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# PREFACE

THE author's aim has been to produce a book suitable to the beginner who wishes to acquire a sound knowledge of the more elementary parts of the subject, and also sufficient for the candidate for a mathematical scholarship. The syllabus for Honour Moderations at Oxford has been taken as a maximum limit.

The main principles observed in its construction are: (1) to utilize the previous knowledge of the student; (2) to make the subject self-dependent; (3) to arrange the bookwork in such logical order that any portion can be readily found; (4) to illustrate difficulties by worked-out problems, each selected with a definite object; (5) to graduate the exercises and to select only those which can be done by the preceding bookwork. The solutions of illustrative examples are not always the most elegant possible; the probable capacity of the student at each stage has been carefully considered.

During twelve years' daily experience of teaching this subject the author has noted the difficulties common to students. For example, the average student has no idea of 'the *form* of an equation': thus, asked to find the equation of a line through a given point perpendicular to a given line, he begins, 'Let $y = mx + c$ be the line, then its "$m$"', &c., whereas, if he had a clear understanding of form, he would readily write such an equation down.

In order to give the reader confidence in analytical methods, familiar properties are used as illustrations and well-known facts are noted wherever they arise naturally out of the analysis. Thus the circle is fully dealt with; methods and ideas are thereby illustrated earlier than usual. A large number of exercises are given in this part of the work so that the pupil can make the foundations sure; the reader with special mathe-

matical ability can omit many of these. Some of the work on the circle, especially that dealing with the circles of the triangle, is, the author believes, new.

The author has tried to avoid obtaining analytical results by quoting geometrical results with which the reader may be acquainted in Geometrical Conics. This process makes some pupils lose confidence in Analytical Geometry; others welcome it as a dodge enabling them to avoid a real understanding of the principles of the subject; in either case the result is bad. All properties of the conic are developed by analytical processes following on definitions. Thus, for instance, the equation of the axes of the general conic, either in Cartesian or Areal coordinates, is obtained from the simple definition of an axis as a straight line about which the conic is symmetrical: the foci are subsequently shown to lie on the axes. This equation of the axes is *not* deduced from those giving the foci by making the statement that the foci lie on the axes, a statement which, most probably, the reader would fail to justify except by an appeal to Geometrical Conics.

Briefly, this book attempts to answer the question, 'What do the general equations of the first and second degree represent?' rather than, 'What equations represent certain known curves?' The chapter on the circle, however, comes before the general discussion of the equation of the second degree; the purpose being to familiarize the student with the work before the more serious attack, and to cater for those examinations which limit their syllabus to the line and circle.

A few details may be noted: abridged notation is insisted upon as probably the best introduction to quite general coordinates. The author's treatment of the parabola $\sqrt{ax} + \sqrt{by} = 1$ is original, and will, he hopes, commend itself to teachers who have realized the difficulties boys find with the usual work. Parametric coordinates are given their rightful prominence. In the first draft of this book point and line coordinates were treated concurrently: convinced, however, of the relative importance of the former, the author changed his scheme: it is hoped that the introductory chapter on line coordinates will prove useful. Special care has been given to the introduction of imaginary points, points at infinity, and the line at infinity. The last chapter is devoted

to Areal coordinates, and here tangential equations are freely used; many of the proofs given are new.

The author's first and unlimited thanks are to Mr. A. E. Jolliffe, M.A., Fellow and Tutor of Corpus Christi College, Oxford: whenever a difficulty, either of arrangement or of method, has arisen he has given most helpful advice, and it is largely due to his aid and encouragement that this work has been completed.

Mr. Jolliffe also read through and thoroughly criticized both the manuscript and the proofs. The author is entirely responsible for the form and accuracy of the work, but it is right to state that Mr. Jolliffe most generously placed a quantity of his own work at the author's disposal; thus the practical methods of drawing conics and some of the best paragraphs in the later chapters are adapted from his manuscript. Chapter IV was inserted at his suggestion, and he kindly submitted this part of the work to other mathematical authorities for their criticism.

Mr. P. H. Wykes, M.A., spent much time and care reading the manuscript, and his suggestions were often adopted.

Miss Isabella Thwaites, scholar of Girton College, Cambridge, and Mr. W. E. Paterson, M.A., have kindly read the proofs.

The author is glad also to recognize the unfailing courtesy and kindness extended to him by the Clarendon Press.

The author hopes he has produced a book that will not only make the subject interesting to schoolboys, but will be a valuable companion to which later on the undergraduate will often refer and from which he will not readily part.

A. C. J.

BRADFORD, 1912.

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# CHAPTER I

## THE POINT

§ 1. The method of algebraical analysis involves three distinct processes:—

(*a*) The conditions of a geometrical problem are represented by algebraical expressions and equations.

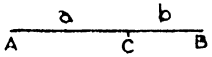
(*b*) The processes of algebra are applied to these expressions and equations to obtain new results.

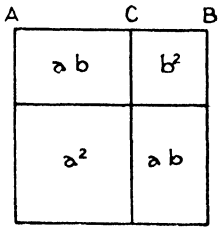
(*c*) These new results are translated back into geometrical language.

The object of the bookwork given is to enable the student to perform readily the first and third operations. The tendency in this subject is to lose sight of the geometrical significance: the student should take the greatest pains to acquire the habit of connecting every algebraical detail with its geometrical interpretation.

One of the most elementary relations between Geometry and Algebra is the expression for the area of a rectangle. The number of square units in the area of a rectangle, whose sides are $a$ and $b$ units of length respectively, is the product $ab$. The algebraical expression $ab$ may thus be said to represent the geometrical quantity, the area of a rectangle.

Algebraical proofs of the propositions in Euclid, Bk. II, are based on this idea. The logic of the process is here illustrated.




*If a straight line be divided into any two parts the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the two parts.*

If $AB$ is the straight line divided at $C$, let $AC$, $CB$ be $a$ and $b$ units of length respectively.

($a$) The area of the square on $AB$ is represented algebraically by the product $(a+b)\times(a+b)$.

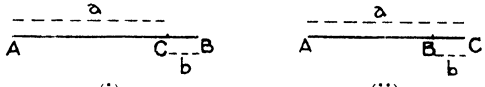
($b$) Now $(a+b)\times(a+b) = (a+b)^2$

$$= a^2+b^2+2ab.$$

($c$) But this result represents the square on $AC$+the square on $CB$+twice the rectangle contained by $AC$ and $CB$.

Hence 'the square on $AB$ is equal to', &c.

The algebraical idea of sign is represented geometrically by direction; thus if the length of $CB$ measured from $C$ to $B$ is $+b$, the length measured from $B$ to $C$ is $-b$.




(i) (ii)

Thus in Fig. (i) $AB$ is of length $(a+b)$, in Fig. (ii) of length $(a-b)$. The absence of this idea in Euclid accounts for the number of pairs of propositions which are algebraically equivalent: e.g. II. 4 and II. 7; II. 5 and II. 6; II. 12 and II. 13.

For instance, in the example given above, if $b$ were negative proposition II. 7 is derived. Using Fig. (ii)

($a$) The square on $AB$ is represented by $(a-b)\,(a-b)$.

($b$) $(a-b)^2 = a^2+b^2-2ab$.

($c$) This represents the sum of the squares on $AC$ and $BC$ less twice the rectangle contained by $AC$ and $BC$.

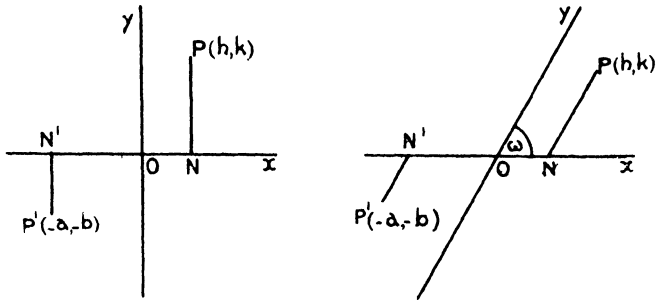
The drawing of graphs is a further useful step towards connecting the subjects of Geometry and Algebra. Graph drawing is now taught in all schools, and we assume that the reader has some knowledge of the process.

### § 2. Cartesian Coordinates. Rectangular and Oblique Axes.

The position of a point $P$ in a plane is indicated by its distances measured in fixed directions from two chosen intersecting straight lines $Ox$, $Oy$ (**the coordinate axes** or **axes of reference**); the axes are called rectangular when $Ox$, $Oy$ are perpendicular, otherwise oblique. In both cases, if $PN$ be drawn parallel to $Oy$ (**the axis of $y$**), the lengths $ON, NP$ are called the $x$ and $y$ coordinates. Thus, if $ON = h$ and $NP = k$, $P$ is the point $(h, k)$.

The same convention with regard to sign is made as in trigonometry: lines measured upwards ($NP$) are positive, downwards ($N'P'$)

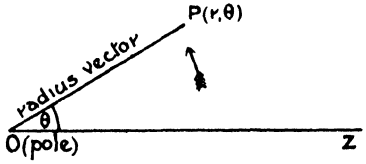
negative, and again to the right ($ON$) positive, to the left ($ON'$) negative: e. g. the point $P'$ is $(-a, -b)$ where $ON' = a$ and $N'P' = b$. The angle $xOy$ is usually called $\omega$.




The $x$-coordinate of a point is called its *abscissa*, the $y$-coordinate its *ordinate*. We shall refer to 'the point $P$ whose coordinates are $a, b$' as 'the point $P(a, b)$'.

**Note.** In the majority of problems it is more convenient to use rectangular axes, especially when the lengths of lines or the magnitude of angles (i. e. metrical properties) are involved, because the expressions for these quantities are much simpler when $\omega$ is a right angle: it may happen, however, that the solution of a problem is so much simplified in other ways by the use of oblique axes that the inconvenience due to the more clumsy formulae is outweighed; consequently the student should make himself familiar with the formulae in the more general case. It may be well to note here that as a rule the student has free choice of axes of reference in any problem; the first step is to decide what lines in the figure will make the most convenient axes of reference, the only restriction being that they must be fixed; a variable line must not be chosen for an axis nor a variable point for origin.

## § 3. Polar Coordinates.



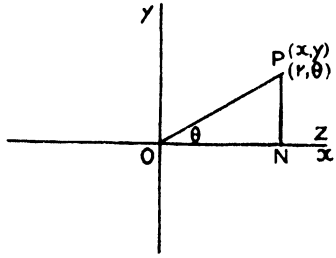

The position of a point $P$ is indicated by

(i) its distance $OP$ ($r$) from a fixed point $O$, called the **pole,**

(ii) the angle ($\theta$) which $OP$ (**the radius vector**) makes with a fixed axis $OZ$.

When $\theta$ is positive, to find the position of $P$, the radius vector must start from the position $OZ$ and revolve about $O$ in a direction opposite to that of the hands of a clock through the angle $\theta$; the distance $r$, if positive, is measured along this radius from $O$; if negative, in the opposite direction. It should be noted that with this convention of sign the points indicated by $(r, \theta)$, $(-r, \pi+\theta)$, $(-r, -\pi+\theta)$, $(r, \theta-2\pi)$ are identical.

§ 4. *The polar coordinates of a point referred to the line OZ are connected by simple relations with the Cartesian coordinates referred to rectangular axes through O, along and perpendicular to OZ.*




Let $P$ be the point $(r, \theta)$ or $(x, y)$:

then $x = ON = r\cos\theta$; $y = NP = r\sin\theta$.

Conversely $r = \sqrt{x^2+y^2}$; and $\theta = \tan^{-1}\frac{y}{x}$.

Thus with these lines of reference the graphs of

$$x = 2y-1 \quad \text{and} \quad r\cos\theta = 2r\sin\theta - 1$$

are identical.

### Examples I a.

1. With rectangular axes mark the positions of the points (2, 1·5), (0, 3), (4, 0), (8, −3) (−2, 4·5). Note graphically that they are collinear. What equation do they all satisfy?

2. With axes inclined at 120°, note the positions of the points (0, 0), (6, 1), (−2, −3), (5, 4), (2, −1) (−3, 4).

Which of these points are collinear?

3. Mark the positions of the following points:—

$$(a, \tfrac{1}{3}\pi), (a, \tfrac{2}{3}\pi), (a, -\tfrac{1}{4}\pi), (-a, -\tfrac{1}{3}\pi),$$

and find their Cartesian coordinates referred to rectangular axes through the pole, one of which coincides with the polar axis.

4. Find the polar coordinates, referred to $Ox$, of the following points, whose Cartesian coordinates referred to rectangular axes are:—

$$(3, 4), (5, 5), (2\sqrt{3}, 2) (-2\sqrt{3}, 2), (-2\sqrt{3}, -2), (2\sqrt{3}, -2).$$

5. The sides of a square are 4 units long; choose coordinate axes so as to make the coordinates of the corners as symmetrical as possible, and state their coordinates.

6. The sides of a parallelogram are 4 and 6 units of length; find the coordinates of its corners referred to suitable axes.

7. The diagonals of a rhombus are 2 and 4 units of length; find the coordinates of its corners referred to suitable rectangular axes.

8. Draw the graphs of (i) $x = 4$, (ii) $y = 3$, (iii) $x = y$, (iv) $x + a = 0$, (v) $y = 4x$, (vi) $x + y = 0$, referred to (*a*) rectangular, (*b*) oblique axes.

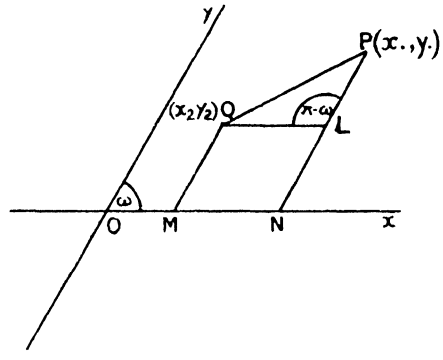
9. Draw graphs of (i) $r\cos\theta = 1$, (ii) $r\sin\theta + 2 = 0$, (iii) $3r - \sec\theta = 0$, (iv) $\theta = \frac{1}{3}\pi$, (v) $2r + 5\operatorname{cosec}\theta = 0$.

10. The polar coordinates of a point are $(a, \alpha)$, its Cartesian coordinates referred to rectangular axes through the pole, the $x$-axis making an angle 30° with the polar axis, are $(x, y)$: prove

(i) $x^2 + y^2 = a^2$; (ii) $\tan\alpha = (y\sqrt{3} + x)/(x\sqrt{3} - y)$.

§ 5. *To find the distance between two points whose coordinates are given.*

**I. Cartesian Coordinates, Oblique axes.**




Let the points be $P(x_1, y_1)$ and $Q(x_2, y_2)$. Draw $PN$, $QM$ parallel to $Oy$, and $QL$ parallel to $Ox$.

then $\angle PLQ = \pi - \omega$; $QL = ON - OM = x_1 - x_2$.

$LP = NP - MQ = y_1 - y_2$;

therefore in the triangle $PLQ$, $PQ^2 = LP^2 + QL^2 - 2LP.QL\cos PLQ$.

$\therefore \quad PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega$,

or $PQ = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega\}}$.

**Note** i. When the axes are rectangular, since $\omega = \frac{1}{2}\pi$,

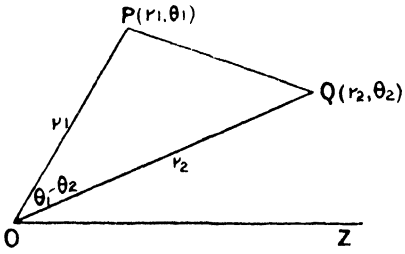
$$PQ = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}}.$$

**Note ii.** If the point $P(x, y)$ is always at a distance $c$ from a fixed point $(a, b)$, then the coordinates of $P$ always satisfy the equation

$$(x-a)^2+(y-b)^2+2(x-a)(y-b)\cos\omega = c^2$$

(oblique axes) and $(x-a)^2+(y-b)^2=c^2$ (rectangular axes). The equations therefore represent a circle whose centre is the point $(a, b)$, and whose radius is $c$.

**II. Polar Coordinates.**




Let the points be $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$, then the $\angle POQ = \theta_1 \quad \theta_2$, and from the triangle $POQ$

$$PQ^2 = r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2),$$

or

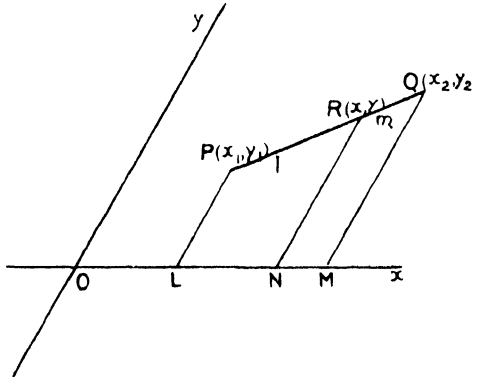
$$PQ = \sqrt{\{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)\}}.$$

**Note.** If the point $(r, \theta)$ moves so as to be always a distance $c$ from the point $(a, \alpha)$, then its coordinates satisfy the equation

$$r^2+a^2-2ra\cos(\theta-\alpha) = c^2,$$

which equation therefore represents in polar coordinates the circle whose centre is $(a, \alpha)$ and whose radius is $c$.

§ 6. *To find the coordinates of a point which divides the distance between two given points in a given ratio.*




The method and result are the same for rectangular and oblique axes.

Let the given points be $P(x_1y_1)$ and $Q(x_2y_2)$, and let the point $R\,(x,\ y)$ divide $PQ$ internally in the ratio $l : m$.

Now draw $PL$, $QM$, $RN$ parallel to $Oy$,

then
$$\frac{PR}{RQ} = \frac{LN}{NM}, \quad \text{i. e. } \frac{l}{m} = \frac{x-x_1}{x_2-x}.$$
$$\therefore \quad lx_2 - lx = mx - mx_1.$$
$$\therefore \quad x = \frac{lx_2 + mx_1}{l+m}.$$

By drawing parallels through $P$, $Q$, $R$ to $Ox$, we can show similarly that
$$y = \frac{ly_2 + my_1}{l+m}.$$

Hence $R$ is the point $\left(\frac{lx_2 + mx_1}{l+m},\ \frac{ly_2 + my_1}{l+m}\right)$.

**Note i.** The mid-point of $PQ$ is $[\frac{1}{2}(x_1+x_2),\ \frac{1}{2}(y_1+y_2)]$.

**Note ii.** When the point $R$ lies outside $PQ$, i. e. when it divides $PQ$ externally in the ratio $l/m$, this ratio must be considered negative, and $m$ written negative in the result. For in the ratio $PR/RQ$ the length $RQ$ is measured in the opposite direction to the length $PR$.

**Note iii.** If the points $C$, $D$ divide a straight line $AB$ internally and externally in the same ratio, the points $CD$ are said to be *Harmonic conjugates* of $A$, $B$: also the four points $A$, $B$, $C$, $D$ are said to form a *Harmonic Range*. From Note (ii) it follows that the points
$$\left(\frac{lx_2 + mx_1}{l+m},\ \frac{ly_2 + my_1}{l+m}\right), \quad \left(\frac{lx_2 - mx_1}{l-m},\ \frac{ly_2 - my_1}{l-m}\right)$$
are harmonic conjugates of the points $(x_1, y_1)$, $(x_2, y_2)$, and these four points form a harmonic range.

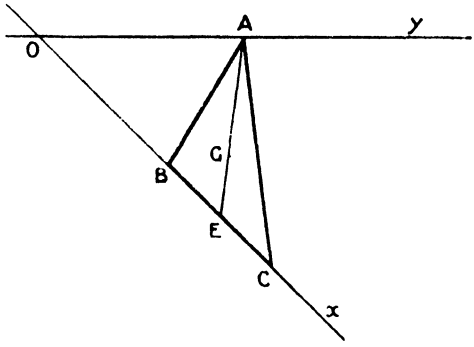
**Note iv.** If $(x, y)$ is a point dividing the distance between $P\ (x_1, y_1)$, $Q\ (x_2, y_2)$ in the ratio $l : m$, we have shown that
$$\frac{l}{m} = \frac{x-x_1}{x_2-x} = \frac{y-y_1}{y_2-y}.$$

This is true for all values of the ratio $l : m$ provided the point $(x, y)$ is on the line $PQ$. Therefore, provided $(x, y)$ is on the straight line joining $P$, $Q$, its coordinates satisfy the equation
$$\frac{x-x_1}{x_2-x} = \frac{y-y_1}{y_2-y},$$
or
$$x\,(y_1-y_2) - y\,(x_1-x_2) + x_1y_2 - x_2y_1 = 0:$$
this equation is then the condition that the point $(x, y)$ should lie on the straight line $PQ$, and is called the equation of the straight line $PQ$.

**Note v.** An investigation of the problem in this paragraph in the case of polar coordinates will well illustrate how inconvenient a particular system of coordinates may be in any special case.

### Illustrative Examples.

**Ex. 1.** *The base of a triangle is fixed and its vertex moves along a fixed straight line: find the locus of its centroid.*




Take the fixed base produced as axis of $x$ and the given fixed line as axis of $y$. Let the coordinates of the fixed points $B$, $C$ be $(l, 0)$, $(m, 0)$, and suppose the variable vertex is $(0, \lambda)$.

The coordinates of $E$, the mid point of $BC$, are $\{\frac{1}{2}(l+m), 0\}$.

The coordinates of $G$ which divides $AE$ in the ratio $2:1$ are $\{\frac{1}{3}(l+m), \frac{1}{3}\lambda\}$.

Hence for all positions of $A$, the $x$-coordinate of $G$ is constant and equal to $\frac{1}{3}(l+m)$, i.e. $G$ lies on the line* $x = \frac{1}{3}(l+m)$, which is parallel to the $y$-axis, i.e. to the given fixed straight line.

**Ex. 2.** *Show that the points* $P(2, -4)$, $Q(4, -2)$, $R(7, 1)$ *lie on a straight line, and find* (i) *the ratio* $PQ : QR$, (ii) *the harmonic conjugate of* $Q$ *with respect to* $P$ *and* $R$.

If $Q$ lies on the line $PR$, its coordinates must be of the form

$$\left\{\frac{7l+2m}{l+m}, \quad \frac{l-4m}{l+m}\right\}.$$

If $\dfrac{7l+2m}{l+m} = 4$, then $3l = 2m$.

Also if $\dfrac{l-4m}{l+m} = -2$, then $3l = 2m$;

i.e. $Q$ lies on $PR$, and $PQ : QR = 2:3$.

The harmonic conjugate of $Q$ with respect to $P$ and $R$ divides $PR$ in the ratio $2 : -3$; its coordinates are therefore

$$\left\{\frac{14-6}{2-3}, \quad \frac{2+12}{2-3}\right\},$$

i.e. $\{-8, -14\}$.

* The abbreviation 'the line . . .' is used throughout for 'the line whose equation is . . .'.

## Examples I b.

(The axes are rectangular unless otherwise stated.)

1. Find the coordinates of the centroid of the triangle whose vertices are (1, 5), (−3, 7), (5, −9).

2. Show that the points (2, 4), $(2+\sqrt{3}, 5)$, (2, 6) are the vertices of an equilateral triangle.

3. Show that the point $\{2a\cos^2\frac{1}{2}\theta,\ 2a\cos^2(\frac{1}{4}\pi+\frac{1}{2}\theta)\}$ is at a constant distance from the point $(a, a)$ for all values of $\theta$.

4. The distance of a point $(x, y)$ from (12, 8) is double its distance from (3, 2). What equation must the coordinates $x$, $y$ satisfy?

5. Find an equation satisfied by the coordinates of all points distant 5 units from (4, 3).

6. Find the ratio in which the point (2, −3) divides the distance between the points $A$ $(-7, 1\frac{1}{2})$, $B$ (−14, 5), and find its harmonic conjugate with respect to $A$ and $B$.

7. Find the coordinates of the mid-point of the line joining (4, 3) and (−2, 1). Where is its harmonic conjugate with respect to these points situated?

8. Straight lines are drawn from a fixed point to meet a fixed straight line. Show that their points of trisection lie on one of two fixed straight lines.

9. Show that the point $(1, 5\frac{2}{7})$ lies on the line joining (5, 3) and (−2, 7). In what ratio does it divide this distance?

What is the harmonic conjugate of this point with respect to the other two?

10. Find the sides of the triangle whose vertices are (1, 3), (3, 1), (6, 4), and its greatest angle.

11. Find the coordinates of the centre of gravity of five equal particles situated at the points (2, 4), (−1, 7), (8, 11), (−8, −5), (4, 8).

12. Find the distance of the mid-point of the line joining $(a, -b)$, $(b, a)$ from the origin.

13. Prove that the points (1, 1), (4, 4), (4, 8), (1, 5) are the corners of a parallelogram, and find the lengths of its diagonals.

14. Prove algebraically that the joins of the mid-points of the sides of a quadrilateral form a parallelogram. (Take the diagonals for axes.)

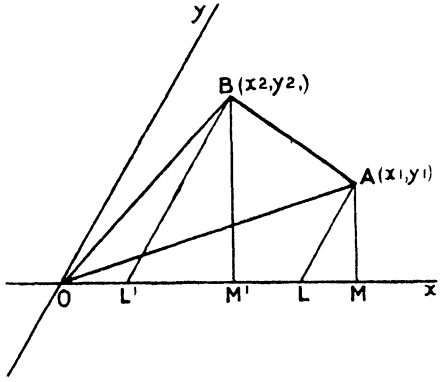
15. Find the condition that the points $(a_1, b_1)$, $(a_2, b_2)$ should lie on a straight line through the origin. (Oblique coordinates.)

16. Find the polar coordinates of the six vertices of a regular hexagon (side $a$) taking one vertex and side as pole and initial line.

17. Points $X$, $Y$ are taken in the side $BC$ of a triangle and in $BC$ produced respectively so that $BX : XC = BY : YC = AB : AC$. Show that $XAY$ is a right angle.

§ 7. *To find the area of any rectilineal figure, given the coordinates of its vertices.*

I. *If $A$, $B$ be the points $(x_1, y_1)$, $(x_2, y_2)$, to find the area $OAB$.*




Let $\angle AOx = \theta_1$, $BOx = \theta_2$; draw $AL$, $BL'$ parallel to $Oy$; $AM$, $BM'$ perpendicular to $Ox$. Then since $\angle AOB = \theta_2 - \theta_1$

$$\begin{aligned}\text{area } OAB &= \tfrac{1}{2}\, OA \,.\, OB \sin(\theta_2 - \theta_1)\\ &= \tfrac{1}{2}\{OB \sin\theta_2 \,.\, OA\cos\theta_1 - OA\sin\theta_1 \,.\, OB\cos\theta_2\}\\ &= \tfrac{1}{2}\{BM' \,.\, OM - AM \,.\, OM'\}\\ &= \tfrac{1}{2}\{y_2 \sin\omega\,(x_1 + y_1\cos\omega) - y_1\sin\omega\,(x_2 + y_2\cos\omega)\}\\ &= \tfrac{1}{2}\{x_1y_2 - x_2y_1\}\sin\omega\,;\end{aligned}$$

or in determinant notation

$$\text{area } OAB = \tfrac{1}{2}\sin\omega \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

**Note.** So many results in analytical geometry are simplified by the determinant notation that the student is recommended to acquaint himself with it before proceeding with the subject: sufficient knowledge of determinants for this purpose can be acquired in two or three hours. *Vide* Hall and Knight's *Higher Algebra*, Chap. XXXIII.

**Cor.** If the points $O$, $A$, $B$ are collinear the triangle $OAB$ has zero area: hence the condition that the points $(0, 0)$, $(x, y)$, $(x_1, y_1)$ should be collinear is $xy_1 - x_1y = 0$.

This condition must be satisfied by the coordinates $(x, y)$ of *any* point on the straight line joining $(0, 0)$ and $(x_1, y_1)$: and conversely any point whose coordinates satisfy this equation lies on this straight line. Hence $xy_1 - x_1y = 0$ is called the equation of the straight line joining the origin and $(x_1, y_1)$.

In the above method of finding the area of the triangle $OAB$ no question of sign arises, because the angle $\theta_2$ is taken greater than $\theta_1$ in the figure. But when the coordinates of two points $A$, $B$ are given in a general form

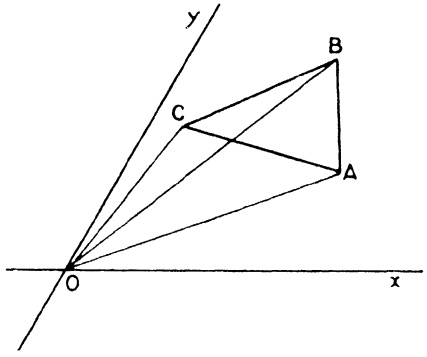
such as $(x_1, y_1)$ $(x_2, y_2)$, we do not know which of the angles $AOx$, $BOx$ is the greater, and the result for the area will differ in *sign* according to which angle is chosen the greater in the figure.

We adopt the following convention:—

When the expression for the area of the triangle $OAB$ is written down, the first term contains the $x$-coordinate of $A$. Thus $A$ and $B$ being the points $(x_1, y_1)$, $(x_2, y_2)$,

$$\text{area } OAB = \tfrac{1}{2} \sin \omega \, (x_1 y_2 - x_2 y_1);$$
$$\text{area } OBA = \tfrac{1}{2} \sin \omega \, (x_2 y_1 - x_1 y_2); \text{ and consequently}$$
$$\text{area } OAB = -\text{area } BOA.$$

II. *To find the area of the triangle whose vertices are* $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$.




Let $ABC$ be the points; join $OA$, $OB$, $OC$, then remembering that as drawn in the figure the area $OAC$ is positive and therefore area $OCA$ negative,

$$\Delta ABC = \Delta OAB + \Delta OBC + \Delta OCA$$
$$= \tfrac{1}{2} \sin \omega \, \{(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3)\},$$

or in determinant notation

$$\tfrac{1}{2} \sin \omega \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

**Note.** This expression gives the area positively if, as we go round the triangle in the order $A$, $B$, $C$, the rotation is positive in accordance with the convention used in trigonometry, and vice versa.

**Cor. i.** When the axes are rectangular the area of the triangle whose vertices are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is

$$\tfrac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

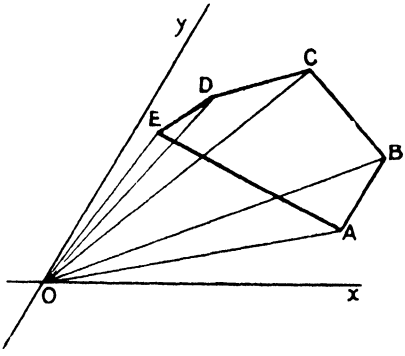
**Cor. ii.** If $(x, y)$ are the coordinates of any point $P$ on the straight line joining the points $Q(x_1, y_1)$, $R(x_2, y_2)$, then the area of the triangle $PQR$ is zero. Hence

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

or $$x(y_1-y_2)-y(x_1-x_2)+x_1y_2-x_2y_1 = 0.$$

This is therefore the condition that the point $(x, y)$ should lie on the straight line joining $(x_1, y_1)$ $(x_2, y_2)$; in other words, it is the equation of this straight line. (Cf. § 7, Cor.)

III. The method applies similarly to the area of a figure of any number of sides. Thus a pentagon, for example :—




Area $ABCDE = \Delta OAB + \Delta OBC + \Delta OCD + \Delta ODE + \Delta OEA.$

$= \frac{1}{2} \sin\omega \{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_5-x_5y_4)$
$+(x_5y_1-x_1y_5)\}.$

IV. The case when the polar coordinates $(r_1, \theta_1)$, $(r_2, \theta_2)$, $(r_3, \theta_3)$ of the vertices are given is left as an exercise for the student: the consequent formula is of no value: particular cases should be worked out from first principles.

§ 8. We have seen that the coordinates of a point lying on the straight line joining two given points satisfy a certain equation, and conversely that all points whose coordinates satisfy the equation lie on the straight line. This straight line is called the *locus* of points which satisfy the equation. Any equation containing the variables $x$ and $y$ represents a locus. To every value of $x$ definite corresponding values of $y$ can be found, one value if $y$ occurs only in the first power, two values if $y$ occur in the second power. Thus the coordinates of any number of points can be found which can be indicated

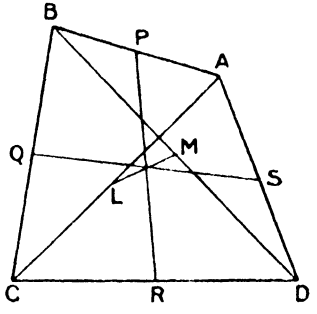
on a graph. If $x$ is made to increase continuously, the corresponding values of $y$ will as a rule change continuously; a line can be drawn which passes through all these points, and the curve so drawn is called the locus of the point whose coordinates satisfy the given equation. In the present work all forms of equations of the first and second degree are examined, and the properties of the corresponding loci found. At present, when the student is asked in examples to find 'the equation of the locus' of a point under given conditions, the equation is all the answer required; the student is not of course at this stage expected to recognize the locus which the equation represents: its form can be roughly found by drawing its graph. We may also note that when the coordinates of a point are given in some special *form*, this is equivalent to giving an equation satisfied by the coordinates $(x, y)$. Thus the point $(a\cos\theta, a\sin\theta)$, for different values of $\theta$, is on the locus $x^2+y^2=a^2$.

Again, points whose coordinates are of the form $(c\lambda, c/\lambda)$ lie on the locus $xy=c^2$, whatever value $\lambda$ may have. The idea conveyed by the word 'form' is of the highest importance.

### Illustrative Examples.

**Ex. 1.** *The joins of the mid-points of the opposite sides of a quadrilateral and the join of the mid-points of its diagonals bisect each other.*

**Choice of axes.** It is an advantage in this case to indicate the four vertices by quite general coordinates: for, if $ABCD$ is the quadrilateral and $P, Q, R, S$ are the mid-points of the sides, and $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$ the vertices $ABCD$, then, having found the coordinates of the mid-point of $PR$, those of $QS$ follow by changing the suffixes 1, 2, 3, 4 in cyclic order.




The coordinates of $P$ and $R$ are

$$\{\tfrac{1}{2}(x_1+x_2), \tfrac{1}{2}(y_1+y_2)\},\ \{\tfrac{1}{2}(x_3+x_4), \tfrac{1}{2}(y_3+y_4)\}.$$

Therefore those of the mid-point of $PR$ are

$$\{\tfrac{1}{4}(x_1+x_2+x_3+x_4), \tfrac{1}{4}(y_1+y_2+y_3+y_4)\};$$

The same result is therefore true for the mid-point of $QS$.

The coordinates of $L$ and $M$ are similarly

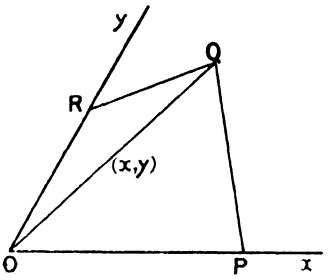
$$\{\tfrac{1}{2}(x_1+x_3), \tfrac{1}{2}(y_1+y_3)\},\ \{\tfrac{1}{2}(x_2+x_4), \tfrac{1}{2}(y_2+y_4)\}.$$

Therefore those of the mid-point of $LM$ are

$$\{\tfrac{1}{4}(x_1+x_2+x_3+x_4), \tfrac{1}{4}(y_1+y_2+y_3+y_4)\};$$

hence the mid-points of $PR$, $QS$, $LM$ are identical.

**Ex. 2.** *Two sides OP, OR of a quadrilateral OPQR are fixed: if the area of the quadrilateral is constant, find the equation of the locus of the mid-point of OQ.*



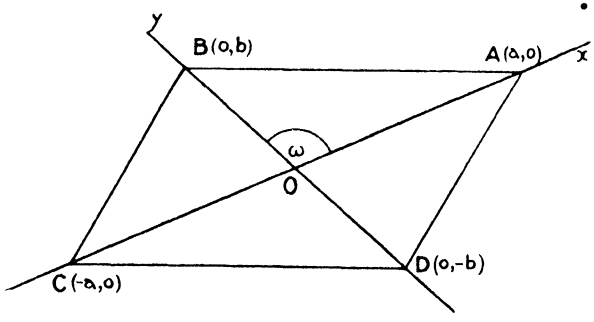

Take $OP$, $OR$ as axes of $x$ and $y$ and let $P$ be the point $(a, 0)$, $R$ the point $(0, b)$. If the coordinates of the mid-point of $OQ$ be $(x, y)$, those of $Q$ are $(2x, 2y)$.

Area $OPQ = \frac{1}{2} \sin\omega \,.\, 2ay = ay \sin\omega$.

Area $OQR = \frac{1}{2} \sin\omega \,.\, 2bx = bx \sin\omega$.

Hence $ay + bx$ is constant, and the equation of the locus is $bx + ay = c$, where $c \sin\omega$ is the area of the quadrilateral.

**Ex. 3.** *$ABCD$ is a parallelogram whose diagonals meet at $O$: if $P$ is any point, prove that $PA^2 + PB^2 + PC^2 + PD^2 = AB^2 + BC^2 + 4PO^2$.*




We give two solutions of this question to illustrate the effect of the use of different coordinate axes:—

(i) Take the diagonals for axes of reference; since the diagonals of a parallelogram bisect each other, the coordinates of the vertices are $(a, 0)$, $(0, b)$, $(-a, 0)$, $(0, -b)$; let $P$ be the point $(x, y)$, then

$$PA^2 = (x-a)^2 + y^2 + 2y(x-a)\cos\omega,$$
$$PB^2 = x^2 + (y-b)^2 + 2x(y-b)\cos\omega,$$
$$PC^2 = (x+a)^2 + y^2 + 2y(x+a)\cos\omega,$$
$$PD^2 = x^2 + (y+b)^2 + 2x(y+b)\cos\omega.$$

$\therefore\ PA^2 + PB^2 + PC^2 + PD^2 = 4(x^2 + y^2 + 2xy\cos\omega) + 2(a^2 + b^2)$.

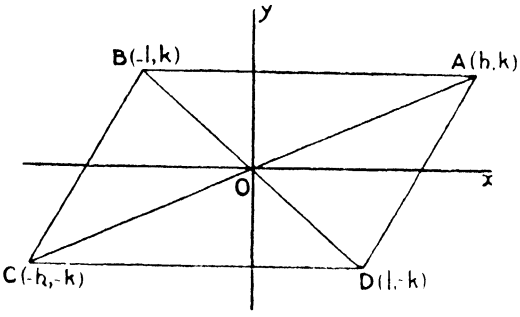
$= 4(x^2 + y^2 + 2xy\cos\omega) + (a^2 + b^2 + 2ab\cos\omega) + (a^2 + b^2 - 2ab\cos\omega)$

$= 4PO^2 + AB^2 + BC^2$.

(ii) Take lines through the intersection of the diagonals parallel and perpendicular to a side respectively as axes. The symmetry of the figure gives for the vertices $(h, k)$, $(-l, k)$, $(-h, -k)$, $(l, -k)$

$$PA^2 = (x-h)^2 + (y-k)^2,$$
$$PB^2 = (x+l)^2 + (y-k)^2,$$
$$PC^2 = (x+h)^2 + (y+k)^2,$$
$$PD^2 = (x-l)^2 + (y+k)^2.$$

$$\therefore \quad PA^2 + PB^2 + PC^2 + PD^2 = 4(x^2+y^2) + 2h^2 + 2l^2 + 4k^2$$
$$= 4(x^2+y^2) + (h+l)^2 + \{(h-l)^2 + 4k^2\}$$
$$= 4PO^2 + AB^2 + BC^2.$$




## Examples I c.

(The axes are rectangular unless otherwise stated.)

1. Find the coordinates of the points of trisection of the straight line joining the point $P(1, 2)$ to $Q(3, -2)$.

Also the coordinates of a point $R$ dividing $PQ$ externally so that $3PR = QR$.

2. The straight line joining the points $(-2, -4), (3, 1)$ is divided into five equal parts. Find the coordinates of the points of division.

3. The coordinates of three points $P$, $Q$, $R$ are (1, 1), (3, 5), and (6, 11). Show that $Q$ is a point lying between $P$ and $R$ on the straight line $PR$. Find also the coordinates of a point $S$ on the straight line between $P$ and $R$ such that $PQ = 3\,SR$.

4. Find the coordinates of the six vertices of a regular hexagon referred to rectangular axes through its centre, one of the axes lying along a diagonal.

5. Squares are described on the sides $AB$, $AC$, $BC$ of a triangle right-angled at $A$. Find the coordinates of the corners of these squares and the points of intersection of their diagonals when $AB$, $AC$ are axes of reference. $AB = c$, $AC = b$.

6. Find the polar coordinates of the six corners of a regular hexagon, when the centre is the pole and a diagonal the initial line.

7. Find the area of a triangle whose vertices are the points (5, 7), $(-2, -1)$, (0, 8).

8. Find the area of the quadrilateral whose vertices are (4, 5), (2, −6), (−14, 6), (−5, −7).

9. Find the cosine of the angle which the straight line joining the points $(a, b)$, $(a', b')$ subtends at the origin.

10. The coordinate axes being inclined at 60°, prove that $(a, 0)$, $(0, 2a)$, $(2a, a)$ are the corners of an equilateral triangle.

11. The centre of a circle is (3, 5), one end of a diameter is (7, 3); what are the coordinates of the other end of this diameter? Find the radius.

12. Prove analytically that the three straight lines joining the vertices of a triangle to the mid-points of the opposite sides have one point of trisection in common.

13. Find the coordinates of a point which is equidistant from $(a, b)$ and $(2a, b)$, and whose distance from the origin is $\frac{5}{2}a$.

14. Find the coordinates of a point which is equidistant from (0, 0), $(0, a)$, and $(3a, 4a)$.

15. Draw the graphs of the equations (i) $x = 5$, (ii) $y = -7$, and find the distances of the point (8, 8) from them.

16. Prove that the point $(a\cos\theta, a\sin\theta)$ is at the same distance from the origin whatever value $\theta$ may have.

Show that the point $(a + r\cos\theta, b + r\sin\theta)$ lies on a fixed circle whose centre is $(a, b)$ whatever value $\theta$ may have.

What equation is satisfied by coordinates of this form, $a$, $b$, and $r$ being fixed?

17. Show that the distance between the points $\{a+5, b+9\}$, $\{a+2, b+5\}$ is the same for all values of $a$ and $b$. Find the distance.

18. Draw the graphs of (i) $x^2 - 2xy + y^2 + x - y = 0$;

$$\text{(ii) } r\cos(\theta + \tfrac{1}{3}\pi) = 2.$$

19. The coordinates of three points $A, B, C$ are (8, 6), (7, 7), (0, 6); show that they are equidistant from the point (4, 3). Find an equation satisfied by the coordinates of any point on the circle $ABC$.

20. Find the distance of the point $(a\tan^2\theta, 2a\tan\theta)$ from $(a, 0)$, and from the locus represented by $x + a = 0$.

What equation will the coordinates in this form satisfy, $a$ being a constant?

21. Find the ratio of the distances between the pairs of points

(i) $(a\sin\theta, b\cos\theta)$, (0, 0); (ii) $(a\cos\theta, b\sin\theta)$, $\{(a^2 - b^2/a)\cos\theta, 0\}$.

22. Express algebraically that the point $(xy)$ is equidistant from $(a, b)$ and $(-a, b)$.

23. What angle does the line joining $(a\sqrt{3}, a)$ and $(b, b\sqrt{3})$ subtend at the origin?

24. Write down an equation satisfied by the coordinates of a point which moves so that the difference of its distances from two intersecting straight lines is constant.

(Take the lines as axes: if they are inclined at 30° trace the graph of the locus.)

25. The polar coordinates of two points are $(r_1, \alpha_1)$, $(r_2, \alpha_2)$: the line through the pole bisecting the angle which they subtend at the pole meets their join in $P$. Find the polar coordinates of $P$.

26. The distances of the collinear points $P, R, Q$ from the origin form

a harmonic series. If the coordinates of $P$, $Q$ are $(a, 0)$, $(b, 0)$, find those of $R$. Prove that $P$, $Q$ divide $OR$ internally and externally in the same ratio.

27. $P$ and $Q$ are two points whose coordinates are $(am^2, 2am)$, $(am^{-2}, -2am^{-1})$, and $S$ is the point $(a, 0)$. Prove (i) that $PSQ$ are collinear, (ii) $1/SP + 1/SQ$ is constant for all values of $m$.

28. Choose the most convenient axes to represent the vertices of an equilateral triangle $ABC$, and find the coordinates of the centre $(P)$ of its circumcircle. If $Q$ is any point in the plane, find the ratio of

$$QA^2 + QB^2 + QC^2 - BC^2 \text{ to } QP^2.$$

29. Find the area of the triangle whose vertices are $(a, 70°)$, $(2a, 40°)$ and the origin.

Also of the triangle whose vertices are $(a, 10°)$, $(3a, 40°)$, $(5a, 100°)$.

30. Show that the points $(a, a)$, $(ka, -ka)$ subtend a right angle at the origin. Prove also that the triangle whose vertices are $(3, -2)$, $(-5, +4)$, $(9, 6)$ is right-angled.

31. Prove that the point $(a + a\tan^2\theta, 2a\tan\theta)$ is equally distant from the point $(2a, 0)$, and the axis of $y$ for all values of $\theta$.

32. Show that the middle points of the non-parallel sides of a trapezium and the middle points of its diagonals lie on a straight line parallel to the parallel sides.

33. Find the equation of the locus of a point $P$ which moves so that the area of the rectangle formed by the axes and the perpendiculars from $P$ to them is of constant area.

34. Prove that the distance of the point $(a\cos\theta, b\sin\theta)$ from the point $(ae, 0)$ is $a + ae\cos\theta$, if $a^2 - b^2 = a^2e^2$.

35. Find the condition that the coordinates of the middle point of the straight line joining $(a, b)$, $(2b, 2a)$ should satisfy the equation $2x + 2y = 3c$.

36. Find the equation of the locus of the middle point of straight lines drawn from a given point to any point in a given straight line.

37. In any triangle $ABC$, if $D$ is the mid-point of $BC$, prove analytically that $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

38. Find the coordinates of the centroid of the triangle $ABC$ whose vertices are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, and hence prove

$$AB^2 + BC^2 + CA^2 = 3\,(AG^2 + BG^2 + CG^2).$$

39. The sum of the squares on the diagonals of any quadrilateral is double the sum of the squares on the lines joining the middle points of the opposite sides.

40. The diagonals of a trapezium are drawn and the mid-points of the parallel sides are joined: show analytically that there is a point common to these straight lines which divides each of them in the same ratio.

(Take the third line and one parallel side for axes.)

41. A variable line cuts any two lines intersecting at $O$ in the points $P$, $Q$ so that $OP + OQ = 2l$. Find the equation of the locus of its middle point.

42. If $P$, $Q$ are two points $(x_1, y_1)$, $(x_2, y_2)$, and $PL$, $QM$ be drawn parallel

to the axis of $y$, find the area of $PQLM$ when (i) the axes are rectangular, (ii) oblique.

43. $P$, $Q$, $R$ are the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, and $PL$, $QM$, $RN$ are drawn perpendicular to the axis of $x$. Find the areas $PLMQ$, $RNMQ$, $PLNR$, and deduce an expression for the area of the triangle $PQR$.

44. Show that the three points $(-1, -2)$, $(4, 1)$, $(9, 4)$ lie on a straight line, and find the ratio of the distances of the second from the other two.

45. Find the condition that the point $(x, y)$ should lie on the straight line joining $(3, 4)$ $(-5, 2)$.

46. $ABCD$ is a rectangle, $P$ any point in the plane.

Prove $PA^2 + PC^2 = PB^2 + PD^2$.

47. The coordinates of two points $P$, $Q$ are $(2, 3)$, $(10, 15)$.

Show that the straight line $PQ$ passes through the origin.

Find the coordinates of a point $R$ such that $RQ = 3QP$ and the ratio $PR/RQ$.

48. The points $C$, $D$ divide the line $AB$ harmonically.

Show that (i) the lengths $AC$, $AB$, $AD$ are in $H.P$;

(ii) if $O$ is the mid-point of $AB$, $OA^2 = OC \cdot OD$;

(iii) $2/CD = 1/BD + 1/AD$.

49. The coordinates of four points are $(1, 3)$, $(3, 5)$, $(4, 6)$, $(7, 9)$.

(i) Prove they are collinear.

(ii) Prove they form a harmonic range.

(iii) In what ratio is the line joining the first and third divided by the second?

(iv) Find the equations of the straight lines joining the origin to the points.

(v) Find the coordinates of the points on each of these straight lines whose abscissae are 10, and prove that these points form another harmonic range.

50. Find the condition that the three points whose polar coordinates are $(r_1, \theta_1)$, $(r_2, \theta_2)$, $(r_3, \theta_3)$ should be collinear.

51. Find the ratio of the areas of two triangles whose angular points are

(i) $(a\lambda^2, 2a\lambda)$, $(a\mu^2, 2a\mu)$, $(a\nu^2, 2a\nu)$;

(ii) $\{a\lambda\mu, a(\lambda+\mu)\}$, $\{a\mu\nu, a(\mu+\nu)\}$, $\{a\nu\lambda, a(\nu+\lambda)\}$.

52. Express algebraically that the distance of the point $(x, y)$ from $(3, 4)$ is 5 units.

What is the graph of the equation?

53. $A$, $B$ are fixed points on the axes: find the equation of the locus of a point $P$ such that $2OP^2 = AP^2 + BP^2$, given that $OA = a$, $OB = b$.

54. Two sides of a quadrilateral $OP$, $OR$ are fixed, and the diagonal $OQ$ is of constant length $c$: find the equation of the locus of the middle point of the join of the mid-points of the diagonals, given that $OP = a$, $OR = b$.

55. Two vertices of a triangle are $(x_1, y_1)$, $(x_2, y_2)$ and the centroid is $(x, y)$: find the coordinates of the other vertex.

56. Find two points $C$, $D$ which are harmonic conjugates of the points $A(x_1, y_1)$, $B(x_2, y_2)$: if perpendiculars are drawn from the four points to the

axis of $x$, prove the feet of the perpendiculars form a harmonic range. What general property can you deduce from this result?

57. $ABCD$ is a parallelogram: show that $D$, a point of trisection of $AC$ and the mid-point of $AB$, are collinear.

58. The vertices of a triangle are $(a\cos\theta,\ b\sin\theta)$, $(a\cos\overline{\theta+\frac{2}{3}\pi},\ b\sin\overline{\theta+\frac{2}{3}\pi})$, $(a\cos\overline{\theta+\frac{4}{3}\pi},\ b\sin\overline{\theta+\frac{4}{3}\pi})$. (i) Prove its centroid is the origin. (ii) Find its area. (iii) If $a=b$ show that the triangle is equilateral.

59. A straight line of length $2l$ has its ends on the axes of reference: find the equation of the locus of its middle point.

Draw the graph when $l=3$.

60. A point $P$ has coordinates which satisfy the equation $x+\sqrt{3}y-2r=0$, and its distance from the origin is $r$: find the angle $POx$.

61. Find the distance between two points whose polar coordinates are $(6,\ \frac{1}{3}\pi)$, $(-3,\ \frac{2}{3}\pi)$.

62. $P$ is a point inside a parallelogram $ABCD$ such that the area $PBCD$ is double the area $PBAD$: find the equation of the locus of $P$.

63. A point moves so that the ratio of its distances from the points $(-a,\ 0)$, $(a,\ 0)$ is $2:3$: find the equation of its locus.

Where does the locus meet the axis of $x$?

64. $A,\ B,\ C \ldots K$ are $n$ points whose coordinates are $(x_1,\ y_1)$, $(x_2,\ y_2)$, $(x_3,\ y_3) \ldots (x_n,\ y_n)$.

