CONTRIBUTIONS TO THE THEORY OF GAMES

VOLUME II

K. J. ARROW
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D. BLACKWELL
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Edited by H. W. Kuhn and A. W. Tucker

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PREFACE

It has been said of the THEORY OF GAMES AND ECONOMIC BEHAVIOR by John von Neumann and Oskar Morgenstern that "posterity may regard this book as one of the major scientific achievements of the first half of the twentieth century."[1] At the beginning of the second half of the twentieth century the theory of games continues to be an object of vigorous and expanding research, as the papers assembled in the present Study amply demonstrate. The authors of these papers have not been content just to solve outstanding problems and elaborate existing results but have gone on to raise fresh problems and extend the theory in new directions.

This Study is a sequel to CONTRIBUTIONS TO THE THEORY OF GAMES, Volume I (Annals of Mathematics Study No. 24, Princeton, 1950). The continuity of research bridging the two volumes is evidenced by the fact that some of the problems posed in the Preface to Study 24 are now solved in this Study. On the other hand, some reorganization has taken place. The simple division of Study 24 into two parts, dealing with finite and with infinite games, has been replaced by a division of this Study into four parts: finite zero-sum two-person games, infinite zero-sum two-person games, games in extensive form, and general n-person games. (The mounting interest in n-person games and extensive games is quite striking; scarcely touched in Study 24, they now occupy fully half of this Study.) Each of the four parts is prefaced by an introduction that describes the papers therein and their interconnections. At the same time these introductions are designed to indicate how the four parts relate to one another. The newcomer to game theory is referred to the Preface in Study 24 for a general expository introduction to the problems of contemporary research in the field.

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[1] A. H. Copeland, Bulletin of the American Mathematical Society 51 (1945) p. 498.

PREFACE

The Bibliography at the end of this Study supplements the Bibliography in Study 24. Recent publications have been added and some omissions filled in, but the previous listings have not been repeated. Many of the new items, particularly those in economics journals, have been obtained from a bibliography on the theory of games prepared by O. Morgenstern.

The editing and preparing of this Study have been done at Princeton University, in the Department of Mathematics, as part of the work of a Logistics Project sponsored by the Office of Naval Research. Members of the Project who have participated in the task have been C. H. Bernstein, D. B. Gillies, I. Glicksberg, R. C. Lyndon, H. Mills, H. Rogers Jr., R. J. Semple, L. S. Shapley, G. L. Thompson, and the undersigned. Papers for the Study were refereed by members of the Project with the generous assistance of D. Gale, B. R. Gelbaum, G. K. Kalisch, J. Laderman, J. P. Mayberry, J. F. Nash, E. D. Nering, and J. von Neumann. The typing of the master copy has been the painstaking work of Mrs. S. H. Robinson. The good services of the Princeton University Press have been ever available through its Science Editor, H. S. Bailey, Jr. To all these individuals, for their friendly cooperation, we express our sincere thanks.

H. W. Kuhn

December 1952

A. W. Tucker

CONTENTS

Preface v

Part I. FINITE ZERO-SUM TWO-PERSON GAMES

Introduction 1

Paper 1. A Certain Zero-sum Two-person Game Equivalent to the Optimal Assignment Problem 5
By John von Neumann

2. Two Variants of Poker 13
By D. B. Gillies, J. P. Mayberry and J. von Neumann

3. The Double Description Method 51
By T. S. Motzkin, H. Raiffa, G. L. Thompson and R. M. Thrall

4. Solutions of Convex Games as Fixed-points 75
By M. Dresher and S. Karlin

5. Admissible Points of Convex Sets 87
By K. J. Arrow, E. W. Barankin and D. Blackwell

Part II. INFINITE ZERO-SUM TWO-PERSON GAMES

Introduction 93

Paper 6. Games of Timing 97
By Max Shiffman

7. Reduction of Certain Classes of Games to Integral Equations 125
By Samuel Karlin

8. On a Class of Games 159
By Samuel Karlin

9. Notes on Games over the Square 173
By I. Glicksberg and O. Gross

10. On Randomization in Statistical Games with k Terminal Actions 183
By David Blackwell

This paper was translated from German notes of J. von Neumann (made in 1929) and arranged for publication by J. P. Mayberry and D. B. Gillies with the active collaboration of Professor von Neumann.

The "double description method" treated in PAPER 3 is a procedure devised by T. S. Motzkin for calculating the full set of solutions of a system of linear inequalities and developed independently by H. Raiffa, G. L. Thompson and R. M. Thrall as an algorithm for computing the solutions of a zero-sum two-person game. Here the four authors have joined forces to explain their common method. To find the optimal mixed strategies of player I in a game with matrix $A = (a_{ij})$, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$, they seek those points of the $(m - 1)$-dimensional simplex

$$\xi_1 \geq 0, \ \ldots, \ \xi_m \geq 0; \quad \xi_1 + \ldots + \xi_m = 1$$

over which the $(m - 1)$-dimensional hypersurface

$$z = \min_j \left(\Sigma_i \, \xi_i a_{ij}\right)$$

attains its maximum elevation. The vertices (extreme points) of this concave polyhedral surface are effectively calculated by the double description method. The virtues of the method lie in the simplicity of the individual steps, each of which amounts to the calculation of the intersection of a straight line with a hyperplane, and in the fact that all optimal mixed strategies are obtained. In a final section the method is applied to a general system of linear inequalities.

The fact that solutions of zero-sum two-person games can be obtained as fixed points of appropriate mappings has been known since the original existence proof of von Neumann. M. Dresher and S. Karlin utilize this fact in PAPER 4 to construct an algorithm for the solution of any "convex game:" namely, a game determined by a bilinear payoff function $A(r,s)$ of mixed strategies r and s, that range over compact convex sets in Euclidean m-space and n-space, for players I and II, respectively. To such games the authors extend theorems known for finite zero-sum two-person games and polynomial-like games, both of which are readily seen to constitute subclasses of the class of convex games.

In PAPER 5 K. J. Arrow, E. W. Barankin, and D. Blackwell establish the following theorem concerning a closed convex set S in Euclidean n-space. Call a point $s = (s_1, \ldots, s_n)$ of S "admissible" if there is no point $t = (t_1, \ldots, t_n)$ in S that is distinct from s and such that $t_i \leq s_i$ for $i = 1, \ldots, n$, and denote by B the set of all points of S through which pass at least one supporting hyperplane whose normal has all components positive. Then every point of B is

admissible and every admissible point of S is a limit of points in B. It is a simple by-product of this theorem that every pure strategy which achieves the value of a zero-sum two-person game against all opposing optimal mixed strategies appears with positive probability in some optimal mixed strategy.

H. W. K.
A. W. T.

A CERTAIN ZERO-SUM TWO-PERSON GAME EQUIVALENT TO THE OPTIMAL ASSIGNMENT PROBLEM[1]

John von Neumann

The optimal assignment problem is as follows: given n persons and n jobs, and a set of real numbers a_{ij}, each representing the value of the i^{th} person in the j^{th} job, what assignments of persons to jobs will yield maximum total value? A solution can be expressed as a permutation of n objects, or, equivalently, as an $n \times n$ permutation matrix. (Such a matrix can be expressed by $\delta_{iP,j}$, where δ_{ij} is the Kronecker symbol and iP is the image of i under permutation P.) The value of a particular assignment (i.e., permutation) P will be $\sum_i a_{i,iP}$. Without further investigation, a direct solution of the problem appears to require $n!$ steps, -- the testing of each permutation to find the optimal permutations giving the maximum $\sum_i a_{i,iP}$.

We observe that the solution is invariant under the matrix transformation $a_{ij} \longrightarrow a_{ij} + u_i + v_j$, where u_i and v_j are any sets of constants. It is clear that $\sum_i u_i + \sum_j v_j$ will be added to each assignment value, and that thus the order of values, particularly the maxima, will be preserved. This enables us to transform a given assignment problem with possibly negative a_{ij} to an equivalent one with strictly positive a_{ij}, by adding large enough positive u_i and v_j.

We shall now construct a certain related 2-person game and we shall show that the extreme optimal strategies can be expressed in terms of the optimal permutation matrices in the assignment problem. (The game matrix for this game will be $2n \times n^2$. From this it is not difficult to infer how many steps are needed to get significant approximate solutions with the method of G. W. Brown and J. von Neumann. [Cf.: "Contributions to the Theory of Games," Annals of Mathematics Studies, No. 24, Princeton University Press, 1950 -- pp. 73-79, especially § 5.] It turns out that this number is a moderate power of n, i.e., considerably smaller than the "obvious" estimate $n!$ mentioned earlier.)

---

[1]Editors' Note: This is a transcript, prepared under Office of Naval Research sponsorship by Hartley Rogers, Jr., of a seminar talk given by Professor von Neumann at Princeton University, October 26, 1951.

We first construct a simple preliminary game, the 1-dimensional game: We may think of the game as played with a set of n cells or boxes indexed $i = 1, \ldots, n$.

Move 1: Player I 'hides' in a cell.

Move 2: Player II, ignorant of I's choice, attempts to 'find' player I by similarly choosing a cell.

This is a play. The payoff is determined by a set of α_i (positive). If player I is 'found' in cell i, he pays player II the amount α_i; otherwise he pays 0.

What are the optimal strategies for I? Let his strategy be to choose cell i with probability x_i. Then player II will obtain expected payoff $\alpha_i x_i$ by choosing i. Hence he will choose a cell i for which $\alpha_i x_i$ is maximum. The value for him will thus be $\max_i (\alpha_i x_i)$.

Now let $x = (x_i)$ be optimal for I. Assume that an $\alpha_j x_j < \max_i (\alpha_i x_i)$. Choose $\varepsilon > 0$ such that $\alpha_j (x_j + \varepsilon) = \max_i (\alpha_i x_i)$. Define

$$x_i' \begin{cases} = x_j + \varepsilon & \text{for } i = j \\ = x_i & \text{otherwise} \end{cases}.$$

Then $\max_i (\alpha_i x_i') = \max_i (\alpha_i x_i)$, and $\sum_i x_i' = \sum_i x_i + \varepsilon = 1 + \varepsilon$. Hence the

$$x_i'' = \frac{x_i'}{1 + \varepsilon}$$

can be used as probabilities, and

$$\max_i (\alpha_i x_i'') = \frac{\max_i (\alpha_i x_i')}{1 + \varepsilon} < \max_i (\alpha_i x_i),$$

i.e., $x = (x_i)$ was not optimal. Thus necessarily all $\alpha_j x_j = \max_i (\alpha_i x_i)$, i.e., $\alpha_1 x_1 = \ldots = \alpha_n x_n = A$. Now $\sum_i x_i = 1$ implies $A = 1/\sum_i \frac{1}{\alpha_i}$, and, of course, $x_i = \frac{A}{\alpha_i}$. The value of the game (for II) is clearly A.

We now introduce the game in which our particular interest lies. We call it the 2-dimensional game; it is a generalization of the 1-dimensional game as follows:

The cells are doubly indexed from 1 to n. (They may be thought of as fields in an $n \times n$ matrix.)

Move 1: Player I hides as before.

Move 2: Player II now attempts to 'find' I by guessing either of the indices of the cell in which player I has hidden.

He must state which index he is guessing. (I.e., II attempts to pick the row, or the column, of I.) Player I, if so 'found' in cell i,j pays to

II the amount α_{ij}, where the α_{ij} are a given set of positive numbers; otherwise he pays 0.

Thus player I has n^2 pure strategies and player II has $2n$.

We now discuss optimal strategies for player I. Let his mixed strategy be $x = (x_{ij})$, $\sum_{ij} x_{ij} = 1$, where each x_{ij} represents the probability of his hiding in cell i,j. Then player II's pure strategies will give a return of $\sum_j \alpha_{ij} x_{ij}$ for row choice i, or $\sum_i \alpha_{ij} x_{ij}$ for column choice j. As in the 1-dimensional game, he can now simply play pure strategies giving the maximum such return. Player I will try to choose x minimising this return. Thus the value of the game (for II) will be:

$$\operatorname*{Min}_{x} \operatorname*{Max}_{i',j'} \left(\sum_j \alpha_{i'j} x_{i'j} , \sum_i \alpha_{ij'} x_{ij'} \right) .$$

The characterization of I's strategies is not quite as easy as before. The simple direct compensatory adjustment of the 1-dimensional game cannot be made.

For further progress, we obtain certain results on the geometry of convex bodies.

We define:

R = Set of all vectors $z = (z_{ij})$ (in n^2 dimensions), such that

$$z_{ij} \geq 0, \ \sum_j z_{ij} = 1 , \ \sum_i z_{ij} = 1 .$$

S = Set of all vectors $z = (z_{ij})$ (in n^2 dimensions), such that

$$z_{ij} \geq 0, \ \sum_j z_{ij} \leq 1 , \ \sum_i z_{ij} \leq 1 .$$

T = Set of all vectors $z = (z_{ij})$ (in n^2 dimensions), such that $z_{ij} = \delta_{iP,j}$ for some permutation P of the integers $1, \ldots, n$. ($\delta_{iP,j}$ thus consists of the $n \times n$ permutation matrices.)

We prove two lemmas:

LEMMA 1. S = Set of all z such that $z \leq$ some $w \in R$.

PROOF. $S \supseteq$ this set is immediate.

$S \subseteq$ this set is shown as follows: For any $z \in S$ let $N(z)$ be the number of all i with $\sum_j z_{ij} \leqq 1$ plus the number of all j with $\sum_i z_{ij} \leqq 1$. Clearly $N(z) = 0, 1, 2, \ldots$ and R is the set of those $z \in S$ for which $N(z) = 0$.

Now consider a $z \in S$. If $z \notin R$, then there is either $\sum_j z_{ij} < 1$ for some i, or $\sum_j z_{ij} < 1$ for some j. However, all $\sum_j z_{ij} \leqq 1$, all $\sum_i z_{ij} \leqq 1$, and $\sum_i (\sum_j z_{ij}) = \sum_j (\sum_i z_{ij})$. Hence $\sum_j z_{ij} < 1$ for some i implies $\sum_i z_{ij} < 1$ for some j, and conversely. Therefore both $\sum_j z_{ij} < 1$ for some i, say $i = i'$, and $\sum_i z_{ij} < 1$ for some j, say $j = j'$. Choose

$$\varepsilon = 1 - \operatorname{Max}\left(\sum_j x_{i'j} , \sum_i x_{ij'} \right) .$$

Then $\varepsilon > 0$. Define

$$z'_{ij} \left\{ \begin{array}{ll} = z_{i'j'} + \varepsilon & \text{for } i = i' , \ j = j' \\ = z_{ij} & \text{otherwise} \end{array} \right| .$$

Then $z' = (z'_{ij}) \in S$, also always $z_{ij} \leqq z'_{ij}$, i.e., $z \leqq z'$; and either $\sum_j z'_{i'j} = 1$ or $\sum_i z'_{i'j} = 1$. Hence $N(z) > N(z')$.

Iterating this process gives a sequence $z \leqq z' \leqq z'' \leqq \ldots$ in S with $N(z) > N(z') > N(z'') > \ldots$, which therefore must terminate. It can only terminate with a $z^{(m)} \in R$. Hence $w = z^{(m)}$ has all desired properties.

LEMMA 2. R = Convex of T.

(This theorem is due to G. Birkhoff, Rev. Univ. Nac. Tacuman, Series A, Vol. 5 [1946], pp. 147-148. Cf. also G. Birkhoff, "Lattice Theory," Revised Edition, Amer. Math. Soc. Coll. Series, Vol. 25 [1948], example 4* on p. 266. The proof that follows is more direct than Birkhoff's.)

PROOF. R is clearly convex. $R \supseteq T$ is immediate. Hence $R \supseteq$ Convex T.

$R \subseteq$ Convex T is demonstrated, if it is established, that all extreme points of the convex R belong to T. Actually they form precisely the set T.

That every point of T is an extreme point of R is clear: A $z \in T$ belongs to R, and if it were not extreme, then $z = tz' + (1 - t)z''$ with $z', z'' \in R$; $z' \neq z''$; $0 < t < 1$. Choose $z'_{ij} \neq z''_{ij}$, say $z'_{ij} < z''_{ij}$. Then $z_{ij} = tz'_{ij} + (1 - t)z''_{ij}$, hence $z'_{ij} < z_{ij} < z''_{ij}$. Now either $z_{ij} = 0$, implying $z'_{ij} < 0$, or $z_{ij} = 1$, implying $z''_{ij} > 1$ -- and both are impossible.

There remains, therefore, only this: To prove that every extreme point of R belongs to T. This is shown as follows:

For a $z \in R$ call a pair i,j inner, if $z_{ij} \neq 0, 1$. Clearly $z \in T$ (for a $z \in R$) means that all $z_{ij} = 0, 1$, i.e., that no i,j is inner. Hence $z \notin T$ means that inner i,j exist.

If a line i (or a column j) contains at most one inner element z_{ij}, then $z_{ij} = 1 - \sum_{j'} z_{ij'}$ (or $z_{ij} = 1 - \sum_{j'} z_{i'j}$) is necessarily $= 1, 0, -1, -2, \ldots$. Since $z_{ij} \geqq 0$, therefore $z_{ij} = 0, 1$, i.e., i,j is not inner either. In other words: If i,j is inner, then there exists an inner i,j' (i',j) with $j' \neq j$ $(i' \neq i)$.

Now consider a $z \in R$, such that $z \notin T$. Let i,j be inner. Choose $j' \neq j$ such that i,j' is inner, then $i' \neq i$ such that i',j' is inner, then $j'' \neq j'$ such that i',j'' is inner, then $i'' \neq i'$ such that i'',j'' is inner, etc. In this way two sequences $i, i', i'', \ldots$ and $j, j', j'', \ldots$ arise, such that $i^{(m)},j^{(m)}$ is inner, $i^{(m)},j^{(m+1)}$ is inner, and $i^{(m)} \neq i^{(m+1)}$, $j^{(m)} \neq j^{(m+1)}$ (all this for all $m = 0, 1, 2, \ldots$). Hence $i^{(p)} = i^{(q)}$ or $j^{(p)} = j^{(q)}$ must occur sometime for $p \neq q$. Choose such a pair with $p < q$, and with q as small as possible, and with p (for this q) as large as possible. Hence $i^{(p)}, i^{(p+1)}, \ldots, i^{(q-1)}, i^{(q)}$ are pairwise different, also $j^{(p)}, j^{(p+1)}, \ldots, j^{(q-1)}, j^{(q)}$ are pairwise different, with the possible exception of $i^{(p)} = i^{(q)}$ or $j^{(p)} = j^{(q)}$ (or both). For $j^{(p)} = j^{(q)}$ define $i_0 = i^{(p)}$, $j_0 = j^{(p)}$ $(= j^{(q)})$, $i_1 = i^{(p+1)}$, $j_1 = j^{(p+1)}$, $\ldots$, $i_{q-p-1} = i^{(q-1)}$, $j_{q-p-1} = j^{(q-1)}$. For $j^{(p)} \neq j^{(q)}$, hence necessarily $i^{(p)} = i^{(q)}$, define $i_0 = i^{(p)}$ $(= i^{(q)})$, $j_0 = j^{(q)}$, $i_1 = i^{(p+1)}$, $j_1 = j^{(p+1)}$, $\ldots$, $i_{q-p-1} = i^{(q-1)}$, $j_{q-p-1} = j^{(q-1)}$. Thus two sequences $i_0, i_1, \ldots, i_{s-1}$ and $j_0, j_1, \ldots, j_{s-1}$ $(s = q - p$, also define $j_s = j_0)$ arise, with the following properties: $i_0, i_1, \ldots, i_{s-1}$ are pairwise different, also $j_0, j_1, \ldots, j_{s-1}$ are pairwise different, i_t,j_t is inner, i_t,j_{t+1} is is inner (all this for $t = 0, 1, \ldots, s - 1$). I.e., the quantities

$$(1)\quad z_{i_0j_0}, z_{i_1j_1}, \cdots, z_{i_{s-1}j_{s-1}},$$

$$(2)\quad z_{i_0j_1}, z_{i_1j_2}, \cdots, z_{i_{s-1}j_s} \qquad (j_s = j_0)$$

are all $> 0, < 1$.

Now let ε be the minimum of the quantities in the lines (1), (2). Then $\varepsilon > 0$. Define $z' = (z'_{ij})$ and $z'' = (z''_{ij})$ as follows:

$$z'_{ij}(z''_{ij}) \begin{cases} = z_{ij} + \varepsilon\,(z_{ij} - \varepsilon) & \text{for the } i,j \text{ of line (1)} \\ = z_{ij} - \varepsilon\,(z_{ij} + \varepsilon) & \text{for the } i,j \text{ of line (2)} \\ = z_{ij} & \text{otherwise} \end{cases} .$$

Then $z' \in R$ and $z'' \in R$ are readily verified. Also clearly $z' \neq z''$ and $z = \frac{1}{2} z' + \frac{1}{2} z''$.

Hence z is not an extreme point in R, q.e.d. ———

We now return to the 2-dimensional game and a characterization of player I's optimal strategies. Let $x = (x_{ij})$ be an optimal strategy and let A be the value of the game (for player II). We define

$$z_{ij} = \frac{\alpha_{ij} x_{ij}}{A} , \quad z = (z_{ij}) .$$

Clearly all $\sum_j \alpha_{ij} x_{ij} \leqq A$ and all $\sum_i \alpha_{ij} x_{ij} \leqq A$, i.e., all $\sum_j z_{ij} \leqq 1$ and all $\sum_i z_{ij} \leqq 1$. Hence $z = (z_{ij})$ belongs to S.

Now Lemma 1 implies the existence of a $w = (w_{ij})$ belonging to R, with $z \leqq w$. Form

$$w_{ij} = \frac{\alpha_{ij} u_{ij}}{A} , \quad u = (u_{ij}) .$$

Then all $\sum_j \alpha_{ij} u_{ij} = A$ $\sum_j w_{ij} = A$, and all $\sum_i \alpha_{ij} u_{ij} = A$ $\sum_i w_{ij} = A$, hence $\operatorname{Max}_{i',j'} (\sum_j \alpha_{i',j} u_{i',j}, \sum_i \alpha_{ij'} u_{ij'}) = A$. Also $z_{ij} \leqq w_{ij}$, hence $x_{ij} \leqq u_{ij}$. Hence $\sum_{ij} x_{ij} \leqq \sum_{ij} u_{ij}$, $\theta = \sum_{ij} x_{ij} / \sum_{ij} u_{ij} \leqq 1$. Put $v_{ij} = \theta u_{ij}$.

Then $\sum_{ij} v_{ij} = \sum_{ij} x_{ij} = 1$, i.e., the v_{ij} can be used as probabilities, like the x_{ij}. Also $\operatorname{Max}_{i',j'} (\sum_j \alpha_{i',j} v_{i',j}, \sum_i \alpha_{ij'} v_{ij'}) = \theta A$. If $\theta < 1$, then this contradicts the optimality of $x = (x_{ij})$. Hence $\theta = 1$, i.e., $\sum_{ij} x_{ij} = \sum_{ij} u_{ij}$. Since $x_{ij} \leqq u_{ij}$, this implies $x_{ij} = u_{ij}$, i.e., $z_{ij} = w_{ij}$. Hence $z = w \in R$.

Now Lemma 2 implies that z is the center of gravity of certain points of T. I.e.,

$$z = \sum_\nu t_\nu z^\nu , \quad z^\nu \in T , \quad \sum_\nu t_\nu = 1 , \quad t_\nu \geqq 0 .$$

The $t = 0$ may be omitted, i.e., $t > 0$ may be assumed. Form

$$z^\nu_{ij} = \frac{\alpha_{ij} x^\nu_{ij}}{A} , \quad x^\nu = (x^\nu_{ij}) .$$

Then the optimality of x implies that one of all x^ν. Also $z^\nu_{ij} = \delta_{iP^\nu,j}$ for a suitable permutation P^ν. Hence

$$x^\nu_{ij} = \frac{A}{\alpha_{ij}} \delta_{iP^\nu,j} .$$

In other words: All optimal strategies are centers of gravity of optimal strategies of the special form

(*) $$x_{ij} = \frac{A}{\alpha_{ij}} \delta_{i^P,j} \qquad (P \text{ a permutation})$$

Consider, therefore, the strategies of the form (*). For such a strategy all $\sum_j \alpha_{ij}x_{ij} = A$ and all $\sum_i \alpha_{ij}x_{ij} = A$. Hence

$$\operatorname*{Max}_{i',j'} \left(\sum_j \alpha_{i',j}x_{i',j} , \sum_i \alpha_{ij',}x_{ij'} \right) = A .$$

Hence the optimal ones among these strategies are those that give the minimum A.

Now, since the x_{ij} are probabilities, $\sum_{ij} x_{ij} = 1$, i.e., $\sum_i \frac{A}{\alpha_{i,i^P}} = 1$, i.e., $A = 1/\sum_i \frac{1}{\alpha_{i,i^P}}$. Hence the minimum A corresponds to the maximum $\sum_i \frac{1}{\alpha_{i,i^P}}$. I.e., precisely those permutations P give the optimal strategies in question, for which $\sum_i \frac{1}{\alpha_{i,i^P}}$ assumes its maximum value.

To sum up:

THEOREM. The extreme optimal strategies (i.e., those, of which all others are centers of gravity) of the 2-dimensional game are precisely the following ones:

Consider those permutations P which maximize

$$\sum_i \frac{1}{\alpha_{i,i^P}} .$$

For each P of this class form the strategy $x = (x_{ij})$ according to (*) above.———

Note, that this means that player I plays only those cells where the permutation matrix (of P) has a 1. (Here line guesses (i) and column guesses (j) correspond to each other equivalently under the relation $j = i^P$.) His play among these cells is then determined by the 1-dimensional game. ———

The condition expressed in the above Theorem for P is exactly the optimal assignment problem with $a_{ij} = 1/\alpha_{ij}$, where the a_{ij} are the elements of the assignment matrix (which, we saw, could be considered as all positive).

Several further remarks can be made.

1) A transformation of $a_{ij} \longrightarrow a_{ij} + u_i + v_j$ in the assignment matrix leaves the solution unchanged, and hence the game will be invariant (in its P's) under the corresponding

$$\alpha_{ij} \longrightarrow \frac{\alpha_{ij}}{1 + \alpha_{ij}(u_i + v_j)} .$$

That the game should be so invariant is not at all clear initially from the game itself. (Note, this is not complete invariance. The 1-dimensional solutions for a particular P may change, though the P remains the same.)

2) Various extensions of the optimum assignment problem are possible and can be settled in essentially the same manner. Thus one can specify certain many-to-many assignment patterns between persons and jobs, and the like.

In addition, certain formal generalizations of the game are possible -- to various k-dimensional forms with $k = 3, 4, \ldots$. These seem to be interesting, but present serious difficulties.

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TWO VARIANTS OF POKER[1]

D. B. Gillies, J. P. Mayberry, and J. von Neumann

INTRODUCTION

Although the minimax theorem for zero-sum two-person games asserts that there always exist good strategies for such games, their calculation may be a most formidable problem. Methods for simplifying this problem are certainly necessary if the complex games which simulate economic situations are to be attacked. In "The Theory of Games and Economic Behavior" (hereafter designated as [1], in accordance with the bibliography at the end of this paper) several idealizations of actual games of Poker are discussed, complete solutions are given to two of them, and some information is given about the solutions to others. This paper supplements that discussion with the solutions of two of the other variants mentioned there. Part I deals with the discrete case of sections 19.4-19.6 of [1] while Part II completes the discussion of the continuous variant of section 19.13.

PART I

§ 1. DESCRIPTION OF THE GAME

We deal with the game treated in [1], sections 19.4-19.6, pp. 190-196. The players 1 and 2 each obtain by a chance device a "hand," (i.e., an integer from among 1, 2, ..., S); for either player, each of these hands is to have equal probability, independently of the opponent's hand. Then each player, being informed of his own hand but not of his opponent's, elects to bet either the amount a (the high bet) or the amount b (the low bet), where $a > b > 0$. If both bet a [both bet b], the holder of the higher hand (larger integer) receives the amount a [the amount b]

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[1]Editors' Note: This is a supplement to THE THEORY OF GAMES AND ECONOMIC BEHAVIOR, pp. 186-219 (as promised in a footnote on page 196). It was prepared under Office of Naval Research sponsorship by D. Gillies (Part II) and J. P. Mayberry (Part I) from notes of Professor von Neumann, with his advice and collaboration.

from his opponent; (if the hands are equal, no payment is made). If one has bet high and the other low, the latter may choose either to "see," or to "pass." If he chooses to see, payment is made as if both had bet high originally; if he chooses to pass, he must forfeit the sum b, regardless of the hands held.

For any s with $s = 1, \ldots, S$ the player has three strategic choices, described by a numerical index $i_s = 1, 2, 3$; $i_s = 1$ meaning a "high" bid; $i_s = 2$ meaning a "low" bid with subsequent "seeing" (if the occasion arises); $i_s = 3$ meaning a "low" bid with subsequent "passing" (if the occasion arises). Thus the (pure) strategy is a specification of such an index i_s for every $s = 1, \ldots, S$ -- i.e., of the sequence $i_1, \ldots, i_S$.

This applies to both players. Accordingly we shall denote the above strategy by $\Sigma_1(i_1, \ldots, i_S)$ for player 1, and a corresponding one by $\Sigma_2(i_1, \ldots, i_S)$ for player 2.

§ 2. DEFINITIONS

Mixed strategies are introduced as in [1], pp. 192-194. I.e., instead of introducing a separate probability $\xi_{i_1 \ldots i_S}$ [$\eta_{i_1 \ldots i_S}$] for player 1's [player 2's] using the pure strategy $\Sigma_1(i_1 \ldots i_S)$ [$\Sigma_2(i_1 \ldots i_S)$], it suffices to use the probabilities ρ_i^s [σ_i^s] for player 1's [player 2's] choosing $i_s = i$ when his hand is s -- and, of course, doing this for each $s = 1, \ldots, S$. These probability-systems are then subject to the conditions $\rho_i^s \geqq 0$ [$\sigma_i^s \geqq 0$] for all s,i, and $\sum_i \rho_i^s = 1$ [$\sum_i \sigma_i^s = 1$] for all s.

If players 1 and 2 use the mixed strategies $\rho = (\rho_i^s)$ and $\sigma = (\sigma_i^s)$, then the expected payoff of the game is the $K(\rho|\sigma)$ of formula (19:6) in [1], p. 195 (there designated $K(\vec{\rho}^1, \ldots, \vec{\rho}^S|\vec{\sigma}^1, \ldots, \vec{\sigma}^S)$). Clearly (cf. also (19:7) in [1], p. 195)

$$K(\rho|\sigma) \equiv \frac{1}{S} \sum_{s,i} \gamma_i^s \sigma_i^s \tag{2.1}$$

where $\gamma_i^s \equiv \gamma_i^s(\rho)$ is the expected payoff for player 1 of his strategy ρ against player 2's actual hand s and actual choice $i_s = i$. Equations (19:9:a) - (19:9:c) in [1], p.196 -- or, alternatively, a common sense evaluation of the meaning of the γ_i^s -- state

$$\gamma_1^s = \frac{1}{S} \left\{ \sum_{t<s} (-a\rho_1^t - a\rho_2^t - b\rho_3^t) - b\rho_3^s + \sum_{t>s} (a\rho_1^t + a\rho_2^t - b\rho_3^t) \right\}, \tag{2.2.a}$$

$$\gamma_2^s = \frac{1}{S} \left\{ \sum_{t<s} (-a\rho_1^t - b\rho_2^t - b\rho_3^t) + \sum_{t>s} (a\rho_1^t + b\rho_2^t + b\rho_3^t) \right\}, \tag{2.2.b}$$

$$(2.2.c)\quad \gamma_3^s = \frac{1}{S}\left\{\sum_{t<s}(b\rho_1^t - b\rho_2^t - b\rho_3^t) + b\rho_1^s + \sum_{t>s}(b\rho_1^t + b\rho_2^t + b\rho_3^t)\right\}.$$

Since the game is symmetric, a strategy ρ is good if and only if it is optimal against itself (cf. [1], top of p.195), i.e., if

$$(2.3)\qquad K(\rho|\sigma) \text{ assumes its } \sigma\text{-minimum at } \sigma = \rho .$$

In view of (2.1), and of the domain of variability of the σ_1^s, (2.3) can also be stated like this (cf. also (19:A) in [1], p.196):

$$(2.4)\qquad \gamma_i^s > \gamma_j^s \text{ implies } \rho_i^s = 0 \quad (\text{for all } s,i,j) .$$

Because of the symmetry $K(\rho,\rho) = 0$, hence (2.3) means that $\min_{\sigma} K(\rho|\sigma) = 0$. Again, because of $K(\rho|\rho) = 0$, this is equivalent to $K(\rho|\sigma) \geqq 0$ for all σ. Owing to (2.1), this can also be stated like this:

$$(2.5)\qquad \sum_s \gamma_{i_s}^s \geqq 0 \qquad (\text{for all } i_1, \ldots, i_S) .$$

§3. PLAN OF ATTACK

In (4) we show that there exist strategies $\rho = (\rho_i^s)$ where all $\rho_2^s = 0$. We direct our attention to their discovery.

In (5) we establish a characterization for good strategies among the ones to which we are now limited -- where all $\rho_2^s = 0$, i.e., where the choice is restricted to $i_s = 1, 3$.

In (6) we obtain a rather specialized, but still somewhat redundant, normal form for these strategies, in terms of certain parameters P,Q $(= 1, \ldots, S)$ and $\varepsilon(\geqq -1, \leqq 1)$, $\eta(\geqq 0, < 1)$.

In (7), (8) this normal form is discussed and specialized further, until an actual catalogue of all good strategies (as restricted in (4), cf. above) is established in (8).

In (9) the restriction of (4) is reconsidered.

In (10), (11) certain general interpretive conclusions are drawn.

§4. PROOF THAT THERE EXIST GOOD STRATEGIES WITH ALL $\rho_2^s = 0$

Suppose we have a good strategy $\rho = (\rho_i^s)$, in which some $\rho_2^s > 0$. Let s^* be the largest such s, i.e., $\rho_2^{s^*} > 0$ and $\rho_2^s = 0$ for $s > s^*$. By (2.2.a), (2.2.b) with $s = s^*$, there results $\gamma_2^{s^*} > \gamma_1^{s^*}$ if ever $\rho_2^s > 0$ for $s < s^*$ or ever $\rho_3^s > 0$ for $s \geqq s^*$. By (2.4) $\gamma_2^{s^*} > \gamma_1^{s^*}$

implies $\rho_2^{s*} = 0$, which is not the case. Hence necessarily $\rho_2^s = 0$ for $s < s^*$ -- and, since this is the case for $s > s^*$, too (cf. above), therefore $\rho_2^s = 0$ for $s \neq s^*$ -- and $\rho_3^s = 0$ for $s \geqq s^*$. Summarizing:

(4.1) If $\rho_2^s > 0$ occurs at all, then it occurs for precisely one s, say $s = s^*$. In this case $\rho_2^s = 0$ for $s < s^*$; $\rho_3^s = 0$ for $s = s^*$; $\rho_2^s = \rho_3^s = 0$ (hence $\rho_1^s = 1$) for $s > s^*$.

Next define $\bar{\rho} = (\bar{\rho}_i^s)$ as follows:

$$(4.2) \qquad \bar{\rho}_i^s \begin{cases} = \rho_1^{s*} + \rho_2^{s*} & \text{for} \quad s = s^*, \ i = 1, \\ = 0 & \text{for} \quad s = s^*, \ i = 2, \\ = \rho_i^s & \text{otherwise.} \end{cases}$$

(Note, that $\rho_3^{s*} = 0$ [cf. (4.1)] implies, that actually $\bar{\rho}_1^{s*} = 1$, $\bar{\rho}_2^{s*} = \bar{\rho}_3^{s*} = 0$.) Let this substitution take γ_i^s into $\bar{\gamma}_i^s$. Now (2.2.a) - (2.2.c) give $\bar{\gamma}_1^s \geqq \gamma_1^s$ for all s,i, except for $s > s^*$, $i = 2$, and in this case $\bar{\gamma}_2^s \geqq \gamma_1^s$. I.e., for every s and i_s there exists a j_s, such that $\bar{\gamma}_{i_s}^s \geqq \gamma_{j_s}^s$. Hence the validity of (2.5) for ρ (i.e., the γ) implies its validity for $\bar{\rho}$ (i.e., the $\bar{\gamma}$) -- i.e., the goodness of ρ implies that one of $\bar{\rho}$. Summarizing:

(4.3) If ρ is a good strategy according to (4.1) (i.e., with $\rho_2^{s*} > 0$), then the $\bar{\rho}$ of (5.2) (which has all $\bar{\rho}_2^s = 0$) is also a good strategy.

Thus the statement in the title of (4) is correct. Accordingly, throughout (5) - (8), $\rho = (\rho_1^s)$ will be assumed to be a strategy with all $\rho_2^s = 0$, and its goodness will be the subject of the investigation.

§ 5. ALGEBRAIC CRITERIA FOR GOOD STRATEGIES

Consider the criterium of goodness (2.4). Since $\rho_2^s = 0$, it need not be applied for $i = 2$. Consider now $j = 2$. (2.2.a), (2.2.b) show that always $\gamma_1^s \leqq \gamma_2^s$. Hence the case $i = 1$, $j = 2$ need not be considered. The above also implies $\gamma_3^s - \gamma_1^s \geqq \gamma_3^s - \gamma_2^s$. Hence the case $i = 3$, $j = 2$ is disposed of if the case $i = 3$, $j = 1$ is. Thus there remain only the cases with $i, j \neq 2$, i.e., $i = 1$, $j = 3$ and $i = 3$, $j = 1$. These cases state, that $\gamma_1^s > \gamma_3^s$ implies $\rho_1^s = 0$, and that

$\gamma_1^s < \gamma_3^s$ implies $\rho_3^s = 0$, i.e., $\rho_1^s = 1$. Put

$$u^s = \frac{S}{a+b}(\gamma_3^s - \gamma_1^s) \quad (5.1)$$

Then these conditions become

$$\begin{cases} u^s > 0 & \text{implies} \quad \rho_1^s = 1, \\ u^s < 0 & \text{implies} \quad \rho_1^s = 0. \end{cases} \quad (5.2)$$

Thus the goodness of ρ is expressed by (5.2).
Next, (2.2.a), (2.2.c) give

$$u^s = \sum_{t<s} \rho_1^t + \frac{b}{a+b}(\rho_1^s + \rho_3^s) + \sum_{t>s}\left(-\frac{a-b}{a+b}\rho_1^t + \frac{2b}{a+b}\rho_3^t\right).$$

Since $\rho_3^t = 1 - \rho_1^t$, this becomes

$$u^s = \sum_{t<s} \rho_1^t - \sum_{t>s} \rho_1^t + \frac{b}{a+b}(2(S-s)+1). \quad (5.3)$$

From (5.3)

$$u^{s+1} - u^s = \rho_1^s + \rho_1^{s+1} - \frac{2b}{a+b}. \quad (5.4)$$

§ 6. DERIVATION OF THE BASIC STRUCTURAL PROPERTIES

(5.3) yields $u^S > 0$, hence by (5.2) $\rho_1^S = 1$.

If $\rho_1^s = 1$ (for some $s = 1, \ldots, S-1$), then (5.2) gives $u^s \geqq 0$, (5.3) gives $u^{s+1} - u^s > 0$, hence $u^{s+1} > 0$, and so by (6.2) $\rho_1^{s+1} = 1$.

Thus s with $\rho_1^s = 1$ exist (e.g., $s = S$), and if $s = R$ is the smallest one, then $s = R+1, \ldots, S$, too, belong to this class. Hence they are precisely $s = R, R+1, \ldots, S$:

$$\begin{cases} \rho_1^s = 1 \text{ occurs precisely for } s = R, R+1, \ldots, S, \text{ where } R \\ \text{is a suitable number } = 1, \ldots, S. \end{cases} \quad (6.1)$$

For $s = 1, \ldots, R-1$ $\rho_1^s \neq 1$, hence by (5.2) $u^s \leqq 0$. It is therefore of interest to determine those $s = 1, \ldots, R-1$, for which $u^s = 0$. Let $\mathcal{A}$ be the set of these s.

Consider an $s \in \mathcal{A}$. Assume $s-1 \notin \mathcal{A}$ $(s \neq 1)$. Then $u^s = 0$, $u^{s-1} < 0$ (cf. above), hence $\rho_1^{s-1} = 0$ (by 5.2) and $u^s - u^{s-1} > 0$,

hence $\rho_1^{s-1} + \rho_1^s - \frac{2b}{a+b} > 0$ (by (5.4)), i.e., $\rho_1^s > \frac{2b}{a+b}$. Hence $\rho_1^s + \rho_1^{s+1} - \frac{2b}{a+b} > 0$, i.e., $u^{s+1} - u^s > 0$ (by (5.4)), $u^{s+1} > 0$, $\rho^{s+1} = 1$ (by (5.2)). Hence $s + 1 \geqq R$, and since $s \leqq R - 1$, this gives $s = R - 1$. Thus $s \in \mathscr{S}$ implies $s - 1 \in \mathscr{S}$ if $s \neq R - 1$ (and $s \neq 1$).

Now let P be the largest number among $0, 1, \ldots, R - 1$, such that all $s = 1, \ldots, P$ belong to $\mathscr{S}$. Then, according to the above, $\mathscr{S}$ consists of the $s = 1, \ldots, P$, and possibly $s = R - 1$. If $P = R - 1$, then the addendum $s = R - 1$ is not needed; if $P = R - 2$, then the addendum $s = R - 1$ would require $P = R - 1$. I.e., this addendum can only occur for $P \neq R - 2, R - 1$.

Assume $P \neq 0$ and $R \neq 1, 2$. Then $u_1^1 = 0$, $u_1^2 \leqq 0$, hence $u_1^2 - u_1^1 \leqq 0$, i.e., $\rho_1^1 + \rho_1^2 - \frac{2b}{a+b} \leqq 0$ (by (5.4)), hence $\rho_1 \leqq \frac{2b}{a+b}$. Also $\rho_1^1 \geqq 0$, hence $\rho_1^1 = \frac{b}{a+b}(1 + \varepsilon)$ with $-1 \leqq \varepsilon \leqq 1$. Next, let s, $s + 1 = 1, \ldots, P$. Then $u^s = u^{s+1} = 0$, $u^{s+1} - u^s = 0$, i.e., $\rho_1^s + \rho_1^{s+1} - \frac{2b}{a+b} = 0$ (by 5.4)), hence $\rho_1^{s+1} = \frac{2b}{a+b} - \rho_1^s$. Summarizing:

(6.2) If $P \neq 0$ and $R \neq 1, 2$, then there exists an ε with $-1 \leqq \varepsilon \leqq 1$, such that for $s = 1, \ldots, P$

$$\rho_1^s = \frac{b}{a+b}(1 \pm \varepsilon) \quad \text{for } s \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}.$$

Next, assume $P \neq 0$ and $R = 1, 2$. This means $P = 1$, $R = 2$. If $\rho_1^s \leqq \frac{2b}{a+b}$, then $\rho_1^s \geqq 0$ gives $\rho_1^s = \frac{b}{a+b}(1 + \varepsilon)$ with $-1 \leqq \varepsilon \leqq 1$, i.e., (6.2) still applies. If $\rho_1^s > \frac{2b}{a+b}$, we redefine P : $P = 0$, and view $s = 1 = R - 1$ as the addendum referred to above. Thus this addendum, $s = R - 1$, has at any rate $\rho_1^s > \frac{2b}{a+b}$. Also $\rho_1^s < 1$, hence $\rho_1^s = 1 - \frac{a-b}{a+b}\eta$ with $0 < \eta < 1$.

Finally, for $P = 0$ (6.2) is vacuous.

Summarizing:

(6.3) For $s = 1, \ldots, R - 1$ necessarily $u^s \leqq 0$. Let $\mathscr{S}$ be the set of all $s = 1, \ldots, R - 1$ with $u^s = 0$. The elements of $\mathscr{S}$ can be represented in this way:

(a) All $s = 1, \ldots, P$.

(b) Possibly $s = R - 1$.

P is a certain number $= 0, 1, \ldots, R - 1$. If (b) occurs, then $P \neq R - 2, R - 1$, except that $P = 1$, $R = 2$ is not excluded. The ρ_1^s, for $s \in \mathscr{S}$ are represented as follows:

(a) There exists a fixed ε with $-1 \leqq \varepsilon \leqq 1$, such that for $s = 1, \ldots, P$

(6.3)
$$\begin{cases} \rho_1^s = \frac{b}{a+b}(1 \pm \varepsilon) \quad \text{for } s \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}. \\ \text{(b) There exists a fixed } \eta \text{ with } 0 < \eta < 1, \text{ such that for } s = R - 1 \text{ (if the case (b) arises)} \\ \qquad \rho_1^s = 1 - \frac{a-b}{a+b}\eta . \end{cases}$$

The case (b) can be made to arise with certainty, by doing this: If case (b) does not arise, replace R by $R + 1$, and say that (b) has arisen with $\eta = 0$.

The set of all $s = 1, \ldots, R - 1$ not in $\mathfrak{s}$ consists now precisely of the $s = P + 1, \ldots, R - 2$. Put $Q = R - P - 2 = 0, 1, \ldots,$ then these are the $s = P + 1, \ldots, P + Q$. For these (5.2) gives (this was used earlier) $\rho_1^s = 0$. I.e.:

(6.4) For $s = P + 1, \ldots, P + Q$ always $\rho_1^s = 0$.

Combining (6.1), the revised form of (6.3), the definition of Q, and (6.4) we can summarize as follows:

(6.5)
$$\begin{cases} \text{The basic parameters of a solution are these: } P, Q, \varepsilon, \eta. \text{ They are subject to these conditions:} \\ P, Q = 0, 1, 2, \ldots ; S \geqq P + Q + 1 ; \\ -1 \leqq \varepsilon \leqq 1 \quad ; 0 \leqq \eta < 1 . \\ P \neq 0, Q = 0 \text{ imply } \eta = 0 . \\ \text{They determine the } \rho_1^s \ (s = 1, \ldots, S) \text{ in this way:} \\ \rho_1^s \begin{cases} = \frac{b}{a+b}(1 \pm \varepsilon) & \text{for } s \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}, \text{ and } s = 1, \ldots, P , \\ = 0 & \text{for } s = P + 1, \ldots, P + Q , \\ = 1 - \frac{a-b}{a+b}\eta & \text{for } s = P + Q + 1 , \\ = 1 & \text{for } s = P + Q + 2, \ldots, S . \end{cases} \end{cases}$$

Note, that $\mathfrak{s}$ now consists of the $s = 1, \ldots, P$, and, if $\eta \neq 0$, also $s = P + Q + 1$.

§ 7. ALGEBRAIC CRITERIA FOR GOOD STRATEGIES (CONTINUED)

We can now determine all good strategies (in the sense of the remark at the end of (4)), by applying (5.2) to (6.5). We will also check that $u_1^s = 0$ for all $s \in \mathscr{S}$, in the sense of the definition of $\mathscr{S}$.

These conditions must hold for all $s = 1, \ldots, S$. We deal with these in three segments:

$s = 1, \ldots, P$: These belong to $\mathscr{S}$, hence we must require $u^1 = \ldots = u^P = 0$. This satisfies (5.2). By (5.4) $u^{s+1} - u^s = 0$ for $s = 1, \ldots, P - 1$, i.e., $u^1 = \ldots = u^P$ automatically. Hence the only condition that we need to state is

$$u^1 = 0 \qquad \text{if} \qquad P \neq 0 \,. \tag{7.1}$$

$s = P + 1, \ldots, P + Q$: (5.2) requires $u^s \leqq 0$ for $s = P + 1, \ldots, P + Q$. By (5.4) $u^{s+1} - u^s \leqq 0$ for $s = P$ (if $P \neq 0$), $P + 1, \ldots, P + Q - 1$, hence $u^P \geqq u^{P+1}$ (if $P \neq 0$) and $u^{P+1} \geqq \ldots \geqq u^{P+Q}$. The above requirement is therefore vacuous, if $Q = 0$; it follows from $u^P = 0$, which is a consequence of (7.1) (and of $u^1 = \ldots = u^P$, cf. above), if $P \neq 0$, $Q \neq 0$; it would follow from $u^{P+1} \leqq 0$, i.e., $u^1 \leqq 0$, if $P = 0$, $Q \neq 0$. Hence the only condition that we need to state is

$$u^1 \leqq 0 \qquad \text{if} \qquad P = 0 \,,\ Q \neq 0 \,. \tag{7.2}$$

$s = P + Q + 1, \ldots, S$: $s = P + Q + 1$ belongs to $\mathscr{S}$ if $\eta \neq 0$, hence we must require $u^s = 0$ in this case. This satisfies (5.2). If $\eta = 0$, then (5.2) requires $u^s \geqq 0$ in this case. I.e.,

$$u^{P+Q+1} \geqq 0 \text{ with } = \text{ if } \eta \neq 0 \,. \tag{7.3}$$

(5.2) requires $u^s \geqq 0$ for $s = P + Q + 2, \ldots, S$. By (5.4) $u^{s+1} - u^s \geqq 0$ for $s = P + Q + 1, \ldots, S - 1$, i.e., $u^{P+Q+1} \leqq \ldots \leqq u^S$. Hence the above condition is automatically satisfied owing to (7.3).

Thus (7.1) - (7.3) are a complete characterization for good strategies.

Now (5.3) gives for $P \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}$:

$$\begin{aligned} u^1 &= -\frac{b}{a+b}\,(P - 1 - \{{}^0_\xi\}) - 1 + \frac{a-b}{a+b}\,\eta \\ &\quad - (S - P - Q - 1) + \frac{b}{a+b}\,(2S - 1) \\ &= -\frac{a-b}{a+b}\,S + Q + \frac{a}{a+b}\,P + \frac{b}{a+b}\,\{{}^0_\xi\} + \frac{a-b}{a+b}\,\eta \,, \end{aligned}$$

with these additional observations: For $P = Q = 0$ this case is immaterial. For $P = 0$, $Q \neq 0$, the $-\frac{b}{a+b}(P - 1 - \{\frac{0}{\varepsilon}\})$ -term in the first form of the right hand side above should be omitted. This is best achieved by putting $\varepsilon = -1$ in this case.

Next, (5.3) gives for $P \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}$:

$$u^{P+Q+1} = \frac{b}{a+b}(P + \{\frac{\varepsilon}{0}\}) - (S - P - Q - 1) + \frac{b}{a+b}(2(S - P - Q - 1) + 1)$$

$$= -\frac{a-b}{a+b}S + \frac{a-b}{a+b}Q + \frac{a}{a+b}P + \frac{b}{a+b}\{\frac{\varepsilon}{0}\} + \frac{a}{a+b}.$$

Hence (7.1), (7.2) can be stated like this:

(7.4)
$$\begin{cases} P \leqq \frac{a-b}{a}S - \frac{a+b}{a}Q - \frac{b}{a}\{\frac{0}{\varepsilon}\} - \frac{a-b}{a}\eta \\ \text{for } P \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}; \text{ with } = \text{ if } P \neq 0; \text{ with } \varepsilon = -1 \text{ if } P = 0, \\ Q \neq 0; \text{ with the condition omitted if } P = Q = 0. \end{cases}$$

(7.3) can be stated like this:

(7.5)
$$\begin{cases} P \geqq \frac{a-b}{a}S - \frac{a-b}{a}Q - \frac{b}{a}\{\frac{\varepsilon}{0}\} - 1 \\ \text{for } P \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}; \text{ with } = \text{ if } \eta \neq 0. \end{cases}$$

§ 8. CATALOGUE OF GOOD STRATEGIES

(7.4), (7.5) characterize the good strategies. It is convenient to restate them with the aid of the quantity

(8.1) $$L = P + Q$$

in place of P. The statements of (6.5) concerning P,Q now become

(8.2) $$L = 0, 1, \ldots, S - 1; \quad Q = 0, 1, \ldots, L.$$

(7.4), (7.5) become:

(8.3)
$$\begin{cases} L \leqq \frac{a-b}{a}S - \frac{b}{a}(Q + \{\frac{0}{\varepsilon}\}) - \frac{a-b}{a}\eta \\ \text{for } L - Q \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}; \text{ with } = \text{ if } L \neq Q; \text{ with } \varepsilon = -1 \text{ if} \\ L = Q \neq 0; \text{ with the condition omitted if } L = Q = 0. \end{cases}$$

$$(8.4)\qquad \begin{cases} L \geqq \frac{a-b}{a} S + \frac{b}{a} (Q - (\delta)) - 1 \\ \text{for } L - Q \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} ; \text{ with } = \text{ if } \eta \neq 0 . \end{cases}$$

It is now convenient to distinguish four cases:

I: $L = 0$ -- i.e., $P = Q = 0$:
(8.3) is omitted, (8.4) becomes $0 \geqq \frac{a-b}{a} S - 1$, i.e.,

$$(8.5)\qquad S \leqq \frac{a}{a-b} .$$

Thus (8.5) characterizes this case.

II: $L = Q \neq 0$ -- i.e., $P = 0$, $Q \neq 0$:
(8.3) becomes

$$L \leqq \frac{a-b}{a} S - \frac{b}{a} (L - 1) - \frac{a-b}{a} \eta ,$$

i.e.

$$L \leqq \frac{a-b}{a+b} S + \frac{b}{a+b} - \frac{a-b}{a+b} \eta .$$

(8.4) becomes

$$L \geqq \frac{a-b}{a} S + \frac{b}{a} L - 1 ,$$

i.e.

$$L \geqq S - \frac{a}{a-b} ,$$

with $=$ if $\eta \neq 0$. Hence necessarily

$$(8.6)\qquad \begin{cases} L \leqq \frac{a-b}{a+b} S + \frac{b}{a+b} , \\ L \geqq S - \frac{a}{a-b} , \end{cases}$$

and if these inequalities are fulfilled, then the original ones are, too, with $\eta = 0$.

Thus (8.6) (for an $L = 1, 2, \ldots$) characterizes this case.

Note, that if an L according to (8.6) exists at all, it is unique. Indeed, the opposite would require that the length of the interval of (8.6) be $\geqq 1$. This means

$$\frac{a-b}{a+b} S + \frac{b}{a+b} \geqq (S - \frac{a}{a-b}) + 1 ,$$

which gives $S \leqq \frac{a}{a-b}$. Then the first relation of (8.6) gives $L \leqq 1$,

hence no solution other than $L = 1$ exists.

Note further, that (8.6) necessitates

$$\frac{a-b}{a+b} S + \frac{b}{a+b} \begin{cases} \geqq S - \frac{a}{a-b} , \\ \geqq 1 , \end{cases}$$

i.e.

$$\frac{a}{a-b} \leqq S \leqq \frac{a^2 + 2ab - b^2}{2b(a-b)} . \tag{8.7}$$

III: $L \neq Q = 0$ -- i.e., $P \neq 0$, $Q = 0$:

(8.3) becomes:

$$L = \frac{a-b}{a} S - \frac{b}{a} \{{}^{0}_{\xi}\} - \frac{a-b}{a} \eta ,$$

for $L \left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}$. (8.4) becomes

$$L \geqq \frac{a-b}{a} S - \frac{b}{a} \{{}^{\xi}_{0}\} - 1 ,$$

for $L \left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}$, with $=$ for $\eta \neq 0$.

The proviso made in (6.5) refers to this case and implies $\eta = 0$. Consequently we have the conditions

$$L = \frac{a-b}{a} S - \frac{b}{a} \{{}^{0}_{\xi}\} ,$$

$$L \geqq \frac{a-b}{a} S - \frac{b}{a} \{{}^{\xi}_{0}\} - 1 ,$$

for $L \left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}$. For an odd L the first condition becomes

$$L = \frac{a-b}{a} S , \tag{8.8}$$

while the second one is then automatically satisfied. For an even L the first condition becomes $L = \frac{a-b}{a} S - \frac{b}{a} \xi$, i.e.

$$\frac{a-b}{a} S - \frac{b}{a} \leqq L \leqq \frac{a-b}{a} S + \frac{b}{a} , \tag{8.9}$$

and the second one is again automatically satisfied.

Thus (8.8) (for an $L = 1, 3, \ldots$) or (8.9) (for an $L = 2, 4, \ldots$) characterizes this case.

Note, that if an L according to (8.8) or (8.9) exists at all, it is unique, because of $\frac{b}{a} < 1$: If (8.8) gives an odd L, (8.9) cannot

contain an even L, and (8.9) cannot contain two different even L.

IV: $L \neq Q \neq 0$ -- i.e., $P \neq 0$, $Q \neq 0$:

Since $Q = 1, 2, \ldots,$ therefore $Q + \{^0_\xi\}$ and $Q - \{^\xi_0\} \geqq 0$. Hence (8.3) implies $L \leqq \frac{a-b}{a} S$ and (9.4) implies $L \geqq \frac{a-b}{a} S - 1$. Hence

$$L = \frac{a-b}{a} S - K , \tag{8.10}$$

where

$$\frac{a-b}{a} S - K = 2, 3, \ldots ; \quad 0 \leqq K \leqq 1 . \tag{8.11}$$

Clearly (8.11) can be satisfied by a suitable K if and only if $\frac{a-b}{a} S \geqq 2$, i.e.

$$S \geqq \frac{2a}{a-b} . \tag{8.12}$$

Also, the K of (8.11) is unique, unless $\frac{a-b}{a} S = 3, 4, \ldots,$ in which case $K = 0, 1$ will do. In this case, however, (8.8) (with $L = 3, 5, \ldots$) or (8.9) (with $L = 2, 4, \ldots$) can be fulfilled. Hence we can ignore it.

Now (8.3) becomes

$$\begin{cases} K = \frac{b}{a} (Q + \{^0_\xi\}) + \frac{a-b}{a} \eta \\ \text{for } L - Q \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} . \end{cases} \tag{8.13}$$

(8.4) becomes

$$\begin{cases} K \leqq - \frac{b}{a} (Q - \{^\xi_0\}) + 1 \\ \text{for } L - Q \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} ; \text{ with } = \text{ if } \eta \neq 0 . \end{cases} \tag{8.14}$$

It is best to discuss (8.13), (8.14) separately for $\eta = 0$ and for $\eta \neq 0$.

$\eta = 0$: (8.13) becomes

$$Q + \{^0_\xi\} = \frac{a}{b} K \qquad \text{for } L - Q \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} . \tag{8.15}$$

(8.14) becomes

$$Q - \{^\xi_0\} \leqq \frac{a}{b} (1 - K) \qquad \text{for } L - Q \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} . \tag{8.16}$$

$\eta \neq 0$: Here $0 < \eta < 1$, and in view of this (8.13) becomes

$$(8.17)\qquad Q + \begin{Bmatrix} 0 \\ \varepsilon \end{Bmatrix} \begin{cases} < \frac{a}{b} K \\ > \frac{a}{b} K - \frac{a-b}{b} \end{cases} \text{ for } L - Q \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} .$$

(8.14) becomes

$$(8.18)\qquad Q - \begin{Bmatrix} \varepsilon \\ 0 \end{Bmatrix} = \frac{a}{b} (1 - K) \qquad \text{for } L - Q \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} .$$

The second relation in (8.17) can be transformed by adding (8.18) to it. This gives

$$2Q \mp \varepsilon > 1 .$$

Since $Q = 1, 2, \ldots,$ this is always true, except when $Q = 1$, $\varepsilon = \pm 1$ for $L \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$. I.e., the second relation in (8.17) can be replaced by forbidding this.

Hence all these cases (expressed by (8.15), (8.16) for $\eta = 0$ and by (8.17), (8.18) for $\eta \neq 0$) can be summarized as follows:

$$(8.19)\qquad \begin{cases} \text{Either } Q + \begin{Bmatrix} 0 \\ \varepsilon \end{Bmatrix} = \frac{a}{b} K , \quad Q - \begin{Bmatrix} \varepsilon \\ 0 \end{Bmatrix} \leqq \frac{a}{b} (1 - K) , \\ \text{or} \qquad Q + \begin{Bmatrix} 0 \\ \varepsilon \end{Bmatrix} < \frac{a}{b} K , \quad Q - \begin{Bmatrix} \varepsilon \\ 0 \end{Bmatrix} = \frac{a}{b} (1 - K) , \\ \text{for } L - Q \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} . \text{ For the second alternative, } Q = 1, \\ \varepsilon = \pm 1 \text{ for } L \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix} \text{ is excluded.} \end{cases}$$

The first alternative, with $L - Q$ odd, requires that $\frac{a}{b} K$ be an integer, and of opposite parity to L. Then $Q = \frac{a}{b} K$, and $Q - \varepsilon \leqq \frac{a}{b} (1 - K)$ in addition. This is possible if and only if

$$Q - 1 \leqq \frac{a}{b} (1 - K) , \quad \frac{a}{b} K - 1 \leqq \frac{a}{b} (1 - K) ,$$

$$\frac{a}{b} K \leqq \frac{1}{2} \left(\frac{a}{b} + 1\right) = \frac{a+b}{2b} , \quad K \leqq \frac{a+b}{2a} .$$

I.e.: This occurs when

(8.20) $\frac{a}{b} K$ is an integer, of opposite parity to L, and $K \leqq \frac{a+b}{2a}$.

The second alternative, with $L - Q$ even, requires that $\frac{a}{b} (1 - K)$ be an integer, and of the same parity as L. The exclusion made there is irrelevant, since the conditions that apply to this case cannot

force a choice $\varepsilon = -1$. Then $Q = \frac{a}{b}(1 - K)$, and $Q - \varepsilon < \frac{a}{b} K$ in addition. This is possible if and only if

$$Q - 1 < \frac{a}{b} K , \quad \frac{a}{b}(1 - K) - 1 < \frac{a}{b} K ,$$

$$\frac{a}{b} K > \frac{1}{2}\left(\frac{a}{b} - 1\right) = \frac{a-b}{2b} , \quad K > \frac{a-b}{2a} .$$

I.e.: This occurs when

$$(8.21) \quad \begin{cases} \frac{a}{b}(1 - K) \text{ is an integer, of the same parity as } L, \text{ and} \\ K > \frac{a-b}{2a} . \end{cases}$$

Assume, finally, that both of these cases do not take place, i.e., that the first alternative holds with $L - Q$ even, or the second alternative with $L - Q$ odd. The excluded case $Q = 1$, $\varepsilon = 1$ can only be forced by the second alternative in (8.19) when $K = 1$. This possibility has the same consequences as the one considered immediately after (8.12). Hence the exclusion of $Q = 1$, $\varepsilon = 1$ can be disregarded.

In view of these remarks, (8.19) now becomes this:

$$(8.22) \quad \begin{cases} \text{Either } Q + \varepsilon = \frac{a}{b} K , Q \text{ of the same parity as } L, \text{ and} \\ Q \leqq \frac{a}{b}(1 - K), \text{ or } Q - \varepsilon = \frac{a}{b}(1 - K) , Q \text{ of opposite} \\ \text{parity to } L, \text{ and } Q < \frac{a}{b} K . \end{cases}$$

The first alternative means, that the smallest integer of the same parity as L, which is $\geqq \frac{a}{b} K - 1$, is also $\leqq \frac{a}{b}(1 - K)$. I.e.:

$$(8.23) \quad \begin{cases} \text{There exists an integer } Q \text{ of the same parity as } L, \text{ which is} \\ \geqq \frac{a}{b} K - 1 \text{ and } \leqq \frac{a}{b}(1 - K) . \end{cases}$$

The second alternative means, that the smallest integer of opposite parity to L, which is $\geqq \frac{a}{b}(1 - K) - 1$, is also $< \frac{a}{b} K$. I.e.:

$$(8.24) \quad \begin{cases} \text{There exists an integer } Q \text{ of opposite parity to } L, \text{ which is} \\ < \frac{a}{b} K \text{ and } \geqq \frac{a}{b}(1 - K) - 1 . \end{cases}$$

Adding 1 to the integer referred to in (8.24), this assumes the following, equivalent, form:

(8.25) $\left\{\begin{array}{l}\text{There exists an integer } Q+1 \text{ of the same parity as } L, \text{ which}\\ \text{is } < \frac{a}{b}K+1 \text{ and } \geqq \frac{a}{b}(1-K)\ .\end{array}\right.$

Now let Q^* be the unique integer of the parity of L, which is $\geqq \frac{a}{b}K-1$ and $< \frac{a}{b}K+1$. Then (23) is solved by Q if and only if this is $= Q^*, Q^*+2, Q^*+4, \ldots$ and $\leqq \frac{a}{b}(1-K)$; and (25) is solved by $Q+1$ if and only if this is $= Q^*, Q^*-2, Q^*-4, \ldots$ and $\geqq \frac{a}{b}(1-K)$. The former is possible when $Q^* \leqq \frac{a}{b}(1-K)$; the latter is possible when $Q^* \geqq \frac{a}{b}(1-K)$.

One more thing remains to be determined: Whether the Q according to (8.19) fulfills the requirements of (8.2) and $Q \neq L$, i.e., whether $Q < L$. If this were not the case, there would be $Q \geqq L$ in (8.19). Now by (8.4)

$$L \geqq \frac{a-b}{a}S + \frac{b}{a}(Q-1) - 1\ ,$$

hence $Q \geqq L$ gives

$$Q \geqq \frac{a-b}{a}S + \frac{b}{a}(Q-1) - 1\ ,$$

and so

$$S \leqq Q + \frac{a+b}{a-b}\ .$$

Adding the two relations of either alternative of (8.19) together, gives

$$2Q + \{^{-}_{\varepsilon}^{\xi}\} \leqq \frac{a}{b}\ ,$$

i.e.

$$2Q - 1 \leqq \frac{a}{b}\ ,\quad Q \leqq \frac{a+b}{2b}\ .$$

Hence

$$S \leqq \frac{a+b}{2b} + \frac{a+b}{a-b} = \frac{(a+b)^2}{2b(a-b)}\ .$$

In other words:

(8.26) $$S > \frac{(a+b)^2}{2b(a-b)}$$

suffices to guarantee $Q < L$ -- i.e., the requirements of (8.2) and $Q \neq L$.

Note, that (8.26) implies (8.12).

§ 9. REMARKS CONCERNING GOOD STRATEGIES WHERE $\rho_2^S \neq 0$ OCCURS

Consider a good strategy $\rho = (\rho_1^S)$ where $\rho_2^S \neq 0$ occurs. Apply (4.1) - (4.3), and form, in particular, the good strategy $\bar{\rho} = (\bar{\rho}_1^S)$ referred to there. For this always $\bar{\rho}_2^S = 0$, and (5) - (8) apply to it. The quantities and concepts that were introduced in (5) - (8) should therefore be understood in the balance of this section as referring to $\bar{\rho}$ (and not to ρ).

Consider the s^* of (4.1). According to the remark immediately following (4.2), $\bar{\rho}_1^{S^*} = 1$. Hence by (6.5)

$$(9.1) \qquad s^* \geqq P + Q + 1 \ , \qquad \text{with } > \text{ for } \eta \neq 0 \ .$$

It is easily seen, that the other inferences that can be drawn from (4.3) (with (4.2)) are automatically satisfied owing to (9.1).

For any s^* satisfying (9.1), $\rho = (\rho_1^S)$ can be derived from $\bar{\rho} = (\bar{\rho}_1^S)$ according to (4.2). Here $\rho_2^{S^*}$ can be chosen freely, subject only to

$$(9.2) \qquad 0 < \rho_2^{S^*} < \bar{\rho}_1^{S^*} \ ,$$

all other ρ_1^S are then determined. There remains the task to establish, which of the strategies ρ that obtain in this manner are good. This will not be undertaken here.

§ 10. CONCLUDING REMARKS

From the general existence proof (i.e., the minimax theorem, cf. the reference at the beginning of the introduction.) we know that good strategies must always exist -- i.e., for all values of a,b $(a > b > 0)$ and of $S(= 1, 2, \ldots)$.

Our present results, as formulated in (8) (together with (4.1) - (4.3), cf. (9) above), do not go quite as far in all cases, but what concerns us here, is the additional information that they give for most cases.

To be specific:

Put $\omega = \frac{b}{a}$ -- hence $0 < \omega < 1$.

Case I $(P = Q = 0)$ takes care precisely of the $S \leqq \frac{1}{1-\omega}$ (cf. (8.5)). Case II $(P = 0, Q \neq 0)$ takes care of certain

$$S \geqq \frac{1}{1-\omega} \ , \quad \leqq \frac{1+2\omega-\omega^2}{2\omega(1-\omega)}$$

(cf. (8.7)), possessing certain special arithmetical properties (cf. (8.6)). Case III (i.e., $P \neq 0$, $Q = 0$) takes care of certain S (cf. (8.8), (8.9)) that form an unbounded set, which has an asymptotic density that is in general < 1 (cf. below). Case IV takes care of certain S, that comprise all the S above a certain limit, namely all those

$$S > \frac{(1+\omega)^2}{2\omega(1-\omega)}$$

(cf. (8.26)).

Thus Cases III and IV represent the "general" solution, and IV usually more than III. To elaborate this further, note that IV comprises all the S above a certain size, while III as a rule does not. In this respect the two following observations are relevant:

Regarding III: If ω is irrational, then the set of (8.8) is empty, and the set of (8.9) has the asymptotic density ω. If ω is rational, with the (irreducible) denominator n ($= 2, 3, \ldots$), then the set of (8.8) has the asymptotic density $\frac{1}{2n}$, and the set of (8.9) has the asymptotic density $\omega + \frac{1}{2n}$. Hence Case III has the total asymptotic density ω or $\omega + \frac{1}{n}$, respectively. This is < 1, unless ω is (rational and) $= \frac{n-1}{n}$ ($n = 2, 3, \ldots$).

Regarding IV: The actual value of S for Poker is $2{,}598{,}960 \sim 2.6 \times 10^6$ (cf. (1)). This lies outside the domain of Case I (i.e., it is $> \frac{1}{1-\omega}$) unless $\omega > 1 - 3.9 \times 10^{-7}$, it lies outside the domain of Case II (i.e., it is $> \frac{1+2\omega-\omega^2}{2\omega(1-\omega)}$) unless $\omega > 1 - 7.8 \times 10^{-7}$ or $\omega < 1.9 \times 10^{-7}$. I.e., if we disregard altogether implausible and extreme values of $\omega = \frac{b}{a}$, Case IV is valid, and Cases I, II are not.

§ 11. CONCLUDING REMARKS (CONTINUED)

The observations of (10) justify our limiting our interpretations to the Cases III and IV of (8), with a special emphasis on Case IV.

In both Cases III and IV values $\rho_1^S \neq 0, 1$ occur (for $i = 1, \ldots, P$ there is $\rho_1^S = \frac{\omega}{1+\omega}(1 \pm \varepsilon)$, cf. (6.5), remembering $\omega = \frac{b}{a}$), i.e., these are truly mixed strategies. Regarding their detailed properties, the following things can be said.

Consider first the asymptotic behavior, i.e., that for $S \longrightarrow \infty$ -- this is, of course, approximately valid and typical for large S, e.g., for the "real" case $S \sim 2.6 \times 10^6$ (cf. the end of (10)).

As observed immediately before (8.26), $L - P = Q$ is bounded ($\leqq \frac{1+\omega}{2\omega}$), and by (8.8), (8.9), (8.10) $(1 - \omega) S - L$ is also bounded. Thus $(1 - \omega) S - P$ and Q are both bounded, and therefore

(11.1) $$\frac{P}{S} \longrightarrow 1 - \omega, \quad \frac{Q}{S} \longrightarrow 0 .$$

Hence the behavior of ρ_1^s, for $s = P + 1, \ldots, P + Q$ is irrelevant. For $s = P + Q + 1, \ldots, S$ $\rho_1^s = 1$. For $s = 1, \ldots, P$ ρ_1^s alternates between the values $\frac{\omega}{1+\omega}(1 \pm \varepsilon)$ (cf. above), i.e., it fluctuates by $\frac{2\omega}{1+\omega}|\varepsilon|$, but its average is $\frac{\omega}{1+\omega}$.

Thus the mean behavior converges to the one determined in equation (18.21) of [1], p. 202, for a continuum of hands. This is unaffected by the possible introduction of the ρ_2^s according to (4) and (9), since this affects only a single s $(= s^*)$. The mean referred to above may be taken for any s-interval whose length increases asymptotically proportionately to S, as $S \longrightarrow \infty$.

On the other hand, the exact oscillation of the ρ_1^s does not subside asymptotically, as $S \longrightarrow \infty$. Indeed, the oscillation of ρ_1^s (in the domain $s = 1, \ldots, P$, between neighboring values of s) is $\frac{2\omega}{1+\omega}|\varepsilon|$ (cf. above), and $S \longrightarrow \infty$ does not imply $\varepsilon \longrightarrow 0$. A specific inspection of Cases III and IV in (8) shows, that ε is very sensitively determined by the precise arithmetical relationships between ω (i.e., a,b) and S, and that it sweeps its entire domain quasi-periodically, no matter how large S gets.

Using a terminology that is familiar from function theory, one might say: as $S \longrightarrow \infty$, the good strategies of the discrete-hand-poker converge to those of the continuous-hand-poker in the sense of "weak" convergence only. This phenomenon is therefore heuristically instructive, regarding what one should expect more generally, when a "continuous game" is investigated in detail in its relationship to the "discrete games" of which it might be viewed as a limiting case.

PART II

§ 1. DESCRIPTION OF THE GAME

The situation to be dealt with is described in "Theory of Games," section 19.13 (pp. 209-211): Each of two players receives a "hand," which is a randomly equidistributed number x in the interval $0 \leqq x \leqq 1$, and declares a bid α in the interval $a \geqq \alpha \geqq b$ $(a > b > 0)$, this declaration being made with the knowledge of his own hand, but in ignorance of his opponent's hand and bid. If the bids are of equal size, then the hands are compared, and the player with the weaker hand pays his opponent the amount of the bid. If the bids are unequal, the player who has made the smaller bid pays this amount to his opponent. He has no option of "seeing" his opponent (i.e., of matching the bid), nor are hands compared.

There are two cases to consider:

(I) All bids α in $a \geqq \alpha \geqq b$ are allowed.

(II) Only a finite number of possible bids:

$$a = a_1 > a_2 > \cdots > a_{m-1} > a_m = b$$

are allowed.

The whole discussion that follows will be more informal, and in parts (2) and (4) more heuristic, than the discussions in "Theory of Games" or in the preceding paper. The definitely formulated conclusions and their proofs are, nevertheless, rigorous.

§ 2. CASE (I): A CONTINUUM OF ALLOWED BIDS

2.1. Let us analyze a potential good strategy -- since the game is symmetric, it does not matter for which player.

Assume that for all hands $x > x_0$, where x_0 is a constant to be determined, the strategy provides always betting a, i.e., the highest possible bid. For the other bids α' (i.e., $a > \alpha' \geqq b$), let $\phi(\alpha)d\alpha$ be the a priori probability (i.e., the one integrated over the random equidistribution of hands x) that α' is in the (infinitesimal) interval from α to $\alpha + d\alpha$.

Assume that for $x \leqq x_0$ the expected value of the game is the same for all choices of the bid α', if the opponent plays this strategy. The expected value in question is

$$(2.1.1) \qquad \int_b^{\alpha'} \alpha\phi(\alpha)d\alpha - \alpha' \left[\int_{\alpha'}^{a} \phi(\alpha)d\alpha + (1 - x_0) \right] .$$

The independence from α' is best expressed by stating that the α'-derivative of this quantity is zero. This gives

$$(2.1.2) \qquad 2\alpha'\phi(\alpha') - \left[\int_{\alpha'}^{a} \phi(\alpha)d\alpha + (1 - x_0) \right] = 0 .$$

Another differentiation (and writing α for α') gives

$$2\alpha\phi'(\alpha) + 3\phi(\alpha) = 0 ,$$

i.e.

$$\frac{\phi'(\alpha)}{\phi(\alpha)} = -\frac{3}{2}\,\frac{1}{\alpha} ,$$

whence

$$(2.1.3) \qquad \phi(\alpha) = k\alpha^{-\frac{3}{2}} .$$

Substituting (2.1.3) into (2.1.2) gives

$$(2.1.4) \qquad 2ka^{-\frac{1}{2}} - (1 - x_0) = 0 .$$

The definition of $\phi(\alpha)$ implies

$$\int_{b}^{a} \phi(\alpha)d\alpha + (1 - x_0) = 1 ,$$

i.e.

$$\int_{b}^{a} \phi(\alpha)d\alpha - x_0 = 0 .$$

Substituting (2.1.3) gives

$$(2.1.5) \qquad 2kb^{-\frac{1}{2}} - 2ka^{-\frac{1}{2}} - x_0 = 0 .$$

From (2.1.4), (2.1.5) by addition

$$2kb^{-\frac{1}{2}} - 1 = 0 , \quad k = \frac{1}{2}\,b^{\frac{1}{2}} ,$$

so that (2.1.3) becomes

$$(2.1.6) \qquad \phi(\alpha) = \frac{1}{2}\,b^{\frac{1}{2}}\alpha^{-\frac{1}{2}} ,$$

and then (from (2.1.4))

$$\left(\frac{b}{a}\right)^{\frac{1}{2}} - (1 - x_0) = 0 ,$$

i.e.

$$(2.1.7) \qquad x_0 = 1 - \left(\frac{b}{a}\right)^{\frac{1}{2}} .$$

2.2. With the choices (2.1.6), (2.1.7), the expression (2.1.1) is independent of α', hence its value for all α' is the one that it has for $\alpha' = b$, which is obviously $-b$. I.e., for any hand $x \leqq x_0$ and any bid α' the expected value of the game is the same: $-b$.

The expected value of the game is still given by the expression (2.1.1) for a hand $x < x_0$, provided that the bid α' is $< a$. I.e., it is still $-b$ in this case. For a hand $x > x_0$ and the bid $\alpha' = a$, however, (2.1.1) must be replaced by

$$\left[\int_b^a \alpha\phi(\alpha)d\alpha + a(x - x_0)\right] - a(1 - x) .$$

This expression has the x-derivative $2a$, and for $x = x_0$ it goes over into (2.1.1) (with $\alpha' = a$), i.e., its value is then $-b$. Hence its general value is $-b + 2a(x - x_0)$.

Summing up:

(2.2.1) The expected value of the game for the hand x and the bid α (if the opponent plays the strategy under consideration) is $-b$, except when $x > x_0$, $\alpha = a$, in which case it is $-b + 2a(x - x_0)$.

2.3. (2.2.1) implies this: In order that a strategy be optimal against the one considered above, it may allow any set of bids α for hands $x \leqq x_0$, but it must prescribe the (highest) bid $\alpha = a$ exclusively for hands $x > x_0$.

The strategy considered above (i.e., the one introduced in (2.1), with the determinations (2.1.6), (2.1.7)) itself fulfills this condition. Hence it is optimal against itself, i.e., it is a good strategy.

Note, that the definition of (2.1) actually allowed for a whole family of strategies, all of which, according to the above conclusion, are good. Indeed, for bids $\alpha < a$, only the a priori probability density $\phi(\alpha)$ is specified, but not the conditional probability density $\phi(\alpha,x)$, for a given value of the hand x. This, more detailed, structure of the bidding-rule has thus no influence on the goodness of the strategy.

This phenomenon may seem strange, but it is quite understandable. All bids $\alpha < a$ are (in these strategies) induced by probability densities, i.e., the probability of any one individual value $\alpha < a$ is 0. Hence the probability for two opponents, whose bids α_1, α_2 happen to be both $< a$, to make (accidentally) equal bids $\alpha_1 = \alpha_2$, is also 0. It is in this case only, however, that their respective hands affect the outcome.

In the case (II), where only a finite number of bids are possible, individual bid-values with non-0-probability will be the rule. In this case

the hand and the bid will therefore interact substantially in determining good strategies, as will appear in the course of (3), especially in (3.3.2) and in the concluding section (3.9). This will lead to very definite "fine structures" for good strategies, which can exhibit various peculiar oscillatory phenomena when the number m of possible strategies tends to ∞. These are in some ways analogous to the "weak convergence" phenomena for poker with a finite number of hands, as discussed in the preceding paper.

§ 3. CASE (II): A FINITE NUMBER m OF ALLOWED BIDS

3.1. Let us again analyze a potential good strategy -- since the game is symmetric, it does not matter for which player.

Let $\theta_\nu(x)$ be the probability of the bid a_ν ($\nu = 1, \ldots, m$) when the hand is x. Hence

$$\theta_\nu(x) \geqq 0 \ , \quad \sum\nolimits_\nu \theta_\nu(x) = 1 \ . \tag{3.1.1}$$

The a priori probability of the bid a_ν is then

$$\theta_\nu = \int_0^1 \theta_\nu(x)\,dx \ , \tag{3.1.2}$$

of course

$$\theta_\nu \geqq 0 \ , \quad \sum\nolimits_\nu \theta_\nu = 1 \ . \tag{3.1.3}$$

Consider the expected value $\gamma_\nu(x)$ of the game for the hand x and the bid a_ν, if the opponent plays this strategy. Clearly

$$\gamma_\nu(x) = \sum\nolimits_{\mu>\nu} a_\mu \theta_\mu - a_\nu \sum\nolimits_{\mu<\nu} \theta_\nu + a_\nu \left(\int_0^x \theta_\nu(y)\,dy - \int_x^1 \theta_\nu(y)\,dy \right) . \tag{3.1.4}$$

From (3.1.4)

$$\gamma_\nu(x') - \gamma_\nu(x'') = 2a_\nu \cdot \int_{x''}^{x'} \theta_\nu(y)\,dy \ . \tag{3.1.5}$$

This makes it also clear, that $\gamma_\nu(x)$ is continuous in x.

3.2. From this point on rigor requires the use of (Lebesgue-) measure theory. We will try to do this without disrupting the intuitive pattern of the deduction too badly.

We call x a ν-point, if for every open interval I with $x \in I$ the $y \in I$ with $\theta_\nu(y) \neq 0$ have a measure $\neq 0$.

Let x' be a ν-point, and $x'' < x'$ not a ν-point. The $x \geqq x''$ which are ν-points form a closed set. This set contains x', hence it has a minimum element x^*. The set does not contain x'', hence

$x^* > x''$. Thus x^* is a ν-point, while no x in $x^* > x \geqq x''$ is a ν-point. By virtue of the usual covering theorems the y in $x^* > y \geqq x''$ with $\theta_\nu(y) \neq 0$ have measure 0.

Consider an open interval I with $x^* \in I$. Then the $y \in I$ with $\theta_\nu(y) \neq 0$ have measure $\neq 0$.

Consider an open sub-interval J of the open interval $x^* > y > x''$. Then (3.1.1) implies the existence of a μ such that the $y \in J$ with $\theta_\mu(y) \neq 0$ have measure $\neq 0$. Hence $\mu \neq \nu$.

Now the goodness of the opponent's strategy excludes $\gamma_\mu(y) > \gamma_\nu(y)$ on a set with measure $\neq 0$ where $\theta_\nu(y) \neq 0$, hence there exists a $y' \in I$ with $\gamma_\mu(y') \leqq \gamma_\nu(y')$. Similarly, it excludes $\gamma_\nu(y) > \gamma_\mu(y)$ on a set with measure $\neq 0$ where $\theta_\mu(y) \neq 0$, hence there exists a $y'' \in J$ with $\gamma_\mu(y'') \geqq \gamma_\nu(y'')$.

Hence

$$\gamma_\mu(y') - \gamma_\nu(y') \leqq 0 \ , \quad \gamma_\mu(y'') - \gamma_\nu(y'') \geqq 0 \ ,$$

therefore

$$[\gamma_\mu(y') - \gamma_\mu(y'')] - [\gamma_\nu(y') - \gamma_\nu(y'')] =$$

$$[\gamma_\mu(y') - \gamma_\nu(y')] - [\gamma_\mu(y'') - \gamma_\nu(y'')] \leqq 0 \ ,$$

and so by (3.1.5)

$$(3.2.1) \qquad 2 \cdot \int_{y''}^{y'} \left(a_\mu \theta_\mu(x) - a_\nu \theta_\nu(x) \right) dx \leqq 0 \ .$$

Now let I converge to x^* and J to x''. Then $y' \longrightarrow x^*$, $y'' \longrightarrow x''$. $\mu\ (\neq \nu)$ may vary, but some value of $\mu\ (\neq \nu)$ must be assumed infinitely often in this convergent sequence. Restricting ourselves to this subsequence, we may assume μ to be fixed. Hence (3.2.1) becomes

$$(3.2.2) \qquad 2 \cdot \int_{x''}^{x^*} \left(a_\mu \theta_\mu(x) - a_\nu \theta_\nu(x) \right) dx \leqq 0 \ .$$

We saw above, that the y in $x^* > y \geqq x''$ with $\theta_\nu(y) \neq 0$ have measure 0. Hence (3.2.2) becomes

$$2 \cdot \int_{x''}^{x^*} a_\mu \theta_\mu(x) dx \leqq 0 \ , \qquad \int_{x''}^{x^*} \theta_\mu(x) dx \leqq 0 \ .$$

Owing to (3.1.1) this means, that the y in $x^* > y \geqq x''$ with $\theta_\mu(y) \neq 0$ have measure 0. This contradicts the fact, that there exist open sub-intervals J of $x^* > y > x''$, such that the $y \in J$ with

$\theta_\mu(y) \neq 0$ have measure $\neq 0$.

In view of this contradiction our original assumption must have been wrong. I.e.:

(3.2.3) If x' is a ν-point, then every $x'' < x'$ is a ν-point, too.

Assume that ν-points exist. They form a closed set, hence it has a maximum element ξ_ν, of course $0 \leqq \xi_\nu \leqq 1$. If no ν-points exist, put $\xi_\nu = 0$. At any rate, no x in $\xi_\nu < x \leqq 1$ is a ν-point. By virtue of the usual covering theorems the x in $\xi_\nu < x \leqq 1$ with $\theta_\nu(x) \neq 0$ have measure 0. In view of the two-way definition of ξ_ν, at least every x in $0 < x \leqq \xi_\nu$ is a ν-point.

Now consider two x', x'' with $0 < x' \leqq \xi_\nu$, $0 < x'' \leqq \xi_\mu$. Consider two open intervals I', I'' with $x' \in I'$, $x'' \in I''$. Then the $y \in I'$ with $\theta_\nu(y) \neq 0$ have measure $\neq 0$ and the $y \in I''$ with $\theta_\mu(y) \neq 0$ have measure $\neq 0$. The goodness of the opponent's strategy excludes $\gamma_\mu(y) > \gamma_\nu(y)$ on a set with measure $\neq 0$ where $\theta_\nu(y) \neq 0$, hence there exists a $y' \in I'$ with $\gamma_\mu(y') \leqq \gamma_\nu(y')$. Similarly, it excludes $\gamma_\nu(y) > \gamma_\mu(y)$ on a set with measure $\neq 0$ where $\theta_\mu(y) \neq 0$, hence there exists a $y'' \in I''$ with $\gamma_\mu(y'') \geqq \gamma_\nu(y'')$.

From this point, the argument used further above, leads to (3.2.1) for these y', y''. Then, letting I' converge to x' and I'' to x'' gives $y' \longrightarrow x'$, $y'' \longrightarrow x''$, and hence (3.2.1) gives

$$(3.2.4) \qquad 2 \cdot \int_{x''}^{x'} \left(a_\mu \theta_\mu(x) - a_\nu \theta_\nu(x) \right) dx \leqq 0 \ .$$

Assume next $0 < x', x'' \leqq \text{Min}\,(\xi_\mu, \xi_\nu)$. Then (3.2.4) holds for μ, ν as they stand, and also with them interchanged. Hence

$$2 \cdot \int_{x''}^{x'} \left(a_\mu \theta_\mu(x) - a_\nu \theta_\nu(x) \right) dx = 0 \ ,$$

i.e.

$$(3.2.5) \qquad \int_{x''}^{x'} a_\mu \theta_\mu(x) dx = \int_{x''}^{x'} a_\nu \theta_\nu(x) dx \ .$$

(3.2.5) can also be stated as follows: For every open sub-interval K of the interval $0 \leqq x \leqq \text{Min}\,(\xi_\mu, \xi_\nu)$, the two functions $a_\mu \theta_\mu(x)$ and $a_\nu \theta_\nu(x)$ have the same integral over K. From this, the usual theorems about integration permit us to conclude, that the x in $0 \leqq x \leqq \text{Min}\,(\xi_\mu, \xi_\nu)$ with $a_\mu \theta_\mu(x) \neq a_\nu \theta_\nu(x)$ have measure 0.

Summing up:

(3.2.6) The ξ_ν $(\nu = 1, \ldots, m)$ have the following properties:

(a) $0 \leqq \xi_\nu \leqq 1$.

(b) The x in $\xi_\nu < x \leqq 1$ with $\theta_\nu(x) \neq 0$ have measure 0.

(c) The x in $0 \leqq x \leqq \operatorname{Min}(\xi_\mu, \xi_\nu)$ with $a_\mu \theta_\mu(x) \neq a_\nu \theta_\nu(x)$ have measure 0.

3.3. Let A be the sum of the sets defined in (3.2.6), (b) and (c). By (3.2.6) A has measure 0.

Assume that $\operatorname{Max}_\nu \xi_\nu < 1$. Then there exists an $x \notin A$ with $\operatorname{Max}_\nu \xi_\nu < x \leqq 1$. Hence by (3.2.6), (b) all $\theta_\nu(x) = 0$, contradicting (3.1.1). Therefore $\operatorname{Max}_\nu \xi_\nu = 1$, i.e.,

$$\text{Some } \xi_\nu = 1 \,. \tag{3.3.1}$$

Consider an $x \notin A$. By (3.3.1) ν with $\xi_\nu \geqq x$ exist. By (3.2.6), (c) $a_\nu \theta_\nu(x)$ has the same value for all such ν, say $f(= f(x))$. Hence (3.2.6), (b) and (c) give

$$\theta_\nu(x) \begin{cases} = \dfrac{f}{a_\nu} & \text{for} \quad \xi_\nu \geqq x \,, \\ = 0 & \text{for} \quad \xi_\nu < x \,. \end{cases}$$

Now (3.1.1) gives

$$f \sum_{\xi_\mu \geqq x} \left(\frac{1}{a_\mu}\right) = 1 \,, \quad f = \frac{1}{\sum_{\xi_\mu \geqq x} \left(\frac{1}{a_\mu}\right)} \,,$$

i.e.

$$\theta_\nu(x) \begin{cases} = \dfrac{1}{\sum_{\xi_\mu \geqq x} \left(\frac{1}{a_\mu}\right)} \dfrac{1}{a_\nu} & \text{for} \quad \xi_\nu \geqq x \,, \\ = 0 & \text{for} \quad \xi_\nu < x \,. \end{cases} \tag{3.3.2}$$

Since A has measure 0, the $\theta_\nu(x)$ for $x \in A$ can be redefined in any manner, without changing the strategy relevantly. Now (3.3.2) is a possible definition of the $\theta_\nu(x)$ by (3.3.1), and it fulfills (3.1.1) -- no matter what x. Hence we can use (3.3.2) to redefine the $\theta_\nu(x)$ for $x \in A$.

This has the effect, that (3.3.2) holds for all x. This will be assumed from now on.

Thus the strategy is completely determined by (3.3.2), and (3.3.2) is stated wholly in terms of the ξ_ν. Hence there remains only the task of determining the ξ_ν.

3.4. Assume that $\xi_\nu = 1$ for some $\nu \neq 1$. Then by (3.3.2) always $\theta_\nu(x) \neq 0$, and also $\theta_\nu \neq 0$. Hence the goodness of the strategy allows $\gamma_1(x) > \gamma_\nu(x)$ only for a set of measure 0. I.e., with such exceptions

$$\gamma_1(x) \leqq \gamma_\nu(x) \ . \tag{3.4.1}$$

By continuity, (3.4.1) holds without exceptions.

From (3.4.1), $\gamma_1(1) \leqq \gamma_\nu(1)$, hence

$$\sum_\mu a_\mu \theta_\mu \leqq \sum_{\mu \geqq \nu} a_\mu \theta_\mu - a_\nu \sum_{\mu < \nu} \theta_\nu \ ,$$

i.e.

$$\sum_{\mu < \nu} (a_\mu + a_\nu)\theta_\mu = 0 \ ,$$

i.e. $\theta_\mu = 0$ for $\mu < \nu$. Again from (3.4.1), $\gamma_1(0) \leqq \gamma_\nu(0)$, hence

$$\sum_{\mu > 1} a_\mu \theta_\mu - a_1 \theta_1 \leqq \sum_{\mu > \nu} a_\mu \theta_\mu \quad a_\nu \sum_{\mu \leqq \nu} \theta_\nu \ ,$$

i.e. (since $\theta_\mu = 0$ for $\mu < \nu$, and since $\nu \neq 1$)

$$2a_\nu \theta_\nu \leqq 0 \ .$$

Hence $\theta_\nu = 0$, contradicting $\theta_\nu \neq 0$ (cf. above).

Hence the original assumption must have been wrong. I.e.:

$$\xi_\nu < 1 \quad \text{for} \quad \nu = 2, \ldots, m \ . \tag{3.4.2}$$

Now (3.3.1) necessitates

$$\xi_1 = 1 \ . \tag{3.4.3}$$

Assume next, that some $\xi_\nu = 0$. Choose the minimal ν with this property. By (3.4.3) $\nu \neq 1$, hence $\xi_\nu = 0$, $\xi_{\nu-1} > 0$. Then by (3.3.2) $\theta_{\nu-1}(x) \neq 0$ for $0 \leqq x \leqq \xi_{\nu-1}$, and also $\theta_\nu = 0$, $\theta_{\nu-1} > 0$.

Hence the goodness of the strategy allows $\gamma_\nu(x) > \gamma_{\nu-1}(x)$ for these x only for a set of measure 0. I.e., with such exceptions

(3.4.4) $$\gamma_\nu(x) \leqq \gamma_{\nu-1}(x) \quad \text{for} \quad 0 \leqq x \leqq \xi_{\nu-1} .$$

By continuity, (3.4.4) holds without exceptions.

From (3.4.4), $\gamma_\nu(0) \leqq \gamma_{\nu-1}(0)$, hence by (3.1.4)

$$\sum_{\mu>\nu} a_\mu \theta_\mu - a_\nu \sum_{\mu\leqq\nu} \theta_\mu \leqq \sum_{\mu\geqq\nu} a_\mu \theta_\mu - a_{\nu-1} \sum_{\mu<\nu} \theta_\mu .$$

In view of $\theta_\nu = 0$ this means

$$(a_{\nu-1} - a_\nu) \sum_{\mu<\nu} \theta_\mu \leqq 0 .$$

Hence $\theta_\mu = 0$ for $\mu < \nu$, contradicting $\theta_{\nu-1} > 0$.

Hence the original assumption must have been wrong. This means that all $\xi_\nu \neq 0$, i.e.:

(3.4.5) $$0 < \xi_\nu \leqq 1 \quad \text{for all} \quad \nu .$$

3.5. For $0 \leqq x \leqq \operatorname{Min}_\lambda \xi_\lambda$ all $\theta_\nu(x) \neq 0$ (cf. (3.3.2)), hence the goodness of the strategy allows $\gamma_\mu(x) > \gamma_\nu(x)$ for these x only for a set of measure 0. I.e., with such exceptions

(3.5.1) $$\gamma_\mu(x) \leqq \gamma_\nu(x) \quad \text{for} \quad 0 \leqq x \leqq \operatorname{Min}_\lambda \xi_\lambda .$$

By continuity (and in view of (3.4.5)) (3.5.1) holds without exceptions.

Putting $x = 0$ in (3.5.1), and considering that it holds for all pairs μ,ν $(= 1, \ldots, m)$, gives

(3.5.2) $$\gamma_1(0) = \ldots = \gamma_m(0) .$$

We will use (3.5.2) to determine the remaining unknown quantities, i.e., $\xi_1, \ldots, \xi_m$ (cf. the remark at the end of (3.3)). Before we do this, however, we will discuss the general significance of (3.5.2).

3.6. The assumed good strategy under consideration, i.e., the $\theta_\nu(x)$, is defined by (3.3.2); the auxiliary quantities θ_ν are then defined by (3.1.2). (3.3.2) implies (3.1.1), (3.1.2) then implies (3.1.3). In order that (3.3.2) be meaningful (for all x) (3.3.1) must be required. We will also use (3.4.5). (We will, however, not need (3.4.2) and (3.4.3); we will rederive these.) In addition the condition (3.5.2) (based on (3.1.4)) holds for every good strategy.

We will see in (3.9) that these requirements determine a unique strategy (i.e., a unique system $\xi_1, \ldots, \xi_m$, cf. the remarks at the end

of (3.3) and of (3.5)). In view of this the existence theorem for good strategies implies, that the strategy in question, which is the only one that can be good, is indeed good. However, the proof of the existence theorem -- "Theory of Games," section 17.6 (pp. 153-155) -- does not cover the case of a continuum of hands. It is true that its extension to families of games that cover this case is well known and not at all difficult. Nevertheless, the situation, as outlined, is complex enough, that it seems natural to prove the goodness of a strategy fulfilling the above conditions directly.

This proof is quite simple. It runs as follows.

It suffices to show, that $\gamma_\nu(x) > \gamma_\mu(x)$ implies $\theta_\mu(x) = 0$ (we could, of course, except a set of measure 0, but we do not need this now) -- this means that the strategy in question is optimal against itself, i.e., good.

Assume $\gamma_\nu(x) > \gamma_\mu(x)$. Then by (3.5.2) $\gamma_\nu(x) - \gamma_\nu(0) > \gamma_\mu(x) - \gamma_\mu(0)$, hence by (3.1.5) (with $x,0$ for x',x'') $\theta_\nu(y) > \theta_\mu(y)$ for some $y \leqq x$. By (3.3.2) $\theta_\nu(y) > \theta_\mu(y)$ occurs only if $\xi_\nu \geqq y$, $\xi_\mu < y$. Hence $\xi_\mu < x$. Now (3.3.2) implies $\theta_\mu(x) = 0$. This completes the proof.

3.7. We proceed now to the complete determination of the strategy under consideration, on the basis of (3.5.2) (cf. (3.5) and (3.6)).

(3.5.2) is equivalent to

$$\gamma_\nu(0) = \gamma_{\nu+1}(0) \quad \text{for} \quad \nu = 1, \ldots, m-1 . \tag{3.7.1}$$

In view of (3.1.4) this means

$$\sum_{\mu>\nu} a_\mu\theta_\mu - a_\nu \sum_{\mu\leqq\nu} \theta_\mu = \sum_{\mu>\nu+1} a_\mu\theta_\mu - a_{\nu+1} \sum_{\mu\leqq\nu+1} \theta_\mu ,$$

i.e.

$$a_{\nu+1}\theta_{\nu+1} + a_{\nu+1}\theta_{\nu+1} - (a_\nu - a_{\nu+1}) \sum_{\mu\leqq\nu} \theta_\mu = 0 ,$$

i.e.

$$\theta_{\nu+1} = \frac{a_\nu - a_{\nu+1}}{2a_{\nu+1}} \sum_{\mu\leqq\nu} \theta_\mu . \tag{3.7.2}$$

Adding $\sum_{\mu\leqq\nu} \theta_\mu$ to both sides produces the equivalent

$$\sum_{\mu\leqq\nu+1} \theta_\mu = \frac{a_\nu + a_{\nu+1}}{2a_{\nu+1}} \sum_{\mu\leqq\nu} \theta_\mu ,$$

i.e.

(3.7.3) $$\sum_{\mu \leqq \nu} \theta_\mu = \frac{2a_{\nu+1}}{a_\nu + a_{\nu+1}} \sum_{\mu \leqq \nu+1} \theta_\mu \ .$$

Now (3.1.3) states, that $\sum_{\mu \leqq m} \theta_\mu = 1$, hence (3.7.3) is again equivalent to

(3.7.4) $$\sum_{\mu \leqq \nu} \theta_\mu = \frac{2a_m}{a_{m-1} + a_m} \cdot \ \dots \ \cdot \frac{2a_{\nu+1}}{a_\nu + a_{\nu+1}} \ .$$

This is valid for $\nu = 1, \dots, m$.

(3.7.4) can be rewritten to express the θ_ν themselves: For $\nu = 2, \dots, m$ θ_ν obtains from (3.7.4) for ν minus (3.7.4) for $\nu - 1$; for $\nu = 1$ θ_ν obtains from (3.7.4) for ν. Hence

(3.7.5) $$\theta_\nu \begin{cases} = \dfrac{2a_m}{a_{m-1} + a_m} \cdot \dots \cdot \dfrac{2a_2}{a_1 + a_2} & \text{for } \nu = 1 \\ = \dfrac{2a_m}{a_{m-1} + a_m} \cdot \dots \cdot \dfrac{2a_{\nu+1}}{a_\nu + a_{\nu+1}} \cdot \dfrac{a_{\nu-1} - a_\nu}{a_{\nu-1} + a_\nu} & \text{for } \nu = 2, \dots, m \ . \end{cases}$$

Thus (3.5.2) is equivalently expressed by the actual determination of the θ_ν by (3.7.5). We must now pass from this to the determination of the ξ_ν.

3.8. Assume $\xi_\nu \geqq \xi_\mu$. Then (3.3.2) gives $a_\nu\theta_\nu(x) > a_\mu\theta_\mu(x)$ for $\xi_\nu \geqq x > \xi_\mu$, and $a_\nu\theta_\nu(x) = a_\mu\theta_\mu(x)$ otherwise. Hence by (3.1.2) $a_\nu\theta_\nu > a_\mu\theta_\mu$ if $\xi_\nu > \xi_\mu$ and $a_\nu\theta_\nu = a_\mu\theta_\mu$ if $\xi_\nu = \xi_\mu$. Interchanging μ, ν gives $a_\nu\theta_\nu < a_\mu\theta_\mu$ if $\xi_\nu < \xi_\mu$.

Summing up:

(3.8.1) $\xi_\mu \gtreqless \xi_\nu$ if and only if $a_\mu\theta_\mu \gtreqless a_\nu\theta_\nu$, respectively.

Thus the θ_ν, which are given by (3.7.5), give in turn the ordering of the ξ_ν by (3.8.1).

Let $\pi_1, \dots, \pi_m$ be the permutation of $1, \dots, m$, which establishes a monotone non-decreasing ordering of the $a_\nu\theta_\nu$:

(3.8.2) $$a_{\pi_1}\theta_{\pi_1} \leqq \dots \leqq a_{\pi_m}\theta_{\pi_m} \ .$$

(If some $a_\nu\theta_\nu$ are equal, then this definition of the π_ρ is non-unique to that extent, but this is irrelevant.)

(3.8.1) now guarantees

(3.8.3) $$\xi_{\pi_1} \leqq \dots \leqq \xi_{\pi_m} \ .$$

The determination of the ξ_ν must be based on (3.3.2) with (3.1.2). (3.1.2) can be written in this form:

$$a_\nu \theta_\nu = \int_0^1 a_\nu \theta_\nu(x)dx \qquad (\nu = 1, \ldots, m) ,$$

or equivalently

$$a_{\pi_\rho} \theta_{\pi_\rho} = \int_0^1 a_{\pi_\rho} \theta_{\pi_\rho}(x)dx \qquad (\rho = 1, \ldots, m) ,$$

or, still equivalently

$$(3.8.4) \quad \left\{ \begin{array}{l} a_{\pi_\rho} \theta_{\pi_\rho} - a_{\pi_{\rho-1}} \theta_{\pi_{\rho-1}} = \\ \qquad \int_0^1 (a_{\pi_\rho} \theta_{\pi_\rho}(x) - a_{\pi_{\rho-1}} \theta_{\pi_{\rho-1}}(x))dx \\ \qquad\qquad (\rho = 1, \ldots, m) , \end{array} \right.$$

where a_{π_0}, θ_{π_0}, $\theta_{\pi_0}(x)$ are taken to be 0.

Now it is clear from (3.3.2), that the right-hand side of (3.8.4) is

$$(3.8.5) \qquad \frac{\xi_{\pi_\rho} - \xi_{\pi_{\rho-1}}}{\sum_{\xi_{\pi_\sigma} \geqq \xi_{\pi_\rho}} \frac{1}{a_{\pi_\sigma}}} ,$$

where ξ_{π_0}, too, is taken to be 0. Note, that for $\xi_{\pi_\rho} > \xi_{\pi_{\rho-1}}$, the condition $\xi_{\pi_\sigma} \geqq \xi_{\pi_\rho}$ can be replaced by $\sigma \geqq \rho$, while for $\xi_{\pi_\rho} = \xi_{\pi_{\rho-1}}$, the expression (3.8.5) is 0 anyhow. With this proviso then, (3.8.5) transforms into

$$a_{\pi_\rho} \theta_{\pi_\rho} - a_{\pi_{\rho-1}} \theta_{\pi_{\rho-1}} = \frac{\xi_{\pi_\rho} - \xi_{\pi_{\rho-1}}}{\sum_{\sigma \geqq \rho} \frac{1}{a_{\pi_\sigma}}}$$

i.e.

$$(3.8.6) \quad \left\{ \begin{array}{l} \xi_{\pi_\rho} - \xi_{\pi_{\rho-1}} = \left(\sum_{\sigma \geqq \rho} \frac{1}{a_{\pi_\sigma}} \right) \left(a_{\pi_\rho} \theta_{\pi_\rho} - a_{\pi_{\rho-1}} \theta_{\pi_{\rho-1}} \right) \\ \qquad\qquad (\rho = 1, \ldots, m) . \end{array} \right.$$

By summation this goes over into the equivalent system

$$(3.8.7) \quad \left\{ \begin{array}{l} \xi_{\pi_\rho} = \sum_{\tau \leqq \rho} \left(\sum_{\sigma \geqq \tau} \frac{1}{a_{\pi_\sigma}} \right) \left(a_{\pi_\tau} \theta_{\pi_\tau} - a_{\pi_{\tau-1}} \theta_{\pi_{\tau-1}} \right) \\ \qquad\qquad (\rho = 1, \ldots, m) . \end{array} \right.$$

Abelian re-summation of (3.8.7) gives

$$\xi_{\pi_\rho} = \Big(\sum_{\sigma \geqq \rho} \frac{1}{a_{\pi_\sigma}}\Big) a_{\pi_\rho} \theta_{\pi_\rho} + \sum_{\tau<\rho} \Big(\sum_{\sigma \geqq \tau} \frac{1}{a_{\pi_\sigma}} - \sum_{\sigma \geqq \tau+1} \frac{1}{a_{\pi_\sigma}}\Big) a_{\pi_\tau} \theta_{\pi_\tau} .$$

The general term of the sum $\sum_{\tau<\rho}$ is $\frac{1}{a_{\pi_\tau}} \cdot a_{\pi_\tau} \theta_{\pi_\tau} = \theta_{\pi_\tau}$. Hence

$$\xi_{\pi_\rho} = \Big(\sum_{\sigma \geqq \rho} \frac{1}{a_{\pi_\sigma}}\Big) a_{\pi_\rho} \theta_{\pi_\rho} + \sum_{\tau<\rho} \theta_{\pi_\tau} ,$$

i.e.,

$$(3.8.8) \qquad \left\{ \quad \xi_{\pi_\rho} = \sum_{\tau \leqq \rho} \theta_{\pi_\tau} + \Big(\sum_{\sigma>\rho} \frac{1}{a_{\pi_\sigma}}\Big) a_{\pi_\rho} \theta_{\pi_\rho} \right.$$

$$\left. \qquad\qquad\qquad\qquad\qquad\qquad (\rho = 1, \ldots, m) . \right.$$

Thus (3.8.7) and (3.8.8) are equivalent to each other and to (3.1.2.).

(3.8.7) makes it clear, that $0 < \xi_{\pi_1} \leqq \cdots \leqq \xi_{\pi_m}$. (3.8.8) and (3.1.3) (which follows from (3.7.4), and hence from (3.7.5)) give $\xi_{\pi_m} = 1$. Hence

$$(3.8.9) \qquad 0 < \xi_{\pi_1} \leqq \cdots \xi_{\pi_{m-1}} \leqq \xi_{\pi_m} = 1$$

follows from our present definitions. Thus (3.3.1) and (3.4.5) are verified, as well as (ex post) the relevant property of the permutation $\pi_1, \ldots, \pi_m$ (i.e., (3.8.3)).

