SYSTEMS AND ROOTS

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BY

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Preface

This work is a revision of the author's "Differential Systems," which was published in 1937 by the American Mathematical Society in its colloquium series. Except for sections (231), (232), (240), which are the chief product of the 1957 summer spent with the International Business Machines Corporation in Poughkeepsie, N. Y., the scope of the text has been purposely and rigidly maintained. The present form is the result of reworking the details for presentation in class over 25 years. The bibliography, originally limited to works directly cited, has been brought up to date and also made much more extensive, although by no means exhaustive. Much thought has been given to the notation, which is judged to have advantages compensating its unusual nature.

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SYSTEMS AND ROOTS

CHAPTER

1

Preliminaries

(1) Sets. The sets f, g, h are manipulated by Boolean algebra, some of whose useful relations follow.

$$(f^{\vee})^{\vee} = f$$

$$0^{\vee} = 1 \qquad 1^{\vee} = 0$$

$$f + 0 = f \qquad f1 = f$$

$$f + 1 = 1 \qquad f0 = 0$$

$$f + f^{\vee} = 1 \qquad ff^{\vee} = 0$$

$$f + f = f \qquad ff = f$$

$$f + g = g + f \qquad fg = gf$$

$$(f + g) + h = f + (g + h) \qquad (fg)h = f(gh)$$

$$f(g + h) = fg + fh \qquad f + gh = (f + g)(f + h)$$

$$(f + g)^{\vee} = f^{\vee}g^{\vee} \qquad (fg)^{\vee} = f^{\vee} + g^{\vee}$$

$$0 \leqq f \qquad f \leqq 1$$

$$f \leqq f$$

$$f \leqq g \quad \text{and} \quad g \leqq f \rightarrow f = g$$

$$f \leqq g \quad \text{and} \quad g \leqq h \rightarrow f \leqq h$$

$$f \leqq g \rightleftarrows f + g = g \qquad g \leqq f \rightleftarrows fg = g$$

$$f \leqq g \rightleftarrows g^{\vee} \leqq f^{\vee}$$

The signs are read as follows:

	$+$	"union"
product or	$\cdot$	"intersection"
	N	"complement"
	$=$	"equals"
	$\leq$	"is in"
	$\rightarrow$	"implies"
	$\leftarrow$	"is implied by"
	$\rightleftarrows$	"is equivalent to"

Symbols 0, 1 are here not numbers but the empty and the all-inclusive set, respectively. It is convenient to tolerate this ambiguity of notation.

Two relations on the same line of the above table are (2) *dual.* A relation written alone is (3) *self-dual.* Simultaneous interchange of

$$0 \text{ and } 1$$
$$+ \text{ and } \cdot$$

in the members of each relation interchanges dual relations and leaves invariant self-dual relations.

(4) Members. A (5) *point* is $(a_1, \cdots, a_n)$, where the a's are numbers. Since the chief applications here are to the field of analytic functions, it is perhaps best to think of the numbers as complex. For many of the results an integrity domain suffices as the number source. At the moment the essential is that numbers 0, 1 have their usual properties, in particular,

$$1 \neq 0, \qquad 0f = 0, \qquad fg = 0 \rightleftarrows f = 0 \quad \text{or} \quad g = 0.$$

At each point each function equals a number, unless otherwise noted.

Function $f(y_1, \cdots, y_n)$ determines an equation $f(y_1, \cdots, y_n) = 0$, for convenience to be called (6) *equation* f, and an inequation $f(y_1, \cdots, y_n) \neq 0$, to be called (7) *inequation* f^N.

A product of functions each of which is labeled equation or inequation is a (8) *member.*

Point $(a_1, \cdots, a_n)$ is (9) *root* of equation f or of inequation f^N according as $f(a_1, \cdots, a_n)$ is 0 or not; and of a member if and only if it is root of one of the factors of that member.

(10) Systems. A (11) *system* S is a set of members. A root of S is a root of all its members. The (12) *solution* of S is the set s of all its roots.

Since both systems and solutions are sets, the Boolean laws are directly applicable to them. In the case of solutions, the meanings given above will be adopted for the signs.

If for systems also $+$ has its set-theoretic meaning and if the solutions of S_1, S_2 are respectively s_1, s_2, the solution of $S_1 + S_2$ is $s_1 s_2$. This suggests that the solution of $S_1 S_2$ be made $s_1 + s_2$, as can be done by stipulating that the relations among systems are had by dualizing the relations among their solutions.

Since the set of relations in (1) satisfied by the solutions is invariant under dualization, the relations in (1) are thus imposed on systems, the meaning of each sign to be had from the dual relation between the solutions.

Thus for systems S_1, S_2 the relation

$$S_2 \leqq S_1 ,$$

means

$$s_1 \leqq s_2 ,$$

that is, "the solution of S_1 is in the solution of S_2", or rephrased again, "every root of S_1 is a root of S_2". The original relation between systems is read "S_1 implies S_2". It is convenient to note that, except in the case of equality, the larger number of essential members is on the right of $S_2 \leqq S_1$ and the larger number of roots on the right of the dual $s_1 \leqq s_2$.

Various consequences of the definition just given are to be kept in mind.

The equation $S_1 = S_2$ means $s_1 = s_2$.

System S^N has for solution the complement of s, that is, every point not a root of S.

In the set relations applied to systems 1 represents the system with no root and 0 the system with every point for root.

Since $+$ means union for systems, write

$$S = f_1 + \cdots + f_n ,$$

where f_i means the system composed of the single member f_i. Where confusion might result, a notation such as (f) can be used to distinguish system (f) from member f. The set f_1, f_2, $\cdots$ on the right is a *base* for S.

The system which consists of one non-zero member f equals that which consists of the infinite set $f, f^2, f^3, \cdots$ because both have the same solution. Hence there is the equality

$$f = f + f^2 + f^3 + \cdots$$

so that every system equals a system with an infinite number of formally different members and has an infinite base.

For simplicity it is supposed that each system considered here has a finite base.

If S is a system and f a function, then

$$S = (S + f)(S + f^N).$$

Consequently, it is always possible to factor a system and all that follows depends in large measure upon such factorization.

Multiplication verifies that

$$S + fg^N = (S + f)(S + f^N + g^N)$$

and that

$$S + fg_1^N \cdots g_k^N = (S + f)(S + f^N + g_1^N)(S + f^N + g_1 + g_2^N) \cdots \cdot (S + f^N + g_{k-1} + g_k^N).$$

Consequently, every system is the product of systems whose only members are equations and inequations. For convenience of language and notation it is generally assumed that the system under consideration has only equations and inequations for members.

(13) Ordering. Set I is (14) *ordered* if for each pair i, j in I exactly one of the three relations

$$i < j, \qquad j < i, \qquad i = j$$

is true, the sign $=$ being given its set-theoretic meaning "identical with" and $<$ being required only to have the transitive property

$$i < j, \quad j < k \quad \rightarrow \quad i < k.$$

Relation $i < j$ is read either "i precedes j" or "j follows i". The reversed sign is not used.

Once I has been ordered, it can be used to order set J whose elements are in one-to-one correspondence with those of I. The elements of I are then (15) *ordinals* and ordering J amounts to assigning each of its elements an ordinal from I.

The positive integers are ordered by reading $<$ "is less than". A countable set can be ordered by taking the positive integers as ordinals. This writes the set as a sequence.

If in the symbol $i_1 \cdots i_n$ each component i_k is in ordered set I_k, equality

$$i_1 \cdots i_n = j_1 \cdots j_n$$

is to mean equality of corresponding components

$$i_1 = j_1 , \cdots , i_n = j_n$$

and precedence

$$i_1 \cdots i_n < j_1 \cdots j_n$$

the existence of a positive integer m such that the first $m - 1$ i's equal their correspondents and the m-th precedes its correspondent, with no commitment about the rest:

$$i_1 = j_1 , \cdots , i_{m-1} = j_{m-1} , i_m < j_m .$$

This ordering is described by the adjective "dictionary". The resulting ordinals are hypercomplex numbers.

If sets I_k are infinite, symbol $i_1 \cdots i_n$ may have an infinite number of predecessors so that even when the number of $i_1 \cdots i_n$ is countable, dictionary order does not write the symbols as a sequence.

(16) Partial ordering. Set I with elements i, j, k is (17) *partially ordered* if

$$i \leqq i,$$

$$i \leqq j \quad \text{and} \quad j \leqq i \rightarrow i = j,$$

$$i \leqq j \quad \text{and} \quad j \leqq k \rightarrow i \leqq k.$$

The sign $\leqq$ is read "does not follow".

Each ordered set becomes partially ordered if each $i < j$ is replaced by $i \leqq j$ and each $i = j$ by the two relations $i \leqq j$ and $j \leqq i$.

(18) Derivatives. Symbol $(i_1 \cdots i_n j)'$ means the (19) *derivative* of function u, taken i_1 times re (that is, with respect to) $x_1 , \cdots , i_n$ times re x_n . It is also written $(mj)'$, with m representing the differential operator or even the corresponding monomial in the x's. At times, in numerical work it is conveniently further abbreviated to $i_1 \cdots i_n j$ and in theoretical work to d.

Symbol $(i_1 \cdots i_n j)'_S$ means the derivative evaluated for the solution of system S. Symbol $(i_1 \cdots i_n j)'_0$ means the derivative evaluated at the origin, that is, $(i_1 \cdots i_n j)'_S$ with $S = x_1 + \cdots + x_n$.

The non-negative integer $i_1 + \cdots + i_n$, called the (20) *order*, orders the set of all derivatives if there is only one function and one independent variable. In all other cases it only partially orders that set.

For later developments orderings with the properties

(21) $$d_1 < md_1 ,$$

(22) $$d_1 < d_2 \rightarrow md_1 < md_2 ,$$

where m is any differential operator except 1, are needed.

These requirements with $<$ replaced by $\leqq$ are made of partial orderings of the same sets. The word "ordinal" will apply to the real or hypercomplex number establishing the order in either case.

The first of these orderings, sufficient for most purposes, is the dictionary order in which derivative $(i_1 \cdots i_n j)'$ has ordinal

$$(i_1 + \cdots + i_n)i_n \cdots i_1 j,$$

called (23) *canonical.* It is sequential. It orders the whole set of derivatives.

(24) Monomials. Unless otherwise indicated the monomials have coefficient 1.

The monomials to be considered are of two types:

Monomial $x_1^{i_1} \cdots x_n^{i_n} u_j$ corresponds to derivative $(i_1 \cdots i_n j)'$ and is given its canonical ordinal $(i_1 + \cdots + i_n)i_n \cdots i_1 j$.

Monomial $x_1^{i_1} \cdots x_n^{i_n}$ corresponds to the differential operator producing that derivative and is given canonical ordinal $(i_1 + \cdots + i_n)i_n \cdots i_1$.

The derivatives of u, for fixed j are in one-to-one correspondence with the monomials in $x_1 , \cdots , x_n$, namely,

$$(i_1 \cdots i_n j)' \rightleftarrows x_1^{i_1} \cdots x_n^{i_n}$$

so that in canonical ordering

$$d_1 < d_2 \rightleftarrows m_1 < m_2 ,$$

that is, the relative ordering of the set of all the derivatives of a given function is the same for each function and is the same as that of the corresponding monomials in x.

If D, M are respectively sets of derivatives and monomials, their (25) *derived sets* D', M' consist of all products dp, mp, where d, m are in D, M, and p is an arbitrary differential operator or an arbitrary monomial in x.

(26) Dot product. If A, X are two sets whose elements have ordinals from the same set of ordinals, the (27) *dot product* $A \cdot X$ is the sum of all products of elements with equal ordinal. The product is interpreted as 0 if no element of A has ordinal equal to that of an element of X. Product $1 \cdot X$ means sum X.

The Maclaurin series for function u is written

$$u = A \cdot X,$$

where A is the set of all derivatives, each evaluated at the origin and divided by the appropriate factorials, and X is the set of all monomials in $x_1, \cdots, x_n$.

If A, B are sets of functions, it is convenient to let AB without the dot be the sum $a_i b_k$, where a_i is in A and b_k is in B.

Let f be a function of variables Y. The set of all first derivatives of f can be written $(Yf)'$ and the set of all first derivatives of Y re a particular x as $(xY)'$. The (28) *indirect derivative* of f re x is then

$$[xf]' = (xY)' \cdot (Yf)'.$$

(29) Monomial sets. Let finite set M of monomials in $x_1, \cdots, x_n$ have LCM (least common multiple) $j_1 \cdots j_n$. Set M^0 consists of all divisors of $j_1 \cdots j_n$, that is, $k_1 \cdots k_n$ is in M^0 if and only if

$$0 \leqq k_1 \leqq j_1, \cdots, 0 \leqq k_n \leqq j_n. \tag{30}$$

Write

$$M^0 = M^1 + M^2.$$

where M^1 is the (31) *parametric set* and M^2 the (32) *principal set*, a monomial of M^0 being principal if and only if divisible by a monomial of the original M. All monomials of M are, of course, principal.

If the exponents of x_k in monomial p of M^0 and in the LCM are equal, that variable x_k is (33) *multiplier* for p; otherwise it is (34) *non-multipler*. The exponent of a multiplier is thus always maximum.

If x_i is non-multiplier for m, the exponent of both m and $x_i m$ satisfy (30) so that $x_i m$ belongs to M^0. Hence

(35) *Every product of a monomial of M^0 by a non-multiplier belongs to M^0. Every product of a monomial of the principal set M^2 by a non-multiplier belongs to M^2.*

On the other hand, the product of a monomial of M^1 by a non-multiplier may belong either to M^1 or M^2.

The set M^0 always contains the monomial 1. Hence an arbitrary monomial p in $x_1, x_2, \cdots, x_n$ is divisible by at least one monomial of M. Among all the monomials in M dividing p let m be one having maximum degree. Suppose the quotient p/m involves an x_i which is a non-multiplier for M. By (35) mx_i belongs to M. As it divides p and has higher degree than m, there is a contradiction. Hence p/m involves only multipliers of m.

Moreover, there is only one m which is in M^0, which has maximum degree and which divides p. For if there are two, say m, m', then

$$p = mq = m'q'.$$

Suppose the exponent of x_1 is greater in m than in m'. Then x_1 is a divisor of q' and therefore a multiplier of m'. This is impossible, for the exponent of x_1 in m' is less than in m and therefore less than maximum in M^0. Hence the exponent of x_1 is the same in m and m'. As the same argument applies to all the x's, $m = m'$. The unique monomial m is called the (36) *generator* of all of its multiples by multipliers.

(37) *With an arbitrary monomial p there is associated in M a unique monomial m called its generator and characterized by being the monomial of highest degree in M^0 which divides p. The monomial p is the product of its generator by multipliers of the generator.*

If M^0 is empty, there is no LCM. Hence M^1, M^2 are as yet undefined. In this case, let M^1 be 1 with $x_1, \cdots, x_n$ as multipliers and let M^2 be empty.

(38) Cuts. If derivative $(mu)'$ is a given Maclaurin series, then $(pmu)'$, where p is an arbitrary monomial in $x_1, \cdots, x_n$, is found by termwise differentiation and has the same region of convergence. The coefficient of any term in u divisible by m can be computed from $(pmu)'_0$.

Next is needed a description of the terms whose monomials are divisible by no monomial of M and therefore cannot be calculated as above.

The Maclaurin series for u can be rewritten

$$u = u^1 + u^2 = A^1 \cdot X^1 + A^2 \cdot X^2,$$

where X^1 are all monomials divisible by no monomial of M and X^2 are the others. This Riquier calls a (39) *cut*.

The two parts of u are described as (40) *parametric* and (41) *principal*. Each can be written as a finite sum. For example,

$$A^1 \cdot X^1 = M^1 \cdot J^1,$$

where the element of J^1 corresponding to monomial p in M^1 is a function (power series) of the multipliers of p. Specification of the finite set of functions J^1 determines the parametric part of u.

It is convenient at times to transfer one or more terms from the principal to the parametric part. This is done by adjoining the monomial of the term to M^1 with no multipliers and its coefficient as a (constant) function to J^1. The resulting separation can still be used to describe the series, although it ceases to be a cut in the precise sense of the definition.

The parametric part u^1 can be described in another useful way. Specifying it is equivalent to specifying

$$(M^1u)'_{S^1} ,$$

where S^1 is the system whose equations are the non-multipliers of the members of M^1. There is the equality

$$(M^1u)'_{S^1} = (M^1u^1)'_{S^1} .$$

(42) The complete and complementary sets. The parametric and principal sets seem best adapted to the proof of existence theorems. In calculations, however, they are conveniently replaced by Janet's complementary and complete sets into which they can be collapsed. The initial determination and the passivity conditions can be more compactly, although less symmetrically, expressed in terms of the contracted sets.

Consider in M^0 a pair of monomials m, x_0m satisfying the condition: (43) x_0 is non-multiplier for m and multiplier for x_0m; and x_0m is not in the original M.

The two terms $mb + x_0mc$ can be written as one md, if m is given the multiplier x_0 .

Let the reduction be made wherever possible, beginning with the x_0m of least ordinal $i_n \cdots i_1$ (an ordinal sometimes given the name (44) *rank*). In this way, for the given order of the independent variables, there is obtained a unique set $M_R = M_1 + M_2$, where M^1, M^2 have collapsed into M_1 , M_2 , respectively. Set M_2 contains the unequal monomials of M and possibly some of their multiples.

The sets M_1 , M_2 were called by M. Janet [65] the (45) *complementary* and the (46) *complete* sets. C. Riquier [103] had previously employed the equivalent of M_1 in the description of the parametric part of the development, which he called the residue. Riquier seems nowhere to make effective use of the representation of the principal part as a finite sum, although except for interpretation he is in possession of the formal results. Compare, for example, the process [75, 166] which he only stated for the parametric part. A unique process for obtaining each principal coefficient by means of the complete set without considering the question of passivity seems Janet's contribution. This process in particular makes it possible for him to formulate for orthonomic systems passivity conditions which are much simpler than Riquier's [75, 357] although equivalent to them. For M_1, M_2 results (35), (37) need to be modified slightly so as to read

(47) *The product of a monomial p by one of its non-multipliers is equal to the product of a unique monomial m of the complete set by multipliers alone. The monomial m is of higher rank than p.*

(48) Finiteness. Let monomial mp, when m is in M^1 and p is a monomial in k of the multipliers of m, be adjoined to M^0.

If $k = 0$ so that mp equals m and is therefore in M^1, the LCM is unchanged. There is a new set, in which m has changed from parametric to principal monomial but has maintained its set of multipliers.

If k is positive, the LCM is changed. In the new cut, parametric m is replaced by certain principal monomials and a number of parametric monomials, each of which has fewer than k multipliers.

The adjunction operation can therefore be applied only a finite number of times. This is effectively Tresse's theorem [107].

(49) Inequality system. For fixed x, u

$$(50) \qquad 1 < x_i \, , \qquad 1 \leqq u_l \, ,$$

monomial $i_1 \cdots i_n j$ becomes a positive number which can serve as ordinal for derivative $(i_1 \cdots i_n j)'$ since (22), (23) are satisfied. The result is a partial ordering of the set of all derivatives. Some subsets are ordered. This type of ordering and partial ordering is (51) *numerical.*

Let a finite set of derivatives D be subject to order relations of the type

$$T_1 \qquad d_L < d_R \, .$$

There is a corresponding (52) *inequality system*

$$I \qquad \begin{gathered} 1 < x_i \, , \qquad 1 \leqq u_l \, , \\ m_L u_L < m_R u_R \end{gathered}$$

with one inequality of the type on the second line corresponding to each order relation, the sign $<$ being now read "is less than" in the numerical sense.

System T_1 is satisfied if and only if set D is numerically ordered by a root of the inequality system. Therefore D can be numerically ordered in the way prescribed if and only if the inequality system is consistent. Because of (23), (50) this numerical ordering serves for the derived set D'.

Let D be a set of derivatives all of order h and let T_1 order D canonically. The inequality system is then consistent.

The proof will be made for the system

$$(53)\qquad 0 < u_j\,, \qquad 1 < x_i\,,$$
$$k m_L u_L < m_R u_R\,,$$

where k is a positive number, arbitrary for each left number: the inequality system is implied by making all k's 1.

The inequalities in (53) with $m_L = m_R$ are first satisfied by taking

$$\max(ku_1\,, u_1) < u_2\,,$$
$$\max(ku_2\,, u_2) < u_3\,,$$
$$\cdots \qquad \cdots$$
$$\max(ku_{r-1}\,, u_{r-1}) < u_r\,.$$

The rest then amount to

$$1 < x_i\,, \qquad k m_L < m_R\,,$$

where m_L precedes m_R and the previously determined u's have been absorbed into the k's. A typical inequality is

$$k x_1^{i_1} \cdots x_n^{i_n} < x_1^{j_1} \cdots x_n^{j_n}$$

with the last non-zero exponent difference $-i_g + j_g = i$ positive. Consequently, the set is a sum of systems

$$z < x_g^i\,,$$

where for fixed right member z runs over a set involving only the k's and powers (positive and negative) of $x_1\,, \cdots\,, x_{g-1}$, so that indeterminates $x_1\,, \cdots\,, x_n$ can be had successively.

It is to be noted that the root of the inequality system can be made to have integral components since $u_j\,, x_i$ can be made integers.

(54) Order system. For the same derivative set D let T_1 be replaced by

$$T_2 \qquad d_L \leqq d_R\,.$$

It is convenient to make the substitution

$$u_j = e(c_j), \qquad x_i = c_i$$

where e is the Napierian base, in the new inequality system which thereby becomes linear in indeterminates c_j :

$$O \qquad 0 \leqq c_j\,, \qquad o_{jk} + c_k \leqq o_j + c_j\,,$$

the o's now to be defined.

Let the members of T_2 be given ordinals $1, 2, \cdots$.

In the j-th member of T_2 let d_R have order (that is, degree of x-monomial) o_j and unknown u_i . If the unknown in d_L of the k-th member is u_i , let o_{jk} be the order of d_L ; if the unknown in that d_L is not u_i , put $o_{jk} = -\infty$. If the g-th and j-th members have the same unknown u_i on the right,

$$o_{jk} = o_{gk} .$$

System O is the (55) *order system* for T_2 . If c_j is a root of O, the result of differentiating the j-th member of T_2 c_j times is a like system T_2^* satisfying

$$(56) \qquad o_{jk}^* \leqq o_j^* ,$$

that is, the order in any indeterminate u_i is the maximum order of the right members in that indeterminate.

For a canonically ordered set D the order system is consistent. By differentiation, the order of all d_R with the same u_i can be made equal to the original maximum. Then (56) are satisfied and values for c_i are the number of differentiations used.

CHAPTER

2

Certain Linear Systems

(57) Linear Diophantine systems. Here coefficients and roots are integers. Consider the linear form

$$ax + by.$$

If integer a is negative, reflection

$$x^* = -x, \qquad y^* = y$$

gives linear form

$$(-a)x^* + by^*$$

in which the coefficient of x^* is positive. Hence both a, b are supposed positive.

If $a = b$, they equal their GCD (greatest common divisor) denoted by (a, b). The form can be written

$$(a, b)(x + y).$$

If $a \neq b$, write the form as the determinant

$$\begin{vmatrix} x & -b \\ y & a \end{vmatrix}.$$

Then replace it by the equal determinant

$$\begin{vmatrix} x+y & -b+a \\ y & a \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} x & -b \\ x+y & a-b \end{vmatrix}$$

according as $a < b$ or $b < a$, that is, replace a row by the sum of the rows so as to preserve the sign in each position. Since the operation decreases a positive integer or increases a negative integer, it ends after a finite number of applications with the form

$$\begin{vmatrix} px + qy & -(a, b) \\ rx + sy & (a, b) \end{vmatrix}$$

so that the original form equals

$$(a, b)(u + v),$$

where in matrix notation

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In addition

$$\det \begin{bmatrix} p & q \\ r & s \end{bmatrix} = 1$$

since the determinant arises from that of the identity by addition and substraction of rows. Hence linear transformation.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

takes the form into another with equal coefficients.

If two coefficients in linear form

$$A \cdot U \tag{58}$$

are unequal, a change of variables such as the above replaces them by their GCD. Hence a reversible linear transformation with integral coefficients takes (58) into

$$a(1 \cdot U^*),$$

where a is the GCD of A.

System S with one equation

$$-b + A \cdot U$$

is consistent in the ring of integers if and only if a divides b. If consistent, S is equivalent to

$$-a^{-1}b + 1 \cdot U^*. \tag{59}$$

If there are other equations in S, equation (59) can be used to eliminate one unknown from S. If S is linear, this reduction process solves S.

For properties of the solution see [96].

(60) Binomial inequalities. For sake of concreteness each member of a system has so far been interpreted as the statement that a function value is zero or not.

The systems S next treated have members which are inequalities of the form

$$m < p$$

rather than equations or inequations, m, p being monomials in $x_1, \cdots, x_n$ with positive coefficients which need no longer be 1. If f denotes that inequality, then f^N is

$$p \leqq m.$$

The signs $<$, $\leqq$ now have their arithmetical sense and reading.