$AB$ is bisected at $P_1$, $P_1C$ is divided at $P_2$ so that $2P_1P_2 = P_2C$, $P_2D$ is divided at $P_3$ so that $3P_2P_3 = P_3D$, and so on until $P_{n-2}K$ is divided at $P_{n-1}$ so that $(n-1)\,P_{n-2}P_{n-1} = P_{n-1}K$. Find the coordinates of $P_{n-1}$.

# CHAPTER II

## THE EQUATION OF THE FIRST DEGREE

§ 1. When two points on a straight line are known, the straight line is completely determined. We have shown in Chapter I, § 6, that the coordinates $(x, y)$ of *any* point on the straight line which joins the points $(x_1, y_1)$, $(x_2, y_2)$ satisfy the equation

$$x(y_1-y_2)-y(x_1-x_2)+(x_1y_2-x_2y_1)=0,$$

or in determinant notation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

This equation is of the first degree in the variable coordinates $(x, y)$: we proceed to show conversely that every equation of the first degree in $x$ and $y$ represents a straight line, or, in other words, that every pair of values of $x$ and $y$ which satisfy a given equation of the first degree represents a point on a definite straight line.

The most general form of the equation of the first degree is

$$Ax+By+C=0,$$

where the coefficients or constants of the equation $A$, $B$, $C$ can have any values, positive or negative.

§ 2. *The equation* $Ax+By+C=0$ *represents a straight line.*

**Method I.** Let $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ be *any* three points whose coordinates satisfy the equation: then

$$\begin{aligned} Ax_1+By_1+C&=0, \\ Ax_2+By_2+C&=0, \\ Ax_3+By_3+C&=0. \end{aligned}$$

Eliminate the constants $A$, $B$, and $C$, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Hence the triangle formed by joining *any* three points on the locus represented by

$$Ax+By+C=0$$

has zero area, i.e. any three points on the locus lie on a straight line: the locus is therefore a straight line.

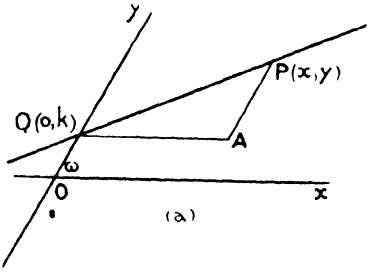
**Method II.** If $B$ is not zero the point $\left(0, -\frac{C}{B}\right)$ satisfies the equation

$$Ax+By+C=0,$$

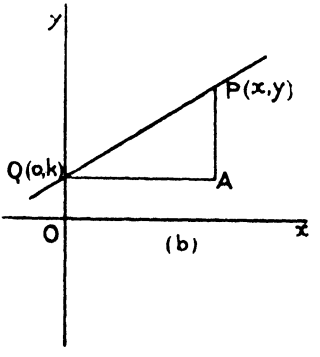
and lies on the axis of $y$: it is therefore the point where the locus represented by this equation cuts the axis of $y$. Let this point be $Q$, and for convenience let $k = -\frac{C}{B}$.

Clearly $-\frac{C}{B}$ can be positive or negative; if due regard is paid to the signs the following argument holds whether we take this quantity positive or negative.

(a) Oblique axes.




(b) Rectangular axes.




Suppose $P(x, y)$ is any other point whose coordinates satisfy

$$Ax+By+C=0.$$

Join $PQ$ and draw $QA$, $PA$ parallel to the coordinate axes, then

$$QA = x, \quad AP = y-k.$$

The given equation can be written

$$y + \frac{C}{B} = -\frac{A}{B}x, \quad \text{or} \quad \frac{y-k}{x} = -\frac{A}{B},$$

hence

$$\frac{AP}{QA} = -\frac{A}{B};$$

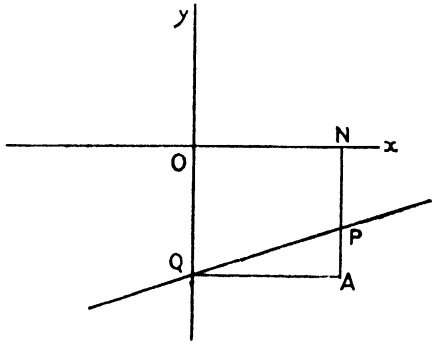
consequently for all positions of $P$ the ratio $\frac{AP}{QA}$ is constant, hence $P$ lies on a straight line. [Euclid VI.]

If $B = 0$, the equation reduces to

$$Ax+C=0,$$

which evidently represents a straight line parallel to the axis of $y$.

**Note.** If $k$ were negative and $P$ in some other quadrant, say the fourth, then the length of $OQ$ is $+\frac{C}{B}$, and since the $y$-coordinate of $P$ is negative,




the numerical value of the length $NP$ is $-y$. hence the length $AP$ is $k-(-y)=k+y$ and $QA=x$.

But, as above, $$y+\frac{C}{B}=-\frac{A}{B}x;$$

i.e. $$\frac{y+k}{x}=-\frac{A}{B} \text{ and } \frac{AP}{QA}=-\frac{A}{B}.$$

The student should go through the proof with the point $P$ in the other quadrants, remembering that the general coordinates $(x, y)$ contain the signs: thus $(x, y)$ might represent, for example, $(-3, 4)$ or $(-5, -6)$.

**Cor.** Let the straight line $PQ$ make an angle $\theta$ with the axis of $x$; i.e. $\angle PQA=\theta$; then $\angle QPA=\omega-\theta$. In the triangle $PQA$

$(a)$ $$\frac{QA}{AP}=\frac{\sin(\omega-\theta)}{\sin\theta}=-\frac{B}{A};$$

hence $$\sin\theta\,(A\cos\omega-B)=A\sin\omega\,.\cos\theta.$$

Therefore $$\tan\theta=\frac{A\sin\omega}{+A\cos\omega-B},$$

or $(b)$ in rectangular coordinates

$$\tan\theta=\frac{AP}{QA}=-\frac{A}{B}.$$

It should be carefully noted that the value of $\tan\theta$ does not contain the constant $C$: hence the direction of the straight line

$$Ax+By+C=0$$

is independent of $C$, and depends *only* on the ratio of the coefficients of $x$ and $y$. Hence if $A$ and $B$ are kept constant while different values are given to $C$, the resulting equations will represent a series of parallel straight lines all making an angle $\tan^{-1}\left(-\frac{A}{B}\right)$ with the axis of $x$: the position of any particular line is fixed by the value of $C$, for this determines the position of $Q$.

### Examples II a.

(The axes are rectangular except in questions marked with an asterisk.)

1. *Find the equation of the straight line joining the points $(a+1, b+3)$, $(a, b)$.

2. *Find the equations of the straight lines joining the following pairs of points: (a) (1, 1), (5, 5); (b) (−2, 3), (−7, −9);
(c) (0, 1), (1, 0); (d) (0, 0), (6, 3);
(e) (0, 0), (0, $a$); (f) (0, 0), ($b$, 0).

3. *What are the equations of the coordinate axes?

4. Draw the graphs of:
$$3x+4y=-12,$$
$$3x+4y=-6,$$
$$3x+4y=0,$$
$$3x+4y=6,$$
$$3x+4y=12;$$
and show that the distance between each pair of consecutive lines is the same.

5. What angles do the following lines make with the axes of $x$ and $y$ respectively?
(a) $y+\sqrt{3}x=7$;
(b) $\sqrt{3}y-x=$ ,
(c) $x-y=-9$;
(d) $x+y=-8$. Draw these lines.

6. *If the axes of coordinates are inclined at 60°, what angles do the following lines make with the $x$-axis?
(a) $y-x=0$;
(b) $y+x=-2$;
(c) $2x+y=1$;
(d) $(\sqrt{3}-1)y-2x=7$.

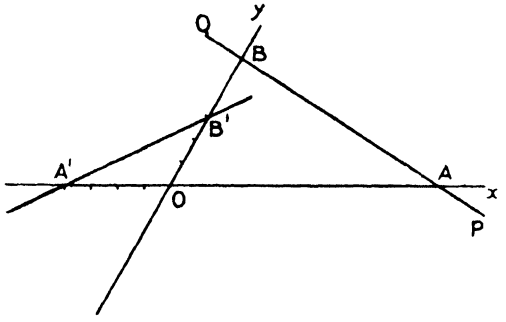
7. What is the equation of a straight line parallel to $2x+3y=6$ which passes (i) through the point (0, 4); (ii) through the origin?

8. Find the angle between the pairs of lines
(a) $y-\sqrt{3}x=0$, $\sqrt{3}y-x=0$;
(b) $y+\sqrt{3}x=5$, $\sqrt{3}y-x=1$.

## § 3. Some special forms of the equation of a straight line.

Special attention should be paid to the results found for rectangular axes: as before stated, oblique axes are not often necessary in the more elementary parts of the subject. The results for oblique axes are, however, worked out for future reference.

**Form I.** *The equation of the straight line in terms of the lengths which it intercepts on the coordinate axes.*




Let the straight line $PQ$ cut the coordinate axes in $A$ and $B$ so that $OA = a$, $OB = b$. Now the points $(a, 0)$, $(0, b)$ are on the straight line; its equation therefore is

$$bx + ay = ab,$$

or

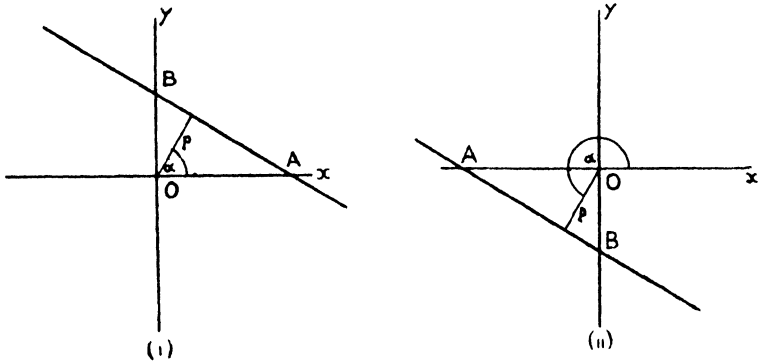
$$\frac{x}{a} + \frac{y}{b} = 1. \qquad \text{(i)}$$

The usual convention with regard to sign must be observed; thus the equation of the straight lin $A'B'$ making intercepts $-4$ and 3 on the axes is $-\frac{1}{4}x + \frac{1}{3}y = 1$.

N.B.—This result is true for rectangular *and* oblique axes.

**Form II.** *The equation of the straight line in terms of the perpendicular on it from the origin and the angle this perpendicular makes with the* x*-axis.*

(Rectangular Coordinates.)




Let the length of the perpendicular be $p$ and the angle made by it with the $x$-axis be $\alpha$. This angle must be measured from $Ox$ as

initial line as in polar coordinates: thus the polar coordinates of the foot of the perpendicular on the straight line are $(p, \alpha)$.

Let the straight line meet the axes in $A$ and $B$.

Hence
$$OA = p \sec \alpha,$$
$$OB = p \operatorname{cosec} \alpha.$$

Therefore by (i) the equation is
$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1,$$
or
$$x \cos \alpha + y \sin \alpha = p. \qquad \text{(ii)}$$

In this form of the equation $p$ is supposed to be positive and $\alpha$ to vary from 0 to $2\pi$.

If the line does not lie in the positive quadrant, as in Fig. (ii), it will be found that the signs of the intercepts are given correctly by the signs of the sine and cosine of $\alpha$: thus in the third quadrant $\sin \alpha$ and $\cos \alpha$ are both negative, and the intercepts $OA$, $OB$ are both negative.

**Cor. i.** In the case of oblique axes, if the perpendicular makes angles $\alpha, \beta$ with $Ox$ and $Oy$, it can be shown in exactly the same way that the equation of the line is $x \cos \alpha + y \cos \beta = p$.

**Cor. ii.** The general equation $Ax + By + C = 0$ can be put in this form.

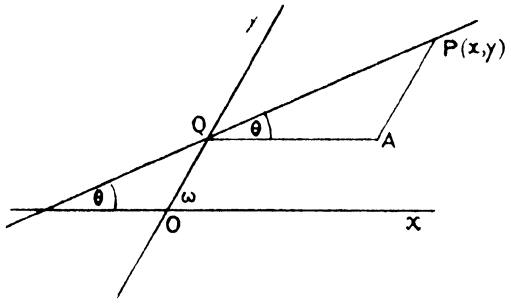
Let
$$A = r \cos \alpha,$$
$$B = r \sin \alpha,$$
then
$$r^2 = A^2 + B^2;$$
consequently
$$\cos \alpha = \frac{A}{\sqrt{A^2 + B^2}},$$
$$\sin \alpha = \frac{B}{\sqrt{A^2 + B^2}}.$$
Hence
$$\frac{A}{\sqrt{A^2 + B^2}} x + \frac{B}{\sqrt{A^2 + B^2}} y = -\frac{C}{\sqrt{A^2 + B^2}}$$
is in the required form, if the sign of the radical is chosen so as to make the right-hand side positive.

**Note.** It follows that the length of the perpendicular from the origin to the straight line $Ax + By + C = 0$ is
$$-\frac{C}{\sqrt{A^2 + B^2}}.$$

**Form III.** *The equation of the straight line in terms of the angle which it makes with the* x*-axis, and the length of the intercept on the axis of* y.

Let the angle be $\theta$ and the intercept $c$.

(a) **Oblique Coordinates.**




Let $P(x, y)$ be any point on the straight line, $Q$ the point where it cuts the $y$-axis; therefore $OQ = c$.

Now $$AP = y - c, \quad QA = x,$$

hence in the triangle $APQ$,

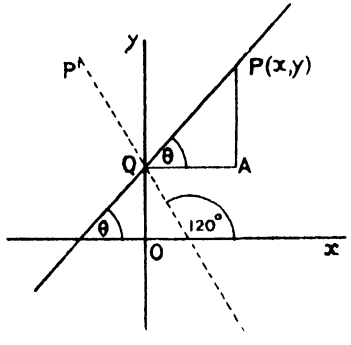
$$\frac{y-c}{x} = \frac{\sin\theta}{\sin(\omega-\theta)},$$

or the coordinates of any point $(x, y)$ on the straight line satisfies the equation

$$y = x\frac{\sin\theta}{\sin(\omega-\theta)} + c.$$

**Cor.** The equation $y = mx + c$ represents a straight line cutting off a length $c$ from the axis of $y$ and making an angle $\theta$ with the axis of $x$ given by $\sin\theta = m\sin(\omega - \theta)$.

(b) **Rectangular Coordinates.**




Since $\omega = \frac{1}{2}\pi$ in this case, the equation becomes $y = x\tan\theta + c$, and $y = mx + c$ represents a straight line inclined at an angle $\tan^{-1} m$ to the axis of $x$ and cutting off a length $c$ from the axis of $y$. The angle $\theta$ is that formed by a straight line starting from the initial position $Ox$ and revolving in the positive direction about $O$ until it is parallel to the given line. Thus $P'Q$ in the figure is the straight line $y = x\tan\frac{2}{3}\pi + c$.

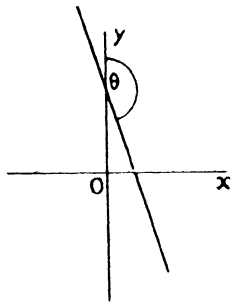
**Cor. i.** If the straight line make an angle $\theta$ with the axis of $y$ and cut off a length $c$ from the axis of $x$, its equation is

$$x = y \tan\theta + c,$$

and conversely the equation

$$x = my + c$$

makes an angle $\tan^{-1} m$ with the axis of $y$, and cuts off a length $c$ from the axis of $x$. This angle must be measured in a similar way: a straight line starting from the position $Oy$ and revolving in the *negative* direction until parallel to the given line moves through the angle $\theta$.




**Cor. ii.** The straight lines

$$\left.\begin{aligned} y &= x\tan\theta + c \\ y &= x\tan(\theta \pm \tfrac{1}{2}\pi) + d \end{aligned}\right\}$$

are evidently perpendicular to each other.

These can be written

$$\left.\begin{aligned} y &= x\tan\theta + c \\ y &= -x\cot\theta + d \end{aligned}\right\};$$

or, with the notation of this paragraph,

$$\left.\begin{aligned} y &= mx + c \\ y &= -\frac{x}{m} + d \end{aligned}\right\}.$$

Thus, for example, the straight lines

$$\left.\begin{aligned} ax + by + c &= 0 \\ bx - ay + d &= 0 \end{aligned}\right\}$$

are perpendicular.

This can be expressed otherwise thus: the straight lines

$$\left.\begin{aligned} y &= mx + c \\ y &= m'x + d \end{aligned}\right\}$$

are perpendicular if $mm' = -1$,

or the straight lines

$$\left.\begin{aligned} Ax + By + C &= 0 \\ A'x + B'y + C' &= 0 \end{aligned}\right\}$$

are perpendicular if $AA' + BB' = 0$.

It should be noted, as in § 2 of this chapter, that the direction of the lines depends only on the ratios of the coefficients of $x$ and $y$.

## Examples II b.

1. Find the equation of a straight line which passes through the origin and makes an angle (i) 45°, (ii) 30°, (iii) 120°, with the axis of $x$.

2. Draw the straight lines

$$\tfrac{1}{3}x + \tfrac{1}{4}y = 1, \qquad \tfrac{1}{3}x - \tfrac{1}{4}y = -1,$$
$$\tfrac{1}{3}x - \tfrac{1}{4}y = 1, \qquad \tfrac{1}{3}x + \tfrac{1}{4}y = -1.$$

In what points do they intersect?

3. What angles do the straight lines $x + y = 2$, $x - y = 4$ make with the axis of $x$ and with each other?

4. Find the lengths of the perpendiculars from the origin on the straight

lines $3x+4y=5$, $12x-5y=10$, $x+y+4=0$, $3x-a=0$, $2y-5b=0$. Draw the line in each case.

5. Find the equation of the straight line which makes intercepts on the axes twice as long as those made by $7y-9x=63$.

6. Find the equations of the sides of an equilateral triangle referred (i) to a pair of sides, (ii) to the bisectors of one angle as axes.

7. Two straight lines make intercepts $a$, $b$ and $ka$, $kb$ respectively on the coordinate axes. Show that they are parallel.

8. Two straight lines make intercepts $a$, $b$ and $b$, $-a$ on the axes. Show that they are perpendicular.

9. If the axes are inclined at 120°, what angle does $x=2y+3$ make with the $x$-axis?

10. A straight line makes intercepts $\sqrt{3}$ and 1 on the axes of $x$ and $y$; what angle does it make with the axes? also if the intercepts are $5\sqrt{3}$ and 5?

11. The angle $A$ of a triangle is 75°, the perpendicular from $A$ on the base $BC$ is of length 3, and divides the angle $A$ in the ratio 2 : 3. Find the equation of $BC$ referred to $AB$, $AC$ as axes.

12. Prove that the straight lines $y=3x+7$, $5y=8x+35$, and $x=0$ are concurrent.

13. Put the equations (i) $3x+4y=12$;
(ii) $2y-x+4=0$
into each of the standard forms I, II, and III.

14. For what value of $m$ are the straight lines $y=3x+7$, $y=mx+5$ (i) parallel, (ii) perpendicular?

15. Find the equations of the straight lines making an intercept 3 on the axis of $y$ and inclined at an angle 30° to $y=\sqrt{3}x+4$.

16. Draw the straight lines
(i) $x\cos\frac{1}{3}\pi+y\sin\frac{1}{3}\pi=a$;
(ii) $x\cos\frac{2}{3}\pi+y\sin\frac{2}{3}\pi=a$;
(iii) $x\cos\frac{1}{5}\pi+y\sin\frac{1}{5}\pi=a$, when $a=2$ and $-2$ respectively.

17. Draw the lines (i) $y=x\tan\frac{2}{3}\pi+c$;
(ii) $y=x\tan\frac{5}{4}\pi+c$;
(iii) $x=y\tan\frac{3}{4}\pi+c$;
when $c=3$ and $-3$ respectively.
What intercepts does each make on the axes?

18. Show that the straight line
$$x+y\tan\alpha=3\sec\alpha$$
touches a fixed circle, whatever value $\alpha$ may have.

19. If the straight lines $3x+4my+7=0$,
$3my-9x+8=0$,
are perpendicular, find the value of $m$.

20. Find the equation of the straight line perpendicular to
$$y=x\sqrt{3}+5$$
which (i) cuts off a length 4 units from the $x$-axis;
(ii) cuts off $-4$ units from the $y$-axis;
(iii) is distant 5 units from the origin.

§ 4. The coefficients or constants, $A$, $B$, and $C$, in the general equation of a straight line $Ax+By+C=0$, involve only two independent quantities, viz. two of the ratios $A : B : C$, for the equation represents the same straight line if each term is multiplied or divided by the same quantity.

Thus
$$3x+4y+5=0,$$
$$15x+20y+25=0$$
represent the same straight line.

We have seen that when certain pairs of conditions are given the equation of the straight line can be formed. Thus

(*a*) If two points $(x_1, y_1)$, $(x_2, y_2)$ on the straight line are given, its equation is
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

(*b*) If one point and the direction of the line, i. e. the angle it makes with some given straight line such as the axis of $x$ are given, the equation is of the form $y = mx+c$.

(*c*) If the direction of the line and its distance from the origin are given, the equation is then of the form $x\cos\alpha+y\sin\alpha=p$.

In each of these cases there are two independent constants in the equation of the straight line corresponding to the two given conditions.

The form $Ax+By+C=0$ is used for a general discussion of the straight line. Any one of the equations
$$lx+my+1=0, \quad \text{(i)}$$
$$x+my+n=0, \quad \text{(ii)}$$
$$lx+\ y+n=0 \quad \text{(iii)}$$
represents a straight line and contains two independent constants. No one of them, however, is suitable for a general discussion; the first cannot represent a straight line through the origin, the second cannot represent a straight line parallel to the axis of $x$, the third cannot represent a straight line parallel to the axis of $y$.

It is evident then that two conditions are necessary to fix a straight line, and further that in the cases above given these two conditions are sufficient: we shall see later that two conditions are not always sufficient.

When one condition (e. g. a point on the line, or the direction of the line) is given, a relation between the constants can be found. Thus if we know that the point $(a, b)$ lies on the straight line, we have
$$Aa+Bb+C=0.$$

It is then possible to find the value of one of the independent ratios or constants in terms of the other, and the equation of the line can be found in a form which involves only one unknown or undetermined constant. Thus, in the example given above, since $C = -Aa - Bb$, the equation

$$Ax + By + C = 0$$

can be written $$Ax + By - Aa - Bb = 0,$$

or $$x - a + \frac{B}{A}(y - b) = 0,$$

which involves only the one undetermined constant $\frac{B}{A}$. A second condition is necessary in order to find the value of this constant and so fix the equation of the straight line.

We have seen that the general equation $Ax + By + C = 0$ can be made to represent any straight line by giving suitable values to the constants. When one condition is given, and therefore only one undetermined constant is left in the equation, the equation can no longer be made to represent **any** straight line, but only one of a **definite group** of straight lines. Thus, in the example given above, the equation $A(x - a) + B(y - b) = 0$ must represent one of a group or system of straight lines all of which pass through the point $(a, b)$.

By giving definite values to the remaining constant $\frac{B}{A}$, the equation can be made to represent any one of this system of straight lines. Hence we see that if an equation of the first degree contains only one undetermined constant, this equation for different values of the constant represents some definite group of straight lines; a further condition, if sufficient, will determine any particular line of the group. In the following work we shall for convenience use the Greek letter $\mu$ for the undetermined constant.

Two conditions are not always sufficient to determine a straight line. If a point on the line be given together with the condition that the line touches a given circle, there are two straight lines which fulfil both conditions. If a single insufficient condition like the latter is given, it will be found that the corresponding equation in the constants representing this condition algebraically contains one constant in the second or a higher power, so that more than one value of it in terms of the other constant can be found: such cases will be fully discussed at a later stage.

**Groups or Systems of Straight Lines.**

(i) *Straight lines which pass through a given point* $(h, k)$.

The equation $y-k=\mu(x-h)$ represents a straight line, and is satisfied by the coordinates $(h, k)$ of the given point, whatever the value of $\mu$ may be: it therefore represents for different values of $\mu$ the system of straight lines which meet at the point $(h, k)$.

Such an equation is said to be of a particular form: thus, for instance, an equation of the form $y=\mu(x-3)$ represents a system of straight lines through the point $(3, 0)$.

(ii) *Straight lines inclined at a given angle* $(\alpha)$ *to the* x-*axis*.

The direction of a straight line depends only on the ratio of the coefficients of $x$ and $y$. The equation $y=x\tan\alpha+\mu$ represents for different values of $\mu$ the system of parallel straight lines which make an angle $\alpha$ with $Ox$.

In particular, $y=mx+\mu$ represents a system of straight lines parallel to a given straight line $y=mx$.

(iii) *Straight lines perpendicular to a given straight line.*

Let the given straight line be $y=mx+c$, then (p. 35) the equation $my+x+\mu=0$ represents a straight line perpendicular to it: it therefore for different values of $\mu$ represents a system of parallel straight lines all perpendicular to the given straight line.

If the given line is $ax+by+c=0$, the required form of the equation is $bx-ay+\mu=0$.

(iv) *System of straight lines passing through the intersection of the two given straight lines* $Ax+By+C=0$; $A'x+B'y+C'=0$.

The coordinates of the intersection of these straight lines satisfy simultaneously both the equations

$$Ax+By+C=0, \quad A'x+B'y+C'=0.$$

Hence they satisfy also the equation

$$Ax+By+C+\mu(A'x+B'y+C')=0.$$

This equation is of the first degree in $x$ and $y$ and represents a straight line; it therefore represents a straight line through the intersection of the given lines; e.g. $x+\mu y=0$ represents a system of straight lines through the intersection of $x=0$ and $y=0$, i.e. through the origin.

(v) *System of straight lines distant* p *from the origin.*

The equation $x\cos\alpha+y\sin\alpha=p$ represents a straight line whose distance from the origin is $p$, whatever value $\alpha$ may have; $\alpha$ is in

this case the undetermined constant. The equation can be written

$$x(\cos^2\tfrac{1}{2}\alpha-\sin^2\tfrac{1}{2}\alpha)+y\,.\,2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha=p(\cos^2\tfrac{1}{2}\alpha+\sin^2\tfrac{1}{2}\alpha)$$

or
$$x(1-\tan^2\tfrac{1}{2}\alpha)+2y\tan\tfrac{1}{2}\alpha=p(1+\tan^2\tfrac{1}{2}\alpha).$$

Put $\mu$ for $\tan\frac{1}{2}\alpha$ and we obtain

$$\mu^2(x+p)-2\mu y+p-x=0,$$

i. e. the undetermined constant appears in the second degree.

The method illustrated in this section is generally convenient for numerical examples; the use of the single constant $\mu$ does not, however, give quite satisfactory results. For example, we have shown in (iv) that

$$Ax+By+C+\mu(A'x+B'y+C')=0 \qquad \text{(i)}$$

represents, for different values of $\mu$, a pencil of lines passing through the point of intersection of the straight lines $Ax+By+C=0$ and $A'x+B'y+C'=0$. If we give $\mu$ increasingly large values, the straight line represented by (i) approaches the position of the straight line $A'x+B'y+C'=0$. It is sometimes said that for an infinite value of $\mu$ it coincides with this straight line. It is safer and more correct, however, to write equation (i) in the form

$$\lambda(Ax+By+C)+\mu(A'x+B'y+C')=0. \qquad \text{(ii)}$$

This equation contains only one independent constant, viz. the ratio $\lambda/\mu$ or the ratio $\mu/\lambda$; but the equations of every straight line in the pencil can be found by giving finite values to $\lambda$ and $\mu$. When $\lambda$ is zero, the equation reduces to $A'x+B'y+C'=0$, and when $\mu$ is zero it reduces to $Ax+By+C=0$. This method of writing the equation is preferable for two reasons: we avoid the necessity of giving an infinitely large value to the constant, and we get a more symmetrical equation.

### Examples II c.

Write down the equation of the straight line in the form in which it represents

(i) A system of straight lines intersecting at (2, 1).

(ii) A system of straight lines parallel to $3x+4y=6$.

(iii) A system of straight lines perpendicular to (a) $7x-y=6$, (b) $x-3=0$.

(iv) A system of straight lines concurrent with $3x+y-5=0$ and $x=0$.

(v) Straight lines touching a circle of unit radius whose centre is at the origin.

(vi) A group of parallel straight lines making an angle of 30° with the straight line $y=\sqrt{3}x+12$.

§ 5. *Determination of the equation of a straight line when two conditions are given.*

I. *The straight line passing through the two points* $(x_1, y_1)$, $(x_2, y_2)$.

This equation has been already found in the form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

but is given again as an example of the present method.

Since the line passes through $(x_1, y_1)$ its equation is of the form

$$x - x_1 = \mu (y - y_1).$$

The second condition, viz. that $(x_2, y_2)$ lies on the line, fixes the value of $\mu$. Thus we have $x_2 - x_1 = \mu (y_2 - y_1)$.

Substituting for $\mu$, the required equation is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

II. *The straight line parallel to* $ax + by + c = 0$ *and passing through the point* $(x_1, y_1)$.

Since it is parallel to $ax + by + c = 0$, the straight line is one of the group $ax + by + \mu = 0$.

The point $(x_1, y_1)$ lies on it; hence the value of $\mu$ for the particular line of the group required is given by $ax_1 + by_1 + \mu = 0$.

With this value of $\mu$, $a(x - x_1) + b(y - y_1) = 0$ is the required equation.

This equation is instructive: it should be noted that (§ 4. i) its form implies that it passes through $(x_1, y_1)$ and (§ 4. ii) its form also implies that it is parallel to $ax + by + c = 0$.

When the student is familiar with the forms of the equation corresponding to given geometrical conditions, such results can be at once written down.

III. *The straight line perpendicular to* $ax + by + c = 0$ *and passing through the intersection of the straight lines*

$$px + qy + r = 0, \quad p'x + q'y + r' = 0.$$

The second condition gives that the required equation is in the form $(px + qy + r) + \mu (p'x + q'y + r') = 0$, and since it is perpendicular to $ax + by + c = 0$, the ratio of the coefficients of $x$ and $y$ is $b : -a$;

hence
$$\frac{p + \mu p'}{b} = \frac{q + \mu q'}{-a},$$

or
$$\mu (ap' + bq') = -(ap + bq),$$

and the required equation is

$$(ap' + bq')(px + qy + r) - (ap + bq)(p'x + q'y + r') = 0.$$

IV. *To find the equation of the straight line through the point $(h, k)$ which makes an angle $\alpha$ with the straight line $ax+by+c=0$.*

The straight line $ax+by+c=0$ makes an angle $\tan^{-1}\left(-\frac{a}{b}\right)$ with the $x$-axis, hence the required line makes an angle $\tan^{-1}\left(-\frac{a}{b}\right)\pm\alpha$ with the $x$-axis; it is therefore parallel to

$$y = x\tan\left\{\tan^{-1}\left(-\frac{a}{b}\right)\pm\alpha\right\},$$

i. e.
$$y = x\,\frac{-\frac{a}{b}\pm\tan\alpha}{1\pm\frac{a}{b}\tan\alpha},$$

i. e.
$$(b\pm a\tan\alpha)\,y+x\,(a\mp b\tan\alpha)=0.$$

And since the line passes through $(h, k)$ its equation is (*vide* II)

$$(y-k)\,(b\pm a\tan\alpha)+(x-h)\,(a\mp b\tan\alpha)=0.$$

There are evidently two straight lines which satisfy the conditions

V. *The straight line through $(x_1, y_1)$ distant p from the origin.*

Since the distance of the line from the origin is $p$, the equation of the straight line is in the form (§ 4. v)

$$\mu^2(x+p)-2y\mu+p-x=0.$$

Since $(x_1, y_1)$ lies on this, to determine the value of $\mu$ we have

$$\mu^2(p+x_1)-2y_1\mu+p-x_1=0.$$

This is a quadratic equation in $\mu$, hence there are two straight lines which satisfy both the given conditions: these conditions are therefore not sufficient to determine one definite straight line.

**Note.** The above results are not to be regarded as formulae; each numerical example should be worked out on similar lines.

### Examples II d.

Write down the equations of the straight lines which satisfy the following conditions:—

1. The straight line passes through the point $(-2, 3)$ and is (a) parallel, (b) perpendicular to $3x-y=6$.

2. The straight line passes through the origin and is perpendicular to $4x+7y=8$.

3. The straight line joining the points (a) $(3, 2), (1, 1)$, (b) $(-4, 5), (-2, -3)$, (c) $(a, b), (a-6, b-6)$, (d) $(a\cos\theta, a\sin\theta), (a\cos\phi, a\sin\phi)$, (e) $(am^2, 2am), (an^2, 2an)$, (f) $(a, b), (a+r\cos\theta, b+r\sin\theta)$.

4. The straight line cuts off a length 5 from the axis of $x$ and makes an angle of 75° with it.

5. The straight line cuts off a length 10 from the $y$-axis and is perpendicular to $11y - 10x = 21$.

6. Find the equation of the straight line drawn through the point $(x_1, y_1)$ perpendicular to the straight line joining the points $(x_2, y_2)$, $(x_3, y_3)$.

7. Find the equation of a straight line passing through the intersection of $ax + by + c = 0$, $a'x + b'y + c' = 0$ which is parallel to $px + qy + r = 0$.

8. What is the length of the perpendicular from the origin on the straight line $3x + 4y = 2$?

Find the tangent of the angle this perpendicular makes with the axis of $x$.

The centre of an equilateral triangle is at the origin, and one side of the triangle lies in the straight line $3x + 4y = 2$. Find the equations of the other two sides.

9. What angle is formed by the straight lines whose equations are

$$x \cos \alpha + y \sin \alpha = p,$$
$$x \cos \beta + y \sin \beta = q?$$

10. Find the equation of a straight line inclined to the straight line $x \cos \alpha + y \sin \alpha = p$ at an angle $\alpha$, whose distance from the origin is $2p$.

11. Find the equation of the straight line joining the point of intersection of the lines $ax + by + c = 0$, $a'x + b'y + c' = 0$ to the origin.

12. A square has its centre at the origin: one side of the square is the straight line $x - \sqrt{3}y - 4 = 0$; find the equations of the other three sides.

13. Find the equation of the straight line which passes through the points of intersection of each of the pairs of straight lines

$$\left.\begin{aligned} ax + by + c &= 0, \\ ax - by + c &= 0, \end{aligned}\right\}' \qquad \left.\begin{aligned} ax + by + d &= 0 \\ -ax + by + d &= 0 \end{aligned}\right\}.$$

14. The perpendicular from the origin to a straight line meets it at the point $(h, k)$: show that the equation of the straight line is

$$hx + ky = h^2 + k^2.$$

15. Show that the straight lines joining the points $(1, 1)$, $(2, 2)$ respectively to the point of intersection of the straight lines

$$19x + 3y - 29 = 0, \quad 13x + 11y - 27 = 0$$

are at right angles.

16. Find the equations of the medians of the triangle whose vertices are $(2, 1)$, $(3, -3)$, $(-5, 2)$.

17. The sides of a quadrilateral taken in order are

$$x + 2y = 5, \quad 3x + y = 10, \quad x + 6y + 8 = 0, \quad 4x - y + 7 = 0;$$

find the equations of its diagonals.

18. Find the equation of a straight line through the intersection of $3x + 5y - 7 = 0$ and $4x + 6y - 5 = 0$ parallel (i) to the axis of $x$, (ii) to the axis of $y$.

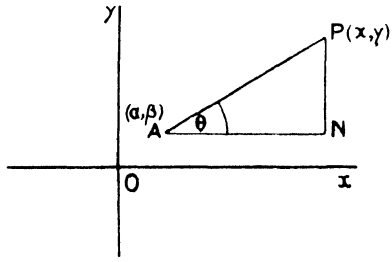
19. Show that the two straight lines $x + y + c = 0$, $\sqrt{3}x + y + d = 0$ are inclined at an angle of 15°.

20. Find the condition that the straight lines joining the origin to the points of intersection of the pairs of straight lines

$$\left.\begin{aligned} x + ay + c &= 0 \\ ax + y + c &= 0 \end{aligned}\right\}' \qquad \left.\begin{aligned} lx + my + l &= 0 \\ (l + l')x + (m + m')y + l &= 0 \end{aligned}\right\}$$

should be (i) perpendicular, (ii) coincident.

§ 6. *To find the coordinates of a point whose distance from a fixed point* $(\alpha, \beta)$ *is* r *and which lies on a straight line through* $(\alpha, \beta)$ *inclined at an angle* $\theta$ *to the* x*-axis.*




Let $A$ be the point $(\alpha, \beta)$ and $P(x, y)$ the required point.

Draw $AN$, $PN$ parallel to the axes.

Then
$$AN = x - \alpha = r\cos\theta$$
$$PN = y - \beta = r\sin\theta,$$

i.e. the coordinates of $P$ are $(r\cos\theta + \alpha,\ r\sin\theta + \beta)$.

**Note.** The length $r$ is measured from $A$ in the positive direction; if $r$ is negative $P$ lies on the other side of $A$.

This result can be written
$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r,$$

and if $P$ be regarded as any point $(x, y)$ on the straight line $AP$, this may be regarded as the equation of the straight line. The equation in this form is often very useful, and the meaning of the constants should be carefully noted. $(\alpha, \beta)$ is a fixed point on the straight line, $\theta$ is the angle the straight line makes with the $x$-axis, $(x, y)$ is any point on the straight line, and the distance of $(x, y)$ from $(\alpha, \beta)$ is $r$. Two examples will illustrate the type of problems in which this form of the equation is useful: we shall refer to it again when equations of a higher degree are discussed.

**Ex.** i. *A straight line is drawn through the point* $P(2, 3)$ *parallel to the straight line* $y = \sqrt{3}x$ *to meet the straight line* $2x + 4y = 27$ *in the point* $Q$. *Find the length* $PQ$.

The straight line $y = \sqrt{3}x$ makes an angle $\frac{1}{3}\pi$ with the $x$-axis: hence the equation of the straight line through $(2, 3)$ parallel to it is
$$\frac{x-2}{\cos\frac{1}{3}\pi} = \frac{y-3}{\sin\frac{1}{3}\pi} = r,$$

or
$$\frac{x-2}{\frac{1}{2}} = \frac{y-3}{\frac{\sqrt{3}}{2}} = r,$$

where $r$ is the distance between the point $(x, y)$ and the fixed point

(2, 3). Now if $r$ is equal to $PQ$, then $(x, y)$ must be the point $Q$ which lies on the line $2x+4y=27$; the coordinates of $Q$ are therefore $\left\{\frac{1}{2}r+2,\ \frac{\sqrt{3}}{2}r+3\right\}$ where $r=PQ$.

Hence $$2\left(\frac{1}{2}r+2\right)+4\left(\frac{\sqrt{3}}{2}r+3\right)=27,$$

i.e. $$(2\sqrt{3}+1)r=11,$$

hence $$r=\frac{11}{2\sqrt{3}+1}=2\sqrt{3}-1=2{\cdot}46,$$

which is the required length of $PQ$.

**Ex. ii.** *To find the perpendicular distance of the point* $(a, b)$ *from the straight line* $Ax+By+C=0$.

The equation of a straight line through $(a, b)$ perpendicular to the given line is $B(x-a)-A(y-b)=0$,

i.e. $$\frac{x-a}{A}=\frac{y-b}{B}=\frac{\sqrt{(x-a)^2+(y-b)^2}}{\pm\sqrt{A^2+B^2}}$$

$$=\frac{r}{\pm\sqrt{A^2+B^2}},$$

where $r$ is the distance between $(a, b)$ and $(x, y)$.

If $r$ is equal to the length of the required perpendicular, the point $(x, y)$ is on $Ax+By+C=0$.

Hence $$A\left(\frac{Ar}{\pm\sqrt{A^2+B^2}}+a\right)+B\left(\frac{Br}{\pm\sqrt{A^2+B^2}}+b\right)+C=0,$$

i.e. $$r=\pm\frac{Aa+Bb+C}{\sqrt{A^2+B^2}}.$$

## Examples II e.

1. Find the perpendicular distance of (2, 5) from $12x+5y=40$ and from $3x+4y=16$.

2. $A$, $B$, $C$, $D$ is a parallelogram; $A$ is the point $(-5, 2)$; $BC$, $CD$ are the straight lines $x-y=7$ and $3x+4y=28$. Find the lengths of the sides of the parallelogram.

3. Through a point $P(h, k)$ a straight line is drawn, making an angle $\theta$ with the axis of $x$, to meet the straight line $ax+by=1$ at $Q$.

Find the coordinates of a point on $PQ$ distant $PQ/n$ from $P$.

Hence find the locus of such points when $\theta$ varies.

4. A straight line is drawn through the point $A(a, b)$ making an angle $\theta$ with the axis of $x$ to meet the locus represented by $x^2+y^2=c^2$. Show that it meets it in two points $P$, $Q$, and that the rectangle $AP.AQ$ is independent of $\theta$.

If $AP$, $AQ$ are equal, prove that $(a\sin\theta-b\cos\theta)^2=c^2$.

§ 7. *To find the perpendicular distance of any point from a straight line whose equation is given.*

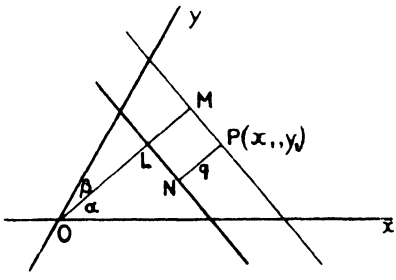
(a) **Oblique Coordinates.**

Let the equation of the straight line be given in the form

$$x\cos\alpha + y\cos\beta = p \qquad \text{(i)}$$

and let the point be $P(x_1, y_1)$.

Through $P$ draw a straight line parallel to $x\cos\alpha + y\cos\beta = p$.




And from $O$ draw $OLM$ perpendicular to these lines. Draw $PN$ perpendicular to the given line: then $PN$ is the required length, call it $q$.

Now by § 3 of this chapter $OL = p$, hence $OM = p + q$ and the equation of $PM$ is

$$x\cos\alpha + y\cos\beta = p + q.$$

But $(x_1, y_1)$ lies on this, hence

$$x_1\cos\alpha + y_1\cos\beta = p + q;$$

therefore

$$q = x_1\cos\alpha + y_1\cos\beta - p.$$

Hence, if the equation of a straight line is given in the form

$$x\cos\alpha + y\cos\beta - p = 0,$$

the length of the perpendicular from any point on it is found by substituting its coordinates in the left-hand side of the equation. It follows from this result that if we can put the equation of any straight line in this particular form, the length of the perpendicular on it from any point can be at once written down.

Compare the equations

$$Ax + By + C = 0 \qquad \text{(i)}$$

and

$$x\cos\alpha + y\cos\beta - p = 0. \qquad \text{(ii)}$$

If they represent the same straight line,

$$\frac{\cos\alpha}{A} = \frac{\cos\beta}{B} = \frac{-p}{C}.$$

Hence

$$\frac{\cos^2\alpha}{A^2} = \frac{\cos^2\beta}{B^2} = \frac{\cos\alpha\cos\beta}{AB}$$

$$= \frac{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos(\alpha+\beta)}{A^2 + B^2 - 2AB\cos(\alpha+\beta)}$$

$$= \frac{\sin^2(\alpha+\beta)}{A^2 + B^2 - 2AB\cos(\alpha+\beta)}.$$

But $\alpha+\beta=\omega$, therefore

$$\frac{\cos\alpha}{A}=\frac{\cos\beta}{B}=\frac{-p}{C}=\frac{\sin\omega}{\sqrt{A^2+B^2-2AB\cos\omega}}.$$

Hence equation (i) written in the form (ii) is

$$\frac{A\sin\omega\,.\,x+B\sin\omega\,.\,y+C\sin\omega}{\sqrt{A^2+B^2-2AB\cos\omega}}=0.$$

Therefore the perpendicular from $(x_1, y_1)$ on the line $Ax+By+C$ is

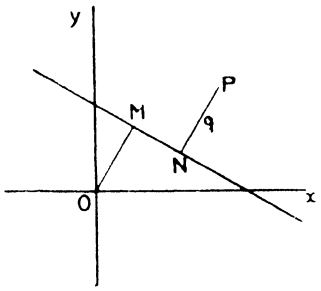
$$\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2-2AB\cos\omega}}\,.\,\sin\omega.$$

(b) **Rectangular Coordinates.**

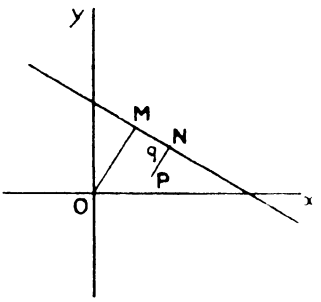
The student should work out in the same way the more simple case: the results are

(i) The length of the perpendicular from $(x_1, y_1)$ on the straight line $x\cos\alpha+y\sin\alpha-p=0$ is $x_1\cos\alpha+y_1\sin\alpha-p$.

(ii) The length of the perpendicular from $(x_1, y_1)$ on the straight line $Ax+By+C=0$ is $\dfrac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$.




(i)




(ii)

In the two figures given $P$ is placed on opposite sides of the line: the equation of the straight line through $P$ parallel to the given line is

Fig. (i) $\quad x\cos\alpha+y\sin\alpha-(p+q)=0,$

Fig. (ii) $\quad x\cos\alpha+y\sin\alpha-(p-q)=0;$

and as above, since $(x_1, y_1)$ is on the line,

Fig. (i) $\quad x_1\cos\alpha+y_1\sin\alpha-(p+q)=0,$

Fig. (ii) $\quad x_1\cos\alpha+y_1\sin\alpha-(p-q)=0.$

Hence the length $q$ of the perpendicular is

Fig. (i) $\quad q=x_1\cos\alpha+y_1\sin\alpha-p,$

Fig. (ii) $\quad q=p-x_1\cos\alpha-y_1\sin\alpha,$

and these results are of different sign.

The sign of the perpendicular changes as the point $P$ crosses the line, being of zero length when $P$ is on the line. When dealing with more than one perpendicular we must call those drawn from points on one side of the line positive and those from points on the other negative. It follows similarly that all points whose coordinates make $Ax+By+C$ positive lie on one side of the line $Ax+By+C=0$, and those whose coordinates make $Ax+By+C$ negative lie on the other side.

This can also be shown independently

The result only gives us a means of discovering whether two points are on the same or opposite sides of a straight line: this question of sign arises in such problems as 'find a point equidistant from three intersecting straight lines'; there are four such points, the in-centre and the three ex-centres of the triangle formed by the lines. Thus, if the sides are

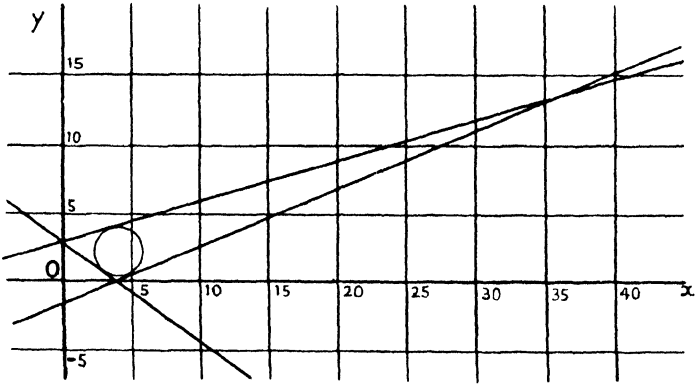
$$a_1x+b_1y+c_1=0,\quad a_2x+b_2y+c_2=0,\quad a_3x+b_3y+c_3=0,$$

the four points are given by the equation

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}=\pm\frac{a_3x+b_3y+c_3}{\sqrt{a_3^2+b_3^2}}.$$

To determine in any special case which solution corresponds for example to the in-centre, the student should draw the graph of the lines.

**Example.** *To find the in-centre of the triangle formed by the straight lines* (i) $3x+4y-12=0$, (ii) $5x-12y-20=0$, (iii) $24y-7x-72=0$.




We see from the figure that the in-centre and the origin are on opposite sides of the line (i), and hence the perpendiculars from these points must be taken of opposite sign.

Since the substitution of $x=0$, $y=0$ in $3x+4y-12$ gives us a negative

value, the substitution of the coordinates $(x', y')$ of the in-centre will give a positive result: hence the perpendicular from the in-centre $(x', y')$ on $3x+4y-12=0$ is numerically equal to

$$\tfrac{1}{5}(3x'+4y'-12).$$

The origin and the in-centre are on the same side of the line (ii): $x=0$, $y=0$ makes $5x-12y-20$ negative, $x=x'$, $y=y'$ will therefore also make it negative.

The numerical value of the perpendicular from $x', y'$ on this line is therefore

$$-\tfrac{1}{13}(5x'-12y'-20') = \tfrac{1}{13}(12y'-5x'+20).$$

Similarly, the perpendicular from $(x', y')$ on $24y-7x-72=0$ is

$$-\tfrac{1}{25}(24y'-7x'-72) = \tfrac{1}{25}(7x'-24y'+72).$$

Hence the in-centre will be given by

$$\tfrac{1}{5}(3x+4y-12) = \tfrac{1}{13}(12y-5x+20) = \tfrac{1}{25}(7x-24y+72);$$

i.e. $$8x-y=32, \text{ and } 2x+11y=33,$$

and the in-centre is $(4\tfrac{5}{18}, 2\tfrac{2}{9})$.

### Examples II f.

1. In the above example work out similarly the coordinates of each of the ex-centres, giving a reason for the signs chosen in each case.

2. Find the centre of the circle inscribed in the triangle whose sides are

$$x-y+1=0,\ x+y-7=0,\ x-3y+5=0.$$

3. The equations of the sides of a triangle are $x=0$, $y=0$, $3x+4y=12$. Find the coordinates of the centre of the circle escribed to the side $y=0$.

§ 8. *Relations between two straight lines whose equations are given.*

(i) *To find the coordinates of the point of intersection of two straight lines.*

This is equivalent to finding a pair of values of the coordinates $x$ and $y$ which satisfy simultaneously the equations of both the straight lines; in other words, to solving the equations simultaneously. e. g. to find the point of intersection of

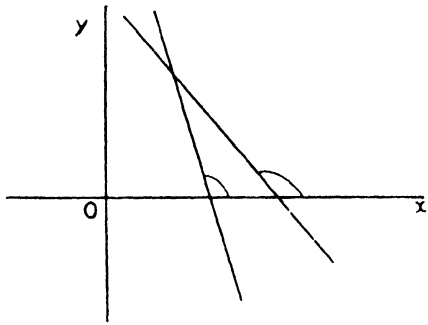
$$5x+4y-7=0,$$
$$3x-7y+6=0,$$

by cross multiplication

$$\tfrac{1}{28}x = \tfrac{1}{51}y = \tfrac{1}{-47};$$

$\therefore$ the point of intersection is $(\tfrac{28}{47}, \tfrac{51}{47})$.

(ii) *To find the angle between two straight lines whose equations are given in rectangular coordinates.*




Let the equations be $Ax+By+C=0$,

$A'x+B'y+C'=0$;

these straight lines make angles

$$\tan^{-1}\left(-\frac{A}{B}\right) \text{ and } \tan^{-1}\left(-\frac{A'}{B'}\right)$$

with the axis of $x$; the angle between them is therefore

$$\tan^{-1}\left(-\frac{A'}{B'}\right)-\tan^{-1}\left(-\frac{A}{B}\right)$$

$$=\tan^{-1}\frac{\frac{A}{B}-\frac{A'}{B'}}{1+\frac{AA'}{BB'}}=\tan^{-1}\frac{AB'-A'B}{AA'+BB'}.$$

If the equations are given in the form

$$y=mx+c, \qquad y=m'x+c',$$

the angle between them is $\tan^{-1}\dfrac{m-m'}{1+mm'}$.