3.9. (3.7.5), (3.8.2), (3.8.8) contain a complete definition of the θ_ν and ξ_ν, hence with the help of (3.3.2) the $\theta_\nu(x)$ are completely defined. I.e., we have completed the determination of the good strategy, and shown that it exists and is unique.

We observe further: By (3.7.5)

$$a_1 \theta_1 = \frac{2a_m}{a_{m-1} + a_m} \cdot \ldots \cdot \frac{2a_2}{a_1 + a_2} \cdot a_1 ,$$

and for $\nu = 2, \ldots, m$

$$a_\nu \theta_\nu = \frac{2a_m}{a_{m-1} + a_m} \cdot \ldots \cdot \frac{2a_\nu}{a_{\nu-1} + a_\nu} \cdot \frac{a_{\nu-1} - a_\nu}{2} ,$$

hence

$$\frac{a_1\theta_1}{a_\nu\theta_\nu} = \frac{2a_{\nu-1}}{a_{\nu-2} + a_{\nu-1}} \cdot \ldots \cdot \frac{2a_2}{a_1 + a_2} \cdot \frac{2a_1}{a_{\nu-1} - a_\nu}$$

$$\geqq \frac{2a_{\nu-1}}{2a_{\nu-2}} \cdot \ldots \cdot \frac{2a_2}{2a_1} \cdot \frac{2a_1}{a_{\nu-1} - a_\nu}$$

$$= \frac{2a_{\nu-1}}{a_{\nu-1} - a_\nu} > 1 ,$$

i.e.

$$a_1\theta_1 > a_\nu\theta_\nu .$$

Now (3.8.1) gives

(3.9.1) $$\xi_1 > \xi_\nu \quad \text{for} \quad \nu = 2, \ldots, m ,$$

and therefore (3.8.2) prescribes (unambiguously, cf. there)

(3.9.2) $$\pi_m = 1 .$$

This verifies (3.4.2) and (3.4.3).

§ 4. GENERAL CONSIDERATIONS

4.1. It is of interest to see, how the case (II), corresponding to a finite number of allowed bids (treated in (3)), converges to the case (I), corresponding to a continuum of allowed bids (treated in (2)), when the allowed bids of the former (cf. (II) in (1)) spread densely over the entire allowed interval of the latter (cf. (I) in (1)).

Consider first the general situation, where a finite number m of bids, according to (II) in (1), is allowed, and where the mesh-width

(4.1.1) $$\varepsilon = \operatorname*{Max}_{\nu \neq 1} (a_{\nu-1} - a_\nu)$$

of this bid-system converges to 0.

Let an α' with $a \geqq \alpha' \geqq b$ be given. Choose ν with $a_\nu \geqq \alpha' > a_{\nu+1}$. Then by (3.7.4)

$$\sum_{a_\mu \geqq \alpha'} \theta_\mu = \sum_{\mu \leqq \nu} \theta_\mu = \prod_{\lambda > \nu} \frac{2a_\lambda}{a_{\lambda-1} + a_\lambda}$$

$$= e^{-\sum_{\lambda > \nu} \ln \frac{a_{\lambda-1} + a_\lambda}{2a_\lambda}}$$

$$= e^{-\sum_{\lambda>\nu} \ln\left(1 + \frac{1}{2}\frac{a_{\lambda-1} - a_\lambda}{a_\lambda}\right)}$$

Replacing $\ln(1 + \frac{1}{2}\frac{a_{\lambda-1} - a_\lambda}{a_\lambda})$ by $\frac{1}{2}\ln(1 + \frac{a_{\lambda-1} - a_\lambda}{a_\lambda})$ changes it by $O((a_{\lambda-1} - a_\lambda)^2)$, i.e. by $O(\varepsilon(a_{\lambda-1} - a_\lambda))$. Hence the entire $\sum_{\lambda>\nu}$ is changed by $O(\varepsilon(a_\nu - a_m))$, i.e. by $O(\varepsilon(a - b))$, i.e. by $O(\varepsilon)$. Thus

$$\begin{aligned}
\sum_{a_\mu \geqq \alpha'} \theta_\mu &= e^{-\sum_{\lambda>\nu} \frac{1}{2}\ln\left(1 + \frac{a_{\lambda-1} - a_\lambda}{a_\lambda}\right) + O(\varepsilon)} \\
&= e^{-\sum_{\lambda>\nu} \frac{1}{2}\ln\frac{a_{\lambda-1}}{a_\lambda} + O(\varepsilon)} \\
&= e^{-\frac{1}{2}\ln\frac{a_\nu}{a_m} + O(\varepsilon)} \\
&= (1 + O(\varepsilon))\left(\frac{a_\nu}{a_m}\right)^{-\frac{1}{2}} \\
&= (1 + O(\varepsilon))\left(\frac{a_m}{a_\nu}\right)^{\frac{1}{2}} .
\end{aligned}$$

Now $a_m = b$, $a_\nu = \alpha' + O(\varepsilon) = (1 + O(\varepsilon))\alpha'$, hence

$$\sum_{a_\mu \geqq \alpha'} \theta_\mu = (1 + O(\varepsilon))\left(\frac{b}{\alpha'}\right)^{\frac{1}{2}} .$$

I.e.:

$$\sum_{a_\mu \geqq \alpha'} \theta_\mu \longrightarrow \left(\frac{b}{\alpha'}\right)^{\frac{1}{2}} \quad \text{for} \quad \varepsilon \longrightarrow 0 . \tag{4.1.2}$$

The right hand side of (4.1.2) is equal to

$$\int_{\alpha'}^{a} \phi(\alpha)\,d\alpha + (1 - x_0) , \tag{4.1.3}$$

with the $\phi(\alpha)$ of (2.1.6) and the x_0 of (2.1.7). I.e. the a priori probabilities θ_ν for the bids a_ν define a distribution that converges (for $\varepsilon \longrightarrow 0$) to the distribution with the cumulative distribution function (4.1.3) -- i.e. with the probability density $\phi(\alpha)$ in $a > \alpha \geqq b$ and the point probability $1 - x_0$ at $\alpha = a$. According to (2.3) this is precisely the distribution of a priori probabilities of bids that characterizes the good strategies that were obtained in (2) for the continuum case, i.e. (I) in (1).

In other words: If the mesh-width of the bid system (cf. (4.1.1))

converges to 0, the distribution of a priori probabilities of the bid system for case (II) converges to that one for case (I).

To this extent, then, we have continuity.

4.2. The behavior of the a priori probabilities, i.e. of the θ_ν (or, rather, of the $\sum_{a_\nu \geqq \alpha} \theta_\nu$), turned out to be continuous. Let us now consider the $\theta_\nu(x)$ themselves, or, which in view of (3.3.2) amounts to the same thing, the ξ_ν.

Consider again the set up of 4.1, more specifically $\varepsilon \longrightarrow 0$ for the ε of (4.1.1). We will prove, that this is compatible with a wide variety of behaviors for the ξ_ν.

We observe first, that (3.7.3) implies

$$\sum_{\mu \leqq \nu} \theta_\mu > \frac{2a_{\nu+1}}{2a_\nu} \sum_{\mu \leqq \nu+1} \theta_\mu ,$$

i.e.

$$a_\nu \sum_{\mu \leqq \nu} \theta_\mu > a_{\nu+1} \sum_{\mu \leqq \nu+1} \theta_\mu ,$$

i.e.

(4.2.1) $a_\nu \sum_{\mu \leqq \nu} \theta_\mu$ is a monotone decreasing function of ν.

In addition to this, obviously,

(4.2.2) $\sum_{\mu \leqq \nu} \theta_\mu$ is a monotone increasing function of ν.

From these (with $\nu - 1$ in place of ν):

(4.2.3) $\begin{cases} (a_\nu - a_{\nu-1}) \sum_{\mu \leqq \nu-1} \theta_\mu \text{ is a monotone } \{\substack{\text{increasing} \\ \text{decreasing}}\} \text{ function of} \\ \nu\ (= 2, \ldots, m), \text{ if } a_\nu - a_{\nu-1} \ \{\substack{= \delta \\ = \delta\, a_{\nu-1}}\} \\ (\delta \text{ constant}), \text{ i.e. if the } a_\nu \text{ form an } \{\substack{\text{arithmetic} \\ \text{geometric}}\} \text{ progression.} \end{cases}$

Now by (3.7.2) (with $\nu - 1$ in place of ν)

(4.2.4) $a_\nu \theta_\nu = \frac{1}{2} (a_\nu - a_{\nu-1}) \sum_{\mu \leqq \nu-1} \theta_\mu$ for $\mu = 2, \ldots, m$.

Combining (4.2.3) and (4.2.4) with (3.8.1) and (3.9.1) gives therefore:

$$(4.2.5)\qquad \begin{cases} \begin{cases} \xi_2 < \cdots < \xi_m < \xi_1 \\ \xi_m < \cdots < \xi_2 < \xi_1 \end{cases} & \text{if the } a_\nu \text{ form an} \\ \{\substack{\text{arithmetic} \\ \text{geometric}\} \text{ progression.} \end{cases}$$

Hence (3.8.2) prescribes (unambiguously, cf. there):

$$(4.2.6)\qquad \begin{cases} \begin{cases} \pi_1 = 2, \ \ldots, \ \pi_{m-1} = m, \ \pi_m = 1 \\ \pi_1 = m, \ \ldots, \ \pi_{m-1} = 2, \ \pi_m = 1 \end{cases} & \text{if the } a_\nu \text{ form an} \\ \{\substack{\text{arithmetic} \\ \text{geometric}\} \text{ progression.} \end{cases}$$

The arithmetic progression is given by the formula

$$(4.2.7.a)\qquad a_\nu = a\,\frac{m-\nu}{m-1} + b\,\frac{\nu-1}{m-1},$$

and the geometric progression is given by the formula

$$(4.2.7.b)\qquad a_\nu = a^{\frac{m-\nu}{m-1}}\, b^{\frac{\nu-1}{m-1}}$$

As $m \longrightarrow \infty$, both bid-systems (4.2.7.1) and (4.2.7.b) spread densely over the entire interval $a \geqq \alpha \geqq b$ -- i.e. in both cases the mesh-width ε of (4.1.1) converges to 0. Nevertheless, (4.2.5) shows, that the two bid-systems command altogether different good strategies (i.e. "bluffing" procedures): The hands $x \leqq \xi_1$, are the ones which do not command the highest bid (a_1) exclusively (cf. (3.3.2)) -- i.e. these form the range for "bluffing," in the sense of the interpretations of "Theory of Games," section 19.10 (pp. 204-207). (3.3.2) and (4.2.5) show, that in this region weak hands favor high bids when the bid-system (4.2.7.a) is valid, while strong hands favor high bids when the bid-system (4.2.7.b) is valid.

(4.2.6) permits calculating the ξ_ν in both cases. This can be used to illustrate the points made above further, but we will not go into this in detail at this occasion.

4.3. Actually, the permutation $\pi_1, \ldots, \pi_m$ of $1, \ldots, m$ (cf. (3.8.2) and the text that precedes it) can be prescribed entirely at will, as long as the necessary condition $\pi_m = 1$ (cf. (3.9.2)) is fulfilled. This can be shown as follows.

Assume $\nu = 1, \ldots, m-1$. In view of $\sum_{\mu \leqq m} \theta_\mu = 1$ (cf. (3.1.3)), (4.2.2) implies $\sum_{\mu \leqq \nu} \theta_\mu < 1$, and (4.2.1) implies $a_\nu \sum_{\mu \leqq \nu} \theta_\mu > a_m$. Since $a_\nu \leqq a_1 = a$, $a_m = b$, there follows

$$\frac{b}{a} < \sum_{\mu \leqq \nu} \theta_\mu < 1 \quad \text{for} \quad \nu = 1, \ldots, m-1 . \tag{4.3.1}$$

Replace ν by $\nu - 1$ in (4.3.1). Then (4.2.4) gives

$$\left\{ \frac{b}{a} \cdot \frac{1}{2} (a_\nu - a_{\nu-1}) < a_\nu \theta_\nu < \frac{1}{2} (a_\nu - a_{\nu-1}) \right. \quad \text{for} \quad \nu = 2, \ldots, m . \tag{4.3.2}$$

Let $\rho_1, \ldots, \rho_m$ be the inverse permutation (of $1, \ldots, m$) to $\pi_1, \ldots, \pi_m$. Assume that

$$a_\nu - a_{\nu-1} = h \left(\frac{a}{b}\right)^{\rho_\nu} \quad \text{for} \quad \nu = 2, \ldots, m, \tag{4.3.3}$$

for a suitable constant h (> 0). Then (4.3.2) becomes

$$\left\{ \frac{1}{2} h \cdot \left(\frac{a}{b}\right)^{\rho_\nu - 1} < a_\nu \theta_\nu < \frac{1}{2} h \cdot \left(\frac{a}{b}\right)^{\rho_\nu} \right. \quad \text{for} \quad \nu = 2, \ldots, m . \tag{4.3.4}$$

Now $\rho_\nu > \rho_\mu$ ($\mu, \nu = 2, \ldots, m$) implies $\rho_\nu - 1 \geqq \rho_\mu$, and thence by (4.3.4) $a_\nu \theta_\nu > a_\mu \theta_\mu$. I.e. the $a_\nu \theta_\nu$ are monotone increasing functions of ρ_ν for $\nu = 2, \ldots, m$, i.e. for $\nu \neq 1$. Therefore the $a_{\pi_\rho} \theta_{\pi_\rho}$ are monotone increasing functions of ρ for $\pi_\rho \neq 1$, i.e. (because of the stipulation of $\pi_m = 1$, cf. above) for $\rho \neq m$. The exclusion of $\rho = m$ may be removed because of (3.9.1) (and the stipulation of $\pi_m = 1$) So we have:

$$a_{\pi_1} \theta_{\pi_1} < \ldots < a_{\pi_m} \theta_{\pi_m} . \tag{4.3.5}$$

In view of (3.8.2) this means, that these $\pi_1, \ldots, \pi_m$ are indeed the (unique) $\pi_1, \ldots, \pi_m$ of the present example.

There remains the task of safeguarding (4.3.3). Since $a_1 = a$, (4.3.3) is equivalent to

$$a_\nu = a - h \sum_{\mu \leqq \nu} \left(\frac{a}{b}\right)^{\rho_\mu} . \tag{4.3.6}$$

This condition $a = a_1 > a_2 > \ldots > a_{m-1} > a_m = b$ is now automatically fulfilled, except for $a_m = b$. This means

$$h \sum_{\rho \leqq m} \left(\frac{a}{b}\right)^{\rho_\mu} = a - b ,$$

i.e., since $\rho_1, \ldots, \rho_m$ is a permutation of $1, \ldots, m$,

$$h \sum_{\rho \leqq m} \left(\frac{a}{b}\right)^{\rho} = a - b .$$

Hence

$$h = \frac{a - b}{\sum_{\rho \leqq m} \left(\frac{a}{b}\right)^{\rho}} \tag{4.3.7}$$

meets this requirement.

4.4. Examples of various kinds could be multiplied beyond this point. Instead of pursuing this, we make the following general remarks about the ordering of the ξ_ν according to size.

The position of ξ_1, in this ordering is determined a priori by (3.9.1): It is the largest one. Hence we need to consider the ξ_ν with $\nu = 2, \ldots, m$ only. Combining (3.8.1) and (4.2.4) shows, that the ordering of these ξ_ν coincides with that one of the

$$(a_\nu - a_{\nu-1}) \sum_{\mu \leqq \nu-1} \theta_\mu . \tag{4.4.1}$$

Now let the bid-system of (II) in (1) spread densely over the entire interval $a \geqq \alpha \geqq b$ -- i.e. let the mesh-width ε of (4.1.1) converge to 0. Consider the two factors of (4.4.1) separately.

The second factor of (4.4.1) was dealt with in (4.1), and it was seen to converge to a simple function of α' -- the connection between α' and ν being given by $a_{\nu-1} \geqq \alpha' > a_\nu$ (this obtains from the corresponding condition in (4.1) by replacing ν by $\nu - 1$). This function of α' is given in (4.1.2), and interpreted in (4.1.3). It is continuous in α', hence asymptotically infinitesimally slowly varying in ν.

It follows therefore, that the ordering of the quantities (4.4.1) (i.e. of the ξ_ν referred to above) is determined primarily (i.e. apart from "long range effects" in ν, i.e. effects that manifest themselves only on the α'-scale) by the first factor of (4.4.1), which is $a_{\nu-1} - a_\nu$. In the relevant comparisons of size, of course, only the relative sizes of the $a_{\nu-1} - a_\nu$ matter, those belonging to "remote" ν-regions (i.e. to regions relevantly separated on the α'-scale) being weighted against each other asymptotically by the limit of the second factor of (4.4.1). This is the function of α' given in (4.1.2). As a weight function this amounts to $(\alpha')^{-\frac{1}{2}}$, or, which comes asymptotically to the same thing, to $a_\nu^{-\frac{1}{2}}$.

With these reservations, then, the ordering of the ξ_ν in question corresponds to the ordering of the $a_{\nu-1} - a_\nu$. This quantity is the excess over a_ν of the next higher bid, $a_{\nu-1}$ -- i.e. the minimum overbid over a_ν.

This result lends itself to an easy and plausible interpretation. The minimum overbid over a given bid (a_ν) expresses in some sense the degree of defensibility of this particular bid. Indeed, it specifies the price (as an increase in risk when met by still higher bids, or by the same bid combined with the differential danger of a stronger hand) at which the bid in question can be defeated. It is plausible, that the strategic desirability of any particular bid should be reduced at will by allowing "cheap" overbids over it. The remarkable thing about our result is, that it furnishes (by way of (3.3.2), together with (3.8.2), (3.8.8)) a quantitative implementation for this qualitative insight. The asymptotically defined long range comparison principle for the minimum overbids $(a_{\nu-1} - a_\nu)_1$ with the help of the weight function introduced further above (a_ν^{-2}) is a particularly striking phase of this, but it is only one phase. The complete result is the one as summarized at the beginning of (3.9).

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THE DOUBLE DESCRIPTION METHOD[1]

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INTRODUCTION

The purpose of this paper is to present a computational method for the determination of the value and of all solutions of a two-person zero-sum game with a finite number of pure strategies, and for the solution of general finite systems of linear inequalities and corresponding maximization problems.

Although a system of linear equations forms a particular case of a system of linear inequalities, the comparative efficiency of numerical methods of solution is very different for the two. In the case of linear equations it is as yet undecided whether direct or iterative methods are generally preferable, the aim of both types of methods being to solve a system which is supposed to be non-singular and, therefore, to possess a unique solution.

For inequalities all numerical methods hitherto used, including probabilistic methods and descent methods with or without distance check, are essentially approximative and reach, after a finite or infinite number of steps, a particular solution. However, systems of linear inequalities correspond, in general, to singular systems of equations and may have none, one, or infinitely many solutions. If it is desired to determine the totality of solutions (in parametric form) these methods will not do.

The totality of solutions of a finite system of linear inequalities occupies a bounded or unbounded convex polyhedral set in Euclidean space whose description may be of theoretical interest. In applications when minimization in various directions is required, as for example, in optimization problems within an otherwise fixed economy with variable price vector, the full determination of the solution set may be necessary or time-saving.

This determination can be effected by any one of the direct

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methods. Among them determinants are impractical already for equations. The other direct methods amount to progressive reduction of the coefficient matrix by left or right multiplication.

A particular procedure of left multiplication is the elimination of the unknowns, one by one, and this method (which for equations, though very much older, is called the Gauss method, and is one of the best bets in the race for maximum efficiency) was proposed for linear inequalities by Fourier [1][2] but seems to be entirely out of question because of the huge number of elementary operations involved, which grows extremely fast with the number of unknowns and inequalities.

Right multiplication on the other hand, as represented by the successive solution of one, two, etc., inequalities until arriving at the whole system seems to be within the range of machine, and sometimes of hand, computation, as we hope to corroborate in the present paper. The advisability of the same method adapted to systems of equations is under consideration.

The idea was, as far as we know, first used in the dissertation of T. S. Motzkin [2, pp. 24-26] for purely theoretical purposes (a proof of a generalization of Minkowski's theorem on the finiteness of a base for all solutions of a finite system of linear inequalities).

Independently Raiffa, Thompson, and Thrall [6,7] developed the corresponding computational scheme for the case of finite games.[3] The exposition of this scheme is given in Parts I and II. The number of steps in determining the solutions for one player is equal to the number of pure strategies of the other player. Frequently the solutions for the first player can be used to shorten the computations for the solutions for the second player. A technique which enables one to determine optimal strategies for player II utilizing the computations for player I is being investigated.

Again independently, the computational scheme for general systems of linear inequalities was given by T. S. Motzkin [3]. This procedure forms the content of Part III.

PART I. INTUITIVE CONCEPTS OF THE GAMES

§1. NOTATION

Let $A = ||a_{ij}||$ be the matrix of a two-person zero-sum game in

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[2]Numbers in square brackets refer to the bibliography at the end of the paper.

[3]This procedure has since been adapted for Linear Programming.

which player I has m pure strategies $\alpha_1, \alpha_2, \ldots, \alpha_m$ and player II has n pure strategies $\beta_1, \beta_2, \ldots, \beta_n$ with payoffs a_{ij} to I and $-a_{ij}$ to II if the strategies (α_i, β_j) are chosen. A row vector $\xi = (\xi_1, \ldots, \xi_p)$ will be called a probability vector of dimension p if $\xi \geq 0$ (i.e., each $\xi_i \geq 0$) and if $\xi_1 + \ldots + \xi_p = 1$. We denote by S_p the set of all probability vectors of dimension p. We admit as mixed strategies for player I the set of all probability vectors ξ having m components and for player II the set of all probability vectors η having n components. The corresponding payoffs will be

$$E_I(\xi,\eta) = \xi A \eta^T = \sum_{\nu,\mu} \xi_\nu a_{\nu\mu} \eta_\mu \tag{1}$$

to I and $E_{II}(\xi,\eta) = -E_I(\xi,\eta)$ to II. (Here η^T denotes the transpose of η.)

Let the row and column vectors of the matrix A be denoted by

$$R_i = ||a_{i1}, \ldots, a_{in}|| , \qquad C_j = \left|\left| \begin{matrix} a_{1j} \\ \vdots \\ a_{mj} \end{matrix} \right|\right| \tag{2}$$

We then have

$$E_I(\xi,\eta) = \sum_\nu \xi_\nu (R_\nu \eta^T) = \sum_\mu (\xi C_\mu) \eta_\mu \tag{3}$$

The vector $x = x(\xi) = (x_1, \ldots, x_n)$ where $x_j = x_j(\xi) = \xi C_j$ is called the marginal vector corresponding to the strategy ξ for player I and similarly $y = y(\eta) = (y_1, \ldots, y_m)$ where $y_i = y_i(\eta) = R_i \eta^T$ is called the marginal vector corresponding to the strategy η for player II.

If player I chooses the strategy ξ then his payoff will be at least the minimum component of the vector $x(\xi)$. It is well known (cf. [7], p. 159) that the objective of player I is to select a strategy ξ so as to maximize the minimum component, thus he can be assured a payment of at least

$$v_1 = \max_\xi \min_j x_j(\xi) = \max_\xi \min_\eta E_1(\xi,\eta) . \tag{4}$$

(We use here the fact that $\min_\eta x(\xi)\eta = \min_j x_j(\xi)$.)

Similarly player II can play so as to be assured that the payoff to I is no more than

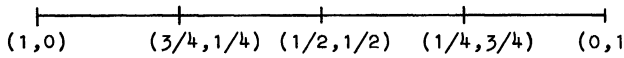
$$v_2 = \min_{\eta} \max_{i} y_i(\eta) = \min_{\eta} \max_{\xi} E_1(\xi,\eta) \quad . \tag{5}$$

The minimax theorem states that $v_1 = v_2 = v$ (for proofs see [5], p. 128, ff. and [10], p. 19).

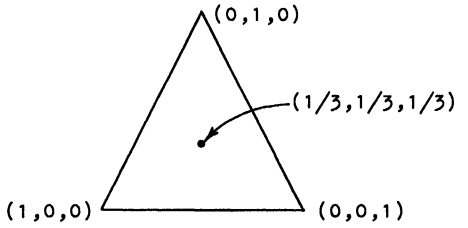
We say that $f(\xi) = \min_j x_j(\xi)$ is the minimum function for player I and call those ξ which maximize $f(\xi)$ optimal strategies for I. Similarly we call $g(\eta) = \max_i y_i(\eta)$ the maximum function for player II and call those η which minimize $g(\eta)$ optimal strategies for II.

§ 2. A GEOMETRIC MODEL

Let E_p be real cartesian space of dimension p and let E_{p-1} be the subspace of E_p consisting of those points with last coordinate zero. We choose barycentric coordinates in E_{p-1} based on some set of p scattered points (for instance the origin and the unit points on the first $p-1$ coordinate axes would do). A point p in E_{p-1} belongs to the closed simplex with these points as vertices if and only if its barycentric coordinates $c_1, \ldots, c_p$ constitute a probability vector. Thus we can realize S_p as a closed simplex in E_{p-1}, with the probability vector ξ represented by the point whose barycentric coordinates are $\xi_1, \ldots, \xi_p$. For examples, S_2 is realized by a line segment (with some typical coordinates)



and S_3 by a triangle, viz:



Any point P in E_p can be specified by the p barycentric coordinates $\xi_1, \ldots, \xi_p$ of its projection in E_{p-1} together with its final ordinary Cartesian coordinate x and we write $P = (\xi_1, \ldots, \xi_p; x)$ or for short $P = (\xi;x)$. (Here ξ need not be a probability vector.) We denote by Γ_p the set of all points $P = (\xi;x)$ in E_p for which $\xi \in S_p$

and call Γ_p the game space of dimension p.

Suppose that $h(\xi)$ is any real valued function defined for all vectors $\xi \in S_p$. We can represent $h(\xi)$ geometrically by the surface in Γ_p consisting of all points $(\xi;h(\xi))$ for $\xi \in S_p$.

Since only Γ_2 and Γ_3 can be realized pictorially, we now embark on a discussion of 2xn and 3xn games to motivate further algebraic considerations.

§ 3. THE 2 x 2 GAME

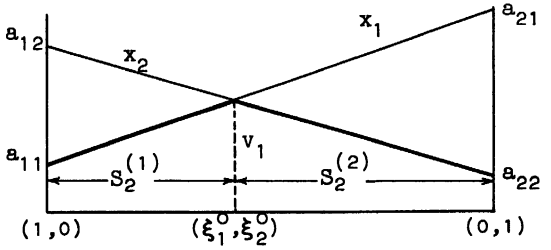
Let the play matrix be

$$A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$$

and corresponding to $(\xi_1,\xi_2) \in S_2$ we can find the marginals, viz:

$$x_1 = a_{11}\xi_1 + a_{21}\xi_2, \qquad x_2 = a_{12}\xi_1 + a_{22}\xi_2 .$$

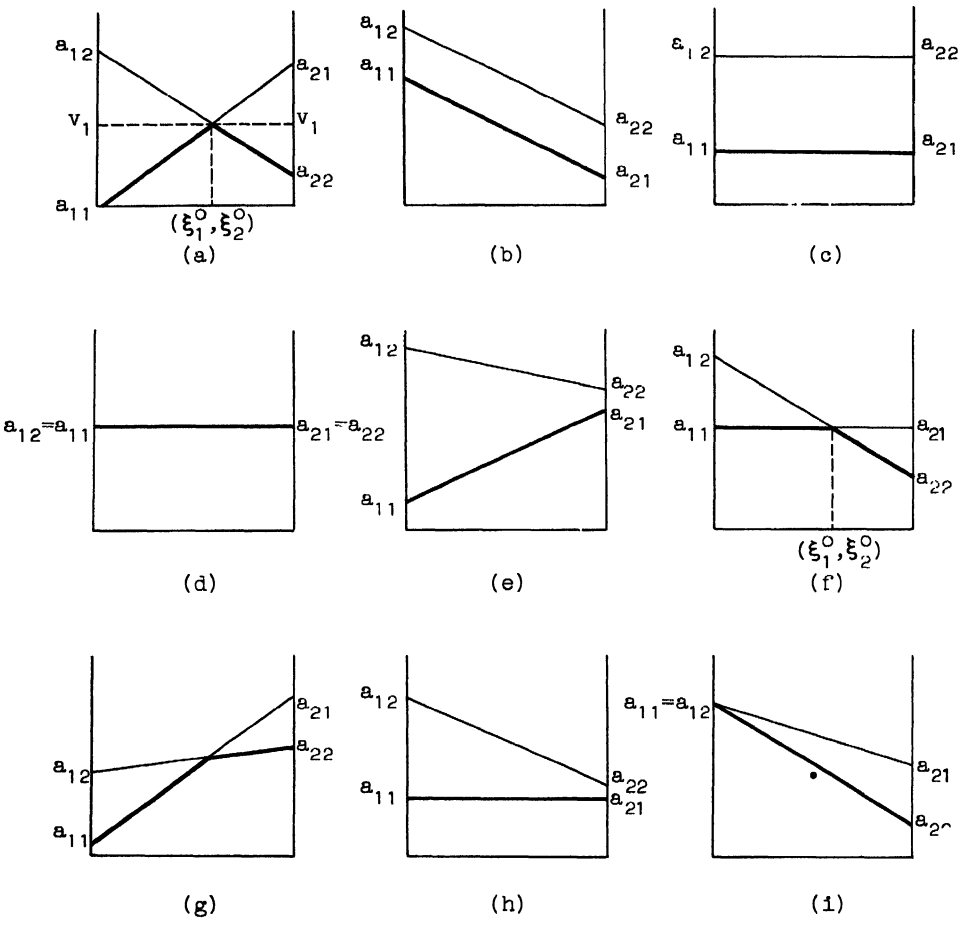
Geometrically, these marginals can be plotted, viz:



The two straight lines represent the marginals as a function of ξ, i.e., $x_1 = (\xi\ C_1)$, $x_2 = (\xi\ C_2)$. The dark line represents the minimum function, $f(\xi) = \min_j (\xi\ C_j)$ of player I. The maximum of this minimum function is v_1 which occurs at the optimal solution (ξ_1^o,ξ_2^o). Note that S_2 is decomposed into subsets $S_2^{(1)}$ and $S_2^{(2)}$. The above diagram incorporates all the essential features of the game when the optimal strategy (ξ_1^o,ξ_2^o) is an interior point of S_2.

We illustrate in the figures below typical examples of 2 x 2

games including all possible ordering relationships among the elements of the game matrix.



In the cases (c), (d), and (h) all strategies for I are optimal. In case (f) there is a subset of optimal strategies. In cases (b), (e), (g), and (i) the optimal strategies occur on the boundaries and thus I has a unique pure strategy. In all cases the set of optimal strategies forms a convex subset of S_2. Those optimal strategies which occur at the intersection of two of the four lines $x = \xi C_1$, $x = \xi C_2$, $\xi_1 = 0$, $\xi_2 = 0$ are called basic strategies. In each case the convex set generated by the basic strategies comprises the set of optimal strategies. Algebraically then, the basic strategies are the solutions of a proper system of linear equations.

To find the point of intersection of the marginal lines for player I we solve the system:

$$a_{11}\xi_1 + a_{21}\xi_2 - x = 0$$

$$a_{12}\xi_1 + a_{22}\xi_2 - x = 0$$

$$\xi_1 + \xi_2 = 1$$

The determinant of the system is

$$\phi(A) = \begin{vmatrix} a_{11} & a_{21} & -1 \\ a_{12} & a_{22} & -1 \\ 1 & 1 & 0 \end{vmatrix} = a_{11} - a_{12} + a_{22} - a_{21}$$

which is easily seen to be the sum of the elements of the adjoint of A. By Cramer's rule we have, provided $\phi(A) \neq 0$,

$$\xi_1 = \frac{a_{22}-a_{21}}{\phi(A)}, \qquad \xi_2 = \frac{a_{11}-a_{12}}{\phi(A)}, \qquad x = \frac{\det A}{\phi(A)}.$$

If we define the special matrices

$$1^R = \|1 \quad 1\|, \qquad 1^C = \begin{Vmatrix} 1 \\ 1 \end{Vmatrix}$$

then we can write the solution as

$$\xi = \frac{1^R(\text{adj } A)}{\phi(A)}, \qquad x = \frac{\det A}{\phi(A)}.$$

By a similar analysis, the point of intersection of the marginals for II is

$$\eta = \frac{(\text{adj } A)1^C}{\phi(A)}, \qquad \cdot\cdot = \frac{\det A}{\phi(A)}.$$

In cases (b), (c), and (d) the marginal lines are parallel and $\phi(A) = 0$. In cases (e) and (h), $\phi(A) \neq 0$ but the point of intersection lies outside of the game space and does not represent a basic solution. In case (g) the point of intersection lies in the game space but again it is not a basic solution as it is not the maximum of the minimum function.

If we now include the lines $\xi_1 = 0$ and $\xi_2 = 0$ as possible equations we get additional points of intersection, namely $(1,0;\ a_{11})$, $(1,0;\ a_{21})$, $(0,1;\ a_{12})$, $(0,1;\ a_{22})$. Algebraically, then, a method of procedure would be to find all points of intersection (which we call critical points). Those critical points which lie in the game space and lie on

the maximum of the minimum function represent the basic solutions. The entire set of optimal solutions consists of convex linear combinations of these basic solutions.

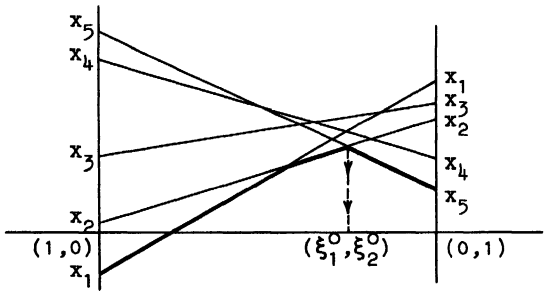
We will indicate subsequently why this procedure of computing all critical points is not computationally economical. However, this method generalizes to give the results in [8], page 35, concerning the determination of the basic solutions.

§ 4. THE 2 x n GAME

Here the play matrix is of the form

$$\left\| \begin{matrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \end{matrix} \right\|$$

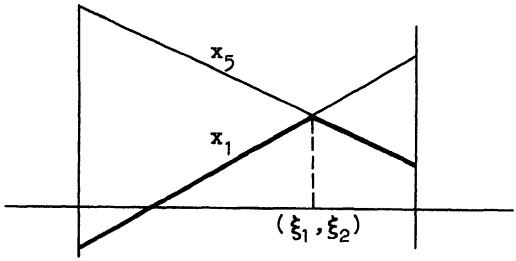
and the marginals are $x_j(\xi) = (\xi C_j)$ for $j = 1, 2, \ldots, n$. Again, we plot these marginal lines in Γ_2. As a typical example let us consider the diagram for a 2 x 5 case:



The minimum function for player I is given by the broken dark line, the optimal strategy is (ξ_1^0, ξ_2^0), and the value of the game is v. In the above diagram there are nineteen critical points involved, some of which are obviously unimportant. Since in higher dimensional cases we cannot have the support of a diagram and hence cannot see which points are important, we want now to devise a computational procedure from purely algebraic considerations. In Part II we will discuss the formal details of the computational procedure, but for the present we use the above diagram to illustrate our procedure.

Since $x_1(\xi)$ and $x_5(\xi)$ take on minimum values on the boundary

planes $\xi_1 = 0$ and $\xi_2 = 0$ we first consider the minimum function of these two marginal lines, viz:



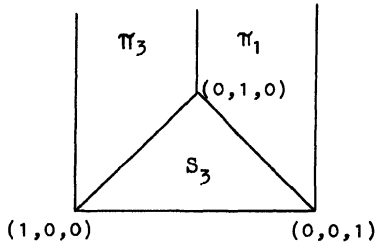
Now evaluating x_3 and x_4 at the critical points $(1,0)$, (ξ_1, ξ_2), and $(0,1)$ we note that these lines lie above the minimum function and so may be discarded. But x_2 at (ξ_1, ξ_2) lies below the critical point and hence we must find the points of intersection of x_1 and x_2 and of x_2 and x_5. We then discard the critical point arising from the intersection of x_1 and x_5. The critical point with the largest x value then gives the optimal strategy and the value of the game.

§ 5. THE 3 x n GAME

In the 3 x n game

$$\begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \end{Vmatrix}$$

we consider the game space Γ_3, viz:



Each marginal function $x_j(\xi)$ can be represented as a plane in Γ_3. It is now possible to represent geometrically I's minimum function and to find the optimal strategies and value, v, of the game. A basic strategy, $\xi^{(o)}$, is an optimal solution such that $(\xi^{(o)}, v)$ is a point of intersection of 3 planes (we include boundary planes π_1; $\xi_1 = 0$,

$\pi_2;\ \xi_2 = 0,\ \pi_3;\ \xi_3 = 0)$.

The geometrical model for the three dimensional case can be constructed analogously to the two dimensional model with planes playing the role of straight lines. It is suggested the the reader keep this model in mind while reading the next part of this paper.

PART II. THE COMPUTATIONAL PROCEDURE

§ 6. SUMMARY OF PROBLEM

As we have seen, the goal of player I is to find those ξ which maximize his minimum function $f(\xi)$. Since the marginals $x_j = \xi C_j$ are hyperplanes in Γ_m, the minimum surface $x = f(\xi)$ in E_m is a polyhedron and so can be completely described by giving its vertices which are, of course, finite in number. We now develop a systematic computational procedure for determining these vertices. Suppose the vertices are $Q_1 = (\xi^{(1)};\ x^{(1)}),\ \ldots,\ Q_s = (\xi^{(s)};\ x^{(s)})$. Then $v = \max_j x^{(j)}$ is the value of the game and the $\xi^{(j)}$ for which $x^{(j)} = v$ are basic optimal strategies in the sense that the set of all optimal strategies is just the set of points in the smallest convex subspace of S_m which contains these maximizing $\xi^{(j)}$, see [8], page 28, 30. It can be shown that the two definitions of basic strategies are equivalent, see [6], appendix.

§ 7. THE SOLUTIONS FOR PLAYER I

A basic component of the computational technique, to be discussed, will be to find the intersection of the line segment joining two points of Γ_m and a hyperplane in Γ_m. To this end, we introduce the following lemma.

LEMMA. The line segment joining the two points $P_i = (\xi^{(i)};\ x^{(i)})$ and $P_j = (\xi^{(j)};\ x^{(j)})$ in Γ_m intersects the hyperplane: $\{(\xi;\xi C_k);\ \xi \in S_m\}$ if and only if $d_i^{(k)}\, d_j^{(k)} \leq 0$ where $d_i^{(k)} = \xi^{(i)} C_k - x^{(i)}$ and $d_j^{(k)} = \xi^{(j)} C_k - x^{(j)}$. In particular, if $d_i^{(k)}\, d_j^{(k)} < 0$ then the point of intersection, P, is unique, and given by the relation:

$$P = \frac{-d_j^{(k)} P_i + d_i^{(k)} P_j}{d_i^{(k)} - d_j^{(k)}}$$

Proof. The line segment joining P_i and P_j can be parametrized

by the relations $t\,P_i + (1 - t)P_j$ where $0 \leq t \leq 1$. This line segment will intersect the hyperplane if and only if there exists a t in the interval $[0,1]$ such that

$$(t\,\xi^{(i)} + (1 - t)\xi^{(j)})C_k - (t\,x^{(i)} + (1 - t)x^{(j)}) = 0 ,$$

or if

$$t\,d_i^{(k)} + (1 - t)d_j^{(k)} = 0 . \tag{7.1}$$

If $d_i^{(k)} = 0$, then the point P_i lies in the hyperplane. If $d_j^{(k)} = 0$ then the point P_j lies in the hyperplane. If $d_i^{(k)} = d_j^{(k)} = 0$ then the line segment lies in the hyperplane. If both d_i and d_j are different from zero and of the same sign then there is no solution of 7.1 for $0 \leq t \leq 1$. If, however, $d_i^{(k)}$ and $d_j^{(k)}$ are different from zero and of different sign (i.e., $d_i^{(k)} d_j^{(k)} < 0$) then there is a unique solution of (7.1) given by

$$t = \frac{-d_j^{(k)}}{d_i^{(k)} - d_j^{(k)}} , \qquad 1 - t = \frac{d_i^{(k)}}{d_i^{(k)} - d_j^{(k)}} ;$$

and then the point of intersection is

$$P = \frac{-d_j^{(k)}\,P_i + d_i^{(k)}\,P_j}{d_i^{(k)} - d_j^{(k)}}$$

which completes the proof of the lemma.

For computational purposes we introduce the column vector

$$C_k' = \begin{Vmatrix} C_k \\ -1 \end{Vmatrix}$$

and then $d_i^{(k)} = P_i C_k'$, $d_j^{(k)} = P_j C_k'$.

In computation of points it is necessary to know not only the coordinates of the point but also all the planes (or hyperplanes) which pass through the point. Let π_i denote the boundary plane (of Γ_m) which has the equation $\xi_i = 0$ $(i = 1, 2, \ldots, m)$. The planes in question are the m boundary planes π_i plus the n marginal planes $C_j (j = 1, 2, \ldots, n)$. A point in Γ_m is determined by the intersection of m planes and we adopt the notation

$$\tilde{P} = \left(\frac{\xi_1, \ldots, \xi_m; x}{T_1, \ldots, T_m} \right) = \left(\frac{P}{T_1, T_2, \ldots, T_m} \right)$$

where $T_1, \ldots, T_m$ are m of the $n + m$ planes mentioned above. We may later wish to add other planes to this symbol if they contain the point P but this will become clear in the following discussion and in the examples. We call $\tilde{P}$ the labelled point corresponding to P.

A "corner point" of a plane C_j is the piercing point of the plane and a line which is the intersection of $m-1$ boundary planes. For example in the 2×2 case

$$\left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\|$$

the corner points of C_1 with the appropriate labels are

$$\tilde{P}_1 = \left(\frac{1,0;a_{11}}{C_1, \pi_2} \right) \quad \text{and} \quad \tilde{P}_2 = \left(\frac{0,1;a_{21}}{C_1, \pi_1} \right)$$

At a "corner" of Γ_m it is clear which plane is part of the minimum function -- namely the plane or planes which contains the row minima corresponding to that corner. Hence we assume the matrix A is written so that the columns containing the row minima have the lowest index since these planes will certainly occur in the minimum function. Preferably, we consider those columns containing the most row minima first since they probably are the most important.

The first step in the procedure is to write down the labelled corner points of C_1; i.e.,

$$\tilde{P}_1 = \left(\frac{1,0,\ldots,0;a_{11}}{C_1, \pi_2, \ldots, \pi_m} \right)$$

$$\tilde{P}_2 = \left(\frac{0,1,\ldots,0;a_{21}}{\pi_1, C_1, \pi_3, \ldots, \pi_m} \right)$$

$$\cdots \quad \cdots$$

$$\tilde{P}_m = \left(\frac{0,0,\ldots,1;a_{m1}}{\pi_1, \pi_2, \ldots, \pi_{m-1}, C_1} \right)$$

These m points completely determine C_1. They form the initial set of critical points in our computational procedure.

Next evaluate $d_i^{(2)} = \xi^{(1)} C_2 - x^{(1)} = P_1 C_2'$ for $i = 1, 2, \ldots, m$. Now we want to add new critical points to those obtained above according to the following rule;

Find the piercing point of the line segment joining P_{i_1} and P_{i_2} and the plane C_2 if

1. $d_{i_1}^{(2)}\, d_{i_2}^{(2)} < 0$, and
2. the line segment joining P_{i_1} and P_{i_2} is an edge of the minimum surface bounded by $\pi_1, \ldots, \pi_m$ and C_1.

This critical point is to be added to the above set and labelled with C_2 and the planes common to P_{i_1} and P_{i_2}.

A geometrical interpretation of the above step is the following. Condition 2 insures that the two points lie on an edge of the minimum surface determined by the planes already considered in the process. Condition 1 insures that C_2 is above one of the points and below the other point so the line between them pierces C_2. Hence this piercing point is important in the construction of $f(\xi)$.

To all labelled points $\tilde{P}_i$ such that $d_i^{(2)} = 0$ we must add the plane C_2 to the label since it may be important later on to remember that C_2 passed through these points (e.g., in satisfying Condition 2 above for later steps in the process).

Next we eliminate all critical points $\tilde{P}_i$ with $d_i^{(2)} < 0$ since plane C_2 goes below them.

Finally we add all corner points of C_2 which are relative row minima of the payoff matrix, e.g., those whose y coordinates satisfy

$$a_{i2} < a_{i1} .$$

We take account of the results of the last two operations by means of index sets defined as follows: let K_1 be the set of integers $1, 2, \ldots, m$; let K_2 be the set of integers $i (1 \leq i \leq k_2)$ (where k_2 is the total number of points considered in the first two stages) which are the indices of those points not eliminated in the first two stages.

We now have found all the important critical points of $f(\xi)$ which arise from the first two columns. Next take the column C_3 and repeat the above process, then C_4 etc. up to C_n.

In order to avoid introduction of extraneous points which are not vertices we need a criterion which will tell us, at any stage, whether or not two vertices are adjacent (i.e., connected by an edge). A necessary condition at the end of the h-th stage for P_i and P_j to be adjacent is that they shall have at least m-1 planes of the set $\Sigma^{(h)} = (\pi_1, \ldots, \pi_m, C_1, \ldots, C_h)$ in common. However, it may happen that certain planes will be dependent and then the number of common planes can be greater than m-1. Suppose that P_i and P_j have the subset $\Sigma_{ij}^{(h)} = (T_1, \ldots, T_r)$ of $\Sigma^{(h)}$ as their set of common planes. Then P_i and P_j

will be adjacent if and only if no other vertex P_h is on all of the planes of $\Sigma_{ij}^{(h)}$.

In the practical determination of adjacent vertices the plan followed here is first to check for all pairs of vertices with at least m-1 planes in common and then to determine which of these pairs are actually adjacent.

We summarize the procedure now in a minimal set of rules given in an inductive form.

STEP 1: 1. Label columns containing the row minima of A with lowest index.

2. Compute the initial set of critical points consisting of the m corner points of C_1, and initial index set $K_1 = \{1, 2, \ldots, m\}$.

STEP j+1: Assume the j^{th} step $(j = 1, 2, \ldots, n-1)$ to be completed with critical points $P_1, \ldots, P_{k_j}$ and index set K_j obtained.

1. For each critical point $P_i = (\xi^{(i)}; x^{(i)})$, $i \in K_j$ compute $d_i^{(j+1)} = P_i C_{j+1}'$.
2. Add the symbol C_{j+1} to the label of all points $\tilde{P}_i$ such that $d_i^{(j+1)} = 0$.
3. Find the piercing point of the line between P_{i_1} and P_{i_2} and the plane C_{j+1} if
   (a) $d_{i_1}^{(j+1)} \; d_{i_2}^{(j+1)} < 0$
   (b) $m-1$ of the planes in each of the labelled points $\tilde{P}_{i_1}$ and $\tilde{P}_{i_2}$ are the same, and the vertices P_{i_1} and P_{i_2} are adjacent.

   This point is to be added to the set of critical points and the planes in its symbol are the planes which the two points above have in common together with C_{j+1}.
4. Add to the set of critical points all corner points of C_{j+1} which are relative row minima, i.e., whose x values satisfy

$$a_{i,j+1} < \min \{a_{i1}, a_{i2} \cdots, a_{ij}\} .$$

5. Eliminate all critical points P_i with $d_i^{(j+1)} < 0$.
6. On the basis of rules 3, 4, and 5 compute the index set K_{j+1} from the set K_j.

At the conclusion of n such steps a finite set of critical points which completely describe $f(\xi)$ is obtained.

§ 8. EXAMPLE

The computational technique for the game

$$A = \begin{Vmatrix} 1 & 3 & 2 & 13/5 & 10/3 \\ 2 & 0 & 5/2 & 1 & 1/2 \\ 3 & 2 & 1 & 8/5 & 1 \end{Vmatrix}$$

is given in Table I.

	P	ξ			x	C_1	C_2	C_3	C_4	C_5	1	2	3	d^2	d^3	d^4	d^5
	1	1	0	0	1	x						x	x	2	1	8/5	7/3
Step 1	2	0	1	0	2	x					x		x	-2			
	3	0	0	1	3	x					x	x		-1			
	4	1/2	1/2	0	3/2	x	x						x		3/4	3/10	5/12
	5	1/3	0	2/3	7/3	x	x					x			-1		
Step 2	6	0	0	1	2		x				x	x			-1		
	7	0	1	0	0		x				x		x		5/2	1	1/2
	8	0	2/7	5/7	10/7		x	x	x		x					0	-4/7
	9	2/3	0	1/3	5/3	x		x				x				3/5	8/9
Steps 3,4	10	3/7	2/7	2/7	13/7	x	x	x	x	x						0	0
	11	0	0	1	1			x			x	x				3/5	0
Step 5	12	0	2/3	1/3	2/3		x			x	x						

TABLE I

STEP 1. Relative to column C_1 we get points P_1, P_2 and P_3

together with the bookkeeping symbols denoted by x's in the appropriate columns.

STEP 2. Fill in column $d^{(2)}$ with entries $d_i^{(2)} = P_i C_2'$ for $i = 1, 2, 3$. Compute piercing points P_4 and P_5 by formula given in Lemma. Label appropriate planes in bookkeeping section. Add the corner points P_6 and P_7. Strike out P_2 and P_3 so that $K_2 = \{1, 4, 5, 6, 7\}$.

STEP 3. Fill in column $d^{(3)}$ with entries $d_i^{(3)} = P_i C_3'$, $i \in K_2$. Compute piercing points P_8, P_9, P_{10}. Label appropriate planes in bookkeeping section. Add corner point P_{11}. Strike out P_5 and P_6 so that $K_3 = \{1, 4, 7, 8, 9, 10, 11\}$.

STEP 4. Fill in columns $d^{(4)}$ with entries $d_i^{(4)} = P_i C_4'$, $i \in K_3$. Since $d_i^{(4)} \geq 0$ for all $i \in K_3$ there are no new points. Add C_4 to labelled points $\tilde{P}_8$ and $\tilde{P}_{10}$. Now $K_4 = K_3$.

STEP 5. Fill in columns $d^{(5)}$ with entries $d_i^{(5)} = P_i C_5'$, $i \in K_4$. Compute piercing point P_{12} and delete point P_8. Add C_5 to labelled points $\tilde{P}_{10}$ and $\tilde{P}_{11}$.

The value of the game is given by $\tilde{P}_{10}$. The optimal strategy for player I is $(3/7, 2/7, 2/7)$ yielding a value of $13/7$. Planes C_1, C_2, C_3, C_4, and C_5 all pass through this point.

§ 9. THE SOLUTION FOR PLAYER II

To obtain the entire minimum function for player II it is necessary to take the negative transpose of A and repeat the process as described above. However, if one is only interested in obtaining all basic solutions for player II then once the basic solutions for player I have been determined the solution for player II is (possibly) made somewhat easier. If for any basic solution ξ^0 of I the marginal $\xi^0 C_j$ is greater than v, then eliminate the column C_j from A. After all such eliminations have been made call the matrix so obtained B. The columns of B are the totality of all non-boundary planes which are common to the symbols of all basic points of $f(\xi)$. Suppose that B is an m by p matrix where $p \leq n$. The solutions of the game with B as payoff matrix are the same as those with A as payoff matrix, since II will not play with positive probability any column of A which has marginal $> v$ for any basic solution of I.

To find the solution for player II it is now only necessary to take the negative transpose of B and solve the resulting game according to the process described above. The basic solutions for the game with A are then obtained by extending the basic solutions for the game with B to n-dimensional vectors by putting 0's in the n-p places corresponding to columns eliminated according to the paragraph above.

The minimum function for player II for the example of section 8 is discussed in [6], page 34. There are three basic strategies for player II, namely (7/21, 4/21, 10/21, 0, 0), (3/7, 0, 5/14, 0, 3/14), and (21/63, 0, 22/63, 20/63, 0).

PART III. GENERAL INEQUALITIES AND SECOND VARIANT OF THE COMPUTATIONAL PROCEDURE

§ 10. ROUGH PROCEDURE

Let $\sum a_{jk}x_k \geq 0$, $j = 1, \ldots, h$; $k = 1, \ldots, n$; be a system of h inequalities in n unknowns. Suppose that the general solution of the system formed by the first $h - 1$ inequalities is given by $x_k = \sum p_{kl}\lambda_l$, with fixed p_{kl}, λ_l taking on all non-negative values. Then the h^{-th} inequality will determine a half-space (or the whole space) both in x-space and λ-space. Putting $b_l = \sum a_{hk}p_{kl}$, the common part of the halfspace $\sum b_l\lambda_l \geq 0$ and of the positive λ-orthant will be the convex cone generated by the rays from 0 toward e_l (e_l denoting the l^{-th} unit vector) for every l with $b_l \geq 0$, and $(|b_{l_2}|e_{l_1} + |b_{l_1}|e_{l_2})$ for every l_1 and l_2 with $b_{l_1}b_{l_2} < 0$. Denoting p_{kl} for $b_l \geq 0$ by q_{km}, $m = m(l) = 1, \ldots, s$, and $p_{kl_1}|b_{l_2}| + p_{kl_2}|b_{l_1}|$ for $b_{l_1}b_{l_2} < 0$ by q_{km}, $m = m(l_1, l_2) = s + 1, \ldots, t$, the complete solution of the h inequalities will be given by $x_k = \sum_{m=1}^{t} q_{km}\mu_m$; $\mu_m \geq 0$.

§ 11. A GEOMETRIC INTERPRETATION. REDUCTION TO ESSENTIALS

The parametric solution just established means that the solutions occupy a convex polyhedral cone in n-space with the origin as vertex. The induction step corresponds to finding the subcone cut off by an additional hyperplane through 0. Some basic rays are the same as before this step; others are obtained by joining two rays on each side of the hyperplane by a plane and cutting this plane with the hyperplane. It is not supposed that the base before this step contains a minimal number of rays, and even if this is the case the use of all pairs of rays as indicated is likely to introduce many superfluous rays in the new base. It is clearly desirable -- and for numerical computations even imperative -- to have a shortest description of the polyhedral cone and to stay with it during the whole procedure.

First consider a pointed polyhedral cone (that is, a cone not containing a straight line, or equivalently a cone with a unique vertex) in n-dimensional space and not in a lower-dimensional space. Here a concise

description consists of an enumeration of the extremal rays, that is, those rays which are not between any two rays of the cone, and of the hyperplanes containing the (n-1)-dimensional faces of the cone. These extremal rays and hyperplanes are uniquely determined by the cone, and the double and self-dual description consisting of their coordinates and coefficients, respectively, furnishes much of the useful information needed to determine, for instance, maximizing rays or hyperplanes, the dual cone, the intersection and convex hull of two cones, the extension of the cone by an additional ray, and dually, as in the above induction step, the common part of the cone and a halfspace.

If the cone while still n-dimensional contains a maximal linear variety, say of d dimensions, the description will be the same as far as hyperplanes are concerned but the rays indicated will fall into two subsets, the central part and the extremal part. The central part describing the linear variety will consist of $d + 1$ rays $a_0, a_1, \ldots, a_d$ or perhaps more conveniently of d pairs of opposite rays $a_1, -a_1, \ldots, a_d, -a_d$. The extremal part will contain one ray for each $(d + 1)$-dimensional face of the cone. Every ray of this kind is only determined modulo an arbitrary vector of the d-dimensional variety of vertices.

If the cone has less than n, say l, dimensions this will not affect the description by rays. But now the hyperplanes will fall into a central part and an extremal part. The central part ($n - l$ hyperplanes) describes the smallest linear variety containing the cone. Every highest-dimensional face within this variety is singled out by one hyperplane in the extreme part.

The full double description scheme at every stage looks like this:

	C	D
A	0	0
B	0	M

where A, B, C, D, M, and the 0's are matrices.

Each row of A contains the coordinates of a ray of the central part, while the rows of B give the rays of the extremal part. Every column of C consists of the coefficients of a hyperplane of the central part, while D is in the same way composed from those belonging to the extremal part. In the intersection of each row of A or B and each column of C or D stands the scalar product of both, which is always non-negative. A and C consist of those rows and columns respectively for which all scalar products in which they enter are zero, since the rays of A lie in all hyperplanes, and the hyperplanes of C contain all rays. Thus M has no zero row or column.

Two rows of B correspond to adjacent $(d + 1)$-dimensional faces of the cone if and only if the entire set of columns of D for which both give zero product does not simultaneously give zero products for any other

row of B (cf. page 63). In fact two faces of lowest dimension are adjacent if and only if the dimension of the face joining them exceeds their dimension by one unit only, and if and only if this joining face contains only the two given faces.

To reduce the case of inhomogeneous inequalities to the above scheme, homogenize by writing an additional unknown ξ beside the constants and adding the inequality $\xi \geq 0$. The points with $\xi = 0$ lie in the hyperplane at infinity.

Now suppose that a full double description has been obtained at a certain stage, and a new step is to be taken corresponding to a hyperplane H. We write the coefficients of H as an additional column next to C and D and compute the inner products with the rows of A and B. If the products involving H are all non-negative, then the cone defined by H and the hyperplanes of C and D is the same as the cone defined by C and D above, which by supposition is fully described. Hence H is superfluous (it either gives no face at all or a face already indicated) and should be omitted.

Secondly, if the products involving H and rows of A are all zero, while there exist rows of B for which the product is negative, each such row should be thrown out. However, before throwing a row out it should be combined with each row of B corresponding to an adjacent face, as explained before, and giving a positive product. By combining we mean determining the intersection ray of the plane corresponding to the two rows and of H and writing the row obtained as an additional row to B. The determination can be effected by the formula $p_{kl_1} | b_{l_2} | + p_{kl_2} | b_{l_1} |$ of Section 10, or as in the computing instruction below.

Finally we have to consider the case where not all products of rows of A involving H vanish. We choose as normal form of A a description by d pairs of opposite vectors; hence the product with H is positive for a certain row a of A (if there are several such rows choose one). Then one has to: 1) delete a and -a and adjoin a to B; 2) delete every row $a^* \neq a$ of A giving a positive product while replacing $-a^*$ by its "combination" defined as before with a and by the negative of that combination; 3) replace every row of B giving a negative or positive product by its combination with a or -a, respectively.

§ 12. SPECIAL CASE: $x \geqq 0$

If the inequalities $x_1 \geqq 0, \ldots, x_n \geqq 0$ occur in the given system, as for instance in the game case treated in Parts I and II, these inequalities can be taken first and since their solution is the positive orthant, d is already zero, the central part A is void and the last-

mentioned possibility in the procedure of Section 11 does not occur. If we suppose that the given system is non-degenerate in the sense that no $n + 1$ of the (inhomogeneous) linear functions vanish at the same point, which is always the case after a small change of the coefficients, then also the condition for adjacency takes on an especially simple form and a computing instruction for such a system would be as given below.

The computation at the end (see Tables II and III) concerns a submatrix of the diet-matrix in [9]. In this example the matrices A and C of the schematic diagram above are absent. The rows L 1, ..., L n (L i represents the i-th row below L), corresponding to the inequalities $x_1 \geq 0$, ..., $x_n \geq 0$, are the initial entries in the matrix D which is written as rows rather than columns for convenience in tabulation. The additional rows L n + 1, ..., L n + h of D, corresponding to the other given inequalities, are used successively in the computation. The entries in L n + h + 1 are the coefficients of the function to be minimized. Column 0 of the table contains the constants of the linear inequalities and, below P, the homogenizing coordinates of the vertices. Columns 1 through 6 contain the coefficients of the linear inequalities and of the function to be minimized and the coordinates of the vertices.

The rows P 1, ..., P n + 1 (P i means the i-th row under P) constitute the matrix B at the beginning of the computation. During the computation rows P n + 2, ... are incorporated in B, while some P-rows may cease to belong to B. The scalar products in the matrix M used in the computations appear in columns n + 1, ..., n + h + 1. The arrows indicate the correspondence between the L-rows and these latter columns. Columns -3, -2, and -1 are used to record labels and side calculations.

The final polyhedron is described by B in its final form, namely, it has vertices P 1 to P 35 except P 1, P 8, P 10, P 11, P 12, P 20; and by D in its final form, namely, by the five-dimensional sides L 0 to L 10 except L 8, where L 0 denotes the hyperplane at infinity.

Since row P 19 gives the smallest value in column 11, it is the only solution of the minimization problem.

ABBREVIATIONS. L 3 means the third row below L.
P 3 means the third row below P.
P 3 7 means the number in row P 3 and column 7.

ORDER OF STEPS. Perform step 1 A, then all steps 1 B, then step 2 A, then steps 2 B, step 3 A, ..., until step h B.

STEP s A. Compute entries P k n+s in the $n+s^{th}$ column as far down as possible by formula 1, except that if P k 0 = 0 write P k n+s = ∞; and omit P k n+s if P k n+h+1 is an x.

STEP s B. For every k for which P k n+s $<$ 0 and for every l for which P l n+s $>$ 0 or P l n+s = ∞, and for which P k and P l

-4 -3 -2 -1	0 1 2 ... n	n+1 ... n+h n+h+1
L		
1	0 1 0 ... 0	
2	0 0 1 ... 0	
...	...	
n	0 0 0 ... 1	
n+1 ... n+h	given	
n+h+1	0 1 1 ... 1	
P		
1	1 0 0 ... 0	
2	0 1 0 ... 0	
...	...	
n+1	0 0 0 ... 1	
n+2 n+3 ...	to be computed	

TABLE II

have n-1 common zeros perform substep n+s k 1. When through all substeps n+s k 1 belonging to the same P k n+s $<$ 0 (there may be none at all), make an x in P k n+h+1 and in all free places in columns n+1 to n+h+1 in row P k.

SUBSTEP n+s k 1. Start a new row by its number, say m, in column -4. Wrtie k in column -3, 1 in column -2, and 1000 |P k n+s|/(|P k n+s| + P 1 n+s) in column -1 (to be omitted if P 1 n+s = ∞); write 1000 in column 0 (n.b., we multiply our computations by 1000 to avoid decimals); and write 0 in each of the n-1 columns in which P k and P 1 have common zeros and in column n+s. The remaining P m j, j = 1, ..., n, compute by formula 2, except that if P 1 n+s = ∞, use formula 3. (Other entries in row P m coming before column n+s are unnecessary for subsequent calculation and are marked by an x.)

FORMULA 1. P k n+s = (P k 0 . L n+s 0) + (P k 1 . L n+s 1) + ... + (P k n . L n+s n).

FORMULA 2. P m j = P k j + 1/1000[P m (-1)(P 1 j - P k j)].

FORMULA 3. P m j = 1000 |L n+s 0| / L n+s j unless L n+s j = 0 in which case write P m j = 1000 |L n+s 0|.

-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
L															
1				0	1	0	0	0	0	0					
2				0	0	1	0	0	0	0					
3				0	0	0	1	0	0	0					
4				0	0	0	0	1	0	0					
5				0	0	0	0	0	1	0					
6				0	0	0	0	0	0	1					
7				-3	25	12	26	18	6	2					
8				-70	786	393	655	651	283	94					
9				-1	1	15	1	4	10	3					
10				-12	203	183	72	245	6	27					
11				0	1	1	1	1	1	1					
P															
1				1	0	0	0	0	0	0	-3	x	x	x	x
2				0	1	0	0	0	0	0	∞	∞	∞	∞	∞
3				0	0	1	0	0	0	0	∞	∞	∞	∞	∞
4				0	0	0	1	0	0	0	∞	∞	∞	∞	∞
5				0	0	0	0	1	0	0	∞	∞	∞	∞	∞
6				0	0	0	0	0	1	0	∞	∞	∞	∞	∞
7				0	0	0	0	0	0	1	∞	∞	∞	∞	∞
8	1	2		1000	120	0	0	0	0	0	0	24320	-880	x	x
9	1	3		1000	0	250	0	0	0	0	0	28250	2750	33750	250
10	1	4		1000	0	0	115	0	0	0	0	5325	-885	x	x
11	1	5		1000	0	0	0	166	0	0	0	38066	-336	x	x
12	1	6		1000	0	0	0	0	500	0	0	71500	4000	-9000	x
13	1	7		1000	0	0	0	0	0	1500	0	71000	3500	28500	1500
14	8	2		1000	1000	0	0	0	0	0	x	x	0	191	1000
15	8	9	242	1000	91	61	0	0	0	0	0	x	0	17636	152
16	8	12	180	1000	98	0	0	0	90	0	0	x	0	8434	188
17	8	13	200	1000	96	0	0	0	0	300	0	x	0	15588	396
18	10	4		1000	0	0	1000	0	0	0	x	x	0	60000	1000
19	10	9	243	1000	0	61	87	0	0	0	0	x	0	5427	148
20	10	12	181	1000	0	0	94	0	91	0	0	x	0	-4686	x
21	10	13	202	1000	0	0	92	0	0	303	0	x	0	2832	395
22	11	5		1000	0	0	0	250	0	0	x	x	0	49250	250
23	11	9	108	1000	0	27	0	148	0	0	0	x	0	29201	175
24	11	12	077	1000	0	0	0	153	39	0	0	x	0	25719	192
25	11	13	087	1000	0	0	0	152	0	131	0	x	0	28777	283
26	12	6		1000	0	0	0	0	2000	0	x	x	x	0	2000
27	12	9	211	1000	0	53	0	0	394	0	0	x	x	0	447
28	12	13	240	1000	0	0	0	0	380	360	0	x	x	0	740
29	12	16	516	1000	51	0	0	0	288	0	0	x	x	0	339
30	12	24	259	1000	0	0	0	40	381	0	0	x	x	0	421
31	20	16	357	1000	35	0	60	0	91	0	0	x	x	0	186
32	20	18	072	1000	0	0	159	0	84	0	x	x	0	0	243
33	20	19	463	1000	0	28	91	0	49	0	0	x	0	0	168
34	20	21	627	1000	0	0	93	0	34	189	0	x	0	0	316
35	20	24	154	1000	0	0	80	24	83	0	0	x	0	0	187

TABLE III

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SOLUTIONS OF CONVEX GAMES AS FIXED-POINTS[1]

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§ 1. INTRODUCTION

In a game with a finite number of pure strategies the choice of a mixed strategy is equivalent to the selection of a point in a simplex. If the strategies are constrained in any way, then the choice is no longer made from a simplex but from an arbitrary convex set. Many infinite games, e.g., polynomial and polynomial-like, are essentially finite games over general convex sets since the choosing of a mixed strategy is equivalent to choosing a point in a finite dimensional convex set.

In this paper we study games played over arbitrary convex sets. Interpreting the solutions of a game as the fixed-points in a continuous mapping, we obtain some general results on the dimensionality and continuity of solutions. Dimensional relationships for games played over simplices were first derived in [1] and [3]. Some dimensional and continuity relationships for polynomial-like games were first obtained in [2]. The general convex game treated here can be formulated as a polynomial-like game by spanning the convex set with a Peano space-filling curve. However, the complicated nature of such a curve makes this formulation impractical for theoretical and computational purposes. We also describe a method of computing the solutions by mapping one convex set onto another. The method is applicable to both finite games and infinite games with polynomial or polynomial-like payoffs.

§ 2. CONVEX GAMES

We define a finite dimensional convex game as follows: Player I chooses a point $r = (r_1, r_2, \ldots, r_m)$ from a convex set R lying in Euclidean m-space. Player II chooses a point $s = (s_1, s_2, \ldots, s_n)$ from a convex set S in Euclidean n-space. R and S are bounded and closed. The payoff from Player II to Player I is given by a bilinear form

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$$(1) \quad A(r,s) = \sum_{j=1}^{n} f_j(r)s_j + f_0(r) = \sum_{i=1}^{m} g_i(s)r_i + g_0(s) = \sum r_i a_{ij} s_j$$

where $f(r)$ and $g(s)$ are linear functions of r and s, respectively.

If Player I has m pure strategies and Player II has n pure strategies, then R and S are $(m-1)$-dimensional and $(n-1)$-dimensional simplices, respectively, and $A(r,s)$ is the mixed strategy payoff to Player I. However, if a game has a finite number of pure strategies and the mixed strategies are subject to some finite number of linear constraints, then the sets R and S are polyhedral convex sets. If a game is continuous and the payoff is polynomial-like, i.e., the payoff to Player I from Player II if they choose pure strategies x and y, respectively, is given by

$$M(x,y) = \sum_{i,j=1}^{m,n} a_{ij} r_i(x) s_j(y)$$

where r_i and s_j are continuous functions, then R is the convex set spanned by the curve

$$r_i = r_i(x), \qquad 0 \leq x \leq 1, \qquad i = 1, 2, \ldots, m,$$

traced out in m dimensions, and S is the convex set spanned by the curve

$$s_j = s_j(y), \qquad 0 \leq y \leq 1, \qquad j = 1, 2, \ldots, n,$$

traced out in n dimensions [2]. The payoff $A(r,s)$ is given by (1).

§3. SOLUTIONS OF CONVEX GAMES

The existence of <u>optimal strategies</u> r^0, s^0 and a game <u>value</u> v, such that

$$(2) \qquad \max_{r \in R} A(r,s^0) = \min_{s \in S} A(r^0,s) = v$$

can be established in two fundamental ways. We may use properties of convex sets, specifically the result that two non-overlapping convex sets can be separated by a hyperplane. It also follows from the Kakutani fixed-point theorem [4].

The optimal strategies (one set for each player), or <u>solutions</u> are the generalized fixed-points in the upper semicontinuous mapping described as follows: Let $r^0 \in R$. Define the image of r^0 on S to be the set of points $S(r^0) \subseteq S$ where $\min_{s \in S} A(r^0,s)$ is assumed. $S(r^0)$, which is the intersection of a hyperplane with the boundary of S or coincides with

S, is a convex set. Let $s^0 \in S$ and let the image of s^0 on R be the set of points $R(s^0) \subseteq R$ where $\max_{r \in R} A(r,s^0)$ is assumed. If $r^0 \in R(s^0)$ and $s^0 \in S(r^0)$, then r^0, s^0 satisfies (2) and is therefore a solution of the game. Further, since r^0 is an image of s^0 and s^0 is an image of r^0, r^0 and s^0 are fixed points in R and S, respectively. It is clear that if R^0, S^0 are the sets of optimal strategies of the two players, then every point of R^0 is an image of every point of S^0 and every point of S^0 is an image of every point of R^0.

We can also formulate the solutions as fixed-points of the point-set mapping F which takes a point (r,s) in the product space $R \otimes S$ into the non-void set $(R(s),S(r)) = F(r,s)$ in $R \otimes S$, where R(s) and S(r) are defined above.

§4. CONTINUITY OF SOLUTIONS

To develop the theory of convex games, we first study the solutions for continuity. The following theorem, first proven in [2], is repeated for completeness.

THEOREM 1. The solution of a convex game is a lower semi-continuous function of the payoff.

PROOF. Let G be an open set containing the optimal strategies R^0 of player I in the game with payoff $A = A(r,s)$. Suppose each element of A is perturbed by at most ε, let A_ε denote the resulting payoff and R_ε denote the set of optimal strategies of player I. We assert that, for a sufficiently small ε, R_ε is in G. For, let us assume the contrary. Then there exists a sequence $\{\varepsilon_n\}$ tending to zero with a corresponding sequence of payoffs A_{ε_n} such that each R_{ε_n} is not contained in G. Let r_n be in R_{ε_n} but not in G. Since R is compact we may suppose that r_n tends to r as $n \to \infty$, where r is not in G but in R. Further, it is readily verified that a limit point of optimal strategies for the payoffs A_{ε_n} is an optimal strategy for the payoff A. But r is not in G and hence not in R^0 and so we arrive at a contradiction.

By an identical argument one can show that if the strategy spaces R_n converge to R, then the sets of solutions are lower semi-continuous. A set of spaces R_n is said to converge to R if every point of R is a limit point of points of R_n and there exist no other limit points of points of R_n.

§ 5. GAMES WITH UNIQUE SOLUTIONS

In this section we derive a dimensional relationship for games with unique solutions. It is shown that the relevant property of the polyhedral face containing the optimal strategy is the dimension of the face rather than the number of vertices.

Let r^0,s^0 be the unique solution of the game with payoff $A(r,s)$ over the strategy space (R,S). Assume that R and S are polyhedral convex sets. Let r^0 be interior to a k-dimensional face R^0 of R, and s^0 be interior to an l-dimensional face S^0 of S. R^0 and S^0 are polyhedral faces. Since s^0 is optimal, it maps onto some maximal face $R' \supseteq R^0$. Similarly, r^0 maps onto some maximal face $S' \supseteq S^0$.

LEMMA 1. The game with payoff $A(r,s)$ over the reduced strategy spaces (R',S') has the unique solution r^0,s^0.

PROOF. Since r^0 maps onto $S' \supseteq S^0 \supseteq s^0$ and s^0 maps onto $R' \supseteq R^0 \supseteq r^0$, it follows that r^0,s^0 is a solution of the game over the strategy space (R',S'). To show it is unique, let us assume that $s' \in S'$ is another optimal strategy for Player II. Then s' maps onto some face containing R^0. Let $\bar{s} = \epsilon s' + (1 - \epsilon)s^0$ be a solution close to s^0, and therefore maps onto a face containing R^0. Now for the original game over (R,S) we defined R' as the maximal face upon which s^0 maps. Hence a point $\bar{s}$ sufficiently close to s^0 will map into some part of R' containing r^0. Therefore $\bar{s}$ is another solution of the full game, which contradicts the uniqueness assumption.

We may now confine ourselves to the reduced polyhedral game over (R',S'). In this game r^0 is interior to R^0, and r^0 maps onto S' Similarly, s^0 is interior to S^0 and s^0 maps onto R'.

LEMMA 2. If r^0,s^0 is the unique solution of the polyhedral game (R',S'), then

$$R^0 = R' \quad \text{and} \quad S^0 = S'.$$

PROOF. Suppose s^0 is on the boundary of S', or $S^0 \subset S'$. Consider a sequence S_n, of polyhedra interior to S' excluding S^0 which expand out to all of S' as n increases. We can construct S_n by using any inner point c of S' and taking the set of all points on the segment

$$\lambda c + (1 - \lambda)x \qquad 0 \leq \lambda \leq 1 - \epsilon_n$$

and x any point of S'.

Consider the game over the spaces (R', S_n). Let (X_n, Y_n) denote the solutions to this game. Then from the lower semi-continuity of the solutions it follows that every sequence of solutions s_n tends to s^0. Now since r^0 is interior to R^0, then for n sufficiently large the solutions r_n lie interior to a polyhedron having R^0 as a face. Therefore, for n large, s_n maps into a polyhedron containing R^0. But r^0 maps into all of S', it follows that (r^0, s_n) is a solution of the game over (R', S_n). It also follows that (r^0, s_n) is a solution of the game over (R', S'). This contradicts the hypothesis, since s_n is interior to S'. Therefore $S' = S^0$, and similarly $R' = R^0$.

REMARK. Using a similar argument, we can generalize Lemma 2 to games with non-unique solutions. Let R^0 be the smallest polyhedral face containing the set of solutions of Player I. Now every optimal strategy of Player II will map onto some polyhedral face of R; some of the optimal strategies will map onto R^0. Let R' be the maximal intersection of these polyhedral faces. Then, by an argument identical to above, it can be readily shown that $R' = R^0$. Similarly, if S^0 is the smallest polyhedral face containing the optimal strategies of Player II, and S' is the intersection of all polyhedral faces into which are mapped Player I's optimal strategies, then $S' = S^0$. Again, we may confine ourselves to the reduced game over the space (R^0, S^0).

THEOREM 2. If a polyhedral game has a unique solution, then the two optimal strategies lie in polyhedra of the same dimension.

PROOF. Since r^0 maps into an l-dimensional polyhedron in S, we must have

$$f_j(r) = 0 \qquad j = 1, 2, \ldots, l,$$

or some l linear relations must be satisfied. These relations determine a manifold of points in R mapping onto S^0 and having dimension, at least, $m - l$. Now the manifold and the k-dimensional polyhedron have only r^0 in common, otherwise the uniqueness of the solution would be contradicted. Therefore,

$$m - l + k \leq m$$

or

$$k \leq l.$$

Similarly we can show $l \leq k$, and therefore $k = l$.

§ 6. GAMES WITH MANY SOLUTIONS

We can reduce the general case of polyhedral games with many solutions to the special case of games with unique solutions. Suppose X, the set of optimal strategies of Player I, is k-dimensional and is interior to a polyhedron, R^0, which is u-dimensional. Suppose Y, the set of optimal strategies of Player II, is l-dimensional and is interior to a polyhedron, S^0, which is v-dimensional.

THEOREM 3. For any polyhedral game, the set of solutions and their containing polyhedra satisfy the dimensional relationship

$$u - k = v - l.$$

PROOF. We have shown that X maps onto S^0 and Y maps onto R^0. Consider the reduced game over the spaces R^0, S^0. The payoff now becomes

$$A(r,s) = \sum_{j=1}^{v} f_j(r)s_j + f_0(r) = \sum_{i=1}^{u} g_i(s)r_i + g_0(s).$$

The common zeros of $f_1(r), f_2(r), \ldots, f_v(r)$ in R^0 correspond to the optimal strategies of Player I and the common zeros of $g_1(s), g_2(s), \ldots, g_u(s)$ in S^0 correspond to the optimal strategies of Player II.

Form the factor space of E^u/X^u, where E^u is the Euclidean space containing R^0 and X^u is the linear extension of X in E^u. By taking a cross section T in E^u perpendicular to the manifold X, the polyhedron R^0 becomes a new polyhedron in T. Since X was interior to R^0, then X becomes a unique interior point r^0 of T. In a similar way we construct the factor space E^v/Y^v, and by taking a cross section, the polyhedron S^0 becomes a polyhedron U and Y becomes a unique interior point s^0. We obtain the induced mappings of $f_1, f_2, \ldots, f_v$ and $g_1, g_2, \ldots, g_u$ on T to U, which are well defined.

For this mapping it is clear that r^0, s^0 are optimal strategies, as X^u and Y^v constitute the zeros of f_i and g_j, respectively. They are also unique, since any other strategy r' of Player I must cover all of T and hence must be a common zero of the induced mappings $f_1, f_2, \ldots, f_v$. In terms of E^u, this implies that r' belongs to the coset X^u. From Theorem 2 it follows that

$$\dim T = \dim U$$

or

$$u - k = v - l.$$

§ 7. UNIQUENESS OF SOLUTION AND PERTURBATION OF PAYOFF

We shall demonstrate that if a finite polyhedral game has a unique solution, then for any sufficiently small change in the payoff the solution remains unique.

Let the solution r^0, s^0 of a polyhedral game be interior to a k and l dimensional face R^0 and S^0, respectively. We have proven that $k = l$. Let T be the manifold of all points in R mapping into S^0. Then T is defined by l linear relations, say

$$\sum_{i=1}^{m} b_{ij} r_i = 0, \qquad j = 1, 2, \ldots, l$$

where b_{ij} is a linear function of the matrix A. Let the dimension of T be w, then

$$w \geq m - l.$$

Since T and R^0 intersect in a unique point r^0, we have

$$m \geq w + k = w + l,$$

and thus

$$w = m - l.$$

The last relationship implies that the matrix (b_{ij}) has full rank. If the payoff is perturbed by a sufficiently small amount, the dimension of T will not change -- the rank is preserved. We can also preserve the rank of the intersection of the manifold T and R^0. Therefore we can obtain an $(m - l)$-dimensional manifold which intersects R^0 in a unique point r'. In a similar way we can obtain a unique point s' in S^0. Since r' and s' are unique points of intersections, it follows that r', s' is the unique solution of the game with the perturbed payoff.

If the game is a general convex game, not necessarily polyhedral, then the uniqueness of a solution is no longer preserved under small perturbations. For example, in the game with payoff $M(x,y) = xy - x^2$, both players have unique optimal strategies lying on the boundaries of their respective spaces. If the payoff is perturbed to $xy - x^2 + \varepsilon x$, the uniqueness is destroyed. However, if both players possess interior unique

optimal strategies, then the planes $f_j(r) = 0$ intersect in a unique interior point and the planes $g_i(s) = 0$ also intersect in a unique interior point, and small perturbations of the payoff preserving this intersection property will preserve the uniqueness as well.

§8. INTERIOR AND "IDENTICALLY v" SOLUTIONS

We can interpret geometrically two general types of solutions:

1. Solutions interior to R and S.
2. Strategies which yield identically v to a player independent of strategies of the other player.

Let r^0 be an interior optimal strategy of Player I and s^0 some optimal strategy of Player II. Then s^0 maps onto a set $R(s^0)$ containing no interior point of R unless $g_i(s^0) = 0$ for all i. Therefore every optimal strategy of Player II is on the intersection of the planes $g_i(s) = 0$, and thereby yields identically v. The interior solution, r^0, of Player I need not have any special position relative to the planes $f_j(r) = 0$. However, an optimal strategy r^0 yields identically v if and only if it lies on the common portion of all the planes $f_j(r) = 0$.

§9. SYMMETRIZING A CONVEX GAME

A finite convex game can be symmetrized in a manner similar to the simplex game. Let the payoff be represented by

$$(Ar,s) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} r_i \right) s_j .$$

Define

$$p(r_0,s_0) = \max_{r \in R} (Ar,s_0) - \min_{s \in S} (Ar_0,s).$$

It can be verified that $p(r_0,s_0) \geq 0$, for all r_0,s_0 and $p(r_0,s_0) = 0$ if and only if (r_0,s_0) is a solution of the game. Furthermore,

$$\begin{aligned} p(r_0,s_0) &= \max_{r \in R} (Ar,s_0) + \max_{s \in S} (r_0,-A's) \\ &= \max_{\substack{r \in R \\ s \in S}} [(Ar,s_0) + (r_0,-A's)]. \end{aligned}$$

Form the product space $E_{n+m} = E_n \otimes E_m$ and the product convex

set $R \otimes S$. Consider the linear operator $\{A,-A'\}\ \{r,s\}$ defined over E_{n+m}. The payoff over the product space becomes

$$(\{A,-A'\}\ \{r,s\},\ \{s_0,r_0\}) = (Ar,s_0) + (r_0,-A's).$$

This is a symmetric game in which Player I picks (r,s) and Player II picks (s,r), each from the space $R \otimes S$. The solutions of this symmetric game are those (r_0,s_0) for which $p(r_0,s_0) = 0$.

Every finite convex game can be symmetrized to a new convex game in $(n + m)$-dimensional space whose payoff matrix is given by

$$\begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix}$$

and where both players choose points from the product set $R \otimes S$. If R and S are both simplices, then $R \otimes S$ is no longer a simplex. In order to play on a simplex set it is necessary to add an extra dimension.

§10. COMPUTATION

We can compute the set of solutions R^0,S^0 by an orderly examination of the spaces R and S for fixed-points in a continuous mapping. The method consists of dividing the space R into a finite number of convex subsets $R_1, R_2, \ldots, R_\alpha$ by means of the hyperplanes $f_j(r) = 0$, $j = 1, 2, \ldots, n$, and the boundaries of S, and similarly dividing the space S into a finite number of convex subsets $S_1, S_2, \ldots, S_\beta$ by means of the hyperplanes $g_i(s) = 0$, $i = 1, 2, \ldots, m$, and the boundaries of R. The division of R is such that each of its subsets R_i maps onto some maximal boundary of S and each of the subsets S_j maps onto some maximal boundary of R. The subsets, R_i, will overlap, but their union is the full strategy space, R. From the previous discussion, it follows that R^0,S^0 are the fixed-points in the mappings of $R_1, R_2, \ldots, R_\alpha$ onto S and $S_1, S_2, \ldots, S_\beta$ onto R. Further, $R^0 = R_i$ for some i, and $S^0 = S_j$ for some j.

We illustrate the method by two examples -- first, a game with a finite number of pure strategies, and second, a game with a continuum of strategies with a polynomial payoff function.

EXAMPLE 1. FINITE GAME WITH CONSTRAINTS

Let the payoff corresponding to the pure strategies be defined by the matrix

$$\begin{pmatrix} 3 & 39 & 30 \\ 33 & 9 & 0 \\ 28 & 4 & 25 \end{pmatrix}$$

We wish to determine the optimal strategies if the mixed strategies are subject to the following additional constraints:

$$1 \leq 10x_1 \leq 8 \qquad 1 \leq 10y_2 \leq 20y_1$$

$$1 \leq 20x_2 \leq 10 \qquad 1 \leq 6y_3.$$

$$18 - 40x_1 \leq 15x_3.$$

From the constraints it is evident that R is a pentagon and S is a triangle. We may write the payoff as follows:

$$A(x,y) = (10x_2 - 10x_1 + 1)y_1 + (10x_2 + 10x_1 - 7)y_2 + \tfrac{1}{3}(5x_1 - 25x_2 + 25)$$

$$= (-10y_1 + 10y_2 + \tfrac{5}{3})x_1 + (10y_1 + 10y_2 - \tfrac{25}{3})x_2 + (y_1 - 7y_2 + \tfrac{25}{3})$$

$$= p_1y_1 + p_2y_2 + p_3 = q_1x_1 + q_2x_2 + q_3.$$

We now divide the spaces R and S into the following convex sets:

$R_1(p_1 \geq 0,\ p_2 \geq 0)$, $R_2(p_1 \geq 0,\ p_2 \leq 0)$, $R_3(p_1 \leq 0,\ p_2 \geq 0)$,

$R_4(p_1 \leq 0,\ p_2 \leq 0)$, $R_5(p_1 = p_2 \leq 0)$, $R_6(p_1 = p_2 = 0)$.

$S_1(q_1 \geq 0,\ q_2 \leq 0)$, $S_2(q_1 \leq 0,\ q_2 \geq 0)$, $S_3(q_1 = 0,\ q_2 \leq 0)$,

$S_4(q_1 = q_2 = 0)$.

Some of these sets may be void, in which case they will not appear in the later mappings. Some of the sets overlap. Mapping each R_i into S and each S_j into R, we have

$$R_1 \longrightarrow (S_1) \longrightarrow (R_3), \qquad R_2 \longrightarrow (S_1) \longrightarrow (R_3), \qquad R_3 \longrightarrow (S_2) \longrightarrow (R_2)$$

$$R_4 \longrightarrow (S_2) \longrightarrow (R_2), \qquad R_5 \longrightarrow S_4 \longrightarrow R_5.$$

where (S_1) represents a point in S_1 and (R_3) is a point in R_3. Therefore $R^0 = R_5$ and $S^0 = S_4$, or the optimal strategies are given by

$$x_1 = \frac{2}{5}, \qquad \frac{1}{20} \leq x_2 \leq \frac{3}{10}, \qquad x_3 = 1 - x_1 - x_2$$

$$y_1 = \frac{1}{2}, \qquad y_2 = \frac{1}{3}, \qquad y_3 = \frac{1}{6}.$$

The value of the game is $v = 19.5$.

EXAMPLE 2. INFINITE GAME WITH POLYNOMIAL PAYOFF

Let the pure strategy payoff be given by

$$M(x,y) = 21x + 18x^2 - 24x^3 - 16y - 36xy - 9x^2y + 18x^3y + 60y^2 - 36y^3$$

where $0 \leq x \leq 1$, $0 \leq y \leq 1$. If $F(x)$ and $G(y)$ are mixed strategies of the two players, define

$$r_i = \int_0^1 x^i dF(x) \qquad i = 1, 2, 3$$

$$s_j = \int_0^1 y^j dG(y) \qquad j = 1, 2, 3.$$

The mixed strategy payoff becomes

$$A(r,s) = 21r_1 + 18r_2 - 24r_3 - 16s_1 - 36r_1s_1 - 9r_2s_1 + 18r_3s_1 + 60s_2 - 36s_3.$$

R, the strategy space of Player I, is a 3-dimensional solid whose boundary consists of the curve C (defined parametrically by $r_1 = t$, $r_2 = t^2$, $r_3 = t^3$, $0 \leq t \leq 1$), lines connecting the origin with each point of C and lines connecting (1, 1, 1) with each point of C. S is defined in exactly the same way (S is the convex hull of the curve defined by $s_1 = u$, $s_2 = u^2$, $s_3 = u^3$ where $0 \leq u \leq 1$).

Let

$$f = -16 - 36r_1 - 9r_2 + 18r_3,$$

then we divide the spaces R and S into the following convex subsets

$$R_1(f < -28), \qquad R_2(f > -28), \qquad R_3(f = -28)$$

$$S_1(s_1 > \tfrac{2}{3}), \qquad S_2(s_1 < \tfrac{2}{3}), \qquad S_3(s_1 = \tfrac{2}{3}).$$

We now have the following mappings of R onto S and S onto R:

$$R_1 \longrightarrow (1, 1, 1)$$

$$R_2 \longrightarrow u = \frac{10 - \sqrt{100 + 3f}}{18}$$

$$R_3 \longrightarrow \beta(\tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{27}) + (1 - \beta)(1, 1, 1) \quad \text{where} \quad 0 \leq \beta \leq 1$$

$$S_1 \longrightarrow (0, 0, 0)$$

$$S_2 \longrightarrow t = \frac{-3(2 - s_1) - \sqrt{3(75s_1^2 - 150s_1 + 68)}}{6(3s_1 - 4)}$$

$$S_3 \longrightarrow \alpha(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}) + (1 - \alpha)(0, 0, 0) \quad \text{where} \quad 0 \leq \alpha \leq 1.$$

From the above, it can be verified that

$$R_3 \longrightarrow S_3 \longrightarrow R_3 \quad \text{if} \quad \alpha = \tfrac{2}{3}, \ \beta = \tfrac{1}{2}.$$

This yields the unique solution

$$F^*(x) = \tfrac{1}{3} I_0(x) + \tfrac{2}{3} I_{\frac{1}{2}}(x)$$

$$G^*(y) = \tfrac{1}{2} I_{\frac{1}{3}}(y) + \tfrac{1}{2} I_1(y)$$

where I_a is a step-function with a jump at a. The value of the game is $v = 6$.

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ADMISSIBLE POINTS OF CONVEX SETS

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§1. SUMMARY

A point s of a closed convex subset S of k-space is admissible if there is no $t \in S$ with $t_i \leq s_i$ for all $i = 1, \ldots k$, $t \neq s$. An example is given in which the set A of admissible points is not closed. Let P be the set of vectors $p = (p_1, \ldots, p_k)$ with $p_1 > 0, \sum_1^k p_i = 1$, let $B(p)$ be the set of $s \in S$ with $(p,s) = \min_{t \in S} (p,t)$, and let $B = \sum B(p)$, so that B consists of exactly those points of S at which there is a supporting hyperplane whose normal has positive components.

THEOREM 1. $B \subset A \subset \overline{B}$. If S is determined by a finite set, there is a finite set $p_1, \ldots, p_N$, $p_j \in P$, such that $B = \sum_{j=1}^{N} B(p_j)$, so that, since $B(p)$ is closed for fixed p, $B = A = \overline{B}$.

That $B \subset A \subset \overline{B}$ when S is determined by a finite set was noted, in the language of convex cones, by Gale [2].[1]

A consequence of Theorem 1 is that, in a two-person zero-sum game in which each player has only a finite number of pure strategies, every pure strategy which yields the value of the game against every minimax strategy of the opponent enters with positive probability into some minimax strategy, a result due to Bohnenblust, Karlin, and Shapley [1].

§2. PROOF OF THEOREM 1

Suppose first that S is bounded. If $s \in S$, $s \notin A$, there is a $t \in S$ with $t_i \leq s_i$ for all i and $t \neq s$. Then $(p,t) < (p,s)$ for every $p \in P$, so that $s \notin B$. Thus $B \subset A$.

To show that $A \subset \overline{B}$, let $a \in A$. We may achieve

[1] Numbers in square brackets refer to the bibliography at the end of the paper.

$a = (0, 0, \ldots, 0)$ by translating the coordinate system. For any ε, $0 < \varepsilon \leq \frac{1}{k}$, let $P(\varepsilon)$ be the set of $p \in P$ with $p_i \geq \varepsilon$ for $i = 1, \ldots, k$. For any two closed bounded convex sets R,S of k-space, there are vectors $r^* \in R$, $s^* \in S$ with $(r^*,s) \geq (r^*,s^*) \geq (r,s^*)$ for all $r \in R$, $s \in S$; this fact may be proved as follows. Let $r_n \in R$, $s_n \in S$ be sequences dense in R,S respectively. The finite game whose NxN matrix is $(r_i,s_j), 1 \leq i,j \leq N$, has a value $V(N)$ and good strategies $(\lambda_{N1}, \ldots, \lambda_{NN})$, $(\mu_{N1}, \ldots, \mu_{NN})$; thus if $r^*_N = \sum_{i=1}^{N} \lambda_{Ni} r_i$, $s_N^* = \sum_{j=1}^{N} \mu_{Nj} s_j$, we have $(r_N^*, s_j) \geq V(N) \geq (r_i, s_N^*)$ for $1 \leq i,j \leq N$. Letting $N \to \infty$ through a subsequence for which r_N^*, s_N^* converge, say to r^*,s^*, yields $(r^*,s_j) \geq (r_i,s^*)$ for all i,j, and r^*,s^* have the required property. We apply this result to $P(\varepsilon),S$, obtaining $p(\varepsilon) \in P(\varepsilon)$, $s(\varepsilon) \in S$ such that $(p(\varepsilon),s) \geq (p(\varepsilon), s(\varepsilon)) \geq (p,s(\varepsilon))$ for all $p \in P(\varepsilon)$, $s \in S$, and note that $(p(\varepsilon), s(\varepsilon)) \leq (p(\varepsilon), a) = 0$. Choose a sequence $\varepsilon_n \to 0$ for which $s(\varepsilon_n)$ converges, say to $s^* \in S$. For all $p \in P$, $(p,s(\varepsilon_n)) \leq 0$ for sufficiently large n, so that $(p,s^*) \leq 0$. Thus $s_i^* \leq 0$ for all i and, since $a = (0, 0, \ldots, 0)$ is admissible, $s^* = a$. Finally $s(\varepsilon_n) \in B(p(\varepsilon_n)) \subset B$, so that $a \in \bar{B}$.

For unbounded S, we use the

LEMMA. Let C be a convex set, N a neighborhood, and s_0 minimize (p,s) for $s \in C \cap N$. Then s_0 minimizes (p,s) for $s \in C$.

PROOF. Suppose, for some $s_1 \in C$, $(p,s_1) < (p,s_0)$. Let $s_2 = \alpha s_0 + (1-\alpha)s_1$; for $0 < \alpha < 1$, $s_2 \in C$, $(p,s_2) < (p,s_0)$. But for α sufficiently small, $s_2 \in N$, which contradicts the hypothesis.

It follows from the Lemma that Theorem 1 holds without the condition that S be bounded. Let S be any closed convex set, a an admissible point of S. Let N be a closed neighborhood of a. Then a is admissible in $S \cap N$. By Theorem 1, $a = \lim_{n\to\infty} s_n$, where s_n minimizes (p_n,s) for $s \in S \cap N$ for some p_n in P. Then, by the Lemma, s_n minimizes (p_n,s) for $s \in S$ for the same p_n.

Now suppose that S is determined by a finite set $s_1, \ldots, s_m$. A subset U of $s_1, \ldots, s_m$ will be called <u>usable</u> if there is a $p \in P$ for which $U \subset B(p)$. Let $U_1, \ldots, U_N$ be the usable subsets of $s_1, \ldots, s_m$, and let $p_1, \ldots, p_N$ be corresponding p's, so that $U_j \subset B(p_j)$. Let $s \in B$, say $s \in B(p)$, $s = \sum_1^m \lambda_i s_i$. Then if U is the set of s_i for which $\lambda_i > 0$, $U \subset B(p)$, since $(p,s_i) > \min_{t\in S} (p,t)$ for $s_i \in U$ would imply $(p,s) = \sum \lambda_i (p,s_i) > \min_{t\in S} (p,t)$. Thus U is usable, say $U = U_j$. Then $s \in B(p_j)$ and $B = \sum_1^N B(p_j)$. This completes the proof.

§3. AN EXAMPLE

If $k = 2$, it is easily shown that A is closed; in our example $k = 3$ and A is not closed. Let U be the closed arc consisting of all points $(x,y,1)$ with $(x - 1)^2 + (y - 1)^2 = 1$, $0 \leq x,y \leq 1$, let $e = (1,0,0)$, and let S be the convex set determined by U and e. The point $f = (1,0,1) \in S$ is not admissible; we show that every point $s_0 = (x_0,y_0,1)$ with $x_0,y_0 > 0$ and $(x_0 - 1)^2 + (y_0 - 1)^2 = 1$ is admissible, in fact, that it is an element of B.

There is a vector $p_0 = (u_0,v_0,0)$ with $u_0 > 0$, $v_0 > 0$, $u_0 + v_0 = 1$ such that $(p_0,s) > (p_0,s_0)$ for all $s \in S$, $s \neq s_0$. Let $p = (1 - \varepsilon)p_0 + \varepsilon(0,0,1)$, $0 < \varepsilon < 1$. Then $p \in P$, and $(p,s) = (1 - \varepsilon)(p_0,s) + \varepsilon$ for $s \in U$, so that $(p,s_0) < (p,s)$ for $s \neq s_0$, $s \in U$. Also $(p,e) = (1 - \varepsilon)(p_0,f) > (1 - \varepsilon)(p_0,s_0) + \varepsilon = (p,s_0)$ for ε sufficiently small. Thus $s_0 \in B(p)$ for ε sufficiently small, completing the proof.

We remark incidentally that $e \in A$ but $e \notin B$, so that $A \neq B$, $\overline{B} \neq A$ in our example.

§4. AN APPLICATION

A consequence of our theorem is the following result of Bohnenblust, Karlin, and Shapley [1].

> THEOREM 2. Let $A = ||a_{ij}||$ be an $m \times n$ matrix, considered as the payoff of a zero-sum two-person game, and let D be the set of all i for which $A(i,q) = \sum_{j=1}^{n} a_{ij}q_j = v$ for every minimax strategy q for II, where v is the value of the game. Then there is a minimax strategy p for I with $p_i > 0$ for $i \in D$.

PROOF. If we delete from A the rows for which $i \notin D$, the resulting game has value v, and every minimax strategy for I in the new game is minimax in the original game. Moreover $A(i,q) = v$ for every $i \in D$ and every minimax strategy q for II in the new game, for if $A(i_0,q_0) < v$ for some $i_0 \in D$ and some minimax q_0 in the new game, $A(i_0,q^*) < v$ for every $\varepsilon > 0$, where $q^* = \varepsilon q_0 + (1 - \varepsilon)q_1$ and q_1 is minimax in the original game. Since q_1 may be chosen so that $A(i,q_1) < v$ for $i \notin D$, q^* will be minimax in the original game for sufficiently small $\varepsilon > 0$, and $i_0 \notin D$. Thus we may suppose $i \in D$ for $i = 1, \ldots, m$.

Let S be the convex set in m-space determined by the columns

of A. The point $s_0 = (v, v, \ldots, v) \in S$ and is admissible, so that, by Theorem 1, there is a $p \in P$ with $s_0 \in B(p)$. That is, $(p,s_0) = v \leq (p,s)$ for all $s \in S$. Thus p is a minimax strategy for I, and $p_i > 0$ for $i \in D$.

§5. IMPLICATIONS FOR STATISTICS AND ECONOMICS

Suppose we have a statistical problem with a finite number of possible states of nature. To each possible strategy of the statistician, there can be assigned a vector whose components are the risks of the strategy under each of the possible states of nature. Let S be the set of such vectors; S will be convex if mixed strategies of the statistician are admitted and closed under very general assumptions. Then if we define the distance between two strategies as the Euclidean distance between their risk vectors, Theorem 1 asserts that every admissible strategy is the limit of a sequence of strategies each of which is a Bayes solution against an a priori probability distribution which assigns positive probability to every state of nature. If S is determined by a finite set (for example, if the sample size is bounded, the variables observed take on only a finite set of values, and the number of actions among which choice is to be made is finite), then there is a finite set of such a priori distributions such that a strategy is admissible if and only if it is Bayes against one of these distributions.

In an economic context, it is assumed that production is made up by carrying on different activities at varying levels of intensity. An activity is characterized by a vector a whose components are amounts of the different commodities passing through the activity, being positive for outputs and negative for inputs. If the activity is carried on at level x, the amount of commodity i produced is xa_i (which is negative for inputs). If there are a finite number of activities a^j, then the amount of commodity i produced is $x_j a_i^j$. Let commodities $1, \ldots, k$ be final for desired outputs, $k+1, \ldots, m$ be primary inputs. If the negative of the total available supply of commodity $i (i > k)$ is η_i, then the activity levels x_j must satisfy the conditions,

$$\sum_j a_i^j x_j \geq \eta_i, \quad (i > k).$$

The image of this set in x-space under the transformation $y_i = \sum_j a_i^j x_j (i = 1, \ldots, k)$ is then the set S of all possible combinations of final commodities possible with the given technology and resource limitations. S is clearly closed, bounded and convex. A point s of S is said to be efficient if there is no $t \in S$ such that $t_i \geq s_i$ for all i, $t \neq s$. By Theorem 1 applied to efficient points, there is a finite set of

vectors p, with $p_i > 0$ for all i, such that s is efficient if and only if s maximizes (p,s) for $s \in S$ for some one of those p's. The vector p can easily be interpreted as a set of prices for the final commodities, and (p,s) is the profit arising from the choice of activity levels leading to s. Hence, any efficient point can be arrived at by instructing producers to maximize profits for a suitable set of positive prices. (See Koopmans [3], especially Theorem 4.3).

In the more general case where the number of activities is not finite, the activity levels will in general be replaced by a measure over the space of activities. The set S is still convex, and, under suitable assumptions, closed. Then Theorem 1 tells us that we can approximate any given efficient point arbitrarily closely by instructing producers to maximize profits given a suitably chosen set of positive prices.

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Part II

INFINITE ZERO-SUM TWO-PERSON GAMES

Among the zero-sum two-person games that have infinite sets of pure strategies available to the contestants, the "games on the unit square" continue to attract the main attention. These games generalize matrix games by replacing the elements a_{ij} of a payoff matrix by a real-valued function $K(x,y)$ defined on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. The pure strategies available to players I and II are indexed by the real numbers x and y, while their mixed strategies become probability distributions over the unit intervals $0 \leq x \leq 1$ and $0 \leq y \leq 1$. This may seem entirely straightforward, but a large gallery of counter-examples has shown that such games can possess quite pathological deviations from the sound norm established by matrix games. So research continues in the direction of isolating classes of games on the unit square that will have suitably regular and manageable properties.

The symmetric "games of timing" discussed by M. Shiffman in PAPER 6 can be interpreted as duels in which the opponents have equal strategic resources. Their pure strategies, x and y, represent the times at which they act (the later, the better). Shiffman shows that an optimal strategy is characterized either as a density from some point a to 1 in the unit interval, satisfying an integral equation with positive kernel, or as such a density combined with a jump at 0. Existence, uniqueness, and method of construction of the optimal strategy are obtained thereby. The jump at 0 is excluded or not according as an eigenvalue of the integral equation attains the value 1 or not. In certain cases, the integral equation can be replaced by a system of linear differential equations.

In PAPER 7, S. Karlin continues the study of "games of timing" with the restriction to equal strategic resources removed. Even so, he finds unique solutions which consist of absolutely continuous distributions in the interior of the unit interval with possible jumps at the endpoints. The densities occur as solutions of certain integral equations and are

expressed either as Neumann series or as eigenfunctions of integral operators. In addition Karlin solves another type of game in which the payoff $K(x,y)$ is concave in each variable separately on each side of the diagonal $x = y$; the solutions are densities absolutely continuous over the closed unit interval. (Some particular games of this type, called "butterfly games," in which the payoff is monotonic away from the diagonal, have been treated by I. Glicksberg and O. Gross.)

The class of games considered by Karlin in PAPER 8 is a natural extension of the notion of convex game (see Study 24, Paper 15). The payoffs $K(x,y)$ considered are assumed to have positive nth partial derivatives with respect to y. The author carries out a detailed analysis for $n = 1, 2, 3, 4$ and shows that player I has an optimal strategy which is a step-function with at most n steps, while player II needs at most $\frac{n}{2}$ steps in some step-function strategy (jumps at the ends of the interval being counted as half jumps). It is asserted that the theorem holds for arbitrary n.

Polynomial games (see Study 24, Paper 14) form a class of infinite games that have finite-dimensional sets of mixed strategies, just as finite games do. But any hope that this characteristic might extend to games with rational payoff functions has been dashed by I. Glicksberg and O. Gross in PAPER 9, where they exhibit some of the pathology of such games. They match the known fact that polynomial games always have step-function solutions with an example of a game which has a continuous rational payoff but which has only densities as optimal strategies. This example involves an interesting application of Tarski's system of "elementary algebra." They also construct a rational-payoff example that has unique optimal strategies based on dense, countable sets. Finally, to point up the exceptional perversity of games on the square, they propose a method which constructs a game in C^{∞} having an arbitrarily given pair of distributions as unique optimal strategies.

PAPER 10 by D. Blackwell deals with a type of infinite game, not on the square, that has a statistical context. The payoff is a function $L(\omega, x, i)$, where ω is an element chosen by player I from a set Ω, finite or infinite, x is a point chosen from a space X by a chance move defined by a probability measure P_{ω} over X depending on ω, and i is an integer chosen from $1, 2, \ldots, k$ by player II, who is informed of x but not of ω. A pure strategy for player II is a partition of X into k mutually disjoint sets, while a mixed strategy for II is a probability measure defined on such partitions. An alternative method of randomizing, directly related to Kuhn's "behavior strategies" (see Paper 11, this Study), is to assign a probability distribution on the integers $1, 2, \ldots, k$ for

each x in X. Blackwell shows that, from the point of view of payoff, these two methods of randomizing are equivalent to one another and to a denumerable mixture of certain canonical pure strategies.

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GAMES OF TIMING[1]

Max Shiffman

SUMMARY

We introduce the notion of a symmetric game of timing. It is defined here as a continuous game involving the bilinear functional

$$\int_0^1 \int_0^1 K(x,y)\,dF(x)\,dG(y) , \qquad K(x,y) = -K(y,x)$$

in which for $x < y$, $K(x,y)$ is a strictly increasing function of x and a strictly decreasing function of y (more precisely defined in § 1). If $K(1^-,1) \leq 0$, there is an optimum pure strategy at 1; if $K(0,1) \geq 0$, there is an optimum pure strategy at 0. Aside from these trivial cases, it is proved that there is a unique optimal strategy which is either a density from some point a to 1, or is a jump at 0 and a density from a to 1. If the quantity $K(y^-,y)$ varies in sign as y varies, let b be the value such that $K(b^-,b) = 0$, while $K(y^-,y) > 0$ for $b < y \leq 1$. In this case, the optimal strategy is a density from a to 1 where $a > b$. It is shown that the determination of the density function depends on the solution of a certain integral equation with positive kernel, and the theory of such integral equations is discussed and applied. It is also shown that for a general category of cases the optimal strategy can likewise be obtained in terms of a system of ordinary linear differential equations.

§ 1. PRELIMINARIES

We shall consider a class of symmetric continuous games involving the bilinear functional

$$\int_0^1 \int_0^1 K(x,y)\,dF(x)\,dG(y)$$

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[1]This paper was prepared as a report (Sept. 6, 1949) for the RAND Corporation.

where x, y range over the real numbers from 0 to 1 inclusive, the symmetry of the game reflecting itself in the skew-symmetry of the kernel $K(x,y)$:

$$K(x,y) = -K(y,x) \ .$$

Concerning the kernel $K(x,y)$, we will suppose that for $x < y$ $K(x,y)$ is a strictly increasing function of x and a strictly decreasing function of y. This property likewise holds for $x > y$ by virtue of the skew-symmetry of $K(x,y)$. But across the main diagonal $x = y$ this property may cease, i.e., there may be a jump of $K(x,y)$ and $K(a+\delta,a)$ may be smaller than $K(a-\delta,a)$ for small positive δ's.

We shall call such a game a game of timing, by virtue of the following important interpretation. The variables x and y may represent the times at which the players I and II take certain specific actions; and it is profitable for each player to delay action as long as possible, provided his action is prior to his opponent's action. If the times x, y at which players I, II take action are near each other, there is a decided difference in the outcome according as $x < y$ or $x > y$. Each player is thus subject to the following motive: he wishes to delay action so as to increase his reward, but at the same time not to delay so long that his opponent can with effectiveness precede him.