The system of two inequalities

$$m \leqq p, \qquad p \leqq m$$

is equivalent to the equation

$$m = p.$$

In formal manipulative processes an equation is to be regarded as an abbreviation for the two inequalities.

The identity

$$(p - m)(t - q) = pt - mq - m(t - q) - q(p - m)$$

shows that

$$m < p, \quad q < t \quad \rightarrow \quad mq < pt,$$

$$m < p, \quad q \leqq t \quad \rightarrow \quad mq < pt,$$

$$m < p, \quad q = t \quad \rightleftarrows \quad mq < pt,$$

$$m \leqq p, \quad q \leqq t \quad \rightarrow \quad mp \leqq pt,$$

$$m \leqq p, \quad q = t \quad \rightleftarrows \quad mq \leqq pt,$$

$$m = p, \quad q = t \quad \rightarrow \quad mq = pt.$$

If the inequalities on the left of each line are denoted by f, g, that on the right is by definition fg.

Similarly, kf stands for the result of multiplying both sides of f by positive number k, by a monomial k or by a k whose reciprocal is a monomial dividing both sides of f, and f^k stands for the result of raising both sides of f to the positive k power.

(61) Reduction. The reduction processes for binomial inequalities are accordingly:

(62) $$fg \leqq f + g,$$

(63) $$kf = f,$$

(64) $$f^k = f,$$

the calculus being that of systems so that (62), for example, is read "f and g imply fg".

(65) Solution. For definiteness, the sign $<$ is written but the methods apply to systems with any distribution of signs until the contrary is stated.

Consider first a system in the single unknown u. Let two members be

(66) $$a < cu^i, \qquad du^j < b.$$

They are equivalent by (63), (64) to

$$(ac^{-1})^{1/i} < u, \qquad u < (bd^{-1})^{1/j}$$

and are consistent if and only if

$$(ac^{-1})^{1/i} < (bd^{-1})^{1/j}.$$

Raise both sides to power ij:

$$a^j c^{-j} < b^i d^{-i}.$$

Multiply both sides by $c^j d^i$:

(67) $$a^j d^i < b^i c^j.$$

Form (67) of the condition for consistency can be had by the obvious method for elimination of u. This method amounts to multiplication if i, j are integers.

In general, when there is just one unknown, system S can be written

$$S = S_L + S_R ,$$

where the members of S_L , S_R are of the type

$$a < u, \qquad u < b,$$

respectively, and either set may be empty. Thus S is the sum of the following two systems each containing just one inequality

(68) $$\max a < u, \qquad u < \min b,$$

for an empty set max being $-\infty$ and min $+\infty$.

The condition of consistency of the system S in one unknown is accordingly

(69) $$\max a < \min b.$$

Let now there be r unknowns. The subset S_r of S whose members when reduced contain u_r with positive exponent are of types (66) with $u = u_r$ and a, b, c, d monomials in $u_1, \cdots, u_{r-1}$ alone.

Adjoin all conditions (67) to $S - S_r$ to form S_{r-1}, a system in $u_1, \cdots, u_{r-1}$ alone. For each root of S_{r-1} there is a segment (max a, min b), each point u_r of which gives a root $(u_1, \cdots, u_r)$ of S.

Repetition finally gives system S_0 in which there are no u's.

If the coefficients of the monomials in S are given positive numbers, S_0 consists of inequalities among positive numbers and S is consistent if and only if the inequalities of S_0 are true.

If the coefficients of S are indeterminates, then S_0 is a system of binonomial inequalities in those indeterminates as unknowns. This system S_0 is described (see (73)) as unit, since its coefficients are all obviously 1.

(70) Consistency in terms of homogeneous set. System S can also be written

$$S = S_L + S_H + S_R,$$

where $m < p$ goes in S_L, S_H or S_R according as

$$\deg m < \deg p, \qquad \deg m = \deg p, \qquad \deg p < \deg m,$$

that is, according as the degree on the left is lower, the degrees are equal, or the degree on the right is lower.

(71) *If either S_L or S_R is empty, system S is consistent if and only if the homogeneous set is consistent.*

(72) Consistency in terms of unit system. In the next result, the restriction to sign $<$ is necessary.

System S^1 got from S by replacing each coefficient by 1 is the (73) *unit system* for S.

(74) *If the unit system S^1 for system S with all signs $<$ is consistent, so is S.*

The unit system is consistent only if the consistency set for S is empty: if that set contains $m < p$, the corresponding set for S^1 contains $1 < 1$.

(75) Equivalence to linear systems. Although in the preceding discussion the use of the terms "monomial" has supposed that

$$m = au_1^{i_1} \cdots u_r^{i_r},$$

where the i's are non-negative integers, all of the operations, except the inconsequential operation of removing the highest common factor, are valid if the i's are non-negative numbers.

Now it is convenient to write the inequalities in the form

$$(76) \qquad 1 < cu_1^{a_1} \cdots u_r^{a_r}.$$

Since $\log x$ is real and increasing for positive x, this is equivalent to

$$0 < a_1 \log u_1 + \cdots + a_r \log u_r + \log c.$$

Hence linear inequality

$$(77) \qquad 0 < a_1u_1 + \cdots + a_ru_r + b$$

is solved by finding the positive solutions of (76) in which $c = c(b)$ and then by replacing u by $\log u$.

It is immediately seen that the reductions applied to inequalities of the form (76) when translated into operations upon inequalities of the form (77) become the operations of elimination used by Dines [29]. In illustration set

$$x_3 = \log x, \qquad x_4 = \log y, \qquad x_1 = \log z, \qquad x_2 = \log t$$

in Dines [29, 198]. The right members of the form (76) are

$$xy^{-1}zt^{-3}, \qquad xy^2z^2t^{-3}, \qquad x^3yz^3t^{-2}, \qquad x^{-1}y^2z^{-2}t^2.$$

The system is unit. The solution is

$$yz^{-1}t^3, \qquad y^{-2}z^{-2}t^3, \qquad y^{-1/3}z^{-1}t^{2/3} < x < y^2z^{-2}t^2,$$

$$zt, \qquad t^{1/4}, \qquad z^{3/7}t^{-4/7} < y,$$

a result easily identified with the second solution in [29, 198].

The method of elimination can also be illustrated by solving the following system of equations (Hall and Knight, [52, 110]):

$$(78) \qquad x^3y^2z = 12,$$

$$(79) \qquad x^3yz^3 = 54,$$

$$(80) \qquad x^7y^3z^2 = 72.$$

Let (79)′ denote (79) with members written in the opposite order:

(81)	= (78)(79)′	$9y = 2z^2$
(82)	= (78)′(80)	$x^4yz = 6$
(83)	= (79)′(80)	$3x^4y^2 = 4z$
(84)	= (78)′(82)	$2x = y$
(85)	= (83)(84)^2	$3x^6 = z$
(86)	= (78)(84)^2(85)	$x^{11} = 1$

The solution is therefore

(87) $$x^{11} = 1, \qquad y = 2x, \qquad z = 3x^6.$$

(88) Rational roots. A result useful in applications is

(89) *A consistent system of linear inequalities, all of which have the sign* $<$, *has a rational root. If the inequalities are homogeneous, there is an integral root.*

Let u_i be a real root of (77). Let $\lim u_{ij} = u_i$, where u_{ij} are sequences of rational numbers. Then

(90) $$0 < b + \lim A \cdot U_j\,.$$

Hence there is an n such that

$$0 < b + A \cdot U_j$$

for $n < j$. For each of the other inequalities there is a similar n. If $\max n < j$, a rational root U_j is had.

If u_i is a rational root of a homogeneous system and c is the positive LCM of the denominators of the u's, then cu_i is an integral root.

The example $0 < -u$, $0 < 1 + u$ shows that a consistent non-homogeneous system does not necessarily have an integral root.

(91) Consistency of order system. The system

(92) $$0 \leqq c_j\,, \qquad o_{jk} + c_k \leqq o_j + c_j\,,$$

to be solved in integers, can now be treated.

Rather than convert (92) into a binomial system apply the elimination process directly to (92). Since the coefficients of the c's are equal to unity, this process amounts to adding an inequality with a given c on the left to one with the same c on the right. It is convenient to denote by (j, k) the second inequality in (92) and by $(j, k) + (k, l)$ the result of adding two inequalities member for member.

In view of (71), in forming the conditions of consistency the inequalities $0 \leqq c_i$ can be ignored.

Making $j = k$ gives

(93) $$o_{jj} \leqq o_j \, .$$

If $1 < l$ and $i_1 \cdots i_l$ are any set of integers from the range of j, then

(94) $$(i_1 \, , i_2) + (i_2 \, , i_3) + \cdots + (i_{l-1} \, , i_l) + (i_l \, , i_1)$$

is free of the c's and hence is a condition of consistency. Moreover, the totality of conditions (94) for all possible distinct cycles $i_1 i_2 \cdots i_l$, taken together with (93) which correspond to the cycles of length one, contains all the conditions of consistency. To see this, remark that if $(i_1 \, , i_2)$, $(i_2 \, , i_3)$ are added, there results an inequality with right member containing c_{i_3}, which must subsequently be eliminated by adding an inequality $(i_3 \, , i_4)$. And so on. The last inequality added must eliminate both the remaining c's.

Consequently

(95) *System* (92) *is consistent in the non-negative integers if and only if*

$$\text{sum } o_{ji} \leqq \text{sum } o_i \; ,$$

where there is an inequality corresponding to each cycle and j has the value immediately following that of i in the cycle.

(96) Further results. The early work in linear inequalities is H. Minkowski's [90]. L. L. Dines [29] gave the elimination method of (75) whose dual for binomial inequalities was developed in [132].

Ruth W. Stokes put the linear theory in geometric terms [116]. The condition for consistency so expressed is that the origin be outside the the convex body representing the system. Related treatments are in [28], [41].

Inequalities of higher degree in one unknown are treated by C. F. Gummer [49] and by B. E. Meserve [89].

Applications of the linear theory are in [121], [129], [130] and F. E. Clark [20].

There is an enormous recent literature on inequalities called "linear programming".

CHAPTER

3

Algebraic Systems

The problem here is to resolve a system with given finite base into factors which are easier to solve than the original system. The ultima Thule consists of factors whose solution is described by an established existence theorem.

The fundamental operations available are addition and multiplication by any function. Consequently from the postulational standpoint the functions are a ring of which each system is an ideal. Except in Chapters 7, 8, 9 multiplication is commutative and the set of functions as well as the subset of constants is an integrity domain.

If in a system some of the unknowns are required to be derivatives of others, the system is (97) *differential.* If not, it is simply a (98) *function system.* A system is (99) *algebraic* if its members are polynomials in the unknowns.

(100) Reduction of equations in one unknown. Consider the system of two equations

$$S = f + g,$$

where

$$f = 2u^3 - u^2 + u - 2, \qquad g = 3u^2 - 2u - 1.$$

Obviously

$$S = (3f - 2g) + g,$$

S being thus expressed by a base to be regarded as simpler than the original because one member has been replaced by a polynomial of lower degree and the other is unchanged.

This (101) *reduction of f re g* can be repeated. Ultimately, the two polynomials in the base are associates and another application gives

$$S = 0 + h = h,$$

that is, S has been replaced by a base with a single equation, whose polynomial is a GCD of f, g denoted by (f, g).

Calculation of (f, g) can be formalized as follows.

(102)		$2 - 1 + 1 - 2$
(103)		$3 - 2 - 1$
(104)	$= 3(102)$	$6 - 3 + 3 - 6$
(105)	$= 2(103)$	$6 - 4 - 2$
(106)	$= (104) - (105)$	$1 + 5 - 6$
(107)	$= 3(106)$	$3 + 15 - 18$
(108)	$= -(103) + (107)$	$17 - 17$
(109)	$= 17^{-1}(108)$	$*1 - 1$
(110)	$= (106) - (109)$	$6 - 6$
(111)	$= 6^{-1}(110)$	$*1 - 1$

The GCD and final base are thus $u - 1$. This formal process can be described as follows. There are two lines (polynomials) at the start. The left numbers (initials) are made equal by applying appropriate multipliers. If the new lines are identical, they are a GCD. If the new lines are not identical, subtraction gives a third line shorter than at least one of the others, 0's at the beginning of a line being ignored and the first non-zero coefficient being flush left. The third line replaces one of the original pair, the longer, if such is, otherwise either. At any stage removing any factor common to all the numbers of a line is legitimate and may simplify the arithmetic.

This reduction process applies immediately to replace any finite non-empty system of equations in a single unknown by a single equation.

The increase to two unknowns greatly complicates matters. The difficulty lies in this. To make the theory for a single unknown useful, an equation in u, v must be written as a polynomial, say $f(v)$, whose coefficients themselves are polynomials in u. The nature of $f(v)$ is deeply influenced by the value assigned u. In particular, even the degree of $f(v)$ depends on u. This, in fact, is perhaps the chief difficulty, which can be described as that of the vanishing initial.

To illustrate, suppose S has a single member, the equation

$$f = u^2v^2 + u + v + 1.$$

Although the base has just one polynomial, it must be reduced because no existence theorem such as the fundamental theorem of algebra is directly applicable. The equality

$$S = (f + u)(f + u^N)$$

factors S into two systems

$$u^N, \qquad u^2v^2 + v + u + 1 \qquad S_1$$

$$u, \qquad v + 1 \qquad S_2$$

to whose members the fundamental theorem of algebra is applicable: the solution of u^N is the complex field less 0 and any root of the first member of S_1 introduced into the second gives a polynomial which has two complex roots v.

The system S consisting of the single equation

$$f = v^2 + 3uv + 2u + 1$$

has roots whose existence is stated by the fundamental theorem, but those roots become equal for $u^2 = 4$ and as analytic functions of u have $u = +2$, $u = -2$ as singular points. The factorization

$$S = S_1S_2S_3 ,$$

where

$$u - 2, \qquad v + 3 \qquad S_1$$

$$u + 2, \qquad v - 3 \qquad S_2$$

$$(u^2 - 4)^N, \qquad v^2 + 3uv + 2u^2 + 1 \qquad S_3$$

gives factors with unequal roots.

Even when a system all of whose members are equations is proposed, inequations force their way into the discussion. Hence it seems best to admit the inequation on equal footing with the equation at the start.

(112) Reduction of inequations. Return for the moment to a single indeterminate and consider the system

$$S = f + g^N,$$

where f, g are as before, namely,

$$f = 2u^3 - u^2 + u - 2, \qquad g = 3u^2 - 2u - 1,$$

and consider reduction of f re g. It is true that

$$(3f - 2g)^N + g^N \leqq S,$$

but since, for example, 0 is a root of the left side and not of the right, there is no longer equality. Direct application of reduction does not give a new, let alone a simpler, base.

On the other hand,

$$S = f(f, g)^{-1}$$

because the root of f which offends by making g also zero is eliminated. Moreover, the final result of appropriate application of the reduction process gives a base, although the intermediate stages do not. Ultimately, (145), explicit formulas are given for the base.

The GCD is found as before to be $u - 1$. Then f is reduced by (f, g) as follows.

$$\begin{array}{l} 2 - 1 + 1 - 2 \\ 2 - 2 \\ \\ 1 + 1 - 2 \\ 1 - 1 \\ \\ 2 - 2 \\ 2 - 2 \end{array}$$

The successive leading pairs of equals are the coefficients in the quotient, which is $2 + 1 + 1$, and

$$S = 2u^2 + u + 2.$$

Calculation of the coefficients by synthetic division appears thus, changing signs having replaced subtraction by addition:

$$\begin{array}{r|l} & \underline{2 + 1 + 2} \\ 1 + 1 & 2 - 1 + 1 - 2 \\ & + 2 + 1 + 2 \end{array}$$

(113) Formulas for reduction. Let polynomials f^*, g^*, $[f, g]$ be defined by

$$f^*(f, f') = f, \qquad g^*(g, g') = g, \qquad [f, g](f, g) = fg,$$

where f' is the derivative of f. Polynomials f^*, g^* have the roots of f, g respectively as simple roots and $[f, g]$ is an LCM for f, g.

System S whose members are polynomials in the single unknown u

can be reduced by the following equalities among systems:

$$f + g = (f^*, g^*),$$
$$f + g^N = f^*(f^*, g^*)^{-1},$$
$$f^N + g^N = [f^*, g^*]^N.$$

If S is non-empty, this gives a base which is an equation except when all members of S are inequations.

There are also the equalities:

$$(f^*, g^*) \equiv (f, g)^* = (f, g),$$
$$f^*(f^*, g^*)^{-1} \equiv f^*(f^*, g)^{-1},$$
$$[f^*, g^*]^N = ([f, g][f, g]^{*-1})^N = (fg)^N,$$

with degree non-decreasing from left to right, the sign $\equiv$ meaning identical as polynomials and the sign $=$ meaning equal as systems. If low degree is desired, use the polynomial farthest to the left.

Use of the last gives

$$f^N + g^N = (fg)^N,$$

a relation appearing in (1).

For illustration, use the systems formed from

$$f = 1 - 9 + 31 - 51 + 40 - 12 + 0,$$

$$g = 1 - 11 + 46 - 92 + 88 - 32 + 0,$$

the polynomials being written with detached coefficients, that is, $f = u^6 - 9u^5 + \cdots$. The calculations of (f, f') and f^* are given for convenience of comparison. The factor u of f is ignored.

$$1 - 9 + 31 - 51 + 40 - 12$$

$$5 - 36 + 93 - 102 + 40$$

$$5 - 45 + 155 - 255 + 200 - 60$$

$$9 - 62 + 153 - 160 + 60$$

$$45 - 324 + 837 - 918 + 360$$

$$45 - 310 + 765 - 800 + 300$$

$$14 - 72 + 118 - 60$$

$$7 - 36 + 59 - 30$$

$$
\begin{array}{r}
35 - 252 + 651 - 714 + 280 \\
35 - 180 + 295 - 150 \phantom{{}+280} \\
72 - 356 + 564 - 280 \phantom{{}+280} \\
18 - 89 + 141 - 70 \phantom{{}+280} \\
126 - 648 + 1062 - 540 \phantom{{}+280} \\
126 - 623 + 987 - 490 \phantom{{}+280} \\
25 - 75 + 50 \phantom{{}-000+280} \\
1 - 3 + 2 \phantom{{}-000+280} \\
7 - 21 + 14 \phantom{{}-000+280} \\
15 - 45 + 30 \phantom{{}-000+280} \\
1 - 3 + 2 \phantom{{}-000+280}
\end{array}
$$

Polynomial (f, f') is therefore

$$(f, f') = 1 - 3 + 2.$$

Next find the quotient $f^* = f(f, f')^{-1}$.

$$
\begin{array}{l}
1 - 9 + 31 - 51 + 40 - 12 \\
1 - 3 + 2 \\
-6 + 29 - 51 + 40 - 12 \\
-6 + 18 - 12 \\
11 - 39 + 40 - 12 \\
11 - 33 + 22 \\
-6 + 18 - 12 \\
-6 + 18 - 12
\end{array}
$$

Hence, factor u being restored,

$$f^* = 1 - 6 + 11 - 6 + 0.$$

By the same method are found:

$$
\begin{aligned}
(g, g') &= 1 - 4 + 4, \\
g^* &= 1 - 7 + 14 - 8 + 0, \\
(f^*, g^*) &= 1 - 3 + 2 + 0, \\
f^*(f^*, g^*)^{-1} &= 1 - 3, \\
g^*(f^*, g^*)^{-1} &= 1 - 4, \\
[f^*, g^*] &= 1 - 10 + 35 - 50 + 24 + 0.
\end{aligned}
$$

Polynomials f, g are factored thus:

$$f = (1 - 0)(1 - 1)^2(1 - 2)^2(1 - 3),$$
$$g = (1 - 0)(1 - 1)(1 - 2)^3(1 - 4).$$

These can be used to check the calculations.

(114) Consistency of two equations. If $S = f + g$, where

(115)
$$a_0u^m + \cdots + a_m = f,$$
$$b_0u^n + \cdots + b_n = g$$

has a root, polynomials f, g have by the factor theorem a factor h

(116)
$$f = hf_1\,, \qquad g = -hg_1\,.$$

Seek an h of positive degree at least equal to p satisfying (116) for given f, g.

A necessary condition is the existence of two polynomials f_1, g_1 which have degree not exceeding $m - p$, $n - p$, respectively, and which satisfy

(117)
$$fg_1 + f_1g = 0, \qquad f_1g_1 \neq 0.$$

This condition is also sufficient: not all the n linear factors of g can divide g_1 whose degree does not exceed $n - p$, so that at least p linear factors of g divide f. This argument assumes that the degree of g is actually n, that is, constant $b_0 \neq 0$. If $a_0 \neq 0$, a similar argument applies.

(118) Sylvester's elimination. Consider first a cubic and a quadratic polynomial,

$$a_0u^3 + a_1u^2 + a_2u + a_3 = f, \qquad b_0u^2 + b_1u + b_2 = g.$$

Multiply these equations by u, 1; and by 1, u, u^2, respectively, and display the results in the form:

(119)
$$\begin{aligned}
a_0u^4 + a_1u^3 + a_2u^2 + a_3u \qquad &= uf,\\
a_0u^3 + a_1u^2 + a_2u + a_3 &= f,\\
b_0u^2 + b_1u + b_2 &= g,\\
b_0u^3 + b_1u^2 + b_2u \qquad &= ug,\\
b_0u^4 + b_1u^3 + b_2u^2 \qquad\qquad &= u^2g.
\end{aligned}$$

This linear system in unknowns u^4, u^3, u^2, u, 1 has determinant

$$R(f, g) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_0 & b_1 & b_2 \\ 0 & b_0 & b_1 & b_2 & 0 \\ b_0 & b_1 & b_2 & 0 & 0 \end{vmatrix}$$

called the resultant of f, g. Solving for unknown 1 gives

$$(120) \qquad R = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & uf \\ 0 & a_0 & a_1 & a_2 & f \\ 0 & 0 & b_0 & b_1 & g \\ 0 & b_0 & b_1 & b_2 & ug \\ b_0 & b_1 & b_2 & 0 & u^2g \end{vmatrix} \cdot$$

By expanding re the last column this equation can be written

$$(121) \qquad R = Pf + Qg,$$

where P, Q are polynomials in u. Since R is independent of u, it is an eliminant of f, g. A necessary condition for f, g to have a non-constant factor in common is

$$(122) \qquad R = 0.$$

To examine the sufficiency of this condition write

$$(123) \qquad f_1 = u_1u + u_3u + u_2 , \qquad g_1 = u_0u + u_1 ,$$

and seek to determine u_0 , $\cdots$, u_4 to satisfy (117). Substitution gives

$$(124) \qquad \begin{aligned} 0 &= a_0u_0 & & & &+ b_0u_4 , \\ 0 &= a_1u_0 + a_0u_1 & & &+ b_0u_3 &+ b_1u_4 , \\ 0 &= a_2u_0 + a_1u_1 &+ b_0u_2 &+ b_1u_3 & &+ b_2u_4 , \\ 0 &= a_3u_0 + a_2u_1 &+ b_1u_2 &+ b_2u_3 , \\ 0 &= \qquad a_3u_1 &+ b_2u_2 , \end{aligned}$$

a system whose determinant is the transpose of R. There is a non-trivial root since the determinant is zero.

The resultant is made the first term $R_0 = R$ in a sequence of determinants with R_j for positive j formed by deleting the first and last rows and the first and last columns from R_{j-1}.

Omit the first and last equations from (119). The determinant of the resulting linear system in unknowns u^3, u^2, u is R_1 and solving for unknown u gives

$$R_1 u = \begin{vmatrix} a_0 & a_1 & f - a_3 \\ 0 & b_0 & g - b_2 \\ b_0 & b_1 & ug \end{vmatrix}.$$

By expanding and isolating f, g this can be rewritten

$$(125) \qquad R_1 u + S = \begin{vmatrix} a_0 & a_1 & f \\ 0 & b_0 & g \\ b_0 & b_1 & 0 \end{vmatrix}.$$

The GCD of f, g divides the left side.

If $R_0 \neq 0$, the GCD has degree 0 and can be taken as R_0.

If $R_0 = 0$, $R_1 \neq 0$, the GCD has degree 1 and is (125), say the right side.

If $R_0 = R_1 = 0$, $R_2 = b \neq 0$, the GCD has degree 2 and is g.

Accordingly, under the assumption $a_0 b_0 \neq 0$, the GCD has degree equal to the index on the first non-zero R_j of the resultant sequence. The GCD is found by replacing each element of the last column in R_j by the element on the same row and the last column of (120).

(126) The general case. Consider now the two polynomials (115) and assume $n \leqq m$. Multiply the first by u^{n-1}, u^{n-2}, $\cdots$, u, 1 and the second by 1, u, $\cdots$, u^{m-1} to obtain a linear system in unknowns u^{m+n-1}, u^{m+n-2}, $\cdots$, u, 1 whose determinant R is the (127) *resultant* of the two polynomials.

