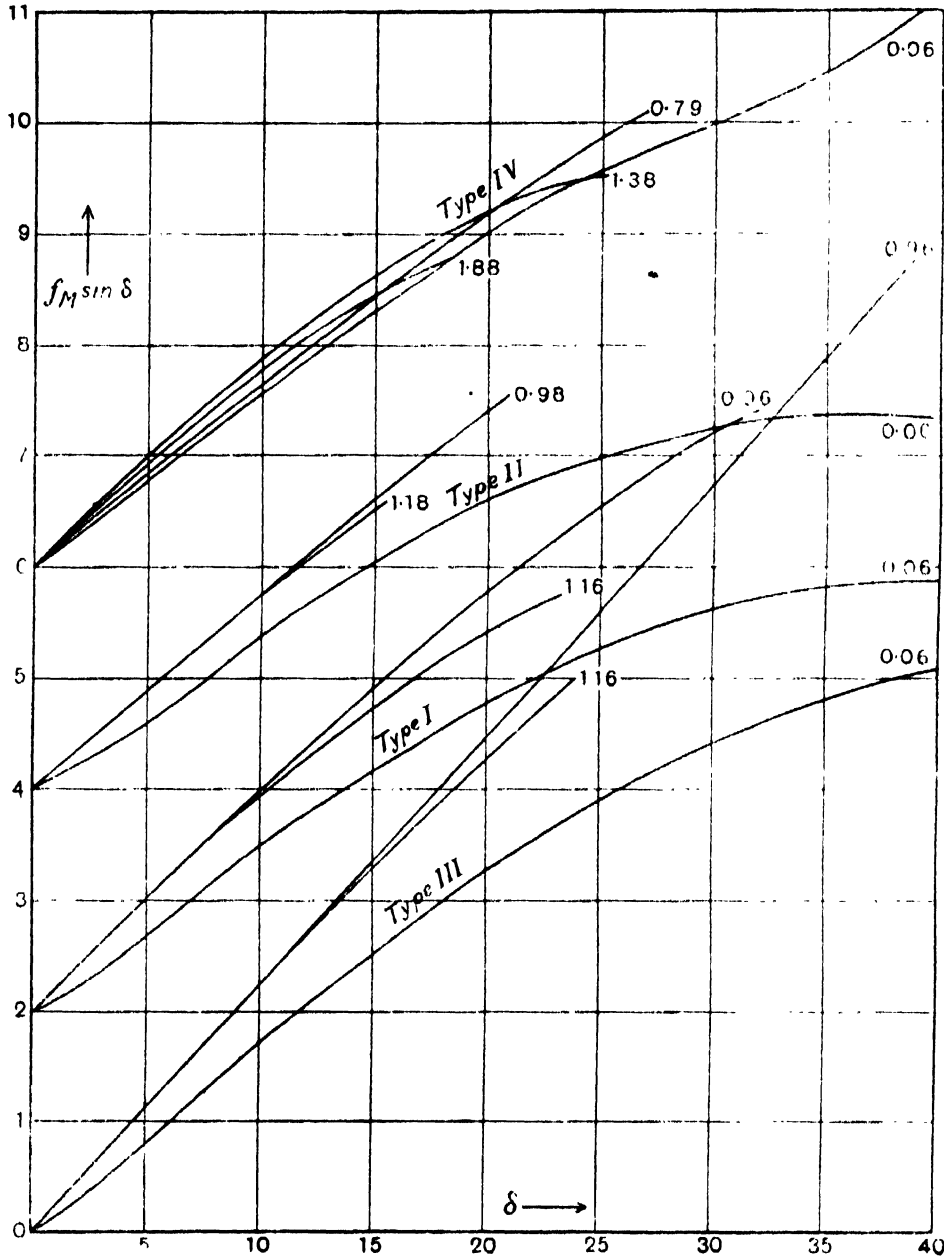


Fig. 2. Curves of the complete couple coefficient $f_M(v/a, \delta) \sin \delta$ against δ for various values of v/a (as shown against the curves). The curves stop at the greatest value of δ for which observations are available.

The origins of all the curves except for type III, are displaced upwards and can be recovered from the fact that all curves pass through their respective origins. The scale of δ is in degrees.

velocity (wind channel) value (fig. 2); the curve of the couple coefficient against yaw has here a large curvature downwards. It is, on the other hand, almost straight in the region of the velocity of sound.

We may notice also the peculiar behaviour of rounds III. (11-13), fig. 3,* in which the motion seems to become quite irregular from round to round after the first maximum yaw is attained. As these are the only rounds in which a yaw of 35 degrees or more is developed, it seems likely that some peculiar change in the type of the airflow occurs at or about this angle, analogous to the phenomenon of "burbling."†

Rounds IV. (10-12) have the smallest velocity of any group fired; the agreement of f_M for these rounds with wind channel results for the whole range of δ , figs. 1 and 2, is very satisfactory.

The new values of $f_M(v/a, 0)$ can be used to correct slightly the mean curves of f_M , and from the modified curves the value of f_N , the coefficient of the force normal to the shell's axis at small yaw, may be re-determined by the method used in (A), § 1.13. The re-determined curve for f_N is shown in fig. 1. The values differ only slightly from the former values, and the main features of the f_N curve are fully confirmed.

The final results of the complete experiment, stable and unstable rounds alike, corrected for the effect of the cards, have been combined together to give the final values of $f_M(v/a, 0)$ for shells of types I. and IV. and $f_N(v/a, 0)$ for type I. shown in Table I., which replaces the corresponding Table I. of (A).

TABLE I.—Final values of $f_M(v/a, 0)$ and $f_N(v/a, 0)$ for shells of type I. and $f_M(v/a, 0)$ for shells of type IV., embodying the results of the whole experiment. The effect of the cards has been corrected for as far as possible, and this table supersedes Table I. of (A).

v/a .	Shells of type I.		Type IV.
	$f_M(v/a, 0)$.	$f_N(v/a, 0)$.	$f_M(v/a, 0)$.
Wind channel	8.57	3.34	8.95
0.7	8.6	3.35	9.0
0.8	9.05	4.0	9.7
0.9	10.35	5.2	11.1
1.0	11.55	5.25	11.75
1.1	11.55	4.7	11.6
1.2	11.25	3.9	11.4
1.3	10.9	3.7	11.25
1.4	10.55	3.85	11.1
1.5	10.3	4.0	11.0
1.6	10.05	4.15	10.95
1.7	9.85	4.3	10.9
1.8	9.65	4.5	10.8
1.9	9.4	—	10.75
2.0	9.15	—	—

* Also (A), p. 349, fig. 12B.

† This term is commonly applied to the sudden increase of turbulence behind an aerofoil at the critical angle.

§ 4. *The Solution of the Equations of Motion.*

We shall now solve the equations of motion (1) and (2), assuming that M is of form (3). We take first the case of rosette motion, in which zero values of δ can occur, so that we may assume the initial conditions

$$\delta = 0, \quad \delta' = b\Omega.$$

Eliminating ϕ' and writing $\sin \frac{1}{2}\delta = y$ we get

$$4y'^2 = \Omega^2 \{b^2(1-y^2) - y^2 + y^2(1-y^2)(1/s - 4qy^2)\} \dots \dots \dots (5)$$

The right-hand side is a cubic in y^2 whose roots are such that it may be written in the form

$$4y'^2 = 4q\Omega^2 (h^2 + y^2)(\alpha^2 - y^2)(f^2 - y^2), \quad (1/f^2 < 1 < 1/\alpha^2) \dots \dots \dots (6)$$

Formulae connecting* h , α and f with b , q and s may be obtained most conveniently by putting $y^2 = 1$, $y^2 = 0$, and by differentiating with respect to y^2 and putting $y^2 = 0$. The resulting formulae are

$$4q(1+h^2)(1-\alpha^2)(f^2-1) = 1, \dots \dots \dots (7)$$

$$4q\alpha^2 h^2 f^2 = b^2, \dots \dots \dots (8)$$

$$4q(\alpha^2 f^2 - \alpha^2 h^2 - h^2 f^2) = -b^2 + 1/s - 1. \dots \dots \dots (9)$$

A solution of (6) is obtained by assuming, in the usual notation of Jacobian elliptic functions,

$$y^2 = \frac{\alpha^2 \operatorname{cn}^2 u}{1 - g^2 \operatorname{sn}^2 u}, \dots \dots \dots (10)$$

where the constant g and the modulus k of the elliptic functions remain to be determined. If we solve (10) for $\operatorname{cn}^2 u$ and differentiate, we get

$$-\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u u' = \frac{\alpha^2(1-g^2)yy'}{(\alpha^2 - g^2 y^2)^2},$$

leading to

$$y'^2 = \frac{u'^2}{\alpha^4(1-g^2)} (\alpha^2 - g^2 y^2)(\alpha^2 - y^2) \{ \alpha^2(1-k^2) + (k^2 - g^2)y^2 \}.$$

Comparing this with (6) we may write $u' = \pm \lambda\Omega$, λ constant, and obtain

$$f^2 = \alpha^2/g^2, \dots \dots \dots (11)$$

$$h^2 = \frac{\alpha^2(1-k^2)}{k^2 - g^2}, \dots \dots \dots (12)$$

$$q = \frac{\lambda^2 g^2 (k^2 - g^2)}{\alpha^4 (1 - g^2)} \dots \dots \dots (13)$$

* This α has, of course, no connection with the velocity of sound.

Since $y = 0$ when $t = 0$, we may write $u = K - \lambda \Omega t$, where K is the complete elliptic integral of the first kind to modulus k . Treating Ωt as independent variable, the final form of our solution contains four constants a, λ, k and g , which must be completely determinable in terms of s, q and b , so that there must be an independent relation between a, λ, k and g by which any one can be found when the other three are known. To determine them it would be necessary to solve (5), the original cubic in y^2 . To analyse the experiments, however, we have to solve the inverse problem of determining s, q and b when a, λ, k and g are known. In practice g is small so that, as a first approximation, we may use the following simplified form of (10):—

$$y = a \operatorname{cn} (K - \lambda \Omega t), \dots \dots \dots (14)$$

where a, k and λ may be treated as independent. By fitting a curve of this type to the curve of observed values of y ($\sin \frac{1}{2} \delta$) against Ωt , we can determine the constants a, λ and k . It is at once clear that $a = \sin \frac{1}{2} \alpha$, where α is the maximum yaw, but we shall continue to call this constant a for shortness. The value of g can then be obtained from the identical relation in terms of a, k, λ , and the curve re-calculated by formula (10) if g is large enough to make it worth while to do so. After that the values of a, λ and k could be re-adjusted and the process repeated. Theoretically, we could presumably arrive at the precise solution in this manner by a limiting process. Practically, in nearly every case, the first approximation with $g = 0$ is all that is required.

The values of s, q and b are given by simple formulæ in terms of a, λ, k and g . From (8) and (9) we get

$$1/s - 1 = 4q (a^2 h^2 f^2 + a^2 f^2 - a^2 h^2 - h^2 f^2);$$

on substituting for q, h^2 and f^2 from (11)–(13), this reduces to

$$1/s - 1 = 4\lambda^2 \left\{ -\cos 2\kappa + \frac{\cos^2 \kappa (a^2 - 3g^2)}{1 - g^2} \right\} \dots \dots \dots (15)$$

where $k = \sin \kappa$.* In practice either a or $\cos \kappa$, or both, are small and g is of the same order as a ; the second term inside the bracket may then be neglected in determining s , in which case the value of g is not required. This, as we shall see, is really a consequence of the smallness of b , its mean value in practice being about 0.015. For q write equation (7) in the form

$$4q + \frac{1}{1 - a^2} = 4q (h^2 f^2 - h^2 + f^2),$$

and substitute for q, h^2 and f^2 in the right-hand side, getting

$$4q = -\frac{1}{1 - a^2} + \frac{4\lambda^2}{a^2} \left\{ \sin^2 \kappa + \frac{\cos^2 \kappa (a^2 - 2g^2)}{1 - g^2} \right\} \dots \dots \dots (16)$$

* This κ will not be confused with the κ of (A), *loc. cit.*, p. 328, which is the damping coefficient depending on the cross-wind force.

Finally equation (8) gives

$$b^2 = \frac{4\lambda^2 a^2 \cos^2 \kappa}{1-g^2} \dots \dots \dots (17)$$

The equation for g is obtained by substituting for q , h^2 and f^2 in (7), which becomes

$$4\lambda^2(1-\alpha^2)(\alpha^2-g^2)\{\alpha^2-g^2+(1-\alpha^2)k^2\}-\alpha^4(1-g^2)=0: \dots \dots (18)$$

this is a quadratic for g^2 , whose solution may be written

$$g^2 = \sin^2 \frac{1}{2}\alpha \left(1 - \frac{\tan \frac{1}{2}\theta}{2\lambda}\right), \quad (\sin \frac{1}{2}\alpha = \alpha), \dots \dots \dots (19)$$

where θ is given by

$$\cot \theta = \lambda k^2 \cot^2 \frac{1}{2}\alpha - (\tan^2 \frac{1}{2}\alpha)/4\lambda. \dots \dots \dots (20)$$

The ambiguity is settled in practice by the fact that g^2 must be small if equation (14) is to be taken as a first approximation to the solution. In practice, as we have said, b is small. Valuable information as to the nature of the solution is, therefore, obtainable by considering its limit as $b \rightarrow 0$. This gives us a guide as to the actual relative order of all terms.

Let us suppose then that $b \rightarrow 0$, that s and q are definite constants, and let us assume that g is of the same order as a , which by (19) and (20) must be the case unless $\lambda \rightarrow 0$. Then equation (17) shows that $\lambda^2 a^2 \cos^2 \kappa / (1-g^2) \rightarrow 0$, and, therefore, in the limit,

$$1/s-1 = -4\lambda^2 \cos 2\kappa. \dots \dots \dots (21)$$

Equation (21) shows that, if $s \neq 1$, $\lambda \rightarrow 0$ is impossible. Hence in all cases ($s \neq 1$) our assumption as to g is justified and $a \cos \kappa \rightarrow 0$. This also justifies our previous statement concerning (15). There are now two cases according as $s < 1$ or $s > 1$.

Case (i).— $s < 1$. We are supposing that $s-1$ is fixed, so that as $b \rightarrow 0$, $s-1$ is large compared to b . To satisfy the signs of (21) we must have $\kappa > 45$ degrees. This implies $\sin^2 \kappa > \frac{1}{2}$, so that (16) becomes in the limit

$$4q = -\frac{1}{1-\alpha^2} + \frac{4\lambda^2 \sin^2 \kappa}{\alpha^2}.$$

It follows that $a \rightarrow 0$ is impossible, and therefore $\cos \kappa \rightarrow 0$, $\kappa \rightarrow 90$ degrees, and a and λ tend to definite non-zero limits. The limiting forms for $1/s$, q and g^2 are easily found to be

$$1/s-1 = 4\lambda^2, \dots \dots \dots (22)$$

$$4q = -\sec^2 \frac{1}{2}\alpha + 4\lambda^2 \operatorname{cosec}^2 \frac{1}{2}\alpha, \dots \dots \dots (23)$$

$$g^2 = \sin^2 \frac{1}{2}\alpha \{1 - (\tan^2 \frac{1}{2}\alpha)/4\lambda^2\}, \dots \dots \dots (24)$$

in which we have replaced a by $\sin \frac{1}{2}\alpha$. These formulæ are good approximations in practice, when s is not too close to unity. The half-period ΩT^* ($= K/\lambda$) tends to infinity, but so slowly that no difficulty occurs in practice. Since α , the maximum yaw, does not tend to zero with b , the initial disturbance, we have here what may be called *the unstable case*.

Case (ii).— $s > 1$. We now must have $\kappa < 45$ degrees and, therefore, as $\cos^2 \kappa > \frac{1}{2}$, $a \rightarrow 0$ at the same rate as b ; this is *the stable case* in the usual sense. Equation (16) shows further that $\kappa \rightarrow 0$, and, therefore, by (15), λ has a definite limit determined by

$$1 - 1/s = 4\lambda^2.$$

For given q equations (16)–(20) determine the limiting ratios of $b : \sin \frac{1}{2}\alpha : k : g$.

The case in which $s = 1$ and $b \rightarrow 0$ can be treated in a similar way. It is found that $\kappa \rightarrow 45$ degrees and a and λ both tend to zero like \sqrt{b} .

§ 5. *Rounds with a Non-Zero Minimum Yaw.*

We shall only consider cases in which the minimum yaw β is small, and shall take as initial conditions

$$\delta = \beta, \quad \delta' = 0, \quad \phi' \sin \delta = b_1 \Omega.$$

The equations of motion become

$$\phi' \sin^2 \delta - \Omega b_1 \sin \beta + \Omega (\cos \beta - \cos \delta) = 0, \quad \dots \dots \dots (25)$$

$$\delta'^2 + \phi'^2 \sin^2 \delta - \Omega^2 b_1^2 + \int_{\beta}^{\delta} \frac{2\mu}{B} d \cos \delta = 0. \quad \dots \dots \dots (26)$$

If we write $y_1^2 = \sin^2 \frac{1}{2}\delta - \sin^2 \frac{1}{2}\beta$, y_1 vanishes initially and the equation for y_1 may be written

$$4y_1'^2 = \Omega^2 \{ b_1^2 (\cos \beta - y_1^2) - y_1^2 - b_1 \sin \beta + (\sin^2 \frac{1}{2}\beta + y_1^2) (\cos^2 \frac{1}{2}\beta - y_1^2) [1/s - 4q (\sin^2 \frac{1}{2}\beta + y_1^2)] \}. \quad \dots \dots (27)$$

We identify (27) with the equation

$$4y_1'^2 = 4q\Omega^2 (h_1^2 + y_1^2) (\alpha_1^2 - y_1^2) (f_1^2 - y_1^2), \quad \dots \dots \dots (28)$$

in which $\alpha_1^2 = \sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\beta$. Equations (10)–(13) retain their form, and (7)–(9) become

$$(b_1 \sin \frac{1}{2}\beta + \cos \frac{1}{2}\beta)^2 = 4q (h_1^2 + \cos^2 \frac{1}{2}\beta) (f_1^2 - \cos^2 \frac{1}{2}\beta) \cos^2 \frac{1}{2}\alpha, \quad \dots \dots (29)$$

$$b^2 = (b_1 \cos \frac{1}{2}\beta - \sin \frac{1}{2}\beta)^2 = 4q (h_1^2 - \sin^2 \frac{1}{2}\beta) (f_1^2 + \sin^2 \frac{1}{2}\beta) \sin^2 \frac{1}{2}\alpha, \quad \dots \dots (30)$$

$$-b_1^2 + (1/s) \cos \beta - 1 - 4q \sin^2 \frac{1}{2}\beta (2 \cos^2 \frac{1}{2}\beta - \sin^2 \frac{1}{2}\beta) = 4q (\alpha_1^2 f_1^2 - h_1^2 \alpha_1^2 - h_1^2 f_1^2). \quad \dots (31)$$

* T is the time interval between a zero or minimum and an adjacent maximum of the yaw.

The solution is

$$y_1^2 = \frac{\alpha_1^2 \operatorname{cn}^2 u}{1 - g^2 \operatorname{sn}^2 u}, \quad (u = K - \lambda \Omega t),$$

or

$$\sin^2 \frac{1}{2} \delta - \sin^2 \frac{1}{2} \beta = \frac{(\sin^2 \frac{1}{2} \alpha - \sin^2 \frac{1}{2} \beta) \operatorname{cn}^2 u}{1 - g^2 \operatorname{sn}^2 u} \dots \dots \dots (32)$$

The values of s and g in terms of λ , $\sin \frac{1}{2} \alpha$, k and g will differ from the values they had before by terms of the order $\sin^2 \frac{1}{2} \beta$ or $b \sin \frac{1}{2} \beta$ which are negligible in practice compared to α_1^2 . Hence the previous solution may be applied provided that (32) replaces (10) in calculations of the curve of $\sin \frac{1}{2} \delta$ as a function of Ωt . For convenience in computing, (32) may be put in the form (neglecting g^2)

$$\sin \frac{1}{2} \delta = \frac{\sin \frac{1}{2} \alpha \cos \chi}{\sin \theta} = \frac{\sin \frac{1}{2} \beta \sin \chi}{\cos \theta}, \dots \dots \dots (33)$$

where

$$\tan \theta = \sin \frac{1}{2} \alpha \cot \chi / \sin \frac{1}{2} \beta, \quad \cos \chi = \operatorname{cn} u.$$

Formula (17) remains a valid approximation for b^2 provided b^2 is defined by (30).

§ 6. *A Discussion of the Probable Effects of Damping, and Other Factors Omitted in the Foregoing Solution.*

Up to this point we have assumed that the motion in yaw is exactly periodic with half-period ΩT . This would be exactly true if the couple M were a function of δ only, OP a fixed straight line, and no other couples existed. In actual fact, M is a function of the forward velocity and therefore of the time; OP changes direction under the influence of gravity and the cross-wind force, and other couples besides M act on the shell, depending on the angular velocity of the shell. All these factors cause progressive changes in the curve of yaw from period to period; for the case of the stable rounds they have been discussed at length in (A), where it is shown that they do not appreciably affect the determination of M at any velocity for small values of δ . In particular, the effect of gravity is almost entirely allowed for by using (as we do) the true yaw and not the angle between the axis of the shell and some *fixed* straight line. As explained in (A)* the shape of the hole in the cards determines the true yaw and not the angle between the axis of the shell and the normal to the card.

There is no reason to expect that any of these damping effects will be relatively more important for an unstable than for a stable shell, except for the change of M with the velocity. Although the change in velocity over a single period is always small, yet when s is less than or nearly equal to unity a small change in M will cause a fairly

* *Loc. cit.*, p. 318, footnote.

large change in the type of motion. This is shown clearly in fig. 3, round III. 16, where the decrease in amplitude and the change from an unstable to a stable type is quite marked, but still not sufficient to introduce any error in the determination of the couple for a single half-period. The effect is illustrated by the change in s in successive half-periods (Table III.), which is in general in the direction, and roughly of the amount, required by theory.

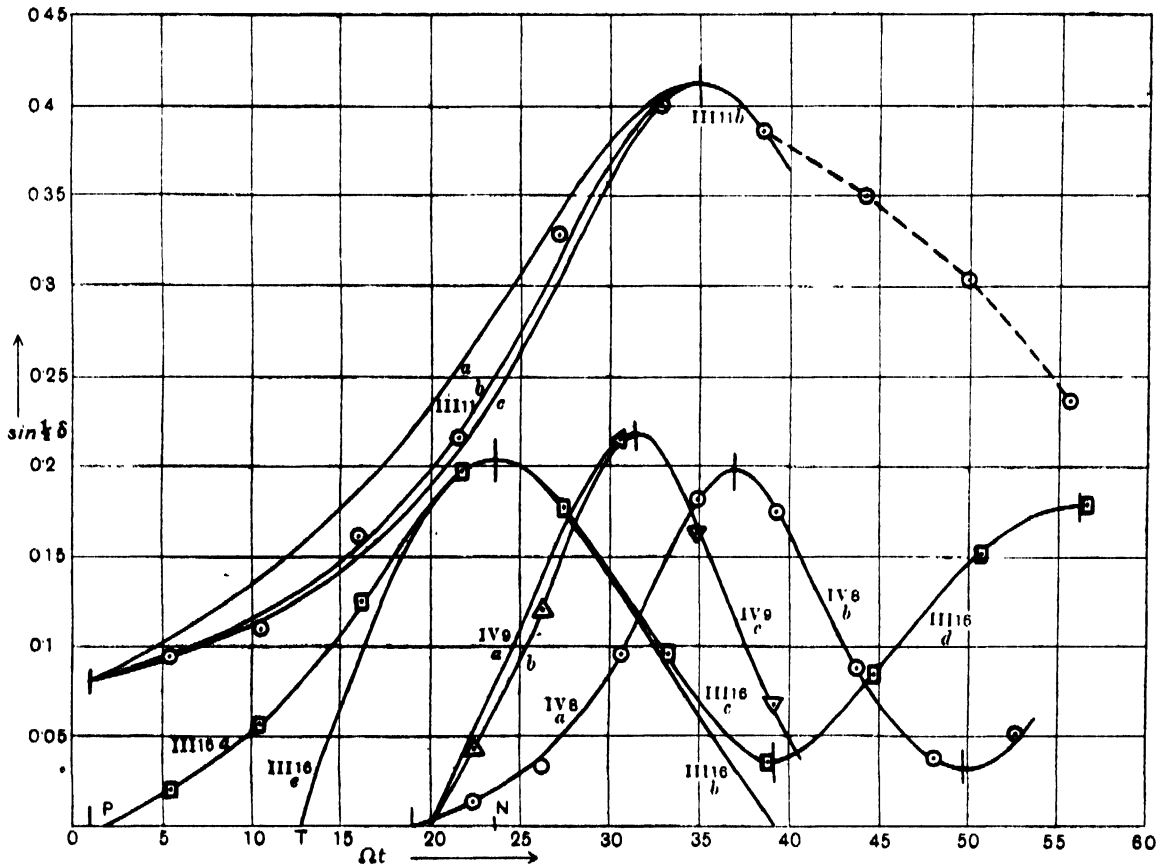


Fig. 3. The observed and calculated motion in yaw, compared for selected rounds. The plotted points show the observed values of $\sin \frac{1}{2}\delta$ plotted against Ωt for rounds III. (11, 16) and IV. (8, 9). The continuous curves are the result of calculations described in detail in §§ 6, 8. Short vertical lines mark the positions of maxima and minima, and the origins of co-ordinates. The values of the constants used are as follows:—

III. 11. (a) $\kappa = 80$ degrees; (b) $\kappa = 85$ degrees, $g^2 = 0$; (c) $\kappa = 85$ degrees, $g^2 = -0.185$.

III. 16. (a) $\kappa = 80$ degrees; (b) $\kappa = 60$ degrees, $\beta = 0$; (c) $\kappa = 60$ degrees, $\sin \frac{1}{2}\beta = 0.037$; (d) $\kappa = 40$ degrees, $\sin \frac{1}{2}\beta = 0.037$; (e) $\kappa = 0$, with third order contact with (a) at maximum.

IV. 8. (a) $\kappa = 85$ degrees; (b) $\kappa = 75$ degrees, $\sin \frac{1}{2}\beta = 0.034$.

IV. 9. (a) $\kappa = 60$ degrees; (b, c) $\kappa = 70$ degrees.

It appears that a change in M with v cannot alter an initial rosette motion into one with non-zero minimum yaw. This alteration, as in the stable case, must be due to the other couples depending on the angular velocity of the axis, and to the sideways

motion of the centre of gravity, which function as damping forces as explained in (A).^{*} No means of dealing with these effects theoretically has yet been devised for the case of large yaw ; the observed changes are clearly of the general type which one's experience of the stable case would lead one to predict.

§ 7. *The Motion in ϕ .*

It is not difficult to write down a formal expression for the motion in ϕ . If we take equation (25) and substitute for $\sin \frac{1}{2}\delta$ from equation (32) we obtain after reduction

$$\begin{aligned} \phi' &= \frac{\Omega}{1 + \cos \delta} + \frac{b\Omega \sin \frac{1}{2}\beta}{(1 + \cos \delta) y^2}, \dots \dots \dots (33)' \\ &= \frac{\Omega}{1 + \cos \delta} + \frac{b\Omega \sin \frac{1}{2}\beta}{1 + \cos \delta} \frac{1 - g^2 \sin^2 u}{(1 - g^2) \sin^2 \frac{1}{2}\beta \sin^2 u + \sin^2 \frac{1}{2}\alpha \cos^2 u}, \end{aligned}$$

where b is defined by (30). Now even when α is as big as 30 degrees there is still only a maximum difference of 7 per cent. between $1 + \cos \delta$ and 2, and this maximum is only effective for a short part of each period. Hence, for almost all purposes, we are still justified in replacing $1 + \cos \delta$ by 2†. In order to integrate this equation we notice that $\lambda\Omega dt = -du$ and that $t = 0$ or $u = K$ corresponds to the minimum. Thus,

$$\phi = \phi_0 + \frac{1}{2}\Omega t + \frac{b \sin \frac{1}{2}\beta}{2\lambda} \int_u^K \frac{(1 - g^2 \sin^2 u) du}{(1 - g^2) \sin^2 \frac{1}{2}\beta \sin^2 u + \sin^2 \frac{1}{2}\alpha \cos^2 u} \dots \dots \dots (34)$$

This equation contains an elliptic integral of the third kind which can be evaluated in θ -functions. We have not, however, made this evaluation or calculated any actual ϕ -curves from (34) mainly because it does not appear that any further information as to the forces acting on the shell would be obtained thereby. We shall content ourselves in this paper with a statement of sufficient theoretical results to show that the observed ϕ -curves are qualitatively of the form to be expected from (34). A more detailed discussion of these curves, however, would, we think, be of some interest.

It is convenient to treat the motion by using the variable $\psi = \phi - \phi_0 - \frac{1}{2}\Omega t$. When β is zero, ϕ' will be constant and equal to $\frac{1}{2}\Omega$ to our present approximation ; with y and ψ as polar co-ordinates the motion then consists of an oscillation in a straight line through the origin, for $\psi' = 0$. In the general case we may eliminate dt between equations (33)' and (28), and on substituting for b from (30) obtain an equation for $dy/d\psi$ in the form‡

$$\left(\frac{dy}{d\psi}\right)^2 = y^2 \left(\frac{y^2}{\sin^2 \frac{1}{2}\beta} - 1\right) \left(1 - \frac{y^2}{\sin^2 \frac{1}{2}\alpha}\right) \left(1 + \frac{y^2}{h_1^2 - \sin^2 \frac{1}{2}\beta}\right) \left(1 - \frac{y^2}{f_1^2 + \sin^2 \frac{1}{2}\beta}\right), \dots \dots \dots (35)$$

If we assume that h_1^2 and f_1^2 are large compared to $\sin^2 \frac{1}{2}\alpha$, the last two brackets

* In particular see p. 313.

† It is easy to estimate the precise effect of this approximation in the simple case of the rosette motion. The error caused is always very small.

‡ In deducing (35) we replace $1 + \cos \delta$ by 2.

in (35) are effectively unity, and the equation reduces to that of an ellipse in polar co-ordinates with axes $\sin \frac{1}{2}\alpha$, $\sin \frac{1}{2}\beta$. This tends to a straight line as the limiting form when $\beta \rightarrow 0$.

The same (simplified) relation between δ and ψ was obtained in (A)* for the case of small yaw only. The elliptic motion may be calculated most easily by recognising that ψ is the auxiliary angle θ of equation (33) so that

$$\phi = \phi_0 + \frac{1}{2}\Omega t + \theta.$$

The conditions under which this approximation is valid may be seen by reference to equations (11) and (12) which are satisfied by f_1^2 , h_1^2 and a_1^2 . We notice that f_1^2 is never small compared to unity and tends to infinity as g , and therefore g tends to zero; while h_1^2 may be comparable with a_1^2 unless k is small. Thus the approximation is really only valid for $s > 1$, in which case it applies even if β is not small compared to α . Finally, if $1 - k^2$ is small, equation (12) shows that h_1^2 will be small compared to a_1^2 , and the third bracket in (35) will be more important than the second near a minimum of y ; this indicates that the shape of the (y, ψ) curve there approximates to an hyperbola instead of to an elongated ellipse; the curve may also no longer be re-entrant, the total change of ψ in one period differing from $\pm\pi$.

Examples of all these results may be seen in fig. 4, in which the observations for three different rounds are plotted with (y, ψ) as polar co-ordinates. For round I. (5), for which β is small and the shell just stable, we find the expected elongated ellipse-like curve, with a slowly-developing minor axis caused by the damping factors. In III. (16) $\kappa = 80$ degrees for the first half-period, diminishing to 40 degrees for the third; a considerable minimum yaw develops and the curve is less like an ellipse, though the maxima are still nearly 180 degrees apart. Finally, in IV. (8) $\kappa = 85$ degrees, falling to 75 degrees, and the shape of the curve near the minimum clearly resembles an hyperbola: we may guess then the angle between the two maxima is somewhere about 100 degrees instead of 180 degrees.

Lastly, a word must be said about the observational determination of Ω , which is, of course, theoretically determined by the muzzle velocity, twist of rifling and moments of inertia of the shell. In all cases the slope of the ϕ -curve over the first half-period, or rather more, is uniform and well determined. Since β appears to be really very small initially one may expect from theory the slope of this part of the curve to be $\frac{1}{2}\Omega$ whatever the value of κ . The agreement between this observed slope and the calculated value of Ω is satisfactory.

§ 8. *The Method of Analysis.*

The method of analysis of the $\sin \frac{1}{2}\delta$ -curves will now be explained with reference to fig. 3. After the observed values of $\sin \frac{1}{2}\delta$ have been plotted against Ωt , the values

* *Loc. cit.*, p. 346, equation 4.06.

of Ωt corresponding to the ends of half-periods can be determined from the symmetry of the curve, with the exception of the start of the first half-period. This must, of course, be near the muzzle, but need not be actually at the muzzle, and its precise

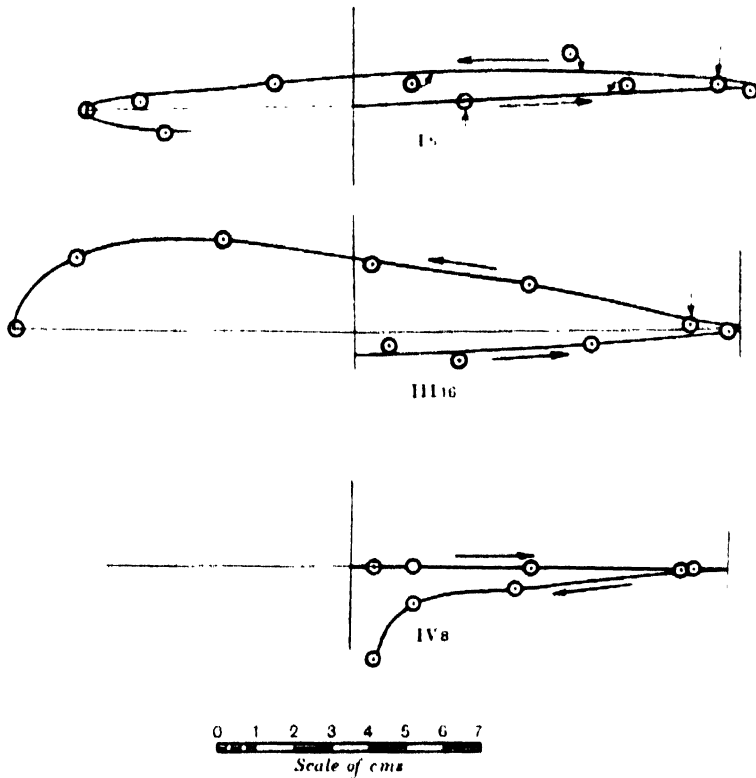


Fig. 4. Curves showing the observed angular motion of the axis with $(\sin \frac{1}{2}\delta, \psi)$ as polar co-ordinates ($\psi = \phi - \phi_0 - \frac{1}{2}\Omega t$). The circles show the observed points and the curves are drawn freehand through them.

For I. (5) 1 cm. represents 0.01 in $\sin \frac{1}{2}\delta$.

For III. (16) and IV. (8) 1 cm. represents 0.02 in $\sin \frac{1}{2}\delta$.

position must be guessed. The values of $\sin \frac{1}{2}\alpha$ can be obtained from a rough curve drawn freehand through the observations. Each half-period is then analysed separately and the only additional constant (when $\beta = 0$) required for computing a curve, by the approximate formula

$$\sin \frac{1}{2}\delta = \sin \frac{1}{2}\alpha \operatorname{cn}(K - \lambda\Omega t, k),$$

is the value of k or $\sin \kappa$. This may be approximated to with the help of the following artifice. Draw by eye a cosine curve (fig. 3, III. 16, first half-period) to have third order contact with the observed $\sin \frac{1}{2}\delta$ -curve at the maximum and to cut the axis at T. Then the ratio of NT to the half-period NP must be $\pi/2K$. For all curves of the form $y = A \operatorname{cn}(K - \beta x)$, for different values of k , have third-order contact at the maximum ordinate, and a cosine curve is the limiting case, in which $K = \frac{1}{2}\pi$. A first estimate of κ can usually be made by this method to the nearest 5 degrees, when

$\kappa > 45$ degrees. If necessary, similar curves are drawn for other values of κ (rounds III. 11, III. 16) and the true values of κ , $\sin \frac{1}{2}\alpha$ and ΩT (the length of the half-period) finally settled by interpolation.

If required a curve can be calculated by the exact formula (10) after g has been determined by (19), but the change of shape is negligible unless $\sin \frac{1}{2}\alpha$ and, therefore, g is very large. An example of the effect of including g is shown in fig. 3 for III. (11), but round III. (11) and its companions are the only ones in which g had any sensible effect. Rounds IV. (8 and 9) illustrate the effect of a considerable alteration in b (representing the initial disturbance) in causing, when $s < 1$, a considerable change in period but only a small change in amplitude. The fit obtained between calculated and observed curves is generally good. The selected curves in fig. 3 are a fair sample of the whole.

When the minimum yaw β is not zero the curve is first calculated as above, as if $\beta = 0$, and then corrected by (33), using the observed value of β , which is obtained like α from a rough curve. An example of such curves will be found in fig. 3 for the second half-period of IV. (8) and the second and third half-periods of III. (16). After the β correction has been put on, the value of κ may require readjustment to obtain the proper fit.

The values of κ , $\sin \frac{1}{2}\alpha$ and ΩT so determined are given for each half-period of each round in Table II., together with $\sin \frac{1}{2}\beta$ and λ obtained from the equation $\lambda = K/\Omega T$. We then obtain s , g and b from equations (15)–(17).* From the values of s and g and the other observational data we can calculate $f_M \sin \delta$ as a function of v/a and δ from formulæ (3) and (4). The results are shown in figs. 1 and 2, and Tables I. and III., and have already been discussed.

The damping effects appear in the variation of the various constants from one half-period to another. In general, s increases approximately at the rate required by theory, *i.e.*, inversely as the square of the velocity at the middle point of the half-period.†

* When κ is much less than 45 degrees the method breaks down, as κ cannot be determined satisfactorily from the observational curves. The method explained in (A), p. 343, could then be employed. This is equivalent to assuming $g = 0$ and using formula (16) to determine κ given $\sin \frac{1}{2}\alpha$ and λ . Under these conditions $\sin \frac{1}{2}\alpha$ is so small that the value of g does not appreciably affect either the value of s or the shape of the curve of f_M against δ .

† Theoretically, s should increase while g should remain roughly constant. But $\sin \frac{1}{2}\alpha$ is common to the first and second half-period, while λ is determined mainly by the shape of the curve near the maximum. Hence, in general, κ alone varies between the first and second half-periods. It appears that the result of varying κ only in formulæ (15, 16) is to produce a fictitious decrease in g while the increase of s is diminished as may be seen in Table II. For this reason mean values for the whole period are used in constructing figs. 1 and 2.

TABLE II.—General Table of Results.

- Column 1. Number of round and number of half-period for the round (in brackets).
 „ 2. Muzzle velocity, f.s., for round or mean for group.
 „ 3. Air density ρ , lb./ $(ft.)^3$, and temperature, ° F.
 „ 4. Ω ($= AN/B$), radians/sec.
 „ 5. ΩT , radians, where T is the observed duration (sec.) of each half-period.
 „ 6. Observed values of $\sin \frac{1}{2}\alpha$, where α is the maximum yaw.
 „ 7. Observed values of $\sin \frac{1}{2}\beta$, where β is the minimum yaw.
 „ 8. Values of κ , degrees, where k ($= \sin \kappa$) is the modulus of the elliptic function which fits the observations.
 „ 9. Values of λ ($= K/\Omega T$), where K is the complete elliptic integral of the first kind for the value of κ in (8).

1	2	3	4	5	6	7	8	9
I. 8 (1)	1049	0.0792	84.2	17.5	0.276	—	79	0.172
I. 9 (1)	1084	43°	86.9	19.8	0.251	—	83	0.154
(2)				16.2	0.251	—	70	0.155
I. 10 (1)	1084		86.9	15.2	0.245	—	75	0.182
I. 17 (1)	1312	0.0812	105.2	16.6	0.192	—	75	0.167
(2)		40°		13.8	0.192	0.020	65	0.167
I. 18 (1)				13.5	0.203	—	75	0.205
(2)				12.0	0.203	0.049	70	0.209
I. 5 (1)	1582	0.0782	125.4	13.8	0.108	—	40	0.130
(2)		45°		15.6	0.108	—	57	0.133
(3)				12.0	0.070	—	20?	0.135
I. 6 (1)				13.6	0.118	—	45	0.136
(2)				14.4	0.118	—	35	0.120
I. 7 (1)				24.0	0.090	—	56	0.086
(2)				19.5	0.090	0.018	0?	0.082
II. 11 (1)	1107	0.0807	81.9	18.2	0.161	—	57	0.114
(2)		42°		14.6	0.161	0.055	20?	0.112
II. 12 (1)				20.8	0.148	—	52	0.095
(2)				16.4	0.148	0.075	35	0.105
II. 13 (1)				22.0	0.183	—	70	0.114
(2)				14.4	0.183	0.046	53	0.138

TABLE II. (continued).

1	2	3	4	5	6	7	8	9
II. 14 (1)	1334	0.0812 40°	98.6	18.4	0.135	—	60	0.119
(2)				19.8	0.135	0.014	55	0.105
II. 15 } (1)				16.6	0.122	—	48	0.117
* 16 } (2)				15.9	0.122	0.012	35	0.111
III. 11 (1)	1077	0.0807 42°	99.0	34.0	0.332	—	84	0.107
III. 12 (1)				32.0	0.290	—	85	0.120
III. 13 (1)				28.4	0.301	—	85	0.135
III. 14 (1)	1312	0.0812 40°	120.4	22.0	0.204	—	78	0.135
(2)				17.2	0.204	—	70	0.146
III. 15 (1)				18.0	0.207	—	75	0.154
(2)				14.5	0.207	0.053	55	0.141
(3)				13.9	0.187	0.053	55	0.147
III. 16 (1)				21.8	0.204	—	80	0.145
(2)				15.6	0.204	0.037	60	0.138
(3)				17.0	0.179	0.037	40	0.105
IV. 10 (1)				884	0.0807 42°	62.5	16.5	0.233
(2)	13.3	0.233	—				50	0.145
IV. 11 (1)	21.0	0.225	—				78	0.142
(2)	15.0	0.225	—				50	0.129
IV. 12 (1)	20.0	0.207	—				77	0.145
(2)	14.5	0.207	—				62	0.153
IV. 7 (1)	1553	0.0779 47°	109.7	15.5	0.213	—	83	0.226
IV. 8 (1)				18.0	0.199	—	85	0.213
(2)				13.0	0.199	0.034	75	0.213
IV. 9 (1)				11.5	0.218	—	70	0.218
IV. 1 (1)	2130	0.0782 45°	150.6	12.5	0.160	—	66	0.187
(2)				12.9	0.160	0.018	65	0.179
IV. 2 (1)				18.0	0.157	—	82	0.187
(2)				14.2	0.157	0.013	70	0.176
IV. 3 (1)				17.0	0.160	—	80	0.185
(2)				14.8	0.160	—	75	0.187
IV. 5 (1)				15.5	0.160	—	74	0.175
(2)				13.9	0.160	0.015	70	0.180

* II. (15 and 16) are practically indistinguishable. These values refer to their mean.

TABLE III.—Mean values, for each group, of s , q and \bar{v} ; the corresponding values of \bar{v}/a and $f_M(\bar{v}/a, 0)$; and the percentage spread in s . Each half-period is treated separately in each group.

Group.	s .	Percentage spread in s .	q .	\bar{v} .	\bar{v}/a .	f_M .
I. 8-10 (1)	0.896	0.8	0.213	1063	0.965	11.80
I. 9 (2)	0.932	—	0.084	1053	0.956	11.56
I. 17, 18 (1)	0.892	4.3	0.578	1297	1.181	11.61
I. 17, 18 (2)	0.908	5.5	0.597	1264	1.150	12.01
I. 5-7 (1)	1.000	2.3	0.398	1560	1.413	10.56
I. 5-7 (2)	1.007	5.6	0.507	1517	1.375	11.08
I. 5 (3)	1.060	—	—	1494	1.354	10.86
II. 11-13 (1)	0.977	3.1	0.068	1094	0.994	10.34
II. 11-13 (2)	1.014	5.5	-0.04	1075	0.977	10.30
II. 14-16 (1)	0.987	2.1	0.284	1315	1.198	10.20
II. 14-16 (2)	1.006	3.2	0.066	1286	1.171	10.47
III. 11-13 (1)	0.946	2.6	-0.119	1063	0.966	13.08
III. 14-16 (1)	0.931	1.8	0.203	1296	1.180	13.10
III. 14-16 (2)	0.959	3.6	0.118	1263	1.150	13.43
III. 15, 16 (3)	0.990	3.7	—	1239	1.128	13.52
IV. 10-12 (1)	0.934	0.3	0.146	877	0.797	10.00
IV. 10-12 (2)	0.975	3.8	0.033	866	0.787	9.79
IV. 7-9 (1)	0.852	4.5	0.793	1532	1.385	11.40
IV. 8 (2)	0.864	—	0.820	1488	1.345	11.93
IV. 1-3, 5 (1)	0.897	3.5	0.987	2102	1.905	10.80
IV. 1-3, 5 (2)	0.907	3.5	0.885	2052	1.860	11.20

The values of f_M here given are not corrected for the effect of the cards.

§ 9. *The Values of b.*

The value of b represents the initial angular velocity of the axis of the shell, for at the beginning of the first half-period rosette motion may be assumed and $\delta' = b\Omega$. The values of b for each round are given in Table IV.

TABLE IV.—Values of b ($= \delta'_0/\Omega$), where δ'_0 is the initial angular velocity of the axis of the shell.

Group.	Values of b .			Mean.	Group.	Values of b .			Mean.
I. 8-10	0.018	0.011	0.023	0.017	III. 11-13	0.007	0.006	0.007	0.007
I. 17, 18	0.017	0.022		0.019	III. 14-16	0.011	0.016	0.011	0.013
I. 5-7	0.021	0.023	0.009	0.018	IV. 10-12	0.024	0.013	0.013	0.017
II. 11-13	0.020	0.017	0.014	0.017	IV. 7-9	0.012	0.007	0.032	0.017
II. 14-16	0.016	0.018	0.018	0.017	IV. 1-3, 5	0.024	0.008	0.010	
						0.015			0.014

They represent the size of the initial disturbance which upsets the nose-on motion, and vary irregularly in any one group, as might be expected. Their general size, however, is remarkably consistent from group to group. The corresponding values for the stable rounds and rough values for some of the rounds here analysed have been discussed by us in a previous paper.* The further results here given confirm the statements of that paper; the mean value of b for any group is practically constant somewhere between 0.01 and 0.02 for all groups fired.

§ 10. Allowance for the Effect of Cards.

In obtaining figs. 1 and 2 and Table I., but not in other cases, a small correction has been made to the value of the couple to allow for the impulsive action on the shell when it strikes a card. The amount of this correction was calculated from a few special rounds fired with cards on the far screens only. The following argument indicates that the effect should be roughly proportional to the couple due to the air and independent of other factors except the number of cards and the muzzle velocity. The effect of each card may be considered as an impulsive force whose moment about the centre of gravity is roughly proportional to $\sin \delta$. If the cards are (as they were) uniformly distributed in space and therefore in time, there will be a total impulse per second which is proportional to $\sin \delta$, and if the cards are not too far apart this is equivalent to a steady couple roughly proportional to the couple due to the air. The correction worked out in the case of the stable rounds at from 3 per cent. to 4 per cent. depending on the muzzle velocity. In view of the preceding argument the same corrections are applied here.

§ 11. Concluding Remarks.

In view of possible future experiments it is of interest to compare the merits of the stable and unstable rounds for the purpose of this analysis. A large proportion of the rounds analysed in (A) had a stability coefficient s of about 1.8. The advantages of this are that the theory of the motion is practically complete, and in addition to the values of f_M as a function of v/a rough values of the damping factors were obtained, which could be greatly improved if a longer range were available. The maximum yaw, however, is small, so that small errors in the determination of the yaw are important, while the periods are short so that, unless the cards are close together, it is difficult to draw curves through the observation points with sufficient certainty to determine the periods accurately.

For the unstable rounds here analysed the theory is imperfect as regards the determination of damping. But a small change in the value of the couple M will

* 'Proc. Camb. Phil. Soc.,' vol. XX, p. 311, 1921.

make a large change in the type of motion, so that the method is now a very sensitive one for determining the couple, and no great accuracy of observation is required. The yaw developed is also much larger, so that values of the couple are determined over a larger range of yaws.

From the point of view of general aerodynamical theory the results form a preliminary contribution to the problem of determining the force system impressed by the air on a body moving unsymmetrically through a fluid at velocities at which the compressibility of the fluid produces marked effects.

VIII. BAKERIAN LECTURE.—*Optical Rotatory Dispersion.—Part II. Tartaric Acid and the Tartrates.*

By THOMAS MARTIN LOWRY, F.R.S., and PERCY CORLETT AUSTIN.

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I. OPTICAL ROTATORY POWER OF TARTARIC ACID AND THE TARTRATES.

THE optical rotatory power of tartaric acid was discovered in 1832 by BIOT ('*Mém. Acad. Sci., Paris, 1835, vol. 13, Table G, p. 168*; paper read November 5, 1832), who devoted one of his longest memoirs ('*Mém. Acad. Sci., 1838, vol. 15, pp. 93-279*; paper read January 11, 1836) to a detailed account of its properties when mixed with water, with alcohol and with wood-spirit.