To be specific, a game will be called a symmetric game of timing if the kernel $K(x,y)$ satisfies the following conditions:

(1 a) $$K(x,y) = \begin{cases} A(x,y) & \text{for } x < y \\ 0 & \text{for } x = y \\ -A(y,x) & \text{for } x > y \end{cases}$$

where $A(x,y)$ is continuous in $x \leq y$.

(1 b) $A(x,y)$ is a strictly increasing function of x and a strictly decreasing function of y.

In what follows, we shall make the following additional hypothesis:

(1 c) $A(x,y)$ has continuous first derivatives in $x \leq y$, and the set of points where $A_x(x,y) = 0$ or $A_y(x,y) = 0$ contains no linear intervals $x = \text{constant}$,

$$\beta_1 < y < \beta_2 \quad \text{or} \quad y = \text{constant}, \quad \alpha_1 < x < \alpha_2.^2$$

As a consequence of (1 b), we can assert that

$$A_x(x,y) \geq 0, \quad A_y(x,y) \leq 0 \qquad \text{for} \quad x \leq y .$$

The condition (1 c) makes a mild limitation on the places where either of these derivatives is zero.

We shall show that, aside from trivial cases, the optimal strategy of a game of timing is unique and consists either of a density function from some point a to 1, or consists of a jump at 0 and a density from some point a to 1. The optimum strategy will be obtained as the solution of a certain integral equation with a positive kernel. In a wide category of cases, this integral equation is equivalent to a certain linear differential equation or system of linear first order differential equations.

§ 2. GENERAL CONDITIONS ON AN OPTIMUM STRATEGY

It is easy to show the following: if $A(1,1) \leq 0$, a pure strategy at 1 is the unique optimum strategy; if $A(0,1) \geq 0$, a pure strategy at 0 is the unique optimum strategy. We exclude these trivial cases and suppose henceforth that

$$A(0,1) < 0, \qquad A(1,1) > 0 . \tag{2.1}$$

We shall first suppose that there is an optimum strategy $F(x)$ for the game, and derive necessary conditions satisfied by $F(x)$. Thus

$$V(y) = \int_0^1 K(x,y)\,dF(x) \geq 0 \qquad \text{for all} \quad y , \tag{2.2}$$

while

$$\int_0^1 V(y)\,dF(y) = \int_0^1 \int_0^1 K(x,y)\,dF(x)\,dF(y) = 0 \tag{2.3}$$

by the skew-symmetry of $K(x,y)$.

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[2] The reader may if desired make the further simplification $A_x(x,y) > 0$, $A_y(x,y) < 0$ for $x < y$.

LEMMA 1. $F(x)$ cannot have a jump at an interior point x_0, $0 < x_0 < 1$, unless $A(x_0,x_0) = 0$. $F(x)$ does not have a jump at $x = 1$.

PROOF. If $F(x)$ had a jump at such a point x_0, $0 < x_0 < 1$, we would have

$$V(x_0) = \int_0^1 K(x,x_0)dF(x) = 0$$

by virtue of (2.2), (2.3). Letting $y \longrightarrow x_0^{\pm}$ in (2.2),

$$V(x_0^{\pm}) = \int_0^1 K(x,x_0^{\pm})dF(x) \geq 0 ,$$

so that

$$\int \left\{ K(x,x_0^{\pm}) - K(x,x_0) \right\} dF(x) \geq 0 .$$

But $K(x,x_0^{\pm}) - K(x,x_0) = 0$ except for $x = x_0$, and we have

$$\pm A(x_0,x_0) \left\{ F(x_0^{+}) - F(x_0^{-}) \right\} \geq 0 .$$

The first part of the lemma is proved.

The above argument also shows that the following statement can be made at the end-points 0,1:

$$A(0,0)F(0^{+}) \geq 0 , \qquad A(1,1)(1 - F(1^{-})) \leq 0 . \tag{2.4}$$

Because of $A(1,1) > 0$, the second part of the lemma is proved.

The quantity $V(y)$ is continuous, except at $y = 0$ if $A(0,0)F(0^{+}) \neq 0$. For, if $x_0 \neq 0$,

$$V(x_0^{\pm}) - V(x_0) = \int \left\{ K(x,x_0^{\pm}) - K(x,x_0) \right\} dF(x)$$

$$= \pm A(x_0,x_0) \left\{ F(x_0^{+}) - F(x_0^{-}) \right\} = 0 ,$$

while

$$V(0^{+}) - V(0) = A(0,0)F(0^{+}) \geq 0 .$$

It follows from (2.2) and (2.3) that

(2.5) $V(y) = 0$ *for all points* y *in the spectrum of* $F(x)$.[3]

LEMMA 2. The spectrum of $F(x)$ either is an interval from a to 1, or is the point 0 and an interval from a to 1, where $a < 1$.

PROOF. Denote the closed point set which is the spectrum of $F(x)$ by S. If S is not the entire set from 0 to 1, let $p < y < q$ be one of the open intervals composing the complement of S. For any point y in this interval,

$$V(y) = \int_0^{p+} A(x,y)dF(x) - \int_{q-}^1 A(y,x)dF(x) ,$$

so that, by hypothesis (1 b), $V(y)$ *is a strictly decreasing function* of y in this interval. By virtue of (2.2) and (2.5), the left-hand end-point p of the interval must be 0. Otherwise p would belong to S, $V(p) = 0$, and by the continuity of $V(y)$ at p, we would then have $V(p + \delta) < 0$ for small positive δ. This contradicts (2.2), so that $p = 0$. The right-hand end-point q cannot be 1. Otherwise, 1 would not belong to S by Lemma 1, and 0 would be the only point in S; then (2.2) contradicts (2.1). The lemma is proved.

The hypothesis (1 b) makes no assertion concerning the behavior of $A(x,x)$, except that for x near 1 we have $A(x,x) > 0$ by (2.1). If there are points x on the main diagonal where $A(x,x) = 0$, let b be the maximum of all such points x. Thus $A(b,b) = 0$, while for $b < x \leq 1$ we have $A(x,x) > 0$. If there are no points x where $A(x,x) = 0$, we set $b = 0$. The interval $b \leq x \leq 1$ will be called the *basic interval*, by virtue of the following lemma.

LEMMA 3. The spectrum of $F(x)$ lies completely in the basic interval $b \leq x \leq 1$.

PROOF. This lemma makes an assertion only if $b > 0$. Consider a new game in which the pay-off is $K(x,y)$ but in which x, y are limited to the interval $b \leq x \leq 1$, $b \leq y \leq 1$. This is again a game of timing, and we shall show later in the paper that a game of timing has a solution.

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[3]By the spectrum of $F(x)$ is meant the set of points x_0 such that either $F(x_0 + \delta) > F(x_0)$ for all positive δ, or $F(x_0 - \delta) < F(x_0)$ for all positive δ.

Assuming this, let the solution to this game be $\phi(x)$, $b \leq x \leq 1$, with $\phi(1) = 1$, $\phi(b) = 0$. Extend $\phi(x)$ below b by setting $\phi(x) = 0$, for $x < b$. Set

$$\overline{V}(y) = \int_0^1 K(x,y)d\phi(x) \ .$$

We have: $\overline{V}(y) \geq 0$ for $b \leq y \leq 1$ since $\phi(x)$ is the solution to the game in the interval $b \leq x \leq 1$. We have

$$\overline{V}(b^-) - \overline{V}(b) = \int_0^1 \left\{ K(x,b^-) - K(x,b) \right\} d\phi(x) = 0$$

since $A(b,b) = 0$. Thus $\overline{V}(b^-) = \overline{V}(b) \geq 0$. In the interval from 0 to b, $\overline{V}(y)$ is a strictly decreasing function of y, so that we may assert that

$$\overline{V}(y) > 0 \quad \text{in} \quad 0 \leq y < b \ .$$

Suppose now that there were points of the spectrum of $F(x)$ in the interval $0 \leq x < b$. Then we would have

$$\int_0^1 \overline{V}(y)dF(y) > 0 \ .$$

But $\int \overline{V}(y)dF(y) = \int_0^1\int_0^1 K(x,y)d\phi(x)dF(y) = -\int_0^1\int_0^1 K(x,y)dF(x)d\phi(y) = -\int_0^1 V(y)d\phi(y) \leq 0$ since $V(y) \geq 0$. This contradiction establishes Lemma 3.

Lemma 3 shows that we need only consider the game over the basic interval $b \leq x \leq 1$. Henceforth, we consider only the basic interval, and ignore the values before this basic interval. With no loss of generality, we can let this basic interval be the interval from 0 to 1, so that we have

(2.6) $$A(x,x) > 0 \quad \text{for} \quad 0 < x \leq 1 \ .$$

LEMMA 4. In $a < x \leq 1$, $F(x)$ has a continuous derivative $F'(x) = f(x)$.

PROOF. Set $F(0^+) = \alpha$, $0 \leq \alpha < 1$. Let y be any point in $a < y \leq 1$. Then (2.5) gives

(2.7) $$0 = \alpha A(0,y) + \int_{a^+}^{y} A(x,y)dF(x) - \int_y^1 A(y,x)dF(x) \ .$$

Suppose first that $A(x,y)$ has continuous second derivatives. Then

transform (2.7) by integration by parts into

$$0 = \alpha A(0,y) - \alpha A(a,y) - A(y,1) + 2F(y)A(y,y)$$

$$- \int_{a+}^{y} F(x)A_1(x,y)dx + \int_{y}^{1} F(x)A_2(y,x)dx$$

where the subscript 1, 2 means differentiation with respect to the first or second argument respectively. This equation can be solved for $F(y)$, since $A(y,y) \neq 0$, and shows that $F(y)$ has a continuous derivative in $a < y \leq 1$.

If $A(x,y)$ does not have continuous second derivatives, we can proceed directly from (2.7). Rewrite (2.7) in the form

$$0 = \alpha A(0,y) + \int_{a+}^{y} \{A(x,y) - A(y,y)\}dF(x)$$

$$- \int_{y}^{1} \{A(y,x) - A(y,y)\}dF(x) + A(y,y)\{2F(y) - \alpha - 1\}$$

and solve for $F(y)$. The result shows that $F(y)$ has a continuous derivative with respect to y by virtue of the following remarks:

If $B(x,y)$ is continuous and has continuous first derivatives, if $B(y,y) = 0$, and if $F(x)$ is a continuous distribution function, then

$$\int_{a}^{y} B(x,y)dF(x)$$

has a continuous derivative with respect to y. For

$$\frac{1}{h}\left\{\int_{a}^{y+h} B(x,y+h)dF(x) - \int_{a}^{y} B(x,y)dF(x)\right\} =$$

$$\int_{a}^{y} \frac{B(x,y+h) - B(x,y)}{h} dF(x) + \int_{y}^{y+h} \frac{B(x,y+h)}{h} dF(x) .$$

The first integral approaches $\int_a^y B_y(x,y)dF(x)$ as $h \longrightarrow 0$, while the second integral can be estimated by

$$\left|\int_{y}^{y+h} \frac{B(x,y+h) - B(y+h,y+h)}{h} dF(x)\right| \leq M\left|F(y+h) - F(y)\right| ,$$

where M is a bound in the first derivatives, and this approaches zero as $h \longrightarrow 0$. Thus the derivative exists and

$$\frac{d}{dy}\int_{a}^{y} B(x,y)dF(x) = \int_{a}^{y} B_y(x,y)dF(x) .$$

This shows that the derivative is continuous.

The lemma is proved.

§ 3. THE INTEGRAL EQUATION FOR f(x)

In (2.5), set $F'(x) = f(x)$, where $f(x) \geq 0$:

$$0 = \alpha A(0,y) + \int_a^y A(x,y)f(x)dx - \int_y^1 A(y,x)f(x)dx, \quad a \leq y \leq 1 . \tag{3.1}$$

Differentiation of (3.1) gives

$$f(y) = \alpha p(y) + \int_a^1 L(x,y)f(x)dx, \quad a \leq y \leq 1 , \tag{3.2}$$

where

$$p(y) = -\frac{A_2(0,y)}{2A(y,y)} , \quad \alpha \geq 0 , \tag{3.3}$$

and

$$L(x,y) = \begin{cases} -\dfrac{A_2(x,y)}{2A(y,y)} & \text{for} \quad x < y \\ \dfrac{A_1(y,x)}{2A(y,y)} & \text{for} \quad x > y \end{cases} \tag{3.4}$$

as the fundamental integral equation which $f(y)$ satisfies.

The kernel $L(x,y)$ and the function $\alpha p(y)$ are everywhere ≥ 0 by virtue of hypothesis (1 b) and equation (2.6). Also, in case $A(0,0) = 0$, we see from (3.1) that $a > 0$; otherwise, by letting $y \longrightarrow 0$ in (3.1) we would obtain a contradiction. Thus, $A(y,y) > 0$ everywhere in the closed interval $a \leq y \leq 1$, and the functions $L(x,y)$, $\alpha p(y)$ are uniformly bounded in the interval of integration.

Equation (3.2) is the fundamental integral equation satisfied by $f(x)$. But we must also have

$$\int_a^1 f(x)dx = 1 - \alpha \tag{3.5}$$

and, setting

$$W = \alpha A(0,y) + \int_a^y A(x,y)f(x)dx - \int_y^1 A(y,x)f(x)dx , \quad a \leq y \leq 1 , \tag{3.6}$$

which must be constant for $a \leq y \leq 1$ by virtue of (3.2), we must have

$$W = 0 . \tag{3.7}$$

Conversely, if for some α and a, $0 \leq \alpha < 1$, $0 \leq a < 1$, there is found a non-negative function $f(x)$ satisfying (3.2), (3.5), (3.7), then

$$F(x) = \begin{cases} 0 & \text{for } x = 0 \\ \alpha & \text{for } 0 < x < a, \\ \alpha + \int_a^x f(\xi)\,d\xi & \text{for } a < x \leq 1 \end{cases} \tag{3.8}$$

is an optimal strategy. For, from (3.7), it follows by multiplying (3.6) by $f(y)$ and integrating from a to 1 that

$$0 = \alpha \int_a^1 A(0,y) f(y)\,dy . \tag{3.9}$$

Then we have, for $V(y)$ defined in (2.2),

$$V(y) = \begin{cases} 0 & \text{for } y = 0 \text{ in case } \alpha \neq 0 \\ \text{positive and monotonic decreasing for} & \\ 0 < y < a & (\text{for } 0 \leq y < a \text{ if } \alpha = 0) \\ 0 & \text{for } a \leq y \leq 1 \end{cases}$$

Therefore, $V(y)$ satisfies the inequality

$$V(y) \geq 0$$

and this means that $F(x)$ is an optimal strategy. The following theorem has been demonstrated:

> THEOREM. There is an optimal strategy $F(x)$ if and only if there are numbers α, a, $0 \leq \alpha < 1$, $0 \leq a < 1$ and a non-negative function $f(x)$, defined in $a \leq x \leq 1$, which satisfies the equations (3.2), (3.5), (3.7). Then $F(x)$ is given in (3.8).

§ 4. ON INTEGRAL EQUATIONS WITH A POSITIVE KERNEL[4]

Consider the homogeneous integral equation of the second kind,

[4]The theorems obtained in § 4 are related to similar questions studied by

$$\int_p^q L(x,y)f(y)dy = \lambda f(x) , \qquad p \leq x \leq q , \tag{4.1}$$

or, in operator form

$$\mathcal{L}f(x) = \lambda f(x)$$

where $\mathcal{L}f(x)$ is the integral operation on the left-hand side of (4.1). We will suppose that the kernel $L(x,y)$ is non-negative, and we are interested in non-negative solution $f(x)$ of (4.1), i.e.,

$$f(x) \geq 0 , \qquad 0 < \int f(x)dx < \infty , \tag{4.2}$$

(throughout this digression, all integral signs will be understood between the fixed limits p, q). The class of functions satisfying (4.2) will be denoted by P_1. To be specific, the assumptions on the kernel $L(x,y)$ are as follows:

$$L(x,y) \geq 0 , \tag{4 a}$$

the points where $L = 0$ forming on each line $x =$ constant or line $y =$ constant a point set containing no intervals.

$$L(x,y) \leq M \tag{4 b}$$

$$\int \left| L(x_2,y) - L(x_1,y) \right| dy \longrightarrow 0 \quad \text{and} \tag{4 c}$$

$$\int \left| L(x,y_2) - L(x,y_1) \right| dx \longrightarrow 0 \quad \text{if}$$

$$|x_2 - x_1| \longrightarrow 0 \quad \text{or} \quad |y_2 - y_1| \longrightarrow 0 \quad \text{respectively} .$$

Weaker assumptions can be made if desired. It follows from (4 a), (4 b), (4 c) that if $f(y)$ is in P_1, then

$$\int L(x,y)f(x)dx \quad \text{and} \quad \int L(x,y)f(y)dy$$

are continuous functions of y and x respectively, and if $f(y)$ is in addition continuous they have a positive minimum.

We shall prove the following theorems.

Perron and by Frobenius for finite matrices and by Jentzsch for integral equations. The maximum property of the eigenvalue was told to me by S. Karlin, and is related to unpublished work of Bohnenblust and Karlin. See Perron, Math. Ann. 65 (1907), pp. 248-263; Frobenius, Berl. Ber. (1908), pp. 471-476; Jentzsch, J. Reine Angew. Math. 141 (1912), pp. 235-244.

1. There exists a unique $f(x)$ in P_1, normalized by $\int f(x)dx = 1$, satisfying (4.1) for some positive λ.

2. If μ is a positive number such that

$$\int L(x,y)g(y)dy \geq \mu g(x) , \qquad a \leq x \leq b \tag{4.3}$$

for some $g(x)$ in P_1, then $\mu \leq \lambda$ and $\mu = \lambda$ only if $g(x)$ is a multiple of $f(x)$. (Maximum property of the eigenvalue λ.)

3. The inhomogenous equation

$$\phi(x) = h(x) + \rho\int L(x,y)\phi(y)dy \tag{4.4}$$

where $h(x)$ is a given function in P_1 and $\rho > 0$ has a solution $\phi(x)$ in P_1 if and only if $\rho < \frac{1}{\lambda}$. The solution is then unique and is given by (4.5) $\phi(x) = h(x) + \sum_{\nu=1}^{\infty} \rho^{\nu}\mathcal{L}^{\nu} h(x)$, where $\mathcal{L}^{\nu}$ means the ν^{th} iterate of $\mathcal{L}$.

4. If Ψ is absolutely integrable, and

$$\Psi(x) \leq h(x) + \rho\int L(x,y)\Psi(y)dy , \quad \text{where} \quad \rho < \frac{1}{\lambda} ,$$

then

$$\Psi(x) \leq \phi(x)$$

everywhere.

PROOF. (1) and (2). Consider the set Ω of μ's satisfying (4.3) for some $g(x)$ in P_1. Setting $g(x) \equiv 1$, shows that there are positive μ's in Ω, namely $\mu = \operatorname{Min}_x \int L(x,y)dy$. Also, integrating (4.3) with respect to x and using condition (b) shows that

$$\mu \leq M .$$

Normalizing the functions $g(x)$ in (4.3) by supposing $\int g(x)dx = 1$, we see that

$$g(x) \leq \frac{M}{\mu} \text{ for every } x .$$

Let $\lambda = \operatorname{Sup}_{\mu \text{ in } \Omega} \mu$. There is a sequence $\mu_i \longrightarrow \lambda$ with $g_i(x)$, normalized, satisfying

$$\mathcal{L}g_i(x) \geq \mu_i g_i(x) . \tag{4.6}$$

But then also,

$$\mathcal{L}(\mathcal{L}g_i(x)) \geq \mu_i(\mathcal{L}g_i(x)) . \tag{4.7}$$

The functions $\mathcal{L}g_i(x)$ are positive continuous functions, and are equicontinuous. For,

$$\left|\int L(x_1,y)g_i(y)dy - \int L(x_2,y)g_i(y)dy\right| \leq$$
$$\frac{M}{\mu_i}\int \left|L(x_1,y) - L(x_2,y)\right| dy$$

and equicontinuity follows from hypothesis (4 c) concerning the kernel $L(x,y)$. The theorem of Arzela shows that a subsequence can be found such that $\mathcal{L}g_i(x)$ converge uniformly to a continuous non-negative $f(x)$. From (4.7) we find

$$\mathcal{L}f(x) \geq \lambda f(x) , \tag{4.8}$$

while integrating (4.6) shows that $\int f(x)dx \geq \lambda$, so that $f(x)$ is not identically 0. Thus λ belongs to the set Ω. But in fact, equality holds in (4.8). For, both sides of (4.8) are continuous functions, and a strict inequality at any point x would show that

$$\mathcal{L}(\mathcal{L}f(x) - \lambda f(x)) \geq \delta > 0$$

for some positive δ. But then

$$\mathcal{L}(\mathcal{L}f(x)) \geq \lambda \mathcal{L}f(x) + \delta \geq (\lambda + \tfrac{\delta}{N})\mathcal{L}f(x)$$

where $N = \operatorname{Max}_x \mathcal{L}f(x)$. This shows that $\lambda + \frac{\delta}{N}$ belongs to the set Ω, contrary to the definition of λ. Therefore

$$\mathcal{L}f(x) = \lambda f(x) \tag{4.9}$$

and Theorems (1) and (2) have been established, except for the uniqueness of $f(x)$.

The equality (4.9) shows that $f(x)$ has a positive minimum. The above establishes the existence of a solution $g(y)$ in P_1 to the transposed equation

$$\int L(x,y)g(x)dx = \lambda' g(y) \ .$$

The function $g(y)$ is also continuous and has a positive minimum. Multiplying this equation by $f(y)$ and integrating with respect to y yields

$$\lambda \int f(x)g(x)dx = \lambda' \int f(y)g(y)dy$$

or $\lambda = \lambda'$ since $\int f(x)g(x)dx > 0$. This shows that there cannot be a solution in P_1 of (4.1) for any other value of λ. Suppose now that there is another solution $h(x)$ of (4.9) in P_1. Let

$$c = \operatorname*{Min}_x \frac{h(x)}{f(x)} \ ,$$

so that $h(x) - cf(x) \geq 0$ and $= 0$ in at least one point. But since

$$\mathcal{L}(h(x) - cf(x)) = \lambda(h(x) - cf(x))$$

it follows that $h(x) - cf(x)$ has a positive minimum (unless it is identically zero). This shows that, indeed,

$$h(x) \equiv cf(x) \ ,$$

and the uniqueness in (1) is established.

REMARK. The proof we have given depended on Arzela's theorem. Other proofs and other methods can be given depending on notions of weak convergence and applicable to different classes of kernels $L(x,y)$.

(3). Let (4.4) have a solution in P_1. Let $g(x)$ be the positive eigenfunction for the transposed homogeneous equation (with the same eigenvalue λ). Multiplying by $g(x)$ and integrating gives

$$\int \phi(x)g(x)dx = \int h(x)g(x)dx + \rho\lambda \int \phi(x)g(x)dx$$

or

$$(1 - \rho\lambda) \int \phi(x)g(x)dx = \int h(x)g(x)dx \ .$$

The integrals appearing are both positive and this requires that

$$1 - \rho\lambda > 0 \quad \text{or} \quad \rho < \frac{1}{\lambda} .$$

For $\rho < \frac{1}{\lambda}$, let $S_n(x)$ be the first $n+1$ terms of the right-hand side of (4.5),

$$S_n(x) = h(x) + \sum_{\nu=1}^{n} \rho^\nu \mathcal{L}^\nu \, h(x) .$$

Multiplying by $g(x)$ and integrating gives

$$\int S_n(x)g(x)dx = \int h(x)g(x)dx \cdot \left(1 + \sum_{\nu=1}^{n} \rho^\nu \lambda^\nu\right)$$

or

$$\lim_{n\to\infty} \int S_n(x)g(x)dx = \frac{1}{1 - \rho\lambda} \cdot \int h(x)g(x)dx .$$

The sequence $S_n(x)$ is an <u>increasing</u> sequence of (positive) functions which converges to a function $\phi(x)$ (with $+\infty$ admitted as a value). Since $g(x) > 0$ it follows from a standard theorem that $\phi(x)g(x)$ is integrable (so that ∞ is taken as value at most on a set of measure zero) and

$$\int \phi(x)g(x)dx = \frac{1}{1 - \rho\lambda} \int h(x)g(x)dx .$$

But, since $g(x)$ has a positive minimum, $\phi(x)$ is also integrable.

Concerning $S_n(x)$ we know that

$$S_n(x) = h(x) + \rho\mathcal{L}S_{n-1}(x) .$$

A passage to the limit yields

$$\phi(x) = h(x) + \rho\mathcal{L}\phi(x) ,$$

and indeed a solution is obtained to (4.4).

Let $\Psi(x)$ be any solution to (4.4), so that

$$\Psi(x) - \phi(x) = \rho\int L(x,y)(\Psi(y) - \phi(y))dy .$$

Therefore

$$|\Psi(x) - \phi(x)| \leq \rho\int L(x,y) \, |\Psi(y) - \phi(y)| \, dy .$$

Since $\frac{1}{\rho} > \lambda$, we see that $|\Psi(x) - \phi(x)|$ must be a null function, or $\Psi(x) = \phi(x)$.

(4). Successive iteration of the inequality gives

$$\psi \leq h + \rho\mathcal{L}h + \ldots + \rho^{n-1}\mathcal{L}^{n-1}h + \rho^n\mathcal{L}^n\psi$$

so that

$$\psi \leq \phi + \rho^n\mathcal{L}^n\psi .$$

Now

$$|\rho^n\mathcal{L}^n\psi| \leq \rho^n\mathcal{L}^n|\psi|$$

and this approaches zero almost everywhere, by the argument in the proof of 3 above (convergence of the series (4.5)). Thus, $\psi \leq \phi$ almost everywhere. But the inequality then gives

$$\psi \leq h + \rho\mathcal{L}\phi = \phi$$

so that $\psi \leq \phi$ everywhere. This completes the proof of the entire theorem.

§ 5. THE DEPENDENCE ON LIMITS OF INTEGRATION

Suppose that $L(x,y)$ is defined over some rectangle, say $0 \leq x \leq 1$, $0 \leq y \leq 1$, and satisfies conditions (4 a), (4 b), (4 c) there, and that $p(y)$ is defined in $0 \leq y \leq 1$ and is continuous and of class $\mathcal{B}_1$. Consider the integral equations in $\mathcal{B}_1$,

$$\int_a^1 L(x,y)f_a(x)dx = \lambda(a)f_a(y) , \qquad a \leq y \leq 1 , \tag{5.1}$$

and

$$\phi_a(y) = p(y) + \int_a^1 L(x,y)\phi_a(x)dx , \qquad a \leq y \leq 1 , \tag{5.2}$$

where $0 \leq a < 1$. We shall discuss the dependence of the positive eigenvalue $\lambda(a)$, the non-negative normalized eigenfunction $f_a(x)$, and the solution $\phi_a(x)$ of the inhomogeneous integral equation on the parameter a. A similar consideration applies if the upper limit is varied instead of the lower limit.

THEOREM 5.1. The eigenvalue $\lambda(a)$ is a strictly decreasing continuous function of a with $\lambda(a) \longrightarrow 0$ as $a \longrightarrow 1$.

PROOF. For $a' < a$, define

$$h(x) = \begin{cases} 0 , & a' \leq x < a \\ f_a(x) , & a \leq x \leq 1 \end{cases} .$$

Then

$$\int_{a'}^{1} L(x,y)h(x)dx = \int_{a}^{1} L(x,y)f_a(x)dx = \begin{cases} \text{non-negative} , & a' \leq y < a \\ \lambda(a)f_a(y) , & a \leq y \leq 1 \end{cases}$$

and

$$\int_{a'}^{1} L(x,y)h(x)dx \geq \lambda(a)h(y) ,$$

so that

$$\lambda(a') \geq \lambda(a) .$$

Furthermore, equality can hold only if $h(y)$ is an eigenfunction for the interval $a' \leq y \leq 1$. But $h(y) = 0$ on a subinterval, and an eigenfunction has a positive minimum. Therefore,

$$\lambda(a') > \lambda(a) .$$

Over the smaller interval $a \leq x \leq 1$, use the function $f_{a'}(x)$. We have

$$\begin{aligned} \int_{a}^{1} L(x,y)f_{a'}(x)dx &= \lambda(a')f_{a'}(y) - \int_{a'}^{a} L(x,y)f_{a'}(x)dx \\ &\geq \lambda(a')f_{a'}(y) - \frac{M^2}{\lambda(a')} (a - a') \\ &\geq \left[\lambda(a') - \frac{M^2}{m\,\lambda(a')} (a - a')\right] f_{a'}(y) \end{aligned}$$

where $m = \min_{y} f_{a'}(y) > 0$, and we have used

$$|f_{a'}(y)| \leq \frac{M}{\lambda(a')} .$$

Therefore

$$\lambda(a) \geq \lambda(a') - \frac{M^2}{m\,\lambda(a')} (a - a') ,$$

and this establishes the continuity of $\lambda(a)$.

By integrating (5.1) with respect to y, we obtain

$$\lambda(a) \leq \operatorname{Max}_x \int_a^1 L(x,y)dy \leq M(1 - a) .$$

Hence $\lambda(a) \longrightarrow 0$ as $a \longrightarrow 1$. This completes the desired proof.

THEOREM 5.2. The eigenfunction $f_a(x)$, $a \leq x \leq 1$, converges uniformly[5] to $f_{a'}(x)$, $a' \leq x \leq 1$, if $a \longrightarrow a'$.

PROOF. From (5.1) we find

$$f_a(y) \leq \frac{M}{\lambda(a)} \quad \text{for all } y \text{ in } a \leq y \leq 1 . \tag{5.3}$$

Also,

$$\lambda(a)\,|f_a(y_1) - f_a(y_2)| \leq \frac{M}{\lambda(a)} \int |L(x,y_1) - L(x,y_2)|\,dx .$$

Thus as long as a is bounded away from 1, the functions $f_a(y)$ are equicontinuous, and as $a \longrightarrow a'$ a subsequence can be found which converges uniformly to a function which must coincide with $f_{a'}(y)$ since equation (5.1) for a' is satisfied by the limit function. Since this is true for every infinite set of a's $\longrightarrow a'$, the theorem is proved.

THEOREM 5.3. The function $\phi_a(x)$, $a \leq x \leq 1$, converges uniformly[5] to $\phi_{a'}(x)$ as $a \longrightarrow a'$ if $\lambda(a') < 1$. If $\lambda(a_1) = 1$, then

$$\int_a^1 \phi_a(x)dx \longrightarrow \infty \quad \text{and} \quad \frac{\phi_a(x)}{\int_a^1 \phi_a(x)dx}$$

converges uniformly[5] to $f_{a_1}(x)$.

PROOF. Let $g_a(x)$ be the eigenfunction solving the transposed equation to (5.1). Then,

$$\int_a^1 \phi_a(y)g_a(y)dy = \frac{1}{1 - \lambda(a)} \int_a^1 p(y)g_a(y)dy .$$

If a lies in the range $a_1 + \delta \leq a \leq 1 - \delta$ for some positive δ, then by the uniform convergence of $g_a(y)$ as a varies there is a positive lower bound m of $g_a(y)$, and the same upper bound as $f_a(x)$ in the

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[5]The sense is clear; for example, by transforming the differing intervals of definition of the functions into a common interval by linear transformations.

proof of Theorem 5.2:

$$0 < m \leq g_a(y) \leq \frac{M}{\lambda(a)} .$$

Therefore

$$(5.4) \qquad \frac{m\lambda(a)}{M(1-\lambda(a))} \int_a^1 p(y)dy \leq \int_a^1 \phi_a(y)dy \leq \frac{M}{m\lambda(a)(1-\lambda(a))} \int_a^1 p(y)dy .$$

Equation (5.2) gives

$$(5.5) \qquad \phi_a(y) \leq p(y) + \frac{M^2}{m\lambda(a)(1-\lambda(a))} \int_a^1 p(y)dy \leq M_1$$

and therefore

$$|\phi_a(y_1) - \phi_a(y_2)| \leq |p(y_2) - p(y_1)| + M_1 \int_a^1 |L(x,y_2) - L(x,y_1)| \, dx .$$

This proves the equicontinuity of $\phi_a(y)$, and as in the proof of Theorem 5.2 the uniform convergence of $\phi_a(y)$ to $\phi_{a'}(y)$ as $a \longrightarrow a'$.

If $a \longrightarrow a_1$, where $\lambda(a_1) = 1$, the unboundedness of $\int_a^1 \phi_a(x)dx$ is stated in (5.4). Also,

$$\frac{\phi_a(y)}{\int_a^1 \phi_a(x)dx} \leq \frac{p(y)}{\int_a^1 \phi_a(x)dx} + M \leq M_2 ,$$

and

$$\frac{|\phi_a(y_2) - \phi_a(y_1)|}{\int_a^1 \phi_a(x)dx} \leq \frac{|p(y_2) - p(y_1)|}{\int_a^1 \phi_a(x)dx} + M_2 \int_a^1 |L(x,y_2) - L(x,y_1)| \, dx .$$

Again this establishes the uniform convergence of

$$\frac{\phi_a(x)}{\int_a^1 \phi_a(x)dx}$$

to the solution $f_{a_1}(x)$ of the homogeneous equation.

It also follows from Theorem 4 of § 4 that $\phi_a(y) \geq \phi_{a'}(y)$ if $a < a'$.

THEOREM 5.4.[6] For $a_1 < a < 1$, there is a non-

[6] By virtue of a supplementary remark at the end of the paper, this theorem is not necessary for the application to games of timing.

negative function $\psi_a(x)$, $a \leq x \leq 1$ such that

$$\int_{\bar{a}}^{1} \left| - \frac{\phi_{a+h}(x) - \phi_a(x)}{h\,\phi_a(a)} - \psi_a(x) \right| dx \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0 ,$$

where $\bar{a}$ is the maximum of a, $a+h$. This function $\psi_a(x)$ satisfies the integral equation

$$\psi_a(y) = L(a,y) + \int_a^1 L(x,y)\psi_a(x)dx , \qquad a \leq y \leq 1 . \tag{5.6}$$

If

$$\frac{1}{h}\int_a^{a+h} L(x,y)dx \longrightarrow L(a,y) \quad \text{as} \quad h \longrightarrow 0 \quad \text{for each} \quad y \geq a$$

then $\frac{\partial}{\partial a}\phi_a(x)$ exists and

$$\psi_a(x) = - \frac{1}{\phi_a(a)} \frac{\partial}{\partial a} \phi_a(x) .$$

PROOF. Subtraction of the equations satisfied by $\phi_a(y)$, $\phi_{a+h}(y)$ gives

$$- \frac{\phi_{a+h}(y) - \phi_a(y)}{h\,\phi_a(a)} = \int_{\bar{a}}^{1} L(x,y) \left(- \frac{\phi_{a+h}(x) - \phi_a(x)}{h\,\phi_a(a)} \right) dx + \frac{1}{h}\int_a^{a+h} L(x,y) \frac{\phi_{\tilde{a}}(x)dx}{\phi_a(a)}$$

where $\tilde{a} = \min(a, a+h)$. Define $\psi_a(y)$ as the non-negative solution of (5.6), and set

$$- \frac{\phi_{a+h}(y) - \phi_a(y)}{h\,\phi_a(a)} - \psi_a(y) = \chi(y) , \tag{5.7}$$

$$\frac{1}{h}\int_a^{a+h} L(x,y) \frac{\phi_{\tilde{a}}(x)}{\phi_a(a)} dx - L(a,y) - \int_a^{\bar{a}} L(x,y)\psi_a(x)dx = \eta(y) . \tag{5.8}$$

We have

$$\chi(y) = \eta(y) + \int_{\bar{a}}^{1} L(x,y)\chi(x)dx$$

or

$$|\chi(y)| \leq |\eta(y)| + \int_{\bar{a}}^{1} L(x,y)\ |\chi(x)|\ dx\ . \tag{5.9}$$

Theorem 4 of § 4 and (5.4) gives

$$\int_{\bar{a}}^{1} |\chi(y)|\ dy \leq \frac{M}{m\lambda(a)(1-\lambda(a))} \int_{\bar{a}}^{1} |\eta(y)|\ dy\ . \tag{5.10}$$

But

$$\int_{\bar{a}}^{1} |\eta(y)|\ dy \leq \left(\frac{\frac{1}{h}\int_{a}^{a+h} \phi_{\tilde{a}}(x)dx}{\phi_{\tilde{a}}(a)} - 1 \right) \int_{\bar{a}}^{1} L(a,y)dy$$

$$+ \frac{1}{h} \int_{a}^{a+h} \frac{\phi_{\tilde{a}}(x)}{\phi_{\tilde{a}}(a)} \left[\int_{\bar{a}}^{1} |L(x,y) - L(a,y)|\ dy \right] dx$$

$$+ \int_{\bar{a}}^{1} \int_{a}^{\bar{a}} L(x,y) \psi_{a}(x) dxdy\ .$$

By virtue of hypothesis (4 c), the uniform convergence of $\phi_{a}(x)$, as in Theorem 5.3, and the bound (5.5) for $\psi_{a}(x)$, it follows from this estimate that

$$\int_{\bar{a}}^{1} |\eta(y)|\ dy \quad \text{and therefore} \quad \int_{\bar{a}}^{1} |\chi(y)|\ dy \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0\ ,$$

and this proves the main portion of the theorem. If

$$\frac{1}{h} \int_{a}^{a+h} L(x,y)dx \longrightarrow L(a,y) \quad \text{for each} \quad y \geq a\ ,$$

this means that $\eta(y) \longrightarrow 0$ for each $y \geq a$, and therefore from

$$|\chi(y)| \leq |\eta(y)| + M \int_{\bar{a}}^{1} |\chi(x)|\ dx$$

we see that $\chi(y) \longrightarrow 0$. q.e.d.

We can also assert that

$$\frac{\phi_{a+h}(y) - \phi_{a}(y)}{h}$$

remains uniformly bounded as $h \longrightarrow 0$. For, the uniform boundedness of $\psi_{a}(x)$ is obtained as in (5.5), of $\eta(y)$ from its definition in (5.8), of $\chi(y)$ from (5.9) and (5.10), and finally of the desired quantity from (5.7).

§ 6. THE OPTIMAL STRATEGY

The theory of §§ 4, 5 will now be applied to the particular integral equation (3.2) with kernel $L(x,y)$ given by (3.4). The conditions (4 a), (4 b), (4 c) are easily verified. The parameter a is to vary over the range $0 \leq a < 1$, except that if $A(0,0) = 0$ the case $a = 0$ is to be excluded. As in §§ 4, 5 denote the positive eigenvalue of the homogeneous equation by $\lambda(a)$, and the positive eigenfunction by $f_a(x)$, $a \leq x \leq 1$.

The integral equation (3.2) has a non-negative solution for $\alpha = 0$ only when $\lambda(a) = 1$, and for $\alpha > 0$ only when $\lambda(a) < 1$. There are two cases: either there is a value a_1 such that $\lambda(a_1) = 1$, or $\lambda(a) < 1$ for all a.

Case I. $\lambda(a_1) = 1$. When $a = a_1$ and $\alpha = 0$, the function $f_{a_1}(x)$ satisfies the integral equation (3.2) and the condition (3.5). The corresponding quantity W defined in (3.6) vanishes, since this is obtained by multiplying (3.6) by $f_{a_1}(y)$ and integrating from a_1 to 1, the double integral on the right-hand side vanishing by the skew-symmetry of the kernel. The theorem of § 3 shows that we have an optimum strategy. That this is a unique optimum strategy follows from the discussion in Case II below.

Case II. $\lambda(a) < 1$ for all a. We consider here also Case I, but then limit the range of a to $a_1 < a < 1$. For the positive solution of (3.2), set

$$f(x) = \alpha \phi_a(x)$$

where $\phi_a(x)$ satisfies

$$\phi_a(y) = p(y) + \int_a^1 L(x,y)\phi_a(x)dx , \qquad a \leq y \leq 1 . \tag{6.1}$$

The condition (3.5) becomes

$$\int_a^1 \phi_a(x)dx = \frac{1 - \alpha}{\alpha} \tag{6.2}$$

which gives a unique determination $\alpha(a)$, $0 < \alpha(a) < 1$, in terms of a. Denote the quantity defined in (3.6) by $W(a)$, and set

$$\frac{W(a)}{\alpha(a)} = U(a)$$

where

$$(6.3)\qquad U(a) = A(0,y) + \int_a^y A(x,y)\phi_a(x)dx - \int_y^1 A(y,x)\phi_a(x)dx\ ,$$

$$a \leq y \leq 1\ ,$$

the value being independent of y when y varies in the indicated range. The parameter a must be determined by the remaining condition (3.7).

LEMMA. $U(a)$ is a strictly decreasing, continuous function of a with the following properties:

$$\lim_{a \to 1} U(a) = A(0,1) < 0$$

$$\lim_{a \to a_1^+} U(a) \leq 0 \quad \text{in case} \quad \lambda(a_1) = 1$$

$$U(0) = \frac{A(0,0)}{1 + \int_0^1 \phi_0(y)dy} > 0 \quad \text{in case} \quad \lambda(a) < 1 \quad \text{for all} \quad a \quad \text{and} \quad A(0,0) > 0\ .$$

PROOF. The continuity of $U(a)$ is a consequence of Theorem 5.3. The first limiting relation is clear. To obtain the second and third relations, use two equivalent forms of (6.3). First multiply (6.3) by $\phi_a(y)$ and integrate with respect to y from a to 1:

$$(6.4)\qquad U(a)\int_a^1 \phi_a(y)dy = \int_a^1 A(0,y)\phi_a(y)dy$$

the double integral terms vanishing because of the skew-symmetry of the kernel. Also, set $y = a$ in (6.3).

$$(6.5)\qquad U(a) = A(0,a) - \int_a^1 A(a,x)\phi_a(x)dx\ .$$

Divide (6.4) by $\int_a^1 \phi_a(y)dy$, let $a \longrightarrow a_1^+$, in case $\lambda(a_1) = 1$, and use theorem (5.3). There results

$$\lim_{a \to a_1^+} U(a) = \int_a^1 A(0,y)f_{a_1}(y)dy\ .$$

But

$$\int_{a_1}^1 A(0,y)f_{a_1}(y)dy \leq \int_{a_1}^1 A(a_1,y)f_{a_1}(y)dy\ ,$$

and this second integral is zero since it is the negative of the corresponding quantity W in Case I, which was proved to be zero in Case I. This proves the second limiting relation.

To obtain the third relation set $a = 0$ in (6.4) and (6.5). The two integrals on the right-hand sides are then identical, and their elimination gives

$$U(0) = \frac{A(0,0)}{1 + \int_0^1 \phi_0(x)dx} .$$

There remains to establish the strictly decreasing character of $U(a)$.[7] We have by differentiation of (6.3):

$$U'(a) = -A(a,y)\phi_a(a) + \int_a^y A(x,y) \frac{\partial \phi_a(x)}{\partial a} dx - \int_y^1 A(y,x) \frac{\partial \phi_a(x)}{\partial a} dx , \qquad a \leq y \leq 1 , \tag{6.6}$$

which is independent of y in the indicated range. Again, obtain two expressions for $U'(a)$, first by multiplying by

$$\frac{\partial \phi_a(y)}{\partial a}$$

and integrating from a to 1, and second by setting $y = a$:

$$U'(a) \int_a^1 \frac{\partial \phi_a(y)}{\partial a} dy = - \phi_a(a) \int_a^1 A(a,y) \frac{\partial \phi_a(y)}{\partial a} dy$$

$$U'(a) = -A(a,a)\phi_a(a) - \int_a^1 A(a,x) \frac{\partial \phi_a(x)}{\partial a} dx .$$

The two integrals on the right-hand sides are identical, and their elimination gives

$$U'(a) = \frac{-A(a,a)\phi_a(a)}{1 - \frac{1}{\phi_a(a)} \int_a^1 \frac{\partial \phi_a(y)}{\partial a} dy} < 0 ,$$

---

[7] This is used below to establish the uniqueness of an optimal strategy, but a simpler proof is obtained in accordance with the supplementary remark at the end of the paper. The decreasing character of $U(a)$ is then additional information about $U(a)$.

since $\frac{\partial \phi_a(y)}{\partial a} \leq 0$ as in Theorem 5.4. This completes the proof of the lemma.

This basic lemma allows us to complete our discussion. In Case I, the lemma shows that $U(a) < 0$ for all a in the range $a_1 < a < 1$, and establishes the uniqueness of the optimal strategy obtained in Case I.

In Case II, under the supposition $A(0,0) > 0$, we see from the lemma that there is a unique value of a which causes $U(a)$ to vanish. This gives an optimal strategy and also establishes its uniqueness.

There remains the case when $A(0,0) = 0$, which is not yet covered by the lemma because of the unboundedness of the kernel. The following lemma completes the theory.

LEMMA. If $A(0,0) = 0$, there is a positive a_1 with $\lambda(a_1) = 1$.

PROOF. Suppose that the lemma were false, so that $\lambda(a) < 1$ for all a in $0 < a < 1$. Formula (6.4) shows that $U(a) < 0$ since $A(0,y) < 0$ for all $y > 0$ because of $A(0,0) = 0$. There is therefore no solution of the game with $a > 0$. But we have already established in a discussion near the beginning of § 3, that a cannot be 0 in the case $A(0,0) = 0$. There would therefore be no optimal strategy and if we suppose that in this case an optimal strategy must exist, the resulting contradiction would establish the lemma.[8]

We now give another proof of the lemma independent of the general theory of games. Again suppose that the lemma were false, so that $\lambda(a) < 1$ for all a in $0 < a < 1$. Take a positive ε, and consider the game with the same kernel $K(x,y)$ but with x,y limited to the range $\varepsilon \leq x \leq 1$, $\varepsilon \leq y \leq 1$. For this range the whole preceding theory is valid since $A(\varepsilon,\varepsilon) > 0$. Also, the kernel in the new integral equation corresponding to (3.2) is the same, so that the eigenvalues are the same, but the term $p(y)$ in (3.2) and (3.3) is now given by

$$p(y) = \frac{-A_2(\varepsilon,y)}{2A(y,y)} .$$

[8] It is possible to show from very general considerations in the theory of games that in the present case, with $A(0,0) = 0$, an optimal strategy exists. In these general considerations, difficulties occur at the end points 0,1. Here, no difficulties occur at 0 because of $A(0,0) = 0$ (continuity at 0), nor at 1 by the argument in Lemma 1.

That is, all previous equations remain valid with 0 in the argument always replaced by the new lower limit ε.

Let the solution of the game, corresponding to $f(x)$ in (3.2) be $\psi_\varepsilon(x)$, with the values a_ε, α_ε of a and α. We have

$$\int_{a_\varepsilon}^{1} \psi_\varepsilon(x)dx = 1 - \alpha_\varepsilon$$

and since the corresponding W and U are zero, we have corresponding to (6.4)

$$\int_{a_\varepsilon}^{1} A(\varepsilon,y)\psi_\varepsilon(y)dy = 0 \tag{6.7}$$

and

$$\alpha_\varepsilon A(\varepsilon,1) + \int_{a_\varepsilon}^{1} A(x,1)\psi_\varepsilon(x)dx = 0 \tag{6.8}$$

corresponding to (6.3) with $y = 1$. From (6.7), since $A(0,y) < 0$ for $y > 0$, we see that $a_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Select a fixed $b > 0$ so that $A(b,1) < 0$. We have, from (6.7) for sufficiently small ε,

$$\int_{a_\varepsilon}^{b} A(\varepsilon,y)\psi_\varepsilon(y)dy + \int_{b}^{1} A(\varepsilon,y)\psi_\varepsilon(y)dy = 0$$

or

$$\int_{a_\varepsilon}^{b} (A(\varepsilon,y) - A(\varepsilon,a_\varepsilon))\psi_\varepsilon(y)dy + \int_{b}^{1} (A(\varepsilon,y) - A(\varepsilon,b))\psi_\varepsilon(y)dy$$

$$= -A(\varepsilon,a_\varepsilon)\left(1 - \alpha_\varepsilon - \int_{b}^{1} \psi_\varepsilon(y)dy\right) - A(\varepsilon,b)\int_{b}^{1} \psi_\varepsilon(y)dy .$$

The left-hand side is ≤ 0, while the right-hand side approaches $-A(0,b) \lim_{\varepsilon\to 0} \int_b^1 \psi_\varepsilon(y)dy$ which is > 0 unless the limit expression is zero. Therefore $\lim_{\varepsilon\to 0} \int_b^1 \psi_\varepsilon(y)dy = 0$.

From (6.8), we now have

$$\alpha_\varepsilon A(\varepsilon,1) + \int_{a_\varepsilon}^{b} [A(x,1) - A(b,1)]\psi_\varepsilon(x)dx + A(b,1)\left(1 - \alpha_\varepsilon - \int_{b}^{1} \psi_\varepsilon(x)dx\right)$$

$$+ \int_{b}^{1} [A(x,1) - A(1,1)]\psi_\varepsilon(x)dx + A(1,1)\int_{b}^{1} \psi_\varepsilon(x)dx = 0 .$$

Take the limit as $\varepsilon \longrightarrow 0$. All the integrals above except for the last are ≤ 0, with $\int_b^1 \psi_\varepsilon(x)dx \longrightarrow 0$, and $A(0,1) < 0$, $A(b,1) < 0$. If

$\alpha_\varepsilon \longrightarrow \alpha_0$, $0 \leq \alpha_0 \leq 1$, we see that the limit of the left-hand side of the above equation is definitely < 0, contrary to the equation. This is a contradiction and the lemma is established.

This completes the proof of the main theorem.

§ 7. REDUCTION TO LINEAR DIFFERENTIAL EQUATIONS

In a wide category of cases, the integral equation (3.2) is equivalent to a system of ordinary linear differential equations. This occurs if the function $A(x,y)$ has the following special form:

$$A(x,y) = \sum_{i=1}^{n} p_i(x)q_i(y) \ .$$

Set

$$\int_y^1 p_i(x)f(x)dx = \xi_i(y) \ , \qquad \int_y^1 q_i(x)f(x)dx = \eta_i(y)$$

or

$$(7.1) \quad \begin{cases} \xi_j'(y) = -p_j(y)f(y) \ , & \xi_j(1) = 0, \quad j = 1, \ldots, n \\ \eta_j'(y) = -q_j(y)f(y) \ , & \eta_j(1) = 0, \quad j = 1, \ldots, n \ . \end{cases}$$

The integral equation becomes

$$(7.2) \quad 2A(y,y)f(y) = -\alpha \sum_i p_i(0)q_i'(y) - \sum_i q_i'(y)(\xi_i(a) - \xi_i(y)) + \sum_i p_i'(y)\ \eta_i(y) \ .$$

Temporarily set $\alpha p_i(0) + \xi_i(a) = c_i$, where c_i are arbitrary coefficients. Substitution of (7.2) into (7.1) gives a system of ordinary linear differential equations for $\xi_j(y)$, $\eta_j(y)$, depending linearly and homogeneously on the parameters c_i. The solutions of this system also depend linearly and homogeneously on c_i. The equation

$$c_i = \alpha p_i(0) + \xi_i(a)$$

gives a system of n linear equations for the determination of c_i in terms of a,α and the conditions (3.5), (3.7) then determine α,a. The question of the unique solvability for the parameters is answered by our

main theorem, provided the monotonicity conditions of the theorem are satisfied.

SUPPLEMENTARY REMARK

The proof of the uniqueness of an optimal strategy can be given in a simpler form than presented in the body of the text. The existence of a solution is a consequence of the lemma in § 6, the uniqueness following from the decreasing character of U(a). Instead of the decreasing character of U(a), one can also use the following lemma: if there are two optimal strategies, they must have the same spectrum. This is proved as in the proof of Lemma 3.

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REDUCTION OF CERTAIN CLASSES OF GAMES TO INTEGRAL EQUATIONS[1]

Samuel Karlin

In this paper the solution of certain classes of games is reduced to the problem of solving associated integral equations. The games treated share the important property of a diagonal discontinuity in the payoff kernel or its derivatives and it is precisely this that leads to the integral equations. In all but the final class considered the optimal strategies are unique.

Part I deals with the general game of timing which was analyzed in the symmetric case by M. Shiffman [2]. In these games the diagonal discontinuity of the payoff (1.1) manifests the dependence of the result of a play on the order in which the players act; the optimal strategies are distributions which are (mainly) density functions, thus reflecting the lack of perfect information in these games. In the final portion of Part I the analagous game of perfect information is considered as a limiting case.

Part II examines payoff kernels which are concave on each side of the diagonal and which the discontinuity appears in the derivatives of the payoff. The optimal strategies obtained here are similar to those obtained in Part I. Several examples are given to illustrate the general theory.

PRELIMINARIES AND NOTATION

We introduce some standard notation which will be used throughout this paper. The payoff kernel $L(x,y)$ shall be defined on the unit square. The following strategy (= distribution) $F = (\alpha I_0, \phi_{ab}(x), \beta I_1)$ shall mean that F has a jump of value α at zero and a jump of value β at one while $\phi_{ab}(x)$ represents a continuous density function over the interval $[a,b]$. Precisely, $F(x) = \alpha + \int_0^x \phi_{ab}(x)dx$ for $0 \leq x < 1$ and $F(1) = \alpha + \int_0^1 \phi_{ab} + \beta = 1$. Whenever the notation ϕ_a is used then it is to be understood as ϕ_{a1}. Finally $[a,b]$ is taken to be the smallest interval outside of which ϕ_{ab} is identically zero. The spectrum of F shall include all points x for which every neighborhood of x has posi-

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tive F measure. The point spectrum of F shall consist of those points at which F possesses a jump. The jump of F at a point x_0 shall be denoted by $\sigma_{x_0} F$. The definition of spectrum implies that the spectrum is closed. Similar notation shall be used for quantities associated with the G strategy.

PART I

The kernel investigated in this section is of the following form:

$$L(x,y) = \begin{cases} K(x,y) & x < y \\ \Phi(x) & x = y \\ M(x,y) & x > y \end{cases} \tag{1.1}$$

satisfying the following conditions:

(a) The functions $K(x,y)$ and $M(x,y)$ are defined on $x = y$ and have continuous second partial derivatives defined in the closed triangles $x \leq y$ and $x \geq y$ respectively.

(b) The value $\Phi(1)$ lies between $K(1,1)$ and $M(1,1)$ and $\Phi(0)$ lies between $K(0,0)$ and $M(0,0)$ while the value assigned to $\Phi(x)$ for $0 < x < 1$ is bounded but otherwise arbitrary.

(c) The most important requirement is that

$$K_x(x,y) > 0 \quad \text{and} \quad M_x(x,y) > 0 \quad \text{for} \quad x < 1$$

$$K_y(x,y) < 0 \quad \text{and} \quad M_y(x,y) < 0 \quad \text{for} \quad y < 1$$

(in their respective domains of definition).

In particular, K and M, in their respective regions of definition, are strictly increasing in x and strictly decreasing in y.

A solution to the game defined by the payoff kernel $L(x,y)$ is a pair of distribution functions $F(x)$ and $G(y)$ where $0 \leq x \leq 1$ and $0 \leq y \leq 1$ together with a real number v such that

$$\int_0^1 L(x,y)dF(x) \geq v \quad \text{for all} \quad y \quad \text{and}$$

$$\int_0^1 L(x,y)dG(y) \leq v \quad \text{for all} \quad x.$$

If $F(x)$ is a strategy of the form $(\alpha I_0, \phi_{ab}(x), \beta I_1)$, then the explicit expression for $\int_0^1 L(x,y)dF(x)$ reads, for $0 < y < 1$,

$$\int_0^1 L(x,y)dF(x) = \alpha K(0,y) + \int_a^y K(x,y)\ \phi_{ab}(x)dx + \int_y^b M(x,y)\ \phi_{ab}(x)dx + \beta M(1,y).$$

It is of course understood that for $y < b$ the integral $\int_a^y K(x,y)\ \phi_{ab}(x)dx$ does not appear with a similar remark applicable to the integral $\int_y^b M(x,y)\ \phi_{ab}(x)dx$ for $y > b$. For $y = 0$, $\alpha K(0,0)$ must be replaced by $\alpha \Phi(0)$, and for $y = 1$, $\beta M(1,1)$ by $\beta \Phi(1)$.

The procedure we follow will be to try to find a solution of the following form: an absolutely continuous distribution with continuous derivative on the interior of the unit interval with additional possible jumps at the ends of the interval. Such solutions will be exhibited and afterwards their uniqueness will be established. The following lemma serves only as a guide to the subsequent theory. It will not be explicitly used hereafter.

LEMMA 1.1. If both players possess optimal strategies of the form $F = (\alpha I_0, \phi_{ab}(x), \beta I_1)$ and $G = (\gamma I_0, \psi_{cd}(y), \delta I_1)$, then the form of the density is given by $\phi_{ab} = \psi_{a1}$ and $\psi_{cd} = \psi_{a1}$ i.e., the spectrum of the absolutely continuous part of both distributions begin at a common value $0 \leq a \leq 1$ and extend to the upper end of the unit interval.

PROOF. For any distribution of the type indicated above, we see that $\int_0^1 L(x,y)dF(x)$ is continuous and non-increasing for $0 < y < 1$. If $\phi(x)$ is zero in any interval, then $\int L(x,y)dF(x)$ is strictly decreasing in that portion by condition (c). Since $F(x)$ represents an optimal strategy and c is in the spectrum of G, it therefore follows that the yield $\int L(x,y)dF(x) \equiv v$ for $c \leq y < 1$, and hence that the spectrum of F includes the interval $[c,1]$. A similar argument applies to G and we establish that $c = a$.

THE MAIN THEOREM

The main theorem of Part I deals with a complete description of the optimal strategies for the payoff kernel (1.1). Unfortunately, many different type of solutions may occur depending upon the nature of $K(x,x)$, $M(x,x)$ and the values of $L(x,y)$ at the extreme points of the unit square. It is therefore necessary to present a mutually exclusive classification of the various possibilities. The analysis of the kernel $L(x,y)$ of (1.1) subdivides into three main parts. In the case where there exists an x_0 such that $K(x_0,x_0) = M(x_0,x_0)$, then the type of optimal strategies that

appear occur under the headings of B, C and D of Theorem 1. More precisely, the essential feature is to study the spectral radius $\lambda(a)$ and $\mu(a)$ of the two integral equations (1.2) and (1.3) given below. The type of solutions as described in B, C, and D always appear when either $\lambda(a) \geq 1$ or $\mu(a) \geq 1$ for some a in the unit interval. Furthermore, if $K(x_0,x_0) = M(x_0,x_0)$ for some x_0 with $0 \leq x_0 \leq 1$, then it follows that both $\lambda(a) > 1$ and $\mu(a) > 1$ for some a in the unit interval. The second and third parts of Theorem 1 suppose that both $K(x,x) > M(x,x)$ for $0 \leq x \leq 1$ and $\lambda(0) < 1$ and $\mu(0) < 1$. They are distinguished by the assumption of $K(0,1) < M(1,0)$ and $K(0,1) \geq M(1,0)$ respectively. In both cases the value $\Phi(0)$ plays a very fundamental role.

The statement of the theorem to be established in Part I is summarized in the following array:

THEOREM 1.1. The optimal strategies for the pay-off kernel (1.1) are unique and are enumerated in the following table:

	Kernel	Optimal F	Optimal G
A	$K(1,1) \leq M(1,1)$	I_1	I_1
B	$K(x,x) > M(x,x)$, $a \leq x \leq 1$ $\lambda(a) = 1$ $\mu(a) < 1$	(ϕ_a)	$(\psi_a, \delta I_1)$
C	$K(x,x) > M(x,x)$, $a \leq x \leq 1$ $\lambda(a) = 1$ $\mu(a) = 1$	(ϕ_a)	(ψ_a)
D	$K(x,x) > M(x,x)$, $a \leq x \leq 1$ $\lambda(a) < 1$ $\mu(a) = 1$	$(\phi_a, \beta I_1)$	(ψ_a)
	If $K(x_0,x_0) = M(x_0,x_0)$ for some $0 \leq x_0 \leq 1$	B, C, or D	B, C, or D
	$K(x,x) > M(x,x)$, $0 \leq x \leq 1$ $\lambda(0) < 1$ $\mu(0) < 1$ $K(0,1) < M(1,0)$		
E	$\Phi(0) = K(0,0)$	$(\alpha I_0, \phi_0)$	$(\psi_0, \delta I_1)$

	Kernel	Optimal F	Optimal G
F	$S_0 < \Phi(0) < K(0,0)$	$(\alpha I_0, \phi_a)$	$(\gamma I_0, \Psi_a, \delta I_1)$
G	$\Phi(0) = S_0$	$(\alpha I_0, \phi_a)$	$(\gamma I_0, \Psi_a)$
H	$M(0,0) < \Phi 0) < S_0$	$(\alpha I_0, \phi_a, \beta I_1)$	$(\gamma I_0, \Psi_a)$
I	$M(0,0) = \Phi(0)$	$(\phi_0, \beta I_1)$	$(\gamma I_0, \Psi_0)$
	$K(x,x) > M(x,x), 0 \leq x \leq 1$ $\lambda(0) < 1 \quad \mu(0) < 1$ $K(0,1) \geq M(1,0)$		
J	$K(0,1) \geq \Phi(0) \geq M(1,0)$	I_0	I_0
K	$K(0,0) > \Phi(0) > K(0,1)$	$(\alpha I_0, \phi_a)$	$(\gamma I_0, \Psi_a, \delta I_1)$
L	$K(0,0) = \Phi(0)$	$(\alpha I_0, \phi_0)$	$(\Psi_0, \delta I_1)$
M	$M(1,0) > \Phi(0) > M(0,0)$	$(\alpha I_0, \phi_a, \beta I_1)$	$(\gamma I_0, \Psi_a)$
N	$\Phi(0) = M(0,0)$	$(\phi_0, \beta I_1)$	$(\gamma I_0, \Psi_0)$

Furthermore, the densities ϕ and Ψ are obtained as the solution to certain integral equations. These solutions are either Neumann series or eigenfunctions of integral operators.

The proof of this theorem shall be divided into a series of lemmas. The aim of the first series of lemmas, Lemma 1.2 - 1.16 is purely to show the existence of the solutions indicated in the shorthand of Theorem 1.1, in the various cases A through N. The uniqueness question is settled by Lemmas 1.17 - 1.20.

LEMMA 1.2. If $K(1,1) \leq M(1,1)$, then the point $(1,1)$ is a saddle point of the kernel $L(x,y)$.

PROOF. The statement is an immediate consequence of conditions (b) and (c).

THE INTEGRAL EQUATIONS

On account of Lemma 1.2 we can suppose in all that follows that $K(1,1) > M(1,1)$. The continuity of K and M provides a non-empty interval $a \leq x \leq 1$ for which $K(x,x) > M(x,x)$. We introduce now the following integral equations:

Let

$$f(\xi) - \int_a^1 T(x,\xi)f(x)dx = \alpha p_1(\xi) + \beta p_2(\xi) \tag{1.2}$$

where

$$T(x,\xi) = \begin{cases} \dfrac{-K_y(x,\xi)}{K(\xi,\xi)-M(\xi,\xi)} & a \leq x \leq \xi \\ \dfrac{-M_y(x,\xi)}{K(\xi,\xi)-M(\xi,\xi)} & a \leq \xi \leq x \leq 1, \end{cases}$$

$$p_1(\xi) = \frac{-K_y(0,\xi)}{K(\xi,\xi)-M(\xi,\xi)}, \qquad p_2(\xi) = \frac{-M_y(1,\xi)}{K(\xi,\xi)-M(\xi,\xi)},$$

and α and β are constants.

For convenience, we denote this equation by $(I - T_a)f = \alpha p_1 + \beta p_2$, I being the identity operator and T_a denoting the integral operator defined by the kernel $T(x,\xi)$ and lower limit a.

Let

$$g(\eta) - \int_a^1 U(\eta,y)g(y)dy = \gamma q_1(\eta) + \delta q_2(\eta) \tag{1.3}$$

where

$$U(\eta,y) = \begin{cases} \dfrac{M_x(\eta,y)}{K(\eta,\eta)-M(\eta,\eta)} & a \leq y \leq \eta \\ \dfrac{K_x(\eta,y)}{K(\eta,\eta)-M(\eta,\eta)} & \eta < y \leq 1 \end{cases}$$

and

$$q_1(\eta) = \frac{M_x(\eta,0)}{K(\eta,\eta)-M(\eta,\eta)}, \qquad q_2(\eta) = \frac{K_x(\eta,1)}{K(\eta,\eta)-M(\eta,\eta)}.$$

Again, we denote this equation by $(I - U_a)g = \gamma q_1 + \delta q_2$. The Banach space upon which these operators T_a, U_a act shall consist of all continuous functions defined on the interval $a \leq x \leq 1$. Whenever a function f in the domain of T_a is considered on an interval extending beyond $a \leq x \leq 1$, then it is to be understood that f is taken to be zero outside of

$a \leq x \leq 1$. Equations (1.2) and (1.3) are considered over an interval $[a,1]$ in which $K(x,x) > M(x,x)$. Consequently, the operators T_a and U_a are strictly positive completely continuous operators. We enumerate some of the properties of such operators.

I. If $\lambda(a)$ denotes the radius of the smallest circle containing the spectrum of T_a, then $\lambda(a)$ is an eigenvalue and has a unique positive eigenfunction solution. The strict positivity guarantees the uniqueness.

II. The quantity $\lambda(a)$ is a monotone continuous function of a. When $a \longrightarrow 1$, then $\lambda(a) \longrightarrow 0$. The eigenfunction solution f_a converges uniformly for any preassigned η on $a_0 + \eta \leq x \leq 1$ to f_{a_0} as a tends to a_0.

III. Finally, if $\lambda > \lambda(a)$, then $(I - \frac{1}{\lambda} T_a)^{-1}$ exists and is given by $\sum_{n=0}^{\infty} (\frac{T_a}{\lambda})^n$. This series applied to a continuous function converges uniformly on the interval $[a,1]$. This series representation for the operator $(I - \frac{T_a}{\lambda})^{-1}$ defines thus a strictly positive transformation which maps continuous positive functions into strictly positive continuous functions. In addition, $(I - \frac{T_a}{\lambda})^{-1}$ varies continuously with a in the sense described in II provided that $\lambda \geq \lambda(a) + \varepsilon_0$. Similar results apply to the operator U_a. There we employ the notation of $\mu(a)$ in place of $\lambda(a)$.

The results enunciated above are deduced elsewhere as special applications of a general theory of positive operators developed by F. H. Bohnenblust and the author [1]. We shall employ them throughout the remainder of this paper.

Case B, C, and D.

LEMMA 1.3. If there exists an x_0 with $0 \leq x_0 < 1$ and $K(x_0,x_0) = M(x_0,x_0)$, then there exists an $a > x_0$ such that $\lambda(a) = 1$. Similarly, $a^* > x_0$ exists with $\mu(a^*) = 1$.

PROOF. We treat only the case involving $\lambda(a) = 1$. Let x_0 denote the largest x for which $K(x,x) = M(x,x)$. Therefore, for $x_0 < x \leq 1$, we have $K(x,x) > M(x,x)$. Furthermore, let $f_a(x)$ with $a > x_0$ denote the eigenfunction associated with T_a and $\lambda = \lambda(a)$. Let $\varepsilon > 0$ be chosen so that $x_0 < 1 - \varepsilon$. For a such that $x_0 < a < 1 - \varepsilon$, we may assume f_a has been normalized so that $\int_a^{1-\varepsilon} f_a(x)dx = 1$.

Now by condition (c) there exists a $k > 0$ for which

$$-K_y(x,y) > k, \quad x \leq y \leq 1 - \varepsilon,$$

$$-M_y(x,y) > k, \quad y \leq x \leq 1 - \varepsilon,$$

so that, for $x_0 < a \leq \eta \leq 1 - \varepsilon$

$$0 < k = k \int_a^{1-\varepsilon} f_a(x)dx \leq \int_a^{\eta} - K_y(x,\eta)f_a(x)dx + \int_{\eta}^{1-\varepsilon} - M_y(x,\eta)f_a(x)dx$$

$$\leq \int_a^{\eta} - K_y(x,\eta)f_a(x)dx + \int_{\eta}^{1} - M_y(x,\eta)f_a(x)dx \ .$$

But since $T_a f_a = \lambda f_a$, this last expression is $[K(\eta,\eta) - M(\eta,\eta)]$. $f_a(\eta)\lambda(a)$, so that

$$0 < \frac{k}{[K(\eta,\eta) - M(\eta,\eta)]} \leq \lambda(a) f_a(\eta) \ . \tag{1.4}$$

Since the partial derivatives of K and M are bounded we have a constant C for which

$$K(\eta,\eta) - M(\eta,\eta) = [K(\eta,\eta) - M(\eta,\eta)] - [K(x_0,x_0) - M(x_0,x_0)] \leq C(\eta - x_0) \ .$$

Consequently, $0 < \frac{k'}{\eta - x_0} \leq \lambda(a) f_a(\eta)$, and integration over $(a, 1 - \varepsilon)$ yields

$$k' \int_a^{1-\varepsilon} \frac{1}{\eta - x_0} d\eta \leq \lambda(a) \int_a^{1-\varepsilon} f_a(\eta)d\eta = \lambda(a) \ .$$

Hence $\lambda(a)$ becomes unbounded as a tends to x_0, and since $\lambda(a) \longrightarrow 0$ as $a \longrightarrow 1$, by Property II we have an a for which $\lambda(a) = 1$.

LEMMA 1.4. Under the assumption of Lemma 1.3 there exist optimal strategies for both players of the form: an absolutely continuous distribution over an interval $[a,1]$ with a possible jump at 1 for one of the players.

PROOF. Lemma 1.3 implies the existence of an a $(a > x_0)$ for which either $\lambda(a) = 1$ and $\mu(a) < 1$, $\lambda(a) = 1$ and $\mu(a) = 1$, or $\lambda(a) < 1$ and $\mu(a) = 1$. We begin with the first case. Since $\lambda(a) = 1$, we obtain a $\phi_a(x) \geq 0$ normalized by $\int_a^1 \phi_a = 1$ for which $T_a\phi_a = \phi_a$. This implies that

$$u(a) = \int_a^y K(x,y)\phi_a(x)dx + \int_y^1 M(x,y)\phi_a(x)dx \quad \text{for} \quad a \leq y \leq 1 \tag{1.5}$$

where $u(a)$ is a constant depending only on a, since differentiation of equation (1.5) yields precisely the relation $T_a\phi_a = \phi_a$. If we consider $\phi_a(x)$ extended beyond the interval $[a,1]$ so that $\phi_a(x) = 0$ elsewhere, then on differentiating the right-hand side of (1.5) we obtain, in view of condition (c), a negative value when $0 \leq y < a$. Thus the expression

$\int_a^1 L(x,y)\phi_a(x)dx > u(a)$ for $0 \leq y < a$ by virtue of the continuity of this integral. Moreover, as $\mu(a) < 1$, according to Property III we can write $(I - U_a)^{-1}q_2 = \psi_2$ with $\psi_2 \geq 0$. Let $\int_a^1 \psi_2(y)dy = c$. Define $\sigma = \frac{1}{1+c}$ so that $\int \sigma\psi_2(y)dy = 1 - \sigma$. Put $\psi_a(y) = \sigma\psi_2(y)$. The equation $(I - U_a)\psi_a = \sigma q_2$ becomes on integration

$$(1.6) \qquad w(a,\sigma) = \sigma K(x,1) + \int_a^x M(x,y)\psi_a(y)dy + \int_x^1 K(x,y)\psi_a(y)dy$$
$$\text{for } a \leq x \leq 1 .$$

An analysis similar to that applied to equation (1.5) shows that $\int_a^1 L(x,y)dG(y) < w(a,\sigma)$ for $0 \leq x < a$ where $G(y) = (\psi_a, \sigma I_1)$. Integration of (1.5) with respect to G and (1.6) with respect to F now yields $u(a) = w(a) = v$. This establishes the proof for the case $\lambda(a) = 1$, $\mu(a) < 1$. A symmetrical argument can be applied to the case $\lambda(a) < 1$, $\mu(a) = 1$. Finally, when $\lambda(a) = \mu(a) = 1$ both the solution and proof are simpler. Both players use only absolutely continuous distributions which are the eigenfunction solution to $T_a\phi_a = \phi_a$ and $U_a\psi_a = \psi_a$. The details of this case are a straightforward adaptation of the arguments already given.

In the future we may assume that $K(x,x) > M(x,x)$ for $0 \leq x \leq 1$. The operators T_a and U_a may be now considered for any a in the unit interval.

LEMMA 1.5. If there exists an a for which either $\lambda(a)$ or $\mu(a) = 1$ with $0 \leq a \leq 1$, then Case B, C, or D occur with optimal strategies existing of the form indicated in Theorem 1.1.

PROOF. Since $\lambda(a)$ and $\mu(a)$ tend monotonically to zero as a tends to one, by virtue of the hypothesis it is clear that there exists an a' with either $\lambda(a') = 1$ $\mu(a') < 1$, $\lambda(a') = \mu(a') = 1$, or $\lambda(a') < 1$ $\mu(a') = 1$. The remainder of the proof follows as in Lemma 1.4.

$\underline{\lambda(0) < 1, \ \mu(0) < 1, \text{ and } K(0,1) < M(1,0)}$. This section considers the case where $\lambda(0) < 1$ $\mu(0) < 1$ and $K(0,1) < M(1,0)$. These conditions also imply that $K(x,x) > M(x,x)$ for $0 \leq x \leq 1$, by Lemma 1.3. Since $\lambda(0) < 1$ and $\mu(0) < 1$, we can form $(I - T_a)^{-1}$ and $(I - U_a)^{-1}$ for every a with $0 \leq a \leq 1$ and, in view of Property III, these operators transform positive continuous functions into strictly positive continuous functions. Furthermore, as $a \longrightarrow 1$, then $(I - T_a)^{-1}p = \phi_a \longrightarrow p$ and hence $\int_a^1 \phi_a \longrightarrow 0$. An analogous remark applies to $(I - U_a)^{-1}$. If we set $(I - T_a)^{-1}p_i = \phi_i$ for $i = 1, 2$, then $(I - T_a)^{-1}(\alpha p_1 + \beta p_2) = \alpha\phi_1 + \beta\phi_2 = \phi_a$. Let us choose α and β non-negative satisfying

$\alpha \int_a^1 \phi_1 + \beta \int_a^1 \phi_2 = 1 - \alpha - \beta$, so that $F = (\alpha I_0, \phi_a, \beta I_1)$ is a strategy. Now

$$(1.7)\quad u(a;\alpha,\beta) = \alpha K(0,y) + \beta M(1,y) + \int_a^y K(x,y)\phi_a(x)dx + \int_y^1 M(x,y)\phi_a(x)dx$$

for $a \leq y \leq 1$ (where $u(a;\alpha,\beta)$ is independent of y), since differentiating this expression with respect to y yields $\alpha p_1 + \beta p_2 = (I - T_a)\phi_a$. Moreover, in view of condition (c) we have, for $0 \leq y < a$,

$$(1.7a)\qquad u(a;\alpha,\beta) < \alpha K(0,y) + \beta M(1,y) + \int_a^1 M(x,y)\phi_a(x)dx$$

so that $u(a;\alpha,\beta) \leq \int L(x,y)dF(x)$ for $0 < y < 1$.

Similarly, if we set $(I - U_a)^{-1}q_i = \psi_i$, $i = 1, 2$, and choose $\gamma, \delta \geq 0$, satisfying $\gamma \int_a^1 \psi_1 + \delta \int_a^1 \psi_2 = 1 - \gamma - \delta$, we have for $\psi_a = \gamma\psi_1 + \delta\psi_2$ that $G = (\gamma I_0, \psi_a, \delta I_1)$ is a strategy and

$$(1.8)\quad w(a;\gamma,\delta) = \gamma M(x,0) + \delta K(x,1) + \int_a^x M(x,y)\psi_a(y)dy + \int_x^1 K(x,y)\psi_a(y)dy$$

for $a \leq x \leq 1$; and again in view of (c),

$$w(a;\gamma,\delta) > \gamma M(x,0) + \delta K(x,1) + \int_a^1 K(x,y)\psi_a(y)dy$$

for $0 \leq x < a$. Consequently $w(a;\gamma,\delta) \geq \int L(x,y)dG(y)$ for $0 < x < 1$.

We now have available the five parameters $\alpha, \beta, \gamma, \delta$ and a (subject only to the conditions $\alpha, \beta, \gamma, \delta \geq 0$, $0 \leq a < 1$,

$$\alpha \int_a^1 \phi_1 + \beta \int_a^1 \phi_2 = 1 - \alpha - \beta, \quad \gamma \int_a^1 \psi_1 + \delta \int_a^1 \psi_2 = 1 - \gamma - \delta)$$

which, as we shall show in this and the following section, may, in each of the remaining cases of Theorem 1.1, be chosen to insure the following conditions:

$$u(a;\alpha,\beta) = w(a;\gamma,\delta)$$
$$\int L(0,y)dG(y) \quad \text{and} \quad \int L(1,y)dG(y) \leq w(a;\gamma,\delta)$$
$$\int L(x,0)dF(x) \quad \text{and} \quad \int L(x,1)dF(x) \geq u(a;\alpha,\beta)\ .$$

When these conditions are fulfilled we shall evidently have, for all x and y,

$$\int L(x,y)dF(x) \geq u(a;\alpha,\beta) = v = w(a;\gamma,\delta) \geq \int L(x,y)dG(y)\ ,$$

i.e., that (F,G) is a solution.

In the remainder of Part I it is to be understood that in writing $u(a;\alpha,\beta)$ we tacitly assume that $\alpha, \beta \geq 0$, $0 \leq a < 1$ and $\alpha \int_a^1 \phi_1 + \beta \int_a^1 \phi_2 = 1 - \alpha - \beta$. We note specifically that we may always take α (or β) to be zero, and that then $\beta = (1 + \int_a^1 \phi_2)^{-1}$ is a continuous function of a. We may also require α and β to have a fixed ratio r, $0 \leq r \leq +\infty$, in which case $u(a;\alpha,r)$* is a continuous function of a. Similar remarks of course apply to γ and δ.

LEMMA 1.6. For any ratio r there exists an a for which $u(a;\alpha,0) = w(a;\gamma,r\gamma)$. Similarly there exists an a' for which $u(a';\alpha,r\alpha) = w(a';\gamma,0)$.

PROOF. If in (1.7) and (1.8) we set $x = y = 1$, then for any $\varepsilon > 0$ and a sufficiently close to 1 we have

$$|u(a;\alpha,\beta) - [\alpha K(0,1) + \beta M(1,1) + (\int_a^1 \phi_a) K(1,1)]| < \varepsilon/3$$

$$|w(a;\gamma,\delta) - [\gamma M(1,0) + \delta K(1,1) + (\int_a^1 \psi_a) M(1,1)]| < \varepsilon/3 .$$

Since $\int_a^1 \phi_a$ and $\int_a^1 \psi_a \longrightarrow 0$ as $a \longrightarrow 1$, we conclude that for a near 1

$$|u(a;\alpha,\beta) - w(a;\gamma,\delta) - \{\alpha K(0,1) + \beta M(1,1) - \gamma M(1,0) - \delta K(1,1)\}| < \varepsilon ,$$

for any appropriate values of $\alpha, \beta, \gamma, \delta$. Now by assumption $K(0,1) < M(1,0)$ and $M(1,1) < K(1,1)$ while monotonicity implies that $K(0,1) < K(1,1)$ and $M(1,1) < M(1,0)$. Since

$$1 - \alpha - \beta = \alpha \int_a^1 \phi_1 + \beta \int_a^1 \phi_2$$

and

$$1 - \gamma - \delta = \gamma \int_a^1 \psi_1 + \delta \int_a^1 \psi_2$$

both tend to zero as $a \longrightarrow 1$, we have clearly

$$\alpha K(0,1) + \beta M(1,1) - \gamma M(1,0) - \delta K(1,1) < 0$$

for a sufficiently close to 1, for any appropriate values of $\alpha, \beta, \gamma, \delta$. Consequently $u(a;\alpha,\beta) - w(a;\gamma,\delta) < 0$ for a near 1.

*Here we make the convention that $u(a;\alpha,\infty\cdot\alpha) = u(a;0,\beta)$.

Let us now substitute the values $a = 0$, $x = 0$, and $a = 0$, $x = 1$ in (1.8); similarly set $a = 0$, then let $y \longrightarrow 0$ and $a = 0$, and then let $y \longrightarrow 1$ in (1.7). We obtain the equations

$$u(0;\alpha,\beta) = \alpha K(0,0) + \beta M(1,0) + \int_0^1 M(x,0)\phi_0(x)dx \tag{1.9}$$

$$w(0;\gamma,\delta) = \gamma M(0,0) + \delta K(0,1) + \int_0^1 K(0,y)\psi_0(y)dy \tag{1.10}$$

$$u(0;\alpha,\beta) = \alpha K(0,1) + \beta M(1,1) + \int_0^1 K(x,1)\phi_0(x)dx \tag{1.11}$$

$$w(0;\gamma,\delta) = \gamma M(1,0) + \delta K(\ ,1) + \int_0^1 M(1,y)\psi_0(y)dy\ . \tag{1.12}$$

Also, if we multiply (1.7) by $\psi_0(y)$ and integrate we obtain

$$(1 - \gamma - \delta)u(0;\alpha,\beta) = \alpha\int_0^1 K(0,y)\psi_0(y)dy + \beta\int_0^1 M(1,y)\psi_0(y)dy + \int_0^1\int_0^1 L(x,y)\phi_0(x)\psi_0(y)dxdy\ . \tag{1.13}$$

Similarly, multiplying (1.8) by $\phi_0(x)$ and integrating we obtain

$$(1 - \alpha - \beta)w(0;\gamma,\delta) = \gamma\int_0^1 M(x,0)\phi_0(x)dx + \delta\int_0^1 K(x,1)\phi_0(x)dx + \int_0^1\int_0^1 L(x,y)\phi_0(x)\psi_0(y)dxdy\ . \tag{1.14}$$

Multiplying (1.9), (1.10), (1.11), and (1.12) by γ, α, δ, and β respectively and eliminating the integrals yields finally

$$u(0;\alpha,\beta) - w(0;\gamma,\delta) = \alpha\gamma[K(0,0) - M(0,0)] + \beta\delta[M(1,1) - K(1,1)]\ . \tag{1.15}$$

Now if either β or δ is zero, say β, we have $u(0;\alpha,0) - w(0;\gamma,\delta) \geq 0$ since $K(0,0) > M(0,0)$. Since $u(a;\alpha,0) - w(a;\gamma,r\gamma)$ is a continuous function of a, the conclusion of the lemma is evident. We note for further use that (1.15) does not depend on the assumption $K(0,1) < M(1,0)$.

LEMMA 1.7. Statement E in Theorem 1.1.

PROOF. If in (1.7) and (1.8) we set $\beta = \gamma = 0$, then (1.15) shows that $u(0;\alpha,0) = w(0;0,\delta) = v$. Consequently, we have only to show $\int L(x,0)dF(x)$, $\int L(x,1)dF(x) \geq v$ and $\int L(0,y)dG(y)$, $\int L(1,y)dG(y) \leq v$, where $F = (\alpha I_0,\phi_0)$ and $G = (\psi_0,I_1)$. The assumption $\Phi(0) = K(0,0)$, implies $\int L(x,0)dF(x) = v$, and since $\beta = 0$, $\int L(x,1)dF(x) = v$. Similarly since $\gamma = 0$, $\int L(0,y)dG(y) = v$, and

$$\int L(1,y)dG(y) = \delta\Phi(1) + \int_0^1 M(1,y)\psi_0(y)dy \leq \delta K(1,1) + \int_0^1 M(1,y)\psi_0(y)dy = v\ .$$

In a similar manner one can establish

LEMMA 1.8. Statement I in Theorem 1.1.

LEMMA 1.9. There exists a value S_o with $M(0,0) < S_o < K(0,0)$ for which $\Phi(0) = S_o$ implies the existence of optimal strategies of the form indicated in Case G of Theorem 1.1.

PROOF. We take $\beta = \delta = 0$, and abbreviate $u(a;\alpha,0)$, $w(a;\gamma,0)$ by $u(a)$, $w(a)$. Lemma 1.6 implies the existence of an a for which $u(a) = w(a)$, and (1.15) implies $a > 0$. Consequently $\int L(x,y)dG(y) < v = w(a)$ for $0 < x < a$ and $\int L(x,y)dF(x) > v$ for $0 < y < a$, where $F = (\alpha I_o, \phi_a)$, $G = (\gamma I_o, \psi_a)$. Since $\beta = \delta = 0$, we have

$$\int L(x,1)dF(x) = v = \int L(1,y)dG(y) .$$

Now

$$\int L(x,0)dF(x) = \alpha\Phi(0) + \int_a^1 M(x,0)\phi_a(x)dx ,$$

and (1.7a) implies

$$v < \alpha K(0,0) + \int_a^1 M(x,0)\phi_a(x)\alpha x .$$

If $\Phi(0) = M(0,0)$, since we then have

$$v > \gamma M(0,0) + \int_a^1 K(0,y)\psi_a(y)dy = \int L(0,y)dG(y) ,$$

and $\alpha > 0$, we obtain $\iint L(x,y)dF(x)dG(y) < v$, or $\gamma \int_o^1 L(x,0)dF(x) + \int_a^1\int_o^1 L(x,y)\alpha F(x)\psi_a(y)dy = \gamma \int_o^1 L(x,0)dF(x) + v(1-\gamma) < v$ or $\int_o^1 L(x,0)dF(x) < v$. Hence for some value S_o, $M(0,0) < S_o < K(0,0)$, for $\Phi(0)$ we have $\int_o^1 L(x,0)dF(x) = v$. This in turn implies $\int_o^1 L(0,y)dG(y) = v$ since then

$$v = \int_o^1\Big(\int_o^1 L(x,y)dF(x)\Big)\alpha G(y) = \alpha\int_o^1 L(0,y)dG(y) + \int_a^1\Big(\int_o^1 L(x,y)dG(y)\Big)\phi_a(x)dx$$
$$= \alpha\int_o^1 L(0,y)dG(y) + (1 - \alpha)v .$$

It is assumed in the next lemma that the value a_o of a used in Lemma 1.9 is the least a for which $u(a;\alpha,0) = w(a;\gamma,0)$. Hence for $0 \le a < a_o$ we have $u(a;\alpha,0) > w(a;\gamma,0)$.

LEMMA 1.10. Statement F in Theorem 1.1.

PROOF. The function f defined by

$$f(a) = \frac{1}{\alpha}\left[u(a;\alpha,0) - \int_0^1 M(x,0)\phi_a(x)dx\right] ,$$

where $\alpha = (1 + \int_a^1 \phi_1)^{-1}$, is continuous. By (1.9) $f(0) = K(0,0)$, and as we have seen in Lemma 1.9, $f(a_0) = S_0$. Consequently, if $S_0 < \mathfrak{F}(0) < K(0,0)$ there exists an a, $0 < a < a_0$, for which $f(a) = \mathfrak{F}(0)$, or, for $F = (\alpha I_0, \phi_a)$,

$$\int L(x,0)dF(x) = \alpha\mathfrak{F}(0) + \int_0^1 M(x,0)\phi_a(x)dx = u(a;\alpha,0) .$$

Since $\beta = 0$, we also have $\int L(x,1)dF(x) = u(a;\alpha,0)$.

In view of the remark preceding this lemma, $u(a;\alpha,0) > w(a;\gamma,0)$ for this a. However, taking $\gamma = 0$ and $G = (\Psi_a, \delta I_1)$ yields

$$\begin{aligned} u(a;\alpha,0) &= \int_0^1 \left(\int_0^1 L(x,y)dF(x) \right) dG(y) \\ &= \alpha\int_0^1 L(0,y)dG(y) + \int_a^1 \int_0^1 L(x,y)dG(y)\ \phi_a(x)dx \\ &= \alpha\int_0^1 L(0,y)dG(y) + (1 - \alpha)w(a;0,\delta) . \end{aligned}$$

Since

$$\int L(0,y)dG(y) = \delta K(0,1) + \int_0^1 K(0,y)\Psi_a(y)dy < w(a;0,\delta) ,$$

we must have $u(a;\alpha,0) < w(a;0,\delta)$. Clearly then there exists γ, $\delta > 0$ for which $u(a;\alpha,0) = w(a;\gamma,\delta) = v$.

We shall now verify that $F = (\alpha I_0, \phi_a)$ and $G(\gamma I_0, \Psi_a, \delta I_1)$ are optimal. We already have

$$\int L(x,0)dF(x) = \int L(x,1)dF(x) = v .$$

Now

$$\begin{aligned} \int L(1,y)dG(y) &= \gamma M(1,0) + \delta\mathfrak{F}(1) + \int_a^1 M(1,y)\Psi_a(y)dy \\ &\leq \gamma M(1,0) + \delta K(1,1) + \int_a^1 M(1,y)\Psi_a(y)dy = v , \end{aligned}$$

and since $v = \int_0^1 (\int_0^1 L(x,y)dF(x))dG(y)$,

$$v = \alpha\int_0^1 L(0,y)dG(y) + (1 - \alpha)v$$

so that $\int L(0,y)dG(y) = v$, and the proof is complete. A similar proof yields

LEMMA 1.11. Statement H of Theorem 1.1.

$\lambda(0) < 1$, $\mu(0) < 1$, and $K(0,1) - M(1,0) \geq 0$. It remains only to consider the case where $K(0,1) - M(1,0) \geq 0$. If $M(1,0) \leq \Phi(0) \leq K(0,1)$, then it is easily verified that a saddle-point exists at $(0,0)$. This establishes

LEMMA 1.12. Statement J of Theorem 1.1.

LEMMA 1.13. Statement L of Theorem 1.1.

The proof of this proposition is identical with the argument of Lemma 1.7.

In a similar manner to Lemma 1.8, we have

LEMMA 1.14. Statement N of Theorem 1.1.

LEMMA 1.15. Statement K of Theorem 1.1.

PROOF. If we set $\beta = y = a = 0$ in equation (1.7) we obtain

$$u(0;\alpha,0) = \alpha K(0,0) + \int_0^1 M(x,0)\phi_0(x)dx;$$

consequently the continuous function f defined by

$$f(a) = \frac{1}{\alpha}\left[u(a;\alpha,0) - \int_0^1 M(x,0)\phi_a(x)dx\right],$$

where $\alpha = (1 + \int_a^1 \phi_1)^{-1}$, has $f(0) = K(0,0)$. Inserting the values $\beta = 0$, $y = 1$ in (1.7) yields

$$u(a;\alpha,0) = \alpha K(0,1) + \int_a^1 K(x,1)\phi_a(x)dx,$$

so that

$$f(a) = K(0,1) + \frac{1}{\alpha}\int_a^1 (K(x,1) - M(x,0))\phi_a(x)dx.$$

As $a \longrightarrow 1$, $\alpha \longrightarrow 1$, and the integral in this last equation tends to zero, so that $f(a) \longrightarrow K(0,1)$. Hence if $K(0,1) < \Phi(0) < K(0,0)$ there exists an a, $0 < a < 1$, for which $f(a) = \Phi(0)$, or

$$u(a;\alpha,0) = \alpha\Phi(0) + \int_a^1 M(x,0)\phi_a(x)dx = \int L(x,0)dF(x)$$

where $F = (\alpha I_0, \phi_a)$. Since $\beta = 0$, $\int L(x,1)dF(x) = u(a;\alpha,0)$ also.

Now for this same a let $G = (\gamma I_0, \psi_a)$, where $\gamma = (1 + \int_a^1 \psi_1)^{-1}$. Since $\int L(x,y)dF(x) = u(a;\alpha,0)$ for $a \leq x \leq 1$ and $x = 0$, we have

$$\int\Big(\int L(x,y)dF(x)\Big)\, dG(y) = u(a;\alpha,0) \ . \tag{1.16}$$

On the other hand

$$\int\Big(\int L(x,y)dG(y)\Big)\, dF(x) = \alpha\int L(0,y)dG(y) + (1 - \alpha)w(a;\gamma,0) \tag{1.17}$$

and

$$\int L(0,y)dG(y) = \gamma\Phi(0) + \int_a^1 K(0,y)\psi_a(y)dy > \gamma M(1,0) + \int_a^1 M(1,y)\psi_a(y)dy \tag{1.18}$$

since $\Phi(0) > K(0,1) \geq M(1,0)$ and $K(0,y) \geq K(0,1) \geq M(1,0) \geq M(1,y)$. By (1.8) (with $\delta = 0$ and $x = 1$) the last quantity in (1.18) is $w(a;\gamma,0)$, so that combining (1.16), (1.17), and (1.18) yields $u(a;\alpha,0) > w(a;\gamma,0)$.

However taking $\gamma = 0$ produces the inequality $u(a;\alpha,0) < w(a;0,\delta)$, for if $G = (\psi_a, \delta I_1)$ then

$$\begin{aligned} u(a;\alpha,0) &= \int\int L(x,y)dF(x)dG(y) = \alpha\int L(0,y)dG(y) + (1 - \alpha)w(a;0,\delta) \\ &= \alpha[\delta K(0,1) + \int_a^1 K(0,y)\psi_a(y)dy] + (1 - \alpha)w(a;0,\delta) \\ &< \alpha w(a;0,\delta) + (1 - \alpha)w(a;0,\delta) = w(a;0,\delta) \ . \end{aligned}$$

Since the two extreme choices of γ and δ yield opposite inequalities there exists a pair of values γ and δ for which $u(a;\alpha,0) = w(a;\gamma,\delta)$. The verification of the optimal nature of the strategies $F = (\alpha I_0, \phi_a(x))$ and $G = (\gamma I_0, \psi_a(y), \delta I_1)$ is now straightforward and similar to the last part of the argument in Lemma 1.10.

LEMMA 1.16. Statement M of Theorem 1.1.

PROOF. Similar to Lemma 10.

Uniqueness. The optimal strategies of the specified types have now been exhibited and there remains only the question of uniqueness. In Cases A and J it is easily verified via condition (c) that the solutions are unique. From the remaining cases we shall select Case (F) as typical, and show uniqueness in this case. Any other case can be handled similarly.

LEMMA 1.17. If F represents any optimal strategy for player I, then the spectrum of F on the interior of the unit interval is continuous.

PROOF. Since we already have an optimal strategy $G = (\gamma I_0, \Psi_a, \delta I_1)$ for player II for which $\int L(x,y)dG(y) < v$ for $0 < x < a$, the spectrum of F must be confined to the interval $[a,1]$ and the end point 0. Consider any y_0 with $a < y_0 < 1$; then $\int L(x,y_0)dF = v$ while $\int L(x,y)dF(x) = v$ for a neighborhood of y_0. By bounded convergence, we get

$$\lim_{y \to y_0+} \int L(x,y)dF = \int L(x,y_0^+)dF = v .$$

Similarly $\int L(x,y_0^-)dF(x) = v$. The difference gives $[K(y_0,y_0) - M(y_0,y_0)]\, \sigma'_{y_0} F = 0$. As $K(y_0,y_0) > M(y_0,y_0)$, we have $\sigma'_{y_0} F = 0$. The case where $y_0 = a$ can be handled similarly. In fact, one has $[\phi(a) - K(a,a)]\, \sigma'_a F = 0$ and $[\phi(a) - M(a,a)]\, \sigma'_a F \leq 0$. Since $K(a,a) > M(a,a)$ we deduce from these two relations that $\sigma'_a F = 0$.

LEMMA 1.18. Under the conditions of Lemma 1.17, then F is absolutely continuous for $a \leq x < 1$ and $F' = f$ is continuous.

PROOF. By virtue of Lemma 1.17, we know that $F = (\alpha I_0, F_0(x), \beta I_1)$ and the spectrum of F_0 is continuous and confined to $a \leq x \leq 1$. Also,

$$(1.20) \qquad v = \int L(x,y)dF(x) = \alpha K(0,y) + \int_a^1 L(x,y)dF_0(x) + \beta M(1,y)$$

for $a \leq y < 1$. An integration by parts furnishes with the aid of the continuity of F_0 for this interval that

$$(1.21) \qquad \int L(x,y)dF_0(x) = \int_a^y K_x F_0 dx + \int_y^1 M_x F_0(x)dx + [K(y,y) - M(y,y)]\, F_0(y) + c .$$

On account of $K(y,y) > M(y,y)$, this equation and (1.20) show that $F_0(y)$ is absolutely continuous. Upon differentiation of (1.21), one finds that $F_0' = f$ is also continuous for $a \leq x < 1$.

LEMMA 1.19. The optimal strategy for player I is unique.

PROOF. Since we are considering Case (F), we know that optimal strategies of the form $F = (\alpha I_0, \phi_a)$, $G = (\gamma I_0, \psi_a, \delta I_1)$ exist with $\delta \neq 0$, $a < 1$. By the preceding lemmas, any other optimal strategy F^* for player I has the form $F^* = (\alpha_1 I_0, F_0, \beta_1 I_1)$, where F_0 is absolutely continuous, is confined to the interval $[a,1]$, and has a continuous derivative $F_0' = f$ on $a \leq x < 1$. Now β_1 must be zero, for if not, then

$$v = \int_0^1 L(x,1)dF^*(x) = \alpha_1 K(0,1) + \beta_1 \Phi(1) + \int_a^1 K(x,1)dF_0(x)$$

$$v = \int_0^1 L(1,y)dG(y) = \gamma M(1,0) + \delta \Phi(1) + \int_a^1 M(1,y)\psi_a(y)dy$$

since both δ and β_1 are non-zero; on the other hand

$$v = \alpha_1 K(0,1) + \beta_1 M(1,1) + \int_a^1 K(x,1)dF_0(x)$$

$$v = \gamma M(1,0) + \delta K(1,1) + \int_a^1 M(1,y)\psi_a(y)dy$$

by continuity. But the first and third equations imply $\Phi(1) = M(1,1)$ and the second and fourth $\Phi(1) = K(1,1)$, which is impossible since $K(1,1) > M(1,1)$.

Thus we see that $f = F_0'$ satisfies the integral equation (1.2) with $\beta = 0$, and since the solution of this equation is unique (α is uniquely determined by $\int \phi_a = 1 - \alpha$), we must have $F^* = F$.

LEMMA 1.20. The solution for player II is unique.

PROOF. In a similar manner it is easy to deduce that the form of any optimal strategy is $G = (\gamma I_0, \psi_a(y), \delta I_1)$ with $\psi_a(y)$ a density function for $a \leq y \leq 1$. It follows that γ and δ are determined by $\int \psi_a = 1 - \gamma - \delta$ and $\gamma\phi(0) + \int_a^1 K(0,y)\psi_a(y) + \delta K(0,1) = v$ where $\psi_a = \gamma\psi_1 + \delta\psi_2$. If the solution to these equations were one dimensional, then there would exist a solution with $\gamma = 0$, $\delta > 0$. But the monotonicity criteria yields that

$$\int_a^1 K(0,y)\psi(y) + \delta K(0,1) < v$$

which is a contradiction. Thus the solution of the above linear equations is unique. This immediately implies the conclusion of the lemma.

PROOF OF THEOREM 1.1. Lemma 1.2 to Lemma 1.20.

Additional remarks.

A. If $K(x,y)$ and $M(x,y)$ are both separable functions ($K(x,y) = \sum_{i=1}^n r_i(x)s_i(y)$ etc.) then it can be shown that the integral equations producing the continuous portion of the optimal strategies may be

replaced by differential equations.

B. If the kernel is symmetric i.e., $K(x,y) = -M(y,x)$ and $\Phi(x) = 0$, then the only type strategies that appear are (A), (C), (G) or (J).

C. If K_{xx}, $M_{xx}(x,y) \leq 0$, $K_x(x,x) > M_x(x,x)$, and $K'(\eta,\eta) > M'(\eta,\eta)$ then the transformation U_a transforms positive continuous functions into continuous monotonic decreasing functions. Indeed, we differentiate

$$U_a g = \int_a^\eta \frac{M_x(\eta,y)}{K(\eta,\eta)-M(\eta,\eta)}\, g(y)dy + \int_\eta^1 \frac{K_x(\eta,y)}{K(\eta,\eta)-M(\eta,\eta)}\, g(y)dy$$

getting

$$\frac{d(U_a g)}{d\eta} = \left[\frac{1}{K(\eta,\eta)-M(\eta,\eta)}\right]\left\{\int_a^\eta M_{xx}g + \int_\eta^1 K_{xx}g(y) + \left[M_x(\eta,\eta)-K_x(\eta,\eta)\right]g(\eta)\right\}$$
$$- \frac{(\int M_x g + K_x g)}{[K(\eta,\eta)-M(\eta,\eta)]^2}\left[K'(\eta,\eta) - M'(\eta,\eta)\right].$$

The hypothesis clearly imply that $\frac{d}{d\eta}(U_a g) < 0$. Since $U_a g$ is continuous and positive, we thus deduce that the optimal $\psi_a(y)$ is positive continuous and decreasing. In fact, $\psi_a(y) = \sum_{n=0} U_a^n q$ each term of which has the desired property.

In order to insure that the optimal $\phi_a(x)$ shall be positive and decreasing it is sufficient to assume that $K_{yy} \geq 0$, $M_{yy} \geq 0$, $K_y(y,y) > M_y(y,y)$ and $K'(\xi,\xi) > M'(\xi,\xi)$.

An example where these conditions are fulfilled is

$$L(x,y) = \begin{cases} K(x,y) = x - (1 - x)y & x < y \\ \Phi(x) = 0 & \\ M(x,y) = -y + (1 - y)x & x > y\ . \end{cases}$$

Thus, one can conclude that the absolutely continuous part of the optimal strategy for either player for this kernel is continuous and decreasing. In this particular example the absolutely continuous distributions consist of inverse cubics.

Many other such examples can be constructed.

D. We remark that if all the inequalities in condition (c) are reversed, then the form of the optimal strategies are obtained by inverting the unit interval. (Replacing x by $1 - x$ and y by $1 - y$.)

E. Concerning the statements E to I of Theorem 1, some information can be given about the change of the interval $[a,1]$. Implicitly contained in the proof of Theorem 1 is the fact that as $\Phi(0)$ traverses

the range from $K(0,0)$ to S_0 to $M(0,0)$ the point a increases from 0 to a maximum value $a_0 < 1$ and then decreases from a_0 to 0. In a similar manner it can be shown that pertaining to statements J to N as $\Phi(0)$ traverses $K(0,0)$ to $K(0,1)$ to $M(1,0)$ to $M(0,0)$, then a increases from 0 to 1 remains at 1 while $M(1,0) \leq \Phi(0) \leq K(0,1)$ and then afterwards decreases from 1 to 0.

F. A remark concerning the result when $\Phi(1)$ and $\Phi(0)$ do not lie between $K(1,1)$, $M(1,1)$ and $K(0,0)$, $M(0,0)$ respectively is in order. It turns out that in this circumstance optimal strategies may not exist although the value is determined in the sense that inf sup = sup inf.

G. The kernel (1.1) represents the payoff of a game of timing with some kind of partial information as to the time or method of action. It is therefore not surprising that the optimal strategy should involve partly a density. However, if the game of timing has complete information then the optimal strategies behave very much like pure strategies. A game of complete information gives rise to a kernel of the following form:

$$L(x,y) = \begin{cases} m(x) & x < y \\ L(x,x) & x = y \\ n(y) & x > y \end{cases} \tag{1.22}$$

where $C \geq m'(x) \geq \delta > 0$ and $-C \leq n'(x) \leq -\delta < 0$. Also, $m(1) > n(1)$ while $m(0) < n(0)$ and $L(x,x)$ is bounded but otherwise arbitrary.

The monotonicity properties of m and n imply the existence of a unique a_0 with $0 < a_0 < 1$ so that $m(a_0) = n(a_0)$. We now establish the following proposition:

> THEOREM 1.2. The game (1.22) has a value $v = m(a_0) = n(a_0)$ and the optimal strategies F and G are as follows: If $L(a_0,a_0) > v$ ($L(a_0,a_0) < v$), then $F = I_{a_0}$ ($G = I_{a_0}$) while the player II (player I) has no optimal attainable strategy. Moreover if $L(a_0,a_0) = v$, then the point (a_0,a_0) is a saddle point.

REMARK. We shall treat the kernel (1.22) as a limiting case of the kernel (1.1). A different proof of Theorem 1.2 was given by written communication by I. Glicksberg. It is felt that the procedure followed here is more natural. In addition, we shall drop below the assumptions about the values of m and n at 0 and 1 and indicate the form of solutions for the more general game.

PROOF. Let $L^*(x,y)$ be any game of type (1.1), and consider the new games defined by the kernels $L_k(x,y) = L(x,y) + \frac{1}{k} L^*(x,y)$ for $k = 1, 2, \ldots$. It is clear that $L_k(x,y)$ has the properties of the kernel of equation (1.1). Explicitly

$$L_k(x,y) = \begin{cases} m(x) + \frac{1}{k} K(x,y) & x < y \\ L(x,x) + \frac{1}{k} \Phi(x) & x = y \\ n(y) + \frac{1}{k} M(x,y) & x > y \, . \end{cases}$$

The assumptions on $m(x)$ and $n(y)$ require that for sufficiently large k we have an x_k, $0 < x_k < 1$, for which $m(x_k) + \frac{1}{k} L(x_k,x_k) = n(x_k) + \frac{1}{k} M(x_k,x_k)$, and clearly $x_k \longrightarrow a_o$. In view of Theorem 1.1 (we have Case A, B, or C) the form of the unique optimal strategies for the game $L_k(x,y)$ is $F_k = (\phi_{a_k}, \beta_k I_1)$, $G_k = (\psi_{a_k}, \delta_k I_1)$ where $x_k < a_k < 1$ and only one of the pair β_k, δ_k is non-zero. For definiteness we may assume that $\beta_k = 0$ for an infinite set of indices, and, replacing our sequence by a subsequence we may assume that $\beta_k = 0$ for all k, $a_k \longrightarrow a$, and that, if v_k denotes the value of the game $L_k(x,y)$, $v_k \longrightarrow v$.

Since $x_k < a_k < 1$ and $x_k \longrightarrow a_o$, $a_o \leq a \leq 1$; moreover since $v_k \longrightarrow v$ we may conclude that v is the value of the game $L(x,y)$, that is

$$(*) \quad \inf_G \sup_F \iint L(x,y)dF(x)dG(y) = v = \sup_F \inf_G \iint L(x,y)dF(x)dG(y) \, .$$

for, since $L_k(x,y)$ converges uniformly to $L(x,y)$,

$$\int L(x,y)dF_k(x) \geq v - \varepsilon, \; v + \varepsilon \geq \int L(x,y)dG_k(y)$$

for $k \geq k(\varepsilon)$, and (*) follows.

We may now show that the value v of the game $L(x,y)$ is $m(a_o) = n(a_o)$. By (1.2) for $a_k \leq y < 1$ we have

$$(1.23) \qquad n'(y) \int_y^1 \phi_{a_k}(x)dx + (m(y) - n(y))\phi_{a_k}(y) + \rho_k(y) = 0$$

where $\rho_k(y)$ tends uniformly to zero as $k \longrightarrow \infty$. In view of Helly's Selection Theorem we may (again replacing our sequence by a subsequence if necessary) assume that there exists a distribution F_o for which $F_k(x) \longrightarrow F_o(x)$ for every point x of continuity of F_o. Consequently, for $y \geq a + \delta$, $\delta > 0$, we have

$$\int_y^1 \frac{-n'(t)}{m(t) - n(t)} (1 - F_k(t))dt = 1 - F_k(y) + \varepsilon_k(y)$$

where $\varepsilon_k(y)$ tends uniformly to zero, and hence

$$\int_y^1 \frac{-n'(t)}{m(t) - n(t)} (1 - F_0(t))dt = 1 - F_0(y)$$

at every point y of continuity of F_0. Since $\frac{-n'(t)}{m(t) - n(t)}$ is bounded for $t \geq a + \delta > a_0$, this implies by a classical argument that $F_0(y) = 1$ for $y \geq a + \delta$. Indeed

$$1 - F_0(y) \leq M \int_y^1 (1 - F_0(t))dt$$

yields, upon iteration (valid since the points of discontinuity of F_0 form a set of measure zero),

$$(1 - F_0(y)) \leq \frac{M^n(1 - y)^{n-1}}{(n - 1)!} .$$

Since $\delta > 0$ is arbitrary $F_0(y) = 1$ for $y > a$. For $y < a$, $F_k(y) = 0$ for $k \geq k(y)$, hence $F_0(y) = 0$, and $F_0 = I_a$. Moreover, we may now assert that $a = a_0$, for otherwise, since $m(y) > n(y)$ for $y \geq a > a_0$, we conclude from (1.23) that the functions ϕ_{a_k} are bounded, so that F_k cannot converge to I_a.