It is clear that in the general case we cannot decide whether the acute angle between the lines is $\tan^{-1}\dfrac{m-m'}{1+mm'}$ or $\tan^{-1}\dfrac{m'-m}{1+mm'}$; one value is positive and corresponds to the acute angle, one is negative and corresponds to the obtuse angle.

Corresponding results can be obtained for oblique axes in the same way (§ 2); these results are

$$\tan^{-1}\frac{(AB'-A'B)\sin\omega}{AA'+BB'-(AB'+A'B)\cos\omega} \text{ and } \tan^{-1}\frac{(m-m')\sin\omega}{1+(m+m')\cos\omega+mm'}.$$

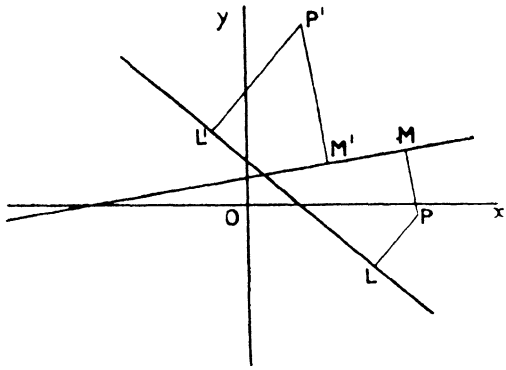
(iii) The condition that the straight lines should be perpendicular follows. Since the tangent of a right angle is infinite, in rectangular coordinates $AA'+BB'=0$ or $1+mm'=0$,

and in oblique coordinates

$$AA'+BB'-(AB'+A'B)\cos\omega=0 \text{ or } 1+(m+m')\cos\omega+mm'=0.$$

(iv) *To find the equations of the bisectors of the angles between the straight lines whose equations in rectangular coordinates are*

$$Ax+By+C=0, \qquad A'x+B'y+C'=0.$$




If $(x_1, y_1)$ is any point on either of the bisectors, the perpendiculars from it to the straight lines are of equal length. It is evident (see figure) that for points on one bisector these perpendiculars are of the same sign, and for points on the other bisector they are of opposite sign.

Hence
$$\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}=\pm\frac{A'x_1+B'y_1+C'}{\sqrt{A'^2+B'^2}},$$

and any point on a bisector lies on one of the lines

$$\frac{Ax+By+C}{\sqrt{A^2+B^2}}=\frac{A'x+B'y+C'}{\sqrt{A'^2+B'^2}}, \qquad \frac{Ax+By+C}{\sqrt{A^2+B^2}}=-\frac{A'x+B'y+C'}{\sqrt{A'^2+B'^2}},$$

which are consequently the equations of the bisectors.

(v) It has been shown in § 4 (iv) that the equation of any straight line through the point of intersection of the straight lines

$$Ax+By+C=0, \qquad A'x+B'y+C'=0,$$

is of the form

$$Ax+By+C+\lambda(A'x+B'y+C')=0,$$

whatever coordinate axes are employed.

It is often convenient in order to save the constant repetition of such an equation as $Ax+By+C=0$ to refer to it in the following way: let $u$ stand for the expression $Ax+By+C$, i.e. $u \equiv Ax+By+C$, then $u=0$ represents the straight line, and for the sake of brevity we can refer to 'the straight line $u$'. This notation (usually called abridged) is of much greater importance than appears at first sight. There is a large class of problems in geometry of a purely descriptive character (i.e. not dealing with magnitudes such as length, or size of angle) which can be solved in a quite general manner: in such work it is quite sufficient to recognize that $u=0$ represents a straight line where $u$ is an abbreviation for any expression which, equated to zero, represents a straight line without any regard to the actual system of coordinates employed.

This idea is introduced early in the work and will be developed when occasion offers, and it is hoped by this means to help the student at the later stages to pass to generalized coordinates more readily.

If $u=0$, $v=0$ represent two straight lines, any line through their point of intersection is $u+\lambda v=0$.

If $u' \equiv Ax'+By'+C$, then $u'=0$ is the condition that the point $(x', y')$ should lie on the straight line $u=0$; and similarly we interpret $v'=0$.

If the straight line $u+\lambda v=0$ passes through some given point $(x', y')$, then $u'+\lambda v'=0$, hence $uv'-u'v=0$ represents the straight line joining the point $(x', y')$ to the intersection of $u=0$, $v=0$.

Again, $u-u'=0$ represents a straight line through $(x', y')$ parallel to $u=0$.

In the particular case when the equations of straight lines are in the form

$$x\cos\alpha+y\sin\alpha-p=0,$$

if

$$u \equiv x\cos\alpha+y\sin\alpha-p,$$

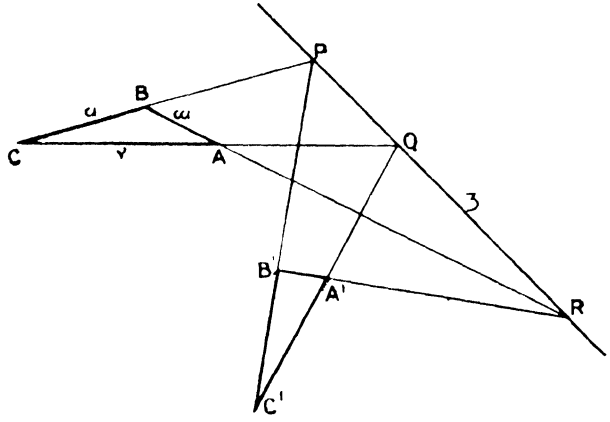
$$v \equiv x\cos\beta+y\sin\beta-q,$$

then $u'$ and $v'$ are the actual values of the perpendiculars from $(x', y')$ to the straight lines $u=0$, $v=0$.

Hence the equations of the bisectors of the angle between the straight lines $u=0$, $v=0$ are $u-v=0$ and $u+v=0$.

A point can be determined as the intersection of two given straight lines: if then $u=0$, $v=0$ represent straight lines, their point of intersection can be referred to as the point $(u, v)$.

**Example.** *If two triangles $ABC$, $A'B'C'$ are so placed that the points of intersection of $AB$, $A'B'$; $BC$, $B'C'$; $CA$, $C'A'$ lie on a straight line, then the joins of corresponding vertices $AA'$, $BB'$, $CC'$ will meet in a point.* (*Coaxal triangles are also copolar.*)




Let $CB$, $C'B'$ meet at $P$; $CA$, $C'A'$ at $Q$; $AB$, $A'B'$ at $R$, where $PQR$ is a straight line.

Let the sides of the triangle $ABC$ be the straight lines $u = 0$, $v = 0$, $w = 0$, and the straight line $PQR$ be $z = 0$.

Now $B'C'$ is a straight line through the point of intersection of $u = 0$, $z = 0$. Its equation is therefore of the form $au + z = 0$, where $a$ is a constant which can be determined for any chosen system of coordinates; for, when the axes are fixed, since $BC$, $B'C'$ are given straight lines their equations are completely known. Without reference to any particular coordinate axes, therefore, we can consider the constant $a$ as known. Similarly, $C'A'$ passes through the intersection of the given straight lines $CA$, $PQ$ and its equation is $bv + z = 0$, where $b$ is a known constant. $A'B'$ in the same way is represented by $cw + z = 0$. Hence the sides of the triangle $A'B'C'$ are the straight lines whose equations are

$$au + z = 0,\ bv + z = 0,\ cw + z = 0.$$

Now the equation $bv + z - (cw + z) = 0$ represents a straight line through the point of intersection of the straight lines $bv + z = 0$, $cw + z = 0$, i.e. through the point $A'$.

But $(bv + z) - (cw + z) = 0$ is equivalent to $bv - cw = 0$, and this represents a straight line through the point of intersection of the straight lines $v = 0$, $w = 0$, i.e. through the point $A$. Hence $A$ and $A'$ both lie on $bv - cw = 0$: this then is the equation of the straight line $AA'$.

In exactly the same way $cw - au = 0$ represents the straight line $BB'$, and $au - bv = 0$ the straight line $CC'$.

Hence the equations of the three straight lines joining corresponding vertices, viz. $AA'$, $BB'$, $CC'$,

are
$$bv - cw = 0 \quad \text{(i)}$$
$$cw - au = 0 \quad \text{(ii)}$$
$$au - bv = 0 \quad \text{(iii)}$$

But $(bv - cw) + (cw - au) = 0$, which represents a straight line through the point of intersection of (i) and (ii), is the same equation as $au - bv = 0$, i.e. the straight line $CC'$ (iii) passes through the intersection of $AA'$ and $BB'$: in other words, $AA'$, $BB'$, $CC'$ are concurrent.

## Examples II g.

1. Find the length of the perpendicular drawn from the point (2, 4) to the straight line joining the points (3, 1), (7, 4).

2. Find the equation of the straight line which joins the intersection of the lines $3x - 4y + 1 = 0$ and $5x + y - 1 = 0$ to the point (1, 3). Draw the lines and find the tangent of the angle between them.

3. Find the coordinates of the feet of the perpendiculars drawn from the point (1, 1) to the straight lines $x - 2y + 2 = 0$, $2x - y + 1 = 0$. Find also the length of the perpendicular drawn from the point (1, 1) to the straight line joining these feet.

4. Find the length of the perpendicular from the origin on
$$(x \cos\theta)/a + (y \sin\theta)/b = 1.$$

5. Find the equations of the straight lines through the intersection of the straight lines $2x - y + 5 = 0$, $x + 3y - 6 = 0$ respectively perpendicular and parallel to the straight line $5x + 8y - 10 = 0$.

6. Prove that (2, −1), (0, 2), (3, 0), (−1, 1) are the angular points of a parallelogram, and find the angle between the diagonals.

7. Find the condition that the straight line $lx + my + n = 0$ should touch the circle whose centre is $(a, b)$ and radius $r$.

8. Find the equations of the sides of a rectangle which has (1, 2), (4, 3) as coordinates of the extremities of one diagonal, and whose other diagonal is parallel to $x + 3y = 0$.

9. Find the coordinates of the orthocentre of the triangle whose sides are $y = 2x$, $2y = x$, $x + y = 9$.

10. Show that the triangle formed by the straight lines whose equations are $4x - 3y - 8 = 0$, $3x - 4y + 6 = 0$, $x + y - 9 = 0$ is isosceles: find the length of the equal sides.

11. One side of an equilateral triangle is the straight line $x - \sqrt{3}y + 4 = 0$, and the opposite vertex is the point (2, 1).

Find the equations of the other sides.

12. Find the conditions that the straight lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ should form an isosceles right-angled triangle, the last line being the hypotenuse.

Give, with numerical coefficients, the equations of three such lines.

13. The algebraical sum of the perpendiculars from the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$ on a straight line is zero: show that the line must pass through the mean centre of the four points.

Extend your proof to the case of any $n$ fixed points.

14. A straight line is drawn through (5, 9) inclined at 45° to the axis of $x$. The straight line is cut in $P$, $Q$ by $x+3y=20$, $7x+y=120$, which pair intersect at $T$. Show that $PQT$ is isosceles; give the lengths of the equal sides and the tangent of the angle at the vertex.

15. If the straight line $PQ$ drawn through the point (8, 4) at right angles to the line joining (6, 2) (4, 1) intersects the axes in $P$ and $Q$, find the area of the triangle $POQ$, and the distance of (4, 2) from $PQ$.

16. Find the coordinates of the centre of the circle inscribed in the triangle formed by the lines $x=1$, $3x+4y-5=0$, $5x-12y+16=0$; also those of the centre of the circle escribed to the side $x=1$.

17. Find the coordinates of the foot of the perpendicular from $(a, 0)$ on the line $\lambda y = x + a\lambda^2$, and show that when $\lambda$ varies all these feet lie on a fixed straight line.

18. Show that the two straight lines

$$(x\cos\theta)/a + (y\sin\theta)/b = 1, \qquad (x\sin\theta)/b - (y\cos\theta)/a = (ae\sin\theta)/b$$

are perpendicular, and that the distance of their point of intersection from the origin is independent of $\theta$ if $a^2(1-e^2) = b^2$.

19. Straight lines are drawn from the point (3, 2) to meet the straight line $6x+7y=30$, and these straight lines are bisected: find the equation of the locus of the points of bisection.

20. The product of the perpendiculars from $(ae, 0)$, $(-ae, 0)$ on

$$(x\cos\theta)/a + (y\sin\theta)/b = 1$$

is $b^2$: prove $$b^2 = a^2(1-e^2).$$

21. The equations of the sides of a triangle are

$$x + ly - l^2 = 0, \quad x + my - m^2 = 0, \quad x + ny - n^2 = 0.$$

Find the coordinates of its orthocentre.

22. The connector of any point $P(x', y')$ with the origin $O$ meets the straight line $ax+by+1=0$ in $Q$: show that $PQ : OQ = ax' + by' + 1$.

23. If the sides of a parallelogram are the straight lines $u=0$, $v=0$, $u=a$, $v=b$, find the equations of (i) the diagonals, (ii) lines through the intersection of the diagonals parallel to the sides.

24. Find the equation of the straight line which joins the point of intersection of $u=0$, $v=0$, to that of $u+w=0$, $v+w=0$.

25. The four sides of a quadrilateral have equations $u+v=0$, $v+w=0$, $u-v=0$, $v-w=0$. Find the equations of its diagonals.

26. Find the locus of the middle point of that portion of a straight line passing through the fixed point $(h, k)$ which is intercepted between the axes of coordinates.

**§ 9. Relations between three straight lines whose equations are given.**

(i) *To find the condition that the three straight lines*

$$Ax+By+C=0,\quad A'x+B'y+C'=0,\quad A''x+B''y+C''=0$$

*should be concurrent.*

If they are concurrent they have a common point; let its coordinates be $(x_1, y_1)$. Then

$$\begin{aligned} Ax_1 + By_1 + C &= 0,\\ A'x_1 + B'y_1 + C' &= 0,\\ A''x_1 + B''y_1 + C'' &= 0. \end{aligned}$$

Eliminating $(x_1, y_1)$ we find the required condition to be

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0.$$

Since any two straight lines meet in a point, one condition is necessary and sufficient that a third line should pass through the same point.

(ii) *To find the area of the triangle enclosed by the three straight lines*

$$A_1x+B_1y+C_1=0,\quad A_2x+B_2y+C_2=0,\quad A_3x+B_3y+C_3=0.$$

**Method i.** The equations can be solved in pairs and the coordinates of the vertices of the triangle found, and hence, Chapter I, § 7, the area. This method is often useful in the case of numerical examples.

**Method ii.** The following method obtains the result in determinate form.

Let the equations of the sides of the triangle be

$$a_1x+b_1y+c_1=0, \tag{i}$$
$$a_2x+b_2y+c_2=0, \tag{ii}$$
$$a_3x+b_3y+c_3=0. \tag{iii}$$

Suppose (ii) and (iii) intersect at $(x_1, y_1)$. (iii) and (i) at $(x_2, y_2)$, (i) and (ii) at $(x_3, y_3)$. Then

$$\frac{x_1}{b_2c_3-b_3c_2} = \frac{y_1}{c_2a_3-c_3a_2} = \frac{1}{a_2b_3-a_3b_2}$$

or, let us say

$$\frac{x_1}{A_1} = \frac{y_1}{B_1} = \frac{1}{C_1}$$

We have similarly

$$\frac{x_2}{A_2} = \frac{y_2}{B_2} = \frac{1}{C_2}$$

$$\frac{x_3}{A_3} = \frac{y_3}{B_3} = \frac{1}{C_3}$$

where $A_2$, $A_3$, &c., are formed from $A_1$ by permuting the suffixes in cyclic order.

Then the area of the triangle $= \frac{1}{2C_1C_2C_3}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$.

The product of this determinant and of the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, which we denote by $\Delta$, is

$$\begin{vmatrix} a_1A_1+b_1B_1+c_1C_1, & a_2A_1+b_2B_1+c_2C_1, & a_3A_1+b_3B_1+c_3C_1 \\ a_1A_2+b_1B_2+c_1C_2, & a_2A_2+b_2B_2+c_2C_2, & a_3A_2+b_3B_2+c_3C_2 \\ a_1A_3+b_1B_3+c_1C_3, & a_2A_3+b_2B_3+c_2C_3, & a_3A_3+b_3B_3+c_3C_3 \end{vmatrix}$$

or $\begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix}$; so that its value is $\Delta^2$, and we have for the area the expression $\frac{\Delta^2}{2C_1C_2C_3}$, where $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $C_1$, $C_2$, $C_3$ are the minors of $c_1$, $c_2$, $c_3$ in the determinant $\Delta$.

Here the coordinates have been taken to be rectangular. For oblique coordinates we multiply by $\sin\omega$.

(iii) *If* $A_1x+B_1y+C_1=0$, $A_2x+B_2y+C_2=0$, $A_3x+B_3y+C_3=0$ *are three straight lines which are not concurrent, the equation of any straight line can be expressed in the form*

$$l(A_1x+B_1y+C_1)+m(A_2x+B_2y+C_2)+n(A_3x+B_3y+C_3)=0.$$

Let $Ax+By+C=0$ be the equation of the straight line which is to be put into the required form.

Compare the coefficients with those of the above equation, then

$$\begin{aligned} lA_1+mA_2+nA_3 &= A, \\ lB_1+mB_2+nB_3 &= B, \\ lC_1+mC_2+nC_3 &= C. \end{aligned}$$

This gives three equations to determine the three constants $l$, $m$, $n$: they are in general sufficient to find these constants. If the given straight lines are concurrent we have the relation

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = 0,$$

in which case we do not find finite the values of $l$, $m$, and $n$, and there is no solution.

**Cor.** If values of $l$, $m$, and $n$ can be found so that the relation

$$l(A_1x+B_1y+C_1)+m(A_2x+B_2y+C_2)+n(A_3x+B_3y+C_3)=0$$

is *identically* true, we must have

$$\begin{aligned} lA_1+mA_2+nA_3&=0,\\ lB_1+mB_2+nB_3&=0,\\ lC_1+mC_2+nC_3&=0; \end{aligned}$$

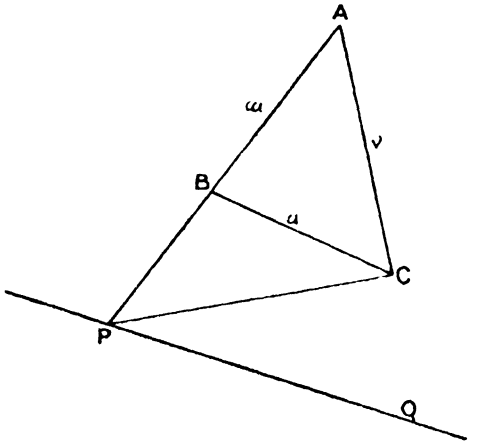
and consequently

$$\begin{vmatrix} A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3 \end{vmatrix}=0,$$

i.e. the straight lines are concurrent.

Thus, for instance, we can see from their form that the straight lines (§ 8) $u-v=0$, $v-w=0$, $w-u=0$ are concurrent.

Proposition iii of this section can also be proved in the following manner.




Let the three equations in abridged notation be $u=0$, $v=0$, $w=0$, and let $BC$, $CA$, $AB$ represent these lines.

If $PQ$ is any other line, let $AB$ meet it in $P$ and join $PC$.

Since $PC$ is a straight line through the intersection of $u=0$, $v=0$, its equation is of the form (p. 52) $lu+mv=0$.

Hence $PQ$ is a straight line through the intersection of the straight lines $lu+mv=0$ and $w=0$; its equation is therefore of the form

$$(lu+mv)+nw=0,$$

or

$$lu+mv+nw=0;$$

hence if $u=0$, $v=0$, $w=0$ are any three straight lines, the equation of any other straight line can be expressed in the form

$$lu+mv+nw=0.$$

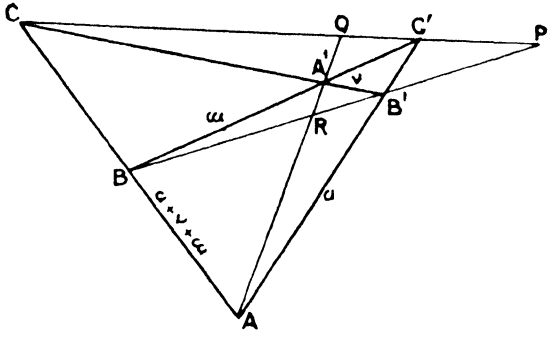
**Cor.** If $u = 0$, $v = 0$, $w = 0$, $lu + mv + nw = 0$ are four straight lines, multiply the first three equations by $l$, $m$, and $n$ respectively, and let

$$U \equiv lu, \quad V \equiv mv, \quad W \equiv nw,$$

then $U = 0$, $V = 0$, $W = 0$ represent the three straight lines, and in this case the fourth line is represented by the equation

$$U + V + W = 0.$$

This result is useful when descriptive properties of a quadrilateral are considered: the following example illustrates the method, which will be referred to again later.




If the four sides of a complete quadrilateral are represented by the equations $u = 0$, $v = 0$, $w = 0$, $u + v + w = 0$, to find the equations of its three diagonals.

Let $ABA'B'$ be the quadrilateral.

Now the diagonal $AA'$ passes through the intersection of the pairs of straight lines

$$\left.\begin{matrix} v = 0 \\ w = 0 \end{matrix}\right\} \text{ and } \left.\begin{matrix} u = 0 \\ u + v + w = 0 \end{matrix}\right\};$$

| | | |
|---|---|---|
| but the equation | $v + w = 0$ | [straight line through $A'$] |
| can also be written | $(u + v + w) - u = 0$ | [straight line through $A$] |

and therefore represents the straight line $AA'$.

| | | |
|---|---|---|
| Similarly the equation | $w + u = 0$ | [straight line through $C'$] |
| can be written | $(u + v + w) - v = 0$ | [straight line through $C$] |

and represents $CC'$.

| | | |
|---|---|---|
| Also the equation | $u + v = 0$ | [straight line through $B'$] |
| can be written | $(u + v + w) - w = 0$ | [straight line through $B$] |

and represents $BB'$.

Hence $v + w = 0$, $w + u = 0$, $u + v = 0$ are the three diagonals of the quadrilateral.

## Examples II h.

1. What is the value of $a$ if the three straight lines $x + y - 4 = 0$, $3x + 2 = 0$, $x - y + 3a = 0$ are concurrent?

2. Find the area of the triangle formed by the three straight lines $2y + x - 5 = 0$, $y + 2x - 7 = 0$, $x - y + 1 = 0$.

3. For what values of $a$ are the three straight lines $2x+y+1=0$, $3x+2ay+4=0$, $x+y-3a=0$ concurrent?

4. The equations of the sides of a triangle are $3x+4y=12$, $5x-12y=20$, $24y-7x=72$: find its area.

5. Prove that the straight lines

$$(b+c)x-bcy=a(b^2+bc+c^2),$$
$$(c+a)x-cay=b(c^2+ca+a^2),$$
$$(a+b)x-aby=c(a^2+ab+b^2)$$

are concurrent. What is the point of intersection?

6. Find the condition that the three straight lines

$$\lambda x+y=2a\lambda+a\lambda^3,$$
$$\mu x+y=2a\mu+a\mu^3,$$
$$\nu x+y=2a\nu+a\nu^3$$

should be concurrent.

7. Find the area of the triangle formed by the axes and the line $\lambda^2 x+y=c\lambda$, when the axes are inclined at an angle $\omega$.

8. Prove that the perpendiculars of a triangle are concurrent.

9. Show, using abridged notation, that the three bisectors of the angles of a triangle are concurrent.

10. Prove analytically that the bisectors of the vertical angle and the bisectors of the exterior base angles of a triangle are concurrent.

11. The equations of three straight lines are

$$u\equiv 3x+4y-7=0,\ v\equiv 4x+5y-6=0,\ w\equiv x-y+1=0,$$

and that of a fourth line is $59x+7y-21=0$.

Express the last equation in the form $lu+mv+nw=0$.

Find the equations of the three diagonals of the quadrilateral formed by these four lines.

12. The six vertices of a complete quadrilateral are $AA'$, $BB'$, $CC'$, and the diagonals form the triangle $PQR$. If the equations of the sides are $u=0$, $v=0$, $w=0$, $u+v+w=0$, find the equations of $PA'$, $QB'$, $RC'$, and of $PA$, $QB$, $RC$. (See figure, p. 59.)

Show that $PA'$, $QB'$, $RC'$ are concurrent.

13. If the sides of a quadrilateral are the four straight lines $u+v+w=0$, $-u+v+w=0$, $u-v+w=0$, $u+v-w=0$, find the equations of its diagonals.

## § 10. Relations between four straight lines whose equations are given.

### Anharmonic Ratios.

**Definition.** If $A$, $B$, $C$, $D$ are four points on a straight line, then the value of the expression $AC.DB/CB.AD$, in which the signs as well as the magnitudes of the segments are taken into

consideration, is an Anharmonic Ratio of the four points $A, B, C, D$. It is frequently denoted by $(AB, CD)$.*

There are as many anharmonic ratios of four points as there are permutations of the letters $A, B, C, D$, that is 24; but if one ratio is known all the others can be completely and uniquely determined (*vide* Russell's *Treatise on Pure Geometry*, Chap. IX). For the sake of preciseness we shall refer to the above expression as *the* anharmonic ratio of the points $A, B, C, D$.

It should be noted that, if $C$ divides the segment $AB$ in the ratio $l : m$, and $D$ divides the segment $AB$ in the ratio $l' : m'$, these ratios having 'sign' as well as 'magnitude' as in Chapter I, § 6, then the anharmonic ratio of $A, B, C, D$ is $\frac{l}{m} \div \frac{l'}{m'}$.

In Chapter I a harmonic range was defined as four points, two of which divide the distance between the other two internally and externally in the same ratio: i. e.

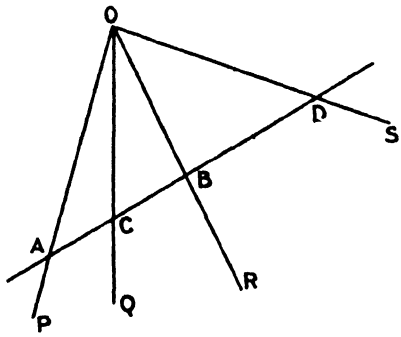
$$\frac{AC}{CB} = \frac{AD}{BD} \text{ or } \frac{AC \cdot BD}{AD \cdot CB} = 1,$$

which, with due regard to sign, gives

$$\frac{AC \cdot DB}{CB \cdot AD} = \{AB, CD\} = -1;$$

thus, when the range is harmonic, the value of the anharmonic ratio is $-1$.

I. *If any four concurrent straight lines OP, OQ, OR, OS (called a pencil of rays, vertex O) are cut by any straight line (called a transversal) in four points A, B, C, D, the anharmonic ratio* $\{AB, CD\}$ *is the same for all positions of the transversal.*




* This notation is not universal. Many writers use $(ABCD)$ to denote the anharmonic ratio that we should denote by $(AC, BD)$.

The ratio

$$\frac{AC.BD}{AD.BC} = \frac{\Delta OAC.\Delta OBD}{\Delta OAD.\Delta OBC} \qquad \text{(Euc. VI. i.)}$$

$$= \frac{OA.OC.\sin\angle AOC.OB.OD.\sin\angle BOD}{OA.OD.\sin\angle AOD.OB.OC.\sin\angle BOC}$$

$$= \frac{\sin\angle AOC.\sin\angle BOD}{\sin\angle AOD.\sin\angle BOC},$$

and this quantity is independent of the position of the transversal $ABCD$.

This proves the proposition so far as the magnitude of the ratio is concerned; it can easily be verified, by drawing transversals in different positions, that 0, 2, or 4 of the segments $AC$, $CB$, $AD$, $DB$ change their sign; the sign of the ratio is therefore unaltered. The constant anharmonic ratio determined on any transversal by the pencil is called the anharmonic ratio of the pencil.

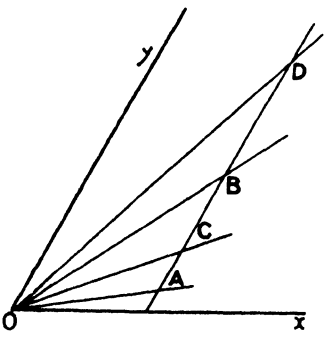
**Cor. i.** Since the anharmonic ratio of a pencil of four rays depends only on the sines of the angles between the rays, the anharmonic ratio of any parallel pencil is the same.

Hence the anharmonic ratio of any pencil of four rays is equal to that of a pencil formed by four parallel straight lines through the origin: consequently we need only consider pencils with vertices at the origin.

**Cor. ii.** The pencil formed by joining any point to four points forming a harmonic range is a harmonic pencil.

II. (a) *To find the anharmonic ratio of the pencil formed by the four lines* $y = px$, $y = qx$, $y = rx$, $y = sx$.

Let the pencil be cut by a transversal $x = h$ in the points $A$, $B$, $C$, $D$.




The coordinates of these points are $(h, ph)$, $(h, qh)$, $(h, rh)$, $(h, sh)$: hence

$$\frac{AC.BD}{AD.BC} = \frac{(rh-ph)(sh-qh)}{(sh-ph)(rh-qh)}$$

$$= \frac{(r-p)(s-q)}{(s-p)(r-q)}$$

$$= \frac{(p-r)(q-s)}{(p-s)(q-r)}.$$

(b) *If* $u = 0$, $v = 0$ *are any two straight lines, to find the anharmonic ratio of the pencil formed by the lines* $u - k_1 v = 0$, $u - k_2 v = 0$, $u - k_3 v = 0$, $u - k_4 v = 0$.

Let the lines cut any transversal in the points $A$, $B$, $C$, $D$; let $A$ be the point $(x_1, y_1)$, $B$ the point $(x_2, y_2)$, and let $u_1$, $u_2$ and $v_1$, $v_2$ be the values of $u$ and $v$ when the coordinates of these points are substituted in them.

Let $\frac{AC}{CB} = \lambda$, $\frac{AD}{DB} = \mu$; then the coordinates of

$$C \text{ are } \left\{\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}\right\}, \text{ and of } D \left\{\frac{x_1+\mu x_2}{1+\mu}, \frac{y_1+\mu y_2}{1+\mu}\right\}.$$

Now, since the points $A$, $B$, $C$, $D$ lie on the four given lines respectively, we have

$$u_1 - k_1 v_1 = 0 \text{ (i)}, \qquad u_2 - k_2 v_2 = 0 \text{ (ii)},$$
$$u_1 - k_3 v_1 + \lambda (u_2 - k_3 v_2) = 0 \text{ (iii)},$$
$$(u_1 - k_4 v_1) + \mu (u_2 - k_4 v_2) = 0 \text{ (iv)}.$$

Substituting from (i) and (ii) in (iii) and (iv) we have

$$(k_1 - k_3) v_1 + \lambda (k_2 - k_3) v_2 = 0,$$
$$(k_1 - k_4) v_1 + \mu (k_2 - k_4) v_2 = 0.$$

Hence
$$\frac{\lambda}{\mu} = \frac{(k_1 - k_3)(k_2 - k_4)}{(k_1 - k_4)(k_2 - k_3)}.$$

But $\frac{\lambda}{\mu} = \frac{AC}{CB} \div \frac{AD}{DB}$ = the anharmonic ratio of the four lines.

This result is a complete analytical proof of proposition I of this section; for the value of $\lambda/\mu$ found is independent of the position of the transversal. This proof, though more difficult, is preferable because it deals simultaneously with the magnitude and the sign of the ratio.

**Cor. i.** An important result follows from this: the anharmonic ratio of the pencil formed by the four lines $u = 0$, $v = 0$, $u + \lambda v = 0$, $u + \lambda' v = 0$ is $\frac{\lambda}{\lambda'}$. For a harmonic pencil $\frac{\lambda}{\lambda'} = -1$, hence the four straight lines $u = 0$, $v = 0$, $u + \lambda v = 0$, $u - \lambda v = 0$ form a harmonic pencil. But given any four concurrent straight lines, if $u = 0$, $v = 0$ are two of them, the equations of others can be put in the form $lu + mv = 0$, $l'u + m'v = 0$.

Hence the equations of the rays of any harmonic pencil are of the form $u = 0$, $v = 0$, $lu + mv = 0$, $lu - mv = 0$.

**Cor. ii.** The condition that the four straight lines $y = px$, $y = qx$, $y = rx$, $y = sx$ should form a harmonic pencil is

$$(p - r)(q - s) + (p - s)(q - r) = 0,$$

or
$$(p + q)(r + s) = 2(pq + rs).$$

**Ex. i.** *Find the locus of a point $P$ which is such that the lines joining it to the points $(a, 0)$, $(-a, 0)$ are harmonic conjugates of the lines joining it to the points $(0, b)$, $(0, -b)$.*

Let $P$ be the point $(\xi, \eta)$, then the equations of the four straight lines are

$$\frac{x-a}{\xi-a}=\frac{y}{\eta},\ \frac{x+a}{\xi+a}=\frac{y}{\eta},\ \frac{x}{\xi}=\frac{y-b}{\eta-b},\ \frac{x}{\xi}=\frac{y+b}{\eta+b}.$$

These are parallel to

$$y=\frac{\eta}{\xi-a}x,\ y=\frac{\eta}{\xi+a}x,\ y=\frac{\eta-b}{\xi}x,\ y=\frac{\eta+b}{\xi}x.$$

Hence, **Cor. ii** above,

$$\left(\frac{\eta}{\xi-a}+\frac{\eta}{\xi+a}\right)\left(\frac{\eta-b}{\xi}+\frac{\eta+b}{\xi}\right)=2\left\{\frac{\eta^2}{\xi^2-a^2}+\frac{\eta^2-b^2}{\xi^2}\right\},$$

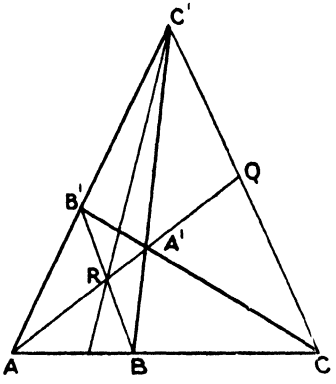
i. e. $$\eta^2\xi^2=(\xi^2-a^2)(\eta^2-b^2);$$

i. e. the locus of $(\xi, \eta)$ is $$b^2x^2+a^2y^2=a^2b^2.$$

**Ex. ii.** *Each diagonal of a complete quadrilateral is divided harmonically by the other two.*

Let the sides $AB'$, $B'A'$, $A'B$, $BA$ be $u=0$, $v=0$, $w=0$, $u+v+w=0$. Then it was shown in § 9 that the equations of $AA'$, $BB'$, $CC'$ are

$$v+w=0,\ u+v=0,\ u+w=0.$$




Now $C'R$ is a straight line through the intersection of $AB'$ and $BA'$, i. e. of $u=0$ and $w=0$.

It also passes through the intersection of $AA'$ and $BB'$, i. e. of $v+w=0$, $u+v=0$; its equation is therefore $u-w=0$, for this can also be written $(u+v)-(v+w)=0$.

Hence the pencil $C'A$, $C'A'$, $C'R$, $CC'$ is $u=0$, $w=0$, $u-w=0$, $u+w=0$, which from its form is seen to be harmonic.

## Examples II i.

1. Find the equation of a straight line which is the harmonic conjugate of $y-mx=0$ with respect to the axes.

2. Find the anharmonic ratio of the pencil $x+ly=0$, $x+my=0$, $x+ny=0$, $x+py=0$.

3. Find the condition that the four straight lines joining the point $(x, y)$ to the four corners of a square whose centre is the origin, and whose sides are parallel to the axes, should form a harmonic pencil.

4. Find the fourth harmonic of the pencil $y-px=0$, $y-qx=0$, $y-rx=0$.

5. The equations of two intersecting straight lines are $X=0$, $Y=0$: find the value of the anharmonic ratio of the pencil formed by $l_1X+m_1Y=0$, $l_2X+m_2Y=0$, $l_3X+m_3Y=0$, $l_4X+m_4Y=0$.

## § 11. Polar Coordinates.

We have shown in Chapter I that if the initial line and a perpendicular through the pole are taken as axes of Cartesian coordinates, then $(x, y)$ and $(r, \theta)$ being the same point

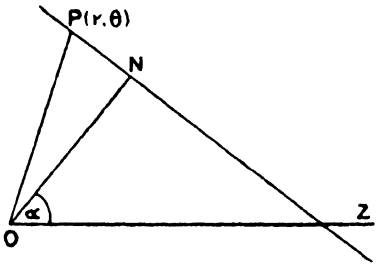
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Since the equation of a straight line is linear in $x$ and $y$, the general equation of a straight line in polar coordinates is

$$Ar\cos\theta + Br\sin\theta + C = 0. \tag{i}$$

(i) *To find the equation of a straight line in polar coordinates, given the length* p *of the perpendicular to it from the pole and the angle* $\alpha$ *which this perpendicular makes with the initial line.*

Let $P(r, \theta)$ be any point on the line, $ON$ the perpendicular to the line.




Then

$POZ = \theta$, $NOZ = \alpha$, $PON = (\theta - \alpha)$;

hence in the triangle $OPN$

$$r\cos(\theta - \alpha) = p, \tag{ii}$$

and this equation, being true for the coordinates of any point on the line, is the equation of the line.

Expanding $\cos(\theta - \alpha)$ the equation becomes

$$r\cos\theta \,.\, \cos\alpha + r\sin\theta \,.\, \sin\alpha - p = 0,$$

which is of the form (i) found above. (Cf. § 3, II.)

**Cor. i.** The equation of any straight line parallel to (ii) is of the form

$$r\cos(\theta - \alpha) = q.$$

**Cor. ii.** The equation of any straight line perpendicular to (ii) is of the form

$$r\sin(\theta - \alpha) = q,$$

since $\alpha$ is in this case increased or decreased by $\frac{1}{2}\pi$.

**Cor. iii.** The equation of any straight line through the pole is of the form $\theta = \alpha$, for any point $(r, \alpha)$ lies on it whatever value $r$ may have.

Let $PQ$ be any straight line

$$Ar\cos\theta + Br\sin\theta + C = 0, \tag{i}$$

and $OR$ a straight line through the origin parallel to it.

Since the equation (i) is equivalent to

$$Ax + By + C = 0,$$

the straight line $PQ$ makes an angle $\tan^{-1}\left(-\frac{A}{B}\right)$ with the initial line $OZ$.

The straight line $OR$, parallel to it, makes the same angle with $OZ$; consequently its equation is

$$\theta = \tan^{-1}\left(-\frac{A}{B}\right), \qquad \text{(ii)}$$

which may also be written

$$A\cos\theta + B\sin\theta = 0. \qquad \text{(iii)}$$

In the same way we can show that the equation of the straight line through the pole, making an angle $\alpha$ with

$$Ar\cos\theta + Br\sin\theta + C = 0,$$

is

$$A\cos(\theta \pm \alpha) + B\sin(\theta \pm \alpha) = 0 \qquad \text{(iv)}$$

according as the angle is made towards the initial line or away from it.

**Example.** *To find the angle between the two straight lines*

$$\frac{l}{r} = \cos\theta + \cos\overline{\theta - \alpha} \qquad \text{(i)}$$

*and*

$$\frac{l}{r} = \cos\theta + \cos\overline{\theta - \beta}. \qquad \text{(ii)}$$

These straight lines are parallel to the lines

$$\cos\theta + \cos\overline{\theta - \alpha} = 0,$$

$$\cos\theta + \cos\overline{\theta - \beta} = 0,$$

which pass through the pole.

These can be written

$$2\cos\tfrac{1}{2}\alpha \,.\, \cos(\theta - \tfrac{1}{2}\alpha) = 0,$$

and

$$2\cos\tfrac{1}{2}\beta \,.\, \cos(\theta - \tfrac{1}{2}\beta) = 0,$$

i. e.

$$\theta = \tfrac{1}{2}\pi + \tfrac{1}{2}\alpha,$$

and

$$\theta = \tfrac{1}{2}\pi + \tfrac{1}{2}\beta.$$

The angle between these straight lines is $\frac{1}{2}(\alpha - \beta)$, and this is also the angle between the given lines which are parallel to them.

## § 12. Envelopes.

The equation $lx + my + n = 0$ can represent any chosen straight line provided that the values which can be assigned to $l$, $m$, and $n$ are unrestricted. If, however, there is some given relation between $l$, $m$, and $n$ (e.g. $l = m - 2n$) the equation $lx + my + n$ can represent only one or other of a group of straight lines.

In the simple case $l = m - 2n$, we can replace $l$ by its value in terms of $m$ and $n$ and the equation then becomes

$$m(x + y) = n(2x - 1),$$

and this represents for different values of $m$ and $n$ any one of a pencil of lines passing through the intersection of $x + y = 0$ and $2x - 1 = 0$, i. e. the point $(\frac{1}{2}, -\frac{1}{2})$.

The equation $lx + my + n = 0$ contains only two independent constants (*vide* § 4); for the sake of simplicity we will discuss the

equation in the form $lx+my=1$, which can represent any straight line which does not pass through the origin.

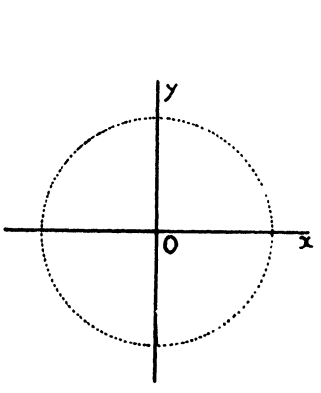
A relation then between $l$ and $m$ means that $lx+my=1$ represents a definite system of lines: if we take a series of values for $l$, gradually increasing by small quantities, and find the corresponding value or values of $m$, the line represented by

$$lx+my=1$$

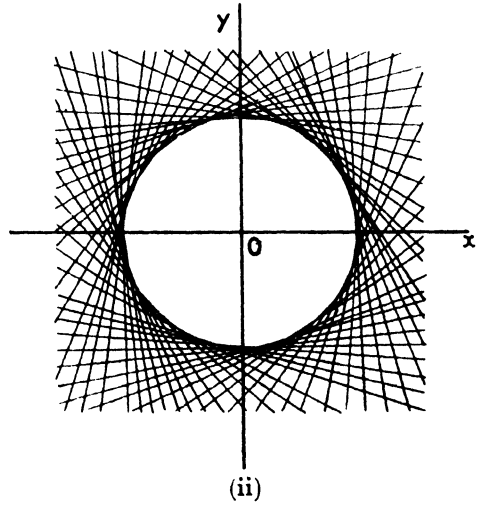
for these values of $l$ and $m$ will take up a series of different positions, each differing slightly from the one before.

Compare the case of a point whose coordinates $(x, y)$ are connected by some given equation (e.g. $3x+4y=1$); if series of values are given to $x$, increasing by very small quantities, and the corresponding values of $y$ are found, the corresponding point will take up a series of positions each differing slightly from the one before.

If a point continuously changes its position, a curve is formed on which all the points lie; the more points we plot the more clearly this curve is indicated. So if the line continuously changes its position, a curve is formed which all the lines envelope or touch; the more lines we draw the more clearly this curve is indicated. The curve thus formed by a moving line is called an envelope, the curve formed by a moving point is called a locus. It should be clearly understood that the envelope of a line moving under a constraint or condition is as simple an idea as the locus of a point moving under a constraint or condition.




(i)




(ii)

The figures illustrate (i) the locus of a point whose distance from the origin is constant, (ii) the envelope of a line whose distance from the origin is constant.

In (i) two points of the series lie on any given straight line, and in (ii) two lines of the series pass through any given point.

In (i) the line joining two consecutive and very near points is said to touch the locus, and in (ii) the intersection of two consecutive and very near lines is said to lie on the envelope.

To consider this algebraically: if the coefficients $l$ and $m$ are connected by any relation, we can in general find $l$ in terms of $m$ and substitute in $lx+my=1$; we thus obtain an equation containing only one arbitrary constant.

We shall consider only the two cases, where the undetermined quantity occurs in the equation (a) in the first degree only, (b) in the second degree at most.

For case (a) we have already shown that all the straight lines meet at a point, i. e. the envelope is a point.

The most general equation in case (b) is

$$\lambda^2(ax+by+c)+\lambda(a'x+b'y+c')+(a''x+b''y+c'')=0,$$

where all the coefficients except $\lambda$ are supposed known.

Now through any point $(x_1, y_1)$ two lines of the system pass, the values of $\lambda$ being given by the equation

$$\lambda^2(ax_1+by_1+c)+\lambda(a'x_1+b'y_1+c')+(a''x_1+b''y_1+c'')=0.$$

These two lines will be coincident if

$$(a'x_1+b'y_1+c')^2=4(ax_1+by_1+c)(a''x_1+b''y_1+c'');$$

but in this case the point $(x_1, y_1)$ is on the envelope; thus the equation

$$(a'x+b'y+c')^2=4(ax+by+c)(a''x+b''y+c'')$$

is satisfied by the coordinates of all points on the envelope, i. e. is the equation of the envelope.

**Note.** In abridged notation the envelope of the line $\lambda^2u+\lambda v+w=0$ is $v^2=4uw$.

**Ex.** i. *To find the envelope of a line whose distance from the origin is constant.*

The equation of the line can be written

$$x\cos\alpha+y\sin\alpha-p=0,$$

where $p$ is constant.

This can be written $\lambda^2(x+p)-2\lambda y+(p-x)=0,$

where $\lambda\equiv\tan\frac{1}{2}\alpha.$

Hence the envelope is $y^2=(x+p)(p-x),$

or $x^2+y^2=p^2.$

**Ex. ii.** *To find the envelope of the straight line* $\frac{x}{a}+\frac{y}{b}=1$ *when the coefficients are connected by the condition* $a+b=c$, *where* $c$ *is a constant.*

Since $a+b=c$, the equation of the straight line can be written

$$\frac{x}{a}+\frac{y}{c-a}=1,$$

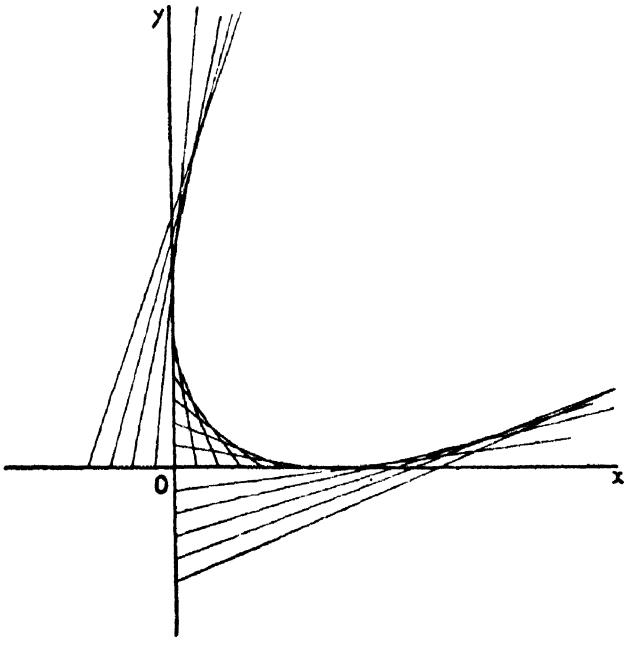
or
$$a^2-a(x-y+c)+cx=0,$$

where $a$ is an undetermined constant.

The values of $a$ for lines of the system which pass through a particular point $(x', y')$ are given by $a^2-a(x'-y'+c)+cx'=0$.

This is quadratic in $a$; hence two lines of the system pass through any point $(x', y')$. If these are coincident, $(x', y')$ lies on the envelope.

The condition for this is $(x'-y'+c)^2=4cx'$.




Hence the equation of the locus of $(x', y')$, i.e. the envelope of the given system of lines, is

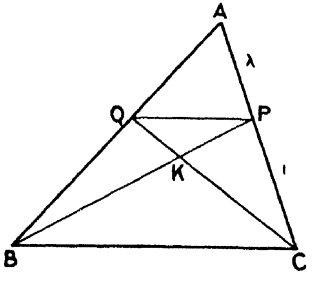
$$(x-y+c)^2=4cx.$$

This is shown in the figure.

### Illustrative Examples.

(i) *A straight line parallel to the base BC of a triangle meets the sides AC, AB in P and Q respectively. Show that the area of the triangle formed by the straight lines BP, CQ, PQ bears to the area of the triangle ABC the ratio* $\lambda^2 : (\lambda+1)^2(2\lambda+1)$, *where* $\lambda$ *is the ratio of AP to PC.*

Take $AB$, $AC$ for axes of $x$ and $y$ and let $B$, $C$ be the points $(b, 0)$, $(0, c)$.




Then the coordinates of $P$ and $Q$ are

$$\left(0, \frac{\lambda c}{\lambda+1}\right) \left(\frac{\lambda b}{\lambda+1}, 0\right).$$

Hence $BP$ is the line

$$\frac{x}{b} + \frac{(\lambda+1)y}{\lambda c} = 1, \qquad \text{(i)}$$

and $CQ$ is the line

$$\frac{(\lambda+1)x}{\lambda b} + \frac{y}{c} = 1. \qquad \text{(ii)}$$

The coordinates of their point of intersection are given by these equations: by subtraction $\frac{x}{\lambda b} - \frac{y}{\lambda c} = 0$, i.e. $\frac{x}{b} = \frac{y}{c} = l$ (say).

Substitute in (i) to find $l$, hence

$$l + \frac{\lambda+1}{\lambda} l = 1 \quad \text{or} \quad l = \frac{\lambda}{2\lambda+1},$$

i.e. $K$ is the point $\left(\frac{\lambda b}{2\lambda+1}, \frac{\lambda c}{2\lambda+1}\right)$.