BIOT found that tartaric acid, when "dissolved in different fluid media, exercises on the planes of polarisation of light a special power, which distinguishes it from all the other substances hitherto studied." These had agreed with quartz in obeying, at least approximately, BIOT'S LAW, according to which "the rotation of the different simple rays is reciprocal to the square of their wave-lengths" ('*Mém. Acad. Sci., 1817, vol. 2, pp. 49, 57 and 135*; paper read September 22, 1818). This agreement had been verified in the case of turpentine, alone and mixed with ether, and of cane-sugar dissolved in water (i.) by comparing the tints with those produced by equivalent plates of quartz, and (ii.) by eliminating the effects of rotatory polarisation with the help of a quartz

plate of opposite sign acting as a compensator ('Mém. Acad. Sci.,' 1817, vol. 2, pp. 103-114). When tartaric acid was compared with quartz, however, no such parallelism was observed, the rotations for the chief colours being as follows:—

	Red.	Orange.	Yellow.	Green.	Blue.	Indigo.	Violet.
Quartz	18·99	21·40	23·99	27·86	32·31	36·13	40·88
Tartaric acid	38·7	40·29	42·51	46·11	44·40	42·9	39·38

('Mém. Acad. Sci.,' 1838, vol. 15, p. 236). Similar phenomena were observed when tartaric acid was dissolved in alcohol (*ibid.*, p. 245); but BIOT found that "when it combines with basic substances in the same media, it loses its special action and imprints on the products the properties common to all other bodies endowed with rotatory power" ('Mém. Acad. Sci.,' 1838, vol. 16, p. 229; paper read November 27, 1837).

BIOT's experiments on tartaric acid and the tartrates covered so wide a range that the whole of the work described in the present paper may be regarded as a logical extension of his investigations, aided on the physical side by modern optical theories and by modern apparatus, and on the chemical side by structural formulæ and by stereochemical notions which were only struggling for recognition even in the closing years of BIOT's life. The contributions which he made to the fundamental problem of determining the form of the curves of rotatory dispersion are discussed below in a separate paragraph, but reference may be made here to a few of the many topics covered by his investigations.

(a) *Influence of Water on the Rotatory Power of Tartaric Acid.*

BIOT discovered that "in aqueous solutions of tartaric acid at a given temperature, the rotatory power of the acid calculated for each simple ray is always of the form $A + Be$, where e represents the proportion by weight of water in the solution" ('Mém. Acad. Sci.,' 1838, vol. 15, p. 216; compare *ibid.*, p. 207), the specific rotation being therefore a linear function of the concentration. This linear law was first described in a sealed communication deposited in the Archives of the Academy on August 25, 1834, and opened on December 7, 1835. A note added on the latter date ('Comptes Rendus,' 1835, vol. 1, p. 459) stated that the law was an approximation which did not apply to dilute solutions. BIOT made use of the linear law to construct a diagram ('Mém. Acad. Sci.,' 1838, vol. 15, fig. 4, facing p. 652), in which the rotatory power for light of different colours is shown as a series of straight lines inclined at different angles to the axis of concentration. This diagram shows a marked similarity to the "characteristic diagram" constructed 75 years later by ARMSTRONG and WALKER ('Roy. Soc. Proc.,' 1913, series A, vol. 88, pp. 388-403), the chief difference being that the rotations for green light are

represented by a line inclined at an arbitrary angle, instead of at 45 degrees, to the horizontal axis; the anomalous dispersion in concentrated solutions is shown by the intersection of the lines showing the rotatory power of the acid for light of different colours, and the removal of some of the more obvious anomalies on diluting the solutions is shown by the gradual separation of the lines until they follow the normal sequence of the primary colours of the visible spectrum.

(b) *Rotatory Power of Amorphous Tartaric Acid.*

Our own investigations (pp. 266 to 271) have shown that BIOT's linear law is only an approximation, and that an equation with five arbitrary constants would probably be required to express completely the relationship between rotatory power and concentration. With the help of his diagram and formula, BIOT was, however, able to calculate by extrapolation the rotatory power of pure anhydrous tartaric acid at different temperatures and for light of different colours. In particular, he concluded that the rotatory power A of the anhydrous acid for the red light transmitted through glass coloured by cuprous oxide would change sign at 23° C., being positive above this temperature and negative below it ('Mém. Acad. Sci.,' 1838, vol. 16, p. 269). This prediction was verified dramatically some years later when LAURENT in 1849 discovered a method by which moistened tartaric acid could be fused and cooled to a transparent glass in thicknesses up to 76 mm. ('Ann. Chim. Phys.,' 1850, vol. 28, p. 353). BIOT then found that the hot, pasty acid produced a strong dextrorotation, which became negative on cooling. A 70 mm. column of the acid at $+3.5^{\circ}$ C. gave $\alpha_{\text{red}} = -3.28^{\circ}$, when $[\alpha]_{\text{red}} = -2.787^{\circ}$, agreeing very closely with the value $[\alpha]_{\text{red}} = -2.752^{\circ}$ calculated by extrapolation from the rotatory power of concentrated aqueous solutions of the acid (*ibid.*, p. 366).

(c) *Rotatory Power of the Tartrates.*

Whilst his first long memoir on tartaric acid ("Méthodes mathématiques et expérimentales, pour discerner les Mélanges et les Combinaisons, définies ou non définies, qui agissent sur la Lumière Polarisée; suivies d'applications aux combinaisons de l'acide tartrique avec l'eau, l'alcool, et l'esprit de bois," 'Mém. Acad. Sci.,' 1838, vol. 15, pp. 93-279) dealt with solutions of the acid in water, alcohol, and wood spirit, his second memoir ("Mémoire sur plusieurs Points Fondamentaux de Mécanique Chimique"; 'Mém. Acad. Sci.,' 1838, vol. 16, pp. 229-396) described the changes which are produced in the rotatory power of tartaric acid by adding other acids (sulphuric, hydrochloric and citric, *loc. cit.*, 271-304), alkalis (potash, soda and ammonia, *loc. cit.*, 307-377) and earths (alumina and beryllia, *loc. cit.*, 377-385). The action of alkalis was of special interest in that the tartrates derived from them showed none of the anomalies of the acid, their rotatory dispersion conforming approximately to the law of inverse squares and agreeing generally with that of quartz. Our own measurements have shown that this conclusion

is broadly correct, since aqueous solutions of sodium tartrate show a closer approximation to the requirements of BIOT'S Law than in the case of any other substance that we have yet investigated ; the agreement is, however, even here not exact, and the complex character of the rotatory dispersion of the acid is shown, although in a much less striking way, in the salts derived from it.

(d) Influence of Boric Acid.

Of wider general interest is BIOT'S discovery of the remarkable exaltation of rotatory power which is produced by the addition of boric acid to tartaric acid. The properties of the boro-tartaric solutions were first described in outline in his sealed note to the Academy ('Comptes Rendus,' 1835, vol. 1, p. 458, compare 'Mém. Acad. Sci.,' 1838, vol. 16, p. 271), but a detailed description was given ten years later as part of a long memoir "On the employment of polarised light to study various questions of chemical mechanics" ('Ann. Chim. Phys.,' 1844, vol. 11, pp. 82-112; see also 'Ann. Chim. Phys.,' 1860, vol. 59, pp. 229-256). It is characteristic of the thoroughness with which BIOT worked, that he extended his observations to include not only aqueous solutions, but also glassy amorphous mixtures of the two anhydrous acids ('Ann. Chim. Phys.,' 1850, vol. 28, p. 368). The exaltation of rotatory power which BIOT observed on the addition of boric to tartaric acid appears in many other substances which resemble tartaric acid in containing two hydroxyl-groups attached to adjacent carbon-atoms. It is generally accompanied by an increase of electrical conductivity, and in recent years has been made the basis of a general method of determining the configuration of hydroxylic-compounds of the sugar group (see especially MAGNANINI, 'Zeitschr. physikal. Chem.,' 1890, vol. 6, p. 67; 'Berichte Deut. Chem. Ges.,' 1891, vol. 24 ref., p. 894; BOESEKEN 'Berichte,' 1913, vol. 46, p. 2612; IRVINE, 'Trans. Chem. Soc.,' 1914, vol. 105, p. 898; 1915, vol. 107, pp. 1221 and 1230).

Later workers, whilst confirming the accuracy of his general conclusions, have added many details to the broad outlines of BIOT'S work. Thus ARNDTSEN ('Ann. Chim. Phys.,' 1858, vol. 54, p. 411), during a summer spent in Paris, showed that BIOT'S maximum in the green disappeared at the violet end of the spectrum at concentrations below 20 per cent. of tartaric acid; KRECKE ('Archives Néerlandaises,' 1872, vol. 7, p. 107) showed that even in a 50 per cent. solution the maximum vanished at the violet end of the spectrum when the temperature was raised to 50° C.; and WENDELL ('Wiedemann's Ann. Phys. Chem.,' 1898, vol. 66, pp. 1149-1161) confirmed the observations of ARNDTSEN and of KRECKE as to the displacement of the maximum towards the violet by dilution and by heating, and the increase of rotatory power which accompanies these changes. LEPESCHKIN ('Berichte Deut. Chem. Ges.,' 1899, vol. 32, pp. 1180-1184), working in the opposite direction, observed in a supersaturated solution at 20° C. a negative rotatory power, analogous with that which had been recorded by

BIOT for the anhydrous acid, and by ARNDTSEN ('Ann. Chim. Phys.,' 1858, vol. 54, p. 415) for very strong alcoholic solutions; this negative rotation for dark-blue light became positive for light of longer wave-lengths, reaching a maximum dextrorotation in the yellow region of the spectrum. WINTHER ('Zeitschr. physikal. Chem.,' 1902, vol. 41, pp. 181-189), who made a detailed study of the effects of temperature and concentration on the rotatory power of the acid in aqueous and in alcoholic solutions, extrapolated to 100 per cent. tartaric acid, and concluded that the maximum might be displaced still further, disappearing at the red end of the spectrum when the temperature of the anhydrous acid fell below 20° C. This conclusion was not confirmed by BRUHAT ('Trans. Faraday Soc.,' 1914, vol. 10, p. 89), whose extrapolation gave a maximum between the yellow and the red, whilst his observations of the glassy acid at 15° C. indicated that a maximum could still be seen in the red region of the spectrum.

New ground has been broken in the detailed investigation by WINTHER and by PATTERSON of the tartaric esters which show similar anomalies to those which BIOT had discovered in tartaric acid.* Very noteworthy also is the achievement of BRUHAT (*loc. cit.*), who by means of special apparatus succeeded in measuring the rotatory power of fused and superfused tartaric acid for several wave-lengths at temperatures from 180° to 15° C., thereby bringing the acid into line with its esters as they had been investigated 12 years earlier by WINTHER. The investigation of tartar emetic and of the related compounds of arsenic and bismuth has also given interesting results, which are described in detail below (pp. 284 and 285).

2. ROTATORY DISPERSION IN QUARTZ AND IN TARTARIC ACID.

BIOT's measurements of the optical rotatory power of tartaric acid indicated the existence of two types of rotatory dispersion, which may be described provisionally as the "quartz type" and the "tartaric acid type." Since these correspond to some extent with the later classification of rotatory dispersion as normal and anomalous, or as simple and complex, it will be desirable to set out the essential features of these various methods of classification.

(a) BIOT's *Two Types*.

BIOT divided optically active substances into two groups according as they obeyed the law of inverse squares, $\alpha = k/\lambda^2$, or showed large deviations from this law. The maximum in the optical rotatory power of tartaric acid, which he observed in the green region of the spectrum, was merely an incidental feature in the exceptional behaviour of the acid, and did not receive any of the emphasis which has since been placed upon

* For a study of the form of the dispersion-curves for ethyl and methyl tartrates see LOWRY and DICKSON, 'Trans. Chem. Soc.,' 1915, vol. 107, pp. 1173-1187; LOWRY and ABRAM, *ibid.*, pp. 1187-1195.

it.* On the other hand, the extreme sensitiveness of the rotatory power of tartaric acid to changes of temperature, concentration and solvent was regarded by BIOT as one of the chief anomalies in the behaviour of this exceptional substance. BIOT had laid stress on the fact that the optical rotatory power† of a substance was usually independent of the conditions under which it was observed; thus the rotatory power of turpentine was not affected by diluting it with other essential oils ('Mém. Acad. Sci.,' 1817, vol. 2, p. 115), or with ether (*ibid.*, p. 116), and it even retained its optical activity when examined as a vapour in a column 30 metres in length (*ibid.*, pp. 126-133, compare GERNEZ, 'Ann. de l'Ecole Norm.,' vol. 1, p. 1); cane sugar, too, showed a very similar rotatory power when examined in the amorphous solid state as "barley sugar" and when this same product was dissolved in water ('Mém. Acad. Sci.,' 1835, vol. 13, pp. 126-132). The marked influence on the rotatory power of tartaric acid of dilution with water or of addition of boric acid was therefore attributed to chemical changes just as definite as those involved in the conversion of the acid into its salts. This early view, which had fallen into disrepute for many years, has now become prominent again, more especially as an explanation of variations of rotatory power in those cases of anomalous rotatory dispersion of which tartaric acid is still the chief type.

(b) *Normal and Anomalous Rotatory Dispersion.*‡

Only a short time elapsed before BIOT's law of inverse squares was recognised as being inexact. BIOT had suspected almost from the first that small variations of dispersive power might exist in different compounds; in 1836 he obtained clear evidence of this fact by balancing against one another columns of turpentine and of oil of lemon which produced approximately equal and opposite rotations, when he found that "the compensation of the deviations, although very close for all the rays, was, however, neither complete nor general" ('Comptes Rendus,' 1836, vol. 2, p. 543); cane-sugar

* For early examples of this emphasis, see ARNDTSEN, 'Ann. Chim. Phys.,' 1858, vol. 54, p. 409; KRECKE, 'Arch. Néerlandaises,' 1872, vol. 7, p. 114; LANDOLT, 'Liebig's Annalen,' 1877, vol. 189, p. 274.

† BIOT described as the *molecular rotatory power of substances* ('Mém. Acad. Sci.,' 1838, vol. 15, p. 95; compare 'Mém. Acad. Sci.,' 1835, vol. 13, p. 116) the rotation that would be produced by a column 1 mm. thick and of unit density; this is one-hundredth part of what is now called the *specific rotatory power* of the substance.

‡ The term "normal dispersion" was used by ARNDTSEN in 1858 ('Ann. Chim. Phys.,' 1858, vol. 54, p. 412) to describe the case in which "the angle of rotation increases continuously with the refrangibility of the rays." The term "anomalous (rotatory) dispersion" appears to have been introduced in 1877 by LANDOLT ('Liebig's Ann. der Chemie,' 1877, vol. 189, p. 274), who described under this heading (1) a maximum which travels from the violet to the green region of the spectrum as the concentration of the aqueous solution of tartaric acid increases, and (2) a reversal of sign in the rotatory power of the anhydrous acid and of its alcoholic solution. KRECKE, five years earlier ('Arch. Néerland.,' 1872, vol. 7, pp. 98, 110 and 114), had referred less specifically to the "remarkable anomalies" which are observed in the optical properties of tartaric acid.

and invert-sugar gave, on the other hand, a very exact compensation ('Ann. Chim. Phys.,' 1844, vol. 10, p. 35). After BIOT's death the law of inverse squares was generally abandoned, even as a first approximation. The result was most unfortunate, since the experimenters who proved the inaccuracy of BIOT's formula did not possess the mathematical skill that was required to replace it by another formula that was more exact. There can be little doubt that, as more exact methods of measurement were developed, BIOT himself would have investigated the deviations from this law, and might well have discovered the small but important correction which DRUDE introduced many years later when he wrote $\alpha = k/(\lambda^2 - \lambda_0^2)$ instead of $\alpha = k/\lambda^2$.

This discovery was extremely likely in view of the fact that, as early as 1817, BIOT, in applying the law of inverse squares to the rotatory power of quartz, had used a graphical method in which virtually the reciprocal of the rotatory power was plotted against the square of the wave-length* ('Mém. Acad. Sci.,' 1817, vol. 2, plate 3 facing p. 136). This device of plotting $1/\alpha$ against λ^2 is, however, the simplest method of checking the validity of DRUDE's formula, and has been used extensively in recent years as a convenient test for this purpose (LOWRY and DICKSON, 'Trans. Chem. Soc.,' 1913, vol. 103, p. 1075; LOWRY and ABRAM, 'Trans. Faraday Soc.,' 1914, vol. 10, p. 104; compare also FRANKLAND and GARNER, 'Trans. Chem. Soc.,' 1919, vol. 15, p. 640, footnote, and RUPE and AKERMANN, 'Ann. der Chem.,' 1920, vol. 420, p. 12). The mere plotting out on BIOT's original plan of a series of accurate experimental data would therefore have disclosed to him both the existence and the magnitude of DRUDE's correction.

In the absence of BIOT's mathematical genius, however, nearly all the work on rotatory dispersion during the next half-century became semi-qualitative in character, the data being represented by curves of unknown form, instead of by mathematical equations. This fact affords an explanation of the exaggerated importance which was attached to the more conspicuous anomalies, as well as of the utter confusion into which all attempts to classify rotatory dispersion fell. Thus, in the absence of any precise knowledge of the real form of the dispersion curves, KRECKE seized upon "the anomaly that, in concentrated solutions of tartaric acid, the green rays are turned more than the red and violet rays" ('Arch. Néerlandaises,' 1872, vol. 7, p. 114). LANDOLT in 1877 recognised two anomalies, namely, (i.) that in aqueous solutions of tartaric acid "if one increases the concentration, the maximum rotation wanders [from the violet] towards the red end of the spectrum, and in solutions containing 50 per cent. tartaric acid, the green rays are most strongly deflected," (ii) "that tartaric acid in the anhydrous state must deflect the rays C D E to the right, b F e to the left, and that for a certain kind of light, whose wave-length lies between the lines E and C, there can be no rotation at all" ('Liebig's Ann. der Chem.,' 1877, vol. 189, p. 274).

* The lengths of the columns of quartz required to produce a rotation of $\pi/2$ were plotted against the square of the wave-length as a series of straight lines diverging from the origin where $l = 1/\alpha = 0$ and $\lambda^2 = 0$; DRUDE's equation gives straight lines diverging from $1/\alpha = 0$, $\lambda^2 = \lambda_0^2$.

where the "dispersion constants" λ_n^2 , corresponding with the natural free periods $1/\lambda_n$ of the electrons, were deduced from measurements of refractive dispersion as expressed by the equation

$$N^2 = E + \sum \frac{M_n}{\lambda^2 - \lambda_n^2}.$$

in which N is the refractive index and E is the dielectric constant of the medium.

In those cases in which only one electron need be considered, the magnetic rotation could be expressed by an equation

$$\alpha = N \left(\frac{a'}{\lambda^2} + \frac{b'}{\lambda^2 - \lambda_1^2} \right),$$

which involved four constants, of which two could be derived from measurements of refraction.

The importance of these equations did not appear immediately. Very few experimental data were available, and DRUDE applied his formulæ only to the magnetic rotations of carbon disulphide and of creosote and to the natural rotatory power of quartz, which he expressed by the two-term equation

$$\alpha = \frac{k_1}{\lambda^2 - \lambda_1^2} - \frac{k_2}{\lambda^2},$$

containing two arbitrary constants k_1 , k_2 , and one constant, λ_1^2 , derived from measurements of refraction; the dispersion constant of the second term was omitted as being negligible in comparison with λ^2 . Perhaps on account of the lack of suitable data, the new formulæ were not applied to any single member of the vast array of optically active organic compounds which have been prepared and studied, more especially from the time of PASTEUR onwards.

In view of the very limited application of these formulæ by DRUDE himself, the indefinite number of arbitrary constants which they contained, and the fact that a complete knowledge of the refractive dispersion of the medium was presupposed, it is not surprising that DRUDE'S formulæ remained almost barren so far as their immediate application to measurements of rotatory dispersion was concerned. Fifteen years of work on rotatory dispersion ('Phil. Trans.,' A, 1912, vol. 212, p. 261; 'Trans. Chem. Soc.,' 1913, vol. 103, p. 1062 *et seq.*) have, however, provided ample data for testing the validity of these equations, and have established beyond question the fact that they are adequate to meet all the requirements of the most diverse and of the most exact measurements of rotatory dispersion. In the case of some scores of organic compounds,

both the optical and the magnetic rotations* can be expressed by a single term of the general equation (LOWRY and DICKSON, 'Trans. Chem. Soc.,' 1913, vol. 103, pp. 1067-1075; LOWRY and ABRAM, 'Trans. Chem. Soc.,' 1919, vol. 115, p. 300; RUPE and AKERMANN, 'Ann. der Chem.,' 1920, vol. 420, p. 4); liquids such as ethyl and methyl tartrates and aqueous solutions of tartaric acid, which show anomalous rotatory dispersion, require two terms of opposite sign (LOWRY and DICKSON, 'Trans. Chem. Soc.,' 1915, vol. 107, pp. 1173-1187; LOWRY and ABRAM, 'Trans. Chem. Soc.,' 1915, vol. 107, pp. 1187-1195); quartz, although showing no obvious anomalies, requires three terms of DRUDE's equation in order to express the most recent measurements that have been made of its rotatory dispersion (LOWRY, 'Phil. Trans.,' A, 1912, vol. 212, p. 261). This determination of the exact form of the curves has led to an extremely easy and convenient classification of rotatory dispersion, as *simple* when one term of DRUDE's equation is sufficient and *complex* when two or more terms are required (LOWRY and DICKSON, 'Trans. Faraday Soc.,' 1914, vol. 10, p. 102). The complex curves are only anomalous when they cross the axis of rotations and exhibit a reversal of sign; but a simple mathematical analysis has established the conditions under which a complex curve, expressed by two terms of DRUDE's equation, ceases to be *normal* and becomes *anomalous* in the sense of the exact definition already referred to ('Trans. Chem. Soc.,' 1915, vol. 107, p. 1198).

3. THE ORIGIN OF ANOMALOUS ROTATORY DISPERSION.

(a) *Anomalous Rotatory Dispersion as a Problem in Chemical Mechanics.*

The present paper follows one on rotatory dispersion in quartz. It may be regarded as supplementing that paper by extending the new series of exact measurements from the first to the second of BIOT's types of rotatory dispersion. It also carries the work forward from optically active crystals to optically active liquids, and so opens up again the complex chemical problems which led BIOT to describe most of his work on tartaric acid as a study in chemical mechanics rather than as an investigation of the physical properties of the acid. It is this underlying chemical interest that more especially

* The magnetic rotatory dispersion in carbon disulphide can be expressed, at least as well by a simple two-constant equation $\alpha = k/(\lambda^2 - 0.055)$, as by the four-constant equation used by DRUDE. Thus the five lines quoted by DRUDE ('Theory of Optics,' 1907, p. 431) give for k the values:—

	C.	D.	E.	F.	G.
$k =$	0.2224	0.2221	0.2226	0.2236	0.2224

whilst the observed and calculated dispersion-ratios compare as follows:—

Observed.	0.592	0.760	1.000	1.234	1.704
Cal. (four constants)	0.592	0.762	0.999	1.232	1.704
Cal. (two constants)	0.593	0.762	1.000	1.228	1.706

Later measurements ('Trans. Chem. Soc.,' 1913, vol. 103, p. 1074) agree still more closely with a simple constant equation.

distinguishes the work on tartaric acid from that on quartz, and gives to it its own peculiar importance. From the physical point of view the contrast is not great, and almost the whole of the advantage rests with the quartz, the large rotatory power of which renders possible an accuracy of measurement that is probably at least 100 times greater than in the case of any other medium. But whereas the rotatory power of quartz is influenced only slightly by the experimental conditions, tartaric acid is subject to drastic changes of rotatory power as a result of very small changes of circumstance.

In recent years this extreme sensitiveness has been regarded by many writers as an inherent quality of the physical property of optical rotatory power; but all the new evidence goes to show that BIOT was right in attributing it to chemical changes in the optically active liquid. One point must, however, be made clear immediately: whatever the nature of these chemical changes may be, they must proceed to equilibrium with very great rapidity, since even the most careful experiments ('Trans. Chem. Soc.,' 1915, vol. 107, pp. 1177 and 1189) have failed to detect any lag in the adjustment of rotatory power in the tartaric esters when conditions have been altered. In this respect these changes may be compared with the dissociation of nitrogen peroxide, with the dissociation and association of water, or even with the phenomena of ionisation in aqueous solutions, all of which appear to depend on very rapid or instantaneous reactions. Under these circumstances, only static methods of investigation are available, *i.e.*, the liquid must be proved to be complex by recognising in it the attributes of a mixture, rather than by isolating its components and watching them change gradually into one another with lapse of time.

(b) Anomalous Rotatory Dispersion observed in Natural and Artificial Mixtures.

What evidence is there, then, that tartaric acid gives rise to a complex product when it is dissolved or melted in order to convert it into an isotropic, optically active medium? BIOT found evidence of widespread chemical change in the progressive alterations of rotatory power that were produced alike by the addition of water or alcohol, of soda or potash, of ammonia, and of sulphuric and boric acids; but this argument is obviously limited to the changes which accompany dissolution and cannot be applied to the mere fusion of the anhydrous acid. A more general argument is afforded by the experiments in which BIOT attempted to neutralise the optical rotatory power of lævorotatory turpentine by compensating it with a column of dextrorotatory oil of lemon ('Comptes Rendus,' 1836, vol. 2, p. 543): not only was the compensation "neither complete nor general," but "when the principal section of the analysing prism coincided with the original plane of polarisation, there was produced an extraordinary image of blue-violet colour, dark, and sensibly free from red; and, on turning the prism a little to the right or to the left of this position, the tint of the image varied in a contrary sense to the refrangibility, things which are entirely different from those which a single one of the two essences could produce alone." Similar effects were observed in

artificial mixtures of lævorotatory turpentine with dextrorotatory camphor, as well as in some natural turpentine-oils.

On the basis of BIOT'S experiments, as well as of his own observations of the unequal dispersive power of different liquids, ARNDTSEN ('Ann. Chim. Phys.,' 1858, vol. 54, p. 421) put forward for the first time a precise explanation of "the singular dispersion of the planes of polarisation of tartaric acid," as follows:—

"If one should imagine two active substances which do not act chemically upon one another, of which one turns the plane of polarisation to the right, the other to the left, and, in addition that the rotation of the first increased (with the refrangibility of the light) more rapidly than that of the other, it is clear that, on mixing these substances in certain proportions, one would have combinations which would show optical phenomena precisely similar to those of tartaric acid, as M. BIOT has already proved by his researches on different mixtures of turpentine and natural camphor. One might then regard tartaric acid as a mixture of two bodies differing only as regards their optical properties, of which one had a negative rotatory power, the other a positive rotatory power, and of which the rotations varied in different proportions with the refrangibility of the light."

This hypothesis, made more than 60 years ago, appears to us to afford a correct explanation of the peculiar rotatory dispersion of tartaric acid and its derivatives. In its support we submit (i.) the mathematical evidence that the form of the dispersion-curves is in harmony with this view, and (ii.) the chemical evidence that mixtures of isomerides in equilibrium actually exist, *e.g.*, in the case of nitrocamphor, and that their behaviour is in accordance with that which must be postulated for the "two kinds of optically-active molecules" assumed by ARNDTSEN.

(c) *Anomalous Rotatory Dispersion produced by the Partial Compensation of two Simple Dispersions.*

The measurements described below afford strong support to ARNDTSEN'S theory, since it has been established by visual and photographic readings that the dispersion curves for tartaric acid, like those of its esters ('Trans. Chem. Soc.,' 1915, vol. 107, pp. 1173-1195), can be represented over a wide range of wave-length and to a close degree of approximation by two terms of DRUDE'S equation, thus

$$\alpha = \frac{k_1}{\lambda^2 - \lambda_1^2} - \frac{k_2}{\lambda^2 - \lambda_2^2}.$$

This equation is a direct mathematical expression of the view that the anomalous rotatory dispersion of tartaric acid is produced by the counterbalancing action of two components of opposite rotatory power and unequal dispersion. It is, however, not in itself a conclusive argument for the presence of two kinds of optically active molecules

in these liquids since the two compensating terms in the equation may be due to (i.) two electrons with opposite influence on the rotatory power, as in the case of quartz, where, however, they do not give rise to anomalous dispersion, (ii.) two radicals of opposite activity united in one molecule, as in the cases of *l*-menthyl *d*-camphor β -sulphonate (TSCHUGAEFF, 'Ber. Deutsch. Chem. Gesell.,' 1911, vol. 44, p. 2023; 1912, vol. 45, p. 2759; compare 'Trans. Faraday Soc.,' 1914, vol. 10, p. 73) and *l*-menthyl *d*-diphenylmethylacetoacetate (RUPE and KÄGI, 'Ann. der Chem.,' 1920, vol. 420, p. 38); (iii.) two molecules of opposite rotatory power, either easily convertible, as in the case of the isomeric nitro-camphors (LOWRY, 'Trans. Faraday Soc.,' 1914, vol. 10, p. 100), or fixed, as in the artificial mixtures of BIOT ('Comptes Rendus,' 1836, vol. 2, p. 543), and of VON WYSS ('WIEDEMANN'S Ann. Phys. Chem.,' 1888 [2], vol. 33, p. 567). To decide between these three possibilities, further consideration is needed both of the chemical and of the physical properties of the solution as set out in the following paragraphs.

(d) Dynamic Isomerism as an Explanation of Anomalous Rotatory Dispersion.

The view that the two terms in the equations showing the effect of wave-length on the rotatory power of tartaric acid and its derivatives are due to two electrons, as in the case of quartz, is rendered improbable by the fact that substances of similar type do not show this effect. Thus, if tartaric acid be regarded as a dicarboxylic acid of the sugar-group, belonging to the C_4 series and containing two asymmetric carbon atoms, it might be expected that the methyl-glucosides, which belong to the C_6 series and contain five asymmetric carbon atoms, would give even more complex dispersion-curves; actually, however, their dispersion can be expressed accurately by a simple one-term formula. The hypothesis of two radicals of opposite optical activity is even less easy to apply to tartaric acid, unless some form of molecular rearrangement is first postulated, since the two active radicals of which it is composed are not only of the same sign, but are of identical structure, and would therefore give identical rotatory dispersions.

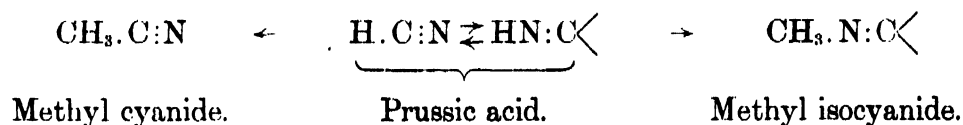
It is, indeed, impossible to discover, in the simple structural formula



commonly assigned to tartaric acid, any justification for its anomalous rotatory power, and some change of molecular structure appears to be inevitable if its peculiar physical properties are to be accounted for, since the complex dispersion of the acid is just as exceptional amongst simple organic compounds as would be the appearance of a bright blue or green colour in a simple compound of the alcohol or sugar group. If then, some form of molecular rearrangement must be assumed, no simpler hypothesis can be adopted than that of ARNDTSEN, which suggests that the rearrangement is incomplete, so that one of the two compensating factors required to account for the complex or anomalous dispersion of the medium is merely the original

form of the acid, whilst the other is a product of change, of opposite sign and unequal dispersion, but of a character sufficiently commonplace to give rise to a simple, instead of a complex, dispersion-curve when studied as a separate entity. This product might be an ion, a hydrate, a polymer, or an isomer of the original form of the acid; but, since the amorphous acid and its liquid esters exhibit a full range of anomalies in the absence of any solvent, the first two possibilities are ruled out, and the alternatives are reduced to two, involving (i.) polymerisation or depolymerisation as suggested by WENDELL ('WIEDEMANN'S Ann. Phys. Chem.,' 1898, vol. 66, p. 1156), or (ii.) isomeric change. These alternatives may even be combined, since when isomeric change takes place, it is not unusual for one or other of the isomers to undergo association if the physical and chemical conditions are favourable.

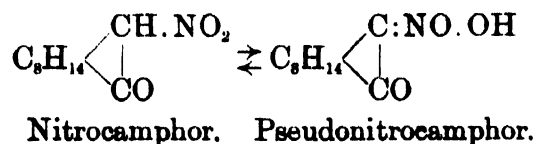
ARNDTSEN'S hypothesis may then be interpreted in the light of modern knowledge, by suggesting that tartaric acid and its esters afford yet another example of "dynamic isomerism" or reversible isomeric change. This phenomenon was discovered in 1877 by BUTLEROW ('Liebig's Ann. der Chem.,' 1877, vol. 189, p. 77), who made use of it in order to account for the production of two types of derivatives from prussic acid, a result that could be explained most readily by assuming that the liquid acid was a mixture of the two parent-compounds, thus



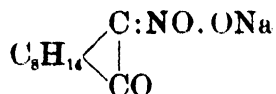
These two isomerides have not yet been isolated, although BUTLEROW was able to demonstrate the existence of a similar equilibrium between two isomeric olefines when isodibutylene was dissolved in strong sulphuric acid. If tartaric acid gave rise, when fused or dissolved, to a similar mixture of isomerides with suitable optical properties, ARNDTSEN'S hypothesis would afford a complete explanation of the anomalous dispersion of the acid.

(e) *Plastic and Fixed Derivatives of Nitrocamphor and of Tartaric Acid.*

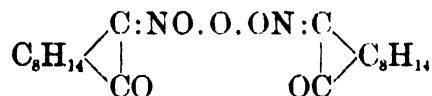
Nitrocamphor, which exists in solution in two optically active forms, of opposite sign and unequal dispersion, affords a still more striking example of dynamic isomerism. The ordinary form of the compound is lævorotatory, but its rotatory power in freshly prepared solutions changes from left towards right, giving rise to the phenomenon of mutarotation, or change of rotatory power with time (LOWRY, 'Trans. Chem. Soc.,' 1899, pp. 75, 211) as the result of a reversible isomeric change, which can be expressed by the balanced equation:



In the case of nitrocamphor, only one form of the parent substance is known, although derivatives of both types have been prepared, including dextrorotatory salts, such as

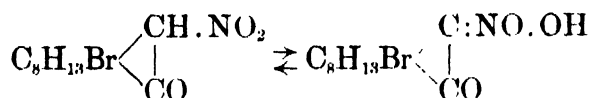


and a dextrorotatory anhydride



derived from the acidic pseudo-form of the nitro-compound.

In the case of π -bromonitrocamphor, however, both isomers have been isolated and have been found to change in opposite directions to an equilibrium mixture of intermediate rotatory power (*ibid.*, p. 225).



Nitrocamphor agrees with tartaric acid in that its optical rotatory power is exceptionally sensitive to changes of conditions, no doubt in part as a result of the displacement of the point of equilibrium, *e.g.*, in different solvents. As in the case of tartaric acid, the rotations may even exhibit a change of sign under some extreme conditions; thus in certain oxygenated solvents nitrocamphor, which is usually lævorotatory, gives small dextrorotations, just as dextro-tartaric acid, when dissolved in an excess of alkali, may give small lævorotations (see pp. 280-282 below).

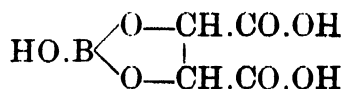
It is, however, characteristic of nitrocamphor that it is able also to give derivatives of fixed type (compare the conversion of prussic acid into methyl cyanide and methyl isocyanide) in which the plasticity of the parent substance has disappeared. Thus by chlorination nitrocamphor is converted into two derivatives of the normal nitro type, whilst in the formation of the anhydride and of the salts a complete conversion into the acid pseudo-nitro type takes place. The chloro-nitrocamphor corresponding to nitrocamphor itself gives $[\alpha]_D -5^\circ$ in chloroform, whilst the anhydride gives $[\alpha]_D +167^\circ$ in chloroform.

In the case of tartaric acid, it has been less easy to discover fixed derivatives to correspond with these compounds, since (as has already been noted) the esters show just the same anomalies as the acid itself, and any molecular rearrangement which involves the hydrogen atoms of the carboxyl groups is therefore ruled out as an explanation of the flexible rotatory power of the acid. BIOT in 1835 ('Comptes Rendus,' 1835, vol. 1, pp. 458-459) appeared to have found a solution of this problem when he stated, in his sealed note to the Academy, that "The combinations of tartaric acid with solid bases, also with boric acid, give products endowed with rotation towards the right;

but the relative intensity of these rotations for the different simple rays, obeys the general law of this phenomenon, to which tartaric acid alone is a marked exception, at least among all the bodies which I have been able to study hitherto."

Preliminary observations appeared to confirm BIOT'S observations, since the dispersions of the tartrates in aqueous solutions not only approximated very closely to the requirements of the simple dispersion formula, but, in the case of sodium tartrate, showed a surprisingly close agreement with the law of inverse squares. The conclusion that the rotatory dispersion of the tartrates is "simple" was, however, open to grave suspicion on account of the extreme smallness of the dispersion-constants, corresponding with absorption-bands not far removed from zero wave-length. Moreover, on calculating a simple dispersion formula for sodium tartrate from the data for Hg 5461 and Hg 4359, selected from a particularly long series of readings (including 16 wave-lengths in the visual and 6 wave-lengths in the photographic region of the spectrum, instead of the short series of four or five wave-lengths which we have generally used in investigating cases of simple rotatory dispersion), we obtained an unmistakable series of positive differences from red to green, negative between green and violet, and positive again beyond the violet mercury line. These differences were observed in two solutions of different concentrations, and could not therefore be accidental. It was therefore clear that the curves for the metallic tartrates were not really simple, but complex, with a large positive and a small negative term, so that the anomalies would be pushed right out into the ultra-violet region of the spectrum where the solutions are too opaque for observation by ordinary methods.

In the case of boric acid we were more fortunate, since when an excess of boric acid was added to tartaric acid the aqueous solution of boro-tartaric acid gave a simple dispersion curve, with a normal value, 0.0246, for the dispersion-constant λ_0^2 . Boro-tartaric acid then appears to be a "fixed" derivative of tartaric acid, in which the acid has been locked up in one of its two alternative forms. A similar result was obtained with tartar emetic, which gives very large dextrorotations, but a perfectly simple dispersion, with a dispersion-constant $\lambda_0^2 = 0.0494$. Boro-tartaric acid, which has many analogies amongst the polyhydric alcohols of the sugar group (see p. 252) is probably



and it is possible that the simple character of its rotatory dispersion may be due to the bridge between the two asymmetric carbon atoms which is shown in this formula.

The search for a "fixed" derivative of the elusive levorotatory modification of the acid proved even more difficult than in the case of the dextrorotatory component. The negative rotations discussed on pp. 280 to 284 of this paper are usually complex in their dispersion, but we were fortunate in discovering that the levorotatory solutions obtained by dissolving tartar emetic in an excess of alkali are not only comparable in

rotatory power with the parent substance, but like it exhibit a perfectly simple rotatory dispersion, with a dispersion-constant $\lambda_0^2 = 0.0627$.

After the mathematical evidence set out under (c) above, the discovery of these fixed compounds is the most important evidence that has yet been put forward in support of ARNDTSEN'S hypothesis. If it is difficult to discover in the formula commonly assigned to tartaric acid any physical basis for the anomalous optical properties of the acid and of so many of its derivatives, it would be at least equally difficult to discover either in boro-tartaric acid or in tartar emetic any factor which would account for the disappearance of the anomalies, apart from the view which has already been advanced that in these compounds the plastic acid has been fixed in one of its labile forms.

4. EXPERIMENTAL METHODS.

The experimental work described in the present paper was undertaken with the object of applying to the problems investigated by BIOT the exact methods of measuring rotatory dispersion which have been developed during the past 15 years, and which have already been applied (as described in the preceding paper of this series) to the exact determination of the optical rotatory power of quartz. Thus Section 5 describes a critical investigation of the relationship between the rotatory power and concentration of aqueous solutions of tartaric acid, and a detailed study of the deviations from BIOT'S linear law, as revealed by exact measurements for a series of eight wave-lengths in the visible region of the spectrum. Section 6 describes an investigation of the relationship between optical rotatory power and wave-length for a series of aqueous solutions of tartaric acid; this was undertaken in order to test in the case of tartaric acid the equations which had already been proved to be adequate to express the anomalous rotatory dispersion of the tartaric esters. The remaining sections of the paper deal with the influence of various chemical agents on the rotatory power of tartaric acid, in extension of the early pioneering work of BIOT.

The optical apparatus used in the experiments on tartaric acid was the same as that which has already been described in the preceding paper of the present series. An important improvement has, however, been effected in the matter of light-sources. In the experiments on quartz the cadmium lines were read with the help of light derived from an open arc burning between electrodes of a cadmium-silver alloy (LOWRY, 'Phil. Mag.,' 1909, vol. 18, pp. 320-327). Five years later a description was given of "An enclosed cadmium arc for use with the polarimeter" (LOWRY and ABRAM, 'Trans. Faraday Soc.,' 1914, vol. 10, pp. 103-106); this arc was of an experimental type, requiring the continuous use of a Gaede pump to maintain the vacuum, and merely served to show how very valuable an efficient lamp of this type would be in all optical experiments which demand intense sources of monochromatic light. During the period covered by the experiments now described we have had the privilege of using the enclosed cadmium arc designed by Dr. SAND and described by him at the Manchester meeting of the British Association ('B.A. Report,' 1915, vol. 85, p. 386). This lamp has proved to be per-

factly adapted to the work we have been doing, and in some respects is even more convenient than the mercury arcs which are now available for experimental work. The importance of this development may be shown by the statement that whereas twenty years ago the sodium light was the only practical light-source for use in polarimetry, this line is now usually left to the last because it is more troublesome and in every way less satisfactory to read than the lines derived from the enclosed mercury and cadmium arcs.

Photographic observations form an essential feature of the experiments which are now described. The method used was described in outline in 1908 ('Roy. Soc. Proc.,' 1908, A, vol. 81, 472-474), but has reached its highest development in connection with observations of the rotatory power of quartz for ultra-violet light, of which a detailed account will be given in the third paper of the present series.

5. SPECIFIC ROTATORY POWER OF TARTARIC ACID FOR LIGHT OF DIFFERENT WAVE-LENGTHS.

A long series of observations was made, in order to establish standard values for the specific rotatory power of tartaric acid in aqueous solutions at 20° C. Similar observations have been made by a number of observers using monochromatic sodium light (PRIBRAM and GLÜCKSMANN, 'Monatshefte,' 1898, vol. 19, p. 136), or patches picked out from a continuous spectrum, with the help of a spectroscope or by means of light-filters (WENDELL, 'Wied. Ann.,' 1898, vol. 66, p. 1153; WINTHER, 'Zeitschr. Physikal. Chem.,' 1902, vol. 41, p. 166); but no values have been given hitherto for the pure light-sources, derived from the spectra of mercury and of cadmium, which have now become the common standards in polarimetric work. The values placed on record in Table I. make good this deficiency; they also provide data for testing, for a considerable series of pure monochromatic light-sources, the linear relationship between concentration and rotatory power which was put forward by BIOT in 1834, as well as the parabolic relationship used by WINTHER in 1902.

In order to secure exact values for the specific rotatory power of the acid, all the solutions for these experiments were prepared from exactly-weighed quantities of acid and water. Two samples of acid were used, one of them (for which we are indebted to Messrs. Bennet, Lawes & Co.) being a specially pure sample, containing less than 0.05 per cent. of ash, and practically no trace of lead. In order to avoid changes of concentration caused by evaporation, the precaution was taken of using solutions which had not been filtered; small traces of insoluble matter were found to come almost exclusively from the Wedgwood mortar used to powder the crystals, and, when the further precaution was taken of merely crushing the crystals in an agate mortar, the amount of solid undissolved was negligible. The rotations of the solutions were determined in a polarimeter-tube 6 decimetres long, which had been calibrated to check the length of the column, and was maintained at a constant temperature by a rapid flow of water at 20° C. Two, three, or even four series of readings were taken, as shown in Table I., with solutions prepared independently of one another, until it appeared probable that

the casual errors had been reduced to a few hundredths of a degree. Table I. shows the observed rotations, α , and the specific rotatory powers $[\alpha]$ derived from them, for 11 solutions containing from 5 to 55 grams of acid in each 100 grams of solution; the corresponding molecular rotatory powers are also shown in the table.

The data used in testing BIOT'S linear law and WINTHER'S parabolic formula are shown in Table II. BIOT'S linear law, although based originally upon somewhat rough measurements, is very good as a first approximation. Thus, in fig. 1, in which the

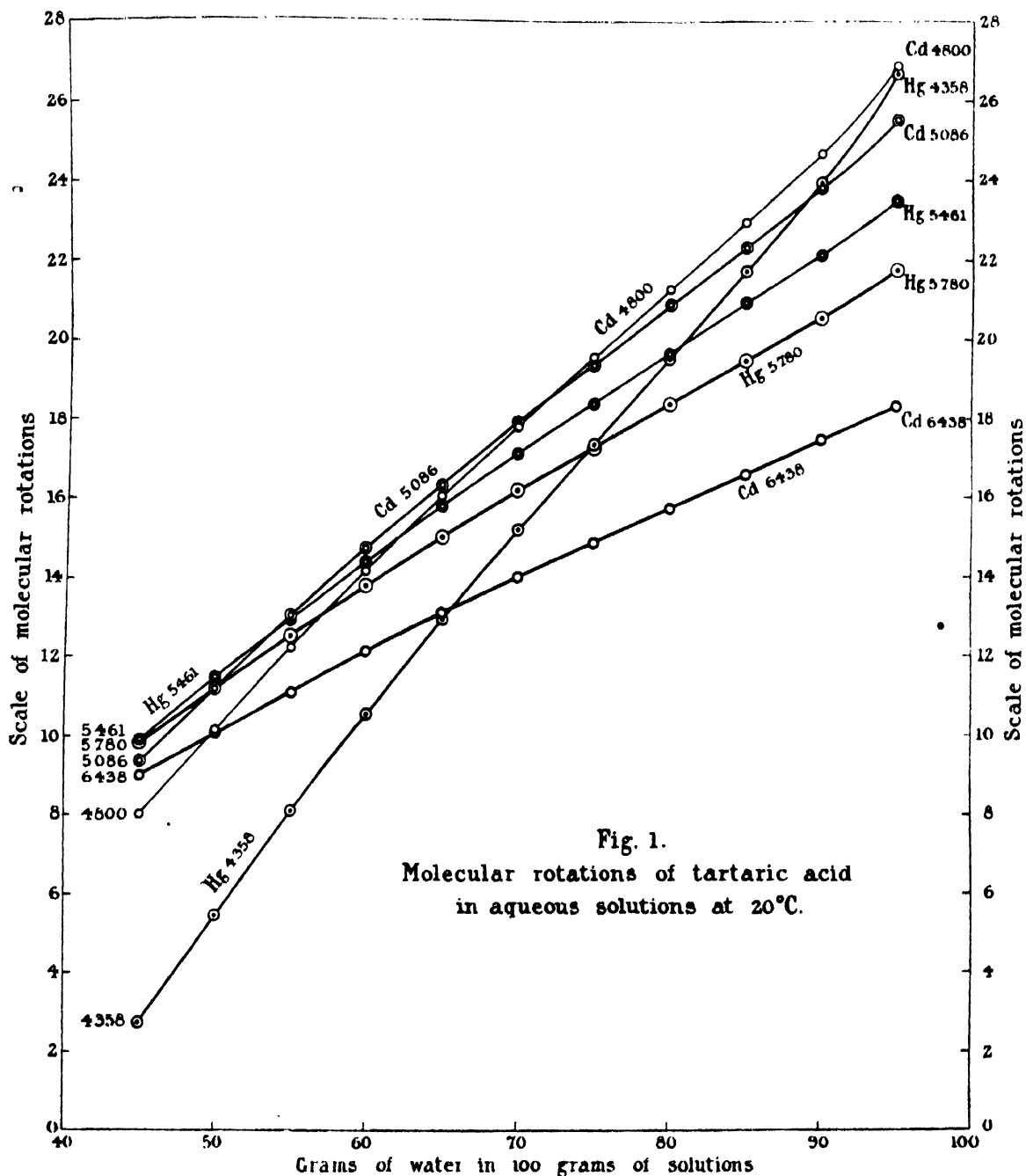


Fig. 1.
Molecular rotations of tartaric acid
in aqueous solutions at 20°C.

molecular rotatory power of the acid is plotted against the percentage by weight of water in the solution, the curves are seen to be inflected, but the deviations from the linear law, though quite real, are by no means conspicuous. More remarkable still is the fact, which BIOT discovered in 1850, that extrapolation by means of the linear law leads to substantially correct values for the specific rotatory power of the anhydrous glassy or amorphous acid. In our own calculations we have used, for each of a series of nine wave-lengths, a linear formula based on the values for the specific rotatory powers of the acid in solutions containing 55 and 85 per cent. of water; the rotatory powers of the anhydrous acid, as shown under $e = 0$ in Table II., are derived by interpolation from the observations of BRUHAT ('Trans. Faraday Soc.,' 1914, vol. 10, p. 89), and the differences between the observed and calculated values are:—

$$1.2, 1.5, 1.6, 1.7, 1.7, 1.7, 1.8, 0.6^\circ, \text{Mean } 1.5^\circ.$$

This agreement is remarkably close, having regard to the facts that (i.) the linear law is only an approximation, (ii.) the extrapolation covers nearly half of the total range of concentration, and amounts in the case of the violet mercury line to an extension of over 12 degrees in the range of rotatory powers.

In view of the fact that the linear law is valid to this extent over the wide gap between the anhydrous acid and its saturated solutions in water, we were prepared to find that, although closer examination would show marked deviations in the values for dilute solutions, no substantial errors would occur in the case of the more concentrated solutions. This anticipation was, however, by no means correct. Basing the linear formula again on the rotatory powers of the acid in solutions containing 55 and 85 per cent. of water, we find that the deviations which are produced by increasing the concentration to *water* 45 per cent., *acid* 55 per cent., are even greater than those which result from diluting to *water* 95 per cent., *acid* 5 per cent.; and this is true, not only for one wave-length, but for the whole range of wave-lengths shown in Table II. The linear law is thus shown to be even less exact in concentrated than in dilute solutions.