The resultant can be written as follows: Write the coefficients of the first polynomial on a number of rows equal to the degree of the second, starting each new row one place farther to the right. Then write the coefficients of the second polynomial on a number of rows equal to the degree of the first, putting the last coefficient on the first of these rows at the extreme right and starting each new row one place farther to the left. Use 0's to complete the square.

The above does not define the resultant when both polynomials have degree 0 or when $fg = 0$. In the first case put $R = 1$ and leave it undefined in the second.

The substitution

$$f_1 = u_{m+n-1}u^{m-1} + \cdots + u_n, \qquad g_1 = u_0u^{n-1} + \cdots + u_{n-1}$$

in (117) gives a linear homogeneous system whose determinant is the transpose of R. In the presence of $a_0 \neq 0$ or $b_0 \neq 0$ condition $R = 0$ is necessary and sufficient for the existence of a non-constant common factor.

(128) *Two polynomials with non-zero initials have a common factor of degree at least one if and only if their resultant is zero.*

The definition of the resultant sequence in (118) applies to the general case. If the above expressions for f_1, g_1 and $h = u - c$ are substituted in (116) and the resultant thus put in terms of c, u, , elementary transformations which make the last column of $R(f, g)$ consist of 0's take $R_1(f, g)$ into $R(f_1, g_1)$. Repeated use of this gives

(129) *The degree of the* GCD *of two polynomials with non-zero initials is the index of the first non-zero term in the resultant sequence.*

Omission of the first p and the last p equations of the linear system in $u^{m+n-1}, \cdots, 1$ gives a linear system in $u^{m+n-p-1}, \cdots, u^p$ with determinant R_p. Solving for unknown u^p gives

$$(130) \qquad R_pu^p + S_p = P_pf + Q_pg,$$

where S_p is a polynomial of grade $p - 1$. If the degree of the GCD is p, the left side of (130) can be taken as the GCD.

(131) If the GCD of two polynomials f, g has degree p, it is had by replacing the last column of the resultant with index p in the following way. On the last row of the coefficients of f place f, on the next to last uf, then u^2f, and so on. On the first row of the coefficients of g place g, on the next ug, then u^2g, and so on.

To have the coefficient of u^{p-k} in the GCD, replace each element in the last column of R_p by the element on the same row and k columns to the right in R_0.

The quantities in (130) are all had by ring operations. In the solution of the system like (124) for the $u_0, \cdots, u_{m+n-2p+1}$ one initial may be chosen arbitrarily. If $u_0 = -R_pb_0$, then $u_{m+n-2p+1} = R_pa_0$. Let h be the left side of (130); it differs from the h in (116) by a constant factor, which can be had by comparing the initial of f with that of hf_1. Since the latter is $R_p^2a_0$, system (116) is equivalent to

$$(132) \qquad R_p^2f = hf_1, \qquad R_p^2g = -hg_1.$$

The advantage of these over (116) is that their coefficients can be computed from those of f, g by ring operations.

(133) Discriminants. The first column of $R(f, f')$, where f' is the derivative of f, is divisible by a_0. Hence $a_0^{-1}R(f, f')$ is a polynomial in the coefficients of f, called the (134) *discriminant* of f and denoted by $D(f)$.

(135) Commutative polynomials. Consider now polynomials in r indeterminates $u_1, \cdots, u_r$. Most of the results are true for coefficients from an integrity domain, but it is perhaps best to think of the coefficients as complex numbers since a positive existence theorem is here proved only for the complex field.

The (136) *ordinal* of u_j is the positive integer j.

The formula defining polynomial f gives for it the following:

(137) the *grade in* u_i is the maximum exponent on u_i;

(138) the *index* k is the maximum ordinal of an indeterminate in f;

(139) the *right indeterminate* is that with maximum ordinal and the other indeterminates are left;

(140) the *initial* is the coefficient (for $1 < r$, a polynomial) of the maximum power of the right indeterminate (this useful name was coined by Ritt [106]);

(141) the *ordinal* is the complex number (index, grade in right indeterminate); for constant $f \neq 0$ it is $(0, 0)$ and for 0, undefined.

The initial a_0 of polynomial f is a polynomial of lower index, and therefore lower ordinal, than f. For any root of inequation a_0 polynomial f is a polynomial in the single indeterminate u_k, where k is the index, with degree equal to its grade.

(142) Reduction of a polynomial. Use the notation (115) with $u = u_k$ and grade $f \leqq$ grade g. Polynomial g_1 given by

$$g_1 = a_0 g - b_0 u_k^{n-m} f$$

is of lower grade in u_k than g. If grade $f \leqq$ grade g_1, repetition is possible. Finally,

$$g_l = a_0^l g - qf, \qquad \text{grade } g_l < \text{grade } f \quad \text{in} \quad u_k .$$

If k is the index of f, polynomial g_l is (143) *g reduced re f*. As systems,

$$f + g + a_0^N = f + g_l + a_0^N,$$

$$f + g^N + a_0^N = f + g_l^N + a_0^N.$$

(144) Resultant and discriminant sequences. Two polynomials of index

k can be written as in (115)

$$a_0 u_k^m + \cdots + a_m = f,$$

$$b_0 u_k^n + \cdots + b_n = g.$$

Since the coefficients are in an integrity domain, namely, that of all polynomials with index less than k, the GCD theory applies. The following sequences

$$D_p^2 f = h_p f_p^*, \qquad R_p^2 f = (f, g)_p f_p, \qquad R_p^2 g = (f, g)_p g_p,$$

$$R_p^1[f, g]_p = (f, g)_p f_p g_p,$$

found by ring operations, are available. Placed only with a base implying whichever of the systems

$$D_0 + \cdots + D_{p-1} + D_p^N$$

and

$$R_0 + \cdots + R_{p-1} + R_p^N$$

is relevant, abbreviated symbols

$$D, \qquad R, \qquad f^*, \qquad (f, g), \qquad f(f, g)^{-1}, \qquad [f, g]$$

mean their counterparts endowed with the proper p.

(145) Reduction of algebraic systems. At each stage of the reduction system S appears as the product of a finite number of factors. A factor to which the reduction process is inapplicable is simple. Certain priorities are stated. Some of them are essential but applicability forces them automatically to be respected: for example, if S does not imply a_0^N, then (146) must precede (147). The non-simple factor farthest to the right is given priority. The process stops when all factors are simple.

In a factor, the set of polynomials with maximum index has priority for reduction. In the tabulation below among those polynomials f is an equation with minimum grade, g is an equation, F is an inequation with minimum grade and G is an inequation. The initials of f, F are a_0, A_0, respectively.

The seven reduction processes are as follows:

(146) $$S = (S + a_0)(S + a_0^N).$$

(147) $$S = (S + D)(S + D^N - f + f^*).$$

(148) $$S = (S + R)(S + R^N - f - g + (f, g)).$$

(149) $$S = (S + R)(S + R^N - f - F^N + f(f, F)^{-1}).$$

(150) $S = (S + A_0)(S + A_0^N)$.

(151) $S = (S + D)(S + D^N - F^N + F^{*N})$.

(152) $S = S - F^N - G^N + [F, G]^N$.

The process with lower number has priority.

(153) Simple systems. A finite number of ring operations therefore factors S thus

$$S = S_l \cdots S_1 ,$$

where $i \neq j$ implies $S_i + S_j = 1$, that is, no two factors have a root in common and in addition
(154) S_i contains at most one polynomial of each index;
(155) if f of index k is in S_i , a root of the subsystem of all members of S_i with smaller index substituted in f gives a polynomial in u_k with non-zero initial and discriminant and with degree equal to its grade.

The integer in (155) is the (156) *degree* of S_i in u_k .

The factors are (157) *simple* and factorization of S by the above process is a (158) *split.*

Factorization into simple systems is not unique. Each of the three systems

$$S_1 = u^N, \qquad S_2 = u - 1, \qquad S_3 = (u^2 - u)^N$$

is simple and

$$S_1 = S_2S_3 .$$

This same example shows that a set of simple factors is not necessarily a split for their product.

Let f, g be polynomials of systems S, T respectively. According to the nature of these members system ST has members of the types

(159) $fg, \qquad fg^N, \qquad f^Ng^N.$

The first is equation. The others are neither equation nor inequation.

That in general the product ST can not be replaced by a single factor containing only equations and inequations is illustrated by

(160) $S = u, \qquad T = v^N.$

Let ST imply g, where

$$g = b_0v^n + \cdots + b_n .$$

If g is equation, for fixed u its number of roots is infinite. Hence each of polynomials b has that u for root, consequently has an infinite number of roots and is 0. If g is inequation, that is, if ST implies g^N, since (0, 0) is root of ST inequation $(ub_n)^N$ has an infinite number of roots. If $(b_j)^N$ for some $j < n$ has one of those roots, system $g + v^N$ has a root. This contradiction shows that $b_j = 0$ for $j < n$. If in b_n a positive power of u has a non-zero coefficient, the same contradiction is had. Hence ST implies no non-trivial equation or inequation.

For systems of equations J. F. Ritt [105], [106] has given a different factorization, which is unique.

(161) Examples. To save parentheses u^N, $(u + 1)^N$ are written uN, $u + 1N$.

Consider the system of three equations

$$S \qquad u^2 + vw + 1, \qquad v^2 - uw + 1, \qquad w^2 + uv - 3.$$

The resultant of the first two gives the reducible equation

$$(u + v)(u^2 - uv + v^2 + 1).$$

It is convenient to take advantage of this circumstance and write S as the product of two factors

$$S + (u + v), \qquad S + (u^2 - uv + v^2 + 1).$$

The first two equations of S reduced re $u + v$ become one and the first factor is

$$-uw + u^2 + 1, \qquad u + v, \qquad w^2 - u^2 - 3.$$

The resultant of the first and third gives the equation $u^2 - 1$ which implies u^N. Hence finally

$$u^2 - 1, \qquad v + u, \qquad uw - 2$$

is a simple factor.

In the presence of the equation $u^2 - uv + v^2 + 1$ the second equation of S yields the reducible equation

$$u(u - v + w),$$

whence the two factors

$$u, \qquad v^2 + 1, \qquad vw + 1, \qquad w^2 - 3;$$

and

$$u^2 - uv + v^2 + 1, \qquad u^2 + vw + 1, \qquad u - v + w, \qquad uv + w^2 - 3.$$

Both these factors are 1 so that S has one simple consistent factor, the others being suppressed. The result is tabulated thus:

$$(162)\qquad \begin{array}{llll} S & u^2 + uw + 1, & v^2 - uw + 1, & w^2 + uv - 3 \\ S_1 & u^2 - 1, & v + u, & uw - 2 \end{array}$$

Next apply the present method to an example used by van der Waerden [142, 9] to illustrate Kronecker's method.

$$(163)\qquad \begin{array}{lll} S & u^2 + uv = f, & uv + v^2 + u + v = g \\ S_1 & uN, & uv + u^2 \\ S_2 & u, & v^2 + v \end{array}$$

$$R(f, g) = 0, \qquad R_1(f, g) = u, \qquad (f, g) = f$$

Accidentally, the polynomials are reducible. Use of this fact gives immediately another factorization.

$$(164)\qquad \begin{array}{lll} S & u(u + v), & (u + v)(v + 1) \\ S_1 & u, & v + 1 \\ S_2 & & v + u \end{array}$$

A less immediate system is the next.

$$(165)\qquad \begin{array}{llll} & w^2 + 2(v - u)w + v^2 - 2uv = f & & \\ S & w^2 + (3v - u)w + 3uv - 2u^2 = g & & \\ & w^2 + (v - 2u)w - 2uv = h & & \\ S_1 & uN, & v^2 - uv, & (u + v)w + 4uv - 2u^2 \\ S_2 & u, & v, & w \end{array}$$

The quantities needed in the above are next given.

$$R(f, g) = -4v^4 + 20uv^3 - 28u^2v^2 + 12u^3v$$

$$R_1(f, g) = v + u$$

In the factor $S + R + R_1N$ the GCD of f, g is

$$(f, g) = (v + u)w - v^2 + 5uv - 2u^2$$

$$R((f, g), h) = 2v^4 - 18uv^3 + 30u^2v^2 - 14u^3v$$

$$R_1((f, g), h) = v + u$$

The rather long direct calculations of the GCD of $R(f, g)$, $R((f, g), h)$ can be avoided since they can accidentally be factored by inspection:

$$R(f, g) = -4v(v - u)^2(v - 3u),$$
$$R((f, g), h) = 2v(v - u)^2(v - 7u).$$

Hence the factor can be taken as $v^2 - uv$. This reduces (f, g) to

$$(f, g) = (v + u)w + 4uv - 2u^2.$$

The members

$$v^2 - uv, \qquad v + uN$$

have resultant $-2u^2$. For $u = 0$ the system of two members is v, vN and is inconsistent. For $u \neq 0$, the inequation is implied by the equation. Hence S_1 is found.

To treat the factor for which $R_1 = v + u = 0$, it is easiest to reduce the three original polynomials by that equation. The first two become

$$w^2 - 4uw + 3u^2, \qquad w^2 - 4uw - 5u^2,$$

whence $u = 0$ so that S_2 is as stated.

Further examples are tabulated without comment.

(166)
S $u^4 + u^2 + v^4 + v^2 - 22,$ $u^2 - uv + v^2 - 3$
S_1 $u^4 - 5u^2 + 4,$ $uv - 2$

(167)
S $2u^2 + uv - 1,$ $u^2v + 2u - 3v - 4$
S_1 $2u^4 - 9u^2 + 4u + 3,$ $uv + 2u^2 - 1$

(168)
S $u^2v + 2u - 3v - 5,$ $uv^2 - u + v - 1$
S_1 $u^5 + u^4 - 8u^3 + 9u^2 - 22u + 24,$ $(u^2 - 3)v + 2u - 5$

(169)
S $u^2 + v^2 + 4u - 2v - 20,$ $9u^2 + 16v^2 + 36u - 32v - 236$
S_1 $u^2 + 4u - 12,$ $v^2 - 2v - 8$

(170)
S $u^2 + v^2 - 4u + 2v,$ $u^2 - uv + v^2 - 5u + 4v + 4$
S_1 $u(u - 1)(u - 3)(u - 4),$ $(u - 2)v + u - 4$

(171)
S $uv + u^2 - 1,$ $uv - v^2 - 2$
S_1 $2u^4 - u^2 + 1,$ $uv + u^2 - 1$

(172)

$$\begin{array}{llll} S & vw + v + w + 1, & wu + w + u - 3, & uv + u + v + 1 \\ S_1 & u + 1N, & v + 1, & (u + 1)w + u - 3 \end{array}$$

(173)

$$\begin{array}{llll} S & 2u^2 - vw - wu - 4uv, & & 2u^2 - 2vw - wu - 5uv \\ S_1 & uN, & v - u, & w + u \\ S_2 & uN, & v, & w - 2u \\ S_3 & u, & vN, & w \\ S_4 & u, & v & \end{array}$$

(174)

$$\begin{array}{lll} S & uv^3 + (u^2 - u - 1)v^2 + (u^2 - 3u + 4)v + 2u - 4 & \\ & 3uv^3 - (2u^2 + 3u + 3)v^2 - (2u^2 + 4u - 12)v - 4u - 12 & \\ S_1 & uN, & uv^2 + (u - 1)v + 2 \\ S_2 & u, & v - 2 \end{array}$$

(175) Reduction of coefficients. Let S be simple. Any polynomial g by (143) can be reduced re an equation f of S. If f has index k, the resulting polynomial has grade in u_k less than the degree of f in u_k. If g is reduced in this way re all the equations of S, then g is (176) *reduced re* simple S.

If g or G^N is a member of S, this becomes an eighth reduction operation, namely,

(177) $$S = S - g + g_l\,, \qquad S = S - G^N + G_l^N,$$

the significance of the subscript being given by (143). This like (152) does not factor S.

After use of (177) the simple factors have the additional property:

(178) *The equation of index k has degree in u_k greater than the grade in u_k of any other member of the simple factor.*

(179) Polynomial factorization. The last of the ten operations to be specifically listed are

(180) $$fg \leqq S \rightarrow S = (S + f)(S + g),$$

(181) $$(fg)^N \leqq S \rightarrow S = S + f^N + g^N.$$

Both of these depend on factorization of a member. As already seen in the examples they can be extremely useful in simplifying calculations. To illustrate further this point, note that (165) implies

$$v(v + w), \qquad (v + w)(-2u + v + w), \qquad (v + w)(-2u + w)$$

which can be made to give almost immediately the factorization

(182)
$$\begin{array}{llll} T_1 & uN, & v, & w - 2u; \\ T_2 & v - u, & w + u, & \end{array}$$

(183) Existence. Assume that each equation with one indeterminate and positive degree has roots equal in number to its degree and that each inequation with one indeterminate and non-negative degree has an infinite number of roots. Under this assumption, a simple factor which contains neither 1 as equation nor 0 as inequation has a root, whose existence can be proved constructively as follows.

If a simple factor contains no member of index k, indeterminate u_k is (184) *parametric* for that factor.

Let f be the member of least index.

If f is equation, its initial is a non-zero constant. Its degree is positive for all values of the parametric indeterminates. Let the parametric indeterminates be given arbitrary values. The resulting equation in the single right indeterminate has roots by the assumed fundamental theorem.

If f is inequation, it can be replaced by equation $f + u$, where u is a parameter which can have all values except 0, and the previous argument applies.

Let g be the member with next to least index. A root of f introduced into g makes the fundamental theorem applicable after any indeterminates of intermediate index (necessarily parametric) have been given arbitrary values.

Let $S(k)$ consist of all members of simple S with index not greater than k. Let $(a_1, \cdots, a_{k-1})$ be a root of $S(k-1)$. That root introduced into the member of index k in S, if such there be, gives a member with non-zero initial, appropriate degree and non-zero discriminant. The desired result is had by induction.

(185) *System S is consistent if and only if $S \neq 1$, that is, at least one of its simple factors is not* 1.

It is useful to note that root $(a_1, \cdots, a_{k-1})$ introduced into the k-th member of S can be said to give d_k unequal roots a_k, where d_k is the degree of the equation with index k or infinity according as there is such an equation or not. Moreover, there is always a root

$(a_1, \cdots, a_{k-1}, a_k, \cdots, a_r)$ of S containing a prescribed root $(a_1, \cdots, a_{k-1})$ of $S(k-1)$.

(186) Implication by a simple system. With slightly modified notation let polynomial f reduced re simple S give non-zero f_1. There is then a sequence in which f_i is the (necessarily not identically 0) initial of f_{i-1} and the last term is a non-zero constant. Let f_i have ordinal (k_i, n_i). By induction, it is to be proved that *if* $(u_1, \cdots, u_{k_i-1})$ *is a root of* $S(k_i - 1)$, *not all the roots of the* k_i*-th member of* S *are roots of* f_i. The last member of the sequence, being a non-zero constant, serves as the start. The initial of f_i is f_{i-1}. By hypothesis, there is a root of $S(k_{i-1}) + f_{i-1}^N$. Since $k_{i-1} < k_i$, that root is contained in a root of $S(k_i - 1)$. For the root f_i has degree n_i, which by the argument leading to (178) is less than the degree of the k_i-th member of S. Hence the theorem is true. Applied to f_1 it gives

(187) *Simple* S *implies equation* f *if and only if* f *reduced* re S *is* 0.

If on the other hand S is to imply f^N, it is necessary that f reduced re S give non-zero f_1. If S has no member with the same index as f_1, the above argument shows that $S + f_1$ is consistent. Hence a necessary condition is that S have a member with index k equal to that of f_1. If that member is inequation g^N, in the presence of $S(k-1)$ there is the relation $f^N \leqq g^N$ which by (1) is equivalent to $g \leqq f$, so that $S(k-1) + g \leqq S(k-1) + f$, but lacking information about the initial of f one can not say that $g \leqq g_1$ in the notation of (142). The possibility that the member of S is equation presents even more complications.

If, however, it is question of the equality of simple systems S, T, these difficulties disappear. It is possible to prove

(188) *Simple systems* S, T *are equal if and only if they can be put in one-to-one correspondence so that* (i) *equation corresponds to equation, inequation to inequation*; (ii) *each equation reduced* re *the other system is* 0; *and* (iii) *if* f^N *in* S *corresponds to* g^N *in* T, *then* f *reduced* re $S - f + g$ *is* 0 *and vice versa.*

Let k be the index of the inequation f^N with least index among all the inequations of both systems. If f^N in S corresponds to equation g in T, then with coefficients evaluated for a root of $S(k-1)$ the impossible relation $g \leqq f^N$ is had.

In illustration, let $S = S_1$ in (165) and $T = T_1$ in (182). The systems are not equal because equation v in T is as it stands reduced re S and is not 0. It is apparent, however, that $S \leqq T$. A simple system equal to S is had if its last equation is replaced by $(u + 2v)w + 5uv - 2u^2$.

(189) Equal systems. The conditions that S, not necessarily simple, imply member with polynomial f are

$$(190)\qquad \begin{aligned} f &\leqq S \rightleftarrows S + f = S, \\ f^N &\leqq S \rightleftarrows S + f = 1. \end{aligned}$$

Let systems S, T to be tested for equality be split

$$S = S_t \cdots S_1, \qquad T = T_t \cdots T_1 .$$

Each equation reduced re a factor of the other system must be 0. This disposes of the first row of (190). The other conditions are

$$S_1 + g = 1, \qquad T_1 + f = 1,$$

where f^N, g^N run through the inequations of S, T, respectively. All simple factors of all of these systems are to be 1.

(191) Existence in complex field. The condition that equation f have complex root $u + iv$ is a system of two equations in the real field. Proving the fundamental theorem by treating this system seems to have some interest.

Suppose that S is a single equation $f(a, z)$ of grade n in z with indeterminate complex coefficients a, . Replace z by $u + iv$, where $i^2 = -1$, and segregate the terms of odd degree in v to get

$$f(a, z) = g(a, u, v^2) + iuh(a, u, v^2).$$

Replace a by $b + ci$ to get

$$(192)\qquad f(a, z) = g(b, u, v^2) - uh(c, u, v^2) + i[g(c, u, v^2) + uh(b, u, v^2)].$$

The complex roots of f are got from the real roots (u, v) of the system with two equations and real coefficients

$$(193)\qquad g(b, u, v^2) - uh(c, u, v^2), \qquad g(c, u, v^2) + uh(b, u, v^2),$$

where b, c are the real components of given complex a.

On the other hand, any root (u, v), real or imaginary, of (192) makes $u + iv$ a complex root of f.

To settle the existence question, it is first proved that every non-constant f with real coefficients has a complex root.

For real coefficients $a = b$, $c = 0$ so that (193) becomes

$$g(a, u, v^2), \qquad uh(a, u, v^2),$$

a system implied by

$$g(a, u, v^2), \qquad h(a, u, v^2).$$

The resultant of g, h as polynomials in v^2 is a polynomial R_u in u, called the (194) u-resolvent of f, g [134]. Its degree is $n(n-1)/2$. To prove this, write for n even and equal to $2p$

$$(195) \qquad \begin{aligned} g(u, v^2) &= f_0 v^n + f_2 v^{n-2} + \cdots + f_n , \\ h(u, v^2) &= f_1 v^{n-2} + f_3 v^{n-4} + \cdots + f_{n-1} , \end{aligned}$$

and for n odd and equal to $2p + 1$

$$(196) \qquad \begin{aligned} g(u, v^2) &= f_1 v^{n-1} + f_3 v^{n-3} + \cdots + f_n , \\ h(u, v^2) &= f_0 v^{n-1} + f_2 v^{n-3} + \cdots + f_{n-1} . \end{aligned}$$

The degree is equal to the weight in the f's.

The degree of non-constant polynomial f can be written $2^a(2b + 1)$. The polynomials can be put in sets of ordinal (a, b). The theorem to be proved is that each polynomial of ordinal (a, b) with real coefficients has a complex root. This is done by induction on (a, b). The start is the set with ordinal $(0, b)$, that is the polynomials of odd degree. Continuity shows that each of these has a real root since $f(-\infty)f(\infty) < 0$. For positive a, polynomial f with ordinal (a, b) has R_u with ordinal $(a - 1, c)$. By hypothesis, R_u has root u. For that root g, h are polynomials in v^2 with GCD of positive degree which (195), (196) show to be less than $n/2$. Hence $f(z)$, whose coefficients are real, has a factor with complex coefficients and degree less than $n/2$ and finally is reducible $f = f_1 f_2$ in the real field. Let the ordinals of f_1, f_2 be (a_1, b_1), (a_2, b_2) with $a_1 \leqq a_2$. Since $\deg f = \deg f_1 + \deg f_2$, it follows that $a_1 \leqq a$ and, if $a_1 = a$, that $b_1 < b$. Hence $(a_1, b_1) < (a, b)$ and by the induction hypothesis that f_1 has a complex root, which is also a root of f.

Reverting to the original meaning of a, let a^C be the conjugate of a. Then $f(a, z)f(a^C, z)$ has real coefficients and therefore a root r, which is a root of $f(a, z)$ or of $f(a^C, z)$. If the latter, then r^C is a root of $f(a, z)$ and the proof is complete.