The area of the triangle $PKQ$ is then

$$\tfrac{1}{2}\sin A \begin{vmatrix} 0 & \frac{\lambda c}{\lambda+1} & 1 \\ \frac{\lambda b}{\lambda+1} & 0 & 1 \\ \frac{\lambda b}{2\lambda+1} & \frac{\lambda c}{2\lambda+1} & 1 \end{vmatrix}$$

$$= \tfrac{1}{2} \frac{\lambda^2 bc \sin A}{(\lambda+1)^2(2\lambda+1)} \times \begin{vmatrix} 0 & 1 & \lambda+1 \\ 1 & 0 & \lambda+1 \\ 1 & 1 & 2\lambda+1 \end{vmatrix}$$

$$= \tfrac{1}{2} \frac{\lambda^2 bc \sin A}{(\lambda+1)^2(2\lambda+1)}.$$

$\therefore \quad \Delta PKQ : \Delta ABC = \lambda^2 : (\lambda+1)^2(2\lambda+1).$

(ii) *The area included by the lines*

$$bx\cos\alpha + ay\sin\alpha = ab,$$
$$bx\cos\beta + ay\sin\beta = ab,$$
$$bx\cos\gamma + ay\sin\gamma = ab$$

*is* $ab\tan\frac{1}{2}(\beta-\gamma)\tan\frac{1}{2}(\gamma-\alpha)\tan\frac{1}{2}(\alpha-\beta)$.

$$\text{The area} = \tfrac{1}{2}\begin{vmatrix} b\cos\alpha & a\sin\alpha & ab \\ b\cos\beta & a\sin\beta & ab \\ b\cos\gamma & a\sin\gamma & ab \end{vmatrix}^2$$

$$\div \begin{vmatrix} b\cos\alpha & a\sin\alpha \\ b\cos\beta & a\sin\beta \end{vmatrix} \times \begin{vmatrix} b\cos\beta & a\sin\beta \\ b\cos\gamma & a\sin\gamma \end{vmatrix} \times \begin{vmatrix} b\cos\gamma & a\sin\gamma \\ b\cos\alpha & a\sin\alpha \end{vmatrix} \quad (c.\ \S\ 9)$$

$$= \tfrac{1}{2}a^4b^4\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}^2 \div a^3b^3\sin(\beta-\alpha)\sin(\gamma-\beta)\sin(\alpha-\gamma)$$

$$= \tfrac{1}{2}ab\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ 2\sin\frac{1}{2}(\beta-\alpha)\sin\frac{1}{2}(\beta+\alpha) & 2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta+\alpha) & 0 \\ 2\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\gamma+\alpha) & 2\sin\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\alpha+\gamma) & 0 \end{vmatrix}^2$$

$$\div 8\sin\tfrac{1}{2}(\beta-\alpha)\cos\tfrac{1}{2}(\beta-\alpha)\sin\tfrac{1}{2}(\gamma-\beta)\cos\tfrac{1}{2}(\gamma-\beta)\cdot\sin\tfrac{1}{2}(\alpha-\gamma)\cos\tfrac{1}{2}(\alpha-\gamma)$$

$$= ab\sin\tfrac{1}{2}(\beta-\alpha)\sin\tfrac{1}{2}(\gamma-\alpha)\begin{vmatrix} \sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta+\alpha) \\ \sin\frac{1}{2}(\gamma+\alpha)\cos\frac{1}{2}(\gamma+\alpha) \end{vmatrix}^2$$

$$\div \sin\tfrac{1}{2}(\gamma-\beta)\cos\tfrac{1}{2}(\gamma-\beta)\cos\tfrac{1}{2}(\alpha-\gamma)\cos\tfrac{1}{2}(\beta-\alpha)$$

$$= ab\tan\tfrac{1}{2}(\beta-\gamma)\tan\tfrac{1}{2}(\gamma-\alpha)\tan\tfrac{1}{2}(\alpha-\beta),$$

for $\sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\alpha+\gamma) - \sin\frac{1}{2}(\gamma+\alpha)\cos\frac{1}{2}(\beta+\alpha) = \sin\frac{1}{2}(\beta-\gamma)$.

(iii) *If straight lines drawn through the points A, B, C parallel respectively to the lines MN, NL, LM are concurrent, then straight lines through LMN parallel respectively to BC, CA, AB are concurrent.*

This is a good example of the advantage of using quite general coordinate axes in dealing with a general proposition.

Let $A$, $B$, $C$ be the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ and $L$, $M$, $N$ the points $(\xi_1, \eta_1)$, $(\xi_2, \eta_2)$, $(\xi_3, \eta_3)$.

The line $MN$ is $x(\eta_2-\eta_3) - y(\xi_2-\xi_3) = \eta_2\xi_3 - \eta_3\xi_2$.

Hence the line through $A(x_1, y_1)$ parallel to this is

$$x(\eta_2-\eta_3) - y(\xi_2-\xi_3) = x_1(\eta_2-\eta_3) - y_1(\xi_2-\xi_3). \qquad \text{(i)}$$

From symmetry the equations of the lines through $B$ and $C$ parallel to $NL$, $LM$, are

$$x(\eta_3-\eta_1) - y(\xi_3-\xi_1) = x_2(\eta_3-\eta_1) - y_2(\xi_3-\xi_1), \qquad \text{(ii)}$$
$$x(\eta_1-\eta_2) - y(\xi_1-\xi_2) = x_3(\eta_1-\eta_2) - y_3(\xi_1-\xi_2). \qquad \text{(iii)}$$

The condition that (i), (ii), and (iii) should be concurrent, is

$$\begin{vmatrix} \eta_2-\eta_3 & \xi_2-\xi_3 & x_1(\eta_2-\eta_3) - y_1(\xi_2-\xi_3) \\ \eta_3-\eta_1 & \xi_3-\xi_1 & x_2(\eta_3-\eta_1) - y_2(\xi_3-\xi_1) \\ \eta_1-\eta_2 & \xi_1-\xi_2 & x_3(\eta_1-\eta_2) - y_3(\xi_1-\xi_2) \end{vmatrix} = 0.$$

Add the second and third row to the top, then

$$\begin{vmatrix} 0 & 0 & \Sigma x_1(\eta_2-\eta_3)-\Sigma y_1(\xi_2-\xi_3) \\ \eta_3-\eta_1 & \xi_3-\xi_1 & x_2(\eta_3-\eta_1)-y_2(\xi_3-\xi_1) \\ \eta_1-\eta_2 & \xi_1-\xi_2 & x_3(\eta_1-\eta_2)-y_3(\xi_1-\xi_2) \end{vmatrix} = 0.$$

Hence either $$\Sigma x_1(\eta_2-\eta_3)-\Sigma y_1(\xi_2-\xi_3) = 0,$$

or $$(\eta_3-\eta_1)(\xi_1-\xi_2)-(\eta_1-\eta_2)(\xi_3-\xi_1) = 0;$$

which cannot be true unless $L$, $M$, $N$ are collinear: therefore the required condition is

$$\Sigma x_1(\eta_2-\eta_3)-\Sigma y_1(\xi_2-\xi_3) = 0. \qquad \text{(iv)}$$

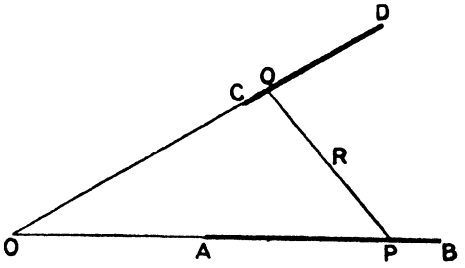
From symmetry the condition that the lines through $L$, $M$, $N$ parallel to $BC$, $CA$, $AB$ should be concurrent is

$$\Sigma \xi_1(y_2-y_3)-\Sigma \eta_1(x_2-x_3) = 0. \qquad \text{(v)}$$

But the conditions (iv) and (v) are identical; this can be at once seen by expanding the terms.

Hence the required proposition is established.

(iv) *A point $P$ divides a fixed straight line $AB$ in the ratio $p+m:p-m$, and $Q$ divides another fixed line $CD$ in the ratio $p+n:p-n$ where $m$ and $n$ are constants and $p$ is variable. Find the locus of the middle point of $PQ$.*




Take $AB$, $CD$ for coordinate axes and let $A$, $B$, $C$, $D$ be the points $(a, 0)$, $(b, 0)$, $(0, c)$, $(0, d)$. The coordinates of $P$ and $Q$ are

$$\left\{\frac{a(p-m)+b(p+m)}{2p}, 0\right\}, \quad \left\{0, \frac{c(p-n)+d(p+n)}{2p}\right\}.$$

Hence if $R$, the mid-point of $PQ$, is $(x, y)$,

$$x = \frac{a+b}{4} - \frac{m(a-b)}{4p}; \quad y = \frac{c+d}{4} - \frac{n(c-d)}{4p}.$$

By eliminating $p$ we get

$$\frac{4x-a-b}{m(a-b)} = \frac{4y-c-d}{n(c-d)},$$

the equation of a straight line.

(v) *A straight line moves so that the sum of the intercepts made by it on the axes is constant and equal to c.*

*Show that the locus of the point of intersection of two such lines at right angles to each other is the straight line* $(x+y)(1+\cos\omega) = c\cos\omega$, *the axes of coordinates containing the angle* $\omega$.

Let the equations of the two straight lines be

$$\frac{x}{a} + \frac{y}{c-a} = 1\,; \quad \frac{x}{b} + \frac{y}{c-b} = 1,$$

or
$$(c-a)x + ay - a(c-a) = 0, \qquad \text{(i)}$$
$$(c-b)x + by - b(c-b) = 0. \qquad \text{(ii)}$$

Cross multiplying to find the coordinates of their point of intersection

we get
$$\frac{x}{ab(b-a)} = \frac{y}{(c-a)(c-b)(b-a)} = \frac{1}{c(b-a)},$$

or
$$\frac{x}{ab} = \frac{y}{(c-a)(c-b)} = \frac{1}{c}. \qquad \text{(iii)}$$

But since (i) and (ii) are perpendicular

$$(c-a)(c-b) + ab = \{b(c-a) + a(c-b)\}\cos\omega,$$

[Ch. II, § 8 (ii).]

i.e.
$$\{(c-a)(c-b) + ab\}\{1+\cos\omega\} = c^2\cos\omega.$$

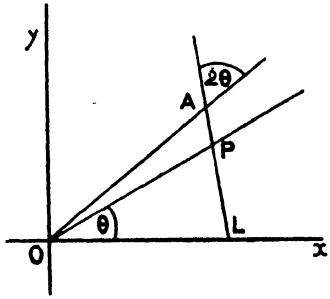
Hence (iii)
$$(x+y)(1+\cos\omega) = c\cos\omega,$$

which is the required locus.

(vi) *A straight line OP is drawn through the origin making an angle* $\theta$ *with the axis of x; a straight line AP is drawn through the point A* $(h, k)$ *making an angle* $2\theta$ *(in the same sense as* $\theta$*) with OA produced. Prove that the locus of P, the intersection of these straight lines, is*

$$2\frac{h^2+k^2}{x^2+y^2} = \frac{h}{x} + \frac{k}{y},$$

*the axes being rectangular.*




Let the angle $LOA = \phi$ so that $\tan\phi = \frac{k}{h}$, and let $P$ be the point $(x, y)$; so that $\tan\theta = \frac{y}{x}$. Then the angle $OPL = \theta + \phi$,

and $$\frac{OA}{OP} = \frac{\sin \angle OPA}{\sin \angle OAP} = \frac{\sin \angle OPL}{\sin \angle OAP} = \frac{\sin(\theta+\phi)}{\sin 2\theta},$$

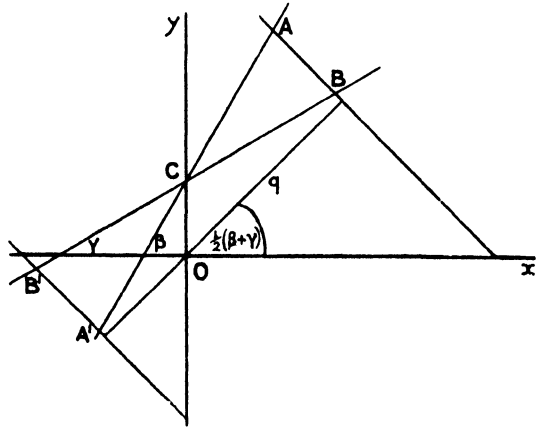
i. e. $$\frac{OA}{OP} = \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{2\sin\theta\cos\theta},$$

i. e. $$2\frac{OA}{OP} = \frac{\cos\phi}{\cos\theta} + \frac{\sin\phi}{\sin\theta}.$$

$\therefore$ $$2\frac{OA^2}{OP^2} = \frac{OA\cos\phi}{OP\cos\theta} + \frac{OA\sin\phi}{OP\sin\theta},$$

or $$2.\frac{h^2+k^2}{x^2+y^2} = \frac{h}{x} + \frac{k}{y}.$$

(vii) *Find the equations of two straight lines which make equal angles with the lines* $y = x\tan\beta + c$; $y = x\tan\gamma + c$ *and form triangles with them whose areas are* $c^2$, *the axes of coordinates being rectangular.*




The given lines $AA'$, $BB'$ pass through the point $(0, c)$, and make angles $\beta$, $\gamma$ respectively with the $x$-axis Let $AB$, $A'B'$ be the required lines inclined at an angle $\theta$ to the $x$-axis and equally inclined to $AA'$, $BB'$.

If $p$ be the perpendicular from the vertex $C$ on these lines, the area of the triangle $CAB$ or $CA'B'$ is

$$p^2\tan\tfrac{1}{2}(\beta-\gamma).$$

Now $$\theta = \beta + \angle CAB$$
$$= \beta + \angle ABC$$
$$= \beta + \gamma + \pi - \theta.$$
$$\therefore\ \theta = \tfrac{1}{2}\pi + \tfrac{1}{2}(\beta+\gamma).$$

Hence $AB$ is the straight line

$$x\cos\tfrac{1}{2}(\beta+\gamma) + y\sin\tfrac{1}{2}(\beta+\gamma) - q = 0,$$

if the perpendicular from (0, 0) on it is equal to $q$,

i. e. $$\pm p = c\sin\tfrac{1}{2}(\beta+\gamma) - q.$$

But $$c^2 = p^2\tan\tfrac{1}{2}(\beta-\gamma) \quad \text{or} \quad p = c\sqrt{\cot\tfrac{1}{2}(\beta-\gamma)}.$$

Hence $q = c\sin\frac{1}{2}(\beta+\gamma) \pm c\sqrt{\cot\frac{1}{2}(\beta-\gamma)}$,

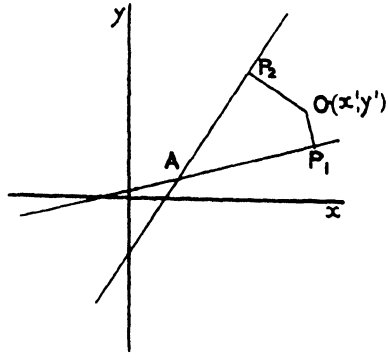
and the required equations are

$$x\cos\tfrac{1}{2}(\beta+\gamma) + y\sin\tfrac{1}{2}(\beta+\gamma) - c\sin\tfrac{1}{2}(\beta+\gamma) = \pm c\sqrt{\cot\tfrac{1}{2}(\beta-\gamma)}.$$

There are two other straight lines perpendicular to these, which satisfy the conditions of the problem.

(viii) *If $P_1$ and $P_2$ are the feet of the perpendiculars drawn from the point $O(x', y')$ to the straight lines $l_1x+m_1y+n_1=0$, $l_2x+m_2y+n_2=0$, find the value of $OP_1^2+OP_2^2-P_1P_2^2$; hence show that $O$ lies in the obtuse or acute angles between the straight lines according as the expressions $(l_1x'+m_1y'+n_1)\times(l_2x'+m_2y'+n_2)$ and $l_1l_2+m_1m_2$ have the same or opposite signs.*

Since the question is one of sign, it is better to proceed in the most straightforward manner possible, and thus avoid any unnecessary complication of signs. Let the straight lines intersect at $A$.




The equation of $AP_1$ is

$$l_1x+m_1y+n_1=0,$$

so that the equation of $OP_1$ is

$$(x-x')/l_1 = (y-y')/m_1.$$

The coordinates of $P_1$ are therefore of the form $x'+l_1k$, $y'+m_1k$, and since $P_1$ lies on $AP_1$ we find that $k$ is

$$-(l_1x'+m_1y'+n_1)/(l_1^2+m_1^2).$$

We shall use the following abbreviations:

$$L_1 \equiv l_1x'+m_1y'+n_1,$$
$$L_2 \equiv l_2x'+m_2y'+n_2.$$

Then the coordinates of $P_1$ are

$$x' - \frac{l_1L_1}{l_1^2+m_1^2} \quad \text{and} \quad y' - \frac{m_1L_1}{l_1^2+m_1^2}.$$

Similarly, the coordinates of $P_2$ are

$$\left\{x' - \frac{l_2L_2}{l_2^2+m_2^2},\quad y' - \frac{m_2L_2}{l_2^2+m_2^2}\right\}.$$

Now $$OP_1^2 = \frac{L_1^2}{l_1^2+m_1^2}, \quad OP_2^2 = \frac{L_2^2}{l_2^2+m_2^2}.$$

Hence $OP_1^2+OP_2^2-P_1P_2^2$

$$= \frac{L_1^2}{l_1^2+m_1^2} + \frac{L_2^2}{l_2^2+m_2^2} - \left\{\frac{l_1L_1}{l_1^2+m_1^2} - \frac{l_2L_2}{l_2^2+m_2^2}\right\}^2 - \left\{\frac{m_1L_1}{l_1^2+m_1^2} - \frac{m_2L_2}{l_2^2+m_2^2}\right\}^2$$

$$= \frac{2L_1L_2(l_1l_2+m_1m_2)}{(l_1^2+m_1^2)(l_2^2+m_2^2)}.$$

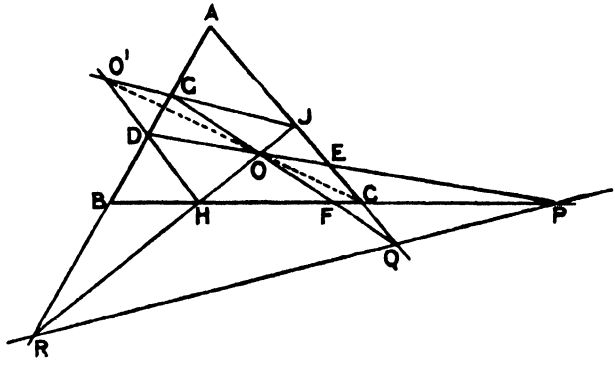
Now the angle $P_1OP_2$ is obtuse or acute, and consequently $P_1AP_2$ is acute or obtuse according as $OP_1^2+OP_2^2-P_1P_2^2$ is negative or positive, i. e. according as $L_1L_2$ and $l_1l_2+m_1m_2$ have different or the same signs.

(ix) *A straight line moves so that the product of the perpendiculars on it from two fixed points is constant: find the equation of its envelope.*

Let the two fixed points be $(a, 0)$, $(-a, 0)$, and the equation of the straight line
$$lx+my+1=0. \qquad \text{(i)}$$
The condition gives us that
$$\frac{(la+1)}{\sqrt{l^2+m^2}} \cdot \frac{(-la+1)}{\sqrt{l^2+m^2}} = \text{constant} = c^2 \text{ (say)},$$
i.e.
$$1-l^2a^2 = c^2l^2+c^2m^2,$$
or
$$l^2(c^2+a^2)+c^2m^2=1. \qquad \text{(ii)}$$
Now the values of $l$ and $m$ for those straight lines of the system which pass through some particular point $(x_1, y_1)$ are given by
$$lx_1+my_1+1=0 \qquad \text{(iii)}$$
and
$$l^2(c^2+a^2)+c^2m^2=1. \qquad \text{(iv)}$$
The values of $l$ (and note that from (iii) to any one value of $l$ there corresponds one and only one value of $m$) are thus given by
$$l^2y_1^2(c^2+a^2)+c^2(lx_1+1)^2=y_1^2,$$
i.e.
$$l^2\{(c^2+a^2)y_1^2+c^2x_1^2\}+2c^2lx_1+c^2-y_1^2=0.$$
If the two values of $l$ given by this equation are coincident, two coincident straight lines of the system intersect at $(x_1, y_1)$, and $(x_1, y_1)$ is on the envelope.

The condition for this is
$$c^4x_1^2 = \{(c^2+a^2)y_1^2+c^2x_1^2\}\{c^2-y_1^2\},$$
i.e.
$$0 = c^2(c^2+a^2)y_1^2-(c^2+a^2)y_1^4-c^2x_1^2y_1^2,$$
or
$$c^2x_1^2+(c^2+a^2)y_1^2 = c^2(c^2+a^2).$$
Hence the equation of the envelope is
$$c^2x^2+(c^2+a^2)y^2 = c^2(c^2+a^2).$$

(x) *ABC is a triangle, and any straight line cuts the sides BC, CA, AB in the points P, Q, R. O is any other point, and OP cuts AB, AC in D and E, OQ cuts BC, BA in F and G, OR cuts CA, CB in J, H. The straight lines JG, HD intersect in O'. Prove that C, O, O' are collinear.*




No special details of any of the lines in this question are given: there is nothing metrical; this property of the triangle is purely descriptive; we shall therefore use the abridged notation.

Let $PQR$ be the straight line $x = 0$, and let the sides $BC$, $CA$, $AB$ of the triangle be $u = 0$, $v = 0$, $w = 0$.

Now $DE$, $GF$, $JH$ are straight lines through the intersection of the pairs of lines $(u, x)$, $(v, x)$, $(w, x)$: let their equations be

$$lu + x = 0,$$
$$mv + x = 0,$$
$$nw + x = 0.$$

Now the equation $mv + nw + x = 0$

can be written in either of the forms

$$mv + (nw + x) = 0,$$
$$nw + (mv + x) = 0;$$

it therefore represents a straight line through the points of intersection of $v = 0$ and $nw + x = 0$, and also of $w = 0$ and $mv + x = 0$, i.e. through the points $J$ and $G$.

Hence the equation of $JG$ is $mv + nw + x = 0$.

In the same way the equation of $HD$ is $lu + nw + x = 0$.

Now the equation $(lu + nw + x) - (mv + nw + x) = 0$ represents a straight line through the intersection of $JG$ and $HD$, i.e. through $O'$. But it reduces to $lu - mv = 0$, i.e. it represents a straight line through the intersection of $u = 0$ and $v = 0$, i.e. through $C$. Hence the equation of $CO'$ is

$$lu - mv = 0.$$

But $O$ is the point of intersection of $DE$ and $GF$,

i.e. of $lu + x = 0$, $mv + x = 0$,

and therefore $(lu + x) - (mv + x) = 0$

is a straight line through $O$.

But this equation reduces to $lu - mv = 0$, i.e. $O$ lies on the straight line $CO'$. In other words $COO'$ is the straight line $lu - mv = 0$.

## Examples II j.

1. Show that the feet of the perpendiculars from the origin to the straight lines $x + y - 4 = 0$, $x + 5y - 26 = 0$, $15x - 27y - 424 = 0$ are collinear.

2. Through the origin a straight line is drawn making an angle of 30° with the axis of $x$. A second straight line is drawn making intercepts 3 and 5 on the positive directions of the axes of $x$ and $y$ respectively. Determine the distance of their point of intersection from the origin.

3. Find the locus of a point which moves in such a way that its distances from the straight lines $2x - y + 5 = 0$, $4x - 2y - 3 = 0$ are equal.

4. Find the acute angle between the straight lines $12x - 5y - 5 = 0$, $6 - 3x - 4y = 0$ and the equation of the straight line which bisects the obtuse angle between them. In which angle does the origin lie?

5. Through the origin three straight lines are drawn, making angles 30°, 120°, 150° with the axis of $x$, of length 4 units. Find the coordinates of their extremities and those of the centroid of the triangle of which these extremities are the vertices.

6. Find the length of the perpendicular drawn from the point (2, 2) to the straight line joining (3, 1), (7, 4).

7. Show that the straight lines joining (3, 0), (3, 4) to the point of intersection of the straight lines $19x+3y-29=0$, $13x+11y-27=0$ are at right angles.

8. Find the equation of the two straight lines which pass through the intersection of the lines $x-y+2=0$, $2x-y+3=0$, and are such that the perpendicular from the point $(-1, -1)$ on each is of length 6/5.

9. Prove that the line joining (1, 1) and (23, 4) passes through the intersection of $x+y-7=0$ and $2x-3y-6=0$.

10. Find in what ratio the straight line $3x+2y=7$ divides the distance between the points (6, 5), $(-3, 2)$.

11. The coordinates of $ABCD$ are (3, 4), (6, 3), (5, 7), and (4, 6).

Find the equation of the straight line which joins the mid-points of $AC$ and $BD$. Show that it cuts $AD$, $CB$ in $P$, $Q$ respectively such that $AP/PD = CQ/QB = 5$.

12. Find the area of the quadrilateral whose vertices are (2, 1), (4, −3), (2, −5), $(-1, 4)$, the axes being inclined at 60°.

13. Find the polar coordinates of the foot of the perpendicular from (3, 0) on $2x-\sqrt{3}y+1=0$, the line $Ox$ being the initial line.

14. A straight line moves in such a way that the sum of the intercepts it makes on the axes is 4. Find the locus of either of the points of trisection of the portion intercepted between the axes.

15. Find in their simplest form the equations of the straight lines joining the following pairs of points:—

(i) $(am^2, 2am)$, $(an^2, 2an)$;

(ii) $(am, a/m)$, $(an, a/n)$;

(iii) $(a\cos\theta, b\sin\theta)$, $(a\cos\phi, b\sin\phi)$.

16. Find the equations of the straight lines from the vertices of a triangle perpendicular to the opposite sides, given the vertices (3, 4), (1, 5), (6, 7). At what point do they intersect?

17. Find the equations of the medians of a triangle whose vertices are (3, −2), (−6, 5), (4, −7), and find the point where they intersect.

18. A straight line is drawn from the point $P(a, b)$ in a direction inclined at an angle $\alpha$ to the axis of $x$ to meet the straight line $x/a+y/b=1$ at $Q$. Find the length of $PQ$.

19. Show that the straight line $b\sin\frac{1}{2}(\alpha+\beta)\,x = a\cos\frac{1}{2}(\alpha+\beta)\,y$ bisects the distance between the points $(a\cos\alpha, b\sin\alpha)$, $(a\cos\beta, b\sin\beta)$.

20. The equations of three lines are $5x-y=7$, $y=7x-5$, $y+4x=2$; find the length intercepted on the third by the other two.

21. Find the locus of a point $(x_1, y_1)$ which is such that the straight line $xx_1+yy_1=a^2$ passes through the fixed point $(h, k)$ for all values of $(x_1, y_1)$.

22. A straight line of given length slides between two lines at right angles; find the locus of a point dividing it in a given ratio.

23. Find the coordinates of the centre of the circle circumscribing the triangle whose vertices are (2, 3), (3, 4), (6, 8).

24. If $Ax+By=1$, $ax+by=1$ are parallel, find the distance between them, given $a^2+b^2=c^2$ and $A=ka$.

25. Determine the locus of the intersections of the straight lines given by $tx/a-y/b+t=0$, $x/a+ty/b-1=0$, $t$ being a variable.

26. Two straight lines cut the axis of $x$ at distances $a$ and $-a$, and the axis of $y$ at distances $b$ and $b'$ from the origin. Find the point of intersection.

27. If a number of such pairs of lines be drawn with $a$ the same for all and $bb'=a^2$, find the locus of their intersection.

28. Find the area of $OLM$, where $LM$ are the feet of the perpendiculars let fall from the origin $O$ on $x\cos\alpha_1+y\sin\alpha_1=p_1$,
$x\cos\alpha_2+y\sin\alpha_2=p_2$.

29. Find the locus of the orthocentre of the triangle whose sides are $\lambda y=x+a\lambda^2$, $\mu y=x+a\mu^2$, $\nu y=x+a\nu^2$, for different values of $\lambda$, $\mu$, $\nu$.

30. Find the coordinates of the circumcentre of the triangle formed by the lines $x+2y+3=0$, $2x+3y+1=0$, $3x+5y+2=0$.

31. Find the coordinates of a point which is such that the line joining it to the point (7, 4) is bisected at right angles by the line $3x-y=7$.

32. Find the orthocentre of the triangle whose sides are

$$x/m+y/p-1=0,\ x/n+y/p-1=0,\ y=0.$$

33. Two straight lines have equal angular velocities $\omega$ in opposite directions about two points $O$, $A$. Find the polar equation of the locus of their intersection when $O$ is pole, $OA$ initial line, and $OA=a$, and each line is initially inclined at $\frac{1}{4}\pi$ to $OA$.

34. Find the angles which the straight lines

(i) $r\sin(\theta-\beta)=q\cos\alpha$;
(ii) $l/r=\cos\theta-\cos(\theta-\alpha)$;
(iii) $l/r=\cos\theta+e\cos(\theta+\alpha)$

make with the initial line.

35. Find the polar equation of the locus of the feet of the perpendiculars from the pole on a straight line which passes through the fixed point $(\rho, \alpha)$.

36. Find the polar coordinates of a point $P'$ which is distant $d$ from a point $Q$ $(\rho, \alpha)$ when $PQ$ makes an angle $\beta$ with the initial line.

37. Find the polar equations of the straight lines bisecting the angles between the following pairs of lines: —

(i) $\theta=\alpha$, $\theta=\beta$;
(ii) $r\cos\theta=p$, $r\sin\theta=q$;
(iii) $\sin\theta-\sin(\theta+\alpha)=0$, $\cos\theta+\cos(\theta-\beta)=0$;
(iv) $r\cos(\theta-\alpha)=p$, $r\cos\theta=p\cos\alpha$.

38. The line joining the feet of the perpendiculars from a point $P$ on the two fixed straight lines $y=0$, $y=x$ is parallel to the line joining the feet of the perpendiculars from $P$ on $3x+4y-12=0$ and $4x-3y+8=0$. Find the equation of the locus of $P$.

39. Find the area of the triangle formed by the straight lines

$$r\cos(\theta-\alpha)=p,\ r\cos\theta=p\sec\alpha,\ r\sin(\theta-\alpha)=p.$$

40. The vertices of a triangle lie on the lines $y = x\tan\theta_1$, $y = x\tan\theta_2$, $y = x\tan\theta_3$, the circumcentre being at the origin: prove that the locus of the orthocentre is the right line $x\Sigma\sin\theta - y\Sigma\cos\theta = 0$.

41. If $A$, $P$ be two points on $Ox$, and $B$, $Q$ two points on $Oy$, $A$ and $B$ being fixed, and $P$ and $Q$ varying in such manner that $1/OA - 1/OP = 1/OB - 1/OQ$, show that $PQ$ passes through a fixed point.

42. The lines
$$l(l-a)x + bly = l - a,$$
$$m(m-a)x + bmy = m - a$$
make a constant angle $\alpha$ with one another when $l$ and $m$ vary.

Show that the locus of their intersection is
$$(bx+ay)^2\tan^2\alpha = (ax-by+1)^2 - 4ax.$$

43. Find the perpendicular distance of the point $(r', \theta')$ from

(i) $r\cos(\theta-\alpha) = p$, (ii) $l/r = \cos\theta + e\cos(\theta-\alpha)$.

44. If a straight line moves so that the sum of the perpendiculars let fall on it from two fixed points (3, 4), (7, 2) is equal to three times the perpendicular from a third fixed point (1, 3), show that this line passes through one of four fixed points.

45. A straight line $AB$ makes intercepts $OA$, $OB$ on the axes of $x$ and $y$ of lengths 4, 3 respectively; if $P$ is the mid-point of $AB$, find the equation of $OP$ referred to $BA$ as axis of $x$, and the perpendicular from (7, 8) upon $BA$ as axis of $y$.

46. The vertices $A$, $B$ of a given isosceles triangle $ABC$, right-angled at $C$, move one on each of two fixed perpendicular straight lines $OA$, $OB$. $OC$ meets $AB$ at $X$, and the feet of the perpendiculars from $A$, $B$ on $OC$ are $MN$. Prove that (i) $CO \cdot CX$ is constant; (ii) $OM = NC$.

47. $P$ is any point on the straight line whose equation is $y = mx$, and through $P$ any two straight lines are drawn meeting the axis of $x$ in points $A_1$ and $A_2$, and the axis of $y$ in $B_1$, $B_2$. Prove that the point of intersection of $A_1B_2$, $A_2B_1$ lies on the straight line whose equation is $y + mx = 0$.

48. The ends $BC$ of the base of a triangle are $(a, 0)$, $(-a, 0)$.

Find the locus of the vertex when (i) $AB^2 - AC^2 = c^2$; (ii) $AB^2 + AC^2 = 2c^2$.

49. If the coordinates of two points $A$ and $B$ be (2, 3) and $(-1, 4)$, find the coordinates of the points $P$ and $Q$ in $AB$ and in $AB$ produced respectively for which $AP/PB = AQ/QB = \lambda$.

Hence find the ratio in which $AB$ is divided by $x + y = 6$.

50. If $\alpha = 0$, $\beta = 0$, $\gamma = 0$ represent in Cartesian coordinates three straight lines such that an identical relation $l\alpha + m\beta + n\gamma = 0$ holds for all values of $(x, y)$, then these lines meet in a point.

Hence show that the bisectors of the angles of a triangle meet by threes in four points.

51. Find the equations to

(i) The two straight lines through the point $(-4, 3)$ which make an angle 45° with $3x - y + 5 = 0$.

(ii) The straight lines through the origin, each of which forms with the two lines $x + y = 0$, $2x - 3y = 4$ a triangle whose area is 5.

52. Prove that (1, 2) is the centre of one of the four circles touching the three straight lines $3x+4y-16=0$, $5x-12y+6=0$, $4x+3y-15=0$, and find which of the four circles it is.

53. Given two straight lines $3x-4y+5=0$, $3x-7y-8=0$, determine the equation to the straight line through their point of intersection making the same angle with the first straight line as the second does on the opposite side of it.

54. A variable straight line through the fixed point $(f, g)$ meets the axes of coordinates in $P$, $Q$. Prove that the points of trisection of $PQ$ lie on one or other of the loci whose equations are $3xy-2gx-fy=0$, $3xy-gx-2fy=0$.

55. $AB$, $CD$ are two finite straight lines: $P$, $Q$ are their middle points. Prove that $PQ$ divides $AC$ and $BD$ in the same ratio.

56. Find the conditions that the straight line joining the origin to the intersection of the straight lines $ax+by+c=0$, $a'x+b'y+c'=0$ should bisect the angle between them.

57. $ABCD$ is a rhombus, and the polar coordinates of $ABC$ are $(4, \frac{1}{3}\pi)$, $(\sqrt{3}, \frac{1}{2}\pi)$, $(4, \frac{2}{3}\pi)$. Find the coordinates of the remaining corner, and the equations of the sides and diagonals.

58. Find the locus of the intersection of two straight lines which pass through $(a, 0)$, $(-a, 0)$ respectively and include an angle of 45°.

59. Prove that if the three straight lines

$$ax\sec\theta - by\operatorname{cosec}\theta = c^2,$$
$$ax\sec\phi - by\operatorname{cosec}\phi = c^2,$$
$$ax\sec\psi - by\operatorname{cosec}\psi = c^2$$

are concurrent, then $\sin(\theta+\psi)+\sin(\phi+\theta)+\sin(\psi+\phi)=0$.

60. The equations of two parallel straight lines are $4x+3y=12$, $4x+3y=3$; obtain the equations to the straight lines which pass through the point $(-2, -7)$ and have a length 3 intercepted on them between the parallel straight lines.

61. Find the coordinates of that point on the straight line $2x-y-5=0$ the sums of whose distances from the points (19, 13) and (9, 3) is least.

62. The vertices of a triangle lie on three fixed concurrent straight lines, and two sides pass each through a fixed point: prove that the third side passes through a fixed point.

63. Show that the lines

$$\left.\begin{aligned} 4x+3y-25&=0\\ 3x-2y+11&=0\end{aligned}\right\} \text{ and } \left.\begin{aligned} 2x-7y+47&=0\\ 10x-y-3&=0\end{aligned}\right\}$$

are concurrent, and find the anharmonic ratio of the pencil they form.

64. A triangle is formed by the axis of $x$ and by the straight lines whose equations are $\frac{1}{4}x+\frac{1}{10}y=1$ and $\frac{1}{3}x-\frac{1}{10}y+1=0$; find the equation of the locus of the centre of the rectangle inscribed in the triangle and having one side on the axis of $x$.

65. $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are three points $A$, $B$, $C$, and are such that the straight line $BC$ is $xx_1+yy_1=a^2$, and $CA$ is $xx_2+yy_2=a^2$. Prove that the equation of $AB$ is $xx_3+yy_3=a^2$.

66. What curve do all lines of the form $a\lambda^2 - \lambda y + x = 0$ touch?

67. What curve do all lines of the form $\lambda^2 x - a\lambda + y = 0$ touch?

68. What curve do all lines of the form $(x\cos\lambda)/a + (y\sin\lambda)/b = 1$ touch?

69. Find the envelope of a straight line $PQ$ which meets the axis of $y$ in $Q$, and is always perpendicular to $SQ$, where $S$ is the point $(a, 0)$.

70. $ABC$ is a triangle, $D, E, F$ the feet of the perpendiculars from $A, B, C$ respectively on any straight line. Prove algebraically that the perpendicular from $D, E, F$ on $BC, CA, AB$ respectively meet in a point.

71. The sides of a triangle $ABC$ are $u = 0$, $v = 0$, $w = 0$; find the equation of the straight line joining $A$ to the intersection of $lu + mv = 0$, $lu + nw = 0$, and the equation of the harmonic conjugate of this line with respect to $v = 0$ and $w = 0$.

72. Find the equation of the straight line which passes through the intersection of the two pairs of lines

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} u + v + w &= 0 \\ u - v + 2w &= 0 \end{aligned}\right\},$$

where $u$, $v$, $w$ are abridged forms of the equations of straight lines.

73. When the axes are oblique ($\omega$) find the equations of lines through $(p, q)$ perpendicular and parallel to the axes.

74. Show that, if the point $(h, k)$ is the foot of the perpendicular drawn from the point $(x', y')$ to the straight line $lx + my + n = 0$, then

$$\frac{h - x'}{l} = \frac{k - y'}{m} = -\frac{(lx' + my' + n)}{l^2 + m^2}.$$

75. If a triangle $ABC$ remains similar to a given triangle, and if the point $A$ be fixed, and $B$ move along a straight line, find the equation of the locus of $C$ (polars).

76. Find the lengths of the sides of the triangle formed by the lines

$$x\cos\alpha + y\sin\alpha = p, \quad x\cos\beta + y\sin\beta = q, \quad x\cos\gamma + y\sin\gamma = r.$$

Prove that the area is

$$\frac{\{p\sin(\beta-\gamma) + q\sin(\gamma-\alpha) + r\sin(\alpha-\beta)\}^2}{2\sin(\beta-\gamma)\cdot\sin(\gamma-\alpha)\cdot\sin(\alpha-\beta)}.$$

77. $A$ and $B$ are fixed points, $LM$ a fixed straight line. Points $P$, $Q$ are taken in $LM$ such that $PQ$ is of constant length. $AP$, $BQ$ meet in $R$. Find the locus of $R$ as $PQ$ moves along $LM$.

78. $P$, $Q$ are two points one on each of two fixed straight lines at right angles, such that $PQ$ subtends a right angle at a fixed point.

Find (a) the locus of the middle point of $PQ$; (b) the envelope of $PQ$.

79. If the coordinates of the vertices of the triangle $ABC$ are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ respectively, and those of the triangle $A'B'C'$ $(\xi_1, \eta_1)$, $(\xi_2, \eta_2)$, $(\xi_3, \eta_3)$, show that the perpendiculars from $A$, $B$, $C$ to $B'C'$, $C'A'$, $A'B'$ are concurrent if $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ \xi_1 & \xi_2 & \xi_3 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} = 0.$

What can we infer from the symmetry of this result in the two sets of coordinates?

80. Four straight lines $a, b, c, d$ being given, show that in general one and only one straight line can be drawn meeting them respectively in points $A, B, C, D$ (in this order), so that $AB = BC = CD$.

Discuss exceptional cases.

81. $P, Q, R$ are three points in the sides $BC, CA, AB$ of the triangle $ABC$ such that

$$BP : PC = l : m ; \; CQ : QA = m : n ; \; AR : RB = n : p.$$

$AP, BQ, CR$ are joined: show that the area of the triangle formed by these lines

$$= \Delta ABC . \frac{m^2 n^2 (p-l)^2}{(mn+mp+np)(nl+mn+mp)(nl+ml+mn)}.$$

82. The triangle $ABC$ is formed by the lines

$$a_1 x + b_1 y + c_1 = 0, \; a_2 x + b_2 y + c_2 = 0, \; a_3 x + b_3 y + c_3 = 0,$$

and the lines joining the vertices to the origin meet the opposite sides in $D, E, F$. Show that the sides of the triangle $DEF$ intersect the corresponding sides of $ABC$ on the straight line

$$x (a_1/c_1 + a_2/c_2 + a_3/c_3) + y (b_1/c_1 + b_2/c_2 + b_3/c_3) + 3 = 0.$$

83. $PQ$, a bar of fixed length, slides between two bars intersecting in $O$, and from any one point in $PQ$ lines are drawn in all directions and move in rigid connexion with $PQ$: show that there is a point in each of these lines which will trace out a straight line as $PQ$ moves.

84. One side of a quadrilateral is fixed and its length is $2k$, the adjacent sides are each of length $a$, and the opposite side is of length $2c$. Prove that the equation of the locus of the middle point of the last side may be written in the form $r^2 (r^2 + b^2)^2 - 4k^2 x^2 (r^2 + b^2) = 4c^2 k^2 y^2 - 4k^4 x^2$, where $b^2 = c^2 + k^2 - a^2$, $r^2 = x^2 + y^2$, and the fixed side is taken as axis of $x$, and a perpendicular line through its mid-point as axis of $y$.

85. The triangle $AOB$ has the angle at $O$ equal to $\omega$ and $x_0, y_0$ for its orthocentre: show that the equation of $AB$ referred to $OA$, $OB$ as axes of $x$ and $y$ is $x/(x_0 + y_0 \sec \omega) + y/(y_0 + x_0 \sec \omega) = 1$, and the further end of the diameter through $O$ of the circumcircle of the triangle $OAB$ is $(x_1, y_1)$ where $(x_0 + y_0 \cos \omega)/(y_1 + x_1 \cos \omega) = (y_0 + x_0 \cos \omega)/(x_1 + y_1 \cos \omega) = \cos \omega$.

# CHAPTER III

## EQUATIONS OF HIGHER DEGREES. CHANGE OF AXES

§ 1. The equations of several straight lines can be combined into a single equation: thus

$$(3x+2y-1)\ (5x-3y+2) = 0,$$

or
$$15x^2+xy-6y^2+x+7y-2 = 0,$$

is evidently satisfied by the coordinates of any point on either of the straight lines
$$3x+2y-1 = 0,$$

and
$$5x-3y+2 = 0,$$

and conversely the coordinates of no point can satisfy the equation unless it is on one of these lines.

Any equation in the variables $x$ and $y$,

$$f(x,\ y) = 0,$$

will then represent straight lines, if, and only if, $f(x, y)$ breaks up into factors of the first degree.

A single equation may represent partly straight lines and partly some curve. If the graph of $f(x, y) = 0$ is drawn, there will be a straight line in the figure corresponding to every linear factor of $f(x, y)$.

§ 2. *A homogeneous equation of the* n$^{\text{th}}$ *degree represents* n *straight lines through the origin.*

Let
$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n = 0$$

be any homogeneous equation of the $n^{\text{th}}$ degree.

Divide the equation by $y^n$, thus

$$a_0\left(\frac{x}{y}\right)^n + a_1\left(\frac{x}{y}\right)^{n-1} + a_2\left(\frac{x}{y}\right)^{n-2} + \ldots + a_n = 0;$$

this is an equation of the $n^{\text{th}}$ degree in $\left(\frac{x}{y}\right)$ and consequently has $n$ roots, real or imaginary.

If the roots of the equation are $\alpha_1$, $\alpha_2 \ldots \alpha_n$, then the equation can be written

$$a_0\left(\frac{x}{y}-\alpha_1\right)\left(\frac{x}{y}-\alpha_2\right)\ldots\left(\frac{x}{y}-\alpha_n\right)=0;$$

hence the original equation represents the $n$ straight lines

$$x-\alpha_1 y=0,\ x-\alpha_2 y=0,\ \ldots\ x-\alpha_n y=0,$$

i. e. $n$ straight lines, real or imaginary, all passing through the origin.

**Cor. i.** If $u=0$, $v=0$ represent two straight lines, a homogeneous equation of the $n^{\text{th}}$ degree in $u$ and $v$ must represent $n$ straight lines through the point of intersection of these two straight lines.

In particular the equation

$$a_0(x-a)^n+a_1(x-a)^{n-1}(y-b)+a_2(x-a)^{n-2}(y-b)^2+\ldots+a_n(y-b)^n=0$$

represents $n$ straight lines through the point $(a, b)$.

**Cor. ii.** An important method arises from the result of this paragraph. Consider the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0; \qquad \text{(i)}$$

whatever locus it represents, the straight line

$$lx+my+n=0 \qquad \text{(ii)}$$

cuts it in two points, which may be found by solving the equations (i) and (ii). Without thus solving we can at once write down the equation of the straight lines joining the origin to these points of intersection; for the equation

$$ax^2+2hxy+by^2-2(gx+fy)\frac{(lx+my)}{n}+\frac{c(lx+my)^2}{n^2}=0 \qquad \text{(iii)}$$

is homogeneous in $x$ and $y$, and is of the second degree: it therefore represents two straight lines through the origin.

But the coordinates of any point common to the loci (i) and (ii) satisfy the equation (iii): hence this equation represents the straight lines joining the origin to the points of intersection of (i) and (ii).

The same method can be used with equations of a higher degree.

§ 3. Consider the equation

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)(a_3x+b_3y+c_3)\ldots(a_nx+b_ny+c_n)=0; \qquad \text{(i)}$$

when the factors are multiplied together the terms of the highest (i. e. $n^{\text{th}}$) degree are obtained from the product

$$(a_1x+b_1y)(a_2x+b_2y)(a_3x+b_3y)\ldots(a_nx+b_ny)=0. \qquad \text{(ii)}$$

Now $a_1x+b_1y=0$ is a straight line through the origin parallel to $a_1x+b_1y+c_1=0$. Hence the equation (ii) represents $n$ straight lines through the origin parallel to the $n$ straight lines represented by equation (i). This result is of great importance: we see that if any equation represents straight lines, the terms of the highest degree

equated to zero represent a system of straight lines parallel to them through the origin. Any question dealing only with the directions of straight lines given by a single equation can thus be at once simplified by considering parallel straight lines through the origin.

**Example.** *To find the equation of two straight lines through the point* $(2, -3)$ *parallel to the straight lines*

$$15x^2+xy-6y^2+x+y-2=0. \qquad \text{(i)}$$

The equation $15x^2+xy-6y^2=0$ represents two straight lines through the origin parallel to those represented by equation (i); hence

$$15(x-2)^2+(x-2)(y+3)-6(y+3)^2=0$$

represents straight lines through the point $(2, -3)$ parallel to (i).

§ 4. Most of the properties of equations representing straight lines can be investigated by comparing the equations with the product of linear factors, and the majority of problems on them are little more than algebraical exercises in the comparison of coefficients.

When the axes are rectangular the directions of the lines can be found by substituting $y = x \tan\theta$ in the equation to parallel straight lines through the origin; this gives an equation for $\tan\theta$, and the values of $\theta$ thus found give the angles which the straight lines make with the axis of $x$. We give some illustrations of these points.

**Ex.** i. *What is the equation of* n *straight lines through the point* $(h, k)$ *perpendicular respectively to the* n *straight lines given by the equation*

$$p_0x^n+p_1x^{n-1}y+p_2x^{n-2}y^2+\ldots+p_ny^n=0. \qquad \text{(i)}$$

Let
$$p_0x^n+p_1x^{n-1}y+p_2x^{n-2}y^2+\ldots$$
$$\equiv(a_1x+b_1y)(a_2x+b_2y)(a_3x+b_3y)\ldots(a_nx+b_ny). \qquad \text{(ii)}$$

The straight line perpendicular to $a_1x+b_1y=0$ is $a_1y-b_1x=0$; hence the equation of $n$ straight lines parallel to those required is

$$(a_1y-b_1x)(a_2y-b_2x)(a_3y-b_3x)\ldots(a_ny-b_nx)=0,$$

which is obtained from the right-hand side of (ii) by substituting $y$ for $x$, and $-x$ for $y$.

Since the right-hand side is identically equal to the left, the equation of $n$ straight lines through the origin perpendicular to the given straight lines is obtained by making the same substitution in (i); this gives

$$p_0y^n-p_1y^{n-1}x+p_2y^{n-2}x^2-\ldots+p_n(-1)^nx^n=0,$$

and straight lines through the point $(h, k)$ parallel to these are

$$p_0(y-k)^n-p_1(y-k)^{n-1}(x-h)+p_2(y-k)^{n-2}(x-h)^2+\ldots+p_n(-1)^n(x-h)^n=0.$$

**Ex.** ii. *Find the conditions that two of the straight lines*

$$ax^3 + bx^2y + cxy^2 + dy^3 = 0$$

*should be perpendicular to one another.*

The equation of the three straight lines perpendicular respectively to the given straight lines is found by substituting $y$ for $x$ and $-x$ for $y$,

i. e. $$dx^3 - cx^2y + bxy^2 - ay^3 = 0.$$

Hence the two expressions

$$ax^3 + bx^2y + cxy^2 + dy^3,\ dx^3 - cx^2y + bxy^2 - ay^3$$

have a common quadratic factor, since each of the two perpendicular straight lines becomes the other in the equation of the perpendiculars.

Add $a$ times the first to $d$ times the second, and take $a$ times the second from $d$ times the first: then the common quadratic factor is also a factor

of both $$x\{(a^2+d^2)x^2 + (ab-cd)xy + (ca+bd)y^2\}$$

and $$y\{(bd+ac)x^2 - (ab-cd)xy + (a^2+d^2)y^2\};$$

consequently the quadratic factors in these are identical, if $a^2+d^2 \neq 0$,

i. e. $$\frac{a^2+d^2}{bd+ac} = \frac{ab-cd}{-(ab-cd)} = \frac{ac+bd}{a^2+d^2},$$

i. e. $$a^2 + d^2 + bd + ac = 0.$$

The solution can also be obtained as follows:—

put $y = x\tan\theta$ in the equation, then

$$d\tan^3\theta + c\tan^2\theta + b\tan\theta + a = 0.$$

This must give two values of $\theta$ which differ by a right angle,

i. e. two values of $\tan\theta$ whose product is $-1$.

Hence if the roots of the cubic

$$dt^3 + ct^2 + bt + a = 0$$

are $t_1, t_2, t_3$ we have $t_2t_3 = -1$.

But $t_1t_2t_3 = -\frac{a}{d}$. $\therefore t_1 = \frac{a}{d}$;

and since $t_1$ is a root of the equation, we have

$$d\cdot\frac{a^3}{d^3} + c\cdot\frac{a^2}{d^2} + b\cdot\frac{a}{d} + a = 0,$$

or $a^3 + a^2c + abd + ad^2 = 0$; or $a^2 + ac + bd + d^2 = 0$,

unless $a = 0$; this special case is left for the reader's consideration.

**Ex.** iii. *Find the angles which the straight lines*

$$5y(x^2+y^2)^2 - 20y^3(x^2+y^2) + 16y^5 = 0$$

*make with the axis of* $x$.

Put $y = x\tan\theta$ in the equation, then

$$5\tan\theta(1+\tan^2\theta)^2 - 20\tan^3\theta(1+\tan^2\theta) + 16\tan^5\theta = 0.$$

Multiply throughout by $\cos^5\theta$, then

$$5\sin\theta - 20\sin^3\theta + 16\sin^5\theta = 0,$$

hence $$\sin\theta\{16\sin^4\theta - 20\sin^2\theta + 5\} = 0,$$

i.e. $$\sin\theta\{4(1-\cos 2\theta)^2 - 10(1-\cos 2\theta) + 5\} = 0;$$

$$\therefore \sin\theta\{4 - 8\cos 2\theta + 2(1+\cos 4\theta) - 10 + 10\cos 2\theta + 5\} = 0;$$

$$\therefore \sin\theta + 2\cos 2\theta\sin\theta + 2\cos 4\theta\sin\theta = 0;$$

$$\therefore \sin\theta + \sin 3\theta - \sin\theta + \sin 5\theta - \sin 3\theta = 0,$$

i.e. $$\sin 5\theta = 0,$$

hence $5\theta = 180n°$ ($n$ being an integer),
and the values of $\theta$ are 0, 36°, 72°, 144°, 288°.

N.B.—It should be carefully noted that any other value of $n$ gives a value of $\theta$ corresponding to one of these directions.

## Examples III a.

1. Represent the following loci in a figure :—
(i) $xy = 0$; (ii) $x^2 - 4y^2 = 0$; (iii) $x^2 - 7xy + 10y^2 = 0$; (iv) $x^2 - xy = 0$;
(v) $x^2 - 2xy + y^2 = 0$; (vi) $x^2 - 2\sqrt{3}xy + y^2 = 0$.