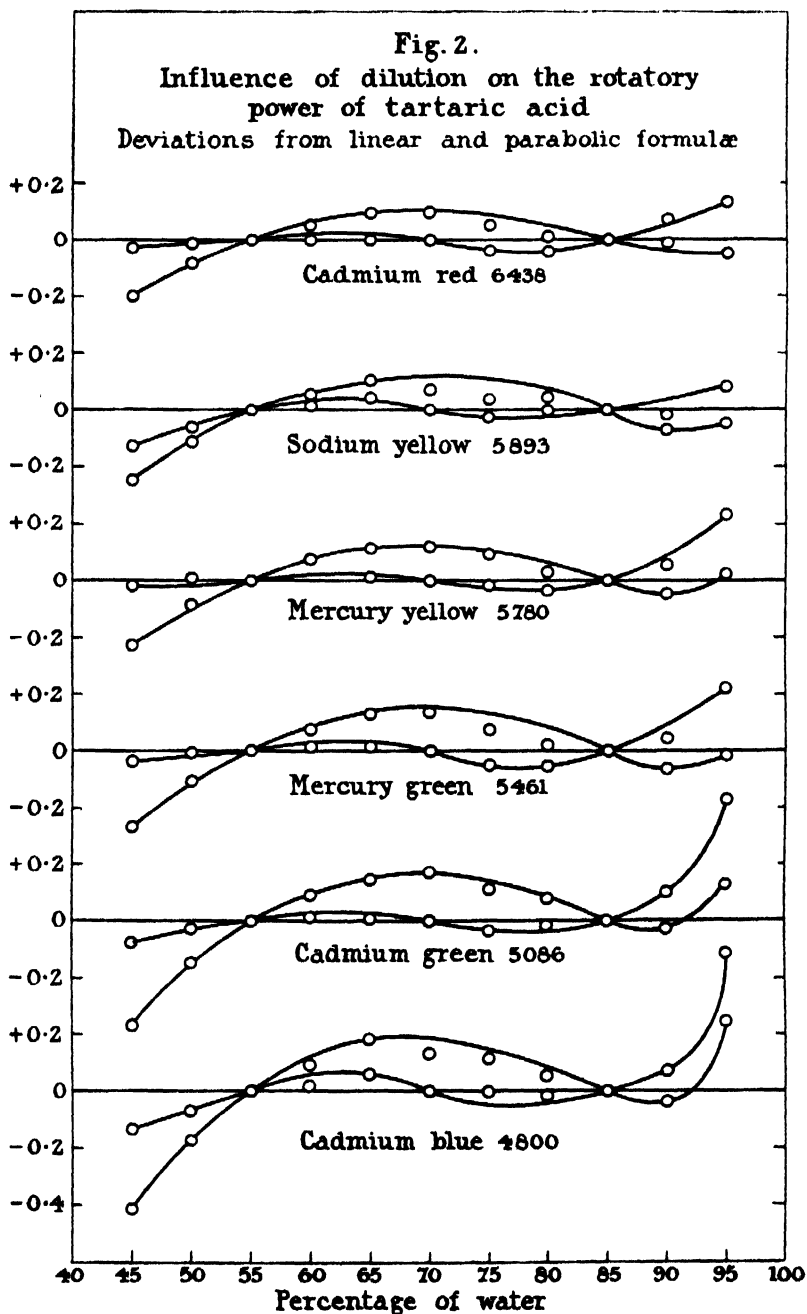
A natural sequel to the recognition of the fact that the linear law is inexact, is the introduction of a third term into the equation, which thus changes from

$$[\alpha_1] = A_1 + B_1e \quad \text{to} \quad [\alpha_2] = A_2 + B_2e + C_2e^2.$$

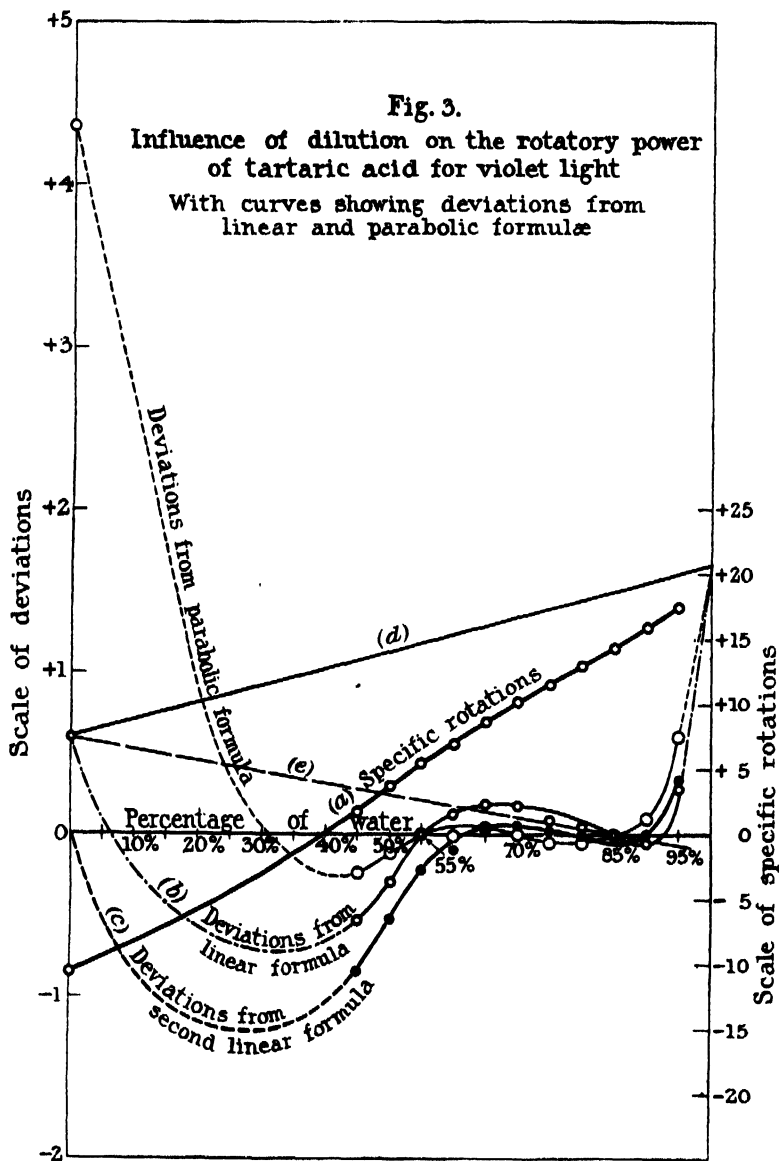
This method of expressing the specific rotatory powers of the acid was adopted by WINTHER in 1902. Its application to the data now recorded is shown in Table II. The parabolic formulæ were all based on the readings for solutions containing 55, 70 and 85 per cent. of water; in a few cases these readings may have been less exact than others that might have been selected; but no great advantage would have been obtained from any laborious attempt to smooth the values or to adjust the curves, since the general results were perfectly obvious when the errors were examined over the whole of the series of nine wave-lengths. Within the range from 55 to 85 per cent. of water, the

agreement was much better than with the linear formula, as was inevitable since there was exact agreement at three points instead of two; in the more concentrated solutions, also, the parabolic formula gave rather better results than the linear law; but in the more dilute solutions the errors were increased very considerably by the introduction of a third term, with the result that outside the limit of 55 to 85 per cent. of water, the parabolic was distinctly worse than the linear formula.

The real character of the deviations was only revealed when the errors of the various formulæ were plotted against the concentrations as in figs. 2 and 3. As in the case of



solution volumes (BOUSFIELD and LOWRY, 'Phil. Trans.,' A, 1905, vol. 204, p. 283) the errors in the measurement of specific-rotatory power increase with dilution, so that the course of the curves becomes uncertain as the proportion of water increases towards 100 per cent. It is, however, quite evident that the curve of errors, fig. 3*b*, for the linear formula intersects the zero-line at three points between $e = 0.45$ and $e = 1$.



Any attempt to flatten this curve, by the addition of a third term containing e^2 , is obviously doomed to failure, since the curvature from the straight line is in opposite directions at the two ends, and cannot be covered by a section of a parabola. The type of formula used by WINTHER is therefore only of value for interpolation over a narrow range, and is even worse than a simple linear formula for extrapolation. This is true not only for dilute solutions, where the errors immediately become larger, because the curvature

is in the wrong direction, but also for more concentrated solutions, as may be seen by comparing BRUHAT'S experimental values for the anhydrous acid, $e = 0$, on the one hand, with the values of A_1 in the linear equation, which showed deviations amounting, on the average, to 1.5° , and on the other hand, with the values of A_2 in the parabolic equation, which gives errors as follows :—

3.3, 2.9, 4.1, 4.5, 5.1, 4.5, 4.5, 4.3, Mean 4.1° .

The complete course of the curve of rotatory power against concentration is shown, for the violet mercury line, by the heavy curve a in fig. 3. The ordinates of curve b represent the errors of a linear formula which is correct at 55 and 85 per cent. of water ; the curve c shows the errors of a parabolic formula which is correct at 55, 70 and 85 per cent. of water. The error ($+0.6$ degrees) in extrapolating by means of the linear formula to the anhydrous condition is shown on the left-hand side of the diagram, where a broken line is used to cover the interval between 0 and 45 per cent. of water in which no experimental values are available. A similar prolongation of the curve of errors on the right-hand side of the diagram indicates the most probable value for infinite dilution. If the aqueous solutions contained only an anhydrous acid and a fully-hydrated acid, both of constant specific rotatory power, the relationship between rotatory power and concentration should be expressed from end to end, if not by the straight line d , at least by a simple uninflected curve. The actual deviations from this ideal straight line are shown in fig. 3 by measuring the ordinates downwards from d instead of from the horizontal axis of zero rotation. It will be seen that this curve of deviations (which are all negative in sign) shows one minimum and two maxima, so that the exact relationship between rotation and concentration could only be expressed by an equation containing at least five arbitrary constants.

In view of the complexity disclosed by this preliminary analysis, we have not thought it worth while to pursue the subject further, except to point out that in the case of each wave-length it is possible to use a linear formula (as indicated by the straight line e , and the curve of errors c , in fig. 3), which is substantially correct both for the anhydrous acid and for aqueous solutions from, say, $e = 0.6$ to 0.9 , but widely divergent outside these limits. A series of values calculated from such a linear formula, for the mercury violet line, is given under $[\alpha_0]$ in Table II. ; this formula is correct for the anhydrous acid $e = 0$, and gives a \pm error at three other points lying near $e = 0.63$, $e = 0.80$ and $e = 0.91$.

6. ROTATORY DISPERSION IN AQUEOUS SOLUTIONS OF TARTARIC ACID.

In order to determine the exact form of the curves of rotatory dispersion in aqueous solutions of tartaric acid, eight solutions were originally prepared and examined, containing from 5 up to 70 grams of tartaric acid in 100 c.c. of solution. In two cases the readings were confined to the seven visible cadmium and mercury lines ; in four other

cases the series was extended to wave-length about 4150, by the inclusion of photographic readings, and in the case of two of the strongest solutions, readings were taken for 21 and 26 lines, extending to wave-lengths 4005 and 3941 respectively.

These solutions were filtered, in order to facilitate the reading of the more difficult blue and violet lines. The concentrations, as well as the specific and molecular rotations derived from them, were therefore less certain than those set out in the preceding section, and, when the latter had been completed, the earlier observations were regarded as obsolete, with the exception of the two long series which covered a very wide range of the spectrum, and were specially well adapted for testing the form of the dispersion-curves. The rotations for these two solutions are set out in Table III. (a) and (b). In order to bring them into line with the standard series of readings of Table I. the true concentrations of these two solutions were found by interpolation from the standard series, the concentrations deduced in this way being $e = 0.4590$ and 0.5875 , or 69.90 and 49.96 grams of tartaric acid in 100 c.c. of solution, as compared with the nominal values of 70 and 50 per cent. After making these very small corrections, the specific and molecular rotations of the two long series were included with the shorter standard series of Table I. amongst the data used in discussing the relation between rotatory power and wave-length.

The calculated rotations shown in Tables III. and IV. are derived from an equation of the DRUDE type containing one positive and one negative term, thus

$$\alpha = \frac{k_1}{\lambda^2 - \lambda_1^2} - \frac{k_2}{\lambda^2 - \lambda_2^2}.$$

But whereas in the case of quartz the data are now so extensive and so accurate that *five* arbitrary constants can be determined exactly, the range and accuracy of the data in the case of tartaric acid and other optically-active organic compounds are only sufficient for the exact determination of *three* arbitrary constants; in other words, the effect of a small alteration in any one of four constants can be eliminated almost entirely by suitable alterations in the other three. This limitation has already been discussed in the case of ethyl tartrate, the rotatory-dispersion of which can be calculated almost equally well from formulæ in which the dispersion-constants are:—

- | | | |
|-------|------------------------|------------------------|
| (i) | $\lambda_1^2 = 0.035,$ | $\lambda_2^2 = 0.065,$ |
| (ii) | $\lambda_1^2 = 0.030,$ | $\lambda_2^2 = 0.070.$ |
| (iii) | $\lambda_1^2 = 0.025,$ | $\lambda_2^2 = 0.080,$ |

where the sum of the dispersion-constants λ_1^2 and λ_2^2 is almost constant (LOWRY and DICKSON, 'Trans. Chem. Soc.,' 1915, vol. 107, p. 1186). The best results are therefore obtained by assuming a steady value for one of the two dispersion-constants in a series of related compounds; thus, in 22 independent series of observations of tartaric acid and its esters, the value $\lambda_1^2 = 0.030$ may be maintained for the smaller of the two

dispersion-constants. This may be regarded as substantially correct for the whole of the series, since, in case after case, independent calculations have given numbers lying within a narrow range on either side of this average value.

Having thus established a fixed value for the smaller dispersion-constant, the magnitude of the larger dispersion-constant λ_2^2 may be deduced with a very fair degree of accuracy. Thus in the case of methyl and ethyl tartrates the values were as follows —

	<i>Methyl ester.</i>	<i>Ethyl ester.</i>
Pure ester	0·054	0·056
In ethylene chloride or bromide	0·058	0·061
In formaldehyde	0·070	0·070

The concordance in the values for the two esters when examined under similar conditions is a very fair indication of the accuracy of the dispersion-constants deduced in this way.

In the case of the two concentrated solutions of tartaric acid, the values of the second dispersion-constant deduced independently were 0·074 and 0·073. With these values, a close agreement was found between the observed and calculated rotations as set out in Table III., both in the visual and in the photographic regions. This agreement, following upon a similar concordance in 12 long series of observations of methyl and ethyl tartrate, is sufficient to establish the general validity of the two-term equation as an expression of the complex rotatory dispersion both of tartaric acid and of its esters.* It is then a simple problem to study the variations of the four constants of this equation as the concentration of the solution is increased or diminished. Preliminary observations indicated that the dispersion-constant λ_2^2 for tartaric acid in aqueous solutions might, with advantage, be reduced from 0·074 for solutions containing 70 grams of acid in 100 c.c. to 0·065 for solutions containing 5 grams in 100 c.c. ; but this diminution could not be confirmed in the more exact, though rather less extensive, observations set out in Table I. We have, therefore, preferred to make use of constant values, $\lambda_1^2 = 0·030$, $\lambda_2^2 = 0·074$, for the two dispersion-constants of this series of solutions rather than to introduce small variations which would merely have led to irregularities in the values of the "rotation-constants" k_1 and k_2 . The observed and calculated values of the molecular rotatory powers at eleven different concentrations are set out in Table IV. Table V. shows, for the 13 solutions for which data are given in Tables III. and IV., the magnitude of the four constants of the two-term equation, together with the wave-lengths at which the principal anomalies are found, namely, the inflection of the curve at λ_p , the maximum at λ_μ and the reversal of sign at λ_r . These latter have been calculated from the equations, as a convenient method of interpolation ; wave-lengths

* In the case of the acid there are indications of a predominance of negative errors between the green and violet mercury lines ; this is, perhaps, an effect of ionisation, since nothing of the sort has been noticed in the case of the esters.

which depend on extrapolation beyond the range of the experimental data are printed in italics.

From Table V., and from the graphical representation of fig. 4, the following conclusions may be drawn :—

- (1) The reversal of sign (which has been observed hitherto only in the anhydrous acid, in alcoholic and in super-saturated aqueous solutions or in the ultra-violet region of the spectrum) is here recorded with the help of the camera, in the extreme violet region of the spectrum in unsaturated aqueous solutions containing only 50 grams of tartaric acid in 100 c.c. of the solution ($e = 0.5875$ or $P = 41.25$ per cent.).
- (2) The maximum rotation, which has usually been said to vanish on dilution, is here retained on the less refrangible side of the violet mercury line even at a concentration of only 5 grams per 100 c.c. This fact is of interest in connection with the statement of ARNDTSEN that at concentrations below 20 per cent. by weight “the dispersion becomes normal in so far as the angle of rotation increases continuously with the refrangibility of the light; yet the rotation increases so little from the ray F to the ray e , that one cannot doubt that it would have a maximum in the violet part of the spectrum” (‘Ann. Chim. Phys.,’ 1858, vol. 54, p. 412). Our observations prove the reality of this hypothetical maximum, whilst, at the same time, they show the futility of any definition of “normal dispersion” which depends merely on the removal of the maximum from the easily-visible into the violet or ultra-violet region of the spectrum; a similar usage of this term by WINTHER has already been criticised (LOWRY, ‘Trans. Chem. Soc.,’ 1915, vol. 107, p. 1195) on the ground that it implies a physiological definition of a purely physical property. Anomalous dispersion depending on the imperfect compensation of two simple rotatory dispersions of opposite sign may, as a matter of actual fact, be rendered normal by displacing the maximum to infinite wave-length in the infra-red; but any displacement in the opposite direction only exaggerates the invisible maximum in the ultra-violet region.*

7. TARTRATES OF THE ALKALI-METALS.

In his sealed note, communicated to the Academy on August 25, 1834, BIOT referred for the first time to the rotatory power of the tartrates in the following terms :—“The combinations of tartaric acid with solid bases give products endowed with rotation towards the right; but the relative intensity of these rotations for the different simple rays obeys the general law of this phenomenon, to which tartaric acid alone is a marked exception, at least, among all the bodies which I have been able to study hitherto” (‘Comptes Rendus,’ 1835, vol. 1, p. 458). A detailed study of the tints produced by

* For a series of curves illustrating this point, see ‘Trans. Chem. Soc.,’ 1915, vol. 107, p. 1200.

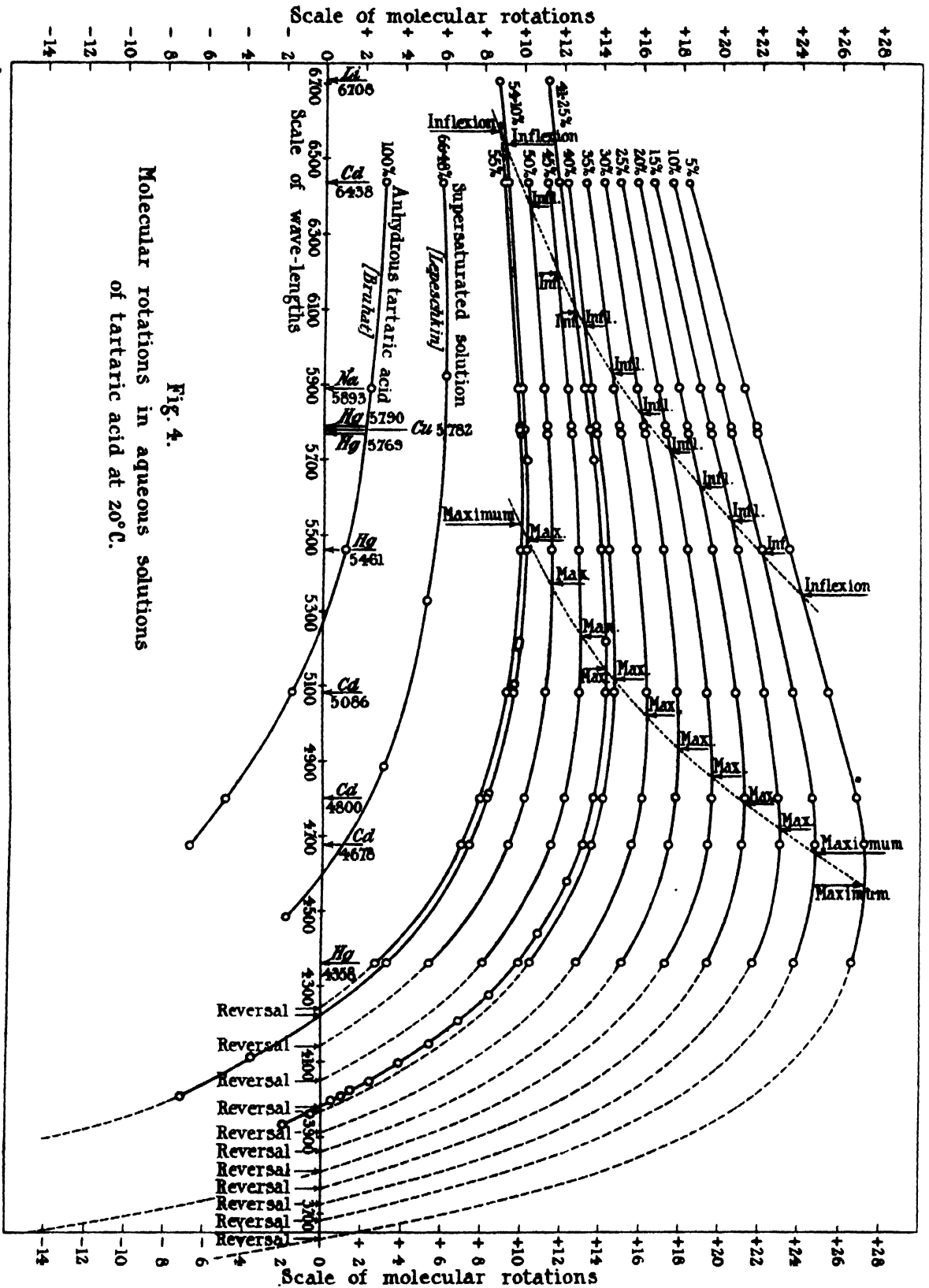


Fig. 4.
Molecular rotations in aqueous solutions
of tartaric acid at 20°C.

aqueous solutions of potassium, sodium and ammonium tartrates ('Mém. Acad. Sci.,' 1838, vol. 16, pp. 304-317) showed that these agreed very closely with those produced by quartz plates of suitable thickness, and with those calculated from BIOT'S law. A similar conclusion was reached by KRECKE, who measured the rotatory powers of the potassium, sodium and ammonium salts over the range from 0° to 100° C., and concluded that "the tartrates examined obey the laws of BIOT" ('Arch. Néerland.,' 1872, vol. 7, p. 114). KRECKE'S figures actually show small regular deviations in the case of potassium and sodium tartrates, and very marked deviations in the figures for ammonium tartrate and for Rochelle salt. Thus his figures for $[\alpha]_{\lambda}^2$ are as follows:—

		C.	D.	E.	b.	F.
	°					
Tartaric acid 50%	0	2,427	2,227	1,602	1,533	1,331
" " 50%	100	5,792	5,289	4,942	4,928	4,646
Potassium tartrate 20%... ..	0-100*	9,407	9,245	9,166	9,285	9,479
Sodium tartrate 20%	0-100*	9,067	8,977	8,863	8,798	9,166
Ammonium tartrate 10%	25	13,376	12,860	11,948	12,081	12,608
Rochelle salt 20%	0-100*	8,179	7,944	7,542	7,508	7,673

The figures for tartaric acid, which are given for comparison, show the sensitiveness of the free acid to changes of temperature, as well as its complete liberation from the requirements of BIOT'S law. Thus the specific rotatory power of potassium tartrate for sodium light decreases by 3 per cent. only, from 27·223° to 26·415°, on raising the temperature from 0° to 100° C.; that of the sodium salt increases by 2 per cent., from 25·702° to 26·256° between 0° and 75° C., and then diminishes again to 25·211° at 100° C.; whilst in the case of Rochelle salt there is a regular increase of 9 per cent. from 21·820 at 0° to 23·993° at 100°. The rotatory power of the free acid, however, is practically three times as great at 100° as at 0° C.

Our own observations of the rotatory power of the tartrates of the alkalis are set out in Tables VI., VII., VIII. and IX. As in the two long series of readings of tartaric acid, filtered solutions (made up volumetrically) were used to facilitate the reading of the more difficult wave-lengths; their exact concentrations were then determined by comparing the readings for Hg 5461 with those made with unfiltered solutions, prepared gravimetrically from samples of the various salts in which the proportion of moisture had been checked by analysis. In the case of the ammonium salt, which decomposes on drying even at moderate temperatures, standard solutions for the determination of concentrations were made by adding the calculated amounts of a titrated solution of ammonia to weighed amounts of tartaric acid in a stoppered flask and diluting the solution to a known weight. Densities of all the solutions were also determined. The data used in these calculations are shown in Table X., where also are given a number

* Average values.

of molecular rotations for sodium light, by means of which a comparison may be made with rotations interpolated from the earlier data of THOMSEN (' Jour. Prakt. Chem.,' 1886, new series, vol. 34, p. 74) and of PATTERSON (' Trans. Chem. Soc.,' 1904, vol. 85, p. 1120).

Our measurements of the molecular rotations of the tartrates show a close general agreement with the requirements of BIOT'S law, but with small deviations of a different type from those recorded by KRECKE. Thus whilst his values for $[M]\lambda^2$ always passed through a minimum in the central portion of the spectrum, ours either pass through a maximum as in Table VI., or rise progressively as the wave-length diminishes as in Tables VII., VIII. and IX. In the case of sodium tartrate (22.54 gr. of $\text{Na}_2\text{H}_2\text{C}_4\text{O}_6 \cdot 2\text{H}_2\text{O}$ in 100 c.c. of solution, Table VI. (a)) the agreement of our numbers with the requirements of BIOT'S law is remarkably close; thus the product $[M]\lambda^2$ has the value 20.29 in the visible red region at wave-lengths 6708 to 6438, and the value 20.31 in the extreme violet at wave-lengths 4072 to 4005, rising in the intermediate region to a shallow maximum 20.68 in the blue at wave-length 4678; if all the 22 readings are considered the average value of $[M]\lambda^2$ is 20.40, the maximum errors are +0.28 and -0.23, and the average error is only ± 0.11 or 0.5 per cent.

In view of the smallness of the deviations from BIOT'S law, it seemed probable that a complete agreement might be obtained between the observed and calculated values by using a "simple" dispersion formula containing a second arbitrary constant. In each case, therefore, the constants of the "simple" formula $[M] = k/(\lambda^2 - \lambda_0^2)$ were calculated in the usual way from the rotations for the two dominant mercury lines Hg_{5461} and Hg_{4358} , as shown in Table XI.

Special attention should be given to the values of the "dispersion-ratio" $\alpha_{4358}/\alpha_{5461}$, shown in the last column of Table XI., which increases and diminishes with the magnitude of the "dispersion-constant" λ_0^2 . The latter constant being the square of a real quantity must always be positive, so that the smallest value for this dispersion-ratio is that given by BIOT'S law, where $\lambda_0^2 = 0$, namely,

$$\frac{\alpha_{4358}}{\alpha_{5461}} = \frac{(5461)^2}{(4358)^2} = 1.570.$$

The ratios shown above come nearer to this minimum than in any case hitherto investigated, and in one of the solutions of sodium tartrate, where λ_0^2 falls to 0.00032, this minimum is almost attained. Some of the lowest values previously recorded are shown in Table XII.

These figures show that the dispersion-ratio rarely falls below 1.630 or the dispersion-constant below 0.018; in other words, the absorption-band which determines the position of the vertical asymptote of the dispersion-curve may be pushed out so far as $\lambda = \sqrt{0.018} = 0.135\mu$ or 1350 Å.U., but it never goes much beyond this, even in the case of the most transparent of the substances quoted in Table XII. When dealing

with the natural rotatory power of optically active liquids, the asymptote rarely goes beyond $\lambda = \sqrt{0.024} = 0.15\mu$ or 1500 \AA.U. In view of this strict limitation of transparency, in carbon-compounds generally as well as in water and in silica, it seemed improbable that the metallic tartrates could be transparent to light of excessively short wave-length, especially as the magnetic rotations in ethyl tartrate are controlled by an absorption-band in exactly the same position as in the simple alcohols. The validity of the simple formula, therefore, lay open to suspicion, on account of the extreme smallness of the dispersion-constant, and it appeared much more probable that the dispersion-ratios in the tartrates had been brought down into close proximity with the minimum value 1.570 by the influence of a second (negative) term in DRUDE'S equation, similar to that which brings the dispersion-ratios in tartaric acid down below this minimum and gives rise to the anomalous dispersion of the acid.

The evidence which first convinced us that the simple dispersion-formula, like BIOT'S law, is merely an approximate expression of the rotatory dispersion in the tartrates, is shown in Tables VI. (a) and (b). In these two tables a "simple" formula is shown which has been calculated to fit the experimental values for the green and violet mercury lines. The average error is only a fraction of 1 per cent.; but without exception, all the errors between the green and violet lines are positive in sign, whilst all those beyond the violet are negative; below the green, the errors are again negative in eight out of ten cases. This regular distribution of the errors, which was confirmed in the case of ammonium tartrate, Table IX., for 14 out of 15 visual readings in one series and for 12 out of 13 in a second series which has not been reproduced, proves clearly that the deviations are not accidental but systematic, calling for some further modification of the formula used to express the observations.

When once it has been recognised that the dispersion curves for the alkali-tartrates cannot be represented by the "simple" formula $\alpha = k/(\lambda^2 - \lambda_0^2)$, no difficulty is experienced in securing a satisfactory agreement between the observed and calculated rotations, by using a DRUDE equation with one positive and one negative term, as is shown in the last columns of Tables VI., VII., VIII. and IX., where the errors in the visual readings are seen to be small and for the most part distributed quite irregularly. On account of the smallness of the deviations from the simple law, it is difficult to determine the exact magnitude of the two additional constants which serve to express them in the equation. The dispersion-constants selected for these calculations were $\lambda_1^2 = 0.038$, $\lambda_2^2 = 0.06$, but since the sum of these two constants 0.098 is almost identical with the sum of the constants $0.030 + 0.074 = 0.104$ used in the case of tartaric acid, there can be little doubt that the latter pair would have given equally satisfactory results.

The negative rotation-constants, k_2 , as set out in Table XIII., are larger than might have been expected, approaching almost to one-half of the values for the positive rotation-constant k_1 . This is due to the fact that the dispersion-constant λ_2^2 of the negative term is but little greater than the dispersion-constant λ_1^2 of the positive term; the

negative term must therefore be weighted somewhat heavily to produce any marked alteration in the form of the curve.

8. NEGATIVE ROTATIONS IN SOLUTIONS OF *d*-TARTARIC ACID AND ITS SALTS.

Although the common form of tartaric acid, and the tartrates derived from it, are usually dextrorotatory, negative rotations are occasionally observed, especially at high concentrations and low temperatures, and in the more refrangible portions of the spectrum. BIOT, who had predicted this phenomenon in 1838 ('Mém. Acad. Sci.,' 1838, vol. 16, p. 269), detected it twelve years later in the cold, glassy acid ('Ann. Chim. Phys.,' 1850, vol. 28, p. 353); the more elaborate work of BRUHAT ('Trans. Faraday Soc.,' 1914, vol. 10, p. 89) on the rotatory power of the anhydrous acid has shown that, whilst the rotations are positive in the red and yellow regions of the spectrum, they become negative for wave-lengths less than 5600 at 15° C. None of the aqueous or alcoholic solutions examined by BIOT gave negative rotations, but LEPESCHKIN ('Ber. Deutsch. Chem. Gesell.,' vol. 32, p. 1180-1184) detected a negative rotation $[\alpha]_{4492}^{20} = -1.22^\circ$ in the dark blue region of the spectrum when working with a supersaturated aqueous solution containing 66.5 per cent. by weight of tartaric acid. Our own observations include negative readings in the violet region for unsaturated solutions containing 50 grams of tartaric acid in 100 c.c. of solution; and NUTTING ('Physical Review,' vol. 17, p. 11) has observed very large negative rotations, up to $[\alpha]_{2760}^{19} = -296.8^\circ$, in the ultra-violet, in a solution containing 28.62 grams per cent. of tartaric acid. GROSSMANN ('Trans. Faraday Soc.,' 1914, vol. 10, p. 63) observed only positive rotations in methyl alcohol, but negative values have been recorded by ARNDTSEN ('Ann. Chim. Phys.,' 1858, vol. 54, p. 415) for more concentrated solutions. GROSSMANN showed, however, that solutions in ethyl alcohol, even at a concentration as low as 5 per cent., give negative rotations in the blue region beyond 4700, whilst similar solutions in propyl alcohol are levorotatory beyond 5200. Negative rotations for sodium light have been observed in solutions of tartaric acid in acetone mixed with ether or with chloroform (LANDOLT, 'Ber.,' 1880, vol. 13, p. 2333), in water mixed with butyl alcohol (PRIBRAM, 'Monatshefte,' 1888, vol. 9, p. 485) and in alcohol mixed with benzene, toluene, chlorobenzene or ethyl bromide (PRIBRAM, 'Ber.,' 1889, vol. 22, pp. 6-11).

This depression of rotatory power and ultimate reversal of sign may be regarded as a result of getting the tartaric acid into solution in much the same condition as that in which it exists in the anhydrous amorphous state, without developing the great increase of dextrorotatory power which results from hydration, or the lesser increase which results from interaction with the lowest members of the series of alcohols. A similar depression of dextrorotatory power is produced by formic acid, acetic acid, and especially propionic acid (GROSSMANN, *loc. cit.*, p. 65) as well as by mineral acids. In the special case of sulphuric acid, BIOT found that the addition of 20 per cent. of the strong acid to an aqueous solution of tartaric acid lowered its specific rotatory power by about one-third

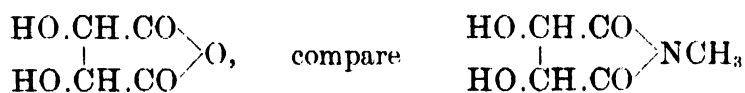
from $\alpha_{\text{RED}} = 15.4$ to 10.90 ('Mém. Acad. Sci.,' 1838, vol. 16, p. 280); but a concentrated solution containing

$\text{C}_4\text{H}_6\text{O}_6$, 22.68 gram, SO_3 , 65.02 gram, H_2O 95.90 gram

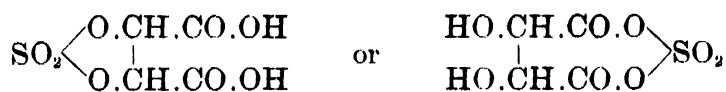
gave a much lower rotation and a novel type of dispersion in which the maximum disappeared into the infra-red, whilst a reversal of sign occurred in the blue region, giving rise to lævorotations in the violet thus :

red $+2.200^\circ$, yellow $+0.850^\circ$, green $+0.550^\circ$, violet -4.950°

(*ibid.*, p. 301). GROSSMANN has recently examined solutions of tartaric acid in sulphuric acid of higher concentrations (including the anhydrous acid containing 100 per cent. H_2SO_4), and has shown that lævorotations are found for violet light of wave-length 4620 from 45 to 75 per cent. of sulphuric acid only; when sulphuric acid containing less than 35 per cent. of water is used as a solvent, the tartaric acid increases again in rotatory power, and finally gives a specific rotation from $3\frac{1}{2}$ to 7 times as great as in aqueous solutions ('Trans. Faraday Soc.,' 1914, vol. 10, p. 67). This case is of special interest on account of the clear evidence which it affords of the formation of some new chemical compound of high dextrorotatory power, perhaps an anhydride :



for which $[\alpha]_{\text{D}} = +124^\circ$, $[\text{M}]_{\text{D}} = +281^\circ$, or a sulphate :



Of special interest is the fact, which is disclosed by plotting $1/\alpha$ against λ^2 , that, whatever the nature of this compound may be, it is sufficiently "fixed" to give rise to a rotatory dispersion which, in the case of five readings out of six, appears to obey the "simple" law $\alpha = k/(\lambda^2 - \lambda_0^2)$. In view of the importance of these observations we made several attempts to repeat them, in order to test the validity of the "simple" dispersion law by means of fresh data extending over a wider range and including a larger number of wave-lengths; but, up to the present, we have not succeeded in preparing solutions sufficiently clear to use for such a test.

In quite a different category must be placed BIOT'S observation that aluminium tartrate, which is strongly dextrorotatory in dilute solutions, becomes lævorotatory when the solution is concentrated ('Comptes Rendus,' 1835, vol. 1, p. 459; 'Mém. Acad. Sci.,' 1838, vol. 16, Tables 12, 13, 14, at end of volume). It is possible that trivalent aluminium behaves in some respects like boron or antimony, and that concentration is accompanied by a change of structure analogous with that which is produced

by the addition of an excess of potash to tartar emetic (see below, p. 285). A detailed investigation of the aluminium tartrates from this point of view would be of great interest, but was not undertaken in the course of the present research.*

Confirmation has, however, been obtained of BIOT's observation that the alkali-tartrates become lævorotatory when dissolved in an excess of strong alkali; thus whilst the rotatory power of potassium hydrogen tartrate in dilute solutions is about twice as great as that of the free acid, and the rotatory power of the neutral potassium salt is about three times as great, the further addition of potash (perhaps producing some displacement of hydrogen by potassium in the two hydroxyl-groups) causes the rotatory power to diminish and finally to become reversed in sign. BIOT found ('Mém. Acad. Sci.,' 1838, vol. 16, p. 345, and Table 6, facing p. 338) that a solution containing

$K_2C_4H_4O_6$, H_2O , 2.6 per cent. ; K_2O , 36.2 per cent. ; H_2O , 61.26 per cent.

was lævorotatory and gave the remarkable dispersion shown by the following readings,

red -0.433° , yellow -0.617° , green -0.2° ,

where the shallow maximum is of opposite sign to that observed in tartaric acid. Our own observations on a similar, but more concentrated, solution containing

$K_2C_4H_4O_6$, H_2O , 2.10 gr. ; K_2O , 36.43 gr. ; H_2O , 51.89 gr.

in about 60 c.c. (density 1.504) gave a series of very small negative readings from which the molecular rotations shown in Table XIV. (a) may be deduced. The dispersion is obviously not simple, and the high value of the dispersion ratio $\alpha_{4858}/\alpha_{5461} = 3.3$ suggests that the rotation would become positive in the infra-red. BIOT's reversed maximum is not confirmed by our observations.

Sodium tartrate is more readily soluble both in water and in strong alkali. BIOT (*ibid.*, p. 364, and Table 8 at end of volume) found that a solution containing

$Na_2C_4H_4O_6$, $2H_2O$, 13.27 per cent. ; Na_2O , 19.69 per cent. ; H_2O , 67.04 per cent.

gave the following rotations :—

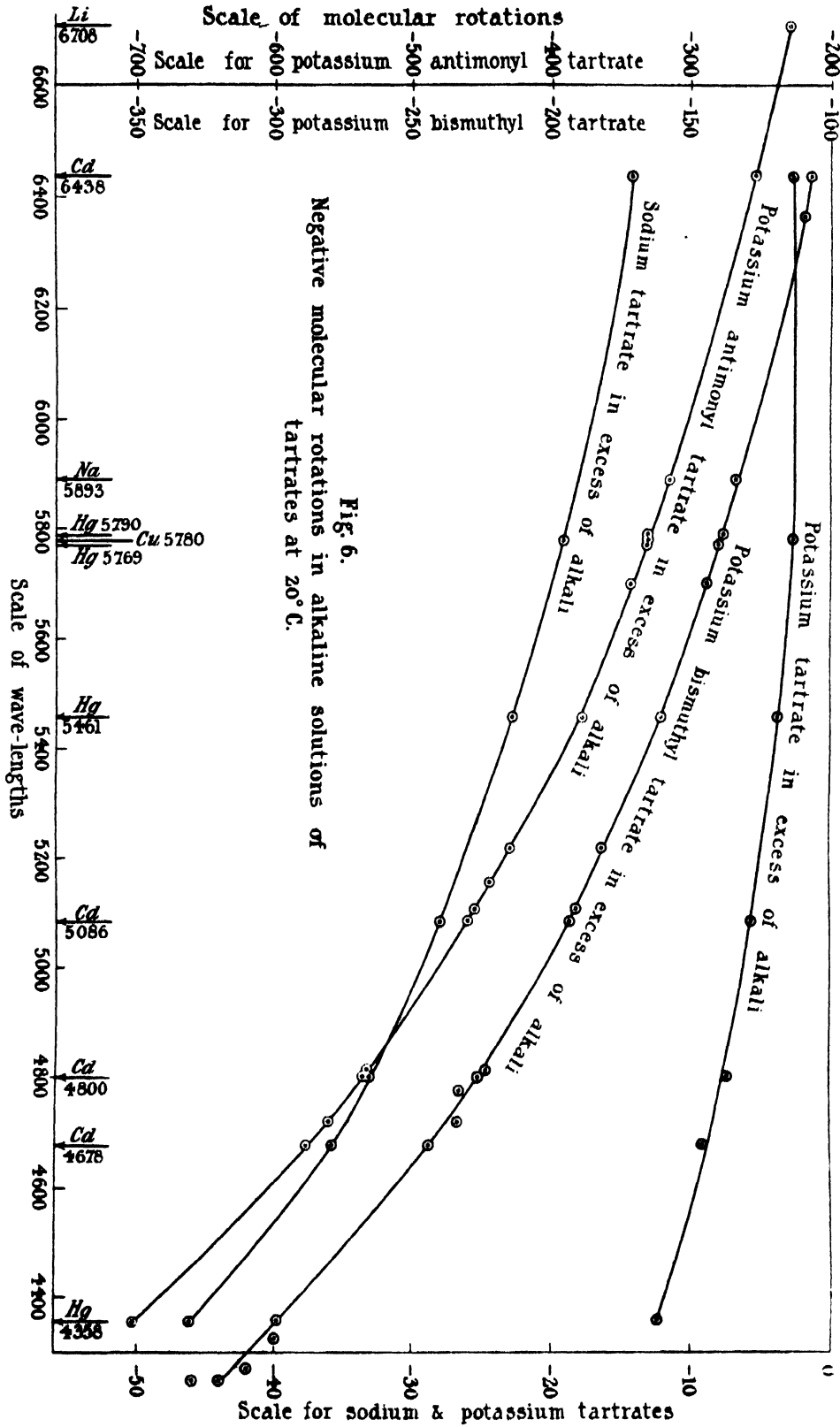
red -2.675° , yellow -4° , green -4.700° .

Our own observations on a stronger solution containing

$Na_2C_4H_4O_6$, $2H_2O$, 13.3 gr. ; Na_2O , 20.41 ; H_2O , 59.18,

in about 66 c.c. (density 1.4093) gave the very substantial negative readings shown in Table XIV (b). These readings are large enough to give smooth values (fig. 6) for the molecular rotatory power of the dissolved salt. They confirm the increase of lævorotatory power with diminishing wave-length already recorded by BIOT, but they do not

* This investigation is now being carried out by Miss GRAHAM (26.1.22).



conform to BIOT'S law since $[M]\lambda^2$ increases from -3.9 at Cd 6438 to -5.7 at Hg 4358; and they cannot be expressed by the simple DRUDE formula $\alpha(\lambda^2 - \lambda_0^2) = \text{const.}$, since the line obtained by plotting $1/\alpha$ against λ^2 has a perfectly definite though very slight curvature.

The effects produced by the addition of a few molecular proportions only of alkali to aqueous solutions of potassium and sodium tartrates are shown in Table XV. (a) and (b). The molecular rotations are a little lower than in Tables VI. and VII., but the solutions are still dextrorotatory, and the dispersions, though certainly complex, can be expressed very closely by a "simple" formula, which, in the case of the sodium salt, is identical with BIOT'S law; the readings are, however, too small to reveal the complexity first disclosed in Table VI. (a).

9. TARTAR EMETIC.

Tartar emetic, $K(\text{SbO}) \text{C}_4\text{H}_4\text{O}_6, \frac{1}{2}\text{H}_2\text{O}$, although one of the most interesting of the tartrates, does not appear to have been examined by BIOT. The rotatory dispersion in tartar emetic was measured, apparently for the first time, in 1872 by KRECKE ('Arch. Néerland., 1872, vol. 7, p. 114), who observed a very close agreement with the requirements of BIOT'S law. Thus he gives for $[\alpha]\lambda^2$ the following numbers:—

	C	D	E	b	F
At 0°	52,405	51,253	50,966	51,308	53,142
At 100°	48,436	45,193	45,796	45,851	49,307

He calls attention to the fact that "the specific rotatory power of tartar emetic is extraordinarily great, and diminishes with rise of temperature," and includes this salt in his general statement that "the tartrates examined follow the law of BIOT."* Even more interesting than the high rotatory power of the salt is the fact discovered by GROSSMANN ('Zeitschr. Physikal. Chem.,' 1907, vol. 57, pp. 533-556) that when sodium hydroxide is added, the specific rotation of the salt assumes a large negative value, probably because "in the alkaline solution an antimonyl alkali tartrate is present, in which the hydrogen atoms of the alcoholic hydroxyl groups are also displaced."

Our own experiments give no support to KRECKE'S view that tartar emetic obeys BIOT'S law. Thus the first preliminary series of observations gave for the product $[M]\lambda^2$ values which increased progressively from 156.17 at wave-length 6708 to 203.85 at wave-length 3917. But a "simple" dispersion formula $[M] = 140.67/(\lambda^2 - 0.0477)$, based upon the readings for the two dominant mercury lines, showed an agreement that was very satisfactory, especially in view of the fact that the observed rotations had been multiplied by nearly 12 to convert them into molecular rotations. Moreover, the

* For further observations on tartar emetic, see LONG, 'Amer. Journ. Sci.,' 1889, series (3), vol. 38, p. 264; 1890, vol. 40, p. 275.

magnitude of the dispersion-constant 0.0477 did not present the same grounds for suspicion as in the case of the tartrates of the alkali metals. In view, however, of our experience with these tartrates, we considered it necessary to make a series of measurements at least as complete and as extensive as that which had disclosed the real complexity of the rotatory dispersion in sodium tartrate. The solution used for these measurements was made up to be approximately M/60 or 5.53 grams of tartar emetic per 100 c.c.; a comparison with the rotations of unfiltered solutions made up gravimetrically, Table X., gave the concentration as 5.32 grams per 100 grams of solution of density 1.0368, or 5.516 grams per 100 c.c., and this figure was adopted in calculating the molecular rotations. The rotations of this solution are set out in Table XVI., which gives the results of visual readings for 18 lines and photographic readings for five lines, as compared with 16+6 readings in the case of sodium tartrate. Although the observed rotations are multiplied by 10 in order to convert them into molecular rotations, their agreement with the calculated values is very satisfactory; even in the photographic region the largest error is only 1 per cent., and all the errors appear to be distributed quite casually, without any of the long series of positive or negative errors observed in the case of the alkali-tartrates. We therefore conclude that the rotatory dispersion in tartar emetic is definitely "simple" in character, and that in this compound we have for the first time succeeded in eliminating completely that complexity which is so strongly developed in the acid and its esters.

Similar remarks may be applied to the alkaline solutions of tartar emetic. The preliminary series of observations for seven lines in the visible spectrum showed that $[M]\lambda^2$ increased from -105.8 at wave-length 6438 to -133.3 at wave-length 4358, but that the dispersion could be represented satisfactorily by a simple formula. The simplicity of the rotatory-dispersion was, however, confirmed by a series of 17 visual readings as shown in Table XVII. Photographic readings are not included, as the solutions are not stable and show a marked diminution of rotatory power on keeping; the visual readings could be taken before any serious alteration had occurred, but the photographic readings would have occupied too much time for this to be done successfully.

10. COMPOUNDS OF ARSENIC AND BISMUTH.

The arsenyl compounds $H(AsO)C_4H_4O_6$, $K(AsO)C_4H_4O_6$, $Na(AsO)C_4H_4O_6$, corresponding with tartar emetic were examined by LANDOLT ('Ber. Deutsch. Chem. Gesell., 1873, vol. 6, p. 1077, and "Optical Rotatory Power," tr. 1902, p. 553). No measurements of dispersion were made, but the data showed that the arsenyl compounds have much the same rotatory power as the simple alkali-salts and do not exhibit any of the special qualities of tartar emetic. GROSSMANN (*loc. cit.*) states that the addition of sodium hydroxide to sodium arsenyl tartrate appeared to cause complete decomposition of the salt into neutral disodium tartrate and optically-inactive sodium arsenite. Our own observations on the rotatory dispersion in an alkaline solution of the arseno-tartrate,

Table XVIII. (a), are in agreement with GROSSMANN'S view. The rotations are positive and of the same order of magnitude as in the case of an alkaline solution of the potassium salt. The dispersion shows the same rough conformity with BIOT'S law, and the same close approximation to the requirements of a simple formula with an abnormally small dispersion-constant, as in Table XV. (a), but the observed rotations are too small in magnitude and too few in number to provide conclusive evidence of the complexity of the dispersion. In the absence of an alkali, the formation of an arseno-tartaric acid is indicated by the increase of rotatory power, Table XVIII. (b), which is produced by saturating a solution of tartaric acid with arsenious oxide and filtering off the excess of arsenic. The dispersion-curve is not "simple," like that of tartar emetic, since the dispersion-ratio $\alpha_{4368}/\alpha_{5461} = 1.476$ is less than the minimum value 1.570 for $\lambda_0^2 = 0$; it is probable, therefore, that the arseno-tartaric acid is already partially dissociated, and that the addition of an excess of alkali merely completes the hydrolysis already initiated by the water in the solution.

The dispersion of the corresponding compounds of bismuth do not appear to have been investigated hitherto*, but we have been able to prepare an alkaline solution of a bismuthyl compound which shows all the characteristics of alkaline solutions of tartar emetic. Its rotatory power is strongly negative, though less than in the case of tartar emetic, and its rotatory dispersion, as set out for 21 wave-lengths in Table XIX., agrees very well with a simple dispersion formula $[M] = -37.414/(\lambda^2 - 0.0645)$. The dispersion-constant in this formula agrees closely with the value 0.06275 given by an alkaline solution of tartar emetic; it is therefore clear that in alkaline solutions bismuth yields a levorotatory compound with tartaric acid of exactly the same type as the compound formed from tartar emetic, and that both compounds differ from the acid and from its simple salts in giving simple rotatory dispersion. The simple dispersion in these more complex tartrates is indeed the most striking discovery that we have made in the course of the present investigation.

11. BORO-TARTARIC ACID.

Boric acid was mentioned in BIOT'S sealed communication of August 25, 1834, as a substance which (like the alkalis) combined with tartaric acid, giving dextrorotatory solutions which obeyed the general laws of rotatory dispersion as they had been established in the case of quartz and of several optically-active liquids ('Comptes Rendus,' 1835, vol. 1, p. 458; compare 'Mém. Acad. Sci.,' 1838, vol. 16, p. 271). The action of boric acid was described in detail ten years later ('Ann. Chim. Phys.,' 1844, vol. 11, pp. 82-112; see also 1860, vol. 59, pp. 229-256) in a long memoir "On the Employment of Polarised Light to Study Various Questions of Chemical Mechanics." The acid produced a marked increase in the dextrorotatory power of tartaric acid; and this could be represented by a hyperbolic formula tending to a constant value when the

* For rotations of sodium light in potassium bismuthyl tartrates, see ROSENHEIM, VOGELSANG, and GROSSMANN, 'Zeitschr. für anorg. Chem.,' 1906, vol. 48, p. 209.

boric acid was in large excess, or less exactly by a linear formula representing a tangent to the hyperbola. This linear formula was found to apply also to the enhancement of rotatory power in a glassy mixture of amorphous tartaric and boric acids; four such mixtures gave rotations (for the neutral tint) ranging from $+31^\circ$ to $+13^\circ$, and these when extrapolated gave for pure glassy tartaric acid a negative rotation -2.9° , agreeing closely with the value -3.28° observed experimentally ('Ann. Chim. Phys.,' 1850, vol. 28, p. 374).