Now for $y > a_0$, $\int L(x,y)dF_k(x) \longrightarrow v$; consequently

$$v = \int L(x,y)dF_0(x) = L(a_0,y) = m(a_0) = n(a_0) .$$

Now suppose F is optimal for the game $L(x,y)$. Then we may assert that $F(x) = 0$ for $x < a_0$. For if $x_0 < a_0$, $F(x_0) > 0$, then since $a_k \longrightarrow a_0$, $\int L(x,y)dG_k(y) = m(x) \leq m(x_0)$ for $x \leq x_0$ and $k \geq k_0$. Moreover, since $L_k(x,y)$ converges uniformly to $L(x,y)$ and $v_k \longrightarrow v$, for any $\varepsilon > 0$ we have, for $k \geq k(\varepsilon)$,

$$\int L(x,y)dG_k(y) \leq v + \varepsilon$$

for all x. Now $\delta(a_0 - x_0) \leq m(a_0) - m(x_0)$, and if we set $\varepsilon = \frac{1}{2} F(x_0)\delta(a_0 - x_0) > 0$ we have

$$\int\int L(x,y)dG_k(y) \leq m(a_0) - \delta(a_0 - x_0) = v - \delta(a_0 - x_0)$$

for $x \leq x_0$. Consequently, for $k \geq k(\varepsilon)$,

$$\int\int L(x,y)dG_k(y)dF(x) \leq F(x_0)(v - \delta(a_0 - x_0)) + (1 - F(x_0))(v + \varepsilon)$$

$$< F(x_0)(v + \varepsilon - \delta(a_0 - x_0)) + (1 - F(x_0))(v + \varepsilon)$$

$$< v + \varepsilon - F(x_o)\delta(a_o - x_o)$$

$$< v - \tfrac{1}{2} F(x_o)\delta(a_o - x_o) < v$$

so that F could not be optimal.

Since $F(x) = 0$ for $x < a_o$, we may write $F = \alpha I_{a_o} + F_1$, where F_1 is continuous at a_o, $F_1(a_o) = 0$ $F_1(1) = 1 - \alpha$. For $y > a_o$

$$\alpha m(a_o) + \int_{a_o}^{y} m(x)dF_1(x) + n(y)(1 - \alpha - F_1(y)) \geq v = m(a_o) = n(a_o)$$

so that

$$\int_{a_o}^{y} (m(x) - m(a_o))dF_1(x) + (n(y) - n(a_o))(1 - \alpha - F_1(y)) \geq 0 .$$

Since $C(y - a_o) \geq m(y) - m(a_o) \geq \delta(y - a_o)$ and F_1 is continuous at a_o we have $\int_{a_o}^{y} (m(x) - m(a_o))dF_1(x) = o(y - a_o)$; since $n(y) - n(a_o) \leq -\delta(y - a_o) < 0$, we have

$$0 \leq \delta(y - a_o)(1 - \alpha - F_1(y)) \leq -(n(y) - n(a_o))(1 - \alpha - F_1(y))$$

$$\leq \int_{a_o}^{y} (m(x) - m(a_o))dF_1(x) = o(y - a_o)$$

so that $1 - \alpha - F_1(y) \longrightarrow 0$ as $y \longrightarrow a_o$. Thus $F_1(y) = 1 - \alpha$ for $y > a_o$, and $F_1 = (1 - \alpha)I_{a_o}$, $F_o = \alpha I_{a_o} + (1 - \alpha)I_{a_o} = I_{a_o}$.

A completely analagous argument applies to player II and shows that only I_{a_o} can serve as a solution. The remaining statements of Theorem 1.2 now follow readily from these facts.

REMARK. Let $L(x,y)$ satisfy the conditions of Theorem (1.2) except that no requirement is imposed about the values of $m(x)$ and $n(x)$ at 0 and 1. Suppose moreover that $L(0,0)$ and $L(1,1)$ lie between the respective values of m and n at these end points. The optimal strategies are described as follows: (a) $(1,1)$ is a saddle point if and only if $m(1) < n(1)$, (b) $(0,0)$ is a saddle point if and only if $m(0) > n(0)$ and (c) for the remaining case the solution is given in Theorem 1.2.

PART II

We analyze the following class of kernels

$$(*) \qquad L(x,y) = \begin{cases} K(x,y) & x \leq y \\ M(x,y) & x \geq y \end{cases} \qquad K(x,x) = M(x,x)$$

satisfying the following conditions.

(a) In their respective domains both $K(x,y)$ and $M(x,y)$ have continuous third partial derivatives.

(b) $K_{xx}(x,y)$, $K_{yy}(x,y)$ are strictly negative for $x \leq y$ and M_{xx}, M_{yy} are strictly negative for $x \geq y$. In other words K and M are each strictly concave in each variable separately on each side of the diagonal. The kernel $L(x,y)$ is not however necessarily concave in each variable separately throughout the unit square.

In all that follows we assume (a) and (b) are satisfied.

Butterfly Kernel. We examine completely here first a specific case of the above kernel (*) which arises frequently in applications. This class of kernels was first introduced by Glicksberg and Gross.

DEFINITION 2.1. A kernel of type (*) is said to be butterfly if in addition to the requirements (a) and (b), we assume

(c) The function $K(x,y)$ is strictly increasing in y and strictly decreasing in x. Also, $M(x,y)$ is strictly increasing in x and decreasing in y. In other words the kernel is monotonic away from the diagonal.

We establish here the following principal theorem.

THEOREM 2.1. Both players possess unique solutions of the following form $F = (\alpha I_0, \phi, \beta I_1)$ and $G = (\gamma I_0, \Psi, \delta I_1)$ where ϕ and Ψ are absolutely continuous over the full interval and are obtained as the unique solutions to a pair of integral equations.

$$\alpha p_1 + \beta p_2 = \phi - T\phi \tag{2.1}$$

$$\gamma q_1 + \delta q_2 = \Psi - U\Psi \tag{2.2}$$

where

$$T\phi = \int_0^y \frac{-K_{yy}(x,y)}{K_y(y,y)-M_y(y,y)}\,\phi(x)dx + \int_y^1 \frac{-M_{yy}(x,y)}{K_y(y,y)-M_y(y,y)}\,\phi(x)dx$$

$$U\Psi = \int_0^x \frac{-M_{xx}(x,y)}{M_x(x,x)-K_x(x,x)}\,\Psi(y)dy + \int_x^1 \frac{-K_{xx}(x,y)}{M_x(x,x)-K_x(x,x)}\,\Psi(y)dy$$

$$p_1 = \frac{-K_{yy}(0,y)}{(K_y-M_y)(y,y)},\quad p_2 = \frac{-M_{yy}(1,y)}{(K_y-M_y)(y,y)},\quad q_1 = \frac{-M_{xx}(x,0)}{(M_x-K_x)(x,x)},\quad q_2 = \frac{-K_{xx}(x,1)}{(M_x-K_x)(x,x)}$$

The solution of these integral equations can be obtained by the

use of Neumann Series. For clarity of understanding, we shall divide the proof into a series of lemmas.

REMARK. The operators T and U constitute strictly positive operators which transform positive continuous functions into continuous strictly positive functions. The properties enunciated in Part I for such operators are valid for T and U. Finally, the operator T_{ab} etc. shall denote the same operator T except that the integration is extended over the interval $a \leq x \leq b$.

LEMMA 2.1. The radius of the spectrum $\lambda(T_{01})$ $[\mu(U_{01})]$ of the operator T [U] is less than 1.

PROOF. If we assume the contrary, then $\lambda(T_{01}) \geq 1$. As we let $a \longrightarrow 1$, then $\lambda(T_{a1}) \longrightarrow 0$. By Property II of positive operators there exists an a for which $\lambda(T_{a1}) = 1$. Property I guarantees a non-negative eigenfunction f(x) such that $T_{a1}f = f$. Upon integration of $Tf = f$, we get

$$\int_a^y K_y(x,y)f(x)dx + \int_y^1 M_y(x,y)f(x)dx = c$$

for $a \leq y \leq 1$. Setting $y = a$, as $M_y(x,a) < 0$, we have $c < 0$. On the other hand if we put $y = 1$, then as $K_y(x,1) > 0$, we secure that $c > 0$. This contradiction establishes the assertion of this lemma. A similar argument applies to the operator U.

Lemma 2.1 enables us to invert (2.1) and (2.2) for any non-negative α, β, γ, δ and we produce positive continuous functions. Explicitly, let $(I - T)^{-1}p_i = \phi_i$ $i = 1, 2$ and $(I - U)^{-1}q_i = \phi_i$ $i = 1, 2$. Put $\phi = \alpha\phi_1 + \beta\phi_2$ and $\psi = \gamma\psi_1 + \delta\psi_2$.

LEMMA 2.2. There exists optimal strategies of the following form $F = (\alpha I_0, \phi(x), \beta I_1)$ and $G = (\gamma I_0, \psi, \delta I_1)$.

PROOF. We suppose that all the quantities have been defined as above. Integration once of (2.1) gives that

$$(2.3) \qquad c = \alpha K_y(0,y) + \beta M_y(1,y) + \int_0^y K_y(x,y)\phi(x)dx + \int_y^1 M_y(x,y)\phi(x)dx .$$

We choose α and β so that $\int \phi = \alpha \int \phi_1 + \beta \int \phi_2 = 1 - \alpha - \beta$. There exists in fact an entire interval of values for which this is fulfilled. Supposing first $\alpha = 0$ and then letting $y = 0$, we obtain

$$c = \beta M_y(1,0) + \int_0^1 M_y(x,0)\phi(x)dx < 0 .$$

In a similar manner, if we choose $\beta = 0$ and look at $y = 1$, we secure that $c > 0$. Continuity thus produces a choice of $\alpha > 0$ and $\beta > 0$ so that $c = 0$. Analogously, we find the existence of positive constants γ and δ so that

$$(2.4) \qquad 0 = \gamma M_x(x,0) + \delta K_x(x,1) + \int_0^x M_x(x,y)\Psi(y)dy + \int_x^1 M_y(x,y)\Psi(y)dy$$

An integration once more of (2.3) and (2.4) yields that

$$\int_0^1 L(x,y)dF(x) = b \qquad \text{and} \qquad \int L(x,y)dG(y) = d$$

where

$$F = (\alpha I_0, \phi(x), \beta I_1) \qquad \text{and} \qquad G = (\gamma I_0, \Psi(x), \delta I_1) .$$

Multiplying the first by dG and the second by dF and integrating shows that $b = d = v$. It remains only to establish the uniqueness.

LEMMA 2.3. If F is optimal for player I, then on the interior of the unit interval F is continuous.

PROOF. Let y_0 be any point where $0 < y_0 < 1$. Since F is optimal, we have $\int L(x,y)dF = v$ for all y. Thus

$$\int \frac{L(x,y)-L(x,y_0)}{y-y_0}\, dF = 0 \quad \text{for} \quad y \neq y_0 .$$

As the right-hand derivative exists everywhere and is bounded, then with the aid of bounded convergence, we get $\int L_{y+}(x,y_0)dF = 0$. Similarly, we obtain $\int L_{y-}(x,y_0)dF = 0$. Since $L_{y+}(x,y_0) = L_{y-}(x,y_0)$ except for $x = y_0$, we have

$$0 = \int (L_{y+}-L_{y-})(x,y_0)dF(x) = [K_y(y_0,y_0) - M_y(y_0,y_0)]\,\sigma_{y_0}F .$$

Thus $\sigma_{y_0}F = 0$ and the proof is complete.

LEMMA 2.4. If F is optimal, then F is absolutely continuous and $F' = f$ is continuous for the points where $0 < x < 1$.

PROOF. Since F is optimal, we have

$$(2.5)\qquad v \equiv \int L(x,y)\,dF(x) = \alpha_0 K(0,y) + \beta_0 M(1,y) + \int_0^y K(x,y)\,dF_0(x) + \int_y^1 M(x,y)\,dF_0(x)$$

where $F_0(x)$ represents the continuous part of $F(x)$ concentrated on the interval of 0 to 1. An integration by parts of the last two integrals gives

$$(2.6)\qquad \int_0^y K(x,y)\,dF_0(x) + \int_y^1 M(x,y)\,dF_0(x) = -\int_0^y K_x(x,y)F_0(x)\,dx - \int_y^1 M_x(x,y)F_0(x)\,dy + M(1,y)F_0(1)\ .$$

This formula is valid by virtue of Lemma 2.3 which established the continuity of $F_0(x)$. Differentiating (2.5) after (2.6) has been substituted into (2.5) yields

$$(2.7)\qquad 0 = \alpha_0 K_y(0,y) + (\beta_0 + F_0(1))M_y(1,y) - \int_0^y K_{xy}F_0\,dx - \int_y^1 M_{xy}F_0\,dx - [K_x(y,y) - M_x(y,y)]\,F_0(y)\ .$$

Since $K_x - M_x \geq \delta > 0$ we see from (2.7) by assumption (a) that $F_0(y)$ is absolutely continuous. Upon differentiating once more which is valid by assumption (a), we obtain that $F_0'(y) = f(y)$ is continuous.

LEMMA 2.5. The optimal strategy for player I is unique.

PROOF. Lemma 3 shows that any optimal strategy for player I has the form $F(\alpha,f(x),\beta)$ with $f(x)$ continuous for $0 < x < 1$. Differentiating $\int L(x,y)\,dF = v$ twice establishes that f satisfies equation (1) i.e.,

$$\alpha p_1 + \beta p_2 = f - Tf\ .$$

Clearly $f = \alpha\phi_1 + \beta\phi_2$. It remains to show that there exists a unique pair of positive quantities α, β satisfying

$$(2.8)\qquad \alpha\int \phi_1 + \beta\int \phi_2 = 1 - \alpha - \beta$$

and

$$(2.9)\qquad \alpha K_y(0,y) + \beta M_y(1,y) + \int_0^y K_y(x,y)f(x)\,dx + \int_y^1 M_y(x,y)f(x)\,dx = 0$$

with $f(x) = \alpha\phi_1 + \beta\phi_2$. If the solution to these two equations is one dimensional, then there is a solution of the form $\alpha > 0$, $\beta = 0$. However, this is impossible for then by taking $y = 1$, we obtain that the relation (2.9) is strictly positive.

PROOF OF THEOREM 2.1. This is the content of Lemma 2.2 and Lemma 2.5 with a dual proof to Lemma 2.5 establishing the uniqueness of the optimal G strategy.

We close this section with some additional remarks.

If $L(x,y)$ is a butterfly kernel and $L(x,y) = L(y,x)$, then the unique optimal strategy for each player coincide. This is due to the fact that the integral equations which produce the solution are the same.

We now present a few examples of butterfly kernels, specifically,

(a) $$L(x,y) = |x - y| - \lambda(x - y)^2 \quad \text{where} \quad 0 < \lambda \leq \tfrac{1}{2}$$

(b) $$L(x,y) = \max(m(x)n(y), n(x)m(y))$$

with $m(x), n(x) \geq k > 0$, $m'(x) \leq -k < 0$, $n'(x) \geq k \geq 0$, and $m''(x)$, $n''(x) \leq -k < 0$.

Kernels concave on each side of the diagonal. In this section, we drop assumption (c) and suppose only that $(K_y - M_y)(y,y) > 0$ and $(M_x - K_x)(x,x) > 0$. This includes the case wherein the kernel has butterfly shape only in the neighborhood of the diagonal.

> THEOREM 2.2. Under the condition stated above all the optimal strategies have the following form: There exists a unique interval $[a,b]$ such that any maximizing strategy $F = (\alpha I_a, \phi_{ab}(x), \beta I_b)$ and any minimizing $G = (\gamma I_0, \psi_{ab}(y), \delta I_1)$ where $\alpha > 0$ when $0 < a < 1$ and $\beta > 0$ when $0 < b < 1$. Generally, α, β, γ, $\delta \geq 0$ and ϕ_{ab}, ψ_{ab} are obtained as the Neumann Series or eigenfunctions to associated integral equations. There exists at most a one dimensional set of solutions for each player.

The proof of this theorem is subdivided into a series of lemmas. Although an analagous method to that used in the study of butterfly kernels can be employed to treat this case, we follow a different procedure which is also applicable to the preceding section. Both approaches have their individual merits and are useful for different particular examples.

> LEMMA 2.6. There exists a unique interval $[a,b]$ which is the spectrum of any optimal strategy for the

maximizing player. If this interval is non-degenerate, it is, with the possible addition of 0 and 1, the spectrum of any optimal strategy for the minimizing player. If $a = b$, then the spectrum of any optimal strategy for the minimizing player is contained in $\{0,a,1\}$.

PROOF. We shall let F and G represent optimal strategies for the maximizing and minimizing players, respectively. If (c,d) is any open interval of F measure zero with $0 < c < d < 1$, it is easily seen that $\int L(x,y)dF(x)$, which is $\geq v$, is strictly concave for $c < y < d$, hence $> v$ in this interval. Consequently the spectrum of any optimal G cannot meet this interval, and the spectrum of any optimal G is contained in the spectrum of any optimal F, with the possible exception of the end points 0 and 1.

Now suppose c and d are in the spectrum of F, $c < d$, and (c,d) is of F measure zero. Then (c,d) is of G measure zero for any optimal G, so that $\int L(x,y)dG(y)$ is strictly concave for $c < x < d$. Since this integral is a continuous function of x and must assume the value v at c and d we have $\int L(x,y)dG(y) > v$ for $c < x < d$, which contradicts the optimality of G. Hence the spectrum of any optimal F is some closed interval $[a,b]$.

Suppose that for some optimal F this interval is non-degenerate, i.e., $a < b$. Then $[a,b]$ is contained in the spectrum of any optimal G since $\int L(x,y)dG(y) = v$ for $a \leq x \leq b$ so that no open subinterval is of G measure zero. Combined with the fact that the spectrum of any G is contained (disregarding 0 and 1) in the spectrum of any F, this implies that the spectrum of any optimal F' coincides with (again disregarding 0 and 1) that of G, so that $[a,b]$ is the spectrum of all optimal F, and similarly of all optimal G (again disregarding 0 and 1).

If on the other hand $[a,b]$ is degenerate for all optimal F, it is still unique. For suppose I_a and $I_{a'}$ are optimal, $a \neq a'$. Then since the spectrum of an optimal G is contained in $\{0,a,1\}$ and $\{0,a',1\}$, the spectrum of any optimal G is contained in $\{0,1\}$ so that $\int L(x,y)dG(y)$ is strictly concave for $0 < x < 1$, $\leq v$ and yet equal to v at a and a', which is clearly absurd.

LEMMA 2.7. The maximizing optimal strategies have the form $F = (\alpha I_a, \phi_{ab}(x), \beta I_b)$ with $\alpha > 0$ when $0 < a < 1$, $\beta > 0$ when $0 < b < 1$, and $\alpha \geq 0$, $\beta \geq 0$ otherwise.

PROOF. The identical arguments of Lemmas 2.3 and 2.4 apply here and show with the aid of Lemma 2.6 for x interior to the closed interval $[a,b]$ that $F(x)$ is absolutely continuous and $F'(x) = f(x)$ is continuous over the interval $[a,b]$. We now verify that if $0 < a < 1$, then F possesses an additional point spectrum at a. To this end, if we assume the contrary, then the derivative of $h(y) = \int L(x,y)dF(x)$ exists and is zero at a. Since $h(y)$ is strictly concave for $0 < y < a$, this implies the fact that $\int L(x,y)dF(x) < v$ for $0 < y < a$ which is contrary to the optimal character of F. A similar argument applies to the end point b if it is interior to the unit interval.

LEMMA 2.8. The minimizing strategy G has the form $G = (\gamma I_0, \Psi_{ab}(y), \delta I_1)$ with $\gamma, \delta \geq 0$.

PROOF. As in the previous lemma it is easy to deduce that G is absolutely continuous over the interior of the interval $[a,b]$. In view of Lemma 2.6 it remains only to show that if $0 < a < 1$, then G is continuous at a. As in Lemma 2.3, it follows that $\int L_{x+}(a,y)dG(y) = 0$. On the other hand, we have $\int L(a,y)dG(y) = v$ and $\int L(x,y)dG \leq v$ for $x < a$. Consequently, $\int \frac{L(x,y)-L(a,y)}{x-a} dG(y) \geq 0$ for $x < a$. Proceeding to a limit gives that $\int L_{x-}(a,y)dG \geq 0$. Therefore, $0 \geq \int (L_{x+} - L_{x-})(a,y)dG(y) = (M_x - K_x)(a,a)\sigma_a G$. Since $M_x - K_x > 0$ and $\sigma_a G \geq 0$, we deduce $\sigma_a G = 0$. An analogous argument applies at the point b if $0 < b < 1$. Indeed, $\int L_{x-}(b,y)dG(y) = 0$ and $\int L(b,y)dG(y) = v$ while $\int L(x,y)dG(y) \leq v$ for $x > b$. Whence $\int L_{x+}(b,y) \leq 0$. Therefore,

$$0 \geq \int [L_{x+}(b,y) - L_{x-}(b,y)]\, dG(y) = (M_x - K_x)\sigma_b(G) .$$

Again, this gives that $\sigma_b(G) = 0$. This completes the proof of the lemma.

Upon two differentiations of $\int_0^1 L(x,y)dF(x) = \int_0^1 L(x,y)dG(y) = v$ for $a \leq x, y \leq b$, we obtain for $\phi = \phi_{ab}$ and $\Psi = \Psi_{ab}$ that

$$0 = \alpha K_{yy}(a,y) + \beta M_{yy}(b,y) + \int_a^y K_{yy}(x,y)\phi(x)dx + \int_y^b M_{yy}(x,y)\phi(x)dx + (K_y - M_y)(y,y)\phi(y) . \tag{2.8}$$

$$0 = \gamma M_{xx}(x,0) + \delta K_{xx}(x,1) + \int_a^x M_{xx}(x,y)\Psi(y)dy + \int_x^b K_{xx}(x,y)\Psi(y)dy + (M_x - K_x)(x,x)\Psi(x) . \tag{2.9}$$

Expressing these integral equations differently, we get

$$\alpha p_1 + \beta p_2 = \phi - T_{ab}\phi \tag{2.10}$$

$$\gamma q_1 + \delta q_2 = \Psi - U_{ab}\Psi \tag{2.11}$$

where

$$T_{ab}\phi = \int_a^y - \frac{K_{yy}(x,y)}{(K_y - M_y)(y,y)}\,\phi(x)dx + \int_y^b - \frac{M_{yy}(x,y)}{K_y - M_y(y,y)}\,\phi(x)dx\,,$$

$$p_1 = \frac{-K_{yy}(a,y)}{(K_y - M_y)(y,y)}\,, \qquad p_2 = \frac{-M_{yy}(b,y)}{K_y - M_y(y,y)}$$

and

$$U_{ab}\Psi = \int_a^x - \frac{M_{xx}(x,y)}{M_x(x,x) - K_x(x,x)}\,\Psi(y)dy + \int_x^b - \frac{K_{xx}(x,y)}{M_x(x,x) - K_x(x,x)}\,\Psi(y)dy$$

$$q_1 = \frac{-M_{xx}(x,0)}{(M_x - K_x)(x,x)}\,, \qquad q_2 = \frac{-K_{xx}(x,1)}{(M_x - K_x)(x,x)}\,.$$

The two integral equations (2.10) and (2.11) are to be used to determine $\phi(x)$ and $\Psi(x)$. The fact that p_i and q_i are positive implies that $\phi \geq T_{ab}\phi$ and $\Psi \geq U_{ab}\Psi$. Using again the general theory of positive operators [3] yields that $\lambda(T_{ab}) \leq 1$ and $\mu(U_{ab}) \leq 1$. If either a or b are interior to the interval $[0,1]$, then $\lambda(T_{ab}) < 1$. In fact, Lemma 2.7 guarantees that (2.10) is satisfied with at least α or β non zero and one concludes from this (since $p_i \geq \delta > 0$) that $T_{ab}\phi \leq \lambda\phi$ with $\lambda < 1$. Thus $\lambda(T_{ab}) \leq \lambda < 1$. Concerning $\mu(U_{ab})$, we have that both cases where $\mu(U_{ab}) = 1$ and $\mu(U_{ab}) < 1$ may occur.

Let $0 < a < b < 1$.

CASE 1. If $\mu(U_{ab}) = 1$, then the optimal strategies have the form $F = (\alpha I_a, \phi_{ab}(x), \beta I_b)$ and $G = (\Psi_{ab}(y))$ with $\alpha > 0$ and $\beta > 0$.

PROOF. Lemmas 2.6 to 2.8 have established that the optimal solutions have the form $F = (\alpha I_a, \phi_{ab}(x), \beta I_b)$ and $G = (\gamma I_o, \Psi_{ab}(x), \delta I_1)$. If either γ or δ were not zero, then equation (2.11) would imply that $\mu(U_{ab}) < 1$ contradicting the hypothesis.

Let $0 < a < b < 1$.

CASE 2. If $\mu(U_{ab}) < 1$, then the form of the optimal strategies is as follows: $F = (\alpha I_a, \phi_{ab}(x), \beta I_b)$ and $G = (\gamma I_o, \Psi_{ab}(y), \delta I_1)$ with $\alpha, \beta > 0$ and at least γ or δ positive.

PROOF. The demonstration is similar.

Let $0 < a < b = 1$.

CASE 3. If $\mu(U_{ab}) = 1$, then $F = (\alpha I_a, \phi_{ab}(x), \beta I_b)$ and $G = (\psi_{ab}(x))$ with $\alpha > 0$ and $\beta \geq 0$ and at least γ or δ positive.

The proof of this is similar to the preceding. An analogous set of results presents itself when $0 = a < b < 1$ and $0 = a < b = 1$. We finally remark that when $\gamma > 0$ in any of the above strategies, then this imposes a condition on the yield $\int L(x,y)dF(x)$; namely $\int L(x,0)dF(x) = v = \int L(x,a)dF(x)$. A similar requirement results when $\delta > 0$, i.e., $\int L(x,1)dF(x) = \int L(x,b)dF(x) = v$. These conditions determine a and b uniquely.

We now discuss the question of uniqueness. Whenever at least a or b is interior to the unit interval and there exists an optimal minimizing strategy using both end points 0 and 1, then we show that the maximizing strategy is unique. The proof of Theorem 2.2 has shown that any optimal strategy for player I has the form $F = (\alpha I_a, \phi_{ab}(x), \beta I_b)$ with $\phi_{ab}(x) = \alpha\phi_1(x) + \beta\phi_2(x)$ where $(I - T_{ab})^{-1}p_i = \phi_i$ and $i = 1, 2,$. We must verify that the choice of α and β are unique. For any values of α and β which supply optimal strategies it follows that

$$0 = \alpha K_y(a,y) + \beta M_y(b,y) + \int_a^y K_y(x,y)\phi_{ab}(x)dx + \int_y^b M_y(x,y)\phi_{ab}(x)dx$$

for $a \leq y \leq b$.

This has the form $\alpha R(y) + \beta S(y) = 0$. Explicitly $R(y) = K_y(a,y) + \int_a^b L_y(x,y)\phi_1(x)dx$ and $S(y) = M_y(b,y) + \int_a^b L_y(x,y)\phi_2(x)dx$. From the properties of the integral operators it follows that $R(y) \equiv c_1$ and $S(y) \equiv c_2$. If $R(y)$ or $S(y)$ is not zero, we get a one dimensional set of solutions for α and β. The normalization $1 - \alpha - \beta = \alpha \int \phi_1 + \beta \int \phi_2$ produces a unique choice of (α,β) in this one dimensional set. We may hence assume that $R(y) = S(y) = 0$ for $a \leq y \leq b$. The assumption on the minimizing strategy implies for any optimal F that $\int L(x,0)dF(x) = v = \int L(x,y)dF(x)$ for $a \leq y \leq b$. Expanding this equality, we get $\alpha k_1 + \beta k_2 = 0$ with another similar relation $\alpha l_1 + \beta l_2 = 0$, resulting from consideration of the point 1. It is to be emphasized that the constants k_1 and k_2 do not depend upon α or β. These equations cannot possess a two dimensional set of solutions for then there would exist an optimal F strategy with no jump at either a or b which contradicts Lemma 2.7. The normalization again selects thus a unique pair.

If $\mu(U_{ab}) = 1$, the uniqueness of the optimal strategy for the minimizing player follows from the fact that any solution of the form $(\gamma I_0, \psi_{ab}(y)), \delta I_1)$ with say $\gamma > 0$ implies $\mu(U_{ab}) < 1$ and the eigen-

function is unique (Property I of strictly positive operators). We have been able to verify in every particular example the uniqueness. However, the uniqueness question in general remains open. The fact that we get at most a one dimensional set of solutions follows since they all satisfy the same integral equation. In fact, α, β range only over a one dimensional set since the additional normalization restricts one of the variables.

An example. We illustrate the theory of the preceding section by the following example. Let

$$L(x,y) = \begin{cases} K(x,y) = y - x - \lambda(x - y)^2 & y \geq x \\ M(x,y) = x - y - \lambda(x - y)^2 & x \geq y \end{cases} \qquad \lambda > 0 .$$

It is easily verified that $K_{xx} = K_{yy} = M_{xx} = M_{yy} = -2\lambda$ and $K_y = 1 + 2\lambda(x - y)$, $M_y = -1 + 2\lambda(x - y)$. Furthermore, it is easy to show that this kernel is invariant under the transformation $x \longrightarrow 1 - x$ and $y \longrightarrow 1 - y$. It therefore follows that there exists optimal strategies $F(x)$ and $G(y)$ which are unchanged under the mapping $x \longrightarrow 1 - x$ and $y \longrightarrow 1 - y$ respectively. Consequently, we search for a symmetric interval about $\frac{1}{2}$ to play the role of $[a,b]$. The solution to the integral equation

$$\lambda = \phi(y) - \int_{\frac{1}{2} - \frac{c}{2}}^{\frac{1}{2} + \frac{c}{2}} \frac{2\lambda}{2} \phi(x)dx$$

implies that $\phi(x) = \frac{\lambda}{1-\lambda c}$.

If we consider the equation $\alpha p_1 + \beta p_2 = \Phi - T_{ab}\Phi$ with $\beta = \alpha$, we get $\Phi = \lambda$ and $\alpha = \frac{1-\lambda c}{2}$. The condition $\int L(x,0)dF = \int L(x,a)dF$ implies after a simple calculation that either $c = 1$ or $c = \frac{2-\lambda}{\lambda}$. This gives that $a = 1 - \frac{1}{\lambda}$ and $b = \frac{1}{\lambda}$ when $c = \frac{2-\lambda}{\lambda}$ and $a = 0$, $b = 1$ when $c = 1$. For $0 < \lambda \leq 1$ the operator $T\phi = \int \lambda\phi$ has $\lambda(T) \leq 1$ while for $1 < \lambda \leq 2$, we have $\lambda(T) > 1$ thus to generate a positive operator of spectral radius less than 1 it suffices to take the interval $(1 - \frac{1}{\lambda}, \frac{1}{\lambda})$ for $1 < \lambda < 2$. If we compute α for the two cases we get $\alpha = \frac{1-\lambda}{2}$ when $0 < \lambda < 1$ and $\alpha = \frac{\lambda-1}{2}$ when λ is between 1 and 2. We have therefore found the form of the strategies to be for $0 < \lambda \leq 1$

$$F = (\tfrac{1-\lambda}{2} I_0, \phi_{01}(x), \tfrac{1-\lambda}{2} I_1)$$

$$G = (\tfrac{1-\lambda}{2} I_0, \psi_{01}(y), \tfrac{1-\lambda}{2} I_1)$$

$$\phi_{01} = \psi_{01} = \lambda ,$$

and for $1 < \lambda < 2$

$$F = \left(\frac{\lambda-1}{2} I_{1-\frac{1}{\lambda}} \phi_{1-\frac{1}{\lambda}, \frac{1}{\lambda}}(x) \frac{\lambda-1}{2} I_{\frac{1}{\lambda}} \right)$$

$$\phi_{1-\frac{1}{\lambda}, \frac{1}{\lambda}} = \psi_{1-\frac{1}{\lambda}, \frac{1}{\lambda}} = \lambda$$

$$G = \left(\frac{\lambda-1}{2} I_0, \psi_{1-\frac{1}{\lambda}, \frac{1}{\lambda}}(y), \frac{\lambda-1}{2} I_1 \right) .$$

For $\lambda \geq 2$ one verifies directly that the optimal strategies are

$$F = (I_{\frac{1}{2}}) \quad \text{and} \quad G = (\tfrac{1}{2} I_0, \tfrac{1}{2} I_1)$$

and the value is equal to $\frac{1}{2} - \lambda(\frac{1}{2})^2$. In all these cases of this example it can easily be shown that these optimal strategies are unique.

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ON A CLASS OF GAMES*

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The purpose of this note is to describe qualitatively the nature of optimal strategies for a payoff kernel $K(x,y)$ defined on the unit square satisfying $K_{y\ldots y}(x,y) \geq 0$ with n partial derivatives taken with respect to y. We present a complete analysis for $n \leq 4$. However, the method employed seems to extend easily and should enable one, by enumerating cases, to analyze the situation for general n. Specifically, we show that for $n = 3$ and $n = 4$ the maximizing player has optimal strategies involving respectively at most 3 points and at most 4 points of increase. For general n, it can be shown that the maximizing player has optimal strategies using at most n points. For the minimizing player the statement of the nature of an optimal strategy is more precise. There always exist for the general case optimal solutions using at most $\frac{n}{2}$ points, with the understanding that the end points 0 or 1 when used are each counted only half. For example when n is odd, then $\frac{n}{2}$ is a half integer and hence must use a single end point if a full optimal strategy exists employing $\frac{n}{2}$ points. This counting procedure applies only to the minimizing strategies.

§ 1. OPTIMAL STRATEGIES FOR PLAYER II

We assume throughout this section that $K_{y\ldots y}(x,y) \geq 0$ in the unit square with n partial derivatives taken with respect to y and that K possesses continuous n^{th} partial derivatives.

LEMMA 1. If $K_{y\ldots y}(x,y) \geq 0$, then there exists an optimal strategy for player II with at most $\frac{n}{2}$ points of increase.

REMARK 1. It is sufficient to prove Lemma 1 when $K_{y\ldots y}(x,y) \geq \delta > 0$. Indeed, if we perturb K by $K^n(x,y) = K(x,y) + \frac{1}{n} L(x,y)$ where $L_{y\ldots y}(x,y) \geq \delta > 0$,

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then $K^n(x,y)$ converges uniformly to $K(x,y)$ and $K^n_{y\ldots y}(x,y) \geq \frac{1}{n}\,\delta > 0$ for each n. The function $L(x,y)$ can be chosen to be an n^{th} degree polynomial in x and y with the coefficient of y^n a strictly positive polynomial for x ranging in the unit interval. Each kernel K^n possesses an optimal solution G_n for player II of at most $\frac{n}{2}$ points in the spectrum. We can select a limit distribution G_0 which also involves at most $\frac{n}{2}$ points of increase. Similarly, let F denote any limit distribution of optimal strategies F_n of the games $K^n(x,y)$. It follows easily that G_0 and F are optimal strategies for players II and I respectively for the game with kernel $K(x,y)$.

PROOF OF LEMMA 1. In view of Remark 1 we may assume that $K_{y\ldots y}(x,y) > 0$ for x,y traversing the unit square. Let $F_0(x)$ denote an optimal strategy for player I, then the hypothesis and bounded convergence easily yields that $h(y) = \int_0^1 K(x,y)dF_0(x)$ has $h^{(n)}(y) > 0$. This implies that $h(y)$ can achieve a value at most n times since otherwise there exists a y_0 with $h^n(y_0) = 0$. Consequently, the minimum m of $h(y)$ can only be achieved at most $\frac{n}{2}$ times, end points being counted half as described in the introduction; for otherwise $m + \epsilon$ with ϵ sufficiently small would be taken on at least $n + 1$ times. Since an optimal strategy for the minimizing player can only concentrate at these points, the conclusion of the lemma is now evident.

§ 2. OPTIMAL STRATEGIES FOR PLAYER I WITH $n = 1$ AND 2

The case $n = 1$ yields a saddle point. In fact if x_0 denotes a point where $K(x_0,0) = \max_x K(x,0)$, then it is easily verified that $x = x_0$ and $y = 0$ is a saddle point.

LEMMA 2. Let X be compact and Y an n dimensional simplex. If $M(x,y)$ is a continuous payoff kernel defined on $X \times Y$ which for each x is a convex function in y, then there exists an optimal strategy for player I involving at most $n + 1$ points of increase.

This lemma comprises the essential result of [1]. For a more direct and illuminating proof see a forthcoming paper by the author on the general theory of infinite games.

The content of Lemma 2 contains the case of $n = 2$ being analyzed

in this paper. Player II has a pure optimal strategy while player I possesses an optimal solution using at most 2 points in its spectrum.

§ 3. OPTIMAL STRATEGIES FOR PLAYER I WITH $n = 3$

Throughout this section we assume that $K_{yyy}(x,y) \geq 0$ and we establish the following theorem.

THEOREM 1. If K possesses continuous third partial derivatives and $K_{yyy}(x,y) \geq 0$ for $0 \leq x,y \leq 1$, then player I has an optimal strategy F with at most 3 points of increase.

The proof of this theorem will consist in analyzing the various possibilities. It is no loss of generality to suppose that $K_{yyy}(x,y) > 0$. An argument analogous to that used in remark 1 establishes this fact. Moreover, any optimal minimizing strategy involves at most 3/2 points. The various possible minimizing strategies are therefore of the following forms, where I_y denotes a pure distribution with full weight at y: (a) $\alpha I_0 + (1-\alpha)I_{y_0}$ with $0 < y_0 < 1$, (b) $\alpha I_0 + (1-\alpha)I_1$, (c) I_{y_0} with $0 < y_0 < 1$, (d) I_0 and (e) I_1. The strategies listed above present an enumeration of all the possible ways of obtaining $\frac{1}{2}$, 1, or $1\frac{1}{2}$ points, with the exception of one. It remains only to show why an optimal strategy of the form $\alpha I_{y_0} + (1-\alpha)I_1$ with $0 < y_0 < 1$ cannot occur.

(A) In fact, if $F(x)$ is optimal for player I, then $H(y) = \int_0^1 K(x,y)dF(x)$ has the properties that $h(1) = h(y_0) = v$, $h(y) \geq v$, and $h'''(y) > 0$. This implies that h is convex in the neighborhood of y_0 where h attains a minimum and further, since $h(1) = v$, that h is concave at $y_1 > y_0$. Thus $h''(y_0) > 0$ and $h''(y_1) < 0$. But since $h'''(y) > 0$, it follows that $h''(y)$ is increasing which is incompatible with the preceding remark.

In general, once the yield $h(y) = \int_0^1 K(x,y)dF(x)$ is convex then it cannot turn back and become concave.

To establish the assertion of Theorem 1, we shall show for each of the possible forms of an optimal solution for player II that player I possesses an optimal strategy of the desired type.

Case (a). Since $G_0 = \alpha I_0 + (1-\alpha)I_{y_0}$ is optimal for player II, any value x occurring with positive probability in an optimal strategy F must imply that $\int_0^1 K(x,y)dG_0(y) = v$. Let X be the set of all x for which $\int_0^1 K(x,y)dG_0(y) = v$. Evidently, the set X is compact. We now introduce the following auxiliary game defined for x in X and z

ranging over a 2 dimensional simplex Z spanned by z_1, z_2, and z_3. For x in X, set $M(x,z_1) = v_1 - v$, where v_1 is the value at 1 of the tangent line to $K(x,y)$ at y_0, $M(x,z_2) = v_2 - v$, where v_2 is the value at 0 of the tangent line to $K(x,y)$ at y_0, and $M(x,z_3) = K(x,0) - v$. Extend $M(x,z)$ linearly over Z for each x. It is easy to verify that the hypothesis of Lemma 2 are satisfied. Also if F is optimal for the game $K(x,y)$, then $\int_X M(x,z)dF(x) = 0$. Therefore, there exists an optimal $F^*(= \lambda_1 I_{x_1} + \lambda_2 I_{x_2} + \lambda_3 I_{x_3})$ for which $\int_X M(x,z)dF^*(x) \geq$ value ≥ 0. We consider $h(y) = \int_0^1 K(x,y)dF^*(x)$. The interpretation of the game $M(x,z)$ shows that the tangent to $h(y)$ at y_0 lies above v at 0 and 1. Thus $h(y_0) \geq v$. Also, $h(0) \geq v$. However, since x_1, x_2, and x_3 belong to X, we obtain that $v \leq \int_0^1 h(y)dG_0(y) = \int_0^1 dF^*(x) \int_0^1 K(x,y)dG_0(y) = v$. This yields that $h(0) = h(y_0) = v$ and hence the slope $h'(y_0) = 0$. The argument of paragraph (A) shows that $h(y)$ is convex at y_0 and $h(y) \geq v$ throughout.

Case (b). Let X consist of all x for which $v = \int_0^1 K(x,y)dG_0(y)$ where $G_0(y)$ is optimal and of the form $\alpha I_0 + (1-\alpha)I_1$. Evidently, X is compact. Let $M(x,z)$ be defined over the set X and the two dimensional simplex z spanned by z_1, z_2, and z_3 defined as follows. Put $M(x,z_1) = K(x,0) - v$, $M(x,z_2) = K(x,1) - v$ and $M(x,z_3) = -K_y(x,1)$ with $M(x,z)$ extended linearly over z. If $F_0(x)$ is any optimal strategy for player I of the game corresponding to the payoff kernel $K(x,y)$, then $\int_X M(x,z)dF(x) \geq 0$. In view of Lemma 2 we can find an optimal strategy $F^*(x)$ using at most 3 points x in X. If $h(y) = \int_0^1 K(x,y)dF^*(x)$, then $h(0) \geq v$, $h(1) \geq v$ and $h'(1) \leq 0$. Since $G_0(y)$ concentrates only at 0 and 1 and $\int_0^1 dG_0(y) \int_0^1 K(x,y)dF^*(x) = v$ we conclude that $h(0) = h(1) = v$. Furthermore $h'(y)$ is negative at 1 and $h'''(y) > 0$ requires therefore that $h(y)$ is greater than v throughout the unit interval.

Case (c). In this case an optimal minimizing strategy exists concentrating at the point y_0. Let X comprise all x where $K(x,y_0) = v$. Again, we introduce the auxiliary game $M(x,z)$ with x in X and z in the two dimensional simplex. Put $M(x,z_1) = +K_y(x,y_0)$, $M(x,z_2) = -K_y(x,y_0)$, and $M(x,z_3) = K(x,0) - v$ with the usual linear extension. An analogous reasoning produces an optimal strategy $F^*(= \alpha I_{x_1} + \beta I_{x_2} + \gamma I_{x_3})$ with $\int_X M(x,z)dF^*(x) \geq 0$. The function $h(y) = \int_0^1 K(x,y)dF^*(x)$ has the properties that $h(y_0) = v$, $h'(y_0) = 0$ and $h(0) \geq v$. The argument in (A) and $h(0) \geq v$ show that h is convex at y_0 and since $h(0) \geq v$ we easily find that $h(y) \geq v$ for $0 \leq y \leq 1$.

Case (d). The form of G_0 is I_0. We select for X all x where $K(x,0) = v$, and for some optimal F_0 we set $m(y) = \int K(x,y)dF_0(x)$. Evidently $m'''(y) > 0$, so that m'' is strictly increasing, and we may

divide the analysis into the following exhaustive subcases:

$$(d_1)\quad m''(y) \leq 0 \qquad 0 \leq y \leq 1\ ,$$

$$(d_2)\quad m''(y) \geq 0 \qquad 0 \leq y \leq 1\ ,$$

$$(d_3)\quad m''(0) < 0 < m''(1)\ .$$

(d_1). We define an auxiliary game $M(x,z)$ as follows: $M(x,z_1) = -K_{yy}(x,1)$, $M(x,z_2) = K(x,1) - v$. In this case Z is one dimensional, and by Lemma 2 we can find an optimal strategy F^* using two values x in X for the auxiliary game. Now $h(y) = \int_0^1 K(x,y)dF^*(x)$ satisfies $h''(1) \leq 0$, $h(0) = v$ and $h(1) \geq v$. Since h'' is increasing, h is concave. Since $h(y)$ is no less than v at both ends of the unit interval, $h(y) \geq v$ for all y, and F^* is optimal for the original game.

(d_2). Since $m(y) \geq v$ and $m(0) = v$, we must have $m'(0) \geq 0$. Consequently, defining the auxiliary game by $M(x,z_1) = K_{yy}(x,0)$, $M(x,z_2) = K_y(x,0)$, Z is again one dimensional and $\int M(x,z)dF_0(x) \geq 0$. By Lemma 2 we have an optimal F^* for the auxiliary game using two points in X, and for $h(y) = \int K(x,y)dF^*(x)$ we have $h''(0) \geq 0$, $h'(0) \geq 0$ and $h(0) = v$. Since h'' is increasing, h is convex; combined with the fact that $h(0) = v$, $h'(0) \geq 0$, this yields $h(y) \geq v$, or the optimality of F^* in the original game.

(d_3). Consider the graph of m. Since m'' is strictly increasing we have a unique point of inflection y_0, $0 < y_0 < 1$, $m''(y_0) = 0$. For some y_1, $y_0 \leq y_1 \leq 1$, the tangent line to the curve at y_1 will have its value at 0 as well as its value at $1 \geq v$. For if $m'(y_0) \geq 0$, this will be true for $y_1 = y_0$ since the tangent at y_0 lies above the concave portion of the curve (where $y \leq y_0$), and thus the height of the tangent line at 0 will be at least $v = m(0)$; since the slope of the tangent is non-negative this is also true at 1. On the other hand if $m'(y_0) < 0$, then either we have a point y_1, $y_0 < y_1 \leq 1$, where $m'(y_1) = 0$ (and hence where the entire tangent line is above the height v since $m(y_1) \geq v$) or $m'(y) < 0$ for $y_0 \leq y \leq 1$; since $m(1) \geq v$, the tangent line at 1 will do in this case.

We now define an auxiliary game over X and a two dimensional simplex Z via the usual linear extension and $M(x,z_1) = K_{yy}(x,y_1)$, $M(x,z_2) = v_1 - v$, $M(x,z_3) = v_2 - v$, where v_1 and v_2 are, for fixed x, the heights at 0 and 1, of the tangent line to the curve $K(x,y)$ at y_1. We have chosen y_1 so that $\int M(x,y)dF_0(x) \geq 0$, and thus by Lemma 2 we have an $F^* = \alpha I_{x_1} + \beta I_{x_2} + \gamma I_{x_3}$ for which $\int M(x,z)dF^*(x) \geq 0$. But for $h(y) = \int K(x,y)dF^*(x)$ we have the heights at 0 and 1 of the tangent line to the curve $h(y)$ at $y_1 \geq v$; since $h''' > 0$, the convex

portion of the curve occurs over an interval $[a,1]$ which includes y_1 since $h''(y_1) \geq 0$. The convex portion clearly lies above the tangent line at y_1, and since the concave portion of the curve, which occurs over the interval $[0,a]$, has both endpoints above the height v, $h(y) \geq v$ for all y, and F^* is optimal in the original game.

Case (e). This can be analyzed similarly to case (d). Since these five cases exhaust all the possibilities the proof of Theorem 1 is now complete.

§ 4. OPTIMAL STRATEGIES FOR PLAYER I WITH $n = 4$

Throughout this section we assume that $K_{yyyy}(x,y) \geq 0$. The following result is demonstrated.

THEOREM 2. If K possesses continuous fourth partial derivatives and $K_{yyyy}(x,y) \geq 0$ for $0 \leq x, y \leq 1$, then player I has an optimal strategy with at most 4 points of increase.

The method of proof of this theorem is analogous to that employed in Theorem 1. Again, without restricting generality we may assume that $K_{yyyy}(x,y) > 0$. Furthermore an optimal strategy for player II involves at most $\frac{4}{2} = 2$ points. The various possible minimizing strategies are therefore of the form (a) $\alpha I_{y_0} + (1-\alpha)I_{y_1}$, with $0 < y_0 < y_1 < 1$, (b) $\alpha I_0 + (1-\alpha)I_{y_0}$ with $0 < y_0 < 1$, (c) $\alpha I_1 + (1-\alpha)I_{y_0}$ with $0 < y_0 < 1$, (d) $\alpha I_0 + (1-\alpha)I_1$, (e) I_{y_0} with $0 < y_0 < 1$, (f) I_0, and (g) I_1. We have omitted one additional possible strategy with total weight 2; namely $\alpha I_0 + \beta I_{y_0} + (1-\alpha-\beta)I_1$, where $0 < y_0 < 1$. This form for an optimal strategy is not possible.

(B) Indeed, let us suppose that player II has an optimal strategy of the above form. Let $F_0(x)$ be optimal for player I, then $m(y) = \int_0^1 K(x,y)dF_0(x)$ has the following properties: $m(0) = v$, $m(1) = v$, $m(y_0) = v$, $m'(y_0) = 0$ and $m(y) \geq v$ for $0 \leq y \leq 1$. It follows then that m is convex at y_0 while concave somewhere between 0 and y_0 and somewhere between y_0 and 1. This contradicts the fact that $m''(y)$ is convex which is required by $m''''(y) > 0$.

We shall frequently use this general fact that the yield $m(y)$ of any strategy F (i.e., $m(y) = \int_0^1 K(x,y)dF(x)$) cannot be concave in two portions and convex in between.

The proof of Theorem 2 will consist in verifying for each of the possible forms of the optimal strategies for player II that an optimal

strategy for player I of the desired type exists.

Case (a). Let the minimizing player have an optimal solution of the form $G_o = \alpha I_{y_o} + (1-\alpha)I_{y_1}$, where $0 < y_o < y_1 < 1$. Let the set X consist of all x for which $\int_0^1 K(x,y)dG_o(y) = v$. We construct now the following auxiliary game $M(x,z)$ defined for x in X and z in Z where Z is a 3 dimensional simplex spanned by the points z_1, z_2, z_3 and z_4. For x in X, set $M(x,z_1) = v_1 - v$, where v_1 is the tangent line to the curve $K(x,y)$ at y_o evaluated at 0, $M(x,z_2) = v_2 - v$, where v_2 is the tangent line to the curve $K(x,y)$ at y_o evaluated at 1, and $M(x,z_3)$ and $M(x,z_4)$ are defined similarly in terms of curve $K(x,y)$ at the point y_1. The function $M(x,z)$ is defined over Z by linear extension. The kernel $M(x,z)$ satisfies all the conditions of Lemma 2. Furthermore, if $F_o(x)$ is optimal for player I and the game $K(x,y)$, then $\int_X M(x,z)dF_o(x) \equiv 0$. Lemma 2 provides an optimal strategy $F^*(x)$ with at most 4 points in its spectrum such that $\int_X M(x,z)dF^*(x) \geq 0$. Put $h(y) = \int_0^1 K(x,y)dF^*(x)$. The optimal nature of $F^*(x)$ for the game $M(x,z)$ implies that the tangent lines to $h(y)$ at y_o and y_1 lie everywhere above the height v. Consequently, $h(y_o) \geq v$ and $h(y_1) \geq v$. Hence, $\int_0^1 h(y)dG_o(y) \geq v$. But since every x involved in $F^*(x)$ is included in X, we conclude that $\int_0^1 dF^*(x) \int_0^1 K(x,y)dG_o(y) = v$. Thus, $h(y_o) = h(y_1) = v$. Therefore also, $h'(y_o) = h'(y_1) = 0$. In view of the remark (B), we deduce that $h(y) \geq v$ throughout the unit interval.

Case (b). For this case we construct the auxiliary game over X and Z as follows: The values of $M(x,z_1)$ and $M(x,z_2)$ are exactly as in case (a). Put $M(x,z_3) = K(x,0) - v$ and $M(x,z_4) = + K_y(x,0)$ and $M(x,z)$ for z in Z is defined by linear extension. The set X consists of all x for which $\int K(x,y)dG_o(y) = v$ where $G_o(y) = \alpha I_o + (1-\alpha)I_{y_o}$. By Lemma 2 the game $M(x,z)$ with x in X and z in Z has an optimal strategy $F^*(x)$ with at most 4 points of increase such that $\int_X M(x,z)dF^*(x) \geq 0$. This gives for $h(y) = \int_0^1 K(x,y)dF^*(x)$ that $h(0) \geq v$ and $h(y_o) \geq v$. It can be deduced in a similar manner to case (a) that $h(0) = h(y_o) = v$ and $h'(y_o) = 0$ while $h'(0) \geq 0$. With the aid of remark (B) we conclude easily that $h(y) \geq v$ for $0 \leq y \leq 1$.

Case (c). If the minimizing optimal strategy has the form $G_o = \alpha I_1 + (1-\alpha)I_{y_o}$, then an argument similar to that employed in case (b) applies here to furnish the desired kind of optimal strategy for player I.

Case (d). Let $G_o(y) = \alpha I_o + (1-\alpha)I_1$ and let X consist of all x in the unit interval for which $\int K(x,y)dG_o(x) = v$. Put for x in X $M(x,z_1) = K(x,0) - v$, $M(x,z_2) = K_y(x,0)$, $M(x,z_3) = K(x,1) - v$ and $M(x,z_4) = -K_y(x,1)$ with a linear extension. Again, one can find an optimal strategy $F^*(x)$ involving at most 4 points for which $\int M(x,z)dF^*(x) \geq 0$.

It follows easily for $h(y) = \int_0^1 K(x,y)dF^*(x)$ that $h(0) \geq v$, $h(1) \geq v$, $h'(0) \geq 0$ and $h'(1) \leq 0$. Employing the definition of X yields that $h(0) = h(1) = v$. On account of remark (B) it follows that $h(y) \geq v$ for y in the unit interval.

Case (e). The form of the minimizing solution is I_{y_0} with $0 < y_0 < 1$. We subdivide this case into four subcases. The analysis of case (e) is as follows:

Case (e_1) treats the situation where there exists a solution $F(x)$ whose yield $m(y)$ is convex for all $0 \leq y \leq 1$. The other alternative is that $m(y)$ is concave somewhere in the unit interval. For definiteness, we take $m(y)$ to be concave to the left of y_0 and consider all resulting possibilities. An entirely analagous investigation applies if $m(y)$ is concave in a portion to the right of y_0. Under the given supposition in view of remark (B) and the fact that $m(y)$ is always convex at y_0, we deduce that $m(y)$ is convex for $1 \geq y \geq y_0$.

Three possibilities arise in this circumstance that $m(y)$ is concave in a portion to the left of y_0.

Case (e_2) examines the situation where $m(y)$ is concave at 0.

Case (e_3) treats the circumstance where there exists a $0 \leq y_1 < y_0$ for which $m''(y_1) > 0$ and $m'(y_1) = 0$ and case (e_4) considers the possibility that $m''(0) > 0$ while $m'(0) > 0$.

(e_1). Suppose there exists an optimal $F_0(x)$ for which $m(y) = \int_0^1 K(x,y)dF_0(x)$ is convex for all y in the unit interval. Let X be composed of all x where $K(x,y_0) = v$. We now define a new game $M(x,z)$ over $X \times Z$ as follows. For x in X, set $M(x,z_1) = v_1 - v$, where v_1 is the value of the tangent line to the curve $K(x,y)$ at y_0 evaluated at 0, and $M(x,z_2) = v_2 - v$, where v_2 is the value of the tangent line to the curve $K(x,y)$ at y_0 evaluated at 1. Finally for the segment $[z_3,z_4]$ which we choose to be the unit interval $[0,1]$, we define $M(x,z) = K_{yy}(x,z)$. Since the fourth derivative $K_{yyyy} \geq 0$, we get that $M(x,z)$ is convex on the segment $[z_3,z_4]$ for each x. The function $M(x,z)$ is defined on the segment $[z_3,z_4]$ and at the vertices z_1 and z_2. We first extend $M(x,z)$ to the face L spanned by z_2, z_3 and z_4. Every point p of L not on $[z_3,z_4]$ is a convex combination of a point z of the segment $[z_3,z_4]$ and z_2. Explicitly $p = \alpha z + (1 - \alpha)z_2$. Put $M(x,p) = \alpha M(x,z) + (1 - \alpha)M(x,z_2)$. Thus M is defined on L. In an analagous way M can now be extended to the full simplex Z, and is easily seen to be convex. It now obviously satisfies the requirements of Lemma 2. Furthermore, the nature of $m(y)$ implies that $\int_X M(x,z)dF_0(x) \geq 0$. On account of Lemma 2, we can select an optimal strategy $F^*(x)$ which involves at most 4 points in its spectrum for which $\int_X M(x,z)dF^*(x) \geq 0$. Let $h(y) = \int_0^1 K(x,y)dF^*(x)$. An interpretation of

of the auxiliary game introduced yields that $h''(y) \geq 0$, $h'(y_0) = 0$ and $h(y_0) = v$. Consequently, $h(y) \geq v$ for $0 \leq y \leq 1$.

(e_2). Suppose there exists an optimal strategy $F_0(y)$ for which $m(y) = \int_0^1 K(x,y)dF_0(x)$ is concave at 0. The set X is chosen as we did before for (e_1). We construct $M(x,z_1)$, $M(x,z_2)$ as before while $M(x,z_3) = K(x,0) - v$ and $M(x,z_4) = -K_{yy}(x,0)$ with $M(x,z)$ extended linearly over Z for each x in X. An optimal strategy using at most 4 points for the game corresponding to $M(x,z)$ turns out to be an optimal strategy also for $K(x,y)$. The details are similar to the preceding cases.

(e_3). We assume now that there exists an optimal strategy $F_0(y)$ for which $m(y) = \int_0^1 K(x,y)dF_0(x)$ has the following properties: $m'(y_0) = 0$, $m(y_0) = v$, $m(y) \geq v$ for $0 \leq y \leq 1$, there exists a $0 < y_1 < y_0$ for which $m''(y_1) > 0$ and $m'(y_1) = 0$. Let X comprise those x for which $K(x,y_0) = v$. Let Z be a 4 dimensional simplex spanned by z_1, z_2, z_3, z_4 and z_5. Let for x in X $M(x,z_1)$ and $M(x,z_2)$ be given as before. Also, put $M(x,z_3) = v_3 - v$, where v_3 is the tangent line to the curve $K(x,y)$ at y_1 evaluated at 0, $M(x,z_4) = v_4 - v$, where v_4 is the tangent line to the curve $K(x,y)$ at y_1 evaluated at 1, and $M(x,z_5) = K_{yy}(x,y_1)$. Furthermore $M(x,z)$ is extended linearly over Z. It follows easily that $\int_X M(x,z)dF_0(x) \geq 0$. We now verify that the optimal strategies for player II for the game with payoff $M(x,z)$ cannot possess an interior pure strategy I_{z_0}. Otherwise, $\int_X M(x,z)dF_0(x) \equiv 0$ for it is simple to show that the value is zero and therefore $m(y) = \int K(x,y)dF_0(x)$ satisfies $m(y) \geq v$ throughout and $m(y_1) = v$, $m'(y_1) = 0$ and $m''(y_1) = 0$ which contradicts the assumption made on $m(y)$. In view of [1] and the corollary to Theorem 4 of [2], we can conclude, since I_{z_0} is on the boundary of Z, that there exists an optimal strategy $F^*(x)$ using at most 4 points. Let $h(y) = \int_0^1 K(x,y)dF^*(x)$. It follows that $h'(y_0) = 0$, $h(y_0) = v$, the tangent line to $h(y)$ at y_1 lies above v for $0 \leq y \leq 1$ and $h(y)$ is convex at y_1. Now with the aid of remark (B) it is easy to show that $h(y) \geq v$ throughout the unit interval.

It remains only to consider the case where $m(y) = \int K(x,y)dF_0(x)$ satisfies the same properties as in (e_3) except that $y_1 = 0$. We then define $M(x,z_1)$ and $M(x,z_2)$ as before while $M(x,z_3) = K(x,0) - v$ and $M(x,z_4) = K_y(x,0)$ with x any point in the set X where $K(x,y_0) = v$. A simple analysis with the help of the game $M(x,z)$, shows in this circumstance the existence of an optimal strategy of the desired kind.

(e_4). The final case consists of a solution $F_0(x)$ such that $m(y) = \int_0^1 K(x,y)dF_0(x)$ is convex at y_0 and at 0 in such a way that $m'(0) \geq 0$. The auxiliary game used here is the same employed in case (b). An optimal strategy $F^*(x)$ using at most 4 points exists with

$h(y) = \int K(x,y)dF^*(x)$ such that $h'(y_0) = 0$, $h(y_0) = v$, $h(0) \geq 0$ and $h'(y_0) \geq 0$. It follows on account of (A) that $h(y) \geq v$ in the unit interval.

Case (f) and Case (g). The arguments for these two cases are similar to the preceding.

The proof of Theorem 2 is now complete in view of the fact that we have treated every possibility. It is interesting to note that the proof of Theorem 2 introduced some new techniques in order to exhibit the desired strategies. In particular, we emphasize the proof of case (e).

In a future paper we intend to present a generalization of this result to the case where x ranges over a compact space and y traverses a k dimensional set with the condition $K_{y\ldots y}(x,y) \geq 0$ replaced by the requirement that the n^{th} term of the Taylor expansion in y should be non-negative. Also, the question of uniqueness of the set of optimal strategies can be analyzed.

§ 5. DIMENSIONAL RELATIONS

In this section we analyze further properties of the optimal strategies for the two cases considered in sections 3 and 4. We deal first with the case where $K_{yyy}(x,y) > 0$. Here any optimal minimizing strategy must be confined at most to two points, 0 and y_0 or 0 and 1, and we can therefore speak about the dimension of the set of solutions for the minimizing players. The two possible cases are 0 and 1 dimensional sets. If the spectrum for the optimal minimizing strategies is restricted to one point, then clearly the solution is unique for player II.

Case (a). Let us suppose that player II has a one dimensional set of optimal strategies mixing the pure strategies 0 and y_0, with $0 < y_0 < 1$. There exist at least two optimal strategies of the form $G = \lambda I_0 + (1 - \lambda)I_{y_0}$ and $G' = \lambda' I_0 + (1 - \lambda')I_{y_0}$ with $\lambda' \neq \lambda$. We consider the set X of all x for which $\int K(x,y)dG(y) = \int K(x,y)dG'(y) = v$. Explicitly, we obtain

$$\lambda [K(x,0) - v] + (1 - \lambda)[K(x,y_0) - v] = 0$$

$$\lambda'[K(x,0) - v] + (1 - \lambda')[K(x,y_0) - v] = 0 .$$

As $\lambda \neq \lambda'$, we find that $K(x,0) = K(x,y_0) = v$. We now construct the auxiliary game where z consists of a one dimensional simplex spanned by z_1 and z_2. Put for x in X $M(x,z_1) = -K_y(x,y_0)$, $M(x,z_2) = K_y(x,y_0)$ and let $M(x,z)$ be defined on the remaining points of z by linear extension. There exists an optimal F^* using only two points at most. In view of the nature of X it is easily verified that F^* is an optimal

strategy for the game given by $K(x,y)$

Case (b). We assume that player II has a one dimensional set of strategies using only the points 0 and 1 It can be shown as above that there exists an optimal maximizing strategy consisting of at most two points of increase.

We have thus demonstrated the following theorem:

THEOREM 3. If K satisfies $K_{yyy}(x,y) > 0$ and player II possesses a one dimensional set of optimal strategies then player I can find a solution using at most two points of increase.

In a similar manner we can obtain

THEOREM 4. If K satisfies $K_{yyyy}(x,y) > 0$ and player II possesses a one dimensional set of optimal strategies, then player I can find an optimal strategy with at most three points in the spectrum.

Every case is easily handled but one. Suppose the minimizing strategy has a one dimensional set of solutions using y_0 and y_1 with $0 < y_0 < y_1 < 1$. Let X consist of all x where $K(x,y_0) = K(x,y_1) = v$. An argument as in Theorem 3 shows that these are the only points x which need be considered. We construct the following auxiliary game defined over X and Z where Z is a two dimensional simplex. Put $M(x,z_1)$ = the tangent line to $K(x,y)$ at y_0 evaluated at 1 minus the tangent line to $K(x,y)$ at y_1 evaluated at 1. $M(x,z_2)$ = the same as above except the evaluation takes place at 0 and $M(x,z_3)$ = the tangent line to $K(x,y)$ at y_1 evaluated at $\frac{y_0+y_1}{2}$ minus v. If F is optimal for the game corresponding to $K(x,y)$ then $\int_X M(x,y)dF(x) \equiv 0$. Thus Lemma 2 provides an optimal strategy F^* using at most 3 points of X with $\int M(x,y)dF^* \geq 0$. One can easily verify that F^* is optimal for $K(x,y)$. This completes the proof.

In the general case where $K_{y\ldots y}(x,y) > 0$ for n partial derivatives with respect to y, it seems that if the y player has a k-dimensional set of solutions, then the x player can find an optimal strategy involving at most $n - k$ points in its spectrum.

§ 6. AN APPLICATION

We close this note with an application of Theorems 1 and 2 to a class of matrices. First, it is important to note that the conclusions of

Theorems 1 and 2 remain valid when x traverses any compact set and where y ranges over the unit interval. We consider now a class of matrices (a_{ij}) $1 \leq i \leq n$, $1 \leq j \leq m$, satisfying the requirement that $\Delta^k a_{ij} \geq 0$ for each i where k is a fixed integer ≤ 4. We define $\Delta a_r = a_r - a_{r-1}$ for $r \geq 0$ with a_{-1} taken to be zero, and Δ^k is defined by k successive applications of the operator Δ. It is easy to show that if there exists a finite sequence of numbers a_r $(r = 0, 1, \ldots, n)$ which satisfy $\Delta^k a_r \geq 0$ for each r, then the function $f(y)$ defined as $f(\frac{r}{m}) = a_r$, $r = 0, 1, \ldots, m$ and extended by linear interpolation between the successive values of r/m possesses the property that $\Delta^k f(y) \geq 0$ with the difference increment taken to be $\frac{1}{m}$ and enjoys properties similar to functions for which $f^{(k)}(y) \geq 0$. Indeed, we sketch the proof by induction. When $k = 1$, then the above interpolation yields a monotonic increasing function. If $k = 2$, it is also trivial that the linear interpolation produces a convex function. We treat the case $k = 3$. Let $b_r = \Delta a_r$, then $\Delta^2 b_r \geq 0$. Let $g(y)$ denote the interpolated convex function with $g(\frac{r}{m}) = b_r$. Set $g(y) \equiv 0$ for $y < 0$ and define $f(y) = \sum_{r=0}^{m} g(y - \frac{r}{m})$. It easily follows that $f(y) - f(y - \frac{1}{m}) = g(y)$, $f(\frac{r}{m}) = a_r$ and f is piecewise linear between the points $\frac{r}{m}$. Since $g(y)$ is convex and the piecewise linear character in equal lengths of $f(y)$ implies that a convex portion of $f(y)$ cannot be followed by a concave part. This was the essential fact that enabled us to carry through Theorem 1. A similar analysis applies to the case $k = 4$. We perform the above extension for every row of the matrix and we secure a function $K(x,y)$ with x ranging over a finite number of points $i = 1, \ldots, n$ and y over the unit interval. The conclusions of Theorem 1 and 2 remain valid for such a piecewise linear kernel. Consequently, we can find optimal strategies of the game $K(x,y)$ for player I and II using at most k points and $k/2$ points respectively. Due to the linear nature of $K(x,y)$ any point y_0 can be obtained as a convex combination of two values $\frac{r}{m} < y_0 < r + \frac{1}{m}$ with $K(x,y_0) = \lambda K(x,\frac{r}{m}) + (1 - \lambda)K(x,r + \frac{1}{m})$. Thus in terms of the original matrix both players I and II possess solutions employing at most k rows and k columns.

THEOREM 5. If a_{ij} is a matrix such that for each i $\Delta^k a_{ij} \geq 0$ for all j with $k \leq 4$, then both players have solutions using at most k rows and columns respectively.

Finally, we remark that all the analogous results hold for the situations where $K_{y \ldots y}(x,y) \leq 0$, $K_{x \ldots x}(x,y) \geq 0$ and $K_{x \ldots x}(x,y) \geq 0$.

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NOTES ON GAMES OVER THE SQUARE*

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INTRODUCTION

In attempts to find methods of solving a fairly wide class of (2-person, zero-sum) games over the unit square, real success has been attained in only two cases: (1) Games in which the problem is essentially a finite-dimensional one, notably those with polynomial or polynomial-like payoffs [1], and (2) games having solutions which are absolutely continuous (or very nearly so) and can be solved via differential or integral equations [2,3]. These latter games all feature discontinuities in the payoffs or their derivatives. Attempts to find a tractable though wider class than that of polynomial games have led, naturally, to investigation of the class with rational payoff.

The object of these notes is to present some simple examples and results which, though of meager intrinsic interest, may serve as some guide in future investigations. In § 1, we give a simple consequence, due to one of the writers (O. Gross), of Tarski's decision theory of elementary algebra [4], together with an example, to show that rational games do not share with polynomial games the property of always having step function solutions. In § 2, we give an example of a game with rational payoff in which the unique optimal strategy for each player is based on a dense countable set. From this we conclude that neither of the methods mentioned above is completely applicable to the class of rational games. In the remaining section, we give explicit constructions to show that even in the class of games with payoffs in $C^{(\infty)}$ one may obtain any given pair (f,g) of distributions as the unique solution of an element of the class.

§1. ALGEBRAIC GAMES WITH TRANSCENDENTAL VALUES

Let Γ be a 2-person zero-sum game for which the pure strategy spaces, S_1 and S_2, and the payoff function M, defined over $S_1 \times S_2$, are definable in Tarski's system of "elementary algebra" [4]. Suppose,

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also, that Γ has a value which is a transcendental number. We can then conclude that there is no optimal strategy for either player consisting of a step function (i.e., a distribution in which the probabilities are all concentrated on a finite set of points). For suppose the contrary for one of the players, say the maximizing one. Then, for some positive integer m, the value of is given by

$$v = \max_{\substack{\langle \alpha_1, \ldots, \alpha_m \rangle \in \mathfrak{S}_m \\ x_1, \ldots, x_m \in S_1}} \min_{y \in S_2} \sum_{i=1}^{m} \alpha_i M(x_i, y),$$

where $\mathfrak{S}_m$ is the set of all m-tuples $\langle \mu_1, \ldots, \mu_m \rangle$ such that $\mu_i \geq 0$ for $i = 1, \ldots, m$, and $\sum_{i=1}^{m} \mu_i = 1$ But, according to [4], v would be algebraically definable, and it is a principal result of [4] that every algebraically definable number is algebraic.

In particular, our result applies to any game over the square with transcendental value, in which M is a continuous rational function with integral coefficients.

EXAMPLE. Take

$$M(x,y) = \frac{(1 + x)(1 + y)(1 - xy)}{(1 + xy)^2},$$

$$S_1 = \{x \mid 0 \leq x \leq 1\},$$

and

$$S_2 = \{y \mid 0 \leq y \leq 1\}.$$

Here, $v = \frac{4}{\pi}$, and a pair of distribution functions yielding this value is given by

$$f^*(x) = \frac{4}{\pi} \arctan \sqrt{x}$$

$$g^*(y) = \frac{4}{\pi} \arctan \sqrt{y} \qquad 0 \leq \frac{x}{y} \leq 1.$$

Thus, in this game, there is no step function optimal strategy, for π is a transcendental number. We mention, as an aside, that one can show by slightly more laborious methods, that the above solution is the only one for this game.

§ 2. RATIONAL GAMES WITH PATHOLOGICAL UNIQUE SOLUTIONS

In this section we shall construct a game with rational payoff in which the solution is uniquely given by a pair of distribution functions, each of which is based on a countable dense set in $(0,1)$. The following

simple facts form the germ of the construction:

(a) If, for any continuous function N on the unit square, and any pair (f,g) of strategies, one forms the function M by means of

$$M(x,y) = N(x,y) - \int N(x,y)df(x) - \int N(x,y)dg(y),$$

then the game with payoff M has (f,g) as a solution.

(b) If R is any rational function on the square, then

$$\int [R(x,y) - \tfrac{1}{2} R(\phi(x),y)]\, d\left(\sum_0^\infty \frac{1}{2^{n+1}} I_{\phi^{(n)}(x_0)}(x)\right)$$

(where ϕ: $[0,1] \longrightarrow [0,1]$ is a polynomial, $\phi^{(n)}$ its n-th iterate, and I_a is a function which is 0 for $x < a$ and 1 for $x \geq a$) is again a rational function; for it is precisely

$$\sum_0^\infty \frac{1}{2^{n+1}} [R(\phi^{(n)}(x_0),y) - \tfrac{1}{2} R(\phi^{(n+1)}(x_0),y)] = \tfrac{1}{2} R(x_0,y)$$

since the series telescopes.

(c) The polynomial $\phi(x) = 4x(1 - x)$ will provide us with a dense set of iterates for certain x_0.

Only (c) requires any proof, and if we set $x_k = \phi^{(k)}(x_0)$, then

$$x_{k+1} = 4x_k(1 - x_k) = 1 - (1 - 2x_k)^2,$$

and

$$1 - 2x_{k+1} = 2(1 - 2x_k)^2 - 1;$$

consequently if we set $u_k = 1 - 2x_k$, $u_0 = \cos 2\pi\theta_0$, then

$$u_k = \cos 2^k(2\pi\theta_0),$$

or

$$x_k = \frac{1 - \cos 2^k(2\pi\theta_0)}{2}.$$

Clearly then we need only select an x_0 for which the numbers $\{2^k\theta_0\}$ are, modulo 1, dense in $(0,1)$. This is equivalent to the occurrence of every finite sequence of 0's and 1's as a segment in the binary expansion of θ_0 (i.e., if $\theta_0 = \frac{a_1}{2} + \frac{a_2}{4} + \ldots$ then any sequence of n zero's and one's may be found as $(a_{k+1}, \ldots, a_{k+n})$ for some k);

such numbers θ_0 (and x_0) are dense in $(0,1)$, and there are a continuum of them.

Now let R be a rational function of a single variable. If we form

$$\Psi(x,y) = R(xy) - \tfrac{1}{2} R(\phi(x)y) - \tfrac{1}{2} R(x\phi(y)) + \tfrac{1}{4} R(\phi(x)\phi(y))$$

where $\phi(x) = 4x(1 - x)$, then by (b)

$$\int \Psi(x,y)\alpha \left(\sum_0^\infty \frac{1}{2^{n+1}} I_{\phi(n)(x_0)}(x) \right) = \tfrac{1}{2} R(x_0 y) - \tfrac{1}{4} R(x_0 \phi(y)).$$

Consequently the game with payoff M defined by

$$M(x,y) = \Psi(x,y) - \int \Psi(x,y)df'(x) - \int \Psi(x,y)dg(y),$$

where $f = \sum \frac{1}{2^{n+1}} I_{\phi(n)(x_0)}$, $g = \sum \frac{1}{2^{n+1}} I_{\phi(n)(y_0)}$, is a game with rational payoff having the pair (f,g) as a solution.

For definiteness, let us specifically set $R(t) = \frac{1}{2 + t}$, so that M has the form

$$M(x,y) = \frac{1}{2 + xy} - \tfrac{1}{2} \frac{1}{2 + 4xy(1-x)} - \tfrac{1}{2} \frac{1}{2 + 4xy(1-y)} + \tfrac{1}{4} \frac{1}{2 + 16xy(1-x)(1-y)}$$

$$- \tfrac{1}{2} \frac{1}{2 + x_0 y} + \tfrac{1}{4} \frac{1}{2 + 4x_0 y(1-y)} - \tfrac{1}{2} \frac{1}{2 + xy_0} + \tfrac{1}{4} \frac{1}{2 + 4xy_0(1-x)}$$

where we choose x_0 and y_0 so that their iterates are dense. This, then, is the example and we need only show (f,g) to be the unique solution.

Since, in the example, g places positive weight on a dense set of points, we must have, for any optimal f', $\int M(x,y)df'(x) = v$ for all y. Consequently

$$\int \Psi(x,y)df'(x) - \int \Psi(x,y)df(x) = \iint \Psi(x,y)df'(x)dg(y) + v = c$$

or $\int \Psi(x,y)d(f' - f)(x) = c$. But

$$\Psi(x,y) = R(xy) - \tfrac{1}{2} R(\phi(x)y) - \tfrac{1}{2} R(x\phi(y)) + \tfrac{1}{4} R(\phi(x)\phi(y))$$

$$= - \sum_0^\infty (-\tfrac{1}{2})^{n+1} [(xy)^n - \tfrac{1}{2}(\phi(x)y)^n - \tfrac{1}{2}(x\phi(y))^n + \tfrac{1}{4}(\phi(x)\phi(y))^n]$$

$$= - \sum_0^\infty (-\tfrac{1}{2})^{n+1} [x^n - \tfrac{1}{2}(\phi(x))^n][y^n - \tfrac{1}{2}(\phi(y))^n]$$

and thus

$$- \sum (-\tfrac{1}{2})^{n+1} h_n[y^n - \tfrac{1}{2}(\phi(y))^n] = c$$

for all y, where $h_n = \int [x^n - \frac{1}{2}(\phi(x))^n]d(f' - f)(x)$. Hence, for all m and y,

$$- c = \sum_0^\infty (-\tfrac{1}{2})^{n+1} h_n[(\phi^{(m)}(y))^n - \tfrac{1}{2}(\phi^{(m+1)}(y))^n]$$

so that

$$\begin{aligned} - 2c &= \sum_{m=0}^\infty \frac{1}{2^m} \sum_{n=0}^\infty (-\tfrac{1}{2})^{n+1} h_n[(\phi^{(m)}(y))^n - \tfrac{1}{2}(\phi^{(m+1)}(y))^n] \\ &= \sum_{n=0}^\infty (-\tfrac{1}{2})^{n+1} h_n \sum_{m=0}^\infty \frac{1}{2^m}[(\phi^{(m)}(y))^n - \tfrac{1}{2}(\phi^{(m+1)}(y))^n] \\ &= \sum_{n=0}^\infty (-\tfrac{1}{2})^{n+1} h_n y^n , \end{aligned}$$

for all y, and $h_n = 0$ for all n (obviously $h_0 = 0$). However, this implies $f' = f$, for the sequence of functions $\{x^n - \frac{1}{2}(\phi(x))^n\}_{n=0, 1, \ldots}$ is closed in $C(0,1)$ (i.e., any continuous function defined on $[0,1]$ may be uniformly approximated to within $\varepsilon > 0$ by a finite linear combination of these functions). This is most easily seen by noting that for a given continuous F on $[0,1]$ there exists a continuous function G satisfying $F(x) = G(x) - \frac{1}{2} G(\phi(x))$ given by

$$G(x) = \sum_{n=0}^\infty (\tfrac{1}{2})^n F(\phi^{(n)}(x)).$$

Consequently, if $\varepsilon > 0$ be given, one can find coeficients $a_0, \ldots, a_N$ for which

$$|G(x) - \sum_{n=0}^N a_n x^n| < \varepsilon \qquad 0 \leq x \leq 1,$$

by the Weierstrass Approximation Theorem, so that

$$|G(\phi(x)) - \sum_{n=0}^N a_n(\phi(x))^n| < \varepsilon$$

and

$$|F(x) - \sum_{n=0}^N a_n[x^n - \tfrac{1}{2}(\phi(x))^n]| < (1 + \tfrac{1}{2})\varepsilon$$

for all x.

The same arguments show that g is the unique optimal strategy for the minimizing player; thus the solution of the game is uniquely given by the "pathological" pair (f,g).

§ 3. GAMES WITH GIVEN UNIQUE SOLUTIONS

In this section we shall construct for each strategy pair (f,g), a game having (f,g) as its unique solution.

Case I. Neither f nor g a step function.

One need only set

$$M(x,y) = \sum_{n=1}^{\infty} 2^{-n}(x^n - f_n)(y^n - g_n)$$

where f_n and g_n are the usual power moments of f and g. For clearly, f and g are optimal and $v = 0$ for the game with payoff M; moreover if f' is any optimal strategy for the first player then

$$\int M(x,y)df'(x) = \sum 2^{-n}(f'_n - f_n)(y^n - g_n) = 0$$

for y in the spectrum of g (which by assumption is an infinite set) and thus for all y, so that $f'_n = f_n$ or $f' = f$. Similarly g is unique.

This simple construction must be modified if either f or g (or both) are step functions. Indeed, we have several cases.

Case II. f not a step function, g a step function with a step interior to $(0,1)$.

Let

$$g = \sum_{j=0}^{k} \beta_j I_{y_j}, \quad 0 = y_0 < y_1 < \cdots < y_k = 1, \ \beta_1 \neq 0.$$

We define the payoff M of our game by

$$M(x,y) = \sum_{n=1}^{\infty} 2^{-n}(x^n - f_n)[(y - y_1)^{2n-1} - g_{2n-1}] + e^{-\prod_j (y-y_j)^{-2}}$$

where $f_n = \int x^n df(x)$, $g_{2n-1} = \int (y - y_1)^{2n-1} dg(y)$. Obviously f and g are optimal and $v = 0$. Now if f' is optimal for player I then

$$\int M(x,y)df'(x) = \sum_{n=1}^{\infty} 2^{-n}(f'_n - f_n)[(y - y_1)^{2n-1} - g_{2n-1}] + e^{-\prod_j (y-y_j)^{-2}} \geq 0$$

and since $\beta_1 \neq 0$,

$$\int M(x,y_1)df'(x) = \sum 2^{-n}(f'_n - f_n)(-g_{2n-1}) = 0$$

so that

$$\sum 2^{-n}(f'_n - f_n)(y - y_1)^{2n-1} + e^{-\prod_j (y-y_j)^{-2}}$$

is nonnegative and vanishes at y_1. But clearly this quantity changes sign

at y_1 if any coefficient in the series is non-zero; hence $f' = f$, and f is the unique optimal strategy for player I.

Since $\int M(x,y)df(x) = e^{-\pi(y-y_j)^{-2}}$, which vanishes only at the points y_j, any g' optimal for player II must be based on these points, and

$$g' = \sum_{j=0}^{k} \beta'_j I_{y_j}.$$

Since f is not a step function we conclude, as in Case I, that

$$0 = \sum 2^{-n}(x^n - f_n)(g'_{2n-1} - g_{2n-1}) = \int M(x,y)dg'(y),$$

hence $g'_{2n-1} = g_{2n-1}$ for all $n > 0$, that is

$$(1) \qquad (\beta'_0 - \beta_0)y_1^{2n-1} = \sum_{j=2}^{k} (\beta'_j - \beta_j)(y_j - y_1)^{2n-1}.$$

Now if $y_j \neq 2y_1$, $j = 2, 3, \ldots, k$, this implies $\beta'_j = \beta_j$, $j = 0, 2, \ldots, k$, or $g' = g$; for since the positive numbers $y_1, y_2 - y_1, \ldots, y_k - y_1$, are all distinct, if we divide (1) by the $2n-1^{st}$ power of the largest of these and let $n \longrightarrow \infty$, we obtain $\beta'_{j_0} - \beta_{j_0} = 0$. Deleting this term and repeating the argument yields $\beta'_{j_1} - \beta_{j_1} = 0$, and thus finally $\beta'_j = \beta_j$, $j = 0, 2, \ldots, k$.

In case $y_j = 2y_1$ for some j we need only find a (1-1) continuous mapping ρ for which $\rho(y_j) = y'_j \neq 2y'_1$. For, forming the game with unique solution (f,g'), $g' = \sum \beta_j I_{y'_j}$ and payoff M as has been done above, we find that the game with payoff $M(x,\rho^{-1}(y))$ has the unique solution (f,g). In the following cases we shall assume that $y_j \neq 2y_1$.

Case III. Both f and g step functions with a step interior to $(0,1)$.

In this case we have $f = \sum_{i=1}^{k} \alpha_i I_{x_i}$, $g = \sum_{j=0}^{l} \beta_j I_{y_j}$, $0 = x_0 < x_1 < \ldots < x_k = 1$, $0 = y_0 < y_1 < \ldots < y_l = 1$, $\alpha_1 \neq 0 \neq \beta_1$. We set

$$M(x,y) = \sum 2^{-n}[(x-x_1)^{2n-1} - f_{2n-1}][(y-y_1)^{2n-1} - g_{2n-1}]$$
$$+ e^{-\pi(y-y_j)^{-2}} - e^{-\pi(x-x_i)^{-2}}$$

where

$$f_{2n-1} = \int (x - x_1)^{2n-1}\, df(x),\quad g_{2n-1} = \int (y - y_1)^{2n-1}\, dg(y).$$

Here, we have

$$\int M\,df = e^{-\prod_j (y-y_j)^{-2}},\quad \int M\,dg = e^{-\prod_i (y-y_j)^{-2}}$$

so that optimal strategies may only be based on the x_i or y_j. If f' is optimal, then

$$0 \leq \sum 2^{-n}(f'_{2n-1} - f_{2n-1})[(y-y_1)^{2n-1} - g_{2n-1}] + e^{-\prod_j (y-y_j)^{-2}}$$

and since $\beta_1 \neq 0$

$$0 = \sum 2^{-n}(f'_{2n-1} - f_{2n-1})(-g_{2n-1}).$$

Consequently

$$0 \leq \sum 2^{-n}(f'_{2n-1} - f_{2n-1})(y-y_1)^{2n-1} + e^{-\prod_j (y-y_j)^{-2}}$$

and as before we conclude $f'_{2n-1} = f_{2n-1}$. Since we may assume $x_i \neq 2x_1$, we conclude $f' = f$ as in Case II. The same argument shows g to be unique.

Case IV. f a step function with step interior to $(0,1)$ and $g = \beta I_0 + (1 - \beta)I_1$, $0 < \beta < 1$.

Here we set

$$M(x,y) = \sum_{n=1}^{\infty} 2^{-n}[(x-x_1)^{2n-1} - f_{2n-1}](y^n \sin \tfrac{1}{y} - (1 - \beta)\sin 1)$$

$$+ e^{-\frac{1}{y^2(1-y)^2}} - e^{-\prod_i (x-x_i)^{-2}}$$

Again (f,g) is a solution, $v = 0$, and player I must play the points x_i and player II must play $[0,1]$. For any optimal g' the previous argument yields $(1 - \beta')\sin 1 = (1 - \beta)\sin 1$ or $g' = g$. Moreover, if f' is optimal, since $0 < \beta < 1$,

$$\sum 2^{-n}(f'_{2n-1} - f_{2n-1})(-(1 - \beta)\sin 1) = 0$$

and thus

$$\sum 2^{-n}(f'_{2n-1} - f_{2n-1})y^n \sin \tfrac{1}{y} + e^{-\frac{1}{y^2(1-y)^2}} \geq 0.$$

Now the first non-vanishing term of the series dominates all other terms of the sum in a neighborhood of zero; but by setting

$$\frac{1}{y_m} = 2m\pi + \frac{\pi}{2}, \quad \frac{1}{y'_m} = 2m\pi - \frac{\pi}{2},$$

we find that the first non-vanishing term has the form

$$2^{-n}(f'_{2n-1} - f_{2n-1})y_m^n \quad \text{and} \quad -2^{-n}(f'_{2n-1} - f_{2n-1})y_m'^n$$

at y_m and y'_m respectively, so that the whole sum changes sign near zero if $f'_{2n-1} \neq f_{2n-1}$ for some n. Thus $f'_{2n-1} = f_{2n-1}$ for all n and f is again unique.

Case V. f not a step function, $g = \beta I_0 + (1 - \beta)I_1$, $0 < \beta < 1$.

In this case we modify the payoff of the previous case to

$$M(x,y) = \sum_{n=1}^{\infty} 2^{-n}(x^n - f_n)(y^n \sin \frac{1}{y} - (1 - \beta)\sin 1) + e^{-\frac{1}{y^2(1-y)^2}}$$

where f_n is the ordinary moment, $\int x^n df(x)$. Again (f,g) is a solution, $v = 0$ and player II may play only $\{0,1\}$. Since any optimal g' must yield $0 = \int M(x,y)dg(y)$ as in Case I, g is unique as in the preceding case. f is the unique optimal strategy for player I by the arguments of the preceding case.

Case VI. $f = \alpha I_0 + (1 - \alpha)I_1$, $g = \beta I_0 + (1 - \beta)I_1$, $0 < \alpha, \beta < 1$.

We set

$$M(x,y) = -\beta x + (2 - \alpha)y + x^2 + (1 - \alpha - \beta)xy - y^2.$$

Observe that M is <u>strictly</u> convex in x and <u>strictly</u> concave in y. Hence any optimal strategy for either player must be based on the points $0,1$. Consequently any solution $(\alpha' I_0 + (1 - \alpha')I_1, \beta' I_0 + (1 - \beta')I_1)$ of the game with this payoff must yield a solution $((\alpha', 1 - \alpha'), (\beta', 1 - \beta'))$ of the finite game with payoff given by the 2×2 matrix

$$\begin{pmatrix} M(0,0) & M(0,1) \\ M(1,0) & M(1,1) \end{pmatrix} = \begin{pmatrix} 0 & 1 - \alpha \\ 1 - \beta & 1 - \alpha - \beta \end{pmatrix}$$

However, it is easily verified that this game has the unique solution $((\alpha, 1 - \alpha), (\beta, 1 - \beta))$.

Case VII. f arbitrary, $g = I_0$.

Here we let

$$M(x,y) = \sum 2^{-n}(x^n - f_n)y^n \sin \frac{1}{y} + e^{-y^{-2}}$$

where f_n is the ordinary n-th moment of f. Again (f,g) is a solution, $v = 0$, and g is unique since $\int M df = e^{-y^{-2}}$. Moreover, if f' is optimal, we conclude as in Case IV that $f'_n = f_n$, or $f' = f$.

Since we may interchange players and the end points 0 and 1, all cases have been covered.

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ON RANDOMIZATION IN STATISTICAL GAMES WITH k TERMINAL ACTIONS[1]

David Blackwell

§1. SUMMARY

In a two-person zero-sum game in which a player receives partial, i.e., statistical, information about his opponent's strategy after which he takes one of k terminal actions, any randomized strategy is equivalent to a mixture of a countable number of pure strategies in the proportions $\lambda_n = (\frac{k-1}{k})^{n-1} \frac{1}{k}$, $n = 1, 2, \ldots, .$ The proof uses the fact that for any k non-negative numbers $z_1, \ldots, z_k$ with $\sum_1^k z_i = 1$ there is a partition of the set of positive integers into disjoint sets $S_1, \ldots, S_k$ such that $\sum_{n \in S_j} \lambda_n = z_j$.

§2. SPECIAL AND GENERAL RANDOMIZATION

Many zero-sum two-person games, including statistical multi-decision problems with a fixed sampling plan, have the following structure. Player I chooses a point ω from a set Ω. A point x is then chosen from a space X according to a probability distribution P_ω, defined on a Borel field $\mathcal{B}$ of subsets of X, and x is announced to Player II. Player II then chooses one of k terminal actions, which we denote simply by the integers $1, \ldots, k$ and incurs a loss $L(\omega,x,i)$, supposed $\mathcal{B}$-measurable for fixed ω,i.

Thus, a pure strategy for I is an $\omega \in \Omega$, a strategy for II is a partition $\mathcal{A} = (A_1, \ldots, A_k)$ of X into k disjoint sets $A_1, \ldots, A_k$, $A_i \in \mathcal{B}$, where A_j specifies the set of x's for which II proposes to take action j. II's loss is then

$$R(\omega,\mathcal{A}) = \sum_{j=1}^k \int_{A_j} L(\omega,x,j)\,dP_\omega(x).$$

A mixed strategy for II is a probability distribution Q over some Borel

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field $\mathcal{D}$ of subsets of the set Y of $\mathcal{B}$-measurable partitions α of X. II's loss from Q is

$$R(\omega,Q) = \int R(\omega,\alpha)\,dQ(\alpha)$$
$$= \sum_{j=1}^{k} \int [a_j(x,\alpha)L(\omega,x,j)\,dP_\omega(x)]\,dQ(\alpha).$$

where $a_j(x,\alpha)$ is the characteristic function of the set A_j of the partition α. We shall suppose that $a_j(x,\alpha)$ is measurable with respect to each measure $P_\omega xQ$, i.e., that Q assigns, for each P_ω, a definite overall probability of taking terminal action j. Then

$$R(\omega,Q) = \sum_{j=1}^{k} \int L(\omega,x,j)[\int a_j(x,\alpha)\,dQ(\alpha)]\,dP_\omega(x)$$
$$= \sum_{j=1}^{k} \int L(\omega,x,j)\,\phi_j(x,Q)\,dP_\omega(x),$$

where $\phi_j(x,Q) = \int a_j(x,\alpha)\,dQ(\alpha)$ is the probability under Q of taking action j when x is observed.

Thus any mixed strategy Q determines a set Φ of $\mathcal{B}$-measurable, non-negative functions $\{\phi_j(x)\}$ with $\sum_1^k \phi_j(x) = 1$ for each x such that

$$R(\omega,Q) = R(\omega,\Phi) = \sum_{j=1}^{k} \int L(\omega,x,j)\,\phi_j(x)\,dP_\omega(x).$$

Now any family Φ, whether arising from a Q or not, can be considered as constituting a mixed strategy, in which the mixing is done after x is observed, with action j being chosen with probability $\phi_j(x)$, and the expected loss from Φ is the function $R(\omega,\Phi)$ defined above. The relation between Q-mixing, i.e., mixing before x is observed, and Φ-mixing, i.e., mixing after x is observed, has been investigated by Wald [2],[2] Wald and Wolfowitz [3], and Kuhn [1], who have shown, under various hypotheses, that the two kinds of mixing are equivalent. For the special case considered here, we obtain the even more precise result that both are equivalent to mixtures of a countable number of pure strategies in the proportions $\lambda_n = (\frac{k-1}{k})^{n-1}\frac{1}{k}$ and that the pure strategies may be chosen independently of $L(\omega,x,j)$. We have already seen above that every Q-mixture is equivalent to a Φ-mixture, with $\phi_j(x) = \int a_j(x,\alpha)\,dQ(\alpha)$ independent of $L(\omega,x,j)$, so that it suffices to consider Φ-mixtures.

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[2]Numbers in square brackets refer to the bibliography at the end of this paper.

§3. THE MAIN THEOREM

For Φ-mixtures, we prove the

THEOREM. For any $\Phi = (\phi_1, \ldots, \phi_k)$ there is a sequence $\Psi_n = (\psi_{n1}, \ldots, \psi_{nk})$, where each ψ_{nj} is the characteristic function of a $\mathcal{B}$-measurable set and $\sum_{j=1}^{k} \psi_{nj} = 1$ for all n,x, such that $\sum_{n=1}^{\infty} \lambda_n \psi_{nj}(x) = \phi_j(x)$ for all j,x, where $\lambda_n = (\frac{k-1}{k})^{n-1} \frac{1}{k}$, so that, for all $L(\omega,x,j)$,

$$R(\omega,\Phi) = \sum_{j=1}^{k} \int L(\omega,x,j)\psi_j(x)dP_\omega(x)$$

$$= \sum_{1}^{\infty} \lambda_n (\sum_{j=1}^{k} \int L(\omega,x,j)\psi_{nj}(x)dP_\omega(x))$$

$$= \sum \lambda_n R(\omega,\Psi_n) = R(\omega,\phi^*),$$

where ϕ^* is the mixture of the pure strategies $\Psi_1, \Psi_2, \ldots$ in proportions $\lambda_1, \lambda_2, \ldots$.

PROOF. We define ϕ_{nj}, ψ_{nj} inductively as follows: $\phi_{1j} = \phi_j$, $\phi_1^* = \max_j \phi_{1j}$, j_1 = smallest j with $\phi_{1j} = \phi_1^*$, $\psi_{1j_1} = 1$, $\psi_{1j} = 0$, $j \neq j_1$. If ϕ_{ij}, ψ_{ij} have been defined for $i < n$, define $\phi_{nj} = \phi_j - \sum_{i=1}^{n-1} \lambda_i \psi_{ij}$, $\phi_n^* = \max_j \phi_{nj}$, j_n = smallest j with $\phi_{nj} = \phi_n^*$, $\psi_{nj_n} = 1$, $\psi_{nj} = 0$ for $j \neq j_n$. Since $\sum_1^k \phi_j = 1 = \sum_{j=1}^{k} (\sum_{i=1}^{\infty} \lambda_i \psi_{ij})$, it is sufficient to show that $\phi_j \geq \sum_{i=1}^{\infty} \lambda_i \psi_{ij}$ for $j = 1, \ldots, k$, which is equivalent to $\phi_{nj} \geq 0$ for all n,j. We prove $\phi_{nj} \geq 0$ for all n,j by induction on n; clearly $\phi_{1j} \geq 0$ for all j; suppose $\phi_{nj} \geq 0$ for all j. We have $\phi_{n+1\,j} = \phi_{nj} - \lambda_n \psi_{nj}$. Since $\psi_{nj} = 0$ for $j \neq j_n$, $\phi_{n+1\,j} = \phi_{nj} \geq 0$ for $j \neq j_n$. For $j = j_n$, $\phi_{n+1\,j} = \phi_n^* - \lambda_n$, and we must show $\phi_n^* \geq \lambda_n$. We have $\sum_{j=1}^{k} \phi_{nj} = \sum_n^{\infty} \lambda_i = k\lambda_n$, so that $\max_j \phi_{nj} = \phi_n^* \geq \lambda_n$. This completes the proof.

The sequence $\lambda_n = \frac{1}{k} (\frac{k-1}{k})^{n-1}$ is not the only sequence which can be used; it does have the advantage that the only random mechanism required

is one which will select one of the integers $1, 2, \ldots, k$ with equal probabilities $1/k$; the probability that integer 1 is selected for the first time at the nth trial is exactly λ_n. Moreover, if μ_n is any other usable sequence, $\sum_1^N \mu_n \leq \sum_1^N \lambda_n = 1 - (\frac{k-1}{k})^N$ for all N, so that $\sum\lambda_n$ converges faster than $\sum\mu_n$. Formally stated we have:

> If $\mu_n \geq 0$, $\sum_1^\infty \mu_n = 1$, and for every set of k non-negative numbers $x_1, \ldots, x_k$ with $\sum x_i = 1$, the series $\sum\mu_n$ can be split into k disjoint subseries whose sums are $x_1, \ldots, x_k$ respectively, then $\sum_1^N \mu_n \leq 1 - (\frac{k-1}{k})^N$ for all N.

PROOF. We may suppose $\mu_1 \geq \mu_2 \geq \cdots$. If N is the smallest integer for which $\sum_1^N \mu_n > 1 - (\frac{k-1}{k})^N$, we have $\mu_N > \frac{1}{k}(\frac{k-1}{k})^{N-1}$. Consider $x_1 = 1 - (\frac{k-1}{k})^N$, $x_i = \frac{1}{k}(\frac{k-1}{k})^{N-1}$ $i = 2, \ldots, k$. Since $\mu_1 \geq \cdots \geq \mu_N > x_i$ for $i \geq 2$, the set x_i cannot be represented, for $\mu_1, \ldots, \mu_N$ could go only into the subseries for x_1 and their sum exceeds x_1.

§4. AN EXAMPLE

We give here an example of a game of the type described above in which any minimax strategy for II actually requires the mixture of a countable number of pure strategies. We choose $\Omega = \{0 \leq \omega \leq 1\}$, $X = \{0 \leq x \leq \frac{1}{2}\}$, $P_\omega[\min(\omega, 1-\omega)] = 1$, $k = 2$, $L(\omega,x,1) = 2$, 1, or 0 according as ω is $<$, $=$, or $> \frac{1}{2}$, $L(\omega,x,2) = \frac{1-2\omega}{1-\omega}$, 1, or $\frac{1}{\omega}$ according as ω is $<$, $=$, or $> \frac{1}{2}$. The game is one in which Player II, seeing x, knows that I has chosen one of the two strategies $\omega = x$, $\omega = 1 - x$, and has available two actions, 1 and 2. The payoff is chosen so that the resulting game has value 1 and II must mix 1 and 2 in proportions x, $1 - x$ for $x < \frac{1}{2}$ in order to achieve this value. No mixture of a finite number of pure strategies can achieve this result. Formally, any Φ-mixture for II is specified by $\phi(x) = \phi_1(x)$, since $\phi_2 = 1 - \phi_1$. Any ϕ with $0 \leq \phi \leq 1$ for all x determines a Φ, the characteristic functions ϕ correspond to pure strategies, and finite linear combinations of characteristic functions correspond to mixtures of a finite number of pure strategies. We have

$$R(\omega,\phi) = 2\phi(\omega) + \frac{1-2\omega}{1-\omega}[1 - \phi(\omega)],\ 1,\ \frac{1-\phi(1-\omega)}{\omega}$$

according as ω is $<$, $=$, or $>$ $\frac{1}{2}$. The choice $\phi^*(x) = x$ yields $R(\omega,\phi^*) = 1$ for all ω, and, since $R(\frac{1}{2},\phi) = 1$ for all ϕ, the value of the game is 1. Thus any minimax ϕ for II must have $R(\omega,\phi) \leq 1$ for all ω. For $\omega < \frac{1}{2}$, this yields $\phi(\omega) \leq \omega$; for $\omega > \frac{1}{2}$, this yields $\phi(1 - \omega) \geq 1 - \omega$, so that $\phi(x) = x$ for $x < \frac{1}{2}$. Clearly $\phi(x)$ is not representable as a finite linear combination of characteristic functions, since any finite linear combination of characteristic functions assumes only a finite number of values.

To exhibit ϕ^* as $\sum_{n=1}^{\infty} \lambda_n \phi_n(x)$, where $\lambda_n = 2^{-n}$ and $\phi_n(x)$ is a characteristic function, we choose for $\phi_n(x)$ the n-th digit in the binary representation of x.

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Part III

GAMES IN EXTENSIVE FORM

The core of our intuitive notion of a game seems to be the successive presentation of alternatives to the players with varying degrees of information, or to chance devices with fixed probabilities, culminating in terminal situations with preassigned payoffs. Von Neumann has applied the name "games in extensive form" to his original formalization of this concept in terms of personal and chance moves, patterns of information, plays and their associated payoffs. Through the introduction of pure strategies this natural game apparatus disappears and only payoff functions defined over strategy vectors remain. The resulting form of a game, which is an ordinary matrix with real entries in the zero-sum two-person case, is called the "normalized form." It is peculiarly suited to the proof of theorems which hold for all games, while the extensive form serves to separate games into characteristic types. Thus, the normalized form has been a tool of great theoretical importance in establishing the existence and classification of solutions for all zero-sum two-person games, while the only hope of utilizing the individual properties of such games as Chess and Poker for their solution lies in their extensive form.

With one major exception, games in extensive form have been neglected in the press of research on games in normalized form. This exception is the theorem which states that every zero-sum two-person game with perfect information possesses a solution in pure strategies (see Theory of Games and Economic Behavior, p. 112, and for historical interest a paper of E. Zermelo listed in the Bibliography at the end of this Study). This perfect-information theorem still retains its topical importance (Papers 11, 12 and 13 all deal, in part, with its extensions and ramifications) and also points the way to the role of extensive games in analyzing the effects of varying the patterns of information. In PAPER 11, H. W. Kuhn introduces a new formalization of extensive games with precisely this object in mind. His is essentially a geometric model, mirroring the successive presentation of alternatives by the branching of a topological

tree and reproducing the patterns of information as partitions of the vertices of the tree. With this scheme, he extends the theorem cited above to general n-person games, proving that every such game with perfect information has a Nash equilibrium point in pure strategies. In addition, another problem of information is treated, that of the efficacy of behavior strategies. A "behavior strategy" can be viewed as a local randomization on the occasion of a choice rather than the total randomization of pure strategies made before a play by means of a mixed strategy. Kuhn shows that, when "perfect recall" prevails, behavior strategies can be used with the same effect on the payoff as mixed strategies. Thus, games such as Poker can be legitimately solved by means of behavior strategies -- with evident computational advantages.

Using the same geometric model, N. Dalkey considers in PAPER 12 the effect on the normalized form of a game of varying the information partitions. In a game without perfect information the rules do not inform a player of the precise nature of the situation that prevails when he is called upon to make a choice. Instead, they inform him that the situation is one of several possible ones. By an "inflation" Dalkey narrows the number of possibilities, increasing the accuracy of the information, and incidentally, the number of pure strategies. The only restriction on the inflation is that the resulting information be no more precise than the information given by the rules, combined with whatever knowledge of the situation the player can gain through the use of a pure strategy (in general, the choices dictated by his strategy may prevent many situations from arising!). The result of the successive application of this process until no further inflation is possible is called the "complete inflation" of a game. Dalkey shows that two games which differ only in information have the same normalized form (up to repetition of strategies) if and only if they have identical complete inflations. He also finds the necessary condition for the validity of the perfect-information theorem -- namely, that the complete inflation of the game in question permits each player to know everything his opponents knew and did, previous to his own choice.

Since the known proofs of the perfect-information theorem are based firmly on the finiteness of the games involved, it is appropriate to ask if the result carries over to the infinite case. D. Gale and F. M. Stewart answer this question negatively in PAPER 13. Precisely, they exhibit an infinite game with perfect information that possesses no solution in pure strategies. Positive results are obtained by observing that the set of plays of a game possesses a natural topology. The authors then consider "win-lose" games, that is, games in which player I wins a constant amount on a subset S_I of the plays and loses the same amount on the complement. If S_I is open or closed in the natural topology of the plays, or in any finite intersection of open and closed sets, then the game (with

perfect information) has a solution in pure strategies. A number of other, more detailed, results are obtained and several open problems are proposed.

PAPER 14 by G. L. Thompson continues the study of behavior strategies in an arbitrary game. He introduces the notion of a "signaling strategy," which is a pure strategy cut down to the information sets on which the player is required by the rules to forget something that he knew or did. Thompson shows that a combination of signaling strategies and associated behavior strategies suffices to produce any payoff attainable by mixed strategies and hence to provide a means of solution. In PAPER 15 he applies these concepts to the solution of a model of Bridge, considered as a two-person game. The model has the usual four players and four suits, but just two cards (ace and king) in each suit. There is no bidding, so only the playing characteristics of Bridge remain. To achieve the minimax value in no trump it is necessary for East-West, South being the declarer, to use privately randomized signaling with respect to West's initial lead from a pair of kings. If one considers such private signaling as contrary to Hoyle, it would seem necessary to regard Bridge as a four-person game with enforced coalitions or else as a two-person game with announcements of pure signaling strategies included as preliminary moves in the extensive form.

In PAPER 16 J. W. Milnor analyzes a situation which appears in many board games, including Chess and the Japanese game Go. In these games one can measure the "incentive" to move at any particular configuration by imagining the possibility of passing instead. Also, in the end play of such games (particularly Go), the entire game may become a "sum" of essentially independent games, so that the problem confronting a player is not only how to move but in which game. Milnor shows that the "sum" is an axiomatic group operation and that the "incentive" gives a natural metric for the group.

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EXTENSIVE GAMES AND THE PROBLEM OF INFORMATION

H. W. Kuhn[1]

In the mathematical theory of games of strategy as described by von Neumann and Morgenstern,[2] the development is seen to proceed in two major steps: (1) the presentation of an all-inclusive formal characterization of a general n-person game, (2) the introduction of the concept of pure strategy which makes possible a radical simplification of this scheme, replacing an arbitrary game by a suitable prototype game. They called these two descriptions the extensive and the normalized forms of a game. As noted there, the normalized form is better suited to the derivation of general theorems (e.g., the main theorem of the zero-sum two-person game), while the extensive form exposes the characteristic differences between games and the decisive structural features which determine those differences. Since all games are found in extensive form, while it is practical to normalize but a few, it seems desirable to attack the completion of a general theory of games in extensive form.