2. Show that $2x^2 + 3xy - 2y^2 - x + 3y - 1 = 0$ represents a pair of straight lines at right angles, and draw them.

3. Find one equation representing the diagonals and sides of a square referred to convenient axes.

4. The centre of a square lies at the point $(a, a)$, one of its diagonals is parallel to the axis of $x$, and each side is of length $2a$.
Find an equation representing its sides and diagonals.

5. What does the equation $x^2y^2 - a^2x^2 - a^2y^2 + a^4 = 0$ represent?

6. Draw the locus $x^2y^2 - a^2x^2 - b^2y^2 + a^2b^2 = 0$.

7. Draw the locus $(x-a)^2 - (y-b)^2 = 0$.
What is the equation of parallel straight lines through the origin?

8. Find the equation of a pair of straight lines through the origin perpendicular to the lines $x^2 + 2hxy - y^2 = 0$.
What do you conclude from your result?

9. Find the equations of pairs of straight lines through the origin making an angle $\theta$ with (a) $x + y = 0$, (b) $x - y = 0$.

10. Find the angles which the straight lines given by the following equations make with the $x$-axis. Hence write down the separate equation of each line:—
(i) $x^3 - x^2y - 3xy^2 + 3y^3 = 0$;
(ii) $x^3\sin 3\alpha - 3x^2y\cos 3\alpha - 3xy^2\sin 3\alpha + y^3\cos 3\alpha = 0$.

11. If $u = 0$, $v = 0$ represent straight lines, what does the equation $u^2 - m^2v^2 = 0$ represent?

12. Find the equation of two straight lines (a) through the origin, (b) through the point (3, 0), parallel to $x^2 - 3xy + 6y^2 = 0$.

13. Find the equation of a pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$, and passing through the point $(b, a)$.

14. What lines are represented by $x^2 - 2xy\sec\theta + y^2 = 0$? What is the angle between them?

15. If $u \equiv x\cos\alpha + y\sin\alpha - p$, $v \equiv x\cos\beta + y\sin\beta - q$, express in a single equation the bisectors of the angles between $u = 0$, $v = 0$.

16. Find the coordinates of the points where the straight lines

$$8x^2 + xy - 4y^2 - 6x - 22y - 24 = 0$$

cut the axes of coordinates and the line $x + y = 0$. Can you draw the lines from these data? If so, find their separate equations.

17. Find the condition that the two lines $ax^2 + 2hxy + by^2 = 0$ should be perpendicular.

18. Find the conditions that the equation $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$ should represent,

(a) three coincident straight lines;

(b) two coincident lines and another;

(c) three lines equally inclined to each other.

19. Find the equation of three straight lines through the origin which make angles with the axis of $x$:—

(i) $\theta$, $\theta + 60°$, $\theta + 120°$;

(ii) $\theta - 45$, $\theta$, $\theta + 45$.

20. When does $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ represent a pair of perpendicular lines and a straight line bisecting the angles between them?

21. Find the equation of the straight lines joining the origin to the points of intersection of the lines

$$2x + 3y = 1 \text{ and } 3x^2 + 2xy - y^2 - 7x - 8y + 3 = 0.$$

22. Find the condition that the straight lines joining the origin to the points common to the loci $x^2 + y^2 = a^2$ and $lx + my = 1$ should be coincident.

23. Form the equation of four straight lines which make angles $\theta$, $\theta + \frac{1}{4}\pi$, $\theta + \frac{1}{2}\pi$, $\theta + \frac{3}{4}\pi$ with the axis of $x$.

24. If $u = 0$, $v = 0$, $w = 0$ represent straight lines, and $u'$, $v'$, $w'$ are the values of $u$, $v$, $w$ when $x = h$ and $y = k$, show that

$$uvw'^2(uv' - u'v) + vwu'^2(vw' - wv') + wuv'^2(wu' - uw') = 0$$

represents the joins of $(h, k)$ to the vertices of the triangle formed by $u$, $v$, and $w$.

25. Show that the equation of any pair of perpendicular straight lines through the origin can be put in the form $x^2 - y^2 + 2hxy = 0$.

Hence show that

$$x^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4 = 0$$

represents two pairs of perpendicular straight lines if $B + D = 0$ and $E = 1$.

§ 5. The greater part of elementary algebraical geometry is occupied with the properties of the locus represented by the most general equation of the second degree, viz.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

When the left-hand side of this equation has two linear factors the locus represents two straight lines which are parallel to the straight lines through the origin

$$ax^2 + 2hxy + by^2 = 0.$$

We propose here to discuss the properties of this latter equation.

(i) *To find the nature of the lines* $ax^2+2hxy+by^2=0$.

If the equation

$$b\left(\frac{y}{x}\right)^2+2h\left(\frac{y}{x}\right)+a=0 \qquad \text{(i)}$$

is solved, and $m_1$, $m_2$ are the two values of $\frac{y}{x}$ so found, then

$$y-m_1x=0, \quad y-m_2x=0$$

are the two straight lines.

These are therefore real, coincident, or imaginary according as $m_1$, $m_2$ are real, coincident, or imaginary.

Hence, according as

$$h^2-ab \text{ is positive, zero, or negative,}$$

the lines represented by the equation are real, coincident, or imaginary.

The idea of an imaginary line has been adopted in order to preserve continuity: thus, for instance, we say that two tangents can be drawn to a circle from a point which are real, coincident, or imaginary according as the point lies outside, on, or inside the circumference. Imaginary points and lines cannot be represented in the same way as real: their existence is indicated by the algebraical consideration of geometry and has been accepted in Pure Geometry with fruitful results. They offer an explanation of many facts, and their recognition saves the necessity of a long series of exceptions. (See Chap. IV).

(ii) *To find the angle between the straight lines* $ax^2+2hxy+by^2=0$.

When the axes are rectangular, since the equation of a straight line through the origin making an angle $\theta$ with the $x$-axis is $y=x\tan\theta$, the two values of $\left(\frac{y}{x}\right)$ given by the equation are the values of $\tan\theta_1$, $\tan\theta_2$, where $\theta_1$, $\theta_2$ are the angles which the straight lines make with the $x$-axis.

Put therefore $\frac{y}{x}=\tan\theta$ in the equation and we get

$$b\tan^2\theta+2h\tan\theta+a=0.$$

Hence
$$\tan\theta_1+\tan\theta_2=-\frac{2h}{b},$$

and
$$\tan\theta_1\tan\theta_2=\frac{a}{b}.$$

The angle between the lines is $(\theta_1-\theta_2)$, hence

$$\tan(\theta_1-\theta_2)=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$

$$=\frac{\sqrt{\frac{4h^2}{b^2}-\frac{4a}{b}}}{1+\frac{a}{b}}=\frac{2\sqrt{h^2-ab}}{a+b}.$$

The reader should consider the case when $b$ is zero.

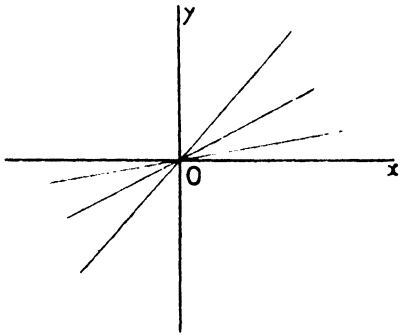
**Cor. i.** If the lines are at right angles, $\theta_1-\theta_2=\frac{1}{2}\pi$ and $\tan(\theta_1-\theta_2)$ is infinite, hence the condition that the lines should be perpendicular is $a+b=0$.

**Cor. ii.** If the lines are coincident, $\theta_1=\theta_2$, and $\tan(\theta_1-\theta_2)$ is zero, i. e. $h^2=ab$.

**Note.** When the coordinates are oblique, it may be shown that the angle between the lines is

$$\tan^{-1}\frac{2\sqrt{h^2-ab}\,.\sin\omega}{a+b-2h\cos\omega}.$$

(iii) *To find the equation of the straight lines which bisect the angles between* $ax^2+2hxy+by^2=0$.




If $\theta_1$, $\theta_2$ are the angles which these straight lines make with the $x$-axis, the bisectors make angles $\frac{1}{2}(\theta_1+\theta_2)$, $\frac{1}{2}\pi+\frac{1}{2}(\theta_1+\theta_2)$ with this axis.

Hence the equation of the bisectors is

$$\left(y-x\tan\frac{\theta_1+\theta_2}{2}\right)\left(y+x\cot\frac{\theta_1+\theta_2}{2}\right)=0,$$

i. e. $$y^2+xy\{\cot\tfrac{1}{2}(\theta_1+\theta_2)-\tan\tfrac{1}{2}(\theta_1+\theta_2)\}-x^2=0,$$

i. e. $$y^2+2xy\,.\,\frac{\cos^2\frac{1}{2}(\theta_1+\theta_2)-\sin^2\frac{1}{2}(\theta_1+\theta_2)}{2\sin\frac{1}{2}(\theta_1+\theta_2).\cos\frac{1}{2}(\theta_1+\theta_2)}-x^2=0.$$

Hence $$(x^2-y^2)\tan(\theta_1+\theta_2) = 2xy,$$

i. e. $$(x^2-y^2)(\tan\theta_1+\tan\theta_2) = 2xy(1-\tan\theta_1\tan\theta_2);$$

or $$(x^2-y^2)\left(\frac{-2h}{b}\right) = 2xy\left(1-\frac{a}{b}\right),$$

i. e. $$h(x^2-y^2) = (a-b)xy.$$

Hence the required equation is

$$hx^2-(a-b)xy-hy^2 = 0.$$

**Note i.** The condition that these lines should be real is that $(a-b)^2+4h^2$ should be positive, which is always the case.

**Note ii.** The equation satisfies the condition that the bisectors should be perpendicular to each other.

(iv) *To find the condition that the straight lines* $ax^2+2hxy+by^2=0$ *should be harmonically conjugate with respect to the straight lines*

$$a'x^2+2h'xy+b'y^2 = 0.$$

If neither $b$ nor $b'$ is zero, let

$$ax^2+2hxy+by^2 \equiv b(y-px)(y-qx)$$

and $$a'x^2+2h'xy+b'y^2 \equiv b'(y-rx)(y-sx).$$

Now the straight lines $y-px=0$, $y-qx=0$ are harmonic conjugates with respect to the straight lines $y-rx=0$, $y-sx=0$ if $$(p-r)(q-s) = -(p-s)(q-r),$$ (§ 10. II)

i.e. if $$2(pq+rs) = (p+q)(r+s).$$

Expressing this relation in terms of the coefficients in the given equations, we find

$$2\left(\frac{a}{b}+\frac{a'}{b'}\right) = \left(-\frac{2h}{b}\right)\left(-\frac{2h'}{b'}\right);$$

$$\therefore \quad ab'+a'b = 2hh'.$$

The reader should prove that this condition holds when either $b$ or $b'$ is zero.

The converse can be easily proved by reversing the steps in the work above.

The equation $ax^2+2hxy+by^2=0$ contains only two independent constants, viz. the mutual ratios $a:h:b$. If $a \neq 0$, we can therefore put

$$ax^2+2hxy+by^2 \equiv a(x+py)(x+qy),$$

since the right-hand side contains the two independent constants $p$, $q$. This is a useful comparison to make in many special cases.

**Ex.** i. *To find the product of the perpendiculars from the point* $(\xi, \eta)$ *on the straight lines* $ax^2+2hxy+by^2=0$.

Let $$ax^2+2hxy+by^2 \equiv a(x+py)(x+qy).$$

Comparing coefficients $p+q=\dfrac{2h}{a}$,

$$pq=\frac{b}{a}.$$

The product of the perpendiculars

$$\begin{aligned}
&= \frac{\xi+p\eta}{\sqrt{1+p^2}} \cdot \frac{\xi+q\eta}{\sqrt{1+q^2}} \\
&= \frac{\xi^2+(p+q)\xi\eta+pq\eta^2}{\sqrt{1+p^2+q^2+p^2q^2}} \\
&= \frac{a\xi^2+a(p+q)\xi\eta+apq\eta^2}{\sqrt{\{a^2+a^2(p+q)^2-2a^2pq+a^2p^2q^2\}}} \\
&= \frac{a\xi^2+2h\xi\eta+b\eta^2}{\sqrt{\{a^2+4h^2-2ab+b^2\}}} \\
&= \frac{a\xi^2+2h\xi\eta+b\eta^2}{\sqrt{\{(a-b)^2+4h^2\}}}.
\end{aligned}$$

**Ex.** ii. *To find the anharmonic ratio of the pencil formed by the two pairs of straight lines whose equations are*

$$ax^2+2hxy+by^2=0 \quad \text{and} \quad a'x^2+2h'xy+b'y^2=0.$$

Let $$ax^2+2hxy+by^2 \equiv a(x+py)(x+qy),$$
$$a'x^2+2h'xy+b'y^2 \equiv a'(x+ry)(x+sy),$$

then $$p+q=\frac{2h}{a},\ pq=\frac{b}{a},$$

$$s+r=\frac{2h'}{a'},\ sr=\frac{b'}{a'}.$$

The anharmonic ratio $$= \frac{(p-r)(q-s)}{(p-s)(q-r)}$$
$$= \frac{pq+rs-(ps+qr)}{pq+rs-(pr+qs)}.$$

Now $$(p+q)(s+r)=ps+qr+pr+qs,$$
$$(p-q)(s-r)=ps+qr-(pr+qs).$$

Hence $$\begin{aligned}
ps+qr &= \tfrac{1}{2}\{(p+q)(s+r)+(p-q)(s-r)\} \\
&= \tfrac{1}{2}\left\{\frac{4hh'}{aa'}+\frac{4\sqrt{h^2-ab}\,.\,\sqrt{h'^2-a'b'}}{aa'}\right\} \\
&= \frac{2}{aa'}\{hh'+\sqrt{h^2-ab}\,.\,\sqrt{h'^2-a'b'}\},
\end{aligned}$$

and similarly $$pr+qs=\frac{2}{aa'}\{hh'-\sqrt{h^2-ab}\,\sqrt{h'^2-a'b'}\};$$

the anharmonic ratio is therefore equal to

$$\frac{ab'+a'b-2hh'-2\sqrt{h^2-ab}\,\sqrt{h'^2-a'b'}}{ab'+a'b-2hh'+2\sqrt{h^2-ab}\,.\,\sqrt{h'^2-a'b'}}.$$

**Ex.** iii. *To find the area of the triangle whose sides are given by the equations* $ax^2+2hxy+by^2=0$, $lx+my=1$.

Let the points of intersection of $lx+my=1$ with the lines

$$ax^2+2hxy+by^2=0 \quad \text{be} \quad (x_1, y_1), (x_2, y_2);$$

then the area of the triangle, since one vertex is at the origin, is

$$\tfrac{1}{2}(x_1y_2-x_2y_1),$$

which

$$= \tfrac{1}{2}y_1y_2\left(\frac{x_1}{y_1}-\frac{x_2}{y_2}\right).$$

The values of $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$ are given by the equation

$$a\left(\frac{x}{y}\right)^2+2h\left(\frac{x}{y}\right)+b=0.$$

Hence

$$\frac{x_1}{y_1}+\frac{x_2}{y_2}=-\frac{2h}{a}, \quad \frac{x_1}{y_1}\frac{x_2}{y_2}=\frac{b}{a},$$

i.e.

$$\frac{x_1}{y_1}-\frac{x_2}{y_2}=\frac{2\sqrt{h^2-ab}}{a}.$$

Again, the values of $y_1$ and $y_2$ are given by

$$\frac{a}{l^2}(1-my)^2+\frac{2hy}{l}(1-my)+by^2=0,$$

i.e.

$$y^2(am^2-2hlm+bl^2)+2y(hl-am)+a=0;$$

hence

$$y_1y_2=\frac{a}{am^2-2hlm+bl^2}.$$

The area of the triangle is therefore

$$\frac{\sqrt{h^2-ab}}{am^2-2hlm+bl^2}.$$

## Examples III b.

1. Prove that the equation $4x^2-12xy+9y^2=0$ represents two coincident straight lines, and $4x^2-12xy+9y^2=1$ two parallel straight lines.

Find the equation of coincident lines perpendicular to them and passing through the point $(1, -1)$.

2. Find the angle between the lines $60x^2-103xy-72y^2=0$.

3. Show that the equation $x^2-2xy\cot 2\alpha-y^2=0$ represents two straight lines, and find their equations.

Draw the locus when $\alpha = 30°$.

4. Show that the two pairs of lines

$$10x^2+8xy+y^2=0 \text{ and } 5x^2-12xy+6y^2=0$$

contain the same angle.

5. Find the area of the figure enclosed by the lines $x=3$, $y=2$, $5x^2-18xy+9y^2=0$.

6. Prove that the two pairs of lines

$$ax^2+acxy+cy^2=0, \quad (3+c^{-1})x^2+xy+(3+a^{-1})y^2=0$$

have the same bisectors of the angles between them.

7. Find the length of the intercept cut off on

$$x+y-1=0 \text{ by } x^2+4xy+y^2=0.$$

8. Find the separate equations of the bisectors of the angles between

$$(a+8)x^2+6xy+ay^2=0.$$

9. Prove that $x-y=0$ bisects the angle between $4x^2-xy+4y^2=0$.

10. Show that the straight lines $(a+\lambda)x^2+2hxy+(b+\lambda)y^2=0$ have the same bisectors whatever value $\lambda$ may have.

11. Find the anharmonic ratio of the pencil formed by the two pairs of lines $3x^2-5xy+y^2=0$ and $x^2+7xy+9y^2=0$.

12. Find the condition that one of the lines $ax^2+2hxy+by^2=0$ should (a) coincide with, (b) be perpendicular to one of the lines

$$a'x^2+2h'xy+b'y^2=0.$$

13. Find the equation of a pair of straight lines which are harmonic conjugates with respect to each of the pairs

$$4x^2+5xy+y^2=0 \text{ and } 3x^2+7xy+4y^2=0.$$

14. Find the equation of a pair of straight lines which are at right angles and have the same bisectors of the angle between them as the straight lines $y-2x=0$, $y-3x=1$.

15. Show that $11y^2+16xy-x^2=0$ represents a pair of lines through the origin inclined at 30° to the line $x+2y=1$.

16. Find the condition that $x\cos\alpha+y\sin\alpha=p$ should be parallel to one of the straight lines $ax^2+2hxy+by^2=0$.

17. Find the angle between the straight lines

$$x^2+y^2=4(x\cos\theta+y\sin\theta)^2.$$

18. Show that the straight lines $a\cos^2\alpha\,x^2+abxy+b\sin^2\alpha\,y^2=0$ form for different values of $\alpha$ pairs of straight lines which are harmonic conjugates of $ax^2+2xy+by^2=0$.

19. Prove that the straight lines joining the origin to the points common to $(x-h)^2+(y-k)^2=c^2$ and $kx+hy=2hk$ will be at right angles if $h^2+k^2=c^2$.

20. Prove that $(a+2h+b)x^2+2(a-b)xy+(a-2h+b)y^2=0$ denotes a pair of straight lines each inclined at 45° to one or other of the lines given by $ax^2+2hxy+by^2=0$.

21. Find the length of the intercept cut off on

$$x\cos\alpha+y\sin\alpha=p \text{ by } ax^2+2hxy+cy^2=0.$$

Interpret the results when $h^2=ac$ or when $\alpha$ satisfies the equation $a\tan^2\alpha-2b\tan\alpha+c=0$.

22. Find the condition that the straight lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ should form a right-angled isosceles triangle.

23. Find the equation of the straight lines which are harmonic conjugates of both the pairs $ax^2+2hxy+by^2=0$, $a'x^2+2h'xy+b'y^2=0$.

24. One of the lines $ax^2+bxy+cy^2=0$ coincides with one of the lines $a'x^2+b'xy+c'y^2=0$. Show that the tangent of the angle between the other two is $\{ac'-a'c\}^2/\{aa'(bc'-b'c)+cc'(ab'-a'b)\}$.

25. Two fixed straight lines whose equation is $x^2+2xy\cot\alpha-y^2=0$ are intersected in $P$, $Q$ by a variable line $lx+my=1$.

Find the area of the triangle formed, and, if the area is constant, find the equation of the locus of the mid-point of $PQ$.

26. Find the condition that the two pairs of straight lines

$$ax^2+2hxy+by^2=0,\ a'x^2+2h'xy+b'y^2=0$$

may form a harmonic pencil (i) when lines of the same pair are conjugate, (ii) when lines of different pairs are conjugate.

27. Show that the conditions that the straight lines $ax^2-2hxy+by^2=0$ should form an equilateral triangle with $x\cos\alpha+y\sin\alpha=p$ are

$$a/(1-2\cos 2\alpha)=h/(2\sin 2\alpha)=b/(1+2\cos 2\alpha).$$

28. A straight line of constant length $2l$ has its extremities one on each of the straight lines $ax^2+2hxy+by^2=0$.

Show that the locus of its middle point is

$$(ax+hy)^2+(hx+by)^2+(ab-h^2)l^2=0.$$

29. If $x\cos\alpha+y\sin\alpha=p$ makes angles $\theta_1$, $\theta_2$ with the lines

$$ax^2+2hxy+by^2=0,$$

then the values of $\tan\theta_1+\tan\theta_2$ and $\tan\theta_1\tan\theta_2$ are

$$2.\ \frac{ht^2-(a-b)t-h}{a+2ht+bt^2} \text{ and } \frac{at^2-2ht+b}{a+2ht+bt^2}.\ (t\equiv\tan\alpha).$$

30. The diagonals of a quadrilateral are $x=c$ and $y=c$, and a pair of opposite sides are $ax^2+by^2=0$. Show that the other two sides intersect at the point $\{2bc/(b-a),\ 2ac/(a-b)\}$ and are parallel to

$$(ax+by)^2+ab(x+y)^2=0.$$

31. Find the length of the intercept on the line $y=mx+c$ made by the lines $ax^2+2hxy+by^2=0$ when the axes are oblique.

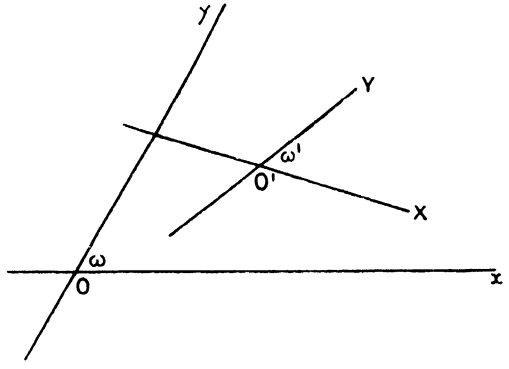
32. Show that, when the axes are oblique and inclined at an angle $\omega$, the lines $ax^2+2hxy+by^2=0$ also include an angle $\omega$, if

$$4ab\cos^2\omega-4h(a+b)\cos\omega+(a+b)^2=0.$$

## § 6. Change of Axes.

It is often convenient to change the position of the axes of coordinates: it is then necessary to find what the equation of any locus referred to the original axes becomes when referred to the new axes.

If the axes are changed from $Ox$, $Oy$ in the figure to $O'X$, $O'Y$, the transformation can be made in two stages.




(i) We can transfer to a pair of axes parallel to $Ox$ and $Oy$ drawn through the point $O'$.

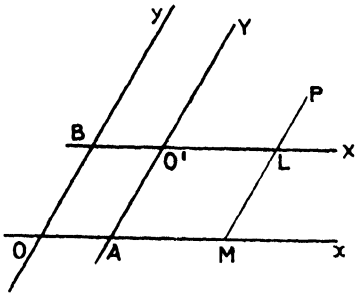
(ii) We can then change the directions of these axes to $O'X$, $O'Y$. The two stages can be examined separately: the first step is simple in all cases; the second, however, in the case of oblique coordinates, is very involved and is rarely required in practice. We need, however, for future work to show that no change of axes alters the degree of the equation of any locus: to do this it is necessary to show that, if the coordinates of a point referred to the original axes are $(x, y)$ and referred to the new axes $(X, Y)$, then the new equation is obtained from the old by some *linear* substitution such as

$$x = lX + mY + n, \quad y = l'X + m'Y + n'.$$

We shall prove this, but otherwise confine our attention to special cases which experience shows are required in the processes of analysis. This part of the work is often omitted by the student. This is a mistake, as many elementary properties become clear if the results of transformation are understood: we intend therefore to give a considerable number of easy exercises to emphasize this part of the work.

I. *To change the origin without changing the direction of the axes.*

Let the new origin be the point $O'(h, k)$. Suppose the coordinates of any point $P$ are $(x, y)$ referred to the axes $Ox$, $Oy$ and $(X, Y)$ referred to parallel axes $O'X$, $O'Y$.




Draw $PLM$ parallel to $Oy$ to meet $O'X$, $Ox$ in $L$ and $M$: and let the new axes meet the original axes in $A$, $B$.

Then
$$x = OM = OA + O'L = h + X,$$
$$y = MP = AO' + LP = k + Y.$$

Hence, if in the equation of any locus referred to the axes $Ox$, $Oy$ we substitute $(h+X)$ for $x$ and $(k+Y)$ for $y$, the equation obtained is that of the same locus referred to $O'X$, $O'Y$.

**Note.** The point $P$ has been taken in the positive quadrant for both sets of axes: the student should draw other figures and see that this method gives results true for all points.

### Examples III c.

1. What does the equation $3x+4y=7$ become when referred to axes through the point (1, 1) parallel to the original axes? Verify the result by drawing the graph of the locus.

2. Find what the equation $4x^2+8xy+3y^2=0$ becomes when referred to parallel axes through the point $(-2, 3)$.

Verify, by finding their separate equations, that the new equation still represents two straight lines.

3. Take any pair of coordinate axes and a pair of parallel axes through the point $(-4, 5)$.

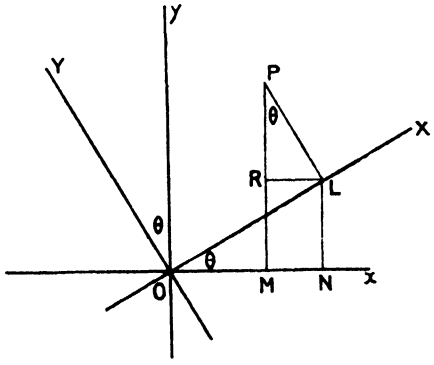
Verify, by drawing, that the coordinates $(x, y)$ of the following points become $(X, Y)$ when referred to the new axes, where $x = X-4$ and $y = Y+5$.

(i) (7, 2). (ii) (6, 5). (iii) $(-2, -8)$.

(iv) $(-6, -8)$. (v) $(-4, -7)$. (vi) $(-6, 8)$.

4. Prove that, when the origin is changed but the directions of the axes are unchanged, the coefficients of the highest powers of $x$ and $y$ in an equation are not altered, e.g. $a$ and $b$ in $ax+by+c=0$, or $a$, $h$, $b$ in
$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

II. *To change from one set of rectangular axes to another set of rectangular axes without changing the origin.*




Let the new axes make an angle $\theta$ with the original axes, and let $P$ be the point whose coordinates referred to $Ox$, $Oy$ are $(x, y)$, and referred to $OX$, $OY$ are $(X, Y)$.

Draw $PL$ perpendicular to $OX$ and $PM$ perpendicular to $Ox$, $LN$, $LR$ perpendicular and parallel to $Ox$.

Then

$$x = OM = ON - RL = OL\cos\theta - LP\sin\theta = X\cos\theta - Y\sin\theta,$$

$$y = MP = NL + RP = OL\sin\theta + LP\cos\theta = X\sin\theta + Y\cos\theta.$$

The equation of any locus referred to the new axes is thus obtained by substituting $(X\cos\theta - Y\sin\theta)$, $(X\sin\theta + Y\cos\theta)$ for $x$ and $y$ in the equation of the locus referred to the original axes.

### Examples III d.

1. What does the equation $x^2 + y^2 = a^2$ become when the axes are turned through an angle $\theta$?

2. What does the equation $x^2 - y^2 = a^2$ become when referred to axes inclined at an angle $\frac{1}{4}\pi$ to the original axes?

3. For the equation $x^2 - y^2 - 2ax + 2by + c^2 = 0$ change the origin to the point $(a, b)$ and turn the axes through an angle $\frac{1}{4}\pi$.

4. What does the expression $ax^2 + 2hxy + by^2$ become when the axes are turned through an angle $\theta$ and the origin is unchanged?

5. What does the equation $x\cos\alpha + y\sin\alpha - p = 0$ become when the axes are turned through an angle $\alpha$?

Draw a figure.

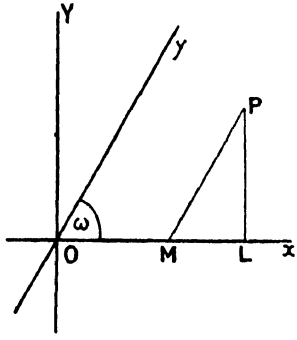
6. Find the angle between the straight lines $x^2 - 2xy\sec 2\theta + y^2 = 0$. If the axes were changed to the bisectors of the angles between these lines, what would the equation become?

7. Transform the equation $x^2+4xy+y^2+6x-3=0$ by turning the axes through 60° and changing the origin to $(1, -2)$.

What do you conclude from your result?

8. If the new axes are inclined at an angle $\theta$, taken in the positive sense, to the old, find the equations of the new axes referred to the old. Hence find the values of $X$ and $Y$ in terms of $x$, $y$, and $\theta$; discuss the signs of the expressions so found.

III. *To change from a pair of oblique axes to a convenient pair of rectangular axes, retaining the same origin.*




Retain the original axis of $x$ and take a line perpendicular to it for axis of $y$.

Let $P$ be $(x, y)$ referred to the original axes and $(X, Y)$ to the new.

Draw $PL$ perpendicular to $Ox$, and $PM$ parallel to $Oy$.

Then $x = OM = OL - ML = OL - PL\cot\omega = X - Y\cot\omega$,

$y = PM = Y\operatorname{cosec}\omega$.

Hence for $x$ and $y$ we substitute $(X - Y\cot\omega)$ and $Y\operatorname{cosec}\omega$.

N.B.—To transfer back to the original axes we must put for $X$ and $Y$ the expressions $x + y\cos\omega$, $y\sin\omega$.

### Examples III e.

1. If the axes be inclined at an angle $\omega$, find by changing to rectangular axes the conditions that the lines

(a) $\begin{matrix} Ax+By+C=0, \\ A'x+B'y+C'=0; \end{matrix}$ (b) $ax^2+2hxy+by^2=0$

should be perpendicular.

2. Referred to oblique axes inclined at an angle $\omega$ the point $P$ is $(x_1, y_1)$, and the line $AB$ $lx+my+n=0$.

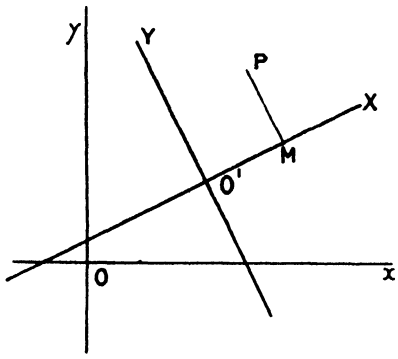
Change to rectangular axes and hence find the length of the perpendicular from $P$ on $AB$.

3. What does the expression $x^2+y^2+2xy\cos\omega$ become when the axes are changed to rectangular axes?

Interpret the result geometrically.

4. When the axes are inclined at $\omega$ the equation of a pair of straight lines is $ax^2+2hxy+by^2=0$. Change the axes to rectangular axes, form the equation of the bisectors of the angles between them, and find what the equation of the bisectors becomes when the axes are changed back to the original axes.

IV. *To change a pair of rectangular axes to another pair of rectangular axes whose equations referred to the original axes are given.*




Let the equation of the new axes referred to the old be reduced to the form, $x\cos\alpha+y\sin\alpha-p=0$, $-x\sin\alpha+y\cos\alpha-q=0$.

Let $P$ be any point whose coordinates referred to the original axes are $(x, y)$ and to the new axes $(X, Y)$. Draw $PM$ perpendicular to $O'X$, then $X = O'M =$ perpendicular from $P$ on $O'Y$

$$= -x\sin\alpha+y\cos\alpha-q, \qquad \text{(i)}$$

$Y = MP =$ perpendicular from $P$ on $O'X$

$$= x\cos\alpha+y\sin\alpha-p. \qquad \text{(ii)}$$

**Note.** In the figure $P$ is placed in the positive quadrant $XO'Y$; $X$ and $Y$ are therefore positive. In the forms chosen the substitution of the coordinates of $O$ in the equations of the lines gives $-p$, and $-q$, hence, since $P$ is on the opposite side of the lines to the origin the substitution of the coordinates of $P$ will give positive results as required. Note that $p$ and $q$ were considered positive in the figure. In any special case under consideration draw a rough figure and determine the signs.

Equations (i) and (ii) give us two linear simultaneous equations from which to find $x$ and $y$ in terms of $X$ and $Y$; the results are

$$x=(Y+p)\cos\alpha-(X+q)\sin\alpha,$$
$$y=(X+q)\cos\alpha+(Y+p)\sin\alpha.$$

**Cor.** In the case of oblique coordinates the expression for the perpendicular from any point on a straight line contains the coordinates only in the

first degree: hence, if the method just explained is used when transforming from any pair of axes to any other pair, we shall get two simultaneous equations of the first degree between the old and new coordinates, and consequently the change is effected by a substitution of the form

$$x = lX + mY + n,$$
$$y = l'X + n'Y + n'.$$

Hence, however the axes may be changed the *degree* of any equation is unaltered.

**Example.**

*What does the equation of the straight lines* $7x^2 + 4xy + 4y^2 = 0$ *become when the axes are the bisectors of the angles between them?*

The equation of the bisectors is

$$2x^2 - 3xy - 2y^2 = 0,$$

i.e.
$$2x + y = 0, \quad x - 2y = 0.$$

Now we know that the equations of two straight lines equally inclined to the $x$-axis are of the forms $y - mx = 0$, $y + mx = 0$; hence the single equation representing the two lines referred to the bisectors of the angles between them as axes contains only the $x^2$ and $y^2$ terms: suppose it is

$$ax^2 + by^2 = 0. \qquad \text{(i)}$$

The coordinates in this case are the perpendiculars from any point on the lines $2x + y = 0$ and $x - 2y = 0$, i.e. in terms of the old coordinates are

$$\frac{2x + y}{\sqrt{5}}, \quad \frac{x - 2y}{\sqrt{5}}.$$

Change equation (i) back to the old axes; it becomes

$$\tfrac{1}{5}\{a(2x + y)^2\} + \tfrac{1}{5}\{b(x - 2y)^2\} = 0,$$

or
$$a(2x + y)^2 + b(x - 2y)^2 = 0;$$

this is therefore equivalent to

$$7x^2 + 4xy + 4y^2 = 0.$$

Hence
$$4a + b = 7,$$
$$4a - 4b = 4,$$
$$a + 4b = 4,$$

which are consistent and give $a = \frac{8}{5}$, $b = \frac{3}{5}$.

Hence the required equation is

$$8x^2 + 3y^2 = 0.$$

The lines are evidently imaginary.

§ 7. *Invariants.* When any equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is transformed by any change of axes to another equation of the second degree, such as

$$a'X^2 + 2h'XY + b'Y^2 + 2g'X + 2f'Y + c' = 0,$$

certain relations between the constants are unaltered by the change; these relations we call *Invariants.*

(i) *To show that when we transform from one set of rectangular coordinates to another the quantities $a+b$ and $ab-h^2$ are unaltered.*

*Firstly.* We have seen (Ex. III c. 4) that changing the origin without changing the direction of the axes does not affect the coefficients $a$, $h$, and $b$. We have only then to deal with a change in direction of the axes.

*Secondly.* No change of axes affects the degree of the terms

$$2gx+2fy+c;$$

hence these terms do not affect the coefficients $a'$, $h'$, and $b'$.

We have then only to find what the terms $ax^2+2hxy+by^2$ become when the direction of the axes is changed. Suppose the axes turned through an angle $\theta$.

Then $ax^2+2hxy+by^2$ becomes

$$a(X\cos\theta-Y\sin\theta)^2+2h(X\cos\theta-Y\sin\theta)(X\sin\theta+Y\cos\theta)+b(X\sin\theta+Y\cos\theta)^2$$

$$=X^2[a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta]+2XY[(b-a)\sin\theta\cos\theta+h(\cos^2\theta-\sin^2\theta)]+Y^2[a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta].$$

Hence, if $ax^2+2hxy+by^2$ becomes $a'x^2+2h'xy+b'y^2$, we have

$$a'=a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta,$$
$$b'=a\sin^2\theta-2h\cos\theta\sin\theta+b\cos^2\theta,$$

i.e. $$a'+b'=a+b;$$

and further, $$h'=h(\cos^2\theta-\sin^2\theta)+(b-a)\sin\theta\cos\theta$$
$$=h\cos 2\theta+\tfrac{1}{2}(b-a)\sin 2\theta;$$
$$\therefore\quad 2h'=2h\cos 2\theta-(a-b)\sin 2\theta.$$

Also we can write

$$2a'=2h\sin 2\theta+a+b+(a-b)\cos 2\theta,$$
$$2b'=-2h\sin 2\theta+a+b-(a-b)\cos 2\theta,$$
$$4a'b'=(a+b)^2-[2h\sin 2\theta+(a-b)\cos 2\theta]^2$$
$$=(a+b)^2-4h^2\sin^2 2\theta-4h(a-b)\sin 2\theta\cos 2\theta-(a-b)^2\cos^2 2\theta$$
$$=4ab-4h^2+(a-b)^2\sin^2 2\theta-4h(a-b)\sin 2\theta\cos 2\theta+4h^2\cos^2 2\theta$$
$$=4ab-4h^2+4h'^2$$

or $$a'b'-h'^2=ab-h^2.$$

**Note.** One point needs careful notice: the proposition states and the proof implies that $a+b$, $ab-h^2$ are invariants if $a'X^2+2h'XY+b'Y^2$ is obtained from $ax^2+2hxy+by^2$ by the processes of transformation. It does not follow when we are told that

$$ax^2+2hxy+by^2+\&c.=0 \text{ and } a'x^2+2h'xy+b'y^2+\&c.$$

represent the same locus referred to different rectangular axes that these relations are true: for either equation may have been simplified by multiplication or division by some constants.

Thus, for example, $x^2-y^2=0$ and $X^2-Y^2-2\sqrt{3}XY=0$ represent the same pair of straight lines (the angle $XOx$ being $30°$), but the values of $ab-h^2$ in these two equations are $-1$ and $-4$ respectively: the fact is that when the process of transformation is completed the second equation appears in the form $\frac{1}{2}X^2-\frac{1}{2}Y^2-\sqrt{3}.XY=0$.

All we can say, then, when we know that

$$ax^2+2hxy+by^2=0 \text{ and } a'x^2+2h'xy+b'y^2=0$$

represent the same locus referred to different axes, is that

$$a+b=\lambda(a'+b'),$$
$$ab-h^2=\lambda^2(a'b'-h'^2)$$

where $\lambda$ is a constant.

In any case, however, $\frac{(a+b)^2}{ab-h^2}$ is an invariant.

### Example.

*What does the equation of the pair of lines* $7x^2+4xy+4y^2=0$ *become when referred to the bisectors of the angles between them?*

We know that the new equation is of the form

$$aX^2+bY^2=0,$$

and we suppose that this equation is the result obtained by changing the axes to the pair of bisectors.

Then
$$a+b=7+4=11,$$
$$ab=7.4-2^2=24;$$
$$\therefore \quad a=8,\ b=3 \text{ or } a=3 \text{ and } b=8,$$

and the equation is $8X^2+3Y^2=0$ or $3X^2+8Y^2=0$.

The two results correspond to two cases when a particular bisector is taken as axis of $X$ or as axis of $Y$.

The new axes being now called the axes of $x$ and $y$ the results can be written $8x^2+3y^2=0$ or $3x^2+8y^2=0$.

(ii) A proof of this invariant property due to Prof. Boole is applicable also to any change of axes. Suppose that we transform an equation from axes inclined at $\omega$ to axes inclined at $\omega'$; and that *on making the substitutions for transformation* the expression $ax^2+2hxy+by^2$ becomes $a'X^2+2h'XY+b'Y^2$: the expression $x^2+2xy\cos\omega+y^2$ represents the (distance)$^2$ of the point $(x, y)$ from the origin, and when transformed must therefore become

$$X^2+2XY\cos\omega'+Y^2.$$

We suppose that the origin is unchanged, for we have shown

that a change of origin only does not alter the coefficients $a$, $h$, $b$, and therefore such a change need not be considered.

It follows then that the equation

$$ax^2+2hxy+by^2+k(x^2+2xy\cos\omega+y^2)=0 \qquad \text{(i)}$$

will become

$$a'X^2+2h'XY+b'Y^2+k(X^2+2XY\cos\omega'+Y^2)=0. \qquad \text{(ii)}$$

Hence if the value of $k$ is such that the first equation represents a pair of coincident straight lines, i. e. if the left-hand side of the equation (i) is a perfect square, the second equation must also represent coincident lines and the left-hand side of (ii) is also a perfect square. The conditions in these cases are

$$(a+k)(b+k)=(h+k\cos\omega)^2, \qquad \text{(i)}$$

$$(a'+k)(b'+k)=(h'+k\cos\omega')^2. \qquad \text{(ii)}$$

Hence any value of $k$ which satisfies (i) also satisfies (ii); these equations are therefore identical. They may be written

$$k^2\sin^2\omega+(a+b-2h\cos\omega)k+ab-h^2=0,$$

$$k^2\sin^2\omega'+(a'+b'-2h'\cos\omega')k+a'b'-h'^2=0.$$

Hence

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}$$

$$\frac{ab-h^2}{\sin^2\omega}=\frac{a'b'-h'^2}{\sin^2\omega'}.$$

The student should work out the case of rectangular axes in the same way.

We again note the words in italics: it is supposed that the second equation is in the form given by the process of transformation without subsequent simplification.

If we merely know that $ax^2+2hxy+by^2=0$

and $$a'x^2+2h'xy+b'y^2=0$$

represent the same locus referred to axes inclined at $\omega$ and $\omega'$ respectively, all we can say is that

$$\frac{(a+b-2h\cos\omega)^2}{(ab-h^2)\sin^2\omega}=\frac{(a'+b'-2h'\cos\omega')^2}{(a'b'-h'^2)\sin^2\omega'}.$$

### Examples III f.

In the following exercises 1-6 it is understood that the general equation of the second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ is transformed by a change of origin or a change in the direction of the axes from one set of rectangular axes to another.

1. Show that it is possible by a change of origin only to remove the term which contains $x$. Find the equation which the coordinates of the new origin referred to the original axes satisfy.

Can this always be done, and in how many ways?

2. Discuss the removal under the same conditions of (a) the $y$ term, (b) the constant term.

What is the geometrical significance of the transformation?

3. When is it possible by a change of origin to remove both the $x$ and $y$ terms? Examine the case when $ab = h^2$. In how many ways can it be done? Where is the new origin?

4. Can the $y$ term and the constant be removed simultaneously? When is this impossible?

5. Show that by changing the direction of the axes the term (a) $x^2$ or (b) $y^2$ or (c) $xy$ can in general be removed. Find the equation giving the value of the angle through which the axes are turned.

In what cases is the transformation impossible?

6. If the equation can be transformed to $y^2 + 2lx + 2my = 0$, what conditions exist among the original constants of the equation?

§ 8. The most general equation of the second degree in $x$ and $y$ is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0;$$

this contains five independent constants, viz. the ratios of $a$, $h$, $b$, $g$, $f$, $c$.

The greater part of analytical geometry is concerned with the loci which this equation represents in the various special forms to which it can be reduced, and under the various conditions which may exist among the independent constants. The student will thus do well to acquire early a knowledge of the notation by which the discussion of the equation is simplified. We shall discuss in the next paragraph the properties of the equation *when it represents a pair of straight lines* and include this notation.

§ 9. If the general equation

$$f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines, then the expression $f(x, y)$ can be resolved into two linear factors. The condition for this is worked out in most text-books on Algebra: we append here the most obvious method because it applies to any system of coordinates, and to equations of a higher degree.

Let
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$
$$\equiv (px + qy + r)(p'x + q'y + r').$$

Then, comparing coefficients,

$$a = pp', \quad b = qq', \quad c = rr', \quad 2f = qr' + q'r,$$
$$2g = pr' + p'r, \quad 2h = pq' + p'q.$$

From the equations $2f = qr' + q'r$, $2g = pr' + p'r$

we obtain $$2(fp-gq) = r(pq'-p'q)$$

and $$2(fp'-gq') = r'(p'q-pq'),$$

therefore $$4(fp-gq)(fp'-gq') = -rr'(pq'-p'q)^2,$$

i.e. $$4\{pp'f^2-(pq'+p'q)fg+qq'g^2\} = -rr'\{(pq'+p'q)^2-4pp'qq'\}.$$

Hence $$af^2-2fgh+bg^2 = -c(h^2-ab),$$

i.e. $$abc+2fgh-af^2-bg^2-ch^2 = 0.$$

This condition is necessary. It includes all cases, whatever values $p, q, r, p', q', r'$ may have.

Conversely, to show that it is sufficient, i.e. if

$$abc+2fgh-af^2-bg^2-ch^2 = 0,$$

then $ax^2+2hxy+by^2+2gx+2fy+c$ can be factorized.

We can always find $p, q, p', q'$, so that

$$ax^2+2hxy+by^2 \equiv (px+qy)(p'x+q'y),$$

where $pp' = a$, $pq'+p'q = 2h$, $qq' = b$; and evidently $p$ and $q$ cannot both be zero, nor can $p'$ and $q'$.

We are given that

$$af^2-2fgh+bg^2 = -c(h^2-ab);$$

thus $$4(fp-gq)(fp'-gq') = -c(pq'-p'q)^2.$$

Now put $$c = rr',\ 2(fp-gq) = r(pq'-p'q),$$

then $$2(fp'-gq') = r'(p'q-pq').$$

Solving these equations for $f$ and $g$, we find

$$2f = qr'+q'r \quad \text{and} \quad 2g = pr'+p'r$$

provided that $p'q-pq'$ is not zero.

If $f$ and $g$ have these values it is evident that

$$ax^2+2hxy+by^2+2gx+2fy+c \equiv (px+qy+r)(p'x+q'y+r').$$

The condition is thus proved to be sufficient except when $pq'-p'q = 0$. In this case, however, we have

$$(fp-gq)(fp'-gq') = 0,$$

i.e. $$fp-gq = 0 \quad \text{or} \quad fp'-gq' = 0.$$

Either of these conditions combined with $pq'-p'q = 0$ gives us that $gx+fy$ is a multiple of $px+qy$; further, since $pq'-p'q = 0$, it is evident that $ax^2+2hxy+by^2$ is a multiple of $(px+qy)^2$. Hence $ax^2+2hxy+by^2+2gx+2fy+c$ is of the form

$$l(px+qy)^2+2m(px+qy)+c,$$

and can therefore be written in the form $l(px+qy+\alpha)(px+qy+\beta)$.

The condition is therefore sufficient in this case also. The factors equated to zero represent *parallel* straight lines.

The reader may examine the special cases when $p$ and $p'$ or $q$ and $q'$ are both zero.

§ 10. In discussion of the general equation the following notation is convenient:—

$$\begin{aligned}
u &\equiv ax^2+2hxy+by^2+2gx+2fy+c,\\
u' &\equiv ax'^2+2hx'y'+by'^2+2gx'+2fy'+c,\\
X &\equiv ax+hy+g, \qquad X' \equiv ax'+hy'+g,\\
Y &\equiv hx+by+f, \qquad Y' \equiv hx'+by'+f,\\
Z &\equiv gx+fy+c, \qquad Z' \equiv gx'+fy'+c,\\
\Delta &\equiv abc+2fgh-af^2-bg^2-ch^2,\\
A &\equiv bc-f^2, \qquad F \equiv gh-af,\\
B &\equiv ca-g^2, \qquad G \equiv hf-bg,\\
C &\equiv ab-h^2, \qquad H \equiv fg-ch.
\end{aligned}$$

The latter can be remembered in the notation of the differential calculus, thus

$$A=\frac{d\Delta}{da}, \qquad B=\frac{d\Delta}{db}, \qquad C=\frac{d\Delta}{dc},$$

$$2F=\frac{d\Delta}{df}, \qquad 2G=\frac{d\Delta}{dg}, \qquad 2H=\frac{d\Delta}{dh}.$$

It is evident that

$$u = xX+yY+Z,$$

and

$$u' = x'X'+y'Y'+Z'.$$

§ 11. Now if the equation $u = 0$ represents *a pair of non-parallel straight lines*, these must intersect at some point $(x', y')$. If, then, the origin of coordinates is changed to the point $(x', y')$, the resulting equation must represent a pair of straight lines through the origin, and is therefore of the form

$$Ax^2+2Hxy+By^2=0,$$

i.e. the constant term and the terms containing $x$, $y$ disappear.

The transformed equation is

$$a(x+x')^2+2h(x+x')(y+y')+b(y+y')^2 \\ +2g(x+x')+2f(y+y')+c=0.$$

Hence, equating the coefficients of $x$ and $y$, and the independent term to zero, we get

$$\begin{gathered}
ax'+hy'+g=0, \quad \text{i.e. } X'=0,\\
hx'+by'+f=0, \quad \text{i.e. } Y'=0,\\
ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0, \quad \text{i.e. } u'=0.
\end{gathered}$$

But, identically, $u' = x'X' + y'Y' + Z'$;

hence, since $X'$ and $Y'$ are zero, we also have $Z' = 0$,

i. e.
$$ax' + hy' + g = 0,$$
$$hx' + by' + f = 0,$$
$$gx' + fy' + c = 0.$$

Eliminating $x'$ and $y'$,

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,$$

which is the condition that $u = 0$ should represent a pair of straight lines.

Now $X' = 0$ and $Y' = 0$; but these are the conditions that the point of intersection $(x', y')$ [referred to the original axes] should be on each of the lines

$$ax + hy + g = 0,$$
$$hx + by + f = 0.$$

This point is therefore given by

$$\frac{x}{hf - bg} = \frac{y}{gh - af} = \frac{1}{ab - h^2},$$

i. e. the point of intersection of the straight lines is $\left(\frac{G}{C}, \frac{F}{C}\right)$ referred to the original axes.

We can obtain other forms by using either $X = 0$ and $Z = 0$, or $Y = 0$ and $Z = 0$: the results are identical because $\Delta = 0$

I. Since the point of intersection of the given straight lines lies on each of the lines $X = 0$, $Y = 0$, $Z = 0$, there must be some linear relation between $X$, $Y$, $Z$, such as $lX + mY + nZ = 0$.

Now we have by Algebra (or from the theory of Determinants) the identities

$$aG + hF + gC = 0,$$
$$hG + bF + fC = 0,$$
$$gG + fF + cC = \Delta = 0.$$

Multiply these equations by $x$, $y$ and 1 respectively, and add; then
$$GX + FY + CZ = 0.$$

II. Since the point of intersection of the straight lines lies on each of the lines $X = 0$, $Y = 0$, i. e. each of the given lines is a straight line through the intersection of $X = 0$, $Y = 0$, their equations must be of the forms $lX + mY = 0$, $l'X + m'Y = 0$, and consequently the equation $u = 0$ must be of the form $pX^2 + qXY + rY^2 = 0$.

We proceed to obtain the equation in this form; now

$$\begin{aligned} bX-hY &= b(ax+hy+g)-h(hx+by+f) \\ &= (ab-h^2)x-(fh-bg) \\ &= Cx-G, \\ -hX+aY &= -h(ax+hy+g)+a(hx+by+f) \\ &= (ab-h^2)y-(hg-af) \\ &= Cy-F. \end{aligned}$$

Hence $$\begin{aligned} bX^2-2hXY+aY^2 &= X(bX-hY)+Y(aY-hX) \\ &= X(Cx-G)+Y(Cy-F) \\ &= C(Xx+Yy)-GX-FY \\ &= C(u-Z)-GX-FY \\ &= Cu-(GX+FY+CZ) \\ &= Cu, \end{aligned}$$

for $GX+FY+CZ=0$ identically.