Preliminary experiments on an equimolecular mixture of tartaric and boric acids (15 grams of tartaric acid and 6.2 grams of boric acid in 100 c.c. of solution) showed that the dispersion did not fulfil the requirements of BIOT'S law, but could be expressed by a simple formula with a normal dispersion constant $[M] = 24.835/(\lambda^2 - 0.0271)$. A more exact series, Table XX.(a), including readings for 18 lines in the visual and five lines in the photographic region of the spectrum, showed, however, that the simple formula was again only an approximation; but the negative term in the complex formula is very small (only about $\frac{1}{20}$ of the positive term), and would probably disappear altogether if a sufficient quantity of boric acid were used to convert the tartaric acid wholly into boro-tartaric acid. Table XX.(b), which shows the effect of $1\frac{1}{2}$ mols. of boric acid on tartaric acid of half the strength used for Table XX.(a) affords further justification for this view; the dispersion is here so nearly simple that the only hint of complexity is that given by a few negative errors in the red and in the extreme violet region of the spectrum.

The study of boro-tartaric acid, like that of tartar emetic, illustrates in a very striking manner the way in which the complex rotatory dispersion of tartaric acid is simplified when it is converted into "fixed" derivatives, even when these are more complex in their chemical structure. The actual structure of these derivatives has been the subject of much speculation, and must be regarded as still very uncertain; but we hope to be able to carry out a chemical study of this problem, which will form a suitable sequel to the physical investigations which are described in the present paper.

12. SUMMARY.

1. The rotatory power of tartaric acid for a series of 9 wave-lengths has been determined in aqueous solutions of 11 different concentrations ranging from 5 to 55 per cent. by weight, and also for 21 and 26 wave-lengths respectively at 2 other concentrations.

2. The optical rotatory power of tartaric acid, like that of its methyl and ethyl esters, is expressed to a close degree of approximation by the formula

$$\alpha = \frac{k_1}{\lambda^2 - \lambda_1^2} - \frac{k_2}{\lambda^2 - \lambda_2^2}.$$

3. The rotatory power of sodium tartrate agrees very closely with BIOT'S law, $\alpha = k/\lambda^2$, but requires for its exact expression a two-term formula similar to that

used for tartaric acid. Potassium and ammonium tartrates and Rochelle salt give similar dispersion curves, but deviate more widely from BIOT'S law.

4. A number of solutions which give negative rotations have been examined for a range of wave-lengths, and the corresponding dispersion-curves have been plotted.

5. In presence of an excess of boric acid the rotatory dispersion of tartaric acid is no longer complex but simple, and can be expressed over a wide range of wave-lengths by the equation

$$\alpha = \frac{k}{\lambda^2 - \lambda_0^2}.$$

Tartar emetic also gives a simple dispersion curve.

6. When tartar emetic is dissolved in an excess of potassium hydroxide, or when a corresponding solution is prepared with bismuth in place of antimony, strongly lævoptatory solutions are obtained, but these are again characterised by a simple rotatory dispersion.

7. It is suggested that *d*-tartaric acid, like nitro-camphor, exists in solution in two labile isomeric forms, and that the anomalous dispersion of the acid and of many of its derivatives is due to the presence of two isomeric compounds of opposite rotatory power and unequal dispersion. Derivatives which give simple rotatory dispersion are assumed (like the salts of nitro-camphor) to be fixed in one of these forms.

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TABLE II.—Specific Rotatory Power of Tartaric Acid, Observed and Calculated.

e = proportion by weight of water.

$[\alpha]$ = observed specific rotation.

$[\alpha_0]$ = $A_0 + B_0e$ (linear law, correct at $e = 0$, for Hg 4358 only).

$[\alpha_1]$ = $A_1 + B_1e$ (linear law, correct at $e = 0.55$ and 0.85).

$[\alpha_2]$ = $A_2 + B_2e + C_2e^2$ (parabolic law, correct at $e = 0.55, 0.70$ and 0.85).

A.	$e = 0.$	e										0.95	Constants of Equations.
		0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90		
Cd red 6438	$[\alpha]$	+2.000	+6.004	7.421	8.073	8.728	9.336	9.886	10.457	11.047	11.645	12.209	$A_1 = + 0.7733$
	$[\alpha_1]$	+0.773	+6.212	7.421	8.025	8.630	9.234	9.838	10.443	11.047	11.651	12.256	$B_1 = + 12.0867$
	$[\alpha_2]$	-1.346	+6.031	7.421	8.082	8.720	9.336	9.929	10.499	11.047	11.572	12.074	$A_2 = - 1.3460$
	$[\alpha] - [\alpha_1]$	+1.227	-0.208	±	+0.048	+0.098	+0.102	+0.048	-0.014	±	-0.006	-0.047	$B_2 = + 18.4333$
	$[\alpha] - [\alpha_2]$	+3.346	-0.027	-0.003	-0.009	+0.008	±	-0.043	-0.042	+0.073	+0.135	$C_2 = - 4.5333$	
Na yellow 5893	$[\alpha]$	+1.600	+6.501	8.219	9.010	9.799	10.500	11.207	11.949	12.642	13.304	14.070	$A_1 = + 0.1100$
	$[\alpha_1]$	+0.110	+6.745	8.219	8.966	9.693	10.431	11.168	11.905	12.642	13.379	14.116	$B_1 = + 14.7433$
	$[\alpha_2]$	-1.329	+6.623	7.429	8.996	9.755	10.500	11.229	11.943	12.642	13.324	13.991	$A_2 = - 1.3292$
	$[\alpha] - [\alpha_1]$	+1.490	-0.244	±	+0.064	+0.106	+0.069	+0.039	+0.044	±	-0.075	-0.046	$B_2 = + 19.0610$
	$[\alpha] - [\alpha_2]$	+2.929	-0.122	-0.063	+0.014	+0.044	±	-0.022	+0.006	-0.020	+0.079	$C_2 = - 3.0888$	
Hg yellow 5790	$[\alpha]$	—	+6.566	8.331	9.162	9.975	10.752	11.481	12.197	12.948	13.687	14.527	$A_2 = - 2.4710$
	$[\alpha_1]$	—	+6.592	7.474	8.331	9.163	9.970	11.509	12.241	12.948	13.630	14.287	$B_2 = + 22.3900$
	$[\alpha] - [\alpha_2]$	—	-0.026	-0.008	-0.001	+0.005	±	-0.028	-0.044	±	+0.057	+0.240	$C_2 = - 5.0000$
Hg yellow 5769	$[\alpha]$	—	+6.577	7.487	8.346	9.184	10.014	11.535	12.246	12.980	13.687	14.527	$A_2 = - 2.7054$
	$[\alpha_1]$	—	+6.583	7.480	8.346	9.187	10.000	11.545	12.276	12.980	13.657	14.306	$B_2 = + 23.1000$
	$[\alpha] - [\alpha_2]$	—	-0.006	+0.007	-0.003	+0.014	±	-0.010	-0.030	±	+0.030	+0.221	$C_2 = - 5.4666$
Hg yellow (mean) 5780	$[\alpha]$	+1.500	+6.571	7.476	8.338	9.173	9.994	10.769	11.508	12.221	12.964	13.687	$A_1 = - 0.1416$
	$[\alpha_1]$	-0.142	+6.796	7.567	8.338	9.109	9.880	10.651	11.422	12.193	12.964	13.735	$B_1 = + 15.4183$
	$[\alpha_2]$	-2.588	+6.587	7.474	8.338	9.175	9.985	10.769	11.527	12.258	12.964	13.644	$A_2 = - 2.5882$
	$[\alpha] - [\alpha_1]$	+1.642	-0.225	-0.091	+0.064	+0.114	+0.118	+0.086	+0.028	±	-0.048	+0.021	$B_2 = + 22.7450$
	$[\alpha] - [\alpha_2]$	+4.088	-0.016	+0.002	-0.002	+0.009	±	-0.019	-0.037	+0.043	+0.230	$C_2 = - 5.2333$	

TABLE II. (continued).

λ .	$e = 0$.	0.45.	0.50.	0.55.	0.60.	0.65.	0.70.	0.75.	0.80.	0.85.	0.90.	0.95.	Constants of Equations.
Hg green 5461	$[a]$	+0.700	7.653	8.645	9.594	10.532	11.417	12.237	13.061	13.925	14.740	15.669	$A_1 = -1.0350$
	$[a_1]$	-1.035	7.765	8.645	9.525	10.405	11.285	12.165	13.045	13.925	14.805	15.685	$B_1 = +17.6000$
	$[a_2]$	-3.778	7.662	8.645	9.598	10.522	11.417	12.282	13.118	13.925	14.702	15.450	$A_2 = -3.7780$
	$[a] - [a_1]$	+1.735	-0.112	\pm	+0.069	+0.127	+0.132	+0.072	+0.016	+0.016	-0.065	-0.016	$B_2 = +25.8134$
	$[a] - [a_2]$	+4.478	-0.037	\pm	-0.004	+0.010	\pm	-0.045	-0.057	-0.057	+0.038	+0.219	$C_2 = -5.8666$
Cd green 5086	$[a]$	-1.000	7.479	8.668	9.784	10.876	11.927	12.904	13.899	14.855	15.856	17.040	$A_1 = -2.6786$
	$[a_1]$	-2.674	7.637	8.668	9.700	10.731	11.762	12.793	13.824	14.855	15.886	16.917	$B_1 = +20.6220$
	$[a_2]$	-6.107	7.509	8.668	9.791	10.878	11.927	12.940	13.916	14.865	15.768	16.623	$A_2 = -6.1069$
	$[a] - [a_1]$	+1.674	-0.158	\pm	+0.084	+0.145	+0.165	+0.111	+0.075	+0.030	-0.030	+0.123	$B_2 = +30.9038$
	$[a] - [a_2]$	+5.107	-0.030	\pm	-0.007	-0.002	\pm	-0.036	-0.017	-0.017	+0.098	+0.417	$C_2 = -7.3444$
Cd blue 4800	$[a]$	-3.300	6.781	8.150	9.430	10.714	11.859	13.032	14.162	15.297	16.446	17.922	$A_1 = -4.9528$
	$[a_1]$	-4.953	6.959	8.150	9.341	10.532	11.723	12.915	14.106	15.297	16.488	17.679	$B_1 = +23.8233$
	$[a_2]$	-7.768	6.853	8.150	9.416	10.653	11.859	13.035	14.181	15.297	16.383	17.498	$A_2 = -7.7682$
	$[a] - [a_1]$	+1.653	-0.178	\pm	+0.069	+0.182	+0.136	+0.117	+0.066	+0.042	-0.042	+0.243	$B_2 = +32.2544$
	$[a] - [a_2]$	+4.468	-0.072	\pm	+0.014	+0.061	\pm	-0.003	-0.019	-0.019	+0.063	+0.484	$C_2 = -6.0222$
Cd blue 4678	$[a]$	-4.500	6.241	7.707	9.056	10.427	11.660	12.969	14.061	15.359	16.558	18.150	$A_1 = -6.3217$
	$[a_1]$	-6.322	6.432	7.707	8.982	10.288	11.533	12.808	14.084	15.359	16.634	17.910	$B_1 = +25.5067$
	$[a_2]$	-8.960	6.333	7.707	9.053	10.370	11.660	12.921	14.184	15.359	16.535	17.684	$A_2 = -8.9605$
	$[a] - [a_1]$	+1.822	-0.191	\pm	+0.074	+0.169	+0.127	+0.161	-0.033	-0.033	-0.076	+0.240	$B_2 = +33.4088$
	$[a] - [a_2]$	+4.460	-0.092	\pm	+0.003	+0.067	\pm	+0.048	-0.103	-0.103	+0.023	+0.466	$C_2 = -5.6444$
Hg violet 4368	$[a]$	-10.600	3.631	5.421	7.025	8.637	10.131	11.570	13.023	14.486	15.952	17.791	$A_0 = -10.6000$
	$[a_1]$	-10.600	4.164	5.641	7.117	8.594	10.070	11.547	13.023	14.499	15.976	17.452	$B_0 = +29.5268$
	$[a_2]$	-11.198	3.910	5.421	6.932	8.443	9.953	11.464	12.975	14.486	15.997	17.508	$A_1 = -11.1982$
	$[a_3]$	-14.886	3.772	5.421	7.031	8.601	10.131	11.622	13.074	14.486	15.859	17.192	$B_1 = +30.2167$
	$[a] - [a_2]$	\pm	-0.533	-0.220	-0.092	+0.043	+0.061	+0.023	\pm	-0.013	-0.024	+0.339	$A_2 = -14.8860$
$[a] - [a_3]$	+0.598	-0.279	\pm	+0.093	+0.194	+0.178	+0.106	+0.048	+0.048	-0.045	-0.045	$B_2 = +41.2610$	
$[a] - [a_1]$	+4.266	-0.236	\pm	-0.006	+0.036	\pm	-0.052	-0.052	-0.051	+0.093	+0.599	$C_2 = -7.8890$	

TABLE III.—Rotatory Dispersion of Tartaric Acid in Aqueous Solutions at 20° C.

(a) 54.1 gr. tartaric acid in 100 gr. solution, or 69.9 gr. " " in 100 c.c. Density = 1.2920. Length of tube = 6 dm.					(b) 41.34 gr. tartaric acid in 100 gr. solution, or 50.15 gr. " " in 100 c.c. Density = 1.2132. Length of tube = 6 dm.				
$[M_1] = \frac{17.188}{\lambda^2 - 0.030} - \frac{12.080}{\lambda^2 - 0.074}$					$[M_1] = \frac{17.960}{\lambda^2 - 0.030} - \frac{11.869}{\lambda^2 - 0.074}$				
λ .	α obs.	$[M]$ obs.	$[M_1]$.	$[M] - [M_1]$.	α obs.	$[M]$ obs.	$[M_1]$.	$[M] - [M_1]$.	
Li 6708	+24.35	8.71	8.80	-0.09	+22.40	11.17	11.20	-0.03	
Cd 6438	25.55	9.14	9.22	-0.08	23.51	11.71	11.85	-0.14	
Na 5893	27.78	9.94	9.97	-0.03	26.43	13.18	13.18	±	
Cu 5782	28.52	10.20	10.07	+0.13	26.94	13.43	13.43	±	
Hg 5780	28.12	10.08	10.08	±	26.94	13.43	13.43	±	
Cu 5700	28.67	10.25	10.14	+0.11	27.32	13.62	13.60	+0.02	
Hg 5461	28.53	10.20	10.20	±	28.14	14.03	14.03	±	
Cu 5219	27.75	9.93	10.02	-0.09	28.55	14.24	14.27	-0.03	
Ag 5209	27.41	9.80	10.00	-0.20	—	—	—	—	
Cu 5154	27.51	9.82	9.90	-0.08	28.61	14.26	14.28	-0.02	
Cu 5105	27.18	9.72	9.80	-0.08	28.62	14.26	14.27	-0.01	
Cd 5086	26.97	9.64	9.75	-0.11	28.78	14.34	14.27	+0.07	
Zn 4811	23.54	8.42	8.60	-0.18	—	—	—	—	
Cd 4800	23.30	8.33	8.53	-0.20	27.41	13.67	13.74	-0.07	
Zn 4722	21.91	7.84	7.98	-0.14	—	—	—	—	
Zn 4680	20.73	7.41	7.63	-0.22	—	—	—	—	
Cd 4678	20.69	7.40	7.61	-0.21	26.51	13.22	13.17	+0.05	
Fe 4580	—	—	—	—	25	12.5	12.5	±	
Fe 4440	—	—	—	—	22	11.0	11.1	-0.1	
Hg 4358	9.13	3.26	3.26	±	19.9	9.92	9.92	±	
Fe 4228	0	0	+0.2	-0.2	—	—	—	—	
Fe 4275	—	—	—	—	17	8.5	8.5	±	
Fe 4209	—	—	—	—	14	7.0	7.0	±	
Fe 4146	—	—	—	—	11	5.5	5.5	±	
Fe 4110	-10	-3.6	-3.5	-0.1	—	—	—	—	
Fe 4096	—	—	—	—	8	4.0	3.8	+0.2	
Fe 4046	—	—	—	—	5	2.5	2.0	+0.5	
Fe 4023	—	—	—	—	3	1.5	1.1	+0.4	
Fe 4005	-20	-7.1	-8.0	+0.9	2	1.0	+0.4	+0.6	
Fe 4000	—	—	—	—	+1	+0.5	-0.1	+0.6	
Fe 3967	—	—	—	—	-1	-0.5	-1.3	+0.8	
Fe 3941	—	—	—	—	-4	-2.0	-2.6	+0.6	

TABLE IV.—Molecular Rotatory Power of Tartaric Acid in Aqueous Solutions (Observed and Calculated).

 e = proportion of water present.

[M] = molecular rotation observed.

[M₁] = molecular rotation calculated.

e .	λ .	Cd 6438	Na 5893	Hg 5790	Hg 5769	Hg 5461	Cd 5086	Cd 4800	Cd 4678	Hg 4358
0.45	[M]	9.005	9.751	9.849	9.866	9.919	9.352	8.022	7.108	2.772
	[M ₁]	9.028	9.730	9.819	9.834	9.919	9.414	8.143	7.204	2.772
	[M]—[M ₁]	-0.023	+0.021	+0.030	+0.032	±	-0.062	-0.121	-0.096	±
0.50	[M]	10.100	11.049	11.198	11.230	11.479	11.218	10.171	9.362	5.446
	[M ₁]	10.106	11.041	11.183	11.210	11.479	11.251	10.249	9.445	5.446
	[M]—[M ₁]	-0.006	+0.008	+0.015	+0.020	±	-0.033	-0.078	-0.083	±
0.55	[M]	11.132	12.328	12.497	12.519	12.967	13.003	12.225	11.560	8.132
	[M ₁]	11.117	12.281	12.477	12.514	12.967	13.028	12.307	11.655	8.132
	[M]—[M ₁]	+0.015	+0.047	+0.020	0.005	±	-0.025	-0.082	-0.095	±
0.60	[M]	12.109	13.514	13.742	13.776	14.391	14.676	14.144	13.584	10.538
	[M ₁]	12.107	13.484	13.726	13.775	14.391	14.699	14.223	13.684	10.538
	[M]—[M ₁]	+0.002	+0.030	+0.016	+0.001	±	-0.023	-0.079	-0.100	±
0.65	[M]	13.092	14.699	14.963	15.021	15.798	16.314	16.071	15.640	12.956
	[M ₁]	13.079	14.667	14.959	15.016	15.798	16.360	16.129	15.712	12.956
	[M]—[M ₁]	+0.013	+0.032	+0.004	+0.005	±	-0.046	-0.058	-0.072	±
0.70	[M]	14.003	15.750	16.129	16.180	17.126	17.890	17.788	17.490	15.197
	[M ₁]	14.004	15.788	16.124	16.192	17.126	17.919	17.912	17.606	15.197
	[M]—[M ₁]	-0.001	-0.038	+0.005	-0.012	±	-0.029	-0.124	-0.116	±
0.75	[M]	14.829	16.811	17.222	17.302	18.355	19.355	19.547	19.453	17.356
	[M ₁]	14.846	16.817	17.196	17.274	18.355	19.376	19.593	19.400	17.356
	[M]—[M ₁]	-0.017	-0.006	+0.026	+0.028	±	-0.021	-0.046	+0.053	±
0.80	[M]	15.685	17.924	18.295	18.370	19.592	20.849	21.243	21.077	19.535
	[M ₁]	15.693	17.854	18.275	18.361	19.592	20.845	21.287	21.208	19.535
	[M]—[M ₁]	-0.008	+0.070	+0.020	+0.009	±	+0.004	-0.044	-0.131	±
0.85	[M]	16.570	18.963	19.423	19.469	20.888	22.283	22.945	23.039	21.730
	[M ₁]	16.595	19.045	19.412	19.507	20.888	22.369	23.030	23.060	21.730
	[M]—[M ₁]	-0.025	-0.082	+0.011	-0.038	±	-0.086	-0.085	-0.021	±
0.90	[M]	17.467	19.956	20.530	20.530	22.109	23.784	24.670	24.837	23.928
	[M ₁]	17.423	19.966	20.474	20.577	22.109	23.824	24.719	24.869	23.928
	[M]—[M ₁]	+0.044	-0.010	+0.056	-0.047	±	-0.040	-0.049	-0.032	±
0.95	[M]	18.314	21.105	21.790	21.790	23.504	25.561	26.883	27.226	26.687
	[M ₁]	18.333	21.105	21.666	21.781	23.504	25.527	26.742	27.068	26.687
	[M]—[M ₁]	-0.019	±	+0.124	+0.009	±	+0.034	+0.141	+0.168	±

The highest number in each series of molecular rotations is shown in heavy type; the wave-lengths at which the true maxima occur are shown in Table V.

TABLE V.—Rotatory Dispersion in Aqueous Solutions of Tartaric Acid at 20° C.

Constants and Anomalies.

$$[M] = \frac{k_1}{\lambda^2 - \lambda_1^2} - \frac{k_2}{\lambda^2 - \lambda_2^2}.$$

P.	λ_1^2 .	λ_2^2 .	k_1 .	k_2 .	λ_σ (Inflection).	λ_μ (Maximum).	λ_ρ (Reversal).
Per cent.							
55	0.03	0.074	17.127	12.093	<i>0.6571</i>	0.5527	<i>0.4239</i>
54.1	0.03	0.074	17.188	12.080	0.6533	0.5485	0.4220
50	0.03	0.074	17.485	12.043	0.6376	0.5373	<i>0.4140</i>
45	0.03	0.074	17.686	11.877	0.6197	0.5232	<i>0.4050</i>
41.25	0.03	0.074	17.960	11.869	0.6092	0.5150	0.3997
40	0.03	0.074	18.053	11.865	0.6058	0.5123	<i>0.3980</i>
35	0.03	0.074	18.367	11.812	0.5929	0.5022	<i>0.3915</i>
30	0.03	0.074	18.709	11.799	0.5822	0.4937	<i>0.3861</i>
25	0.03	0.074	18.936	11.714	0.5722	0.4859	<i>0.3812</i>
20	0.03	0.074	19.160	11.624	0.5628	0.4785	<i>0.3767</i>
15	0.03	0.074	19.485	11.605	0.5543	0.4720	<i>0.3726</i>
10	0.03	0.074	19.657	11.475	0.5458	0.4653	<i>0.3684</i>
5	0.03	0.074	19.592	11.108	0.5341	0.4562	<i>0.3628</i>

P = concentration in grammes of tartaric acid per 100 gr. of solution.

In the last three columns all values which lie outside the range of the observations are given in italics.

TABLE VI.—Rotatory Dispersion of Sodium Tartrate in Aqueous Solution at 20° C.

(a) *First Series*.—22.54 gr. $\text{Na}_2\text{H}_4\text{C}_4\text{O}_6 \cdot 2\text{H}_2\text{O}$ in 100 gr. aqueous solution,
or 25.72 gr. „ „ in 100 c.c.

Density = 1.1411.

Length of tube = 6 dm.

*Simple Formula.**Complex Formula.*

$$[M_1] = \frac{20.152}{\lambda^2 - 0.0029}$$

$$[M_2] = \frac{37.445}{\lambda^2 - 0.038} - \frac{18.025}{\lambda^2 - 0.06}$$

λ .	α .	[M] obs.	[M] λ^2 .	[M ₁].	[M]—[M ₁].	[M ₂].	[M]—[M ₂].
Li 6708	30.23	45.05	20.27	45.07	−0.02	44.67	+0.38
Cd 6438	32.88	49.00	20.31	48.96	+0.04	48.61	+0.39
Na 5893	39.22	58.45	20.30	58.52	−0.07	58.33	+0.12
Hg 5790	40.58	60.48	20.27 ₅	60.64	−0.16	60.50	−0.02
Cu 5782	40.91	60.97	20.38	60.80	+0.17	60.66	+0.31
Hg 5769	40.67	60.62	20.17	61.08	−0.46	61.24	−0.62
Cu 5700	41.95	62.52	20.31	62.58	−0.06	62.47	+0.05
Hg 5461	45.78	68.23	20.35	68.23	±	68.23	±
Cu 5219	50.41	75.13	20.46	74.78	+0.35	74.89	+0.24
Ag 5209	50.57	75.37	20.45	75.07	+0.30	75.18	+0.19
Cu 5154	51.77	77.16	20.50	76.70	+0.46	76.84	+0.32
Cu 5105	53.15	79.21 ₅	20.64 ₅	78.19	+1.02 ₅	78.36	+0.85 ₅
Cd 5086	53.44	79.65	20.60	78.79	+0.86	78.96	+0.69
Cd 4800	59.96	89.36 ₅	20.59	88.58	+0.78 ₅	88.84	+0.52 ₅
Cd 4678	63.41	94.51	20.68	93.32	+1.19	93.58	+0.93
Hg 4358	72.29	107.74	20.46	107.74	±	107.74	±
Fe 4271	75	111.8	20.39	112.3	−0.5	112.0	−0.2
Fe 4210	77	114.8	20.34	115.6	−0.8	115.2	−0.4
Fe 4154	79	117.7	20.32	118.8	−1.1	118.1	−0.4
Fe 4072	82	122.2	20.26	123.7	−1.5	122.6	−0.4
Fe 4033	84	125.2	20.36	126.1	−0.9	124.8	+0.4
Fe 4005	85	126.7	20.32	127.9	−1.2	126.4	+0.3

TABLE VI.—Rotatory Dispersion of Sodium Tartrate in Aqueous Solution at 20° C.
(continued).

(b) *Second Series*.—34.32 gr. $\text{Na}_2\text{H}_4\text{C}_4\text{O}_6 \cdot 2\text{H}_2\text{O}$ in 100 gr. aqueous solution,
or 41.97 gr. „ „ in 100 c.c.

Density = 1.2228.

Length of tube = 6 dm.

*Simple Formula.**Complex Formula.*

$$[M_1] = \frac{19.537}{\lambda^2 - 0.00032}$$

$$[M_2] = \frac{37.108}{\lambda^2 - 0.038} - \frac{18.348}{\lambda^2 - 0.06}$$

λ .	α .	[M] obs.	[M] λ^2 .	[M ₁].	[M]—[M ₁].	[M ₂].	[M]—[M ₂].
Cd 6438	51.50	47.04	19.50	47.17	−0.13	46.80	+0.24
Na 5893	61.42	56.10	19.48	56.31	−0.21	56.12	−0.02
Hg 5780	63.88	58.35	19.49	58.55	−0.20	58.40	−0.05
Hg 5461	71.80	65.58	19.56	65.58	±	65.58	±
Cd 5086	83.08	75.88	19.63	75.62	+0.26	75.80	+0.08
Cd 4800	93.10	85.04	19.59	84.91	+0.13	85.20	−0.16
Cd 4678	98.52	89.99	19.69	89.41	+0.58	89.69	+0.30
Fe 4425	110	100.5	19.67	99.9	+0.6	100.0	+0.5
Hg 4358	112.81	103.04	19.57	103.04	±	103.04	±
Fe 4218	120	109.6	19.50	110.0	−0.4	109.6	±
Fe 4031	130	118.7	19.29	120.5	−1.8	119.1	−0.4
Fe 3874	140	127.9	19.19	130.5	−2.6	127.4	+0.5

TABLE VII.—Rotatory Dispersion of Potassium Tartrate in Aqueous Solution at 20° C.

(a) *First Series*.—33·60 gr. $K_2H_2C_4O_6 \cdot \frac{1}{2}H_2O$ in 100 gr. aqueous solution,
or 41·647 gr. „ „ in 100 c.c.

Density = 1·2395.

Length of tube = 6 dm.

Simple Formula.

Complex Formula.

$$[M_1] = \frac{23'225}{\lambda^2 - 0'0065}$$

$$[M_2] = \frac{41'75}{\lambda^2 - 0'038} - \frac{19'255}{\lambda^2 - 0'06}$$

λ .	α .	[M] obs.	[M] λ^2 .	[M] ₁ .	[M]—[M] ₁ .	[M] ₂ .	[M]—[M] ₂ .
Li 6708	54·66	51·92	23·36	52·37	—0·45	51·96	—0·04
Cd 6438	59·49	56·51	23·42	56·93	—0·42	56·58	—0·07
Na 5893	71·55	67·96	23·60	68·15	—0·19	67·96	±
Hg 5790	74·33	70·60	23·67	70·65	—0·05	70·50	+0·10
Cu 5782	74·49	70·75	23·65	70·85	—0·10	70·71	+0·04
Hg 5769	74·77	71·02	23·64	71·17	—0·15	71·03	—0·01
Cu 5700	76·76	72·91	23·69	72·94	—0·03	72·83	+0·08
Hg 5461	83·81	79·61	23·74	79·61	±	79·61	±
Cu 5219	92·28	87·65	23·87 ₅	87·35	+0·30	87·47	+0·18
Cu 5154	94·74	89·91	23·90 ₅	89·62	+0·29	89·77	+0·14
Cu 5105	96·50	91·66	23·89	91·40	+0·26	91·57	+0·09
Cd 5086	97·26	92·39	23·90	92·10	+0·29	92·28	+0·11
Zn 4811	109·20	103·72	24·01	103·24	+0·48	103·51	+0·21
Cd 4800	109·55	104·06	23·97 ₅	103·73	+0·33	104·00	+0·06
Zn 4722	113·50	107·81	24·04	107·29	+0·52	107·55	+0·26
Zn 4680	115·25	109·50	23·98	109·29	+0·21	109·55	—0·05
Cd 4678	115·42	109·63	23·99	109·38	+0·25	109·65	—0·02
Hg 4358	133·30	126·61	24·05	126·61	±	126·61	±
Fe 4326	136	129·2	24·12	128·6	+0·6	128·5	+0·7
Fe 4187	145	137·7	24·15	137·6	+0·1	137·1	+0·6
Fe 4110	150	142·5	24·07	143·0	—0·5	142·1	+0·4
Fe 4046	155	147·2	24·10	147·7	—0·5	146·5	+0·7

TABLE VII.—Rotatory Dispersion of Potassium Tartrate in Aqueous Solution at 20° C. (continued).

(b) *Second Series.*—32·68 gr. $K_2H_4C_4O_6 \cdot \frac{1}{2}H_2O$ in 100 gr. aqueous solution,
 or 40·255 gr. „ „ in 100 c.c.

Density = 1·2318.

Length of tube = 6 dm.

*Simple Formula.**Complex Formula*

$$[M_1] = \frac{22\cdot93}{\lambda^2 - 0\cdot0068}$$

$$[M_2] = \frac{41\cdot108}{\lambda^2 - 0\cdot038} - \frac{18\cdot886}{\lambda^2 - 0\cdot06}$$

λ .	α .	[M] obs.	[M] λ^2 .	[M ₁].	[M] - [M ₁].	[M ₂].	[M] - [M ₂].
Cd 6438	57·34	55·79	23·12	56·25	-0·46	55·91	-0·12
Na 5893	69·07	67·20	23·34	67·50	-0·30	67·18	+0·02
Hg 5790	71·50	69·57	23·32	69·82	-0·25	69·68	-0·11
Cu 5782	71·95	70·01	23·40	70·01	±	69·88	+0·13
Hg 5769	72·10	70·15	23·35	70·34	-0·19	70·21	-0·06
Cu 5700	74·05	72·05	23·41	72·08	-0·03	71·98	+0·07
Hg 5461	80·88	78·69	23·47	78·69	±	78·69	±
Cu 5219	89·00	86·59	23·59	86·34	+0·25	86·46	+0·13
Cu 5154	91·40	88·93	23·62	88·58	+0·35	88·74	+0·19
Cu 5105	93·15	90·63	23·62	90·34	+0·29	90·52	+0·11
Cd 5086	93·80	91·26	23·61	91·04	+0·22	91·23	+0·03
Zn 4811	105·65	102·79	23·79	102·06	+0·73	102·34	+0·45
Cd 4800	105·67	102·81	23·69	102·55	+0·26	102·82	-0·01
Zn 4722	109·75	106·78	23·81	106·07	+0·71	106·34	+0·44
Cd 4678	111·17	108·68	23·78	108·14	+0·54	108·42	+0·26
Hg 4358	128·70	125·22	23·78	125·22	±	125·22	±
Fe 4337	130	126·5	23·79	126·5	±	126·4	+0·1
Fe 4308	132	128·4	23·84	128·2	+0·2	128·1	+0·3
Fe 4271	134	130·4	23·78	130·6	-0·2	130·4	±
Fe 4251	136	132·3	23·91	131·8	+0·5	131·6	+0·7

TABLE VIII.—Rotatory Dispersion of Rochelle Salt in Aqueous Solution at 20° C.

44·63 gr. $\text{NaKH}_2\text{C}_4\text{O}_6 \cdot 4\text{H}_2\text{O}$ in 100 gr. aqueous solution,
 or 55·93 gr. „ „ in 100 c.c.

Density = 1·2533.

Length of tube = 6 dm.

*Simple Formula.**Complex Formula.*

$$[M_1] = \frac{21'150}{\lambda^2 - 0'0051}$$

$$[M_2] = \frac{38'513}{\lambda^2 - 0'038} - \frac{18'066}{\lambda^2 - 0'06}$$

λ .	α .	[M] obs.	[M] λ^2 .	[M ₁].	[M]—[M ₁].	[M ₂].	[M]—[M ₂].
Cd 6438	61·27	51·49	21·34	51·67	—0·18	51·33	+0·16
Na 5893	73·46	61·72	21·43	61·82	—0·10	61·64	+0·08
Hg 5790	76·31	64·12	21·49	64·07	+0·05	63·93	+0·19
Cu 5782	76·42	64·21	21·47	64·25	—0·04	64·11	+0·10
Hg 5769	76·71	64·45	21·45	64·55	—0·10	64·42	+0·03
Cu 5700	78·76	66·18	21·50	66·14	+0·04	66·04	+0·14
Hg 5461	85·88	72·16	21·52	72·16	±	72·16	±
Cu 5219	94·52	79·42	21·63	79·14	+0·28	79·25	+0·17
Cu 5154	97·19	81·66	21·69	81·18	+0·48	81·32	+0·34
Cu 5105	99·02	83·20	21·68	82·78	+0·42	82·95	+0·25
Cd 5086	99·63	83·72	21·65	83·42	+0·30	83·59	+0·13
Zn 4811	111·47	93·66	21·68	93·44	+0·22	93·70	—0·04
Cd 4800	112·36	94·41	21·75	93·89	+0·52	94·15	+0·26
Zn 4722	116·07	97·53	21·74	97·09	+0·44	97·34	+0·19
Zn 4680	118·07	99·21	21·73	98·88	+0·33	99·15	+0·06
Cd 4678	118·35	99·44	21·76	98·96	+0·48	99·72	—0·28
Hg 4358	136·20	114·45	21·73	114·45	±	114·45	±
Fe 4286	140	117·6	21·61	118·4	—0·8	118·3	—0·7
Fe 4271	142	119·3	21·76	119·3	±	119·1	+0·2
Fe 4135	152	127·7	21·84	127·5	+0·2	126·8	+0·9
Fe 4055	155	130·3	21·42	132·8	—2·5	131·6	—1·3

TABLE IX.—Rotatory Dispersion of Ammonium Tartrate in Aqueous Solution at 20° C.

32·10 gr. (NH₄)₂H₄C₄O₆ in 100 gr. aqueous solution,
or 37·075 gr. „ in 100 c.c.

Density = 1·1550.

Length of tube = 6 dm.

*Simple Formula.**Complex Formula.*

$$[M_1] = \frac{22\cdot173}{\lambda^2 - 0\cdot0094}$$

$$[M_2] = \frac{38\cdot676}{\lambda^2 - 0\cdot038} - \frac{17\cdot117}{\lambda^2 - 0\cdot06}$$

λ .	α .	[M] obs.	[M] λ^2 .	[M ₁].	[M]—[M ₁].	[M ₂].	[M]—[M ₂].
Cd 6438	65·57	54·24	22·48	54·74	—0·50	54·44	—0·20
Na 5893	79·10	65·43	22·72	65·63	—0·20	65·47	—0·04
Hg 5790	82·07	67·88	22·76	68·05	—0·17	67·93	—0·05
Cu 5782	82·45	68·20	22·80	68·24	—0·04	68·12	+0·08
Hg 5769	82·49	68·23	22·71	68·56	—0·33	68·45	—0·22
Cu 5700	85·00	70·31	22·84	70·28	+0·03	70·19	+0·12
Hg 5461	92·81	76·77	22·89	76·77	±	76·77	±
Cu 5219	101·99	84·36	22·98	84·32	+0·04	84·42	—0·06
Cu 5154	104·77	86·66	23·02	86·53	+0·13	86·66	±
Cu 5105	106·75	88·30	23·01	88·27	+0·03	88·41	—0·11
Cd 5086	107·73	89·10	23·05	88·95	+0·15	89·10	±
Zn 4811	120·88	99·98	23·14	99·85	+0·13	100·09	—0·11
Cd 4800	121·52	100·51	23·16	100·33	+0·18	100·57	—0·06
Zn 4722	125·75	104·01	23·19	103·82	+0·19	104·06	—0·05
Zn 4680	128·20	106·04	23·22	105·78	+0·26	106·01	+0·03
Cd 4678	128·28	106·11	23·22	105·87	+0·24	106·11	±
Hg 4358	148·50	122·83	23·33	122·83	±	122·83	±
Fe 4271	155	128·2	23·39	128·2	±	128·0	+0·2
Fe 4199	160	132·3	23·34	132·8	—0·5	132·5	—0·2
Fe 4144	165	136·5	23·44	136·6	—0·1	136·0	+0·5
Fe 4085	170	140·6	23·46	140·8	—0·2	139·9	+0·7
Fe 4046	172·5	142·7	23·36	142·7	±	142·6	+0·1
Fe 4025	175	144·7	23·45	145·3	—0·6	144·1	+0·6
Fe 3970	180	148·9	23·47	149·6	—0·7	148·0	+0·9
Fe 3919	185	153·0	23·50	153·8	—0·8	151·7	+1·3

TABLE X.—Molecular Rotatory Power of Tartrates.

Salt.	Percentage of water.		P.	<i>d.</i>	C.	α_{5461} .	[M] _D .	
	Obs.	Calc.						
Na ₂ H ₄ C ₄ O ₆ ·2H ₂ O ...	15·66	15·65	22·05	1·1379	25·09	44·75	58·45	58·35 (Thomsen)
	15·70	—	<i>22·54</i>	<i>1·1411</i>	<i>25·72</i>	45·78		
			23·02	1·1443	26·34	46·79		
			34·04	1·2207	41·55	71·09		
			<i>34·32</i>	<i>1·2228</i>	<i>41·97</i>	71·80	56·10	56·36 (Thomsen)
			34·99	1·2278	42·96	73·50		
K ₂ H ₄ C ₄ O ₆ · $\frac{1}{2}$ H ₂ O ...	3·94	3·83	32·00	1·2263	39·24	78·76		
	lost at 150°	—	32·68	1·2318	40·255	80·86	67·20	
	3·54	—						
	lost at 120°	—	<i>33·60</i>	<i>1·2395</i>	<i>41·647</i>	83·81	67·96	
			35·44	1·2542	44·45	89·48		
NaKH ₄ C ₄ O ₆ ·4H ₂ O ...	25·53	25·53	44·50	1·2523	55·73	85·47		
			<i>44·63</i>	<i>1·2533</i>	<i>55·93</i>	85·88	61·72	
			44·68	1·2536	56·01	86·01		
			45·01	1·2557	56·52	86·69		
			27·60	1·1333	31·28	77·87		
(NH ₄) ₂ H ₄ C ₄ O ₆ ...	—	—	30·38	1·1466	34·83	87·16		
			32·00	1·1548	36·95	92·50		
			<i>32·10</i>	<i>1·1550</i>	<i>37·075</i>	92·81	65·43	
			33·79	1·1637	39·32	99·03		
KSbOH ₄ C ₄ O ₆ · $\frac{1}{2}$ H ₂ O...	—	—	5·53	1·0382	5·74	59·61		
			5·32	1·0368	5·516	57·20	573·7	

P = grammes of salt in 100 gr. of solution.

d = density.

C = grammes of salt in 100 c.c. of solution.

Values obtained by interpolation are shown in italics.

TABLE XI.—Dispersion-constants and Dispersion-ratios.

Salt.	Concentration.		k .	λ_0^2 .	$\frac{\alpha_{4368}}{\alpha_{5461}}$.
	gr./100 gr.	gr./100 c.c.			
Sodium tartrate (dihydrate) { (a)	22.54	25.72	20.152	0.0029	1.579
(dihydrate) { (b)	34.32	41.97	19.537	0.00032	1.571
Potassium tartrate (hemihydrate) { (a)	33.60	41.65	23.225	0.0065	1.591
(hemihydrate) { (b)	32.68	40.26	22.930	0.0068	1.591
Rochelle salt (tetrahydrate)	44.63	55.93	21.150	0.0051	1.586
Ammonium tartrate (anhydrous)	32.10	37.075	22.173	0.0094	1.600

TABLE XII.—Dispersion-constants and Dispersion-ratios.

	λ^2 .	$\frac{\alpha_{4368}}{\alpha_{5461}}$.
Quartz (opt.-mag.)	0.0173	1.627
Ethyl alcohol (mag.)	0.0170	1.626
Methyl alcohol (mag.)	0.0164	1.624
Other alcohols (mag.)	0.0182	1.630
Water (mag.)	0.0222	1.645
Nine methyl carbinols (opt.)	0.0237	1.651

TABLE XIII.—Constants of DRUDE'S Equation.

	k_1 .	k_2 .	λ_1^2 .	λ_2^2 .
Sodium tartrate—				
(a) 25.72 grams in 100 c.c.	37.445	18.025	0.038	0.060
(b) 41.97 grams in 100 c.c.	37.108	18.348	0.038	0.060
Potassium tartrate—				
(a) 41.647 grams in 100 c.c.	41.750	19.255	0.038	0.060
(b) 40.255 grams in 100 c.c.	41.108	18.886	0.038	0.060
Rochelle salt—				
55.93 grams in 100 c.c.	38.513	18.066	0.038	0.060
Ammonium tartrate—				
37.075 grams in 100 c.c.	38.676	17.117	0.038	0.060

TABLE XIV.—Rotatory Dispersion of Potassium and Sodium Tartrates at 20° C. in presence of an excess of Alkali.

(a) 2.6 gr. of potassium tartrate +36.43 gr. of potassium oxide (K_2O) in 90.32 gr. of aqueous solution. or 4.329 gr. potassium tartrate in 100 c.c. Length of tube = 6 dm. Density = 1.504.			(b) 13.3 gr. of sodium tartrate +20.41 gr. of sodium oxide (Na_2O) in 92.891 gr. of aqueous solution, or 20.178 gr. sodium tartrate in 100 c.c. Length of tube = 6 dm. Density = 1.4093.	
λ .	α .	[M].	α .	[M].
Cd 6438	-0.29	- 2.62	- 7.54	-14.32
Na 5893	-0.30	- 2.71	—	—
Hg 5780	-0.30	- 2.71	-10.09	-19.17
Hg 5461	-0.41	- 3.71	-11.87	-22.55
Cd 5086	-0.61	- 5.52	-14.61	-27.75
Cd 4800	-0.80	- 7.24	-17.43	-33.11
Cd 4678	-0.99	- 8.96	-18.87	-35.85
Hg 4358	-1.34	-12.12	-24.29	-46.14

TABLE XV.—Rotatory Dispersion of Potassium and Sodium Tartrates at 20° C. in presence of Alkali.

(a) 4.7 gr. (0.02 mol.) $K_2H_4C_4O_6 \cdot \frac{1}{2}HO$ in 100 c.c. Length of tube = 6 dm. <i>Simple Formula.</i> $[M_1] = \frac{17.298}{\lambda^2 - 0.013}$						(b) 4.6 gr. (0.02 mol.) $Na_2H_4C_4O_6 \cdot 2H_2O$ in 100 c.c. Length of tube = 6 dm. <i>Simple Formula.</i> $[M_1] = \frac{18.56}{\lambda^2 - 0}$				
λ .	α .	[M] obs.	[M ₁] calc.	[M] - [M ₁].	[M] λ^2 .	α .	[M] obs.	[M ₁] calc.	[M] - [M ₁].	[M] λ^2 .
Cd 6438	+5.13	42.7	43.1	-0.4	17.72	+5.23	43.6	44.8	-1.2	18.08
Hg 5780	6.47	53.9	53.9	±	18.01	6.66	55.5	55.5	±	18.54
Cu 5782	6.46	53.8	53.8	±	18.00	6.65	55.4	55.5	-0.1	18.52
Cu 5700	6.55	54.6	55.4	-0.8	17.74	6.76	56.3	57.1	-0.8	18.29
Hg 5461	7.37	61.4	60.7	+0.7	18.31	7.49	62.4	62.2	+0.2	18.61
Cu 5219	8.12	67.7	66.7	+1.0	18.43	8.25	68.7	68.2	+0.5	18.70
Cu 5154	8.23	68.6	68.5	+0.1	18.21	8.40	70.0	69.9	+0.1	18.59
Cu 5105	8.34	69.5	69.9	-0.4	18.11	8.51	70.9	71.2	-0.3	18.48
Cd 5086	8.50	70.8	70.4	+0.4	18.32	8.61	71.7	71.7	±	18.56
Zn 4811	9.60	80.0	79.2	+0.8	18.52	9.65	80.4	80.2	+0.2	18.61
Cd 4800	9.53	79.4	79.6	-0.2	18.29	9.66	80.5	80.6	-0.1	18.55
Zn 4722	9.84	82.0	82.4	-0.4	18.29	10.05	83.7	83.2	+0.5	18.68
Cd 4678	10.08	84.0	84.0	±	18.38	10.08	84.0	84.8	-0.8	18.38
Hg 4358	11.68	97.3	97.8	-0.5	18.48	11.85	98.7	97.7	+1.0	18.75
Fe 4242	—	—	—	—	—	13	108	103	+5	19.5

TABLE XVI.—Rotatory Dispersion of Tartar Emetic in Aqueous Solution at 20° C.

5.32 gr. $\text{KSbOH}_4\text{C}_4\text{O}_6 \cdot \frac{1}{2}\text{H}_2\text{O}$ in 100 gr. of solution,
 or 5.516 gr. „ „ in 100 c.c.

Density = 1.0368.

Length of tube = 6 dm.

Simple Formula.

$$[M_1] = \frac{142.78}{\lambda^2 - 0.04936}$$

λ .	α .	[M] obs.	[M ₁] calc.	[M]—[M ₁].	[M] λ^2 .	$\frac{[M] \times}{(\lambda^2 - 0.04936)}$.
Li 6708	35.70	358.1	356.4	+1.7	161.1	143.5
Cd 6438	38.87	389.9	391.1	-1.2	161.6	142.4
Zn 6364	40.00	401.3	400.6	+0.7	162.5	142.7
Na 5893	47.95	481.0	479.3	+1.7	167.0	143.3
Hg 5790	49.70	498.6	499.4	-0.8	167.1	142.5
Cu 5782	49.82	499.8	501.1	-1.3	167.1	142.4
Hg 5769	50.07	502.3	503.7	-1.4	167.2	142.4
Cu 5700	51.60	517.6	518.2	-0.6	168.2	142.6
Hg 5461	57.19	573.7	573.7	±	171.1	142.8
Cu 5219	63.78	639.8	640.2	-0.4	174.3	142.7
Cu 5154	65.70	659.1	660.2	-1.1	175.1	142.6
Cu 5105	67.25	674.6	675.9	-1.3	175.8	142.5
Cd 5086	68.00	682.2	682.2	±	176.5	142.8
Zn 4811	78.15	784.0	784.1	-0.1	181.5	142.8
Cd 4800	78.67	789.2	788.7	+0.5	181.8	142.9
Zn 4722	82.00	822.6	822.4	+0.2	183.4	142.8
Cd 4678	83.98	842.5	842.5	±	184.4	142.8
Hg 4358	101.26	1015.8	1015.8	±	192.9	142.8
Fe 4228	110	1103	1103	±	197.3	142.8
Fe 4160	115	1154	1154	±	199.6	142.7
Fe 4113*	120	1204	1192	+12	203.6	144.2
Fe 4035	125	1254	1258	-4	204.2	142.3
Fe 3984	130	1304	1306	-2	207.0	142.6

* This reading appears to be incorrect, but has been retained in the table.

TABLE XVII.—Rotatory Dispersion of Tartar Emetic in presence of excess of Alkali at 20° C.

5.53 gr. ($\frac{1}{60}$ mol.) $\text{KSbOH}_4\text{C}_4\text{O}_6 \cdot \frac{1}{2}\text{H}_2\text{O} + 11.2$ gr. ($\frac{1}{5}$ mol.) KOH in 100 c.c. of solution.

Length of tube = 6 dm.

Simple formula.

$$[M_1] = \frac{-89.48}{\lambda^2 - 0.06275}$$

λ .	α .	$[M]$ obs.	$[M_1]$ calc.	$[M] - [M_1]$.	$[M] \lambda^2$.
Li 6708	-22.90	-229.0	-231.1	+2.1	-103.0
Cd 6438	-25.38	-253.8	-254.4	+0.6	-105.2
Na 5893	-31.37	-313.7	-314.5	+0.8	-108.9
Hg 5790	-32.88	-328.8	-328.4	-0.4	-110.2
Cu 5782	-32.91	-329.1	-329.5	+0.4	-110.0
Hg 5769	-33.13	-331.3	-331.3	±	-110.3
Cu 5700	-34.11	-341.1	-341.3	+0.2	-110.8
Hg 5461	-38.04	-380.4	-380.0	-0.4	-113.4
Cu 5218	-42.72	-427.2	-426.8	-0.4	-116.4
Cu 5153	-44.12	-441.2	-441.2	±	-117.2
Cu 5105	-45.32	-453.2	-452.2	-1.0	-118.1
Cd 5086	-45.78	-457.8	-456.7	-1.1	-118.4
Zn 4811	-53.11	-531.1	-530.4	-0.7	-122.9
Cd 4800	-53.43	-534.3	-533.7	-0.6	-123.1
Zn 4722	-55.82	-558.2	-558.5	+0.3	-124.5
Cd 4678	-57.43	-574.3	-573.2	-1.1	-125.7
Hg 4358	-70.20	-702.0	-703.6	+1.6	-133.2

TABLE XVIII.—Rotatory Dispersion of Potassium and Hydrogen Arsenyl Tartrates at 20° C.