First of all, this paper presents a new formulation of the extensive form which, while appearing quite natural intuitively, covers a larger class of games than that used by von Neumann. The use of a geometrical model reduces the amount of set theoretical equipment necessary and clarifies the delicate problem of information. After the definition of pure strategies, Theorem 1 removes the redundancy found in a direct definition. Theorems 2 and 3 characterize the properties of a natural decomposition of many games into a subgame and a number of difference games. They represent a generalization of the theorem that every zero-sum two-person game with perfect information has a solution in pure strategies. Theorem 4 presents a positive criterion that a game be solvable in behavior strategies, a method that presents extreme computational advantages over mixed strategies in many cases.

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[1]The preparation of this paper was supported by the Office of Naval Research.

[2]von Neumann, J. and Morgenstern, O., The Theory of Games and Economic Behavior, 2nd ed., Princeton, 1947.

Throughout the paper, passages which serve as motivation, interpretation, or heuristic discussion are placed in brackets, [...]. This is done to emphasize the independence of the definitions and proofs of these sections.

§1. THE EXTENSIVE FORM OF A GAME

DEFINITION 1. A game tree K is a finite tree with a distinguished vertex 0 which is imbedded in an oriented plane.[3]

[The concept of a game tree is introduced as a natural geometric model of the essential character of a game as a successive presentation of alternatives. The distinguished vertex and the imbedding are devices which facilitate the arithmetization of the notion of strategy. Before proceeding to the definition of a game, it is necessary to present some general technical terms associated with a game tree; it is important to remark that, although these terms are taken from common parlance, their meaning is given by the definitions.]

TERMINOLOGY: The alternatives at a vertex $X \in K$ are the edges e incident at X and lying in components of K which do not contain 0 if we cut K at X. If there are j alternatives at X then they are indexed by the integers $1, \ldots, j$, circling X in the positive sense of the orientation. At the vertex 0, the first alternative may be assigned arbitrarily. If one circles a vertex $X \neq 0$ in the positive sense, the first alternative follows the unique edge at X which is not an alternative. The function thus defined which indexes the alternatives in K will be denoted by ν; thus, $\nu(e)$ is the index of the alternative e. Those vertices which possess alternatives will be called moves;[4] the remaining vertices will be called plays. The name play will also be used for the unique unicursal path from 0 to a play when no confusion results. The partition[5] of the moves into sets A_j, $j = 1, 2, \ldots$, where A_j contains all of the moves with j alternatives, will be called the alternative partition. A temporal order on K is defined by $X \leq Y$ if X lies on W_Y,

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[3] A graphical representation of a game by a tree has been suggested by von Neumann, loc. cit., p. 77; however, he only treats the case of games with perfect information.

[4] It is important to make a clear distinction between our "moves" and those of von Neumann. Precisely, a von Neumann move is the set of all moves of a given rank in our sense.

[5] In this paper a partition means an exhaustive decomposition into (possibly void) disjoint sets.

the unicursal path joining 0 to Y; it is a partial order. The rank of a move Y is the number of X such that $X \leq Y$ or, equivalently, the number of moves $X \in W_Y$.

DEFINITION 2. A general n-person game Γ is a game tree K with the following specifications:

(I) A partition of the moves into n+1 indexed sets $P_0, P_1, \ldots, P_n$ which will be called the player partition. The moves in P_0 will be called chance moves; the moves in P_i will be called personal moves of player i for $i = 1, \ldots, n$.

(II) A partition of the moves into sets U which is a refinement of the player and alternative partitions (that is, each U is contained in $P_i \cap A_j$ for some i and j) and such that no U contains two moves on the same play. This partition is called the information partition and its sets will be called information sets.

(III) For each $U \subset P_0 \cap A_j$, a probability distribution on the integers $1, \ldots, j$, which assigns positive probability to each. Such information sets are assumed to be one-element sets.

(IV) An n-tuple of real numbers $h(W) = (h_1(W), \ldots, h_n(W))$ for each play W. The function h will be called the pay-off function.

[How is this formal scheme to be interpreted? That is, how is a general n-person game played? To personalize the interpretation, one may imagine a number of people called agents isolated from each other and each in possession of the rules of the game. There is one agent for each information set and they are grouped into players in a natural manner, an agent belonging to the i^{th} player if his information set lies in P_i. This seeming plethora of agents is occasioned by the possibly complicated state of information of our players who may be forced by the rules to forget facts which they knew earlier in a play.[6]

A play begins at the vertex 0. Suppose that it has progressed to the move X. If X is a personal move with j alternatives then the agent whose information set contains X chooses a positive integer not greater than j, knowing only that he is choosing an alternative at one

[6] It has been asserted by von Neumann that Bridge is a two-person game in exactly this manner.

of the moves in his information set. If X is a chance move, then an alternative is chosen in accordance with the probabilities specified by (III) for the information set containing X. In this manner, a path with initial point 0 is constructed. It is unicursal and, since K is finite, leads to a unique play W. At this point, player i is paid the amount $h_i(W)$ for $i = 1, \ldots, n$. The case in which K reduces to the vertex 0 is not excluded. Then Γ is a no-move game, no one does anything, and the payoff is $h(0)$.

The price of the intuition gained with the use of a geometric model is the introduction of a certain amount of redundancy. Suppose Γ_1 and Γ_2 are two games defined with game trees K_1 and K_2 such that:

(1) K_1 and K_2 are homeomorphic.

(2) The homeomorphic mapping σ of K_1 onto K_2 preserves the distinguished vertex and the specifications (I) - (IV).

(3) On each information set, σ effects a permutation of the indices of the alternatives. (To make this more precise, (3) requires the existence of a permutation τ_U of the integers $1, \ldots, j$ for each information set $U \subset A_j$ such that, if ν_1 and ν_2 denote the functions indexing the alternatives in K_1 and K_2, $\nu_2(\sigma(e)) = \tau_U(\nu_1(e))$ for all alternatives e at moves in U.) It is clear that in such a case the games Γ_1 and Γ_2 should be considered equivalent and that a proper definition would define a general n-person game as an equivalence class under this equivalence relation. However, this distinction need not be emphasized; if care is taken to frame definitions which apply to the equivalence class as well as to the representative, one can ignore it entirely in proofs and remark that all of the theorems hold for either class or representative.

Although the majority of the above formalization needs no justification, ample motivation appearing in the book of von Neumann and Morgenstern, several features deserve comment. The first concerns the finiteness of the game tree. Since the rules of most games include a Stop Rule which only insures that every play terminates after a finite number of choices it is not completely obvious that this entails that the game tree be finite. To demonstrate this, following König,[7] we will assume that there are an infinite number of possible plays and then contradict the Stop Rule by constructing a unicursal path starting at 0 and containing an infinite number of edges. Since the choice at 0 is made from a finite set of alternatives, there must be an infinite number of plays with the same first edge e_1. We proceed by induction; assume that edges $e_1, \ldots, e_l$ have been chosen such that $e_1 \ldots e_l$ is the beginning segment

[7] König, D., Über eine Schlussweise aus dem Endlichen ins Unendliche, Acta Szeged 3 (1927), pp. 121-130.

of an infinite number of plays. Then, since the next choice is made from a finite set, an infinite subset of these plays must continue with the same edge, say e_{l+1}. This completes the proof.

A more crucial question is that of formalizing the state of information of a player at the occasion of a decision, i.e., at a move. If one examines the information given to a player by von Neumann's "patterns of information" it is found to consist of several parts. First of all, he is told that it is his move and is told the number of alternatives. In our terminology, this says that the move lies in $P_i \cap A_j$ for some fixed i and j. Secondly, he is told that the move lies on one of a certain set of plays and has been preceded by a fixed number of choices. From this he can deduce that his move lies in a set of moves which all have the same rank. It is this set of moves which forms a U in the information partition; however, we have weakened the requirement that all moves in U have the same rank to the condition that no U contains two moves on the same play.]

§ 2. COMPARISON WITH THE FORMULATION OF VON NEUMANN

[The object of this section is to clarify the relation between our "general n-person game" and a "von Neumann n-person game." Incidentally, it will be shown that the formulation given above is more general than von Neumann's but this is of minor importance beside the major purpose of throwing light on the points of agreement. The best way to do this seems to be to describe the method of transition from one extensive description to the other. Accordingly, we commence by deriving a "general n-person game" from a "von Neumann n-person game."[8]

Take as vertices of K the non-void subsets A_κ of the partitions α_κ, $\kappa = 1, \ldots, \nu + 1$. A vertex A_κ is joined to a vertex $A_{\kappa+1}$ by an edge if $A_{\kappa+1} = A_\kappa \cap C_\kappa$ for some C_κ. Remark that A_κ has j alternatives if it is contained in a D_κ which contains j sets C_κ and that the moves on K are the vertices A_κ, $\kappa = 1, \ldots, \nu$, while the plays are the vertices $A_{\nu+1}$. The player partition of the moves is defined by $P_k = \{A_\kappa | A_\kappa \subset \text{some } B_\kappa(k)\}$ for $k = 0, 1, \ldots, n$. The information partition of the moves is defined by $U = \{A_\kappa | A_\kappa \subset D_\kappa\}$ with one U defined for each $D_\kappa \in \mathfrak{D}_\kappa(k)$, $\kappa = 1, \ldots, \nu$ and $k = 1, \ldots, n$; the chance moves $A_\kappa \subset B_\kappa(0)$ form one-element sets in the information partition. For each $U \subset P_0 \cap A_j$, that is, for each $A_\kappa \subset B_\kappa(0)$, where A_κ contains j sets C_κ of $\mathcal{C}_\kappa(0)$, the probability assigned to the alternative corresponding to $A_\kappa \cap C_\kappa$ is $p_\kappa(C_\kappa)$. Finally, the functions h_k are defined on the

[8] For the comparison, the notation of von Neumann, loc. cit., pp. 73-75, is followed.

plays by $h_k(A_{\nu+1}) = \mathfrak{P}_k(A_{\nu+1})$ for all plays $A_{\nu+1}$ and $k = 1, \ldots, n$.

The imbedding has been left to the last, as it may be, since it is independent of the other specifications. Imbed 0 arbitrarily. Suppose the imbedding has progressed as far as the moves A_κ. We imbed the alternatives at all of the A_κ of a fixed U in the same order in the orientation. This is possible since each of these has the same number of alternatives (the number of C_κ contained in the fixed set D_κ which defines U).

There are two restrictive conditions which are fulfilled on the general n-person game thus obtained.

(A) All plays contain the same number of moves ν.

(B) All moves in a fixed information set U (defined by D_κ) have the same rank (κ).

Condition (A) is trivial and can be fulfilled in all of our games by filling out short plays with "dummy" chance moves with one alternative. Condition (B) is not trivial and will be discussed after the derivation of a "von Neumann n-person game" from a "general n-person game" which satisfies conditions (A) and (B).

(10:A:a) The number ν is the number given by condition (A).

(10:A:b) The finite set $\mathfrak{H}$ is the set of plays in K.

(10:A:c) For every $i = 1, \ldots, n$: The function $\mathfrak{P}_i(W) = h_i(W)$ for $W \in \mathfrak{H}$.

(10:A:d) For every $r = 1, \ldots, \nu$: The partition $\mathfrak{a}_r$ in $\mathfrak{H}$ contains one set A_r for each move X of rank r which is defined by $A_r = \{W | W > X\}$. The partition $\mathfrak{a}_{\nu+1}$ consists of the one-element sets $\{W\}$.

(10:A:e) For every $r = 1, \ldots, \nu$: The partition $\mathfrak{B}_r$ in $\mathfrak{H}$ contains one set $B_r(i)$ for each $i = 0, 1, \ldots, n$ which is defined by $B_r(i) = \{W | W > X$ where X is of rank r and $X \in P_i\}$.

(10:A:f) For every $r = 1, \ldots, \nu$ and every $i = 0, 1, \ldots, n$: The partition $\mathfrak{C}_r(i)$ in $B_r(i)$ contains one set C_r for each alternative e at an information set $U \subset P_i$ which is of rank r. It is defined by $C_r = \{W | W$ follows some $X \in U$ by the alternative $e\}$.

(10:A:g) For every $r = 1, \ldots, \nu$ and every $i = 1, \ldots, n$: The partition $\mathfrak{D}_r(i)$ in $B_r(i)$ contains one set D_r for each $U \subset P_i$ which contains only moves of rank r. It is defined by $D_r = \{W | W > X \in U\}$.

(10:A:h) For every $r = 1, \ldots, \nu$ and every $C_r \in \mathfrak{C}_r(0)$: The number $p_r(C_r)$ is the probability assigned to the alternative e by specification (III).

The proofs that the games thus derived actually satisfy the requirements placed on them (that is, by Definitions 1 and 2 in the first case and by (10:1:a) - (10:1:j) in the second) are omitted. If they are not always immediate, they are all straight-forward and easily obtained. The important question is: What happens if we derive a general n-person game from a von Neumann game again? The answer is quite obvious from the above, namely, the set Ω has been reduced to include only those plays which are allowed by the rules of the game. More precisely, Ω has been reduced so that (1) all partitions of a set are such that the union of the elements of the partition is exactly the set and (2) plays $\pi \in C_K \in \mathcal{C}_K(0)$ such that $p_K(C_K) = 0$ have been dropped. The discussion may be summarized by:

> THEOREM OF CATEGORIZATION: The von Neumann games, excluding illegal or impossible plays, are the general n-person games in which a player's information at the occasion of a decision includes the number of choices preceding that decision.

The following diagram concludes this section with an example[9] of a general n-person game in which a player is not informed of the number of choices preceding his move. The dotted lines enclose information sets.]

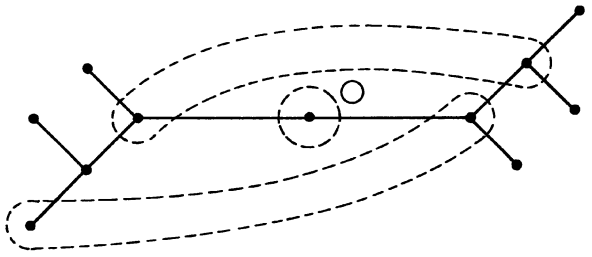


Figure 1.

§ 3. PURE AND MIXED STRATEGIES

DEFINITION 3. Let $\mathcal{U}_i = \{U | U \subset P_i\}$. A <u>pure strategy for player</u> i is a function π_i mapping

[9]This example, as well as substantial contributions to the interpretations, was communicated by L. S. Shapley and J. C. C. McKinsey.

$\mathcal{U}_i$ into the positive integers such that $U \subset A_j$ implies $\pi_i(U) \leq j$. We will say that π_i chooses the alternative e incident at $X \in U$ if $\pi_i(U) = \nu(e)$.

[The interpretation of a pure strategy is that it is a plan formulated before a play by a strategist for player i who then communicates his choices to the agents of player i. One may do this without violating the rules of the game regarding the information possessed by the agents by imagining that the strategist writes a book with one page for each of player i's information sets. If an information set has j alternatives, its page will contain a positive integer not greater than j and is given to the agent who is acting on that information set for player i. If his information set appears in a play then he is to choose the alternative indexed by the integer. This interpretation motivates the following definitions.]

Any n-tuple $\pi = (\pi_1, \ldots, \pi_n)$ of pure strategies for the n players defines a probability distribution on the alternatives in each of the information sets of K as follows:

If e is an alternative at a personal move for player i in the information set U: $p_\pi(e) = \begin{cases} 1 & \text{if } \pi_i(U) = \nu(e) \\ 0 & \text{otherwise.} \end{cases}$

If e is an alternative at a chance move: $p_\pi(e)$ is the probability assigned to $\nu(e)$ by (III). This, in turn, defines a probability distribution on the plays of K by: $p_\pi(W) = \prod_{e \in W} p_\pi(e)$ for all W.

[The interpretation is immediate; if the n strategists choose the pure strategies $\pi_1, \ldots, \pi_n$, then the probability that the play W will result is $p_\pi(W)$.]

DEFINITION 4. The expected payoff $H_i(\pi)$ to player i for pure strategies $\pi_1, \ldots, \pi_n$ is defined by $H_i(\pi) = \sum_W p_\pi(W) h_i(W)$ for $i = 1, \ldots, n$.

[Again redundancy is the price of a facile definition of pure strategies. Its nature is made clear once it is remarked that a pure strategy may make a choice at an early part of a game that makes many later moves impossible and hence renders the choices on those moves irrelevant. However, it can be viewed in another manner, namely, the efficacy of a strategy π_i is to be measured by the payoff H_i and hence two strategies should be considered equivalent if they produce the same payoff against all counter-strategies employed by the other players. Leaving the payoff function h arbitrary for the moment, this can be restated: two strategies

should be considered equivalent if they yield the same probability for every play against all counter-strategies employed by the other players. The rest of this section shows that these two views of the redundancy are the same and revises the definition of pure strategy accordingly.]

If $\pi = (\pi_1, \ldots, \pi_i, \ldots, \pi_n)$ we shall write π/π_i' for $(\pi_1, \ldots, \pi_i', \ldots, \pi_n)$.

DEFINITION 5. The pure strategies π_i and π_i' are equivalent, written $\pi_i \equiv \pi_i'$, if and only if $p_\pi(W) = p_{\pi/\pi_i'}(W)$ for all plays W and all π containing π_i.

DEFINITION 6. A personal move X for player i is called possible when playing π_i if there exists a play W and π containing π_i such that $p_\pi(W) > 0$ and $X \in W$. An information set U for player i is called relevant when playing π_i if some $X \in U$ is possible when playing π_i. We will denote the set of moves which are possible when playing π_i by Poss π_i and the family of information sets which are relevant when playing π_i by Rel π_i.

PROPOSITION 1. A move X for player i is possible when playing π_i if and only if π_i chooses all alternatives on the path W_X from 0 to X which are incident at moves of player i.

PROOF. Let X be possible when playing π_i; then there exists a play W containing X and π containing π_i such that $p_\pi(W) = \prod_{e \in W} p_\pi(e) > 0$. Hence it is clear that π_i chooses all of the alternatives for player i on W, thus certainly those of player i on W_X.

Now assume π_i chooses all the alternatives for player i on W_X. To demonstrate the possibility of X it is necessary to construct a play and strategies for the remaining players. Since no information set contains two moves on the same play, in constructing the strategies, choices on a unicursal path may be assigned independently. The choices for personal moves on W_X are assigned so as to conform to W_X. At X the choice is made by π_i; the construction of a play is continued by making choices arbitrarily except when they are specified by π_i. Call the resulting play W. The unspecified choices are made arbitrarily and the resulting pure strategies are called $\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n$. Then, since the probabilities of chance alternatives are positive,

$p_\pi(W) > 0$ and X is possible when playing π_i.

COROLLARY. Let the information set U contain the first move X for player i on a play W. Then U is relevant for all pure strategies π_i for player i.

THEOREM 1. The pure strategies π_i and π_i' are equivalent if and only if they define the same relevant information sets and coincide on these sets.

PROOF. Suppose $\pi_i \equiv \pi_i'$ and U is relevant when playing π_i. Then there exists a move $X \in U$, a play W, and π containing π_i such that:

$$p_\pi(W) > 0 \quad \text{and} \quad X \in W .$$

Hence $p_{\pi/\pi_i'}(W) = p_\pi(W) > 0$, and U is relevant for π_i'. Moreover, by the definition of $p(W)$, $\pi_i(U) = \pi_i'(U) = \nu(e)$ where e is the alternative at X which lies on W and thus π_i and π_i' coincide on U.

To show that the condition is sufficient, assume that a play W and π containing π_i are given. If $p_\pi(W) = \prod_{e \in W} p_\pi(e) > 0$ then $\pi_i(U) = \nu(e)$ for all alternatives e lying on W and incident at personal moves of player i in U. All of these moves are possible when playing π_i and hence the same conclusion holds for π_i'. But then

$$\prod_{e \in W} p_{\pi/\pi_i'}(e) = \prod_{e \in W} p_\pi(e) .$$

But, interchanging π_i and π_i' in this argument, $p_\pi(W) = 0$ implies $p_\pi(W) = 0$. Consequently, $\pi_i \equiv \pi_i'$.

[The interpretation of a pure strategy as a strategy book can be extended to the equivalence classes defined above by leaving blank those pages which correspond to irrelevant information sets.]

[Even the simplest games, e.g. Matching Pennies, reveal that a player is at a disadvantage if he uses the same pure strategy in each play. Instead he should randomize his choices. Two methods of randomization are described in this paper. In the first, the player uses a probability distribution on his pure strategies, choosing the particular pure strategy that he will employ in a given play of the game according to this distribution. Following von Neumann, it is called a mixed strategy. The second method is studied in Section 5.]

DEFINITION 7. A <u>mixed strategy for player</u> i, μ_i,

is a probability distribution on the pure strategies for player i, which assigns probability q_{π_i} to π_i.

Any n-tuple $\mu = (\mu_1, \ldots, \mu_n)$ of mixed strategies for the n players defines a probability distribution on the plays of K by:

$$p_\mu(W) = \sum_\pi q_{\pi_1} \cdots q_{\pi_n} p_\pi(W) \quad \text{for all } W .$$

DEFINITION 8. The expected payoff $H_i(\mu)$ to player i for mixed strategies $\mu_1, \ldots, \mu_n$ is defined by $H_i(\mu) = \sum_W p_\mu(W) h_i(W)$ for $i = 1, \ldots, n$.

PROPOSITION 2. For each move X, let $c(X)$ be the product of the chance probabilities at alternatives on W_X, the path from 0 to X. Then

$$p_\mu(X) = c(X) \sum_{\substack{X \in \text{Poss}\pi_i \\ i=1,\ldots,n}} q_{\pi_1} \cdots q_{\pi_n} = c(X) \prod_{i=1}^{n} \left(\sum_{X \in \text{Poss}\pi_i} q_{\pi_i} \right)$$

gives the probability that the move X will occur when the players play μ.

PROOF. This is an immediate consequence of Proposition 1 and the interpretation of a mixed strategy.

DEFINITION 9. A personal move X for player i is called possible when playing μ_i if there exists an n-tuple μ of mixed strategies containing μ_i such that $p_\mu(X) > 0$. An information set U for player i is called relevant when playing μ_i if some $X \in U$ is possible when playing μ_i. Again, we shall write Poss μ_i and Rel μ_i for the sets of X and U which are possible and relevant when playing μ_i.

§4. THE DECOMPOSITION OF GAMES

[It often occurs that the moves of a game which are subsequent to a fixed move X form a subgame in a natural manner. They constitute the vertices of a game tree with X as first move while the player partition, the probabilities at the chance moves, and the payoff on the plays in this tree carry over from the original game. This is also true of the information partition if, at every move of the original game, the player choosing

is informed whether or not his move is in the subgame.

If this happens, the moves which do not lie in the subgame also form the moves of a game which is determined up to the payoff at the vertex X (which is a play in this game!). In this section, this decomposition of a game into a pair of games will be studied to show that the equilibrium points of the pair determine the equilibrium points of the original game.]

DEFINITION 10. Given a move X in a game Γ, let K_X be the component of K which contains X if we delete the unique edge (if any) at X which is not an alternative at X. We shall say that Γ <u>decomposes at X</u> into Γ_X and $\Gamma_D(\mu_X)$ if every information set U either is contained in K_X or does not intersect K_X. The game Γ_X is called the <u>subgame</u> and is defined as follows:

The game tree is K_X; as such it is imbedded in the same oriented plane as K and has X as distinguished vertex.

(I_X) (II_X) The player and information partitions of the moves of Γ_X are the respective partitions of the moves of K restricted to K_X. The family of information sets for player i will be denoted by $\mathfrak{X}_i$.

(III_X) For each chance move in Γ_X the probability distribution is that specified by (III) in Γ.

(IV_X) The payoff function h_X for Γ_X is the payoff function h restricted to the plays of K_X.

A game $\Gamma_D(\mu_X)$, called a <u>difference game</u>, is defined <u>for each n-tuple of mixed strategies</u>, μ_X, in Γ_X. Its game tree is $K - K_X$ completed by X and has 0 as its distinguished vertex. The specifications (I_D) - (IV_D) are made as above with the additional definition that $h_D(X) = H_X(\mu_X)$; to emphasize the dependence of the payoff in $\Gamma_D(\mu_X)$ on μ_X, we shall write this payoff as $H_D(\mu_D,\mu_X)$. The family of information sets for player i will be denoted by $\mathfrak{Y}_i$.

[Corresponding to this natural decomposition of Γ into the subgame Γ_X and the difference game Γ_D, there is a natural decomposition of the pure strategies for Γ into pairs of pure strategies for Γ_X and Γ_D. The main burden of our proofs in this section lies in analyzing the effect on the payoff of this decomposition and the analagous splitting of mixed strategies for Γ.]

DEFINITION 11. Let the game Γ decompose at X. Then we shall say that a pure strategy π_i for player i decomposes at X into pure strategies $\pi_{X|i}$ and $\pi_{D|i}$ for player i in Γ_X and Γ_D if

(a) $\pi_{X|i}$ is the restriction of π_i from $\mathcal{U}_i$ to $\mathcal{X}_i$ and

(b) $\pi_{D|i}$ is the restriction of π_i from $\mathcal{U}_i$ to $\mathcal{D}_i$. Since $\mathcal{U}_i$ is the disjoint union of $\mathcal{X}_i$ and $\mathcal{D}_i$, we can also compose a pure strategy π_i from pure strategies $\pi_{X|i}$ and $\pi_{D|i}$ which will be denoted by $\pi_i = (\pi_{X|i}, \pi_{D|i})$.

LEMMA 1. If π_i decomposes into $\pi_{X|i}$ and $\pi_{D|i}$ for $i = 1, \ldots, n$, then

$$p_\pi(Y) = p_{\pi_D}(Y) \quad \text{for all} \quad Y \in K_D$$

and

$$p_\pi(Y) = p_{\pi_D}(X)p_{\pi_X}(Y) \quad \text{for all} \quad Y \in K_X$$

where $\pi = (\pi_1, \ldots, \pi_n)$, $\pi_X = (\pi_{X|1}, \ldots, \pi_{X|n})$, and $\pi_D = (\pi_{D|1}, \ldots, \pi_{D|n})$.

PROOF. The verification is immediate, using the definition of $p_\pi(Y)$ and the fact that all paths from 0 to a Y in K_X pass through X.

DEFINITION 12. Let the game Γ decompose at X. Then we shall say that a mixed strategy μ_i for player i decomposes at X into mixed strategies $\mu_{X|i}$ and $\mu_{D|i}$ for player i in Γ_X and Γ_D if

(a) $q_{\pi_{D|i}} = \sum_{D(\pi_i)=\pi_{D|i}} q_{\pi_i}$ for all $\pi_{D|i}$, where $D(\pi_i)$ denotes the restriction of π_i from $\mathcal{U}_i$ to $\mathcal{D}_i$.

(b) When $X \in \text{Poss}\ \mu_i$,

$$q_{\pi_{X|i}} = \sum_{\substack{X(\pi_i)=\pi_{X|i} \\ X\in\text{Poss}\pi_i}} q_{\pi_i} \Big/ \sum_{X\in\text{Poss}\pi_i} q_{\pi_i} \quad \text{for all } \pi_{X|i},$$

where $X(\pi_i)$ denotes the restriction of π_i from $\mathcal{U}_i$ to $\mathcal{X}_i$.

When $X \notin \text{Poss } \mu_i$,

$$q_{\pi_{X|i}} = \sum_{X(\pi_i)=\pi_{X|i}} q_{\pi_i} \quad \text{for all } \pi_{X|i} .$$

LEMMA 2. Every pair, $\mu_{X|i}$ and $\mu_{D|i}$, of mixed strategies for player i in Γ_X and Γ_D is obtained from the decomposition of some μ_i in Γ.

PROOF. Let $(\pi_{D|i},\pi_{X|i})$ denote the pure strategy for Γ obtained by composing $\pi_{D|i}$ and $\pi_{X|i}$. We then set

$$q_{(\pi_{D|i},\pi_{X|i})} = q_{\pi_{D|i}} q_{\pi_{X|i}} \quad \text{for all } \pi_{D|i} \text{ and } \pi_{X|i} .$$

Then (a) and (b) are easily verified once it is remarked that $X \in \text{Poss } \mu_i$ if and only if $X \in \text{Poss } \mu_{D|i}$. It should be noticed that our composition and decomposition of mixed strategies is a consistent extension of the original definitions for pure strategies.

THEOREM 2. If Γ decomposes at X then there is a mapping of the n-tuples μ of mixed strategies for Γ onto the pairs (μ_D,μ_X) of n-tuples of mixed strategies for Γ_D and Γ_X in such a way that

(1) $$H(\mu) = H_D(\mu_D,\mu_X)$$

if (μ_D,μ_X) corresponds to μ under the mapping.

PROOF. The mapping is the decomposition of Definition 10. Lemma 2 says that it is a mapping onto all pairs. To prove (1) we consider the members of the equation separately. First,

(2) $$H(\mu) = \sum_W p_\mu(W)h(W) = \sum_{W\in K-K_X} p_\mu(W)h(W) + \sum_{W\in K_X} p_\mu(W)h(W) .$$

Second,

(3) $$H_D(\mu_D,\mu_X) = \sum_{W\in K_D} p_{\mu_D}(W)h_D(W) = \sum_{W\in K-K_X} p_{\mu_D}(W)h_D(W) + p_{\mu_D}(X)H_X(\mu_X) .$$

Noting that, if $W \in K - K_X$,

$$p_\mu(W) = \sum_\pi q_{\pi_1} \cdots q_{\pi_n} p_\pi(W)$$

$$= \sum_{\pi_D} \Big(\sum_{D(\pi)=\pi_D} q_{\pi_1} \cdots q_{\pi_n} \Big) p_{\pi_D}(W)$$

$$= \sum_{\pi_D} q_{\pi_{D|1}} \cdots q_{\pi_{D|n}} p_{\pi_D}(W) = p_{\mu_D}(W)$$

and hence, comparing (2) and (3), we need only show

$$\sum_{W \in K_X} p_\mu(W) h(W) = p_{\mu_D}(X) H_X(\mu_X) \quad . \tag{4}$$

However, since

$$H_X(\mu_X) = \sum_{W \in K_X} p_{\mu_X}(W) h_X(W) = \sum_{W \in K_X} p_{\mu_X}(W) h(W) \quad ,$$

to prove (4) it is sufficient to show

$$p_\mu(W) = p_{\mu_D}(X) p_{\mu_X}(W) \quad \text{for all} \quad W \in K_X \quad . \tag{5}$$

(It should be remarked that this is the analogue of the second half of Lemma 2, stated for mixed strategies, and that our definition of the decomposition of mixed strategies has been framed intentionally to preserve this property.)
Noting that

$$p_{\mu_D}(X) = \sum_{\pi_D} q_{\pi_{D|1}} \cdots q_{\pi_{D|n}} p_{\pi_D}(X)$$

$$= c(X) \prod_{i=1}^{n} \Big(\sum_{X \in \mathrm{Poss} \pi_{D|i}} q_{\pi_{D|i}} \Big) = c(X) \prod_{i=1}^{n} \Big(\sum_{X \in \mathrm{Poss} \pi_i} q_{\pi_i} \Big)$$

where $c(X)$ is the product of the probabilities of chance alternatives on the path from 0 to X (the void product is taken to be unity) and

$$p_{\mu_X}(W) = \sum_{\pi_X} q_{\pi_{X|1}} \cdots q_{\pi_{X|n}} p_{\pi_X}(W)$$

$$= \sum_{\pi_X} \left\{ \prod_{i=1}^{n} \Big(\sum_{\substack{X(\pi_i)=\pi_{X|i} \\ X \in \mathrm{Poss} \pi_i}} q_{\pi_i} \Big/ \sum_{X \in \mathrm{Poss} \pi_i} q_{\pi_i} \Big) \right\} p_{\pi_X}(W)$$

we have

$$p_{\mu_D}(X)p_{\mu_X}(W) = c(X)\sum_{\pi_X}\left\{\prod_{i=1}^{n}\left(\sum_{\substack{X(\pi_i)=\pi_{X|i}\\ X\in \mathrm{Poss}\,\pi_i}} q_{\pi_i}\right)\right\} p_{\pi_X}(W)$$

$$= \sum_{\pi} q_{\pi_1}\cdots q_{\pi_n}\, p_{\pi_D}(X)p_{\pi_X}(W)$$

$$= \sum_{\pi} q_{\pi_1}\cdots q_{\pi_n}\, p_{\pi}(W) = p_{\mu}(W)\ .$$

This completes the proof.

[The basic consequence of Theorem 2 is that solutions for Γ can be composed from solutions to Γ_X and Γ_D if we take equilibrium points as our definition of a solution to an n-person game.]

DEFINITION 13. An n-tuple $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n)$ of mixed strategies for a game Γ is called an <u>equilibrium point</u>[10] if

$$H_i(\bar{\mu}) \geqq H_i(\bar{\mu}/\mu_i) \quad \text{for} \quad i = 1, \ldots, n$$

for all μ_i, where $\bar{\mu}/\mu_i$ denotes the n-tuple of mixed strategies obtained by replacing $\bar{\mu}_i$ by μ_i in μ.

THEOREM 3. Let Γ decompose at X, let $\bar{\mu}_X$ be an equilibrium point in the subgame Γ_X, and let $\bar{\mu}_D$ be an equilibrium point in $\Gamma_D(\bar{\mu}_X)$. If $\bar{\mu}$ is any n-tuple of mixed strategies for Γ which decomposes into $\bar{\mu}_X$ and $\bar{\mu}_D$, then $\bar{\mu}$ is an equilibrium point for Γ.

PROOF. Let μ_i be any mixed strategy for player i which decomposes into $\mu_{X|i}$ and $\mu_{D|i}$. Then it is clear that $\bar{\mu}/\mu_i$ decomposes into $\bar{\mu}_X/\mu_{X|i}$ and $\bar{\mu}_D/\mu_{D|i}$, and

[10] See J. Nash, "Non-cooperative games," Ann. of Math. 54 (1951), pp. 286-295.

$$H_i(\bar{\mu}) = H_{D|i}(\bar{\mu}_D,\bar{\mu}_X) \geqq H_{D|i}(\bar{\mu}_D/\mu_{D|i},\bar{\mu}_X)$$

$$= \sum_{W\in K-K_X} p_{\bar{\mu}_D/\mu_{D|i}}(W)h_{D|i}(W) + p_{\bar{\mu}_D/\mu_{D|i}}(X)H_{X|i}(\bar{\mu}_X)$$

$$\geqq \sum_{W\in K-K_X} p_{\bar{\mu}_D/\mu_{D|i}}(W)h_{D|i}(W) + p_{\bar{\mu}_D/\mu_{D|i}}(X)H_{X|i}(\bar{\mu}_X/\mu_{X|i})$$

$$= H_{D|i}(\bar{\mu}_D/\mu_{D|i},\bar{\mu}_X/\mu_{X|i}) = H_i(\bar{\mu}/\mu_i) \ .$$

[The computational consequences of Theorem 3 are clear; it is generally easier to solve two small games than one large game. We present two applications which derive directly or indirectly from this remark.]

(A) THE THEOREM OF ZERMELO - VON NEUMANN. It is well-known that a zero-sum two-person game with perfect information always has a saddle-point in pure strategies.[11] In our formalization, a game with perfect information is one in which all information sets are one element sets and a saddle-point is the zero-sum two-person specialization of the concept of an equilibrium point.

COROLLARY 1. A general n-person game Γ with perfect information always has an equilibrium point in pure strategies.

PROOF. The proof is an induction on the number of moves in Γ. For a game with no moves it is trivially true. For a game with one move it is immediate, since if that move is a personal move of player i then he should choose the alternative that maximizes his payoff, while if that move is a chance move then the theorem is again vacuously satisfied. For a game with m moves, since it assumed to have perfect information, it can be decomposed into two games with less than m moves. By the induction hypothesis, these have pure strategy equilibrium points whose composition is again a pure strategy and, by Theorem 3, an equilibrium point for Γ.

(B) SIMULTANEOUS GAMES. A class of games, introduced by G. Thompson as a natural extension of games with perfect information and called simultaneous games, can easily be solved through the use of Theorem 3. These are zero-sum two-person games which may be described verbally as consisting of a sequence of simultaneous moves by the two players; following each such move both are informed of the choices. Since our formal system does not handle simultaneous moves (even Matching Pennies has two successive

[11] E. Zermelo, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels, Proc. Fifth Int. Cong. Math., Cambridge (1912), Vol. II, p. 501.

moves) we must describe the games as follows: Player 1 has a_{2k-1} alternatives at the moves of rank $2k - 1$ and Player 2 has a_{2k} alternatives at the moves of rank $2k$ where $k = 1, \ldots, K$. Player 1 has perfect information at all of his moves, while Player 2 is informed at his moves of rank $2k$ of everything except the choices by Player 1 at the moves of rank $2k - 1$. Clearly we can decompose a simultaneous game at any move of Player 1.

§ 5. BEHAVIOR STRATEGIES

[In this section, another natural method of randomization is investigated. Using this method, a player chooses a probability distribution on the alternatives in each of his information sets, thus randomizing on the occasion of a choice as he knows it. It is explicitly assumed that the choices of alternatives at different information sets are made independently. Thus it might be reasonable to call them "uncorrelated" or "locally randomized" strategies; however, since these are the distributions that one would measure in attempting to observe the behavior of a player, we have called them behavior strategies.]

DEFINITION 14. To each $U \subset \mathcal{U}_i$ such that $U \subset A_j$, a behavior strategy for player i, β_i, assigns j non-negative numbers $b(U,\nu)$, $\nu = 1, \ldots, j$, such that $\sum_{\nu} b(U,\nu) = 1$.

Any n-tuple $\beta = (\beta_1, \ldots, \beta_n)$ of behavior strategies for the n players defines a probability distribution on the plays of K as follows:

If e is an alternative at a personal move $X \in U \in \mathcal{U}_i$ then $p_\beta(e) = b(U,\nu(e))$.

If e is an alternative at a chance move then $p_\beta(e)$ is the probability assigned to $\nu(e)$ by (III).

Finally, $p_\beta(W) = \prod_{e \in W} p_\beta(e)$.

DEFINITION 15. The expected payoff $H_i(\beta)$ to player i for behavior strategies $\beta_1, \ldots, \beta_n$ is defined by $H_i(\beta) = \sum_W p_\beta(W) h_i(W)$ for $i = 1, \ldots, n$.

[From our interpretation of behavior strategies, it is clear that each mixed strategy determines a behavior strategy. The next definition establishes this correspondence and the lemma following shows that we can achieve every behavior by some mixed strategy.]

DEFINITION 16. The behavior β_i of a mixed

strategy $\mu_i = (q_{\pi_i})$ for player i is a behavior strategy defined by:

If $U \in \text{Rel}\,\mu_i$ then

$$b(U,\nu) = \sum_{\substack{U \in \text{Rel}\pi_i \\ \pi_i(U)=\nu}} q_{\pi_i} \Big/ \sum_{U \in \text{Rel}\pi_i} q_{\pi_i} .$$

If $U \notin \text{Rel}\,\mu_i$ then

$$b(U,\nu) = \sum_{\pi_i(U)=\nu} q_{\pi_i} .$$

LEMMA 3. Given a behavior strategy β_i for player i, define a mixed strategy $\mu_i = (q_{\pi_i})$ for player i by:

(6) $$q_{\pi_i} = \prod_{U \in \mathcal{U}_i} b(U, \pi_i(U)) .$$

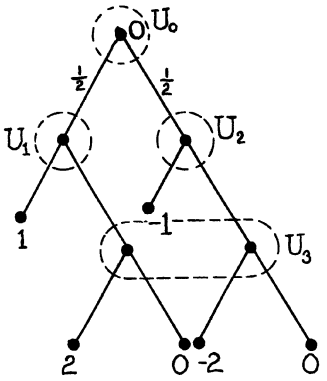
Then β_i is the behavior of μ_i.

PROOF. This lemma is a direct consequence of Definition 16 and (6).

[To apply these notions to a concrete example, consider the following game:

> A PARTNER GAME. In this zero-sum, two-person game, player 1 consists of two agents, called the Dealer and Partner, respectively. Two cards, one marked "High," the other "Low," are dealt to the Dealer and player 2 -- the two possible deals occuring with equal probabilities. The agent with the High card then receives one dollar from the agent with the Low card and has the alternatives of terminating or continuing the play. If the play continues, the Partner, not knowing the nature of the deal, can instruct the Dealer to change cards with player 2 or to hold his card. Again, the holder of the High card receives a dollar from the holder of the Low card.

In our formalization, this game is described by the following diagram (remark that the possible plays are labeled with the payoff to player 1).



$$\mathcal{U}_0 = \{U_0\},\ \mathcal{U}_1 = \{U_1, U_3\},\ \mathcal{U}_2 = \{U_2\}\ .$$

For simplicity's sake, we will denote the pure strategies π_1 for player 1 by $(\pi_1(U_1), \pi_1(U_3))$ and the pure strategies π_2 for player 2 by $(\pi_2(U_2))$. Then the game matrix of expectations $H_1(\pi_1, \pi_2)$ is:

	(1)	(2)
(1,1)	0	$-\frac{1}{2}$
(1,2)	0	$\frac{1}{2}$
(2,1)	$\frac{1}{2}$	0
(2,2)	$-\frac{1}{2}$	0

and the "solution" $q_{(1,1)} = q_{(2,2)} = 0$, $q_{(1,2)} = q_{(2,1)} = \frac{1}{2}$, and $q_{(1)} = q_{(2)} = \frac{1}{2}$ insures player 1 the expectation $\frac{1}{4}$ while player 2 can expect to lose no more than $\frac{1}{4}$. On the other hand, if we let $x = b(U_1, 1)$, $1 - x = b(U_1, 2)$ and $y = b(U_3, 1)$, $1 - y = b(U_3, 2)$ be the behavior strategies for player 1, we have:

$$\text{Player 1's expectation against } \pi_2 = \begin{cases} (1) \\ \\ (2) \end{cases} \text{ is } \begin{cases} -\frac{1}{2} + \frac{1}{2}x + y - xy \\ \\ \frac{1}{2}x \qquad - xy \end{cases}.$$

Hence the maximum amount that player 1 can assure himself is

$$\max_{\substack{0\leq x\leq 1\\ 0\leq y\leq 1}} \min \{-\tfrac{1}{2} + \tfrac{1}{2}x + y - xy,\ \tfrac{1}{2}x - xy\} = 0 .$$

Thus, behavior strategies may do a poorer job than mixed strategies. Remark that the mixed strategy $\mu_1 = (q_{(1,1)}, q_{(1,2)}, q_{(2,1)}, q_{(2,2)})$ has as its behavior $\beta_1 = (x,y) = (q_{(1,1)} + q_{(1,2)}, q_{(1,1)} + q_{(2,1)})$. Hence, if we consider the optimal mixed strategy $(0, \frac{1}{2}, \frac{1}{2}, 0)$ for player 1, the associated behavior is $x = y = \frac{1}{2}$ and, while the optimal mixed strategy assures 1 the amount $\frac{1}{4}$, even its associated behavior only assures 1 the amount 0. This discrepancy is due, of course, to the uncorrelated nature of the behavior strategy. To obtain positive results on the use of behavior strategies we must restrict the nature of the information partition.

DEFINITION 17. A game Γ is said to have <u>perfect recall</u> if $U \in \text{Rel}\ \pi_i$ and $X \in U$ implies $X \in \text{Poss}\ \pi_i$ for all U, X and π_i.

[The reader should verify that this condition is equivalent to the assertion that each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves. This obviates the use of agents; indeed the only games which do not have perfect recall are those, such as Bridge, which include the description of the agents in their verbal rules.]

LEMMA 4. Let Γ be a game with perfect recall for $i = 1, \ldots, n$. If i has a move on W, let the last alternative e for i on W be incident at $X \in U$ and set $T_i(W) = \{\pi_i \mid U \in \text{Rel}\ \pi_i$ and $\pi_i(U) = \nu(e)\}$; otherwise, $T_i(W)$ is the set of all π_i. Finally, let $c(W)$ be the product of the probabilities of the chance alternatives on W or 1 if there are none. Then, for all π and all W,

$$p_\pi(W) = \begin{cases} c(W) & \text{if } \pi_i \in T_i(W) \text{ for } i = 1, \ldots, n . \\ 0 & \text{otherwise .} \end{cases}$$

PROOF. It is clear that we need only show that $\pi_i \in T_i(W)$ implies that π_i chooses all of the alternatives for i on W (if any such exist). But $\pi_i \in T_i(W)$ implies $U \in \text{Rel}\ \pi_i$ and hence, since Γ has perfect recall, $X \in \text{Poss}\ \pi_i$. Therefore, by Proposition 1, π_i chooses all of the alternatives for i on W.

LEMMA 5. Let e be an alternative on a play W incident at $X \in U \in \mathcal{U}_i$ and let the next move for player i, if any, be $Y \in V$. Further, let

$$S = \{\pi_i \mid U \in \operatorname{Rel} \pi_i \text{ and } \pi_i(U) = \nu(e)\}$$

and

$$T = \{\pi_i \mid V \in \operatorname{Rel} \pi_i\} .$$

Then $S = T$.

PROOF. Let $\pi_i \in S$. Then $U \in \operatorname{Rel} \pi_i$ and hence, since Γ has perfect recall, $X \in \operatorname{Poss} \pi_i$ and therefore, by Proposition 1, π_i chooses all alternatives for i on the path 0 to X. But $\pi_i(U) = \nu(e)$ and hence π_i chooses all alternatives for i on the path 0 to Y. Therefore $Y \in \operatorname{Poss} \pi_i$, $V \in \operatorname{Rel} \pi_i$, and $\pi_i \in T$.

Let $\pi_i \in T$. Then $V \in \operatorname{Rel} \pi_i$ and hence, since Γ has perfect recall, $Y \in \operatorname{Poss} \pi_i$, and therefore $X \in \operatorname{Poss} \pi_i$ and $\pi_i(U) = \nu(e)$. That is, $\pi_i \in S$ and the lemma is proved.

THEOREM 4. Let β be the behavior associated with an arbitrary n-tuple of mixed strategies μ in a game Γ (in which all moves possess at least two alternatives). Then a necessary and sufficient condition that

$$H_i(\beta) = H_i(\mu) \quad \text{for} \quad i = 1, \ldots, n$$

and all μ and for all assignments of the payoff function h is that Γ have perfect recall.

PROOF. Assume that Γ has perfect recall; then we need only show

$$p_\beta(W) = p_\mu(W) \quad \text{for all} \quad W .$$

If there is an alternative e on W belonging to player i which is incident at a move in an irrelevant information set for μ_i then both sides are clearly zero. Hence we may assume that all such information sets are relevant for μ_i. Working with each side separately, first:

$$p_\beta(W) = \prod_{e \in W} p_\beta(e) .$$

Considering those alternatives e on W which belong to i, their probabilities are given by the fractions of Definition 16. The first denominator is clearly 1 while each numerator is the denominator of the next fraction by Lemma 5. Hence

$$p_\beta(W) = c(W) \prod_{i=1}^{n} \Big(\sum_{\pi_i \in T_i(W)} q_{\pi_i} \Big)$$

where $c(W)$ and the $T_i(W)$ are defined as in Lemma 4. On the other hand,

$$p_\mu(W) = \sum_{\pi} q_{\pi_1} \cdots q_{\pi_n} p_\pi(W)$$

$$= \sum_{\text{all} \pi_i \in T_i(W)} q_{\pi_1} \cdots q_{\pi_n} c(W)$$

by Lemma 4. Comparing the two expressions for $p(W)$, it is seen that the sufficiency is proved.

For the necessity, if Γ does not have perfect recall then there must be a pure strategy π_i and two moves X and Y in an information set U with $X \in \text{Poss } \pi_i$ and $Y \notin \text{Poss } \pi_i$. Choose a π_i' for which Y is shown to lie in $\text{Poss } \pi_i'$ by the play W and the n-tuple π'. If one sets $\mu_i = \frac{1}{2} \pi_i + \frac{1}{2} \pi_i'$, then

$$p_{\pi'/\mu_i}(W) = \tfrac{1}{2} c(W) .$$

However, there is an alternative e lying on the path from O to Y which is chosen by π_i' but not by π_i and thus is assigned probability $\frac{1}{2}$ by the behavior of μ_i. If we assume $\pi_i'(U) \neq \pi_i(U)$ then the behavior of μ_i assigns the probability $\frac{1}{2}$ to the alternative at Y on W. Hence

$$p_\beta(W) \leqq \tfrac{1}{4} c(W)$$

and the proof is completed.

EXAMPLES. To illustrate the efficacy of behavior strategies, it is possible to draw three examples from the literature. It is mere coincidence that they are all variants of Poker; the essential common property is that they all have perfect recall.

EXAMPLE 1. von Neumann and Morgenstern give a Poker example[12] in which the number of pure strategies for each player is 3^S, S being the number of possible "hands." Hence the dimension of the set of mixed

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[12] Loc. cit., pp. 190-196.

strategies is $3^S - 1$. However, the dimension of the set of behavior strategies is $2S$; when S is a large number the difference is considerable.

EXAMPLE 2. In the example given by the author,[13] domination arguments reduce the number of pure strategies from 27 to 8 for player 1 and from 64 to 4 for player 2. Nevertheless, the computation of solutions is still a tedious matter. With behavior strategies the payoff function has 3 variables for player 1 and 2 variables for player 2; moreover it has no terms of degree higher than 2 and so the computation of solutions is a mere exercise in elementary calculus.

EXAMPLE 3. In the Simple 3-Person Poker of Nash and Shapley,[14] domination arguments result in a game in which the three players have, respectively, 17, 19, and 31 dimensions of mixed strategies. However, they each have 5 dimensions of behavior strategies and this reduction makes possible the discovery of the unique equilibrium point for their game.

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[13] Kuhn, H., "A simplified two-person Poker," Annals of Math. Study 24 (1950), pp. 97-103.

[14] Nash, J. and Shapley, L. S., "A simple three-person poker game," Annals of Math. Study 24 (1950), pp. 105-116.

EQUIVALENCE OF INFORMATION PATTERNS AND ESSENTIALLY DETERMINATE GAMES[1]

Norman Dalkey

INTRODUCTION

In the first sections (1-5) we examine the equivalence of games in extensive form, using the model proposed by Kuhn.[2] There are several kinds of equivalence that might be explored, depending in part on what one considers reasonable methods of play. We are concerned with equivalence with respect to mixed strategies. A quite different notion of equivalence would be needed, for example, if the play were limited to behavior strategies.[2]

The notion of equivalence we evolve is only distantly related to the idea of strategic equivalence introduced by von Neumann and Morgenstern ([5], pp. 245 - 248). They are concerned mainly with variations in the payoff function which leave a solution invariant; we shall be concerned with variations in the structure of a game in extensive form which leave the major strategic properties of the game invariant irrespective of the pay-off function.[3]

A complete treatment of equivalence under variations in the structure of a game in extensive form is not given, but only equivalence under variations in the pattern of information. Roughly speaking two information patterns for the same player are equivalent if they differ at a given position in the game only in the knowledge which that player has of his own previous moves.

In the later sections, these results are applied to furnish a necessary and sufficient condition for a general game to have an equilibrium point in pure strategies independently of the particular pay-off function

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[1]The preparation of this paper was supported by The RAND Corporation.

[2]cf. Kuhn [2].

[3]The mode of approach is quite similar to that of Krental, Quine, McKinsey [1], and our results may be considered as an extension to general games of their results for two-person, zero-sum games.

or of the particular probability distributions assigned to chance moves. We call such games essentially determinate, since the question whether or not they will have an equilibrium point in pure strategies is completely determined by the information pattern alone.

The condition, which we have labelled effectively perfect information, is that at any move a player know all preceding moves of his opponents, and know at least as much as his opponents knew when they made those moves. In the special case that there are no chance moves, this condition is simply that the game be equivalent with respect to information to a game of perfect information.[4]

§ 2. GAMES IN EXTENSIVE FORM

We shall follow rather closely the definition of games in extensive form given by Kuhn, with some minor notational modifications.

DEFINITION 1. A general n-person game Γ in extensive form is defined by

(P1) A game tree, K, which is an ordered set of positions $\{x, y, z \ldots\}$.

(P2) An information pattern $\mathcal{U} = \{\underline{u}_0, \underline{u}_1, \ldots \underline{u}_n\}$ where each $\underline{u}_i = \{U, V, \ldots\}$ partitions a subset of K.

(P3) An n-component, real vector function $h = (h_1(w), h_2(w), \ldots, h_n(w))$ defined on a designated subset W of K.

(P4) A set function $p(\nu,U)$, defined on $U \in \underline{u}_i$, $\nu = 1, 2, \ldots, m(U)$, $0 < p(\nu,U) \leq 1$.

When we wish to indicate the dependency of Γ on these entities, we shall write

$$\Gamma = \Gamma(K,\mathcal{U},h,p) \ .$$

Rather than axiomatize this set of primitives, we shall follow Kuhn in giving them a geometrical interpretation.

K is a finite tree, embedded in an oriented Euclidean plane, with

[4] von Neumann and Morgenstern first proved that perfect information is a sufficient condition for a two-person zero-sum game to have a saddlepoint in pure strategies (see [4], Sec. 15). Kuhn extended this result to equilibrium points in pure strategies for general games. Shapley [5] gave a necessary and sufficient condition for a saddlepoint in pure strategies for a restricted class of two-person zero-sum games which is similar to the condition for general games we give below.

a distinguished vertex 0. The set W of end points of K are called plays, the remaining vertices moves. (We shall sometimes call both plays and moves by the common name positions.) The unique unicursal path leading from 0 to a play w will also be called a play, and represented by $\underline{w}$.

The m positions immediately succeeding a position x are indexed by positive integers $\nu = 1, 2, \ldots, m$ where m depends on x. x_ν will designate the ν'th position immediately following x. m(x) designates the total number of possible choices (alternatives) at x. The rank of x in K, i.e., the number of positions which preceed x, will be designated by r(x). D(x) (descendants of x) will represent the set of all positions that follow x, and $D(\nu,x)$ the set which follow by the ν'th alternative.

The information pattern $\mathcal{U}$ first partitions the moves of K into n+1 exclusive subsets, and then further subdivides each of these subsets into information sets. $\sum_{U\in\mathcal{U}_i} U = P_i$ are those positions where player i "has the move." $P = \{P_0, P_1, \ldots, P_n\}$ is called the player partition. At each $x \in U \in \mathcal{U}_i$, player i is informed that he is at one of the positions in U. For each information set U, the number of alternatives is the same for all positions in U, hence we can write m(U), the total number of alternatives available at any position in U. Information sets are further restricted by the rule that no information set intersects the same play more than once.

The set $\mathcal{U}_0$ is reserved for the "chance player," i.e., every $x \in P_0$ is a chance move. We allow $U \in \mathcal{U}_0$ to contain more than one move; this furnishes a convenient way of identifying the probability distributions at different chance moves.

$p(\nu,U)$ is assigned so that for $U \notin \mathcal{U}_0$, $p(\nu,U) = 1$ for every ν. For $U \in \mathcal{U}_0$, $0 < p(\nu,U) < 1$, $\sum_{\nu=1}^{m(U)} p(\nu,U) = 1$. Hence for chance moves, $p(\nu,U)$ assigns a probability distribution over the m possibilities. We set $p(\nu,x) = p(\nu,U)$, $x \in U$.

h is the pay-off function. To each play w it assigns an n-tuple of real numbers determining how much each player is to receive at that point.

We shall call a pair $(K,\mathcal{U})$ a game structure, and correspondingly $(K,\underline{\mathcal{U}})$ will be called the structure of $\Gamma(K,\underline{\mathcal{U}},h,p)$. $\underline{\mathcal{U}}_i$ will be called the information pattern for player i.

DEFINITION 2. A pure strategy for player i is a function $\pi_i(U)$ which maps each $U \in \mathcal{U}_i$ onto a positive integer $\nu \leq m(U)$. (We shall sometimes also use ρ_i, τ_i to denote pure strategies). We define the choice at a position x of a strategy π_i as $\pi_i(x) = \pi_i(U)$ where $x \in U$.

It is clear that a game structure $(K,\underline{\mathcal{U}})$ completely determines the set of all possible pure strategies (as well as mixed and behavior strategies) for each personal player.

Note that our definition specifies strategies for the chance player as well as for the personal players. A chance strategy corresponds to von Neumann and Morgenstern's "umpire's choice." ([4], p.81).

Let $\pi^* = (\pi_0, \pi_1, \pi_2, \ldots, \pi_n)$ designate an n+1-tuple of strategies, one for each player, and $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ an n-tuple of strategies, one for each personal player. It is clear that each π^* determines a unique play w, which we will designate by $w(\pi^*)$. Let $\pi^*(U) = \pi_i(U)$ where $U \in \underline{\mathcal{U}}_i$ and π_i is contained in π^*. We define $\underline{w}(\pi^*)$ recursively.

DEFINITION 3. $\underline{w}(\pi^*)$ is the set of positions such that

1. $0 \in \underline{w}(\pi^*)$
2. $x \in \underline{w}(\pi^*)$ implies $x_\nu \in \underline{w}(\pi^*)$ where $\pi^*(x) = \nu$

$w(\pi^*)$ is the final member of $\underline{w}(\pi^*)$. In case $x \in \underline{w}(\pi^*)$, π^* is said to <u>realize</u> x. More generally, if $\bar{\pi}$ is an m-tuple of strategies $m \leq n+1$, $\bar{\pi}$ is said to <u>realize</u> x if there is some π^* containing $\bar{\pi}$ and $x \in \underline{w}(\pi^*)$.

An immediate consequence of Definition 3 is

LEMMA 1. $w(\pi^*) = w(\rho^*)$ if and only if for every $x \in \underline{w}(\pi^*)$, $\pi^*(x) = \rho^*(x)$, or, equivalently, if and only if for every U that intersects $\underline{w}(\pi^*)$, $\pi^*(U) = \rho^*(U)$.

Let π^*/ρ_i designate the n-tuple which results when ρ_i is substituted for π_i in π^*; i.e., if $\pi^* = (\pi_0, \pi_1, \ldots, \pi_i, \ldots, \pi_n)$ then $\pi^*/\rho_i = (\pi_0, \pi_1, \ldots, \rho_i, \ldots, \pi_n)$.

THEOREM 1. $w(\pi^*/\rho_i) = w(\pi^*/\tau_j)$ implies

$$w(\pi^*) = w(\pi^*/\rho_i) = w(\pi^*/\tau_j) = w(\pi^*/\rho_i/\tau_j)$$

PROOF. If no $U \in \underline{\mathcal{U}}_i$ intersects $\underline{w}(\pi^*/\rho_i)$, then by Definition 3

$w(\pi^*/\rho_i)$ is independent of ρ_i, and hence $w(\pi^*)$ is independent of π_i; thus $w(\pi^*/\rho_i) = w(\pi^*)$. Similarly $w(\pi^*/\tau_j) = w(\pi^*/\tau_j/\rho_i)$. If $U \in \underline{\mathcal{U}}_i$ intersects $\underline{w}(\pi^*/\rho_i)$, then by hypothesis and Lemma 1, $\pi^*/\rho_i(U) = \pi^*/\tau_j(U)$, i.e., $\pi_i(U) = \rho_i(U)$. If no $V \in \underline{\mathcal{U}}_j$ intersects $\underline{w}(\pi^*/\tau_j)$, then $w(\pi^*/\tau_j)$ is independent of τ_j. If $V \in \underline{\mathcal{U}}_j$ intersects $\underline{w}(\pi^*/\tau_j)$, then $\pi^*/\rho_i(V) = \pi^*/\tau_j(V)$, i.e., $\pi_j(V) = \tau_j(V)$. Since for $U \in \underline{\mathcal{U}}_k$, $k \neq i$, $k \neq j$, $\pi^*(U) = \pi^*/\rho_i(U) = \pi^*/\tau_j(U)$, the equalities follow by Lemma 1.

Theorem 1 plays no role in the subsequent development. However, it illustrates a form of structural relationship among plays and strategies which is lost by a transformation to the normal form of a game.

In the same way that an n+1-tuple π^* of strategies defines a single play, an n-tuple of strategies π defines a subtree $K(\pi)$, where each branch point is a chance move. The end points of $K(\pi)$ are designated by $W(\pi) = \{w|$ there is a π_0 such that $w = w(\pi_0,\pi)\}$.

LEMMA 2. $W(\pi) = W(\rho)$ if and only if $w(\pi_0,\pi) = w(\pi_0,\rho)$ for every π_0.

PROOF. Sufficiency is immediate. Necessity: Assume $W(\pi) = W(\rho)$ but for some π_0, $w(\pi_0,\pi) \neq w(\pi_0,\rho)$. Then there is some τ_0 such that $w(\pi_0,\pi) = w(\tau_0,\rho)$. But then by Lemma 1, for every $U \in \underline{\mathcal{U}}_0$ that intersects $\underline{w}(\tau_0,\rho)$, $\tau_0(U) = \pi_0(U)$, and for every other U that intersects $\underline{w}(\tau_0,\rho)$, $\pi(U) = \rho(U)$. Whence $\underline{w}(\pi_0,\pi) = \underline{w}(\pi_0,\rho)$, a contradiction.

DEFINITION 4. Let $\underline{w}_x$ designate the subplay extending from 0 to x, then for $x \neq 0$, $p(x) = \prod_{y_\nu \in \underline{w}_x} p(\nu,y)$, $p(0) = 1$.

DEFINITION 5. The <u>expected pay-off</u> $H(\pi)$ for an n-tuple of pure strategies π is defined as

$$H(\pi) = \sum_{w \in W(\pi)} p(w)h(w) .$$

In case $\underline{w}$ contains no chance moves, $p(w) = 1$; otherwise $p(w)$ is the product of the probabilities of those alternatives at chance moves in $\underline{w}$ which lead to further members of $\underline{w}$. Note that $p(w)$ depends only on $p(\nu,U)$, $U \in \underline{\mathcal{U}}_0$ and is independent of the strategies of personal players; however, the relation holds

$$\sum_{w \in W(\pi)} p(w) = 1 .$$

§ 3. THE REDUCED NORMAL FORM OF A GAME

A list of pure strategies for each personal player of Γ and a specification of the expected pay-off $H(\pi)$ for each n-tuple of pure strategies is called by von Neumann and Morgenstern the normal form of Γ. When $H(\pi)$ is expressed in matrix notation as an n-dimensional array it is called the pay-off matrix of Γ.

In the transformation of games from extensive to normal form a certain kind of redundancy often appears, namely duplications in the pay-off matrix of rows or columns ("hyper-rows" in the case of n-person games.) We shall show below that this redundancy is generally the result of superfluous information on the part of one or more players. It is clear that a player loses no strategic advantages if duplications are deleted. This motivates the definition of a reduced normal form of a game. We first make precise the notion of duplication by defining an equivalence relation for strategies.

DEFINITION 6. $\pi_i \equiv \rho_i$ if and only if $H(\pi) = H(\pi/\rho_i)$ for every π containing π_i.

LEMMA 3. $\equiv$ is an equivalence relation (transitive, symmetrical and reflexive).

PROOF. Immediate from the definition.

Let s_i, called an equivalence strategy, designate an equivalence class of pure strategies for player i, and S_i the set of all such equivalence classes. The cartesian product $S = S_1 \times S_2 \times \ldots \times S_n$ denotes the set of all n-tuples of equivalence strategies. Let $s = (s_1, s_2, \ldots, s_n)$ denote a member of S. Let $H(s) = H(\pi)$ where $\pi \in s$.

DEFINITION 7. The reduced normal form of a game Γ is a list S_i of equivalence strategies for each personal player i, and a function $H(s)$ assigning an n-tuple of real numbers to each n-tuple of equivalence strategies.

$H(s)$, when expressed as an n-dimensional array, corresponds to a pay-off matrix where repititions of hyper-rows have been deleted.

§ 4. EQUIVALENCE OF GAMES

Under the presumption that mixed strategies are to be allowed, all strategical considerations with respect to a game Γ are summed up in

the reduced normal form. This leads us to consider two different games as being equivalent if their reduced normal forms are identical except for possible permutations of the lists of strategies.

DEFINITION 8. Γ is equivalent to Γ' -- in symbols, $\Gamma \equiv \Gamma'$ -- if and only if there is a one-one correspondence between S_i and S_i' for each $i \neq 0$, such that under this correspondence $H(s) = H'(s')$.

LEMMA 4. $\equiv$ is an equivalence relation for games.

PROOF. Immediate from the definition (since the one-one correspondences and equality of pay-off are equivalence relations).

§ 5. EQUIVALENCE OF INFORMATION PATTERNS

Definitions 6 and 8 characterize a kind of equivalence which is not very profound in the sense that it depends on the pay-off functions and the particular probability distributions at chance moves. A more revealing analysis is afforded if we deal with equivalences (to be called essential equivalences) that hold irrespective of the pay-off and probabilities at chance moves.

Consider two game structures $(K,\underline{\mathcal{U}})$, $(K',\underline{\mathcal{U}}')$ where $K = K'$ and $\underline{\mathcal{U}}_0 = \underline{\mathcal{U}}_0'$. For convenience, we shall say that p is identical with p' when $p(\nu,U) = p'(\nu,U')$ for every $U \in \underline{\mathcal{U}}_0$. This identification seems reasonable in light of the fact that for any $U \notin \underline{\mathcal{U}}_0$, $p(\nu,U) = 1$ for every ν. With this convention in mind, we define.

DEFINITION 9. Let $(K,\underline{\mathcal{U}})$, $(K',\underline{\mathcal{U}}')$ be two game structures where $K = K'$, $\underline{\mathcal{U}}_0 = \underline{\mathcal{U}}_0'$. $(K,\underline{\mathcal{U}})$ is said to be essentially equivalent to $(K',\underline{\mathcal{U}}')$ -- in symbols, $(K,\underline{\mathcal{U}}) \simeq (K',\underline{\mathcal{U}}')$ -- if and only if $\Gamma(K,\underline{\mathcal{U}},h,p) \equiv \Gamma(K',\underline{\mathcal{U}}',h,p)$ for every h and p for which $\Gamma(K,\underline{\mathcal{U}},h,p)$, $\Gamma(K',\underline{\mathcal{U}}',h,p)$ are games.

LEMMA 5. $\simeq$ is an equivalence relation for game structures.

PROOF. From Definition 9 and Lemma 4.

THEOREM 2. $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$ implies $P_i = P_i'$ for every i.

PROOF. The theorem states that the moves assigned to a given player by $\underline{\mathcal{U}}$ must be assigned to the corresponding player by $\underline{\mathcal{U}}'$. The theorem holds by assumption (Definition 9) for P_0, P_0'. Suppose $P_i \neq P_i'$ for some i. Then there is an $x \in P_i$ such that $x \in P'_k$, $k \neq i$. We may assume that there are at least two alternatives at x (otherwise x is a trivial move and can be eliminated). Let $x \in U \in \underline{\mathcal{U}}_i$ and $x \in V \in \underline{\mathcal{U}}'_k$. Let h be as follows:

1) $h_j(w) = 0$ for all w and for all $j \neq i$
2) $h_i(w) = 0$ for all w such that $x_1 \notin \underline{w}$ and $x_2 \notin \underline{w}$.
3) $h_i(w) = 1$ for all w such that $x_1 \in \underline{w}$
4) $h_i(w) = -1$ for all w such that $x_2 \in \underline{w}$.

With this h it is immediately clear that we can never have $H_i(\pi) > 0$, $H_i(\rho) < 0$ where $\pi_i = \rho_i$. Let π_k' and ρ_k' be strategies for player k in $(K,\underline{\mathcal{U}}')$ such that $\pi_k'(V) = 1$ and $\rho_k'(V) = 2$, and both π_k' and ρ_k' realize x. Let τ be any n-tuple of strategies that realizes x. $H_i(\tau/\pi_k') > 0$, $H_i(\tau/\rho_k') < 0$, hence there is no equivalence strategy in $\Gamma(K,\underline{\mathcal{U}},h,p)$ corresponding to the s_i' in $\Gamma(K,\underline{\mathcal{U}}',h,p)$ containing τ_i.

Theorem 2 assures that there is no loss of generality in specializing to the case $P = P'$. For the next step, we specialize even further, and consider the case of a fixed (but arbitrary) information pattern $\underline{\mathcal{U}} - \underline{\mathcal{U}}_i$ for all players but one, and examine the effect of varying $\underline{\mathcal{U}}_i$. In this uncomplicated case it is convenient to overlook the fact that we are dealing with two different game structures, and identify all the components, except the information pattern and strategies for the player i. With this convention it becomes meaningful to write, for example, π/ρ_i', where π is an n-tuple of strategies originally defined for $(K,\underline{\mathcal{U}})$ and ρ_i' is a strategy for player i in $(K',\underline{\mathcal{U}}')$. This convention has the virtue that it saves most of the labor of trivial proofs of one-one correspondences.

DEFINITION 10. Let $(K,\underline{\mathcal{U}})$, $(K,\underline{\mathcal{U}}')$ be two game structures where $\underline{\mathcal{U}} - \underline{\mathcal{U}}_i = \underline{\mathcal{U}}' - \underline{\mathcal{U}}_i'$. A pure strategy π_i for player i in $(K,\underline{\mathcal{U}})$ is said to be <u>essentially equivalent</u> to a pure strategy π_i' for player i in $(K,\underline{\mathcal{U}}')$ -- in symbols $\pi_i \simeq \pi_i'$ -- when $H(\pi) = H(\pi/\pi_i')$ for every π containing π_i and every h and p for which $\Gamma(K,\underline{\mathcal{U}},h,p)$, $\Gamma(K,\underline{\mathcal{U}}',h,p)$ are games.

LEMMA 6. Let $(K,\underline{\mathcal{U}})$ and $(K,\underline{\mathcal{U}}')$ be as in Definition 10. Then $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$ if and only if for every pure strategy π_i for player i in $(K,\underline{\mathcal{U}})$ there is a strategy π_i' for player i in $(K,\underline{\mathcal{U}}')$ such that $\pi_i \simeq \pi_i'$ and vice versa.

PROOF. Sufficiency. If $\tau_j \equiv \rho_j$, $j \neq i$, in $\Gamma(K,\underline{\mathcal{U}},h,p)$ then $\tau_j \equiv \rho_j$ in $\Gamma(K,\underline{\mathcal{U}}',h,p)$. For, assume the contrary, then $H(\pi) = H(\pi/\rho_j)$ for every π containing τ_j, but for some π' containing τ_j, $H(\pi') \neq H(\pi'/\rho_j)$. Let π_i' be the strategy of the i'th player in π'. By hypothesis there is a π_i such that $H(\pi) = H(\pi/\pi_i')$ for every π containing π_i; in particular $H(\pi'/\pi_i) = H(\pi')$. But by hypothesis $H(\pi'/\pi_i) = H(\pi'/\pi_i/\rho_j)$, since π' contains τ_j and $\tau_j \equiv \rho_j$. But also by hypothesis $H(\pi'/\pi_i/\rho_j) = H(\pi'/\pi_i/\rho_j/\pi_i') = H(\pi'/\rho_j)$. Whence $H(\pi') = H(\pi'/\rho_j)$, which is a contradiction. Similarly, $\tau_j \equiv \rho_j$ in $\Gamma(K,\underline{\mathcal{U}}',h,p)$ implies $\tau_j \equiv \rho_j$ in $\Gamma(K,\underline{\mathcal{U}},h,p)$. Whence, the equivalence strategies S_j in $\Gamma(K,\underline{\mathcal{U}},h,p)$ for $j \neq i$ correspond one-for-one with the equivalence strategies S_j' in $\Gamma(K,\underline{\mathcal{U}}',h,p)$. We define a correspondence $s_i \leftrightarrow s_i'$ by the rule: if $\pi_i \in s_i$ then $\pi_i' \in s_i'$ where $\pi_i' \simeq \pi_i$. Clearly, if $\rho_i \equiv \pi_i$, then for every $\rho_i' \simeq \rho_i$, $\rho_i' \equiv \pi_i'$, whence the correspondence is one-one. The equality $H(s) = H(s')$ for this set of correspondences follows immediately from the definition of $\simeq$ for strategies.

Necessity. Suppose that there is a π_i in $(K,\underline{\mathcal{U}})$ such that there is no π_i' in $(K,\underline{\mathcal{U}}')$ for which $\pi_i \simeq \pi_i'$. Then there is some h and p and some π containing π_i such that $H(\pi) \neq H(\pi/\pi_i')$. This violates the assumption that $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$.

THEOREM 3. Let $(K,\underline{\mathcal{U}})$, $(K,\underline{\mathcal{U}}')$ be two game structures with $\underline{\mathcal{U}} - \underline{\mathcal{U}}_i = \underline{\mathcal{U}}' - \underline{\mathcal{U}}_i'$. The following are necessary and sufficient conditions for $\pi_i \simeq \rho_i$, where π_i is a strategy for player i in $(K,\underline{\mathcal{U}})$ and ρ_i is a strategy for player i in $(K,\underline{\mathcal{U}}')$.

1) $W(\pi) = W(\pi/\rho_i)$ for every π containing π_i.

2) $w(\pi^*) = w(\pi^*/\rho_i)$ for every π^* containing π_i.

PROOF. (i) 1) implies $\pi_i \simeq \rho_i$. This is immediate, since if $W(\pi) = W(\pi/\rho_i)$ then $H(\pi) = H(\pi/\rho_i)$ for every π. (ii) $\pi_i \simeq \rho_i$ implies 2). Suppose there is some π^* containing π_i such that $w(\pi^*) \neq w(\pi^*/\rho_i)$. Then there is no τ^* such that $w(\tau^*/\rho_i) = w(\pi^*)$, otherwise, by Definition 3, $\tau^*/\rho_i(x) = \pi^*(x)$ for every $x \in \underline{w}(\pi^*)$ and hence, by Definition 3 again $w(\tau^*/\rho_i) = w(\pi^*/\rho_i) = w(\pi^*)$. Let p be arbitrary and set $h_i(w) = 0$ for every $w \neq w(\pi^*)$, and set $h_i(w(\pi^*)) = 1$. Then $H_i(\pi) > 0$ for some π containing π_i, whereas $H_i(\pi/\rho_i) = 0$. (iii) 2) implies 1). Immediate.

DEFINITION 11. A set of positions B (not necessarily an information set) is said to be realizable by π_i if there is a π^* containing π_i and B intersects $\underline{w}(\pi^*)$. Let $U \in \underline{\mathcal{U}}_i$ be an information set and B

a subset of U. B is said to be isolated in U when, for every π_i, if B is realizable by π_i, then U - B is not realizable by π_i.

LEMMA 7. Let U_ν designate the set of positions which follow any move in U by choice of the ν'th alternative.. B is isolated in $U \in \underline{\mathcal{U}}_i$ if and only if for every $x \in B$, $y \in U - B$, there is a $V \in \underline{\mathcal{U}}_i$ such that $x \in V_\nu$, $y \in V_\eta$ and $\nu \neq \eta$.

PROOF. Sufficiency. Consider any π_i, and assume π_i realizes some $x \in B$ and $y \in U - B$. By hypothesis, there is a $V \in \underline{\mathcal{U}}_i$ such that $x \in V_\nu$, $y \in V_\eta$, $\nu \neq \eta$. But if π_i realizes x, then $\pi_i(V) = \nu$ by Definition 3, whereas if π_i realizes y, then $\pi_i(V) = \eta$, which contradicts the definition of strategy.

Necessity. Assume that for some $x \in B$, $y \in U - B$, for every $V \in \underline{\mathcal{U}}_i$ such that $x \in V_\nu$, $y \in V_\eta$ then $\nu = \eta$. Let π_i realize x and ρ_i realize y. We construct a τ_i so that for every $V \in \underline{\mathcal{U}}_i$ such that $\underline{w}_x$ intersects V, $\tau_i(V) = \pi_i(V)$ and for every $V \in \underline{\mathcal{U}}_i$ such that $\underline{w}_y$ intersects V, $\tau(V) = \rho_i(V)$. Then τ_i realizes both x and y, hence B is not isolated in U.

DEFINITION 12. a) Let $(K,\underline{\mathcal{U}})$, $(K,\underline{\mathcal{U}}')$ be two game structures, where $P_i = P_i' \cdot \underline{\mathcal{U}}_i$ is said to be an immediate inflation of $\underline{\mathcal{U}}_i'$ when there is a $V \in \underline{\mathcal{U}}_i'$ and $U_1, U_2 \in \underline{\mathcal{U}}_i$ such that $\underline{\mathcal{U}}_i - \{U_1, U_2\} = \underline{\mathcal{U}}_i' - \{V\}$ and U_1, U_2 are isolated in V.

b) $\underline{\mathcal{U}}_i$ is an inflation of $\underline{\mathcal{U}}_i'$ when there is a finite sequence $\underline{\mathcal{V}}_i^1, \underline{\mathcal{V}}_i^2, \ldots, \underline{\mathcal{V}}_i^l$ such that $\underline{\mathcal{U}}_i = \underline{\mathcal{V}}_i^1$ and $\underline{\mathcal{U}}_i' = \underline{\mathcal{V}}_i^l$ and $\underline{\mathcal{V}}_i^{j+1}$ is an immediate inflation of $\underline{\mathcal{V}}_i^j$, $j = 1, 2, \ldots, l - 1$.

c) $\underline{\mathcal{U}}_i$ is completely inflated when there is no $\underline{\mathcal{V}}_i$ such that $\underline{\mathcal{V}}_i$ is an inflation of $\underline{\mathcal{U}}_i$.

d) $\underline{\mathcal{U}}_i$ is a complete inflation of $\underline{\mathcal{U}}_i'$ if $\underline{\mathcal{U}}_i$ is an inflation of $\underline{\mathcal{U}}_i'$ and $\underline{\mathcal{U}}_i$ is completely inflated.

e) $\underline{\mathcal{U}}$ is an inflation (complete inflation) of $\underline{\mathcal{U}}'$ when $\underline{\mathcal{U}}_i$ is an inflation (complete inflation) of $\underline{\mathcal{U}}_i'$ for every i.

THEOREM 4. Let $(K,\underline{\mathcal{U}})$, $(K,\underline{\mathcal{U}}')$ be two game structures where $\underline{\mathcal{U}} - \underline{\mathcal{U}}_i = \underline{\mathcal{U}}' - \underline{\mathcal{U}}_i'$, and $\underline{\mathcal{U}}_i$ is an immediate inflation of $\underline{\mathcal{U}}_i'$, then $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$.

PROOF. Let $V \in \underline{\mathcal{U}}_i'$, $U_1, U_2 \in \underline{\mathcal{U}}_i$ be the sets required by Definition 12a. Let π_i' be any strategy for player i in $(K,\underline{\mathcal{U}}')$. Let π_i be the strategy in $(K,\underline{\mathcal{U}})$ such that for every $V' \neq V$, $\pi_i(V') = \pi_i'(V')$ and $\pi_i(U_1) = \pi_i(U_2) = \pi_i'(V)$, then $\pi_i \simeq \pi_i'$; for, let π^* contain π_i, then $w(\pi^*) = w(\pi^*/\pi_i')$ by Lemma 1, and hence $\pi_i \simeq \pi_i'$ by Theorem 3. Consider any π_i in $(K,\underline{\mathcal{U}})$. If π_i does not realize V, then there is a π_i', $\pi_i'(U) = \pi_i(U)$ for $U \neq V$, and by Lemma 1 $\pi_i \simeq \pi_i'$. If π_i realizes V, then, since U_1 and U_2 are isolated in V, π_i can realize either U_1 or U_2, but not both. Let it realize U_1 and let $\pi_i'(U) = \pi_i(U)$ for every $U \neq V$, and $\pi_i'(U_1) = \pi_i(V)$. Since, by Lemma 1, π_i' does not realize U_2, $\pi_i' \simeq \pi_i$. Whence, the theorem follows by Lemma 6.

COROLLARY 4a. If $(K,\underline{\mathcal{U}})$, $(K,\underline{\mathcal{U}}')$ are identical game structures except that $\underline{\mathcal{U}}_i$ is an inflation of $\underline{\mathcal{U}}_i'$ for some i, then $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$.

PROOF. Theorem 4 and Lemma 5.

COROLLARY 4b. If $(K,\underline{\mathcal{U}})$, $(K,\underline{\mathcal{U}}')$ are game structures such that for every i either $\underline{\mathcal{U}}_i = \underline{\mathcal{U}}_i'$ or $\underline{\mathcal{U}}_i$ is an inflation of $\underline{\mathcal{U}}_i'$ or $\underline{\mathcal{U}}_i'$ is an inflation of $\underline{\mathcal{U}}_i$, then $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$.

PROOF. By repeated applications of Corollary 4a.

THEOREM 5. If $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$ where $\underline{\mathcal{U}} - \underline{\mathcal{U}}_i = \underline{\mathcal{U}}' - \underline{\mathcal{U}}_i'$, and both $\underline{\mathcal{U}}$ and $\underline{\mathcal{U}}'$ are completely inflated, then $\underline{\mathcal{U}}_i = \underline{\mathcal{U}}_i'$. (Two equivalent completely inflated information patterns are identical.)

PROOF. Let $(K,\underline{\mathcal{U}})$, $(K,\underline{\mathcal{U}}')$ be as in the hypothesis, and assume $\underline{\mathcal{U}}_i \neq \underline{\mathcal{U}}_i'$. There is no loss of generality in assuming that there is a $V \in \underline{\mathcal{U}}_i'$ and $U_1, U_2 \in \underline{\mathcal{U}}_i$ such that both U_1 and U_2 intersect V. Consider any strategy π_i' in $(K,\underline{\mathcal{U}}')$. π_i' cannot realize both $V \cap U_1$ and $V \cap U_2$, for suppose it did; then there is a π_i in $(K,\underline{\mathcal{U}})$ such that $\pi_i \simeq \pi_i'$, hence π_i realizes both $U_1 \cap V$ and $U_2 \cap V$. We define a ρ_i such that for every $U' \in \underline{\mathcal{U}}_i$, $U' \neq U_1$ and $U' \neq U_2$, $\rho_i(U') = \pi_i(U')$ and $\rho_i(U_1) \neq \rho_i(U_2)$. Now ρ_i realizes both $U_1 \cap V$ and $U_2 \cap V$, and there is no ρ_i' in $(K,\underline{\mathcal{U}}')$ such that $\rho_i' \simeq \rho_i$, since there is some π^* containing ρ_i such that $w(\pi^*) \in V_\nu$ and another $\bar{\pi}^*$ containing ρ_i such that $w(\bar{\pi}^*) \in V_\eta$. Since U_1 and U_2 are any $U \in \underline{\mathcal{U}}_i$ that intersect V, each is isolated in V, which contradicts the assumption that $\underline{\mathcal{U}}_i'$ is completely inflated.

COROLLARY 5a. An information pattern $\underline{\mathcal{U}}_i$ for player i has a unique complete inflation.

PROOF. Immediate from Theorems 4 and 5.

THEOREM 6. $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$ where $\underline{\mathcal{U}} - \underline{\mathcal{U}}_i = \underline{\mathcal{U}}' - \underline{\mathcal{U}}_i'$ if and only if the complete inflation of $\underline{\mathcal{U}}_i$ is identical with the complete inflation of $\underline{\mathcal{U}}_i'$.

PROOF. Corollary 5a assures that the unique complete inflations of $\underline{\mathcal{U}}_i$ and $\underline{\mathcal{U}}_i'$ exist. Sufficiency follows from Theorem 4, necessity from Theorem 5.

COROLLARY 6a. $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$ (where $\underline{\mathcal{U}}_o = \underline{\mathcal{U}}_o'$) if and only if the complete inflation of $\underline{\mathcal{U}}_j$ is identical with the complete inflation of $\underline{\mathcal{U}}_j'$ for every j.

PROOF. Theorem 6 and Corollary 4b.

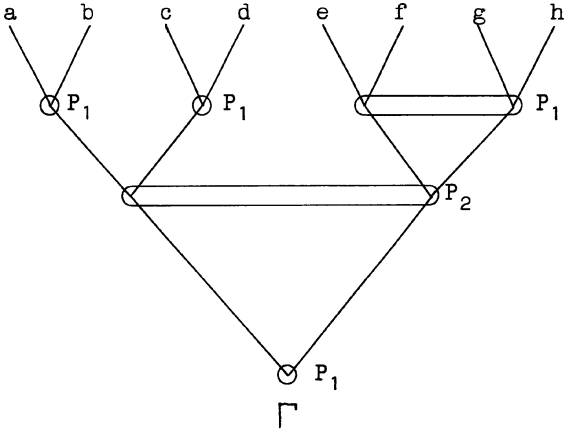
REMARK. The rather tedious route whereby we arrive at Theorem 6 and Corollary 6a can be bypassed intuitively by the following heuristic discussion. Theorem 4, which is not used directly in the proof, indicates that inflation is essentially a process of adding to a player's information at a given move, some further knowledge about his own previous moves (where this additional information has no implications for previous moves of other players.) But if a player is proceeding according to a preformulated pure strategy, and has complete information about the structure of the game, then at any move where it is his turn to play, his strategy will tell him what previous moves he has made. Hence, information solely about his own previous moves is superfluous. Presumably, a player's memory of a strategy will be equally complete whether it results from a direct choice (pure strategy in the strict sense) or from a prior selection of a pure strategy by some chance device (mixed strategy). However, it is clear that if the player is using behavior strategies -- i.e. mixes by some chance selection at each of his moves -- there is no guarantee that he will remember all previous selections. This is true particularly where a player is represented by a team of persons, each assigned to different information sets. For this reason -- as is readily verifiable -- Theorem 6 does not hold for behavior strategies.

§ 6. DEFLATION

So far our emphasis has been on the inflation of information patterns. However, from the practical point of view -- i.e., from the point of view of simplifying games -- inflation is not as interesting as its converse, deflation.

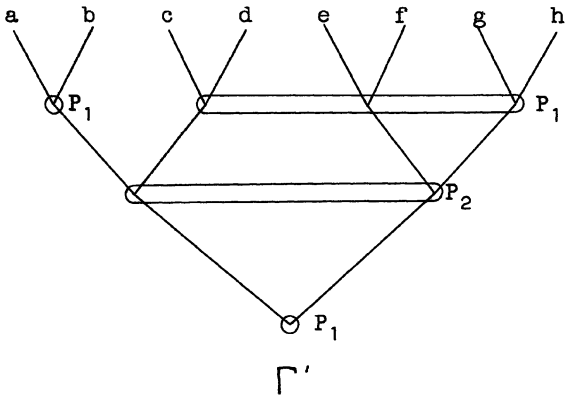
To illustrate the order of reduction in size of pay-off matrices by deflation, in tic-tac-toe the first player has roughly $10^{31,000}$ pure strategies (if we ignore the effect of the stop-rule and assume that each play of the game results in completely filling the open squares.) In essentially reduced form, the first player has only 10^{127} pure strategies. Although this second figure is still ridiculously large, it is clear that a tremendous reduction in size of the game is achieved. The pleasant feature is that this reduction can be carried out by operations directly on the game structure, without first constructing a cosmic sized matrix and then eliminating duplications.

Unfortunately, we cannot assert that every information pattern has a unique complete deflation. Consider the game Γ illustrated below. The ellipses represent information sets for the indicated players.

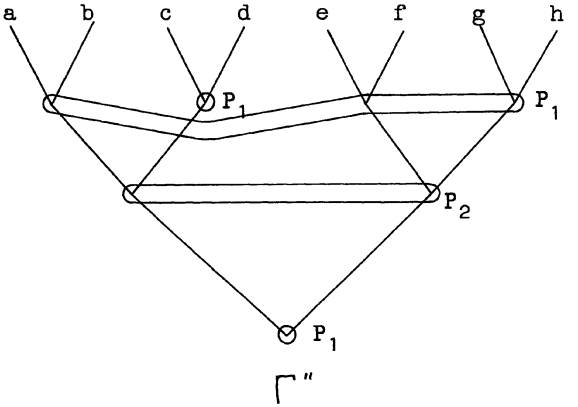


Γ

As it stands $\underline{u}_1$ is inflated; player 1 at his second move knows which first move he made. In fact, $\underline{u}_1$ is an inflation of $\underline{u}_1'$, $\underline{u}_1''$ in Γ' and Γ'' respectively.



Γ'



Γ''

Both $\underline{u}_1'$ and $\underline{u}_1''$ are completely deflated. It is easy to verify that in this case there remains a certain amount of redundancy in the information pattern, in the sense that in both Γ' and Γ'' there are two or more strategies that are essentially equivalent.

§ 7. ESSENTIALLY DETERMINATE GAMES

In the preceeding sections we were guided by a more or less intuitive notion of equivalence, based on the presumption that the normal form of a game completely exhausts the strategic possibilities. It would be desirable to base our investigations on a set of rules for playing a game in a rational fashion. Unfortunately at present the rules for rational play can be considered as being in a satisfactory state only for one-person and two-person zero-sum games. There is, however, a form of weak solution

for general games that has been suggested by Nash [3] which at least can be used for exploring some of the properties of information patterns, and which, in the zero-sum two-person case reduces to the accepted mini-max rule. This is the notion of equilibrium point. Since we have defined only pure strategies above, we limit ourselves to equilibrium points in pure strategies.

DEFINITION 13. A pure strategy n-tuple π is an equilibrium point if and only if $H_i(\pi) \geq H_i(\pi/\rho_i)$ for every i and every ρ_i.

DEFINITION 14. $(K,\underline{\mathcal{U}})$ is said to be essentially determinate if $\Gamma(K,\underline{\mathcal{U}},h,p)$ has an equilibrium point in pure strategies for every h and p for which $\Gamma(K,\underline{\mathcal{U}},h,p)$ is a game.

LEMMA 8. If $(K,\underline{\mathcal{U}}) \simeq (K,\underline{\mathcal{U}}')$ then $(K,\underline{\mathcal{U}})$ is essentially determinate if and only if $(K,\underline{\mathcal{U}}')$ is essentially determinate.

PROOF. Immediate from the definitions of $\simeq$ and essentially determinate.

Kuhn has shown [2] that a sufficient condition for a game structure $(K,\underline{\mathcal{U}})$ to be essentially determinate is that $(K,\underline{\mathcal{U}})$ have perfect information.[5]

DEFINITION 15. $(K,\underline{\mathcal{U}})$ is said to have perfect information if every information set U is a unit set.

It is clear that no essential modification results if we restrict the information sets in Definition 15 to personal information sets.

It is easy to find examples of games which are essentially determinate and which do not have perfect information. In fact, any one-person game is essentially determinate irrespective of the nature of the information pattern (see Lemma 10 below). Hence, perfect information is not a necessary condition for essential determinateness.

Let $U < V$ mean there is a play $\underline{w}$ which intersects both U and V and the intersection of U and $\underline{w}$ preceeds the intersection of V

[5] Strictly speaking, he proved something slightly different from this; but the step to showing essential determinateness is trivial, amounting, in fact, only to pointing out that both h and p can be chosen arbitrarily.

and $\underline{w}$. We shall also use the notation occasionally $x < U$, which means there is a play $\underline{w}$ containing x that intersects U, and x preceeds the intersection of U and $\underline{w}$.

DEFINITION 16. $(K,\underline{\mathcal{U}})$ is said to have <u>effectively perfect information</u> when, for every pair of personal information sets U, V such that $U \in \underline{\mathcal{U}}_i$, $V \in \underline{\mathcal{U}}_k$, $k \neq i$, if $U < V$ then $V \subset U_\nu$ for some ν.

Note that the definition says nothing about the relation of information sets of the same player, nor about the relation of personal information sets to chance moves. In intuitive language, a player is said to have effectively perfect information if at any move where it is his "turn" he "remembers" every previous move of each of his personal opponents and knows at least as much as the opponents knew when they made those moves.

LEMMA 9. Every one-person game has effectively complete recall.

PROOF. Immediate from the definition.

LEMMA 10. Every one-person game is essentially determinate.

PROOF. Since $H(\pi)(= H_i(\pi_i))$ is a function of only one variable over a finite domain, it admits a simple maximum.

We first show that a game with effectively perfect information can be decomposed into a directed set of subgames (ordered by inclusion), each of which can be examined for equilibrium points independently of all succeeding subgames. The notion of subgame involved is somewhat wider than the direct application of Definition 1.

DEFINITION 17. Let B be a set of positions (not necessarily an information set) and $D(B)$ the set of all positions that follow any position in B, i.e. $D(B) = \{y|$ there is an $x \in B$ and $y \in D(x)\}$. Analogously, $D(\nu,B)$ is the set of positions that follow B via the ν'th alternative, i.e. $D(\nu,B) = \{y|$ there is an $x \in B$ and $y \in D(\nu,x)\}$. We will write $x \leq U$ to mean $x < U$ or $x \in U$. By the greatest lower bound (g.l.b) of B is meant the x of highest rank that is contained in every $\underline{w}$ that intersects B. $V \leq B$ means $V < B$ or V intersects B.

A. A set of positions B is said to define a subgame Γ_B when:

1) For every personal information set $V < B$, $B \subset V_\nu$ for some ν.

2) For every personal information set $V \geq B$, $V \subset D(B) \cup B$.

B. A pure strategy for a player i in Γ_B is a function, π_i^B, that maps every $V \in \mathcal{U}_i$, $V \geq B$, onto an integer $\nu \leq m(V)$. We shall also use the expression π_i^B to designate the strategy π_i limited to Γ_B (i.e. π_i^B is the strategy for player i in Γ_B such that $\pi_i^B(V) = \pi_i(V)$, $V \geq B$, $V \in \mathcal{U}_i$.)

C. $W(\pi^B) = \{w \mid w \in D(B)$ and for every personal move $y \geq B$, $y \in \underline{w}$ implies $y_\nu \in \underline{w}$ where $\pi^B(y) = \nu\}$

D. Let $p(B) = \sum_{y \in B} p(y)$. For $x \geq B$, $p_B(x) = \frac{p(x)}{p(B)}$.

E. $H(\pi^B) = \sum_{w \in W(\pi^B)} p_B(w)h(w)$.

A rough justification for calling Γ_B a subgame is the following (a more precise justification is found in Lemmas 11 - 13 below): Definition 17 A2 states that B isolates a certain portion of K, namely $B \cup D(B)$; no personal information set overlaps $B \cup D(B)$ and the rest of K. 17 A1 assures that one and only one set of choices on all personal information sets $V < B$ will realize a position in $B \cup D(B)$. Thus, given a π which realizes B, the selection of a strategy within Γ_B can be made independently of the remaining part of Γ. Note that $\{0\}$ defines a subgame $\Gamma_{\{0\}}$, namely, Γ itself.

The significant elements for a subgame Γ_B, namely strategies and expected pay-off, are defined in a manner quite analogous to those for a complete game, with g.l.b. B replacing 0. We cannot, however, simply define a subtree, say K_B, which begins at g.l.b. B and let K_B with its information pattern define a subgame structure, since in the first place, there may be many personal information sets between g.l.b. B and B which overlap K_B and the rest of K, and secondly, K_B may be much more extensive than the part of K which is isolated from B onward.

LEMMA 11. If B defines a subgame, then for x, $y \in B$, g.l.b. $\{x,y\}$ is a chance move, or B is a unit set.

PROOF. Suppose B is not a unit set and $x, y \in B$. Then g.l.b. $\{x,y\} \notin B$. But g.l.b. $\{x,y\}$ cannot belong to a personal information set by Definition 17 A1. Whence g.l.b. $\{x,y\}$ is a chance move.

LEMMA 12. If B defines a subgame Γ_B, then for every π that realizes B, $H(\pi) = p(B)H(\pi^B) + T(\pi)$ where $T(\pi)$ is independent of π^B.

PROOF. If π realizes B, then $W(\pi^B) = W(\pi) \cap D(B)$. Thus

$$H(\pi) = \sum_{w \in W(\pi^B)} p(w)h(w) + T(\pi) \quad \text{where}$$

$$T(\pi) = \sum_{w \in W(\pi) - W(\pi^B)} p(w)h(w), \quad \text{whence}$$

$$H(\pi) = p(B) \sum_{w \in W(\pi^B)} p_B(w)h(w) + T(\pi)$$

$$= p(B)\, H(\pi^B) + T(\pi) .$$

If $w \in W(\pi) - W(\pi^B)$, then $\underline{w}$ does not intersect B, and hence, by Definition 17 A2, does not intersect any personal $V \geq B$. Whence, if π and ρ are identical for all personal $V \geq B$, $T(\pi) = T(\rho)$.

LEMMA 13. π is an equilibrium point for Γ if and only if π^B is an equilibrium point for every Γ_B which is a subgame of Γ, and for which $K(\pi)$ intersects B.

PROOF. Sufficiency is immediate since Γ is a subgame of itself. Necessity. If π^B is not an equilibrium point of Γ_B, there is an i and a ρ_i^B such that $H_i(\pi^B | \rho_i^B) > H_i(\pi^B)$. Let ρ_i be identical with π_i except in Γ_B, where it is identical with ρ_i^B. Then by Lemma 12 $H_i(\pi | \rho_i) > H_i(\pi)$, and π is not an equilibrium point in Γ.

DEFINITION 18. If U and V are personal information sets, U is said to be <u>connected</u> with V when there is a sequence of personal information sets $U^1, U^2, \ldots, U^l$ such that $U = U^1$, $V = U^l$ and for each i there is a $\underline{w}$ that intersects U^i and U^{i+1}, and $U^i \not\subset U^{i+1}_\nu$, $U^{i+1} \not\subset U^i_\nu$ for any ν.

LEMMA 14. <u>Connected</u> is an equivalence relation.

PROOF. a) Transitivity. If U is connected with V and V is connected with V', then there is a sequence $U^1, U^2, \ldots, U^l$ connecting U and V and another sequence $V^1, V^2, \ldots, V^h$ connecting V and V'. The sequence $U^1, U^2, \ldots, U^l, V^2, \ldots, V^h$ is a sequence of the required sort connecting U and V'.

b) Symmetry. Immediate from the definition.
c) Reflexivity. Immediate.

DEFINITION 19. Denote the set of equivalence classes of information sets under connected by $\mathcal{C} = \{C_1, C_2 \ldots\}$. Let $\sum C = \{x|$ there is a $U \in C$ and $x \in U\}$, then $B(C) = \{x | x \in \sum C$ and there is no $y \in \sum C, y < x\}$ i.e., $B(C)$ consists of the minimal points of $\sum C$.

LEMMA 15. If $C \in \mathcal{C}$ then $B(C)$ defines a subgame.

PROOF. (1) Let U be any personal information set, $U \geq B(C)$, i.e., there is some $\underline{w}$ that intersects U and $B(C)$. Let $\underline{w}$ intersect $B(C)$ in V. If $U \not\subset V_\nu$ for some ν, then $U \in C$, and hence every $\underline{w}$ that intersects U intersects $B(C)$. If $U \subset V_\nu$ for some ν then again every $\underline{w}$ that intersects U intersects $B(C)$.

(2) Let U be any personal information set, $U < B(C)$. $U \notin C$, since $B(C)$ is minimal. There is a $\underline{w}$ that intersects U and intersects $B(C)$ in say $x \in V \in C$, hence $V \subset U_\nu$ for some ν, otherwise $U \in C$. Let V' be any other information set in C. By definition V and V' are connected by a sequence $V^1, V^2, \ldots, V^l, V = V^1, V' = V^l$. We have shown that $V^1 \subset U_\nu$ for some ν. Assume that $V^j \subset U_\nu$. There is a $\underline{w}_o$ that intersects V^j and V^{j+1}, and every $\underline{w}$ that intersects V^j intersects U; hence, $\underline{w}_o$ intersects U and V^{j+1}. Now $U \not\subset V_\eta^{j+1}$ for any η, since $U < B(C)$; whence if $V^{j+1} \not\subset U_\eta$ for some η, $U \in C$. Thus $V^{j+1} \subset U_\eta$ for some η, and $\eta = \nu$ since, by assumption, $\underline{w}_o$ follows the ν'th alternative at U. Therefore there is a ν such that for every $V \in C$, $V \subset U_\nu$. A-fortiori, $B(C) \subset U_\nu$.

DEFINITION 20. An equivalence class C is said to cover an equivalence class C' when $B(C) < B(C')$ and there is no C'' such that $B(C) < B(C'') < B(C')$.

LEMMA 16. If C covers C', then for every $U \in C'$, $U > B(C)$, and for every $V \in C$, if there is a $\underline{w}$ that intersects V and $B(C')$, $B(C') \subset V_\nu$ for some ν.

PROOF. Lemma 15.

Lemma 16 says in effect that if an equivalence class C preceeds another equivalence class C', then $B(C')$ defines a subgame of $\Gamma_{B(C)}$.

LEMMA 17. Two different equivalence classes cannot cover the same equivalence class.

PROOF. Suppose C and C' both cover C''. By Lemma 16 every $\underline{w}$ that intersects B(C'') intersects both B(C') and B(C). Since B(C) and B(C') are not identical, we must have either $B(C) < B(C')$ or $B(C') < B(C)$. In which case either C or C' does not cover C''.

If the first position o of $(K,\underline{\mathcal{U}})$ is a personal move, the C containing o defines the entire game Γ. If $o \in P_o$, it is possible that no C determines the entire game. In this case it is convenient to extend the definition of $\mathcal{C}$ so that $\{o\} \in \mathcal{C}$. In either case, we designate the C containing o by C_o.

THEOREM 6. $\mathcal{C}$ is a tree under the relation covers with a distinguished vertex, C_o.

PROOF. Let $\vdash$ denote the proper ancestral of covers, i.e., $C \vdash C'$ if and only if there is a sequence $C_1, C_2, \ldots, C_n$ such that $C = C_1$, $C' = C_n$ and C_i covers C_{i+1}. $\vdash$ is transitive by definition and asymmetrical by Lemma 16 and the a-cyclicity of the ordering relation on K. $C_o \vdash C$ for every $C \neq C_o$. Finally, if $C_1 \vdash C$, $C_2 \vdash C$, then either $C_1 \vdash C_2$ or $C_2 \vdash C_1$ by Lemma 17.

LEMMA 18. If $(K,\underline{\mathcal{U}})$ has effectively perfect information and $C \in \mathcal{C}$ then $C \subset \underline{u}_i$ for some i.

PROOF. Let $V \in C$, $V \in \underline{u}_i$, be any information set in C. Consider any $U \in C$. U is connected with V by a sequence $U^1, U^2, \ldots, U^h$. But if $U^i \in \underline{u}_i$, then $U^{i+1} \in \underline{u}_i$ since, by assumption, if $U^{i+1} \notin \underline{u}_i$, then either $U^i \subset U^{i+1}_\nu$ or $U^{i+1} \subset U^i_\nu$, and $U^{i+1} \notin C$.

Lemma 18 shows that in the tree $\mathcal{C}$, the transition from a set of subgames to a covering subgame involves only one player.

THEOREM 7. A necessary and sufficient condition that $(K,\underline{\mathcal{U}})$ be essentially determinate is that the complete inflation of $(K,\underline{\mathcal{U}})$ have effectively perfect information.

PROOF. Sufficiency. The restriction to the complete inflation of $(K,\underline{\mathcal{U}})$ is required only for necessity. Hence, assume that $(K,\underline{\mathcal{U}})$ has effectively perfect information. We show sufficiency by exhibiting, for any h and p, a strategy n-tuple $\bar{\pi}$ which is an equilibrium point. By a minimal $C \in \mathcal{C}$ we mean one for which there is no C' such that $C \vdash C'$.

It is clear that a minimal C defines a one-person subgame. To simplify notation, rather than writing $\Gamma_{B(C)}$ and $\pi^{B(C)}$ we will write Γ_C and π^C.

(1) For each minimal C, $\bar{\pi}^C$ $(= \bar{\pi}_i^C$ where $C \subset \underline{U}_i)$ is so chosen that

$$H_i(\bar{\pi}^C) \geq H_i(\bar{\pi}^C/\rho_i^C)$$

(2) For any C, $\bar{\pi}^C(V) = \bar{\pi}^{C'}(V)$ for $V \in C'$, $C \vdash C'$. $\bar{\pi}^C(U)$, $U \in C$, is chosen so that

$$H_k(\bar{\pi}^C) \geq H_k(\bar{\pi}^C/\rho_i^C), \quad C \subset \underline{U}_k$$

where $\rho_i^C(V) = \bar{\pi}^{C'}(V)$ for $V \in C'$, $C \vdash C'$. To show that any $\bar{\pi}$ constructed according to the above recursive rule is an equilibrium point, let ρ_i be any strategy for player i, and let ρ_i^1, ρ_i^2, ... be a sequence of strategies for player i constructed as follows: (a) $\rho_i^1 = \rho_i$, (b) ρ_i^j is identical with ρ_i^{j+1} except for the C -- which we shall call C_j -- of highest rank which intersects $K(\bar{\pi}/\rho_i^j)$ and for which $\rho_i^j(V) \neq \bar{\pi}_i(V)$ for some $V \in C_j$. Note that $C_j \subset \underline{U}_i$. (In case there are two or more C of equal rank of the specified kind, let C_j be any one of these.) (c) For $V \geq B(C_j)$, $\rho_i^{j+1}(V) = \bar{\pi}_i(V)$. Now by Lemma 12 and the definition of $\bar{\pi}$

$$H_i(\bar{\pi}/\rho_i^{j+1}) \geq H_i(\bar{\pi}/\rho_i^j)$$

Since there are only a finite number of C's which intersect $K(\pi/\rho_i)$ and the step from ρ_i^j to ρ_i^{j+1} requires a reduction in rank (or at most a finite number of steps before a reduction in rank), the sequence ρ_i^j must conclude with a ρ_i^l such that $K(\bar{\pi}/\rho_i^l) = K(\bar{\pi})$, whence $H_i(\bar{\pi}/\rho_i^l) = H_i(\bar{\pi})$ and therefore $H_i(\bar{\pi}) \geq H_i(\bar{\pi}/\rho_i)$.

Necessity. Assume that $(K,\underline{U}')$, the complete inflation of $(K,\underline{U})$ does not have effectively perfect information. Let $x_o \in \bar{U} \in \underline{U}_i'$ be the personal position of lowest rank for which there is $z \in D(x_o)$, $z \in V \in \underline{U}_j'$, $i \neq j$, $V \notin \bar{U}_\nu$ for any ν (if there are several positions of the same minimal rank with this property, let x_o be any one of these); and let $x \in \bar{V} \in \underline{U}_k'$, $k \neq i$, be the position of lowest rank in $D(x_o)$ such that $\bar{V} \notin \bar{U}_\nu$ for any ν. Since $x \in D(\eta, x_o)$ for some η, there is a $y \in \bar{V}$ such that $y \notin D(\eta, x_o)$. For simplicity we relabel the persons and alternatives so that $\bar{U} \subset P_1$, $\bar{V} \subset P_2$, $x \in D(1, x_o)$. We distinguish three principal cases:

CASE I. $x_o = \text{g.l.b.}\ \{x,y\}$.

Let $\bar{W} = D(x) \cup D(y)$ and let $m = \max [r(x_1), r(y_1)]$. Define

$h(w)$ as follows, where z is the highest ranking intersection of $\underline{w}$ with $\underline{w}_x$ or $\underline{w}_y$:

(a) $w \notin \overline{W}$

1. $z \in P_o$: $h_i(w) = 0$ for every i.
2. $z \notin P_o$, $z \leq \{x_o\}$: $h_i(w) = \frac{r(z)}{p(z)}$ for every i.
3. $z \notin P_o$, $z > \{x_o\}$: $h_i(w) = \frac{r(z)}{p(z)}$ for every $i \neq 1$, $h_1(w) = \frac{m+1}{p(z)}$ for $z \notin P_1$ otherwise $h_1(w) = \frac{r(z)}{p(z)}$.

(b) $w \in \overline{W}$.