(197) Existence in real field. Let f_0 be a polynomial with one indeterminate u and real coefficients. Let f_1 be the derivative.

Sign being of primary importance in what is now to come, only positive multipliers are to be applied to the polynomial being reduced.

The operation in (142) is accordingly modified to

$$(198) \qquad g_1 = | a_0 | g - (\text{sign } a_0) u^{n-m} f.$$

This operation ultimately gives the remainder in the division of g by f. Either by its use or by division a sequence

(199) $$f_0, f_1, \cdots, f_l$$

in which f_i is the negative of the remainder when f_{i-2} is divided by f_{i-1}. The operation is stopped when the next term would be zero. The last term is a GCD of f_0, f_1.

Suppose first that (199) ends with constant f_l. It is then the (200) *Sturm sequence* for f_0. Let $v(a)$ be the number of variations of sign in the sequence when u is replaced by real a. Sturm's theorem is

(201) *The number of roots of f_0 in $a < u \leqq b$ equals the difference $v(a) - v(b)$. The roots are all simple.*

The proof follows. The relation

$$cf_{j-1} = f_j q_j - f_{j+1} \qquad (0 < j, c)$$

and the fact that f_0, f_1 are relatively prime are used to prove

(202) *In* (199) *the number of variations of sign can change only at a root of f_0.*

This result can be applied to the subsequence obtained by omitting the first term of (199) because the subsequence is generated by the division algorithm applied to f_1, f_2, which are relatively prime. Hence the subsequence can have a change in the number of variations only at a root of f_1. Since no root of f_0 is a root of f_1,

(203) *The number of variations of sign in the subsequence obtained by omitting f_0 from* (199) *does not change at a root of f_0.*

Now use for the first time the fact that f_1 is the derivative. As u increases through a root, f_0^2 decreases to 0 and then increases, that is, the derivative of f_0^2 is negative just before a root and positive just after, see [59]. Hence

(204) *The product $f_0 f_1$ is negative just before a root of f_0 and positive just after.*

Because of (204), just before a root of f_0 the first two terms of (199) show a variation of sign, which is lost at the root. This fact coupled with (202), (203) proves (201).

Suppose that f_l is not necessarily constant. Divide the members of (199) by f_l to form

(205) $$g_0, g_1, \cdots, 1,$$

the *first Sturm sequence* for f_0. It can be generated from g_0, g_1 as (199) was from f_0, f_1. Hence (202), (203) are true of (205). Although g_1 is not the derivative of g, (204) is true of (205) because

$$f_0 f_1 = f_l^2 g_0 g_1 .$$

The roots of g_0 are simple and are the unequal roots of f_0. The loss of variations of (205) on $a < u \leqq b$ is therefore the number of unequal roots of f_0 in that set.

If f_l is not constant, the first Sturm sequence for it is the (206) *second Sturm sequence* for f_0. It is formed by applying the division algorithm to f_l, f_l'.

In general, if h_k is the factor removed in obtaining the k-th Sturm sequence, the first Sturm sequence for h_k is the $(k + 1)$th for f_0. The process stops at the k-th if h_k is constant. Let $V_k(u)$ be the number of variations in the k-th sequence and write

$$V(u) = V_1(u) + \cdots + V_p(u),$$

where the last sequence is the p-th. The number of roots with multiplicity k is

$$V_k(a) - V_{k+1}(a) - V_k(b) + V_{k+1}(b),$$

a V with subscript greater than p being interpreted as 0. Summing gives a generalized form of Sturm's theorem:

(207) *The number of roots on $a < u \leqq b$ is $V(a) - V(b)$, a root of multiplicity k being counted as k roots.*

If neither end point is a root, $f(a)f(b) \neq 0$, the behavior of (199), (205) is the same and the divisions necessary in getting (205) can be avoided. Put the f's in a single (208) *extended Sturm sequence*, whose generation is described as follows:

(209) If the remainder in the division f_{i-2}/f_{i-1} is not zero, the negative of that remainder is f_i.

(210) If that remainder is zero and if f_{i-1}' is not zero, then f_{i-1}' is f_i.

(211) *If $f(a)f(b) \neq 0$ and if a root of multiplicity k is counted as k roots, the number of roots in $a \leqq u \leqq b$ is the same as the loss in variations of the extended Sturm sequence.*

The following example involves three Sturm sequences, written at the right. The extended Sturm sequence is at the left. Positive numerical factors have been introduced where convenient.

$$f_0 = u^6 - 4u^5 + 5u^4 - 2u^3 \qquad g_0 = u^3 - 3u^2 + 2u$$

$$f_1 = 3u^5 - 10u^4 + 10u^3 - 3u^2 \qquad g_1 = 3u^2 - 7u + 3$$

$$f_2 = 5u^4 - 11u^3 + 6u^2 \qquad g_2 = 5u - 6$$

$$f_3 = u^3 - u^2 \qquad g_3 = 1$$

$$f_3 = u^3 - u^2 \qquad u^2 - u$$

$$f_4 = 3u^2 - 2u \qquad 3u - 2$$

$$f_5 = u \qquad 1$$

$$f_5 = u \qquad u$$

$$f_6 = 1 \qquad 1$$

The signs of the first f sequence for $-\infty$, ∞ are, respectively,

$$+ - + -, \qquad + + + +$$

so that f_0 has three unequal real roots. The signs of the three Sturm sequences for the same values are

$$- + - + \qquad + + + +$$

$$+ - + \qquad + + +$$

$$- + \qquad + +$$

so that f_0 has $3 + 2 + 1$ real roots, of which one is simple, one is double and one is triple. The loss of variations in the extended sequence is 6 so that there are six real roots.

The calculations are arranged as follows:

(212)	f_0	1 −	4 +	5 −	2 + 0 + 0 + 0*	
(213) = $2^{-1}(212)'$	f_1	3 −	10 +	10 −	3 + 0 + 0*	
(214) = 3(212)		3 −	12 +	15 −	6 + 0 + 0 + 0	
(215) = (214) − (213)		−2 +	5 −	3 +	0 + 0 + 0	
(216) = 3(215)		−6 +	15 −	9 +	0 + 0 + 0	
(217) = 2(213)		6 −	20 +	20 −	6 + 0 + 0	
(218) = (216) + (217)		−5 +	11 −	6 +	0 + 0	
(219) = −(218)	f_2	5 −	11 +	6 +	0 + 0*	

(220) = 5(213)		$15 - 50 + 50 - 15 + 0 + 0$
(221) = 3(219)		$15 - 33 + 18 + 0 + 0$
(222) = (220) − (221)		$-17 + 32 - 15 + 0 + 0$
(223) = 17(219)		$85 - 187 + 102 + 0 + 0$
(224) = 5(222)		$-85 + 160 - 75 + 0 + 0$
(225) = (223) + (224)		$-27 + 27 + 0 + 0$
(226) = -27^{-1}(225)	f_3	$1 - 1 + 0 + 0$*
(227) = 5(226)		$5 - 5 + 0 + 0$
(228) = (219) − (227)		$-6 + 6 + 0 + 0$
(229) = 6^{-1}(228)		$-1 + 1 + 0 + 0$
(230) = (226) + (229)		$0 + 0 + 0 + 0$

In the above, the members of the sequence are distinguished by an asterisk. The lowest line with an asterisk is the reducer of the moment. When a new minimum length occurs, the line is to be multiplied by -1 and then marked with an asterisk. The multiplier applied to the momentary dividend has to be positive: see (223), for example. Line (230) really should be empty: its length is 0. Its function is to show that the sequence ends with the line bearing the lowest asterisk.

(231) Approximation of roots. If the coefficients are exact, approximations to the roots can be defined and in theory the calculation can be done with arbitrary precision. Practical success is conditioned by ability to handle the large numbers which arise in the numerical work. It seems that processes which avoid such numbers necessarily have field of applicability which is limited and often, what is much worse from the theoretical point of view, ill defined. The hope is that the routine of calculation can be adapted to handle the large numbers.

In applications, the coefficients may themselves be approximations. If the coefficients have parts (real and imaginary) only known to be in given segments, the possible accuracy is limited. If the given segments are too large, definition of "root" in a useful sense becomes difficult if not impossible.

(232) Exact coefficients. Let the coefficients be real and exact. The roots, however, must in general be approximated. An approximate root is a segment containing an exact root and, of course, only one.

For the moment let polynomial f have no multiple roots.

Let u, v be two real unequal numbers, either of which may be the larger. The set of numbers x which satisfy

$$\min(u, v) < x < \max(u, v)$$

is a segment to be denoted here by uv (or by vu). If the order of the end points is known, say if $u < v$, the conventional notation (u, v) can be used instead of uv.

If the three conditions

$$(233) \qquad 0 < |u - v| \leqq h^{-1},$$

$$(234) \qquad f(u)f(v) < 0,$$

$$(235) \qquad 0 \neq f'(x), \quad \text{for} \quad x \quad \text{in} \quad uv,$$

are satisfied, uv is an (236) *approximate root with precision* h for f. Usually, the name will be abbreviated to root uv.

If uv is root of f, by continuity there is an exact root of f in v and by Rolle's theorem only one.

Let now it be asked whether given uv is root of given f. Conditions (233), (234) are easily tested. Testing (235), however, requires some knowledge of the solution in uv of inequation f'^{N}. That solution is the complement re uv of the set of (exact) roots of f'; in other words, the roots of f' must be approximated, at least grossly, before those of f.

All real roots of f are in a segment whose ends can be specified in various ways. G. D. Birkoff [5] has, for example, given the bound

$$(237) \qquad \max_j |a_j({}_nC_j a_0)^{-1}|^{j^{-1}} (2^{n^{-1}} - 1)^{-1}$$

for the modulus of the roots of a polynomial with coefficients a, and degree n.

Let $r_1, \cdots, r_k$ written in increasing order include the real roots of f' and let segment r_1r_k contain all the real roots of f: if the least root of f' is not a lower bound for the roots of f take r_1 equal to some lower bound, say the negative of (237), and similarly for r_k. The approximate roots of f can be taken as the segments r_jr_{j+1} for which

$$(238) \qquad f(r_j)f(r_{j+1}) < 0,$$

the precision being $(r_{j+1} - r_j)^{-1}$.

In practice, however, the roots of f' have to be approximate. Let uv be such a root of f'. In order to simplify language, it is understood once for all that no end of any segment uv is a root of any polynomial appearing in the discussion.

From (204) it then readily follows that polynomial f has constant sign in root uv of f if and only if

$$(239) \qquad 0 < (v - u)f(u)f'(u), \qquad 0 < (u - v)f(v)f'(v).$$

Once roots uv have been calculated, they can be substituted in (239). If one of (239) is not satisfied, the precision must be increased.

Let u_1v_1 , u_2v_2 be consecutive roots of f' which satisfy (239). For substitution in (238) any numbers in the respective intervals will serve. If, for example,

$$f(u_1)f(u_2) < 0,$$

then u_1u_2 is a root of f.

Now let f be general. Its unequal roots are the roots of $f_0 = f(f, f')^{-1}$, all of which are simple. To calculate them, the roots of f_0' are needed. The unequal roots of f_0' are the roots of $f_0'(f_0', f_0'') = f_1$, all of which are simple. Thus is got a sequence f_i in which $f_i' (f_i', f_i'')^{-1} = f_{i+1}$ and the last term is constant. The roots of the polynomials are found in reverse order.

The methods for increasing the precision are legion. Two simple-minded methods are quite effective.

The first doubles the precision by halving the segment. If w is the midpoint of uv which satisfies (233), (234), (235), then $f(w)$ is 0 or has the same sign as one of $f(u)$, $f(v)$ and w can replace the corresponding end of uv.

The second is Horner's. For definiteness suppose $u < v$ and expand $f(x)$ about u. Neglecting terms of higher degree gives

$$f(u) + xf'(u)$$

which suggests the approximation to use, namely,

$$u - f(u)/f'(u).$$

If the precision h of uv exceeds unity, the ratio of the new precision h' to the old satisfies

$$(h - 1)AB^{-1} < h'/h,$$

where A is a lower bound for $| f'(u) |$ and B is an upper bound for the moduli of the coefficients of the neglected terms. For example, (.0004, .0005) is root of

$$x^3 + 1.5x^2 - 5.5x + .0025$$

with precision 10^4. Here $A = 5$, $B = 2$ so that the precision in using Horner's method is at least 10^4 times as great as it is in halving. Approximation by use of the linear terms gives the root (.00045454, .00045455).

On the other hand, halving always doubles the precision, whereas h, A, B must satisfy conditions in order for Horner's method to increase the precision.

(240) Approximate coefficients. Let each coefficient of polynomial f be known simply to be in an interval

$$p_j \leqq a_j \leqq q_j \qquad (0 < j \leqq n),$$

the interval for the initial not to include the origin

$$p_0 \leqq a_0 \leqq q_0 , \qquad 0 < p_0 q_0 ,$$

and let this system of inequalities imply the system of inequations

$$D_0^N + D_1^N + \cdots + D_n^N ,$$

where D_j is the discriminant of the j-th derivative of .

By (241) *approximate polynomial* f is understood the set of all polynomials whose coefficients satisfy these inequalities and by (242) *exact polynomial* f a specific element of that set.

In f replace a_j by p_j for $n - j$ even and by q_j for $n - j$ odd to get exact polynomial m_1 .

In f replace a_j by p_j for all j to get polynomial m_2 .

Let function m equal m_1 for negative x and m_2 for positive x.

In f replace a_j by q_j for $n - j$ even and by p_j for $n - j$ odd to get polynomial M_1 .

In f replace a_j by q_j for all j to get polynomial M_2 .

Let function M equal M_1 for negative x and M_2 for positive x.

Functions m, M are then continuous and for all x satisfy

$$(243) \qquad m(x) \leqq f(x) \leqq M(x).$$

Segment uv is a (244) *root with precision h for approximate f* if it satisfies (233), (235) and

$$(245) \qquad M(u) < 0 < m(v),$$

a condition which replaces (234). Each root uv of approximate f is therefore a root for each of the corresponding exact f's.

The derivative of approximate f is an approximate f' the bounds for whose coefficients can be had from those of f.

To solve approximate f in segment UV, satisfy (235) by supposing

$$f'^N \quad \text{in} \quad UV,$$

this inequation meaning that no exact f' has a root in UV, and seek by the method of (232) a root u_1u_2 for one of the two systems

(246) $$x \leqq 0, \quad M_1\ ; \qquad 0 \leqq x, \quad M_2\ ;$$

and a root v_1v_2 for one of the two systems

(247) $$x \leqq 0, \quad m_1\ ; \qquad 0 \leqq x, \quad m_2\ .$$

Choose u in u_1u_2 and unequal v in v_1v_2 so that (245) is satisfied, if possible.

The roots of approximate f in all UV having been found, in the complement of those roots f^N is satisfied.

To satisfy f'^N in UV, approximate f' must be solved before f. Consequently, to solve approximate f, solve $f^{(n-1)}, \cdots, f', f$ in that order.

As in the case of exact equations, increased precision can be sought by halving or by Horner's method. In contrast with that case, here, as is to be expected, the precision has an upper bound. The least upper bound for h is $|u - v|^{-1}$, where u, v are the exact roots of (exact) (246), (247) which define a root of f. If equality were allowed in definition (245) and the exact roots of (246), (247) were used, h would assume its least upper bound as actually it does not.

All this is illustrated by the following example:

$$f(x) = x + a, \qquad -2 \leqq a \leqq -1$$

$$m_1 = m_2 = x - 2, \qquad M_1 = M_2 = x - 1$$

The solution of (245) is $u < 1,\ 2 < v$. The precision has least upper bound 1.

CHAPTER

4

Riquier's Existence Theorem

(248) Canonical systems. Next are considered systems whose indeterminates are unknowns u, and some of their derivatives re independent variables $x_1, \cdots, x_n$. These systems are differential.

The present chapter is limited to systems effectively solved or immediately solvable for certain of the indeterminates.

System S is (249) *canonical* if and only if:

(250) the unknowns U, the independent variables X, the derivatives D and the members explicitly included are finite in number;

(251) the members have the form

$$-f(X, D_L) + d_R = f_0 ,$$

where d_L, d_R are respectively *left* and *right* derivatives;

(252) no two members have the same right derivative;

(253) each left derivative precedes in canonical order the right derivative in its member;

(254) there is given a *numerical determination* J which is a root of the function system with all X zero;

(255) functions f are analytic and J is an ordinary point for each of them.

Each unknown has a monomial set. It is assumed that J includes a value for each parametric, see (31), derivative, whether that derivative occurs explicitly in the system or not.

Equation f_0 will be cited as "equation f." but the full notation f_0 must be used in formulas, see (275).

The existence problem has four parts:

(256) to determine a Maclaurin series for each unknown;

(257) to prove those series converge;

(258) to find the conditions that they be a root;

(259) to describe the solution.

(260) Determined systems. A system plus an appropriate initial determination is (261) *determined.*

Assign an initial determination U^1 formed from functions holomorphic about 0 and arbitrary except for the terms whose coefficients are in the numerical determination J.

Differentiation and evaluation determine all the coefficients of the terms which are multiples of principal monomials. Since the others are in the initial determination, series for the (262) *tentative root* are determined.

(263) *A determined canonical system has at most one root.*

(264) Convergence. Next it is shown by Riquier's extension of Cauchy's method of dominant functions that the series converge about the origin so that like S the tentative root is composed of analytic functions having the origin for ordinary point.

To simplify matters, the base of S is changed, first to include an equation with each principal derivative as right derivative. Let $h - 1$ be the maximum degree of monomial LCM's for S. In the cuts

$$u_j = u_j^1 + u_j^2$$

let every term of degree less than h in u_j^2 be transferred to u_j^1 to make a new initial determination U^{1*}. Correspondingly, let S be replaced by canonical S^* whose right derivatives correspond to the terms of degree h in U^{2*}. The schism specifies the unique way to form the equations of S^* by differentiation of S.

Since derivatives of order h enter only through differentiation, S^* is linear in those derivatives and has the essentially solved form

$$-P \cdot D_L - q + d_R , \tag{265}$$

where derivatives D_L , d_R are of order h but those in the functions of sets P and the functions q have order less than h.

A translation

$$u_j = v_j + u_j^1 \tag{266}$$

modifies the coefficients in the series for the f's without changing the essential features of the system. If U^1 for the new system consists of 0's, then the corresponding root of S^* has the given initial determination U^1. Hence it is assumed that

$$u_j^1 = 0. \tag{267}$$

Let the series of moduli for P, q converge and have terms bounded by constant a for all the indeterminates equal to positive b. The coefficient of monomial m of total degree k in any of the series P, q has modulus bounded by ab^{-k}.

Consider the function

$$a[1 - b^{-1}(X\cdot 1 + D_L\cdot 1)]^{-1}, \tag{268}$$

where set D_l contains all the derivatives of order less than h and set X all the independent variables. In the series for (268) the coefficient of monomial m is a positive integer times ab^{-k}. Hence each coefficient in (268) is at least equal to the modulus of the corresponding coefficient in any P or q.

As an abbreviation write

$$t = b^{-1}(X\cdot 1 + D_L\cdot 1)$$

and consider system

$$-a(1 - t)^{-1}(L\cdot D + m_R) + am_R + d_R\ , \tag{269}$$

where d_R runs over the set D_R of all right derivatives in (265), where D is the set of all g derivatives of order h and where the other notation will now be explained.

Constant m_R is had by evaluating the monomial of d_R for a root of the inequality system (53) with $k = 2ag$.

The constant l in set L corresponding to d in D is $m_R(2agm)^{-1}$, where m is the monomial of d evaluated as was specified for m_R .

Note that only d, m and the elements of L (through m_R) vary from member to member of (269).

System (269) is said (270) to *dominate* (265) in a sense now to be explained.

If d is left derivative in the equation (265) with right derivative d_R , a member of the inequality system says that $i < l$ in the corresponding equation (269). The coefficient in (269) is greater than its correspondent in (268) and therefore greater than the modulus of its correspondent in (265).

If d is not left derivative in the equation (265) with right derivative d_R , then simply $0 < l$. The coefficient in (269) may be smaller than its correspondent in (268) but it is greater than its correspondent in (265), namely, 0.

Since the constant term in (265) is 0, it is permissible to make its correspondent in (269) 0 as was accomplished by writing the term $+am_R$. This proves convenient later.

The coefficients, say with a temporary change of notation a, of the tentative root are calculated from the coefficients b of the f's by $+\times$ and hence are polynomials in b with positive coefficients:

$$a = g(b).$$

In the dominant system all derivatives of order h occur as left indeterminates. If one of these derivatives is missing from the corresponding place in S^*, multiply that derivative in that place in the dominant system by parameter c. The operations of differentiation and evaluation which calculate the coefficients of the tentative root applied to the dominant system (269) give formulas

$$A = G(B, cA),$$

where B are the coefficients of (269). These equations for $c = 1$ are satisfied by the coefficients of a root of the dominant system and for $c = 0$ become

$$A = g(B).$$

Once the dominant system is known to have a root with non-negative coefficients A

$$|a| \leqq g(|b|) \leqq g(B) = A$$

and the tentative root will be known to converge.

This point established, letters a, b, g revert to their former meanings. The root of the inequality system being $w_1, \cdots, w_s, v_1, \cdots, v_r$, put

$$u_j = v_j u$$

and seek a particular root of the dominating system in which u is function of the single variable

$$x = X \cdot W$$

Then

$$d_l = m_L u^{(h)}, \qquad d_K = m_K u^{(h)},$$

so that the dominant system becomes

$$-(1 - t)^{-1}(u^{(h)} + 2a) + u^{(h)}.$$

It can be solved for $u^{(h)}$ thus

(271) $$-2at(1 - 2t)^{-1} + u^{(h)}.$$

This one equation is a canonical system in the single unknown u. Give it initial determination 0. In its tentative root the coefficients are necessarily non-negative because those of its f are non-negative.

To show that the tentative root converges and satisfies (271), replace (271) by the canonical first order system

$$(272) \qquad -u_2 + u_1', \cdots, -u_h + u_{h-1}', -2at(1-2t)^{-1} + u_h',$$

u_1 being the same as u and the other u's being its derivatives.

Canonical system (272) likewise has a tentative root with non-negative coefficients. To show that that root converges, seek a dominant system for (272). To make that dominant system have a *known* root of simple nature, it is chosen as

$$(273) \qquad -a(1-b^{-1}x)^{-1}(1-b^{-1}u_1)^{-1} \cdots (1-b^{-1}u_h)^{-1} + u_i',$$

where a, b now refer to (273) rather than (265). Again, a root in which all the unknowns are equal, say to a new u, is sought. The h equations all become the same:

$$-a(1-b^{-1}x)^{-1}(1-b^{-1}u)^{-h} + u'.$$

This equation has separable variables and can be solved in terms of the elementary functions. A root which is 0 initially is

$$(274) \qquad u = b - b[1 + \log(1-b^{-1}x)^{a(h+1)}]^{1/(h+1)}.$$

The branch of the logarithm and, for $0 < h$, the branch of the $(h+1)$-th root can be chosen so that 0 is non-singular and so that $u(0) = 0$.

This root dominates the tentative root of (272) and therefore that of (271). The tentative root of (271) accordingly converges and has non-negative coefficients. The equations determining those coefficients say essentially that the result of substituting the tentative root in any equation of the system is a function with 0 coefficients. Hence the tentative root of (271) satisfies (271); this argument can also be formalized as is done in (275) for the general system.

The tentative root of (265) and that of S therefore converge.

(275) Consistency. The coefficients of the tentative root are found, see (38), from the system

$$(276) \qquad (pf_0)' + X \cdot 1$$

for every monomial p in the multipliers of f_0.

Substitution of the tentative root converts f_0 into an analytic function of the independent variables. Henceforth f_0 stands for that function.

For the non-multipliers equal to zero f_0 becomes analytic function e in the multipliers. By (276) e is root of

$$(pe)' + Y \cdot 1,$$

whose independent variables are the set Y of multipliers for f_0. This infinite system says that all coefficients of e are 0 so that f_0 is 0 for its non-multipliers 0.

The tentative root is a root of S, therefore, if and only if f_0 is a root of the first order system

(277) $$(zf_0)',$$

where z is non-multiplier for f_0.

By (35) there is in S a g_0 whose right derivative is that of f_0 differentiated re z Hence, see (26), the identity

$$(zf_0)' = g_0 + g - [zf_0]'.$$

Elimination of right derivatives by means of the equations defining the f_0's gives

$$(zf_0)' = h(x, f_0) + k(x, d),$$

where $h(x, 0) = 0$ and d is a derivative corresponding to a term in U^1.

For satisfaction of (277) it is necessary and sufficient that

(278) $$k(x, d) = 0$$

identically in all their arguments.

System S is (279) *passive* if (278) are true.