Length of tube = 6 dm.

(a) 2.5 gr. ($\frac{1}{60}$ mol.) tartaric acid + 1.65 gr. ($\frac{1}{60}$ mol.) As_2O_3 + 14 gr. ($\frac{1}{4}$ mol.) KOH in 100 c.c.						(b) 15 gr. ($\frac{1}{10}$ mol.) tartaric acid in 100 c.c. Solution saturated with arsenious acid by boiling.	
<i>Simple Formula.</i>							
$[\text{M}_1] = \frac{20.148}{\lambda^2 - 0.017}$							
λ .	α .	$[\text{M}]$.	$[\text{M}] \lambda^2$.	$[\text{M}_1]$.	$[\text{M}] - [\text{M}_1]$.	α .	$[\text{M}]$ approx.
Cd 6438	+5.10	51.0	21.14	50.7	+0.3	+20.81	34.68
Na 5893	—	—	—	—	—	24.93	41.55
Hg 5790	+6.28	62.8	21.06	63.3	-0.5	25.72	42.87
Hg 5769	6.38	63.8	21.23	63.8	±	25.87	43.12
Hg 5461	7.22	72.2	21.53	71.7	+0.5	28.70	47.83
Cd 5086	8.38	83.8	21.68	83.4	+0.4	32.60	54.33
Cd 4800	9.55	95.5	22.00	94.4	+1.1	36.21	60.35
Cd 4678	9.90	99.0	21.66	99.8	-0.8	37.81	63.02
Fe 4378	11.75	117.5	22.10	115.3	+2.2	—	—
Hg 4358	11.49	114.9	21.82	116.5	-1.6	42.37	70.62
Fe 4259	12.25	122.5	22.22	122.5	±	—	—

TABLE XIX.—Rotatory Dispersion of Potassium Bismuthyl Tartrate in presence of excess of Alkali at 20° C.

2.5 gr. ($\frac{1}{80}$ mol.) tartaric acid + 2.16 gr. ($\frac{1}{120}$ mol.) of $\text{Bi}(\text{OH})_3$ + 14 gr. ($\frac{1}{4}$ mol.) KOH in 100 c.c. of solution.

Length of tube = 6 dm.

Simple Formula.

$$[M_1] = \frac{-37.414}{\lambda^2 - 0.0645}$$

λ .	α .	[M] obs.	[M ₁] calc.	[M]—[M ₁].	[M]($\lambda^2 - 0.0645$).
Cd 6438	—10.64	—106.4	—106.9	+0.5	—37.24
Zn 6364	—10.93	—109.3	—109.9	+0.6	—37.22
Na 5893	—13.22	—132.2	—132.2	±	—37.38
Hg 5790	—13.78	—137.8	—138.2	+0.4	—37.31
Cu 5782	—13.90	—139.0	—138.7	—0.3	—37.50
Hg 5769	—13.98	—139.8	—139.4	—0.4	—37.51
Cu 5700	—14.39	—143.9	—143.7	—0.2	—37.47
Hg 5461	—15.99	—159.9	—160.1	+0.2	—37.37
Cu 5219	—18.04	—180.4	—180.2	—0.2	—37.50
Cu 5154	—18.60	—186.0	—186.1	+0.1	—37.41
Cu 5105	—19.01	—190.1	—190.8	+0.7	—37.28
Cd 5086	—19.26	—192.6	—192.6	±	—37.40
Zn 4811	—22.23	—222.3	—224.0	+1.7	—37.12
Cd 4800	—22.55	—225.5	—225.5	±	—37.35
Zn 4722	—23.28	—232.8	—236.1	+3.3	—36.89
Cd 4678	—24.34	—243.4	—242.5	—0.9	—37.57
Hg 4358	—29.89	—298.9	—298.4	—0.5	—37.49
Fe 4326	—30	—300	—305	+5	—36.79
Fe 4271	—31	—310	—317	+7	—36.55
Fe 4251	—32	—320	—322	+2	—37.19
Fe 4249	—33	—330	—322	—8	—38.28

TABLE XX.—Rotatory Dispersion of Boro-tartaric Acid in Aqueous Solutions at 20° C.

(a) 15 gr. (0.10 mol.) tartaric acid + 6.2 gr. (0.10 mol.) boric acid in 100 c.c. solution.					(b) 7.5 gr. (0.05 mol.) tartaric acid + 4.65 gr. (0.075 mol.) boric acid in 100 c.c. solution.				
<i>Complex Formula.</i>					<i>Simple Formula.</i>				
$[M_1] = \frac{25.7}{\lambda^2 - 0.03} - \frac{1.249}{\lambda^2 - 0.065}$					$[M_2] = \frac{24.08}{\lambda^2 - 0.02458}$				
λ .	α .	[M] obs.	[M ₁].	[M]—[M ₁].	λ .	α .	[M] obs.	[M ₂].	[M]—[M ₂].
Li 6708	+34.66	57.8	57.9	−0.1	Li 6708	+16.76	55.9	56.6	−0.7
Cd 6438	37.79	63.0	63.3	−0.3	Cd 6438	18.40	61.3	61.8	−0.5
Na 5893	46.00	76.7	76.6	+0.1	Na 5893	22.43	74.8	74.6	+0.2
Hg 5790	47.83	79.7	79.6	+0.1	Hg 5790	23.24	77.5	77.5	±
Cu 5782	47.90	79.8	79.8	±	Cu 5782	23.39	78.0	77.7	+0.3
Hg 5769	48.06	80.1	80.2	−0.1	Hg 5769	23.42	78.1	78.1	±
Cu 5700	49.50	82.5	82.3	+0.2	Cu 5700	24.13	80.4	80.2	+0.2
Hg 5461	54.38	90.6	90.5	+0.1	Hg 5461	26.42	88.1	88.0	+0.1
Cu 5218	60.04	100.1	100.0	+0.1	Cu 5218	29.28	97.6	97.2	+0.4
Cu 5153	61.86	103.1	102.8	+0.3	Cu 5105	30.65	102.2	102.0	+0.2
Cu 5105	63.06	105.1	105.1	±	Cd 5086	30.92	103.1	102.9	+0.2
Cd 5086	63.52	105.9	106.0	−0.1	Zn 4811	35.09	117.0	116.4	+0.6
Zn 4811	72.11	120.2	120.1	+0.1	Cd 4800	35.17	117.2	117.0	+0.2
Cd 4800	72.30	120.5	120.7	−0.2	Zn 4722	36.50	121.7	121.4	+0.3
Zn 4722	75.30	125.5	125.3	+0.2	Cd 4678	37.24	124.1	124.0	+0.1
Zn 4680	76.50	127.5	127.8	−0.3	Hg 4358	43.62	145.4	145.6	−0.2
Cd 4678	76.85	128.1	128.0	+0.1	Fe 4353	44	147	146	+1
Hg 4358	90.10	150.2	150.7	−0.5	Fe 4288	45	150	151	−1
Fe 4242	96	160	161	−1	Fe 4261	46	153	153	±
Fe 4182	99	165	166	−1	Fe 4210	47	156	158	−2
Fe 4123	102	170	172	−2	Fe 4178	48	160	161	−1
Fe 4072	105	175	177	−2	Fe 4100	50	167	168	−1
Fe 4046	107	178	180	−2					

IX. *On the Mathematical Foundations of Theoretical Statistics.*

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DEFINITIONS.

Centre of Location.—That abscissa of a frequency curve for which the sampling errors of optimum location are uncorrelated with those of optimum scaling. (9.)

Consistency.—A statistic satisfies the criterion of consistency, if, when it is calculated from the whole population, it is equal to the required parameter. (4.)

Distribution.—Problems of distribution are those in which it is required to calculate the distribution of one, or the simultaneous distribution of a number, of functions of quantities distributed in a known manner. (3.)

Efficiency.—The efficiency of a statistic is the ratio (usually expressed as a percentage) which its intrinsic accuracy bears to that of the most efficient statistic possible. It

expresses the proportion of the total available relevant information of which that statistic makes use. (4 and 10.)

Efficiency (Criterion).—The criterion of efficiency is satisfied by those statistics which, when derived from large samples, tend to a normal distribution with the least possible standard deviation. (4.)

Estimation.—Problems of estimation are those in which it is required to estimate the value of one or more of the population parameters from a random sample of the population. (3.)

Intrinsic Accuracy.—The intrinsic accuracy of an error curve is the weight in large samples, divided by the number in the sample, of that statistic of location which satisfies the criterion of sufficiency. (9.)

Isostatistical Regions.—If each sample be represented in a generalized space of which the observations are the co-ordinates, then any region throughout which any set of statistics have identical values is termed an isostatistical region.

Likelihood.—The likelihood that any parameter (or set of parameters) should have any assigned value (or set of values) is proportional to the probability that if this were so, the totality of observations should be that observed.

Location.—The location of a frequency distribution of known form and scale is the process of estimation of its position with respect to each of the several variates. (8.)

Optimum.—The optimum value of any parameter (or set of parameters) is that value (or set of values) of which the likelihood is greatest. (6.)

Scaling.—The scaling of a frequency distribution of known form is the process of estimation of the magnitudes of the deviations of each of the several variates. (8.)

Specification.—Problems of specification are those in which it is required to specify the mathematical form of the distribution of the hypothetical population from which a sample is to be regarded as drawn. (3.)

Sufficiency.—A statistic satisfies the criterion of sufficiency when no other statistic which can be calculated from the same sample provides any additional information as to the value of the parameter to be estimated. (4.)

Validity.—The region of validity of a statistic is the region comprised within its contour of zero efficiency. (10.)

1. THE NEGLECT OF THEORETICAL STATISTICS.

SEVERAL reasons have contributed to the prolonged neglect into which the study of statistics, in its theoretical aspects, has fallen. In spite of the immense amount of fruitful labour which has been expended in its practical applications, the basic principles of this organ of science are still in a state of obscurity, and it cannot be denied that, during the recent rapid development of practical methods, fundamental problems have been ignored and fundamental paradoxes left unresolved. This anomalous state of statistical science is strikingly exemplified by a recent paper (1) entitled "The Funda-

mental Problem of Practical Statistics," in which one of the most eminent of modern statisticians presents what purports to be a general proof of BAYES' postulate, a proof which, in the opinion of a second statistician of equal eminence, "seems to rest upon a very peculiar—not to say hardly supposable—relation." (2.)

Leaving aside the specific question here cited, to which we shall recur, the obscurity which envelops the theoretical bases of statistical methods may perhaps be ascribed to two considerations. In the first place, it appears to be widely thought, or rather felt, that in a subject in which all results are liable to greater or smaller errors, precise definition of ideas or concepts is, if not impossible, at least not a practical necessity. In the second place, it has happened that in statistics a purely verbal confusion has hindered the distinct formulation of statistical problems; for it is customary to apply the same name, *mean*, *standard deviation*, *correlation coefficient*, etc., both to the true value which we should like to know, but can only estimate, and to the particular value at which we happen to arrive by our methods of estimation; so also in applying the term probable error, writers sometimes would appear to suggest that the former quantity, and not merely the latter, is subject to error.

It is this last confusion, in the writer's opinion, more than any other, which has led to the survival to the present day of the fundamental paradox of inverse probability, which like an impenetrable jungle arrests progress towards precision of statistical concepts. The criticisms of BOOLE, VENN, and CHRYSTAL have done something towards banishing the method, at least from the elementary text-books of Algebra; but though we may agree wholly with CHRYSTAL that inverse probability is a mistake (perhaps the only mistake to which the mathematical world has so deeply committed itself), there yet remains the feeling that such a mistake would not have captivated the minds of LAPLACE and POISSON if there had been nothing in it but error.

2. THE PURPOSE OF STATISTICAL METHODS.

In order to arrive at a distinct formulation of statistical problems, it is necessary to define the task which the statistician sets himself: briefly, and in its most concrete form, the object of statistical methods is the reduction of data. A quantity of data, which usually by its mere bulk is incapable of entering the mind, is to be replaced by relatively few quantities which shall adequately represent the whole, or which, in other words, shall contain as much as possible, ideally the whole, of the relevant information contained in the original data.

This object is accomplished by constructing a hypothetical infinite population, of which the actual data are regarded as constituting a random sample. The law of distribution of this hypothetical population is specified by relatively few parameters, which are sufficient to describe it exhaustively in respect of all qualities under discussion. Any information given by the sample, which is of use in estimating the values of these parameters, is relevant information. Since the number of independent facts supplied in

the data is usually far greater than the number of facts sought, much of the information supplied by any actual sample is irrelevant. It is the object of the statistical processes employed in the reduction of data to exclude this irrelevant information, and to isolate the whole of the relevant information contained in the data.

When we speak of the *probability* of a certain object fulfilling a certain condition, we imagine all such objects to be divided into two classes, according as they do or do not fulfil the condition. This is the only characteristic in them of which we take cognisance. For this reason probability is the most elementary of statistical concepts. It is a parameter which specifies a simple dichotomy in an infinite hypothetical population, and it represents neither more nor less than the frequency ratio which we imagine such a population to exhibit. For example, when we say that the probability of throwing a five with a die is one-sixth, we must not be taken to mean that of any six throws with that die one and one only will necessarily be a five; or that of any six million throws, exactly one million will be fives; but that of a hypothetical population of an infinite number of throws, with the die in its original condition, exactly one-sixth will be fives. Our statement will not then contain any false assumption about the actual die, as that it will not wear out with continued use, or any notion of approximation, as in estimating the probability from a finite sample, although this notion may be logically developed once the meaning of probability is apprehended.

The concept of a *discontinuous frequency distribution* is merely an extension of that of a simple dichotomy, for though the number of classes into which the population is divided may be infinite, yet the frequency in each class bears a finite ratio to that of the whole population. In *frequency curves*, however, a second infinity is introduced. No finite sample has a frequency curve: a finite sample may be represented by a histogram, or by a frequency polygon, which to the eye more and more resembles a curve, as the size of the sample is increased. To reach a true curve, not only would an infinite number of individuals have to be placed in each class, but the number of classes (arrays) into which the population is divided must be made infinite. Consequently, it should be clear that the concept of a frequency curve includes that of a hypothetical infinite population, distributed according to a mathematical law, represented by the curve. This law is specified by assigning to each element of the abscissa the corresponding element of probability. Thus, in the case of the normal distribution, the probability of an observation falling in the range dx , is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx,$$

in which expression x is the value of the variate, while m , the mean, and σ , the standard deviation, are the two parameters by which the hypothetical population is specified. If a sample of n be taken from such a population, the data comprise n independent facts. The statistical process of the reduction of these data is designed to extract from them all relevant information respecting the values of m and σ , and to reject all other information as irrelevant.

It should be noted that there is no falsehood in interpreting any set of independent measurements as a random sample from an infinite population ; for any such set of numbers are a random sample from the totality of numbers produced by the same matrix of causal conditions : the hypothetical population which we are studying is an aspect of the totality of the effects of these conditions, of whatever nature they may be. The postulate of randomness thus resolves itself into the question, " Of what population is this a random sample ? " which must frequently be asked by every practical statistician.

It will be seen from the above examples that the process of the reduction of data is, even in the simplest cases, performed by interpreting the available observations as a sample from a hypothetical infinite population ; this is *a fortiori* the case when we have more than one variate, as when we are seeking the values of coefficients of correlation. There is one point, however, which may be briefly mentioned here in advance, as it has been the cause of some confusion. In the example of the frequency curve mentioned above, we took it for granted that the values of both the mean and the standard deviation of the population were relevant to the inquiry. This is often the case, but it sometimes happens that only one of these quantities, for example the standard deviation, is required for discussion. In the same way an infinite normal population of two correlated variates will usually require five parameters for its specification, the two means, the two standard deviations, and the correlation ; of these often only the correlation is required, or if not alone of interest, it is discussed without reference to the other four quantities. In such cases an alteration has been made in what is, and what is not, relevant, and it is not surprising that certain small corrections should appear, or not, according as the other parameters of the hypothetical surface are or are not deemed relevant. Even more clearly is this discrepancy shown when, as in the treatment of such fourfold tables as exhibit the recovery from smallpox of vaccinated and unvaccinated patients, the method of one school of statisticians treats the proportion of vaccinated as relevant, while others dismiss it as irrelevant to the inquiry. (3.)

3. THE PROBLEMS OF STATISTICS.

The problems which arise in reduction of data may be conveniently divided into three types :—

- (1) Problems of Specification. These arise in the choice of the mathematical form of the population.
- (2) Problems of Estimation. These involve the choice of methods of calculating from a sample statistical derivatives, or as we shall call them statistics, which are designed to estimate the values of the parameters of the hypothetical population.
- (3) Problems of Distribution. These include discussions of the distribution of statistics derived from samples, or in general any functions of quantities whose distribution is known.

It will be clear that when we know (1) what parameters are required to specify the

population from which the sample is drawn, (2) how best to calculate from the sample estimates of these parameters, and (3) the exact form of the distribution, in different samples, of our derived statistics, then the theoretical aspect of the treatment of any particular body of data has been completely elucidated.

As regards problems of specification, these are entirely a matter for the practical statistician, for those cases where the qualitative nature of the hypothetical population is known do not involve any problems of this type. In other cases we may know by experience what forms are likely to be suitable, and the adequacy of our choice may be tested *a posteriori*. We must confine ourselves to those forms which we know how to handle, or for which any tables which may be necessary have been constructed. More or less elaborate forms will be suitable according to the volume of the data. Evidently these are considerations the nature of which may change greatly during the work of a single generation. We may instance the development by PEARSON of a very extensive system of skew curves, the elaboration of a method of calculating their parameters, and the preparation of the necessary tables, a body of work which has enormously extended the power of modern statistical practice, and which has been, by pertinacity and inspiration alike, practically the work of a single man. Nor is the introduction of the Pearsonian system of frequency curves the only contribution which their author has made to the solution of problems of specification: of even greater importance is the introduction of an objective criterion of goodness of fit. For empirical as the specification of the hypothetical population may be, this empiricism is cleared of its dangers if we can apply a rigorous and objective test of the adequacy with which the proposed population represents the whole of the available facts. Once a statistic, suitable for applying such a test, has been chosen, the exact form of its distribution in random samples must be investigated, in order that we may evaluate the probability that a worse fit should be obtained from a random sample of a population of the type considered. The possibility of developing complete and self-contained tests of goodness of fit deserves very careful consideration, since therein lies our justification for the free use which is made of empirical frequency formulæ. Problems of distribution of great mathematical difficulty have to be faced in this direction.

Although problems of estimation and of distribution may be studied separately, they are intimately related in the development of statistical methods. Logically problems of distribution should have prior consideration, for the study of the random distribution of different suggested statistics, derived from samples of a given size, must guide us in the choice of which statistic it is most profitable to calculate. The fact is, however, that very little progress has been made in the study of the distribution of statistics derived from samples. In 1900 PEARSON (15) gave the exact form of the distribution of χ^2 , the Pearsonian test of goodness of fit, and in 1915 the same author published (18) a similar result of more general scope, valid when the observations are regarded as subject to linear constraints. By an easy adaptation (17) the tables of probability derived from this formula may be made available for the more numerous cases in which linear con-

straints are imposed upon the hypothetical population by the means which we employ in its reconstruction. The distribution of the mean of samples of n from a normal population has long been known, but in 1908 "Student" (4) broke new ground by calculating the distribution of the ratio which the deviation of the mean from its population value bears to the standard deviation calculated from the sample. At the same time he gave the exact form of the distribution in samples of the standard deviation. In 1915 FISHER (5) published the curve of distribution of the correlation coefficient for the standard method of calculation, and in 1921 (6) he published the corresponding series of curves for intraclass correlations. The brevity of this list is emphasised by the absence of investigation of other important statistics, such as the regression coefficients, multiple correlations, and the correlation ratio. A formula for the probable error of any statistic is, of course, a practical necessity, if that statistic is to be of service: and in the majority of cases such formulæ have been found, chiefly by the labours of PEARSON and his school, by a first approximation, which describes the distribution with sufficient accuracy if the sample is sufficiently large. Problems of distribution, other than the distribution of statistics, used to be not uncommon as examination problems in probability, and the physical importance of problems of this type may be exemplified by the chemical laws of mass action, by the statistical mechanics of GIBBS, developed by JEANS in its application to the theory of gases, by the electron theory of LORENTZ, and by PLANCK'S development of the theory of quanta, although in all these applications the methods employed have been, from the statistical point of view, relatively simple.

The discussions of theoretical statistics may be regarded as alternating between problems of estimation and problems of distribution. In the first place a method of calculating one of the population parameters is devised from common-sense considerations: we next require to know its probable error, and therefore an approximate solution of the distribution, in samples, of the statistic calculated. It may then become apparent that other statistics may be used as estimates of the same parameter. When the probable errors of these statistics are compared, it is usually found that, in large samples, one particular method of calculation gives a result less subject to random errors than those given by other methods of calculation. Attacking the problem more thoroughly, and calculating the surface of distribution of any two statistics, we may find that the whole of the relevant information contained in one is contained in the other: or, in other words, that when once we know the other, knowledge of the first gives us no further information as to the value of the parameter. Finally it may be possible to prove, as in the case of the Mean Square Error, derived from a sample of normal population (7), that a particular statistic summarises the whole of the information relevant to the corresponding parameter, which the sample contains. In such a case the problem of estimation is completely solved.

4. CRITERIA OF ESTIMATION.

The common-sense criterion employed in problems of estimation may be stated thus :— That when applied to the whole population the derived statistic should be equal to the parameter. This may be called the *Criterion of Consistency*. It is often the only test applied : thus, in estimating the standard deviation of a normally distributed population, from an ungrouped sample, either of the two statistics—

$$\sigma_1 = \frac{1}{n} \sqrt{\frac{\pi}{2}} S (|x - \bar{x}|) \quad (\text{Mean error})$$

and

$$\sigma_2 = \sqrt{\frac{1}{n} S (x - \bar{x})^2} \quad (\text{Mean square error})$$

will lead to the correct value, σ , when calculated from the whole population. They both thus satisfy the criterion of consistency, and this has led many computers to use the first formula, although the result of the second has 14 per cent. greater weight (7), and the labour of increasing the number of observations by 14 per cent. can seldom be less than that of applying the more accurate formula.

Consideration of the above example will suggest a second criterion, namely :—That in large samples, when the distributions of the statistics tend to normality, that statistic is to be chosen which has the least probable error.

This may be called the *Criterion of Efficiency*. It is evident that if for large samples one statistic has a probable error double that of a second, while both are proportional to n^{-1} , then the first method applied to a sample of $4n$ values will be no more accurate than the second applied to a sample of any n values. If the second method makes use of the whole of the information available, the first makes use of only one-quarter of it, and its efficiency may therefore be said to be 25 per cent. To calculate the efficiency of any given method, we must therefore know the probable error of the statistic calculated by that method, and that of the most efficient statistic which could be used. The square of the ratio of these two quantities then measures the efficiency.

The criterion of efficiency is still to some extent incomplete, for different methods of calculation may tend to agreement for large samples, and yet differ for all finite samples. The complete criterion suggested by our work on the mean square error (7) is :—

That the statistic chosen should summarise the whole of the relevant information supplied by the sample.

This may be called the *Criterion of Sufficiency*.

In mathematical language we may interpret this statement by saying that if θ be the parameter to be estimated, θ_1 a statistic which contains the whole of the information as to the value of θ , which the sample supplies, and θ_2 any other statistic, then the

surface of distribution of pairs of values of θ_1 and θ_2 , for a given value of θ , is such that for a given value of θ_1 , the distribution of θ_2 does not involve θ . In other words, when θ_1 is known, knowledge of the value of θ , throws no further light upon the value of θ_2 .

It may be shown that a statistic which fulfils the criterion of sufficiency will also fulfil the criterion of efficiency, when the latter is applicable. For, if this be so, the distribution of the statistics will in large samples be normal, the standard deviations being proportional to $n^{-\frac{1}{2}}$. Let this distribution be

$$df = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{-\frac{1}{1-r^2}\left\{\frac{\theta_1-\theta}{2\sigma_1} - \frac{2r\theta_1-\theta}{2\sigma_1\sigma_2} + \frac{\theta_2-\theta}{2\sigma_2}\right\}^2} d\theta_1 d\theta_2,$$

then the distribution of θ_1 is

$$df = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{\theta_1-\theta}{2\sigma_1}^2} d\theta_1,$$

so that for a given value of θ_1 the distribution of θ_2 is

$$df = \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left\{\frac{r\theta_1-\theta}{\sigma_1} - \frac{\theta_2-\theta}{\sigma_2}\right\}^2} d\theta_2;$$

and if this does not involve θ , we must have

$$r\sigma_2 = \sigma_1;$$

showing that σ_1 is necessarily less than σ_2 , and that the efficiency of θ_2 is measured by r^2 , when r is its correlation in large samples with θ_1 .

Besides this case we shall see that the criterion of sufficiency is also applicable to finite samples, and to those cases when the weight of a statistic is not proportional to the number of the sample from which it is calculated.

5. EXAMPLES OF THE USE OF THE CRITERION OF CONSISTENCY.

In certain cases the criterion of consistency is sufficient for the solution of problems of estimation. An example of this occurs when a fourfold table is interpreted as representing the double dichotomy of a normal surface. In this case the dichotomic ratios of the two variates, together with the correlation, completely specify the four fractions into which the population is divided. If these are equated to the four fractions into which the sample is divided, the correlation is determined uniquely.

In other cases where a small correction has to be made, the amount of the correction is not of sufficient importance to justify any great refinement in estimation, and it is sufficient to calculate the discrepancy which appears when the uncorrected method is applied to the whole population. Of this nature is SHEPPARD'S correction for grouping,

and it will illustrate this use of the criterion of consistency if we derive formulæ for this correction without approximation.

Let ξ be the value of the variate at the mid point of any group, a the interval of grouping, and x the true value of the variate at any point, then the k^{th} moment of an infinite grouped sample is

$$\sum_{p=-\infty}^{p=\infty} \int_{\xi-1/2a}^{\xi+1/2a} \xi^k f(x) dx,$$

in which $f(x) dx$ is the frequency, in any element dx , of the ungrouped population, and

$$\xi = \left(p + \frac{\theta}{2\pi} \right) a,$$

p being any integer.

Evidently the k^{th} moment is periodic in θ , we will therefore equate it to

$$A_0 + A_1 \sin \theta + A_2 \sin 2\theta \dots \\ + B_1 \cos \theta + B_2 \cos 2\theta \dots$$

Then

$$A_0 = \frac{1}{2\pi} \sum_{p=-\infty}^{p=\infty} \int_0^{2\pi} d\theta \int_{\xi-1/2a}^{\xi+1/2a} \xi^k f(x) dx$$

$$A_s = \frac{1}{\pi} \sum_{p=-\infty}^{p=\infty} \int_0^{2\pi} \sin s\theta d\theta \int_{\xi-1/2a}^{\xi+1/2a} \xi^k f(x) dx,$$

$$B_s = \frac{1}{\pi} \sum_{p=-\infty}^{p=\infty} \int_0^{2\pi} \cos s\theta d\theta \int_{\xi-1/2a}^{\xi+1/2a} \xi^k f(x) dx.$$

But

$$\theta = \frac{2\pi}{a} \xi - 2\pi p,$$

therefore

$$d\theta = \frac{2\pi}{a} d\xi,$$

$$\sin s\theta = \sin \frac{2\pi}{a} s\xi,$$

$$\cos s\theta = \cos \frac{2\pi}{a} s\xi,$$

hence

$$A_0 = \frac{1}{a} \int_{-\infty}^{\infty} d\xi \int_{\xi-1/2a}^{\xi+1/2a} \xi^k f(x) dx = \frac{1}{a} \int_{-\infty}^{\infty} f(x) dx \int_{-\infty}^{\infty} \xi^k d\xi.$$

Inserting the values 1, 2, 3 and 4 for k , we obtain for the aperiodic terms of the four moments of the grouped population

$$\begin{aligned} {}_1A_0 &= \int_{-\infty}^{\infty} x f(x) dx, \\ {}_2A_0 &= \int_{-\infty}^{\infty} \left(x^2 + \frac{\alpha^2}{12}\right) f(x) dx, \\ {}_3A_0 &= \int_{-\infty}^{\infty} \left(x^3 + \frac{\alpha^2 x}{4}\right) f(x) dx, \\ {}_4A_0 &= \int_{-\infty}^{\infty} \left(x^4 + \frac{\alpha^2 x^2}{2} + \frac{\alpha^4}{80}\right) f(x) dx. \end{aligned}$$

If we ignore the periodic terms, these equations lead to the ordinary SHEPPARD corrections for the second and fourth moment. The nature of the approximation involved is brought out by the periodic terms. In the absence of high contact at the ends of the curve, the contribution of these will, of course, include the terms given in a recent paper by PEARSON (8); but even with high contact it is of interest to see for what degree of coarseness of grouping the periodic terms become sensible.

Now

$$\begin{aligned} A_s &= \frac{1}{\pi} \sum_{\mu=-\infty}^{\mu=\infty} \int_0^{2\pi} \sin s\theta d\theta \int_{\xi-\frac{1}{2}a}^{\xi+\frac{1}{2}a} \xi^k f(x) dx, \\ &= \frac{2}{a} \int_{-\infty}^{\infty} \sin \frac{2\pi s\xi}{a} d\xi \int_{\xi-\frac{1}{2}a}^{\xi+\frac{1}{2}a} \xi^k f(x) dx, \\ &= \frac{2}{a} \int_{-\infty}^{\infty} f(x) dx \int_{x-\frac{1}{2}a}^{x+\frac{1}{2}a} \xi^k \sin \frac{2\pi s\xi}{a} d\xi. \end{aligned}$$

But

$$\frac{2}{a} \int_{x-\frac{1}{2}a}^{x+\frac{1}{2}a} \xi \sin \frac{2\pi s\xi}{a} d\xi = -\frac{a}{\pi s} \cos \frac{2\pi sx}{a} \cos \pi s,$$

therefore

$${}_1A_s = (-)^{s+1} \frac{a}{\pi s} \int_{-\infty}^{\infty} \cos \frac{2\pi sx}{a} f(x) dx;$$

similarly the other terms of the different moments may be calculated.

For a normal curve referred to the true mean

$${}_1A_s = (-)^{s+1} \frac{2\epsilon}{s} e^{-\frac{s^2\sigma^2}{2\epsilon^2}},$$

$${}_1B_s = 0,$$

in which

$$a = 2\pi\epsilon.$$

The error of the mean is therefore

$$-2\epsilon \left(e^{-\frac{\epsilon^2}{2\sigma^2}} \sin \theta - \frac{1}{2} e^{-\frac{4\epsilon^2}{2\sigma^2}} \sin 2\theta + \frac{1}{8} e^{-\frac{9\epsilon^2}{2\sigma^2}} \sin 3\theta - \dots \right).$$

To illustrate a coarse grouping, take the group interval equal to the standard deviation : then

$$\epsilon = \frac{\sigma}{2\pi},$$

and the error is

$$-\frac{\sigma}{\pi} e^{-2\pi^2} \sin \theta$$

with sufficient accuracy. The standard error of the mean being $\frac{\sigma}{\sqrt{n}}$, we may calculate the size of the sample for which the error due to the periodic terms becomes equal to one-tenth of the standard error, by putting

$$\frac{\sigma}{10\sqrt{n}} = \frac{\sigma}{\pi} e^{-2\pi^2},$$

whence

$$n = \frac{\pi^2}{100} e^{4\pi^2} = 13,790 \text{ billion.}$$

For the second moment

$$B_2 = (-)^2 4 \left(\sigma^2 + \frac{\epsilon^2}{s^2} \right) e^{-\frac{\pi^2 \sigma^2}{s^2}},$$

and, if we put

$$\frac{\sqrt{2}\sigma^2}{10\sqrt{n}} = 4\sigma^2 e^{-2\pi^2},$$

there results

$$n = \frac{1}{800} e^{4\pi^2} = 175 \text{ billion.}$$

The error, while still very minute, is thus more important for the second than for the first moment.

For the third moment

$$A_3 = (-)^3 \frac{6\sigma^3 s}{\epsilon} \left\{ 1 + \frac{\epsilon^2}{s^2 \sigma^2} - \frac{\epsilon^4}{3s^4 \sigma^4} (\pi^2 s^2 - 6) \right\} e^{-\frac{\pi^2 \sigma^3}{s^2}};$$

putting

$$\frac{\sqrt{15}\sigma^3}{10\sqrt{n}} = 12\pi\sigma^3 e^{-2\pi^2},$$

$$n = \frac{1}{960\pi^3} e^{4\pi^2} = 14.7 \text{ billion.}$$

While for the fourth moment

$$B_4 = (-)^{4+1} \frac{8\sigma^4 s^3}{\epsilon^3} \left\{ 1 - (\pi^2 s^2 - 3) \frac{\epsilon^4}{s^4 \sigma^4} - (\pi^2 s^2 - 6) \frac{\epsilon^6}{s^6 \sigma^6} \right\} e^{-\frac{\pi^2 \sigma^4}{s^2}},$$

so that, if we put,

$$\frac{\sqrt{96}\sigma^4}{10\sqrt{n}} = 32\pi^3 \sigma^4 e^{-2\pi^2},$$

$$n = \frac{3}{3200\pi^4} e^{4\pi^2} = 1.34 \text{ billion.}$$

In a similar manner the exact form of SHEPPARD'S correction may be found for other curves ; for the normal curve we may say that the periodic terms are exceedingly minute so long as α is less than σ , though they increase very rapidly if α is increased beyond this point. They are of increasing importance as higher moments are used, not only absolutely, but relatively to the increasing probable errors of the higher moments. The principle upon which the correction is based is merely to find the error when the moments are calculated from an infinite grouped sample ; the corrected moment therefore fulfils the criterion of consistency, and so long as the correction is small no greater refinement is required.

Perhaps the most extended use of the criterion of consistency has been developed by PEARSON in the "Method of Moments." In this method, which is without question of great practical utility, different forms of frequency curves are fitted by calculating as many moments of the sample as there are parameters to be evaluated. The parameters chosen are those of an infinite population of the specified type having the same moments as those calculated from the sample.

The system of curves developed by PEARSON has four variable parameters, and may be fitted by means of the first four moments. For this purpose it is necessary to confine attention to curves of which the first four moments are finite ; further, if the accuracy of the fourth moment should increase with the size of the sample, that is, if its probable error should not be infinitely great, the first eight moments must be finite. This restriction requires that the class of distribution in which this condition is not fulfilled should be set aside as "heterotypic," and that the fourth moment should become practically valueless as this class is approached. It should be made clear, however, that there is nothing anomalous about these so-called "heterotypic" distributions except the fact that the method of moments cannot be applied to them. Moreover, for that class of distribution to which the method can be applied, it has not been shown, except in the case of the normal curve, that the best values will be obtained by the method of moments. The method will, in these cases, certainly be serviceable in yielding an approximation, but to discover whether this approximation is a good or a bad one, and to improve it, if necessary, a more adequate criterion is required.

A single example will be sufficient to illustrate the practical difficulty alluded to above. If a point P lie at known (unit) distance from a straight line AB, and lines be drawn at random through P, then the distribution of the points of intersection with AB will be distributed so that the frequency in any range dx is

$$df = \frac{1}{\pi} \cdot \frac{dx}{1 + (x-m)^2},$$

in which x is the distance of the infinitesimal range dx from a fixed point 0 on the line, and m is the distance, from this point, of the foot of the perpendicular PM. The distri-

bution will be a symmetrical one (Type VII.) having its centre at $x = m$ (fig. 1). It is therefore a perfectly definite problem to estimate the value of m (to find the best value of m) from a random sample of values of x . We have stated the problem in its simplest possible form: only one parameter is required, the middle point of the distribution.

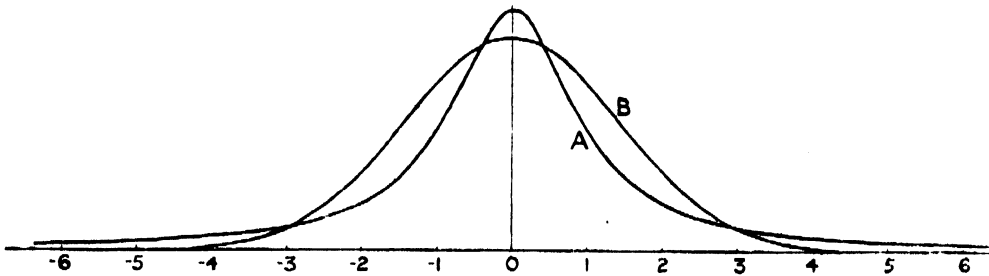


Fig. 1. Symmetrical error curves of equal intrinsic accuracy.

A $df = \frac{1}{\pi} \frac{dx}{1+x^2}$.

B $dy = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}}$

By the method of moments, this should be given by the first moment, that is by the mean of the observations: such would seem to be at least a good estimate. It is, however, entirely valueless. The distribution of the mean of such samples is in fact the same, identically, as that of a single observation. In taking the mean of 100 values of x , we are no nearer obtaining the value of m than if we had chosen any value of x out of the 100. The problem, however, is not in the least an impracticable one: clearly from a large sample we ought to be able to estimate the centre of the distribution with some precision; the mean, however, is an entirely useless statistic for the purpose. By taking the median of a large sample, a fair approximation is obtained, for the standard error of the median of a large sample of n is $\frac{\pi}{2\sqrt{n}}$, which, alone, is enough to show that

by adopting adequate statistical methods it must be possible to estimate the value for m , with increasing accuracy, as the size of the sample is increased.

This example serves also to illustrate the practical difficulty which observers often find, that a few extreme observations appear to dominate the value of the mean. In these cases the rejection of extreme values is often advocated, and it may often happen that gross errors are thus rejected. As a statistical measure, however, the rejection of observations is too crude to be defended: and unless there are other reasons for rejection than mere divergence from the majority, it would be more philosophical to accept these extreme values, not as gross errors, but as indications that the distribution of errors is not normal. As we shall show, the only Pearsonian curve for which the mean

is the best statistic for locating the curve, is the normal or gaussian curve of errors. If the curve is not of this form the mean is not necessarily, as we have seen, of any value whatever. The determination of the true curves of variation for different types of work is therefore of great practical importance, and this can only be done by different workers recording their data in full without rejections, however they may please to treat the data so recorded. Assuredly an observer need be exposed to no criticism, if after recording data which are not probably normal in distribution, he prefers to adopt some value other than the arithmetic mean.

6. FORMAL SOLUTION OF PROBLEMS OF ESTIMATION.

The form in which the criterion of sufficiency has been presented is not of direct assistance in the solution of problems of estimation. For it is necessary first to know the statistic concerned and its surface of distribution, with an infinite number of other statistics, before its sufficiency can be tested. For the solution of problems of estimation we require a method which for each particular problem will lead us automatically to the statistic by which the criterion of sufficiency is satisfied. Such a method is, I believe, provided by the Method of Maximum Likelihood, although I am not satisfied as to the mathematical rigour of any proof which I can put forward to that effect. Readers of the ensuing pages are invited to form their own opinion as to the possibility of the method of the maximum likelihood leading in any case to an insufficient statistic. For my own part I should gladly have withheld publication until a rigorously complete proof could have been formulated; but the number and variety of the new results which the method discloses press for publication, and at the same time I am not insensible of the advantage which accrues to Applied Mathematics from the co-operation of the Pure Mathematician, and this co-operation is not infrequently called forth by the very imperfections of writers on Applied Mathematics.

If in any distribution involving unknown parameters $\theta_1, \theta_2, \theta_3, \dots$, the chance of an observation falling in the range dx be represented by

$$f(x, \theta_1, \theta_2, \dots) dx,$$

then the chance that in a sample of n , n_1 fall in the range dx_1 , n_2 in the range dx_2 , and so on, will be

$$\frac{n!}{\prod (n_p!)} \prod \{f(x_p, \theta_1, \theta_2, \dots) dx_p\}^{n_p}.$$

The method of maximum likelihood consists simply in choosing that set of values for the parameters which makes this quantity a maximum, and since in this expression the parameters are only involved in the function f , we have to make

$$S(\log f)$$

a maximum for variations of $\theta_1, \theta_2, \theta_3,$ &c. In this form the method is applicable to the fitting of populations involving any number of variates, and equally to discontinuous as to continuous distributions.

In order to make clear the distinction between this method and that of BAYES, we will apply it to the same type of problem as that which BAYES discussed, in the hope of making clear exactly of what kind is the information which a sample is capable of supplying. This question naturally first arose, not with respect to populations distributed in frequency curves and surfaces, but with respect to a population regarded as divided into two classes only, in fact in problems of *probability*. A certain proportion, p , of an infinite population is supposed to be of a certain kind, e.g., "successes," the remainder are then "failures." A sample of n is taken and found to contain x successes and y failures. The chance of obtaining such a sample is evidently

$$\frac{n!}{x!y!} p^x (1-p)^y.$$

Applying the method of maximum likelihood, we have

$$S(\log f) = x \log \hat{p} + y \log (1-\hat{p})$$

whence, differentiating with respect to p , in order to make this quantity a maximum,

$$\frac{x}{p} = \frac{-y}{1-\hat{p}}, \quad \text{or} \quad \hat{p} = \frac{x}{n}.$$

The question then arises as to the accuracy of this determination. This question was first discussed by BAYES (10), in a form which we may state thus. After observing this sample, when we know p , what is the *probability* that p lies in any range dp ? In other words, what is the frequency distribution of the values of p in populations which are selected by the restriction that a sample of n taken from each of them yields x successes. Without further data, as BAYES perceived, this problem is insoluble. To render it capable of mathematical treatment, BAYES introduced the *datum*, that among the populations upon which the experiment was tried, those in which p lay in the range dp were equally frequent for all equal ranges dp . The probability that the value of p lay in any range dp was therefore assumed to be simply dp , before the sample was taken. After the selection effected by observing the sample, the probability is clearly proportional to

$$p^x (1-p)^y dp.$$

After giving this solution, based upon the particular datum stated, BAYES adds a *scholium* the purport of which would seem to be that in the absence of all knowledge save that supplied by the sample, it is reasonable to assume this particular *a priori* distribution of p . The *result*, the *datum*, and the *postulate* implied by the *scholium*, have all been somewhat loosely spoken of as BAYES' Theorem.

The postulate would, if true, be of great importance in bringing an immense variety of questions within the domain of probability. It is, however, evidently extremely arbitrary. Apart from evolving a vitally important piece of knowledge, that of the exact form of the distribution of values of p , out of an assumption of complete ignorance, it is not even a unique solution. For we might never have happened to direct our attention to the particular quantity p : we might equally have measured probability upon an entirely different scale. If, for instance,

$$\sin \theta = 2p - 1,$$

the quantity, θ , measures the degree of probability, just as well as p , and is even, for some purposes, the more suitable variable. The chance of obtaining a sample of x successes and y failures is now

$$\frac{n!}{2^n x! y!} (1 + \sin \theta)^x (1 - \sin \theta)^y;$$

applying the method of maximum likelihood,

$$S(\log f) = x \log (1 + \sin \theta) + y \log (1 - \sin \theta) - n \log 2,$$

and differentiating with respect to θ ,

$$\frac{x \cos \theta}{1 + \sin \theta} = \frac{y \cos \theta}{1 - \sin \theta}, \quad \text{whence} \quad \sin \theta = \frac{x - y}{2n},$$

an exactly equivalent solution to that obtained using the variable p . But what *a priori* assumption are we to make as to the distribution of θ ? Are we to assume that θ is equally likely to lie in all equal ranges $d\theta$? In this case the *a priori* probability will be $d\theta/\pi$, and that after making the observations will be proportional to

$$(1 + \sin \theta)^x (1 - \sin \theta)^y d\theta.$$

But if we interpret this in terms of p , we obtain

$$p^x (1 - p)^y \frac{dp}{\sqrt{p(1 - p)}} = p^{x-1/2} (1 - p)^{y-1/2} dp,$$

a result inconsistent with that obtained previously. In fact, the distribution previously assumed for p was equivalent to assuming the special distribution for θ ,

$$df = \frac{\cos \theta}{2} d\theta,$$

the arbitrariness of which is fully apparent when we use any variable other than p .

In a less obtrusive form the same species of arbitrary assumption underlies the method

known as that of inverse probability. Thus, if the same observed result A might be the consequence of one or other of two hypothetical conditions X and Y , it is assumed that the probabilities of X and Y are in the same ratio as the probabilities of A occurring on the two assumptions, X is true, Y is true. This amounts to assuming that before A was observed, it was known that our universe had been selected at random for an infinite population in which X was true in one half, and Y true in the other half. Clearly such an assumption is entirely arbitrary, nor has any method been put forward by which such assumptions can be made even with consistent uniqueness. There is nothing to prevent an irrelevant distinction being drawn among the hypothetical conditions represented by X , so that we have to consider two hypothetical possibilities X_1 and X_2 , on both of which A will occur with equal frequency. Such a distinction should make no difference whatever to our conclusions; but on the principle of inverse probability it does so, for if previously the relative probabilities were reckoned to be in the ratio x to y , they must now be reckoned $2x$ to y . Nor has any criterion been suggested by which it is possible to separate such irrelevant distinctions from those which are relevant.

There would be no need to emphasise the baseless character of the assumptions made under the titles of inverse probability and BAYES' Theorem in view of the decisive criticism to which they have been exposed at the hands of BOOLE, VENN, and CHRYSTAL, were it not for the fact that the older writers, such as LAPLACE and POISSON, who accepted these assumptions, also laid the foundations of the modern theory of statistics, and have introduced into their discussions of this subject ideas of a similar character. I must indeed plead guilty in my original statement of the Method of the Maximum Likelihood (9) to having based my argument upon the principle of inverse probability; in the same paper, it is true, I emphasised the fact that such inverse probabilities were relative only. That is to say, that while we might speak of one value of p as having an inverse probability three times that of another value of p , we might on no account introduce the differential element dp , so as to be able to say that it was three times as probable that p should lie in one rather than the other of two equal elements. Upon consideration, therefore, I perceive that the word probability is wrongly used in such a connection: probability is a ratio of frequencies, and about the frequencies of such values we can know nothing whatever. We must return to the actual fact that one value of p , of the frequency of which we know nothing, would yield the observed result three times as frequently as would another value of p . If we need a word to characterise this relative property of different values of p , I suggest that we may speak without confusion of the *likelihood* of one value of p being thrice the likelihood of another, bearing always in mind that likelihood is not here used loosely as a synonym of probability, but simply to express the relative frequencies with which such values of the hypothetical quantity p would in fact yield the observed sample.

The solution of the problems of calculating from a sample the parameters of the hypothetical population, which we have put forward in the method of maximum likeli-

hood, consists, then, simply of choosing such values of these parameters as have the maximum likelihood. Formally, therefore, it resembles the calculation of the mode of an inverse frequency distribution. This resemblance is quite superficial: if the scale of measurement of the hypothetical quantity be altered, the mode must change its position, and can be brought to have any value, by an appropriate change of scale; but the optimum, as the position of maximum likelihood may be called, is entirely unchanged by any such transformation. Likelihood also differs from probability* in that it is not a differential element, and is incapable of being integrated: it is assigned to a particular point of the range of variation, not to a particular element of it. There is therefore an absolute measure of probability in that the unit is chosen so as to make all the elementary probabilities add up to unity. There is no such absolute measure of likelihood. It may be convenient to assign the value unity to the maximum value, and to measure other likelihoods by comparison, but there will then be an infinite number of values whose likelihood is greater than one-half. The sum of the likelihoods of admissible values will always be infinite.

Our interpretation of BAYES' problem, then, is that the likelihood of any value of p is proportional to

$$p^x (1-p)^y,$$

and is therefore a maximum when

$$p = \frac{x}{n},$$

which is the best value obtainable from the sample; we shall term this the *optimum* value of p . Other values of p for which the likelihood is not much less cannot, however, be deemed unlikely values for the true value of p . We do not, and cannot, know, from the information supplied by a sample, anything about the probability that p should lie between any named values.

The reliance to be placed on such a result must depend upon the frequency distribution of x , in different samples from the same population. This is a perfectly objective statistical problem, of the kind we have called problems of distribution; it is, however, capable of an approximate solution, directly from the mathematical form of the likelihood.

When for large samples the distribution of any statistic, θ , tends to normality, we

* It should be remarked that likelihood, as above defined, is not only fundamentally distinct from mathematical probability, but also from the logical "probability" by which Mr. KEYNES (21) has recently attempted to develop a method of treatment of uncertain inference, applicable to those cases where we lack the statistical information necessary for the application of mathematical probability. Although, in an important class of cases, the likelihood may be held to measure the degree of our rational belief in a conclusion, in the same sense as Mr. KEYNES' "probability," yet since the latter quantity is constrained, somewhat arbitrarily, to obey the addition theorem of mathematical probability, the likelihood is a quantity which falls definitely outside its scope.

may write down the chance for a given value of the parameter θ , that θ_1 should lie in the range $d\theta_1$, in the form

$$\Phi = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\theta_1 - \theta)^2}{2\sigma^2}} d\theta_1.$$

The mean value of θ_1 will be the true value θ , and the standard deviation is σ , the sample being assumed sufficiently large for us to disregard the dependence of σ upon θ .

The likelihood of any value, θ , is proportional to

$$e^{-\frac{(\theta_1 - \theta)^2}{2\sigma^2}},$$

this quantity having its maximum value, unity, when

$$\theta = \theta_1;$$

for

$$\frac{\partial}{\partial \theta} \log \Phi = \frac{\theta_1 - \theta}{\sigma^2}.$$

Differentiating now a second time

$$\frac{\partial^2}{\partial \theta^2} \log \Phi = -\frac{1}{\sigma^2}.$$

Now Φ stands for the total frequency of all samples for which the chosen statistic has the value θ_1 , consequently $\Phi = S'(\phi)$, the summation being taken over all such examples, where ϕ stands for the probability of occurrence of a certain specified sample. For which we know that

$$\log \phi = C + S(\log f),$$

the summation being taken over the individual members of the sample.