1. $i \neq 1$, $i \neq 2$: $h_i(w) = \frac{m}{p(z)}$.
2. $i = 1$ or $i = 2$:

	$h_1(w)$	$h_2(w)$
$w \in D(x_1)$	$\frac{m}{p(x)}$	$\frac{m+1}{p(x)}$
$w \in D(x_2)$	$\frac{m+1}{p(x)}$	$\frac{m}{p(x)}$
$w \in D(y_1)$	$\frac{m+1}{p(y)}$	$\frac{m}{p(y)}$
$w \in D(y_2)$	$\frac{m}{p(y)}$	$\frac{m+1}{p(y)}$

With h as defined, no π is an equilibrium point. There are two possibilities:

A. $K(\pi) \cap \overline{W} = \theta$.
Let $\overline{z} \in U \in \mathcal{U}_k$' be the highest ranking intersection of $K(\pi)$ with $\underline{w}_x$ or $\underline{w}_y$; and let ρ_k be identical with π_k except that $\rho_k(U) = \nu$ where $\overline{z}_\nu \in \underline{w}_x$ or $\underline{w}_y$. Then $H_k(\pi/\rho_k) \geq r(\overline{z}_\nu) > r(\overline{z}) = H_k(\pi)$

B. $K(\pi) \cap \overline{W} \neq \theta$.
The pay-off to any player is independent of choices made above $\overline{V}$; hence there are essentially four cases, depending on $\pi_1(\overline{U})$ and $\pi_2(\overline{V})$; these are summed up in the matrices

$H_1(\pi)$

$\pi_1(\bar{U})$ \ $\pi_2(\bar{V})$	1	2
1	m	m+1
2	m+1	m

$H_2(\pi)$

$\pi_1(\bar{U})$ \ $\pi_2(\bar{V})$	1	2
1	m+1	m
2	m	m+1

For any pair of choices $\pi_1(\bar{U})$, $\pi_2(\bar{V})$, there is a choice by one of the players that increases his pay-off.

CASE II. $x_0 \neq$ g.l.b. $\{x,y\}$, $\bar{U}$ intersects $\underline{w}_y$.

g.l.b. $\{x,y\} \in P_0$, otherwise x_0 would not be the minimal position for which effectively perfect information later fails. Let $y_0 \in \underline{U}$ be the lowest ranking position in $\underline{w}_y$ such that $\underline{U}$ intersects $\underline{w}_x$ and $\bar{V} \not\subset \underline{U}_\nu$ for any ν. Because of the minimality condition on $\bar{V}$, $\underline{U} \in \mathcal{U}_1'$. Let $\bar{W}$, z, and m be as in Case I, and set $t = r(x_0) - r(y_0)$. Define h(w) as in Case I for (a) 1,2 and (b) 1.

(a)3. $z \notin P_0$, $z > \{x_0\}$: $h_i(w) = \frac{r(z)}{p(z)}$ for every $i \neq 1$,

$$z \in \underline{w}_x: \quad h_1(w) = \frac{2m+1+t}{p(z)}$$

$$z \in \underline{w}_y: \quad h_1(w) = \frac{2m+1-t}{p(z)}$$

for $z \notin P_1$

$$h_1(w) = \frac{r(z)}{p(z)} \quad \text{for} \quad z \in P_1$$

(b)2.

	$h_1(w)$	$h_2(w)$
$w \in D(x_1)$	$\frac{2m}{p(x)}$	$\frac{2m+1}{p(x)}$
$w \in D(x_2)$	$\frac{2m+1+t}{p(x)}$	$\frac{2m}{p(x)}$
$w \in D(y_1)$	$\frac{2m+1-t}{p(y)}$	$\frac{2m}{p(y)}$
$w \in D(y_2)$	$\frac{2m}{p(y)}$	$\frac{2m+1}{p(y)}$

The proof that no π is an equilibrium point proceeds as in Case I except that a more complicated set of subcases must be examined.

A. $K(\pi) \cap \overline{W} = \theta$. Let $\overline{z}$ be as in Case IA.

1. $\overline{z} <$ g.l.b. $\{x,y\}$: Argument as in Case IA.

2. $\overline{z} >$ g.l.b. $\{x,y\}$: Let $z^X \in U^X$ and $z^Y \in U^Y$ be the highest ranking intersections of $K(\pi)$ with $\underline{w}_X$ and $\underline{w}_Y$ respectively. There are two sub-sub-cases.
(i) Either $U^X \not< z^Y$ or $U^Y \not< z^X$: Assume $U^X \not< z^Y$. Let $U^X \in \underline{\mathcal{U}}_i'$ and let ρ_i be identical with π_i except that $\rho_i(U^X) = \nu$ where $z^X_\nu \in \underline{w}_X$. Then

$$H_i(\pi/\rho_i) \geq r(z^X_\nu) + r(z^Y) > r(z^X) + r(z^Y) = H_i(\pi)$$

a similar argument holds in case $U^Y \not< z^X$.
(ii) $U^X < z^Y$ and $U^Y < z^X$. In this case both U^X, $U^Y \in \underline{\mathcal{U}}_1'$, otherwise the minimality conditions on $\overline{U}$ and $\overline{V}$ are violated. Let ρ_1 be identical with π_1 except that for every $U \in \underline{\mathcal{U}}_1'$, $x > U \geq z^X$, $\rho_i(U) = \nu$ where $x \in D(\nu,U)$. Then $H_1(\pi/\rho_1) \geq 2m+r(y_o) > r(z^X) + r(z^Y) = H_1(\pi)$. The central inequality holds since $r(z^Y) - r(y_o) < m$ and $r(z^X) < m$.

B. $K(\pi) \cap \overline{W} \neq \theta$. At most one of x and y is in $K(\pi)$. Let α denote the one that is in $K(\pi)$ and β denote the other. Then α_o denotes x_o or y_o respectively, and $z^\beta \in U^\beta \in \underline{\mathcal{U}}_i'$ denotes the highest ranking intersection of $K(\pi)$ with $\underline{w}_\beta$. Note that $z^\beta \leq \{\beta_o\}$. Let w_o represent any $w \in K(\pi) \cap \overline{W}$.

1. $z^\beta < \{\beta_o\}$ U^β does not intersect w_α because of the minimality condition on α_o. Let ρ_i be identical with π_i except that $\rho_i(U^\beta) = \nu$ where $z^\beta_\nu \in \underline{w}_\beta$. Then

$$H_i(\pi/\rho_i) \geq p(\alpha)h_i(w_o) + r(z^\beta_\nu) > p(\alpha)h_i(w_o) + r(z^\beta) = H_i(\pi)$$

2. $z^\beta = \beta_o$.
There are four possibilities, depending on the choices at $\overline{U}$, $\underline{U}$, and $\overline{V}$. The apparent eight resulting from two choices at each of three information sets is reduced by the fact that either $\overline{U} = \underline{U}$ or else four of the possibilities give $K(\pi) \cap \overline{W} = \theta$ and are treated above. Label the choices at $\underline{U}$ so that $x \in D(1,\underline{U})$, $y \in D(2,\underline{U})$. $\overline{U}$, $\underline{U} \in \underline{\mathcal{U}}_1'$ because of the minimality condition on $\overline{V}$, therefore the choices at $\underline{U}$, $\overline{U}$ are completely under the control of player 1 and we have the sub-matrices

$H_1(\pi)$

$\pi(\bar{U})$ \ $\pi(\bar{V})$	$\pi(\underline{U})$	1	2
1	1	$2m+r(y_0)$	$2m+1+r(x_0)$
2	2	$2m+1+r(y_0)$	$2m+r(x_0)$

$H_2(\pi)$

$\pi(\bar{U})$ \ $\pi(\bar{V})$	$\pi(\underline{U})$	1	2
1	1	$2m+1+r(y_0)$	$2m+r(y_0)$
2	2	$2m+r(x_0)$	$2m+1+r(x_0)$

For any one of the four possibilities, one of the players can choose a strategy which will increase his pay-off.

Note that no essential modification is needed in the preceeding proof if we relax the minimality condition on x_0, requiring merely that it is the lowest ranking position for which effectively perfect information later fails and belongs to an information set which intersects both $\underline{w}_x$ and $\underline{w}_y$.

CASE III. $x_0 \neq$ g.l.b. $\{x,y\}$, $\bar{U}$ does not intersect $\underline{w}_y$. There is no loss of generality in assuming that for every $U \notin \underline{\mathcal{U}}_0'$, if U intersects $\underline{w}_y$ and $\underline{w}_x$, then $\{x,y\} \subset U_\nu$ for some ν. This assumption is justified by the following:

1. There is a $z \in \bar{V} - D(\bar{U})$, such that for every $U \in \underline{\mathcal{U}}_2'$, $U \neq \bar{V}$, that intersects $\underline{w}_x$ and $\underline{w}_y$, $\{x,z\} \subset U_\nu$ for

some ν. For assume that for every $z' \in \overline{V}$, $z' \notin D(\overline{U})$, there is a $U \in \underline{\mathcal{U}}_2'$, $\{x,z\} \not\subset U_\nu$ for any ν. By the minimality condition on x, $U < x_o$. Whence, by the minimality condition on x_o, $\overline{U} \subset U_\nu$ for some ν, and hence $D(U) \subset U_\nu$ for some ν. But then, for every $z'' \in D(\overline{U}) \cap \overline{V}$, and every $z' \in \overline{V} - D(\overline{U})$, there is a $U \in \underline{\mathcal{U}}_2'$ such that $z'' \in U_\nu$, $z' \in U_\eta$, $\nu \neq \eta$. Thus by Lemma 7 $D(\overline{U}) \cap \overline{V}$ is isolated in $\overline{V}$, contrary to the hypothesis that $\underline{\mathcal{U}}'$ is completely inflated. We can take the z thus proved to exist to be y.

2. There is no $U \notin \underline{\mathcal{U}}_o' \cup \mathcal{U}_1' \cup \underline{\mathcal{U}}_2'$ such that U intersects $\underline{w}_x$ and $\underline{w}_y$ and $\{x,y\} \not\subset U_\nu$ for any ν by the minimality conditions on x_o and x.

3. There is no $U \in \underline{\mathcal{U}}_1'$, $U < x_o$ such that U intersects $\underline{w}_x$ and $\underline{w}_y$ and $\{x,y\} \not\subset U_\nu$ for any ν, by the minimality condition on x_o.

4. If there is a $U \in \underline{\mathcal{U}}_1'$, $U > x_o$ and U intersects $\underline{w}_x$, $\underline{w}_y$, $\{x,y\} \not\subset U_\nu$, then we have essentially Case II as noted at the end of the proof for that section.

Let m and z be as in the previous cases. Let $\underline{W} = \overline{W} \cup D(2,x_o)$. Define an h as follows:

(a) $w \notin \underline{W}$

1. $z \in P_o$: $h_i(w) = 0$ for every i

2. $z \notin P_o$, $z > \{x_o\}$: $h_i(w) = \dfrac{r(z)}{p(z)}$

3. $z \notin P_o$, $z > \{x_o\}$: $h_i(w) = \dfrac{r(z)}{p(z)}$ for $i \neq 1$.

$$h_1(w) = \frac{m+2}{p(z)} \quad \text{for} \quad z \notin P_1$$

$$h_1(w) = \frac{r(z)}{p(z)} \quad \text{for} \quad z \in P_1$$

(b) $w \in \underline{W}$

1. $h_i(w) = \dfrac{m}{p(z)}$ for every $i \neq 1$, $i \neq 2$

2.

	$h_1(w)$	$h_2(w)$
$w \in D(2,x_o)$	$\dfrac{m+1}{p(x_o)}$	$\dfrac{m}{p(x_o)}$
$w \in D(x_1)$	$\dfrac{m}{p(x)}$	$\dfrac{m+2}{p(x)}$
$w \in D(x_2)$	$\dfrac{m+2}{p(x)}$	$\dfrac{m}{p(x)}$

$w \in D(y_1)$	$\frac{m}{p(y)}$	$\frac{m}{p(y)}$
$w \in D(y_2)$	$\frac{m}{p(y)}$	$\frac{m+1}{p(y)}$

Let $W^* = [D(2,x_o) \cup D(x)] \cap D(y)$. We consider two cases.

A. $K(\pi) \cap W^* = \theta$.

The argument is the same as in Case II A 1, 2(i).

B. $K(\pi) \cap W^* \neq \theta$.

Here there are four cases, depending on the choice at $\overline{U}$ and $\overline{V}$, summed up in the matrices

$H_1(\pi)$

$\pi_1(\overline{U})$ \ $\pi_2(\overline{V})$	1	2
1	2m	2m+2
2	2m+1	2m+1

$H_2(\pi)$

$\pi_1(\overline{U})$ \ $\pi_2(\overline{V})$	1	2
1	2m+2	2m+1
2	2m	2m+1

and for any pair of choices, there is another choice open to one of the players which increases his payoff.

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INFINITE GAMES WITH PERFECT INFORMATION

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INTRODUCTION

The notion of a game with perfect information has been defined by von Neumann and Morgenstern [2].[1] The concept corresponds intuitively to games like checkers and chess in which the moves and positions of a player are known to his opponents at all times. The principal result of the theory states that a finite two-person zero-sum game with perfect information is strictly determined, that is, that such games have a value which can be achieved by pure strategies. Recently Kuhn [1] has obtained a generalization of this result for n-person games which are not necessarily zero-sum. In this paper we consider only the two-person zero-sum case, but relax the condition of finiteness. Our first result shows that in this case the von Neumann theorem no longer holds and such games may be indeterminate.

To illustrate this somewhat paradoxical result we consider the "binary game." In this game players I and II alternately choose binary digits 0 or 1. A play of the game consists of a countable number of such moves or choices. Thus each play consists of a sequence, $s(0)$, $s(1)$, $s(2)$, ... of digits which determines the real number $s = \sum_{n=1}^{\infty} s(n)/2^n$. Player I wins a dollar from player II if s belongs to a certain subset T of the unit interval, while player II wins a dollar from player I if s does not belong to T. It will be shown that the set T can be chosen so that no matter what strategy player I uses, there is a strategy by which player II can win, while, contrariwise, for each strategy player II may elect to use there is a strategy by which player I can win.

Since infinite games in extensive form have not previously been considered in the literature, it is necessary to devote the first section of this paper to setting up definitions and notation. In the second section a proof is given of the general indeterminacy theorem already mentioned. As might be suspected this proof depends heavily on non-constructive methods such as well-ordering. It is therefore natural to inquire if a game can still be indeterminate if the pay-off function is "well-behaved." For ex-

[1] Numbers in brackets refer to the bibliography at the end of this paper.

ample in the binary game, how pathological must the set T be for the game to be indeterminate? The remainder of the paper is devoted to investigation of such questions, that is, under what circumstances is a game strictly determined? For this purpose we introduce in section 3 the notion of the natural topology of a game and the concept of subgame. Certain simple topological conditions are sufficient to insure strict determinacy. In terms of the notion of subgame we define an absolutely determined game as one all of whose subgames together with certain other related games are strictly determined. Using these concepts we are able to give the statements of all the positive results which have been obtained. These are listed at the end of section 3. As an example of these results we mention that the binary game is strictly determined if the set T belongs to the Boolean Algebra generated by the open subsets of the unit interval. In section 4 we give proofs of the results listed at the end of section 3. In the final section a number of unsolved problems are listed. As an example we mentioned the following open question: Is the binary game determined if the set T is a G_δ or an F_σ?

Finally a word should be said about the reasons for considering games of the type described. Certainly such a game cannot be played in its extensive form, but it is hoped that the techniques used here may have applications to infinite games in the normalized form. Furthermore, consideration of the counter example and the positive results in the infinite case sheds light on the von Neumann proof for finite games. Lastly, in certain tactical and chase problems one has a great deal, but not perfect, information about the opponents' past moves. For such problems some extensive form may be more useful than the normal form. We hope that this study of games with perfect information may suggest methods helpful in the study of some games with partial information.

§1. DEFINITIONS AND NOTATION

A fundamental problem which must be faced by anyone writing on games in extensive form is that of finding a notation which will adequately represent the situation and at the same time will not be too cumbersome to work with. It seems traditional that everyone who works in this subject invents a new notation and we shall be no exception. Since we are discussing only a special case of extensive games, namely two-person with perfect information, we have chosen a representation of such games which is as simple as possible, and have not attempted to find a representation which will apply to more general cases. Also we have eliminated the possibility of "chance moves," again for the sake of simplicity. What we do in this section is to describe a mathematical model for the games to be discussed. It would be possible simply to define this model and proceed to prove

theorems about it without further explanation. The significance of the results, however, lies in their interpretation as theorems about games and for this reason we have chosen to justify each definition by its game theoretic interpretation as we go along. This procedure which seems essential to giving a clear understanding of the results, will necessitate a certain amount of "talk." The reader may test his understanding of the model by seeing how the binary game fits into the general framework we are about to describe.

A game in extensive form has been likened to a topological tree (see [1] and [2]). Each vertex of the tree corresponds to a particular "position" of the game and the branches issuing from a given vertex correspond to the positions which are possible on the next move. The "root" or initial vertex of the tree corresponds to the initial position of the game. Further, each vertex of the tree is assigned to player I or II according to whose move it is at the position in question. The first object in our model of a game will then be a set X, the set of positions. This set is partitioned into sets X_I and X_{II} to represent the fact that at each position it is the move of player I or player II.

We must next describe how the vertices of the tree are connected together. Given a position[2] $x \in X$ we must give a way of telling which positions x' can be obtained from x on the next move. Such a position x' will be called an immediate successor of x, while x will be called the immediate predecessor of x'. A simple way of describing this situation is to give a function f which carries each position, x, onto its immediate predecessor,[3] $f(x)$. We must also distinguish a particular position, x_0, the initial position of the game. (Note that if $x_0 \in X_{II}$, then player II has the first move.) Every $x \in X$ except x_0 has an immediate predecessor, and x_0 is a (not necessarily immediate) predecessor of every $x \in X$. These properties can be assured by imposing the following conditions on f.

(1.1) f maps $X - x_0$ onto X.

(1.2) For any $x \in X$ there is an integer $n \geq 0$ such that $f^n(x) = x_0$.

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[2]We use the standard notations, $x \in X$ for x is a member of the set X, $X \subset Y$ for X is a subset of Y, and $X \cup Y$ and $X \cap Y$ for the union and intersection of the sets X and Y. The empty set is denoted by Λ. The set of all $d(x)$ such that $s(x)$ is denoted by $\{d(x) \mid s(x)\}$. The supremum of this set is denoted by $\sup \{d(x) \mid s(x)\}$ or $\sup_{s(x)} d(x)$. If f is a function $f|X$ is its restriction to the set X.

[3]We are assuming that each position has a unique immediate predecessor. In an actual game, say chess, one knows that the same disposition of the pieces can be arrived at by different successions of moves. This difficulty can be avoided if we consider a position as consisting not only of the arrangement of the pieces, but also of all previous arrangements.

Now x is a predecessor of y and y is a successor of x if for some integer $n \geq 0$, $x = f^n(y)$. If $n = 1$, x is the immediate predecessor of y and y is an immediate successor of x.

The game is now played as follows. According as x_0 is in X_I or X_{II} player I or II picks a position $x_1 \in f^{-1}(x_0)$ (note that $f^{-1}(x_0)$ is the set of immediate successors of x_0). In general if n positions have been chosen, then according[4] as x_n belongs to X_I or X_{II} player I or II chooses a point $x_{n+1} \in f^{-1}(x_n)$. In this way a sequence of points $\{x_n\}$ is chosen by the two players. Such a sequence is called a play of the game.

More precisely a sequence, $s(0), s(1), \ldots,$ is a <u>play</u> of the game if

$$s(0) = x_0 , \tag{1.3}$$

and

$$\text{For any } i \geq 0,\ s(i) = f(s(i+1)) . \tag{1.4}$$

The set of all plays is called the <u>space of the game</u> and will usually be denoted by S.

To complete the description of the game it is only necessary to give a <u>pay-off function</u>, Φ, that is, a real valued function on the set S. If the players play so that outcome is s then player II pays player I an amount $\Phi(s)$ units. Most of our results are for the special case where Φ is the characteristic function of a set S_I which will be called the <u>winning set for player I</u>; $S_{II} = S - S_I$ will be called the <u>winning set for player II</u>. A play s of the game is a <u>win</u> for player I or player II according as $s \in S_I$ or $s \in S_{II}$.

To summarize

DEFINITION 1. A <u>game</u> Γ is a collection of objects $(x_0, X_I, X_{II}, X, f, S, \Phi)$, such that

$$x_0 \in X, \tag{1.5}$$

$$X_I \cap X_{II} = \Lambda,\quad X_I \cup X_{II} = X, \tag{1.6}$$

(1.7) f is a function satisfying (1.1) and (1.2),

(1.8) S is the set of all sequences satisfying (1.3) and (1.4),

(1.9) Φ is a real valued function on S.

[4] In most games the players move alternately. We shall not require this since its only result would be to complicate the notation.

DEFINITION 2. A win-lose game Γ is a collection of objects $(x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ such that (1.5) - (1.8) and

(1.10) $\quad S_I \cap S_{II} = \Lambda, \quad S_I \cup S_{II} = S.$

A play of Γ is an element of S. It is a win for player I (player II) if it belongs to S_I (S_{II}).

The above description completely defines the rules of the game. In order to discuss questions of determinacy it is now necessary to define the fundamental notion of a strategy. A strategy here is clearly a rule which in any position will tell a player what his next move shall be.

DEFINITION 3. If $\Gamma = (x_0, X_I, X_{II}, X, f, S, \Phi)$ is a game, the set, $\sum_I(\Gamma)$, of strategies for player I consists of all functions, σ, with domain X_I such that

(1.11) $\quad \sigma(x) \in f^{-1}(x).$

The set, $\sum_{II}(\Gamma)$, of strategies for player II consists of all functions τ on X_{II} satisfying

(1.12) $\quad \tau(x) \in f^{-1}(x).$

Now to every pair of strategies σ and τ will correspond a unique play s of the game Γ. This play which we write as $s = \langle\sigma,\tau\rangle$ is the play which results if the two players pick the strategies σ and τ respectively, and will satisfy the following inductive definition.

(1.13) $\quad s(0) = x_0.$

(1.14) $\quad$ If $s(n) \in X_I$ then $s(n+1) = \sigma(s(n))$;

$\quad$ if $s(n) \in X_{II}$ then $s(n+1) = \tau(s(n))$.

We may now consider the pay-off function, Φ, as defined on pairs of strategies, as is usual in game theory. That is $\Phi(\sigma,\tau) = \Phi(\langle\sigma,\tau\rangle)$. The definitions of strict determinacy and of the value of a game now follow in the usual manner, ([2] p. 106) since we have now succeeded in putting the game in the normalized form. Thus Γ is said to be strictly determined and have value v if,

$$\inf_{\tau\in\sum_{II}(\Gamma)} \sup_{\sigma\in\sum_I(\Gamma)} \Phi(\sigma,\tau) = \sup_{\sigma\in\sum_I(\Gamma)} \inf_{\tau\in\sum_{II}(\Gamma)} \Phi(\sigma,\tau) = v$$

As usual with infinite games, we require only the equality of infima and suprema rather than that of maxima and minima.

A win-lose game $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ is equivalent to the game with pay-off, $\Gamma' = (x_0, X_I, X_{II}, X, f, S, \Phi)$ where Φ is the characteristic function of S_I. If Γ' has a value then, since Φ can assume only the values 0 and 1, there is a $\sigma_0 \in \Sigma_I(\Gamma)$ and a $\tau_0 \in \Sigma_{II}(\Gamma)$ such that $\sup_{\sigma \in \Sigma_I(\Gamma)} \Phi(\sigma, \tau_0) = \inf_{\tau \in \Sigma_{II}(\Gamma)} \Phi(\sigma_0, \tau) = \Phi(\sigma_0, \tau_0) = v$. The value, v, is 1, and σ_0 always wins Φ for player I if and only if for all $\tau \in \Sigma_{II}(\Gamma)$, $\langle \sigma_0, \tau \rangle \in S_I$. Similarly v is zero and τ_0 always wins for player II if and only if $\langle \sigma, \tau_0 \rangle \in S_{II}$ for all $\sigma \in \Sigma_I(\Gamma)$.

DEFINITION 4. If $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ is a win-lose game then the set, $\Sigma_I^w(\Gamma)$, (or $\Sigma_{II}^w(\Gamma)$) of <u>winning strategies for player I</u> (or player II) consists of all $\sigma \in \Sigma_I(\Gamma)$ (or $\tau \in \Sigma_{II}(\Gamma)$) such that

(1.15) $\tau \in \Sigma_{II}(\Gamma)$ implies $\langle \sigma, \tau \rangle \in S_I$

(or

(1.16) $\sigma \in \Sigma_I(\Gamma)$ implies $\langle \sigma, \tau \rangle \in S_{II}$.)

DEFINITION 5. A win-lose game Γ is <u>strictly determined</u> if $\Sigma_I^w(\Gamma) \neq \Lambda$ or $\Sigma_{II}^w(\Gamma) \neq \Lambda$.

§ 2. THE INDETERMINACY THEOREM

We now construct an example of a an infinite game with perfect information which is not strictly determined.

At each turn the possible moves are i) choose the digit 0 and ii) choose the digit 1. Thus X, the set of positions, consists of all finite sequences $\{a(1), a(2), \ldots, a(n)\}$ where each $a(i)$ is either 0 or 1. The initial position x_0 is the sequence of length zero. A position $\{a(1), \ldots, a(n)\}$ is in X_I if n is even, and in X_{II} if n is odd.

At each move a player cannot change the previous moves so $f(\{a(1), \ldots, a(n)\}) = \{a(1), \ldots, a(n-1)\}$.

The set, S, of plays consists of all infinite sequences of the form $\{x_0, \{a(1)\}, \{a(1), a(2)\}, \{a(1), a(2), a(3)\}, \ldots\}$.

We now construct subsets S_I and S_{II} of S such that the win-lose game $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ is not strictly determined.

The construction is based on two facts.

A. The number of strategies in $\Sigma_I(\Gamma)$, (and in

$\Sigma_{II}(\Gamma))$ is $2^{\aleph_0}$.

B. For each $\tau_0 \in \Sigma_{II}(\Gamma)$ the number of plays in $S_{\tau_0} = \{\langle\sigma,\tau_0\rangle | \sigma \in \Sigma_I(\Gamma)\}$ is $2^{\aleph_0}$. Similarly for each $\sigma_0 \in \Sigma_I(\Gamma)$, $S_{\sigma_0} = \{\langle\sigma_0,\tau\rangle | \tau \in \Sigma_{II}(\Gamma)\}$ has $2^{\aleph_0}$ members.

PROOF OF A. Each function θ on X_I to the set whose members are 0 and 1 determines and is determined by a strategy, $\sigma_\theta \in \Sigma_I(\Gamma)$ where $\sigma_\theta(\{a(1), \ldots, a(n)\}) = \{a(1), \ldots, a(n), \theta(\{a(1), \ldots, a(n)\})\}$. Since X_I has $\aleph_0$ elements there are $2^{\aleph_0}$ such functions and $2^{\aleph_0}$ strategies.

PROOF OF B. There are $2^{\aleph_0}$ infinite sequences $\{\alpha(0), \alpha(1), \ldots,\}$ where $\alpha(i)$ is 0 or 1. Corresponding to each such sequence there is a play s defined inductively by: i) $s(0) = x_0$; ii) if $s(2n) = \{a(1), \ldots a(2n)\}$ then $s(2n+1) = \{a(1), \ldots a(2n), \alpha(n)\}$; iii) $s(2n) = \tau_0(s(2n-1))$. Distinct sequences determine distinct plays.

For each such play, s, there is a (not unique) strategy σ such that $\langle\sigma,\tau_0\rangle = s$. E.g., let $\sigma(s(2n)) = s(2n+1)$ and $\sigma(x)$ is arbitrary if $x \in X_I$ but $x \neq s(n)$ for any n.

Thus there are at least $2^{\aleph_0}$ elements in $\{\langle\sigma,\tau_0\rangle | \sigma \in \Sigma_I(\Gamma)\}$. Since S itself has only $2^{\aleph_0}$, $S_{\tau_0} = \{\langle\sigma,\tau_0\rangle | \sigma \in \Sigma_I(\Gamma)\}$ must have exactly $2^{\aleph_0}$ elements.

The proof for S_{σ_0} is similar.

We proceed to the construction of S_I and S_{II}.

Let α be the least ordinal such that there are $2^{\aleph_0}$ ordinals less than α. By A we can index the strategies in $\Sigma_I(\Gamma)$ as σ_β, $\beta \in \{\gamma | \gamma < \alpha\}$. Similarly we can index the strategies in $\Sigma_{II}(\Gamma)$ as τ_β, $\beta \in \{\gamma | \gamma < \alpha\}$.

Choose $s_0 \in S_{\tau_0}$. Choose $t_0 \in S_{\sigma_0}$ so that $t_0 \neq s_0$. Proceed inductively. If s_γ, t_γ have been chosen for all $\gamma < \beta < \alpha$ the set $\{t_\gamma | \gamma < \beta\}$ has fewer than $2^{\aleph_0}$ members so $S_{\tau_\beta} - \{t_\gamma | \gamma < \beta\}$ is not empty. Choose one of its elements and call it s_β. Similarly $S_{\sigma_\beta} - \{s_\gamma | \gamma \leq \beta\}$ is not empty. Call one of its members t_β.

We now define $A \subset S$ by the rule $A = \{s_\gamma | \gamma < \alpha\}$, and $B \subset S$ by the rule $B = \{t_\gamma | \gamma < \alpha\}$, and we assert that the sets A and B are disjoint, i.e., for any γ and β, $s_\gamma \neq t_\beta$.

CASE I. If $\gamma \leq \beta$ then $t_\beta \in S_{\sigma_\beta} - \{s_\delta | \delta \leq \beta\}$ so $t_\beta \notin \{s_\delta | \delta \leq \beta\}$ and therefore $t_\beta \neq s_\gamma$.

CASE II. If $\gamma > \beta$ then $s_\gamma \in S_{\tau_\gamma} - \{t_\delta | \delta < \gamma\}$ so $s_\gamma \notin \{t_\delta | \delta < \gamma\}$ and therefore $s_\gamma \neq t_\beta$.

We now partition S into two sets S_I and S_{II} in any manner provided only that $A \subset S_I$ and $B \subset S_{II}$. To show that the game so defined is not strictly determined it suffices to show that $\sum_I^W(\Gamma) = \sum_{II}^W(\Gamma) = \Lambda$, (see DEF. 5). Let σ be in $\sum_I(\Gamma)$. Then σ corresponds to some index, say β. By construction there exists a play t_β such that $t_\beta \in S_{\sigma_\beta}$ and hence there exists $\tau \in \sum_{II}(\Gamma)$ such that $\langle \sigma_\beta, \tau \rangle = t_\beta \in B \subset S_{II}$. Thus $\sigma_\beta \notin \sum_I^W(\Gamma)$ and hence $\sum_I^W(\Gamma) = \Lambda$. Symmetrically one shows that $\sum_{II}^W(\Gamma) = \Lambda$, and hence the theorem is proved.

Thus Γ is not strictly determined and we have

THEOREM 1. There is an infinite game with perfect information which is not strictly determined.

§ 3. THE TOPOLOGY OF INFINITE GAMES

Our positive results about strictly determined games are most easily stated in terms of a natural topology on the set, S, of all plays. Each play is a function from the non-negative integers to the set, X, of positions. The topology imposed on S turns out to be that of pointwise convergence of the functions if X is given a discrete topology.

Throughout this section we will give most of our attention to win-lose games, but the definitions and results are equally applicable, mutatis mutandis, to games with pay-off.

NOTATION. If $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ is a game and $x \in X$ then

$$\mathcal{U}(x) = \{s \mid s \in S \text{ and, for some } i, \ s(i) = x\} .$$

One easily verifies that the i is the same for all $s \in \mathcal{U}(x)$ and is the integer i such that $f^i(x) = x_o$.

DEFINITION 6. If $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ is a game and $s \in S$, a <u>neighborhood of s</u> is any $\mathcal{U}(x)$ containing s.

Note that $\mathcal{U}(x)$ is a neighborhood of s if and only if $x = s(i)$ for some integer $i \geq 0$.

THEOREM 2. If $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ is a game the neighborhoods of points of S determine a Hausdorff topology for S. In this topology S is totally disconnected.

The proof is a routine matter of verification. We will wish, however, to refer to the following facts.

(3.1) If $f^i(x) = y$ then $\mathcal{U}(x) \subset \mathcal{U}(y)$

(3.2) If, for every $i \geq 0$, neither $f^i(x) = y$ nor $f^i(y) = x$ then $\mathcal{U}(x) \cap \mathcal{U}(y) = \Lambda$.

We shall call a win-lose game Γ open or closed according as the set S_I is open or closed.

It is desirable to be able to consider finite games with perfect information as a special case of infinite games. There is a slight difficulty in doing this since by our definition of a game each play involves an infinite number of moves. This can be gotten around, however, by introducing "dummy" moves, that is, positions such that the player whose move it is has only one alternative. Such a position shall be called ineffective. In terms of our notation the position x is ineffective if $f^{-1}(x)$ contains only one element. Thus chess may be considered an infinite game by making all positions which are successors of a checkmate or stalemate ineffective.

Consider now the following three finiteness conditions.

DEFINITION 7. An infinite game $\Gamma(x_0, X_I, X_{II}, X, f, S, \Phi)$ satisfies condition F.1, or F.2, or F.3 if, respectively,

(F.1) X contains only a finite number of effective positions;

(F.2) there is an N such that for all $s \in S$, $n \geq N$ implies $s(n)$ is ineffective;

(F.3) For every $s \in S$ only a finite number of the $s(n)$, $n = 0, 1, \ldots,$ are effective.

Note that (F.1) implies (F.2) and (F.2) implies (F.3).

We now state a result connecting the notion of finiteness with the topology of a game. The proof is straight forward and we omit it.

THEOREM 3. The game Γ satisfies the condition (F.3) if and only if the space of Γ is discrete.

Von Neumann has proved [2, pp. 112-128] that finite games with perfect information are strictly determined. His methods -- in particular section 15.8 -- can be extended to show that every game satisfying (F.3) is

strictly determined. However, this result turns out, in view of Theorem 3, to be just a special case of our Theorem 15. There need not, however, be a strategy which ensures the achievement of the value. This situation is clarified by Theorem 16.

Another concept intimately related to the topology of a game is that of a subgame.

DEFINITION 8. A win-lose game $\Gamma' = (x_0', X_I', X_{II}', X', f', S', S_I', S_{II}')$ is a subgame of $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ if

(3.3) $x_0' = x_0$

(3.4) $X_I' \subset X_I$, $X_{II}' \subset X_{II}$

(3.5) $f' = f|X'$

(3.6) $S_I' = S_I \cap S'$, $S_{II}' = S_{II} \cap S'$.

For games with pay-off (3.6) is replaced by

(3.7) $\Phi' = \Phi|S'$.

Intuitively a subgame is obtained by adding new rules forbidding certain moves. E.g., if two chess players agree to use a particular opening, they play a subgame of chess rather than chess itself.

The connection between subgames and the topology is embodied in the following two theorems whose proofs will be found in section 4.

THEOREM 4. If $\Gamma' = (x_0', X_I', X_{II}', X', f', S', S_{II}')$ is a subgame of $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ then S' is a closed, non-empty subset of S.

THEOREM 5. If $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ is a game and F is a closed non-empty subset of S then there is a unique subgame of Γ whose space is F with its relative topology as a subset of S.

NOTATION. If F is a closed, non-empty subset of S then the corresponding subgame of Γ will be denoted by Γ_F, and X_I^F, X_{II}^F, X^F, f^F, S^F, S_I^F, S_{II}^F will denote the objects such that $\Gamma_F = (x_0, X_I^F, X_{II}^F, X^F, f^F, S^F, S_I^F, S_{II}^F)$. If F is a neighborhood $\mathcal{U}(x)$ we will use Γ_x, X_I^x, etc. instead of $\Gamma_{\mathcal{U}(x)}$, $X_I^{\mathcal{U}(x)}$, etc.

We now define certain simple set theoretic operations on games.

DEFINITION 9. If $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ and $\Gamma' = (x_0, X_I, X_{II}, X, f, S, S_I', S_{II}')$ then

we define the intersection of Γ and Γ' by

$$\Gamma \cap \Gamma' = (x_o, X_I, X_{II}, X, f, S, S_I \cap S_I', S - (S_I \cap S_I'))$$

and the union of Γ and Γ' by

$$\Gamma \cup \Gamma' = (x_o, X_I, X_{II}, X, f, S, S_I \cup S_I', S - (S_I \cup S_I')) .$$

Note that unions and intersections can only be defined for pairs of games having the same x_o, X_I, X_{II}, X and f.

DEFINITION 10. We define the negative of the game $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ to be the game $-\Gamma = (x_o, X_I, X_{II}, X, f, S, S_{II}, S_I)$.

In section 5 we give examples to show that the unions, intersections, and negatives of strictly determined games need not be strictly determined. Accordingly one is led to consider the following stronger condition of determinacy for a game.

DEFINITION 11. A win-lose game Γ is absolutely determined if all subgames of Γ and $-\Gamma$ are strictly determined.

We are now in a position to state our remaining results. The proofs of most of them will be given in the next section.

THEOREM 6. The union of an open game and an absolutely determined game is strictly determined.

COROLLARY 7. An open or closed game is strictly determined.

COROLLARY 8. A discrete game is strictly determined.

COROLLARY 9. The intersection of a closed game and an absolutely determined game is strictly determined.

COROLLARY 10. An open or closed game is absolutely determined.

THEOREM 11. The intersection of an open game with

an absolutely determined game is strictly determined.

COROLLARY 12. The union of a closed game with an absolutely determined game is strictly determined.

COROLLARY 13. The union and intersection of an open or closed game with an absolutely determined game are absolutely determined.

THEOREM 14. If $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ and S_I belongs to the Boolean algebra generated by the open sets then Γ is absolutely determined.

THEOREM 15. If $\Gamma = (x_0, X_I, X_{II}, X, f, S, \Phi)$ is a game with pay-off and Φ is continuous then Γ is strictly determined.

THEOREM 16. If $\Gamma = (x_0, X_I, X_{II}, X, f, S, \Phi)$ is a game with pay-off the following three conditions are equivalent:

i) For every $x \in X$, $f^{-1}(x)$ is a finite set.

ii) S is compact.

iii) For every continuous pay-off function Ψ on S the value, v', of $\Gamma' = (x_0, X_I, X_{II}, X, f, S, \Psi)$ can be achieved. (I.e., there is a $\sigma_0 \in \sum_I(\Gamma')$ and a $\tau_0 \in \sum_{II}(\Gamma')$ such that $\inf_{\tau \in \sum_{II}(\Gamma')} \Psi(\langle \sigma_0, \tau \rangle) = \sup_{\sigma \in \sum_I(\Gamma')} \Psi(\langle \sigma, \tau_0 \rangle) = \Psi(\sigma_0, \tau_0) = v'$.

§ 4. PROOFS

This section consists of the proofs of Propositions 4 through 16.

PROOF OF THEOREM 4. Since Γ' is a game, S' cannot be empty. Let s be any element of $S - S'$. For some n, $s(n) \notin X'$ since S' is the set of all sequences satisfying (1.3) and (1.4) with $f' = f|X'$ in place of f. Now if $t \in \mathcal{U}(s(n))$ then $t(n) = s(n) \notin X'$ so $S - S'$ contains a neighborhood of s. Thus $S - S'$ is open and S' is closed.

PROOF OF THEOREM 5. Let $X^F = \{x | x \in X$ and $\mathcal{U}(x) \cap F \neq \Lambda\}$, $X_I^F = X_I \cap X^F$, $X_{II}^F = X_{II} \cap X^F$, $f^F = f|X^F$, $S^F = F$, $S_I^F = S_I \cap F$, $S_{II}^F = S_{II} \cap F$, and $\Gamma_F = (x_0, X_I^F, X_{II}^F, X^F, f^F, S^F, S_I^F, S_{II}^F)$. Since (3.3) - (3.7) are obviously satisfied, Γ_F is a subgame of Γ if Γ_F is a game. We proceed therefore to verify that Γ_F satisfies (1.5) - (1.9).

Clearly (1.5) and (1.6) are satisfied.

Now (1.1) and $f^F = f|X^F$ entail (1.2) so (1.7) depends only on verifying (1.1) for f^F. Since $\mathcal{U}(x) \subset \mathcal{U}(f(x))$ the range of f^F is contained in X^F. By definition, if $x \in X^F$ then there is an $s \in F$ and an integer $n \geq 0$ such that $x = s(n)$. But $s \in \mathcal{U}(s(n+1))$ so $s(n+1) \in X^F$ and $x = s(n) = f(s(n+1)) = f^F(s(n+1))$ so the range of f^F contains X^F.

As above if $s \in F$ then $s(i) \in X^F$ for every integer $i \geq 0$. Thus (1.3) and (1.4) hold for every $s \in F$. Hence to establish (1.8) we need only show that every sequence, s, satisfying (1.3) and (1.4) belongs to F. If $s(0) = x_0$ and $s(i-1) = f^F(s(i))$ then $s(i) \in X^F$. Hence every neighborhood, $\mathcal{U}(s(i))$, of s contains points of F. Since F is closed, $s \in F$.

Condition (1.9) is clearly satisfied.

Thus Γ_F is a subgame. We now show that it is unique by assuming that $\Gamma' = (x_0', X_I', X_{II}', X', f', F, S_I', S_{II}')$ is a subgame and proving $\Gamma' = \Gamma_F$.

By (3.3), $x_0' = x_0$.

If $x \in X'$ then by (1.1) there is an n such that $f'^n(x) = x_0$. Set $s(i) = f'^{n-i}(x)$ for $0 \leq i \leq n$. Since f is onto (i.e., (1.1)) we can, by induction, define $s(i)$ for $i = n+1, n+2, \ldots$ so that $s(i) = f'(s(i+1))$. By (1.8), $s \in F$. Since $s \in \mathcal{U}(s(n)) = \mathcal{U}(x)$ it follows that $x \in X^F$.

Conversely if $x \in X^F$ then, for some $s \in F$ and some integer $n \geq 0$, $x = s(n)$. Using (1.8) and (1.4) $s(i) \in X'$ for all i, and in particular $x = s(n) \in X'$. Thus $X^F \subset X'$.

Hence $X^F = X'$ and, by (1.6), (3.4) and the definitions of X_I^F and X_{II}^F, $X_I^F = X_I'$ and $X_{II}^F = X_{II}'$.

From (3.5) and (3.6), $f' = f|X^F$ and $S_I' = S_I \wedge F$ and $S_{II}' = S_{II} \wedge F$ which completes the proof.

In the following proofs we need to know something of the relationship between strategies in a game and those in its subgames. If $\Gamma' = (x_0, X_I', X_{II}', f', S', S_I', S_{II}')$ is a subgame of Γ a strategy $\sigma \in \Sigma_I(\Gamma)$ is an <u>extension</u> of $\sigma' \in \Sigma_I(\Gamma')$ if

$$\sigma' = \sigma|X_I' . \tag{4.1}$$

Extensions of $\tau' \in \Sigma_{II}(\Gamma')$ are defined similarly. Clearly every strategy for a subgame has an extension which is a strategy for the original game.

LEMMA 4.1. If

$$\Gamma' = (x_0, X_I', X_{II}', X', f', S', S_I', S_{II}') \text{ is a} \tag{4.2}$$

subgame of Γ

(4.3) $\sigma \in \Sigma_I(\Gamma)$ is an extension of $\sigma' \in \Sigma_I(\Gamma')$

(4.4) $\tau \in \Sigma_{II}(\Gamma)$

(4.5) $\langle \sigma, \tau \rangle \in S'$

then there is a $\tau' \in \Sigma_{II}(\Gamma')$ such that $\langle \sigma', \tau' \rangle = \langle \sigma, \tau \rangle$.

PROOF. Let $\langle \sigma, \tau \rangle = s$. By (4.5), (1.4), and (3.5) $s(i) \in X'$ for all i. Let $\tau'(s(i)) = s(i+1)$ if $s(i) \in X'_{II}$, and let $\tau'(x)$ be any element of $f'^{-1}(x)$ if $x \notin X'_{II}$ and $x \neq s(i)$ for each i. Let $s' = \langle \sigma', \tau' \rangle$. If $s(i) = s'(i) \in X'_I$ then by (4.1) and (4.3) $s(i+1) = \sigma(s(i)) = \sigma'(s'(i)) = s'(i+1)$. If $s(i) = s'(i) \in X_{II}$ then $s(i+1) = \tau(s(i)) = s'(i+1)$. Since $s(0) = x_0 = s'(0)$ we have $s = s'$ by induction.

LEMMA 4.2. If $\Gamma = (x_0, X_I, X_{II}, X, f, \ldots, S_I, \ldots)$ is a game; $Y \subset X$; for each $y \in Y$, there is a $\sigma_y \in \Sigma_I(\Gamma_y)$; and $\sigma_0 \in \Sigma_I(\Gamma)$ then there is a $\sigma \in \Sigma_I(\Gamma)$ such that for each $\tau \in \Sigma_{II}(\Gamma)$ either

(4.6) $\langle \sigma, \tau \rangle = \langle \sigma_0, \tau \rangle \notin \bigcup_{y \in Y} \mathcal{U}(y)$

or

(4.7) For some $y \in Y$ and some $\tau_y \in \Sigma_{II}(\Gamma_y)$, $\langle \sigma, \tau \rangle = \langle \sigma_y, \tau_y \rangle$.

PROOF. If z is a successor of some $y \in Y$ let $\phi(z)$ be the "last" predecessor of z in Y. I.e., $\phi(z) = f^{n(z)}(z)$ where $n(z) = \sup \{i \mid f^i(z) \in Y\}$. Let

$$\sigma(x) = \begin{cases} \sigma_0(x) & \text{if } x \in X_I \text{ and, for all } i \geq 0,\ f^i(x) \notin Y \\ \sigma_{\phi(x)}(x) & \text{if } x \in X_I \text{ and, for some } i \geq 0,\ f^i(x) \in Y. \end{cases}$$

If $\tau \in \Sigma_{II}(\Gamma)$ let $s = \langle \sigma, \tau \rangle$.

If $s \in \bigcup_{y \in Y} \mathcal{U}(y)$ then there is a least integer n such that $s(n) \in Y$. For $x = f^i(s(n)) = s(n-i) \in X_I$, $i = 0, 1, \ldots, n$, $\sigma(x) = s(n-i+1) = \sigma_{s(n)}(x)$ so σ is an extension of $\sigma_{s(n)}$. By Lemma 4.1 there is a $\tau_{s(n)} \in \Sigma_{II}(\Gamma_{s(n)})$ such that $\langle \sigma, \tau \rangle = \langle \sigma_{s(n)}, \tau_{s(n)} \rangle$.

If $s \notin \bigcup_{y \in Y} \mathcal{U}(y)$ then $s(i) \notin Y$ for each i. Hence if $s(i) \in X_I$, $s(i+1) = \sigma(s(i)) = \sigma_0(s(i))$ so $\langle \sigma, \tau \rangle = \langle \sigma_0, \tau \rangle$.

LEMMA 4.3. If Γ' is a subgame of Γ, $\sigma \in \Sigma_I(\Gamma)$ is an extension of $\sigma' \in \Sigma_I(\Gamma')$, and $\tau \in \Sigma_{II}(\Gamma)$ is

an extension of $\tau' \in \Sigma_{II}(\Gamma')$ then $\langle \sigma, \tau \rangle = \langle \sigma', \tau' \rangle$.

PROOF. Let $s = \langle \sigma, \tau \rangle$, $s' = \langle \sigma', \tau' \rangle$. By (3.3) and (1.3), $s(0) = s'(0)$. If $s(i) = s'(i) \in X_I$, then, by (3.5), $s(i+1) = \sigma(s(i)) = \sigma'(s'(i)) = s'(i+1)$. Similarly if $s(i) = s'(i) \in X_{II}$ then $s(i+1) = \tau(s(i)) = \tau'(s'(i)) = s'(i+1)$. In either case $s(i+1) = s'(i+1)$. Hence by induction $s(i) = s'(i)$ for all i, or $s = s'$.

LEMMA 4.4. If $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ is a game, $f(x) \in X_I$, and $\Sigma_I^W(\Gamma_x)$ is not empty then $\Sigma_I^W(\Gamma_{f(x)})$ is not empty.

PROOF. It is clear that Γ_x is a subgame of $\Gamma_{f(x)}$. If $\sigma' \in \Sigma_I^W(\Gamma_x)$ let $\sigma \in \Sigma_I(\Gamma_{f(x)})$ be an extension of σ'. If $\tau \in \Sigma_{II}(\Gamma_{f(x)})$ then, since $f(x) \in X_I$, $\tau' = \tau | X_{II}^x \in \Sigma_{II}(\Gamma_x)$. By Lemma 4.3, $\langle \sigma, \tau \rangle = \langle \sigma', \tau' \rangle \in S_I$ because $\sigma' \in \Sigma_I^W(\Gamma_x)$. Thus $\sigma \in \Sigma_I^W(\Gamma_{f(x)})$.

LEMMA 4.5. If $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ is a game and if $x \in X$, such that $f(y) = x$ implies $\Sigma_I^W(\Gamma_y)$ is not empty, then $\Sigma_I^W(\Gamma_x)$ is not empty.

PROOF. If $x \in X_I$ then $\Sigma_I^W(\Gamma_x)$ is not empty by Lemma 4.4. If $x \in X_{II}$ let $Y = f^{-1}(x)$. For each $y \in Y$ choose $\sigma_y \in \Sigma_I^W(\Gamma_y)$. Let σ_o be any element of $\Sigma_I(\Gamma)$. Let σ be a strategy determined from Y, σ_y, σ_o according to Lemma 4.2. Since $\bigcup_{y \in Y} \mathcal{U}(y) = S^x$, (4.6) can never hold. Thus for any τ, and some $y \in f^{-1}(x)$, $\langle \sigma, \tau \rangle = \langle \sigma_y, \tau_y \rangle$ where $\tau_y \in \Sigma_{II}(\Gamma_y)$. Since $\sigma_y \in \Sigma_I^W(\Gamma_y)$, $\langle \sigma, \tau \rangle \in S_I$.

To avoid unnecessary repetition we do not prove the dual lemmas with the roles of the players interchanged.

PROOF OF THEOREM 6. Let G be open in S and let Γ be absolutely determined. We shall show that

$$\Gamma' = (x_o, X_I, X_{II}, X, f, S, S_I \cup G, S_{II} - G)$$

is strictly determined.

Let

$$\Gamma^* = (x_o, X_I, X_{II}, X, f, S, G, S - G),$$

$$W_I^* = \{x | \Sigma_I^W(\Gamma_x^*) \text{ is not empty}\},$$

$$F = S - \bigcup_{x \in W_I^*} \mathcal{U}(x).$$

If $\mathcal{U}(x) \subset G$ then every $\sigma \in \Sigma_I(\Gamma_x)$ will win for player I, so $x \in W_I^*$. Since G is open it is the union of all neighborhoods contained in it. Hence

$$G \subset \bigcup_{x \in W_I^*} \mathcal{U}(x) \ . \tag{4.8}$$

If $x_0 \in W_I^*$ then any $\sigma \in \Sigma_I^W(\Gamma_{x_0}^*)$ will, a fortiori, win Γ'.

If $x_0 \notin W_I^*$ then, by induction using Lemmas 4.4 and 4.5, there is an $s \in S$ such that $s(i) \notin W_I^*$ for each i. Accordingly $s \in F$, so F is not void and determines a subgame, Γ_F, of Γ.

Since Γ is absolutely determined it remains only to show

A: If $\Sigma_I^W(\Gamma_F) \neq \Lambda$ then $\Sigma_I^W(\Gamma') \neq \Lambda$, and

B: If $\Sigma_{II}^W(\Gamma_F) \neq \Lambda$ then $\Sigma_{II}^W(\Gamma') \neq \Lambda$.

PROOF OF A. For each $x \in W_I^*$ choose $\sigma_x \in \Sigma_I^W(\Gamma_x^*)$. Choose $\sigma_1 \in \Sigma_I^W(\Gamma_F)$ and let σ_0 be any extension of σ_1. By Lemma 4.2 there is a $\sigma \in \Sigma_I(\Gamma)$ such that for each $\tau \in \Sigma_{II}(\Gamma)$ either

$$\langle \sigma, \tau \rangle = \langle \sigma_0, \tau \rangle \in F \tag{4.9}$$

or

$$\text{For some } x \in W_I^*, \text{ and some } \tau_x \in \Sigma_{II}(\Gamma_x) \tag{4.10}$$
$$\langle \sigma, \tau \rangle = \langle \sigma_x, \tau_x \rangle \ .$$

If (4.9) obtains then, by Lemma 4.1, there is a $\tau_1 \in \Sigma_{II}(\Gamma_F)$ such that $\langle \sigma, \tau \rangle = \langle \sigma_0, \tau \rangle = \langle \sigma_1, \tau_1 \rangle$. Because $\sigma_1 \in \Sigma_I^W(\Gamma_F)$, $\langle \sigma, \tau \rangle \in S_I$.

If (4.10) obtains then, since $\sigma_x \in \Sigma_I^W(\Gamma_x^*)$, $\langle \sigma, \tau \rangle \in G$.

In either case $\langle \sigma, \tau \rangle \in S_I \cup G$ so $\sigma \in \Sigma_I^W(\Gamma')$.

PROOF OF B. Choose $\tau_F \in \Sigma_{II}^W(\Gamma_F)$. Choose $\tau(x) = \tau_F(x)$ if $x \in X_{II}^F$; $\tau(x)$ is any element of $f^{-1}(x) - W_I^*$ if such exist. If x is any other element of X_{II} then $\tau(x)$ is any element of $f^{-1}(x)$. Let σ be any element of $\Sigma_I(\Gamma')$. If $\langle \sigma, \tau \rangle \in F$ then by Lemma 4.1 and the fact that

$$\tau_F \in \Sigma_{II}^W(\Gamma_F) \ , \ \langle \sigma, \tau \rangle \in F - S_I = S - \bigcup_{x \in W_I^*} \mathcal{U}(x) - S_I \subset S_{II} - G \ .$$

If $\langle \sigma, \tau \rangle = s \notin F$, then there would be a least integer $n \geq 0$ such that $s(n) \in W_I^*$. By Lemma 4.4 $s(n-1) = f(s(n)) \notin X_I$. But if $s(n-1) \in X_{II} - W_I^*$ then by Lemma 4.5 there would be a $y \in f^{-1}(s(n-1)) - W_I^*$. Accordingly, by the definition of τ,

$s(n) = \tau(s(n-1)) \in f^{-1}(s(n-1)) - W_I^*$, which is impossible.

Thus for every σ, $\langle\sigma,\tau\rangle \in S_{II} - G$.

PROOF OF COROLLARY 7. Apply Theorem 6 to the special case where the absolutely determined game is $\Gamma = (x_o, X_I, X_{II}, X, f, S, \Lambda, S)$. This proves the corollary for open games. If $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ is closed then $\Gamma' = (x_o, X_{II}, X_I, X, f, S, S_{II}, S_I)$ is open. The corollary then follows from the obvious 1-1 correspondences between $\Sigma_I(\Gamma')$ and $\Sigma_{II}(\Gamma)$ and between $\Sigma_{II}(\Gamma')$ and $\Sigma_I(\Gamma)$ which maps winning strategies onto winning strategies.

PROOF OF COROLLARY 8. In a discrete game every set $S_I \subset S$ is open. Hence by Corollary 7, a discrete game is strictly determined.

PROOF OF COROLLARY 9. Let $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ be absolutely determined and let F be a closed subset of S. We must show that $\Gamma' = (x_o, X_I, X_{II}, f, S, S_I \cap F, S - (S_I \cap F))$ is strictly determined.

Consider the game

$$\Gamma'' = (x_o, X_{II}, X_I, X, f, S, S - (S_I \cap F), S_I \cap F)$$

and note that $S - (S_I \cap F) = S_{II} \cup (S - F)$. Since Γ is absolutely determined Γ'' is by Theorem 6 strictly determined, and reversing the roles of the two players again shows that so is Γ'.

PROOF OF COROLLARY 10. Let $S_I \subset S$ be open. If $\Gamma_F = (x_o, X_I^F, X_{II}^F, X^F, f^F, S^F = F, S_I \cap F, S_{II} \cap F)$ is the subgame of Γ determined by the closed subset F then the topology of S^F is just the relative topology of F as a subset of S. By the previous corollaries Γ_F and $-\Gamma_F = (x_o, X_I^F, X_{II}^F, X^F, f^F, S^F, S_{II} \cap F, S_I \cap F)$ are strictly determined.

PROOF OF THEOREM 11. Let $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ be absolutely determined and let G be an open set in S. We must show that $(x_o, X_I, X_{II}, f, S, S_I \cap G, S - (S_I \cap G))$ is strictly determined.

Let

$$Y = \{x \mid \mathcal{U}(x) \subset G \text{ and } \Sigma_I^W(\Gamma_x) \neq \Lambda\}$$

$$Z = \{x \mid \mathcal{U}(x) \subset G \text{ and } \Sigma_{II}^W(\Gamma_x) \neq \Lambda\}.$$

For $y \in Y$ choose $\sigma_y \in {}^{W}_{I}(\Gamma_y)$. For $z \in Z$ choose $\tau_z \in {}^{W}_{II}(\Gamma_z)$. Let $\Gamma^* = (x_o, X_I, X_{II}, X, f, S, \bigcup_{x\in Y}\mathcal{U}(x), S - \bigcup_{x\in Y}\mathcal{U}(x))$. By Corollary 7 Γ^* is strictly determined.

CASE A: $\Sigma_I^W(\Gamma^*) \neq \Lambda$.

Choose $\sigma_o \in \Sigma_I^W(\Gamma^*)$ and determine a strategy using Y, σ_y, and σ_o as in Lemma 4.2.

Let τ be any element of $\Sigma_{II}(\Gamma)$. Since $\langle\sigma_o,\tau\rangle \in \bigcup_{x\in Y}\mathcal{U}(x)$, (4.6) cannot obtain. Hence for some $y \in Y$ and some $\tau_y \in \Sigma_{II}(\Gamma_y)$, $\langle\sigma,\tau\rangle = \langle\sigma_y,\tau_y\rangle$. Since $\sigma_y \in \Sigma_I^W(\Gamma_y)$, $\langle\sigma_y,\tau_y\rangle \in S_I \cap \mathcal{U}(y) \subset S_I \cap G$. Thus $\sigma \in \Sigma_I^W(\Gamma')$.

CASE B: $\Sigma_{II}^W(\Gamma^*) \neq \Lambda$.

Choose $\tau_o \in \Sigma_{II}^W(\Gamma^*)$. We can apply the dual of Lemma 4.2 to the set Z, the strategies τ_z, $z \in Z$, and τ_o to determine a $\tau \in \Sigma_{II}(\Gamma')$ such that for any $\sigma \in \Sigma_I(\Gamma')$ either

$$\langle\sigma,\tau\rangle = \langle\sigma,\tau_o\rangle \notin \bigcup_{z\in Z}\mathcal{U}(z) \tag{4.11}$$

or

(4.12) for some $z \in Z$ and some $\sigma_z \in \Sigma_I(\Gamma_z')$, $\langle\sigma,\tau\rangle = \langle\sigma_z,\tau_z\rangle$.

If (4.11) obtains then, since $\tau_o \in \Sigma_{II}^W(\Gamma^*)$, $\langle\sigma,\tau\rangle = \langle\sigma,\tau_o\rangle \notin \bigcup_{y\in Y}\mathcal{U}(y)$. Using this and (4.11), $\langle\sigma,\tau\rangle \in S - \bigcup_{y\in Y}\mathcal{U}(y) - \bigcup_{z\in Z}\mathcal{U}(z)$. Since Γ is absolutely determined and G is open,

$$S - \bigcup_{y\in Y}\mathcal{U}(y) - \bigcup_{z\in Z}\mathcal{U}(z) = S - \bigcup_{\mathcal{U}(x)\subset G}\mathcal{U}(x) = S - G \subset S - (S_I \cap G).$$

If (4.12) obtains then, since $\tau_z \in \Sigma_{II}^W(\Gamma_z)$, $\langle\sigma,\tau\rangle = \langle\sigma_z,\tau_z\rangle \in S_{II} = S - S_I \subset S - (S_I \cap G)$.

In either case τ defeats σ, so $\tau \in \Sigma_{II}^W(\Gamma')$.

The proof of Corollary 12 is like that of Corollary 9.

PROOF OF COROLLARY 13. Let $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ be absolutely determined and let T be an open subset of S. We must show that $(x_o, X_I, X_{II}, f, S, S_I \cup T, S_{II} - T)$ and $(x_o, X_I, X_{II}, f, S, S_I \cap T, S - (S_I \cap T))$ are absolutely determined.

Let F be a closed subset of S. Now $(S_I \cup T) \cap F = (S_I \cap F) \cup (T \cap F)$ and $(S_{II} - T) \cap F = (S_{II} \cap F) \cap ((S - T) \cap F)$. Since the topology in $S^F = F$ is its relative topology as a subset of S, $(x_o, X_I^F, X_{II}^F, X^F, f^F, S^F, (S_I \cup T) \cap F, (S_{II} - T) \cap F)$ and $(x_o, X_I^F, X_{II}^F, X^F, f^F, S^F, (S_{II} - T) \cap F, (S_I \cup T) \cap F)$, are, by Propositions 6, 9, 11, and 12, strictly determined. I.e., $(x_o, X_I, X_{II}, X, f, S, S_I \cup T, S_{II} - T)$ is absolutely determined.

We omit the proof for $(x_o, X_I, X_{II}, X, f, S, S_I \cap T, S - (S_I \cap T))$.

PROOF OF THEOREM 14. Because the topology of the space of a subgame is the relative topology it will suffice to show that Γ is strictly determined.

Every element of the Boolean algebra generated by the open sets belongs to a Boolean algebra generated by n particular open sets. We assume that the theorem is true if S_I belongs to a Boolean algebra

generated by n open sets and prove that it then holds with n replaced by $n + 1$. Corollary 13 ensures that it is true for $n = 1$.

We assume that S_I belongs to the Boolean algebra generated by the open sets $G_1, \ldots, G_{n+1}$. We give the proof for the case

$$G_1' \cap \ldots \cap G_{n+1}' \subset S_{II} \quad (G_i' \text{ here denotes } S - G_i) \,. \tag{4.13}$$

The proof for the contrary case, $G_1' \cap \ldots \cap G_{n+1}' \subset S_I$ is obtained by interchanging the roles of the players.

Let $Y = \{y | \mathcal{U}(y) \subset G_i$ for some i and $\sum_I^W(\Gamma_y) \neq \Lambda\}$, $Z = \{z | \mathcal{U}(z) \subset G_i$ for some i and $\sum_{II}^W(\Gamma_z) \neq \Lambda\}$. For each $y \in Y$ choose $\sigma_y \in \sum_I^W(\Gamma_y)$ and for each $z \in Z$ choose $\tau_z \in \sum_{II}^W(\Gamma_z)$.

If $\mathcal{U}(x) \subset G_i$ for some i then $S_I^x = S_I \cap \mathcal{U}(x) \subset S_I \cap G_i$. Thus S_I^x belongs to the Boolean algebra generated by $G_1 \cap \mathcal{U}(x), \ldots, G_{i-1} \cap \mathcal{U}(x), G_{i+1} \cap \mathcal{U}(x), \ldots, G_{n+1} \cap \mathcal{U}(x)$. By the inductive hypothesis Γ_x is strictly determined so $x \in Y \cup Z$ and, using (4.13)

$$S_I \subset G_1 \cup \ldots \cup G_{n+1} = \bigcup_{y \in Y} \mathcal{U}(y) \cup \bigcup_{z \in Z} \mathcal{U}(z) \,. \tag{4.14}$$

By Corollary 10, $\Gamma' = (x_o, X_I, X_{II}, X, f, S, \bigcup_{y \in Y} \mathcal{U}(y), S - \bigcup_{y \in Y} \mathcal{U}(y))$ is strictly determined. We thus have two cases to consider.

CASE A: $\sum_I^W(\Gamma') \neq \Lambda$.

Choose $\sigma_o \in \sum_I^W(\Gamma')$ and let σ be determined from Y, σ_y, and σ_o according to Lemma 4.2. Let τ be any element of $\sum_{II}(\Gamma)$.

Since $\sigma_o \in \sum_I^W(\Gamma')$, (4.6) cannot hold. Thus from (4.7) there is a $y \in Y$ and a $\tau_y \in \sum_{II}(\Gamma_y)$ such that $\langle \sigma, \tau \rangle = \langle \sigma_y, \tau_y \rangle$. Since $\sigma_y \in \sum_I^W(\Gamma_y)$, $\langle \sigma, \tau \rangle \in S_I \cap \mathcal{U}(y) \subset S_I$ and $\sigma \in \sum_I^W(\Gamma)$.

CASE B: $\sum_{II}^W(\Gamma') \neq \Lambda$.

Choose $\tau_o \in \sum_{II}^W(\Gamma')$ and let τ be determined from Z, τ_z, and τ_o according to the dual of Lemma 4.2. If $\sigma \in \sum_I(\Gamma)$ then either

$$\langle \sigma, \tau \rangle = \langle \sigma, \tau_o \rangle \notin \bigcup_{z \in Z} \mathcal{U}(z) \,, \tag{4.15}$$

or

$$\text{for some } z \in Z \text{ and some } \sigma_z \in \sum_I(\Gamma_z),\ \langle \sigma, \tau \rangle = \langle \sigma_z, \tau_z \rangle \,. \tag{4.16}$$

If (4.15) holds then, since $\tau_o \in \sum_{II}^W(\Gamma')$, $\langle \sigma, \tau \rangle \notin \bigcup_{y \in Y} \mathcal{U}(y)$. Combining this with (4.15) we see by (4.14) that $\langle \sigma, \tau \rangle \notin S_I$.

If (4.16) obtains then, since $\sigma_z \in \sum_{II}^W(\Gamma_z)$, $\langle \sigma, \tau \rangle \in S_{II} \cap \mathcal{U}(z)$.

In either case $\langle \sigma, \tau \rangle \in S_{II}$ so $\tau \in \sum_{II}^W(\Gamma)$.

PROOF OF THEOREM 15. Let $S_\alpha = \{s | s \in S, \Phi(s) > \alpha\}$. Since Φ

is continuous $\Gamma_\alpha = (x_0, X_I, X_{II}, X, f, S, S_\alpha, S - S_\alpha)$ is strictly determined. Let $v = \sup \{\alpha | \sum_I^W (\Gamma_\alpha) \neq \Lambda\}$. If $v_1 < v$ then there is an α, $v_1 < \alpha$ such that $\sum_I^W (\Gamma_\alpha)$ is not void. Choose $\sigma_0 \in \sum_I^W (\Gamma_\alpha) \subset \sum_I(\Gamma)$. If $\tau \in \sum_{II}(\Gamma) = \sum_{II}(\Gamma_\alpha)$ then $\langle \sigma_0, \tau \rangle \in S_\alpha$ so $\Phi(\langle \sigma_0, \tau \rangle) > \alpha > v_1$. Thus $\sup_{\sigma \in \sum_I(\Gamma)} \inf_{\tau \in \sum_{II}(\Gamma)} \Phi(\langle \sigma_0, \tau \rangle) \geq \inf_{\tau \in \sum_{II}(\Gamma)} \Phi(\langle \sigma_0, \tau \rangle) \geq \alpha > v_1$.

Similarly if $v_2 > v$ then there is an $\alpha < v_2$ such that $\sum_{II}^W (\Gamma_\alpha)$ is not void. There is, as before, a τ_0 such that $\inf_{\tau \in \sum_{II}(\Gamma)} \sup_{\sigma \in \sum_I(\Gamma)} \Phi(\langle \sigma, \tau \rangle) \leq \sup_{\sigma \in \sum_I(\Gamma)} \Phi(\langle \sigma, \tau_0 \rangle) \leq \alpha < v_2$.

Since $\sup_{\sigma \in \sum_I(\Gamma)} \inf_{\tau \in \sum_{II}(\Gamma)} \Phi(\langle \sigma, \tau \rangle) \leq \inf_{\tau \in \sum_{II}(\Gamma)} \sup_{\sigma \in \sum_I(\Gamma)} \Phi(\langle \sigma, \tau \rangle)$ for any Φ, it follows that $\inf_{\tau \in \sum_{II}(\Gamma)} \sup_{\sigma \in \sum_I(\Gamma)} \Phi(\langle \sigma, \tau \rangle) = \sup_{\sigma \in \sum_I(\Gamma)} \inf_{\tau \in \sum_{II}(\Gamma)} \Phi(\langle \sigma, \tau \rangle) = v$.

PROOF OF THEOREM 16. We omit details.

If $f^{-1}(x)$ is infinite then the set $\{\mathcal{U}(y) | y \in f^{-1}(x)\} \cup \{\mathcal{U}(z) | \mathcal{U}(z) \wedge \mathcal{U}(x) = \Lambda\}$ is a family of open sets covering S. No finite subfamily can cover S. Thus ii) implies i).

By setting up a 1-1 mapping from S into a closed, totally disconnected subset of the unit interval one can show that i) implies ii).

We omit the construction of a function Ψ showing that when i) fails so does iii).

To show that i) implies iii) one places on the sets of strategies a topology analogous to that on S. Then $\langle \sigma, \tau \rangle$ is a continuous function of σ and τ. If i) holds then both $\sum_I(\Gamma)$ and $\sum_{II}(\Gamma)$ are compact. Using the S_α, Γ_α of the proof of Theorem 15 one notes that $S - S_\alpha$ is closed. From this and the continuity of $\langle \sigma, \tau \rangle$ it follows that $\sum_{II}^W (\Gamma_\alpha)$ is closed. Now $\alpha > \beta$ implies $\sum_{II}^W (\Gamma_\beta) \subset \sum_{II}^W (\Gamma_\alpha)$. Choose a sequence $\alpha_1 > \alpha_2 > \ldots > v$ converging to v. There is a $\tau_0 \in \sum_{II}^W (\Gamma_{\alpha_n})$ for all $n > 0$. For any $\sigma \in \sum_I(\Gamma)$, $\Phi(\langle \sigma, \tau_0 \rangle) < \alpha_n$ so $\Phi(\langle \sigma, \tau_0 \rangle) \leq v$.

§ 5. EXAMPLES AND QUESTIONS

Simple examples obtained by inserting indeterminate games as subgames show that various plausible conjectures are false. Throughout this section $\Gamma^* = (y_0, Y_I, Y_{II}, Y, g, T, T_I, T_{II})$ will stand for some fixed game which is not strictly determined. The reader will find that sketching the topological tree corresponding to the rather laborious descriptions given makes the ideas comparatively transparent.

EXAMPLE 1. A subgame of a strictly determined game need not be strictly determined.

Let $\{x_0, x_1, \ldots\}$ be any sequence of distinct positions. Let $X_I = Y_I \cup \{x_i | i \text{ is odd}\}$, $X_{II} = Y_{II} \cup \{x_i | i \text{ is even}\}$, $X = X_I \cup X_{II}$, and

let $f(x) = g(x)$ if $x \in Y - y_0$, $f(y_0) = x_0$, $f(x_i) = x_{i-1}$ if $i \geq 1$. Let S_{II} be $\{\{x_0, y_0, y_1, \ldots\} | \{y_0, y_1, \ldots\} \in T_{II}\}$ together with the sequence $\{x_0, x_1, \ldots\}$ and let $S_I = \{\{x_0, y_0, y_1, \ldots\} | \{y_0, y_1, \ldots\} \in T_I\}$, $S = S_I \cup S_{II}$. If $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ then any $\tau \in \Sigma_{II}(\Gamma)$ such that $\tau(x_0) = x_1$ belongs to $\Sigma^w_{II}(\Gamma)$. The subgame determined by the closed set $\{\{x_0, y_0, y_1 \ldots\} | \{y_0, y_1, \ldots\} \in T\} \subset S$ is not strictly determined.

EXAMPLE 2. The game $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ may be strictly determined while $-\Gamma = (x_0, X_I, X_{II}, X, f, S, S_{II}, S_I)$ is not.

Let X_I, X_{II}, X, f be the same as in Example 1. Let S_{II} be $\{\{x_0, y_0, y_1, \ldots\} | \{y_0, y_1, \ldots\} \in T_I\}$ together with $\{x_0, x_1, \ldots\}$ and let S_I be $\{\{x_0, y_0, y_1, \ldots\} | \{y_0, y_1, \ldots\} \in T_{II}\}$. Again $\tau(x_0) = x_1$ implies $\tau \in \Sigma^w_{II}(\Gamma)$. Each strategy in $\Sigma_I(-\Gamma)$ consists of a function on $x_1, x_3, \ldots$, and a function on Y_I. If $\tau \in \Sigma_{II}(-\Gamma)$ and $\tau(x_0) = x_1$ then for any $\sigma \in \Sigma_I(-\Gamma)$, $\langle\sigma,\tau\rangle = \{x_0, x_1, \ldots\} \in S_{II}$ so $\tau \notin \Sigma^w_{II}(-\Gamma)$. On the other hand if $\tau(x_0) = y_0$ then $\langle\sigma,\tau\rangle = s$, where $s(0) = x_0$, and $s(i+1) = t(i)$ and $t = \langle\sigma^*,\tau^*\rangle$ with $\sigma^* \in \Sigma_I(\Gamma^*)$, $\tau^* \in \Sigma_{II}(\Gamma^*)$. Since Γ^* is not strictly determined neither is $-\Gamma$.

EXAMPLE 3. The intersection and union of strictly determined games need not be strictly determined. I.e., there are sets S^1_I and S^2_I such that $\Gamma_i = (x_0, X_I, X_{II}, X, f, S, S^i_I, S - S^i_I)$ are strictly determined, but $\Gamma = (x_0, X_I, X_{II}, X, f, S, S^1_I \cup S^2_I, S - (S^1_I \cup S^2_I))$ is not.

Let Γ_1 be the game of Example 1. Let S^2_I consist only of $\{x_0, x_1, \ldots\}$.

An interchange of the roles of the players shows that union may be replaced by intersection.

EXAMPLE 4. There are games $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ and $\Gamma' = (x'_0, X'_I, X'_{II}, X', f', S', S'_I, S'_{II})$ and a function ϕ such that Γ is strictly determined while Γ' is not, although ϕ is a homeomorphism of S onto S' which carries S_I and S_{II} onto S'_I and S'_{II}. I.e., strict determinacy is not a topological invariant.

Let $\Gamma^*_i = (y^i_0, Y^i_I, Y^i_{II}, g_i, T^i, T^i_I, T^i_{II})$, $i = 1, 2$ be two replicas of Γ^*. Let x_0, x_1, x_2 be three distinct positions and let $\{z^i_0, z^i_1, \ldots\}$ $i = 1, 2$ be two infinite sequences of distinct positions. Let X_I and X'_I both consist of x_0, all z^i_j with j even and all $y \in Y^1_I \cup Y^2_I$. Let X_{II} and X'_{II} consist of x_1, x_2, all z^i_j with j odd and all $y \in Y^1_{II} \cup Y^2_{II}$. Let $X = X' = X_I \cup X_{II}$. Let $f(y^i) = f'(y^i) = g_i(y^i)$ for $y^i \in Y^{i-y^i_0}$, $f(z^i_j) = f'(z^i_j) = z^i_{j-1}$ for $j \geq 1$, $f(z^i_0) = x_1$, $f(y^i_0) = x_2$, $f(x_1) = f(x_2) = f'(x_1) = f'(x_2) = x_0$, and $f'(z^1_0) = f'(y^1_0) = x_1$,

$f'(z_o^2) = f'(y_o^2) = x_2$.

Let S_I be all sequences of the form $\{x_o, x_2, y_o^i, y_1^i, \ldots\}$ with $\{y_o^i, y_1^i, \ldots\} \in T_I^i$, $i = 1, 2$, together with $\{x_o, x_1, z_1^i, \ldots\}$ $i = 1, 2$. Let S and S_{II} be determined accordingly. Let S_I' be all sequences of the form $\{x_o, x_i, y_o^i, y_1^i, \ldots\}$ with $\{y_o^i, y_1^i, \ldots\} \in T_I^i$, $i = 1, 2$, together with $\{x_o, x_i, z_o^i, z_1^i, \ldots\}$, $i = 1, 2$. The homeomorphism ϕ is

$$\phi(\{x_o, x_2, y_o^i, y_1^i, \ldots\}) = \{x_o, x_i, y_o^i, y_1^i, \ldots\} ,$$

$$\phi(\{x_o, x_1, z_o^i, z_1^i, \ldots\}) = \{x_o, x_i, z_o^i, z_1^i, \ldots\} .$$

We now list some questions whose answers we should like to know.

QUESTION 1. Are there examples in which every subgame of $\Gamma = (x_o, X_I, X_{II}, X, f, S, S_I, S_{II})$ is strictly determined, but $-\Gamma = (x_o, X_I, X_{II}, X, f, S, S_{II}, S_I)$ is not?

QUESTION 2. Are the union and intersection of absolutely determined games absolutely determined?

QUESTION 3. How far can Propositions 10, 11, and 17 be extended? E.g., is Γ strictly determined if S_I is a G_δ? Does the family of sets S_I such that Γ is absolutely determined coincide with any of the well known families of sets, e.g., the Borel sets?

QUESTION 4. Is absolute determinacy a topological invariant?

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SIGNALING STRATEGIES IN n-PERSON GAMES[1]

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Behavior strategies have been used successfully to solve a number of games, notably Poker (see, for example, references [1], [3], and [4]). However, the only way to solve games without perfect recall has heretofore been mixed strategies. In the first part of the paper we introduce the idea of signaling strategies, associated behavior strategies and composite strategies in n-person games, and prove that any payoff attainable by means of mixed strategies can also be attained by means of composite strategies. As a consequence any two-person game can be solved by means of composite strategies.

A signaling strategy for a player is a pure strategy for that player restricted to that subset of his information sets which prevent him from having perfect recall. Then a composite strategy consists of a mixture of pure signaling strategies, together with the assignment to each pure signaling strategy of a behavior strategy on the remaining relevant information sets.

In Section 3 we give the definition of an agent of a player in an n-person game. Finally in Section 4 we discuss coalitions in n-person games in light of the previous parts of the paper.

The proofs of all statements in this paper are quite trivial and it is felt that its contribution is conceptual in nature. Elsewhere in this volume (reference [5]) the ideas of this paper have been applied by the author to the solution of a simplified game of Bridge.

§ 1. BASIC CONCEPTS

The description of an n-person game given by Kuhn in [2] will be used. The notation and definitions given there will be followed as closely as possible. The names of the concepts of [2] which will be used here are collected below, but definitions are repeated only when the notation given here is different, or when comparison with new definitions is made. Definitions of new concepts are numbered.

K is the game tree in which each move has at least two alternatives.

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The players are $i = 0, 1, \ldots, n$.

P_i is the set of moves of the i^{th} player.

A_j is the subset of moves in K having exactly j alternatives.

$\mathcal{U}$ is the aggregate of informations sets U. $\mathcal{U}_i$ is the aggregate of information sets of player i.

If U is contained in $P_0 \cap A_j$ then U is a one element set $\{X\}$ and there is a probability distribution with all components positive on the alternatives at X.

The above concepts are sufficient to define a game Γ (except for the payoff which will not enter this paper in an essential manner).

DEFINITION 1. Let U be an information set for player i and let $U_\nu = \{Z \mid Z$ follows some move in U by the ν^{th} alternative$\}$. Then U is a <u>signaling information set</u> for player i if, for some ν and some information set V of player i, $U_\nu \cap V \neq \phi$ and $V \not\subset U_\nu$.

The game tree of a two-person game is given in Figure 1(a). In this game, player 1 (P_1) "forgets what he knew" and player 2 (P_2) "forgets what he did." U_1 is a signaling information set for player 1. U_2 is a signaling information set for player 2.

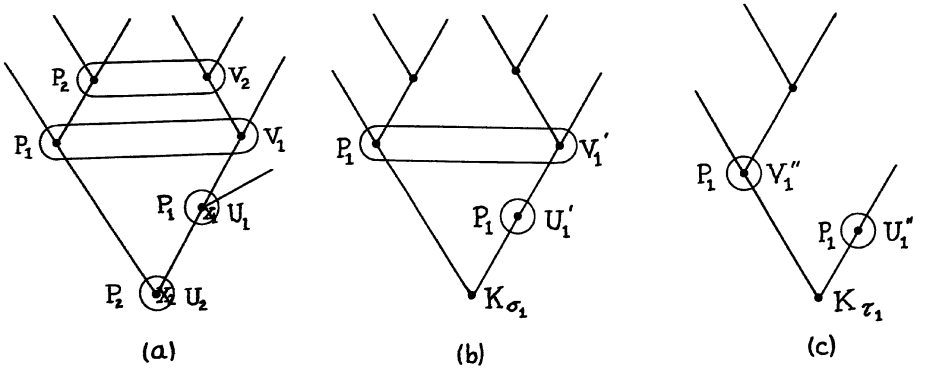

Figure 1

Let $\mathcal{I}_i$ be the set of signaling information sets for player i. Clearly $\mathcal{I}_i \subset \mathcal{U}_i$. From the definition of perfect recall in [2] it follows that Γ has perfect recall for player i if and only if $\mathcal{I}_i = \phi$.

A pure strategy for player i is a single-valued function π_i on $\mathcal{U}_i$ to the positive integers such that if $U \subset A_j \cap P_i$ then $\pi_i(U) \leq j$.

Let Π_i be the set of all π_i.

DEFINITION 2. A pure signaling strategy for player i is a single-valued function σ_i on $\mathcal{S}_i$ to the positive integers such that if $U \in \mathcal{S}_i$ and $U \subset A_j$ then $\sigma_i(U) \leq j$. Let Σ_i be the set of all σ_i. (If $\mathcal{S}_i = \phi$ then Σ_i consists of the single vacuous function.)

In Figure 1(a) there are exactly two pure signaling strategies for player 1, one (call it σ_1) that chooses alternative 1 at X_1 and the other (τ_1) that chooses alternative 2 at X_1. Player 2 also has two pure signaling strategies.

Since $\mathcal{S}_i \subset \mathcal{U}_i$ there is a natural mapping ψ of Π_i onto Σ_i, where $\psi(\pi_i)$ is the strategy π_i restricted to $\mathcal{S}_i$. Thus π_i is mapped by ψ onto that σ_i which chooses the same alternatives as π_i does on the information sets U in $\mathcal{S}_i$. Then ψ^{-1} defines an equivalence relation on Π_i and partitions Π_i into disjoint classes. To each such equivalence class corresponds exactly one pure signaling strategy σ_i. Hence we denote these equivalence classes by the symbol Π_{σ_i}. Then two pure strategies belong to Π_{σ_i} if and only if they make the same choices as σ_i does for all U in $\mathcal{S}_i$.

If $\Sigma_i = \phi$ then ψ maps each π_i onto the null set ϕ and $\psi^{-1}(\phi) = \Pi_i$. In this case we shall speak of the vacuous signaling strategy σ_i which makes no choice on any information set and for which $\Pi_{\sigma_i} = \Pi_i$. This convention will allow us not to have to distinguish the cases $\Sigma_i \neq \phi$ and $\Sigma_i = \phi$ in the statements that follow.

As in [2] a pure strategy vector π defines a function $p_X(\pi)$ on the moves X in K. A move X is possible when playing π_i, symbolically X Poss π_i, if and only if there exists a π such that $p_X(\pi/\pi_i) > 0$. A move X is Poss $(\sigma_i:\pi_i)$ if and only if X Poss π_i and $\pi_i \in \Pi_{\sigma_i}$. An information set $U \in \mathcal{U}_i$ is Rel π_i if and only if some X in U is Poss π_i. An information set $U' \in \mathcal{U}_{\sigma_i}$ is Rel $(\sigma_i:\pi_i)$ if and only if some X in U' is Poss $(\sigma_i:\pi_i)$. Define $T_i(X) = \{\pi_i \mid X \text{ Poss } \pi_i\}$, $T_i(U) = \{\pi_i \mid U \text{ Poss } \pi_i\} = \bigcup_{X \in U} T_i(X)$, $R_{\sigma_i}(X) = \{\pi_i \mid X \text{ Poss } (\sigma_i:\pi_i)\}$, and $R_{\sigma_i}(U') = \{\pi_i \mid U' \text{ Rel } (\sigma_i:\pi_i)\} = \bigcup_{X \in U'} R_{\sigma_i}(X)$.

Let K_{π_i} be the subset of all moves X of K which are possible for π_i. Proposition 1 of [1] proves that K_{π_i} is a tree. Then let $K_{\sigma_i} = \bigcup_{\pi_i \in \Pi_{\sigma_i}} K_{\pi_i}$. K_{σ_i} is also a tree. Intuitively K_{σ_i} is the subset of moves of K which can possibly occur if player i uses the signaling strategy σ_i.

Let $\mathcal{U}_{\sigma_i}$ be the partition $\mathcal{U}$ relativized to K_{σ_i} as follows: If $U \in \mathcal{U}_i$ then the information set $U' = U \cap K_{\sigma_i}$ is in $\mathcal{U}_{\sigma_i}$. We do not define information sets in K_{σ_i} for the other players. The ordering and indexing of moves and assignment of chance probabilities and payoffs in K_{σ_i} is the same as for the corresponding moves and plays in K.

The trees K_{σ_1} and K_{τ_1}, shown in Figure 1(b) and 1(c), correspond to the signaling strategies σ_1 and τ_1 of player 1 in the game of Figure 1(a). It is not hard to construct the trees corresponding to each of the two signaling strategies of player 2 in Figure 1(a).

Although K_{σ_1} is not a game it is possible to speak of signaling information sets and perfect recall for player 1 in K_{σ_1}, since these concepts depend only on the information sets for that player.

PROPOSITION 1. Player 1 has perfect recall in K_{σ_1}.

PROOF. If not, there exist moves $X \in U'$, $Y \in V'$ for player 1 with $X < Y$ and $Y \in U'_\nu$ but $U'_\nu \not\supset V'$. Now $U_\nu \cap V \neq \phi$ and $U_\nu \supset V$ or else U was a signaling information set in K and V' was defined incorrectly. Hence there is a move Z in V', so that $Z \in K_{\sigma_1}$, and a move Z_1 in $U - U'$, so that $Z_1 \notin K_{\sigma_1}$, with $Z_1 < Z$. Since K_{σ_1} is a tree (that is, every move in K_{σ_1} has a unique predecessor) $Z \notin K_{\sigma_1}$, a contradiction, which proves the proposition.

As a corollary to Proposition 1 of [2] we have the following proposition.

PROPOSITION 2. Let the information set U' contain the first move for player 1, X, on a play W in the tree K_{σ_1}. Then $R_{\sigma_1}(U') = \Pi_{\sigma_1}$.

A mixed strategy μ_1 for player 1 is a probability distribution on Π_1 which assigns the probability q_{π_1} to π_1.

DEFINITION 3. A mixed signaling strategy ν_1 for player 1 is a probability distribution on Σ_1 which assigns the probability q_{σ_1} to σ_1.

A mixed strategy vector μ defines a probability distribution on the moves X in K_{σ_1} as follows:

$$p_X(\sigma_1 : \mu) = c(X) \sum_{\substack{\pi_1 \in R_{\sigma_1}(X) \\ \pi_j \in T_j(X),\, j \neq 1}} q_{\pi_1} \cdots q_{\pi_n}$$

where $c(X)$ is the product of the chance moves on the path from 0 to X. Define X Poss $(\sigma_1 : \mu_1)$ if and only if there exists a μ such that $p_X(\sigma_1 : \mu / \mu_1) > 0$; and define, for $U' \in \mathcal{U}_{\sigma_1}$, U' Rel $(\sigma_1 : \mu_1)$ if and only if some X in U' is Poss $(\sigma_1 : \mu_1)$.

DEFINITION 4. An associated behavior strategy for player 1, $\beta_1(\sigma_1)$, is an assignment to each $U' \in \mathcal{U}_{\sigma_1}$

of a probability distribution $b(\sigma_i, U', \delta)$ on the alternatives at U'. Thus, if U' has j alternatives, then $b(\sigma_i, U', \delta) \geq 0$ and $\sum_{\delta=1}^{j} b(\sigma_i, U', \delta) = 1$.

Note that when $\mathcal{A}_i = \phi$, that is, when layer i has perfect recall, this definition of an associated behavior strategy reduces to the concept of a behavior strategy as defined in [2].

PROPOSITION 3. A mixed strategy μ_i for player i defines a mixed signaling strategy $\nu_i(\mu_i) = \{q_{\sigma_i}\}$, and an associated behavior strategy $\beta_i(\sigma_i : \mu_i)$. Conversely, given a mixed signaling strategy $\nu_i^0 = \{q_{\sigma_i}^0\}$ and an associated behavior strategy $\beta_i^0(\sigma_i)$, there is a mixed strategy μ_i such that $\nu_i(\mu_i) = \nu_i^0$ and $\beta_i(\sigma_i : \mu_i) = \beta_i^0(\sigma_i)$.

PROOF. Given $\mu_i = \{q_{\pi_i}\}$ define $\nu_i(\mu_i) = \{q_{\sigma_i}\}$ by

(a) $$q_{\sigma_i} = \sum_{\pi_i \in \Pi_{\sigma_i}} q_{\pi_i}$$

which is clearly a mixed signaling strategy. [N. B. If $\Sigma_i = \phi$ then $q_{\sigma_i} = 1$ for the (vacuous) signaling strategy σ_i in Σ_i. We denote this case by $\nu_i = 1$.] Define for $U' \in \mathcal{U}_{\sigma_i}$

(b) $$b(\sigma_i, \mu_i, U', \delta) = \sum_{\substack{\pi_i \in R_{\sigma_i}(U') \\ \pi_i(U') = \delta}} q_{\pi_i} \Big/ \sum_{\pi_i \in R_{\sigma_i}(U')} q_{\pi_i}$$

if U' Rel $(\sigma_i : \mu_i)$,

(c) $$b(\sigma_i, \mu_i, U')\delta) = \frac{1}{q_{\sigma_i}} \sum_{\substack{\pi_i \in \Pi_{\sigma_i} \\ \pi_i(U') = \delta}} q_{\pi_i}$$

if U' not Rel $(\sigma_i : \mu_i)$ and $q_{\sigma_i} > 0$

(d) $b(\sigma_i, \mu_i, U', \delta)$ is an arbitrary probability distribution if $q_{\sigma_i} = 0$.

It is not hard to show that (b), (c), and (d) are probability distributions on the alternatives at U'.

Conversely, given $\nu_i^0 = \{q_{\sigma_i}^0\}$ and $\beta_i^0(\sigma_i)$, define μ_i by

(e) $$q_{\pi_i} = q_{\sigma_i}^0 \sum_{U' \in \mathcal{U}_{\sigma_i}} b_i^0(\sigma_i, U', \pi_i(U'))$$

if $\pi_i \in \Pi_{\sigma_i}$ (it will belong to exactly one such). The verification of the proposition follows by routine substitution into (a) through (d).

DEFINITION 5. The pair $(\nu_i, \beta_i(\sigma_i))$ is called a composite strategy of player i.

The intuitive interpretation of a composite strategy for the i^{th} player is given as follows: The strategist for player i chooses, with probability q_{σ_i}, a signaling strategy σ_i. Then the agents who control the information sets in $\mathcal{S}_i$ then make their choices as dictated by σ_i, and the agents who control the other information sets U in $\mathcal{U}_i - \mathcal{S}_i$ make their selections with probabilities dictated by $\beta_i(\sigma_i)$ among the alternatives at U. Another possible interpretation of a composite strategy is given in Section 3 of this paper.

The following proposition is related to Proposition 3 of [2].

PROPOSITION 4. Let Γ be any game with game tree K and let W be any play in K. Let $T_i(W) = \{\pi_i \mid W \text{ Poss } \pi_i\}$, and let $c(W)$ be the product of the chance alternatives on W, or 1 if there are no chance alternatives on W. Then for all $\pi = (\pi_1, \ldots, \pi_n)$ and all W

$$p_W(\pi) = \begin{cases} c(W), & \text{if } i \in T_i(W) \text{ for } i = 1, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. By Proposition 1 of [2], W is possible for π_i if and only if π_i chooses all of the alternatives for player i on W. The statement then follows from the definition of $p_W(\pi)$.

A mixed strategy $\mu = (\mu_1, \ldots, \mu_n)$ defines a probability distribution on the plays W as follows:

$$(1) \qquad p_W(\mu) = \sum_{\pi} q_{\pi_1} \cdots q_{\pi_n} p_W(\pi) = c(W) \sum_{\pi_i \in T_i(W), \text{all } i} q_{\pi_1} \cdots q_{\pi_n}$$

where the last step follows from Proposition 4.

Then the payoff vector $H(\mu) = (H_1(\mu), \ldots, H_n(\mu))$ corresponding to a mixed strategy vector μ has components

$$(2) \qquad H_i(\mu) = \sum_{\text{all } W} p_W(\mu)\, h_i(W)$$

which represents the payoff to player i if strategy μ is used.

A strategy vector $\bar{\mu} = (\mu_1, \ldots, \mu_{i-1}, (\nu_i, \beta_i), \mu_{i+1}, \ldots, \mu_n)$, in which the i^{th} component is the composite strategy (ν_i, β_i) corresponding to the mixed strategy μ_i for player i, also defines a probability distribution on the plays W of K. To see this, note that ν_i defines a probability distribution q_{σ_i} on the trees K_{σ_i}. Then for each

play W^{σ_i} in K_{σ_i} [note that W^{σ_i} is the same as some play W in K and hence $c(W^{\sigma_i}) = c(W)$] let $Q_i(W^{\sigma_i}) = \{e \mid e$ is an alternative for player i on $W^{\sigma_i}\}$ and let U' be the information set containing the last move for player i on W^{σ_i}. Then the strategy vector $\bar{\mu}' = (\mu_1, \cdots, \mu_{i-1}, \beta_i, \mu_{i+1}, \cdots, \mu_n)$ defines a probability distribution on the plays W^{σ_i} in K_{σ_i} as follows:

$$(3)\quad p_{W^{\sigma_i}}(\bar{\mu}') = c(W) \sum_{\pi_j \in T_j(W),\, j \neq i} q_{\pi_1} \cdots q_{\pi_{i-1}} q_{\pi_{i+1}} \cdots q_{\pi_n} \cdot \prod_{e \in Q_i(W^{\sigma_i})} b(e)$$

if U' Rel μ_i

$= 0$, otherwise.

In this formula empty products are to be set equal to 1 and W is the play in K which is the same as the play W^{σ_i} in K_{σ_i}. In turn formula (3) defines a probability distribution on the plays W of K by

$$(4)\quad p_W(\bar{\mu}) = \sum_{W^{\sigma_i} = W} p_{W^{\sigma_i}}(\bar{\mu}')\, q_{\sigma_i} .$$

Finally, the payoff vector $H(\bar{\mu})$ has components

$$(5)\quad H_i(\bar{\mu}) = \sum_{\text{all } W} p_W(\bar{\mu})\, h_i(W) .$$

Let $I = \{1, \ldots, n\}$, let $S \subset I$ be any subset of I and let $I - S$ be the complementary set to S in I. Then repeated application of the above reasoning provides the payoff vector if the players i in S use the composite strategies (ν_i, β_i) and the players j in $I - S$ use the mixed strategies μ_j.

§ 2. A THEOREM ON COMPOSITE STRATEGIES

The following lemma is closely related to the sufficiency part of the proof of Theorem 4 in [2].

LEMMA. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a mixed strategy vector and let

$$\bar{\mu} = (\mu_1, \cdots, \mu_{i-1}, (\nu_i, \beta_i), \mu_{i+1}, \cdots, \mu_n)$$

where (ν_i, β_i) is the composite strategy corresponding to μ_i. Then $p_W(\mu) = p_W(\bar{\mu})$ for all W in K.

PROOF. Let W be any move in K. If W contains no move of player i the lemma is obviously true. If W contains a move for player i let X be the first and $Y \in U$ the last move for player i on W. Let σ_i be any signaling strategy with $q_{\sigma_i} > 0$ and such that W is contained in K_{σ_i}, that is, such that $W = W^{\sigma_i}$ for some W^{σ_i} in K_{σ_i}. By (3)

$$(6) \qquad p_{W^{\sigma_i}}(\bar{\mu}') = C \cdot \prod_{e \in Q_i(W^{\sigma_i})} b(e), \quad \text{if } U \in \operatorname{Rel} \mu_i ,$$

$$= 0, \quad \text{otherwise},$$

where C is a constant defined in (3). Consider the quantities $b(e)$ arranged in the same order as the alternatives e for player i on W are arranged. By Proposition 2 the denominator of the first one is exactly $\sum_{\pi_i \in \Pi_{\sigma_i}} q_{\pi_i} = q_{\sigma_i}$. By Proposition 1 of [2] the numerator of the first is cancelled by the denominator of the second, the numerator of the second cancelled by the denominator of the third, etc., until only the numerator of the last fraction remains. The last numerator is equal to $\sum_{\pi_i \in T_i(W^{\sigma_i})} q_{\pi_i}$. Then one can write (6) as

$$(7) \quad p_{W^{\sigma_i}}(\bar{\mu}') = \frac{c(W)}{q_{\sigma_i}} \sum_{\pi_j \in T_j(W^{\sigma_i}),\, j \neq i} q_{\pi_1} \cdots q_{\pi_{i-1}} q_{\pi_{i+1}} \cdots q_{\pi_n} \sum_{\pi_i \in T_i(W^{\sigma_i})} q_{\pi_i}$$

$$= \frac{c(W)}{q_{\sigma_i}} \sum_{\pi_j \in T_j(W^{\sigma_i}),\, \text{all } j} q_{\pi_1} \cdots q_{\pi_n} .$$

Substituting (7) into (4) gives the statement of the lemma.

In the following theorem the sets S and $I - S$ are defined as at the end of Section 1.

THEOREM. Let Γ be a game, S a subset of the players, and $\mu = (\mu_1, \ldots, \mu_n)$ a strategy vector. Let ψ be the function defined as follows:

$$\psi(\mu_i) = \begin{cases} [\nu_i(\mu_i), \beta_i(\sigma_i)] = (\nu_i, \beta_i), & \text{if } i \in S , \\ \mu_i, & \text{if } i \in I - S . \end{cases}$$

Then if $\bar{\mu}$ is the strategy vector $[\psi(\mu_1), \ldots, \psi(\mu_n)]$ we have $H(\bar{\mu}) = H(\mu)$.

PROOF. By repeated applications of the lemma and the remark at the end of Section 1.

Intuitively this theorem means that any payoff which players can obtain by means of mixtures of pure strategies, the can also obtain by means of composite strategies. This theorem together with the fact that the normalized form of the game obscures signaling strategies, explains one reason why the normalized form of the game is not always the best form in which to solve actual games.

COROLLARY. Any zero-sum two-person game can be solved by means of composite strategies.

§ 3. THE DEFINITION OF AN AGENT

In [2] an agent is postulated to control each information set of a player. In many actual games, for example bridge, rummy, canasta, etc., an agent obviously controls more than one information set (see Figure 1 of [5]). To formalize this, we broaden the definition of an agent to a concept similar to that found in practical games.

DEFINITION 7. By the agent partition of player i $(i > 0)$ we shall mean a partition of the information sets $\mathcal{U}_i$ of player i into sets $\mathcal{U}_{i1}, \cdots, \mathcal{U}_{it_i}$, $(t_i \geq 1)$, such, that, if P_{ik} is the set of moves in the information sets contained in $\mathcal{U}_{ik}$, then the moves in P_{ik} satisfy the condition of perfect recall.

The moves in P_{ik} will be called the moves of agent ik meaning "the k^{th} agent of player i."

If each P_{ik} contains exactly one information set then this definition of an agent coincides with the one given in [2]. The requirement of perfect recall is added to the definition of an agent to capture the intuitive idea that an agent is a single personality and thus avoiding the problem of "split personalities," (see [4] p. 53). The agent partition may be given by the rules of the game as in bridge and other card games, or it may be up to the player himself to choose the partition as in certain economic situations.

We can think of player i using a composite strategy (ν_i, β_i) as follows: The agents of player i meet before the game is played and decide (by means of a chance device and the probabilities given by ν_i) which signaling strategy σ_i to use, this decision, of course, being kept secret from the other players in the game. The agents who control information sets relevant for σ_i then are instructed to choose the alternative

dictated by σ_i at those information sets. The agents controlling other information sets make choices as dictated by $\beta_1(\sigma_i)$.

In [5] the author solves a simplified bridge game by means of composite strategies and with the use of this notion of agent.

§ 4. REMARKS ON n-PERSON GAME THEORY

The formulation of the general theory of n-person zero-sum games is given in [4] p. 238, ff. It is assumed there that if a non-empty, proper subset S of the set of players I forms a coalition, then they will "fully cooperate" with each other and become a single player. This could be formulated as follows: The set of agents of the composite player S is the union of the sets of agents of each of the players i in S, and it is assumed that these agents will cooperate as a single player.

It is not entirely clear under what circumstances the agents of the players in a coalition will be allowed to cooperate with each other under the rules of Γ, nor exactly what it means for them to cooperate. The answer to these questions is not difficult. Suppose $S = \{1, \ldots, t\}$ where $0 < t < n$ is a coalition of players in Γ. A pure strategy for the coalition S consists of a vector $\pi = (\pi_1, \ldots, \pi_t)$ taken from the (cartesian) product space of the pure strategy spaces for the players i in S; that is, $\pi \in \mathcal{P}_{i \in S}(\Pi_i)$, where $\mathcal{P}$ denotes the cartesian product. A mixed strategy for the coalition is then an assignment of a probability distribution on these pure strategies. Let M_S be the set of all mixed strategies. We shall say that a mixed strategy μ in M_S is <u>allowable</u> in Γ if it can be used by the coalition S in Γ. Then the first answer to the question posed above is:

> The agents of the players in S can fully cooperate as a single player (coalition) in Γ if and only if each mixed strategy μ in M_S is allowable in Γ.

From this remark it is clear that the specification of the allowable strategy spaces should be included in the definition of a game.

The theorem of the last section allows us to reformulate this remark in a way that is easy to apply to some actual games. Let ν and β be the mixed signaling strategies and associated behavior strategies respectively for the coalition S considered as a single player. Let us say that the strategies ν and β are allowable in Γ if they can be used in any way by the coalition S in Γ. Then we can say:

> The agents of the players in S can fully cooperate as a single player in Γ if and only if each of

the composite strategies (γ,β) for S is allowable in Γ.

In reference [5] there is an example of a simplified two-person bridge game with enforced coalitions in which only pure signaling strategies are allowable.

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BRIDGE AND SIGNALING[1]

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Bridge has long been a favorite parlor game that has held and continues to hold ascendancy over most other card games. There are a number of reasons for this popularity, some of which we discuss below.

First of all, bridge is complicated enough that many different situations can arise, with no set of similar situations appearing so often that the game becomes boring. There is always a new problem to be solved, a new "strategy" to be devised. Second, the payoff in bridge varies with such factors as doubling, vulnerability, the suit that is trump (or no trump), etc. Hence, there is the possibility that the player's strategy should change with each change in payoff. Third, a player in bridge has imperfect information, both as to the cards held by his opponents and as to the cards held by his partner. He must therefore make his strategy depend on the most likely distributions of cards in the other players' hands.

There are two important consequences of this last observation. The players' lack of information about the cards in their opponents' hands means[2] that there probably is no "best" way of playing bridge. Finally, the players' lack of information makes signaling[3] among the players in the game possible and desirable.

It is in the subtlety and importance of signaling that bridge differs most characteristically from other card games, such as rummy or canasta, which have to some extent the phenomena noted above. Both direct signaling (signaling between partners), and inverted signaling (deceptive signaling or bluffing against opponents) occur in bridge. It is the presence of signaling that makes bridge particularly interesting to game theoreticians.

In this note we discuss a simplified bridge model played both as

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[1]The preparation of this paper was supported by the Office of Naval Research.

[2]Cf. Theorem 7 of [2]. Actually it is also necessary to verify that the payoff of bridge is such that mixed strategies are needed to solve the game. This is true for the bridge models discussed here.

[3]The first discussion of signaling occurs in [6] pp. 53, 54. The remarks made in [5] about games without perfect recall in general, and bridge in particular, are also relevant. The definition of a signaling strategy is given in [7].

a two-person and as a four-person game, the latter being closer to ordinary bridge. The models discussed are similar to the play of the last two tricks in a hand of bridge. Good strategies[4] are devised for playing a single hand[5] of each game. Both models exhibit the types of signaling described above although the signaling in the two-person game is more subtle than that in the four-person game. Finally, we observe that the four-person game cannot be directly treated by means of characteristic functions,[6] hence a new definition of solution for that game is defined and used.

In discussing bridge we shall assume that the reader is familiar with the technical terms of bridge (which can be found, for example, in [4], p. 100, ff.). We shall use as few of the technical concepts of game theory as possible (relegating them to footnotes wherever possible) so that almost anyone familiar with bridge can follow the discussion by using his "bridge sense."

§ 1. DESCRIPTION OF THE BRIDGE MODELS TO BE USED

The deck of cards customarily used in bridge contains 52 cards. The number of deals possible with this deck is then approximately $5 \cdot 10^{28}$. Since a complete analysis of bridge requires that every deal be "played out," the amount of labor required to completely analyze ordinary bridge is clearly prohibitive.[7] There are several ways of making simplified models of bridge, one of which we discuss here.

In the model which we shall discuss there are four players[8] who are called North, East, South, and West. East and West together form a pair, East-West; similarly, North and South together form a pair, North-South. The rules concerning the order of deal and play, taking of tricks,

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[4]The mathematical justification of our method of finding the good strategies appears in [7].

[5]Actually, as observed in [6] p. 49, a play of bridge should be a "rubber." The payoff function in the bridge models considered here is not sufficiently close to the payoff function in bridge to make this necessary.

[6]For the definition of characteristic function see [6] p. 238 ff. The technical reasons why the four-person bridge model cannot be solved by characteristic functions are discussed in [7].

[7]Even the amount of labor required to describe the game tree of bridge is enormous. Chapters 1-4 of [1] are almost entirely devoted to an analysis of this problem.

[8]The technical term for what is here called "player" is "agent," a definition of which is given in [7], section 3. The term "player" is used technically to indicate what is here called a "pair," that is, a coalition of two "agents."

trump and no trump, etc., are as in ordinary bridge. In addition we make the following simplifying assumptions:

a. The number of cards in the deck is to be eight, divided into four suits with two cards in each suit. The four suits are called 1) spades, 2) hearts, 3) diamonds, and 4) clubs, ranked in that order. The two cards in each suit are called ace and king with ace ranking over king.

b. The bidding stage of bridge is omitted. Instead the declarer and trump suit (or no trump) are determined by a random device.[9] In particular discussions it will always be assumed that South is declarer, North is dummy, and West leads. By interchanging names one can arrive at any other situation.

c. Scoring is to be done as follows: The payoff to the pair East-West is to be two units if the pair takes both tricks, one unit if the pair takes one trick, and zero units if the pair loses both tricks. North-South is to receive the negative of these amounts. It is also assumed that the players in each pair share equally in the payoff.

d. We shall assume that the shuffling is perfect, that is, the order of cards after shuffling is independent of the order before shuffling.

It is probable that this particular model would have little interest in itself as a game for bridge players. Note, however, the similarity between this model and the play of the last two tricks in ordinary bridge. The main difference is that the information pattern in a play is different from that assumed here. In particular, note that the dummy, North, would have laid down his cards before West leads.

The analysis of the model given here is not trivial and, as we shall see, it has many of the essential properties of bridge.

By the simple rules of combinatorial analysis (see, for example, reference [3]) the number of deals is $8!/(2!)^4 = 2{,}520$ and the number of different hands is $8!/(2!6!) = 28$. These hands are divided as follows: four hands with ace and king in the same suit, twelve hands with ace and king in different suits, and six hands each with two aces or two kings.

Since there is no bidding the only information given to each player is the knowledge of the cards in his hand plus those cards exposed in the dummy's hand and those exposed during the play. When West leads he has knowledge of the two cards in his hand, hence he knows that one of $6!/(2!)^3 = 90$ deals has occured. After West leads, North, as dummy, lays down his hand so that South, who plays for North, knows where all but three of the cards are (one in West's hand and two in East's). These can be distributed in three different ways, hence South knows that one of three

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[9]The determination of the trump suit by a random device is used in the game of whist, a forerunner of bridge, the rules for which are given in [4]. However, whist is not played in no trump.

deals has occurred. At East's move, East knows where all but three of the cards are (one in West's and two in South's hand), hence knows that one of three different deals has occurred. After East plays, South knows all cards but two (one in West's and one in East's hand), hence knows that one of two deals has occurred. To illustrate this information pattern we draw part of the game tree for this model of bridge (for definition of game tree see [5]). Readers not familiar with game thoery can recognize Figure 1 as a graphical picture of the discussion in the paragraph above.