Thus the equation $u=0$ can also be written

$$bX^2-2hXY+aY^2=0.$$

**Note.** This enables us to factorize the equation of a pair of straight lines with numerical coefficients.

III. If the equation $u=0$ represents straight lines, they must be parallel to the pair of straight lines through the origin which are given by $ax^2+2hxy+by^2=0$.

The bisectors of the angles between the lines $u=0$ are therefore straight lines drawn through the point $\left(\frac{G}{C}, \frac{F}{C}\right)$ parallel to

$$h(x^2-y^2)-(a-b)xy=0,$$

i.e. they are the lines

$$h\left[\left(x-\frac{G}{C}\right)^2-\left(y-\frac{F}{C}\right)^2\right]-(a-b)\left(x-\frac{G}{C}\right)\left(y-\frac{F}{C}\right)=0.$$

or $$h[(Cx-G)^2-(Cy-F)^2]-(a-b)(Cx-G)(Cy-F)=0.$$

Using the results in (II), we can write this equation

$$h[(bX-hY)^2-(aY-hX)^2]-(a-b)(bX-hY)(aY-hX)=0,$$

which reduces, on our dividing by $ab-h^2$, to

$$h(X^2-Y^2)-(a-b)XY=0.$$

IV. *To find the condition that the straight lines $u=0$ should be parallel.*

The angles between the straight lines $u=0$ are equal to the angles between the straight lines through the origin

$$ax^2+2hxy+by^2=0.$$

When the straight lines $u = 0$ are parallel these straight lines through the origin are coincident, hence $ab = h^2$.

In this case we have $\Delta = 0$ and $C = 0$; it follows at once that $G = 0$ and $F = 0$.

Thus $bX = hY$ and $hX = aY$; consequently if $u = 0$ represents parallel straight lines,

(a) the equation $bX^2 - 2hXY + aY^2 = 0$ becomes an identity, and (b) the straight lines $X = 0$, $Y = 0$ are identical.

V. *To find the product of the lengths of the perpendiculars from any point* $(x', y')$ *to the straight lines* $u = 0$.

Let

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (px + qy + r)(p'x + q'y + r').$$

Then the product of the perpendicular from $(x', y')$ on

$$px + qy + r = 0, \quad p'x + q'y + r' = 0,$$

is

$$\frac{(px' + qy' + r)(p'x' + q'y' + r')}{\sqrt{p^2 + q^2}\ \sqrt{p'^2 + q'^2}}.$$

The numerator is equal to

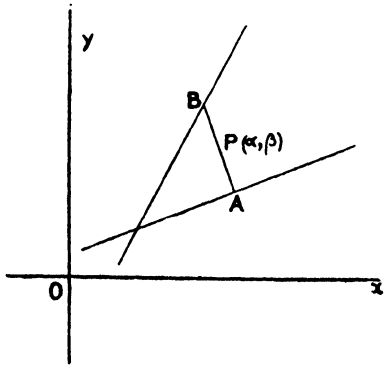
$$ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c, \text{ that is } u'.$$

We have also $pp' = a$, $qq' = b$, $pq' + p'q = 2h$.

Hence
$$\begin{aligned}(p^2 + q^2)(p'^2 + q'^2) &= p^2p'^2 + q^2q'^2 + p^2q'^2 + p'^2q^2 \\ &= p^2p'^2 + q^2q'^2 + (pq' + p'q)^2 - 2pp'qq' \\ &= (pp' - qq')^2 + (pq' + p'q)^2 = (a - b)^2 + 4h^2.\end{aligned}$$

Hence the required product $= -\dfrac{u'}{\sqrt{(a-b)^2 + 4h^2}}$.

VI. *To find the locus of the middle points of the intercepts made by the straight lines* $u = 0$, *on a system of straight lines parallel to* $\dfrac{x}{l} = \dfrac{y}{m}$, *and*




*to deduce the equation of the straight lines bisecting the angles between the straight lines* $u = 0$.

Let the point $P(\alpha, \beta)$ be the mid-point of any one intercept $AB$.

Then the equation of the straight line $AB$ which makes the intercept is $\frac{x-\alpha}{l} = \frac{y-\beta}{m}$, for it is parallel to $\frac{x}{l} = \frac{y}{m}$ and passes through $(\alpha, \beta)$.

This equation can be written

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \sqrt{\frac{(x-\alpha)^2+(y-\beta)^2}{l^2+m^2}}$$

$$= \frac{r}{\sqrt{l^2+m^2}} = kr,$$

where $r$ is the distance of any point $(x, y)$ on the straight line from the fixed point $(\alpha, \beta)$ and $k$ is a constant put for convenience instead of $\frac{1}{\sqrt{l^2+m^2}}$.

If the value of $r$ is either $BP$ or $PA$, then the point $(x, y)$ is on the given locus; its coordinates are then $(klr+\alpha, kmr+\beta)$, and, since it is on the locus,

$$a(klr+\alpha)^2 + 2h(klr+\alpha)(kmr+\beta) + b(kmr+\beta)^2 + 2g(klr+\alpha) + 2f(kmr+\beta) + c = 0.$$

Consequently this quadratic in $r$ gives the values of $PA$ and $PB$: these are to be equal in magnitude and opposite in sign; hence the coefficient of $r$ in the equation must be zero. This gives

$$2k\{la\alpha + hl\beta + hm\alpha + bm\beta + gl + fm\} = 0,$$

i. e.
$$l(a\alpha + h\beta + g) + m(h\alpha + b\beta + f) = 0.$$

Hence $(\alpha, \beta)$ lies on the line

$$l(ax+hy+g) + m(hx+by+f) = 0, \qquad \text{(i)}$$

or, with our previous notation,

$$lX + mY = 0, \qquad \text{(ii)}$$

i. e. a straight line through the intersection of $X = 0$, $Y = 0$.

Now if this equation represented one of the bisectors of the angles between the lines it would be perpendicular to the intercept $AB$, and therefore to $\frac{x}{l} = \frac{y}{m}$.

The condition for this is

$$(la+hm)m - (lh+mb)l = 0,$$

or
$$lm(a-b) + h(m^2-l^2) = 0.$$

But any point on the locus satisfies the equation

$$lX + mY = 0,$$

or
$$\frac{X}{m} = \frac{Y}{-l}.$$

Hence, in the special case considered, a point on the locus, i.e. on one of the bisectors, satisfies

$$-XY(a-b)+h(X^2-Y^2)=0,$$

or

$$\frac{X^2-Y^2}{a-b}=\frac{XY}{h}.$$

This represents a pair of *perpendicular* straight lines (for, if the equation be written in full, it will be seen that the sum of the coefficients of $x^2$ and $y^2$ is zero).

Hence the locus of the middle points of the intercepts made on straight lines parallel to $\frac{x}{l}=\frac{y}{m}$ by

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

when this equation represents a pair of straight lines, is a straight line through their point of intersection; and when $l$ and $m$ are such that the locus is perpendicular to $\frac{x}{l}=\frac{y}{m}$, the locus is one of the perpendicular straight lines

$$\frac{X^2-Y^2}{a-b}=\frac{XY}{h},$$

which equation therefore represents the two bisectors of the angles between the straight lines $u=0$.

When the straight lines $u=0$ are *parallel*, and therefore $ab=h^2$, we have seen that the straight lines $X=0$, $Y=0$ are identical. It follows from equation (i) that the straight line $X=0$ lies midway between the straight lines $u=0$.

## Examples III g.

In the following exercises $u\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ is supposed to represent a pair of straight lines, and consequently the coefficients are connected by the relation $\Delta=0$. It must be carefully noted that the results given are, as a rule, only true in this special case.

1. Prove that $Bu\equiv cX^2-2gXZ+aZ^2$.
2. Show that $G/C=A/G=H/F$, and that $F/C=H/G=B/F$.
3. Prove that $AX+HY+GZ\equiv 0$.
4. Find for what values of $\lambda$ the following equations respectively represent a pair of straight lines:—

(a) $3x^2+7xy+2y^2+8x-7y+\lambda=0$;
(b) $\lambda x^2+3xy-5y^2+x-y+4=0$;
(c) $5x^2-7xy+\lambda y^2-7x+3y-5=0$;
(d) $6x^2+10xy+3y^2+2\lambda x+8y+3=0$;
(e) $18x^2+2\lambda xy+7y^2-12x-10y+1=0$.

5. If all the coefficients in the equation $u = 0$ are known except $g$, show that the equation can represent real straight lines, provided that $CA$ is positive. Examine the case when $CA$ is zero.

6. If $u = 0$ represents two parallel straight lines, show that

$$(a+b)\,d^2 = H/h = -B/a = -A/b,$$

where $2d$ is the distance between them.

7. Show that the necessary and sufficient condition that the triangle formed by the straight lines $u = 0$ and $lx + my = 1$ may be right-angled is $(a+b)\,(al^2 + 2hlm + bm^2) = 0$.

8. Show that the equation

$$(ab-h^2)\,(ax^2 + 2hxy + by^2 + 2gx + 2fy) + af^2 + bg^2 - 2fgh = 0$$

represents a pair of straight lines, and that they form a rhombus with $ax^2 + 2hxy + by^2 = 0$, provided that $(a-b)fg + h\,(f^2 - g^2) = 0$.

9. Find the condition that $u = 0$ should represent (a) two parallel, (b) two perpendicular straight lines.

10. Find the equation of the lines $x^2 + 8\sqrt{2}xy + 5y^2 = 0$ referred to the bisectors of the angles between them as axes.

11. Find the equation of the straight lines $x^2 + xy - y^2 - 3x - 4y + 1 = 0$ referred to the bisectors of the angles between them as axes.

12. Prove that $x^2 + 9y^2 + 6xy + 4x + 12y - 5 = 0$ represents two parallel straight lines, and indicate them in a figure.

13. If $\lambda$, $\mu$ are quantities, the difference of whose reciprocals is constant, and $p$, $q$ are constants, show that $(\lambda px + \mu qy)^2 = (\lambda x^2 + \mu y^2)\,(\lambda p^2 + \mu q^2 - 1)$ represents two straight lines equally inclined to each of two fixed straight lines.

14. Show that the area of the parallelogram formed by the straight lines $u = 0$ and $ax^2 + 2hxy + by^2 = 0$ is equal to $c/(2\sqrt{h^2 - ab})$.

15. Prove that the two straight lines

$$(x^2 + y^2)\,(\cos^2\theta \sin^2\alpha + \sin^2\theta) = (x\tan\alpha - y\sin\theta)^2$$

are inclined at the same angle whatever value $\theta$ may have.

Turn the axes through an angle $\tan^{-1}(\tan\alpha \operatorname{cosec}\theta)$.

16. Show that the equation of the bisectors of the angles between the straight lines $u = 0$ can be written in the form

$$(ab-h^2)\{h\,(x^2 - y^2) - (a-b)\,xy + 2fx - 2gy\} + (a+b)\{x\,(gh - af) - y\,(fh - bg)\} - h\,(f^2 - g^2) - (a-b)\,fg = 0.$$

17. If the axes are oblique and inclined at an angle 30°, sketch the locus

$$6x^2 - 5y^2 - 7xy - 4x + 11y = 2.$$

18. Show that if the straight lines given by $ax^2 + 2hxy + by^2 = 0$ are turned through an angle $\alpha$, their equation in their new position will be

$$ax^2 + 2hxy + by^2 - 2\{(b-a)\,xy + h\,(x^2 - y^2)\}\tan\alpha + (hx^2 - 2hxy + ay^2)\tan^2\alpha = 0.$$

19. If the axes and two pairs of the five lines

$$ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5 = 0$$

contain right angles, prove that the equation of the fifth can be written

$$-x/y = f/a = (a-e)/(b-f) = (b+d)/(c+e).$$

20. Show that the coordinates of the orthocentre of the triangle formed by the straight lines $ax^2+2hxy+by^2=0$ and the straight line $lx+my=1$ are given by $x/l=y/m=(a+b)/(am^2-2hlm+bl^2)$.

21. Show that the four lines $x^4+7x^3y+15x^2y^2+7xy^3-6y^4=0$ form a harmonic pencil.

22. If the two straight lines $u=0$ are equidistant from the origin, show that $f^4-g^4=c(bf^2-ag^2)$.

23. Show that the angle between one of the lines $ax^2+2hxy+by^2=0$ and one of the lines $ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$ is equal to the angle between the other two lines of the set.

24. If $\mu$ is one of the anharmonic ratios of the pencil formed by

$$ax^2+2hxy+by^2=0,\quad a'x^2+2h'xy+b'y^2=0,$$

show that
$$\left(\frac{1+\mu}{1-\mu}\right)^2=\frac{(ab'+a'b-2hh')^2}{4(h^2-ab)(h'^2-a'b')}.$$

25. Find an equation for $\lambda$ so that $X^2+Y^2+\lambda u=0$ may represent a pair of straight lines.

26. If the same straight line occurs in each of the two pairs

$$ax^2+2hxy+by^2=0,\quad a'x^2+2h'xy+b'y^2=0,$$

and $\theta$ is the angle between the other two, then

$$\pm 2\cot\theta=aa'/(ha'-h'a)+bb'/(h'b-hb').$$

27. What is the meaning of the equation $a^{2n}x^{2n}-2a^nb^nx^ny^n+b^{2n}y^{2n}=0$ where $x$ and $y$ are coordinates with respect to oblique axes?

28. The base of a triangle passes through a fixed point $(f, g)$, and its sides are respectively bisected at right angles by the lines

$$ax^2+2hxy+by^2=0.$$

Show that the locus of the vertex is

$$(a+b)(x^2+y^2)+2h(fy+gx)+(a-b)(fx-gy)=0.$$

29. Find the condition that one of the lines $ax^2+2hxy+by^2=0$ may make an angle $\frac{1}{4}\pi$ with one of the lines $a'x^2+2h'xy+b'y^2=0$.

30. Obtain the equation to the bisectors of the angles between the lines $u=0$, in the form

$$[\{(ab-h^2)x-fh+bg\}^2-\{(ab-h^2)y-gh+af\}^2]/(a-b)$$
$$=[\{(ab-h^2)x-fh+bg\}\{(ab-h^2)y-gh+af\}]/h.$$

31. Prove that there is always one real value of $k$, for which the equation

$$ax^2+2hxy+by^2+2gx+2fy+c+k\{a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'\}=0$$

represents straight lines. In this question $\Delta$ is not zero.

32. Find the values of $k$ for which the equation

$$(lx+my+1)(l'x+m'y+1)+kxy=0$$

represents pairs of straight lines.

Give a geometrical explanation.

33. Show that, if $ax^2+2hxy+by^2=0$ and $a_1x^2+2h_1xy+b_1y^2=0$ are transformed by any change of axes, the expression $(ab_1+a_1b-2hh_1)\operatorname{cosec}^2\omega$ is unaltered.

# CHAPTER IV

## ANALYTICAL NOTATION, A REVISION AND EXTENSION

§ 1. The geometrical ideas employed in the previous chapters to obtain our formulae and equations have been those of Euclidean geometry, with the addition of the sign convention used in Trigonometry.

The coordinates of a point, $x$ and $y$, are numbers which are the measures of the distances of the point from two fixed straight lines in terms of some chosen unit of length. Conversely, if any real numbers are chosen for $x$ and $y$, we can, having chosen a unit, plot a point of which they are the coordinates. We have here implicitly assumed that, in any system of units, there is a number which is the measure of any such distance (e.g. the diagonal of a unit square), and thus the idea of number has been used in a wider sense than that of a *rational* number. It is beyond the scope of this book to dwell on this idea. The reader is referred to G. H. Hardy's *Course of Mathematics.*

We have shown that a geometrical property of a point can be expressed by a relation between its coordinates, and, conversely, that a relation between the coordinates of a point expresses the fact that it lies on some locus. Thus, if a point moves on a straight line, there is a relation of the form $lx+my+n=0$ between its coordinates. Conversely, if any set of numbers be assigned to $l$, $m$, and $n$ (excluding the case when $l$ and $m$ are both zero), any points whose coordinates satisfy the relation $lx+my+n=0$ lie on a certain straight line.

### § 2. The points of intersection of two loci.

If we wish to discuss the intersection of two loci, we obtain their equations and find sets of value of $x$ and $y$ which satisfy these equations simultaneously. Each set of values gives the coordinates of one point of intersection.

For example, the point of intersection of the two straight lines $lx+my+n=0$ and $l'x+m'y+n'=0$ is the point whose coordinates are $(mn'-m'n)/(lm'-l'm)$ and $(nl'-n'l)/(lm'-l'm)$. If, however, the straight lines are parallel, their equations are of the form

$lx+my+n=0$ and $lx+my+n'=0$; in this case the method fails, for we cannot find any set of values of $x$ and $y$ which will satisfy these equations simultaneously. This result is to be expected, for in Euclidean geometry parallel straight lines are straight lines in the same plane, which, being produced ever so far in either direction, never meet.

We proceed to investigate the intersections of a straight line with a locus whose equation is of the second degree. The nature of the locus is immaterial for our present purpose: we wish to discover whether the method of solving the equations of two loci always gives satisfactory results.

Consider then the points of intersection of the locus

$$x^2+4xy+3y^2-2x-2y+1=0 \qquad \text{(i)}$$

with the straight line

$$lx+my+n=0. \qquad \text{(ii)}$$

The equation can be solved by substituting $y=-(lx+n)/m$ or $x=-(my+n)/l$, obtained from the equation of the straight line, in the equation of the locus (i). This substitution evidently gives us in general a quadratic equation in either $x$ or $y$. Suppose that the equation in $x$ is $Lx^2+Mx+N=0$.

Three cases may occur:

(i) $L$ is not zero; the equation is quadratic.

(ii) $L$ is zero, $M$ is not zero; the equation is the simple equation $Mx+N=0$.

(iii) $L$ and $M$ are zero; the solution fails entirely.

**Note.** If $m=0$, we substitute for $x$ and get a quadratic in $y$, $L'y^2+M'y+N'=0$; exactly similar cases may then occur.

**Case i.** If the roots of the quadratic in $x$ are real and distinct, we have two distinct real values of $x$ and one value of $y$ corresponding to each satisfying both equations. There are therefore two points in which the straight line meets the locus. Let us examine special cases illustrating the possible results.

(a) The straight line $4y=9$ meets the locus in the two points whose coordinates are $(-2\frac{3}{4}, 2\frac{1}{4})$, $(-4\frac{1}{4}, 2\frac{1}{4})$.

(b) The straight line $y=0$ gives us the quadratic $x^2-2x+1=0$; this gives us only one point of intersection (1, 0).

(c) For the straight line $y=1$ the equation for $x$ is $x^2+2x+2=0$; the sets of values of $x$ and $y$ are then $(-1+\sqrt{-1}, 1)$, $(-1-\sqrt{-1}, 1)$, or, in the usual notation, $(-1+i, 1)$, $(-1-i, 1)$. Evidently it is impossible for us to plot any points whose coordinates are given by either of these sets of values.

In the Euclidean sense therefore the straight line $y = 0$ meets the locus in one point only, and the straight line $y = 1$ does not meet it at all. If we investigate similarly the intersections of the locus with the straight line $y = k$, we find that, unless $k$ lies between 0 and 2, there are two real points of intersection, and the distance between them is $2\sqrt{k^2-2k}$; this distance becomes smaller and smaller as $k$ approaches one of the values 0, 2. It is clear that $y = 0$ is the limiting position of a straight line which meets the locus in two points. Instead then of saying that the straight line $y = 0$ meets the locus in the single point (1, 0), we say that it meets it in two *coincident* points (1, 0), (1, 0).

In the case of the straight line $y = 1$ we found two distinct sets of values of $x$ and $y$ satisfying the equation, but we cannot plot any points to correspond to them. We say that this straight line meets the locus in two *imaginary points.*

Thus by adopting the ideas of 'coincident points' and 'imaginary points' we are able to say that (for all straight lines which come under Case i) a straight line meets the locus in two points which may be real and distinct, real and coincident, or imaginary and distinct.

**Note.** When the coefficients of the equations are real we obtain a quadratic equation with real coefficients; the imaginaries so found are called 'conjugate'; that is to say, if $(a+bi,\ c+di)$ are the coordinates of one point, $(a-bi,\ c-di)$ are the coordinates of the other. Thus one cannot have a real straight line meeting the locus in coincident imaginary points.

**Case ii.** If the straight line is $x+y = 2$, we substitute $y = 2-x$ in the equation of the locus and obtain $4x-9 = 0$.

This straight line then meets the locus in the single point $(2\frac{1}{4},\ -\frac{1}{4})$. This is a single point in a totally different sense to that in which $y = 0$ meets it in a single point. We get simply one point, not two coincident points.

**Case iii.** If we take the straight line $x+y = 0$, we cannot find any values of $x$ and $y$ which satisfy both equations. This straight line does not meet the locus at all. This is a totally different result to that which we found for $y = 1$; there we found sets of values for $x$ and $y$, but could not plot corresponding points; here we find no values for $x$ and $y$ at all.

It is convenient and important in Analytical Geometry to be able to assign complete generality to our results; to say that 'Every two straight lines meet at a single point', 'Every straight line meets

every locus of the second degree in two points', 'Every equation of the first degree represents a straight line', and so on.

To effect this and to include the second and third cases illustrated above, we require, in addition to the non-Euclidean ideas of 'coincident points' and 'imaginary points', the ideas of 'points at infinity' and 'the straight line at infinity'. We proceed to develop these ideas.

### § 3. Homogeneous Coordinates. The straight line.

The general equation of a straight line contains only two independent constants, but we found that in order to represent every straight line by a general equation we had to adopt the form $lx+my+n=0$; we saw further that although the equation in this form apparently contains three constants, in reality it is given by two independent ones; one of the constants, though not always *any* one, can have a purely arbitrary value, other than zero, assigned to it. The constants $l$, $m$, $n$ have no absolute values and no geometrical meaning in themselves, though the ratios of two of them to the third are perfectly determined for any particular straight line, and have precise geometrical meanings.

If we give any set of values to $l$, $m$, $n$ (except simultaneous zero values to $l$ and $m$, a restriction we shall practically remove later) we have an equation, the locus of which is a straight line; we may refer to it as the straight line $(l, m, n)$, and, since a set of values of $l$, $m$, $n$ completely fixes the straight line, we may call $l$, $m$, $n$ the coordinates of the straight line. Such coordinates have no absolute values, although their ratios have. The set of coordinates $kl$, $km$, $kn$ (where $k$ is any number) determines the same straight line as the set $l$, $m$, $n$; e.g. the equations $5x+10y-15=0$ and $x+2y-3=0$ obviously represent the same straight line. Such coordinates are said to be homogeneous.

Any relation between the coordinates $l$, $m$, $n$, expressing some property of the straight line, must be homogeneous in those coordinates. For example, the fact that the straight line passes through the point $(a, b)$ is expressed equally well by the relations $la+mb+n=0$ and $kla+kmb+kn=0$, where $k$ is any number. A non-homogeneous relation, such as $al+bm+cn^2=0$, cannot indicate any property of the straight line. The coordinates of an arbitrary straight line, $x+2y+3=0$ for instance, can be made to satisfy this relation by choosing them to be $(k, 2k, 3k)$ where $k$ is determined by the equation $a+2b+9ck=0$.

Unless we have to deal with a straight line passing through the

origin we can put $n = 1$ and take the equation of a straight line to be $lx+my+1 = 0$; and we can call it the straight line $(l, m)$. By giving arbitrary values to $l$ and $m$ we can obtain the equation of any straight line, *except a straight line through the origin.*

To obtain complete generality, we require the homogeneous system of coordinates.

§ 4. **Homogeneous Coordinates. The Point.**

It may now seem natural to inquire whether we cannot obtain complete generality for Cartesian coordinates by adopting some system of homogeneous coordinates which we may use when the ordinary coordinates appear to involve a loss of generality, as in cases (ii) and (iii) above. We shall see later that there are systems of coordinates, Areal and Trilinear, in which a point is determined uniquely by a set of numbers, the absolute values of which need not be fixed although their ratios are, and that Cartesians may be regarded as a special or rather limiting case of these. Let us take a set of three numbers which we will call $\xi$, $\eta$, $\zeta$, which have themselves no absolute values but are such that, for any particular set, the ratios of two of them to the third are fixed. How can these represent the point whose Cartesian coordinates are $(x, y)$? There is one quite simple way of effecting this. Let the ratios $\xi/\zeta$ and $\eta/\zeta$ be respectively $x$ and $y$. The point $(x, y)$ will then be defined by the set of numbers $(x\zeta, y\zeta, \zeta)$, where $\zeta$ is arbitrary. Conversely, if $\zeta$ *is not zero,* the set of numbers $(\xi, \eta, \zeta)$ define the point whose Cartesian coordinates are $\xi/\zeta$ and $\eta/\zeta$. Our equations in $x$ and $y$ now become homogeneous in $\xi$, $\eta$, $\zeta$. We write $\xi/\zeta$ for $x$ and $\eta/\zeta$ for $y$, and multiply by the power of $\zeta$ necessary to clear the equation of fractions. The general equation of the first degree then becomes $l\xi+m\eta+n\zeta = 0$, and the general equation of the second degree becomes $a\xi^2+b\eta^2+c\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta = 0$.* Any two equations of the first degree

$$l\xi+m\eta+n\zeta = 0 \text{ and } l'\xi+m'\eta+n'\zeta = 0$$

are satisfied by a common set of values of $\xi$, $\eta$, $\zeta$, viz. $mn'-m'n$, $nl'-n'l$, $lm'-l'm$.

We have seen that any set of numbers $(\xi, \eta, \zeta)$ define a point in the Euclidean sense if $\zeta$ is not zero. *We shall now say that such a set continue to define a point even when $\zeta$ is zero.* Such a point

* We may notice that this form of the equation explains the conventional distribution of the coefficients in the expression $ax^2+2hxy+by^2+2gx+2fy+c$.

is not a point in the Euclidean sense; it cannot be plotted. It may, however, be regarded as the limit of a sequence of points which can be plotted. Let us consider a straight line $l(x-a)+m(y-b)=0$. Any point on this straight line has Cartesian coordinates of the form $(a+mt,\ b-lt)$. In homogeneous coordinates we may take the coordinates of this point to be $(m+a\zeta,\ -l+b\zeta,\ \zeta)$, $\zeta$ being the same as $1/t$. Now as the point moves along the straight line further and further from the point $(a,\ b)$, $t$ increases in absolute magnitude, its sign being positive for points moving in one direction and negative for those moving in another. So that, as the point moves further and further from $(a,\ b)$, its homogeneous coordinates take the form $(m+a\zeta,\ -l+b\zeta,\ \zeta)$, where $\zeta$ is continually diminishing. These coordinates tend to the numbers $(m,\ -l,\ 0)$ as a limit. We say then that $(m,\ -l,\ 0)$ is the 'point at infinity' on the straight line $l(x-a)+m(y-b)=0$. Note two things about this 'point at infinity'. Firstly, its coordinates are independent of $a$ and $b$; secondly, we arrive at the same 'point at infinity' in whichever direction we proceed along the straight line. So that a straight line has only one 'point at infinity', and a set of parallel straight lines have the same 'point at infinity'. Thus we may now say that parallel straight lines meet at a 'point at infinity' instead of saying, with Euclidean Geometry, that they do not meet. The equations $lx+my+n=0$, $lx+my+n'=0$ have no common set of solutions, the equations $l\xi+m\eta+n\zeta=0$, $l\xi+m\eta+n'\zeta=0$ have, however, the common set $(m,\ -l,\ 0)$.

We see that all points at infinity possess the common property $\zeta=0$. Now $\zeta=0$ is a form of the equation $l\xi+m\eta+n\zeta=0$, which is the general equation to a straight line. We say then that $\zeta=0$ is the equation to a straight line. We call it the 'straight line at infinity'. We will now arrive at $\zeta=0$ as the equation of a straight line from other considerations. The equation $k(l\xi+m\eta)+n\zeta=0$ represents for all values of $k$, other than zero, a straight line parallel to the straight line $lx+my=0$. The intercepts made by this straight line on the axes of coordinates are $-n/kl$ and $-n/km$. Therefore as $k$ diminishes, the straight line recedes further and further from the origin in one sense or other according to the sign of $k$. Now as $k$ diminishes the equation $k(l\xi+m\eta)+n\zeta=0$ tends to the form $n\zeta=0$, or, what is the same thing, $\zeta=0$.

We see then that the equations of all straight lines, as these straight lines recede from the origin, tend to the same form $\zeta=0$.

There is therefore in the plane one 'line at infinity', and all 'points at infinity' lie on it.

The homogeneous coordinates of a point dividing in the ratio $\lambda : 1$, the distance between the points whose Cartesian coordinates are $(x_1, y_1)$, $(x_2, y_2)$, are $(x_1+\lambda x_2,\ y_1+\lambda y_2,\ 1+\lambda)$. So that the 'point at infinity' on the straight line joining two points may be said to be the point dividing the distance between them in the ratio $-1$. We may notice also that the 'point at infinity' on the straight line joining two points is the harmonic conjugate with respect to them of the middle point of the segment joining them.

§ 5. **Cases (ii) and (iii) rediscussed.**

We may now resume the discussion of the intersection of straight lines with the locus $x^2+4xy+3y^2-2x-2y+1=0$, in cases (ii) and (iii).

Take the straight line $x+y=2$, which previously we found to meet the locus in a single point, and use homogeneous coordinates. The equations of this line and the locus now become

$$\xi+\eta-2\zeta=0 \text{ and } \xi^2+4\xi\eta+3\eta^2-2\xi\zeta-2\eta\zeta+\zeta^2=0.$$

Substituting $\eta=2\zeta-\xi$ in the equation of the locus, we obtain $(4\xi-9\zeta)\zeta=0$. This gives us $4\xi-9\zeta=0$ or $\zeta=0$; combining these results with $\xi+\eta-2\zeta=0$, we get the two sets of values $(9, -1, 4)$ and $(1, -1, 0)$. The former is the point whose Cartesian coordinates are $(2\frac{1}{4}, -\frac{1}{4})$, which we found before; the latter is a 'point at infinity'. So that $x+y=2$ now meets the locus in two points, one of them a 'point at infinity'.

Take now the straight line $x+y=0$, which appeared to have no points of intersections with the locus. Its equation in homogeneous coordinates is $\xi+\eta=0$. Substituting $\xi=-\eta$ in the equation of the locus we obtain $\zeta^2=0$; so that now this straight line meets the locus in two coincident 'points at infinity', the point $(1, -1, 0)$ repeated.

We now have a method of making all our results general. For example, the properties which we have proved for a system of straight lines passing through an ordinary point will be true for a system of parallel straight lines; for a system of parallel straight lines is a system of straight lines passing through a 'point at infinity'. We can in future deal with the nature of a locus at an infinite distance from the origin by making our equations homogeneous, and considering the intersections of the locus with the straight line $\zeta=0$, instead of entering upon an investigation of limiting values.

§ 6. The third coordinate $\zeta$ may present some slight difficulty at first to the reader owing to the fact that no geometrical meaning in the Euclidean sense can be assigned to it. It is, however, impossible to dispense with it if we wish our scheme of Analytical Geometry to be 'Projective'. The Cartesian system of two coordinates is based on Euclidean notions, and is necessarily subject to their restrictions. It might be thought that we can avoid a third coordinate by the use of the symbol $\infty$ (infinity); that we might in fact represent the 'point at infinity' on the straight line $lx+my+n=0$ by the coordinates $(m\infty, -l\infty)$. But while the two sets of homogeneous coordinates $(l, -m, 0)$, $(-l, m, 0)$ define the same point, $(+l\infty, -m\infty)$ and $(-l\infty, +m\infty)$ do not. And we cannot choose $+\infty$ in preference to $-\infty$. Our parallel straight lines would now meet in two points, not in one. Apart from this, 'infinity' is not a number such as 1, 2, 3, . . ., and it is undesirable to regard it as if it were, which we should be doing if we employed the symbol $\infty$ to represent a Cartesian coordinate. It is also undesirable to think of the 'straight line at infinity' except as $\zeta=0$. The equation $C=0$, $C$ being read as 'constant', is sometimes employed. This is open to the very obvious objection that the equation '$C=0$' habitually stands for the statement '$C$ is zero'. It is also open to the much more serious objection that '$C=0$' does not discriminate between $\zeta=0$, $\zeta^2=0$, $\zeta^3=0$, . . ., a discrimination that it is occasionally extremely important to make.

It is most desirable that the reader should understand clearly that 'points at infinity' and the 'straight line at infinity' are conventions of Analytical Geometry. They are not realities. Parallel straight lines do not actually meet. They do not meet for all purposes of Mathematics. In the Integral Calculus and the Theory of Infinite Series there are no 'points at infinity' and no 'straight line at infinity'. They are conventions of Analytical Geometry, and, what is more, of particular types of coordinates in Analytical Geometry. There is no 'straight line at infinity' in Polar Coordinates. It is possible to dispense with 'points at infinity' and 'the straight line at infinity' altogether. We could prove without their use anything that we can prove with their use, but we should only be taking unnecessary trouble, and we should miss many of the beauties of Analytical Geometry.

We have already said that 'points at infinity' are not real points in the Euclidean sense. They are not 'imaginary points' in the sense in which we have already annexed that term to indicate

a point to which we can assign definite, but complex numbers, for its coordinates. A 'point at infinity' may also be an 'imaginary point' in the sense that its coordinates are complex numbers. The terms 'ideal points' and 'fictitious points' are sometimes used, but are not particularly satisfactory. Perhaps the simplest thing for the reader to do is to think of them as 'points at infinity' in inverted commas till he is sufficiently familiar with them to think of them as 'points' without confusion with real Euclidean points.

The notation $\xi$, $\eta$, $\zeta$ has been used for homogeneous coordinates to avoid, at their first introduction, any possible confusion with the $x$ and $y$ of the Cartesian coordinates that might arise from calling them $x$, $y$, and $z$. But $x$, $y$, and $z$ are the symbols used in English writings for general trilinear coordinates, of which our homogeneous coordinates can be considered to be a special case; and $x$, $y$, $z$ are generally employed for all types of homogeneous point coordinates. A very little experience will enable the reader to use $x$, $y$, $z$ without confusion with $x$ and $y$.

# CHAPTER V

## THE CIRCLE

§ 1. **Definition.** A *circle* is the locus of a point which moves so that its distance from a fixed point is constant: the fixed point is called the *centre* and the constant length is called the *radius*.

(A.) To show that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \text{(i)}$$

in rectangular coordinates does under certain conditions represent a circle, and to find the conditions.

Let $C\,(\alpha,\ \beta)$ be a fixed point; then any straight line through $(\alpha,\ \beta)$ is

$$\frac{x-\alpha}{\cos\theta}=\frac{y-\beta}{\sin\theta}=r, \qquad \text{(ii)}$$

where $r$ is the distance of a point $(x,\ y)$ on the line from the fixed point $C$.

Suppose that this line meets the locus represented by (i) in two points $P$ and $Q$.

Then, if $r$ is put equal to either $CP$ or $CQ$, the point $(x,\ y)$ must be on the locus (i).

Thus, if $r$ is equal to either $CP$ or $CQ$, the coordinates of the point

$$\{r\cos\theta+\alpha,\ r\sin\theta+\beta\}$$

satisfy equation (i). Hence, substituting, we find

$$\begin{aligned} r^2\{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta\} \\ +2r\{(a\alpha+h\beta+g)\cos\theta+(h\alpha+b\beta+f)\sin\theta\} \\ +a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0. \end{aligned} \qquad \text{(iii)}$$

This equation is quadratic in $r$, and its roots are the lengths of $CP$ and $CQ$.

The equation (i) can represent a circle, if we can show that the point $C\,(\alpha,\ \beta)$ can, subject to certain relations between the constants of this equation, be selected so that as the line revolves about $C$ (i. e. as $\theta$ varies):

(a) the values of $CP$, $CQ$ shall be always equal and opposite

(b) these values shall be independent of $\theta$.

The locus will then be a circle whose centre is $C$.

The first condition (a) is fulfilled if $(\alpha, \beta)$ is chosen so that

$$\left.\begin{aligned} a\alpha+h\beta+g=0 \\ h\alpha+b\beta+f=0 \end{aligned}\right\}, \qquad \text{(iv)}$$

for in this case the equation (iii) will be of the form

$$Ar^2=B.$$

Let $\alpha, \beta$ have the values given by these equations; we then have

$$CP^2=CQ^2=r^2=-\frac{a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}. \qquad \text{(v)}$$

The second condition (b) then requires that

$$a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta$$

shall be independent of $\theta$. This is so in one and one case only.

Let $$a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta=k$$
$$=k(\cos^2\theta+\sin^2\theta)$$

where $k$ is independent of $\theta$; then $(b-k)\tan^2\theta+2h\tan\theta+a-k=0$ for all values of $\tan\theta$; hence $b=k$, $h=0$, and $a=k$, or $a=b$ and $h=0$.

Hence the general equation of the second degree represents a circle when $a=b$ and $h=0$, and in this case only.

Put $b=a$ and $h=0$ in equation (iv).

Hence $$\alpha=-\frac{g}{a},\ \beta=-\frac{f}{a},$$

i. e. the centre of the circle is $\left(-\frac{g}{a}, -\frac{f}{a}\right)$.

From equation (v) we find the value of the square of the radius.

Thus, putting $b=a$ and $h=0$, and substituting the values of $\alpha$ and $\beta$ just found, $$r^2=\frac{g^2}{a^2}+\frac{f^2}{a^2}-\frac{c}{a}.$$

We thus conclude that the equation of the second degree represents a circle when it is of the form $a(x^2+y^2)+2gx+2fy+c=0$, and that its centre is the point $\left(-\frac{g}{a}, -\frac{f}{a}\right)$, and its radius is

$$\frac{\sqrt{g^2+f^2-ac}}{a}.$$

If the equation is divided throughout by $a$, it becomes

$$x^2+y^2+2\frac{g}{a}x+2\frac{f}{a}y+\frac{c}{a}=0;$$

or, writing $g_1$, $f_1$, and $c_1$ for $\frac{g}{a}$, $\frac{f}{a}$, $\frac{c}{a}$,

$$\left.\begin{aligned} x^2+y^2+2g_1x+2f_1y+c_1=0, \\ \text{which can be written} \\ (x+g_1)^2+(y+f_1)^2=g_1^2+f_1^2-c_1, \end{aligned}\right\} \quad \text{(vi)}$$

where $(-g_1, -f_1)$ is the centre and $\sqrt{g_1^2+f_1^2-c_1}$ the radius.

**Note i.** If the quantity $(g^2+f^2-ac)$ is negative the radius is imaginary, and the equation does not represent a real circle.

**Note ii.** If $r$ is the radius and the centre is at the origin the equation becomes $x^2+y^2=r^2$, which is the simplest form of the equation of a circle.

(B.) Conversely, if the centre and radius of the circle are given we can write down an equation which is always satisfied by the co-ordinates of any point on its circumference.

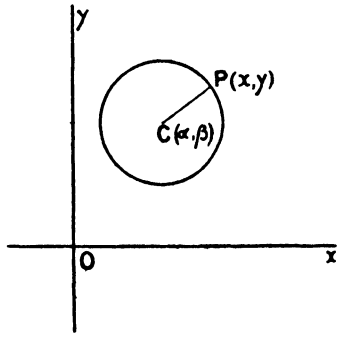
For let $C(\alpha, \beta)$ be the centre and $r$ the radius, and suppose $P(x, y)$ to be any point on the circle.

By the definition

$$CP^2=r^2,$$

i. e. $(x-\alpha)^2+(y-\beta)^2=r^2,$

which corresponds with the form (vi) found above.




This equation is evidently the general equation of a circle, for the centre and radius have been chosen in general. By comparing this equation with the general equation of the second degree, we see that $a=b$ and $h=0$.

The discussion in (A) is given for two reasons: (i) it helps to prepare the way for a more general analysis of the general equation; (ii) the second method is not always convincing to the student.

**Note.** We have used rectangular coordinates: it is rarely necessary to use oblique coordinates in work on the circle. If in (B) the axes of coordinates were inclined at an angle $\omega$, the equation would be

$$(x-\alpha)^2+(y-\beta)^2+2(x-\alpha)(y-\beta)\cos\omega=r^2,$$

so that the more general conditions for a circle referred to any axes are $a=b$ and $h=a\cos\omega$.

### Examples V a.

1. Find the centre and radius of each of the following circles:—
   (i) $x^2+y^2-2x-4y+1=0$;
   (ii) $x^2+y^2-6x=0$;
   (iii) $(x+a)^2+(y+b)^2=c^2$;
   (iv) $12x^2+12y^2-12x-8y+3=0$;

(v) $(x-a)(x-2a)+(y-b)(y-2b)=0$;
(vi) $7x^2+7y^2-3x-2y-3=0$.

2. Write down the equations of the circles whose centres and radii are
(i) (3, 4), 5 units,
(ii) $(-2, 3)$, 1 unit,
(iii) $(2\cos\theta, 2\sin\theta)$, 2 units,
(iv) $(0, 1\frac{1}{2})$, 1 unit,
(v) (0, 0), $a$.
Find the real points where they cut the line $x=y$.

3. Draw the following circles and note any special points on them:—
(i) $x^2+y^2-6x+5y=0$;
(ii) $x^2+y^2-4x=0$;
(iii) $(x-1)(x-2)+(y-3)(y-4)=0$;
(iv) $x^2+y^2-2x-4=0$;
(v) $4x^2+4y^2+12x-8y=11$.

4. Find the equation of a circle whose centre is $(-1, -3)$ and radius 2 units; when the axes are inclined at (a) 60°, (b) 120°, (c) 45°.

5. If $x^2+xy+y^2-8x-7y+6=0$ represents a circle, find the angle between the coordinate axes, and the centre and the radius of the circle.

6. In Question 3 put each of the equations (i), (ii), and (iii) into the form $(x-\alpha)^2+(y-\beta)^2=r^2$.

7. Show that the equation of the circle $x^2+y^2=r^2$ is unaltered if the axes are turned through any angle $\theta$.

8. Write down the equations of the circles given in Question 1 when the axes are changed to any pair of rectangular axes through their centres.

§ 3. The general equation of the circle

$$x^2+y^2+2gx+2fy+c=0$$

contains three independent constants, $g$, $f$, and $c$.

A circle can therefore be drawn to satisfy three conditions, if these conditions give equations from which $g$, $f$, and $c$ may be found. Thus we may be given

(a) The two coordinates of the centre and the radius.
(b) Two points on the circle and its radius.
(c) Three points on the circle.

On the other hand, we see that a circle cannot be made to satisfy more than three conditions: for example, a circle will not in general pass through four given points.

**Example i.** *To find the equation of a circle which passes through the points* (2, 3), (6, −1) *and whose radius is* 4 *units.*

The equation will be of the form

$$(x-\alpha)^2+(y-\beta)^2=16,$$

where $(\alpha, \beta)$ is its centre.

The conditions that the given points may lie on it are

$$(2-\alpha)^2+(3-\beta)^2=16,$$
$$(6-\alpha)^2+(1+\beta)^2=16;$$

or
$$\alpha^2+\beta^2-4\alpha-6\beta=3,$$
$$\alpha^2+\beta^2-12\alpha+2\beta=-21.$$

$\therefore$ by subtraction
$$8\alpha-8\beta=24,$$

or
$$\alpha-\beta=3.$$

Hence
$$\alpha-2=(\beta+1);$$

by substitution
$$(\beta+1)^2+(3-\beta)^2=16,$$

or
$$2\beta^2-4\beta-6=0,$$

i.e.
$$\beta^2-2\beta-3=0.$$

$$\therefore\ \beta=-1 \text{ or } 3 \text{ and } \alpha=2 \text{ or } 6.$$

There are consequently two circles which satisfy the given conditions, viz.:

$$(x-2)^2+(y+1)^2=16,$$
$$(x-6)^2+(y-3)^2=16.$$

**Example ii.** *To find the equation of the circle passing through the three points* $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$.

Suppose the equation of the circle is

$$x^2+y^2+2gx+2fy+c=0. \qquad \text{(i)}$$

The conditions that the three points should lie on this circle are

$$2gx_1+2fy_1+c+x_1^2+y_1^2=0, \qquad \text{(ii)}$$
$$2gx_2+2fy_2+c+x_2^2+y_2^2=0, \qquad \text{(iii)}$$
$$2gx_3+2fy_3+c+x_3^2+y_3^2=0. \qquad \text{(iv)}$$

These three equations give the values of $g$, $f$, and $c$, provided that they are independent.

We fail to obtain definite values of $g$, $f$ and $c$ if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0,$$

i.e. if the three given points lie on a straight line. Hence, in order that the circle may be finite the three given points must not be collinear.

The equation of the required circle can be expressed in determinant notation by eliminating $g$, $f$, and $c$ from equations (i), (ii), (iii), and (iv); for equation (i) is satisfied (by hypothesis) by the coordinates of all points on the required circle. Thus,

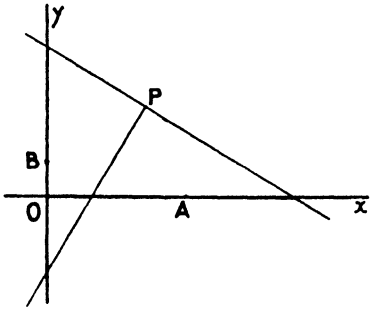
$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0$$

is the equation of the circle through the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$.

**Example iii.** *Two variable straight lines are at right angles and are such that the middle points of their intercepts on the axes are fixed.*

*Find the locus of their point of intersection.*

Let the fixed mid-points of the intercepts be $A(h, 0)$, $B(0, k)$. Suppose $P$ is the point $(\xi, \eta)$: any two perpendicular lines through this point have equations of the form




$$(y-\eta)+m(x-\xi)=0,$$
$$m(y-\eta)-(x-\xi)=0.$$

These meet the axis of $x$ in the points

$$\left\{\xi+\frac{\eta}{m}, 0\right\}, \quad \left\{\xi-m\eta, 0\right\},$$

and the axis of $y$ in the points

$$\left\{0, m\xi+\eta\right\}, \quad \left\{0, \eta-\frac{\xi}{m}\right\}.$$

Hence
$$2\xi+\frac{\eta}{m}-m\eta=2h,$$
$$2\eta+m\xi-\frac{\xi}{m}=2k;$$

thus
$$\eta\left(\frac{1}{m}-m\right)=2h-2\xi,$$
$$\xi\left(m-\frac{1}{m}\right)=2k-2\eta;$$

therefore
$$\xi(2h-2\xi)+\eta(2k-2\eta)=0,$$

and $(\xi, \eta)$ always satisfies the equation

$$x^2+y^2-hx-ky=0,$$

which is a circle whose centre is $(\frac{1}{2}h, \frac{1}{2}k)$ and radius $\frac{1}{2}(\sqrt{h^2+k^2})$, i.e. the circle on $AB$ as diameter; for its centre is the mid-point of $AB$ and its radius is $\frac{1}{2}AB$.

**Example iv.** *To find the equation of the circle circumscribing the triangle whose sides are*

$$l_1x+m_1y+1=0,$$
$$l_2x+m_2y+1=0,$$
$$l_3x+m_3y+1=0.$$

Consider the equation

$$A(l_2x+m_2y+1)(l_3x+m_3y+1)+B(l_3x+m_3y+1)(l_1x+m_1y+1)$$
$$+C(l_1x+m_1y+1)(l_2x+m_2y+1)=0,$$

where $A$, $B$, and $C$ are arbitrary constants.

The coordinates of any points which satisfy the equations of any two of the given lines also satisfy this equation: this equation therefore represents

some locus passing through the vertices of the given triangle: this locus is a circle, provided that

(i) the coefficients of $x^2$ and $y^2$ are equal,

(ii) the coefficient of $xy$ is zero.

Thus $$A(l_2l_3-m_2m_3)+B(l_3l_1-m_3m_1)+C(l_1l_2-m_1m_2)=0,$$
$$A(l_2m_3+l_3m_2)+B(l_3m_1+l_1m_3)+C(l_1m_2+l_2m_1)=0.$$

Cross multiplying to find the ratios $A:B:C$, we have $A, B, C$ proportional to

$$(l_3l_1-m_3m_1)(l_1m_2+l_2m_1)-(l_1l_2-m_1m_2)(l_3m_1+l_1m_3)=(l_1^2+m_1^2)(l_3m_2-l_2m_3),$$

and two symmetrical expressions.

Hence the equation of the circumcircle is

$$(l_1^2+m_1^2)(l_3m_2-l_2m_3)(l_2x+m_2y+1)(l_3x+m_3y+1)$$
$$+(l_2^2+m_2^2)(l_1m_3-l_3m_1)(l_3x+m_3y+1)(l_1x+m_1y+1)$$
$$+(l_3^2+m_3^2)(l_2m_1-l_1m_2)(l_1x+m_1y+1)(l_2x+m_2y+1)=0.$$

## Examples V b.

1. Find the equations of circles whose centres are $(-6, 5)$, $(3, -4)$ and which pass through the point $(0, 1)$.

2. Show that the points $(4, 3)$, $(8, -3)$, $(4\frac{1}{2}, 2\frac{1}{4})$ cannot lie on a circle.

3. Show that the circle whose centre is $(a, b)$ and which passes through the point $(0, b)$ also passes through the point $(2a, b)$.

4. Show that the point $(7, -5)$ lies on the circle $x^2+y^2-6x+4y-12=0$, and find the coordinates of the other end of the diameter through this point.

Show also that the points $\{5\cos\theta+3, 5\sin\theta-2\}$, $\{5\sin\theta+3, 5\cos\theta-2\}$ lie on this circle whatever value $\theta$ may have.

5. Find the equation of the circle which passes through the points $(1, 5)$, $(4, 6)$, $(5, 3)$. What is its radius? Where does the line $x=2$ cut it? Also the line $x-y=0$?

6. Find an equation giving the abscissae of the points of intersection of the circle $x^2+y^2+2gx+2fy+c=0$, and the $x$-axis.

Hence find the general form of the equation of a circle which passes through the points $(a, 0)$, $(-a, 0)$.

7. Find the condition that a circle represented by the general equation should meet the axis of $y$ in coincident points.

8. Find the equation of a circle which passes through the origin and cuts off lengths $a$ and $b$ from the axes.

9. Find the equation of the circle of which the join of $(a, 0)$, $(-a, 0)$ is a diameter.

10. Show that the four points $(2, 2)$, $(5, 3)$, $(6, 0)$, $(3, -1)$ lie on a circle, and find its centre and radius.

11. Prove that points whose coordinates are of the form

$$\{h+a\cos\theta, k+a\sin\theta\}$$

lie on a circle for all values of $\theta$, and find its equation, centre, and radius.

12. What is the locus of the points of intersection of the lines

$$x\cos\theta+y\sin\theta=a\sqrt{2},\quad x\sin\theta-y\cos\theta=a\sqrt{2},$$

where $\theta$ is a variable constant?

13. At what points does the line $x-3y+5=0$ cut the circle

$$x^2+y^2-4x-8y+15=0?$$

Find the length of the intercept made on the line by the circle.

14. Find the locus of a point whose distance from $(a, 0)$ is double its distance from $(-a, 0)$.

Where does the locus cut the axis of $x$?

15. The join of the points $(2, 3)$, $(-1, 2)$ subtends a right angle at $P$. Find the equation of the locus of $P$.

16. Find the equation to the circle whose centre is $(-6, -8)$ and whose radius is 5; determine whether the origin lies inside or outside the circle.