If now we expand $\log f$ in the form

$$\log f(\theta) = \log f(\theta_1) + \overline{\theta - \theta_1} \frac{\partial}{\partial \theta} \log f(\theta_1) + \frac{\overline{\theta - \theta_1}^2}{2} \frac{\partial^2}{\partial \theta^2} \log f(\theta_1) + \dots,$$

or

$$\log f = \log f_1 + a \overline{\theta - \theta_1} + \frac{b}{2} \overline{\theta - \theta_1}^2 + \dots,$$

we have

$$\log \phi = C + \overline{\theta - \theta_1} S(a) + \frac{1}{2} \overline{\theta - \theta_1}^2 S(b) + \dots;$$

now for optimum statistics

$$S(a) = 0,$$

and for sufficiently large samples $S(b)$ differs from $n\bar{b}$ only by a quantity of order $\sqrt{n}\sigma_b$; moreover, $\theta - \theta_1$ being of order $n^{-1/2}$, the only terms in $\log \phi$ which are not reduced without limit, as n is increased, are

$$\log \phi = C + \frac{1}{2} n \bar{b} \overline{\theta - \theta_1}^2;$$

nence

$$\phi \propto e^{\frac{1}{2}n\bar{b}(\theta-\theta_1)^2}.$$

Now this factor is constant for all samples which have the same value of θ_1 , hence the variation of Φ with respect to θ is represented by the same factor, and consequently

$$\log \Phi = C' + \frac{1}{2}n\bar{b}(\theta-\theta_1)^2;$$

whence

$$-\frac{1}{\sigma_{\theta_1}^2} = \frac{\partial^2}{\partial \theta^2} \log \Phi = n\bar{b},$$

where

$$b = \frac{\partial^2}{\partial \theta^2} \log f(\theta_1),$$

θ_1 being the optimum value of θ .

The formulæ

$$-\frac{1}{\sigma_{\theta}^2} = x \frac{\partial^2}{\partial \theta^2} \log f'$$

supplies the most direct way known to me of finding the probable errors of statistics. It may be seen that the above proof applies only to statistics obtained by the method of maximum likelihood.*

For example, to find the standard deviation of

$$\hat{p} = \frac{x}{n}$$

* A similar method of obtaining the standard deviations and correlations of statistics derived from large samples was developed by PEARSON and FILON in 1898 (16). It is unfortunate that in this memoir no sufficient distinction is drawn between the *population* and the *sample*, in consequence of which the formulæ obtained indicate that the likelihood is always a maximum (for continuous distributions) when the *mean* of each variate in the sample is equated to the corresponding mean in the population (16, p. 232, "A_r = 0"). If this were so the mean would always be a sufficient statistic for location; but as we have already seen, and will see later in more detail, this is far from being the case. The same argument, indeed, is applied to all statistics, as to which nothing but their *consistency* can be truly affirmed.

The probable errors obtained in this way are those appropriate to the method of maximum likelihood, but not in other cases to statistics obtained by the method of moments, by which method the examples given were fitted. In the 'Tables for Statisticians and Biometricians' (1914), the probable errors of the constants of the Pearsonian curves are those proper to the method of moments; no mention is there made of this change of practice, nor is the publication of 1898 referred to.

It would appear that shortly before 1898 the process which leads to the correct value, of the probable errors of *optimum* statistics, was hit upon and found to agree with the probable errors of statistics found by the method of moments for *normal* curves and surfaces; without further enquiry it would appear to have been assumed that this process was valid in all cases, its directness and simplicity being peculiarly attractive. The mistake was at that time, perhaps, a natural one; but that it should have been discovered and corrected without revealing the inefficiency of the method of moments is a very remarkable circumstance.

In 1903 the correct formulæ for the probable errors of statistics found by the method of moments are given in 'Biometrika' (19); references are there given to SHEPPARD (20), whose method is employed, as well as to PEARSON and FILON (16), although both the method and the results differ from those of the latter.

in samples from an infinite population of which the true value is p ,

$$\log f = \log p + y \log (1-p),$$

$$\frac{\partial}{\partial p} \log f = \frac{x}{p} - \frac{y}{1-p},$$

$$\frac{\partial^2}{\partial p^2} \log f = -\frac{x}{p^2} - \frac{y}{(1-p)^2}.$$

Now the mean value of x in pn , and of y is $(1-p)n$, hence the mean value of $\frac{\partial^2}{\partial p^2} \log f$ is

$$-\left(\frac{1}{p} + \frac{1}{1-p}\right)n;$$

therefore

$$\sigma_p^2 = \frac{p(1-p)}{n},$$

the well-known formula for the standard error of p .

7. SATISFACTION OF THE CRITERION OF SUFFICIENCY.

That the criterion of sufficiency is generally satisfied by the solution obtained by the method of maximum likelihood appears from the following considerations.

If the individual values of any sample of data are regarded as co-ordinates in hyperspace, then any sample may be represented by a single point, and the frequency distribution of an infinite number of random samples is represented by a density distribution in hyperspace. If any set of statistics be chosen to be calculated from the samples, certain regions will provide identical sets of statistics; these may be called *isostatistical* regions. For any particular space element, corresponding to an actual sample, there will be a particular set of parameters for which the frequency in that element is a maximum; this will be the optimum set of parameters for that element. If now the set of statistics chosen are those which give the optimum values of the parameters, then all the elements of any part of the same isostatistical region will contain the greatest possible frequency for the same set of values of the parameters, and therefore any region which lies wholly within an isostatistical region will contain its maximum frequency for that set of values.

Now let θ be the value of any parameter, $\hat{\theta}$ the statistic calculated by the method of maximum likelihood, and θ_1 any other statistic designed to estimate the value of θ , then for a sample of given size, we may take

$$f(\theta, \hat{\theta}, \theta_1) d\hat{\theta} d\theta_1$$

to represent the frequency with which $\hat{\theta}$ and θ_1 lie in the assigned ranges $d\hat{\theta}$ and $d\theta_1$.

The region $d\hat{\theta} d\theta_1$ evidently lies wholly in the isostatistical region $d\hat{\theta}$. Hence the equation

$$\frac{\partial}{\partial \theta} \log f(\theta, \hat{\theta}, \theta_1) = 0$$

is satisfied, irrespective of θ_1 , by the value $\theta = \hat{\theta}$. This condition is satisfied if

$$f(\theta, \hat{\theta}, \theta_1) = \phi(\theta, \hat{\theta}) \cdot \phi'(\hat{\theta}, \theta_1);$$

for then

$$\frac{\partial}{\partial \theta} \log f = \frac{\partial}{\partial \theta} \log \phi,$$

and the equation for the optimum degenerates into

$$\frac{\partial}{\partial \theta} \log \phi(\theta, \hat{\theta}) = 0,$$

which does not involve θ_1 .

But the factorisation of f into factors involving $(\theta, \hat{\theta})$ and $(\hat{\theta}, \theta_1)$ respectively is merely a mathematical expression of the condition of sufficiency; and it appears that any statistic which fulfils the condition of sufficiency must be a solution obtained by the method of the optimum.

It may be expected, therefore, that we shall be led to a sufficient solution of problems of estimation in general by the following procedure. Write down the formula for the probability of an observation falling in the range dx in the form

$$f(\theta, x) dx,$$

where θ is an unknown parameter. Then if

$$L = S(\log f),$$

the summation being extended over the observed sample, L differs by a constant only from the logarithm of the likelihood of any value of θ . The most likely value, $\hat{\theta}$, is found by the equation

$$\frac{\partial L}{\partial \theta} = 0,$$

and the standard deviation of $\hat{\theta}$, by a second differentiation, from the formula

$$\frac{\partial^2 L}{\partial \theta^2} = -\frac{1}{\sigma_{\hat{\theta}}^2};$$

this latter formula being applicable only where $\hat{\theta}$ is normally distributed, as is often the case with considerable accuracy in large samples. The value $\sigma_{\hat{\theta}}$ so found is in these cases the least possible value for the standard deviation of a statistic designed to

estimate the same parameter ; it may therefore be applied to calculate the efficiency of any other such statistic.

When several parameters are determined simultaneously, we must equate the second differentials of L , with respect to the parameters, to the coefficients of the quadratic terms in the index of the normal expression which represents the distribution of the corresponding statistics. Thus with two parameters,

$$\frac{\partial^2 L}{\partial \theta_1^2} = -\frac{1}{1-r^2_{t_1, t_1}} \cdot \frac{1}{\sigma^2_{t_1}}, \quad \frac{\partial^2 L}{\partial \theta_2^2} = -\frac{1}{1-r^2_{t_2, t_2}} \cdot \frac{1}{\sigma^2_{t_2}},$$

$$\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} = +\frac{1}{1-r^2_{t_1, t_2}} \cdot \frac{r}{\sigma_{t_1} \sigma_{t_2}},$$

or, in effect, σ_{θ}^2 is found by dividing the Hessian determinant of L , with respect to the parameters, into the corresponding minor.

The application of these methods to such a series of parameters as occur in the specification of frequency curves may best be made clear by an example.

8. THE EFFICIENCY OF THE METHOD OF MOMENTS IN FITTING CURVES OF THE PEARSONIAN TYPE III.

Curves of PEARSON'S Type III. offer a good example for the calculation of the efficiency of the Method of Moments. The chance of an observation falling in the range dx is

$$df = \frac{1}{\alpha \cdot p!} \cdot \left(\frac{x-m}{\alpha}\right)^p e^{-\frac{x-m}{\alpha}} dx.*$$

By the method of moments the curve is located by means of the statistic μ_1 , its dimensions are ascertained from the second moment μ_2 , and the remaining parameter p is determined from β_1 . Considering first the problem of location, if α and p were known and we had only to determine m , we should take, according to the method of moments,

$$\mu_1 = m_\mu + \alpha(p+1),$$

where m_μ represents the estimate of the parameter m , obtained by using the method of moments. The variance of m_μ is, therefore,

$$\sigma^2_{m_\mu} = \sigma^2_{\mu_1} = \frac{\mu_2}{n} = \frac{\alpha^2(p+1)}{n}.$$

If, on the other hand, we aim at greater accuracy, and make the likelihood of the sample a maximum for variations of m , we have

$$L = -n \log \alpha - n \log(p!) + pS \left(\log \frac{x-m}{\alpha} \right) - S \left(\frac{x-m}{\alpha} \right),$$

* The expression, $x!$, is used here and throughout as equivalent to the Gaussian $\Pi(x)$, or to $\Gamma(x+1)$, whether x is an integer or not.

and the equation to determine m is

$$\frac{\partial L}{\partial m} = -\rho S\left(\frac{1}{x-\hat{m}}\right) + \frac{n}{a} = 0; \quad \dots \quad (1)$$

the accuracy of the value so obtained is found from the second differential.

$$\frac{\partial^2 L}{\partial m^2} = -\rho S\left(\frac{1}{x-m}\right),$$

of which the mean value is

$$-\frac{n}{a^2(\rho-1)},$$

whence

$$\sigma_m^2 = \frac{a^2(\rho-1)}{n}.$$

We now see that the efficiency of location by the method of moments is

$$\frac{\rho-1}{\rho+1} = 1 - \frac{2}{\rho+1}.$$

Efficiencies of over 80 per cent. for location are therefore obtained if ρ exceeds 9; for $\rho = 1$ the efficiency of location vanishes, as in other cases where the curve makes an angle with the axis at the end of its range.

Turning now to the problem of scaling, we have, by the method of moments,

$$\mu_2 = a^2(\rho+1),$$

whence, knowing ρ , a is obtained. Since

$$\sigma_{\mu_2}^2 = \frac{\beta_2-1}{n} \mu_2^2,$$

we must have

$$\sigma_{a,\mu}^2 = \frac{\beta_2-1}{4n} a^2 = \frac{4+3\beta_1}{8n} a^2 = \frac{\rho+4}{2(\rho+1)n} a^2;$$

on the other hand, from the value of L , we find the equation

$$\frac{\partial L}{\partial a} = -\frac{n}{a}(\rho+1) + \frac{1}{a^2} S(x-m) = 0, \quad \dots \quad (2)$$

to be solved for m and a as a simultaneous equation with (1); whence

$$\frac{\partial^2 L}{\partial m \partial a} = -\frac{n}{a^2},$$

and

$$\frac{\partial^2 L}{\partial a^2} = \frac{n}{a^2}(\rho+1) - \frac{2}{a^3} S(x-m),$$

of which the mean value is

$$-\frac{n(p+1)}{\alpha^2}.$$

The variance of α , determined from this pair of simultaneous equations, is found by dividing

$$\frac{\partial^2 L}{\partial m^2} = -\frac{n}{\alpha^2(p-1)}$$

by the determinant

$$\begin{vmatrix} -\frac{n}{\alpha^2(p-1)} & -\frac{n}{\alpha^2} \\ -\frac{n}{\alpha^2} & -\frac{n(p+1)}{\alpha^2} \end{vmatrix}$$

which reduces to

$$-\frac{2}{p-1} \cdot \frac{n^2}{\alpha^4},$$

whence

$$\sigma_{\alpha}^2 = \frac{\alpha^2}{2n},$$

and the efficiency of scaling by the method of moments is

$$\frac{p+1}{p+4} = 1 - \frac{3}{p+4}.$$

Efficiency of over 80 per cent. for scaling are, therefore, obtained when p exceeds 11. The efficiency of scaling does not, however, vanish for any possible value of p , though it tends to zero, as p approaches its limiting value, -1 .

Lastly, p is found by the method of moments by putting

$$\frac{4}{p+1} = \beta_1.$$

Now

$$\sigma_{\beta_1}^2 = \frac{\beta_1}{n} (4\beta_4 - 24\beta_2 + 36 + 9\beta_1\beta_2 - 12\beta_3 + 35\beta_1),$$

and for curves of Type III,

$$\beta_2 = 3 + \frac{3}{2}\beta_1,$$

$$\beta_3 = 2\beta_1\beta_2 + 4\beta_1 = \beta_1(3\beta_1 + 10),$$

$$\beta_4 = \frac{5}{2}(\beta_3 + 2\beta_2) = \frac{5}{2}(3\beta_1^2 + 13\beta_1 + 6),$$

hence

$$\begin{aligned} \sigma_{\beta_1}^2 &= \frac{3\beta_1}{n} (5\beta_1 + 4)(\beta_1 + 4), \\ &= \frac{\beta_1^2}{n} \cdot \frac{6(p+2)(p+6)}{p+1}, \end{aligned}$$

whence it follows, since n is large, that

$$\sigma_{p^2} = \frac{\overline{p+1}^2}{n} \cdot \frac{6(p+2)(p+6)}{p+1} = \frac{6}{n}(p+1)(p+2)(p+6).$$

From the value of L ,

$$\frac{\partial L}{\partial p} = -n \frac{d}{dp} \log(p!) + S \left(\log \frac{x-m}{a} \right),$$

which equation solved for m , a and p as a simultaneous equation with (1) and (2), will yield the set of values for the parameters which has the maximum likelihood. To find the variance of the value of p , so obtained, observe that

$$\frac{\partial^2 L}{\partial m \partial p} = -S \left(\frac{1}{x-m} \right),$$

of which the mean value is

$$-\frac{n}{ap},$$

$$\frac{\partial^2 L}{\partial a \partial p} = -\frac{n}{a^2},$$

and

$$\frac{\partial^2 L}{\partial p^2} = -n \frac{d^2}{dp^2} \log(p!).$$

The variance of p , derived from this set of simultaneous equations, is therefore found by dividing the minor of $\frac{\partial^2 L}{\partial p^2}$, namely

$$\frac{2}{p-1} \cdot \frac{n^2}{a^4},$$

by the determinant

$$\frac{n^3}{a^4} \begin{vmatrix} \frac{1}{p-1} & 1 & \frac{1}{p} \\ 1 & p+1 & 1 \\ \frac{1}{p} & 1 & \frac{d^2}{dp^2} \log(p!) \end{vmatrix} = \frac{n^3}{a^4} \cdot \frac{1}{p-1} \left\{ 2 \frac{d^2}{dp^2} \log(p!) - \frac{2}{p} + \frac{1}{p^2} \right\};$$

hence

$$\sigma_p^2 = \frac{2}{n \left\{ 2 \frac{d^2}{dp^2} \log(p!) - \frac{2}{p} + \frac{1}{p^2} \right\}}.$$

When p is large,

$$2 \frac{d^2}{dp^2} \log(p!) - \frac{2}{p} + \frac{1}{p^2} = \frac{1}{3} \left(\frac{1}{p^3} - \frac{1}{5p^5} + \frac{1}{7p^7} \dots \right),$$

so that, approximately,

$$\sigma_p^2 = \frac{6}{n} (p^3 + \frac{1}{5}p);$$

for large values of p , the efficiency of the method of moments is, therefore, approximately

$$\frac{p^3 + \frac{1}{5}p}{p+1 \ p+2 \ p+6}.$$

Efficiencies of over 80 per cent. occur when p exceeds 38.1 ($\beta_1 = 0.102$): evidently the method of moments is effective for determining the form of the curve only when it is relatively close to the normal form. For small values of p , the above approximation for the efficiency is not adequate. The true values can easily be obtained from the recently published tables of the Trigamma* function (11). The following values are obtained for the integral values of p from 0 to 5.

p	0	1	2	3	4	5
Efficiency . . .	0	0.0274	0.0871	0.1532	0.2159	0.2727

An interesting point which may be resolved at this stage of the enquiry is to find the variance of m , when a and p are not known, derived from the above set of simultaneous equations; that is to say, to calculate the accuracy with which the limiting point of the curve is determined; such determinations are often stated as the result of fitting curves of limited range, but their probable errors are seldom, if ever, evaluated. To obtain the greatest possible accuracy with which such a point can be determined we must divide the minor of $\frac{\hat{c}^2 L_1}{\hat{c} m^2}$, namely,

$$\frac{n^2}{a^2} \left\{ \frac{1}{p+1} \frac{d^2}{dp^2} \log (p!) - 1 \right\}.$$

by

$$\frac{n^3}{a^4} \cdot \frac{1}{p-1} \left\{ 2 \frac{d^2}{dp^2} \log (p!) - \frac{2}{p} + \frac{1}{p^2} \right\},$$

whence

$$\sigma_{m^2} = \frac{a^2}{n} \frac{p-1 \left\{ \frac{1}{p+1} \frac{d^2}{dp^2} \log (p!) - 1 \right\}}{2 \frac{d^2}{dp^2} \log (p!) - \frac{2}{p} + \frac{1}{p^2}}.$$

The position of the limiting point will, when p is at all large, evidently be determined with much less accuracy than is the position, as a whole, of a curve of known form and size. Let n' be a multiplier such that the position of the extremity of a curve calculated

* It is sometimes convenient to write $F(x)$ for $\frac{d^2}{dx^2} \log(x!)$.

from nn' observations will be determined with the same accuracy as the position, as a whole, of a curve of known form and size, can be determined from a sample of n observations when n is large. Then

$$n' = \frac{\overline{\rho+1} \frac{d^2}{d\rho^2} \log(\rho!) - 1}{2 \frac{d^2}{d\rho^2} \log(\rho!) - \frac{2}{\rho} + \frac{1}{\rho^2}};$$

but, when ρ is large,

$$\overline{\rho+1} \frac{d^2}{d\rho^2} \log(\rho!) - 1 = \frac{1}{2\rho} \left(1 - \frac{2}{3\rho} + \frac{1}{3\rho^2} \dots \right)$$

and

$$2 \frac{d^2}{d\rho^2} \log(\rho!) - \frac{2}{\rho} + \frac{1}{\rho^2} = \frac{1}{3\rho^3} \left(1 - \frac{1}{5\rho^2} \dots \right);$$

therefore

$$\begin{aligned} n' &= \frac{3}{2} \rho^3 \left(1 - \frac{2}{3\rho} \rho + \frac{8}{15\rho^2} \dots \right) \\ &= \frac{3}{2} \rho^2 - \rho + \frac{4}{3}. \end{aligned}$$

For large values of ρ the probable error of the determination of the end-point may be found approximately by multiplying the probable error of location by

$$\left(\rho - \frac{1}{3}\right) \sqrt{\frac{2}{3}}.$$

As ρ grows smaller, n' diminishes until it reaches unity, when $\rho = 1$. For values of ρ less than 1 it would appear that the end-point had a smaller probable error than the probable error of location, but, as a matter of fact, for these values location is determined by the end-point, and as we see from the vanishing of σ_m , whether or not ρ and a are known, when $\rho = 1$, the weight of the determination from this point onwards increases more rapidly than n , as the sample increases. (See Section 10.)

The above method illustrates how it is possible to calculate the variance of any function of the population parameters as estimated from large samples: by comparing this variance with that of the same function estimated by the method of moments, we may find the efficiency of that method for any proposed function. The above examination, in which the determinations of the locus, the scale, and the form of the curve are treated separately, will serve as a general criterion of the application of the method of moments to curves of Type III. Special combinations of the parameters will, however, be of interest in special cases. It may be noted here that by virtue of equation (2) the function of $m + a(\rho + 1)$ is the same, whether determined by moments or by the method of the optimum:

$$m_\mu + a_\mu(\rho_\mu + 1) = \hat{m} + \hat{a}(\hat{\rho} + 1).$$

The efficiency of the method of moments in determining this function is therefore 100 per cent. That this function is the abscissa of the mean does not imply 100 per cent. efficiency of location, for the centre of location of these curves is not the mean (see p. 340).

9. LOCATION AND SCALING OF FREQUENCY CURVES IN GENERAL.

The general problem of the location and scaling of curves may now be treated more generally. This is the problem which presents itself with respect to *error curves* of assumed form, when to find the best value of the quantity measured we must *locate* the curve as accurately as possible, and to find the probable error of the result of this process we must, as accurately as possible, estimate its *scale*.

The *form* of the curve may be specified by a function ϕ , such that

$$df \propto e^{\phi(\xi)} d\xi, \text{ when } \xi = \frac{x-m}{a}.$$

In this expression ϕ specifies the form of the curve, which is unaltered by variations of a and m .

When a sample of n observations has been taken, the likelihood of any combination of values of a and m is

$$L = C - n \log a + S(\phi),$$

whence

$$\frac{\partial L}{\partial m} = S\left(\frac{d\phi}{d\xi} \cdot \frac{d\xi}{dm}\right) = -\frac{1}{a} S(\phi'),$$

since

$$\frac{\partial \xi}{\partial m} = -\frac{1}{a};$$

also

$$\frac{\partial L}{\partial a} = -\frac{1}{a} S(\xi\phi') - \frac{n}{a},$$

since

$$\frac{\partial \xi}{\partial a} = -\frac{\xi}{a}.$$

Differentiating a second time,

$$\frac{\partial^2 L}{\partial m^2} = \frac{1}{a^2} S(\phi'');$$

therefore

$$\sigma_m^2 = -\frac{a^2}{n\phi''}.$$

This expression enables us to compare the accuracy of error curves of different form, when the location is performed in each case by the method which yields the minimum error.

Example :—The curve

$$df = \frac{1}{\pi} \frac{d\xi}{1+\xi^2}$$

referred to in Section 5 has an infinite standard deviation, but it is not on that account an error curve of zero accuracy, for

$$\phi = -\log(1+\xi^2), \quad \phi' = -\frac{2\xi}{1+\xi^2}, \quad \phi'' = -\frac{2(1-\xi^2)}{(1+\xi^2)^2}.$$

Now

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\xi^2}{(1+\xi^2)^3} d\xi = \frac{1}{2},$$

hence

$$\overline{\phi''} = -\frac{1}{2} \quad \text{and} \quad \sigma_m' = \frac{2\alpha^2}{n}.$$

The quantity,

$$-\frac{\overline{\phi''}}{\alpha^2} = \frac{1}{2\alpha^2},$$

which is the factor by which n is multiplied in calculating the weight of the estimate made from n measurements, may be called the *intrinsic accuracy* of an *error curve*. In the above example we see that errors distributed so that

$$df = \frac{a}{\pi} \frac{dx}{a^2 + x^2}$$

have the same intrinsic accuracy as errors distributed according to the normal curve

$$df = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx,$$

provided

$$\sigma^2 = 2\alpha^2.$$

Fig. 1 illustrates two such curves of equal intrinsic accuracy.

Returning now to the general problem in which

$$L = C - n \log a + S(\phi),$$

we have

$$\frac{\partial^2 L}{\partial m \partial a} = \frac{1}{a^2} S(\phi' + \xi \phi'') = \frac{1}{a^2} S(\xi \phi'')$$

and

$$\frac{\partial^2 L}{\partial a^2} = \frac{1}{a^2} S(2\xi \phi' + \xi^2 \phi'') + \frac{n}{a^2} = \frac{1}{a^2} S(\xi^2 \phi'' - 1).$$

The latter expression will directly give the accuracy with which a is determined only if

$$\frac{\partial^2 L}{\partial m \partial a} = 0,$$

and we can always arrange that this shall be so by subtracting from ξ the quantity

$$\frac{\overline{\xi \phi''}}{\overline{\phi''}}.$$

Thus in a Type III. curve where, referred to the end of the range,

$$\overline{\xi \phi''} = -1, \quad \overline{\phi''} = -\frac{1}{\mu-1}.$$

instead of

$$\phi = p \log \xi - \xi$$

we must write

$$\phi = p \log \overline{\xi + p - 1} - \overline{\xi + p - 1} :$$

then

$$\phi' = \frac{p}{\xi + p - 1} - 1, \quad \phi'' = -\frac{p}{\xi + p - 1}^2,$$

hence

$$\begin{aligned} \frac{\partial^2 \mathbf{L}}{\partial a^2} &= \frac{1}{a^2} S (\xi^2 \phi'' - 1) \\ &= \frac{1}{a^2} S \left(-p + \frac{2p \overline{p-1}}{\xi + p - 1} - \frac{p \overline{p-1}^2}{\xi + p - 1} - 1 \right), \end{aligned}$$

of which the mean value is

$$\frac{n}{a^2} (-p + 2 \overline{p-1} - \overline{p-1} - 1) = -\frac{2n}{a^2},$$

hence

$$\sigma_a^2 = \frac{a^2}{2n}.$$

For one particular point of origin, therefore, the variations of the abscissa are uncorrelated with those of a ; this point may be termed the *centre of location*.

Example :- To determine the centre of location of the curve of Type IV.,

$$df \propto e^{-v \tan^{-1} \xi} (1 + \xi^2)^{-\frac{r+2}{2}}.$$

Here

$$\phi = -v \tan^{-1} \xi - \frac{r+2}{2} \log \overline{1 + \xi^2},$$

$$\phi' = -\frac{v}{\overline{1 + \xi^2}} - \frac{r+2}{\overline{1 + \xi^2}},$$

$$\phi'' = \frac{2v\xi}{\overline{1 + \xi^2}^2} + \frac{r+2}{\overline{1 + \xi^2}^2} :$$

from these we find

$$\overline{\phi''} = -\frac{\overline{r+1} \overline{r+2} \overline{r+4}}{\overline{r+4}^2 + v^2},$$

$$\overline{\xi \phi''} = \frac{\overline{r+1} \overline{r+2} v}{\overline{r+4}^2 + v^2},$$

so that

$$\frac{\overline{\xi \phi''}}{\overline{\phi''}} = -\frac{v}{r+4}.$$

The centre of location, therefore, at the distance from the mode,

$$= -\frac{va}{r+4}.$$

Example :—Determine the intrinsic accuracy of an error curve of Type IV. and the efficiency of the method of moments in location and scaling.

Since

$$\overline{\phi''} = - \frac{\overline{r+1} \overline{r+2} \overline{r+4}}{\overline{r+4}^2 + \nu^2},$$

$$\sigma_{\mu}^2 = \frac{\alpha^2}{n} \cdot \frac{\overline{r+4}^2 + \nu^2}{\overline{r+1} \overline{r+2} \overline{r+4}};$$

and the intrinsic accuracy of the curve is

$$\frac{1}{\alpha^2} \frac{\overline{r+1} \overline{r+2} \overline{r+4}}{\overline{r+4}^2 + \nu^2};$$

but

$$\sigma_{m\mu}^2 = \frac{\alpha^2}{n} \cdot \frac{\nu^2 + \nu^2}{\nu^2 \overline{r-1}},$$

therefore the efficiency of the method of moments in location is

$$\frac{\nu^2 \overline{r-1} (\overline{r+4}^2 + \nu^2)}{\overline{r+1} \overline{r+2} \overline{r+4} (\nu^2 + \nu^2)} \dots \dots \dots (3)$$

When $\nu = 0$, we have for curves of Type VII. an efficiency of location

$$1 - \frac{6}{\overline{r+1} \overline{r+2}}.$$

The efficiency of location of these curves vanishes at $r = 1$, at which value the standard deviation becomes infinite. Although values down to -1 give admissible frequency curves, the conventional limit at which curves are reckoned as heterotypic is at $r = 7$. For this value the efficiency is

$$\frac{49}{132} \cdot \frac{121 + \nu^2}{49 + \nu^2},$$

which varies from 91·67 per cent. for the symmetrical Type VII. curve, to 37·12 per cent. when $\nu \rightarrow \infty$ and the curve to Type V.

Turning to the question of scaling, we find

$$\overline{\xi^2 \phi''} - 1 = - \frac{\overline{r+1} (2 \overline{r+4} + \nu^2)}{\overline{r+4}^2 + \nu^2},$$

whence

$$\overline{\xi^2 \phi''} - 1 - \frac{\overline{\xi \phi''}^2}{\overline{\phi''}^2} = - \frac{2 \overline{r+1}}{\overline{r+4}},$$

and

$$\sigma_d^2 = \frac{\alpha^2}{n} \cdot \frac{\overline{r+4}}{2 \overline{r+1}};$$

the intrinsic accuracy of scaling is therefore independent of ν . Now for these curves

$$\beta_2 = \frac{3\sqrt{r-1}}{r-2\sqrt{r-3}} \left(r + 6 - \frac{8\nu^2}{r^2 + \nu^2} \right),$$

so that

$$\frac{\beta_2 - 1}{4} = \frac{\nu^3 \sqrt{r-2} + \nu^2 (r^2 + 10r - 12)}{2\sqrt{r-2}\sqrt{r-3}(r^2 + \nu^2)},$$

and

$$\sigma_{a_\mu}^2 = \frac{\alpha^2}{n} \cdot \frac{\nu^3 \sqrt{r-2} + \nu^2 (r^2 + 10r - 12)}{2\sqrt{r-2}\sqrt{r-3}(r^2 + \nu^2)}.$$

The efficiency of the method of moments for scaling is thus

$$\frac{\sqrt{r-2}\sqrt{r-3}\sqrt{r+4}(r^2 + \nu^2)}{r+1 \left\{ \nu^3 \sqrt{r-2} + \nu^2 (r^2 + 10r - 12) \right\}}; \dots \dots \dots (4)$$

when $\nu = 0$, we have for curves of Type VII. an efficiency of scaling

$$1 - \frac{12}{r^2 + 1}.$$

The efficiency of the method of moments in scaling these curves vanishes at $r = 3$, where β_2 becomes infinite; for $r = 7$, the efficiency of scaling is

$$\frac{55}{2} \cdot \frac{49 + \nu^2}{1715 + 107\nu^2},$$

varying in value from 78.57 per cent. for the symmetrical Type VII. curve, to 25.70 per cent. when $\nu \rightarrow \infty$ and the curve to Type V.

10. THE EFFICIENCY OF THE METHOD OF MOMENTS IN FITTING THE PEARSONIAN CURVES.

The Pearsonian group of skew curves are obtained as solutions of the equation

$$\frac{1}{y} \frac{dy}{dx} = \frac{-(x-m)}{\alpha + bx + cx^2}; \dots \dots \dots (5)$$

algebraically these fall into two main classes,

$$df \propto \left(1 + \frac{x}{\alpha_1} \right)^{m_1} \left(1 - \frac{x}{\alpha_2} \right)^{m_2} dx$$

and

$$df \propto \left(1 + \frac{x^2}{\alpha^2} \right)^{-\frac{r+2}{2}} e^{-\nu \tan^{-1} \frac{x}{\alpha}} dx,$$

according as the roots of the quadratic expression in (5) are real or imaginary.

The first of these forms may be rewritten

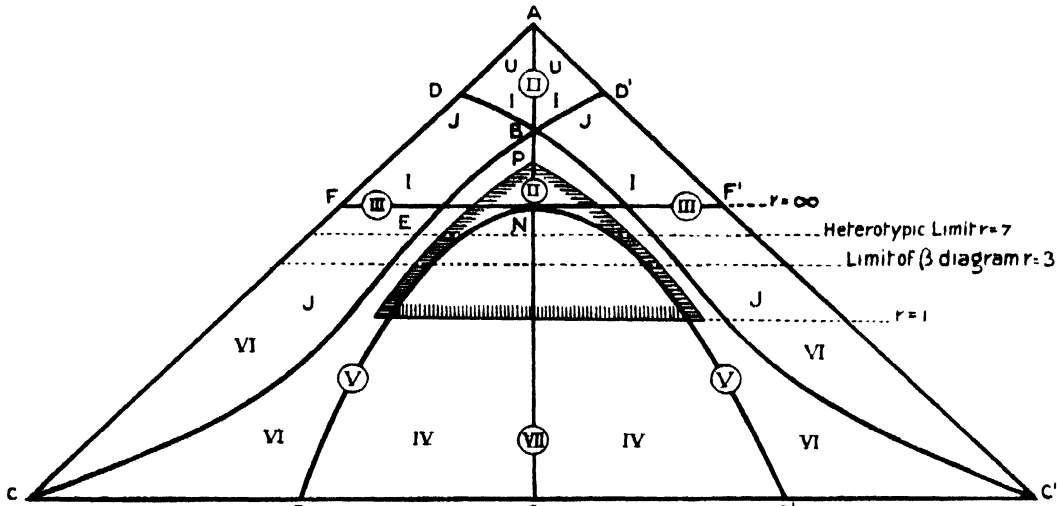
$$d.f \propto \left(1 - \frac{x^2}{a^2}\right)^{-\frac{r+2}{2}} e^{-r \tanh^{-1} \frac{x}{a}} dx,$$

r being negative, showing its affinity with the second class.

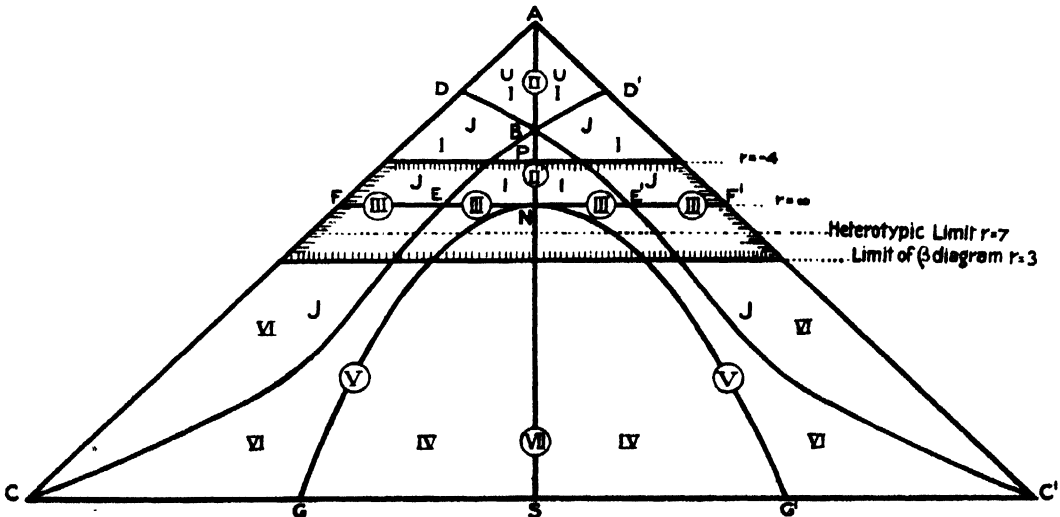
In order that these expressions may represent frequency curves, it is necessary that the integral over the whole range of the curve should be finite; this restriction acts in two ways:—

- (1) When the curve terminates at a finite value of x , say $x = a$, the power to which $a - x$ is raised must be greater than -1 .
- (2) When the curve extends to infinity, the ordinate, when x is large, must diminish more rapidly than $\frac{1}{x}$:

Fig. 2 is shown a conspectus of all possible frequency curves of the Pearsonian type;



A. Showing region of validity of first moment.



B. Showing region of validity of second moment.

Fig. 2. Conspectus of Pearsonian system of frequency curves.

the lines AC and AC' represent the limits along which the area between the curve and a vertical ordinate tends to infinity, and on which m_1 , or m_2 , takes the value -1 ; the line CC' represents the limit at which unbounded curves enclose an infinite area with the horizontal axis; at this limit $r = -1$.

The symmetrical curves of Type II.

$$df \propto \left(1 - \frac{x^2}{a^2}\right)^{-\frac{r+2}{2}}$$

extend from the point N, representing the normal curve, at which r is infinite, through the point P at which $r = -4$, and the curve is a *parabola*, to the point B ($r = -2$), where the curve takes the form of a *rectangle*; from this point the curves are U-shaped, and at A, when the arms of U are hyperbolic, we have the limiting curve of this type, which is the discontinuous distribution of *equal or unequal dichotomy* ($r = 0$).

The unsymmetrical curves of Type I. are divided by PEARSON into three classes according as the terminal ordinate is infinite at neither end, at one end (J curves), or at both ends (U curves); the dividing lines are C'BD and CBD', along which one of the terminal ordinates are finite (m_1 , or m_2 , = 0); at the point B, as we have seen, both terminal ordinates are finite.

The same line of division divides the curves of Type III.,

$$df \propto x^p e^{-x} dx,$$

at the point E ($p = 0$), representing a simple exponential curve; the J curves of Type III. extend to F ($p = -1$), at which point the integral ceases to converge. In curves of Type III., r is infinite; ν is also infinite, but one of the quantities m_1 and m_2 is finite, or zero ($= p$); as p tends to infinity we approach the normal curve

$$df \propto e^{-x^2} dx.$$

Type VI., like Type III., consists of curves bounded only at one end; here r is positive, and both m_1 and m_2 are finite or zero. For the J curves of Type VI. both m_1 and m_2 are negative, but for the remainder of these curves they are of opposite sign, the negative index being the greater by at least unity in order that the representative point may fall above CC' ($r = -1$).

Type V. is here represented by a parabola separating the regions of Types IV. and VI.; the typical equation of this type of curve is

$$df \propto x^{-\frac{r+3}{2}} e^{-\frac{1}{2}x} dx.$$

As r tends to infinity the curve tends to the normal form; the integral does not become divergent until $\frac{r+3}{2} = 1$, or $r = -1$. On curves of Type V., then, r is finite or zero, but ν is infinite.

In Type IV.

$$df \propto \left(1 + \frac{x^2}{\alpha^2}\right)^{-\frac{r+2}{2}} e^{-r \tan^{-1} \frac{x}{\alpha}};$$

we have written ν , not as previously for the difference between m_1 and m_2 , for these quantities are now complex, and their difference is a pure imaginary, but for the difference divided by $\sqrt{-1}$; ν is then real and finite throughout Type IV., and it vanishes along the line NS, representing the symmetrical curves of Type VII.

$$df \propto \left(1 + \frac{x^2}{\alpha^2}\right)^{-\frac{r+2}{2}}$$

from $r = \infty$ to $r = -1$.

The Pearsonian system of frequency curves has hitherto been represented by the diagram (13, p. 66), in which the co-ordinates are β_1 and β_2 . This is an unsymmetrical diagram which, since β_1 is necessarily positive, places the symmetrical curves on a boundary, whereas they are the central types from which the unsymmetrical curves diverge on either hand; further, neither of the limiting conditions of these curves can be shown on the β diagram; the limit of the U curves is left obscure,* and the other limits are either projected to infinity, or, what is still more troublesome, the line at infinity cuts across the diagram, as occurs along the line $r = 3$, for there β_2 becomes infinite. This diagram thus excludes all curves of Types VII., IV., V., and VI., for which $r < 3$.

In the β diagram the condition $r = \text{constant}$ yields a system of concurrent straight lines. The basis of the representation in fig. 2 lies in making these lines parallel and horizontal, so that the ordinate is a function of r only. We have chosen $r = y \cdot \frac{1}{y}$, and have represented the limiting types by the simplest geometrical forms, straight lines and parabolas, by taking

$$4\epsilon = r^2 + \nu^2 = \frac{(\overline{1+y^2+x^2})(\overline{1-y^2-x^2})}{y(x^2+y)}$$

It might have been thought that use could have been made of the criterion,

$$\kappa_2 = \frac{\beta_1(\beta_2+3)^2}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)} = 1 - \frac{r^2}{4\epsilon},$$

by which PEARSON distinguishes these curves; but this criterion is only valid in the region treated by PEARSON. For when $r = 0$, $\kappa_2 = 1$, and we should have to place a variety of curves of Types VII., IV., V., and VI., all in Type V. in order to adhere to the criterion.

This diagram gives, I believe, the simplest possible *conspectus* of the whole of the Pearsonian system of curves; the inclusion of the curves beyond $r = 3$ becomes neces-

* The true limit is the line $\beta_2 = \beta_1 + 1$, along which the curves degenerate into simple dichotomies.

sary as soon as we take a view unrestricted by the method of moments ; of the so-called heterotypic curves between $r = 3$ and $r = 7$ it should be noticed that they not only fall into the ordinary Pearsonian types, but have finite values for the moment coefficients β_1 and β_2 ; they differ from those in which r exceeds 7, merely in the fact that the value of β_2 , calculated from the fourth moment of a sample, has an infinite probable error. It is therefore evident that this is not the right method to treat the sample, but this does not constitute, as it has been called, " the failure of Type IV.," but merely the failure of the method of moments to make a valid estimate of the form of these curves. As we shall see in more detail, the method of moments, when its efficiency is tested, fails equally in other parts of the diagram.

In expression (3) we have found that the efficiency of the method of moments for location of a curve of Type IV. is

$$E = \frac{\gamma^2 r - 1 (\gamma + 4^2 + \nu^2)}{\gamma + 1 \gamma + 2 r + 4 (\gamma^2 + \nu^2)},$$

whence if we substitute for r and ν in terms of the co-ordinates of our diagram, we obtain a general formula for the efficiency of the method of moments in locating Pearsonian curves, which is applicable within the boundary of the zero contour (fig. 3). This may

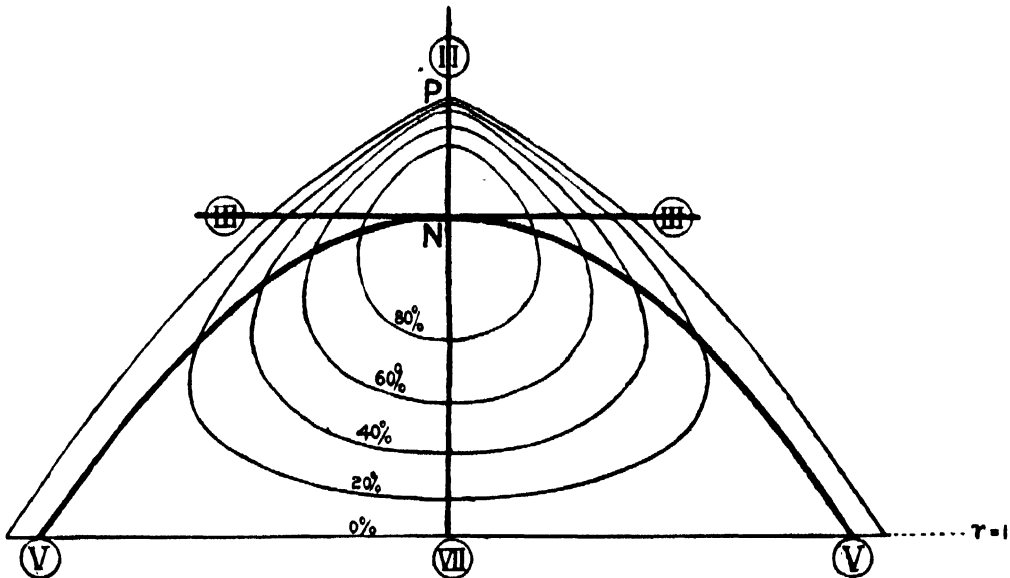


Fig. 3. Region of validity of the first moment (the mean) applied in the location of Pearsonian curves showing contours of efficiency.

be called the region of validity of the first moment ; it is bounded at the base by the line $r = 1$, so that the first moment is valid far beyond the heterotypic limit ; its other boundary, however, represents those curves which make a finite angle with the axis at the end of their range (m_1 , or $m_2 = 1$) ; all J curves (m_1 , or $m_2 < 0$) are thus excluded. This boundary has a double point at P, which thus forms the apex of the region of validity.

In fig. 3 are shown the contours along which the efficiency is 20, 40, 60, and 80 per cent. For high efficiencies these contours tend to the system of ellipses,

$$8x^2 + 6y^2 = 1 - E.$$

In a similar manner, we have obtained in expression (4) the efficiency of the second moment in fitting Pearsonian curves. The region of validity in this case is shown in fig. 4 ; this region is bounded by the lines $r = 3$, $r = -4$, and by the limits

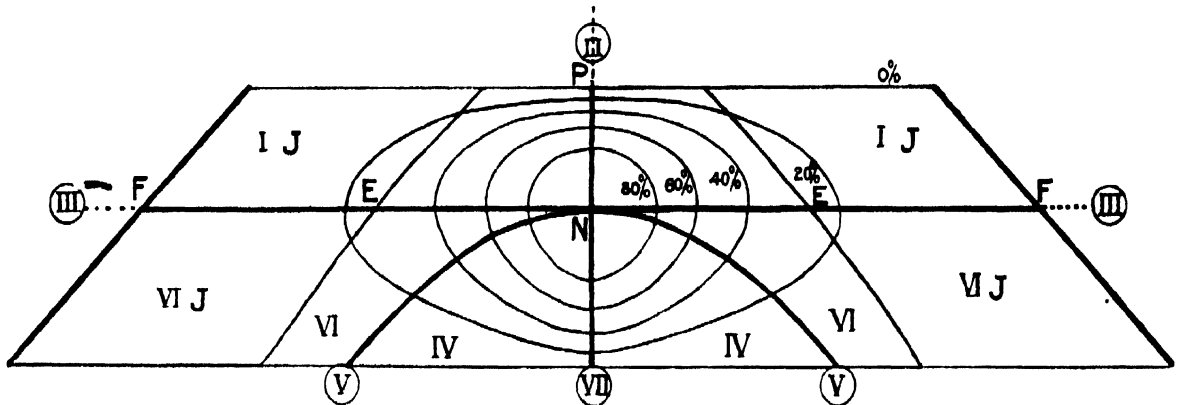


Fig. 4. Region of validity of the second moment (standard deviation) applied in scaling of Pearsonian curves, showing contours of efficiency.

(m_1 , or m_2 , = -1) on which $r^2 + v^2$ vanishes. This statistic is therefore valid for certain J curves, though the maximum efficiency among the J curves is about 30 per cent. As before, the contours are centred about the normal curve (N) and for high efficiencies tend to the system of concentric circles,

$$12x^2 + 12y^2 = 1 - E,$$

showing that the region of high efficiency is somewhat more restricted for the second moment, as compared to the first.

The lower boundary to the efficiencies of these statistics is due merely to their probable errors becoming infinite, a weakness of the method of moments which has been partially recognised by the exclusion of the so-called heterotypic curves ($r < 7$). The stringency of the upper boundary is much more unexpected : the probable errors of the moments do not here become infinite ; only the ratio of the probable errors of the moments to the probable error of the corresponding optimum statistics is great and tends to infinity as the size of the sample is increased.

That this failure as regards location occurs when the curve makes a finite angle with the axis may be seen by considering the occurrence of observations near the terminus of the curve.

Let

$$df = kx^a dx$$

in the neighbourhood of the terminus, then the chance of an observation falling within a distance x of the terminus is

$$f = \frac{k}{\alpha + 1} x^{\alpha+1} = k' x^{\alpha+1},$$

and the chance of n observations all failing to fall in this region is

$$(1-f)^n$$

or, when n is great, and f correspondingly small,

$$e^{-fn}.$$

Equating this to any finite probability, e^{-a} , we have

$$k' x^{\alpha+1} = \frac{a}{n},$$

or, in other words, if we use the extreme observation as a means of locating the terminus, the error, x , is proportional to

$$n^{-\frac{1}{\alpha+1}};$$

when $\alpha < 1$, this quantity diminishes more rapidly than $n^{-\frac{1}{2}}$, and consequently for large samples it is much more accurate to locate the curve by the extreme observation than by the mean.

Since it might be doubted whether such a simple method could really be more accurate than the process of finding the actual mean, we will take as example the location of the curve (B) in the form of a rectangle,

$$df = \frac{dx}{a}, \quad m - \frac{a}{2} < x < m + \frac{a}{2},$$

and

$$df = 0,$$

outside these limits.

This is one of the simplest types of distribution, and we may readily obtain examples of it from mathematical tables. The mean of the distribution is m , and the standard deviation $\frac{a}{\sqrt{12}}$, the error $m_\mu - m$, of the mean obtained from n observations, when n is reasonably large, is therefore distributed according to the formula

$$\frac{1}{a} \sqrt{\frac{6n}{\pi}} e^{-\frac{6nr^2}{a^2}} dx.$$

The difference of the extreme observation from the end of the range is distributed according to the formula

$$\frac{n}{a} e^{-\frac{nr^2}{a^2}} d\xi;$$

if ξ is the difference at one end of the range and η the difference at the other end, the joint distribution (since, when n is considerable, these two quantities may be regarded as independent) is

$$\frac{n^2}{a^2} e^{-\frac{n}{a}(\xi+\eta)} d\xi d\eta.$$

Now if we take the mean of the extreme observations of the sample, our error is

$$\frac{1}{2} \overline{\xi - \eta},$$

for which we write x ; writing also y for $\xi + \eta$, we have the joint distribution of x and y ,

$$\frac{n^2}{a^2} e^{-\frac{n}{a}y} dx dy.$$

For a given value of x the values of y range from $2|x|$ to ∞ , whence, integrating with respect to y , we find the distribution of x to be

$$df = \frac{n}{a} e^{-\frac{2n}{a}|x|} dx,$$

the double exponential curve shown in fig. 5.