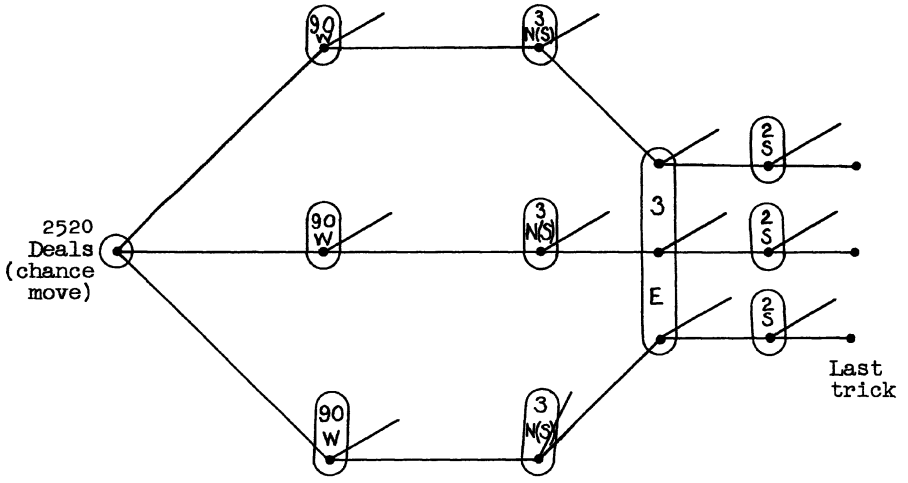

Figure 1

The oval curves indicate information sets and each one is labelled with the player who controls it and the number of moves which it contains. Note that North-South has perfect recall (see ref. [5]) although East-West does not.

In the following sections we shall solve this simplified bridge game. It may be advisable to indicate to those readers who are versed in game theory the theoretical basis on which this will be done. It is proved in [7] that any game can be solved by means of signaling strategies and associated behavior strategies. Only the pair East-West has any signaling strategies (since North-South has perfect recall). We determine all pure signaling strategies for East-West, eliminate some of them by domination, then observe that a mixture of the remaining is optimal. With respect to these signaling strategies we determine the good associated behavior strategies by means of domination.

§ 2. SIGNALING STRATEGIES IN THE NO TRUMP CASE

Suppose South is declarer and the game is to be played in no

trump. It is not hard to see that if West has an ace he should always lead it since it will surely take a trick, whereas if West saves the ace there is only a small chance of making it good. If then East-West agree that West should always lead an ace if he has one, this already constitutes a direct signal between East and West (it also signals indirectly to North-South), in that a king led by West always indicates that his other card is also a king.

If South has two kings, this signal will not reduce the number of moves in East's information set, but in this case the issue will be settled by the distribution of the aces held by North and East. In the other cases the number of moves in East's information set is reduced from three to two (if South has one ace) or one (if South has two aces).

There remains only one case in which this signaling can be improved. It is illustrated in Figure 2(a).

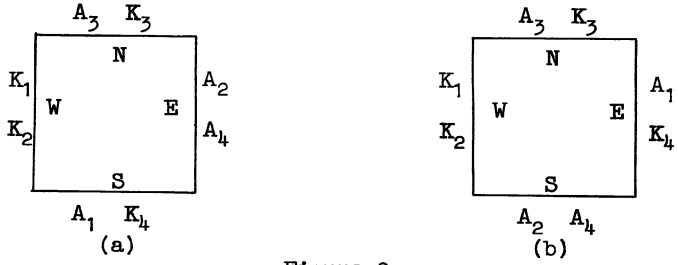

Figure 2

If West leads the king K_1 in 2(a), then East (knowing from his own and the dummy's cards that South will take the trick) is left in a quandry as to which of his aces to discard. If he saves the right one he can take the second trick; if not, he will lose it. He does not know which king is held by South, hence does not know what the last lead will be. Thus if West's king lead could also tell East what the other king in West's hand was, then East could determine which ace to save.

The reader can easily devise ways of signaling this information; for example, always lead the lower ranking suit. It is clear that, since there are six possible two-king hands and only four possible leads, the signaling cannot be perfect; that is, there must be a certain lack of information even if East-West adopts a signaling strategy. Examples of the four possible ways of signaling (we shall call these signals) appear in Figure 3. An x corresponds to the card to be led and the numbers 1, 2, 3, and 4 refer to the four suits.

Type	1 2	1 3	1 4	2 3	2 4	3 4
3111	x	x	x	x	x	x
3210	x	x	x	x	x	x
2220	x	x	x	x	x	x
2211	x	x	x	x	x	x

Figure 3

The numbers indicating the type of signal are obtained by counting the number of hands with the same lead. It is easy to show that there are eight signals each of types 3111 and 2220 and twenty-four signals each of types 3210 and 2211.

It is not hard to see that signals of type 2211 are better than signals of type 2220 and signals of type 3111 are better than type 3210. By examing the possible situations in which signaling is relevant it can be shown that signals of type 2220 are just as good as type 3111, so that signals of type 2211 transmit the most information. It is then clear that only type 2211 signals should be considered for use by East-West.

By a pure signaling strategy[10] for East-West we shall mean an unambiguous command to West which card he shall lead for every possible hand. Certain elements of the signaling strategy have been discussed above but it is instructive to write down completely an example of a pure signaling strategy. The signaling strategy below refers to West's hand only.

a) If the hand contains two aces lead the higher ranking. (The signal of type 3210 illustrated in Figure 3.)
b) If the hand contains ace and king lead the ace.
c) If the hand contains two kings use the type 2211 signal illustrated in Figure 3.

As observed above it is immaterial, as far as the payoff obtained is concerned, what the command at a) is. Moreover, the command b), as given, clearly dominates any other possible command at b). Also, it was observed above that any signal used in c) which is not of type 2211 is dominated by a signal of type 2211. We shall therefore ignore part a) of the strategy, assume that part b) of the strategy is the same for all strategies considered, and then the only difference in the signaling strategies is the choice at c) of one of the 24 possible type 2211 signals.

[10]The precise definition of a signaling strategy is given in [7].

It might seem that East-West could use one of the signals of type 2211, for example the one given in Figure 3, as a pure strategy. If this were an optimal strategy, East-West would not care whether or not North-South found out this strategy. However, Figure 2(b) illustrates a case in which North-South can profit from knowing East-West's strategy. The deal in which the hands of North and South in 2(b) are interchanged is also a case in which North-South profits. Since 2(a) and both variants of 2(b) are equally likely we see that North-South stands to profit twice as much as East-West from such a pure signaling strategy.

Suppose that East-West selects at random (that is, with equal probability) one of the 24 signals of type 2211 to use before each hand is played. Since both East and West know which signal is actually selected and used for the hand, and this knowledge is kept secret from North and South, we see that the desired information is successfully signaled from West to East. That is, North-South cannot draw any conclusions from a king led by West as to the identity of the other king in West's hand, but East can. This concealing of information is an example of inverted signaling. It can be seen from symmetry considerations that no other mixed signaling strategy dominates this one.

The selection of a private signal between East and West violates the rule[11] of bridge stating that all unusual signals between players must be announced. This is equivalent to saying that only pure signaling strategies can be used. We therefore define a two-person bridge game to be one in which any mixed signaling strategy is allowed and a four-person bridge game one in which only pure signaling strategies are allowed.

The above discussion shows that in the four-person game it is unprofitable for East-West to use a pure signaling strategy. Hence in the four-person game West should lead either of the two kings at random.

§ 3. GOOD STRATEGIES FOR THE NO-TRUMP CASE

In this section we devise good strategies for the players by using "bridge sense" and the results of the last section. Then we give a brief discussion of the reasons which make the given strategies optimal.

Good strategies for both the two-person and the four-person bridge game follow.

For West. If West's hand consists of two aces, lead arbitrarily; if it consists of ace and king, lead the ace; if it consists of two kings, use the mixed signaling strategy given in section 2 (for the four-person

[11] The following sentence is found on p. 138 of [4]; "It is improper to employ an unusual convention in play without informing the opponents of its significance." The penalty for violating this rule is "ostracism."

bridge game this part of the strategy is replaced by the phrase "if it consists of two kings, lead at random").

For North (played by South). Follow suit; or discard any worthless card; or discard king before ace; or, if the hand consists of two aces, discard at random; or, if it consists of two kings, discard arbitrarily.

For East. Follow suit; or discard any worthless card; or discard according to West's signal (this phrase is eliminated in the four-person bridge game); or discard king before ace; or discard arbitrarily.

For South. Follow suit; or discard any worthless card; or discard king before ace; or discard arbitrarily.

In applying the strategy the player is to use the first applicable phrase. By a random play we mean that the player is to use a chance device which will select each of the two alternatives (assuming the cards are numbered in the order in which they are dealt) with probability 1/2. By an arbitrary order of play we mean that it is immaterial which card is played first. A card is worthless to a player if he can determine from his information that it will never take a trick or, if discarded, will never give useful information to his opponent. For example, an ace is worthless to North if South has the king of the same suit and neither North nor South can take the first trick.

It is clear that these are strategies, that is, that there is a command to cover every situation. It also is easy (although tedious) to check, by playing out each deal, that the parts of these strategies, exclusive of the signaling, arbitrary playing and random discarding, are as good as any other strategy and that there exist deals for which deviation from these strategies causes over-all loss. The use of the signaling strategy given was justified in the last section. We must still justify the random discard and arbitrary play commands.

If North has two kings then neither will take either trick and, since North is dummy, both are exposed to all players. It is therefore clear that any strategy for discarding, pure or mixed, is good, hence the discard can be arbitrary. If North has two aces the situation is different even though both are exposed. It can be shown (the details are tedious) that if North uses a pure discard strategy of any type then East-West can, by deviating from his optimal strategy and using a suitable countering pure signaling strategy, profit from it. However, North can safely use random discarding in either game. Note that this is inverted signaling.

If East has two kings then a simple analysis of the possibilities shows that no discard strategy, pure or mixed, can give useful information to North-South. If East has two aces then an analysis of the possibilities shows that the only type of deal in which East does not know what to discard (or that it does not matter) is that shown in Figure 2(a). For the two-person game this is precisely the case in which West is signaling to East.

Since the signals of type 2211 are not complete there are still two situations for each signal in which East cannot decide what to discard. It is clear from Figure 2(a) that, regardless of which discard is made, no useful information is imparted to North-South, hence an arbitrary discard is good in the two-person game. For the four-person game Figure 2(a) again shows that the arbitrary discard is as good as any.

The analysis of South's discard strategy is similar to that for North and shows that the arbitrary discard strategy is good.

§ 4. THE VALUES OF THE GAMES IN THE NO-TRUMP CASE

It is evident from the symmetry of the rules, and the randomness of the deal and choice of declarer, that both the two-person and four-person games are fair, i.e., their value is zero. It is still of interest to calculate the expectation of the players once the declarer has been determined. As might be expected, having the lead is an advantage.

If West has two aces or a complete suit then, regardless of the strategies employed by any of the players, East-West will get both tricks. Hence in these cases East-West has an expectation of $\frac{6+4}{28} \cdot 2 = 5/7$. The other expectations are more difficult to compute. The best way of doing it is to assume hands for West and East and then to arrange the other cards in the six possible ways between North and South. Compute in each case the probability of each hand and the amount East-West (say) can get using his optimal strategy. With a little experience one sees how to group similar situations together and thus to shorten the calculations. The two-person game value to East-West is 1.431. Recalling the definition of the payoffs, this number can be interpreted as the expected number of tricks which East-West can take.

The four-person bridge game is one in which there are enforced coalitions and rules disallowing private signaling, hence this game cannot be solved by means of characteristic functions.[6] However, note that the strategies as given in the previous section for the four-person game have the following intuitive characteristics: (1) each pair of players can assure itself of a certain minimal amount regardless of the strategy of the opponents; (2) no pair of players can, by its own efforts or by the individual effort of any player in the pair, increase its expectation <u>within the rules of the game</u>. These criteria are discussed at length for the two-person zero sum case in [6], Chapter III (see particularly pp. 101-102 and p. 159).

If the above criteria are accepted as a definition of the value of the four-person game given, we can compute its value as 1.428. The increase in expectation due to signaling is then .003.

§ 5. GOOD STRATEGIES AND VALUE IN THE TRUMP CASE

Suppose that the declarer and trump suit are chosen at random. For purposes of discussion we assume that the declarer is South. From the game theoretical point of view this game is not as interesting as the no-trump case since there is no opportunity for the more subtle type of signaling previously discussed, hence there is no difference between the two- and four-person games. Good strategies are given below.

For West. If West's hand contains both trump cards lead arbitrarily; if it contains the trump ace and a non-trump card, lead the trump ace; if it contains the trump king and a non-trump card, lead the non-trump card; if it contains two non-trump aces or two non-trump kings, lead at random; if it contains a non-trump ace and king, lead the ace.

For North. Follow suit; if North's hand contains both trumps play arbitrarily; if it contains a trump card and a non-trump card, trump if South cannot take the trick, otherwise play the non-trump card; finally if North's hand contains two non-trump cards, play any worthless card, or discard king before ace, or discard at random.

For East. Follow suit; or trump or overtrump if West does not have the trick; or discard any worthless card; or discard king before ace; or discard at random.

For South. Follow suit; or trump or over-trump if North does not have the trick; or discard any worthless card; or discard king before ace; or discard at random.

The interpretation of these strategies is similar to the no-trump case.

The value of the game is computed as for the previous case, assuming that South has been chosen declarer. Possession of the lead is again important but not as important as in the no-trump case. The value of the game is 1.101.

§ 6. REMARKS

The examples treated in this paper have several possible extensions and raise several questions which we list below.

(a) Elsewhere the author will discuss these bridge models with the bidding stage added.

(b) There are obvious extensions of the problem to considerations of models with more cards in the deck. This would increase the difficulty of the computations and would bring in new types of signaling.

(c) How do the good strategies change if the shuffling is not perfect?

(d) What is the range of validity of the definition of the value

to the four-person game given in section 4?

(e) (Suggested by A. W. Tucker). Find the good strategies for two-trick end-play situations in ordinary bridge. The necessary models can be obtained from the ones used here by suitably changing the information patterns.

In this paper we have dealt with the problem of private signaling between partners by redefining the game to be a four-person game. There are other possible resolutions of this difficulty. In the first place, the wording of the rules of bridge in Hoyle is sufficiently ambiguous that it might be argued that private signaling of the sort described is actually permissible. It is the author's opinion that, at least in the spirit in which the rules are usually interpreted, this is not the case and private signaling is unethical. Accepting this, one could still regard our bridge model as a two-person game if one changed its extensive form to include preliminary moves at which each pair announced its pure signaling strategy, i.e., its leading conventions. However, if one wishes to preserve the extensive form of the model, then the only remaining alternative is the one chosen in this paper, namely, to redefine the game as a four-person game with enforced coalitions. As observed above, this requires a new concept of solution in a four-person game.

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SUMS OF POSITIONAL GAMES[1]

John Milnor

Many common games, such as chess and checkers, exhibit a structure in addition to that which is necessary to define their game theoretic properties. By a position in such a game will be meant merely the physical setup of the board without any specification as to which player is to move.[2] In these games a set of possible moves is defined at each position for each of the players, even though only one of them will actually be able to move from this position in any particular play of the game. In this paper an operation of addition will be defined and studied for games having this structure.

The games to be considered may be described as follows. There is a finite set of possible positions P and a starting position $p_0 \in P$. For each $p \in P$ and each player $i = 1, 2$ there is a set of possible moves $M_i(p) \subset P$. These are to satisfy the following finiteness condition. Every chain $p_1, p_2, p_3, \ldots$ of positions such that

$$p_{j+1} \in M_1(p_j) \cup M_2(p_j) , \quad j = 1, 2, 3, \ldots$$

must terminate. In particular a pass $p \in M_i(p)$ is never possible. For each p such that one player has no possible move, that is such that $M_1(p)$ or $M_2(p)$ is the vacuous set ϕ, the other player shall also have no move[3] and the payoff functions $k_1(p) = -k_2(p)$ shall be defined. The players are to move alternately starting at p_0 until one is unable to

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[1]The preparation of this paper was sponsored (in part) by The RAND Corporation.

[2]This is not quite precise. Chess, for example, is complicated by the rule that if a "configuration," in the sense of the physical setup of the board together with the player whose turn it is to move, is repeated three times then the game is a draw. In order to allow for this it is necessary to define a position as the setup of the board together with the set of configurations which have occurred since the last irreversible move.

[3]This requirement is not too strong since, if it does not hold, it is only necessary to add some fictitious moves and positions to the game in order to satisfy it. For example in chess the move of taking the pieces off the board when the game is stalemated could be added.

move, and then each is to collect his payoff. The information is perfect and there are no chance moves.

For each player i and position p, let $v_i(p)$ denote the value of the game for i if it is i's turn to move at position p. It is characterized by the equations

$$v_i(p) = k_i(p) \quad \text{for } M_i(p) = \phi$$

$$v_i(p) = \operatorname{Max}_{p_1 \in M_i(p)} (-v_{3-i}(p_1)) \quad \text{for } M_i(p) \neq \phi .$$

In particular we have the inequalities

$$v_1(p) \geq -v_2(p_1) \quad \text{for } p_1 \in M_1(p)$$

$$v_2(p) \geq -v_1(p_1) \quad \text{for } p_1 \in M_2(p) . \tag{1}$$

In case the equality holds in one of these inequalities, the move p_1 is called an optimal move. By the value of a game G will be meant $v_i(G) = v_i(p_o)$, where p_o is the starting position of G.

By the incentive $I(p)$ which a player has to move at position p will be meant the amount he would gain by moving rather than passing if he were given this choice. Thus if player 1 moves at position p and both players play optimally the payoff to 1 will be $v_1(p)$. If player 1 were to pass, however, his payoff would be $-v_2(p)$. The amount he gains by moving rather than letting 2 have the first move is therefore given by the symmetrical expression

$$I(p) = v_1(p) + v_2(p) . \tag{2}$$

By the sum of a game G and a game G' will be meant the game in which a player has the choice of moving in G and passing in G' or passing in G and moving in G' on each move, and in which he tries to maximize the sum of his payoffs.[4] Thus a position $p + p'$ in $G + G'$ is a pair of positions $(p,p') \in P \times P'$ and the possible moves are

$$M_i(p + p') = M_i(p) \times p' \cup p \times M_i(p') .$$

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[4] More generally the payoff in the sum game could be defined by

$$k_1(p + p') = f_1(k_1(p), k_1(p'))$$

where f_1 is any function which is monotone in each variable. Generalized sums of three or more games could be defined in the same way.

Note that game addition is an associative, commutative operation.

In order to get some information about $v_1(p + p')$ and $I(p + p')$ it is necessary to construct specific strategies for $G + G'$ in terms of known strategies for G and G'. Thus in order to get an upper bound for $v_1(p + p')$ it is necessary to construct a strategy for player 2. Define such a strategy by the following rules:

a) Always make a move which is optimal in G or G'.

b) If possible move in the game in which player 1 has just moved.[5]

If player 1 is to move at position $p + p'$ and 2 follows these rules, there are two cases. If 2 is always able to follow rule b), then, considering only those moves which were made in game G, the players have moved alternately. Since 1 has moved first and 2 has made optimal moves, the payoff to 1 in game G is at most $v_1(p)$. Similarly his payoff in G' is at most $v_1(p')$. Therefore in this case

$$v_1(p + p') \leq v_1(p) + v_1(p') .$$

It may happen, however, that at some point 2 is unable to follow rule b) since the game in which 1 has just played is over. In this case, considering only the moves made in the other game from the starting position p or p', player 2 is forced either to move twice in a row, or to make the first move. If this happens at position p_1 then it will be shown that the payoff to player 2 will be augmented by $I(p_1)$ and the payoff to player 1 will be bounded by $v_1(p) + v_1(p') - I(p_1)$. Suppose, for example, that 1 has finished the game G' by making the move p_1' and that it is 2's turn to move at $p_1 + p_1'$. Since the players have moved alternately in both games, and since player 2 has moved optimally we have (by repeated application of the inequalities (1) for player 1 and the corresponding equalities for player 2)

$$v_1(p_1) \leq v_1(p)$$

$$v_1(p_1') \leq v_1(p') .$$

Since the game G' is over, the payoff to 1 in G' is $k_1(p_1') = v_1(p_1')$. Since the player 2 will make optimal moves in G the payoff to 1 in G will be at most $-v_2(p_1)$. Therefore

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[5]The following rule would also be sufficient

b') If possible move in the game in which the incentive is highest.

$$v_1(p + p') \leq -v_2(p_1) + v_1(p_1') = -I(p_1) + v_1(p_1) + v_1(p_1')$$

$$\leq -I(p_1) + v_1(p) + v_1(p') .$$

If we define

$$D = \max_{p_1 \in P,\ p_1' \in P'} (0, -I(p_1), -I(p_1')) ,$$

it has thus been proved that

$$v_1(p + p') \leq v_1(p) + v_1(p') + D .$$

Letting player 1 use the same strategy, it follows that

$$\left.\begin{matrix} v_1(p) - v_2(p') - D \\ - v_2(p) + v_1(p') - D \end{matrix}\right\} \leq v_1(p + p') .$$

In many games the incentive can be negative and D is therefore not zero. In chess for example the phenomenon of being forced to move when one would like to pass is called Zugzwang. The preceding inequalities are extremely weak for such games, however, so we will consider only games in which $I(p) \geq 0$ for all $p \in P$, henceforth. For such games $D = 0$ and therefore we have the following statement which will be basic for the remainder of this paper.

$$(3) \qquad \left.\begin{matrix} v_1(p) - v_2(p') \\ - v_2(p) + v_1(p') \end{matrix}\right\} \leq v_1(p + p') \leq v_1(p) + v_1(p') .^{6}$$

In particular, substituting in equation (2) we have the "metric" inequality

$$(4) \qquad |I(p) - I(p')| \leq I(p + p') \leq I(p) + I(p') .$$

Since the left side is non-negative, it follows that in forming sums of

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[6] For the generalized sum of footnote 4 we would have by the same method

$$(3') \qquad \left.\begin{matrix} f_1(v_1(p), -v_2(p')) \\ f_1(-v_2(p), v_1(p')) \end{matrix}\right\} \leq v_1(p + p') \leq f_1(v_1(p), v_1(p')) .$$

Both of these formula remain true if the indices 1 and 2 are interchanged. Similar inequalities hold for generalized sums of three or more games.

games we preserve the condition that I be non-negative.

Define two games G and G' to be <u>equivalent</u> $G \sim G'$ if for every G'' the condition

$$v_1(G + G'') = v_1(G' + G'')$$

is satisfied, where all three games are to satisfy the condition that $I(p)$ be non-negative for all $p \in P$. It is clear that this is an equivalence relation and that sums of equivalent games are equivalent.

Two objections may be raised to this definition of game equivalence. First, it is not symmetric between the two players. Second, no effective procedure for determining whether two games are equivalent is apparent. Both of these objections will be removed by Theorem 1.

Let $-G$ denote the game G with players interchanged so that $v_1(-G) = v_2(G)$ and $I(-G) = I(G)$. The position corresponding to p in $-G$ will be denoted by $-p$. Consider the game $G + (-G)$, which may be abbreviated to $G - G$. Since the player who moves second can do at least as well as the first player by symmetrizing his moves, it follows that $v_1(G - G), v_2(G - G) \leq 0$. Since $I(G - G) \geq 0$, it follows that

$$v_1(G - G) = v_2(G - G) = 0 . \tag{5}$$

THEOREM 1. The following three conditions are equivalent.

a) $G \sim G'$.

b) $v_2(G + G'') = v_2(G' + G'')$ for all G''.

c) $v_1(G - G') = v_2(G - G') = 0$.

It will first be shown that $v_i(G + G'') \leq v_i(G' + G'')$ for all G'' if and only if $v_i(G - G') \leq 0$. Setting $G'' = -G'$ in the first of these two inequalities we have

$$v_i(G - G') \leq v_i(G' - G') = 0 .$$

Conversely it follows by (3) and (5) that if $v_i(G - G') \leq 0$ then

$$v_i(G + G'') = v_i(G + G'') - v_{3-i}(G' - G')$$

$$\leq v_i((G + G'') + (G' - G')) = v_i((G' + G'') + (G - G'))$$

$$\leq v_i(G' + G'') + v_i(G - G') \leq v_i(G' + G'') .$$

The theorem follows from this assertion together with the

observation that $v_1(G - G') = v_{3-i}(G' - G)$. For example, if a) holds then $v_1(G - G') \leq 0$ and $v_1(G' - G) = v_2(G - G') \leq 0$. Since $I(G - G') \geq 0$ it follows that c) holds.

A nontrivial example of two games which are equivalent is given by Figure 1. In this diagram the positions are indicated by vertices, while the possible moves are indicated by arrows which are solid for player 1 and dotted for player 2. The payoff to player 1 is given.

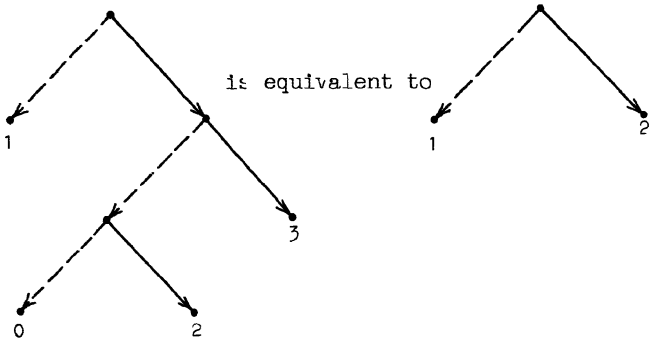

Figure 1

It follows from Theorem 1 that a game G is equivalent to the game zero (the game with one position and payoff zero) if and only if

$$v_1(G) = v_2(G) = 0 .$$

From equation (5) it follows that $G - G \sim 0$, so that $-G$ is indeed an inverse to G. The equivalence classes of games therefore form an abelian group under addition.

Several subgroups of this group are of interest. The symmetric games -- games in which the role of the players is interchangeable -- are characterized by the equation $G \sim -G$ or $G + G \sim 0$. In other words they form the subgroup of elements of order 2.

Another subgroup consists of those games which satisfy $I(G) = 0$. The fact that these form a subgroup follows from the inequalities (4) and the equation $I(G) = I(-G)$. It also follows from (4) that the factor group of games modulo games with incentive zero is a metric group under the metric I. The subgroup itself is isomorphic to the real numbers under addition, the isomorphism being provided by $v_1(G)$ or $v_2(G)$. Suppose, for example, that the games G and G' satisfy $I(G) = I(G') = 0$ and $v_1(G) = v_1(G')$. Then $v_2(G) = v_2(G') = -v_1(G)$ and

$$0 = v_1(G) - v_1(G') = v_1(G) - v_2(-G') \leq v_1(G - G')$$

$$\leq v_1(G) + v_1(-G') = v_1(G) + v_2(G') = 0 .$$

Therefore $v_1(G - G') = 0$ and similarly $v_2(G - G') = 0$, which implies that $G \sim G'$. In particular G and G' are equivalent to the game with only one position in which 2 pays 1 the amount $v_1(G)$.

The fundamental problem involved in the study of sums of games is that of describing optimal strategies for the sum of two or more games in terms of information about the individual games. Some information about this question will be given by Theorem 2, but a precise answer seems fairly difficult.

It will frequently happen that the optimal moves in the sum of two games are not optimal in the individual games. An example of this is given in Figure 2.

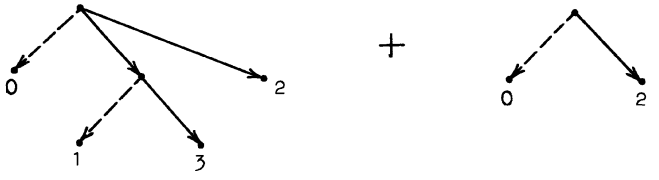

Figure 2

THEOREM 2. If the move $p_1 \in M_1(p)$ is optimal for player 1 at some position $p + p'$, then

$$v_1(p) \leq v_1(p_1)$$

and

$$v_2(p_1) \leq v_2(p) .$$

Furthermore the position p' must satisfy

$$v_1(p) + v_2(p_1) \leq I(p') \leq v_1(p_1) + v_2(p) .$$

These inequalities are obtained by combining the equation $v_1(p + p') = -v_2(p_1 + p')$ which states that $p_1 + p'$ is an optimal move for 1 at the position $p + p'$, with the inequalities (3) as applied to $v_1(p + p')$ and $v_2(p_1 + p')$.

In order to gain some insight into the meaning of these inequalities, several new definitions will be made.

The loss function $r_i(p_1/p) = v_i(p) + v_{3-i}(p_1)$ measures the amount by which the move $p_1 \in M_i(p)$ fails to be optimal. Its fundamental property, which follows from (1) is $r_i(p_1/p) \geq 0$ where equality holds if and only if the move p_1 is optimal.

The gain function $g_i(p_1/p) = v_{3-i}(p) - v_{3-i}(p_1)$ measures the amount by which the move $p_1 \in M_i(p)$ is better than a pass. Its fundamental property is

$$g_i(p_1/p) \leq I(p)$$

where equality holds only for optimal moves. This follows from the equation

$$g_i(p_1/p) = I(p) - r_i(p_1/p) .$$

A move $p_1 + p' \in M_i(p + p')$ will be called sente[7] if

$$I(p_1) \geq I(p') .$$

A sente move tends to force the opponent to reply in the same game. The quantity

$$s_i(p_1,p') = \text{Max } (0, I(p') - I(p_1))$$

measures the amount by which the move $p_1 + p' \in M_i(p + p')$ fails to be sente. It is a non-negative quantity which vanishes only for sente moves. Using these three quantities we may rewrite Theorem 2 as follows.

THEOREM 2'. If the move $p_1 + p' \in M_1(p + p')$ is optimal then the following inequalities hold

$$r_1(p_1/p) \leq \text{Min } (I(p_1), I(p'))$$

$$s_1(p_1,p') \leq g_1(p_1/p) .$$

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[7] From the Japanese game Go. Pronounced sen-teh.

Roughly speaking, the first inequality says that a move which is non-optimal in G can be optimal in $G + G'$ only if it leads to a position of high incentive and if the game G' has high incentive. Apparently it is desirable to place the opponent in a position in which both games have high incentive, and it may be worth the sacrifice of a move which is not optimal in G in order to achieve this end.

The second inequality says that a non-sente move can be optimal in $G + G'$ only if it is much better than a pass, and even a sente move cannot be worse than a pass.

The following inequalities describe the behavior of the loss and gain functions under the formation of sums of games. They are immediate consequences of (3) which are included only for completeness, since they will not be used in the sequel.

$$\left.\begin{array}{l} r_1(p_1/p) - \operatorname{Min}(I(p_1), I(p')) \\ r_1(p_1/p) + s_1(p_1,p') - I(p) \end{array}\right\} \leq r_1(p_1 + p'/p + p') \leq r_1(p_1/p) + I(p')$$

$$\left.\begin{array}{l} g_1(p_1/p) - I(p') \\ g_1(p_1/p) - I(p) \end{array}\right\} \leq g_1(p_1 + p'/p + p') \leq \left\{\begin{array}{l} g_1(p_1/p) + I(p') \\ g_1(p_1/p) + I(p_1) \end{array}\right. .$$

The following theorem gives a characterization of those moves in a game which can be optimal in some sum game.

THEOREM 3. The following three conditions are equivalent.

a) The move $p_1 - p \in M_1(p - p)$ is optimal.

b) The move $p_1 + p' \in M_1(p + p')$ is optimal for some p'.

c) The move $p_1 + p'' \in M_1(p + p'')$ satisfies $g_1(p_1 + p''/p + p'') \geq 0$ for all p''.

Setting $p' = -p$ we have that a) implies b). If $p_1 + p'$ is optimal at $p + p'$ then $(p_1 + p'') + (p' - p'')$ is optimal at $(p + p'') + (p' - p'')$. Since $p_1 + p''$ is an optimal move in some sum game, it follows by Theorem 2' that it has non-negative gain. Therefore b) implies c). If $p'' = -p$ then from c) we have

$$0 \leq g_1(p_1 - p/p - p) = v_2(p - p) - v_2(p_1 - p) = - v_2(p_1 - p) \leq v_1(p - p) = 0$$

which shows that $p_1 - p$ is optimal at $p - p$, and completes the proof.

The following may be described as a converse to (3).

THEOREM 4. If player 1 is to move at $p + p'$, and if he makes the first move in game G, then for any play of $G + G'$ the payoff k_1 of this game will satisfy

$$v_1(p) + v_1(p') - \sum r_1 - \sum s_1 \leq k_1 \leq v_1(p) - v_2(p') + \sum r_2 + \sum s_2$$

where the first two summations are taken over all moves made by 1 in this particular play, and the last two over all moves made by 2.[8]

Thus if player 1 can always make moves which are optimal in G or G' and sente, he will achieve the upper bound of $v_1(p) + v_1(p')$ for his payoff. If 2 can always make optimal sente moves, he will achieve the lower bound of $v_1(p) - v_2(p')$ (or $-v_2(p) + v_1(p')$ if the first move is made in G'). Since these two outcomes are usually incompatible, it follows that it is frequently impossible to make moves in $G + G'$ which are both optimal in G or G' and sente.

Since proofs of these two inequalities are similar, only the first will be carried out. Suppose that player 1 makes the move $p_1 + p' \in M_1(p + p')$ and 2 replies with the move $p_2 + p_2'$, where either $p_1 = p_2$ or $p' = p_2'$. In the first case we have

$$p_1 = p_2, \quad v_2(p') \geq -v_1(p_2'),$$

$$v_1(p_2) + v_1(p_2') = v_1(p_1) + v_1(p_2') \geq v_1(p_1) - v_2(p') =$$

$$v_1(p) + v_1(p') - r_1(p_1/p) - (I(p') - I(p_1)) .$$

In the other case

$$p' = p_2', \quad v_2(p_1) \geq -v_1(p_2),$$

$$v_1(p_2) + v_1(p_2') = v_1(p_2) + v_1(p') \geq -v_2(p_1) + v_1(p') =$$

$$v_1(p) + v_1(p') - r_1(p_1/p) - 0 .$$

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[8] The condition that I be non-negative is not required for this theorem, which depends only on (1).

Combining these two inequalities we have

$$v_1(p_2) + v_1(p_2') \geq v_1(p) + v_1(p') - r_1(p_1/p) - s_1(p_1,p') \ .$$

Taking the corresponding inequalities for each move by player 1 in the play of the game, and combining them we have the required inequality. (If player 2 follows the strategies a) and b') of footnote 5, it is easily shown that equality holds in this expression. This provides an alternative proof of (3).)

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Part IV

GENERAL n-PERSON GAMES

A one-person game can be converted easily into a maximization problem and hence is conceptually quite simple. The direct opposition of interests in a zero-sum two-person game leads to the minimax game-value, which is convincing both in its interpretation and in its mathematical rigor. But, once the game theorist leaves these well-explored areas for the uncharted territory of games with more than two participants or no constant-sum restriction, he has few bearings to guide him through the bewildering maze of paths open to him. The authors of the papers in Part IV attempt to throw light on known problems and to lead the reader into new approaches to the theory of general n-person games.

Is it possible to appraise the equities of the participants in an arbitrary n-person game? (See Study 24, Preface, Problem 10.) In PAPER 17 L. S. Shapley proposes a set of "values" which may be regarded as an a priori evaluation of the players' positions (or seats) in a game, ignoring totally any social structure or standard of behavior. His result can be interpreted by imagining the random formation of a coalition of all of the players, starting with a single member and adding one player at a time. Each player is then assigned the advantage accruing to the coalition at the time of his admission. In this process of computing the expected value for an individual player all coalition formations are considered as equally likely. Shapley shows that there is just one evaluation which is additive with respect to the combination of independent games and which is such that

$$\phi_1 + \phi_2 + \ldots + \phi_n = A \ ,$$

where ϕ_i denotes the value assigned to player $i = 1, 2, \ldots, n$, and A the amount which the entire set of n players can obtain by cooperating.

Still outstanding is the question whether every n-person game has a solution in the sense of von Neumann and Morgenstern. (See Study 24, Preface, Problem 8.) However, Papers 18, 19 and 20 introduce new classes

of n-person games for which solutions are obtained and employ methods which hold promise for further advance. Papers 18 and 19 deal with a class of games, termed "(n, k)-games," that generalize the "direct majority game" of von Neumann and Morgenstern (see Theory of Games and Economic Behavior, p. 431), while Paper 20 develops "quota games," a class of games that includes all constant-sum four-person games.

The (n, k)-game, introduced by R. Bott in PAPER 18, is an n-person game in which any coalition of k or more players, for any fixed $k > n/2$, wins one unit from each of its opponents and any smaller coalition is totally defeated. Here a coalition of exactly n-k+1 players is a "blocking coalition," since it is just large enough to keep its complement from winning. Bott shows that the winning coalitions in a symmetric solution are composed of a number of blocking coalitions and that the members of any one blocking coalition within a winning coalition share equally in an imputation of the solution. Symmetric solutions are thus uniquely characterized. Dropping symmetry, D. B. Gillies exhibits in PAPER 19 a surprising variety of other solutions of (n, k)-games, all derived from Bott's symmetric solutions. Gillies' solutions are obtained by several methods which may carry over to a more general context: (1) by the addition of "bargaining curves" (Theory of Games and Economic Behavior, p. 501), (2) by inflation to larger games (ibid., p. 398), (3) by "discrimination" (ibid., pp. 288-289) in which the non-discriminated players divide their take according to any solution to a smaller game, or (4) by partitioning the players into fixed subsets, assigning the spoils arbitrarily (i.e. in all admissible ways in one solution) among these subsets, and then dividing the spoils in any one subset according to the symmetric solution to a smaller game the players think they are playing.

Contrary to the (n, k)-games whose majority nature is readily described by verbal rules, the "quota games" introduced by L. S. Shapley in PAPER 20 stem from technical properties of the characteristic function v of von Neumann and Morgenstern. An n-person game is called a "quota game" if the amount A which the whole set of n players obtains by cooperating can be split among the n players

$$A = \omega_1 + \omega_2 + \ldots + \omega_n$$

so that

$$\omega_i + \omega_j = v_{ij} \quad \text{for all} \quad i, j \ (i \neq j) \ ,$$

where v_{ij} denotes the amount that the pair of distinct players i and j can obtain by cooperating. Shapley obtains families of solutions for the entire class of quota games, a class that contains some three-person games,

all constant-sum four-person games, and a sizeable swath of all games with more than four players. In a typical imputation in one of these solutions, all but two or three of the players receive their "quotas" ω_i.

In PAPER 21 H. Raiffa considers the challenging conceptual problem of how one should choose an imputation from among the startlingly profuse solution sets of von Neumann and Morgenstern or from the equally abundant equilibrium points of Nash. He proposes "arbitration conventions" as a solution for generalized two-person games, i.e., two-person games in which the restrictions to a transferable utility and a common unit of measure have been dropped. To this end he represents the payoffs x and y to players I and II, which result from a pair of mixed strategies for the two players, as a point (x,y) in a Euclidean plane. The set R^* of all payoff points (x,y) is extended to the minimal closed convex set R^{**} containing R^*. Reflecting the players' preferences in payoffs, a partial ordering of points is made;

$$(x_1,y_1) \prec (x_2,y_2) \quad \text{if} \quad x_1 \leqq x_2,\ y_1 \leqq y_2\ , \quad \text{and} \quad x_1 + y_1 < x_2 + y_2\ ,$$

which leads to the subset $M^{**}_{\prec}$ of points of R^{**} which are maximal with respect to the partial ordering. In terms of this graphical representation, an "arbitration convention" is defined as a mapping $T: R^* \longrightarrow M^{**}_{\prec}$ that satisfies certain "natural" conditions. The mapping T, coupled with the initial mapping of pairs of mixed strategies into payoff points (x,y), assigns to each pair of mixed strategies for the two players a preferred outcome in $M^{**}_{\prec}$, thus transforming the given game into a new game. Raiffa's principal theorem asserts that saddlepoints always exist in this transformed game. Several examples of arbitration schemes especially suited to certain social structures are examined.

H. W. K.
A. W. T.

A VALUE FOR n-PERSON GAMES[1]

L. S. Shapley

§ 1. INTRODUCTION

At the foundation of the theory of games is the assumption that the players of a game can evaluate, in their utility scales, every "prospect" that might arise as a result of a play. In attempting to apply the theory to any field, one would normally expect to be permitted to include, in the class of "prospects," the prospect of having to play a game. The possibility of evaluating games is therefore of critical importance. So long as the theory is unable to assign values to the games typically found in application, only relatively simple situations -- where games do not depend on other games -- will be susceptible to analysis and solution.

In the finite theory of von Neumann and Morgenstern[2] difficulty in evaluation persists for the "essential" games, and for only those. In this note we deduce a value for the "essential" case and examine a number of its elementary properties. We proceed from a set of three axioms, having simple intuitive interpretations, which suffice to determine the value uniquely.

Our present work, though mathematically self-contained, is founded conceptually on the von Neumann-Morgenstern theory up to their introduction of characteristic functions. We thereby inherit certain important underlying assumptions: (a) that utility is objective and transferable; (b) that games are cooperative affairs; (c) that games, granting (a) and (b), are adequately represented by their characteristic functions. However, we are not committed to the assumptions regarding rational behavior embodied in the von Neumann-Morgenstern notion of "solution."

We shall think of a "game" as a set of rules with specified players in the playing positions. The rules alone describe what we shall

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[1]The preparation of this paper was sponsored (in part) by the RAND Corporation.

[2]Reference [3] at the end of this paper. Examples of infinite games without values may be found in [2], pp. 58-59, and in [1], p. 110. See also Karlin [2], pp. 152-153.

call an "abstract game." Abstract games are played by roles -- such as "dealer," or "visiting team" -- rather than by players external to the game. The theory of games deals mainly with abstract games.[3] The distinction will be useful in enabling us to state in a precise way that the value of a "game" depends only on its abstract properties. (Axiom 1 below).

§ 2. DEFINITIONS

Let U denote the universe of players, and define a game to be any superadditive set-function v from the subsets of U to the real numbers, thus:

$$v(0) = 0 , \tag{1}$$

$$v(S) \geq v(S \cap T) + v(S - T) \qquad (\text{all } S, T \subset U) . \tag{2}$$

A carrier of v is any set $N \subset U$ with

$$v(S) = v(N \cap S) \qquad (\text{all } S \subset U) . \tag{3}$$

Any superset of a carrier of v is again a carrier of v. The use of carriers obviates the usual classification of games according to the number of players. The players outside any carrier have no direct influence on the play since they contribute nothing to any coalition. We shall restrict our attention to games which possess finite carriers.

The sum ("superposition") of two games is again a game. Intuitively it is the game obtained when two games, with independent rules but possibly overlapping sets of players, are regarded as one. If the games happen to possess disjunct carriers, then their sum is their "composition."[4]

Let $\Pi(U)$ denote the set of permutations of U -- that is, the one to one mappings of U onto itself. If $\pi \in \Pi(U)$, then, writing πS for the image of S under π, we may define the function πv by

$$\pi v(\pi S) = v(S) \qquad (\text{all } S \subset U) . \tag{4}$$

If v is a game, then the class of games πv, $\pi \in \Pi(U)$, may be regarded as the "abstract game" corresponding to v. Unlike composition, the operation of addition of games can not be extended to abstract games.

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[3] An exception is found in the matter of symmetrization (see for example [2], pp. 81-83), in which the players must be distinguished from their roles.

[4] See [3], §§ 26.7.2 and 41.3.

By the value $\phi[v]$ of the game v we shall mean a function which associates with each i in U a real number $\phi_i[v]$, and which satisfies the conditions of the following axioms. The value will thus provide an additive set-function (an inessential game) $\overline{v}$:

$$\overline{v}(S) = \sum_S \phi_i[v] \qquad \text{(all } S \subseteq U\text{)} , \tag{5}$$

to take the place of the superadditive function v.

AXIOM 1. For each π in $\Pi(U)$,

$$\phi_{\pi i}[\pi v] = \phi_i[v] .$$

AXIOM 2. For each carrier N of v,

$$\sum_N \phi_i[v] = v(N) .$$

AXIOM 3. For any two games v and w,

$$\phi[v + w] = \phi[v] + \phi[w] .$$

COMMENTS. The first axiom ("symmetry") states that the value is essentially a property of the abstract game. The second axiom ("efficiency") states that the value represents a distribution of the full yield of the game. This excludes, for example, the evaluation $\phi_i[v] = v((i))$, in which each player pessimistically assumes that the rest will all cooperate against him. The third axiom ("law of aggregation") states that when two independent games are combined, their values must be added player by player. This is a prime requisite for any evaluation scheme designed to be applied eventually to systems of interdependent games.

It is remarkable that no further conditions are required to determine the value uniquely.[5]

§ 3. DETERMINATION OF THE VALUE FUNCTION

LEMMA 1. If N is a finite carrier of v, then, for $i \notin N$,

$$\phi_i[v] = 0 .$$

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[5]Three further properties of the value which might suggest themselves as suitable axioms will be proved as Lemma 1 and Corollaries 1 and 3 below.

PROOF. Take $i \notin N$. Both N and $N \cup (i)$ are carriers of v; and $v(N) = v(N \cup (i))$. Hence $\phi_i[v] = 0$ by Axiom 2, as was to be shown.

We first consider certain symmetric games. For any $R \subseteq U$, $R \neq 0$ define v_R:

$$v_R(S) = \begin{cases} 1 & \text{if } S \supseteq R , \\ 0 & \text{if } S \not\supseteq R . \end{cases} \tag{6}$$

The function cv_R is a game, for any non-negative c, and R is a carrier.

In what follows, we shall use r, s, n, ... for the numbers of elements in R, S, N, ... respectively.

LEMMA 2. For $c \geq 0$, $0 < r < \infty$, we have

$$\phi_i[cv_R] = \begin{cases} c/r & \text{if } i \in R , \\ 0 & \text{if } i \notin R . \end{cases}$$

PROOF. Take i and j in R, and choose $\pi \in \Pi(U)$ so that $\pi R = R$ and $\pi i = j$. Then we have $\pi v_R = v_R$, and hence, by Axiom 1,

$$\phi_j[cv_R] = \phi_i[cv_R] .$$

By Axiom 2,

$$c = cv_R(R) = \sum_{j \in R} \phi_j[cv_R] = r\phi_i[cv_R] ,$$

for any $i \in R$. This, with Lemma 1, completes the proof.

LEMMA 3.[6] Any game with finite carrier is a linear combination of symmetric games v_R:

$$v = \sum_{\substack{R \subset N \\ R \neq 0}} c_R(v) v_R , \tag{7}$$

N being any finite carrier of v. The coefficients are independent of N, and are given by

[6] The use of this lemma was suggested by H. Rogers.

(8) $$c_R(v) = \sum_{T \subseteq R} (-1)^{r-t} v(T) \qquad (0 < r < \infty) .$$

PROOF. We must verify that

(9) $$v(S) = \sum_{\substack{R \subseteq N \\ R \neq 0}} c_R(v) v_R(S)$$

holds for all $S \subseteq U$, and for any finite carrier N of v. If $S \subseteq N$, then (9) reduces, by (6) and (8), to

$$v(S) = \sum_{R \subseteq S} \sum_{T \subseteq R} (-1)^{r-t} v(T)$$

$$= \sum_{T \subseteq S} \left[\sum_{r=t}^{s} (-1)^{r-t} \binom{s-t}{r-t} \right] v(T) .$$

The expression in brackets vanishes except for $s = t$, so we are left with the identity $v(S) = v(S)$. In general we have, by (3),

$$v(S) = v(N \cap S) = \sum_{R \subseteq N} c_R(v) v_R(N \cap S) = \sum_{R \subseteq N} c_R(v) v_R(S) .$$

This completes the proof.

REMARK. It is easily shown that $c_R(v) = 0$ if R is not contained in every carrier of v.

An immediate corollary to Axiom 3 is that $\phi[v-w] = \phi[v] - \phi[w]$ if v, w, and $v - w$ are all games. We can therefore apply Lemma 2 to the representation of Lemma 3 and obtain the formula:

(10) $$\phi_i[v] = \sum_{\substack{R \subseteq N \\ R \ni i}} c_R(v)/r \qquad (\text{all } i \in N) .$$

Inserting (8) and simplifying the result gives us

(11) $$\phi_i[v] = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{(s-1)!(n-s)!}{n!} v(S) - \sum_{\substack{S \subseteq N \\ S \not\ni i}} \frac{s!(n-s-1)!}{n!} v(S)$$

$$(\text{all } i \in N) .$$

Introducing the quantities

(12) $$\gamma_n(s) = (s-1)!(n-s)!/n! ,$$

we now assert:

THEOREM. A unique value function ϕ exists satisfying Axioms 1 - 3, for games with finite carriers;

it is given by the formula

$$\phi_i[v] = \sum_{S \subseteq N} \gamma_n(s)[v(S) - v(S-(i))] \qquad (\text{all } i \in U) , \tag{13}$$

where N is any finite carrier of v.

PROOF. (13) follows from (11), (12), and Lemma 1. We note that (13), like (10), does not depend on the particular finite carrier N; the ϕ of the theorem is therefore well defined. By its derivation it is clearly the only value function which could satisfy the axioms. That it does in fact satisfy the axioms is easily verified with the aid of Lemma 3.

§ 4. ELEMENTARY PROPERTIES OF THE VALUE

COROLLARY 1. We have

$$\phi_i[v] \geq v((i)) \qquad (\text{all } i \in U) , \tag{14}$$

with equality if and only if i is a dummy -- i.e., if and only if

$$v(S) = v(S - (i)) + v((i)) \qquad (\text{all } S \ni i) . \tag{15}$$

PROOF. For any $i \in U$ we may take $N \ni i$ and obtain, by (2),

$$\phi_i[v] \geq \sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(s) v((i)) ,$$

with equality if and only if (15), since none of the $\gamma_n(s)$ vanishes. The proof is completed by noting that

$$\sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(s) = \sum_{s=1}^{n} \binom{n-1}{s-1} \gamma_n(s) = \sum_{s=1}^{n} \frac{1}{n} = 1 . \tag{16}$$

Only in this corollary have our results depended on the superadditive nature of the functions v.

COROLLARY 2. If v is decomposable -- i.e., if games $w^{(1)}, w^{(2)}, \ldots, w^{(p)}$ having pairwise disjunct carriers $N^{(1)}, N^{(2)}, \ldots, N^{(p)}$ exist such that

$$v = \sum_{k=1}^{p} w^{(k)} ,$$

-- then, for each $k = 1, 2, \ldots, p$,

$$\phi_i[v] = \phi_i[w^{(k)}] \qquad (\text{all } i \in N^{(k)}) .$$

PROOF. By Axiom 3.

COROLLARY 3. If v and w are strategically equivalent -- i.e., if

$$w = cv + \bar{a} , \tag{17}$$

where c is a positive constant and $\bar{a}$ an additive set-function on U with finite carrier[7] -- then

$$\phi_i[w] = c\phi_i[v] + \bar{a}((i)) \qquad (\text{all } i \in U) .$$

PROOF. By Axiom 3, Corollary 1 applied to the inessential game $\bar{a}$, and the fact that (13) is linear and homogeneous in v.

COROLLARY 4. If v is constant-sum -- i.e., if

$$v(S) + v(U - S) = v(U) \qquad (\text{all } S \subseteq U) , \tag{18}$$

-- then its value is given by the formula:

$$\phi_i[v] = 2\Big[\sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(s)v(S)\Big] - v(N) \qquad (\text{all } i \in N) . \tag{19}$$

where N is any finite carrier of v.

PROOF. We have, for $i \in N$,

$$\phi_i[v] = \sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(s)v(S) - \sum_{\substack{T \subseteq N \\ T \not\ni i}} \gamma_n(t+1)v(T)$$

$$= \sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(s)v(S) - \sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(n-s+1)[v(N) - v(S)] .$$

But $\gamma_n(n-s+1) = \gamma_n(s)$; hence (18) follows with the aid of (16).

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[7] This is McKinsey's "S-equivalence" (see [2], p. 120), wider than the "strategic equivalence" of von Neumann and Morgenstern ([3], § 27.1).

§ 5. EXAMPLES

If N is a finite carrier of v, let A denote the set of n-vectors (α_i) satisfying

$$\begin{cases} \sum_N \alpha_i = v(N) , \\ \alpha_i \geq v((i)) \qquad\qquad (\text{all } i \in N) . \end{cases}$$

If v is inessential A is a single point; otherwise A is a regular simplex of dimension n - 1. The value of v may be regarded as a point ϕ in A, by Axiom 2 and Corollary 1. Denote the centroid of A by θ:

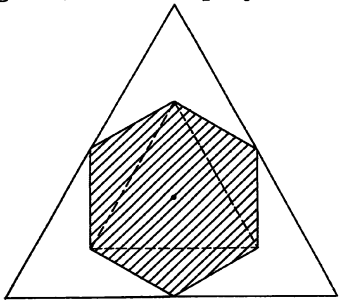
$$\theta_i = v((i)) + \frac{1}{n}\left[v(N) - \sum_{j \in N} v((j))\right] .$$

EXAMPLE 1. For two-person games, three-person constant-sum games, and inessential games, we have

$$\phi = \theta . \tag{20}$$

The same holds for arbitrary symmetric games -- i.e., games which are invariant under a transitive group of permutations of N -- and, most generally, games strategically equivalent to them. These results are demanded by symmetry, and do not depend on Axiom 3.

EXAMPLE 2. For general three-person games the positions taken by ϕ in A cover a regular hexagon, touching the boundary at the midpoint of each 1-dimensional face (see figure). The latter cases are of course the decomposable games, with one player a dummy.



EXAMPLE 3. The quota games[8] are characterized by the existence

[8] Discussed in [4].

of constants ω_i satisfying

$$\begin{cases} \omega_i + \omega_j = v((i,j)) & \text{(all } i,j \in N,\ i \neq j) \\ \sum_N \omega_i = v(N) \ . \end{cases}$$

For $n = 3$, we have

$$\phi - \theta = \frac{\omega - \theta}{2} \ . \tag{21}$$

Since ω can assume any position in A, the range of ϕ is a triangle, inscribed in the hexagon of the preceding example (see the figure).

EXAMPLE 4. All four-person constant-sum games are quota games. For them we have

$$\phi - \theta = \frac{\omega - \theta}{3} \ . \tag{22}$$

The quota ω ranges over a certain cube,[9] containing A. The value ϕ meanwhile ranges over a parallel, inscribed cube, touching the boundary of A at the midpoint of each 2-dimensional face. In higher quota games the points ϕ and ω are not so directly related.

EXAMPLE 5. The weighted majority games[10] are characterized by the existence of "weights" w_i such that never $\sum_S w_i = \sum_{N-S} w_i$, and such that

$$\begin{cases} v(S) = n - s & \text{if } \sum_S w_i > \sum_{N-S} w_i \ , \\ v(S) = -s & \text{if } \sum_S w_i < \sum_{N-S} w_i \ . \end{cases}$$

The game is then denoted by the symbol $[w_1, w_2, \ldots, w_n]$. It is easily shown that

$$\phi_i < \phi_j \text{ implies } w_i < w_j \quad \text{(all } i, j \in N) \tag{23}$$

in any weighted majority game $[w_1, w_2, \ldots, w_n]$. Hence "weight" and "value" rank the players in the same order.

The exact values can be computed without difficulty for particular cases. We have

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[9] Illustrated in [4], figure 1 (page 353 in this volume).

[10] See [3], § 50.1.

$$\phi = \frac{n-3}{n-1}\,(-1, -1, \ldots, -1, n-1)$$

for the game $[1, 1, \ldots, 1, n-2]$,[11] and

$$\phi = \frac{2}{5}\,(1, 1, 1, -1, -1, -1)$$

for the game $[2, 2, 2, 1, 1, 1]$,[12] etc.

§ 6. DERIVATION OF THE VALUE FROM A BARGAINING MODEL

The deductive approach of the earlier sections has failed to suggest a bargaining procedure which would produce the value of the game as the (expected) outcome. We conclude this paper with a description of such a procedure. The form of our model, with its chance move, lends support to the view that the value is best regarded as an a priori assessment of the situation, based on either ignorance or disregard of the social organization of the players.

The players constituting a finite carrier N agree to play the game v in a grand coalition, formed in the following way: 1. Starting with a single member, the coalition adds one player at a time until everyone has been admitted. 2. The order in which the players are to join is determined by chance, with all arrangements equally probable. 3. Each player, on his admission, demands and is promised the amount which his adherence contributes to the value of the coalition (as determined by the function v). The grand coalition then plays the game "efficiently" so as to obtain the amount $v(N)$ -- exactly enough to meet all the promises.

The expectations under this scheme are easily worked out. Let $T^{(i)}$ be the set of players preceding i. For any $S \ni i$, the payment to i if $S - (i) = T^{(i)}$ is $v(S) - v(S - (i))$, and the probability of that contingency is $\gamma_n(s)$. The total expectation of i is therefore just his value, (13), as was to be shown.

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[11] Discussed at length in [3], § 55.

[12] Discussed in [3], § 53.2.2.

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SYMMETRIC SOLUTIONS TO MAJORITY GAMES[1]

Raoul Bott

The purpose of this note is the determination of all symmetric solutions to a family of n-person games [1].[2] A game (n,k) in this class is defined by the value of the game v(S) for all coalitions S. If |S| is the number of players in S, then

$$v(S) = \begin{cases} -|S| & \text{for } |S| < k \\ n - |S| & \text{for } |S| \geq k \end{cases}$$

where k is a fixed integer with $\frac{n}{2} < k < n$.

This means that a coalition of k or more players can obtain the maximum possible amount, and a smaller coalition is totally defeated. An (n-k+1)-player coalition is the smallest coalition which is large enough to block its complement from winning. Such a coalition, if its members act together, must be included in the final (winning) coalition, so we might expect that a number of these "blocking coalitions" would combine to defeat the remaining players, and that all members of a particular blocking coalition would receive the same amount in an imputation of the solution. The systematic theory will confirm this, and will show that this is the unique symmetric standard of behavior. This uniqueness is, however, completely lost when the condition of symmetry is dropped. Mr. D. B. Gillies has found an alarming number of unsymmetric solutions to these games, most of which could not be intuitively anticipated at all.

The game (n,k) is zero sum only if $k = \frac{n+1}{2}$, and in this case (n,k) is the n-person simple majority game. For $k > \frac{n+1}{2}$, there exist pairs of opposing coalitions, both completely defeated, so the games are non-zero sum. These form the symmetric generalization of simple games: every set is either winning or losing, but the game need not be zero sum.

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[1]The preparation of this paper was supported by the Office of Naval Research.

[2]Numbers in square brackets refer to the bibliography at the end of this paper.

A symmetric solution V contains all imputations arising from permutations of the indices of any imputation $a \in V$, and is therefore characterized by its "ordered imputations" $\{a \mid a \in V, a_1 \leq a_2 \leq \cdots \leq a_n\}$. $a \underset{S}{\succ} b$ via S means

$$a_i > b_i \quad \text{for} \quad i \in S, \quad \text{and} \quad |S| \geq k .$$

If $|S| > k$, we can find a subset T of S with $|T| = k$, and $a \underset{T}{\succ} b$ via T, since all k-person sets are effective.[3] Since V is symmetric, we may assume without loss in generality, for $a \in V$, that the largest k components of a majorize the smallest k components of b. These remarks apply, of course, whenever a is a member of a symmetric subset of imputations, and we are deciding whether or not at least one member of this subset dominates b. This leads to the following redefinition of the concept of domination.

Let A be the simplex of ordered imputations in Euclidean n-space E_n, defined by

$$a_i \leq a_{i+1} \tag{1}$$

$$a_1 \geq -1 \tag{2}$$

$$\sum_1^n a_i = 0 \tag{3}$$

Then domination $(\succ)$ is defined by

$$a \succ b \quad \text{if and only if} \quad a_{n-k+\gamma} > b_\gamma \quad \text{for} \quad \gamma = 1, 2, \ldots, k .$$

Since all our proofs require that the dominating imputation be in a symmetric subset of imputations, b can be assumed ordered without loss in generality, and proofs can be constructed relative to A by means of the above definition.

THEOREM. Let $p = n - k + 1$, and write $n = sp + r$, where $0 \leq r < p$. Let $V \subset A$ be the simplex of points $a \in A$ with the following properties.

$$a_1 = -1 \quad \text{if} \quad r \neq 0 . \tag{4}$$

$$a_{r+\gamma} = a_{r+\gamma+1} , \quad \text{if} \quad \gamma \not\equiv 0 \bmod p . \tag{5}$$

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[3] This is proved in [1], p. 439: (49:J).

Then V is the unique solution in A.

PROOF. We break up the proof of this result into a sequence of lemmas.

LEMMA 1. Let $W \subset A$ consist of all points a such that $a_{r+\gamma} = a_{r+\gamma+1}$, $\gamma \not\equiv 0 \bmod p$. Then a solution V of A must certainly be contained in W.

PROOF. Suppose $a \in V$ where V is a solution in A, and suppose $a_{i_0} < a_{i_0+1}$ for $k \leq i_0 < n$. Let a^* be given by

$$a_i^* = a_i + \varepsilon , \quad i \neq i_0 + 1$$

$$a_{i_0+1}^* = a_{i_0+1} - (n - 1)\varepsilon ,$$

where $\varepsilon > 0$ is so small that $a_{i_0}^* < a_{i_0+1}^*$. Since $a^* \not\in a$, and hence cannot be in V, there exists $b \in V$ dominating a^*, and we have $b_{i+n-k} > a_i^*$ $(i = 1, 2, \ldots, k)$. But for $i \leq k < i_0 + 1$, $a_i^* > a_i$, so $b \not\in a$, a contradiction. Thus the last p coordinates of an element in V must be equal. Next assume $a \in V$ and $a_{i_0} < a_{i_0+1}$, $n - 2p + 1 \leq i_0 < n - p + 1$. Define a^* as before. Again there exists $b \in V$ with $b \not\in a^*$, and $b_{i+n-k} > a_i^*$ $(i = 1, \ldots, k)$. Now $a_i^* > a_i$ except for possibly $i = i_0 + 1$. But by our previous result, since $b \in V$, we have

$$b_{i_0+1+n-k} = b_{i_0+2+n-k} .$$

Now $a_{i_0+1} \leq a_{i_0+2} < a_{i_0+2}^* \leq b_{i_0+2+n-k} = b_{i_0+n-k+1}$, so $b \not\in a$ also. Therefore the second block of p components of a must all be equal. The proof now proceeds by induction in a straightforward manner.

LEMMA 2. Any solution in A contains all imputations in W with $a_1 = -1$ if $r \neq 0$.

PROOF. A general imputation in W is of the form

$$a = \{\alpha_0, \alpha_0, \ldots, \alpha_0, \alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_s, \ldots, \alpha_s\}$$

and from our definition of domination we have, for imputations $a, a' \in W$:

$$a \not\in a' \text{ if and only if } \alpha_i > \alpha_i' \text{ for } i = 1, 2, \ldots, s .$$

Hence

$$\sum_1^s \alpha_i > \sum_1^r \alpha_i' \quad (6)$$

But since

$$\sum_1^r a_i = 0 = \sum_1^s a_i' ,$$

we have

$$\alpha_0 + \frac{p}{r}\sum_1^s \alpha_i = \alpha_0' + \frac{p}{r}\sum_1^r \alpha_i' \quad \text{for} \quad r \neq 0 . \quad (7)$$

Combining (6) and (7) we obtain $\alpha_0 < \alpha_0'$ which is impossible if $\alpha_0' = -1$. Hence V contains all imputations with $a_1 = \alpha_0' = -1$.

We have now shown two things:

(1) The set V defined by our theorem is satisfactory, that is, for $a, b \in V$ we cannot have $a \xi b$.

(2) V is contained in every symmetric solution.

The proof that there exists an imputation $a \in V$ dominating any imputation in $A - V$ will show that V is a solution, and that it is unique since no superset of a solution is a solution.

LEMMA 3. For $b \in A$, there exists an $a \in V$ with $a \xi b$, unless $b \in V$.

PROOF. The required imputation a is specified by the quantities $\alpha_1, \alpha_2, \ldots, \alpha_s$ which must satisfy

$$p\sum_1^s \alpha_i = r \quad (\text{since } \sum a_i = 0) . \quad (8)$$

$$\alpha_i > b_{r+(i-1)p+1} \quad \text{for} \quad i = 1, 2, \ldots, s \quad (9)$$

(by the definition of domination).

Summing (9) over i, and substituting in (8) we obtain

$$p\sum_1^s b_{r+(i-1)p+1} < r ,$$

which is sufficient as well as necessary. But since $b_i \leq b_{i+1}$ we have

$$p\sum_{i=1}^s b_{r+(i-1)p+1} = \sum_{i=1}^s p b_{r+(i-1)p+1} \leq \sum_{i=1}^s \Big(\sum_{\gamma=1}^p b_{r+(i-1)p+\gamma}\Big) = -\sum_{i=1}^r b_i .$$

Now $-\sum_1^r b_i \leq r$ since $b_i \geq -1$, with equality only if

$-1 = b_1 = b_2 = \ldots = b_r$. The other inequality becomes equality only if $b_{r+(i-1)p+1} = b_{r+ip}$ for $1 \leq i \leq s$ which defines an imputation in V. Hence unless $b \in V$ we can solve (8), (9) and dominate b.

This completes the proof of the theorem. As mentioned earlier, this unique solution V is entirely in accordance with our intuitive expectation. The "p-coalitions" turn out to be the bargaining units of this game. Further, their bargaining is entirely "free" except of course that they will between them completely exploit the defeated r players which are left over after the maximum possible number of p-coalition receives the same amount.

In the zero-sum case $(n, \frac{n+1}{2})$ the solution consists of a finite set of imputations, formed by permuting the indices of the imputation $(-1, -1, \ldots, -1, \frac{n-1}{n+1}, \frac{n-1}{n+1}, \ldots, \frac{n-1}{n+1})$. This is the main solution to the simple majority game in n players. For $k > \frac{n+1}{2}$, the symmetric solution consists of an infinity of imputations.

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DISCRIMINATORY AND BARGAINING SOLUTIONS TO A CLASS OF SYMMETRIC n-PERSON GAMES[1]

D. B. Gillies

The application of game theory to economics and sociology awaits a general theory of n-person games. As yet, methods have not been found for determining solutions to all n-person games; nor is it known whether every game has a solution. The intensive study of a particular class of games provides empirical data on the nature of solutions, methods which may be applied to other games, and may suggest or disprove conjectures on solutions in general.

R. Bott defines the family (n,k) of n-person games in [1],[2] and determines for each game the unique symmetric solution. All unsymmetric solutions found in this paper can be related to symmetric solutions (to possibly different games of the class) by the systematic use of four devices. We shall study these devices in detail.

(1) Certain symmetric solutions may be modified by the addition of arbitrary functions -- the so-called "bargaining curves" ([2], p. 501) which exist for an infinity of these games.

(2) A new type of discrimination occurs. A group of players may receive a variable amount, but be required to divide it in accordance with the symmetric solution to a smaller game. For example, the game $(4,3)$ has a solution consisting of all imputations: $(z, -1, \frac{1-z}{2}, \frac{1-z}{2})$ and permutations of the last 3 components for all z in $-1 \leq z \leq 3$. That is, player (1) receives some admissible amount z, and the balance is divided in accordance with the symmetric solution to the game $(3,2)$.

(3) Any solution to (n,k) may be inflated ([2], p. 398) to a solution to $(n+t,k)$ by assigning the t additional players fixed amounts, (provided these amounts are not too large, and provided both games are in the class). These are the "discriminatory" solutions ([2], pp. 288-289).

(4) The fixed amounts in (3) are subject to strict inequalities.

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[1]The preparation of this paper was supported by the Office of Naval Research.

[2]Numbers in square brackets refer to the bibliography at the end of this paper.

In some cases one or more equalities are permissible if imputations are added and subtracted from the set, and a new solution obtains.

DEFINITION. The symbol (n,k) denotes an n-person game whose value $v(S)$ for a coalition S, numbering $|S|$ players, is

$$v(S) = \begin{cases} -|S| & \text{for } S < k \\ n - |S| & \text{for } S \geq k, \text{ where } n \geq k > \frac{n}{2} . \end{cases}$$

The symmetric solution $V_{n,k}$ contains every imputation related by a permutation of components to an imputation $a' = (a'_1, a'_2, \ldots, a'_n)$ with

$$a'_i = a'_{i+1} \quad \text{for } i \not\equiv r \pmod{p}$$

$$a'_1 = -1 \quad \text{for } r > 0 ,$$

where $p = n - k + 1$, $n = sp + r$, $0 \leq r < p$.

That is, $a \in V_{n,k}$ means that relative to a there exist s sets of players, each set of size p; the players in any one set receiving the same amount in a. The r players not included in these sets each receive -1.

Bott shows that domination is equivalent to domination via a k-element set, and that all k-element sets are effective. We shall therefore consider only k-element sets in our proofs. A solution V is symmetric over a subset S of components if it is left invariant by any permutation of these components. For such a solution, we may renumber the S-components of a general imputation, since domination of this new imputation implies, by symmetry, domination of the original one.

It is desirable to include also the games (n,n) in our class. For these, only n-person sets are effective, and the unique solution contains all imputations. (This agrees with the formula for $V_{n,n}$).

§ 1. SOLUTIONS WITH BARGAINING

The symmetric solution to (n,k), $V_{n,k}$, can be defined as follows:

An imputation a is in $V_{n,k}$ if and only if there exist s numbers $\{\alpha_i\}$ where $-1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$, and $s+1$ disjunct index sets $S_0, S_1, \ldots, S_s$ such that

(1) $$|S_0| = r$$

$$|S_i| = p \quad \text{if} \quad i \neq 0$$

(2) If $a = (a_1, \ldots, a_n)$ then $a_\nu = \begin{cases} -1 & \text{if} \quad \nu \in S_0 \\ \alpha_i & \text{if} \quad \nu \in S_i \end{cases}$.

(3) $$p \sum_{i=1}^{s} \alpha_i = r \ .$$

Let $r = 0$. S_0 is empty and $\sum_1^n a_\nu = p \sum_1^s \alpha_i = 0$. Let $\{q_\nu(\alpha_1, \ldots, \alpha_s)\}$ be n functions defined for every $a \in V_{n,k}$ $(r = 0)$, for which

(1) $$q_\nu(\alpha_1, \ldots, \alpha_s) + \alpha_1 \geq -1$$

(2) $$\sum_{\nu=1}^{n} q_\nu(\alpha_1, \ldots, \alpha_s) = 0 \ .$$

These imply, for $\alpha_1 = -1$, that $q_\nu(\alpha_1, \ldots, \alpha_q) = 0$ for $\nu = 1, 2, \ldots, n$. Given a, define a vector $q(a) = \{q_\nu(a)\} = \{q_\nu(\alpha_1, \ldots, \alpha_s) + a_\nu\}$. The conditions above are necessary and sufficient for $q(a)$ to be an imputation. Define $V_q = \{q(a) \mid a \in V_{n,k}\}$. We consider additional conditions on the functions q to ensure that V_q is a solution to (n,k).

DEFINITION. Two imputations $a, \bar{a} \in V_{n,k}$ will be called <u>essentially different</u> if they do not differ by just a permutation of components, that is, if

$$\alpha_i \neq \bar{\alpha}_i \quad \text{for some} \quad i \ .$$

THEOREM. A sufficient condition that V_q be a solution is that for any two essentially different imputation $a, \bar{a} \in V_{n,k}$ we have

$$|q_\nu(\alpha_1, \alpha_2, \ldots, \alpha_s) - q_\nu(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_s)| < \frac{1}{s-1} \max_i |\alpha_i - \bar{\alpha}_i| \ .$$

PROOF. We first show that no imputation in V_q dominates another. Assume the contrary, then $q(a) \leftarrow q(\bar{a})$, that is, $q_\nu(\alpha_1, \ldots, \alpha_s) + a_\nu > q_\nu(\bar{\alpha}_1, \ldots, \bar{\alpha}_s) + \bar{a}_\nu$ for some index set of size k. This inequality fails for at most $n-k = p-1$ indices. If $a, \bar{a}$ are not

essentially different, $q_\nu(\alpha_1, \ldots, \alpha_s) = q_\nu(\bar{\alpha}_1, \ldots, \bar{\alpha}_s)$ and $a \leftarrow \bar{a}$, a contradiction. Otherwise we show that there exists an $i = i_o$ such that

$$\bar{\alpha}_{i_o} - \alpha_{i_o} > |q_\nu(\alpha_1, \ldots, \alpha_s) - q_\nu(\bar{\alpha}_1, \ldots, \bar{\alpha}_s)|$$

$$\geq q_\nu(\alpha_1, \ldots, \alpha_s) - q_\nu(\bar{\alpha}_1, \ldots, \bar{\alpha}_s) .$$

If $\max_i |\bar{\alpha}_i - \alpha_i|$ occurs for i_1, then either $\bar{\alpha}_{i_1} - \alpha_{i_1} > 0$, in which case $i_o = i_1$, or

$$\sum_{i \neq i_1} (\bar{\alpha}_i - \alpha_i) = \alpha_{i_1} - \bar{\alpha}_{i_1} > (s-1)|q_\nu(\alpha) - q_\nu(\bar{\alpha})| .$$

Hence for some $i = i_o$, $\bar{\alpha}_{i_o} - \alpha_{i_o} > |q_\nu(\alpha) - q_\nu(\bar{\alpha})|$.
But $q_\nu(\alpha) + \alpha_i > q_\nu(\bar{\alpha}) + \alpha_j$ implies

$$\bar{\alpha}_j - \alpha_i < |q_\nu(\alpha) - q_\nu(\bar{\alpha})| .$$

For $i = j = i_o$ we have seen that this fails (for any ν). Hence for $j \geq i_o$, $i \leq i_o$ it fails a fortiori. That is, for $j \geq i_o$, a necessary condition for the inequality to hold is $i > i_o$. So for the $(s - i_o + 1)p$ components $\{\nu | a_\nu \geq \bar{\alpha}_{i_o}\}$ we can have at most $(s-i_o)p$ inequalities satisfied: $\{\nu | a_\nu > \alpha_{i_o}\}$. Thus the inequality fails for at least p components and $q(a) \not\subset q(\bar{a})$. Hence V_q is a satisfactory set.

To show that any $b \notin V_q$ is dominated by some $q(a) \in V_q$, we consider separately the cases $s = 2$, $s > 2$.

Suppose $s = 2$, then $-\alpha_1 = +\alpha_2 = \beta$ say, where $0 \leq \beta \leq 1$. Suppose b is an imputation which is not dominated by any $q(a) \in V_q$. We obtain a contradiction. Define index sets

$$T_1(\beta) = \{\nu | q_\nu(\beta) + \beta \leq b_\nu\}$$

$$T_2(\beta) = \{\nu | q_\nu(\beta) + \beta > b_\nu \geq q_\nu(\beta) - \beta\}$$

$$T_3(\beta) = \{\nu | q_\nu(\beta) - \beta > b_\nu\} .$$

For $\beta \neq \beta^*$, $|q_\nu(\beta) - q_\nu(\beta^*)| < |\beta - \beta^*|$ so T_1, T_3 are monotone non-increasing sets. Suppose $T_3(\beta)$ is empty for all β, including $\beta = 0$. Then $q_\nu(0) = b_\nu$ and $b \in V_q$. Otherwise let $\bar{\beta}$ be the smallest β for which $T_3(\beta)$ is empty (it is certainly empty at $\beta = 1$). For $\beta < \bar{\beta}$, $|T_1(\beta)| \geq p$, for otherwise $q(a) \succ b$, where

$$q_\nu(a) = \begin{cases} q_\nu(\beta) - \beta & \text{for } \nu = \nu_o \in T_3(\beta) \\ q_\nu(\beta) + \beta & \text{for p other indices in } T_2+T_3 \\ q_\nu(\beta) - \beta & \text{otherwise .} \end{cases}$$

But $|T_1(\beta)|$ is continuous on the left, so $|T_1(\beta)| \geq p$. Let S be a p-element set $\subseteq T_1(\bar{\beta})$ and define

$$\begin{aligned} q_\nu(a) &= q_\nu(\bar{\beta}) + \bar{\beta} \quad \text{for } \nu \in S \\ &= q_\nu(\bar{\beta}) - \bar{\beta} \quad \text{otherwise .} \end{aligned}$$

Then always $q_\nu(a) \leq b_\nu$ so $q(a) = b \in V_q$, the required contradiction.

For $s > 2$, the proof that any imputation is dominated unless it is in the solution, is patterned after an alternative proof for the symmetric solution which we give in detail for the case $r = 0$.

Let b be any imputation, (indexed such that $b_1 \leq b_2 \leq \cdots \leq b_n$). Define $\bar{\alpha}_i = \frac{1}{p} \sum_{(i-1)p+1}^{ip} b_j \geq b_{(i-1)p+1}$, with equality only if all b_j in the sum are equal. Let $\bar{a}$ be the imputation in $V_{n,k}$ defined by

$$\begin{aligned} \bar{a}_1 &= \bar{\alpha}_1 \\ \bar{a}_{1+(i-2)p+\gamma} &= \bar{\alpha}_i \quad \text{if } 1 \leq \gamma \leq p \\ \bar{a}_j &= \bar{\alpha}_1 \quad \text{if } j > k . \end{aligned}$$

Then $\bar{a}_j > b_j$ for the first k indices, unless one or more of the sums are over equal b_j. However not all such sums can consist of equal elements (or $b \in V_{n,k}$) so there is one, say $\bar{\alpha}_{i_o} > b_{(i_o-1)p+1}$.

Let $\bar{\alpha}_{i_o} = b_{(i_o-1)p+1} + s\,\varepsilon$, where $\varepsilon > 0$, and define $\alpha = (\alpha_1, \ldots, \alpha_s)$ by

$$\alpha_i \begin{cases} = \alpha_{i_o} - (s-1)\varepsilon & \text{for } i = i_o \\ = \bar{\alpha}_i + \varepsilon & \text{otherwise .} \end{cases}$$

Then a defined as above, from α, satisfies $a \prec b$, and $a \in V_{n,k}$. This completes the proof.

If $s > 2$, $b \notin V_q$, define a vector function $\Psi(\alpha) = (b_{1'}, -q_{1'}(\alpha), b_{2'}, -q_{2'}(\alpha), \ldots, b_{n'}, -q_{n'}(\alpha))$ where $(1', 2', \ldots, n')$ is the permutation of $(1, 2, \ldots, n)$ such that $\Psi_1(\alpha) \leq \Psi_2(\alpha) \leq \cdots \leq \Psi_n(\alpha)$. (This permutation is a function of α, but the $\Psi_\nu(\alpha)$ are continuous, and satisfy $|\Psi_\nu(\alpha) - \Psi_\nu(\bar{\alpha})| < \frac{1}{s-1} \max_i |\alpha_i - \bar{\alpha}_i|$ for $\alpha \neq \bar{\alpha}$.)

Define a vector function $\phi(\alpha)$ by

$$\phi_i(\alpha) = \frac{1}{p} \sum_{(i-1)p+1}^{ip} \psi_i(\alpha), \quad i = 1, 2, \ldots, s .$$

Then $\sum_{i=1}^{s} \phi_i(\alpha) = 0$ and $|\phi_i(\alpha) - \phi_i(\bar{\alpha})| < \frac{1}{s-1} \max_i |\alpha_i - \bar{\alpha}_i|$ for $\alpha \neq \bar{\alpha}$. By analogy with the previous proof, we solve the vector equation $\phi(\alpha) = \alpha$, if possible.

We shall form a sequence $\alpha^0, \alpha^1, \alpha^2, \ldots$ where $\alpha^{i+1} = \phi(\alpha^i)$ and show that it is a Cauchy sequence. Its limit, $\bar{\alpha}$, satisfies $\phi(\bar{\alpha}) = \bar{\alpha}$, since ϕ is a continuous function. However $\phi(\alpha^i)$ is defined only when α^i can arise from some imputation in $V_{n,k}$ so we must guarantee that $\alpha_1^{i+1} \geq -1$. (Since $\sum \phi_j(\alpha) = 0$, we have $\sum \alpha_j^{i+1} = 0$).

Define α^0, α^1 by:

$$\alpha_j^1 = \frac{1}{p} \sum_{(j-1)p+1}^{jp} b_k$$

$$\alpha_1^0 = -1$$

$$\alpha_i^0 = \alpha_i^1 + \frac{\alpha_1^1 + 1}{s-1} \quad \text{for} \quad i \neq 1 .$$

Then $0 = \frac{1}{p} \sum b_k = \sum \alpha_j^1 = \sum \alpha_j^0$, and $\alpha_1^1 \geq -1$. But $\alpha_1^0 = -1$, so $q_\gamma(\alpha^0) = 0$, and from the definition of ϕ we have $\phi(\alpha^0) = \alpha^1$. Similarly if $\alpha_1^1 = -1$, $\alpha^2 = \phi(\alpha^1) = \alpha^1$ and we have the solution $\bar{\alpha} = \alpha^1$. Assume then $\alpha_1^1 > -1$, or $1 + \alpha_1^1 > 0$. Let $\mu_i = \max_k |\alpha_k^{i+1} - \alpha_k^i|$. Then $\mu_0 = \alpha_1^1 + 1 > 0$. Suppose $\alpha_1^j \geq -1$ for $1 \leq j \leq i$. Then

$$\begin{aligned} \mu_i &= \max_k |\alpha_k^{i+1} - \alpha_k^i| \\ &= \max_k |\phi_k(\alpha^i) - \phi_k(\alpha^{i-1})| \\ &< \left(\frac{1}{s-1}\right) \max_k |\alpha_k^i - \alpha_k^{i-1}| \\ &< \left(\frac{1}{s-1}\right) \mu_{i-1} . \end{aligned}$$

Hence

$$\mu_i < \left(\frac{1}{s-1}\right)^i \mu_0 .$$

But

$$\begin{aligned} \alpha_1^{i+1} &= -1+1 + \alpha_1^1 + (\alpha_1^2 - \alpha_1^1) + (\alpha_1^3 - \alpha_1^2) + \ldots + (\alpha_1^{i+1} - \alpha_1^i) \\ &\geq -1 + (1 + \alpha_1^1) - |\alpha_1^2 - \alpha_1^1| - |\alpha_1^3 - \alpha_1^2| - \ldots - |\alpha_1^{i+1} - \alpha_1^i| \end{aligned}$$

$$\geq -1 + \mu_0 - \mu_1 - \mu_2 - \cdots - \mu_i$$

$$\geq -1 + \mu_0 - \mu_0\left(\frac{1}{s-1} + \left(\frac{1}{s-1}\right)^2 + \cdots + \left(\frac{1}{s-1}\right)^i\right) \geq -1 + \mu_0 - \left(\frac{1}{s-2}\right)\mu_0$$

$$\geq -1 + \left(\frac{s-3}{s-2}\right)\mu_0 \geq -1, \quad \text{since} \quad \mu_0 > 0,\ s > 2 .$$

Hence $\alpha_1^{i+1} \geq -1$ and α^i is defined for all i. Similarly

$$|\alpha_k^i - \alpha_k^{i+m}| \leq |\alpha_k^i - \alpha_k^{i+1}| + |\alpha_k^{i+1} - \alpha_k^{i+2}| + \cdots + |\alpha_k^{i+m-1} - \alpha_k^{i+m}|$$

$$\leq \mu_0\left(\frac{1}{s-1}\right)^i \cdot \frac{s-1}{s-2} .$$

$|\alpha_k^i - \alpha_k^{i+m}| \longrightarrow 0$ as $i \longrightarrow \infty$, and $\{\alpha^i\}$ is a Cauchy sequence, whose limit, $\bar{\alpha}$, satisfies

$$\bar{\alpha}_1 \geq -1$$

$$\sum \bar{\alpha}_i = 0$$

$$\phi(\bar{\alpha}) = \bar{\alpha} .$$

For this $\bar{\alpha}$, we renumber the indices of b relative to a permutation $(1', 2', \ldots, n')$ which makes $\psi_i(\bar{\alpha})$ monotone. Then $\psi_\nu(\bar{\alpha}) = b_\nu - q_\nu(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_s) \leq b_{\nu+1} - q_{\nu+1}(\bar{\alpha}_1, \ldots, \bar{\alpha}_s)$ for $\nu = 1, 2, \ldots, n-1$. If $\psi_{(i-1)p+1} = \psi_{ip} = \bar{\alpha}_i$ (for $i = 1, 2, \ldots, s$), then $b \in V_q$ since $b_{(i-1)p+\gamma} = \alpha_i + q_{(i-1)p+\gamma}(\bar{\alpha}_1, \ldots, \bar{\alpha}_s)$ $(1 \leq \gamma \leq p,\ 1 \leq i \leq s)$. Otherwise, for some $i = i_0$

$$\psi_{(i_0 - 1)p+1} + s\varepsilon = \alpha_{i_0} \quad \text{with} \quad \varepsilon > 0 .$$

Then define

$$\alpha_i = \bar{\alpha}_{i_0} - (s-1)\varepsilon \quad \text{for} \quad i = i_0$$

$$= \bar{\alpha}_i + \varepsilon \quad \text{for} \quad i \neq i_0 .$$

Then $q(a) \leq b$, $q(a) \in V_q$, where $q(a)$ is defined by

$$q_1(a) = q_1(\alpha_1, \ldots, \alpha_s) + \alpha_1$$

$$q_{1+(i-2)p+\gamma}(a) = q_{1+(i-2)p+\gamma}(\alpha_1, \ldots, \alpha_s) + \alpha_i$$

for $1 \leq \gamma \leq p$ and $2 \leq i \leq s$

$$q_\nu(a) = q_\nu(\alpha_1, \ldots, \alpha_s) + \alpha_1 \quad \text{for} \quad \nu > (s-1)p+1) = k .$$

PROOF. We show that domination is via the set $(1, 2, \ldots, k)$. Suppose $q_{1+(i-2)p+\gamma}(\alpha_1, \alpha_2, \ldots, \alpha_s) + \alpha_i \leq b_{1(i-2)p+\gamma}$ for some index $1 + (i-2)p + \gamma$ with $1 \leq \gamma \leq p$. Since

$$|q_{1+(i-2)p+\gamma}(\alpha_1, \ldots, \alpha_s) - q_{1+(i-2)p+\gamma}(\bar{\alpha}_1, \ldots, \bar{\alpha}_s)| < \frac{1}{s-1} \max_i |\alpha_i - \bar{\alpha}_i| = \varepsilon$$

the left side is $> q_{1+(i-2)p+\gamma}(\bar{\alpha}_1, \ldots, \bar{\alpha}_s) - \varepsilon$. Transposing, we have

$$\alpha_i - \varepsilon < \psi_{1+(i-2)p+\gamma}(\bar{\alpha}) \leq \psi_{(i-1)p+1}(\bar{\alpha}) .$$

For $i \neq i_o$, $\alpha_i - \varepsilon = \bar{\alpha}_i$ so we obtain a contradiction. If $i = i_o$, $\alpha_i - \varepsilon = \bar{\alpha}_i - s\varepsilon = \psi_{(i_o-1)p+1}(\bar{\alpha})$ which again gives a contradiction. Hence $q_\nu(a) > b_\nu$ for $\nu \leq k$. This completes the proof of the theorem.

REMARKS

(1) The presence of arbitrary functions $q_\nu(\alpha)$ in the solution bears a resemblance to situations in economics where bargaining occurs.

(2) For $s = 2$, the proof can be strengthened to the case $|q_\nu(\lambda) - q_\nu(\lambda^*)| \leq |\lambda - \lambda^*|$, and it is probable that equality is permissible in the Lipschitz condition for $s > 2$ also, although this cannot be shown by means of a Cauchy sequence.

(3) For $r \neq 0$ I have not found bargaining solutions of full dimension, although a smaller game can often be found whose bargaining solution can be inflated to solve the larger game. In one case: $(2p - 1, p)$ there are certainly no bargaining solutions of full dimension: this is the zero-sum case -- the direct majority game which is a particular case of a simple game. In a simple game, either a set S or its complement is always effective, so two imputations in a solution must have at least one equal component.

§ 2. A NEW TYPE OF DISCRIMINATION

LEMMA. If V is a solution to an n-person game, and $b = (b_1, b_2, \ldots, b_n)$ is a vector satisfying $b_i \geq -1$ for $i = 1, 2, \ldots, n$ and

$$\sum b_i < 0 ,$$

then there exists an imputation $a \in V$ with $a \underset{\sim}{\in} b$.

PROOF. Define $\bar{b}$ by $\bar{b}_i = b_i + \varepsilon$, $i = 1, 2, \ldots, n$, where

$$\varepsilon = \frac{1}{n} \sum(-b_i) > 0 .$$

Then $\bar{b}$ is an imputation. If $\bar{b} \notin V$ there exists $a \in V$ with $a \underset{\sim}{\in} \bar{b}$, and $a \underset{\sim}{\in} b$ a fortiori. If $\bar{b} \in V$, then $\bar{b} \underset{\sim}{\in} b$ via the effective set I of all n players. The symmetric solution $V_{n,k}$ to (n,k), has a very interesting property if $r = p-1$, in the light of this lemma. There are $p-1$ components which are -1, so given any vector b satisfying the conditions of the lemma, we can <u>never</u> have, for $a \in V_{n,k}$: $a_i > b_i$ for <u>more than</u> k indices. The transformation $T(X) : a \longrightarrow a(X)$ defined by $a_i \longrightarrow a_i(1 + \frac{X}{n}) + \frac{X}{n}$, $i = 1, 2, \ldots, n$ has the properties

(1) $a_i = -1$ implies $a_i(X) = -1$

(2) If $X \geq -n$, $a_i > \bar{a}_i \geq -1$ implies $a_i(X) \geq \bar{a}_i(X) \geq -1$, with equality in the transforms only if $X = -n$.

(3) $\sum_{i=1}^{n} a_i(X) = X + (1 + \frac{X}{n}) \sum_{1}^{n} a_i$ $(= X$ if $\sum_{1}^{n} a_i = 0)$.

(4) If $T(X)$ is applied to all imputations $a \in V_{n,k}$, the image $V_{n,k}(X)$ is a solution to the strategically equivalent game defined by $v(I) = -X$:

$$v(S) = \begin{cases} -|S| & \text{if } |S| < k \\ n - X - |S| & \text{if } |S| \geq k, \text{ where } X \geq n . \end{cases}$$

If (n_1,k_1) (n_2,k_2) are 2 games with $r_1 = p_1 - 1 = n_1 - k_1$, $r_2 = n_2 - k_2$, and if $(V_{n_1,k_1}, V_{n_2,k_2})$ is the set of imputations (a,b) in an $(n_1 + n_2)$-dimensional space where $a \in V_{n_1,k_1}$, $b \in V_{n_2,k_2}$, we might expect $(V_{n_1,k_1}(X), V_{n_2,k_2}(-X))$, $-n_1 \leq X \leq n_2$, to be a solution to the game $(n_1 + n_2, k_1 + k_2)$. We prove a more general theorem:

THEOREM. Let (n_1,k_1), (n_2,k_2), $\ldots$, (n_g,k_g) be g games satisfying the conditions $r_i = n_i - k_i = p_i - 1$ for $i = 1, 2, \ldots, g$. Let A be the space of imputations for the $(\sum_1^g n_i)$-player game $(n_1 + \ldots + n_g, k_1 + \ldots + k_g)$, and let $S_1, \ldots, S_g$ be g fixed index

sets for A with

$$|S_i| = n_i$$

$$S_i \cap S_j = \text{the null set} .$$

Number the indices in V_{n_i,k_i} with the same indices as S_i $(i = 1, 2, \ldots, g)$ in some order. If

$$V = \{a(X) = (a^1(X_1), \ldots, a^g(Xg)) \mid a^i \in V_{n_i,k_i}(X_i),\ i = 1, 2, \ldots, g\ ;$$

$$X_i \geq -n_i \text{ and } \sum_1^g X_i = 0\}$$

then V is a solution to $(n,k) = (\sum n_i, \sum k_i)$.

PROOF. We have seen that domination means domination relative to (n_i,k_i) for every index set S_i, and also that there exists $a^i(X) \in V_{n_i,k_i}(X)$ dominating any extended imputation b with $\sum_{j\in S_i} b_j < X_i$. Suppose $a(X) \not\subset a(Y)$ and $a(X), a(Y) \in V$. Now $\sum_1^g X_i = \sum_1^g Y_i$. Then for some $i = i_o$, $X_{i_o} \leq Y_{i_o}$. Then $a^{i_o}(X_{i_o}) \not\subset a^{i_o}(Y_{i_o})$, so the relation $a_i(X) > a_i(Y)$ can hold for at most $\sum_{i\neq i_o} k_i + (k_{i_o} - 1) = \sum k_j - 1$ indices. Hence $a(X) \not\subset a(Y)$. Given $b \notin V$, we determine $\bar{X}_i = \sum_{j\in S_i} b_j$. For at least one "subgame" S_{i_o} we have $a^{i_o}(\bar{X}_{i_o}) \not\subset b^{i_o}$, where b^{i_o} is the extended imputation consisting only of the components whose indices are in S_{i_o}. This gives k_{i_o} strict inequalities, so, for $\varepsilon > 0$ sufficiently small, $a^{i_o}(x_{i_o} - (g-1)\varepsilon) \not\subset b^{i_o}$. We can then determine imputations $a^i(\bar{X}_i + \varepsilon) \not\subset b^i$ for $i \neq i_o$, and

$$(a^1(\bar{X}_1 + \varepsilon), a^2(\bar{X}_2 + \varepsilon), \ldots, a^{i_o}(\bar{X}_{i_o} - (g-1)\varepsilon), a^{i_o+1}(\bar{X}_{i_o+1} + \varepsilon), \ldots)$$

dominates b. (Since $\varepsilon > 0$, $\bar{X}_{i_o} \neq -n_{i_o}$, but this is certainly true, since $a^{i_o}(-n_{i_o}) = (-1, -1, \ldots, -1)$ which does not dominate anything).

REMARKS

(1) (n_1,k_1) can be the game (1,1) or more generally (n_1,n_1). Examples of admissible games are (3,2), (5,3), (5,4), (7,4), (7,6).

(2) In each imputation in V, the players who do not succeed in joining a coalition receive -1. In S_i there are $n_i - k_i$ such players.

Certain amounts X_i are assigned to the sets S_i and are divided among the players in S_i according to the symmetric solution to the smaller games.

(3) For n large, the decomposition of (n,k) into $(\sum n_i, \sum k_i)$ can be attained in a prodigious number of ways satisfying our conditions on (n_i,k_i). The multiplicity of solutions so obtained, and the variety of social structures so described seem surprising in these symmetric games.

§ 3. INFLATION OF SOLUTION

V is called a <u>discriminatory solution</u> to (n+t,k) if there exists a partition of indices into two sets S,T with the properties (1), (2):

(1) $|T| = t \qquad |S| = n$

$T \cup S = I$, the set of all players .

(2) For all $a \in V$, $a_i \equiv c_i$ if $i \in T$, where $\{c_i\}$ are t constants each greater than or equal to -1. Let V^S be the set of constant-sum imputations $\{a^S\}$ in S obtained by projecting $V = \{a\}$ onto S. These have a sum $-\sum_{i\in T} c_i$, since $0 = \sum_{i=1}^{n} a_i = \sum_{i\in T} a_i + \sum_{i\in S} a_i = \sum_{i\in T} c_i + \sum_{i\in S} a_i$.

LEMMA. V^S is a solution to the game strategically equivalent to (n,k) where

$$v(S_1) = -|S_1| \quad \text{if} \quad |S_1| < k$$

$$= n - |S_1| - \sum_{i\in T} c_i \quad \text{if} \quad |S_1| \geq k .$$

PROOF. Let b be an imputation with projections b^S, b^T on S and T and assume $b^T = c$. We show that b^S is either in V^S or dominated via an effective set in S. If $b \in V$, the lemma follows. Otherwise for some $a \in V$, $a_i > b_i$ for k indices. However $c_i = b_i$ for $i \in T$, so the k indices must be in S. Hence $a^S \succ b^S$, which was to be proved.

We propose to reverse this process, and study the <u>inflation</u> of a solution V_n to (n,k) by assigning t additional players fixed amounts. By strategic equivalence, we can consider the game for which $\sum_{i\in S} a_i = 0$ so $V^S = V_n$. The relation: "k-player sets are certainly necessary, and all other sets are certainly unnecessary" is sufficient to define the game (n + t,k), for $k > \frac{n+t}{2}$, is invariant under strategic equivalence, and

is the only relation used in our proofs. Let X be chosen so $V(X)$ has $\sum_{i\in S} a_i = 0$ for $a \in V(X)$, and hence $\sum_{i\in T} a_i = X$. Let $a_i = c_i$ for $i \in T$ (the c_i will differ from their zero-sum analogues in V). Where no confusion could arise imputation will be used in one or other of two senses:

(1) An element $b \in A$ where A is the $(n+t)$-dimensional space of extended imputations over $S \cup T$.

$$b_i \geq -1$$
$$\sum b_i = X \ .$$

(2) A zero-sum imputation over the indices in S.

THEOREM. A sufficient condition that $V(X)$ be a solution to $(n + t,k)$ over A is $\max_{i\in T} c_i < \frac{1}{n-1}$, if $k > \frac{n+t}{2}$.

DEFINITION. An imputation b is certainly unnecessary if we know (owing usually to the domination of certain related imputations) that it can be dominated by an $a \in V$. Otherwise, b is certainly necessary.

LEMMA 1. The imputations b with $b_i < c_i$ for exactly l components $L \subset T$ are certainly necessary if and only if (1) $\sum c_i$ is a maximum for all $|L| = l$

(2) $$b_i = \begin{cases} -1 & \text{for } i \in L \\ c_i & \text{for } i \in T - L \ , \end{cases}$$

for $l = 0, 1, \ldots, t$.

PROOF. It is evident that these are exactly the imputations, for fixed l, for which $\sum_{i\in S} b_i =$ a maximum. Thus given an imputation b^* with $b_i^* < c_i$ for exactly l components we can find an imputation b in the set defined by the lemma, with $b_i = b_i^* + \varepsilon$, $\varepsilon \geq 0$ for all $i \in S$. If $V(X) \supset a \xi\!\!\!-\, b$ via the set U, then $U \cup S$ contains at least $k + t - l$ components $a_i > b_i \geq b_i^*$, and $a_i > b_i$ for l components in T. Hence $a \xi\!\!\!-\, b^*$, so we need not consider b^*. Q.E.D.

For $l = 0$, the theorem now follows from the fact that V is a solution to (n,k).

It remains to find $a \in V_n$ with $a_i > b_i$ for a set of $k - l$ indices $\{i\}$. If L_s is an index set in S with

(1) $|L_s| = l > 0$

(2) $\sum_{i\in L_s} b_i$ is a maximum for all sets $U \subset S$ with $|U| = l$, define b^*, an imputation over the indices S with

$$b_i^* = \begin{cases} -1 & \text{if } i \in L_s \\ b_i & \text{if } i \in S - L_s \end{cases} .$$

If $\sum_{i\in S} b_i^* < 0$, by a previous lemma, there exists $a \in V_n$ with $\bar{a} \xi b^*$

$$\therefore \quad \bar{a}_i > b_i^* \text{ for } k \text{ components}$$

$$\therefore \quad \bar{a}_i > b_i \text{ for at least } k - l \text{ components,}$$

since $b_i^* \neq b_i$ for at most l components. Then if a is defined by

$$a_i = \begin{cases} c_i & \text{for } i \in T \\ \bar{a}_i & \text{for } i \in S \end{cases}$$

then $a \in V(X)$ and $a \xi b$ as required.

In view of this proposed construction, those imputations for which $\sum b_i^*$ is maximized are certainly necessary, and all others are certainly unnecessary. Also $\sum b_i^*$ is maximized if and only if

$$b_i = \text{constant for all } i \in S, \quad (l > 0) ,$$

since b^* obtains from b by replacing the largest l components in S by -1.

Number the indices in $T(1, 2, \ldots, t)$ such that $c_1 \geq c_2 \geq \cdots \geq c_t$. Then $\sum_{i\in T} b_i = \sum_{i=1}^{l} (-1) + \sum_{i=l+1}^{t} c_i = \sum_{1}^{t} c_i - (l + \sum_{i=1}^{l} c_i)$, and for $i \in S$, $b_i = \frac{1}{n}(l + \sum_{i=1}^{l} c_i)$. Then $\sum_{i\in S} b_i^* = (n - l)\frac{1}{n}(l + \sum_{i=1}^{l} c_i) - l$ which is < 0 by hypothesis.

$$\therefore \sum_{i=1}^{l} c_i < \frac{l^2}{n - l} \qquad (l = 1, 2, \ldots, t) .$$

But for $l \geq 2$, this condition follows a fortiori from the relation $c_1 \geq c_2 \cdots \geq c_l$ and $c_1 < \frac{1}{n-1}$. Hence $\max_{i\in T} c_i < \frac{1}{n-1}$ is a sufficient condition.

Thus every solution to (n,k) can be inflated to a solution to $(n + t,k)$ by assigning the t additional players constant amounts less than $\frac{1}{n-1}$ throughout. The inverse transformation $T^{-1}(X)$ maps this solution V into a solution to the normalized game.

The construction of b^* was necessarily crude since we know

nothing about the solution V_n. If V_n is the symmetric solution $V_{n,k}$, then an imputation $a \in V_{n,k}$ should be chosen to majorize the smallest $k - 1$ components of b_i in S. In our former notation, an imputation $a \in V_{n,k}$ is characterized by the s quantities $(\alpha_1, \alpha_2, \ldots, \alpha_s)$, where now we do not insist that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$. If $(q-1)p < k - 1 < qp$, then for $i > q$ we may set $\alpha_i = -1$. Let $(1', 2', \ldots, n')$ $b_{1'} \leq b_{2'} \leq \cdots \leq b_{k-1'} \leq \cdots \leq b_{n'}$ define a numbering of the indices in S. If $(q-1)p + \eta = k - 1$ define $b_{\eta + (i-1)p + \varepsilon} = \alpha_i$, $1 \leq i \leq q$, where ε is chosen so $\sum_{i=1}^{s} \alpha_i = 0$.

If a is defined by

$$a_1 = a_2 \cdots = a_\eta = \alpha_1$$

$$a_{(i-2)p+\eta+1} = \cdots = a_{(i-1)p+\eta} = \alpha_i \quad 2 \leq i \leq q$$

$$a_{(q-1)p+\eta+1} = \cdots = a_{qp} = \alpha_1$$

$$a_\nu = -1 \quad \text{otherwise}$$

(where for simplicity in notation the primes are suppressed) then for $i \leq (q-1)p+\eta = k - 1$ provided $\varepsilon > 0$.

THEOREM. If $V_{n,k}$ is the symmetric solution to (n,k) and $V(X)$ is the set of imputations formed by assigning t additional players fixed amounts, then $V(X)$ is a solution to $(n+t,k)$ provided $\max_i c_i < \frac{\lambda}{n-\lambda}$ where

$$\lambda = \begin{cases} r & \text{if } r \neq 0 \\ p & \text{if } r = 0 \end{cases}.$$

This condition is necessary and sufficient.

LEMMA. If the above construction is planned (and it is the best possible), then the imputations $b \in A$ for which

$$b_i = \begin{cases} c_i \text{ for } i \in T - L & (1) \\ -1 \text{ for } i \in L & (2) \\ -1 \text{ for } i < \eta \text{ in } S & (3) \\ \text{constant otherwise .} & (4) \end{cases}$$

are certainly necessary, and all others are certainly unnecessary.

PROOF. ((1), (2) follow from the proof to the previous theorem and (3), (4) obtain by minimizing ε). $\sum_{i\in S} b_i = l + \sum_{i=1}^{l} c_i$ as before, and the constant value for b_i in all but $(\eta - 1)$ components of S is $\frac{1}{n-\eta+2}(l+\eta-1+\sum_{i=1}^{l} c_i)$. Since $\sum a_i = 0$,

$$pq\left[\frac{1}{n-\eta+1}(l+\eta-1+\sum_1^l c_i)+\varepsilon\right]-(n-pq)=0\ .$$

Imposing the condition $\varepsilon > 0$ and simplifying the resulting expression:

$$\sum_{i=1}^{l} c_i < \frac{l(n-qp)}{qp} \quad \text{for } l = 1, 2, \ldots, t\ .$$

The right hand side is proportional to l if q stays constant, however when q decreases by 1 (as l increases) the constant of proportionality increases. Since $c_1 \geq c_2 \geq \cdots \geq c_t$, the inequalities above are implied by the first one. For $l = 1$, $q = s$ unless $r = 0$ in which case $q = s - 1$.

Hence

$$\max_i c_i < \begin{cases} \frac{r}{n-r} & \text{if } r \neq 0 \\ \frac{p}{n-p} & \text{otherwise}\ . \end{cases}$$

This shows that the condition is sufficient. But the use of certainly necessary imputations and optimal constructions implies necessity also (which can be independently verified by obtaining a contradiction for $\max_i c_i = \frac{r}{n-r}$ or $\frac{p}{n-p}$).

COROLLARY. Any zero-sum imputation b is in at least one solution to (n,k).

PROOF. Index $\{b_i\}$ so $b_1 \leq b_2 \leq \cdots \leq b_n$ and define

$$X = \frac{n\sum_{n-k+1}^{n} b_i}{k+\sum_{n-k+1}^{n} b_i}\ .$$

It can be verified that $b(X)$ has $\sum_{n-k+1}^{n} b_i(X) = 0$. Since $T(X)$ is a monotone transformation

$$\max_{i \leq n-k} b_i(X) = b_{n-k}(X) \leq \frac{1}{n-k}\sum_{n-k+1}^{n} b_i(X) = 0\ .$$

Let $c_i = b_i(X)$ for $i \leq k$, and let V be the set of all imputations (a) with $a_i = c_i$ for $i \leq n - k$. Then V is an inflation of the symmetric solution $V_{k,k}$ to the game (n,k), with $\max_i c_i < \frac{1}{k-1}$. Since $b(X) \in V$, b is in the analogue of this solution in the space of zero-sum imputations.

The inequality obtained for the symmetric solution with $k = n$ is $\max_i c_i < \frac{1}{n-1}$, which agrees with our general theorem. The condition of the general theorem is therefore the best possible (as a function of n only).

§ 4. LIMITING CASES OF INFLATION

Suppose $c_1 = c_2 = \cdots c_{i_o} = \frac{\lambda}{n-\lambda}$ where

$$\lambda = \begin{cases} r & \text{if } r \neq 0 \\ p & \text{if } r = 0 . \end{cases}$$

Then $c_i = \frac{(n-qp)}{qp}$ if $1 \leq \min(i_o, \lambda)$ and $c_i < \frac{(n-qp)}{qp}$ otherwise $(1 \leq t)$. In fact the summation need only be carried out over a set containing l indices in $(1, 2, \ldots, i_o)$. Our previous arguments show that the only imputations in A undominated are the certainly necessary ones: If L is any set in $(1, 2, \ldots, i_o)$ of exactly l indices and L_S is any set in S of $\eta - 1 = \lambda - l$ indices, then the imputation b defined by

$$b_i = \begin{cases} -1 & \text{if } i \in L \cup L_S \\ c_i & \text{if } i \in T - L \\ \frac{\lambda}{n-\lambda} & \text{if } i \in S - L_S \end{cases}$$

is undominated by any $a \in V$, for all choices of L, L_S. Let $\bar{V}$ be the union of these imputations for all choices of L, L_S. (Since $|L| + |L_S| = l + \lambda - l = \lambda < k$, no imputation in V can dominate another (for the effective set must be $L \cup L_S$). Then $V + \bar{V}$ is a solution unless there are imputations $b \in \bar{V}$, $a \in V$ with $b \subsetneq a$.

For $\lambda = r \neq 0$, $\frac{1}{s} \sum_{i=1}^{s} \alpha_i = \frac{\lambda}{n-\lambda}$, so $\alpha_i \geq \frac{\lambda}{n-\lambda}$ for at least one $i = i_o$. But $b \subsetneq a$ is only possible via a k-element set in S, and there are $p = n - k + 1$ elements $a_\nu = \alpha_i$ for which $b_\nu > a_\nu$. Hence $V + \bar{V}$ is a solution if $r \neq 0$.