Functions f_0 for a passive system form a root of system

(280) $$-h(x, f_0) + (zf_0)'.$$

Let the right derivative in f have integer a for numerical ordinal and give $(i_1 \cdots i_r f_0)'$ ordinal $ai_n \cdots i_1$. System (280) fails to be canonical only in that the ordering is not canonical as prescribed by (253). The proof of (263), however, applies to the tentative root of (280). Obviously $f_0 = 0$ satisfies (280) and has initial determination 0. Hence $f_0 = 0$.

(281) *A passive canonical system has a root uniquely determined by arbitrary initial determination.*

(282) Riquier's existence theorem. The definition of canonical system becomes that of (283) *orthonomic system* if canonical ordering is replaced by orthonomic, now to be described.

(284) *Orthonomic ordering* gives derivative $i_1 \cdots i_n j$ the two-component ordinal $(i_1 + \cdots + i_n + c_j)m$, where c_j is a root of the order system and m is the monomial $i_1 \cdots i_n j$ evaluated for a root of the inequality system.

Existence theorem (281) remains true if "canonical" is replaced by "orthonomic".

The order system effectively says that differentiation gives an S^* which is of order h_i in unknown u_i, the right derivatives being the whole set of order h_i whose monomials are multiples of principal monomials by multipliers: in the canonical case $h = h_1 = \cdots = h_r$. The number of times the j-th equation has to be differentiated is c_j. Obviously, if the order system is consistent and t_k is a root of it, so also is $t_k + 1$ so that as in the canonical case S^* can be assumed linear in the derivatives of highest order h_i.

The dominant system is (269), in which sets L, D are to be reinterpreted.

Set D is now the union of r sets, each consisting of the g_i right derivatives of order h_i and unknown u_i.

Constant m_R is had by evaluating the monomial of d_R for a root of the inequality system (53) with $k = 2arg_i$, where u_i is the unknown on the right of the inequality to be modified by factor k.

The constant in L corresponding to d in D is $m_R(2arg_i m)^{-1}$, where m is the monomial of d evaluated for a root of the inequality system as described in connection with m_R.

Modify dominant system (269) by the substitution

$$u_i = u_i v_i, \qquad X \cdot W = x$$

which entails

$$d = m(h_i u_i)', \qquad d_R = m_R(h_R u_R)'.$$

System (269) becomes

$$-(1 - t)^{-1}(1 \cdot U + 2ar) + 2ar + 2r(h_R u_R)',$$

where U is now the set of r unequal $(h_R u_R)'$.

Summation gives

$$-2art + (1 - 2r)1 \cdot U$$

and substitution finally

(285) $$-2at(1 - 2t)^{-1} + (h_R u_R)'.$$

This system can be replaced by a system of the first order in which there are $h_1 + \cdots + h_r$ equations in an equal number of unknowns replacing the h equations (272). The dominant system can be given the form (273) so that it has a root in which all the unknowns are equal. That root has form (274) if h in (274) is interpreted as $h_1 + \cdots + h_r$.

A shorter way of proceeding is to remark that (285) is canonical. Its tentative root therefore converges. Since there is just one equation in each unknown, the tentative root satisfies the system (there are no conditions of passivity for (285)).

The discussion in (275) applies without modification to orthonomic systems so that (281) is indeed extended to them.

E. Delassus [27, 425] gives derivative $(i_1 \cdots i_n j)'$ the $(n + 2)$ components $(i_1 + \cdots + i_n)(r - j + 1)i_1 \cdots i_n$; classification re unknown comes second rather than third as in canonical ordering. Delassus' ordering is a special orthonomic ordering.

C. Riquier [104, 195] describes what is here called orthonomic ordering in a different manner. He assigns a matrix C of non-negative integers called cotes. There is a row belonging to each x and to each u. The total number of columns is left indefinite and is modified from time to time to serve convenience. The numbers in the k-th column are the k-th cotes. The first cote of each x is 1. The cotes of a derivative are found by adding those of the unknown and the variables involved.

In [129] it is shown that two cotes suffice. The equivalence of Riquier's ordering to orthonomic ordering is proved in [130, 287]. Elementary transformations of matrices of cotes is discussed in [129].

The statement in [130, 249] can be misinterpreted. The cotes are essentially the means of ordering the derivatives and cannot be dispensed with. The discussion of (280) really takes advantage of the essential indeterminateness of the number of cotes and introduces an additional cote at the propitious moment. Moreover, it is to be noted that (280) is orthonomic, rather than canonical.

Canonical ordering seems to have three advantages: it is easy to describe; a canonical system is easily recognized; of the forms known to fill the needs of Chapter 6 (and that means necessarily an orthonomic form) the canonical seems the simplest.

(286) Generalizations. In [130] the existence theorem is extended to systems called (287) *orderly*, which can be decomposed into orthonomic components.

C. Riquier [104, 39] gave generalizations subject to conditions of the functions defining the system.

(287) Non-convergent tentative root. To show that the tentative root does not converge for all systems S. Kowalevsky [73, 22] used an example which has become classical

(288) $$2 - 2(1 - x_2)^{-3} - (02)' + (10)'.$$

where there are two independent variables and one unknown. The system is not orthonomic: the order system

$$0 \leqq c_1 , \qquad 2 + c_1 \leqq 1 + c_1$$

is inconsistent so that dominating system (269) is not available.

It is interesting to see why another form of system, say

$$(289) \qquad 3 - 3[1 - 2x_1 - 2x_2 - 2(02)']^{-1} + (10)'$$

also fails. The left side of (289) dominates that of (288). Seeking a root which is a function of $x = x_1 + x_2$ gives

$$(290) \qquad 3 - 3(1 - 2x - 2u'')^{-1} + u'.$$

This equation has a holomorphic root with $u(0) = u'(0) = u''(0) = 0$, $u'''(0) = -1$.

It can be shown by the ratio test that the tentative root of (288) with initial determination 0 diverges except for $x_1 = x_2 = 0$.

A different situation arises in another classical example:

$$-20 - 02 + 11$$

which has consistent order system. The inequality system

$$x_1^2 < x_1x_2 , \qquad x_2^2 < x_1x_2$$

is inconsistent.

Differentiation gives

$$-30 - 12 + 21,$$

$$-21 - 03 + 12$$

and addition

$$30 + 03,$$

that is, a relation between derivatives in the parametric part so that there is not a root with arbitrary initial determination.

See [102, 135], [104, 309–310].

(291) Examples. Consider first a system in a single unknown u and two independent variables x, y. As before, notation 21 means the derivative twice re x and once re y or its x-monomial. A numerical coefficient is separated by a dot.

$$(292) \qquad - \quad 20 - 30 + 21$$

$$(293) \qquad - \quad 10 - 2\cdot 20 - 30 + 12$$

(294)	$-01 - 2\cdot 10 - 3\cdot 20 - 30 + 03$					
M	21	12	03			
LCM	23					
M^1	00	10	01	20	11	02
u_1^1	00	00	00	10	00	00
M^2	*21	*12	*03	22	13	23
u_1^2	10	00	01	10	01	11

The system is canonical.

To have explicit u_1^1 , form the dot product of monomials M^1 by arbitrary functions of the corresponding multipliers indicated on row u_1^1 01 meaning, for example, that the set of multipliers is y. Thus

$$u_1^1 = a_{00} + a_{10}x + a_{01}y + a_{20}(x)x^2 + a_{11}xy + a_{02}y^2.$$

In the same way

$$u_1^2 = a_{21}(x)x^2y + a_{12}xy^2 + a_{03}(y)y^3 + a_{22}(x)x^2y^2 + a_{13}(y)xy^3 + a_{23}(x, y)x^2y^3.$$

Member (293) differentiated once re x and once re y is indicated by 11(293).

20(292)	$- 40 - 50 + 41$	
20(293)	$- 30 - 2\cdot 40 - 50 + 32$	
11(293)	$- 21 - 2\cdot 31 - 41 + 23$	S^*
02(293)	$- 12 - 2\cdot 22 - 32 + 14$	
02(294)	$-2\cdot 12 - 3\cdot 22 - 32 + 05$	

$M^{1*} - M^1$	21	31	12	22	13	03	04
$u^{1*} - u^1$	00	00	00	00	00	00	00
M^{2*}	41	32	23	14	05		
u^{2*}	10	10	11	01	01		

To produce right derivative 32 equation 11(292) could replace 20(293) above:

11(292) $-31 - 41 + 32$

The collapsed sets are:

M_1	00	01	11	02
u_{11}	10	00	00	00
M_2	*21	*12	*03	
u_{12}	11	01	01	

$$u_{11} = a_{00}(x) + a_{01}y + a_{11}xy + a_{02}y^2$$

$$u_{12} = a_{21}(x, y)x^2y + a_{12}(y)xy^2 + a_{03}(y)y^3$$

Accidentally, $M = M_2$. This is not always so; see below.

Differentiate each equation of S re its non-multipliers as shown by row u_{12}.

(295) = 10(293) $\quad - \; 20 - 2\cdot 30 - 40 + 22$

(296) = 10(294) $\quad -11 - 2\cdot 20 - 3\cdot 30 - 40 + 13$

Differentiate to eliminate derivatives of principal derivatives.

(297) = 01(292) $\quad - \; 21 - 31 + 22$

(298) = 01(293) $\quad -11 - 2\cdot 21 - 31 + 13$

(299) = 10(292) $\quad - \; 30 - 40 + 31$

$$(292) - (295) + (297) + (299) = 0$$

$$2\cdot(292) - (296) + (298) + (299) = 0$$

System S is therefore passive.

The system with the principal derivatives for right sides is:

(292) $\quad - \; 20 - 30 + 21$

(293) $\quad - \; 10 - 2\cdot 20 - 30 + 12$

(294) $\quad -01 - 2\cdot 10 - 3\cdot 20 - 30 + 03$

(297) $\quad - \; 21 - 31 + 22$

(298) $\quad - \; 11 - 2\cdot 21 - 31 + 13$

(300) = 02(292) $\quad - \; 22 - 32 + 23$

Differentiation with respect to non-multipliers gives:

01(292) $\quad$ (297)

10(293) $\quad$ (295)

01(293)	(298)
10(294)	(296)
01(297)	(300)

All but the second and fourth of these are equations of the system, elimination of principal derivatives therefore gives 0 and the corresponding passivity conditions are satisfied. The other two (295), (296) are precisely the equations from which principal derivatives were eliminated before.

Consider next a system in two unknowns u, v and two independent variables x, y. Notation 231 means the derivative of $u = u_1$ twice re x and three times re y.

(301) $-\ 001 - 101 + 011$

(302) $-\ 102 - 2\cdot 201 + 121$

(303) $-\ 101 - 301 + 102$

$001 + 111 - 021 - 301 + 102$

The right derivatives in the last two equations are the same. To satisfy (252) the last is chosen arbitrarily for temporary omission.

I $\quad 1 < x < y, \quad x^2u < v < y^2u$

$$o_1 = 1, \quad o_2 = o_3, \quad o_3 = 1$$

	1	2	3
$o_{i,j}$	1	2	3
	$-\infty$	1	$-\infty$

$$1 + c_2 \leqq c_1, \quad 2 + c_3 \leqq c_1, \quad c_3 \leqq c_2, \quad c_1 \leqq 2 + c_2$$

O $\quad c_2 \leqq c_3$

Root of *O* $\quad c_1 - 2 = c_2 = c_3$

The monomial sets are next tabulated.

	u		v
M	01	12	10
LCM	12		10
M^1	00	10	00
u^1	00	10	01

M^2	*01	11	02	*12	*10
u^2	00	10	01	11	11
M_1	00				00
u_1	10				01
M_2	*01	11			*10
u_2	11	10			11

The system with the set of principal derivatives for right derivatives is next given.

(301) $-001 - 101 + 011 = f_{01}$

(302) $-102 - 2\cdot 201 + 121 = f_{02}$

(303) $-101 - 301 + 102 = f_{03}$

(304) = 10(301) $-101 - 201 + 111 = f_{04}$

(305) = 01(301) $-011 - 111 + 021 = f_{05}$

The equations for forming (280) follow.

$$-101 - 201 + 111 = (xf_{01})'$$
$$-011 - 111 + 021 = (yf_{01})'$$
$$-111 - 211 + 121 = (yf_{04})'$$
$$-111 - 211 + 121 = (xf_{05})'$$

Elimination of the principal derivatives eliminates the parametric also. Hence the system, as truncated, is passive. Relations (280) also result.

(306) $-f_{04} + (xf_{01})'$

(307) $-f_{05} + (yf_{01})'$

(308) $-f_{02} - f_{03} + f_{04} + (xf_{04})' + (yf_{04})'$

(309) $-(yf_{04})' + (xf_{05})'$

If now the omitted equation is restored as f_{06}, the additional passivity condition is found to be satisfied, the relation (280) resulting from the elimination being

(310) $f_{01} - f_{03} + f_{05} + f_{06}$.

Two other monomial sets which can be advantageously used to illustrate the theory are

$$xy, \quad xz, \quad z^2;$$

$$x^2y^2, \quad xz, \quad y^2z, \quad z^2.$$

The results for different orderings of the independent variables can be compared.

(311) First order. The systems now to be considered are assumed to satisfy all the requirements for a canonical system with the possible exception of (253). Let the maximum order of a derivative in system S be one. Let S be in solved form with all right derivatives of order one. Finally, what is most important for present purposes, let each unknown and each parametric derivative of order one precede each right derivative and hence be eligible to appear as left derivative in any member of S.

With such a system it is convenient to associate a matrix of n rows and r columns, position ij being associated with the x_i derivative of u_j, that is, with $(x_iu_j)'$, also written ij. Position ij is (312) *hole* or (313) *fill* according as the corresponding derivative is parametric or principal. The matrix is the (314) *array* of S. A hole may be imagined as empty and fill ij as occupied by the equation whose right derivative is ij.

Renumbering the independent variables amounts to permuting the rows of the array, hole and fill being invariant. Renumbering the unknowns amounts similarly to permuting the columns. Let the rows be permuted until the number of holes in a row does not increase with the index of the row. Let the columns be similarly permuted until the number of holes in a column does not increase with the index of the column.

If the hole $(i + 1)j$ is below fill ij, some fill $(i + 1)k$ is below hole ik. The derivative corresponding to any hole must precede that corresponding to any fill. Hence the order relations

$$(i + 1)j < (i + 1)k, \qquad ik < ij$$

which by (22) are equivalent to

$$j < k, \qquad k < j.$$

Hence no hole is below a fill. Similarly, no hole is to the right of a fill. A broken line with horizontal and vertical pieces separates the holes from the fills. The holes are in the upper left and the fills in the lower right region.

Every system with the properties enumerated at the beginning of this section can be made to have an array of the sort just described.

(315) Regular systems. A system in whose array no hole is below or to the right of a fill is (316) *regular.*

For regular S the essential part of the inequality system is

$$u_a < x_b u_c \qquad x_d u_e < x_f u_g \ .$$

That the system has the root

$$x_b = b + r, \qquad u_a = a$$

is seen by substitution because

$$d < f, \qquad e < g.$$

The order system has the root $c_i = 0$ because (56) are satisfied.

Hence giving derivative ij ordinal equal to the integer $(i + r)j$ makes the ordering, and therefore S, orthonomic.

(317) *Every regular system is orthonomic.*

An orthonomic system which is not regular is

$$-u - v + (yv)', \qquad -(yu)' - (xv)' + (xu)'.$$

(318) Special regular systems. If all places in one row of the array are fills and all others are holes, regular S is a (319) *Cauchy-Kowalevsky* system. It is the only non-empty regular system with empty set of passivity conditions.

If all places in the array are fills, S is a (320) *total* system.

If all places in any set of columns are fills and all others are holes, S is a (321) *Koenig* system.

A canonical system of the first order is also a special regular system.

(322) Existence for regular systems. The existence theorem for passive regular systems was first proved by C. Riquier [104, 468]. Previously, J. Koenig [72] had proved the theorem for his systems by a method later used by E. Cartan [12] for his theory of Pfaffian systems (see Chapter 9). Later the same method was used [133] to base a proof of the existence theorem for regular passivè systems upon the Cauchy-Kowalevsky theorem. That proof will now be outlined.

Use the complete set (46) to describe the initial determination I. For each unknown I contains an arbitrary function of the variables corresponding to the holes in the column of that unknown. The unknown reduces to that function when the variables corresponding to the fills in its column are made zero.

Write

$$S = S_1 + S_2 + \cdots + S_n ,$$

where S_k is the k-th row of S imagined written in the form of its array. Let T_k , u_{ik} be what S_k , u_i become in x_1 , $\cdots$, x_k for $x_i = 0, k < i$. For fixed k each T_k is a determined Cauchy-Kowalevsky system in unknowns u_{ik} , the initial determination being $u_{i,k-1}$. By induction on k the existence of a unique root of each determined T_k is proved. As in the general theory previously developed here, the root of $T_n = S_n$ substituted in S_k gives functions f_0 satisfying determined (280) provided S is passive. As before, $f_0 = 0$ so that the root of S_n is root of S.

To illustrate, use a system in three independent variables x, y, z and three unknowns u_1 , u_2 , u_3 :

$$\begin{array}{cccc} & & 13 & \\ & 22 & 23 & S \\ 31 & 32 & 33 & \end{array}$$

Take initial determination I as

(323) $\quad u_1(x, y, 0) = x + y, \qquad u_2(x, 0, 0) = x^2, \qquad u_3(0, 0, 0) = 1.$

Let systems S_i be as below.

$$\begin{array}{ll} -2\cdot 11 \qquad\qquad - \;\; 12 + 13 = f_{01} & S_1 \\ -\;\; 11 + \;\; 21 - \;\; 12 + 22 = f_{02} & S_2 \\ -\;\; 11 - \;\; 21 - \;\; 12 + 23 = f_{03} & \\ -\;\; 11 - 2\cdot 21 - 2\cdot 12 + 31 = f_{04} & \\ -3\cdot 11 + 2\cdot 21 - 2\cdot 12 + 32 = f_{05} & S_3 \\ -5\cdot 11 - 2\cdot 21 - 6\cdot 12 + 33 = f_{06} & \end{array}$$

From (323) for $y = z = 0$

$$u_1 = x, \qquad u_2 = x, \qquad 11 = 21 = \text{unity}, \qquad 12 = 2x$$

and for $z = 0$

$$u_1 = x + y, \qquad 11 = 21 = \text{unity}.$$

For the unknowns in T_i use the notation

$$u_{31} = u_3(x, 0, 0) = u_4 ,$$

$$u_{22} = u_2(x, y, 0) = u_5 ,$$

$$u_{32} = u_3(x, y, 0) = u_6 .$$

Systems T are next tabulated.

$$-2 + 2x + 14 \qquad T_1$$

$$\begin{matrix} -15 + 25 \\ -2 - 15 + 26 \end{matrix} \qquad T_2$$

$$S_3 \qquad T_3$$

The initial determinations are below.

$$1 = u_4(0) \qquad T_1$$

$$x^2 = u_5(x, 0), \qquad u_4 = u_6(x, 0) \qquad T_2$$

$$x + y = u_1(x, y, 0), \qquad u_5 = u_2(x, y, 0), \qquad u_6 = u_3(x, y, 0) \qquad T_3$$

The roots of T_1, T_2 are accidentally polynomials.

$$-1 - 2x + x^2 + u_4 \qquad T_1$$

$$\begin{matrix} -x^2 - 2xy - y^2 + u_5 \\ -1 - 2x - 2y + x^2 - 2xy - y^2 + u_6 \end{matrix} \qquad T_2$$

Relations (280) can be had from the following obtained in the order written. On the left are derivatives of u_1, u_2, u_3. On the right 14 is written for $(f_{04}x_1)'$.

$$\begin{aligned}
- 111 - 2\cdot121 - 2\cdot112 + 131 &= 14 \\
- 111 + 121 - 112 + 122 &= 12 \\
- 3\cdot111 + 2\cdot121 - 2\cdot112 + 132 &= 15 \\
- 2\cdot111 + 121 - 2\cdot112 - 2\cdot221 + 231 &= 2\cdot12 + 24 \\
- 2\cdot111 - 121 - 2\cdot112 + 2\cdot221 + 322 &= -2\cdot12 + 32 + 14 - 24 + 15 \\
- 2\cdot111 - 121 - 2\cdot112 + 2\cdot221 + 232 &= 2\cdot12 + 25 \\
- 5\cdot111 - 2\cdot121 - 6\cdot112 + 313 &= 31 + 2\cdot14 + 15 \\
- 6\cdot111 + 121 - 6\cdot112 - 2\cdot221 + 323 &= 12 + 33 + 14 + 24 + 15 \\
- 5\cdot111 - 2\cdot121 - 6\cdot112 + 133 &= 16 \\
- 6\cdot111 + 121 - 6\cdot112 - 2\cdot221 + 233 &= 26 + 6\cdot12
\end{aligned}$$

System (280) is finally given.

$$-4\cdot 12 + \quad 32 + 14 - 24 + \quad 15 - 25$$

$$12 + \quad 33 + 14 - 24 + \quad 15 - 16$$

$$31 + 2\cdot 14 + 15 - 26 - 6\cdot 12$$

Since the parametric derivatives cancel, S is passive.

CHAPTER

5

Algebraic Differential Systems

(324) Definitions. Let the indeterminates be a finite set of the derivatives re $x_1, \cdots, x_n$ of unknowns $u_1, \cdots, u_r$, to be designated collectively by $v_1, v_2, \cdots$. Let the coefficients be analytic functions of the x's with 0 for ordinary point.

The system of two equations in one unknown u

$$1 + a + u, \qquad 1 - a + u \qquad S$$

implies the equation a. If a is a given function and is not identically 0, immediately $S = 1$. Success in treating the systems discussed in the present chapter is conditioned by knowledge of the identities satisfied by the analytic functions involved.

Likewise the equation

$$-1 + au$$

implies the inequation a^N. There are, of course, other inequations which must be satisfied by the independent variables if the region is to consist of points ordinary for the functions defining the system. Consequently, S is accompanied by a system S_I of inequations with unknowns x. If exceptionally the members of S_I are polynomials in x, system S_I can be factored by the methods of Chapter 3. In general, however, S_I is of the type treated in Chapter 6. This circumstance detracts from elegance in treating the algebraic case separately.

The definitions (23), (136) of ordinal for derivative and unknown are seen to be in harmony since u_j as derivative of order 0 has ordinal $00 \cdots 0j$. Indeterminates v_j are to have the ordinals assigned them as derivatives. Definitions (136) to (141) of grade, index, right indeterminate, initial and ordinal are immediately applicable to the present polynomials. The factorization process is therefore available.

(325) Reduction. The simple factors of Chapter 3, however, have to be further reduced to insure that the components of the algebraic root have the desired differential relationship to each other.

(326) Derived system. If f is an equation of S, equation $[xf]'$ is implied by S and can be adjoined to the base of S.

If f has index k, since

$$(327) \qquad [xf]' = (v_k f)'(xv_k)' + \cdots ,$$

the initial of $[xf]'$ is $(xv_k)'$ and denotes an inequation if f is taken from a simple factor. Thanks to linearity the discriminant is 1. Hence $[xf]'$ is available for reducing. Because it is linear in its right derivative, used as is f in reduction (177) it here actually eliminates $(xv_k)'$.

(328) Passivity. The ninth and final reduction is the adjunction

$$(329) \qquad S = S + \{f, g\},$$

where f, g are equations in S with right indeterminates $(acu)'$, $(bcu)'$, the differential operators a, b are relatively prime and $\{f, g\}$ is the resultant of $[bf]'$, $[ag]'$ re their common right indeterminate $(abcu)'$.

The adjoined polynomial either is 0 or has a right indeterminate which precedes $(abcu)'$. By (48) the second alternative ultimately disappears. The factor is then (330) *passive*, a definition which is seen in the next section to be in essential agreement with (279).

(331) Existence theorem. If the x's are made 0 and the parametric indeterminates are given arbitrary values in a final factor which is passive and not 1, values can be had for the principal indeterminates to complete a root C of the original algebraic system. For that root the initials and discriminants are not 0. Hence by the implicit function theorem [92, 14] near C each factor equals a system, passive in sense (279) and canonical or orthonomic according as the adopted ordinal is canonical or orthonomic.

(332) Examples. Consider first an ordinary system in one unknown u with derivatives

$$(ju)' = u_j \,.$$

$$(333) \qquad -u_1^2 + uu_2$$

$$(334) \qquad -u_2^2 + u_1u_3$$

$$S = (333) + (334)$$

Clearly one passive factor is u_1, u_2. To the other factor reduction (329) is immediately applicable with

$$f = (333), \qquad g = (334), \qquad a = 1, \qquad b = x, \qquad c = 1.$$

$$(333)_1 \qquad -u_1u_2 + uu_3$$

$$\{f, g\} = -uu_2^2 + u_1^2u_2 = -u_2(333)$$

The brace reduced by (333) is 0. The factor is passive.