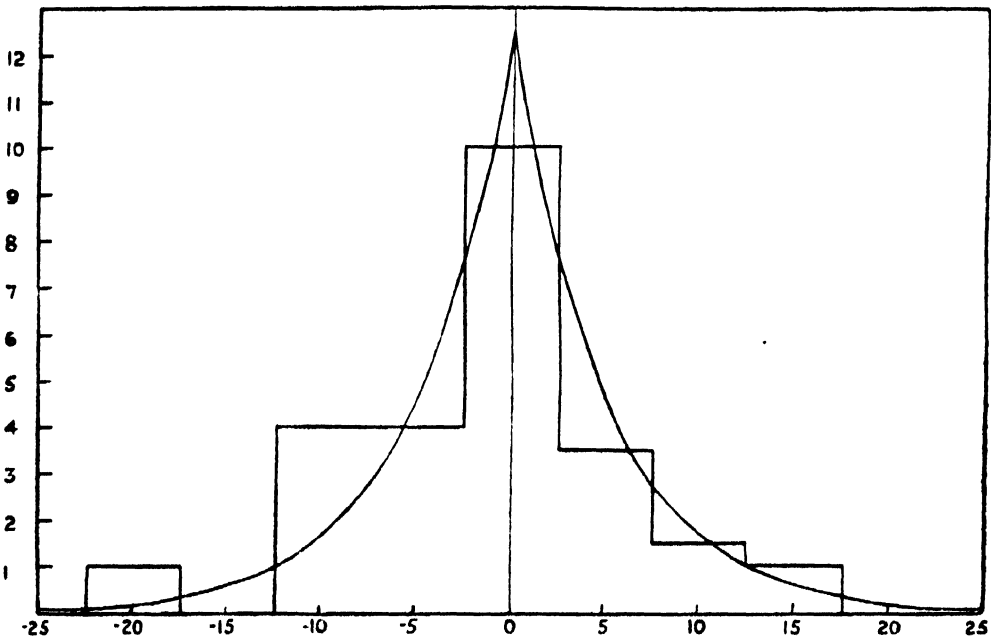


Fig. 5. Double exponential frequency curve, showing distribution of 25 deviations.

The two error curves are thus of a radically different form, and strictly no value for the efficiency can be calculated; if, however, we consider the ratio of the two standard deviations, then

$$\frac{\sigma_{\Delta}^2}{\sigma_{\mu}^2} = \frac{a^2}{2n^2} \div \frac{a^2}{12n} = \frac{6}{n}$$

when n is large, a quantity which diminishes indefinitely as the sample is increased.

For example, we have taken from VEGA (14) sets of digits from the table of Natural Logarithms to 48 places of decimals. The last block of four digits was taken from the logarithms of 100 consecutive numbers from 101 to 200, giving a sample of 100 numbers distributed evenly over a limited range. It is sufficient to take the three first digits to the nearest integer; then each number has an equal chance of all values between 0 and 1000. The true mean of the population is 500, and the standard deviation 289. The standard error of the mean of a sample of 100 is therefore 28.9.

Twenty-five such samples were taken, using the last five blocks of digits, for the logarithms of numbers from 101 to 600, and the mean determined merely from the highest and lowest number occurring, the following values were obtained :—

Digits.	1st hundred.			2nd hundred.			3rd hundred.			4th hundred.			5th hundred.		
	Lowest.	Highest.	$\hat{m}-m$.	Lowest.	Highest.	$\hat{m}-m$.	Lowest.	Highest.	$\hat{m}-m$.	Lowest.	Highest.	$\hat{m}-m$.	Lowest.	Highest.	$\hat{m}-m$.
45-48	24	978	+ 1.0	39	980	+ 9.5	1	999	0	16	983	- 0.5	18	994	+ 6.0
41-44	35.5	993	+ 14.0	3	960	- 18.5	6	997	+ 1.5	1	978	- 10.5	4	979	- 8.5
37-40	9	988	- 1.5	11	999	+ 5.0	31	984	+ 7.5	4	978	- 9.0	2	986	- 6.0
33-36	7	995	+ 1.0	13	997	+ 5.0	4	998	+ 1.0	0	994	- 3.0	3	981	- 8.0
29-32	1	988	5.5	3	988	- 4.5	4	992	- 2.0	1	996	- 1.5	21	977	- 1.0

It will be seen that these errors rarely exceed one-half of the standard error of the mean of the sample. The actual mean square error of these 25 values is 6.86, while the calculated value, $\sqrt{50}$, is 7.07. It will therefore be seen that, with samples of only 100, there is no exaggeration in placing the efficiency of the method of moments as low as 6 per cent. in comparison with the more accurate method, which in this case happens to be far less laborious.

Such a value for the efficiency of the mean in this case is, however, purely conventional, since the curve of distribution is outside the region of its valid application, and the two curves of sampling do not tend to assume the same form. It is, however, convenient to have an estimate of the effectiveness of statistics for small samples, and in such cases we should prefer to treat the curve of distribution of the statistic as an error curve, and to judge the effectiveness of the statistic by the *intrinsic accuracy* of the curve as defined in Section 9. Thus the intrinsic accuracy of the curve of distribution of the mean of all the observations is

$$\frac{12n}{\alpha^2},$$

while that of the mean of the extreme values is

$$\frac{4\nu^2}{e^2},$$

so yielding a ratio $3/n$. It is probable that this quantity may prove a suitable substitute for the efficiency of a statistic for curves beyond its region of validity.

To determine the efficiency of the moment coefficients β_1 and β_2 in determining the form of a Pearsonian curve, we must in general apply the method of Section 8 to the calculation of the simultaneous distribution of the four parameters of those curves when estimated by the method of maximum likelihood. Expressing the curve by the formula appropriate to Type IV., we are led to the determinant

$$\begin{vmatrix} \frac{\overline{r+1} \overline{r+2} \overline{r+4}}{\alpha^2 (\overline{r+4}^2 + \nu^2)} & -\frac{\overline{r+1} \overline{r+2} \nu}{\alpha^2 (\overline{r+4}^2 + \nu^2)} & \frac{\overline{r+1} \overline{r+2}}{\alpha (\overline{r+2}^2 + \nu^2)} & -\frac{\overline{r+1} \nu}{\alpha (\overline{r+2}^2 + \nu^2)} \\ -\frac{\overline{r+1} \overline{r+2} \nu}{\alpha^2 (\overline{r+4}^2 + \nu^2)} & \frac{\overline{r+1} (2\overline{r+4} + \nu^2)}{\alpha^2 (\overline{r+4}^2 + \nu^2)} & -\frac{\overline{r+1} \nu}{\alpha (\overline{r+2}^2 + \nu^2)} & \frac{\overline{r+2} + \nu^2}{\alpha (\overline{r+2}^2 + \nu^2)} \\ \frac{\overline{r+1} \overline{r+2}}{\alpha (\overline{r+2}^2 + \nu^2)} & -\frac{\overline{r+1} \nu}{\alpha (\overline{r+2}^2 + \nu^2)} & \frac{\partial^2}{\partial \nu^2} \log F & \frac{\partial^2}{\partial \nu \partial r} \log F \\ -\frac{\overline{r+1} \nu}{\alpha (\overline{r+2}^2 + \nu^2)} & \frac{\overline{r+2} + \nu^2}{\alpha (\overline{r+2}^2 + \nu^2)} & \frac{\partial^2}{\partial \nu \partial r} \log F & \frac{\partial^2}{\partial r^2} \log F \end{vmatrix}$$

as the Hessian of $-L$, when

$$F = e^{-\frac{1}{2}\nu\pi} \int_0^\pi e^{\nu\theta} \sin^r \theta d\theta.$$

The ratios of the minors of this determinant to the value of the determinant give the standard deviations and correlations of the optimum values of the four parameters obtained from a number of large samples.

In discussing the efficiency of the method of moments in respect of the *form* of the curve, it is doubtful if it be possible to isolate in a unique and natural manner, as we have done in respect of *location* and *scaling*, a series of parameters which shall successively represent different aspects of the process of curve fitting. Thus we might find the efficiencies with which r and ν are determined by the method of moments, or those of the parametric functions corresponding to β_1 and β_2 , or we might use m_1 and m_2 as independent parameters of form; but in all these cases we should be employing an arbitrary pair of measures to indicate the relative magnitude of corresponding contour ellipses of the two frequency surfaces.

For the symmetrical series of curves, the Types II. and VII., the two systems of

ellipses are coaxial, the deviations of r and ν being uncorrelated; in the case of Type VII. we put $\nu = 0$, in the determinant given above, which then becomes

$$\begin{vmatrix} \frac{\overline{r+1} \overline{r+2}}{r+4} & 0 & \frac{r+1}{r+2} & 0 \\ 0 & \frac{\overline{2r+1}}{r+4} & 0 & \frac{1}{r+2} \\ \frac{r+1}{r+2} & 0 & \frac{1}{2} \mathbb{F}\left(\frac{r}{2}\right) & 0 \\ 0 & \frac{1}{r+2} & 0 & \frac{1}{4} \left\{ \mathbb{F}\left(\frac{r-1}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} \end{vmatrix}$$

and falls in the two factors

$$\left[\frac{r+1}{2r+4} \left\{ \mathbb{F}\left(\frac{r-1}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} - \frac{1}{r-2^2} \right] \left[\frac{\overline{r+1} \overline{r+2}}{2r+4} \mathbb{F}\left(\frac{r}{2}\right) - \frac{\overline{r+1}^2}{r+2^2} \right],$$

so that

$$n\sigma_r^2 = \frac{2\overline{r+2}^8}{r+2^3 \mathbb{F}\left(\frac{r}{2}\right) - 2\overline{r+1} \overline{r+4}}$$

and

$$n\sigma_r^2 = \frac{4\overline{r+1} \overline{r+2}^2}{r+1 \overline{r+2}^2 \left\{ \mathbb{F}\left(\frac{r-1}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} - 2\overline{r+4}}$$

The corresponding expressions for the method of moments are

$$n\sigma_{\nu\mu}^2 = \frac{3}{8} \cdot \frac{r^2 \overline{r-2}^2 (r^2 + r + 10)}{r-1 \overline{r-3} \overline{r-5}},$$

and

$$n\sigma_{r\mu}^2 = \frac{2}{3} \cdot \frac{r \overline{r-1}^2 \overline{r-3} (r^2 - r + 18)}{r-5 \overline{r-7}}.$$

Since for moderately large values of r , we have, approximately,

$$\overline{r+2}^3 \mathbb{F}\left(\frac{r}{2}\right) - 2\overline{r+1} \overline{r+4} = \frac{16}{3} \left(1 - \frac{1}{5\overline{r+2}^3} \right),$$

and

$$\overline{r+1} \overline{r+2}^2 \left\{ \mathbb{F}\left(\frac{r-1}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} - 2\overline{r+1} \overline{r+4} = 6 - \frac{4}{\overline{r+2}^3};$$

we have, approximately, for the efficiency of ν_μ ,

$$\frac{(\overline{r+2^3} + \frac{1}{6} \overline{r+2} \dots) \overline{r-1} \overline{r-3} \overline{r-5}}{(r^2+r+10) r^2 \overline{r-2^2}},$$

or, when r is great,

$$1 - \frac{28.8}{r^2};$$

and for the efficiency of τ_μ ,

$$\frac{(\overline{r+2^2} + \frac{2}{3} \dots) \overline{r+1^2} \overline{r-5} \overline{r-7}}{(r^2-r+18) r \overline{r-1^2} \overline{r-3}};$$

or, when r is great,

$$1 - \frac{53.8}{r^2}.$$

The following table gives the values of the transcendental quantities required, and the efficiency of the method of moments in estimating the value of ν and τ from samples drawn from Type VII. distribution.

r .	$\frac{\overline{r+2^3} F\left(\frac{r}{2}\right)}{-2\overline{r+1} \overline{r+4}}$	Efficiency of ν_μ .	$\frac{\overline{r+1^2} \overline{r+2^2}}{\times \left\{ F\left(\frac{r-1}{2}\right) - F\left(\frac{r}{2}\right) \right\}} - 2\overline{r+1} \overline{r+4}$.	Efficiency of τ_μ .
5	5.31271	0		
6	5.31736	0.2572		
7	5.32060	0.4338	5.9473	0
8	5.32296	0.5569	5.9574	0.1687
9	5.32472	0.6449	5.9649	0.3130
10	5.32607	0.7097	5.9706	0.4403
11	5.32713	0.7586	5.9750	0.5207
12	5.32797	0.7963	5.9787	0.5935
13	5.32866	0.8259	5.9810	0.6519
14	5.32919	0.8497	5.9839	0.6990
15			5.9853	0.7376
16			5.9870	0.7694
17			5.9883	0.7959
18			5.9895	0.8182

It will be seen that we do not attain to 80 per cent. efficiency in estimating the form of the curve until r is about 17.2, which corresponds to $\beta_2 = 3.42$. Even for symmetrical curves higher values of β_2 imply that the method of moments makes use of less than four-fifths of the information supplied by the sample.

On the other side of the normal point, among the Type II. curves, very similar formulæ apply. The fundamental Hessian is

$$\begin{vmatrix} \frac{\overline{r-1} \overline{r-2}}{r-4} & 0 & -\frac{\overline{r-1}}{r-2} & 0 \\ 0 & \frac{2 \overline{r-1}}{r-4} & 0 & -\frac{1}{r-2} \\ -\frac{\overline{r-1}}{r-2} & 0 & \frac{1}{2} \mathbb{F} \left(\frac{\overline{r-2}}{2} \right) & 0 \\ 0 & -\frac{1}{r-2} & 0 & \frac{1}{4} \left\{ \mathbb{F} \left(\frac{\overline{r-2}}{2} \right) - \mathbb{F} \left(\frac{\overline{r-1}}{2} \right) \right\} \end{vmatrix}$$

where r is written for the positive quantity, $-r$, whence

$$n\sigma_r^2 = \frac{2 \overline{r-2}^3}{\overline{r-2}^3 \mathbb{F} \left(\frac{\overline{r-2}}{2} \right) - 2 \overline{r-1} \overline{r-4}}$$

and

$$n\sigma_r^2 = \frac{4 \overline{r-1} \overline{r-2}^2}{\overline{r-1} \overline{r-2}^2 \left\{ \mathbb{F} \left(\frac{\overline{r-2}}{2} \right) - \mathbb{F} \left(\frac{\overline{r-1}}{2} \right) \right\} - 2 \overline{r-4}}$$

Now since

$$\mathbb{F} \left(\frac{\overline{r-2}}{2} \right) = \mathbb{F} \left(\frac{\overline{r-4}}{2} \right) - \frac{4}{\overline{r-2}^2},$$

it follows that

$$\overline{r-2}^3 \mathbb{F} \left(\frac{\overline{r-2}}{2} \right) - 2 \overline{r-1} \overline{r-4} = \overline{r-2}^3 \mathbb{F} \left(\frac{\overline{r-4}}{2} \right) - 2 \overline{r-3},$$

which is the same function of $r-4$ as

$$\overline{r+2}^3 \mathbb{F} \left(\frac{r}{2} \right) - 2 \overline{r+1} \overline{r+4}$$

is of r .

In a similar manner

$$\begin{aligned} \overline{r+1}^2 \overline{r-2}^2 \left\{ \mathbb{F} \left(\frac{\overline{r-2}}{2} \right) - \mathbb{F} \left(\frac{\overline{r-1}}{2} \right) \right\} - 2 \overline{r-1} \overline{r-4} \\ = \overline{r-1}^2 \overline{r-2}^2 \left\{ \mathbb{F} \frac{\overline{r-4}}{2} - \mathbb{F} \frac{\overline{r-3}}{2} \right\} - 2 \overline{r-2} \overline{r+1}, \end{aligned}$$

which is the same function of $r-3$ as

$$\overline{r+1}^2 \overline{r+2}^2 \left\{ \mathbb{F} \left(\frac{\overline{r-1}}{2} \right) - \mathbb{F} \left(\frac{r}{2} \right) \right\} - 2 \overline{r+1} \overline{r+4}$$

is of r .

In all these functions and those of the following table, r must be substituted as a positive quantity, although it must not be forgotten that r changes sign as we pass from Type VII. to Type II., and we have hitherto adhered to the convention that r is to be taken positive for Type VII. and negative for Type II.

r .	$r - 2^3 \mathbb{F} \left(\frac{r-2}{2} \right) - 2r - 1 r - 4.$	Efficiency of r_μ .	$\frac{r-1^2 r-2^2}{\times \left(\mathbb{F} \left(\frac{r-2}{3} \right) - \mathbb{F} \left(\frac{r-1}{2} \right) \right) - 2r - 1 r - 4.}$	Efficiency of r_μ .
2	4	0	4	0
3	4.93480	0.0576	5.1595	0.0431
4	5.15947	0.2056	5.5648	0.1445
5	5.23966	0.3590	5.7410	0.2613
6	5.27578	0.4865	5.8305	0.3708
7	5.29472	0.5857	5.8813	0.4653
8	5.30576	0.6615	5.9126	0.5441
9	5.31271	0.7198	5.9331	0.6090
10	5.31736	0.7650	5.9473	0.6624
11	5.32060	0.8005	5.9574	0.7063
12	5.32296	0.8287	5.9649	0.7427
13	5.32472	0.8516	5.9706	0.7731
14	5.32607	0.8702	5.9750	0.7986
15			5.9787	0.8202

In both cases the region of validity is bounded by the rectangle, at the point B (fig. 2, p. 343). Efficiency of 80 per cent. is reached when r is about 14.1 ($\beta_2 = 2.65$). Thus for symmetrical curves of the Pearsonian type we may say that the method of moments has an efficiency of 80 per cent. or more, when β_2 lies between 2.65 and 3.42. The limits within which the values of the parameters obtained by moments cannot be greatly improved are thus much narrower than has been imagined.

11. THE REASON FOR THE EFFICIENCY OF THE METHOD OF MOMENTS IN A SMALL REGION SURROUNDING THE NORMAL CURVE.

We have seen that the method of moments applied in fitting Pearsonian curves has an efficiency exceeding 80 per cent. only in the restricted region for which β_2 lies between the limits 2.65 and 3.42, and as we have seen in Section 8, for which β_1 does not exceed 0.1. The contours of equal efficiency are nearly circular or elliptical within these limits, if the curves are represented as in fig. 2, p. 343, and are ultimately centred round the normal point, at which point the efficiencies of all parameters tend to 100 per cent. It was, of course, to be expected that the first two moments would have 100 per cent. efficiencies at this point, for they happen to be the optimum statistics for fitting the normal curve. That the moment coefficients β_1 and β_2 also tend to 100 per cent. efficiency in this region suggests that in the immediate neighbourhood of the normal

curve the departures from normality specified by the Pearsonian formula agree with those of that system of curves for which the method of moments gives the solution of maximum likelihood.

The system of curves for which the method of moments is the best method of fitting may easily be deduced, for if the frequency in the range dx be

$$y(x, \theta_1, \theta_2, \theta_3, \theta_4) dx,$$

then

$$\frac{\partial}{\partial \theta} \log y$$

must involve x only as polynomials up to the fourth degree; consequently

$$y = e^{-a^2(x^4 + p_1x^3 + p_2x^2 + p_3x + p_4)},$$

the convergence of the probability integral requiring that the coefficient of x^4 should be negative, and the five quantities a, p_1, p_2, p_3, p_4 being connected by a single relation, representing the fact that the total probability is unity.

Typically these curves are bimodal, and except in the neighbourhood of the normal point are of a very different character from the Pearsonian curves. Near this point, however, they may be shown to agree with the Pearsonian type; for let

$$y = Ce^{-\frac{x^2}{2\sigma^2} + k_1\frac{x^3}{\sigma^3} + k_2\frac{x^4}{\sigma^4}}$$

represent a curve of the quartic exponent, sufficiently near to the normal curve for the squares of k_1 and k_2 to be neglected, then

$$\begin{aligned} \frac{d}{dx} \log y &= -\frac{x}{\sigma^2} \left(1 - 3k_1\frac{x}{\sigma} - 4k_2\frac{x^2}{\sigma^2} \right) \\ &= -\frac{x}{\sigma^2 \left(1 + 3k_1\frac{x}{\sigma} + 4k_2\frac{x^2}{\sigma^2} \right)}, \end{aligned}$$

neglecting powers of k_1 and k_2 . Since the only terms in the denominator constitute a quadratic in x , the curve satisfies the fundamental equation of the Pearsonian type of curves. In the neighbourhood of the normal point, therefore, the Pearsonian curves are equivalent to curves of the quartic exponent; it is to this that the efficiency of μ_3 and μ_4 , in the neighbourhood of the normal curve, is to be ascribed.

12. DISCONTINUOUS DISTRIBUTIONS.

The applications hitherto made of the optimum statistics have been problems in which the data are ungrouped, or at least in which the grouping intervals are so small as not to disturb the values of the derived statistics. By grouping, these continuous

distributions are reduced to discontinuous distributions, and in an exact discussion must be treated as such.

If p_s be the probability of an observation falling in the cell (s), p_s being a function of the required parameters $\theta_1, \theta_2 \dots$; and in a sample of N , if n_s are found to fall into that cell, then

$$S(\log f) = S(n_s \log p_s).$$

If now we write $\bar{n}_s = p_s N$, we may conveniently put

$$L = S\left(n_s \log \frac{n_s}{\bar{n}_s}\right),$$

where L differs by a constant only from the logarithm of the likelihood, with sign reversed, and therefore the method of the optimum will consist in finding the *minimum* value of L . The equations so found are of the form

$$\frac{\partial L}{\partial \theta} = -S\left(\frac{n_s}{\bar{n}_s} \frac{\partial \bar{n}_s}{\partial \theta}\right) = 0. \dots \dots \dots (6)$$

It is of interest to compare these formulæ with those obtained by making the Pearsonian χ^2 a minimum.

For

$$\chi^2 = S \frac{(n_s - \bar{n}_s)^2}{n_s},$$

and therefore

$$1 + \chi^2 = S\left(\frac{n_s^2}{\bar{n}_s}\right),$$

so that on differentiating by $d\theta$, the condition that χ^2 should be a minimum for variations of θ is

$$-S\left(\frac{n_s^2}{\bar{n}_s^2} \frac{\partial \bar{n}_s}{\partial \theta}\right) = 0. \dots \dots \dots (7)$$

Equation (7) has actually been used (12) to "improve" the values obtained by the method of moments, even in cases of normal distribution, and the Poisson series, where the method of moments gives a strictly sufficient solution. The discrepancy between these two methods arises from the fact that χ^2 is itself an approximation, applicable only when \bar{n}_s and n_s are large, and the difference between them of a lower order of magnitude. In such cases

$$L = S\left(n_s \log \frac{n_s}{\bar{n}_s}\right) = S\left(\overline{m+x} \log \frac{m+x}{m}\right) = S\left\{x + \frac{x^2}{2m} - \frac{x^3}{6m^2} \dots\right\},$$

and since

$$S(x) = 0,$$

we have, when x is in all cases small compared to m ,

$$L = \frac{1}{2}S\left(\frac{x^2}{m}\right) = \frac{1}{2}\chi^2$$

as a first approximation. In those cases, therefore, when χ^2 is a valid measure of the departure of the sample from expectation, it is equal to $2L$; in other cases the approximation fails and L itself must be used.

The failure of equation (7) in the general problem of finding the best values for the parameters may also be seen by considering cases of fine grouping, in which the majority of observations are separated into units. For the formula in equation (6) is equivalent to

$$S\left(\frac{1}{\bar{n}_s} \frac{\partial \bar{n}_s}{\partial \theta}\right)$$

where the summation is taken over all the observations, while the formula of equation (7), since it involves n_s^2 , changes its value discontinuously, when one observation is gradually increased, at the point where it happens to coincide with a second observation.

Logically it would seem to be a necessity that that population which is chosen in fitting a hypothetical population to data should also appear the best when tested for its goodness of fit. The method of the optimum secures this agreement, and at the same time provides an extension of the process of testing goodness of fit, to those cases for which the χ^2 test is invalid.

The practical value of χ^2 lies in the fact that when the conditions are satisfied in order that it shall closely approximate to $2L$, it is possible to give a general formula for its distribution, so that it is possible to calculate the probability, P , that in a random sample from the population considered, a worse fit should be obtained; in such cases χ^2 is distributed in a curve of the Pearsonian Type III.,

$$df \propto \left(\frac{\chi^2}{2}\right)^{\frac{n'-3}{2}} e^{-\frac{\chi^2}{2}} d\left(\frac{\chi^2}{2}\right)$$

or

$$df \propto L^{\frac{n'-3}{2}} e^{-L} dL,$$

where n' is one more than the number of degrees of freedom in which the sample may differ from expectation (17).

In other cases we are at present faced with the difficulty that the distribution L requires a special investigation. This distribution will in general be discontinuous (as is that of χ^2), but it is not impossible that mathematical research will reveal the existence of effective graduations for the most important groups of cases to which χ^2 cannot be applied.

We shall conclude with a few illustrations of important types of discontinuous distribution.

1. *The Poisson Series.*

$$e^{-m} \left(1, m, \frac{m^2}{2!}, \dots, \frac{m^x}{x!}, \dots \right)$$

involves only the single parameter, and is of great importance in modern statistics. For the optimum value of m ,

$$S \left\{ \frac{\partial}{\partial m} (-m + x \log m) \right\} = 0,$$

whence

$$S \left(\frac{x}{\hat{m}} - 1 \right) = 0,$$

or

$$\hat{m} = \bar{x}.$$

The most likely value of m is therefore found by taking the first moment of the series.

Differentiating a second time,

$$-\frac{1}{\sigma_{\hat{m}}^2} = S \left(-\frac{x}{m^2} \right) = -\frac{n}{m},$$

so that

$$\sigma_{\hat{m}}^2 = \frac{m}{n},$$

as is well known.

2. *Grouped Normal Data.*

In the case of the normal curve of distribution it is evident that the second moment is a sufficient statistic for estimating the standard deviation : in investigating a sufficient solution for grouped normal data, we are therefore in reality finding the optimum correction for grouping ; the SHEPPARD correction having been proved only to satisfy the criterion of consistency.

For grouped normal data we have

$$p_i = \frac{1}{\sigma \sqrt{2\pi}} \int_{x_i}^{x_{i+1}} e^{-\frac{x-m}{2\sigma^2}} dx,$$

and the optimum values of m and σ are obtained from the equations,

$$\frac{\partial L}{\partial m} = S \left(\frac{n_i}{p_i} \frac{\partial p_i}{\partial m} \right) = 0,$$

$$\frac{\partial L}{\partial \sigma} = S \left(\frac{n_i}{p_i} \frac{\partial p_i}{\partial \sigma} \right) = 0 ;$$

or, if we write,

$$z = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{r-m}{2\sigma^2}}$$

we have the two conditions,

$$S \left(\frac{n_s}{p_s} \overline{z_s - z_{s+1}} \right) = 0$$

and

$$S \left\{ \frac{n_s}{p_s} \left(\frac{x_s}{\sigma} z_s - \frac{x_{s+1}}{\sigma} z_{s+1} \right) \right\} = 0.$$

As a simple example we shall take the case chosen by K. SMITH in her investigation of the variation of χ^2 in the neighbourhood of the moment solution (12).

Three hundred errors in right ascension are grouped in nine classes, positive and negative errors being thrown together as shown in the following table :—

0''·1 arc	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9
Frequency . .	114	84	53	24	14	6	3	1	1

The second moment, without correction, yields the value

$$\sigma_v = 2 \cdot 282542.$$

Using SHEPPARD'S correction, we have

$$\sigma_\mu = 2 \cdot 264214,$$

while the value obtained by making χ^2 a minimum is

$$\sigma_{\chi^2} = 2 \cdot 355860.$$

If the latter value were accepted we should have to conclude that SHEPPARD'S correction, even when it is small, and applied to normal data, might be altogether of the wrong magnitude, and even in the wrong direction. In order to obtain the optimum value of σ , we tabulate the values of $\frac{\partial L}{\partial \sigma}$ in the region under consideration; this may be done without great labour if values of σ be chosen suitable for the direct application of the table of the probability integral (13, Table II.). We then have the following values :—

$\frac{1}{\sigma}$	0·43	0·44	0·45	0·46
$\frac{\partial L}{\partial \sigma}$	+15·135	+2·149	-11·098	-24·605
$\Delta^2 \frac{\partial L}{\partial \sigma}$		-0·261	-0·260	

By interpolation,

$$\frac{1}{\hat{\sigma}} = 0.441624$$

$$\hat{\sigma} = 2.26437.$$

We may therefore summarise these results as follows :—

Uncorrected estimate of σ	2.28254
SHEPPARD'S correction	- 0.01833
Correction for maximum likelihood	- 0.01817
" Correction " for minimum χ^2	+ 0.07332

Far from shaking our faith, therefore, in the adequacy of SHEPPARD'S correction, when small, for normal data, this example provides a striking instance of its effectiveness, while the approximate nature of the χ^2 test renders it unsuitable for improving a method which is already very accurate.

It will be useful before leaving the subject of grouped normal data to calculate the actual loss of efficiency caused by grouping, and the additional loss due to the small discrepancy between moments with SHEPPARD'S correction and the optimum solution.

To calculate the loss of efficiency involved in the process of grouping normal data, let

$$v = \frac{1}{a} \int_{t-ta}^{t+ta} f(\xi) d\xi,$$

when $a\sigma$ is the group interval, then

$$\begin{aligned} v &= f(\xi) + \frac{a^2}{24} f''(\xi) + \frac{a^4}{1920} f^{(4)}(\xi) + \frac{a^6}{322,560} f^{(6)}(\xi) + \dots \\ &= f(\xi) \left\{ 1 + \frac{a^2}{24} (\xi^2 - 1) + \frac{a^4}{1920} (\xi^4 - 6\xi^2 + 3) + \frac{a^6}{322,560} (\xi^6 - 15\xi^4 + 45\xi^2 - 15) + \dots \right\}, \end{aligned}$$

whence

$$\log v = \log f + \frac{a^2}{24} (\xi^2 - 1) - \frac{a^4}{2880} (\xi^4 + 4\xi^2 - 2) + \frac{a^6}{181,440} (\xi^6 + 6\xi^4 + 3\xi^2 - 1) - \dots,$$

and

$$\frac{\partial^2}{\partial m^2} \log v = -\frac{1}{\sigma^2} + \frac{1}{\sigma^2} \left\{ \frac{a^2}{12} - \frac{a^4}{720} (3\xi^2 + 2) + \frac{a^6}{30,240} (5\xi^4 + 12\xi^2 + 1) - \dots \right\},$$

of which the mean value is

$$-\frac{1}{\sigma^2} \left\{ 1 - \frac{a^2}{12} + \frac{a^4}{144} - \frac{a^6}{4320} \dots \right\},$$

neglecting the periodic terms ; and consequently

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \left(1 + \frac{\alpha^2}{12} - \frac{\alpha^4}{2880} \dots \right).$$

Now for the mean of ungrouped data

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n},$$

so that the loss of efficiency due to grouping is nearly $\frac{\alpha^2}{12}$.

The further loss caused by using the mean of the grouped data is very small, for

$$\sigma_{v_1}^2 = \frac{v_2}{n} = \frac{\sigma^2}{n} \left(1 + \frac{\alpha^2}{12} \right),$$

neglecting the periodic terms ; the loss of efficiency by using v_1 therefore is only

$$\frac{\alpha^4}{2880}.$$

Similarly for the efficiency for scaling,

$$\begin{aligned} \frac{\partial^2}{\partial \sigma^2} \log v &= \frac{1}{\sigma^2} - \frac{3\xi^2}{\sigma^2} + \frac{1}{\sigma^2} \left\{ \frac{\alpha^2}{12} (10\xi^2 - 3) - \frac{\alpha^4}{360} (9\xi^4 + 21\xi^2 - 5) \right. \\ &\quad \left. + \frac{\alpha^6}{30,240} (26\xi^6 + 110\xi^4 + 36\xi^2 - 7) - \frac{\alpha^8}{1,814,400} (51\xi^8 + 315\xi^6 + 351\xi^4 - 55\xi^2 + 9) + \dots \right\}, \end{aligned}$$

of which the mean value is

$$-\frac{2}{\sigma^2} \left\{ 1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{40} - \frac{\alpha^6}{270} + \frac{83\alpha^8}{129,600} \dots \right\},$$

neglecting the periodic terms ; and consequently

$$\sigma_s^2 = \frac{\sigma^2}{2n} \left\{ 1 + \frac{\alpha^2}{6} + \frac{\alpha^4}{360} - \frac{\alpha^6}{10,800} \dots \right\}.$$

For ungrouped data

$$\sigma_s^2 = \frac{\sigma^2}{2n},$$

so that the loss of efficiency in scaling due to grouping is nearly $\frac{\alpha^2}{6}$. This may be made as low as 1 per cent. by keeping α less than $\frac{1}{4}$.

The further loss of efficiency produced by using the grouped second moment with SHEPPARD'S correction is again very small, for

$$\sigma_{v_2}^2 = \frac{v_4 - v_2^2}{n} = \frac{2\sigma^4}{n} \left(1 + \frac{\alpha^2}{6} + \frac{\alpha^4}{360} \right)$$

neglecting the periodic terms.

Whence it appears that the further loss of efficiency is only

$$\frac{\alpha^2}{10,800}.$$

We may conclude, therefore, that the high agreement between the optimum value of σ and that obtained by SHEPPARD'S correction in the above example is characteristic of grouped normal data. The method of moments with SHEPPARD'S correction is highly efficient in treating such material, the gain in efficiency obtainable by increasing the likelihood to its maximum value is trifling, and far less than can usually be gained by using finer groups. The loss of efficiency involved in grouping may be kept below 1 per cent. by making the group interval less than one-quarter of the standard deviation.

Although for the normal curve the loss of efficiency due to moderate grouping is very small, such is not the case with curves making a finite angle with the axis, or having at an extreme a finite or infinitely great ordinate. In such cases even moderate grouping may result in throwing away the greater part of the information which the sample provides.

3. *Distribution of Observations in a Dilution Series.*

An important type of discontinuous distribution occurs in the application of the dilution method to the estimation of the number of micro-organisms in a sample of water or of soil. The method here presented was originally developed in connection with Mr. CUTLER'S extensive counts of soil protozoa carried out in the protozoological laboratory at Rothamsted, and although the method is of very wide application, this particular investigation affords an admirable example of the statistical principles involved.

In principle the method consists in making a series of dilutions of the soil sample, and determining the presence or absence of each type of protozoa in a cubic centimetre of the dilution, after incubation in a nutrient medium.

The series in use proceeds by powers of 2, so that the frequency of protozoa in each dilution is one-half that in the last.

The frequency at any stage of the process may then be represented by

$$m = \frac{n}{2^x},$$

when x indicates the number of dilutions.

Under conditions of random sampling, the chance of any plate receiving 0, 1, 2, 3 protozoa of a given species is given by the Poisson series

$$e^{-m} \left(1, m, \frac{m^2}{2!}, \frac{m^3}{3!}, \dots \right),$$

and in consequence the proportion of sterile plates is

$$p = e^{-m},$$

and of fertile plates

$$q = 1 - e^{-m}.$$

In general we may consider a dilution series with dilution factor a so that

$$\log p = -\frac{m}{a^x},$$

and assume that s plates are poured from each dilution.

The object of the method being to estimate the number n from a record of the sterile and fertile plates, we have

$$L = S_1 (\log p) + S_2 (\log q)$$

when S_1 stands for summation over the sterile plates, and S_2 for summation over those which are fertile.

Now

$$\frac{\partial p}{\partial \log n} = -\frac{\partial q}{\partial \log n} = p \log p,$$

so that the optimum value of n is obtained from the equation,

$$\frac{\partial L}{\partial \log n} = S_1 (\log p) - S_2 \left(\frac{p}{q} \log p \right) = 0.$$

Differentiating a second time,

$$\frac{\partial^2 L}{\partial (\log n)^2} = S_1 (\log p) - S_2 \left\{ \frac{p \log p}{q} \left(\log p + 1 + \frac{p \log p}{q} \right) \right\};$$

now the mean number of sterile plates is ps , and of fertile plates qs , so that the mean value of $\frac{\partial^2 L}{\partial (\log n)^2}$ is

$$-\frac{1}{\sigma^2_{\log n}} = sS \left\{ p \log p - p \log p \left(\log p + 1 + \frac{p \log p}{q} \right) \right\} = -sS \left\{ \frac{p}{q} (\log p)^2 \right\},$$

the summation, S , being extended over all the dilutions.

It thus appears that each plate observed adds to the weight of the determination of $\log n$ a quantity

$$w = \frac{p}{q} (\log p)^2.$$

We give below a table of the values of p , and of w , for the dilution series $\log p = 2^{-x}$ from $x = -4$ to $x = 11$.

x .	p .	w .	S (w) (per cent.).
-4	0.00000014167	0.000036	0.002
-3	0.0003354626	0.021477	0.906
-2	0.01831564	0.298518	13.485
-1	0.1353353	0.626071	39.865
0	0.3678794	0.581977	64.388
1	0.6065306	0.385355	80.625
2	0.7788009	0.220051	89.897
3	0.8824969	0.117350	94.842
4	0.9394110	0.060565	97.394
5	0.9692333	0.030764	98.690
6	0.9844965	0.015503	99.343
7	0.9922179	0.007782	99.671
8	0.9961014	0.003899	99.836
9	0.9980488	0.001951	99.918
10	0.9990239	0.000976	99.959
11	0.9995119	0.000488	99.979
Remainder		0.000488	
Total		2.373251	

For the same dilution constant the total S (w) is nearly independent of the particular series chosen. Its average value being $\frac{\pi^2}{6 \log \alpha}$, or in this case 2.373138. The fourth column shows the total weight attained at any stage, expressed as a percentage of that obtained from an infinite series of dilutions. It will be seen that a set of eight dilutions comprise all but about 2 per cent. of the weight. With a loss of efficiency of only 2 to 2½ per cent., therefore, the number of dilutions which give information as to a particular species may be confined to eight. To this number must be added a number depending on the range which it is desired to explore. Thus to explore a range from 100 to 100,000 per gramme (about 10 octaves) we should require 10 more dilutions, making 18 in all, while to explore a range of a millionfold, or about 20 octaves, 28 dilutions would be needed.

In practice it would be exceedingly laborious to calculate the optimum value of n for each series observed (of which 38 are made daily). On the advice of the statistical department, therefore, Mr. CUTLER adopted the plan of counting the total number of sterile plates, and taking the value of n which on the average would give that number. When a sufficient number of dilutions are made, $\log n$ is diminished by $\frac{1}{s} \log \alpha$ for each additional sterile plate, and even near the ends of the series the appropriate values of n may easily be tabulated. Since this method of estimation is of wide application, and appears at first sight to be a very rough one, it is important to calculate its efficiency.

For any dilution the variance in the number of sterile plates is

$$spq,$$

and as the several dilutions represent independent samples, the total variance is

$$sS(pq),$$

hence

$$\sigma^2_{\log n} = \frac{(\log \alpha)^2}{s} S(pq).$$

Now $S(pq)$ has an average value $\frac{\log 2}{\log \alpha}$, therefore taking $\alpha = 2$,

$$(\log \alpha)^2 = .480453,$$

and

$$S(pq) = 1$$

being very nearly constant and within a small fraction of unity; whence the efficiency of the method of counting the sterile plates is

$$\frac{6}{\pi^2 \log 2} = 87.71 \text{ per cent.},$$

a remarkably high efficiency, considering the simplicity of the method, the efficiency being independent of the dilution ratio.

13. SUMMARY.

During the rapid development of practical statistics in the past few decades, the theoretical foundations of the subject have been involved in great obscurity. Adequate distinction has seldom been drawn between the sample recorded and the hypothetical population from which it is regarded as drawn. This obscurity is centred in the so-called "inverse" methods.

On the bases that the purpose of the statistical reduction of data is to obtain statistics which shall contain as much as possible, ideally the whole, of the relevant information contained in the sample, and that the function of Theoretical Statistics is to show how such adequate statistics may be calculated, and how much and of what kind is the information contained in them, an attempt is made to formulate distinctly the types of problems which arise in statistical practice.

Of these, problems of Specification are found to be dominated by considerations which may change rapidly during the progress of Statistical Science. In problems of Distribution, relatively little progress has hitherto been made, these problems still affording a field for valuable enquiry by highly trained mathematicians. The principal purpose of this paper is to put forward a general solution of problems of Estimation.

Of the criteria used in problems of Estimation only the criterion of Consistency has hitherto been widely applied ; in Section 5 are given examples of the adequate and inadequate application of this criterion. The criterion of Efficiency is shown to be a special but important case of the criterion of Sufficiency, which latter requires that the whole of the relevant information supplied by a sample shall be contained in the statistics calculated.

In order to make clear the nature of the general method of satisfying the criterion of Sufficiency, which is here put forward, it has been thought necessary to reconsider BAYES' problem in the light of the more recent criticisms to which the idea of " inverse probability " has been exposed. The conclusion is drawn that two radically distinct concepts, both of importance in influencing our judgment, have been confused under the single name of *probability*. It is proposed to use the term *likelihood* to designate the state of our information with respect to the parameters of hypothetical populations, and it is shown that the quantitative measure of likelihood does not obey the mathematical laws of probability.

A proof is given in Section 7 that the criterion of Sufficiency is satisfied by that set of values for the parameters of which the likelihood is a maximum, and that the same function may be used to calculate the efficiency of any other statistics, or, in other words, the percentage of the total available information which is made use of by such statistics.

This quantitative treatment of the information supplied by a sample is illustrated by an investigation of the efficiency of the method of moments in fitting the Pearsonian curves of Type III.

Section 9 treats of the location and scaling of Error Curves in general, and contains definitions and illustrations of the *intrinsic accuracy*, and of the *centre of location* of such curves.

In Section 10 the efficiency of the method of moments in fitting the general Pearsonian curves is tested and discussed. High efficiency is only found in the neighbourhood of the normal point. The two causes of failure of the method of moments in locating these curves are discussed and illustrated. The special cause is discovered for the high efficiency of the third and fourth moments in the neighbourhood of the normal point.

It is to be understood that the low efficiency of the moments of a sample in estimating the form of these curves does not at all diminish the value of the notation of moments as a means of the comparative specification of the form of such curves as have finite moment coefficients.

Section 12 illustrates the application of the method of maximum likelihood to discontinuous distributions. The POISSON series is shown to be sufficiently fitted by the mean. In the case of grouped normal data, the SHEPPARD correction of the crude moments is shown to have a very high efficiency, as compared to recent attempts to improve such fits by making χ^2 a minimum ; the reason being that χ^2 is an expression only approximate to a true value derivable from likelihood. As a final illustration of

the scope of the new process, the theory of the estimation of micro-organisms by the dilution method is investigated.

Finally it is a pleasure to thank Miss W. A. MACKENZIE, for her valuable assistance in the preparation of the diagrams.

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X. BAKERIAN LECTURE.—*On the Spectrum of Hydrogen.*

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[PLATE 3.]

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(1) *Introductory.*

MODERN theoretical investigations have met with signal success in providing an explanation for the production of the primary spectrum of Hydrogen, generally known as the Balmer series, but the many-lined or secondary spectrum has hitherto proved to be a much more formidable problem. There can be little doubt that a completely satisfactory explanation of its genesis will mark an important step in our knowledge of the origin of spectra. The detection of regularities amongst such a vast number of lines is in itself an exceedingly difficult task, and one for which it is essential that the data relating to wave-lengths should be accurate and complete.

Although some of the early investigators were of the opinion that the secondary spectrum was to be referred to impurities in the discharge tube, it is now generally

agreed that both the Balmer series and the secondary spectrum are to be attributed to Hydrogen. The relative intensities of the two spectra vary in a surprising manner. Such traces of Hydrogen as are necessary to exhibit the earlier members of the Balmer series are indeed difficult to eliminate from luminous sources, but the secondary spectrum only appears in sources in which considerable quantities of Hydrogen are present, and its intensity relative to that of the Balmer series is greatly enhanced by the complete removal of all impurities.

It has long been known that in the spectrum of water vapour the lines of the secondary spectrum are very weak, and many investigations have shown that the intensity of the secondary spectrum is greatly reduced by the presence of small traces of mercury vapour and other impurities. The relative intensities of the two spectra are affected by variations in the electrical excitation, but for a high relative intensity of the secondary spectrum purity of the gas is essential. The appearance of the discharge in vacuum tubes containing Hydrogen of the highest degree of purity is indeed strikingly different from that observed in hydrogen tubes prepared without special precautions, the former being of an almost white colour whilst the latter show the familiar red glow, which is due to the predominance of the red line, $H\alpha$, of the Balmer series.* In a recent investigation, WOOD ('Roy. Soc. Proc.,' A, 97, p. 455, 1920; 'Phil. Mag.,' 6, 251, p. 729, 1921) has described a number of interesting phenomena which he has observed in long vacuum tubes containing Hydrogen which were excited by a high potential transformer. Many of these observations cannot at present be explained fully, but the influence of traces of impurities is shown to be an important factor.

It is well known that the secondary spectrum appears under less energetic conditions of excitation than the Balmer series, the latter alone being found in vacuum tubes excited by powerful condensed discharges, and important investigations by FULCHER ('Astrophys. Journ.,' 34, p. 388, 1911; 37, p. 60, 1913) have shown that when Hydrogen is excited by the impact of cathode rays the relative intensity of the secondary spectrum increases as the velocity of the cathode rays is reduced. FULCHER also found similar variations in intensity amongst the lines of the secondary spectrum itself, and identified a number of lines as characteristic of low potential discharges. It was found that these low potential lines exhibited regularities somewhat resembling those associated with band spectra. These regularities and their relation to other phenomena will be discussed in a later section. Any method by which the lines of so complex a spectrum can be separated into different physically related groups, cannot fail to yield results which will prove of assistance in theoretical investigations.

It has been pointed out in a previous communication (MERTON, 'Roy. Soc. Proc.,' A, 96, p. 382, 1920) that the relative intensities of the secondary lines are affected by the pressure in the discharge tube, the Fulcher bands being enhanced at low pressures, but

* These remarks do not apply without amplification to the case of tubes excited by discharges of exceedingly low current density. In the presence of water vapour the intensity of the Balmer lines, relative to the secondary spectrum, increases very rapidly with the current density.

that a much more striking change can be brought about by the admixture of Helium. It was found that in the presence of Helium some of the lines were greatly enhanced and that a number of new lines appeared; another class of lines were apparently unaffected, whilst a third class showed a marked reduction in intensity.*

There are two other methods by which the lines have been classified. DUFOUR ('Ann. Chim. et Phys.' (9), 361, 1906; 'Journ. de Phys.' (4), 8, p. 258, 1909) has investigated the Zeeman effect for the secondary spectrum, and has found that a large number of the lines are not affected in the magnetic field. This classification of the lines has been shown by FULCHER (*loc. cit.*) to be related to the results which he obtained by varying the velocities of the exciting cathode rays, and to the regularities which he found in the spectrum. The Stark effect, the resolution of the lines into components in an electric field, has been studied by TAKAMINE and YOSHIDA ('Mem. Coll. of Sci. Kyoto,' 2, p. 321, 1917), by NITTA (*ibid.*, 2, p. 349, 1917) and by TAKAMINE and KOKUBU (*ibid.*, 3, p. 271, 1919), who have found that the effect is exhibited by 54 lines in the spectrum. Such investigations and those of DUFOUR (*loc. cit.*), relating to the Zeeman effect, are necessarily restricted to the stronger lines of the spectrum, and their value as a means of classification is greatly increased when they can be correlated to changes in the spectrum of a kind which permit of observation for all the lines.

The wave-lengths of the lines in the secondary spectrum have been measured by HASSELBERG ('Mem. Acad.,' St. Petersburg (7), 30, No. 7, 1882; *ibid.* (7), 31, No. 14, 1883; 'Phil. Mag.' (5), 17, p. 329, 1882), AMES ('Phil. Mag.,' 30, p. 33, 1890), FROST ('Astrophys. Journ.,' 16, p. 100, 1902), WATSON ('Roy. Soc. Proc.,' A, 82, p. 189, 1909), PORLEZZA ('Atti Accad. Lincei,' 20 (2), p. 178, 1911), PORLEZZA and NORZI (*ibid.*, 20 (1), p. 822, 1911), and CROZE ('Ann. de Phys.' (9), 1, 48, 1914), but the results obtained by these investigators differ widely in their estimates of the relative intensities of the lines, which is greatly dependent on the particular conditions under which the spectrum is produced, and it would appear also that the tables are by no means complete, more especially in the yellow green regions of the spectrum, for which it has only recently been possible to obtain photographic plates of a sufficiently high degree of sensibility for recording lines of low intensity with a moderately high dispersion.

There has been much difference of opinion as to whether the secondary spectrum is to be attributed to the Hydrogen atom or to the molecule. To the theoretical physicist this is a question of vital importance, for there appears to be little prospect of explaining the origin of the spectrum as due to the Hydrogen atom on the views which are at present accepted with regard to its structure. Evidence on this question has been sought in investigations of the Doppler effect in positive rays by STARK ('Astrophys. Journ.,' 25, pp. 23 and 170, 1907), WILSAR ('Ann. der Phys.,' 37, p. 1251, 1912) and FULCHER ('Astrophys. Journ.,' 35, p. 101, 1912), and more recently by THOMSON ('Phil. Mag.'

* Experiments on the effect of Argon on the secondary spectrum are now in progress. It would appear that if the presence of Argon gives rise to any changes similar to those produced by the presence of Helium, they are at any rate very much less conspicuous.

(6), 40, p. 240, 1920; *ibid.* (6), 41, p. 566, 1921), and VEGARD (*ibid.* (6), 41, p. 558, 1921), but it would appear doubtful whether any conclusive evidence as to the origin of the spectrum can be obtained by these methods.

In their investigations of the widths of spectrum lines BUISSON and FABRY ('Journ. de Phys.' (2), p. 442, 1912) were led to conclude that the secondary spectrum was to be referred to the Hydrogen atom. This conclusion was based on a measurement of the limiting order at which interference fringes could be observed for a line in the secondary spectrum, the relation between the limiting order of interference N and the mass M of the radiating particle (in terms of the Hydrogen atom as unity) being given by the expression $N = k\sqrt{M/\theta}$, where k is a constant and θ the absolute temperature (*cf.* RAYLEIGH, 'Phil. Mag.' (6), vol. 29, p. 274, 1915; SCHÖNROCK, 'Ann. der Phys.', 20, p. 995, 1906).