Find the coordinates of the extremities of that diameter which passes through the origin.

17. Find the equation of the circle circumscribing the triangle whose sides are

$$x+2y=0,$$
$$x-3y+1=0,$$
$$3x+y-5=0.$$

18. Find the condition that the point dividing the join of the points $(2, 5)$, $(3, -4)$ in the ratio $l:1$ should lie on the circle $x^2+y^2=9$.

Deduce the ratios into which this line is divided by the circle.

19. A point $P$ moves in a plane so that its distance from $A$, the point of intersection of $x-2y-4=0$, $7x+11y-3=0$, is a mean proportional between $OA$ and $PN$, $O$ being the origin, $PN$ the perpendicular distance of $P$ from the first line. Prove that $P$ lies on one or other of two fixed circles.

20. A circle passes through two points on the axis of $x$ whose distances from the origin are $a$, $c^2/a$, and through two points on the $y$-axis whose distances from the origin are $b$, $c^2/b$. Find its equation.

21. Find the equation of the line joining the centres of the circles which pass respectively through the points $(2, 1)$, $(3, -2)$, $(4, -3)$, and $(4, 6)$, $(4, -4)$, $(1, 5)$.

22. Find the coordinates of the centre and the radius of

$$8x^2+8y^2+24x-8y+15=0.$$

Find also the coordinates of the point on the circle furthest from the origin.

Determine the length intercepted by the circle on the line $2x-y+3=0$.

23. Write down the equations of circles which satisfy the conditions:—

(i) centre $(a, b)$, and passing through $(h, k)$,

(ii) radius $r$, and touching the $x$-axis at $(a, 0)$,

(iii) radius $r$, and touching the coordinate axes.

24. Prove analytically that if any circle cuts the axes of coordinates at $PQ$ and $P'Q'$ respectively, then $OP.OQ=OP'.OQ'$.

25. Find the equation of the circle through the points $(\alpha, 0)$, $(\beta, 0)$, $(0, \alpha)$, $(0, \beta)$.

26. Find the locus of the centres of circles which pass through the points (3, 2), (5, 4).

27. $P$ is a point in the plane of an equilateral triangle $ABC$ such that $PA^2 = PB^2 + PC^2$. Find the locus of $P$.

28. Show that a circle can be drawn to pass through the origin and the points (1, −3), (2, −4), (5, 5), and find its equation.

29. Find the condition that the circle circumscribing the triangle whose sides are $x = a$, $y = b$, $lx + my = 1$ should pass through the origin.

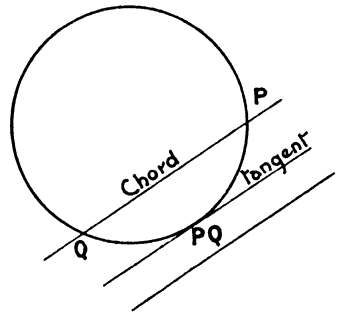
30. $ABCD$ are four concyclic points and $P$ is any other point; prove that

$$PA^2 . \Delta BCD + PC^2 . \Delta ABD = PB^2 . \Delta ACD + PD^2 . \Delta ABC.$$

§ 4. Every straight line meets a circle in two points whose coordinates are obtained by solving simultaneously the equations of the line and circle: these two points may be (i) real and distinct. (ii) coincident, (iii) imaginary.

**Definitions.**

(i) When a straight line meets a curve in two real and distinct points, the join of these points is called a *chord.*




(ii) In the particular case when the points of intersection are coincident, the line joining them is called a *tangent* to the curve at the point. This point is the '*point of contact*'.

(iii) A straight line through the point of contact perpendicular to the tangent is called the *normal* to the curve at this point. Such a straight line cuts the curve at right angles or orthogonally.

(iv) If tangents are drawn at two points $P$ and $Q$ meeting at $T$, then $PQ$ is called the '*chord of contact*' of the tangents from $T$ to the curve.

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points on a circle.

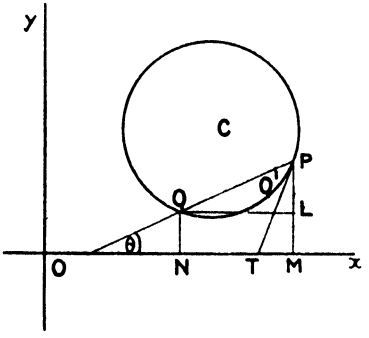
Draw $PM$, $QN$ perpendicular to $Ox$ and $QL$ perpendicular to $PM$.

Then $PL = y_1 - y_2$,

$QL = x_1 - x_2$;

and if $PQ$ makes an angle $\theta$ with the axis of $x$

$$\tan\theta = \frac{y_1 - y_2}{x_1 - x_2}.$$



Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Then, since the two points $P$, $Q$ lie on it,

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0,$$
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0.$$

Subtracting, we get

$$(x_1-x_2)(x_1+x_2+2g)+(y_1-y_2)(y_1+y_2+2f)=0.$$

Hence the direction of the chord can be otherwise expressed, thus

$$\tan\theta=\frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2+2g}{y_1+y_2+2f}. \qquad \text{(i)}$$

Now as the point $Q$ approaches $P$, i.e. as $x_2$ and $y_2$ approach the values $x_1$ and $y_1$ respectively, the fraction $(y_1-y_2)/(x_1-x_2)$ approaches a limit. The chord $PQ$ then becomes ultimately, by our definition, the tangent at $P$. The value of this limit can be found by putting $x_2=x_1$ and $y_2=y_1$ in (i); thus we see that the direction of the tangent at $P(x_1, y_1)$ is given by

$$\tan\theta=-\frac{x_1+g}{y_1+f}. \qquad \text{(ii)}$$

The equation of the chord $PQ$ can be written down at once as the equation of the straight line joining the points $(x_1, y_1)$, $(x_2, y_2)$; since, however, the tangent is defined as a special case of a chord, it is a logical demand that the equation of the chord joining two points should be found in such a form that the equation of a tangent can be deduced by making the points coincident *algebraically*.

*Equation of the chord joining two points* $(x_1, y_1)$, $(x_2, y_2)$ *on the circle* $x^2+y^2+2gx+2fy+c=0$.

We can now obtain the equation of the chord in a form symmetrical with respect to the two points, for the mid-point of the chord is $\{\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2)\}$, and consequently the chord is a straight line passing through this point whose direction is given by equation (i) above: thus the equation is

$$\{x-\tfrac{1}{2}(x_1+x_2)\}(x_1+x_2+2g)+\{y-\tfrac{1}{2}(y_1+y_2)\}(y_1+y_2+2f)\}=0. \qquad \text{(iii)}$$

Now let us call the coordinates of the middle point $A$ of the chord $PQ$ $(\xi, \eta)$, then $\xi=\frac{1}{2}(x_1+x_2)$, $\eta=\frac{1}{2}(y_1+y_2)$; we then see that the equation of the chord $PQ$ can be expressed in terms of its mid-point; for substituting in equation (iii) we obtain

$$(x-\xi)(\xi+g)+(y-\eta)(\eta+f)=0 \qquad \text{(iv)}$$

as the equation of the chord whose mid-point is $(\xi, \eta)$.

We can discover at once from this equation several properties of a chord.

(i) The centre of the circle is $C(-g, -f)$; hence the equation of the line joining the centre to the mid-point $A(\xi, \eta)$ of the chord is

$$\frac{x-\xi}{\xi+g} = \frac{y-\eta}{\eta+f}.$$

Now this line is perpendicular to the chord (iv).

*Hence the straight line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.*

(ii) Suppose the chord to be one of a system of chords parallel to some straight line

$$y = mx. \qquad \text{(i)}$$

The straight line

$$(x-\xi)(\xi+g)+(y-\eta)(\eta+f) = 0$$

is parallel to (i), so that

$$(\xi+g)+m(\eta+f) = 0.$$

Hence the mid-point of any such chord lies on the line

$$(x+g)+m(y+f) = 0,$$

which passes through the centre $(-g, -f)$ and is perpendicular to the line $y = mx$.

*Hence the middle points of all parallel chords of a circle lie on a straight line through the centre of the circle perpendicular to the chords.*

(iii) Again, if the chord

$$(x-\xi)(\xi+g)+(y-\eta)(\eta+f) = 0$$

passes through a fixed point $(h, k)$, we have

$$(h-\xi)(\xi+g)+(k-\eta)(\eta+f) = 0,$$

or

$$\xi^2+\eta^2-\xi(h-g)-\eta(k-f)-hg-fk = 0.$$

Hence the mid-point of any chord which passes through the point $(h, k)$ lies on the locus

$$x^2+y^2-x(h-g)-y(k-f)-hg-fk = 0.$$

This is a circle whose centre is the point $\frac{1}{2}(h-g)$, $\frac{1}{2}(k-f)$, i. e. the point midway between the given fixed point $(h, k)$ and the centre of the given circle $(-g, -f)$: that the two points $(h, k)$, $(-g, -f)$ lie on this circle is evident from the form

$$(h-x)(x+g)+(k-y)(y+f) = 0.$$

Hence '*the locus of the middle points of all chords of a circle which pass*

*through a fixed point is a circle on the straight line joining the fixed point and the centre of the given circle as diameter'.*

(iv) The constant term $c$ of the equation of the circle does not occur in the equation of a chord whose mid-point $(\xi, \eta)$ is given : we conclude from this that *'all concentric circles have the same chord corresponding to a given mid-point'.*

**Note i.** The reader should prove these and similar results in the simple case when the origin is taken as centre of the circle. The equation of the circle is then $x^2+y^2=r^2$,

and the equation of the chord whose mid-point is $(\xi, \eta)$ is

$$(x-\xi)\,\xi+(y-\eta)\,\eta=0,$$

or $$x\xi+y\eta=\xi^2+\eta^2.$$

**Note ii.** If the equation of the circle is given in the form

$$(x-\alpha)^2+(y-\beta)^2=r^2,$$

the equation of the chord whose mid-point is $(\xi, \eta)$ is

$$(x-\xi)\,(\alpha-\xi)+(y-\eta)\,(\beta-\eta)=0.$$

*The equation of the tangent at the point $(x_1, y_1)$ to the circle*

$$x^2+y^2+2gx+2fy+c=0.$$

The equation of the chord joining the points $(x_1, y_1)$, $(x_2, y_2)$ on the circle is

$$\left(x-\frac{x_1+x_2}{2}\right)(x_1+x_2+2g)+\left(y-\frac{y_1+y_2}{2}\right)(y_1+y_2+2f)=0.$$

Hence, putting $x_2=x_1$ and $y_2=y_1$, the equation of the tangent becomes

$$(x-x_1)\,(x_1+g)+(y-y_1)\,(y_1+f)=0,$$

or $$x\,(x_1+g)+y\,(y_1+f)=x_1^2+y_1^2+gx_1+fy_1.$$

But $x_1^2+y_1^2+2gx_1+2fy_1+c=0$ since the point lies on the circle ; hence the equation can be written

$$x\,(x_1+g)+y\,(y_1+f)+gx_1+fy_1+c=0.$$

**Note i.** For the circle $x^2+y^2=r^2$ this becomes $xx_1+yy_1=r^2$.

**Note ii.** In connexion with the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

we adopted in Chapter III the notation

$$X\equiv ax+hy+g, \quad X'\equiv ax'+hy'+g,$$
$$Y\equiv hx+by+f, \quad Y'\equiv hx'+by'+f,$$
$$Z\equiv gx+fy+c, \quad Z'\equiv gx'+fy'+c.$$

For the circle $x^2+y^2+2gx+2fy+c=0$

we have $a=b=1$ and $h=0$;

so that the equation of the tangent at the point $(x', y')$ can be written

$$xX'+yY'+Z'=0,$$

or $$x'X+y'Y+Z=0.$$

This form should be remembered; it will be shown later on to be true for the general equation.

**Note iii.** The equation of the tangent at $(x_1, y_1)$ can also be written

$$(x+g)(x_1+g)+(y+f)(y_1+f)=g^2+f^2-c,$$

where $g^2+f^2-c$ is the square of the radius.

Thus if the equation of a circle is given in the form

$$(x-\alpha)^2+(y-\beta)^2=r^2,$$

the tangent at $(x_1, y_1)$ is

$$(x-\alpha)(x_1-\alpha)+(y-\beta)(y_1-\beta)=r^2.$$

**Definition.** *The angle between a straight line and the tangent at the point where it cuts a circle is called the angle at which the straight line cuts the circle.*

**Ex.** *To find the angle at which the straight line* $lx+my+n=0$ *cuts the circle* $x^2+y^2+2gx+2fy+c=0.$

Let $(x_1, y_1)$ be a point of intersection of the line and circle.

The tangent at $(x_1, y_1)$ is

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0,$$

hence, if $\theta$ is the angle required,

$$\tan\theta=\frac{m(x_1+g)-l(y_1+f)}{m(y_1+f)+l(x_1+g)}.$$

Now
$$\{m(x_1+g)-l(y_1+f)\}^2+\{l(x_1+g)+m(y_1+f)\}^2$$
$$\equiv(l^2+m^2)\{(x_1+g)^2+(y_1+f)^2\}$$
$$=(l^2+m^2)\{g^2+f^2-c\},$$

because $$x_1^2+y_1^2+2gx_1+2fy_1+c=0.$$

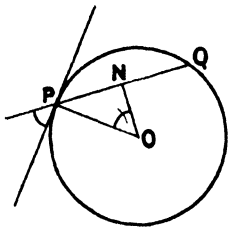
Also $$m(y_1+f)+l(x_1+g)=lx_1+my_1+lg+mf=lg+mf-n,$$

since $(x_1, y_1)$ lies on $$lx+my+n=0.$$

Thus $$\tan\theta=\pm\frac{\sqrt{(l^2+m^2)(g^2+f^2-c)-(lg+mf-n)^2}}{lg+mf-n}.$$

As $(x_1, y_1)$ can be either of the points of intersection, it follows that the straight line cuts the circle at the same angle at both points of intersection, or 'tangents are equally inclined to the chord of contact'.

If $P$ is the point of intersection and $ON$ the perpendicular from the centre $O$ to the line, $OP$ is perpendicular to the tangent and the result can be more easily obtained from the fact that

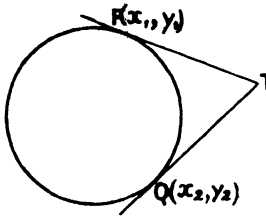



$$\tan\theta = \frac{PN}{ON} = \frac{\sqrt{OP^2 - ON^2}}{ON}$$

where $OP$ is the radius.

*The equation of the chord of contact of tangents which meet at a point* $(\xi, \eta)$.

Let the points of contact of the tangents be $P(x_1, y_1)$ and $Q(x_2, y_2)$, and let these tangents meet at $T(\xi, \eta)$.




If the equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

the equations of the tangents are

$$(x+g)(x_1+g) + (y+f)(y_1+f) = g^2 + f^2 - c$$

and

$$(x+g)(x_2+g) + (y+f)(y_2+f) = g^2 + f^2 - c,$$

and since these pass through $T$ we have the conditions

$$\left.\begin{aligned}(\xi+g)(x_1+g) + (\eta+f)(y_1+f) &= g^2+f^2-c\\ (\xi+g)(x_2+g) + (\eta+f)(y_2+f) &= g^2+f^2-c\end{aligned}\right\}.$$

But these are also the conditions that $(x_1, y_1)$, $(x_2, y_2)$ should be on the straight line

$$(\xi+g)(x+g) + (\eta+f)(y+f) = g^2+f^2-c;$$

hence this must be the equation of the straight line $PQ$.

The equation of the chord of contact of tangents which meet at the point $(\xi, \eta)$ is therefore

$$(x+g)(\xi+g) + (y+f)(\eta+f) = g^2+f^2-c.$$

It should be noted that this is exactly the same form as that of the tangent at a point $(\xi, \eta)$ on the circle, viz. $xX' + yY' + Z' = 0$. The symmetry of the result leads to the following proposition:—

*If the chord of contact of tangents from a point $T$ passes through a point $T'$, then the chord of contact of tangent from $T'$ will pass through $T$.*

For let $T$ be the point $(\xi, \eta)$ and $T'$ be the point $(\xi', \eta')$; the chord of contact of tangents from $T$ is

$$(x+g)(\xi+g) + (y+f)(\eta+f) = g^2+f^2-c, \qquad \text{(i)}$$

and the chord of contact of tangents from $T'$ is

$$(x+g)(\xi'+g)+(y+f)(\eta'+f)=g^2+f^2-c. \qquad \text{(ii)}$$

The conditions that $T$ should lie on (ii) and that $T'$ should lie on (i) are identical, viz.

$$(\xi+g)(\xi'+g)+(\eta+f)(\eta'+f)=g^2+f^2-c.$$

We shall discuss these equations more fully later in the chapter.

*To find the coordinates of the point of intersection of tangents to the circle* $x^2+y^2+2gx+2fy+c=0$ *at its points of intersection with* $lx+my+n=0$.

Let $(\xi, \eta)$ be the point of intersection; then the chord of contact of tangents from $(\xi, \eta)$ to the circle

$$x(\xi+g)+y(\eta+f)+g\xi+f\eta+c=0$$

must be identical with $lx+my+n=0$.

Thus
$$\frac{\xi+g}{l}=\frac{\eta+f}{m}=\frac{g\xi+f\eta+c}{n}$$
$$=\frac{g^2+f^2-c}{lg+mf-n}.$$

The coordinates of the point are therefore

$$\left\{\frac{l(g^2+f^2-c)}{lg+mf-n}-g,\ \frac{m(g^2+f^2-c)}{lg+mf-n}-f\right\}.$$

*To find the length of the tangent from any point* $(\xi, \eta)$ *to the circle* $x^2+y^2+2gx+2fy+c=0$.

Let $(x_1, y_1)$ be the point of contact of the tangent; this lies on the chord of contact of tangents from $(\xi, \eta)$, viz.

$$x(\xi+g)+y(\eta+f)+g\xi+f\eta+c=0;$$

hence
$$x_1(\xi+g)+y_1(\eta+f)+g\xi+f\eta+c=0.$$

Now (length of tangent)$^2$

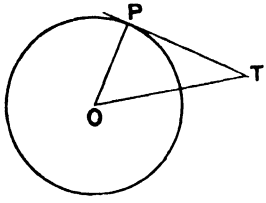
$$=(\xi-x_1)^2+(\eta-y_1)^2$$
$$=\xi^2+\eta^2-2x_1\xi-2y_1\eta+x_1^2+y_1^2$$
$$=\xi^2+\eta^2-2x_1\xi-2y_1\eta-2gx_1-2fy_1-c$$
[since $(x_1, y_1)$ lies on the circle]
$$=\xi^2+\eta^2-2x_1(\xi+g)-2y_1(\eta+f)-c$$
$$=\xi^2+\eta^2+2g\xi+2f\eta+c$$
[since $(x_1, y_1)$ lies on the chord of contact].

**Note i.** If $TP$ is the tangent and $O$ the centre of the circle, then, since $OP$ is perpendicular to $PT$, $PT^2 = OT^2 - OP^2$

$$= (\xi+g)^2+(\eta+f)^2-(g^2+f^2-c)$$
$$= \xi^2+\eta^2+2g\xi+2f\eta+c.$$

**Note ii.** The square of the length of the tangent from $T$ to a circle

$$x^2+y^2+2gx+2fy+c=0,$$




is obtained by substituting the coordinates of $T$ in

$$x^2+y^2+2gx+2fy+c.$$

Hence, if $T$ is outside the circle the result of this substitution is positive, if on the circle zero, if inside the circle negative, the tangent being then imaginary.

This agrees with the condition

$$TO^2 >, =, \text{ or } < (\text{radius})^2.$$

*The equation of the normal at the point* $(x_1, y_1)$ *to the circle*

$$x^2+y^2+2gx+2fy+c=0.$$

The tangent at $(x_1, y_1)$ is

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0,$$

and the normal is the line through $(x_1, y_1)$ perpendicular to this, viz.

$$(x-x_1)(y_1+f)-(y-y_1)(x_1+g)=0,$$

which is the equation of the straight line joining the points $(x_1, y_1)$ and $(-g, -f)$.

Hence the normal at any point of a circle passes through the centre, i. e. is a radius; and conversely all radii cut the circle orthogonally. Hence, also, the join of the centre to the point of contact is perpendicular to the tangent; or, in other words, the perpendicular from the centre on a tangent is equal to the radius.

We have endeavoured so to discuss these preliminary equations as to discover naturally by analytical methods the well-known properties of tangents and chords of a circle: the object of this is to create in the mind of the reader, if possible, a feeling of confidence in the analytical method.

The work can obviously be simplified by assuming these well-known properties (see notes appended above), but if this is done no opportunity occurs of exhibiting in a simple manner the certainty and directness of analysis. When solving problems the student should take the simplest form of the equation of the circle which is allowable under the given conditions.

**Illustrative Examples.**

I. *To find the condition that the line* $lx+my+n=0$ *should touch the circle* $x^2+y^2+2gx+2fy+c=0$.

The perpendicular from the centre $(-g, -f)$ on the line is equal to the radius $\sqrt{g^2+f^2-c}$.

Thus $$(lg+mf-n)^2=(l^2+m^2)(g^2+f^2-c).$$

II. *To find the equations of tangents to the circle*

$$x^2+y^2+2gx+2fy+c=0$$

*which are parallel to the line* $lx+my=1$.

If $(x_1, y_1)$ is the point of contact, then, since the tangent is a line through this point which by hypothesis is parallel to $lx+my=1$, its equation is $$l(x-x_1)+m(y-y_1)=0. \qquad \text{(i)}$$

But the perpendicular on this from the centre $(-g, -f)$ is equal to the radius $\sqrt{g^2+f^2-c}$.

Hence $$\frac{l(g+x_1)+m(f+y_1)}{\sqrt{l^2+m^2}}=\pm\sqrt{g^2+f^2-c},$$

or $$lx_1+my_1=\pm\sqrt{l^2+m^2}\,\sqrt{g^2+f^2-c}-(lg+mf);$$

and substituting in (i) we obtain

$$l(x+g)+m(y+f)=\pm\sqrt{l^2+m^2}\,\sqrt{g^2+f^2-c}$$

as the required equation.

There are then two tangents to a circle parallel to any given line.

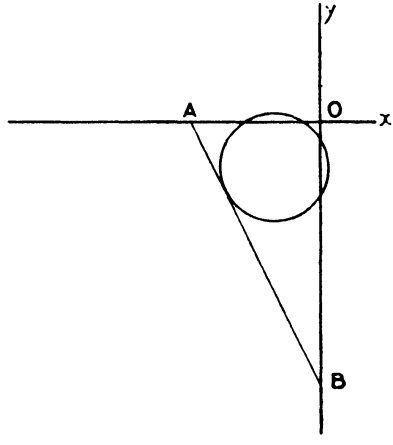
**Note.** If the line is given in the form $x\cos\alpha+y\sin\alpha-p=0$, the equation of the parallel tangents follows more simply, viz.

$$(x+g)\cos\alpha+(y+f)\sin\alpha=\pm\sqrt{g^2+f^2-c}.$$

III. *To find the equation of the circle touching the line*

$$4x+2y+3a\sqrt{5}=0,$$

*and cutting off equal chords, each of length a, from the portions of the coordinate axes between this line and the origin.*




The straight line

$AB$, $4x+2y+3a\sqrt{5}=0$

cuts the axes at points whose coordinates are negative; hence the centre of the circle lying inside the triangle $OAB$ has negative coordinates.

If the circle is

$$x^2+y^2+2gx+2fy+c=0,$$

the values of $g$ and $f$ are consequently positive.

This circle cuts the $x$-axis at points whose abscissae are given by

$$x^2+2gx+c=0.$$

If these are $x_1$ and $x_2$, then by hypothesis

$$x_1-x_2=a.$$

Hence $$4g^2-4c=a^2.$$

Similarly for the $y$-axis $$4f^2-4c=a^2.$$

Hence $$g^2=f^2,$$

and since both are positive $$g=f.$$

Expressing the fact that the perpendicular from the centre $(-g, -f)$ is equal to the radius, we have

$$(4g+2f-3a\sqrt{5})^2=20(g^2+f^2-c).$$

Since $$g=f \text{ and } c=g^2-\tfrac{1}{4}a^2,$$

we have $$(6g-3a\sqrt{5})^2=20g^2+5a^2,$$

i.e. $$16g^2-36ga\sqrt{5}+40a^2=0,$$

or $$4g^2-9ga\sqrt{5}+10a^2=0;$$

$$\therefore\quad g=\tfrac{1}{4}\sqrt{5}\,a \text{ or } 2a\sqrt{5}.$$

The centre is therefore $(-\frac{1}{4}\sqrt{5}\,a, -\frac{1}{4}\sqrt{5}\,a)$,

for the point $(-2a\sqrt{5}, -2a\sqrt{5})$ lies outside the triangle $OAB$.

Hence $$c=\tfrac{5}{16}a^2-\tfrac{1}{4}a^2=\tfrac{1}{16}a^2,$$

and the required equation is

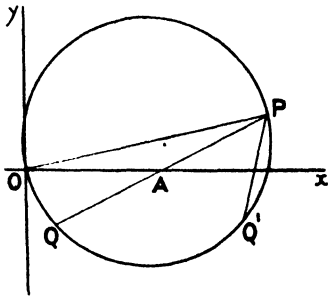
$$x^2+y^2+\tfrac{1}{2}\sqrt{5}\,a(x+y)+\tfrac{1}{16}a^2=0,$$

or $$16(x^2+y^2)+8\sqrt{5}a(x+y)+a^2=0.$$

IV. *Investigate the condition that from the point $P(\alpha, \beta)$ on the circle $x(x-\alpha)+y(y-\beta)=0$ it may be possible to draw two chords each bisected by the axis of $x$, and show that the angle between them is*

$$\tan^{-1}\frac{\sqrt{\alpha^2-8\beta^2}}{3\beta}.$$

Let $A(\xi, 0)$ be the mid-point of a chord $PQ$ of the circle




$$x^2+y^2-\alpha x-\beta y=0.$$

The equation of this chord is

$$(\xi-x)(\xi-\tfrac{1}{2}\alpha)+\tfrac{1}{2}(y\beta)=0.$$

This passes through the point $(\alpha, \beta)$ if

$$(\xi-\alpha)(\xi-\tfrac{1}{2}\alpha)+\tfrac{1}{2}\beta^2=0,$$

i.e. $$\xi^2-\tfrac{3}{2}\xi\alpha+\tfrac{1}{2}(\alpha^2+\beta^2)=0,$$

which equation gives the values of $\xi$.

These values are real if

$$\tfrac{9}{4}\alpha^2>2\alpha^2+2\beta^2,$$

i.e. if $\alpha^2>8\beta^2$, which is the required condition.

If $\xi_1$, $\xi_2$ are the roots of the equation,

$$\xi_1+\xi_2=\tfrac{3}{2}\alpha,\ \xi_1\xi_2=\tfrac{1}{2}(\alpha^2+\beta^2).$$

The equations of the two chords $PQ$, $PQ'$ are then

$$(\xi_1-x)(\xi_1-\tfrac{1}{2}\alpha)+\tfrac{1}{2}y\beta=0,$$
$$(\xi_2-x)(\xi_2-\tfrac{1}{2}\alpha)+\tfrac{1}{2}y\beta=0;$$

thus the angle between them

$$=\tan^{-1}\frac{\frac{2\xi_1-\alpha}{\beta}-\frac{2\xi_2-\alpha}{\beta}}{1+\frac{(2\xi_1-\alpha)(2\xi_2-\alpha)}{\beta^2}}$$

$$=\tan^{-1}\frac{2\beta(\xi_1-\xi_2)}{4\xi_1\xi_2-2\alpha(\xi_1+\xi_2)+\alpha^2+\beta^2}$$

$$=\tan^{-1}\frac{\sqrt{\alpha^2-8\beta^2}}{3\beta};$$

since
$$\xi_1+\xi_2=\tfrac{1}{2}(3\alpha)$$
$$\xi_1\xi_2=\tfrac{1}{2}(\alpha^2+\beta^2),$$

and
$$(\xi_1-\xi_2)^2=(\xi_1+\xi_2)^2-4\xi_1\xi_2.$$

### Examples V c.

1. Find the equation of the chord of the circle $x^2+y^2+6x+8y+9=0$, whose mid-point is $(-2, -3)$.

2. Write down the equations of the tangents at the point $(-2, 4)$ to the circles

(i) $x^2+y^2+4x-10y+28=0$;

(ii) $(x-1)^2+(y-5)^2=10$;

(iii) $(x+5)^2+y^2=25$.

3. Find the equation of the tangent to the circle $x^2+y^2=4$ which is perpendicular to $3x+4y=5$.

4. Show that the line $3x+4y+10=0$ touches the circle

$$x^2+y^2-2x+4y+4=0.$$

Find the point of contact and the equation of the other tangent parallel to this.

5. Is the point $(5, -6)$ inside or outside the circle

$$x^2+y^2-2y-11x-24=0?$$

Show that the point $(1, 2)$ lies inside the circle, and find the equation of the chord of which it is the middle point.

6. Find the coordinates of the point of intersection of tangents at the points where $x+4y=14$ meets the circle $x^2+y^2-3x+2y=5$.

7. Prove that the point $(3, 4)$ lies outside the circle

$$2x^2+2y^2+12x-9y=1,$$

and find the lengths of the tangents from it to the circle.

Show that their points of contact lie on the circle

$$x^2+y^2-6x-8y+\tfrac{1}{2}=0,$$

and on the straight line $24x+7y-2=0$.

8. Find from the definition of a tangent as the limit of a chord the equation of the tangent at (2, 5) to $x^2+y^2=29$.

9. Show that $x+3y-1=0$ touches $x^2+y^2-3x-3y+2=0$.

Find the equations of the tangents to the circle which are perpendicular to the given one.

10. The point (2, 3) is the mid-point of the chord of a circle and the equation of the chord is $5x+2y=16$. Find the locus of the centres of such circles.

11. Find the equation of the circle inscribed in the triangle $3x+4y=12$, $x=0$, $y=0$. Also of the circle escribed to the first side.

12. Prove that the line $(x-1)\cos\theta+(y-2)\sin\theta=3$ touches the circle $(x-1)^2+(y-2)^2=9$ whatever value $\theta$ may have.

Find the coordinates of the point of contact.

13. Show that the lengths of the tangents from any point on the straight line $x-y+1=0$ to the circles

$$x^2+y^2+7x-9y+6=0, \quad x^2+y^2-5x+3y-6=0$$

are equal.

14. A point moves so that the lengths of the tangents from it to the circles $x^2+y^2+6x-4y=12$, $x^2+y^2-4x-4y+5=0$ are equal. Find the equation of its locus.

Find also the equation of the line joining the centres of the circles and show that the locus cuts it orthogonally.

15. Find the equations of tangents to the circle $(x-1)^2+(y-2)^2=4$, which make an angle $\tan^{-1}\frac{3}{4}$ with the axis of $x$.

16. Find the locus of a point $P$ which moves so that $PT^2+PT'^2=$ constant, where $PT$ and $PT'$ are tangents from it to two given circles.

Also when $PT^2-PT'^2=$ constant, and in general when

$$l\,.\,PT^2+m\,.\,PT'^2=\text{constant}.$$

17. A point $P$ moves in the plane $XOY$ so that its distances from the points (5, 1), (3, 2) are in the constant ratio of 2 : 1.

Find the locus of $P$ and show that $2x-y+1=0$ is a tangent to the locus.

18. Find the equation of the circle passing through the points $A(-4, 3)$, $B(-3, -4)$, $C(4, -3)$.

Prove that the tangents at $A$ and $C$ are parallel and each perpendicular to the tangent at $B$.

19. Show that the straight line $x\cos\theta+y\sin\theta-p=0$ meets the circle $x^2+y^2=r^2$ in real, coincident, or imaginary points according as $p$ is $<$, $=$, or $>r$.

20. Find the equations of tangents to the circle $x^2+y^2=r^2$ parallel to $x\cos\alpha+y\sin\alpha=p$.

21. Find the locus of the centres of circles which pass through the points $(a_1, b_1)$, $(a_2, b_2)$.

22. Find the locus of the mid-point of a chord of a circle which is of constant length.

23. Find the locus of the mid-points of all chords of a circle which are at a fixed distance from a given point.

24. Show analytically that two circles can be drawn through a given point to touch each of two given straight lines.

Find the equation to the circle when the straight lines are axes of coordinates and the coordinates of the fixed point are $(3, -6)$.

25. A point $P$ moves so that the squares on the tangents $PT_1$, $PT_2$, $PT_3$... from it to a number of given circles are connected by the relation

$$a_1 PT_1^2 + a_2 PT_2^2 + a_3 PT_3^2 + \ldots = 0.$$

Show that $P$ lies on a circle whose centre is the mean point of the centres of the given circles for the constants $a_1$, $a_2$, $a_3$....

26. A point moves so that its distance from a fixed point is twice the length of the tangent from it to a given circle. Show that its locus is a circle which lies entirely outside the given circle. Find also the distance of the fixed point from the centre of the given circle if the centre of the locus lies on the given circle.

27. Find the locus of the foot of the perpendicular from the origin to tangents to $x^2 + y^2 + 2gx + 2fy + c = 0$.

§ 5. We now propose to discuss various forms into which the equation of the circle can be put, and convenient methods of manipulating these equations; and also to illustrate the types of problems to which each form is specially suitable.

*The centre at the origin.* $x^2 + y^2 = r^2$. (I)

In the majority of problems dealing with one circle the work is simplified by using this, the simplest, form of the equation of the circle. Since the equation is the same for any pair of rectangular axes through the centre, we can further choose the coordinate axes so as to make some other detail of the problem simple, e.g. the axis of $x$ may be taken through some special point.

The various equations already found become in this case:

The chord joining $(x_1, y_1)$, $(x_2, y_2)$

$$x(x_1 + x_2) + y(y_1 + y_2) = x_1 x_2 + y_1 y_2 + r^2.$$

The chord whose mid-point is $(\xi, \eta)$

$$x\xi + y\eta = \xi^2 + \eta^2.$$

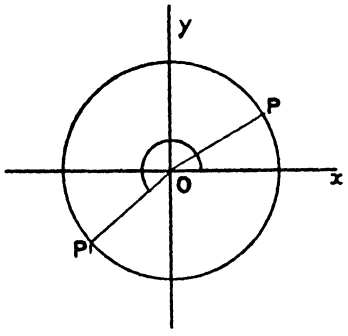
The tangent at $(x_1, y_1)$ $\quad xx_1 + yy_1 = r^2$.

The chord of contact of tangents meeting at $(\xi, \eta)$

$$x\xi + y\eta = r^2.$$

We have hitherto denoted any point on the circle by $(x_1, y_1)$, coupled with the conditional equation $x_1^2 + y_1^2 = r^2$. It is often possible to find a particular form for the coordinates of a point on a given curve containing only one variable, and such that the conditional equation is identically satisfied by the coordinates. Such coordinates are called parametric, and the single variable is called the *parameter*.

Now in the case of the conditional equation $x_1^2+y_1^2=r^2$, since $r^2\cos^2\theta+r^2\sin^2\theta=r^2$ is identically true, the point $(r\cos\theta,\ r\sin\theta)$ satisfies the equation of the circle for all values of $\theta$. We can therefore use $(r\cos\theta,\ r\sin\theta)$ to signify in general a point on the circle provided that *every* point on the circle can be so denoted.




Geometrically, if $P$ is any point on the circle, the position of $P$ is known when the angle $xOP(\theta)$ is known, and provided that the method explained for polar coordinates is adopted to measure the angle $\theta$, one definite point on the circle corresponds to a given value of $\theta$, and, on the other hand, a definite value of $\theta$ corresponds to every point on the circle if $\theta$ is taken between 0 and $2\pi$.

Algebraically, we see from the equation $x^2+y^2=r^2$ that the maximum and minimum real values of $x$ and $y$ are $r$ and $-r$; since $\cos\theta$ and $\sin\theta$ lie between 1 and $-1$, the coordinates $(r\cos\theta,\ r\sin\theta)$ lie between $r$ and $-r$, and can have any value between these limits.

In future we shall refer to the point $(r\cos\theta,\ r\sin\theta)$ as the point on the circle $x^2+y^2=r^2$ whose parameter is $\theta$, or briefly 'the point $\theta$' on the circle.

The equations of the chord joining two points $\theta_1$, $\theta_2$ and of the tangent at the point $\theta$ can be found from the above equations by substitution; we leave this as an exercise for the student and give an independent investigation.

(i) *To find the equation of the chord joining the points $\theta_1$ and $\theta_2$.*

The line joining $(r\cos\theta_1,\ r\sin\theta_1)$, $(r\cos\theta_2,\ r\sin\theta_2)$ is

$$\frac{x-r\cos\theta_1}{r(\cos\theta_2-\cos\theta_1)}=\frac{y-r\sin\theta_1}{r(\sin\theta_2-\sin\theta_1)},$$

i.e. $$\frac{x-r\cos\theta_1}{\sin\frac{1}{2}(\theta_1-\theta_2)\sin\frac{1}{2}(\theta_1+\theta_2)}=\frac{y-r\sin\theta_1}{\sin\frac{1}{2}(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_1+\theta_2)};$$

or, dividing the denominators by $\sin\frac{1}{2}(\theta_1-\theta_2)$, we get

$$x\cos\tfrac{1}{2}(\theta_1+\theta_2)+y\sin\tfrac{1}{2}(\theta_1+\theta_2)=r\cos\tfrac{1}{2}(\theta_1-\theta_2).$$

**Note.** The removal of the factor $\sin\frac{1}{2}(\theta_1-\theta_2)$ should be compared with the corresponding work in § 4, p. 134.

(ii) *The equation of the tangent at the point $\theta$.*

This follows from that of the chord by putting $\theta_1=\theta_2=\theta$;

$$x\cos\theta+y\sin\theta=r.$$

(iii) *The intersection of the line* $lx+my+n=0$ *and the circle*

$$x^2+y^2=r^2.$$

To find the points of intersection of two loci we have to solve simultaneously the equations representing those loci: we may obtain an equation giving the $x$-coordinates of the common points and a second giving the $y$-coordinates.

The following illustrates the advantage of the use of parametric coordinates in such investigations.

If $\theta$ is the parameter of any point common to the circle $x^2+y^2=r^2$ and the line $lx+my+n=0$, the coordinates $(r\cos\theta,\ r\sin\theta)$ of the point must satisfy the equation of the line. Thus

$$lr\cos\theta+mr\sin\theta+n=0.$$

This is true for any common point of the circle and line, hence the values of $\theta$ given by this equation are the parameters of the points of intersection.

The equation can be expressed in terms of a single variable $\tan\frac{1}{2}\theta$ by substituting

$$\cos\theta=\frac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta};\quad \sin\theta=\frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta};$$

in which case it becomes

$$(n-lr)\tan^2\tfrac{1}{2}\theta+2mr\tan\tfrac{1}{2}\theta+(n+lr)=0. \qquad \text{(i)}$$

Each value of $\tan\frac{1}{2}\theta$ given by this equation corresponds to one point only on the circle; for suppose that

$$\tan\tfrac{1}{2}\theta=k,$$

then $$\tfrac{1}{2}\theta=n\pi+\tan^{-1}k,$$

or $$\theta=2n\pi+2\tan^{-1}k,$$

which gives the same point whatever value $n$ has.

The equation (i) is quadratic in $\tan\frac{1}{2}\theta$, and we can now apply our knowledge of quadratic equations to investigate the intersections of the line and circle.

(a) A quadratic equation has two roots; hence a straight line meets a circle in two points.

(b) The roots of the equation are real and distinct, coincident, or imaginary according as

$$m^2r^2 \text{ is } >,\ =,\ \text{or } <n^2-l^2r^2,$$

i.e. as $l^2r^2+m^2r^2-n^2$ is positive, zero, or negative.

Hence the line $lx+my+n=0$ cuts the circle $x^2+y^2=r^2$ if

$l^2r^2+m^2r^2-n^2$ is positive, touches it if the expression is zero, and meets it only in imaginary points if negative.

(This condition corresponds to the perpendicular from the centre to the line being less than, equal to, or greater than the radius.)

(c) If $\theta_1$, $\theta_2$ are the values of $\theta$ which satisfy the above equation,

$$\text{the sum of the roots} = \tan\tfrac{1}{2}\theta_1+\tan\tfrac{1}{2}\theta_2 = -\frac{2mr}{n-lr},$$

$$\text{the product of the roots} = \tan\tfrac{1}{2}\theta_1\tan\tfrac{1}{2}\theta_2 = \frac{n+lr}{n-lr}.$$

Hence $$lr : mr : n$$

$$= 1-\tan\tfrac{1}{2}\theta_1\tan\tfrac{1}{2}\theta_2 : \tan\tfrac{1}{2}\theta_1+\tan\tfrac{1}{2}\theta_2 : -(1+\tan\tfrac{1}{2}\theta_1\tan\tfrac{1}{2}\theta_2)$$

$$= \cos\tfrac{1}{2}(\theta_1+\theta_2) : \sin\tfrac{1}{2}(\theta_1+\theta_2) : -\cos\tfrac{1}{2}(\theta_1-\theta_2).$$

The equation of the straight line in terms of the parameters of its points of intersection with the circle is therefore

$$x\cos\tfrac{1}{2}(\theta_1+\theta_2)+y\sin\tfrac{1}{2}(\theta_1+\theta_2)-r\cos\tfrac{1}{2}(\theta_1-\theta_2) = 0,$$

which agrees with the equation of the chord joining the points $\theta_1$, $\theta_2$ found otherwise.

(iv) *To find the points of contact of the tangents which can be drawn from a given point to the circle.*

Let the given point be $(h, k)$; now the tangent at the point $\theta$ is

$$x\cos\theta+y\sin\theta = r.$$

If this passes through $(h, k)$

$$h\cos\theta+k\sin\theta = r.$$

This equation is true for the parameter of any point the tangent at which passes through $(h, k)$. It can be written in the form

$$(h+r)\tan^2\tfrac{1}{2}\theta-2k\tan\tfrac{1}{2}\theta+(r-h) = 0.$$

Then

(a) Since the equation is quadratic it has two roots; thus two tangents can be drawn from any point to a circle.

(b) The roots are real, coincident, or imaginary as

$$k^2 \text{ is } >, =, \text{ or } < r^2-h^2,$$

i.e. as $h^2+k^2-r^2$ is positive, zero, or negative.

Thus the two tangents which can be drawn from $(h, k)$ to the circle are real, coincident, or imaginary according as

$$h^2+k^2-r^2 \text{ is positive, zero, or negative,}$$

i.e. as $(h, k)$ is outside, on, or inside the circumference.

(c) The square of the length of a tangent from $(h, k)$, (if $\theta$ is the point of contact)

$$\begin{aligned}&= (h - r\cos\theta)^2 + (k - r\sin\theta)^2\\ &= h^2 + k^2 + r^2 - 2r(h\cos\theta + k\sin\theta)\\ &= h^2 + k^2 + r^2 - 2r^2\\ &= h^2 + k^2 - r^2.\end{aligned}$$

(d) If the two values of $\theta$ given by the equation are $\theta_1$, $\theta_2$, then

$$\text{the sum of the roots} = \tan\tfrac{1}{2}\theta_1 + \tan\tfrac{1}{2}\theta_2 = \frac{2k}{r+h},$$

$$\text{the product of the roots} = \tan\tfrac{1}{2}\theta_1 \,.\, \tan\tfrac{1}{2}\theta_2 = \frac{r-h}{r+h}.$$

Hence
$$h = r\,.\,\frac{1-\tan\frac{1}{2}\theta_1\tan\frac{1}{2}\theta_2}{1+\tan\frac{1}{2}\theta_1\tan\frac{1}{2}\theta_2} = r\,.\,\frac{\cos\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1-\theta_2)},$$

$$k = r\,.\,\frac{\tan\frac{1}{2}\theta_1 + \tan\frac{1}{2}\theta_2}{1+\tan\frac{1}{2}\theta_1\tan\frac{1}{2}\theta_2} = r\,.\,\frac{\sin\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1-\theta_2)}.$$

This result evidently gives the coordinates of the points of intersection of the tangents at the points $\theta_1$ and $\theta_2$.

It can be verified by solving the equations

$$x\cos\theta_1 + y\sin\theta_1 = r,$$
$$x\cos\theta_2 + y\sin\theta_2 = r.$$

No new facts have been found in the above analysis; the work is intended to illustrate the application of algebra to geometry, and also to familiarize the student with methods which will be used later on in the book.

### Illustrative Examples.

(i) *To find the angle between the tangents which can be drawn from the point* $(h, k)$ *to the circle* $x^2 + y^2 = r^2$.

Suppose that the points of contact of the tangents are $\theta_1$ and $\theta_2$; then equating the coordinates of the point of intersection of the tangents at these points to the given values

$$\frac{\cos\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1-\theta_2)} = \frac{h}{r}; \quad \frac{\sin\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1-\theta_2)} = \frac{k}{r}. \tag{i}$$

But the equation of the tangent at the point $\theta_1$ is

$$x\cos\theta_1 + y\sin\theta_1 = r,$$

which makes an angle $(\frac{1}{2}\pi + \theta_1)$ with the axis of $x$.

Hence the angle between the tangents is $(\theta_1 - \theta_2)$.

But from equation (i)

$$(h^2+k^2)\cos^2\tfrac{1}{2}(\theta_1-\theta_2)=r^2.$$

$$\therefore\quad 1+\cos\overline{\theta_1-\theta_2}=\frac{2r^2}{h^2+k^2}.$$

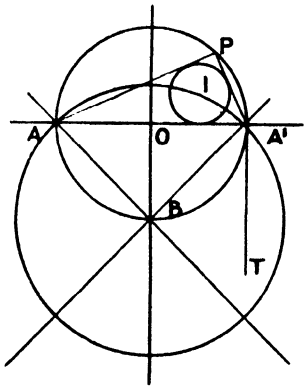
$$\therefore\quad \cos\overline{\theta_1-\theta_2}=\frac{2r^2-h^2-k^2}{h^2+k^2}.$$

**Cor.** The locus of the intersection of tangents to the circle which include a constant angle $2\alpha$ is $x^2+y^2=r^2\sec^2\alpha$, i. e. a concentric circle.

The student should also work the problem geometrically.

(ii) *$AA'$ is the diameter of a semicircle and $P$ is any point on the circumference; to find the locus of the centre of the circle inscribed in the triangle $APA'$.*

The geometrical solution of this question is evidently easy; for if $I$ be the point $IA$ and $IA'$ bisect the angles $PAA'$ and $PA'A$; moreover, the angle $P$ is a right angle. Hence the angle $AIA'$, which is the supplement of $\frac{1}{2}(A+A')$, is 135°. The locus consists therefore of the arc of a circle passing through $AA'$. We have selected this problem, however, because the result obtained by purely algebraical analysis results in an equation which needs discussion and is typical of a variety of problems in which the results are difficult for the student to interpret.




Now let $A'$, $P$, and $A$ be the points $0$, $\theta$, $\pi$ on the circle

$$x^2+y^2=r^2,$$

$\theta$ being $<\pi$.

Then the lines $A'P$, $PA$, and $AA'$ are

$$\begin{aligned} x\cos\tfrac{1}{2}\theta+y\sin\tfrac{1}{2}\theta-r\cos\tfrac{1}{2}\theta&=0,\\ -x\sin\tfrac{1}{2}\theta+y\cos\tfrac{1}{2}\theta-r\sin\tfrac{1}{2}\theta&=0,\\ y&=0.\end{aligned}$$

If $I$, any point on the required locus, be $(x, y)$, since $I$ is the centre of a circle touching $A'P$, $PA$, and $AA'$, the perpendiculars from $I$ on the three straight lines are equal. Hence $(x, y)$ satisfies

$$(x-r)\cos\tfrac{1}{2}\theta+y\sin\tfrac{1}{2}\theta+y=0, \qquad \text{(i)}$$

$$y\cos\tfrac{1}{2}\theta-(x+r)\sin\tfrac{1}{2}\theta+y=0, \qquad \text{(ii)}$$

i. e.
$$\frac{\cos\frac{1}{2}\theta}{y(y+x+r)}=\frac{\sin\frac{1}{2}\theta}{y(y-x+r)}=\frac{1}{r^2-x^2-y^2}. \qquad \text{(iii)}$$

Hence $y^2(y+x+r)^2+y^2(y-x+r)^2=(y^2+x^2-r^2)^2$, which is the equation of the locus required.

This can be simplified thus: it is the same as

$$2y^2(y^2+x^2+2ry+r^2) = (y^2+x^2-r^2)^2,$$

i.e. $$2y^2(y^2+x^2+2ry-r^2) = (y^2+x^2-r^2)^2-4r^2y^2,$$

i.e. $$2y^2(y^2+x^2+2ry-r^2) = (y^2+x^2+2ry-r^2)(y^2+x^2-2ry-r^2),$$

i.e. $$(y^2+x^2+2ry-r^2)(y^2-x^2+2ry+r^2) = 0,$$

i.e. $$(y^2+x^2+2ry-r^2)(y+r+x)(y+r-x) = 0.$$

The locus then consists of three parts:

(a) the circle $x^2+y^2+2ry-r^2=0$, i.e. a circle centre $B(0, -r)$ and radius $r\sqrt{2}$; (b) the line $x+y+r=0$, viz. $AB$; (c) the line $x-y-r=0$, viz. $A'B$.

Solving equations (i) and (ii) for $y$, we find

$$y\left(1+\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right) = r\sin\theta;$$

since $\theta$ is less than $\pi$, $y$ is always positive and therefore the locus is confined to that part of the circle (a) which is above $AA'$.

Now we see from the equation (i) that when $x+y+r=0$, $\theta$ is $\pi$, and when $x-y-r=0$, $\theta$ is 0.

When $P$ approaches $A'$, $AP$ approaches coincidence with $AA'$; and $A'P$ approaches the position where $A'P$ is perpendicular to $AA'$.

In the limit when $P$ coincides with $A'$, then the sides of the triangle are

$$AA',\ AA',\ \text{and}\ TA'.$$

Now the line $BA'$ bisects the angle $AA'T$; hence the perpendiculars from any point on this line to $AA'$ and $TA'$ are equal, i.e. any points on the line fulfil the condition that the perpendiculars from it on the sides of the triangle $APA'$ are equal when $P$ coincides with $A'$.

So also $BA$ corresponds to the position when $P$ coincides with $A$.

**Note.** Had we been finding the locus of the centre of the circle escribed to the triangle $APA'$, the sign of perpendicular on $y=0$ would be changed in our original equations; but in the process of elimination we have squared the $y$ arising from this perpendicular, hence the locus of the escribed centre is contained in the equation found.

## Examples V d.

1. Find the area of the triangle formed by tangents at (1, 18), (6, 17), (10, 15) to the circle $x^2+y^2=325$.

2. Where does the circle $x^2+y^2=125$ meet the line $x+3y=25$?

Find the point of intersection of tangents at these points.

3. Find the coordinates of the points of contact of tangents from (7, 1) to $x^2+y^2=25$.

4. If $AB$ is the chord of contact of tangents from $P$ to a circle centre $O$, prove that $OP$ is perpendicular to $AB$; and if these lines intersect at $N$ then $ON.OP = (\text{radius})^2$.

5. Find the tangents to the circle $x^2+y^2=a^2$ inclined at an angle $\alpha$ to the axis of $x$ and the coordinates of the points of contact.

6. Show that the length of the chord joining the points $\theta_1$, $\theta_2$ on the circle $x^2+y^2=r^2$ is $2r\sin\frac{1}{2}(\theta_1-\theta_2)$.