This second factor can be solved for u_2, u_3 thus:

$$u_1^N, \qquad -u^{-1}u_1^2 + u_2, \qquad -u^{-2}u_1^3 + u_3 .$$

The ordinary system

$$-u_1 + xu_2, \qquad 4 - 4u + u_1^2 \qquad\qquad S$$

can be factored in similar fashion

$$S = S_1S_2,$$

$$S_1 = -1 + u, \qquad S_2 = -1 - x^2 + u.$$

For the partial derivatives of a single unknown re x, y use the Monge abbreviations

(335) $p = 101, \quad q = 011, \quad r = 201, \quad s = 111, \quad t = 021.$

(336) $p^2 - (x + y + 1)p + x + y$

(337) $q^2 - (x + 1)q + x$

$$S = (336) + (337)$$

(338) = 01(336) $\quad (2p - x - y - 1)s - p + 1$

(339) = 10(337) $\quad (2q - x - 1)s - q + 1$

(340) $\quad (1 - x)p + (x + y - 1)q - y$

Polynomial (340) is the resultant of (338) and (339) as polynomials in s. The resultant of (337) and (340) as polynomials in q is found to be $-(1 - x)^2$ times (336). System

(341) $\quad$ (336) + (340)

is simple, passive and equal to S.

Since (336), (337) have rational factors, in the present case there is an alternative procedure.

$$(p - 1)(p - x - y), \qquad (q - 1)(q - x) \qquad\qquad S$$

$$S = S_1S_2S_3S_4$$

$p - 1$	$q - 1$	S_1
$p - 1$	$q - x$	S_2
$p - x - y$	$q - 1$	S_3
$p - x - y$	$q - x$	S_4

Of these, S_1, S_4 are passive. Applying (329) to S_2 gives

$$f = p - 1, \qquad g = q - x, \qquad a = x, \qquad b = y, \qquad c = 1$$

so that

$$\{f, g\} = 0 - 1$$

and the passive base for S_2 is 1, that is, S_2 is inconsistent. So also is S_3.

The solution is found by integrating exact differentials:

$$S_1 = -c_1 - x - y + u, \qquad S_4 = -c_2 - xy - 2^{-1}x^2 + u,$$

$$S = S_1 S_4,$$

where c_1, c_2 are arbitrary constants.

(342) Constant coefficients. Let $A^1, \cdots, A^k$ be sets of constants. Let U be the set of indeterminates $u_1, \cdots, u_n$. Let M be a set of monomials in the u's. Let X be the set of independent variables $x_1, \cdots, x_n$. Let u without index be a single unknown. Let D be the set of derivatives of u re the differential operators got by replacing each u_i in M by the x_i with equal index.

A duality between the algebraic system

$$S = A^1 \cdot M + \cdots + A^k \cdot M$$

and the algebraic differential system

$$T = A^1 \cdot D + \cdots + A^k \cdot D$$

has been remarked by C. Riquier [103] and by J. A. Greenwood [47].

Three theorems illustrating this duality are

(343) $$S = 1 \rightleftarrows T = u$$

(344) $$S = x_1 + \cdots + x_n \rightleftarrows T = -f + u,$$

(345) $$S \leqq (x_1 - b_1) + \cdots + (x_n - b_n) \rightleftarrows T \leqq -e(B \cdot X) + u,$$

where f is a polynomial in the x's, B is a set of constants $b_1, \cdots, b_n$ and e is the exponential function. For simple proofs see [47].

CHAPTER

6

Reduction to Passive Form

(346) Function systems. Analytic function

(347) $$f = a_0 + a_1u_r + a_2u_r^2 + \cdots$$

has root u_r if the sum of the series to n terms has limit 0 for n infinite. The root may be an ordinary or a singular point for f. A limit point of roots of non-constant f is a singular point for f.

Unlike polynomials with complex coefficients a non-constant analytic function may have no root, for example, the exponential $e(u)$.

A closed disk within the circle of convergence contains a finite number (possibly zero) of roots of f. Each root of f^N within the circle of convergence is the center of a disk of roots of f^N.

On the assumption that a root, already (254) called a numerical determination, is known for each f or f^N, the solution in neighborhood A of the numerical determination is now to be discussed for systems whose members have the form (347) with a's analytic functions of $u_1, \cdots, u_{r-1}$.

Root $(u_1, \cdots, u_r)$ of f is (348) *special* if it is a root of the infinite system

(349) $$a_0 + a_1 + a_2 + \cdots;$$

otherwise, the root is *non-special*. The value of u_r in a special root is arbitrary. For example,

$$f = (x + y) + (x - y)^2z + (x + y)^3z^2 + \cdots + [x - (-1)^ny]^nz^n + \cdots$$

has the special root (0, 0, z) with z arbitrary. Interpreted in Euclidean geometry described by Cartesian coordinates this means that the surface composed of the roots of f contains the z-axis.

In system S consider the members of maximum ordinal.

Suppose all of them are inequations. By assumption the roots of each corresponding equation are known as soon as the indeterminates of lower ordinal are given values. Let neighborhood A be restricted so as to exclude such roots. In particular, inequations f^N such that f has no root can be omitted from S.

Let equation f be a member of maximum ordinal. If f has no root, then $S = 1$.

If equation f has only special roots, the members of S with smaller ordinal imply (349) and hence f so that f can be omitted from S without changing the solution.

If f has an ordinary root $(b_1, \cdots, b_r)$, the Weierstrass preparation theorem [92, 86] shows that

$$f = pq,$$

where

(350) p is a polynomial of degree m in u_r;

(351) the coefficients of p are functions of $u_1, \cdots, u_{r-1}$;

(352) the initial of p is 1;

(353) for $(u_1, \cdots, u_{r-1}) = (b_1, \cdots, b_{r-1})$ polynomial $p = (u - b_r)^m$;

(354) root $(b_1, \cdots, b_r)$ of p is an ordinary point for function q and is a root of inequation q^N.

In suitably restricted A the roots of f are the m roots of p. Hence member f can be replaced by polynomial p. The discriminant of p is 0 at b but at no other point of A.

Let g be a second member of S with the same ordinal as f. If g is equation and g has no root in A, then $S = 1$ in A. If g is inequation and g has no root in A, omit g^N from S. If every root of g in A is special, replace member g by member $(f + g)$ labeled equation or inequation according as g is; the roots of $(f + g)$ are non-special. Let, therefore, g have non-special root $(c_1, \cdots, c_r)$ in A. Then in a subregion of A,

$$g = p_1 q_1,$$

where p_1, q_1 have properties (350) to (354) in which p, q, b, m are replaced by p_1, q_1, c, m_1.

The members of ordinal r in the finite base of S can therefore be replaced by polynomials. If the roots like b are excluded from the final A, the polynomials have non-zero initials and discriminants.

Application of the algebraic processes in Chapter 3 gives simple factors for S.

If some of the unknowns are derivatives of the others re $x_1, \cdots, x_n$, the system is differential and the processes of Chapter 5 reduce it to passive form.

To each factorization of S is attached a region A in which it is valid and to which the initial determination must be restricted in order to give a root.

As in (331) a passive factor which is not 1 can be put in canonical or orthonomic form to which the existence theorem is applicable.

Each root of S shows up in some factorization. The totality of factorizations gives the solution of S.

(355) Historical remarks. H. Kistler [70] used the methods developed by Kronecker [74] for algebraic systems in discussing function systems. W. D. Macmillan [83] was perhaps the first to apply a slightly modified form of the Weierstrass theorem to reduce a function system to an algebraic system.

Among the first to consider the consistency of the general differential system were A. Tresse [138] and C. Méray [87]. Tresse considered the passivity conditions but did not give an existence theorem for a passive system. Méray's methods were based on a change of independent variable and were not wholly successful.

C. Riquier collaborated with Méray [87], [88] and later gave a reduction to passive form based on the implicit function theorem as well as his existence theorem for passive orthonomic systems.

Later E. Delassus [27] gave a method likewise based on change of variable. That his canonical form does not have the generality claimed for it was pointed out by N. Gunther [50] and L. B. Robinson [108], [109]. Their example is essentially the following. Let S have one unknown u, three independent variables x, y, z, and right derivatives

$$200, \qquad 110, \qquad 020.$$

In order for Delassus' reduction to apply to S, linear, homogeneous transformation should carry S into canonical form

$$200, \qquad 110, \qquad 101.$$

By (342) this would mean that algebraic systems

(356) $$x^2, \qquad xy, \qquad y^2$$

(357) $$x^2, \qquad xy, \qquad xz$$

should be equivalent under linear, homogeneous transformation. But these systems are, respectively, equivalent to

(358) $$x, \qquad y,$$

(359) $$x.$$

Geometrically phrased, the solution of (356) is a line and that of (357) a plane.

E. Cartan [12] gave the existence theorem for the regular systems occurring in his theory of Pfaffian systems. No generally applicable reduction process leading to passive regular form has, however, been developed. Cartan's prolongation seems inadequate: see [14], [15], [67], [69], [76] and Chapter 9.

L. B. Robinson [107] using Riquier's methods to some extent gave a reduction to passive form involving change of variables.

M. Janet [65] gave an exposition of Riquier's work [104] with fruitful modifications. Janet's article was the starting point for the present writer's study of the problem; see in particular [135].

J. F. Ritt [106] was the first to isolate the algebraic differential case and to find for it a reduction to passive form. The advantage in doing this lies in the fundamental theorem of algebra whose counterpart does not exist in the general case. See [106, 161], [105, preface], [119], and (324) in this connection.

An excellent account of the early history of the problem is in C. Riquier's treatise [104]. Later accounts are L. B. Robinson [107], M. Janet [66], D. L. Bernstein [3], A. Erdélyi [34].

(360) Applications. Let there be a single unknown u, n independent variables $x_1, \cdots, x_n$ and r equations of the form

$$A_i \cdot D,$$

where each A_i is a set of n functions of the x's (vector) and D is the set of first derivatives of u re x (grad u). The passivity conditions have the same form as the system, namely, linear, homogeneous. The system has a non-constant root if and only if the rank of the equivalent passive system is less than the number of independent variables n, see E. Goursat [45, 65–73].

Systems of total differential equations often occur in differential geometry; in fact, many of the early problems amount to the special case of the exact differential. Examples of more complicated situations are found in J. E. Wright [151], O. Veblen [144], J. A. Schouten and W. van der Kulk [112].

A relatively simple example of such a problem is the determination of unknown functions u, v of x, y which satisfy

$$du^2 + dv^2 = y^2\, dx^2 + dy^2$$

in the ordinary differential notation. The system follows:

$$-y^2 + 11 \cdot 11 + 12 \cdot 12$$

$$11\cdot 21 + 12\cdot 22$$

$$-1 + 21\cdot 21 + 22\cdot 22$$

$$y\ 21 + 111$$

$$-y^{-1}\ 11 + 121$$

$$221$$

$$y\ 22 + 112$$

$$-y^{-1}\ 12 + 122$$

$$222$$

The solution can, of course, be had in finite terms and is

$$-y \cos(ex + c) - a + u,$$

$$-y \sin(ex + c) - b + v,$$

where a, b, c are arbitrary constants and $e^2 = 1$. In the Euclidean plane (x, y) can be interpreted as polar and (u, v) as rectangular coordinates.

The equation

$$du^2 + dv^2 = dx^2 + dy^2$$

can be treated similarly. The resulting formulas define the automorphisms of $dx^2 + dy^2$, that is, the group of motions and reflections in the Euclidean plane.

The groups of automorphisms of the following can also be found in finite form:

$$y\,dx^2 + x\,dy^2,$$

$$dx^2 + x\,dy^2,$$

$$2(x^2 + y)^{-1}\,dx\,dy.$$

J. Douglas [31] has ingeniously applied the theory to find integrals having given extremals.

L. Schlaefli [110] stated without proof that a Riemann space of n dimensions can be locally imbedded in a Euclidean space of $n(n + 1)/2$ dimensions. The rest of the present section will be devoted to the subsequent history of this problem.

M. Janet [63] seems to have made the first substantial contribution to the direct study of this problem. For $n = 2$ using his own modifications of Riquier's methods he put the pertinent system in passive form and established the result in all rigor. The same special case has been discussed at length by V. S. Lyuksin [78].

E. W. Herron [58] in addition to proving the theorem for $n = 2$ remarks that an indirect proof is in the older literature (see also L. Zippin, [152]). The Cauchy-Kowalevsky theorem is directly applicable to the Gauss-Codazzi equations and shows the existence of the second fundamental form for an arbitrary first fundamental form. An appeal to the classical result that each root of the Gauss-Codazzi system gives a surface in three-dimensional Euclidean space completes the proof.

Any one of these proofs seems effectively to dispose of A. R. Forsyth's contention [39] that a passivity condition is inconsistent: he claims that two coefficients are not zero but does not give their explicit expression.

For the general case M. Janet [63] gave a reduction not claimed to be complete. Later E. Cartan gave a proof [14] for the general n and still later [15, 199] for $n = 3$. Both of Cartan's proofs depend on prolongation of Pfaffian systems and therefore leave much to be desired. Cartan [14] seemed to think that assuming the metric of the Riemannian space to be a quadratic form with non-zero determinant makes Janet's discussion complete.

E. W. Herron [58] has proved the theorem for $n = 3$ using the theorem for orthonomic systems. His system is not excessively more complicated than Janet's for $n = 2$.

The non-analytic case of the problem has received attention. For $n = 2$ the theorem is proved by A. Wintner [150]. A weakened form of the theorem for general spaces is discussed by J. Nash [91].

(361) Reduction of order. It was early recognized that every system can be reduced to one in which the maximum order of a derivative is one, that is, to what is termed a system of the first order. This is done by an obvious device: If a derivative of order m is present, write it as a derivative of order $m - 1$ of a new unknown. If $2 < m$, repeat the operation.

This obvious process applied to a passive system gives a first order system, which is not necessarily passive. Reduction to a passive system of the first order can, however, be made: C. Riquier [104, 468], R. T. Herbst [57]. Whether such a reduction can be used to simplify the proof of the existence theorem for partial systems is an open question.

Reduction of a single ordinary equation to a passive system of the first order is easy and has been used in the proof of the existence theorem (272). The gain thereby is slight.

J. Drach [32] noticed that every differential system leads to one of order *two* in a single unknown. He uses independent variables $x_{n+1}, \cdots, x_{n+r}$ in addition to the given independent variables $x_1, \cdots, x_n$. He takes as the single unknown

$$u = x_{n+1}u_1 + \cdots + x_{n+r}u_r \tag{362}$$

so that

$$(x_{n+j}u)' = u_j\,, \tag{363}$$

$$(x_i x_{n+j}u)' = (x_i u_j)'. \tag{364}$$

A system S of the first order in $u_1, \cdots, u_r, x_1, \cdots, x_n$ thus becomes one of the second order in $u, x_1, \cdots, x_{n+r}$. To insure that u_j are functions of $x_1, \cdots, x_n$ alone the second order equations.

$$0 = (x_{n+j}x_{n+k}u)' \qquad j, k = 1, \cdots, r$$

are adjoined to give system T, which is of the second order in a single unknown. Each root u of T put in (363) gives a root of S, and all roots of S are so obtained. As in the reduction to first order the passivity conditions are not automatically satisfied.

Consider the system of ordinary first order equations

$$-u_2 + (x_1u_1)', \qquad -u_3 + (x_1u_2)', \qquad -u_1 + (x_1u_3)',$$

which arises in reducing

$$-u_1 + (x_1x_1x_1u_1)' \tag{365}$$

to the first order. Drach's system is

$$\begin{gathered} -(x_3u)' + (x_1x_2u)', \quad -(x_4u)' + (x_1x_3u)', \quad -(x_2u)' + (x_1x_4u)' \\ (x_3x_4u)', \qquad (x_4x_2u)', \qquad (x_2x_3u)'. \end{gathered} \tag{366}$$

What profit can be drawn from this transformation is an open question; see, however, [32].

The ordinary linear, homogeneous differential equation of the second order can be reduced to a single Riccati equation of the first order [60, 23]. Thus order is reduced at the sacrifice of degree. See also the reductions in R. T. Herbst [55] and in [125].

CHAPTER

7

Grassmann Algebra

(367) The Grassmann ring. As before, there is a fundamental integrity domain Z, which is usually best thought of as a field: the elements are the complex numbers in the purely algebraic case and the analytic functions in the differential. To the elements of Z are now adjoined r (368) *marks* $v^1, \cdots, v^r$ to form a (369) *Grassmann ring* $Z[v^1, \cdots, v^r]$ *of degree* r.

Multiplication in this ring obeys the following laws:

(370) $$v^1 \cdots v^r \neq 0,$$

(371) $$v^i v^j = -v^j v^i,$$

(372) $$v^i v^i = 0,$$

(373) $$z \text{ in } Z \rightarrow zv^i = v^i z,$$

(374) $$z \text{ in } Z, \quad zv^1 \cdots v^r = 0 \rightarrow z = 0.$$

(375) Monomials. From (370), (372) two marks are equal only if their indices are equal; and any product containing two factors which are equal marks is 0 and is suppressed. Consequently, a (376) (*pure*) *Grassmann monomial* is the product of unequal marks. The maximum degree of a monomial is r. Permutation of the marks takes monomial $m = v^{i_1} \cdots v^{i_k}$ into $k!$ monomials of degree k which equal $+m$ or $-m$ according as the permutation is even or odd and which are (377) *like* m, all others being unlike m.

Each polynomial f can be written

$$f = z_1 m_1 + \cdots + z_n m_n ,$$

where no two monomials m_i are like and the z's are in Z.

(378) Forms. Each polynomial f can also be written

$$f = f_0 + f_1 + \cdots + f_r,$$

where each f_j is 0 or a form of degree j.

With the summation convention the form of degree p is

(379) $$F = a_{i_1 \cdots i_p} v^{i_1} \cdots v^{i_p}.$$

Subjecting the indices $1 \cdots p$ to the same permutation in all terms of (379) and multiplying by $+1$ or -1 according as the permutation is even or odd is (380) *shifting*. The shift applied either to the pure monomials or to the coefficients gives a form equal to the original.

Shifting the coefficients by all $p!$ permutations, summing and dividing by $p!$ gives the (381) *skew coefficient*. Replacing the original coefficient by the skew coefficient gives an equal form. Shifting a skew coefficient gives an equal coefficient.

In illustration, use the quadratic form in three marks, which written in full is

$$a_{12}v^1v^2 + a_{13}v^1v^3 + a_{21}v^2v^1 + a_{23}v^2v^3 + a_{31}v^3v^1 + a_{32}v^3v^2$$

and in abbreviated notation

(382) $$12\cdot12 + 13\cdot13 + 21\cdot21 + 23\cdot23 + 31\cdot31 + 32\cdot32.$$

Shifting monomials by transposition of the two indices gives

(383) $$-12\cdot21 - 13\cdot31 - 21\cdot12 - 23\cdot32 - 31\cdot13 - 32\cdot23.$$

Each term equals the corresponding term in the original. Shifting coefficients gives

(384) $$-21\cdot12 - 31\cdot13 - 12\cdot21 - 32\cdot23 - 13\cdot31 - 23\cdot32.$$

The first term equals the third term of the original, etc., but the aggregate of terms is the same, as can be seen directly or by comparing (384) with (383). Adding (382), which is the shift by the identity, and (383) or (384) gives

$$(12 - 21)12 + (13 - 31)13 + (21 - 12)21$$
$$+(23 - 32)23 + (31 - 13)31 + (32 - 23)32.$$

This is $2F$. Division by 2 gives F with skew coefficient.

In the summation notation

(385) $$F = a_{ij}v^iv^j$$

the shifts give

$$-a_{ij}v^jv^i, \qquad -a_{ji}v^iv^j,$$

so that the skew coefficient is got by addition

$$2F = (a_{ij} - a_{ji})v^iv^j$$

and is $2^{-1}(a_{ij} - a_{ji})$. The shift on it gives $2^{-1}(-a_{ji} + a_{ij})$, that is, leaves it invariant.

Henceforth, unless the contrary is stated, it is understood that in a formula like (385) the coefficient is skew, that is,

(386) $$a_{ij} + a_{ji} = 0.$$

Note that when the coefficient is skew the suppression of the terms u^iu^i is automatic since $a_{ii} = 0$ by (386).

(387) *Two forms are equal if and only if their skew coefficients are equal.*

A second example is the biquadratic form

$$F = a_{ij}b_{kl}v^iv^jv^kv^l,$$

where a_{ij}, b_{kl} are already skew. In abbreviated notation this is

$$F = ij\cdot kl\cdot ijkl.$$

To get the skew coefficient, it is sufficient to use the six permutations

(388) $$1, \quad (ik), \quad (il), \quad (jk), \quad (jl), \quad (ik)(jl)$$

which give for 6 times the desired coefficient

$$ij\cdot kl - kj\cdot il - lj\cdot ki - ik\cdot jl - il\cdot kj + kl\cdot ij.$$

Set (388) can be had by tabulating the symmetric group of degree 4 on its subgroup

$$1, \quad (ij), \quad (kl), \quad (ij)(kl)$$

whose shift leaves the original coefficient invariant.

(389) Products of forms. The product of a number of forms is either 0 or a form whose degree is the sum of the degrees of its factors. It is convenient to have a technique for changing the relative positions of its factors.

Consider first the product of two monomials f, g of respective degrees p, q. In fg the first mark of g can be put on the left by interchanging it successively with each of the p marks of f, proceeding from right to left. Subsequently, the other marks of g can be passed over those of f. Hence

(390) $$fg = (-1)^{pq}gf.$$

Next consider the product fgh, where h is a third monomial of degree r. The transposition identity $(31) = (23)(21)(23)$ permits comparison of fgh and hgf by the result just established so that

$$(391) \qquad fgh = (-1)^s hgf,$$

where

$$(392) \qquad s = qr + rp + pq.$$

From this it is clear that the multiplier is -1 if and only if an odd number of the three factors has odd degree.

Finally, consider the product FGH of three forms. Any monomial in that product is of type fgh, where f, g, h are monomials from F, G, H, respectively. The monomials of FGH and HGF are in one-to-one correspondence, with fgh and hgf corresponding. Since corresponding monomials are related by (391) and s is the same for all of them, the forms FGH and HGF are also in the relation (391). The presence of additional factors to the right or left does not affect the result. Any permutation of the factors can therefore be effected by

(393) *Interchanging two factors of even degree leaves a product of forms invariant; interchanging two factors of odd degree multiplies it by* -1; *interchanging a factor of even degree and one of odd degree multiplies it by* $(-1)^q$, *where* q *is the total degree of the intervening factors.*

(394) *The square and all higher positive integral powers of a form with odd degree are zero.*

(395) Differentiation. If monomial m contains mark u, the (396) *derivative* of m re u is the result of shifting u to the extreme left and then making it 1. If m does not contain u, the derivative is 0. The derivative of a sum is the sum of the derivatives.

The derivative of monomial m re v^i is denoted by $v^i_{-1}m$. The operator

$$(397) \qquad v^{i_1}_{-1} \cdots v^{i_k}_{-1}$$

gives a derivative of order k; attention must be given to the order in which the separate differentiations are made, the one on the left being made first.

(398) *The Grassmann ring is closed under differentiation, that is, if a polynomial is in the ring, so are all its derivatives.*

(399) *The differential operator* (397) *is invariant under shift of its indices.*

All derivatives of order greater than r are 0.

The processes of differentiation are summarized in

(400) *The rules of the ordinary calculus apply to differentiation in the Grassmann ring provided before each differentiation* (393) *is used to put the factor to be differentiated on the extreme left.*

For example,

$$v_{-1}FG = (v_{-1}F)G - (v_{-1}G)F,$$

if F is linear and G is cubic.

A particular aspect of the rule is that indirect differentiation applies, as is immediately seen.

The rule applied to form (379) gives the following:

(401) $v_{-1}^{i_1} \cdots v_{-1}^{i_k}F = p(p-1) \cdots (p-k+1)a_{i_1 \cdots i_k i_{k+1} \cdots i_p}v^{i_{k+1}} \cdots v^{i_p},$

(402) $$v_{-1}^{i_1} \cdots v_{-1}^{i_k}F = p(p-1) \cdots (p-k+1)F,$$

(403) $$v_{-1}^{i_1} \cdots v_{-1}^{i_p}F = p!\, a_{i_1 \cdots i_p},$$

(404) $$v_{-1}^{i_1} \cdots v_{-1}^{i_{p-1}}F = p!\, F.$$

Formula (402), of which (404) is a special case, is the analogue of Euler's theorem for homogeneous functions.

(405) Sets of linear forms. Formulas

(406) $$w^i = a^i_j v^j \qquad (i = 1, \cdots, n;\ j = 1, \cdots, r)$$

define a set of linear forms. The skew coefficient of the product

$$w^{i_1} \cdots w^{i_p}$$

is the determinant with rows $i_1 \cdots i_p$ and columns $j_1 \cdots j_p$ in the matrix A of the a's.

If Z is a field, if the rank of A is r and if $n = r$, system (406) can be solved for the v's. Every polynomial in the Grassmann ring $Z[v^1, \cdots, v^r]$ is in the Grassmann ring $Z[w^1, \cdots, w^r]$ and vice versa. The two rings are to be regarded as the same ring referred to different bases: see (12).