For the constant k BUISSON and FABRY, following SCHÖNROCK, adopted the value 1.22×10^6 , and the value of N which they found experimentally was in approximate agreement with the view that the spectrum was to be referred to the atom. This result has recently been criticised by SAHA ('Phil. Mag.' (6), 40, p. 159, 1920) on the ground that BUISSON and FABRY obtained a much smaller value for N in the case of the line $H\alpha$, the first member of the Balmer series, and that if the value of k in the formula given above were calculated from the observed limit of interference for the line $H\alpha$, the secondary line would yield a value of M more nearly appropriate to the molecule H_2 . The ground of this criticism does not appear to us to be justified, for it is well known that the line $H\alpha$ is complex and could not therefore be expected to yield results in accordance with the theory for a single line, and moreover SAHA appears to have overlooked the fact that the value of k adopted by BUISSON and FABRY was tested experimentally with lines of the rare gases, and was found to give results in close agreement with the known atomic weights of these gases.

It must, however, be pointed out that any cause of broadening of the lines other than that due to motion in the line of sight will yield too low a value for the mass of the radiating particles; and it follows that measurements of this kind can only set an inferior limit to the mass, unless the possibility of broadening of the lines by any other cause can be excluded.

In the present investigation we have remeasured the wave-lengths of the lines of the secondary spectrum in International Ångström units, and have been able to add a considerable number of lines to those hitherto recorded. We have also investigated the effect of variations in the conditions of electrical excitation, and of the pressure in the discharge-tube, on the relative intensities of the lines, and have compared the results obtained with other methods of classification; previous investigations (MERTON, *loc. cit.*) of the changes in the spectrum which are produced by the admixture of Helium have been extended to the more refrangible regions. The widths of several lines in the secondary spectrum have been measured by a new method with which it is believed that a high degree of precision has been attained, and under conditions of excitation

which have made it possible to establish the conclusion that the secondary spectrum is to be referred to the Hydrogen molecule.

(2) *Experimental.*

As the source of the spectrum vacuum tubes of the H type have been used, and have been designed on lines indicated by the investigations of WOOD (*loc. cit.*) so as to give the secondary spectrum as strongly as possible. The usual capillary tube of a few inches in length was replaced by tubes of from 20 to 50 cm. in length and of about 5 to 8 mm. internal diameter. The electrodes consisted of spirals of aluminium ribbon, and the tubes were provided with palladium tubes, which were sealed through the intermediary of short platinum tubes into side tubes in the usual manner. Pure Hydrogen could be admitted by heating these palladium tubes in a flame or in a current of Hydrogen, or alternatively the Hydrogen in the tubes could be removed by heating the palladium in an atmosphere from which Hydrogen was absent. In some cases the tubes were cleaned before exhaustion by washing them out with a very dilute solution of hydrofluoric acid, followed by distilled water; this procedure was found to be very effective. The tubes were exhausted by means of an oil pump, and the evacuation was completed by means of a bulb containing charcoal, which was cooled with liquid air. For some of the tubes a Gaede mercury pump was used and in all cases the tubes were washed out during the process of exhaustion by the frequent admission of Hydrogen through the palladium tubes. In the case of tubes containing Helium, this gas was prepared by heating powdered Thorianite in a fused silica tube, and was purified before entering the vacuum tubes by passage through a U-tube containing charcoal cooled with liquid air.

The tubes were excited by the current from a large induction coil provided with a mercury jet interrupter, and in some experiments a 15,000 volt $\frac{1}{4}$ kilowatt step-up transformer was used. It should be mentioned that although the utmost care was taken to remove the carbon compounds and other impurities with which vacuum tubes are liable to be contaminated, before the tubes were sealed off, the highest degree of purity, as shown by the intensity of the secondary spectrum relative to that of the Balmer series, was never attained until a discharge had been passed for several hours and the aluminium mirrors, which were deposited on the tubes around the electrodes, had removed the last traces of impurities which had been present in such small quantities when the tubes were sealed off that they could not be detected by any characteristic bands or lines in the spectrum.

(3) *Wave-length Measurements.*

The earliest tables of wave-length of the secondary spectrum are due to HASSELBERG (*loc. cit.*), and although his measurements were made visually, and are not accurate enough for modern requirements, they are more complete than later photographic

records in the green regions of the spectrum, in which the sensibility of the eye is a maximum and that of most panchromatic plates a minimum. FROST (*loc. cit.*) and AMES (*loc. cit.*) have published short lists of some of the stronger lines, but the most comprehensive and accurate table is due to WATSON (*loc. cit.*), who has recorded most of the lines in the red and yellow, and in the ultra-violet, but has not included a considerable number of lines in the green regions of the spectrum. WATSON found no lines of wave-length greater than the Balmer line $H\alpha$. PORLEZZA (*loc. cit.*) and PORLEZZA and NORZI (*loc. cit.*) have published tables which supplement those of WATSON, and CROZE (*loc. cit.*) has measured lines in the infra-red down to $\lambda 8000\text{A}$.

The measurements included in the present investigation extend from $H\alpha$ to the limits imposed in the ultra-violet by the thin glass wall of the discharge-tube, the shortest wave-length recorded being $\lambda 3375\text{A}$, but the continuous spectrum of Hydrogen could be traced on the plates to wave-lengths shorter than $\lambda 3000\text{A}$, and as there were no indications of lines superposed on the background in this region, we did not resort to the use of vacuum-tubes provided with quartz windows. It is intended ultimately to extend the measurements into the infra-red. We have used an Anderson concave grating ruled with 20,000 lines to the inch, and having a radius of curvature of 120 cm., which gave a dispersion of very nearly 10 A per millimetre. The mounting was of the type described by EAGLE ('Astrophys. Journ.,' 31, p. 120, 1910), which involves three adjustments in focussing, of which two determine the angles made respectively by the grating and by the photographic plate with the incident light, whilst the third is used to vary the distance between the grating and the plate. The necessary adjustments for different regions of the spectrum were found from experimentally prepared tables. Plates of especially thin glass were used and were bent to the appropriate curvature in the plate-holder.

The regions from $\lambda 6560$ to $\lambda 5400$ were photographed on Wratten and Wainwright Panchromatic plates, from $\lambda 5400$ to $\lambda 4860$ on Marion's Iso-Record plates which are specially sensitive to this region, and from $\lambda 4860$ to the ultra-violet on Ilford Ordinary plates. The vacuum tubes were used end-on, the light from the capillary being focussed upon the slit of the spectrograph by means of a quartz lens of about 30 cm. focal length. The exposures required to bring up the faintest lines which were measured were five hours with the Panchromatic plates and three and a half hours with the Iso-Record and Ilford plates. The International Secondary standards were used as a comparison spectrum, the source of light being a Pfund ('Astrophys. Journ.,' 27, p. 296, 1908) arc burning with a current of about 3 amps. at 100 volts. The comparison spectrum was limited by a movable stop in the spectrograph to a narrow strip running across the middle of the Hydrogen spectrum. It was found impossible to ensure the absence of very small shifts between the Hydrogen and the comparison spectra, which were photographed consecutively on the same plate, and to eliminate errors, due to these shifts, from the measurements the following procedure was adopted. A tube containing Helium and Hydrogen was substituted for the tube containing pure Hydrogen and a

series of plates taken throughout the spectrum, the series lines of both elements and also most of the secondary Hydrogen spectrum being recorded. The wave-lengths of the first six members of the Balmer series, and of the Parhelium lines included in the range required, were measured on these plates, together with those of about 100 secondary Hydrogen lines.

The deviations of the series line determinations from the values given by CURTIS ('Roy. Soc. Proc.,' A, 46, p. 147, 1920) for Hydrogen and by MERRILL ('Astrophys. Journ.,' 46, p. 357, 1917) for Helium enabled the shifts in the comparison spectrum to be eliminated, and the corrected values obtained for the selected secondary Hydrogen lines were then used as standards in measuring the remaining lines of the spectrum from plates taken with pure Hydrogen tubes. The plates were measured on a Hilger travelling micrometer, with a screw-pitch of 1 mm. and reading by a vernier on the drum to 0.001 mm. Three plates were measured for each region, each plate, following the usual procedure, being measured in both directions to eliminate personal errors in setting. Each series of readings so obtained was repeated before altering the position of the plate on the stage of the micrometer. The two values rarely differed by more than 0.003 mm., and their mean was adopted in subsequent reduction. In all, twelve settings were made on each line, except in the case of some of the faintest lines which were not visible on all the plates; these exceptions have been noted in the tables given.

The reduction of micrometer readings to wave-lengths was simplified by the fact that the dispersion was almost exactly 10 Å per millimetre. In the region $\lambda 6560$ – $\lambda 4860$ the readings were reduced to approximate wave-lengths by the addition of a constant, and the final adjustment was made from an error curve drawn either by means of the iron standards or the Hydrogen standards prepared from them. In the blue and ultra-violet regions a preliminary linear correction was applied so as to reduce the slope of the error curve. The arithmetic mean of the six values calculated in this way was adopted as the final value. The wave-lengths and wave-numbers *in vacuo* have also been tabulated, the corrections for this purpose being taken from the tables of MEGGERS and PETERS ('Bureau of Standards Publications,' p. 698, 1918).

For all but a few of the weakest lines and a few diffuse lines the probable error of the mean wave-length adopted was less than 0.02 Å, and for most lines it was considerably smaller than this. It is believed that the values given can be relied on to two hundredths of an Ångström unit. The weakest lines were much more sharply defined in the visible regions than in the ultra-violet, where they were superposed on the continuous background, and it is possible that this may slightly affect the accuracy of some of these wave-lengths. Our measurements agree in general very closely with those of WATSON (*loc. cit.*), when the latter are transposed into International Units, but FORLEZZA'S (*loc. cit.*) measurements differ from ours, in extreme cases by almost an Ångström unit.

Several hundred new lines have been recorded, these lines occurring for the most

part in the red and yellow, and particularly in the green regions, in which it would have been impossible to record so many faint lines without the use of the Iso-Record plates. It is believed that lines due to impurities have been excluded, all the photographs used for measurement having been taken with tubes which had been run for some hours, and in which the discharge appeared to be white throughout the capillary.

(4) *The Continuous Spectrum.*

In addition to the Balmer series and the secondary spectrum, vacuum tubes containing Hydrogen always emit a continuous spectrum. The intensity and extent of this continuous background depends greatly on the pressure of the Hydrogen in the discharge tube. If the pressure is reduced to the stage at which the glass begins to show a green fluorescence, the continuous spectrum almost disappears and the secondary lines are sharp against a clear background, even in the ultra-violet. At about 50 mm. pressure the continuous spectrum extends into the green, and at still higher pressures it covers the whole spectrum. The introduction of capacity into the discharge circuit has no very marked effect on the continuous spectrum, but appears to weaken it to some extent, an effect which can also be produced by an admixture of Helium. According to LYMAN ('Spectroscopy of the Extreme Ultra-violet'), the continuous spectrum fills the gap between the end of the secondary spectrum and the Hydrogen lines in the Schumann region, but the intensity distribution in this spectrum has not been studied.

(5) *The Classification of the Lines.*

In the tables of wave-lengths, under the heading "Intensity" will be found the estimated intensities of the lines, on the usual scale of 0 to 10, when the discharge tube containing Hydrogen at a pressure of a few millimetres was excited by an uncondensed discharge. Under these conditions the discharge is at its brightest, being intrinsically weaker both at higher and at lower pressures. In the column succeeding those in which the intensities are given, the effects of changes of pressure and other conditions on the relative intensities of the lines are shown. In all such cases the intensity changes have been estimated by examining a standard plate in juxtaposition with a plate taken under the conditions in question and exposed for a time appropriate to the intrinsic brightness of the source. A + denotes that the line is enhanced and ++ that it is greatly enhanced, - and = denoting in the same way that the line is somewhat or greatly weakened as the case may be. The observations referred to under "High Pressure" were made with tubes containing Hydrogen at pressures greater than 50 mm. of mercury, and under these conditions the discharge was much less luminous and of a bluish-white colour, the spectrum lines being superposed on a rather strong continuous background. The changes which were found in the relative intensities of the lines are shown in the tables under the column "High Pressure," and in the succeeding column are given the changes observed at low pressures. By "Low Pressure" we refer to a

pressure at which the walls of the vacuum tube show a vivid green fluorescence and the intrinsic intensity of the light is greatly reduced.

It has long been known that when powerful condensed discharges are passed through vacuum tubes containing Hydrogen, the secondary spectrum disappears and the lines of the Balmer series alone remain, but we have observed that with a condenser and a rather small spark-gap in the circuit a group of lines extending from $\lambda 6000$ to $\lambda 5600$ are very prominent; these lines are intrinsically weakened by the inclusion of the condenser and spark-gap in the circuit, but they are strong in comparison with the remaining lines of the secondary spectrum. If the length of the spark-gap is adjusted carefully, the effect is very striking if the spectrum is examined with a small direct vision instrument of low dispersion. These changes in intensity are given in the column headed "Condensed Discharge."

Under "Helium Effect" are shown the changes of relative intensity which take place when Helium, at pressures up to 40 mm. and more, is admitted to the discharge tube with the Hydrogen. The phenomena which occur have already been described by one of us (T.R.M.) (*loc. cit.*), but in the present investigation we have extended this classification of the lines into the more refrangible portions of the spectrum by adopting a suitable standard of intensity in the comparison-plates.

We have confirmed the previous observation, in the presence of Helium, of a number of lines which do not appear under ordinary conditions, and the wave-lengths of these lines in the tables are given in brackets.

In the next column are given the results obtained by DUFOUR (*loc. cit.*) in his investigations of the Zeeman effect; all the lines examined by DUFOUR have been marked O or Z in the tables, according to whether they show or do not show the Zeeman effect.

The lines which have been arranged by FULCHER into bands have been noted in the succeeding column, and in the last column the results obtained by TAKAMINE and YOSHIDA (*loc. cit.*), NITTA (*loc. cit.*) and TAKAMINE and KOKUBU (*loc. cit.*) in their investigations of the Stark effect are given.

These different methods of classification are related to one another, but there are numerous exceptions to any broad generalization. In the red and yellow regions all the lines which were found by DUFOUR to show the Zeeman effect are "High Pressure" lines, and most of them are strengthened in the condensed discharge; many of these lines also are enhanced by the presence of Helium, but there are exceptions to this rule.

The Fulcher band lines are essentially "Low Pressure" lines, and are weakened in the condensed discharge.

The exceptions to this rule are as follows:—

$\lambda 6197.05$ high pressure line.

$\lambda 6093.83$ somewhat enhanced in condensed discharge.

$\lambda 5989.22$ high pressure line, enhanced by condensed discharge.

$\lambda 5552.52$ enhanced by condensed discharge.

$\lambda 5543.41$ „ „ „ „

$\lambda 5317.90$ high pressure line.

“ Low Pressure ” lines are weakened by the condensed discharge, which enhances the “ High Pressure ” lines, and the lines which are enhanced by the condensed discharge are also enhanced to a smaller extent by admixture of Helium ; but there are exceptions, and the changes in intensity amongst the lines which do conform to these rules vary greatly in magnitude. We are inclined to the view that many of the exceptions may be explained by the assumption that the exceptional lines are in reality close unresolved doublets.

With regard to the lines showing the Stark effect, this attribute appears to bear no relation to any of the other methods of classification, but attention may be drawn to lines at $\lambda \lambda 4185.4$, 4123.9 , 4021.7 , 3927.3 and 3846.0 , which are described as showing the Stark effect, but which do not appear on any of our plates. If these lines are indeed Hydrogen lines, it is possible that they make their appearance only in the powerful electric fields which are necessary for the investigation of the Stark effect.

(6) *Comparison of the Secondary Hydrogen Spectrum with the Solar Spectrum.*

The presence of lines of the secondary Hydrogen spectrum in celestial spectra has not been established, but in view of the fact that we have separated the secondary spectrum into groups, which vary in intensity under different physical conditions, we have made a careful comparison of a number of secondary lines with ROWLAND'S solar wave-lengths, and with the sun-spot lines recorded by HALE and ADAMS (‘ *Astrophys. Journ.*,’ vol. 23, p. 11, 1906). The most prominent lines of each class amongst the secondary lines were selected and were reduced from International Units to ROWLAND'S scale of wave-lengths. A table showing the comparison would be redundant, but it may be stated that there are very few coincidences within the limits of experimental error, and these coincidences appear to be accidental, for the relative intensities of the lines which might seem to be represented are not in harmony with their intensities on any scheme of classification which has been found. It is therefore probable that the secondary spectrum is not represented in the solar spectrum, though the range of wave-lengths relating to the sun-spot spectrum is not as great as might be desired.

(7) *The Widths of Spectrum Lines.*

In a previous section we have referred to the widths of spectrum lines and to their importance in setting an inferior limit to the molecular weight of the radiating particles. The distribution of intensity in a spectrum line, in the case in which the sole cause of broadening is that due to the motion of the radiating particles in the line of sight, is

given by the expression $I = I_0 e^{-kx^2}$, where I is the intensity at a difference of wave-length x from the maximum, where the intensity is I_0 , and k is a constant depending on the mass and the temperature of the radiating particles. RAYLEIGH (*loc. cit.*) has shown that if the "half-width" of the line $\delta\lambda$ be defined as the value of x when $I/I_0 = 0.5$, then $\delta\lambda/\lambda = 3.57 \times 10^{-7} \sqrt{(\theta/m)}$, where θ is the absolute temperature of the gas and m the mass of the radiating particles in terms of the Hydrogen atom as unity. Measurements of the width of a line are usually carried out by determining the limiting order of interference at which fringes can still be seen with such instruments as the Michelson or Fabry and Perot interferometer, in which the difference of path between successive interfering beams can be progressively increased.

The theory of this method has been fully discussed by RAYLEIGH (*loc. cit.*) and by SCHÖNROCK (*loc. cit.*). If N is the limiting order of interference at which the fringes can still be seen it is shown that $N = K \sqrt{(m/\theta)}$, where K is a constant for which RAYLEIGH gives the value 1.427×10^6 , whilst SCHÖNROCK adopts the appreciably smaller value 1.22×10^6 . The exact value of this constant depends on an estimate of the limiting visibility of fringes which can just be seen, and its value can be checked by observations on lines of the rare gases, following BUISSON and FABRY (*loc. cit.*), where the mass of the radiating particles can be assumed. It is doubtful, however, whether a high degree of accuracy can be attained by this method, since the point at which the fringes cease to be visible is necessarily difficult to determine, and might well be affected by the intrinsic intensity of the light, by the wave-length in visual observations, and by other circumstances.

We have therefore endeavoured to avoid the personal errors which are inherent in these methods by adopting a different procedure, in which the determination of the half-widths does not depend on any estimate of visibility in the ordinary sense of the word, but is calculated from the positions of certain definite points on a photographic plate, which can be measured with a micrometer to an accuracy which is limited only by the ordinary instrumental and personal errors which arise in the measurement of spectrum lines.

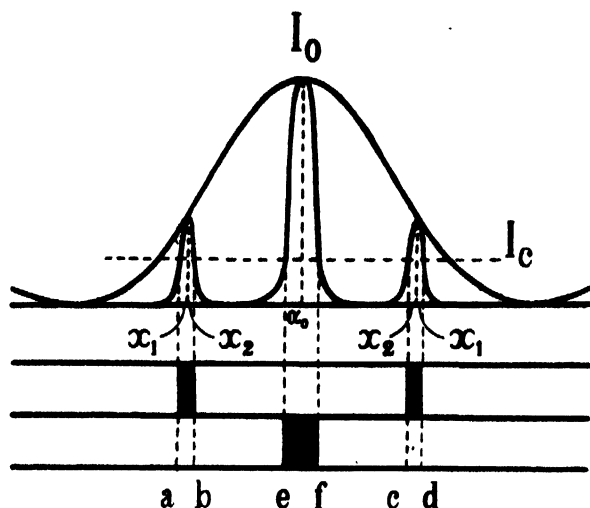
It is of course a first essential that the resolving power of the spectroscope should be sufficiently great, but if this condition be satisfied the method is applicable, with slight modification, to any form of spectroscope. In the present investigation we have used an echelon diffraction grating, consisting of 35 plates of glass each 15 mm. thick, which had a resolving power adequate for the lines in question.

The procedure simply consists in taking photographs of a line under investigation, firstly with the grating in the double-order position and then in the single-order position, with exactly the same times of exposure and without altering the conditions of excitation of the discharge tube. In practice we bracket photographs in the single-order position between photographs in the double-order position, this precaution being taken to provide for the possibility of a gradual change in the luminosity of the discharge tube, and the photographs are taken on adjacent portions of the same plate. The photographs

were taken on Ilford Rapid Process Panchromatic plates, which were developed with a Hydroquinone and Potash developer which gave very great contrast, and by a slight "cutting" of the plates after development with a solution of Potassium Ferricyanide the contrast became so great that the positions of the edges of the black lines could be measured on a photo-measuring micrometer to a few thousandths of a millimetre. The half-widths of the lines were deduced from the measured "apparent widths" in the single- and double-order positions. The theory of the method, as involving the use of the echelon grating, is given in the succeeding section. With other instruments of high resolving power it would be necessary to vary the intensity of the incident light in successive exposures, *e.g.*, by the interposition of a filter of known absorbing power for the radiation in question.

(8) *Theoretical.*

The theory of the method can be seen from the figure in which the upper curve represents the distribution of intensity by the echelon which is given by the equation



$I_n = \sin^2 \alpha / \alpha^2$, where $\alpha = \pi \theta \sigma / \lambda$, σ denoting the step of the grating, λ the wave-length and θ the angle of diffraction.

In the same figure are shown the intensity distribution curve of a spectrum line, as seen in the single-order position, with the two curves showing the line in the double-order position on either side. Although the actual distribution of intensity in the line is given by the equation $I = e^{-kx^2}$, the observed distribution of intensity differs somewhat from this as the distribution of intensity given by the echelon is superposed.

In the two strips below these curves are shown the appearance of the line as seen on the plate, with exposures of equal duration, in the double- and single-order positions respectively. (The absence of disturbance due to irradiation can be seen by an

inspection of the ends of the lines (*cf.* NICHOLSON and MERTON, 'Phil. Trans.,' A, vol. 216, p. 459, 1916.) The edges of the lines at *a*, *b*, *e*, *f*, *c* and *d* denote a certain critical intensity, I_c , which is represented by a dotted line in the upper part of the figure. It will be seen that we assume only that a constant degree of blackening of the plate is produced by a light of constant but entirely unspecified intensity.

Let x_1 and x_2 denote the distances of *a* and *b*, or *c* and *d*, respectively from the true maximum of intensity of the line in the double-order position ($x = 0, \alpha = \pi/2$), and α_0 the distance of *e* and *f* from the maximum in the single-order position ($x = 0, \alpha = 0$). Then expressing α_0 and x in circular measure we have for the single-order position

$$I_c/I_0 = e^{-k\alpha_0^2} \sin^2 \alpha_0/\alpha_0^2, \quad \dots \dots \dots (i.)$$

and for the double-order position

$$I_c/I_0 = e^{-kx_1^2} \sin^2 \alpha_1/\alpha_1^2, \quad I_c/I_0 = e^{-kx_2^2} \sin^2 \alpha_2/\alpha_2^2, \quad \dots (ii.), (iii.)$$

where α_1 and α_2 are the angles corresponding to the points *a* and *b*, or *d* and *c*, respectively.

Putting

$$\sin^2 \alpha_0/\alpha_0^2 = R, \quad \sin^2 \alpha_1/\alpha_1^2 = P, \quad \text{and} \quad \sin^2 \alpha_2/\alpha_2^2 = Q,$$

we have

$$\log (R/P) = k\alpha_0^2 - kx_1^2, \quad \log (R/Q) = k\alpha_0^2 - kx_2^2. \quad \dots (iv.), (v.)$$

Now the plates are measured with a photo-measuring micrometer with which readings of the positions of the points *a*, *b*, *c*, *d*, *e* and *f* are obtained on an arbitrary scale, and since x_1 is not exactly equal to x_2 , the number of micrometer divisions between *a* and *c*, or *b* and *d*, does not correspond to an angle n .

Since it is not possible to measure directly the number of micrometer divisions which are equal to the separation of successive orders, both equations (iv.) and (v.) are required to solve for k .

The measurements give $2\alpha_0$ and $(x_1 + x_2)$ in micrometer divisions, and it is necessary to find a value of $(x_1 - x_2)$ such that equations (iv.) and (v.) give the same value for k , from which the value of $\delta\lambda$, the half-width, at once follows, since the difference in wave-length corresponding to the separation of successive orders is known from the optical constants of the grating. $(x_1 - x_2)$ is very small, and can readily be found by trial of a series of values, which can be plotted against the resulting values of $[k_{(from (iv.))} - k_{(from (v.))}]$ on squared paper.

(9) *Experimental Results.*

We have measured, in the manner described in the preceding section, the half-widths of three lines in the secondary spectrum, $\lambda\lambda$ 6018, 6028 and 6225 A, and also the half-

width of the Helium line $\lambda 5015 \text{ \AA}$ as a check on the accuracy of the method. The choice of lines in the secondary spectrum is limited in the first place by their brightness, and in the second by the possibility of their being isolated completely from neighbouring lines in the spectrum by means of the constant deviation spectroscope which was used for preliminary analysis of the spectrum. It was necessary that the exposures should be comparatively short to avoid disturbances due to the effect of changes of temperature on the echelon grating. An inferior limit to the temperature of the gas in the discharge tube was found in the following manner. The discharge was passed through the vacuum tube for a considerable time, in order to reach a state of equilibrium between the heating of the gas by the discharge and the cooling of the walls of the tube by radiation and convection. The temperature of the walls of the tube was then measured by putting specks of organic substances of known melting point on to the wall of the tube, and observing which of them melted.

The temperature of the outer walls of the tube being thus measured we deduce the temperature of the gas as follows:—It is clear that when thermal equilibrium has been established the temperature of the outer wall of the capillary must lie between that of the radiating gas within the tube and the temperature of the room. An inferior limit to the temperature of the radiating gas is obtained by assuming that the interchange of energy takes place by radiation only, since an undue allowance for the effects of convection currents might lead to too high a value for the temperature of the gas. It is further assumed that the conduction of heat by the glass tube is infinite compared with that of the gas and of the surrounding air. The temperature of the radiating gas in the discharge tube is thus given by $T_{\text{gas}}^1 - T_{\text{glass}}^1 = T_{\text{glass}}^1 - T_{\text{room}}^1$, and it will be noted that the assumptions are such as to lead to an inferior limit for the temperature of the gas. This is an important consideration, for an inferior limit to the theoretical limit to the widths of the lines under the conditions of experiment is required. It is also important to note that any ill-adjustment of the apparatus, resulting in a loss of definition, will give rise to too great a value for the determined half-widths of the lines, and therefore to too small a value for the mass of the radiating particles. It follows that when values for the half-width and the temperature have been determined in the manner described, the masses of the radiating particles must exceed a certain specified amount.

With a discharge of suitable intensity it was found that small specks of cinnamic acid were just melted. The melting point of this substance is 133° C. , and applying the correction in the manner described above, the temperature of the radiating gas is found to be $456^\circ \text{ Absolute.}$ In the following Table are given the results obtained for the three secondary Hydrogen lines and also for the green Parhelium line. In the latter case the temperature of the radiating gas was probably somewhat higher, as the gas was contained in a vacuum tube with a narrow capillary, the walls of which were much thicker than in the case of the Hydrogen tube, but the theoretical half-width has been calculated on the assumption that the temperature was the same in both cases.

	$\delta\lambda$ (found).	$\delta\lambda$ (found) (mean value).	$\delta\lambda$ calculated at 456° Abs. for—	
			Atom.	Molecule.
6018	0·037, 0·036, 0·034, 0·030	0·034	0·046	0·032
6028	0·032, 0·033, 0·032	0·033	0·046	0·032
6225	0·034, 0·033, 0·038, 0·039, 0·032	0·035	0·048	0·034
5015He	0·024, 0·022	0·023	0·019	—

It will be seen that the results are uniformly in agreement with a molecular origin for the secondary lines, the half-widths found being very close to the values calculated for the molecule. In the case of the Helium line the fact that the half-width found is slightly greater than the calculated value is perhaps to be explained in part by the conservative estimate of the temperature in the case of the Helium tube, but it is believed that the limiting theoretical widths of the lines are more nearly attained in the tubes with wide capillaries and in which the current density is consequently lower, than in the tubes with narrow capillaries of the conventional Plucker form. The lines $\lambda\lambda$ 6018 and 6225 are both "Fulcher" lines, are enhanced at low pressures, and are weakened by the condensed discharge. Neither of them shows the Zeeman effect. The line λ 6028 belongs to an entirely different class, being a high-pressure line which shows the Zeeman effect. Since these are the two most important classes of lines it is probable that the whole of the secondary spectrum is due to the Hydrogen molecule.

(10) *The Separation of Gases in Vacuum Tubes.*

In a previous communication (MERTON, 'Roy. Soc. Proc.,' A, vol. 98, p. 255, 1920) an account has been given of a curious effect, which, on further investigation, seems to throw some light on the phenomena observed in vacuum tubes containing Hydrogen. It was found that when a vacuum tube containing Helium at a comparatively high pressure, and also a little Hydrogen, was excited by an uncondensed discharge and was observed through a direct-vision prism, the lines of both Helium and Hydrogen appeared with uniform intensity throughout the capillary. On putting a condenser and a spark gap in the electrical circuit the Hydrogen lines became much weaker in the centre of the capillary, but showed brightly at the two ends. This is in agreement with an observation of CURTIS ('Roy. Soc. Proc.,' A, vol. 89, p. 146, 1914); but it was found that on cutting out the condenser the Hydrogen lines did not immediately reappear with uniform brightness, but gradually extended from bright spots at the ends of the capillary until the intensity became uniform, which took a considerable time to occur, depending

on the total amount of Hydrogen present and the pressure of Helium in the discharge tube.

We have extended these results, and with a vacuum tube containing Helium at a pressure of 56 mm. and a little Hydrogen, in addition to the phenomena described above, it has been observed that on putting in the condenser there is a bright instantaneous flash of the Hydrogen lines throughout the capillary before they appear at the ends of the capillary only.* In addition to the Hydrogen, the proportion of which could be controlled by means of a palladium regulator, the tube showed traces of Mercury, Sulphur, Oxygen, the Ångström Carbon bands and a few other lines due to impurities which have not been completely identified. When the uncondensed discharge was first passed through the tube the Mercury lines were scarcely visible, but they gradually developed, though still very faint and somewhat stronger in the centre of the capillary than at the ends. On putting in the condenser the Mercury lines gradually became brighter, but appeared only in the centre of the capillary. On cutting out the condenser they appeared at once with great brilliance in the centre of the capillary, gradually spreading out towards the ends and at the same time becoming fainter. The Mercury lines behaved in exactly the opposite way to the Hydrogen lines, and it looked as if the effect of the condensed discharge was to collect all the Mercury in the tube to the centre of the capillary. The lines due to Sulphur, Oxygen, &c., behaved in the same manner as the Mercury lines. In Plate 3 (*a*) shows the appearance of the capillary, as photographed in the red and yellow regions of the spectrum while the tube was excited by the condensed discharge; (*b*) shows the appearance immediately after the condenser was cut out, this photograph being obtained by repeatedly putting the condenser in and out, and only exposing the plate immediately after the condenser had been cut out, and in (*c*) the lines are seen uniformly distributed throughout the capillary when the tube was excited by the uncondensed discharge. In (*d*), (*e*) and (*f*) respectively the same phenomena are shown in a more refrangible region, in which the behaviour of the Hydrogen line $H\beta$ and the green Mercury line can be seen. (The Mercury line was too weak for reproduction in (*d*) and (*f*).)

The same phenomena can be observed at lower pressures of Helium in the discharge tube, but the condition of uniform intensity in the capillary after the condenser is cut out is very much less rapidly attained at high pressures.

The possibility of the removal of Hydrogen by absorption by the glass walls of the capillary during the passage of the condensed discharge has been considered; but it is believed that this explanation cannot be upheld, for in this case either it should be possible to reach a steady state in which the phenomena are no longer observed when the tube has been run for some time, or else the whole of the Hydrogen in the tube should rapidly disappear; but there is no evidence of an approach to a steady state, or

* It has also been observed that when the quantity of Hydrogen in the discharge tube is sufficiently great to show the secondary lines, the latter also appear only at the ends of the capillary when the condenser is cut out.

of a rapid disappearance of the gas. In the case of mercury an explanation based on some action of the walls of the capillary is even less satisfactory, as it would be necessary to assume that the glass walls were an almost inexhaustible source of mercury which was rapidly absorbed by the electrodes; for in this case also there is no evidence of any approach to a steady state.

It has not been possible to make any quantitative comparison of the rates at which the Hydrogen lines spread into, and the mercury lines out of, the capillary; but the speed of the Hydrogen lines relative to those of mercury suggests forcibly that the gases actually move in opposite directions, and that the Hydrogen and mercury do not appear at the centre and the ends of the capillary respectively immediately after the condenser is cut out, for the simple reason that they are not there. In the case of lines due to sulphur, &c., there is no doubt that the glass capillary of the vacuum tube may be a source of these impurities, and that their appearance immediately after the condenser is cut out may be due in part to their being liberated from the walls of the capillary by the powerful condensed discharge.* The evidence for a separation of the gases is therefore not clear in such cases. The Angström bands behaved like the Hydrogen lines, but in this case the spectrum is due to a compound which is certainly broken up by the condensed discharge, and which would therefore require some time to reform or accumulate in the capillary when the condenser is cut out.

With this exception all lines due to the heavier elements appear in the centre of the capillary when the condenser is cut out. If a partial separation of the gases takes place it is clear that, whatever the mechanism by which this occurs may be, the degree of separation is not proportional to the total energy flowing through the tube in a given time but must increase rapidly with the current density; for the total energy of the condensed discharge was no greater than that of the uncondensed discharge, and was in fact somewhat smaller. It is, however, to be expected that some separation should be effected by the uncondensed discharge, and in addition to the effect recorded above in the case of the mercury line, we have often noticed that when heavy, uncondensed discharges are passed through Hydrogen tubes containing a little water vapour, the series lines of oxygen appear exclusively in the central portions of the capillary.†

These observations seem to provide an explanation of the greater part of the phenomena described by WOOD (*loc. cit.*), who found that in long Hydrogen tubes which were not absolutely free from impurities, the Balmer series appeared strongly in the central portions of the capillary whilst the secondary spectrum was more strongly developed at

* It has been noticed that when very powerful condensed discharges are employed, "arc" lines of the constituents of the glass walls of the tube appear with great brilliance for a short space of time after the condenser is cut out, and experiments which are now being made seem to show that this may be developed into a convenient method of producing the spectra of many substances.

† The phenomena are evidently of an entirely different character to those recorded by Sir J. J. THOMSON ('Roy. Soc. Proc.', 58, p. 247, 1895). In the latter experiments the discharge was, in the main, unidirectional, and differences were observed in the spectra at the two poles. In our experiments the tubes were excited by high potential alternating discharges, and the spectra at the two electrodes were identical.

the ends. Since the lines of the Balmer series are strongly enhanced relatively to the secondary spectrum by very small traces of impurities, it is evident that the phenomena can be explained by an accumulation of the oxygen or other impurities in the central portions of the discharge tube. When however the total amount of impurity is excessively small, the accumulation of the greater part of it in the centre of the capillary may not be sufficient to weaken the secondary spectrum appreciably, and the capillary thus appears of a uniform white colour throughout its entire length. There are a number of other observations which appear to be related to these effects, but in the absence of any theory we do not venture to discuss them in the present communication.

(11) *On Impurities in Vacuum Tubes.*

It may perhaps be considered remarkable that any profound influence on the spectrum of a gas can be exerted by impurities which are present in such small quantities that their presence cannot be detected by any characteristic lines or bands in the spectrum. The difficulty of eliminating such impurities as those which give rise to the Ångström carbon bands is of course familiar to all who have worked with vacuum tubes, but with the aid of charcoal cooled with liquid air there should be no difficulty in preparing tubes containing Hydrogen or Helium which would show no lines or bands other than those peculiar to these gases. This is indeed the case under the conditions usually obtaining, when the gases are contained in the tubes at pressures of a few millimetres; but it has been found that the difficulties are very much greater when it is desired to obtain vacuum tubes containing Helium at higher pressures up to 60 mm., which show no trace of impurities. The relative intensities of lines and bands due to impurities are enormously enhanced as the pressure increases, and the form in which the impurities appear is also often unusual. The influence of Helium on the secondary spectrum of Hydrogen is by no means unique, and a remarkable instance has been observed in the case of tubes containing Helium at high pressures and a very small trace of some carbon compound. If any considerable quantity of carbon is present the Ångström and Swan bands can be seen, but with a very small trace of carbon the "Comet" bands first observed by FOWLER ('Monthly Notices R.A.S.,' vol. 70, p. 484, 1910) appear quite brightly in the bulbs of the tubes just outside the capillary. This effect was first observed by one of us in an investigation in collaboration with Dr. T. TAKAMINE, to whom we are indebted for a photograph which shows the "Comet" bands almost free from other bands associated with carbon compounds. FOWLER (*loc. cit.*) has found that these bands appear with the greatest relative intensity in tubes in which the pressure is so low (0.01 to 0.005 mm.) that the luminosity of the discharge is very small, and yet we find them here in tubes containing Helium at pressures from 15 to 50 mm. This is only one example of the changes which may occur, and a number of other lines and bands have been observed in the case of other impurities. Further investigation of the phenomena is required, and we do not venture to discuss them in the present communication.

(12) Summary.

(1) A discussion is given of the conditions under which the Balmer series and the secondary spectrum of Hydrogen are produced.

(2) The wave-lengths of about 1,200 lines in the secondary spectrum have been measured.

(3) It has been found possible to classify the lines into different physically related groups under different conditions depending on the pressure of the gas in the discharge tube, the electrical conditions of excitation, and the presence of Helium.

(4) These methods of classifying the lines have been compared with the results obtained by other investigators relating to the Stark and Zeeman effects, and with the regularities observed by FULCHER.

(5) A comparison has been made of the wave-lengths of lines of different classes in the secondary spectrum with the Fraunhofer and Sunspot spectrum. No evidence of the presence of the secondary Hydrogen in the Sun has been obtained.

(6) A new method of measuring the widths of spectrum lines has been developed.

(7) It has been shown that the secondary spectrum is due to the Hydrogen molecule.

(8) Experiments have been made which appear to show that when electrical discharges are passed through vacuum tubes, a partial separation of the gases takes place, and this appears to afford a satisfactory explanation of a number of phenomena which have been observed.

(9) A number of observations relating to the appearance of impurity lines in vacuum tubes are discussed.

We wish to express our thanks to the Department of Scientific and Industrial Research for a grant which has been made to one of us (S. B.) during the course of this investigation.

TABLE of Wave-lengths, &c., in Secondary Hydrogen Spectrum (continued).

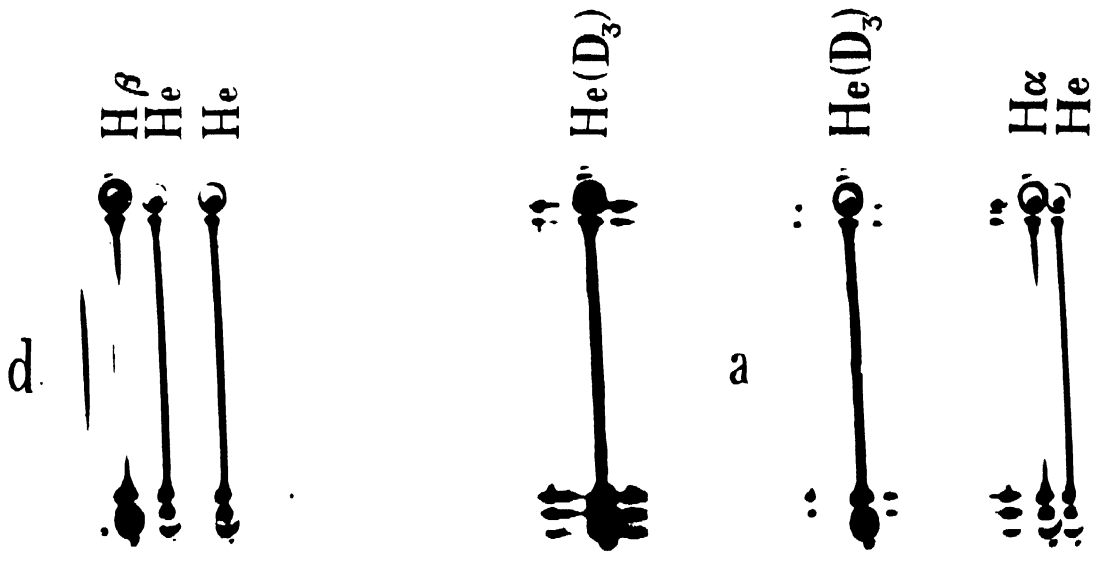
λ (A.)		n.	Intensity.	High Pressure.	Low Pressure.	Condensed Discharge.	Helium Effect.	Zeeman Effect.	Fulcher Lines.	Stark Effect.	λ (A.)		n.	Intensity.	High Pressure.	Low Pressure.	Condensed Discharge.	Helium Effect.	Zeeman Effect.	Fulcher Lines.	Stark Effect.			
In Air.	In Vacuo.										In Air.	In Vacuo.												
6025.76	27.44	16590.79	3			+	+					51.67	84.41	3			+							
23.74	25.90	96.41	7			+	+					50.89	86.69	2			+							
21.28	22.94	16603.19	7	+		+						49.31	91.30	6	+		+							
18.29	19.95	11.43	10	+		+		F				47.29	97.20	0			+							
14.87	16.53	20.88	0			+						43.45	17108.44	0			+							
12.32	13.96	27.92	0			+						42.56	11.04	1			+							
11.38	13.04	30.52	0			+						41.41	14.41	0			+							
10.48	12.14	33.01	3	+		+						39.83	19.03	0			+							
08.77	10.43	37.75	0			+						[39.15	21.04	-			+							
06.91	06.57	42.89	3			+						37.77	25.11	0			+							
04.92	06.58	48.41	2			+		Z				35.98	30.36	9		+	+							
02.81	04.47	54.25	7	+		+						32.89	39.43	5			+							
5998.02	99.68	67.56	2			+						31.00	44.98	3			+							
94.06	96.71	78.59	6			+						[28.78	51.52	-			+							
91.92	93.58	84.52	2			+						27.17	61.00	0			+							
90.51	92.17	86.44	4	+		+						24.53	64.02	2			+							
89.22	90.88	92.04	5	+		+						22.80	69.12	5			+							
[87.30	88.95	97.42	-			+						22.08	71.25	2			+							
82.54	84.19	16710.70	8	+		+						19.31	79.41	2			+							
80.90	82.55	15.38	0			+						17.61	84.43	1			+							
79.11	80.76	20.28	0			+						16.39	88.04	2			+							
78.02	79.67	23.33	0			+						15.00	92.14	6			+							
75.43	77.08	30.58	10			+						12.56	99.36	9			+							
74.12	75.77	34.24	3			+						11.48	17202.55	3			+							
70.96	72.61	43.10	1			+						08.26	12.09	0			+							
70.24	71.89	45.11	3	+		+						07.59	14.07	0			+							
67.31	68.86	53.34	6	+		+						06.09	18.52	5			+							
63.46	65.11	64.15	5	+		+						04.66	22.76	1			+							
59.70	61.35	74.72	8	+		+						03.93	24.96	1			+							
57.26	58.91	81.59	0			+						01.87	31.07	0			+							
56.43	58.07	83.96	3			+						01.14	33.24	3			+							
49.90	51.54	16802.38	9			+						5798.95	800.55	3			+							
47.96	49.60	07.86	0			+						96.59	98.19	1			+							
47.27	48.91	09.80	6			+						94.65	96.25	4			+							
43.57	45.21	20.26	2			+						91.74	61.20	6			+							
43.95	44.59	23.02	1			+						89.85	91.45	1			+							
41.95	43.59	24.86	1			+						88.25	89.85	4			+							
41.23	43.67	26.88	1			+						85.81	87.41	7			+							
38.60	40.24	34.34	9	+		+						85.21	86.81	1			+							

TABLE of Wave-lengths, &c., in Secondary Hydrogen Spectrum (continued).

λ (I.A.)		λ (I.A.)		n. In Vacuo.	Intensity.	High Pressure.	Low Pressure.	Condensed Discharge.	Helium Effect.	Zeeman Effect.	Fulcher Lines.	Stark Effect.
In Air.	In Vacuo.	In Air.	In Vacuo.									
5689.19	90.77	56.97	58.48	20.12	1			+				
84.09	85.66	55.33	56.84	25.62	0			+				
83.05	84.62	52.44	53.95	35.33	0			+				
82.50	84.07	46.70	48.21	54.66	0			+	0			
76.22	77.79	40.17	41.68	76.68	0			+	+			
74.93	76.50	38.82	40.33	81.24	0				+			
[74.04	75.61	37.84.	39.35	84.55	0							
72.73	74.30	34.83	36.34	94.72	0						F	
70.89	72.46	30.86	32.37	18408.17	0			+				
69.36	70.93	[29.97	32.48	07.79	0			+				
67.40	68.97	29.00	30.51	14.48	0			+			F	
62.82	64.39	25.96	27.46	24.83	3				+			
61.72	63.29	25.27	26.77	27.17	0							
*60.83	62.40	23.27	24.77	33.96	0							
*59.50	61.07	22.19	23.69	37.63	0							
57.78	59.35	21.26	22.76	40.80	0							
55.75	57.32	19.90	21.40	45.52	6			+			F	
54.75	56.32	17.80	19.30	52.57	2			+				
52.32	53.88	[16.00	17.50	58.69	0			+				
[47.37	48.93	09.68	11.18	80.26	0			+				
*46.12	47.68	08.79	10.29	83.30	0			+				
42.78	44.34	[07.03	08.53	91.16	0			+				
34.78	36.34	05.38	06.88	94.96	0			+				
*34.31	36.87	01.06	02.56	18509.75	1			+				
32.23	33.79	5399.90	401.40	13.72	0			+				
†30.74	32.30	99.00	400.50	16.80	3			+				
27.45	29.01	98.13	99.63	19.79	1			+				
24.26	26.82	†94.70	96.20	31.56	1			+				
23.04	24.60	92.32	93.82	39.73	3			+				
20.90	22.46	91.22	92.72	43.52	1			+				
16.24	17.80	88.21	89.71	53.88	8			+				
12.55	14.08	86.74	88.24	58.93	0			+				
†04.61	06.16	85.24	87.73	60.70	0			+				
01.66	03.21	85.50	86.99	63.24	3			+				
00.38	01.93	[81.72	83.21	76.26	0			+				
5597.63	99.18	79.80	81.29	83.90	0			+				
91.38	92.93	78.41	79.90	87.71	3			+				
[84.29	85.84	76.55	78.04	94.13	0			+				
83.10	84.65	73.32	74.81	18605.30	2			+				

TABLE of Wave-lengths, &c., in Secondary Hydrogen Spectrum (continued).

λ (L.A.)		λ (L.A.)		Intensity.	High Pressure.	Low Pressure.	Condensed Discharge.	Helium Effect.	Zeeman Effect.	Fulcher Lines.	Stark Effect.
In Air.	In Vacuo.	In Air.	In Vacuo.								
4848.21	49.56	20620.42	62.79	0							
45.52	46.87	31.87	61.39	0							
43.69	45.04	39.66	60.34	0							
43.35	44.70	41.11	57.83	2							
42.45	43.80	44.95	54.01	1							
41.72	43.07	48.06	53.04	0							
38.25	39.60	62.86	50.34	4							
37.02	38.37	68.12	49.47	0							
32.79	34.14	86.20	48.50	3							
31.60	32.96	91.29	45.33	1							
30.69	32.04	95.19	[43.33	1							
24.58	25.92	20721.43	[42.69	2							
22.96	24.30	28.40	[40.47	3							
19.38	20.72	43.79	41.77	0							
17.51	18.85	51.83	39.88	1							
13.61	14.95	68.64	34.66	2							
12.80	14.14	72.14	33.95	0							
11.67	13.01	77.02	31.89	0							
08.75	10.09	89.63	31.44	0							
07.33	08.67	95.77	27.96	0							
04.06	06.40	20809.92	26.94	2							
02.03	03.37	18.72	25.39	0							
4797.76	99.10	37.24	20.72	1							
97.02	98.36	40.45	18.32	4							
95.14	96.48	48.63	17.49	3							
93.95	95.29	53.80	[16.49	0							
92.52	93.86	60.02	14.58	3							
92.15	92.15	67.46	13.10	0							
89.40	90.73	73.65	07.38	1							
86.04	87.37	88.30	05.35	2							
84.85	86.18	93.50	06.64	2							
80.94	82.25	20910.66	99.76	2							
77.42	78.75	25.97	97.22	1							
75.81	77.14	33.02	[93.57	4							
74.29	75.62	39.69	91.84	3							
70.71	72.04	55.39	88.67	0							
69.50	70.83	60.72	85.09	2							
68.10	69.43	66.87	84.48	0							
67.24	68.57	70.65	83.88	1							
			82.58	0							
			81.53	1							
			83.86	4							
			85.16	0							
			86.37	1							
			89.95	0							
			93.12	0							
			94.85	1							
			98.50	1							
			45.22	2							
			40.27	0							
			21707.79	2							
			98.23	3							
			71.34	0							
			55.43	0							
			50.73	3							
			46.86	3							
			35.61	1							
			13.76	4							
			06.53	0							
			28.23	0							
			26.68	4							
			22.01	1							
			19.61	3							
			18.32	1							
			17.78	3							
			15.87	0							
			14.39	0							
			08.67	3							
			06.64	2							
			4598.48	2							
			99.76	2							
			98.50	2							
			94.85	1							
			93.12	1							
			89.95	1							
			86.37	1							
			85.16	0							
			83.88	0							
			82.58	1							
			81.53	4							
			20.67	2							
			15.67	0							
			09.49	0							
			06.63	1							
			21803.73	0							
			86.73	1							
			77.70	1							
			55.43	1							
			45.22	1							
			40.27	2							
			21707.79	0							
			98.23	3							
			71.34	0							
			55.43	3							
			50.73	3							
			46.86	3							
			35.61	1							
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			18.32	1							
			17.78	3							
			15.87	0							
			14.39	0							
			08.67	3							
			06.64	2							
			4598.48	2							
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			98.50	2							
			94.85	1							
			93.12	1							
			89.95	1							
			86.37	1							
			85.16	0							
			83.88	0							
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			09.49	0							
			06.63	1							
			21803.73	0							
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