If $\lambda = p$, $r = 0$, the average value of the α_i is $\frac{1}{s} \sum_{i=1}^{s} \alpha_i = 0$, so we exclude from V all imputations a^* for which all α_i are less than $\frac{\lambda}{n-\lambda}$. Call this restricted set V^*. $V - V^*$ is not

needed for the domination of an imputation having $l > 0$ since by construction the imputation required for domination could have p components equal to -1, hence some p components are greater than or equal to $\frac{\lambda}{n-\lambda}$, as required. For $l = 0$, the arbitrary imputation has either $n - k + 1 = p$ indices in S not less than $\frac{\lambda}{n-\lambda}$, in which case it is dominated by an imputation in $V - V^*$, or has not more than $n - k$ elements not less than $\frac{\lambda}{n-\lambda}$, in which case it is dominated by an imputation in $\overline{V}$. We have proved:

Theorem. If $c_1 = c_2 = c_{i_o} = \frac{\lambda}{n-\lambda}$, and V is the inflated solution to $(n + t,k)$, let $\overline{V}$ be the set of all imputations having l components in $(1, 2, \ldots, i_o)$ equal to -1, $\lambda - l$ components in S equal to -1, $(l = 1, 2, \ldots, \min(\lambda,i_o)$ the other components in T equal to c_i, and the other components in S equal to $\frac{\lambda}{n-\lambda}$. Let V^* be the set of imputations in V with $a_i = \frac{\lambda}{n-\lambda}$ for all $i \in S$ (V^* is empty unless $r = 0$). Then $V + \overline{V} - V^*$ is a solution to $(n + t,k)$.

The discrete solution (a finite number of imputations) to the 3-person game (or more generally to the game $(n, \frac{n+1}{2})$, which is zero sum) can be obtained by inflating $V_{2,2}$ (or $V_{\frac{n+1}{2},\frac{n+1}{2}}$), to the limit.

CONCLUSIONS AND CONJECTURES

(1) Inflation, and inflation to the limit, are very important methods for obtaining new solutions once a discriminatory solution is known. In the latter case, it often happens that much of the original solution is omitted, at the cost of adding a finite number of imputations, and this is very useful in finding finite solutions themselves.

(2) By carrying out the inflation relative to this particular strategically equivalent game, it is evident that the limiting solutions do not depend on possibly carrying out the inflation one component at a time -- all possibilities are subsumed in this general theorem.

(3) The games $(2n+1, n+1)$ are the only zero-sum games in this class (they are the simple majority games), and it is evident that the natural way of determining their solutions is by means of the family $\{(m,k)\}$ of associated non-zero sum games. Finite solutions have been found only for zero-sum games, and the larger the ratio $\frac{k}{m}$ becomes, the higher is the dimensionality of the solution.

(4) The natural way of considering all solutions found so far is in terms of symmetric solutions.

(5) Non-discriminatory bargaining solutions probably do not occur for games with $r \neq 0$. (This is true in the zero-sum case, but remains to be shown otherwise).

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QUOTA SOLUTIONS OF n-PERSON GAMES[1]

L. S. Shapley

§ 1. SUMMARY

Complete sets of solutions for all three-person games have been given by von Neumann and Morgenstern.[2] For higher games the known results are scattered, with no solutions at all for large classes of games and complete sets of solutions for only a few special types.[3] Even the existence of solutions in all cases has not yet been established.

In this note we present a family of solutions for a class $\mathcal{Q}$ of n-person games which embraces all constant-sum four-person games and a not inconsiderable array of higher games.[4] We call them "quota games" because it is possible in them to define a system of individual quotas for the players which determines the effectiveness of the various two-player coalitions. In our solutions most of the players receive their quotas, but there is some latitude for bargaining.[5] The solutions are typically one-dimensional sets, consisting sometimes of n line segments joined at the quota point, sometimes of n-1 disconnected segments. Their behavior under variation of the characteristic function of the game is continuous.

In the final two sections we present some related, more complicated solutions to games in $\mathcal{Q}$, and describe an extension of the earlier results to a wider class of games.

§ 2. PRELIMINARIES

We shall employ the symbols $\in$, $\subseteq$, $\cap$, $\cup$, $-$, in the customary

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[1]The preparation of this paper was sponsored in part by the RAND Corporation and the Office of Naval Research.

[2]J. von Neumann and O. Morgenstern, "Theory of Games and Economic Behavior," Princeton, 1947 ("TGEB" in the sequel): sections 32 and 60.3.

[3]See e.g., TGEB: 36.1.1, 54.2.1, 60.4.2; also other papers in this volume.

[4]See the tables on pages 347 and 359 below.

[5]Details are given in the heuristic accounts which accompany Theorems 3, 4, and 5.

way; we shall use $\{i, j, \ldots, m\}$ for the set consisting of the distinct elements $i, j, \ldots, m$, without regard for order, and $|S|$ for the number of distinct elements in the (finite) set S. Greek letters without subscripts will represent n-vectors; δ^i will stand for the vector whose i-th component is 1, all others 0. By $[\alpha,\beta]$ we shall mean (until section 6) the set of vectors of the form

$$t\alpha + (1 - t)\beta , \qquad 0 \leq t \leq 1 \tag{1}$$

-- geometrically, the straight segment joining α and β.

Let I denote the set of players; $|I| = n$. A general-sum n-person game v is a function from the subsets of I to the reals satisfying

$$v(0) = 0 , \tag{2}$$

$$v(S \cap T) + v(S - T) \leq v(S) , \qquad (\text{all } S, T \subseteq I) . \tag{3}$$

It is constant-sum if there is equality in (3) for $S = I$:

$$v(T) + v(I - T) = v(I) \qquad (\text{all } T \subseteq I) . \tag{4}$$

If equality always holds in (3) for some particular T, then the game is said to be decomposable into the games on the sets of players T and $I - T$; if such a T consists of a single player, he is called a dummy. If all players are dummies, then the game is inessential, and its theory is trivial.

We shall sometimes write v_i, v_{ij}, etc., for $v(\{i\})$, $v(\{i,j\})$, etc.

The space A of "imputations" is defined as the set of α which satisfy

$$\sum_{i \in I} \alpha_i = v(I) \tag{5}$$

$$\alpha_i \geq v_i \qquad (\text{all } i \in I) \tag{6}$$

-- geometrically, an $(n-1)$-dimensional simplex. If

$$\beta_i > \alpha_i \qquad (\text{all } i \in S) , \tag{7}$$

$$\sum_{i \in S} \alpha_i \leq v(S) \tag{8}$$

both hold, we say that β S-dominates α, and write $\alpha \in S\text{-dom } \beta$. It is

easily verified that S-domination can occur between two imputations only if

$$1 < |S| < |I| . \tag{9}$$

We now define the dominion of a single vector:

$$\text{dom } \alpha = \bigcup_{S \subseteq I} \text{S-dom } \alpha , \tag{10}$$

and of a set of vectors:

$$\text{dom } V = \bigcup_{\alpha \in V} \text{dom } \alpha . \tag{11}$$

Finally, V is a solution (of the game v) if and only if

$$V = A - \text{dom } V . \tag{12}$$

§ 3. QUOTA GAMES

By a quota in a game v we shall mean a vector ω having the two properties:

$$\omega_i + \omega_j = v_{ij} \qquad (\text{all } i \neq j) , \tag{13}$$

$$\sum_{i \in I} \omega_i = v(I) . \tag{14}$$

From (13) we see at once that, for $|I| > 2$, the quota, if it exists at all, is unique. Let $\mathcal{Q}$ denote the class of games which possess quotas, the "quota games."

THEOREM 1. The n-person game v is in $\mathcal{Q}$ if and only if

$$v_{ij} + v_{kl} = v_{ik} + v_{jl} \qquad (i, j, k, l \text{ distinct}) \tag{15}$$

always holds, and

$$\sum_{\substack{i,j \in I \\ i \neq j}} v_{ij} = 2(n - 1)v(I) . \tag{16}$$

PROOF. (13) and (14) directly yield (15) and (16). Conversely, given v satisfying (15) and (16) and $n \geq 3$ (the theorem is trivial for $n = 1, 2$), the expression

(17) $$\omega_i = \tfrac{1}{2}\,(v_{ij} + v_{ik} - v_{jk}) \qquad (i,\ j,\ k \text{ distinct})$$

is independent of j and k, and can be used to define a vector ω. (13) now follows from (17). To obtain (14), sum (17) over all distinct (ordered) triples (i, j, k), thus:

(18) $$(n-1)(n-2)\sum_{i\in I}\omega_i = \tfrac{1}{2}\,(n-2)\sum_{\substack{i,j\in I\\ i\neq j}} v_{ij}$$

(the last two sums cancel). An application of (16) now gives the result.

COROLLARY 1. If n is even, (15) follows from (16).

PROOF. Let Π be a partition of I into two-element subsets. Then, by (3),

(19) $$\sum_{S\in\Pi} v(S) \leq v(I)\ .$$

Averaging (19) over all such partitions we obtain, since each S will occur the same number of times:

(20) $$\frac{1}{n-1}\sum_{\substack{S\subset I\\ |S|=2}} v(S) \leq v(I)\ .$$

By (16) we have equality here, and hence also in (19) for every Π. (15) now follows easily.

COROLLARY 2. If n is even, then $v \in \mathcal{Q}$ if and only if

(21) $$v(S \cap T) + v(S - T) = v(S)$$

holds whenever $|S|$, $|S \cap T|$, $|S - T|$ are even. Thus, in even-person quota games,

(22) $$\sum_{i\in S}\omega_i = v(S) \qquad (\text{all } S \subseteq I,\ |S| \text{ even})\ .$$

PROOF. Assume $v \in \mathcal{Q}$. By (3), (13), (14), we have

(23) $$v(S \cap T) + v(S - T) + v(I - S) \geq \sum_{i\in S\cap T}\omega_i + \sum_{i\in S-T}\omega_i + \sum_{i\in I-S}\omega_i$$

$$= v(I)$$

$$\geq v(S) + v(I - S) ,$$

if all the sets in question have even numbers of elements. Then (21) follows from (23) and (3). Conversely, (21) implies (15), permitting us to define ω by (17) (except in the case $n = 2$, for which the corollary is trivial). This vector has the quota properties (13) and (14), as required.

COROLLARY 3. All inessential games, and all constant-sum four-person games, are in Q.

PROOF. For inessential games, put $\omega_i = v_i$, all $i \in I$. For constant-sum four-person games, apply Corollary 2, using (2) and (4).

An idea of the extent of the class Q may be gained by representing each n-person game v as a point in the cartesian space of 2^n dimensions and then comparing the dimension of the (convex) set of quota games with the dimension of the (convex) set of all games. In such representations it is customary to consider just games in "reduced form" and to disregard inessential games.[6] This has the effect of making the convex sets bounded, and of reducing their dimensionality by $n + 1$, without substantial loss of generality. The comparison follows, calculations omitted.

	All games		Quota games	
n	general-sum	constant-sum	general-sum	constant-sum
2	0	(none)	0	(none)
3	3	0	2	(none)
4	10	3	7	3
5	25	10	19	4
6	56	25	31	15
(even)	2^n-n-2	$2^{n-1}-n-1$	$2^{n-1}-1$	$2^{n-2}-1$
(odd)			$2^n-\binom{n}{2}-3$	$2^{n-1}-\binom{n}{2}-2$

A quota is not necessarily an imputation: condition (5) is assured, but not (6). Define the quantities c_i:

[6] Compare TGEB 39. Use of the reduced form in the body of this paper would tend to obscure relations and results, without producing any substantial simplification.

$$c_i = \omega_i - v_i \qquad (i \in I) \tag{24}$$

-- they are effectively the barycentric coordinates of ω in the simplex A. A player o for whom c_o is negative is called weak. A dummy in an essential quota game, for example, is weak. (Indeed, we then have, for each $i \neq o$,

$$c_o = \omega_o - (v_{oi} - v_i) = -\omega_i + v_i = -c_i \ , \tag{25}$$

whence

$$c_o = -\tfrac{1}{n-2} \textstyle\sum_{i \in I} c_i \ . \tag{26}$$

This quantity is negative in an essential game.)[7]

THEOREM 2. In a quota game there is at most one weak player; if $|I|$ is odd, there is none.

PROOF. By (3) and (13) we have, for any $i \neq j$,

$$v_i + v_j \leq v_{ij} = \omega_i + \omega_j \ . \tag{27}$$

Therefore i and j cannot both be weak. Furthermore, if $|I|$ is odd, we have for any i:

$$v(I) - v_i \geq v(I - \{i\}) \geq \sum_{\substack{j \in I \\ j \neq i}} \omega_j = v(I) - \omega_i \ , \tag{28}$$

by applying successively (3), (13) and (3), and (14). Therefore i alone cannot be weak.

§ 4. QUOTA SOLUTIONS

We shall now proceed to construct solutions to games in Q out of the quotas, ω, (when in A) and certain closely related imputations. We define the vectors:

$$\gamma^{ij} = \omega - c_i \delta^i + c_i \delta^j \ ; \tag{29}$$

$$\gamma^{ikj} = \omega - c_i \delta^i - c_k \delta^k + (c_i + c_k) \delta^j \ . \tag{30}$$

---

[7] We return to this game in section 5; it is the only instance of a decomposable quota game.

In referring to these vectors, we shall sometimes speak of j as the "beneficiary" of i.

LEMMA. If there is no weak player, or if i is weak and $j \neq i$, then γ^{ij} is in A. If k is weak and $i, j \neq k$, then γ^{ikj} is in A.

PROOF. Immediate, from the definitions.

THEOREM 3. If ω is in A, and if $b_i \neq i$ is an otherwise arbitrary function from I into itself, then

$$V_b = \bigcup_{i \in I} [\omega, \gamma^{ib_i}] \tag{31}$$

is a solution of the quota game v.

GEOMETRICAL DESCRIPTION. V_b consists of n segments parallel to edges of A radiating from ω, one to each $(n-2)$-dimensional face. If ω is in the boundary, then one or more of the segments is degenerate; but all are degenerate only if A is degenerate as well -- i.e., if the game is inessential. When n is even, it is possible to have $b_{b_i} = i$, all i, whereupon the segments are colinear in pairs, and V_b consists of $n/2$ segments meeting perpendicularly at ω.

VERBAL DESCRIPTION. The standard of behavior attributes to each player a quota and a beneficiary; a player may be the beneficiary of several, or none, of his opponents. In a particular play of the game either all players take their quotas, or one player accepts less and gives the difference to his beneficiary. However, no player ever receives less than the minimum which he can guarantee himself unilaterally -- thus, a player whose quota and minimum are equal will never assume the role of benefactor. When the number of players is even, one solution (mentioned in the preceding paragraph) has them pair off, as if to bargain over the division. In a particular play, any one pair may divide their combined quotas in an arbitrary way, compatible with their minima, while the others settle for their individual quotas exactly.

PROOF OF THEOREM 3. (I) To show $A - \operatorname{dom} V_b \subseteq V_b$. The α in A fall into three categories:

$$\begin{cases} \text{(i)} & \alpha_i < \omega_i \quad (\text{some } i), \quad \alpha_j < \omega_j \quad (\text{some } j \neq i); \\ \text{(ii)} & \alpha_i < \omega_i \quad (\text{some } i), \quad \alpha_j \geq \omega_j \quad (\text{all } j \neq i); \\ \text{(iii)} & \alpha_i \geq \omega_i \quad (\text{all } i) . \end{cases} \tag{32}$$

In the first case we have $\alpha \in \{i, j\}$-dom ω. In the second case we have

$$\alpha_i + \alpha_{b_i} \leq \omega_i + \omega_{b_i} . \tag{33}$$

We can therefore choose t to satisfy one of

$$\begin{cases} \text{(iia)} \quad \alpha_{b_i} - \omega_{b_i} = t = \omega_i - \alpha_i , \\ \text{(iib)} \quad \alpha_{b_i} - \omega_{b_i} < t < \omega_i - \alpha_i . \end{cases} \tag{34}$$

The vector β:

$$\beta = \omega - t\delta^i + t\delta^{b_i} , \tag{35}$$

is in V_b, and we have either $\alpha = \beta$ or $\alpha \in \{i, b_i\}$-dom β. In the third case we have at once $\alpha = \omega$. Hence every imputation is in either V_b or dom V_b. (II) To show $V_b \subseteq A$ - dom V_b. By the lemma, $V_b \subseteq A$. A simple check of the conditions (7), (8), (9) for domination reveals that the assumptions $\alpha \in V_b$, $\beta \in V_b$ and $\alpha \in$ dom β are inconsistent. Hence no imputation is in both V_b and dom V_b.

THEOREM 4. If ω is not in A, let o denote the weak player. Then

$$V_b = \bigcup_{\substack{i \in I \\ i \neq o}} [\gamma^{oi}, \gamma^{iob_i}]$$

is a solution of the quota game v, with b as in Theorem 3.

GEOMETRICAL DESCRIPTION. V_b consists of $n-1$ unconnected segments parallel to edges of A, issuing from points in the face A_o defined by

$$\alpha_o - v_o = 0 , \tag{37}$$

and running one to each of the other $(n-2)$-dimensional faces. Some or all of the segments may lie within A_o, and some or all may degenerate to points in the boundary of A_o; but all degenerate if and only if o is a dummy.[8]

[8] See (41) below.

VERBAL DESCRIPTION. The standard of behavior assigns quotas and beneficiaries as before, but the weak player's quota is below his minimum. In a particular play there is always one benefactor; he may be anyone but the weak player. He first makes up the weak player's deficit out of his own quota, and then, perhaps, gives an additional amount to his assigned beneficiary -- who may also be the weak player. The remaining players take exactly their quotas.

PROOF OF THEOREM 4. (I) To show $A - \operatorname{dom} V_b \subseteq V_b$. Divide the α in A into the categories of (32). In case (i) we must have $o \neq i, j$. Let k be distinct from o, i, and j (possible by Theorem 2!); then $\alpha \in \{i, j\}\text{-dom } \gamma^{ok}$. In case (ii) we must have $o \neq i$ and

$$\alpha_i \leq \omega_i + c_o \,. \tag{38}$$

We can then find, in the manner of the preceding proof, an imputation β in $[\gamma^{oi}, \gamma^{iob_i}]$ such that either $\alpha = \beta$ or $\alpha \in \{i, b_i\}\text{-dom } \beta$. Case (iii) is vacuous, since $\omega \notin A$. Hence every imputation is in either V_b or $\operatorname{dom} V_b$. (II) To show $V_b \subseteq A - \operatorname{dom} V_b$. Proceed as in the proof of Theorem 3.

§ 5. DISCUSSION. THE FOUR-PERSON CONSTANT-SUM CASE

Because of the arbitrariness in the choice of the beneficiary function b, Theorem 3 gives $(n-1)^n$ solutions to each n-person quota game with ω in A. These are all distinct if and only if ω is interior to A. If the game v is varied the quota ω, and hence each γ^{ij}, changes continuously. Therefore the solution V_b is a continuous function of v.

If v is varied so that c_i vanishes for one or more $i \in I$, putting ω in the boundary of A, then those solutions whose beneficiary functions agree except for such i will become indistinguishable, since the segments $[\omega, \gamma^{ij}]$ will have contracted to the single point ω. With ω at a vertex of A there are only $n-1$ distinct solutions (consisting of single edges of A). With ω in an open $(n-2)$-dimensional face of A the number is $(n-1)^{n-1}$.

For ω outside A we turn to Theorem 4 and find, in general, $(n-1)^{n-1}$ solutions, again depending continuously on v. If v is varied so that c_o tends to 0 from below, then ω approaches the boundary of A from the outside, and the points γ^{oi} and γ^{ioj} approach ω and γ^{ij}, respectively, so that each V_b of Theorem 4 goes in the limit into the corresponding V_b of Theorem 3. The transition from one case to the other is perfectly continuous.

If v is now chosen to make .

(39) $$c_i = -c_o > 0$$

for one or more $i \in I - (o)$, then the number of distinct solutions provided by Theorem 4 is reduced as a result of the segments $[\gamma^{oi}, \gamma^{ioj}]$ contracting to the isolated points γ^{oi}. The quota games in which (39) holds for all $i \neq o$ are noteworthy, since they alone have finite quota solutions. It can be shown that, for such a game,

(40) $$v(S) = \sum_{i \in S} v_i - (|S| - 1)c_o \qquad \text{(for } |S| \text{ odd)},$$

$$= \sum_{i \in S} v_i - (|S| - 2)c_o \qquad \text{(for } |S| \text{ even, } o \in S),$$

$$= \sum_{i \in S} v_i - (|S|)c_o \qquad \text{(for } |S| \text{ even, } o \notin S).$$

The game is constant-sum[9] and symmetric in the essential players, o is a dummy, and, of course, $|I|$ is even. The unique quota solution, by Theorem 4, consists of the $n-1$ separate points:

(41) $$\beta^i = \nu - \sum_{\substack{j \in I \\ j \neq o,i}} c_o \delta^j \qquad \text{(all } i \neq o)$$

where ν denotes the vector $\sum_I v_k \delta^k$.

In the four-person case, visualization of the foregoing discussion is aided by a remarkable one-one correspondence that exists between the range of ω (the imputation space A and its environs), and the parameter space of games v. von Neumann regards the essential four-person constant-sum games in reduced form as the points of a certain cube Q.[10] (See Figure 1.)

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[9] Indeed, it is an open question whether finite solutions exist except for constant-sum games.

[10] TGEB 34.2. Figure 1 has been drawn to conform to Figures 61-63 loc. cit.

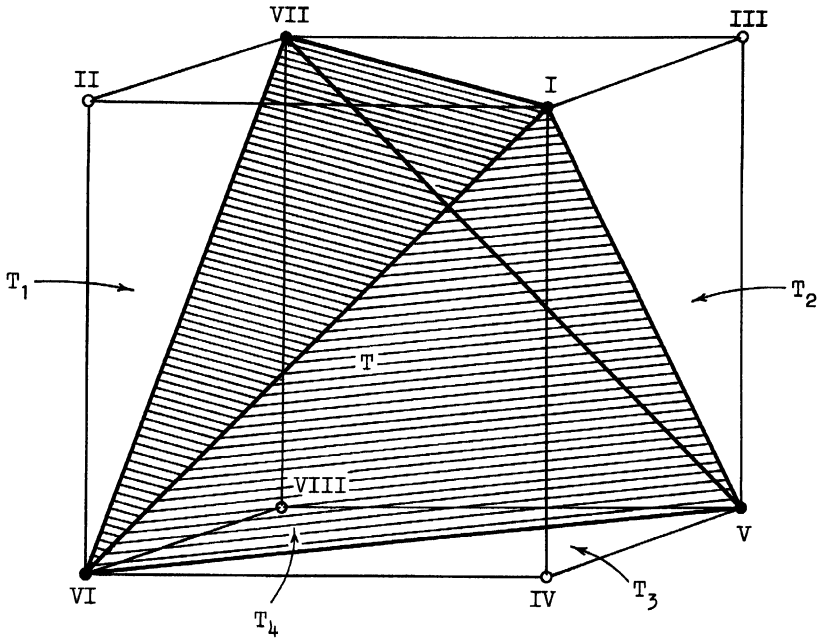

Figure 1. Partition of Q into the tetrahedra T, T_1, T_2, T_3, T_4.

Four of the vertices of Q (o in the figure) are the games with dummies; the other four (•), spanning the inscribed tetrahedron T, are the games in which one player is so strong that only a coalition of all his opponents can defeat him. The four other tetrahedra T_i, named after the dummies in their vertex games (o), form the complement of T in Q.

Now it is possible by a linear transformation mapping the imputation space into the game space to superimpose A on T in such a way that ω coincides with v for every game v. The range of ω is then exactly Q. Theorem 3 (no weak players) applies to games in T, while Theorem 4 (i weak) applies to games in T_i. T comprises one-third of the volume of Q; each T_i, one-sixth. The relation of the weak-player games to the whole set is well illustrated by this representation.

§ 6. FURTHER SOLUTIONS INVOLVING THE QUOTA

A study of the three-person quota games, whose theory is com-

pletely known,[11] reveals that the quota ω belongs to every solution, and that the solutions provided by Theorem 3 are special cases of a continuous family of solutions, consisting generally of arbitrary monotonic curves connecting ω to each edge of the triangle A. (See Figure 2.)

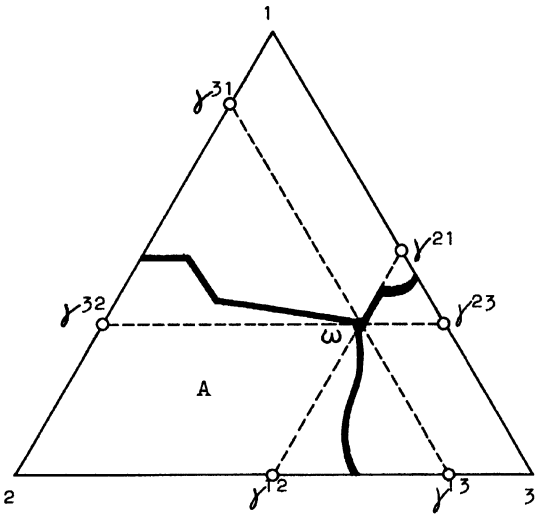

Figure 2. Typical solution of a three-person quota game.

Similar continuous families involving ω exist for higher quota games, but do not in general include all solutions. Their precise nature depends upon detailed properties of the function v, and there are many cases to be distinguished. We shall prove here a result which generalizes the three-person solutions just described to a limited class of higher games, and then give without proof a more complicated example.

First let us revise our definition of $[\alpha,\beta]$: hereafter it will denote an arbitrary curve connecting the points α and β along which all the coordinates, and their sum, vary continuously and monotonically (but not necessarily linearly). In our usage up to this point α and β have never differed in more than two coordinates, therefore the revised definition could have been employed from the start.[12]

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[11] TGEB 60.3.3, with $a_1 + a_2 + a_3 = 0$.

[12] Under the old definition the theorems of this section are still correct, but less general.

THEOREM 5. If $b_i \neq i$ and $b'_i \neq i$ are otherwise arbitrary functions from I into itself; if the vector τ satisfies

$$0 \leq \tau_i \leq 1 \qquad \text{(all } i \in I\text{)} ; \tag{42}$$

and if, finally, the inequality

$$\omega_{b_i} + \omega_{b'_i} + \omega_j > v([b_i, b'_i, j]) \tag{43}$$

holds whenever i, b_i, b'_i, and j are distinct and c_i and c_j are positive; then

$$V_{bb'} = \bigcup_{i \in I} [\omega, (1 - \tau_i)\gamma^{ib_i} + \tau_i \gamma^{ib'_i}] \tag{44}$$

solves the quota game v if ω is in A, and

$$V_{bb'} = \bigcup_{\substack{i \in I \\ i \neq o}} [\gamma^{oi}, (1 - \tau_i)\gamma^{iob_i} + \tau_i \gamma^{iob'_i}] \tag{45}$$

solves it if ω is not in A, o being the weak player.

For $b = b'$, this reduces to Theorems 3 and 4. For $|I| = 3$, (43) is no restriction, and we obtain the results alluded to at the beginning of this section. For four-person games with a weak player o, the theorem yields new solutions if one takes b and b' so that for each i, $b'_i = o$ if and only if $b_i = o$. But the theorem is unproductive for constant-sum games with ω interior to A unless the number of players exceeds five.

VERBAL DESCRIPTION OF THEOREM 5. A player may have two beneficiaries, provided that it is not possible for them to form an effective three-person coalition with some other benefactor. The rule by which the benefits are split is arbitrary (but fixed under the standard of behavior), except that neither beneficiary's share decreases as the other's increases.

PROOF OF THEOREM 5. Suppose that ω is in A, and define for convenience:

$$v^i = [\omega, (1 - \tau_i)\gamma^{ib_i} + \tau_i \gamma^{ib'_i}] . \tag{46}$$

(I) To show A - dom $V_{bb'} \subseteq V_{bb'}$. Divide the α in A into the three categories of (32). In case (i) we have $\alpha \in \{i, j\}$-dom ω. In case (ii)

there is a unique $\beta \in V^i$ with $\beta_i = \alpha_i$, by the monotonicity of V^i. Either

$$(47) \quad \begin{cases} \text{(iia)} \quad \alpha = \beta, \text{ or} \\ \text{(iib)} \quad \alpha_k < \beta_k \text{ for } k = b_i \text{ or } k = b'_i . \end{cases}$$

In case (iia) we have $\alpha \in V^i$. In case (iib) we can find, using monotonicity and the fact that $\beta \neq \omega$, points in V^i near β which (i, k)-dominate α. In case (iii) we have $\alpha = \omega$. Hence every imputation is in either $V_{bb'}$ or dom $V_{bb'}$. (II) To show $V_{bb'} \subseteq A$ - dom $V_{bb'}$. Proceed as in the proof of Theorem 3, observing that the hypothesis (43) exactly excludes the possibility of S-domination for $|S| = 3$. This establishes (44). The proof of (45) is similar.

Theorem 5 provides a family of solutions connecting V_b and $V_{b'}$, assuming condition (43) is fulfilled. The next theorem illustrates one of the more elaborate "connections" that are found in some cases in which (43) does not hold. The solutions "connected" are V_b (at $e = 0$) and $V_{b'}$ (at $e = 1$), where

$$(48) \quad \begin{cases} b_i = k, \quad b'_i = l & \text{(all } i \neq j, k, l), \\ b_j = l, \quad b'_j = k, \\ b_k = m, \quad b'_k = m, \\ b_l = n, \quad b'_l = n . \end{cases}$$

THEOREM 6. If ω is in A, and if

$$(49) \quad \omega_j + \omega_k + \omega_l \leq v_{jkl},$$

for some distinct j, k, l; then, for arbitrary $m \neq k$ and $n \neq l$, the set

$$(50) \quad [\omega, (1 - e)\gamma^{jl} + e\gamma^{jk}] \cup [\omega, \gamma^{km}] \cup [\omega, \gamma^{ln}] \cup \bigcup_{\substack{i \neq j,k,l \\ c_i \geq ec_j}} [(1 - ec_j/c_i)\omega + (ec_j/c_i)\gamma^{ik}, \gamma^{ik}]$$

is a solution of v for $0 \leq e < 1/2$ (Figure 3ab); the set

$$(51)\quad [\omega,\ (1-e)\gamma^{jl} + e\gamma^{jk}] \cup [\omega,\gamma^{km}] \cup [\omega,\gamma^{ln}] \cup \bigcup_{\substack{i\neq j,k,l \\ c_i \geq (1-e)c_j}} [(1-(1-e)c_j/c_i)\omega + ((1-e)c_j/c_i)\gamma^{il},\gamma^{il}]$$

is a solution for $1/2 < e \leq 1$ (Figure 3de); and, if we also have

$$(52)\qquad c_j > c_i \qquad (\text{all } i \neq j,\ k,\ l)\ ,$$

then the set

$$(53)\quad [\omega,\ (1-e)\gamma^{jl} + e\gamma^{jk}] \cup [\omega,\gamma^{km}] \cup [\omega,\gamma^{ln}] \cup \bigcup_{\substack{i\neq j,k,l \\ c_i \geq c_j/2}} \{(1-ec_j/c_i)\omega + (ec_j/c_i)\gamma^{ik}\} \cup \bigcup_{\substack{i\neq j,k,l \\ c_i \geq c_j/2}} \{(1-ec_j/c_i)\omega + (ec_j/c_i)\gamma^{il}\}$$

is a solution for $e = 1/2$ (Figure 3c).

Among the noteworthy features of these solutions are the following: (a) Some of the segments in (50) and (51) are detached from ω; thus there is a positive lower bound, (equal to ec_j in (50)) for some of the benefits. (b) Some of the segments disappear entirely; thus i is not a benefactor in (50) if $0 < c_i < ec_j$, $i \neq k,\ l$, yet he always gets more than his minimum. (c) Isolated points occur in the central case (53) whenever $c_j/2 \leq c_i < c_j$ for some $i \neq k,\ l$; these can lie in the interior of A.

As Figure 3 reveals, the family of solutions is not continuous in the usual sense.[13] Its "connectivity" is similar to the "connectivity" of the set of all solutions to a constant-sum three-person game, a property which has not yet been given an adequate characterization.

[13] If it happens that $c_i \leq c_j/2$ for all $i \neq j,\ k,\ l$, then the family of Theorem 6 is lower semi-continuous.

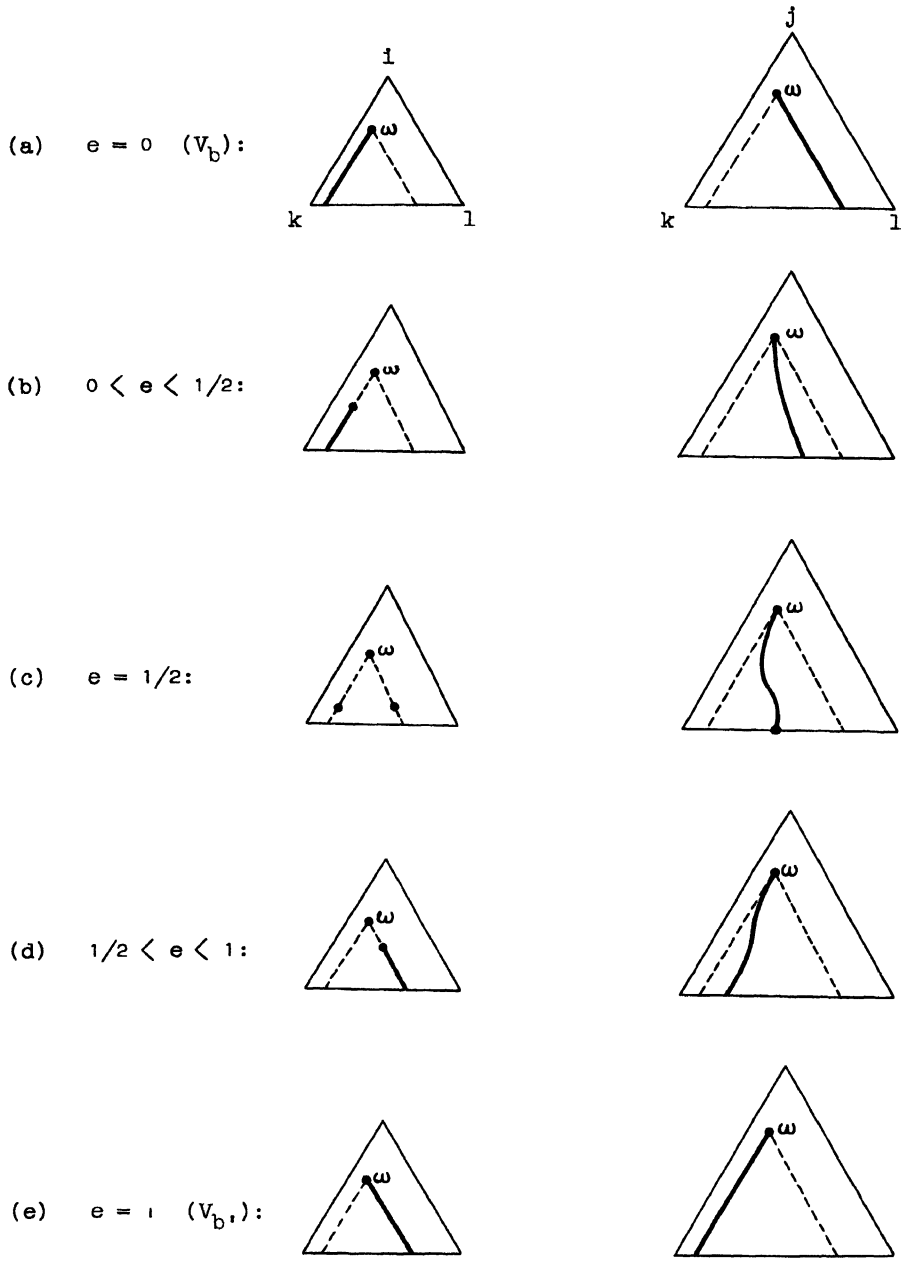

Figure 3. Two-dimensional sections of A through ω, illustrating the connection between V_b and $V_{b'}$ obtained in Theorem 6 by varying e. (The letter "i" stands for a typical member of I - {j, k, l}.)

§ 7. EXTENSION OF RESULTS

By known methods one can readily:

(a) solve a class of strategically equivalent games, given a solution to one of them;

(b) obtain a solution to a decomposable game, given a solution to every component;

(c) obtain a solution to each component, given a decomposable solution to a decomposable game;

(d) given a solution to an n-person game, obtain a solution to its (n+1)-person zero-sum extension;

(e) given a solution to an n-person game lying within a face of A (a "completely discriminatory" solution), obtain a solution to the (n-1)-person game formed by removing the player in question.[14]

Let $\langle \mathfrak{Q} \rangle$ denote the set of ordered pairs $\langle v, V \rangle$, where v is in $\mathfrak{Q}$ and V is a solution of v given by Theorems 3-6. Let $\langle \mathfrak{Q} \rangle^*$ be the smallest set containing $\langle \mathfrak{Q} \rangle$ and closed under the extension operations indicated in (a) - (e) above. The set $\mathfrak{Q}^*$ of games occuring in $\langle \mathfrak{Q} \rangle^*$ then represents a class of games solvable directly or indirectly by the results of this paper.[15] The dimensions of the sets of essential n-person games in reduced form in $\mathfrak{Q}^*$ are as follows, for small values of n:

	Games in $\mathfrak{Q}^*$	
n	General-sum	Constant-sum
2	0	(none)
3	3	0
4	7	3
5	19	7
6	31	19

(Compare the table on page 347 above.) The present sets are not convex. Hence, although substantial improvement over $\mathfrak{Q}$ is apparent in for example the five-person constant-sum case, the dimension numbers do not fully indicate the effectiveness of the extension.

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[14] TGEB 57.5, 44.3, 60.4. Although not mentioned in TGEB, (e) is easily verified from the definitions.

[15] This process of extension can of course be applied to any class $\langle \mathfrak{G} \rangle$ of games with solutions (compare TGEB 54.2.1). In the present case it turns out that (d) and (e) are most powerful, while (b) contributes nothing.

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ARBITRATION SCHEMES FOR GENERALIZED TWO-PERSON GAMES[1]

Howard Raiffa

1.1. INTRODUCTION. John von Neumann and Oscar I. Morgenstern in [9][2] analyze the discrete zero-sum two-person game and indicate that to each such game there exists a unique value. The solution of this type of game has a strong stability in the sense that if both players are made aware of a saddle point (in the generalized space of mixed strategies), there is no tendency for either player to act out of conformity with the predicted mathematical solution. However, in the two-person non-zero-sum game, no attempt is made in [9] to establish the solution as a unique element, but the solution is defined as a family of imputations which is characterized by a set of requirements. It is felt that this multiplicity of solutions is in harmony with realizations of this mathematical model (e.g., the duopoly or the duposony behavior in economics). The actual solution in a given real life situation depends on the psychological behavior of the players in this bargaining problem. In essence, we are faced with the problem of the two players choosing a "best" element from a set of incomparable elements. The inability of the contestants to arrive at a mutually satisfactory return in this bargaining situation often results empirically in the playing of the non-cooperative game. The imputation thus attained is not, in general, an element of the solution set and therefore is dominated by some imputations of the solution set. In situations of this type it behooves the players to offer their case for arbitration. To be objective, the arbiter must seek an imputation from the solution set which satisfies a set of so-called "fair" requirements which is dictated by the sociological, economic or political environment in which the game is embedded. Motivated by the role of arbitration in collective bargaining disputes, we define a scheme of arbitration as a mapping from a given

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[1]The preparation of a more complete version of this work was sponsored by the Office of Naval Research. See item [10] in the bibliography at the end of this paper.

[2]The numbers in square brackets refer to the bibliography at the end of this paper.

generalized two-person game into a unique element of the solution set of that particular game. We shall initially require that the domain of the arbitration mapping shall at least include all two-person games with a finite number of pure strategies. Various conditions are imposed on the arbitration schemes, and this paper shall attempt to construct mappings satisfying these desiderata. No attempt will be made on the formal level to single out a "best" method of arbitration; however, certain provocative games will be arbitrated according to different conventions to aid in the subjective evaluations of these conventions. The results and motivations for this paper were obtained independently of the work of J. F. Nash, Jr., on the two-person non-zero-sum game.

1.2. UTILITY MEASUREMENT. We will follow von Neumann and Morgenstern in hypothesizing a numerical utility determined up to the group of positive linear transformations. However, in this paper two distinct cases will be considered: the existence or non-existence of a common unit of measurement. In the former case we will require that the theory be invariant only up to utility transformations on the players' payoff entries that do not destroy the interpersonal comparison of the unit of measurement. In the latter case we require invariance up to independent utility transformations. Further, in the case where a common unit is hypothesized we will distinguish between the cases where side payments are or are not permitted. For example, the entries may be in monetary units but physical barriers might require that no exchange take place between the individuals. In order not to prejudge the problem, the models introduced will incorporate all such possibilities.[3]

2.1. MIXED STRATEGY SPACES.

DEFINITION 1. Let $\mathcal{A} = \{\alpha\}$ $[\mathcal{B} = \{\beta\}]$ be the set of pure strategies of player I [II]. If the set $\mathcal{A}[\mathcal{B}]$ is finite, denote the pure strategies by $\alpha_1, \alpha_2, \ldots, \alpha_m$ $[\beta_1, \beta_2, \ldots, \beta_n]$.

DEFINITION 2. Let $A(\alpha,\beta)$ $[B(\alpha,\beta)]$ denote the return to player I [II] corresponding to a pair of pure strategies (α,β). In the finite case we abbreviate $A(\alpha_i,\beta_j)$ $[B(\alpha_i,\beta_j)]$ by a_{ij} $[b_{ij}]$.

[3]Even in the case of a monetary unit it is not quite realistic to assume a common unit of measurement. However, it is felt that if one abstracts entirely from the interpersonal comparison of the monetary unit then one alters the basic bargaining structure of the game.

DEFINITION 3. Let

$$R_0 \equiv \{(A(\alpha,\beta), B(\alpha,\beta)) : (\alpha,\beta) \in \mathcal{A} \times \mathcal{B}\} .$$

That is, R_0 is the range of the mapping, $A \times B : \mathcal{A} \times \mathcal{B} \longrightarrow E^{(2)}$ where $E^{(2)}$ represents Euclidean 2-space.

DEFINITION 4. Let F_1 [F_2] be a Borel field of subsets of $\mathcal{A}$ [$\mathcal{B}$] containing all pure strategies.

DEFINITION 5. Let $S_I = \{\xi\}$ [$S_{II} = \{\eta\}$] be the set of all probability measures over $\mathcal{A}$ [$\mathcal{B}$]. Thus each ξ [η] is defined on F_1 [F_2] to the interval [0,1], and ξ [η] is a completely additive measure function. In the finite case we let $\xi(\alpha_i) \equiv \xi_i$ [$\eta(\beta_j) \equiv \eta_j$].

DEFINITION 6. Corresponding to the pair (ξ,η) associate a vector $(A(\xi,\eta), B(\xi,\eta))$ where $A(\xi,\eta) = \int_{\mathcal{B}} \int_{\mathcal{A}} A(\alpha,\beta)d\xi \; d\eta$ ($= \sum_{ij} a_{ij}\xi_i\eta_j$ in the discrete case) $B(\xi,\eta) = \int_{\mathcal{B}} \int_{\mathcal{A}} B(\alpha,\beta)d\xi \; d\eta$ ($= \sum_{ij} b_{ij}\xi_i\eta_j$ in the discrete case).

The expressions $A(\xi,\eta)$ and $B(\xi,\eta)$ represent the expected returns to players I and II respectively corresponding to the pair (ξ,η). We explicitly assume: $A(\alpha,\beta)$, $B(\alpha,\beta)$ are finite and summable with respect to all $(\xi,\eta) \in S_I \times S_{II}$ and the double integrals are equal to the iterated integrals. Without worrying about the logical niceties we will equate the pure strategy α [β] with the mixed strategy ξ [η] such that $\xi(\alpha) = 1$ [$\eta(\beta) = 1$]. In this sense $\mathcal{A} \subset S_I$ and $\mathcal{B} \subset S_{II}$. We take the further liberty of confusing the expressions $A(\xi,\eta)$, $A(\alpha,\beta)$, $A(\xi,\beta)$ etc, since the domain of the function will be clear by the symbols used for the arguments.

DEFINITION 7. Let

$$R_1 = \{(A(\xi,\eta), B(\xi,\eta)) : (\xi,\eta) \in S_I \times S_{II}\} .$$

That is, R_1 is the range of the mapping, $A \times B : S_I \times S_{II} \longrightarrow E^{(2)}$.

In the finite case S_I and S_{II} are closed simplexes in $E^{(m-1)}$ (Euclidean m-1-space) and $E^{(n-1)}$ respectively; thus these spaces

are convex, connected and compact.

In the finite case we can write the game g as a matrix of the form $||(a_{ij}, b_{ij})||$. For example, the game

$$g_1 \equiv \begin{array}{c|cc} & \beta_1 & \beta_2 \\ \alpha_1 & (2, 1) & (-1, -4) \\ \alpha_2 & (-4, -1) & (1, 2) \end{array}$$

has a region, R_1, given below:

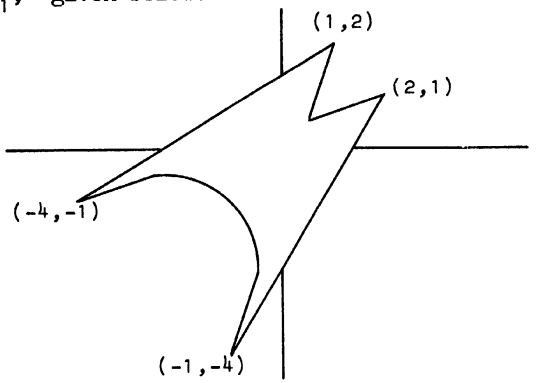

Figure 1

It should be noted that in general the region R_1 of a game is not necessarily convex. It is possible for the players to convexify the region if they use correlated mixed strategies.

2.2. JOINT STRATEGY SPACE.

DEFINITION 8-a. Let $J' = \{\zeta\}$ be the set of all probability measures over $F_1 \times F_2$. In the finite case, let $\zeta(\alpha_i, \beta_j) = \zeta_{ij}$. Thus $\zeta_{ij} \geq 0$ for all i, j and $\sum_{ij} \zeta_{ij} = 1$. The space J' will be restricted to a subspace J in Definition 8-b (which follows Definition 9-b).

Elements ξ and η will be called mixed strategies and elements ζ will be called joint mixed strategies. To every pair (ξ,η) we can associate a joint mixed strategy which is the product measure (i.e., $\xi \times \eta = \zeta$ where $\zeta_{ij} = \xi_i \eta_j$) of ξ and η. It is in this sense that we will consider $S_I \times S_{II} \subset J'$.

DEFINITION 9-a. If $A(\alpha,\beta)$, $B(\alpha,\beta)$ are summable with respect to any $\zeta \in J'$, then for any $\zeta \in J'$ let

$$(A(\zeta), B(\zeta)) = \int_{\alpha x \beta} (A(\alpha,\beta), B(\alpha,\beta)) d\zeta \quad (= \sum_{ij} (a_{ij}, b_{ij}) \zeta_{ij}$$

in the discrete case). Now define

$$R_2' = \overline{(A(\zeta), B(\zeta)) : \zeta \in J'} ;$$

that is, R_2' is the closure of the range of the mapping, $A \times B : J' \longrightarrow E^{(2)}$.

Using the product measure, the product space $S_I \times S_{II} \subset J'$. Further, $R_1 \subset R_2'$. In the finite case we note that R_2' is the minimal convex set containing R_o; and the boundary of R_2' is composed of one-simplexes whose vertices are elements of R_o.

DEFINITION 9-b. Let R_2 be the minimal closed convex set containing R_o (i.e., the convex hull of R_o).

Since in the infinite case it is always possible to define R_o, this definition is non-vacuous and unique by noting that $E^{(2)}$ is closed and convex and contains R_o, and also that the intersection of closed convex sets is closed and convex. Thus for all games we can associate a region R_2.

We can give an equivalent characterization of R_2 as follows: let U be the point set which is the union of all line segments joining any two elements of R_o. Let V be the point set which is the union of all line segments joining any two elements of U. We will now show that V is convex. To this end, let v_1 and v_2 belong to V, and for $0 \leq \alpha \leq 1$ we wish to show $\alpha v_1 + (1 - \alpha) v_2 \in V$. Let u_1, u_2, u_3, u_4 be elements of U used in defining v_1 and v_2, and let $e_1, e_2, \cdots e_8$ be the elements of R_o used in defining u_1, u_2, u_3, u_4. It is easily seen that $\alpha v_1 + (1 - \alpha) v_2$ lies in the minimal convex set containing $e_1, e_2, \ldots, e_8$. Thus $\alpha v_1 + (1 - \alpha) v_2$ lies on a line segment joing two boundary points of this minimal convex set. Hence we conclude that V is convex. Since the closure of a convex set is convex, we finally have $\overline{V} = R_2$.

We now assert that if R_2' is well defined that $R_2' = R_2$. From the above characterization of V, it is seen that V is contained in the image of J' under the mapping $A \times B$. Taking closures, we get $R_2 \subset R_2'$.

That $R_2' \subset R_2$ follows from the observation that the image of J' is contained in R_2. Again taking closures, we get $R_2' \subset R_2$ and consequently $R_2' = R_2$.

Since R_2 is always defined we will have no recourse in the sequel to use the notation R_2'. If the set R_o is compact we will now show that the minimal convex set containing R_o (in the above notation this is the set V) is also compact. Since R_o is bounded the set V is also bounded. It remains to show that V is closed. Let $\{v_n\}$ be a sequence of elements of V which have v for a sequential limit point. To show $v \in V$. For each n, we have

$$v_n = \lambda_{n1}e_{n1} + \lambda_{n2}e_{n2} + \lambda_{n3}e_{n3} + \lambda_{n4}e_{n4}$$

where $\sum_{i=1}^{4} \lambda_{ni} = 1$, $\lambda_{ni} \geq 0$, $e_{ni} \in R_o$. By a diagonalization process there is a subsequence $\{V_{n_m}\}$ such that $\lambda_{n_m i} \longrightarrow \lambda_i$, $e_{n_m i} \longrightarrow e_i$; and further $\sum_1^4 \lambda_i = 1$, $\lambda_i \geq 0$, $e_i \in R_o$. Consequently

$$v = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4$$

and thus $v \in V$ which proves the assertion. Recalling that V is contained in the image of J' we can conclude that if R_o is compact then every element of R_2 has an inverse image in J'. Although in the general case, not every element of R_2 is the image of an element of J', this will not be a serious handicap because of the context in which R_2 is utilized.

The primary role of region R_2 is in the cooperative game in which pre-play communication is allowed. In this case the players must agree on a joint mixed strategy. Although initially we took a rather general viewpoint it should now be clear that we can restrict the space J' to all normalized measures over the ring of sets of $\mathcal{A} \times \mathcal{B}$ which consists of all sets with a finite number of elements. This effectively eliminates any worry over integrability conditions for the expression $\int_{\mathcal{A}\times\mathcal{B}} (A(\alpha,\beta), B(\alpha,\beta))d\zeta$.

> DEFINITION 8-b. Let $J = \{\zeta\}$ be the set of all normalized measures over the ring of sets of $\mathcal{A} \times \mathcal{B}$ which consists of all sets with a finite number of elements.

2.3. COMPOSITE ORDER RELATIONS. We now shall define two partial orderings of $E^{(2)}$.

DEFINITION 10-a. The element (x_2,y_2) compositely dominates (x_1,y_1),

$$(x_1,y_1) \leqq (x_2,y_2) \equiv x_1 \leq x_2,\ y_1 \leq y_2 \quad \text{and} \quad x_1 + y_1 < x_2 + y_2 \ .$$

DEFINITION 10-b. The element (x_2,y_2) compositely uniformly dominates (x_1,y_1),

$$(x_1,y_1) < (x_2,y_2) \equiv x_1 < x_2 \quad \text{and} \quad y_1 < y_2 \ .$$

The symbol $\leqq$ shall be construed as follows:

$$(x_1,y_1) \leqq (x_2,y_2) \equiv x_1 \leq x_2,\ y_1 \leq y_2 \ .$$

An element e of the set $U \subset E^{(2)}$ is said to be <u>maximal</u> ($\leqq$) if there exists no element $e' \in U$ such that $e \leqq e'$. Similarly, we can define maximal ($<$). In case $\mathcal{A} \times \mathcal{B}$ is finite the sets of maximal ($<$ or $\leqq$) elements of R_o, R_1, and R_2 are non-vacuous. In the infinite case to insure that the above sets are non-vacuous, it is quite sufficient to impose a compactness condition on the set R_o. We state without proof (cf. Raiffa [11] for details) that the set of maximal ($\leqq$) points of a compact and convex subset of $E^{(2)}$ is compact and connected. Hence, if R_o is bounded then the set of maximal ($\leqq$) points of R_2 is compact and connected, for: if R_o is bounded then R_2 is compact and convex.

Let the maximal ($\leqq$) elements of R_o, R_1, and R_2 be denoted by M_o, M_1, and M_2 respectively.

The element $(\xi,\eta) \in S_I \times S_{II}$ is said to be <u>admissible</u> if it maps into a maximal element of R_1. The set of admissible elements, $\tilde{M}_1$, is the inverse image of M_1.

Let $(\xi^{(1)},\eta^{(1)})$ and $(\xi^{(2)},\eta^{(2)})$ be two elements of $S_I \times S_{II}$. Then $(\xi^{(1)},\eta^{(1)})$ is said to be <u>equivalent</u> to $(\xi^{(2)},\eta^{(2)})$, $(\xi^{(1)},\eta^{(1)}) \simeq (\xi^{(2)},\eta^{(2)})$, if and only if they have the same image in R_1 (by mapping A x B).

DEFINITION 11. An equivalence class, H, in $S_I \times S_{II}$ is said to be permutable if whenever $(\xi^{(1)},\eta^{(1)})$ and $(\xi^{(2)},\eta^{(2)})$ belong to H, also $(\xi^{(1)},\eta^{(2)})$ and $(\xi^{(2)},\eta^{(1)})$ belong to H.

DEFINITION 12. The strategy $\xi^{(o)}$ $[\eta^{(o)}]$ is said to be a Bayes solution with respect to the strategy

$\eta^{(o)}$ [$\xi^{(o)}$] (or, $\xi^{(o)}$ [$\eta^{(o)}$] is good against $\eta^{(o)}$ [$\xi^{(o)}$]) if $A(\xi^{(o)},\eta^{(o)}) = \sup_{\xi} A(\xi,\eta^{(o)})$ [$B(\xi^{(o)},\eta^{(o)}) = \sup_{\eta} B(\xi^{(o)},\eta)$].

DEFINITION 13. The pair of strategies (ξ^o,η^o) is an equilibrium[4] point if each strategy is good against the other. If a pair of pure strategies, say $(\alpha^{(o)},\beta^{(o)})$, is such that each is good against the other, then the pair is called a saddle point. The totality of saddle and equilibrium points of a given game will be denoted by E_o and E_1 respectively.

To illustrate some of these definitions we return to the example given by Figure 1. The strategy pairs (α_1,β_1) and (α_2,β_2) are both admissible and saddle points. However, these elements are not equivalent. The strategy pair $(\xi^{(o)},\eta^{(o)})$ where $\xi^{(o)} = (3/8,5/8)$ and $\eta^{(o)} = (1/4,3/4)$ is an equilibrium point. The elements (2,1) and (1,2) are the maximal elements of the regions R_o and R_1. The region R_2 is the minimal convex region containing R_o, which consists of the four vertices. The set M_2 of maximal elements of R_2 is the segment from (1,2) to (2,1).

2.4. STRICTLY COMPETITIVE, CONVEX AND LINEAR GAMES. We shall now consider an important subclass of two-person games which we shall call strictly competitive games. These games are characterized by the fact that the elements of R_o are incomparable in the composite ordering ($\leqslant$). If each player simply orders the terminal outcomes of the game according to preference, then in strictly competitive games the orderings of both players are exactly reverse. Hence the players have strictly opposing interests with respect to pairs of pure strategies.

LEMMA 1. In a strictly competitive game the set E_o of saddle points (assuming $E_o \neq \emptyset$) forms a permutable equivalence class.

PROOF. Let (α^1,β^1) and (α^2,β^2) be a pair of saddle points. Since (α^1,β^1) is a saddle point we have

a) $$A(\alpha^2,\beta^1) \leq A(\alpha^1,\beta^1), \qquad B(\alpha^1,\beta^2) \leq B(\alpha^1,\beta^1) .$$

[4]The concept of equilibrium point was first introduced by Nash, [4].

Since the game is incomparable, we have from a)

b) $$B(\alpha^2,\beta^1) \geq B(\alpha^1,\beta^1), \qquad A(\alpha^1,\beta^2) \geq A(\alpha^1,\beta^1) .$$

But (α^2,β^2) is a saddle point and thus

c) $$A(\alpha^2,\beta^2) \geq A(\alpha^1,\beta^2), \qquad B(\alpha^2,\beta^2) \geq B(\alpha^2,\beta^1) .$$

From b) and c) we have

d) $$A(\alpha^2,\beta^2) \geq A(\alpha^1,\beta^1), \qquad B(\alpha^2,\beta^2) \geq B(\alpha^1,\beta^1) .$$

Since the game is strictly competitive, both equalities must hold in d) and therefore equalities must hold in c), b) and a). Consequently, we have

$$(\alpha^1,\beta^1) \simeq (\alpha^1,\beta^2) \simeq (\alpha^2,\beta^1) \simeq (\alpha^2,\beta^2) . \qquad \text{Q.E.D.}$$

A strictly competitive game is said to be convex if $R_0 \subset M_2$ (i.e., every element of R_0 is maximal ($\leq$) in the set R_2). Thus in a convex game all pairs of pure strategies are compositely admissible.

A strictly competitive game is said to be linear if $R_1 = R_2 =$ a segment of a line. Otherwise a game is said to be non-linear. A linear game is also convex, and also $R_1 = M_2$.

The concepts of strictly competitive, convex, and linear games are invariant under independent positive linear transformations of the payoff entries. The linear game is a natural generalization of a two-person zero-sum game. In a linear game no assumption need be made on a common unit of measurement between the two players.

A game is said to have a non-cooperative solution if $E_1 \cap \tilde{M}_1$ forms a permutable equivalence class. The imputation corresponding to this equivalence class will be called the value of the game. J. von Neumann's solution of the zero-sum two-person game would be a non-cooperative solution in our sense.[5]

If both players are aware of the above non-cooperative solution of a game (provided it exists) then there is no advantage for either player to act out of conformity with this solution. Furthermore, even if both players were permitted to enter a cooperative game without side payments and without correlated mixed strategies (but with pre-play communication)

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[5]For a more detailed treatment of the non-cooperative two-person game the above definition is too restrictive and is generalized in Raiffa [10]. These definitions differ with those proposed by Nash [5].

then there still would not be any composite motivation to move from this solution.

If $\xi^{(o)}$ is such that $\max_{\xi \in S_I} \min_{\eta \in S_{II}} A(\xi,\eta) = \min_{\eta \in S_{II}} A(\xi^{(o)},\eta) = v_1$ then $\xi^{(o)}$ and v_1 are said to be the maximin strategy and value respectively for player I. Similarly there will be a maximin strategy $\eta^{(o)}$ and maximin value v_2 for player II. The element (v_1,v_2) will be termed the composite maximin value. To show the dependence on a particular game $\mathcal{G}$ we shall employ the symbolism $(v_1,v_2)(\mathcal{G})$.

We also observe that if $(\xi^{(1)},\eta^{(1)})$ is an equilibrium point then

$$(A(\xi^{(1)},\eta^{(1)}),\ B(\xi^{(1)},\eta^{(1)})) \geq (v_1,v_2)$$

where (v_1,v_2) is the composite maximin value. To verify the above assertion let $\xi^{(o)}$ be a maximin strategy. Now since $\xi^{(1)}$ is good against $\eta^{(1)}$ we have

$$v_1 = \min_{\eta} A(\xi^{(o)},\eta) \leq A(\xi^{(o)},\eta^{(1)}) \leq A(\xi^{(1)},\eta^{(1)}) .$$

Similarly, $v_2 \leq B(\xi^{(1)},\eta^{(1)})$ which proves the assertion.

2.5. REGION CORRESPONDING TO A TRANSFERABLE UTILITY. Thus far in Section 2 we have not assumed any common unit of measurement. However, if there exists a common unit of measurement and side payments are permissible then it is possible for the players to achieve imputations outside of R_2. Let $c = \max_{\alpha,\beta} [A(\alpha,\beta) + B(\alpha,\beta)]$. In the infinite case this is assumed to exist. (For example, it is quite sufficient to assume compactness for $\mathcal{A}$ and $\mathcal{B}$ and continuity for A and B.)

DEFINITION 16. Let $R_3 = \{(x_1,x_2) : x_1 + x_2 \leq c\}$. The set of maximal $(\leq)$ points, M_3, for R_3 is $\{(x_1,x_2) : x_1 + x_2 = c\}$.

To summarize, we have defined four sets in $E^{(2)}$ with the inclusion relations $R_0 \subset R_1 \subset R_2 \subset R_3$. In the non-cooperative game and the cooperative game with independent mixed strategies we must consider R_0 and R_1; in the cooperative case with dependent mixed strategies we must consider R_2; finally, R_3 pertains to the cooperative game with side payments.

§ 3. THE COOPERATIVE GAME

3.1. By using joint strategies $\zeta \in J$, the players can come

arbitrarily close to any element of R_2 (indeed, if R_o is compact all elements of R_2 are attainable). Hence, in the cooperative game with pre-play communication and without side payments, the players should confine themselves to consideration only of the maximal elements, M_2, of R_2, since any element of R_2 not in M_2 is compositely dominated by some element of M_2. Thus if M_2 consists of one element the game is trivially resolved. However, if M_2 consists of more than one element then the elements of this set are compositely incomparable. Define the solution set of the game as the totality of elements of M_2 which dominate ($\leqq$) the composite maximin value (v_1, v_2) of the game. Motivated by our discussion of Section 1.1 we define a scheme of arbitration as a mapping $\mathcal{J}$ from a game into an element of the solution set. More specifically, we have:

$$\text{CONDITION 1.}\quad \mathcal{J} : \mathcal{G} \longrightarrow [V_1(\mathcal{G},\mathcal{J}),\ V_2(\mathcal{G},\mathcal{J})] \in M_2(\mathcal{G}),$$

where the game $\mathcal{G}$ is given by the payoff functions $\{A(\alpha,\beta),\ B(\alpha,\beta)\}$. The pair $[V_1(\mathcal{G},\mathcal{J}),\ V_2(\mathcal{G},\mathcal{J})]$ shall be termed the value of the game relative to the arbitration scheme $\mathcal{J}$ and this value shall be an element of the set[6] M_2, which in turn depends on $\mathcal{G}$. We wish the domain of the mapping to include (at least) all games with a finite number of pure strategies.

$$\text{CONDITION 2.}\quad \text{For all } \mathcal{G},\ [V_1(\mathcal{G},\mathcal{J}),\ V_2(\mathcal{G},\mathcal{J})] \geqq (v_1, v_2)(\mathcal{G}).$$

Condition 2 implies that the value of a zero-sum (linear) game relative to the convention $\mathcal{J}$ agrees with the von Neumann solution.

It might seem natural to impose conditions on the mapping $\mathcal{J}$ such that: if $\mathcal{G}$ has a non-cooperative solution in sense of 2.4 then the arbitrated solution shall agree with this result. If we consider the following game:

$$\mathcal{G}_2 \equiv \begin{array}{c|cc} & \beta_1 & \beta_2 \\ \alpha_1 & (1,\ 10) & (0,\ 0) \\ \alpha_2 & (2,.01) & (0,\ 0) \end{array}$$

we note that (2,.01) is the non-cooperative solution, ((0,.01) is the composite maximin value). However, if this game is to be interpreted with

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[6] If $\mathcal{G}$ is to be interpreted as a game with a common unit of measurement and with side payments allowed, then in Condition 1 we replace $M_2(\mathcal{G})$ with $M_3(\mathcal{G})$

pre-play communication, player II can make a good case for something closer to (1,10) by threatening to play β_2 (not that II wants to actually play β_2). Thus we see that even if the solution of the non-cooperative game is extremely stable it might not be a realistic solution in the case of pre-play communication because of the potential threat powers of the players.

We now exhibit a rather general method for getting a mapping $\mathcal{J}$ which has as its domain all games with a finite number of strategies. This domain can be enlarged, but the writer does not think it adds anything conceptually.

Consider a family $\mathcal{J}' = \{T\}$ of mappings, where each mapping T (generic symbol) has as its domain a region R_1 and as its range a set M_2 (where R_1 and M_2 correspond to some game g), i.e., $T : R_1 \longrightarrow > M_2$, such that T is point to point and

a) T is continuous

b) $T(e) \geq e$ for all $e \in R_1$

c) If $T(e) = T(e')$ then $T(\alpha e + (1 - \alpha)e') = T(e) = T(e')$

for $e, e' \in R_1$ and all $0 \leq \alpha \leq 1$ such that $\alpha e + (1 - \alpha)e' \in R_1$. We further assume that to each distinct pair (R_1, M_2) there exists one and only one T.

For each game g with regions $R_1(g)$, $M_2(g)$ we have a mapping T and we now consider the composite mapping:

$$S_I \times S_{II} \xrightarrow{AxB} \gg R_1 \xrightarrow{T} \gg M_2$$

where

$$(\xi,\eta) \longrightarrow (A(\xi,\eta), B(\xi,\eta) \longrightarrow T(A(\xi,\eta), B(\xi,\eta)) \equiv (A'(\xi,\eta), B'(\xi,\eta)) .$$

Relative to the family $\mathcal{J}'$, we associate to each game $g = (A(\alpha,\beta)\ B(\alpha,\beta))$ a new game $g' = (A'(\xi,\eta), B'(\xi,\eta))$. The set R_0 of game g' is now contained in the set M_2 of g. The game g' is now a convex game and consequently if saddle points exist they form a permutable equivalence class. Theorem 1, below, asserts that for all g with a finite number of pure strategies the induced g' has a saddle point. The value of the game g' will now be termed the solution of g relative to the arbitration scheme $\mathcal{J}'$ (we identify the family of mappings with an arbitration mapping). The motivation for the above section should now be clear. Since we wish the arbitrated solution of g to be an element of $M_2(g)$ we induce a new game whose payoff values are all contained in $M_2(g)$. Thus, given a family $\mathcal{J}'$ satisfying a) - c) above, we get an arbitration scheme which automatically fulfills Condition 1. The validity of Condition 2 follows from consideration of requirement b) for each mapping T of the family $\mathcal{J}'$, as follows: If $\xi^{(o)}, v_1$ are the maximin strategy and value for player I in g then

$\xi^{(o)}$ guarantees a security value at least as large as v_1 in $\mathcal{G}'$ by property b) of T. Hence the composite maximin value (v_1', v_2') of $\mathcal{G}'$ is such that $(v_1', v_2') \geq (v_1, v_2)$. We have also seen that the value corresponding to an equilibrium point is greater than or equal to ($\geq$) the composite maximin value. Hence we see that Condition 2 is satisfied.

To prove Theorem 1 we will utilize the following lemma which was first proved by von Neumann, who utilized the Brouwer Fixed Point Theorem; it was later proved by Kakutani as an immediate consequence of the Kakutani Fixed Point Theorem (cf. Kakutani [2]).

LEMMA. Let S_m and S_n be two bounded, closed convex sets in Euclidean Spaces R_m and R_n respectively. If U and V are two closed subsets of $S_m \times S_n$, such that

$$U_{\xi^{(o)}} \equiv \{\eta \in S_n : (\xi^{(o)}, \eta) \in U\}$$

and

$$V_{\eta^{(o)}} \equiv \{\xi \in S_m : (\xi, \eta^{(o)}) \in V\}$$

are both non-empty, closed and convex subsets of S_n and S_m respectively, for all $\xi^{(o)} \in S_m$ and $\eta^{(o)} \in S_n$, then $U \cap V \neq \emptyset$.

THEOREM 1. Let $\mathcal{G}$ be a two-person game with a finite number of pure strategies. Let $\mathcal{T}'$ be a family of mappings satisfying a) - c) above.[7] The game $\mathcal{G}'$ induced by $\mathcal{T}'$ has at least one saddle point.

PROOF. Let S_m and S_n be the probability spaces of players I and II respectively and let

$$U = \{(\xi, \eta_\xi) : B'(\xi, \eta_\xi) = \max_\eta B'(\xi, \eta)\}$$

and

[7] Property c) of the T mapping is invoked to guarantee that the image sets of a point to set mapping shall be convex as required by the Kakutani Fixed-Point Theorem. Property c) could be weakened appropriately and the Eilenberg-Montgomery Fixed Point Theorem utilized, (cf. Eilenberg, S., and Montgomery, D., [1] .) G. Debreu has recently proved a saddle-point theorem using the Eilenberg-Montgomery results (unpublished).

$$V = \{(\xi_\eta, \eta) : A'(\xi_\eta, \eta) = \max_{\xi} A'(\xi, \eta)\} .$$

That is, U is the set of elements (ξ, η) for which η is "good against" ξ relative to the game $\mathcal{G}'$; and V is the set of elements (ξ, η) for which ξ is "good against" η relative to the game $\mathcal{G}'$. Since the mapping $T[(A \times B)(\xi, \eta)]$ is continuous on $S_m \times S_n$ it follows that U is closed. Similarly V is closed. The set

$$U_{\xi^{(o)}} = \{\eta_{\xi^{(o)}} : B'(\xi^{(o)}, \eta_{\xi^{(o)}}) = \max_{\eta} B'(\xi^{(o)}, \eta)\}$$

is non-empty and closed because of the compactness of S_n and from continuity considerations. As regards convexity, let $\eta'_{\xi^{(o)}}$ and $\eta''_{\xi^{(o)}}$ both belong to $U_{\xi^{(o)}}$ and let $\max_{\eta} B'(\xi^{(o)}, \eta) = m_o$. Then $T[(A \times B)(\xi^{(o)}, \eta'_{\xi^{(o)}})] = T[(A \times B)(\xi^{(o)}, \eta''_{\xi^{(o)}})] = m_o$. From property c) of mapping T, and the restricted linearity of $A \times B$ it follows that $T[(A \times B)(\xi^{(o)}, \alpha\eta'_{\xi^{(o)}} + (1 - \alpha)\eta''_{\xi^{(o)}})] = m_o$ and thus $\alpha\eta'_{\xi^{(o)}} + (1 - \alpha)\eta''_{\xi^{(o)}}$ belongs to $U_{\xi^{(o)}}$. Similarly, the set

$$V_{\eta^{(o)}} = \{\xi_{\eta^{(o)}} : A'(\xi_{\eta^{(o)}}, \eta^{(o)}) = \max_{\xi} A'(\xi, \eta^{(o)})\}$$

is non-empty, closed and convex. Consequently, $U \cap V \neq \emptyset$. Let $(\xi^{(o)}, \eta^{(o)}) \in U \cap V$. Then $\xi^{(o)}$ is "good against" $\eta^{(o)}$ and $\eta^{(o)}$ is "good against" $\xi^{(o)}$ and therefore $(\xi^{(o)}, \eta^{(o)})$ is a saddle point. Q.E.D.

If $\mathcal{G}$ is a linear game then the appropriate T mapping for $\mathcal{G}$ is the identity mapping (follows from condition b); and, consequently, as a corollary to the above theorem we have that all finite linear games have a non-cooperative solution.

We will show that the family $\mathcal{J}'$ of T mappings satisfying properties a) - c) is non-empty by considering various specific $\mathcal{J}'$ conventions. If there exists a common unit of measurement and side payments are permissible then let $\mathcal{J}' = \{T\}$ where each $T : R_1 \longrightarrow M_{\mathcal{G}}$. The induced game $\mathcal{G}'$ is then a constant-sum game and by the same argument has a saddle point; again these saddle points form a permutable equivalence class which yields the arbitrated solution relative to the convention $\mathcal{J}'$.

3.2. If a slight perturbation of the payoff entries would result in a drastic change in the arbitrated solution, then the convention would not be stable enough to be practical. After all, the payoff entries are at best only hazy appraisals of the possible terminal outcomes of the game. The desire to capture this continuity notion motivates this section.

Let $\mathcal{G} = \{(A(\alpha, \beta), B(\alpha, \beta))\}$ be a game defined over the product

space of pure strategies $\mathcal{A} \times \mathcal{B}$. A sequence of games

$$\mathcal{G}_k = \{(A_k(\alpha,\beta),\ B_k(\alpha,\beta))\}$$

each defined over $\mathcal{A} \times \mathcal{B}$ is said to converge to the game $\mathcal{G}$ if for each $(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}$ it follows that $\lim_k (A_k(\alpha,\beta),\ B_k(\alpha,\beta)) = (A(\alpha,\beta),\ B(\alpha,\beta))$.

A sequence of sets $\{H_k\}$ in $E^{(2)}$ is said to converge to the set H provided for every $h \in H$, there exists a sequence of elements $h_k \in H_k$ such that $\lim h_k = h$; and, conversely, for every h' which belongs to the complement of H there is a neighborhood of h' which intersects at most a finite number of H_k.

If the sequence of games $\{\mathcal{G}_k\}$ converges to the game $\mathcal{G}$ then the sequences of sets R_0, R_1, R_2, R_3 of $\mathcal{G}_k$ converge to R_0, R_1, R_2, R_3 of $\mathcal{G}$ respectively. However, it does not follow that the sequences of sets M_0, M_1, M_2 of $\mathcal{G}_k$ converge to M_0, M_1, M_2 of $\mathcal{G}$ respectively.

CONDITION 3. If the sequence $\{\mathcal{G}_k\}$ converges to $\mathcal{G}$ such that $\{M_2(\mathcal{G}_k)\}$ converges to $M_2(\mathcal{G})$ then

$$\lim_k [V_1(\mathcal{G}_k,\mathcal{J}),\ V_2(\mathcal{G}_k,\mathcal{J})] = [V_1(\mathcal{G},\mathcal{J}),\ V_2(\mathcal{G},\mathcal{J})] .$$

An arbitration scheme $\mathcal{J}$ satisfying Condition 3 will be called <u>stable</u>.

From consideration of the sequences of games $\{\mathcal{G}_k\}$, $\{\mathcal{G}'_k\}$,

$$\mathcal{G}_k = \begin{Vmatrix} (0,\ 0) & (\tfrac{1}{k},\ 0) \\ (0,\ 1) & (0,\ 0) \end{Vmatrix} \qquad \mathcal{G}'_k = \begin{Vmatrix} (-\tfrac{1}{k},\ 0) & (-\tfrac{1}{k},\ 0) \\ (0,\ 1) & (-\tfrac{1}{k},\ 0) \end{Vmatrix}$$

which both approach the same game $\mathcal{G}$, one can readily see the dilemma to be faced in deciding on a convention with continuity properties. For this reason we require convergence of $\{M_2(\mathcal{G}_k)\}$ to $M_2(\mathcal{G})$ in Condition 3.

If the arbitration scheme $\mathcal{J}$ is to be identified with a family $\mathcal{J} = \{T\}$ of mappings, then stability of $\mathcal{J}$ requires a bond between the mappings T of the family .

3.3. The game $\mathcal{G}^t = \{(C(\beta,\alpha),\ D(\beta,\alpha))\}$ defined over $\mathcal{B} \times \mathcal{A}$ will be said to be the transpose of the game $\mathcal{G} = \{(A(\alpha,\beta),\ B(\alpha,\beta))\}$ defined over $\mathcal{A} \times \mathcal{B}$ provided that $C(\beta,\alpha) = B(\alpha,\beta)$ and $D(\beta,\alpha) = A(\alpha,\beta)$. Thus the transpose of a game results from an interchange of the roles of the players.

CONDITION 4. $[V_1(\mathcal{G},\mathcal{J}),\ V_2(\mathcal{G},\mathcal{J})] =$

$[V_2(\mathcal{G}^t,\mathcal{J}),\ V_1(\mathcal{G}^t,\mathcal{J})]$ for all $\mathcal{G}$ in the domain of the mapping $\mathcal{J}$. An arbitration scheme, $\mathcal{J}$, satisfying Condition 4 will be said to be symmetric.

3.4. We indicated in Section 1.2 that we would require our theory to be invariant up to various utility transformations, and, further, that we would distinguish between the cases of a common and a non-common unit of measurement. We first take up the case of the common unit of measurement.

Let U_o be a $1 \longleftrightarrow 1$ transformation of $E^{(2)}$ onto itself which takes the following form: for each U_o there exists a real number $\lambda > 0$ and two reals c_1, c_2 such that

$$U_o(x,y) = (\lambda x + c_1, \lambda y + c_2) .$$

A transformation U_o of the above form is said to be a utility transformation preserving the common unit.

CONDITION 5A. $[V_1(U_c\mathcal{G},\mathcal{J}),\ V_2(U_o\mathcal{G},\mathcal{J})] = U_o[V_1(\mathcal{G},\mathcal{J}),\ V_2(\mathcal{G},\mathcal{J})]$, for all $\mathcal{G}$ in the domain of the mapping $\mathcal{J}$, and all utility transformations, U_c, preserving the common unit; where

$$U_o\mathcal{G} \equiv U_c\{(A(\alpha,\beta),\ B(\alpha,\beta))\} \equiv \{U_o(A(\alpha,\beta),\ B(\alpha,\beta))\} .$$

An arbitration mapping $\mathcal{J}$ satisfying Condition 5A is said to be invariant up to a common unit of measurement.

3.5. We shall confine ourselves to the discussion of conventions for which the arbitrated solution is invariant with respect to any permutations of the pure strategies of $\mathcal{A}$ and $\mathcal{B}$. All conventions will be understood to have this invariance property. If permutations [by a permutation of a (possibly) infinite set of elements we mean a 1-1 mapping of the set onto itself] of the elements of $\mathcal{A}$ and $\mathcal{B}$ send α into α^* and β into β^* then we shall let the game induced by these permutations be

$$\mathcal{G}^* = \{(A^*(\alpha^*,\beta^*),\ B^*(\alpha^*,\beta^*))\} = \{(A(\alpha,\beta),\ B(\alpha,\beta))\} .$$

CONDITION 6. If $\mathcal{G}^*$ is a game induced from $\mathcal{G}$, defined over $\mathcal{A} \times \mathcal{B}$, by permutations of the elements of $\mathcal{A}$ and $\mathcal{B}$ then

$$[V_1(\mathcal{G}^*,\mathcal{J}),\ V_2(\mathcal{G}^*,\mathcal{J})] = [V_1(\mathcal{G},\mathcal{J}),\ V_2(\mathcal{G},\mathcal{J})] .$$

A game $\mathcal{G}$ with a common unit of measurement is said to be symmetric if there exist permutations of $\mathcal{A}$ and $\mathcal{B}$ inducing the game $\mathcal{G}^*$ and a utility transformation U_0 such that $U_0\mathcal{G}^*$ is its own transpose. If $\mathcal{G}$ is symmetric then there exists a value c such that the regions R_0, R_1, R_2 of $\mathcal{G}$ are symmetric about the line through $(0,c)$ with direction numbers $(1,1)$. For example, the game

$$\mathcal{G}_3 \equiv \begin{array}{c|cc} & \beta_1 & \beta_2 \\ \alpha_1 & (2,\ 0) & (-10,\ -11) \\ \alpha_2 & (-10,\ -11) & (1,\ 1) \end{array}$$

is symmetric (permute (α_1,α_2) to (α_2,α_1) and let $U_0(x,y) = (x - 1,y)$). For any arbitration scheme $\mathcal{J}$ satisfying Conditions 1, 4, 5A, 6, the arbitrated value for $\mathcal{G}_3$ is $(3/2,1/2)$.

It should be noted that a game $\mathcal{G}$ may be asymmetric although R_0, R_1, and R_2 of $\mathcal{G}$ may be symmetric about some line with direction numbers $(1,1)$. An example of such a game is $\mathcal{G}_1$ of Section 2.1 whose region R_1 is given in Figure 1.

3.6. In this section we shall give an example of an arbitration scheme, $\mathcal{J}_0$, satisfying Conditions 1, 2, 3, 4, 5A, and 6. We shall identify $\mathcal{J}_0$ with a family of mappings $\{T_0\}$ satisfying a) - c) of Section 3.1. We shall first characterize the arbitrated solution for the finite case and then indicate some generalizations to the infinite case.

Let $\mathcal{G}$ have regions R_0, R_1, R_2. Corresponding to the game $\mathcal{G}$ we shall define the point to point mapping $T_0 : R_1 \longrightarrow M_2$. First define T_0 for a subset Q of R_2. Let $e \in Q$ if there exists an $e' \in M_2$ such that the ray e to e' has direction numbers $(1,1)$ and then define $T_0(e) = e'$. Now extend the mapping T_0 from Q to R_2 by requiring that for every $e' \in M_2$ the set $\{e \in R_2 : T_0(e) = e'\}$ is connected. This extension is unique and thus T_0 is uniquely defined on R_2. Now restrict the domain to R_1 and note that T_0 satisfies conditions a) - c) of Section 3.1.

Let us consider the zero-sum game with payoff $K(\alpha,\beta) = A(\alpha,\beta) - B(\alpha,\beta)$ and let us assume that this game has a minimax solution $(v, -v)$ with generic optimal strategies $\xi^{(0)}$ and $\eta^{(0)}$. By the linearity of operators A and B we then have $A(\xi^{(0)},\eta) - B(\xi^{(0)},\eta) \geq v$ for all η and $A(\xi,\eta^{(0)}) - B(\xi,\eta^{(0)}) \leq v$ for all ξ, and $A(\xi^{(0)},\eta^{(0)}) = B(\xi^{(0)},\eta^{(0)}) + v$. Hence, all elements of the form

$$(A(\xi^{(0)},\eta^{(0)}),\ B(\xi^{(0)},\eta^{(0)}))$$

where $(\xi^{(o)},\eta^{(o)})$ are pairs of optimal strategies of $K(\alpha,\beta)$, lie on a ray r_o, with direction numbers $(1,1)$. By referring to Figure 2 it should be clear that depending on the value of v, the ray r_o may or may not intersect M_2.

If we define $T^{(1)}$ and $T^{(2)}$ (corresponding to T) by the relation $T(A \times B)(\xi,\eta) = [T^{(1)}(A \times B)(\xi,\eta), T^{(2)}(A \times B)(\xi,\eta)]$ and if $r_o \cap M_2 \neq \emptyset$ then for the mapping T_o we have

$$T_o^{(1)}(A \times B)(\xi^{(o)},\eta) - T_o^{(2)}(A \times B)(\xi^{(o)},\eta) \geq v, \quad \text{all} \quad \eta$$

and

$$T_o^{(1)}(A \times B)(\xi,\eta^{(o)}) - T_o^{(2)}(A \times B)(\xi,\eta^{(o)}) \leq v, \quad \text{all} \quad \xi.$$

Consequently if $r_o \cap M_2 \neq \emptyset$, then the intersection defines some element, say (m_1,m_2), which is the arbitrated solution of the game $\mathcal{G}$ relative to the convention $\mathcal{J}_o$. By choice of a $\xi^{(o)}$, player I gets at least m_1 and by choice of an $\eta^{(o)}$, player II gets at least m_2. By the incomparability of the elements of M_2, this is an equilibrium situation. If $r_o \cap M_2 = \emptyset$ then the arbitrated solution is one of the end points of the continuum M_2 which is closest to the ray r_o.

For all finite games the above characterization determines the value $[V_1(\mathcal{G},\mathcal{J}_o), V_2(\mathcal{G},\mathcal{J}_o)]$ and in these cases the value of the corresponding zero-sum game can be actually computed.

The above characterization for $\mathcal{J}_o$ can now be generalized to a wide class of infinite games provided the zero-sum game $K(\alpha,\beta) = A(\alpha,\beta) - B(\alpha,\beta)$ is strictly determined in the sense of Wald [12] or Karlin [3]. In general, the arbitrated solution will not be attainable; however, the solution (if it exists) can be approximated arbitrarily close by the players using a joint strategy which utilizes at most two pairs of pure strategies.

We have already indicated that a family of mappings satisfying a) - c) can be identified with an arbitration scheme satisfying Conditions 1 and 2. To verify Condition 3 let $\mathcal{G}_k = \{(A_k(\alpha,\beta), B_k(\alpha,\beta))\}$ approach $\mathcal{G} = \{(A(\alpha,\beta), B(\alpha,\beta))\}$. If v_k is the value of the zero-sum game $A_k(\alpha,\beta) - B_k(\alpha,\beta)$ then we have $\lim v_k = v$ and if we hypothesize $\{M_2(\mathcal{G}_k)\}$ approaches $M_2(\mathcal{G})$ then stability of $\mathcal{J}_o$ follows from the characterization of the arbitrated solution. To verify Condition 4 we note that if $K(\alpha,\beta)$ is the zero-sum game associated with $\mathcal{G}$ and $K^t(\alpha,\beta)$ is the zero-sum game associated with $\mathcal{G}^t$ then $K^t(\beta,\alpha) = -K(\alpha,\beta)$ and consequently the value of the game $K^t(\beta,\alpha)$ is $(-v,v)$. If we now take the reflection of $R_2(\mathcal{G})$ with respect to the line $y = x$ then we get the region $R_2(\mathcal{G}^t)$. The ray r_o through (v,o) reflects into the ray which goes through $(-v,o)$,

still with slope +1. Again from the characterization of the arbitrated solution relative to $\mathcal{J}_0$ we have indicated the validity of Condition 4. If we apply the utility transformation U_0 to $R_2(\mathcal{G})$ then this region is translated and stretched but not rotated. The transformed ray $U_0(r_0)$ still has slope +1 and an x-intercept which corresponds to the value of the zero-sum game induced by $U_0\mathcal{G}$. Hence Condition 5A is valid. Condition 6 follows from knowledge that both $R_2(\mathcal{G})$ and the value of the zero-sum game, $K(\alpha,\beta)$, remain invariant with respect to permutations of pure strategies.

We now consider a further rationalization (metatheoretic) for convention $\mathcal{J}_0$ besides the ones already mentioned.

We first note that the ray r_0 partitions R_2 into two regions denoted by Q_1 and Q_2 in the accompanying figure, viz:

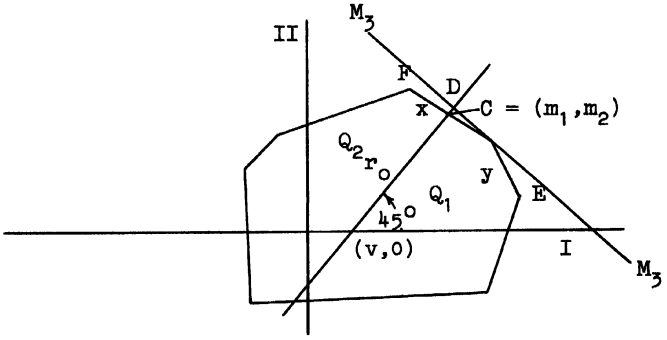

Figure 2

Let us consider the actual bargaining situation involved in this game. If both players are aware of the region R_2 and agree to confine their cooperative behavior to consideration only of elements of M_2, then player II desires point F and player I desires point E. In the ensuing argument each player intimidates his opponent by threatening to play non-cooperatively unless his opponent will agree to his terms. In jockeying about for bargaining advantage a threat by player I to minimize player II's maximum possible gain is not very convincing if at the same time player I has to suffer "more than" player II. For example, if player II desires point x of M_2 (cf. figure) then player I can choose a strategy $\xi^{(o)}$ which will guarantee an imputation on r_0 or in Q_1. Consequently, relative to the point x player II has "more to lose" than player I in a threat of this variety.[8] If player I wishes point y then by the choice of an $\eta^{(o)}$

[8]Even though we postulate a common unit of measurement there is no possibility of formalizing these statements within the undefined terms of the

player II can confine the result of the non-cooperative game to r_0 or Q_2 and thus relatively inflict "a heavier loss" on player I than upon himself.[8] Thus in the argumentative stage the players threaten to play the non-cooperative relative game -- i.e., they consider entries of the form $\{A(\alpha,\beta) - B(\alpha,\beta)\}$. At point C neither player has a bargaining advantage and the corresponding imputation is the arbitrated solution relative to the convention $\mathcal{J}_0$.

In the case side payments are permissible then the arbitrated solution relative to $\mathcal{J}_0$ has a particularly simple characterization, viz:

$$V_1(\mathcal{G},\mathcal{J}_0) - V_2(\mathcal{G},\mathcal{J}_0) = \text{value of zero-sum game } A(\alpha,\beta) - B(\alpha,\beta)$$

$$V_1(\mathcal{G},\mathcal{J}_0) + V_2(\mathcal{G},\mathcal{J}_0) = \sup_{(\alpha,\beta)} \{A(\alpha,\beta) + B(\alpha,\beta)\} .$$

Returning to game $\mathcal{G}_1$ of Section 2.1, we have

$$[V_1(\mathcal{G}_1,\mathcal{J}_0),\ V_2(\mathcal{G}_1,\mathcal{J}_0)] = (2,1) .$$

In this case player I has the bargaining advantage. We also notice that in $\mathcal{G}_1$, α_1 [β_1] is the Bayes solution against the maximin strategy $\eta^{(o)}$ [$\xi^{(o)}$] of player II [I] and further (α_1,β_1) is a saddle point yielding $(2,1)$.

3.7. If the game $\mathcal{G}$ is to be interpreted without a common unit of measurement then we wish the theory to be invariant up to independent linear transformations on the payoff entries. To this end, let U_1 be a $1 \longleftrightarrow 1$ transformation of $E^{(2)}$ onto itself which takes the following form: for each U_1, there exists positive real numbers λ_1, λ_2 and arbitrary reals c_1, c_2 such that

$$U_1(x,y) = (\lambda_1 x + c_1, \lambda_2 y + c_2) .$$

CONDITION 5B. $[V_1(U_1\mathcal{G},\mathcal{J}),\ V_2(U_1\mathcal{G},\mathcal{J})] = U_1[V_1(\mathcal{G},\mathcal{J}),\ V_2(\mathcal{G},\mathcal{J})]$ for all $\mathcal{G}$ in the domain of the mapping $\mathcal{J}$, and all utility transformations U_1.

A game $\mathcal{G}$, understood in the sense of a non-common unit, is said to be symmetric provided there exist permutations of the elements of $\mathcal{A}$ and $\mathcal{B}$, inducing game $\mathcal{G}^*$, and a utility transformation U_1 such that

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utility theory mentioned in Section 1.2. We have gone outside our formal framework to obtain a rationalization of a procedure satisfying our formal requirements. A realistic realization of this rationalization occurs in the case of the monetary payoff.

$U_1\mathcal{G}^*$ is its own transpose. All arbitration schemes $\mathcal{J}$ satisfying Conditions 1, 2, 3, 4, 5B, 6 arbitrate symmetric games in the same way.

3.8. We shall now exhibit a family $\mathcal{J}_1 = \{T_1\}$ of mappings such that the corresponding arbitration scheme satisfies Conditions 1, 2, 3, 4, 5B, 6. The writer's motivation for this family of mappings will be given in Section 3.10.[9]

Let $\mathcal{G}$ be any game with a finite number of pure strategies. Define the mapping $T_1 : R_1(\mathcal{G}) \longrightarrow M_2(\mathcal{G})$ as follows: First define T_1 for a subset $Q \subset R_2$. Let $e \in Q$ if there exists an $e' \in M_2$ and $e' \notin R_0$ such that the ray from e to e' is the negative of the slope of M_2 at e'. Then let $T_1(e) = e'$. Now extend T_1 to all of R_2 such that T_1 is continuous and the inverse image of any element of M_2 is a connected point set. This extension is unique and thus T_1 is uniquely defined on R_2. Now restrict the domain of T_1 to R_1.

Since the negative of the slope of any segment of M_2 is positive, Condition b) of Section 3.1 is satisfied. Conditions a) and c) are obviously satisfied. Hence we can utilize Theorem 1 to show the existence of a saddle point. Since all saddle points form a permutable equivalence class we define the common value to be $[V_1(\mathcal{G},\mathcal{J}_1),\ V_2(\mathcal{G},\mathcal{J}_1)]$.

In order to prove that $\mathcal{J}_1$ satisfies the necessary desiderata we shall first find a characterization of the above arbitrated solution.[10] Let $P_1, P_2, \ldots, P_t$ be the vertices of $M_2(\mathcal{G})$ where $P_i = (c_i,d_i)$ and $c_1 < c_2 < \cdots < c_t$. Let the open one-simplex (P_{i-1},P_i) be denoted by σ_i and let μ_i be the slope of σ_i. Thus $\mu_i < 0$. Let v_i be the value of the zero-sum game $[A(\alpha,\beta) + \frac{1}{\mu_i} B(\alpha,\beta)]$. Hence there exist optimal strategies $\xi^{(i)}$ and $\eta^{(i)}$ such that

$$A(\xi^{(i)},\eta) + \frac{1}{\mu_i} B(\xi^{(i)},\eta) \geqq v_i, \quad \text{all} \quad \eta \in S_{II}, \tag{3.8.1}$$

$$A(\xi,\eta^{(i)}) + \frac{1}{\mu_i} B(\xi,\eta^{(i)}) \leqq v_i, \quad \text{all} \quad \xi \in S_I ; \tag{3.8.2}$$

or since $\mu_i < 0$,

$$B(\xi^{(i)},\eta) \leqq -\mu_i A(\xi^{(i)},\eta) + \mu_i v_i, \quad \text{all} \quad \eta \in S_{II}, \tag{3.8.3}$$

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[9] J. F. Nash, Jr. has utilized the same family of mappings.

[10] In Raiffa [10] scheme $\mathcal{J}_1$ is reduced by successive utility transformations to considerations of scheme $\mathcal{J}_0$. The following procedure is different and proceeds in a manner suggested by R. M. Thrall.

(3.8.4) $\quad B(\xi,\eta^{(i)}) \geqq -\mu_i A(\xi,\eta^{(i)}) + \mu_i v_i, \quad \text{all} \quad \xi \in S_I .$

Conversely, if there exist $\xi^{(i)}$ and $\eta^{(i)}$ satisfying the equations

(3.8.5) $\quad B(\xi^{(i)},\eta) \leqq -\mu_i A(\xi^{(i)},\eta) + \delta, \quad \text{all} \quad \eta \in S_{II}$

(3.8.6) $\quad B(\xi,\eta^{(i)}) \geqq -\mu_i A(\xi,\eta^{(i)}) + \delta, \quad \text{all} \quad \xi \in S_I$

then $\xi^{(i)},\eta^{(i)}$ are optimal strategies and δ/μ_i is the value of the game $\{A(\alpha,\beta) + \frac{1}{\mu_i} B(\alpha,\beta)\}$.

Let r_i be the line $y = -\mu_i x + \mu_i v_i$. We shall now cite three mutually exhaustive cases.

CASE 1. a) If $[V_1(\mathcal{G},\mathcal{J}_1),\ V_2(\mathcal{G},\mathcal{J}_1)] \in \sigma_i$ then $r_i \cap \sigma_i = \{[V_1(\mathcal{G},\mathcal{J}_1),\ V_2(\mathcal{G},\mathcal{J}_1)]\}$.

b) If $r_i \cap \bar{\sigma}_i \neq \emptyset$ then $r_i \cap \bar{\sigma}_i = \{[V_1(\mathcal{G},\mathcal{J}_1),\ V_2(\mathcal{G},\mathcal{J}_1)]\}$.

Regarding a): By definition of the saddle point of the game $\{T_1\ [A(\xi,\eta),\ B(\xi,\eta)]\}$ we can verify that there exist a $\xi^{(i)}$ and $\eta^{(i)}$ satisfying 3.8.5 and 3.8.6 with $\delta = V_2(\mathcal{G},\mathcal{J}_1) + \mu_i V_1(\mathcal{G},\mathcal{J}_1)$. Hence we conclude $T_1^{-1}\ [V_1(\mathcal{G},\mathcal{J}_1),\ V_2(\mathcal{G},\mathcal{J}_1)] \subset r_i$.

Regarding b): If $r_i \cap \bar{\sigma}_i \neq \emptyset$ then player I [II] can confine player II [I] to the region R_1 below [above] or on r_i; hence b) follows.

CASE 2. $\{[V_1(\mathcal{G},\mathcal{J}_1),\ V_2(\mathcal{G},\mathcal{J}_1)]\} = P_i$ $(i = 2, 3, \ldots, t-1)$ if and only if there exist no element of σ_i below r_i and no element of σ_{i+1} above r_{i+1}.

The element P_i is a solution of the arbitrated game relative to $\mathcal{J}_1$ if and only if in the non-cooperative game player I can confine player II (cf. Figure 3) in the region on or below line f and player II can confine player I in the region on or above line g. Noting that r_i is parallel to f and r_{i+1} is parallel to g, Case 2 follows.

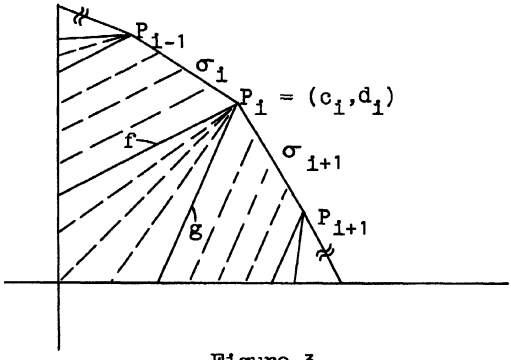


Figure 3

CASE 3. a) $[[V_1(\mathcal{G},\mathcal{J}_1), V_2(\mathcal{G},\mathcal{J}_1)]] = P_1$ if and only if no point of M_2 lies above r_1.

b) $[[V_1(\mathcal{G},\mathcal{J}_1), V_2(\mathcal{G},\mathcal{J}_1)]] = P_t$ if and only if no point of M_2 lies below r_{t-1}.

For player II to get the arbitrated solution value of d_1 it is necessary and sufficient that he have available a strategy $\eta^{(1)}$ which confines the returns of the players in the non-cooperative game on or above the line through P_1 with slope $-\mu_2$. This is possible if and only if no point of M_2 lies above r_1. Part b) is similarly proved.

Conditions 1, 2, and 6 are immediately satisfied. If $\{\mathcal{G}_k\}$ approaches $\mathcal{G}$ then the lines $r_2, \ldots, r_t$ defined above for game approach the lines $r_2^\kappa, \ldots, r_t^\kappa$ for the game $\mathcal{G}$. Since we have characterized the arbitrated solution in terms of these lines and the maximal elements, stability of $\mathcal{J}_1$ follows. Consider the game $\mathcal{G}$ with region R_1 decomposed into equivalence classes induced by mapping T_1 (e is equivalent to e' if $T(e) = T(e')$). The region R_1 of $\mathcal{G}^t$ together with its equivalence classes is the reflection of $R_1(\mathcal{G})$ about the line $y = x$ since a line with slope m gets mapped by the reflection into a line with slope $\frac{1}{m}$. From this, symmetry of the mapping $\mathcal{J}_1$ follows. Finally if $U_1(x,y) = (\lambda_1 x + c_1, \lambda_2 x + c_2)$ then a line with slope m gets mapped by U_1 into a line of slope $\frac{\lambda_2}{\lambda_1} m$. Hence if the slope of one line is the negative of the slope of another line, then this property is preserved under utility transformations. From this it follows that equivalence classes of $R_1(\mathcal{G})$ get mapped by U_1 into equivalence classes of $R_1(U_1\mathcal{G})$ thus giving Condition 5B. We have thus verified that $\mathcal{J}_1$ satisfies our imposed conditions.

If we arbitrate $\mathcal{G}_2$ of Section 3.1 we get

$$[V_1(\mathcal{G}_2,\mathcal{J}_1), V_2(\mathcal{G}_2,\mathcal{J}_1)] = (1.005, 9.995) .$$

If, however, we can interpret $\mathcal{G}_2$ with a common unit of measurement then

$$[V_1(\mathcal{G}_2,\mathcal{J}_0), V_2(\mathcal{G}_2,\mathcal{J}_0)] = (1.82, 1.82) .$$

3.9. In order to show that there exist other arbitration conventions satisfying our conditions we will exhibit another scheme,[11] $\mathcal{J}_2$. Let $\mathcal{G} = \{(A(\alpha,\beta), B(\alpha,\beta))\}$ and define $c_1 = \inf_{(\alpha,\beta)} A(\alpha,\beta)$, $\frac{1}{\lambda_1} = \sup_{(\alpha,\beta)} [A(\alpha,\beta) - c_1]$,

[11] The following scheme has some serious objections in that it is not sensitive enough to intermediate values of the payoff entries. The problem is open how to impose a further condition which would rule out $\mathcal{J}_2$ but not uniquely imply $\mathcal{J}_1$.

$c_2 = \inf_{(\alpha,\beta)} B(\alpha,\beta)$, $\frac{1}{\lambda_2} = \sup_{(\alpha,\beta)} [B(\alpha,\beta) - c_2]$. We explicitly assume that c_1 and c_2 are finite. Since in the case that $\sup [A(\alpha,\beta) - c_1]$ or $\sup [B(\alpha,\beta) - c_2]$ equals zero the game is trivial, we will rule out this possibility. Now for the game, g, define the normalizing utility transformation $\tilde{U}(x,y) = (\lambda_1(x - c_1), \lambda_2(y - c_2))$. The utility transformation $\tilde{U}$ normalizes the players' entries in the sense that (if the infs and sups are attained) the entries 1 and 0 refer to the maximum and minimum that each player can possibly get from the game. We now shall define the scheme $\mathcal{J}_2$, whose domain consists of all g such that $\tilde{U}g$ is in the domain of the mapping $\mathcal{J}_0$. In particular, let[12]

$$[V_1(g,\mathcal{J}_2),\ V_2(g,\mathcal{J}_2)] = \tilde{U}^{-1}\ [V_1(\tilde{U}g,\mathcal{J}_0),\ V_2(\tilde{U}g,\mathcal{J}_0)]\ ,$$

where $\tilde{U}$ is the normalizing transformation which depends on g. The troublesome conditions to verify are Conditions 4 and 5B.

To prove symmetry of $\mathcal{J}_2$, let $\tilde{U}_t$ be the normalizing transformation g^t and let $W(x,y) = (y,x)$. It is then easy to verify the following identities: $\tilde{U}_t = W\,\tilde{U}\,W$ or $\tilde{U}_t^{-1} = W\,\tilde{U}^{-1}\,W$ and $(\tilde{U}g)^t = \tilde{U}_t g^t$. We then have

$$\begin{aligned}
[V_1(g^t,\mathcal{J}_2),\ V_2(g^t,\mathcal{J}_2)] &= \tilde{U}_t^{-1}\ [V_1(\tilde{U}_t g^t,\mathcal{J}_0),\ V_2(\tilde{U}_t g^t,\mathcal{J}_0)] \\
&= W\,\tilde{U}^{-1} W\ [V_1((\tilde{U}g)^t,\mathcal{J}_0),\ V_2((\tilde{U}g)^t,\mathcal{J}_0)] \\
&= W\,\tilde{U}^{-1}\ [V_1(\tilde{U}g,\mathcal{J}_0),\ V_2(\tilde{U}g,\mathcal{J}_0)] \\
&= W\ [V_1(g,\mathcal{J}_2),\ V_2(g,\mathcal{J}_2)] \\
&= [V_2(g,\mathcal{J}_2),\ V_1(g,\mathcal{J}_2)]\ .
\end{aligned}$$

To prove Condition 5B, let U_1 be an arbitrary utility transformation of g and let $\tilde{U}$ and $\tilde{U}_1$ be the normalizing utility transformations of g and $U_1 g$ respectively. Noting that $\tilde{U} = \tilde{U}_1\,U_1$ and $\tilde{U}_1^{-1} = U_1\,\tilde{U}^{-1}$ we then have

$$\begin{aligned}
[V_1(U_1 g,\mathcal{J}_2),\ V_2(U_1 g,\mathcal{J}_2)] &= \tilde{U}_1^{-1}\ [V_1(\tilde{U}_1 U_1 g,\mathcal{J}_0),\ V_2(\tilde{U}_1 U_1 g,\mathcal{J}_0)] \\
&= U_1\tilde{U}^{-1}\ [V_1(\tilde{U}g,\mathcal{J}_0),\ V_2(\tilde{U}g,\mathcal{J}_0)] \\
&= U_1\ [V_1(g,\mathcal{J}_2),\ V_2(g,\mathcal{J}_2)]\ .
\end{aligned}$$

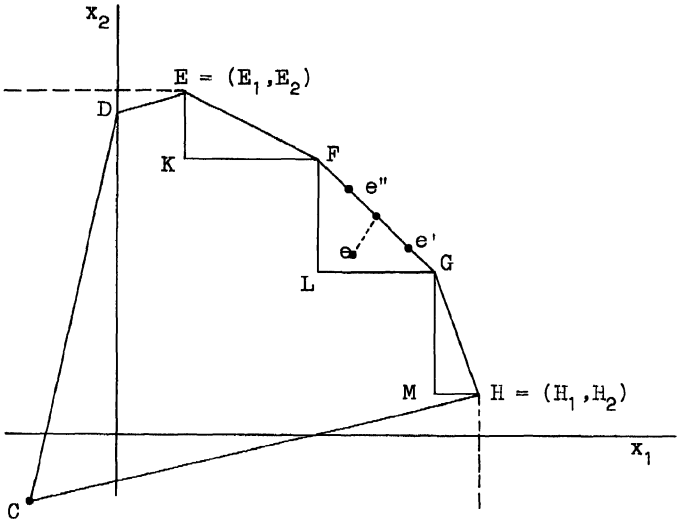
---

[12] We can also define a family of mappings $\mathcal{J}_2 = \{T_2\}$ where $T_2(e) = \tilde{U}^{-1} T_0 \tilde{U}(e)$.

If we arbitrate g_2 of Section 3.1 by convention $\mathcal{J}_2$ we get the solution (4/3, 200/3) which seems (to the writer!) more realistic than the value (1.005, 9.995).

3.10. The author's original motivation for convention $\mathcal{J}_1$ stems from a linear approximation of the non-linear family of mappings to be indicated in this section.

Let us consider a game g with an R_2 of the following form (R_2 is bounded by (C, D, E, F, G, H)):



Let the locus of $M_2(g)$ be given by $(x_1, f(x_1))$ for $E_1 \leq x_1 \leq H_1$ or $(f^{-1}(x_2), x_2)$ for $H_2 \leq x_2 \leq E_2$. Consider a point $e = (e_1, e_2)$ in the subregion (F, L, G). We wish to map e into some element of the segment $e' = (f^{-1}(e_2), e_2)$ to $e'' = (e_1, f(e_1))$. If we desire a symmetric convention it is quite natural to map e into

$$\frac{e' + e''}{2} = \left(\frac{f^{-1}(e_2) + e_1}{2} , \frac{e_2 + f(e_1)}{2} \right) .$$

It is also evident that this mapping will be invariant with respect to utility transformations. With this rationale we can now define the mapping from subregions (E, K, F), (F, L, G) and (G, M, H) onto the maximal elements. In the family of mappings $\mathcal{J}_1$, we merely extend linearly the linear equivalence classes of the above subregions. Now instead of a linear extension we propose the following: decompose the region R_2 into equivalence classes through the medium of the differential equation

$$\frac{dx_2}{dx_1} = \frac{f(x_1) - x_2}{f^{-1}(x_2) - x_1}$$

where we use the convention that $f(x_1) = E_2$ for $x_1 \leq E_1$ and $f^{-1}(x_2) = H_1$ if $x_2 \leq H_2$. Notice that the induced mapping agrees with the mapping $\mathcal{J}_1$ on regions (E, K, F), (F, L, G) and (G, M, H). It is obvious that the mapping via the differential equation can be generalized to a wide class of games with an infinite number of pure strategies. If the region R_2 is bounded, the elements E and H are the non-cut points of the continuum $M_2(g)$. Although this family of mappings is perhaps more realistic (subjective opinion) than the families $\mathcal{J}_1$ and $\mathcal{J}_2$ it has the serious handicap that the equivalence classes of R_2 are in general non-linear. In the existence theorem, this non-linearity is reflected in non-convexity of certain sets which prohibits the use of the Kakutani Fixed Point Theorem.[13] The existence of an arbitrated solution for this non-linear convention, with the support of a computational technique will be left as an open question in this paper.

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