If the rank of A is s and $n = s < r$, then $Z[w^1, \cdots, w^s]$ is still a Grassmann ring but it is a proper subring of $Z[v^1, \cdots, v^r]$.

The theory of linear forms, including the properties of determinants, can be developed by Grassmann multiplication. Here are stated a few of the pertinent results.

(407) *A given independent subset is a base for a set of linear forms if and only if its rank equals that of the whole set.*

(408) *The rank of a set of linear forms is invariant under a change of the base of the Grassmann ring.*

(409) *A set of linear forms whose rank is k determines a subring of degree k any base of which can be made the first k marks in a base for the whole ring.*

(410) *The rank of a set of linear forms in a subring is the same as its rank in the whole.*

(411) *In a set of linear forms let $w^1, \cdots, w^k$ be an arbitrary but fixed subset of rank k. The whole set is of rank s if and only if every product of $s + 1$ forms among which are $w^1, \cdots, w^k$ is zero, whereas some product of s forms among which are $w^1, \cdots, w^k$ is not zero.*

(412) Multilinear forms. The substitution

(413) $$v^{i_j} = x^{i_j},$$

in which the x's are indeterminates, applied to the form (379) with skew coefficient gives a unique (414) *skew p-linear form.* The inverse substitution applied to an arbitrary skew p-linear form gives a unique form of the Grassmann ring.

The multilinear forms can be interpreted as belonging to a polynomial ring whose indeterminates obey the commutative law of multiplication. This correspondence gives a criterion for the consistency of the assumptions on which the present treatment of Grassmann algebra is based. It is also the basis for E. Cartan's view-point, see [11], [13], [15], [118].

(415) Generalized linear dependence. Because of (372) form F of degree p is of degree at most one in each of the marks. Hence

(416) $$F = v^1F_1 + f_1 ,$$

where neither F_1 nor f_1 contains v^1. Repetition gives

(417) $$F = v^1F_1 + \cdots + v^kF_k ,$$

where each F_i is not zero and contains only marks with index greater than i. The number k is identified by

(418) $$v^1 \cdots v^{k-1}F \neq 0, \qquad v^1 \cdots v^kF = 0.$$

This gives E. Cartan's useful result:

(419) *If $w^1, \cdots, w^k$ are linear forms of rank k, relations*

(420) $$F = w^1g_1 + \cdots + w^kg_k$$

and

(421) $$w^1 \cdots w^k F = 0$$

are equivalent.

(422) The associated set. The $(p - 1)$th derivatives of form (379) are linear. If the base of the Grassmann ring is changed by linear transformation, indirect differentiation shows that the derivatives re the new marks are linear homogeneous in the derivatives re the old.

Let F_i be a set of forms. The (423) *associated set* consists of the $(p_i - 1)$th derivatives of F_i. Let $w^1, \cdots, w^k$ be a base for the associated set. The subring $Z[w^1, \cdots, w^k]$ is the (424) *subring of set* F_i and k is the (425) *rank* of that set of forms.

(426) *The subring of a system of forms and their rank are invariant under change of the base of the Grassmann ring.*

Using (409), let the base of the subring of forms F_i be $v^1, \cdots, v^k$. Multiply (404) for $F = F_i$ by the product $v^1 \cdots v^k$. Because the left members of (404) are in the subring, after multiplication the left members are 0. Hence

$$v^1 \cdots v^k F_i = 0.$$

From (420)

(427) $$F_i = v^1 g_{i1} + \cdots + v^k g_{ik}\,.$$

By this is next to be proved

(428) *All the forms of a finite set belong to the subring of that set.*

The proof is by induction on the maximum degree of a form in the set. If the set contains only linear forms, it coincides with its associated set and the theorem is true for it.

To (427) apply a differential operator which contains mark v^i for i fixed and less than $k + 1$ but is otherwise arbitrary of degree $p_i - 1$. Then multiply by $v^1 \cdots v^k$. The single term $v^i g_{ii}$ contributes to the result and its contribution is $v^1 \cdots v^k$ times a derivative of order $p_i - 2$ of g_{ii}. Hence the subring for g_{ii}, which by hypothesis contains g_{ii}, belongs to the subring for F_i. The proof is thus complete.

(429) ***The rank of a set of forms is the minimum number of marks in terms of which the forms can be expressed by change of base.***

If the set were expressed in terms of fewer than k marks, the rank would be less than k.

(430) *A given mark appears in at least one form of a set if and only if it appears in at least one form of the associated set.*

The forms of a set can be subjected to a linear transformation with coefficients from Z. The derivatives then undergo the same linear transformation. Hence

(431) *The subring of a set of forms and the rank of the set are invariant under linear transformation of the forms.*

(432) Factorization. The notion of factor, namely, that G and H are factors of F if $F = GH$, is clearly independent of change of base. The case of linear factors is particularly important. Theorem (419) implies the following:

(433) *Linear w divides F if and only if $wF = 0$.*

(434) *If k independent linear forms divide F, their product divides F.*

(435) *Non-zero F can be written $F = MG$, where M is the product of linear forms or is 1 and no linear form divides G. A linear form divides F if and only if it belongs to the subring of M.*

If $G = 1$ in (435), F is a (436) *monomial form*, so called because change of base makes F equal to a monomial. The degree of G can not be 1.

(437) *Every form of degree r or $r - 1$ in a Grassmann ring of degree r is monomial.*

The result is obvious for degree r. Moreover, the form is monomial for any base.

For degree $r - 1$ use (416). In it F_1 is of degree $r - 2$ in $r - 1$ marks. Change of base makes it $v^2 \cdots v^{r-1}$. Form f_1 is of degree $r - 1$ in the $r - 1$ marks $v^2, \cdots, v^r$. Hence

$$F = v^1v^2 \cdots v^{r-1} + v^2 \cdots v^{r-1}v^r = (v^1 \pm v^r)v^2 \cdots v^{r-1},$$

the possible coefficient of f_1 having been absorbed into v^r.

(438) A quadratic form in the presence of linear forms. Let F be quadratic and $w^1, \cdots, w^h$ linear forms with $w^1 \cdots w^h \neq 0$. There is a least integer k satisfying

(439) $$w^1 \cdots w^hF^{k+1} = 0$$

because the degree of a form cannot exceed r. Equation (417) can be written

(440) $$F = w^1g_1 + \cdots + w^hg_h + w^{h+1}g_{h+1} + G,$$

where G involves only w's of index exceeding $h + 1$ and any g involves only w's with indices exceeding its own. Multiplication gives

(441) $$w^1 \cdots w^h F^{k+1} = (k + 1)w^1 \cdots w^{h+1} g_{h+1} G^k + w^1 \cdots w^h G^{k+1}.$$

Since the first term contains w^{h+1} and the second does not, they must both be zero. Consequently

(442) $$g_{h+1} G^k = 0.$$

Replacing k in (441) by $k - 1$, multiplying by g_{h+1} and using (442) give

(443) $$w^1 \cdots w^h g_{h+1} F^k = 0.$$

If there is no g_{h+1} term in (440), then

$$w^1 \cdots w^h F = 0$$

so that $k = 0$. For positive k, therefore, there is a linear form g_{h+1} satisfying (443) and

(444) $$w^1 \cdots w^h g_{h+1} \neq 0.$$

Let any linear form satisfying (443) and (444) be w^{h+1}, and let l be the least integer satisfying

$$w^1 \cdots w^{h+1} \neq 0, \qquad w^1 \cdots w^{h+1} F^{l+1} = 0.$$

The same process can now be repeated to increase the number of w's and decrease the exponent of F. Since $l < k$, after at most k steps

(445) $$w^1 \cdots w^{h+s} \neq 0, \qquad w^1 \cdots w^{h+s} F = 0, \qquad s \leqq k.$$

Theorem (419) gives

(446) $$F = w^j g, \qquad (j = 1, \cdots, h + s).$$

In the product

(447) $$w^1 \cdots w^h F^k \neq 0$$

every term is of degree $k + h$ in the $s + h$ w's. Hence $s < k$ is untenable. There are exactly $s = k$ steps in obtaining (445), and the exponent of F diminishes by unity each time. Moreover, an appeal to the form (446) resulting from the complete process shows that the non-zero form (447) at any stage is monomial. Any one of the factors not in $Z[w^1, \cdots, w^h]$ can be used as w^{h+1}.

(448) *If F is quadratic, $w^1, \cdots, w^h$ linear and*

$$w^1 \cdots w^h F^k \neq 0, \qquad w^1 \cdots w^h F^{k+1} = 0$$

then the form

$$w^1 \cdots w^h F^k$$

is monomial. If w^{h+1} *is one of its factors satisfying* $w^1 \cdots w^{h+1} \neq 0$, *then*

$$w^1 \cdots w^{h+1} F^{k-1} \neq 0, \qquad w^1 \cdots w^{h+1} F^k = 0.$$

Differentiation of (439) with respect to w^1 gives the useful identity

(449) $$w^2 \cdots w^h F^{k+1} + (-1)^h (k+1) w^1 \cdots w^h (w^1_{-1} F) F^k = 0.$$

(450) Canonical form. If the set of w's in the preceding section is empty at the start ($h = 0$), it follows from (447) that the $2k$ linear forms in (446) are independent. Moreover, the rank of F by (446) does not exceed $2k$. Hence

(451) *The rank of a quadratic form is double the integer* k *defined by*

(452) $$F^k \neq 0, \qquad F^{k+1} = 0.$$

In slightly changed notation (446) is

(453) $$F = U^1U^2 + \cdots + U^{j-1}U^j \qquad (j = 2k).$$

By the process of (438) the U's with odd index are found successively from the linear systems

$$F^kU = 0; \qquad F^{k-1}U^1U = 0; \qquad F^{k-2}U^1U^3U = 0; \qquad \cdots.$$

The right member of (453) is the (454) *canonical form* of F. The U's are (455) *canonical marks.*

(456) Examples. To illustrate the reduction of (438) consider the quadratic form

$$F = 13 - 14 + 23 - 24 + 35 - 36 + 45 - 46,$$

where $13 = v^1v^3$, together with the linear form $1 = v^1$. To find $1F^2$, ignore the first two terms of F and get

$$1F^2 = 4\cdot 1(2345 - 2346),$$

where the factor 4 on the right is numerical. Inspection gives

$$1F^2 = 4\cdot 1234(5 - 6).$$

Choose 3 as the second linear form:

$$13F = 134(2 + 5 - 6).$$

Choose 4 as the third linear form so that

$$134F = 0$$

and equation (446) is

$$F = 17 + 38 + 49.$$

Even with 1, 3, 4 fixed, 7, 8, 9 are by no means unique. One set of values, had by inspection, is shown in

(457) $$F = 1(3 - 4) + 3(-2 + 5 - 6) + 4(2 + 5 - 6).$$

Unlike what is true of the direct reduction to canonical form in (450), the linear forms here are not necessarily independent. It is possible, for example, to regroup thus:

(458) $$F = 3(-1 - 2 + 5 - 6) + 4(-1 + 2 + 5 - 6).$$

To illustrate the direct reduction to canonical form discussed in (450), use the same F and find

$$F^3 = 0, \qquad F^2 = 4\cdot(1 + 2)34(5 - 6).$$

Choose $1 + 2$ as the first linear factor:

(459) $$(1 + 2)F = (1 + 2)(3 + 4)(5 - 6).$$

Choose $3 + 4$ as the second linear form:

$$F = (1 + 2)7 + (3 + 4)8.$$

Form 8 obviously divides $(1 + 2)F$. An arbitrary constant times $(3 + 4)$ can, of course, be added to it and $(1 + 2)$, if present, can be removed by modifying 7. Hence from (459) guess $7 = 5 - 6$:

$$F - (3 + 4)(5 - 6) = (1 + 2)(3 - 4)$$

and a canonical form is had by transposition:

(460) $$F = (1 + 2)(3 - 4) + (3 + 4)(5 - 6).$$

An alternative [13, 53–54] reduction to canonical form is as follows: If $a_{12} \neq 0$, the form

$$f_1 = 2a_{12}F - F_1F_2 ,$$

where F_1, F_2 are the derivatives of F re v^1, v^2, does not involve v^1 or v^2 as can be verified by differentiating re v^1, v^2 by rule (400).

In the present F, coefficient $2a_{13} = 1 \neq 0$.

$$F_1 = 3 - 4, \qquad F_3 = -1 - 2 + 5 - 6$$

$$f_1 = 2a_{13}F - F_1F_3 = 2\cdot(45 - 46)$$

Since the new $a_{45} = 1 \neq 0$, repeat:

$$f_2 = 2f_1 - f_{14}f_{15} = 0.$$

Hence

$$F = (3 - 4)(-1 - 2 + 5 - 6) + 2\cdot 4(5 - 6). \tag{461}$$

The four canonical sets (458), (459), (460), (461) are easily transformed one into the other.

CHAPTER

8

Differential Rings

(462) Differentials. Let Z be the field of analytic functions of variables $u^1, \cdots, u^r$. The (463) *differential* of element z of Z is the element of Grassmann ring $Z[v^1, \cdots, v^r]$ given by

(464) $$z' = z_{,i}v^i,$$

where $z_{,i}$ is the derivative of z re u^i and where the repeated index is summed. A consequence of this definition, when $r = 1$, is that the accent is no longer available to denote the derivative, which is now $z_{,1}$.

Applied to u^i (464) gives

(465) $$u^{i\prime} = v^i$$

so that (464) becomes

(466) $$z' = z_{,i}u^{i\prime}.$$

The (467) *differential* of the Grassmann ring element F given by (379) is another element of the Grassmann ring, namely,

(468) $$F' = a'_{i_1,\cdots,i_p}v^{i_1} \cdots v^{i_p}.$$

The differential of a form of degree p is therefore either 0 or a form of degree $p + 1$. The Grassmann ring is closed under differentiation just as is Z. Such a Grassmann ring is a differential ring.

Note that the differential is had by putting the coefficient on the extreme left and then replacing the coefficient by its differential just as the derivative is had by putting the v involved on the extreme left and then replacing it by its derivative 1. Because of this, rule (400) is immediately seen to apply to the formation of the differential of a form as well as to the formation of its derivative re a mark.

If F, G have respective degrees p, q, from (390)

$$(FG)' = F'G + (-1)^{pq}G'F.$$

Passage of G' over F introduces $p(q+1)$ changes of sign. Hence

(469) $$(FG)' = F'G + (-1)^p FG'.$$

The derivative of $a_{i_1 \cdots i_p}$ re $u^{j_1} \cdots u^{j_k}$ is now to be written $(j_1 \cdots j_k i_1 \cdots i_p)'$. This involves two slight modifications of the notation in (18): first, the single index denoting the function is replaced by a set; second, the symbol indicates the indices of the variables of differentiation rather than the number of differentiations re each variable.

Accordingly, use of (464) in (468) gives

(470) $$F' = (hi_1 \cdots i_p)'v^h v^{i_1} \cdots v^{i_p}.$$

(471) Exact differentials. Replacing form F in (470) by F' as given by (470) gives

(472) $$(F')' = (ghi_1 \cdots i_p)'v^g v^h v^{i_1} \cdots v^{i_p}.$$

If the coefficient of F is skew, the skew coefficient of $(F')'$ is half of

$$(ghi_1 \cdots i_p)' - (hgi_1 \cdots i_p)'.$$

Since the order of differentiation is immaterial, the skew coefficient is 0 so that

(473) $$(F')' = 0.$$

Consider now form G whose differential G' is 0. First, suppose that G does not contain v^1. In the notation of (470)

$$G' = (1i_1 \cdots i_p)'v^1 v^{i_1} \cdots v^{i_p} + \cdots,$$

where the unwritten terms do not involve v^1. Hence

$$(1i_1 \cdots i_p)' = 0,$$

that is, u^1 as well as v^1 is absent from G.

Now suppose simply that $G' = 0$ and seek F to satisfy

$$F' = G.$$

If $r = 1$, integration gives F immediately. For the general r, by (416)

$$G = v^1 G_1 + g.$$

Integrate the coefficients of G_1 re u^1 to get form H. Then

$$H' = v^1 G_1 + h,$$

where h like g does not contain v^1. Form $G - H'$ has 0 differential and does not contain v^1. Hence by the preliminary result it is a form in fewer than r variables and by the induction hypothesis there is a form K satisfying $K' = G - H'$. The desired $F = H + K$.

(474) *Form F is a differential if and only if its differential F' is zero. Form F is then called exact.*

(475) *The solution of $F' = G$ is $F_1 + F_2$, where F_1 is a root of that equation and F_2 is the solution of the corresponding equation $F' = 0$.*

(476) Change of variables. Let variables u and their differentials v be subjected to a transformation

(477) $$u^i = f^i(u^*), \qquad v^i = (ji)'v^{j*},$$

where the f's are analytic, the number of u^*'s is fixed but not necessarily equal to r and j is summed. Let the transformation take F, F' into F^*, $(F')^*$, respectively. Formulas (379), (470) give

(478) $$F^* = a_{i_1 \cdots i_p}(j_1 i_1)' \cdots (j_p i_p)' v^{j_1} \cdots v^{j_p},$$

(479) $$(F')^* = (hi_1 \cdots i_p)'(kh)'(j_1 i_1)' \cdots (j_p i_p)' v^k v^{j_1} \cdots v^{j_p}.$$

If $(F^*)'$ is formed from (478), the right side of (479) results. Hence the fundamental theorem:

(480) $$(F^*)' = (F')^*.$$

The u^* may be new variables, their number equalling r and the Jacobian being different from 0; or the u^* may be the coordinates in a subspace.

(481) Pfaffians. A non-zero monomial form in a differential ring is a (482) *Pfaffian.*

A Pfaffian of degree k has a set of k independent divisors (necessarily linear) which are a (483) *base* for it.

Linear combination of the divisors gives a base of the type

(484) $$v^i + a^i_j v^j \qquad (j = k + 1, \cdots, r).$$

A Pfaffian is (485) *passive* if it has a base composed of differentials.

Since $v^i W = 0$ for Pfaffian W of degree r, the marks are a base for such a Pfaffian and

(486) *A Pfaffian of degree r is passive.*

Base (484) for a Pfaffian of degree $r - 1$ is

$$v^i + a^i v^r \qquad (i = 1, \cdots, r - 1).$$

If z' is to be a differential in the base of the Pfaffian, $f'W = 0$ gives

$$-a^i f_i + f_r .$$

This system is orthonomic; in fact, it is of the Cauchy-Kowalevsky type (319). The initial determination is $z(u^1, \cdots, u^{r-1}, 0)$. Get $r - 1$ roots z corresponding to the initial determinations $u^1, \cdots, u^{r-1}$. These z's have Jacobian initially equal to 1. Hence their differentials are $r - 1$ independent divisors of the Pfaffian.

(487) *A Pfaffian of degree* $r - 1$ *is passive.*

If W is passive, with marks properly chosen

(488) $$W = av^1 \cdots v^k, \qquad W' = a'v^1 \cdots v^k$$

so that W divides W'. Since W' has degree $k + 1$ and k divisors, it has $k + 1$ divisors and is monomial, as is also evident from the formula. Moreover, this W satisfies

(489) $$W' = wW,$$

where w is linear. Under change of base

$$W^* = bW, \qquad W^{*\prime} = b'W - bW'$$

so that W divides W' is equivalent to W^* divides $W^{*\prime}$. Hence (489) is independent of the base. Moreover, the conditions that W divide W', namely $w^iW' = 0$ are equivalent to

(490) $$Ww^{i\prime} = 0.$$

Next (489) is shown sufficient for passivity by induction on the degree of the ring. Let $\deg W = k < r - 1$. Consider the base $w^i - a^i_r v^r$, that is, consider the Pfaffian for constant u^r in a differential ring of degree $r - 1$. By hypothesis there are functions z^i such that $w^i - a^i v^r$ are dependent on $z^{i\prime} - z^i_r v^r$. Consequently, w^i are dependent on a base

(491) $$v^i + a^i v^r \qquad (i = 1, \cdots, k).$$

Substitution in (490) gives

$$v^1 \cdots v^k a^i_j v^j v^r = 0,$$

where subscript j denotes the derivative. Hence $a^i_j = 0$ except for $j = 1, \cdots, k, r$. The Pfaffian of degree k therefore is in $k + 1$ variables and by (487) is passive.

(492) *A Pfaffian is passive if and only if it divides its differential, which is also a Pfaffian. Equivalent conditions are* (490).

(493) *An exact Pfaffian is passive and equals a product of differentials.*

Since $W' = 0$, condition (492) is satisfied. From (488) a is function of $u^1, \cdots, u^k$ only. Find u^* by integrating a re u^1. Then

$$v^* = av^1 + bv^2 + \cdots + cv^k$$

so that $W = v^*v^2 \cdots v^k$.

A passive Pfaffian of degree k accordingly can be written

(494) $$v^1 \cdots v^k$$

or

(495) $$u^1v^1 \cdots v^k$$

according as its differential is zero or not.

If the coefficient of $v^1 \cdots v^k$ in Pfaffian W of degree $k < r - 1$ is not 0, then $v^{k+1} \cdots v^{r-1}W$ is a Pfaffian of degree $r - 1$; it is passive by (487) and has a base of $r - 1$ differentials. Hence Pfaffian W of degree $k \leqq r$ can be expressed in terms of $r - 1$ differentials. Its coefficients, in general, contain all r variables.

(496) *A Pfaffian of degree less than r can be expressed in terms of fewer than r differentials.*

If $k + s$ is the minimum number of differentials in which a Pfaffian of degree k can be expressed, s is the (497) *species*, see [131].

(498) Characteristic Pfaffian. The method used in proving (492) also proves

(499) *If forms F, F' belong to $Z[v^1, \cdots, v^{r-1}]$, they also belong to $Z_r[v^1, \cdots, v^{r-1}]$, where Z_r is the field of analytic functions of $u^1, \cdots, u^{r-1}$.*

Term am in F, whose terms are all unlike, gives term a_rv^rm in F' and that term is unlike the others in F'. Hence $a_r = 0$.

The associated set (423) of sets F_i, F_i' gives the divisors of the (500) *characteristic Pfaffian* of set F_i. The degree of the characteristic Pfaffian is the (501) *class* c of set F_i.

If $c < r$, (496) and (499) can be used to show that the set belongs to a $Z_r[v^1, \cdots, v^{r-1}]$. Repetition is possible until c is the number of variables as well as the degree so the the Pfaffian is passive.

(502) ***The class is the minimum number of variables in terms of which the set of forms can be expressed.***

(503) ***The characteristic Pfaffian is passive.***

(504) Canonical form. For linear form w construct the sequence

(505) $$ww'^h \qquad (h = 0, 1, \cdots, k)$$

stopping when the next term would be 0. Construct also

(506) $$F_h = wv^1 \cdots v^h w'^{k-h},$$

(507) $$G_h = v^1 \cdots v^h, \qquad (h = 0, 1, \cdots, k)$$

with the understanding that F_0 is (505) and $G_0 = 1$.

In (448) let $F = w'$, $w^1 = w$. If $w^2 = v^1, \cdots, w^h = v^{h-1}$ are differentials, then w^{h+1} is to divide F_{h-1} whose differential $F_{h-2}v^{h-1}$ is 0. By (492) Pfaffian F_{h-1} is passive so that w^{h+1} can be taken as a differential v^h. Hence differentials $v^1, \cdots, v^{k+1}$ can be successively determined so that v^h divides F_{h-1} but not G_{h-1}. In particular, v^{k+1} satisfies

$$v^1 \cdots v^{k+1}w = 0, \qquad v^1 \cdots v^{k+1} \neq 0$$

and

(508) $$w = p_i v^i \qquad (i = 1, \cdots, k + 1).$$

If $w'^{k+1} \neq 0$, variables p_i, u^i are independent. The conditions for reducibility to (508) with variables independent are:

(509) $$w'^{k+1} \neq 0, \qquad ww'^{k+1} = 0.$$

If $w'^{k+1} = 0$, form w'^k is monomial and exact. By (493) it can be written as the product of differentials. The same is true of ww'^k and the differentials in w'^k can be included among those for ww'^k. Hence there is a z such that

(510) $$ww'^k = z'w'^k.$$

Subtraction of a differential z' from w therefore gives linear form $w - z'$ satisfying

$$(w - z')'^k \neq 0, \qquad (w - z')(w - z')'^k = 0,$$

that is, (509) with k replaced by $k - 1$. Form $w - z'$ is reduced by the process already given and

(511) $$w = z' + p_i v^i \qquad (i = 1, \cdots, k),$$

the conditions for reducibility to this form being

(512) $$ww'^k \neq 0, \qquad w'^{k+1} = 0.$$

In either case, no p in (508) is 0 and that equality can with a slight change of notation be written

$$w = p_{k+1}(z' + p_i v^i) \qquad (i = 1, \cdots, k).$$

The Pfaffian of degree one with base w therefore also has the (513) *canonical base* $z' + p_i v^i$ $(i = 1, \cdots, k)$.

To recapitulate, the (514) *canonical forms* are given below. Index i has the range $1, \cdots, k$. An inequation means at least one coefficient is not 0 and an equation means all coefficients are 0.

		Class	Canonical Form	Conditions	
Form	w	$2k$	$p_i v^i$	$w'^k \neq 0,$	$ww'^k = 0$
Form	w	$2k + 1$	$z' + p_i v^i$	$ww'^k \neq 0,$	$w'^{k+1} = 0$
Pfaffian	w	$2k + 1$	$z' + p_i v^i$	$ww'^k \neq 0,$	$ww'^{k+1} = 0$

In all three cases, reduction to canonical form involves determination of (508). With $w_i = ww'^i$, the table below displays the conditions successively to be used. A v without index indicates the unknown of the moment.

System	Root
$w_k v + v^N$	v^1
$w_{k-1} v^1 v + (v^1 v)^N$	v^2
$\cdots$	$\cdots$
$wv^1 \cdots v^k v + (v^1 \cdots v^k v)^N$	v^{k+1}

Multiplication of (508) on the left by $v^1 \cdots v^{i-1}$ and on the right by $v^{i+1} \cdots v^{k+1}$ gives

$$(515) \qquad wv^i_{-1} V = p_i V,$$

where

$$(516) \qquad V = v^1 \cdots v^{k+1}.$$

When the v's have been determined, the p's are had by equating coefficients in (515). The coefficients of a single monomial give each p.

(517) Examples. First a linear form of even class is reduced to canonical form. As often, indices are written for variables and numerical coefficients are separated by a dot.

$$w = 2\cdot 51' + (4 + 5)2' + 2\cdot 43' + (-4 + 5)4'$$

$$w' = -2\cdot 1'5' - 2'4' - 2'5' - 2\cdot 3'4' - 4'5'$$

$$ww' = -2 \cdot 51'2'4' + 2 \cdot 41'2'5' - 4 \cdot 51'3'4' + 4 \cdot 41'3'5' - 2 \cdot 41'4'5'$$
$$- 2 \cdot 52'3'4' + 2 \cdot 42'3'5' - 2 \cdot 42'4'5' - 2 \cdot 43'4'5'$$

$w'^2 \neq 0, \qquad ww'^2 = 0$

Hence $k = 1$. The tabulated systems are:

(518) $\qquad w_1 v + v^N \qquad v^1$

(519) $\qquad wv^1 v + (v^1 v)^N \qquad v^2$

To find a root of (518), compute the associated set (423) of w_1. At the left, 12 means differentiate re $1'2'$ and the columns are headed by the differentials whose coefficients are below.

	1′	2′	3′	4′	5′
12				−2·5	2·4
13				−4·5	4·4
14		2·5	4·5		−2·4
15		−2·4	−4·4	2·4	
23				−2·5	2·4
24	−2·5		2·5		−2·4
25	2·4		−2·4	2·4	
34	−4·5	−2·5			−2·4
35	4·4	2·4		2·4	
45	−2·4	−2·4	−2·4		

The combination of rows 24 − 34 is $2 \cdot 5(1' + 2' + 3')$. Hence set

$$v^1 = (1 + 2 + 3)'.$$

(520) $\qquad wv^1 = (-4 + 5)(1'2' + 2 \cdot 1'3' - 1'4' + 2'3' - 2'4' - 3'4')$

Discard the linear factor before forming the associated set. The entries in the table are numbers, not indices.

	1′	2′	3′	4′
1		1	2	−1
2	−1		1	−1
3	−2	−1		−1
4	1	1	1	

$$v^2 = (2 + 2\cdot 3 - 4)'$$

$$V = v^1v^2 = 1'2' + 2\cdot 1'3' - 1'4' + 2'3' - 2'4' - 3'4'$$

(521) $$wv^2 = p_1 V, \qquad wv^1 = -p_2 V$$

Comparing terms containing $1'2'$ in (520) and the second of (521) gives $p_2 = 4 - 5$. The first gives $p_1 = 2\cdot 5$. Hence a canonical form is as below.

(522) $$w = 2\cdot 5(1 + 2 + 3)' + (4 - 5)(2 + 2\cdot 3 - 4)'$$

The reduction can be varied by using other v^1, v^2.

If the Pfaffian is proposed instead of the linear form, the reduction proceeds exactly as above and is completed by dividing by one of the p's.

Next a form of odd class is reduced.

$$w = (2\cdot 5 + 2)1' + (1 + 4 + 5)2' + 2\cdot 43' + (-4 + 5)4'$$

$$w' = -2\cdot 1'5' - 2'4' - 2'5' - 2\cdot 3'4' - 4'5'$$

$$w'^2 = 4\cdot 1'2'4'5' + 8\cdot 1'3'4'5' + 4\cdot 2'3'4'5'$$

$$ww'^2 = 4\cdot(-2\cdot 1 + 2)1'2'3'4'5'$$

$$w'^3 = 0$$

Hence $k = 2$. Since $w'^{k+1} = 0$, the preparatory subtraction of a differential is used before reduction to (508). In writing (510) the terms in $4'$, $5'$ can be ignored because $4'5'$ divides w'^2. The condition on z is

$$z_1 - 2z_2 + z_3 = -2u^1 + u^2$$

and has root $z = u^1u^2$. For this value of z, form $w - z'$ is precisely the form previously treated.

(523) $$w = (12)' + 2\cdot 5(1 + 2 + 3)' + (4 - 5)(2 + 2\cdot 3 - 4)'$$

If the Pfaffian with the above linear form for base is proposed for reduction, the preliminary reduction of class by subtraction is unnecessary. Reduce instead to (508). The tabulated systems are shown.

$$\begin{array}{ll} w_2v + v^N & v^1 \\ w_1v^1v + (v^1v)^N & v^2 \\ wv^1v^2v + (v^1v^2v)^N & v^3 \end{array}$$

A form (508) is found.

$$(-2 - 2\cdot 3 - 4 + 5)4' + (-2 - 2\cdot 1)5'$$
$$+ (12 + 2\cdot 15 + 24 + 25 + 2\cdot 34)'$$

The coefficient p_3 is constant and accidentally the form is (511). The reduction of the form

$$31' + 132' + (12 + 24)3' + (1 + 23 + 35)4' + (2 + 34)5'$$

to (508) requires only roots simply expressible and gives

$$1(23 + 4)' + 2(34 + 5)' + 3(45 + 1)'$$

with non-constant p's.

There are many different ways of reducing to canonical form: see, for example, [46, I, IV].

(524) Generalizations. The theorem at the basis of the results in this chapter is (486). It can be proved for more general Z: the method of successive approximations can be used, for example, if the coefficients have continuous derivatives.

C. J. de la Vallée-Poussin has suggested [140] a criterion for exactness which may broaden the concept of reducing a form. Here the condition that F be exact is $F' = 0$ and requires differentiation of the coefficients. For $r = 2$ and linear F de la Vallée-Poussin suggests assuming the coefficients have differentiable integrals re each variable when the other is fixed. The usual formula for the root of $G' = F$ got by integrating first re u^1 is F_1 and that for the other order F_2. The condition for exactness is $F_1 = F_2$ and is applicable to a more extensive class of coefficients.

CHAPTER

9

Pfaffian Systems

(525) Definitions. System P in sense (11) is a (526) (*generalized*) *Pfaffian system in unknowns* $u_1, \cdots, u_r$ if its members are forms in differential ring $Z[u'_1, \cdots, u'_r]$.

Let $x_1, \cdots, x_d$ be (527) independent variables. None of them appears explicitly in the members of P. Denote the derivative of u_i re x_j by u_j^i.

A (528) *root* of P is a set of functions $u_i(x_1, \cdots, x_d)$ which on substitution make all equations of P equal to 0 and all inequations of P different from 0 for all $(x_1, \cdots, x_d)$ in a subregion of the region of definition for P.

If the rank of the Jacobian matrix u_j^i is d, the (529) *dimension* of the root is d. Clearly, $d \leqq r$.

System P implies Pfaffian system P', where P' contains as equation the differential of each equation of P, so that

$$P = P + P'.$$

Consequently, the base may be supposed to include equation F' if it includes equation F; equation 0, of course, being omitted except in the trivial case $P = 0$. Such a base is (530) *full* and this adjective also describes P conceived as given by a full base.

(531) Existence theorem. Write

$$P = P_0 + P_1 + \cdots + P_r, \tag{532}$$

where the members of P_j have degree j. System P_0 is a function system whose unknowns are the unknowns of P.

With P are associated by (412) function systems S, with members got from those of P_j in the following manner. Corresponding to Grassmann form

$$F = a_{i_1 \cdots i_p} u'_{i_1} \cdots u'_{i_p} \tag{533}$$

which is in P_j and has skew coefficient there are in S_j the ordinary forms

(534) $$F^{j_1 \cdots j_p} = a_{i_1 \cdots i_p} u_{i_1}^{j_1} \cdots u_{i_p}^{j_p},$$

where $j_1 \cdots j_p$ is an arbitrary combination of p unequal numbers from $1, \cdots, j$.

For $0 < j$ system S_j is of the first order and is algebraic in the first derivatives. System S_0 has order 0, that is, is a function system. Any S_i may fail to be algebraic in the unknowns so that these systems in general are not algebraic.

System

(535) $$S = S_0 + S_1 + \cdots + S_r$$

is in sense (97) a differential system which is equivalent to P, that is, has the same solution as P. The theory already developed therefore gives an answer to the existence question for Pfaffian systems.

(536) Classical theory. Historically, Pfaffian systems were studied directly and an interesting, symmetrical theory was developed. A short account of this direct approach is now to be given. It will appear that this at best gives only part of the solution. In addition, the members are limited to equations, at least in the systems as proposed.

To illustrate, consider the system

$$u_1 u_2' + u_3',$$

which exhibited by full base is

$$P: \quad u_1 u_2' + u_3', \qquad u_1' u_2'.$$

For this full base

$$P_0 \qquad 0,$$

$$P_1 \qquad u_1 u_2' + u_3',$$

$$P_2 \qquad u_1' u_2'.$$

A root of P substituted in the above must make the coefficients of x_i' zero. Hence in the usual abbreviated notation every root of P is a root of the equations

(537) $$1(j2)' + (j3)' \qquad (j1)'(k2)' - (j2)'(k1)'.$$

As is true for the general Pfaffian system, the members of S are polynomials in the first derivatives. Here they are also accidentally polynomials in the unknowns.

Every determinant of order two formed from columns one, two in matrix $(jk)'$ is zero. Hence the maximum d is 2. System S is had by letting j, k have values 1, 2. It can be written as an array.

$$\begin{array}{ll} & 1(12)' + (13)' \\ -(12)'(21)' + (11)'(22)' & 1(22)' + (23)' \end{array}$$

If a numerical determination with $(11)' \neq 0$ is assigned, S is in regular form (315). The single passivity condition is found to be identically satisfied. The theory now to be developed shows that this is so in the general case.

Although there is no increase in the dimension of the root, the indices in (537) can be allowed to assume arbitrary values. Let them take the additional value 3, omit the accents to save space and use the solidus to separate factors. Another array is had, which has three passivity conditions. The first column is also implied by the other two.

$$\begin{array}{lll} & & 1/12 + 13 \\ & -12/21 + 11/22 & 1/22 + 23 \\ 21/32 - 22/31 & -12/31 + 11/32 & 1/32 + 33 \end{array}$$

Let $u_3(0, 0, 0)$, $u_2(x_1, 0, 0)$, $u_1(x_1, x_2, x_3)$ be arbitrary except for $u_1^1(0, 0, 0) \neq 0$. Use the Cauchy-Kowalevsky theorem on the rows.

Put $x_2 = x_3 = 0$ in the first row, which then determines $u_3(x_1, 0, 0)$, thereby completing the initial determination for row 2.

Put $x_3 = 0$ in the second row, which gives $u_2(x_1, x_2, 0)$, $u_3(x_1, x_2, 0)$ and completes the initial determination for row 3.

The last row determines $u_2(x_1, x_2, x_3)$, $u_3(x_1, x_2, x_3)$.

(538) Fundamental identities. Equality (379) rewritten is

(539) $$F = F^{j_1 \cdots j_p} x'_{j_1} \cdots x'_{j_p}$$

and in the same way

(540) $$F' = F_1^{j_0 j_1 \cdots j_p} x'_{j_0} x'_{j_1} \cdots x'_{j_p} .$$

Differentiation of (539) and comparison with (540) give

(541) $$(j_0 F^{j_1 \cdots j_p})' - \cdots - (p + 1) F_1^{j_0 j_1 \cdots j_p} = 0,$$

where the unwritten terms arise by performing on the first the signed transpositions $(j_0 j_1)$, $\cdots$, $(j_0 j_p)$. If (541) is applied to F', the last term drops out because $(F')' = 0$ and there results the second fundamental identity

(542) $$(j_{p+1} F_1^{j_0 j_1 \cdots j_p})' - \cdots = 0,$$

where the unwritten terms arise by performing on the first the signed transpositions $(j_{p+1}j_0)$, $(j_{p+1}j_1)$, $\cdots$, $(j_{p+1}j_p)$.

For example, if

$$F = F^{jk}x'_jx'_k\ ,$$

$$F' = F^{ijk}x'_ix'_jx'_k\ ,$$

the identities are

$$(iF^{jk})' - (jF^{ik})' - (kF^{ij})' - 3F^{ijk} = 0,$$

$$(lF_1^{ijk})' - (iF_1^{ljk})' - (jF_1^{ilk})' - (kF_1^{ijl})' = 0.$$

(543) Cartan's existence theorem generalized. Write

$$S = T_1 + \cdots + T_r\ ,$$

where the maximum superscript on u in each equation of T_j is j. For j fixed system T_j is linear and homogeneous in u^j_i. Let every determinant of $m_j + 1$ be identically 0 and some determinant of order m_j be not identically 0 in the matrix of T_j evaluated for the algebraic solution of $T_1 + \cdots + T_{j-1}$. Restrict the region considered so that the determinant is different from 0 throughout the region. System T_j has $r - m_j$ linearly independent roots. Among them are u^k_i for $k = 1, \cdots, j - 1$ provided u^k_i are independent and form a root of $T_1 + \cdots + T_{j-1}$.

Let u^1_k be a non-trivial root of algebraic T_1. Then algebraic T_2 in unknowns u^2_i has root u^1_i. If

$$2 \leqq r - m_2\ ,$$

then T_2 has two independent roots u^1_i, u^2_i. In this way, a set u^1_i, u^2_i, $\cdots$, u^g_i of independent roots is constructed for T_g, where

$$g = r - m_{g+1}\ ,$$

so that g is the maximum dimension of a root of S, rank being m_r. Integer g is the (544) *genus*.

The differences

$$n_j = m_j - m_{j-1}$$

are the (545) *characters*.

The Cauchy-Kowalevsky case of the existence theorem is sufficient to treat differential system T_j. Moreover, a root of T_j can be made to reduce for $x_j = 0$ to a root of T_{j-1}. Treating $T_1, \cdots, T_g$ successively gives respective roots $U_1, \cdots, U_g$ such that U_j reduces to U_{j-1} for $x_j = 0$. Actually, U_j is a root of $T_1 + \cdots + T_g$. To prove this, let

G be the result of substituting U_i in an equation of T_{i-1}. The equation in question is either an F or an F'. In the first case (541) and in the second (542) gives

(546) $$(jG)' + H = 0,$$

where H is a polynomial every term of which involves either a G or the derivative of a G re one of $x_1, \cdots, x_{i-1}$. Moreover, G is 0 for $x_i = 0$. The uniqueness theorem applied to (546) therefore shows that G is identically 0.

This existence theorem was discovered for the linear case by E. Cartan [12]. For the generalized version see [68], [126], [15], [44].

(547) Inequalities on arithmetic invariants. If P has only linear forms, it is easily shown that

(548) $$n_{i+1} \leqq n_i,$$

(549) $$r - n \leqq (n + 1)g,$$

formulas discovered by E. Cartan [12]. Neither of them holds in the general case, for which G. Cerf [17] has given more complicated inequalities.

Other inequalities have been found by D. C. Dearborn [25], J. A. Schouten and W. van der Kulk [112, V].

(550) Singular roots. If the rank of each linear T_i equals the corresponding (maximum) m_i at some point of a root, it equals m_i in the neighborhood of that point. The root is then (551) *ordinary.* The continuation of an ordinary root is ordinary.

A point (numerical determination) where the rank of at least one T_i is less than the maximum m_i is (552) *singular.* A root is (553) *singular* if every point in it is singular. An ordinary root may contain singular points.

The following theorems have been effectively proved:

(554) *Every ordinary root with dimension less than the genus is in an ordinary root with one higher dimension.*

(555) *No ordinary root with dimension equal to the genus is in a root with higher dimension.*

(556) *A root with higher dimension than the genus is necessarily singular.*

(557) Pfaffian with canonical base. Next the existence theorem is applied to the system consisting of a single linear equation whose canonical base (513) is here rewritten

(558) $$w = u'_{2k+1} + u_j u'_{2k+1-j} \ ,$$

the sum of the indices in each term equalling the class $2k + 1$.

It is convenient to have systems T_1, T_2, T_3 for the general w displayed in an array (314).

T_1			$a_j \overset{1}{u}_j$
T_2		$a_j \overset{2}{u}_j$	$a_{ij} \overset{1}{u}_i \overset{2}{u}_j$
T_3	$a_j \overset{3}{u}_j$	$a_{ij} \overset{1}{u}_i \overset{3}{u}_j$	$a_{ij} \overset{2}{u}_i \overset{3}{u}_j$

The contribution of the differential F' of an equation F is had by replacing in F' each Grassmann monomial by the corresponding determinant in the commutative indeterminates. This writes simultaneously two terms in the above entries.

Applied to (558) this process gives

$$\overset{1}{u}_j \overset{2}{u}_{2k+1-j} - \overset{2}{u}_j \overset{1}{u}_{2k+1-j} \ ,$$

where the superscripts represent any pair of unequal positive integers. Written in abbreviated form this is

(559) $$1j/2(2k + 1 - j) - 2j/1(2k + 1 - j).$$

System T_{h+1} is thus

$$(h + 1)(2k + 1) + j/(h + 1)(2k + 1 - j),$$
$$ij/(h + 1)(2k + 1 - j) - (h + 1)j/i(2k + 1 - j),$$

where $i, j = 1, \cdots, h$. The determinant of the coefficients of $(h + 1)(2k + 1 - j)$ is $| u^i_j |$ which is by induction seen to be not identically 0. Hence the rank of T_{h+1} is $h + 1$, the one being contributed by the first row of the table. The number of parametric unknowns in the linear system is therefore $(2k + 1) - (h + 1) = 2k - h$, which must be at least h. Hence $h \leqq k$ and $k = g$.

(560) *The class of a Pfaffian of degree one is* $2g + 1$, *where* g *is the genus of the linear Pfaffian system with the same base.*

(561) Solution of canonical Pfaffian. The differential of (508) is

$$w' = p'_i u^{i\prime}.$$

The formulas

(562) $$x'_1 \cdots x'_e \neq 0,$$

(563) $$x_j = f_j(x_1, \cdots, x_e) \qquad (j = e + 1, \cdots, k + 1),$$

(564) $$p_i = -f^i_j p_j \qquad (i = 1, \cdots, e),$$

where e is in $0 \leqq e \leqq k = g$, the f's are arbitrary and $f_i^{\cdot}$ is the derivative re x_i, define a root with dimension e. Moreover, every root of (508) is so obtained. Verification of this by substitution is immediate. There is no root with dimension greater than the genus.

The root may be ordinary or singular. For example, $z = 1$, $p_1 = 0$ is singular of dimension equal to the genus for $p_1 u_1' + z'$.

The formulas apply to canonical base (513) by making $p_{k+1} = 1$, $x_{k+1} = z$.

(565) *The solution of the canonical base of a Pfaffian of degree one can be had by differentiation.*

(566) Single linear equation. Let w be the base of a Pfaffian of degree one. A root which at some point satisfies (512) is had from a root of canonical form (508) by using the transformation inverse to that producing (508). Such a root of w may be ordinary or singular.

Not all roots of w are had in this way. The linear form

$$w = (1 + 2)1' + 32' - (1 + 2)3'$$

has the surface

$$1 + 2 - 3 = 0$$

as singular root of dimension 2 exceeding the genus 1. By (561) it is not given by the canonical form. It is profitable to examine the situation in detail. It is found that

$$ww' = (-1 - 2 + 3)1'2'3'.$$

A transformation to canonical form

$$2^*1^{*\prime} + 3^{*\prime}$$

is

$$1^* = 1 + 2 \cdot 2, \qquad 2^* = -1 - 2 + 3,$$

$$3^* = 11 + 22 + 2 \cdot 12 - 13 - 23$$

whose Jacobian is $-1 - 2 + 3$. The canonical form does not give the root, which is $2^* = 0$.

A root of w which is also a root of the system whose equations are the coefficients of ww'^k is an (567) *exceptional root.* Such a root may be ordinary or singular for w. It is possible, however, to prove

(568) *No ordinary root with dimension equal to the genus is exceptional.*

This is done by remarking that if all coefficients of ww'^k are 0, there is an algebraic reduction of w' to y,z, with fewer than k terms. The rank therefore increases by at most k, instead of $k + 1$, in the passage from T_1 to T_{k+1}.

Consider now the Pfaffian of class 5 and genus 2:

$$w = 51' + 32' + 14' + 15',$$

$$-2^{-1}ww'^2 = 11'2'3'4'5'.$$

Systems T_1, T_2, T_3 are displayed in a table.

T_1	$5/11 + 3/12 + 0/13 + 1/14 + 1/15$
T_2	$5/21 + 3/22 + 0/23 + 1/24 + 1/25$
	$-14/21 + 13/22 - 12/23 + 11/24 + 0/25$
T_3	$5/31 + 3/32 + 0/33 + 1/34 + 1/35$
	$-14/31 + 13/32 - 12/33 + 11/34 + 0/35$
	$-24/31 + 23/32 - 22/33 + 21/34 + 0/35$

From these are found the following:
(569) $1 = 0$, $2' = 0$, $3 = 4 = 5$ is an ordinary, exceptional root with dimension 1 contained by
(570) $1 = 0$, $2' = 0$, $3 = 4$, which is a singular, exceptional root with dimension equal to the genus 2 and which is in turn contained by
(571) $1 = 0$, $2' = 0$, which is a singular, exceptional root with dimension 3 greater than the genus 2. Since ordinary (569) with dimension one less than the genus is contained in singular (571) with dimension 3 greater than the genus, the statement [46, 360, lines 10–13 up] seems incorrect.

(572) Linear Pfaffian systems. The linear divisors of a Pfaffian of degree p are the base for a Pfaffian system. It has been seen that for $p = 1$ the system has a canonical base, which is helpful in finding roots.

There are known numerous classes of systems with p greater than one and reducible to a canonical form, each equation of which is in the canonical form for the base of a linear Pfaffian.

The first type is the passive linear system with canonical form

$$(573) \qquad 1', \cdots, p'.$$

The conditions for a passive system can be stated in many ways. In addition to (490), (492) there is, for example,

(574) *A Pfaffian is passive if and only if it divides its characteristic Pfaffian.*

(575) Derived system. Let W be a Pfaffian. If its base w^i is changed to y^i by

$$w^i = b^i_j y^j,$$

direct calculation gives

(576) $$Ww'' = (\det b) b^i_j Y y''.$$

Hence the characteristic Pfaffian is independent of the base. Rewrite these equations as

$$W^i = B^i_j Y^j.$$

The forms in set W^i which are not 0 have degree $p + 2$ and the whole set transforms like a vector. Let p_1 of the w's be dependent on the others. It is then possible to find b's which make $Y^1 = \cdots = Y^{p_1} = 0$. The corresponding $y^1, \cdots, y^{p_1}$ constitute the (577) *derived system* P^1 of P. System P^1 has p^1 equations and is independent of the base to which P is referred, that is, $P = Q$ implies $P^1 = Q^1$.

(578) *System P of linear equations is passive if and only if $P = P^1$.*

There is a finite sequence

$$P, P^1, P^2, \cdots$$

generated by $(P^{i-1})^1 = P^i$ and ending when the next term would be empty or would equal its predecessor. Because of (578), in the second case the last derived system is passive.

The numbers of equations in the successive derived systems give the (579) *symbol* $p_0 p_1 p_2 \cdots$, where $p = p_0$.

A non-passive system of class 3 contains only one equation and can be reduced to

$$1' + 23'.$$

There are two non-equivalent, non-passive systems of class 4 with symbols and canonical forms as shown:

21	$1'$	$2' + 34'$
210	$1' + 24'$,	$1' + 34'$

In each case the derived system is the first equation.

A system in whose symbol successive numbers differ by unity has been called (580) *special*. Every special system can be given a canonical form in which each equation has canonical form, see, for example, [46, 328].

Other cases are treated in [48], [80], [112], [131].

The general existence problem can not be handled nor the general system solved in this way. A canonical form of the type here discussed is not even sufficiently general to represent all systems of class 5 and two equations.

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