7. Tangents at the ends of chords of fixed length in a circle intersect on a concentric circle.

8. Find the tangent of the angle between the two tangents from the point $(8\frac{1}{3}, 3\frac{1}{3})$ to the circle $x^2+y^2=29$.

9. Prove analytically that all angles in the same segment of a circle are equal.

10. A straight line cuts two concentric circles in the points $A, B$ and $C, D$ respectively: show that the area of the rectangles $AC.CB$ or $AD.DB$ are independent of the position of the straight line.

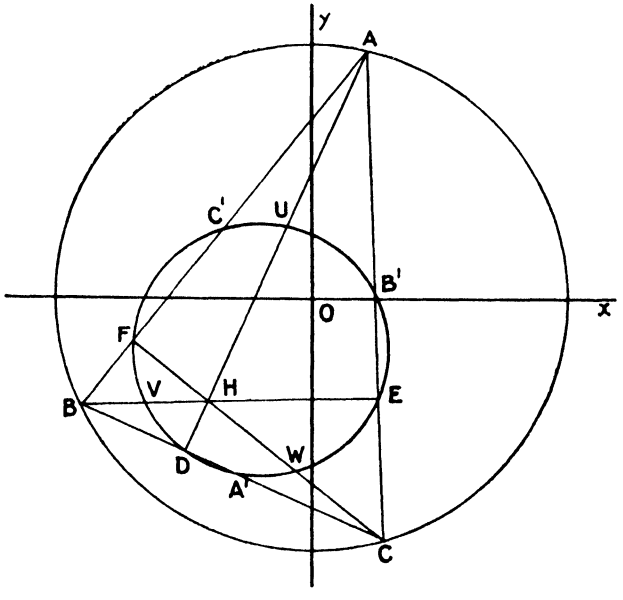
11. $AB$ is a given diameter of a circle and $PQ$ a chord of constant length. If $AP$, $BQ$ meet at $R$, show that the locus of $R$ is a circle.

12. $AC.BD$ are perpendicular chords of a circle through fixed points $A$, $B$ on the circle: prove that $CD$ always touches a circle concentric with the given circle.

13. Tangents from the point $P$ to the circle $x^2+y^2=a^2$ meet the axes of coordinates in four concyclic points. Show that $P$ lies on one or other of two right lines.

14. $AA'$ is a diameter of a circle and $P$ is any point on its circumference. Find the equation of the locus of the foot of the perpendicular from $A$ on the bisector of the angle $APA'$. Describe the locus.

§ 6. A circle can be drawn to circumscribe any triangle, and when its centre is taken as origin of coordinates many of the properties of a triangle can be conveniently investigated.




In this paragraph we shall use the following notation :—
The circumcircle is $x^2+y^2=R^2$; its centre is $O$.

The vertices of the triangle are $A(R\cos\alpha, R\sin\alpha)$; $B(R\cos\beta, R\sin\beta)$; $C(R\cos\gamma, R\sin\gamma)$.

The perpendiculars from the vertices on the sides are $AD$, $BE$, $CF$.

The orthocentre is $H$.

The mid-points of sides are $A'$, $B'$, $C'$.

The centre of gravity, centroid, or centre of mean position of $ABC$ is $G$.

The nine-point centre is $N$.

$$2C \equiv R(\cos\alpha+\cos\beta+\cos\gamma),$$

$$2S \equiv R(\sin\alpha+\sin\beta+\sin\gamma),$$

$$2\sigma \equiv \alpha+\beta+\gamma.$$

The student should work straight through the following examples: the easier questions are left as exercises, and the results obtained are used in the illustrative problems.

### Examples V e.

1. The coordinates of the centre of gravity ($G$) are $\{\frac{2}{3}C, \frac{2}{3}S\}$.
2. The orthocentre ($H$) is the point $\{2C, 2S\}$.
3. The mid-point ($U$) of $HA$ is $\{C+\frac{1}{2}R\cos\alpha,\ S+\frac{1}{2}R\sin\alpha\}$.
4. The mid-point ($A'$) of $BC$ is $\{C-\frac{1}{2}R\cos\alpha,\ S-\frac{1}{2}R\sin\alpha\}$.
5. The mid-point of $A'U$ is $\{C, S\}$.
6. Show that $G$ trisects the line joining $OH$.
7. *The Nine-point Circle.*

The equation of the chord $BC$ (Fig., p. 152) is

$$x\cos\tfrac{1}{2}(\beta+\gamma)+y\sin\tfrac{1}{2}(\beta+\gamma) = R\cos\tfrac{1}{2}(\beta-\gamma). \qquad \text{(i)}$$

The perpendicular to this through $A$, viz. $AD$, is

$$(x-R\cos\alpha)\sin\tfrac{1}{2}(\beta+\gamma)-(y-R\sin\alpha)\cos\tfrac{1}{2}(\beta+\gamma) = 0,$$

i.e.
$$x\sin\tfrac{1}{2}(\beta+\gamma)-y\cos\tfrac{1}{2}(\beta+\gamma) = R\sin\{\tfrac{1}{2}(\beta+\gamma)-\alpha\}. \qquad \text{(ii)}$$

The coordinates of the point ($D$) of intersection of the lines (i) and (ii) are given by solving these equations; thus

$$\begin{aligned} 2x &= R(\cos\beta+\cos\gamma+\cos\alpha-\cos\overline{\beta+\gamma-\alpha}) \\ &= 2C-R\cos(\beta+\gamma-\alpha), \\ 2y &= R(\sin\beta+\sin\gamma+\sin\alpha-\sin\overline{\beta+\gamma-\alpha}) \\ &= 2S-R\sin(\beta+\gamma-\alpha). \end{aligned}$$

Thus
$$\begin{aligned} x-C &= -\tfrac{1}{2}R\cos(\beta+\gamma-\alpha), \\ y-S &= -\tfrac{1}{2}R\sin(\beta+\gamma-\alpha). \end{aligned}$$

Hence the coordinates of the point $D$ satisfy the equation

$$(x-C)^2+(y-S)^2 = \tfrac{1}{4}R^2(\cos^2\overline{\beta+\gamma-\alpha}+\sin^2\overline{\beta+\gamma-\alpha}),$$

i.e.
$$(x-C)^2+(y-S)^2 = \tfrac{1}{4}R^2. \qquad \text{(iii)}$$

The symmetry of the result shows that the coordinates of the points $E$ and $F$ will also satisfy this equation.

Thus equation (iii), which represents a circle, is the circle $DEF$.

This equation is satisfied by the coordinates of $A'$, viz.

$$\{C-\tfrac{1}{2}R\cos\alpha,\ S-\tfrac{1}{2}R\sin\alpha\},$$

and from symmetry by those of $B'$ and $C'$.

So also the point $U$ $\{C+\tfrac{1}{2}R\cos\alpha,\ S+\tfrac{1}{2}R\sin\alpha\}$ lies on this circle, and from symmetry $V$ and $W$.

Hence the nine points $D$, $E$, $F$, $A'$, $B'$, $C'$, $U$, $V$, $W$ lie on a circle whose centre is $(C, S)$ and radius $\tfrac{1}{2}R$.

The nine-point radius is half the circumradius.

8. The orthocentre, nine-point centre, and circumcentre are collinear, the nine-point centre lying midway between the other two.

9. The angles $A$, $B$, $C$ of the triangle are $\tfrac{1}{2}(\gamma-\beta)$, $\pi-\tfrac{1}{2}(\gamma-\alpha)$, $\tfrac{1}{2}(\beta-\alpha)$.

10. Show analytically that $HA = 2R\cos A$.

11. Show that $AB.AC = 2R.AD$.

12. If $ABCD$ are any four points on a circle

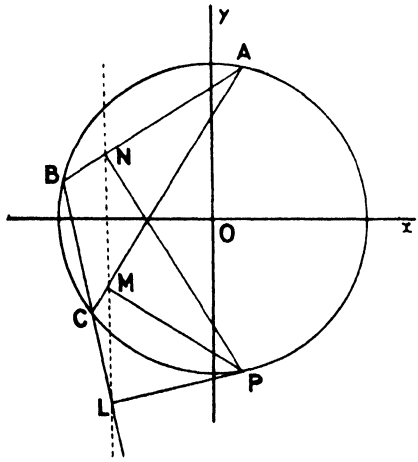
$$AB.DC + AD.BC = AC.BD.$$

13. The feet of the perpendiculars from any point $P$ on the circumcircle to the sides of an inscribed triangle lie on a straight line called the 'Simson line' of the point $P$, or the 'pedal line' of $P$.

Let the circumcircle be $x^2+y^2=R^2$, and the vertices of the triangle be

$$A\,(R\cos\alpha,\ R\sin\alpha),\ B\,(R\cos\beta,\ R\sin\beta),\ C\,(R\cos\gamma,\ R\sin\gamma):$$

let $P(R\cos\theta,\ R\sin\theta)$ be any point on the circumcircle.




Now $BC$ is the line

$$x\cos\tfrac{1}{2}(\beta+\gamma)+y\sin\tfrac{1}{2}(\beta+\gamma)=R\cos\tfrac{1}{2}(\beta-\gamma), \qquad \text{(i)}$$

and the equation of the perpendicular from $P$ to $BC$ is

$$(x-R\cos\theta)\sin\tfrac{1}{2}(\beta+\gamma)-(y-R\sin\theta)\cos\tfrac{1}{2}(\beta+\gamma)=0$$

or

$$x\sin\tfrac{1}{2}(\beta+\gamma)-y\cos\tfrac{1}{2}(\beta+\gamma)=R\sin\{\tfrac{1}{2}(\beta+\gamma)-\theta\}. \qquad \text{(ii)}$$

If then $x$ and $y$ are the coordinates of $L$, the point of intersection of (i) and (ii),

$$x = R\sin\tfrac{1}{2}(\beta+\gamma)\,.\,\sin\{\tfrac{1}{2}(\beta+\gamma)-\theta\}+R\cos\tfrac{1}{2}(\beta+\gamma)\,.\,\cos\tfrac{1}{2}(\beta-\gamma)$$

or $$2x = R\{\cos\theta+\cos\beta+\cos\gamma-\cos(\beta+\gamma-\theta)\};$$

so also $$2y = R\{\sin\theta+\sin\beta+\sin\gamma-\sin(\beta+\gamma-\theta)\}.$$

Hence, using our former notation, we get

$$2x-2C-R\cos\theta = R\{-\cos\alpha-\cos\overline{\beta+\gamma-\theta}\}$$
$$= -2R\cos\tfrac{1}{2}(\alpha+\beta+\gamma-\theta)\cos\tfrac{1}{2}(\alpha+\theta-\beta-\gamma),$$

$$2y-2S-R\sin\theta = R\{-\sin\alpha-\sin\overline{\beta+\gamma-\theta}\}$$
$$= -2R\sin\tfrac{1}{2}(\alpha+\beta+\gamma-\theta)\cos\tfrac{1}{2}(\alpha+\theta-\beta-\gamma);$$

therefore the coordinates $(x, y)$ of the point $L$ satisfy

$$(x-C-\tfrac{1}{2}R\cos\theta)\sin\tfrac{1}{2}(\alpha+\beta+\gamma-\theta)-(y-S-\tfrac{1}{2}R\sin\theta)\cos\tfrac{1}{2}(\alpha+\beta+\gamma-\theta)=0.$$

The result is symmetrical with respect to $\alpha$, $\beta$, $\gamma$, and consequently the coordinates of the feet of the perpendiculars from $P$ to $CA$, $AB$ also satisfy this equation.

Hence $LMN$ is a straight line whose equation is

$$(x-C-\tfrac{1}{2}R\cos\theta)\sin(\sigma-\tfrac{1}{2}\theta)-(y-S-\tfrac{1}{2}R\sin\theta)\cos(\sigma-\tfrac{1}{2}\theta)=0, \qquad \text{(iii)}$$

and this is the pedal line of the point $\theta$.

14. If the pedal lines of $P$, $Q$, $R$ all pass through the nine-point centre of $ABC$, then $PQR$ is an equilateral triangle.

Let $P$ be the point $\theta$, then the pedal line is

$$(x-C-\tfrac{1}{2}R\cos\theta)\sin(\sigma-\tfrac{1}{2}\theta)-(y-S-\tfrac{1}{2}R\sin\theta)\cos(\sigma-\tfrac{1}{2}\theta)=0.$$

The nine-point centre is the point $(C, S)$; hence if the pedal line of the point $\theta$ passes through the nine-point centre we have

$$\cos\theta\,.\,\sin(\sigma-\tfrac{1}{2}\theta)-\sin\theta\,.\,\cos(\sigma-\tfrac{1}{2}\theta)=0 \quad\text{or}\quad \sin(\tfrac{3}{2}\theta-\sigma)=0.$$

Hence $\tfrac{3}{2}\theta-\sigma = n\pi$ or $\theta = \tfrac{2}{3}n\pi+\tfrac{2}{3}\sigma$.

The possible values for the parameters of points $P$, $Q$, $R$ whose pedal lines pass through $(C, S)$ are therefore $\tfrac{2}{3}\sigma$, $\tfrac{2}{3}\pi+\tfrac{2}{3}\sigma$, $\tfrac{4}{3}\pi+\tfrac{2}{3}\sigma$, obtained by giving $n$ the values 0, 1, 2; the value of $\theta$ corresponding to any other value of $n$ gives one or other of the same points $P$, $Q$, $R$.

Hence the angles which $PQ$, $QR$, $RP$ subtend at the centre of the circle are each equal to $\tfrac{2}{3}\pi$, and the triangle $PQR$ is equilateral.

**Note.** If $\alpha$, $\beta$, $\gamma$, $\delta$ are four points on the circle $x^2+y^2=R^2$, and

$$2C' \equiv R(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta),$$
$$2S' \equiv R(\sin\alpha+\sin\beta+\sin\gamma+\sin\delta),$$
$$2\sigma' \equiv \alpha+\beta+\gamma+\delta,$$

then the equation $(x-C')\sin(\sigma'-\theta)-(y-S')\cos(\sigma'-\theta)=0$, when $\theta$ has any one of the values $\alpha$, $\beta$, $\gamma$, or $\delta$, represents the pedal line of the corresponding point with respect to the remaining three points. The symmetry of this result is helpful in problems relating to the pedal lines of four points.

15. The pedal line of $P$ with respect to the triangle $ABC$ bisects the join of $P$ to the orthocentre of the triangle.

16. If the pedal lines of $P$, $P'$ are perpendicular, then $PP'$ is a diameter of the circle.

17. The pedal lines of $PP'$ are inclined at an angle equal to half the angle subtended by $PP'$ at the centre.

18. If $PL$ meets the circumcircle at $R$, $AR$ is parallel to the pedal line of $P$.

19. If the pedal line of $P$ is parallel to $BC$, find $P$.

20. The pedal lines of three points $PQR$ on the circumcircle of a triangle $ABC$ with respect to $ABC$ form a triangle similar to $PQR$.

21. The pedal lines of the extremities of any diameter intersect on the nine-point circle.

22. If $PP'$ is parallel to $BC$, then $P'A$ is perpendicular to the pedal line of $P$.

23. If $A, B, C, D$ are four concyclic points, and the pedal lines of each with respect to the other three be drawn, these four pedal lines meet in a point: prove also that the centre of mean position of $ABCD$ bisects the join of the centre to the point of concurrence of the pedal lines.

24. There are three pedal lines which pass through any given point.

25. If the pedal lines of the points $\theta$, $\phi$, $\psi$ on the circle $x^2+y^2=R^2$ with respect to the points $\alpha, \beta, \gamma$ are concurrent, then $\theta+\phi+\psi=\alpha+\beta+\gamma+2n\pi$.

26. $A, B, C, D$ are fixed points on a circle on which moves a variable point $P$, and the pedal lines of $C$ and $D$ with respect to $ABP$ meet at $Q$. Show that the locus of $Q$ is a circle.

§ 7. *The equation of the circle whose centre* $(\alpha, \beta)$ *and radius* $r$ *are given, is*

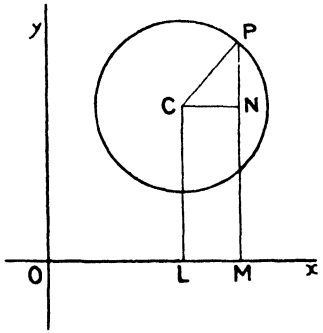
$$(x-\alpha)^2+(y-\beta)^2=r^2. \qquad \text{(II)}$$

This form has occurred previously, and the equations of a chord, tangent, and chord of contact were found.

As in the first form, we can use a parametric system of coordinates for a point on this circle. The coordinates of the point

$$\{\alpha+r\cos\theta,\ \beta+r\sin\theta\}$$

satisfy the equation of the circle for all values of $\theta$.




If $C$ is the centre $(\alpha, \beta)$ of the circle, and $P$ is any point $\theta$ on the circle, then $\theta$ is the angle which the radius $CP$ makes with the axis of $x$.

Thus in the figure the $x$-coordinate of $P$ is

$$OM=OL+LM=\alpha+r\cos\theta,$$

and the $y$-coordinate

$$MP=LC+NP=\beta+r\sin\theta.$$

When $\theta$ is in the second quadrant the $x$-coordinate of $P$ is less than $\alpha$ and the $y$-coordinate greater

than $\beta$, which corresponds to the fact that $\cos\theta$ is negative and $\sin\theta$ positive when $\theta$ lies in the second quadrant. The student should consider the coordinates of $P$ when it lies in the third and fourth quadrants.

*To find the equation of the chord joining two points $\theta$ and $\phi$ on the circle $(x-\alpha)^2+(y-\beta)^2=r^2$.*

The following is a variation of the former method of finding the equation of a chord.

Let the equation of the chord be

$$Ax+By+C=0. \tag{i}$$

Since the given points lie on this

$$A(\alpha+r\cos\theta)+B(\beta+r\sin\theta)+C=0, \tag{ii}$$

$$A(\alpha+r\cos\phi)+B(\beta+r\sin\phi)+C=0. \tag{iii}$$

From (ii) and (iii) by subtraction

$$A(\cos\phi-\cos\theta)=B(\sin\theta-\sin\phi);$$

$$\therefore \quad \frac{A}{\cos\frac{1}{2}(\theta+\phi)}=\frac{B}{\sin\frac{1}{2}(\theta+\phi)}.$$

Now from (i) and (ii) by subtraction the coordinates of any point on the chord satisfy

$$A(x-\alpha)+B(y-\beta)=Ar\cos\theta+Br\sin\theta.$$

Substituting for the ratio $A:B$ we get

$$(x-\alpha)\cos\tfrac{1}{2}(\theta+\phi)+(y-\beta)\sin\tfrac{1}{2}(\theta+\phi)=r\cos\tfrac{1}{2}(\theta-\phi).$$

The equation of the tangent at the point $\theta$ is then

$$(x-\alpha)\cos\theta+(y-\beta)\sin\theta=r.$$

Conversely, all lines whose equations are of this form touch the circle.

Since $\theta$ is the same angle in both forms I and II, these equations follow from those found for I by transferring the origin to the point $(-\alpha, -\beta)$. It is important to notice that when the single variable $(\theta)$ is used to denote a point on a circle, the equation of the tangent (or any line depending on a single point of the circle) takes a definite form and is completely known when $\theta$ is known.

i. *To find the equations of tangents to the circle $(x-\alpha)^2+(y-\beta)^2=r^2$ parallel to the given straight line $y=x\tan\phi$.*

If $(x-\alpha)\cos\theta+(y-\beta)\sin\theta=r$ be such a tangent, then

$$\cot\theta=-\tan\phi=\cot(\tfrac{1}{2}\pi+\phi);$$

$$\therefore \quad \theta=(\tfrac{1}{2}\pi+\phi) \text{ or } (\pi+\tfrac{1}{2}\pi+\phi).$$

The tangents are therefore

$$(x-\alpha)\sin\phi-(y-\beta)\cos\phi+r=0,$$
$$(x-\alpha)\sin\phi-(y-\beta)\cos\phi-r=0.$$

ii. *Two equal circles pass each through the centre of the other: tangents are drawn from points on the first to the second. Show that the feet of the perpendiculars from the centre of the second circle to the chords of contact lie on the straight line perpendicular to the line of centres and midway between them.*

Take the line of centres and the perpendicular line midway between them as axes. Then the centres of the circles are $(a, 0)$, $(-a, 0)$ and the radii are each $2a$. Their equations are therefore

$$(x-a)^2+y^2=4a^2, \quad \text{(i)}$$
$$(x+a)^2+y^2=4a^2. \quad \text{(ii)}$$

Any point on (i) is $(a+2a\cos\theta, 2a\sin\theta)$, and the chord of contact of tangents from it to the second circle is

$$(x+a)(2a+2a\cos\theta)+y\,2a\sin\theta=4a^2,$$

i. e. $$(x+a)(1+\cos\theta)+y\sin\theta=2a$$

or $$(x+a)\cos^2\tfrac{1}{2}\theta+y\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta=a,$$

i. e. $$x\cos^2\tfrac{1}{2}\theta+y\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta=a\sin^2\tfrac{1}{2}\theta,$$

which can be written

$$x+y\tan\tfrac{1}{2}\theta=a\tan^2\tfrac{1}{2}\theta. \quad \text{(iii)}$$

The perpendicular to this from $(-a, 0)$ is

$$y=(x+a)\tan\tfrac{1}{2}\theta. \quad \text{(iv)}$$

Eliminating $\tan\frac{1}{2}\theta$ between equations (iii) and (iv) we get $x=0$ as the equation satisfied by the coordinates of the point common to (iii) and (iv) whatever value $\theta$ may have.

This is then the locus required.

iii. *Chords of one circle are drawn which are tangents to another: find the locus of the points of intersection of the tangents at their extremities. In what case is the locus a circle?*

Let the circle be $x^2+y^2=r^2$ and suppose the chords touch $(x-\alpha)^2+y^2=R^2$, i. e. we have taken the centre of one circle as origin and the line joining the centres as axis of $x$.

The equation of a tangent to the latter circle can be written

$$(x-\alpha)\cos\theta+y\sin\theta=R. \quad \text{(i)}$$

Suppose that $(\xi, \eta)$ is the point of intersection of the tangents to the first circle at the ends of this chord: the equation of the chord must then be

$$x\xi + y\eta = r^2; \qquad \text{(ii)}$$

and therefore (i) and (ii) are the same straight line.

Hence $$\frac{\xi}{\cos\theta} = \frac{\eta}{\sin\theta} = \frac{r^2}{R + \alpha\cos\theta}.$$

Hence $$\frac{\xi}{\cos\theta} = \frac{\eta}{\sin\theta} = \frac{r^2 - \alpha\xi}{R}.$$

Eliminating $\theta$, we get

$$R^2(\xi^2 + \eta^2) = (r^2 - \alpha\xi)^2.$$

Hence the point $(\xi, \eta)$ always lies on the locus

$$R^2(x^2 + y^2) = (r^2 - \alpha x)^2,$$

i. e. $$x^2(R^2 - \alpha^2) + y^2R^2 + 2\alpha r^2 x - r^4 = 0,$$

which is a circle only when $\alpha = 0$, i. e. when the given circles are concentric.

### Examples V f.

1. Find the equation of the tangents to the circle $x^2 + y^2 - 6x - 4y + 5 = 0$ which make an angle of 45° with the axis of $x$. Verify by a figure.

2. Find the equations of the tangents to the circle $(x-a)^2 + (y-b)^2 = r^2$ which make an angle $\alpha$ with the $x$-axis.

3. Find the equations of the circles whose centres are at the origin and which touch $(x-3)^2 + (y-4)^2 = 4$.

4. A tangent to the circle $x^2 + y^2 = R^2$ is perpendicular to a tangent to $(x-a)^2 + (y-b)^2 = r^2$. Find the locus of their intersection.

5. Tangents are drawn to a circle from points on a straight line which does not cut the circle; show that the chords of contact are concurrent.

6. Find the equation of a line inclined at 45° to the axis of $x$, such that the two circles $x^2 + y^2 = 4$, $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal chords on that line.

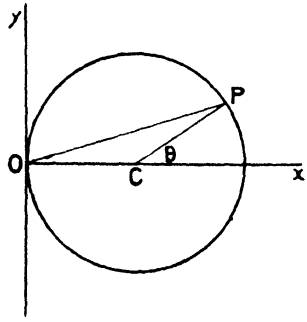
7. Chords of a circle, whose centre is the origin, are tangents to the circle $(x-a)^2 + (y-b)^2 = r^2$; find the equation of the locus of their mid-points.

8. A system of circles of radius $r$ have their centres on a fixed circle of radius $r$. Find the locus of points on these circles the tangents at which are in a given direction.

9. Tangents are drawn one to each of two concentric circles and include an angle of 60°. Find the locus of their intersections.

§ 8. *The equation of a circle referred to a tangent and normal as coordinate axes is* $x^2+y^2=2rx$.

Let the tangent be the axis of $y$: the normal passes through the centre.




Geometrically it is clear that, if the radius is $r$, the centre is $(r, 0)$, and consequently the equation of the circle is

$$(x-r)^2+y^2=r^2$$

or
$$x^2+y^2=2rx. \qquad \text{(i)}$$

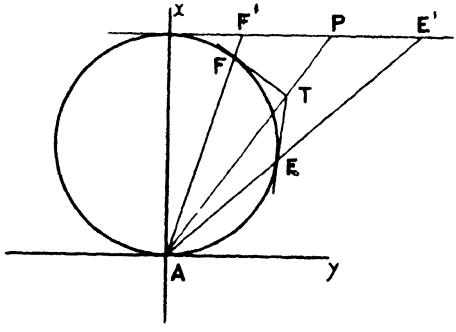
This equation can be obtained algebraically; for, if the equation of the circle is $x^2+y^2+2gx+2fy+c=0$, then, since the line $x=0$ touches the circle at the point (0, 0), the two values of $y$ given by $y^2+2fy+c=0$ must both be zero; hence $f=0$, and $c=0$.

The coordinates of the point $\{r(1+\cos\theta), r\sin\theta\}$ satisfy equation (i) whatever value $\theta$ may have, and these coordinates may be used as those of any point on the circle. The angle $\theta$ is $PCx$ and $\angle POx=\frac{1}{2}\theta$.

**Note.** If two circles touch one another their equations can be written, by a proper choice of axes,

$$x^2+y^2=2rx, \quad x^2+y^2=2Rx.$$

**Ex.** *$AB$ is a diameter of a circle and $P$ any point on the tangent at $B$: a point $T$ is taken on the straight line $AP$; and $TE$, $TF$ are drawn to touch the circle in $E$ and $F$. Show that $AE$ and $AF$ intersect the tangent at $B$ in points equidistant from $P$.*




Take the tangent at $A$ as axis of $y$ and the normal as axis of $x$.

The equation of the circle is

$$x^2+y^2=2rx; \qquad \text{(i)}$$

the tangent at $B$ is
$$x=2r.$$

Let $T$ be the point $(\xi, \eta)$, then the chord of contact $EF$ is

$$x(\xi - r) + y\eta = r\xi. \tag{ii}$$

Now the equation $$x^2 + y^2 = 2rx \cdot \frac{x(\xi - r) + y\eta}{r\xi} \tag{iii}$$

is satisfied by the coordinates of all points which lie on (i) and (ii), and therefore by those of $E$, $F$.

Further, the equation is homogeneous and therefore represents two straight lines through the origin, i.e. represents $AE$, $AF$.

The values of $BE', BF'$ are the values of $y$ which satisfy (iii) when $x = 2r$.

$\therefore$ $BE'$, $BF'$ are given by

$$y^2\xi - 4r\eta y + 4r^2\xi - 8r^2(\xi - r) = 0.$$

$\therefore$ $BE' + BF'$ = the sum of the roots $= \dfrac{4r\eta}{\xi}$.

But $AT$ is the line $y = \dfrac{x\eta}{\xi}$; hence $BP = \dfrac{2r\eta}{\xi}$.

$\therefore$ $BE' + BF' = 2BP$ or $P$ is the mid-point of $E'F'$.

## Examples V g.

1. Show that the equation of the chord joining two points

$$\{r(1 + \cos\theta), r\sin\theta\}, \ \{r(1 + \cos\phi), r\sin\phi\}$$

on the circle $$x^2 + y^2 = 2rx$$

is $$x\cos\tfrac{1}{2}(\theta + \phi) + y\sin\tfrac{1}{2}(\theta + \phi) = 2r\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\phi,$$

and the tangent at the point $\theta$ is

$$x\cos\theta + y\sin\theta = r(1 + \cos\theta).$$

2. The line joining the origin to any point $\theta$ on the circle $x^2 + y^2 = 2rx$ is $y = x\tan\tfrac{1}{2}\theta$.

3. The middle point of the chord joining the origin to the point $\theta$ is

$$(r\cos^2\tfrac{1}{2}\theta, \ r\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta).$$

4. The locus of the middle points of chords of the circle $x^2 + y^2 = 2rx$ which are drawn through the origin is the circle $x^2 + y^2 = rx$.

5. Two circles touch one another, and any straight line through the point of contact cuts the circles in $P$ and $Q$. Show that the locus of the middle points of $PQ$ is a circle which touches the given circles and whose radius is the arithmetic mean of the radii of the given circles.

6. $n$ circles touch one another at the same point $A$ and any straight line through the origin meets them in the points $P_1, P_2, \ldots P_n$. Find the locus of a point $Q$ such that

$$n \cdot AQ = AP_1 + AP_2 + \ldots + AP_n.$$

7. Two circles touch one another, tangents to one of them are chords of the other. Find the locus of the mid-point of these chords.

8. Show that the straight line $3(y\cos\theta - 1) = x\cos 2\theta$ cuts the circles $x^2 + y^2 = 2x$, $x^2 + y^2 = 4x$ in four points which form a harmonic range.

9. Show that any point on the circle $x^2+y^2=2rx$ can be represented by $(2r\cos^2\theta,\ r\sin 2\theta)$.

10. Find the coordinates of the intersection of tangents at the points $\theta$ and $\phi$ in Question 9.

11. Show that the pedal of the origin with regard to a triangle whose vertices are $\alpha, \beta, \gamma$ (see Question 9) on the circle $x^2+y^2=2rx$ is

$$x\cos(\alpha+\beta+\gamma)+y\sin(\alpha+\beta+\gamma)=2r\cos\alpha\cos\beta\cos\gamma.$$

12. If $\alpha, \beta, \gamma, \delta$ be four points on this circle, there are four pedal lines of the origin with respect to these four points taken three at a time. Show that the feet of the perpendiculars from the origin on these four pedal lines are collinear. Find the equation of the line.

13. $AB$ is a diameter of a circle of which $P$ is any point. $AP$ meets at $Q$ the straight line drawn through $B$ such that $BP$ and $BQ$ are equally inclined to the tangent at $B$. Find the equation of the locus of $Q$.

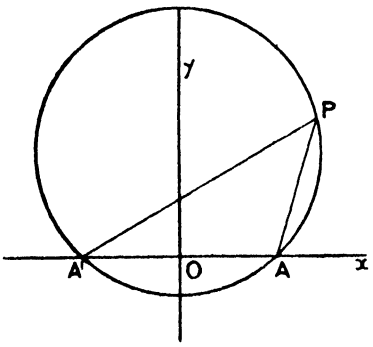
14. $O$ is a fixed point on a circle and tangents to the circle from a point $P$ touch it at $QQ'$. Find the locus of $P$ if $OQ, OQ'$ are harmonic conjugates of a given pair of lines through $O$.

15. Show that there are four chords of the circle $x^2+y^2-4rx-r^2=0$ which subtend right angles at the origin and also touch the circle $x^2+y^2=2rx$; and show that they form a square.

§ 9. *The equation of a system of circles passing through two given points.*

Take $A\ (a,\ 0)$, $A'\ (-a,\ 0)$ for the given points.

If $x^2+y^2+2gx+2fy+c=0$ is the equation of the circle, then the axis $y=0$ meets it at the points $A$ and $A'$.




Thus the values of $x$ which satisfy

$$x^2+2gx+c=0$$

are $a$ and $-a$; therefore $g=0$ and $c=-a^2$.

The required equation is then

$$x^2+y^2+2fy-a^2=0,$$

which for different values of $f$ represents a system of circles passing through $A$ and $A'$: the centres of all these circles lie on the axis of $y$, since $g=0$.

**Ex. i.** *If one side of a triangle inscribed in a circle is fixed, the locus of its centre of gravity is another circle.*

Let $AA'$ be the fixed side and $P\ (x_1, y_1)$ any position of the vertex. The coordinates of the centre of gravity $(G)$ are given by

$$3x=x_1+a-a=x_1,$$
$$3y=y_1.$$

But $$x_1^2 + y_1^2 + 2fy_1 - a^2 = 0$$
if the equation of the circle is
$$x^2 + y^2 + 2fy - a^2 = 0.$$
Hence the locus of $G$ is
$$9x^2 + 9y^2 + 6fy - a^2 = 0,$$
which is a circle.

**Ex. ii.** *To find the locus of a point $P$ which moves so that the angle $APA'$ is constant, where $A$ and $A'$ are fixed points.*

Let $P(x', y')$ be any position of the point.

The lines $AP$ and $A'P'$ are
$$\frac{x-a}{x'-a} = \frac{y}{y'}, \quad \frac{x+a}{x'+a} = \frac{y}{y'}.$$
If $\angle APA' = \alpha$,
$$\pm \tan\alpha = \frac{\dfrac{y'}{x'-a} - \dfrac{y'}{x'+a}}{1 + \dfrac{y'^2}{x'^2 - a^2}} = \frac{2ay'}{x'^2 + y'^2 - a^2}.$$
Hence the locus is the two circles $x^2 + y^2 - a^2 = \pm 2ay \cot\alpha$.

## Examples V h.

1. Find the equation of a circle through the two points $(-a, 0)$, $(a, 0)$ which also
   (i) passes through the point $(h, k)$;
   (ii) touches the line $x + y = 3a$;
   (iii) is such that the tangents at the given points make an angle 30° with the join of the points.

2. $A$, $B$ are fixed points on the circumference of a circle; $AP$, $BQ$ are parallel chords. Prove that the locus of the intersection of $AQ$, $BP$ is the circle through $A$, $B$ and the centre of the given circle.

3. A series of circles pass through two fixed points; find the equation of the locus of the points of contact of tangents to these circles parallel to a fixed straight line.

4. Circles are drawn through the points $(-a, 0)$, $(a, 0)$; find the locus of the point of contact of tangents to these circles which pass through the point $(2a, 0)$.

5. Show that there are two circles of the system $x^2 + y^2 - 2\lambda y = a^2$, where $\lambda$ is a variable constant, which touch the straight line $x\cos\alpha + y\sin\alpha - p = 0$, and that they are real only if $p$ is $=$ or $> a\cos\alpha$. If $\lambda_1$, $\lambda_2$ are the values of $\lambda$ for these circles, show that $\lambda_1\lambda_2 = (a^2 - p^2)\sec^2\alpha$. When the value of $p$ is such that the two circles are coincident, show that their equation is $x^2 + y^2 + 2ay\tan\alpha = a^2$.

6. Two fixed circles intersect in $A$, $B$; $P$ is a variable point on one of

them, and $PA$ meets the other circle in $X$ and $PB$ meets it in $Y$. Prove that $BX$ and $AY$ intersect on a fixed circle.

7. Two circles intersect in $A$ and $B$. A line through $A$ meets one circle again in $P$ and a parallel line through $B$ meets the other circle again in $Q$. Prove that the locus of the middle point of $PQ$ is a circle.

8. The base of a triangle is fixed and its vertical angle is given; find the locus of (i) its centroid, (ii) the inscribed centre, (iii) the orthocentre.

§ 10. *The equation of the circle on the straight line joining* $(x_1, y_1)$, $(x_2, y_2)$ *as diameter is*

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

The centre is the mid-point of the line joining $(x_1, y_1)$, $(x_2, y_2)$,

i. e. $$\{\tfrac{1}{2}(x_1+x_2),\ \tfrac{1}{2}(y_1+y_2)\}.$$

Hence its equation is of the form

$$x^2+y^2-x(x_1+x_2)-y(y_1+y_2)+c=0.$$

But $(x_1, y_1)$ lies on this; hence

$$x_1^2+y_1^2-x_1^2-x_1x_2-y_1^2-y_1y_2+c=0,$$

or $$c=x_1x_2+y_1y_2;$$

the equation is then

$$x^2+y^2-x(x_1+x_2)-y(y_1+y_2)+x_1x_2+y_1y_2=0,$$

or $$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0;$$

e. g. the circle of which the line joining $(-5, 4)$, $(2, -3)$ is a diameter is

$$(x+5)(x-2)+(y-4)(y+3)=0,$$

or $$x^2+y^2+3x-y-22=0.$$

### Examples V i.

1. Find the real points where the circle on the line joining $(13, 5)$, $(-7, 15)$ as diameter cuts the $x$-axis.

2. Find the equation of the circle on the line joining the points $(a\lambda^2, 2a\lambda)$, $(a/\lambda^2-2a/\lambda)$ as diameter.

Where does it meet the line $x=-a$?

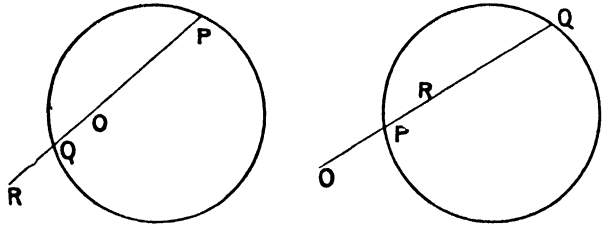
3. Find the equation of the circle on the chord of the circle $x^2+y^2=125$ as diameter whose equation is $x+3y=35$.

4. Find the equation of the circle on the line joining the points $(1, 3)$, $(5, 1)$ as diameter, and the coordinates of the extremities of the perpendicular diameter.

5. A circle is described on the line joining $(3, 7)$, $(9, 1)$ as diameter. Show that it touches $x+y-4=0$.

6. Find the equation of the circle whose diameter is the chord of the circle $x^2+y^2=169$ whose mid-point is $(8, 4)$.

§ 11. *To find an equation giving the lengths of the segments of a chord of a circle drawn through any given point in a given direction.*




If the given point is outside the circle the segments $OP$, $OQ$ are both of the same sign: if the given point is inside the circle the segments $OP$, $OQ$ are of different sign.

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0,$$

and let the given point $O$ be $(\alpha, \beta)$ and let the chord through $O$ make an angle $\theta$ with the axis of $x$. The equation of the chord is

$$\frac{x-\alpha}{\cos\theta}=\frac{y-\beta}{\sin\theta}=r,$$

where $r$ is the distance of any point $(x, y)$ on the line from the fixed point $O$.

Now, if $r$ is equal to $OP$, the point $(x, y)$ is $P$; its coordinates are

$$x=\alpha+r\cos\theta, \quad y=\beta+r\sin\theta,$$

and it lies on the circle. Similarly for $Q$.

Hence, if $r$ is equal to either $OP$ or $OQ$,

$$(\alpha+r\cos\theta)^2+(\beta+r\sin\theta)^2+2g(\alpha+r\cos\theta)+2f(\beta+r\sin\theta)+c=0,$$

i.e. $r^2+2r\{(\alpha+g)\cos\theta+(\beta+f)\sin\theta\}+\alpha^2+\beta^2+2g\alpha+2f\beta+c=0.$

Now, if we write for convenience,

$$f(x, y)\equiv x^2+y^2+2gx+2fy+c,$$

then

$$f(\alpha, \beta)\equiv \alpha^2+\beta^2+2g\alpha+2f\beta+c.$$

Hence the equation may be written

$$r^2+2r\{(\alpha+g)\cos\theta+(\beta+f)\sin\theta\}+f(\alpha, \beta)=0,$$

which is a quadratic in $r$ whose roots are equal to $OP$ and $OQ$.

Hence (i) The product of the roots $= OP.OQ = f(\alpha, \beta)$.

This is constant for a given position of $O$, whatever the value of $\theta$; hence the well-known proposition:

'The rectangle contained by the segments of a chord of a circle which passes through a fixed point is constant.'

Incidentally, we see that if the chord meets the circle in two imaginary points $P$, $Q$ the rectangle of $OP.\ OQ$ has a real value.

Also if $f(\alpha, \beta)$ is positive, $OP.\ OQ$ is positive, and $O$ lies outside the circle; but if $f(\alpha, \beta)$ is negative, $OP.\ OQ$ is negative, and $O$ lies inside the circle.

[This can be shown directly; for the centre $C$ of the circle is $(-g, -f)$; hence $OC^2 = (\alpha+g)^2+(\beta+f)^2$;

$\therefore$ $OC^2-(\text{radius})^2 = \alpha^2+\beta^2+2g\alpha+2f\beta+c = f(\alpha, \beta).$]

(ii) Note that when $O$ is at the centre, i.e.

$$\alpha = -g \text{ and } \beta = -f,$$

the coefficient of $r$ vanishes and we have

$$r^2 = -f(\alpha, \beta),$$

i. e. $OP$, $OQ$ are equal and opposite for all values of $\theta$. (Cf. p. 126.)

(iii) If the values of $r$ given by the equation are equal and opposite, i. e. if $O$ is the middle point of the chord $PQ$, then we must have

$$(\alpha+g)\cos\theta+(\beta+f)\sin\theta = 0.$$

If now we consider the mid-point $O(\alpha, \beta)$ of $PQ$ unknown and $\theta$ constant, i. e. the chord to be drawn in a fixed direction $\theta$, then $(\alpha, \beta)$ must lie on the line

$$(x+g)\cos\theta+(y+f)\sin\theta = 0,$$

which is therefore the locus of the mid-points of all chords of the circle drawn in a fixed direction $\theta$.

(The question of the chord meeting the circle in real points has not arisen; hence the mid-point of the line joining a pair of imaginary points of intersection is real.)

Evidently then 'the locus of the mid-points of parallel chords of a circle is a straight line through the centre perpendicular to the chords'.

(iv) *To deduce the equation of the pair of tangents to the circle from the point* $O(\alpha, \beta)$.

If the chord through $O$, drawn in the direction $\theta$, cuts the circle in $P$ and $Q$, the lengths of $OP$, $OQ$ are given by

$$r^2+2r\{(\alpha+g)\cos\theta+(\beta+f)\sin\theta\}+f(\alpha, \beta) = 0; \qquad \text{(i)}$$

but if this chord touches the circle $P$ and $Q$ coincide and the lengths $OP$, $OQ$ are equal. The condition that the roots of equation (i) should be equal, viz.

$$\{(\alpha+g)\cos\theta+(\beta+f)\sin\theta\}^2 = f(\alpha, \beta), \qquad \text{(ii)}$$

gives us an equation from which to determine the directions $\theta$ in which to draw tangents to the circle.

If we choose either of these values of $\theta$, then any point on the line (which is now a tangent) satisfies

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta}.$$

Hence, substituting for $\cos\theta$ and $\sin\theta$ in (ii), we see that

$$\{(\alpha+g)(x-\alpha)+(\beta+f)(y-\beta)\}^2 = f(\alpha,\beta)\{(x-\alpha)^2+(y-\beta)^2\}$$

is an equation satisfied by any point on a tangent from $O$ to the circle, i.e. is the equation of the tangents from $O$.

This can be at once reduced to the form

$$(x^2+y^2+2gx+2fy+c)(\alpha^2+\beta^2+2g\alpha+2f\beta+c) = \{x(\alpha+g)+y(\beta+f)+\alpha g+\beta f+c\}^2.$$

Now we have shown that the equation

$$x(\alpha+g)+y(\beta+f)+\alpha g+\beta f+c = 0$$

represents the chord of contact of tangents from $O(\alpha, \beta)$ to the circle.

Thus, if $f(x, y) = 0$ is a circle, and $u = 0$ is the equation of the chord of contact of tangents from a point $O(\alpha, \beta)$ to the circle, the equation of these tangents is $f(x, y) \cdot f(\alpha, \beta) = u^2$.

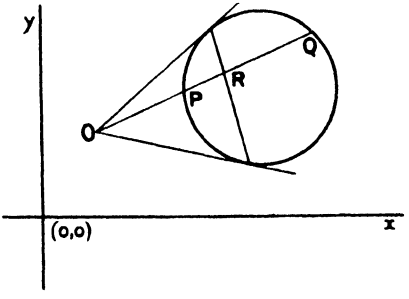
Incidentally, if $\theta$ is the direction of a tangent, equation (i) gives the length $OP$ of this tangent: we see, as previously proved, that

$$OP^2 = f(\alpha, \beta).$$

(v) *If a chord through the point O cuts a circle in the points P and Q, to find the locus of a point R on PQ which is such that*

$$\frac{2}{OR} = \frac{1}{OP} + \frac{1}{OQ},$$

*or, in other words, the point R which is the harmonic conjugate of O with respect to P and Q.*




Let the equation of any such chord be

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r, \qquad \text{(i)}$$

where $O$ is the point $(\alpha, \beta)$; then the lengths of $OP$, $OQ$ are given by

$$r^2+2r\{(\alpha+g)\cos\theta+(\beta+f)\sin\theta\}+f(\alpha,\beta)=0.$$

Hence $$OP+OQ=-2\{(\alpha+g)\cos\theta+(\beta+f)\sin\theta\},$$
$$OP.OQ=f(\alpha,\beta).$$

Hence $$\frac{1}{OP}+\frac{1}{OQ}=-\frac{2\{(\alpha+g)\cos\theta+(\beta+f)\sin\theta\}}{f(\alpha,\beta)}$$
$$=\frac{2}{OR}\text{ (by hypothesis).}$$

Thus $$OR\cos\theta\,(\alpha+g)+OR\sin\theta\,(\beta+f)+f(\alpha,\beta)=0.$$

But since $R$ is on the line $PQ$ (i), if its coordinates are $(x, y)$,

$$x-\alpha=OR\cos\theta,\ y-\beta=OR\sin\theta,$$

i. e. $R$ lies on the locus

$$(x-\alpha)(\alpha+g)+(y-\beta)(\beta+f)+\alpha^2+\beta^2+2g\alpha+2f\beta+c=0,$$

i. e. $$x(\alpha+g)+y(\beta+f)+g\alpha+f\beta+c=0.$$

This is a straight line, and, when $O$ lies outside the circle, it represents the chord of contact of tangents from $O$.

The locus is called in all cases the polar of $O$, and is discussed later.

The above results can be obtained in the following manner :—

Let $T$ be a point $(x, y)$ and let the straight line $OT$ cut the circle at $P$ and $Q$. Since $P$ and $Q$ are points on the straight line joining the points $(\alpha, \beta)$ and $(x, y)$, their coordinates are of the form $\left(\frac{lx+\alpha}{l+1}, \frac{ly+\beta}{l+1}\right)$, where the values of $l$ are given by the condition that $P$ and $Q$ should lie on the circle. Substituting these coordinates in the equation of the circle, we obtain

$$l^2.f(x, y)+2lu+f(\alpha, \beta)=0,$$

where $f(x, y)=0$ is the equation of the circle, and

$$u=(\alpha+g)x+(\beta+f)y+g\alpha+f\beta+c.$$

If $OT$ is a tangent to the circle, $P$ and $Q$ coincide, so that the values of $l$ are equal, hence $f(x, y).f(\alpha, \beta)=u^2$.

This is an equation satisfied by the coordinates of any point $T$ such that $OT$ is a tangent to the circle; it is therefore the equation of the tangents from $O$ to the circle.

If $T$ is the harmonic conjugate of $O$ with respect to $P$ and $Q$, then the values of $l$ are equal and opposite. This gives us $u=0$; this is the equation of the locus found in (v).

### Examples V j.

1. Find the equation of tangents from the point $(h, k)$ to the circle $x^2+y^2=r^2$.

2. Show that, with the notation of Chapter IV, the equation of a pair of tangents from $(x', y')$ to the circle $u=0$ is $uu'=\{xX'+yY'+Z'\}^2$.

3. Find the angle between the tangents from the point (4, 3) to the circle $x^2+y^2-2x-4y=0$.

4. Find the locus of points the tangents from which to the circle

$$x^2+y^2+2gx+2fy+c=0$$

are at right angles.

5. Find the locus of points the tangents from which to the circle $x^2+y^2+4x-6y+12=0$ include an angle of 120°.

6. If $PT$, $PT'$ are two tangents to a circle centre, $C$, $Q$ any point on $PT$, and $QN$ the perpendicular from this point on $TT'$, show that $PT : CP = QN : QT$ and deduce the equation of a pair of tangents in the form found above.

7. Find the locus of the middle points of chords of the circle

$$x^2+y^2+6x-8y+17=0$$

which make an angle $\cos^{-1}\frac{3}{5}$ with the axis of $x$.

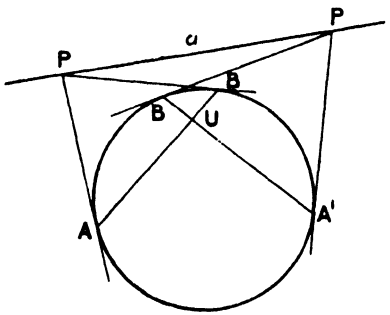
### § 12. Poles and Polars.

(1) *If tangents be drawn to a circle from any point on a given straight line, the chords of contact will all pass through a fixed point.*

(2) *If chords of a circle are drawn through a given point, the tangents at their extremities will meet on a fixed straight line.*

To prove these propositions take the equation of the circle in its simplest form:

$$x^2+y^2=r^2.$$




(1) Let the equation of the given straight line be

$$u \equiv lx+my+n=0.$$

If $P(x', y')$ is any point on this straight line, we have the condition

$$lx'+my'+n=0. \qquad \text{(i)}$$

The equation of the chord of contact of tangents from $(x', y')$ is (p. 145)

$$xx'+yy'=r^2.$$

Using the condition (i) this can be written

$$mxx'-y(n+lx')=mr^2,$$

or

$$x'(mx-ly)=ny+mr^2.$$

This contains one undetermined constant $x'$ in the first degree only, and therefore represents a straight line passing through the fixed point given by

$$mx = ly \text{ and } ny = -mr^2,$$

i. e. the fixed point $U\left(-\frac{lr^2}{n}, -\frac{mr^2}{n}\right)$.

(2) Let the fixed point be $U(\xi, \eta)$. Any straight line through this is

$$y-\eta+k(x-\xi)=0. \qquad \text{(i)}$$

Suppose the tangents at its point of intersection with the circle meet at $(x', y')$, then since

$$xx'+yy'=r^2 \qquad \text{(ii)}$$

is the chord of contact of tangents from $(x', y')$ to the circle, the lines (i) and (ii) are identical. Hence

$$\frac{x'}{k}=\frac{y'}{1}=\frac{r^2}{\eta+k\xi},$$

and therefore

$$=\frac{r^2-x'\xi}{\eta}.$$

Hence $x'\xi+y'\eta=r^2$,

i. e. the point $(x', y')$ always lies on the straight line

$$u \equiv x\xi+y\eta-r^2=0.$$

**Pole and Polar.** If tangents are drawn to a circle from any point on a straight line $u$, their chords of contact all pass through a point $U$.

And if chords of a circle are drawn through a point $U$, the tangents at their extremities intersect on a straight line $u$.

The point $U$ is called the 'pole' of the line $u$, and the line $u$ is called the 'polar' of the point $U$.

**Notes.** (i) If the line $u$ cuts the circle at $A$ and $B$, real tangents cannot be drawn from points between $A$ and $B$ on the line: it can be shown, however, that the chords of contact of pairs of imaginary tangents from these points pass through $U$. When the chosen point is $A$, the tangents from it coincide and their chord of contact is the tangent at $A$; so also at $B$: hence $U$ is the intersection of the tangents at $A$ and $B$. This can also be seen from the above algebraical result.

For if $u \equiv lx+my+n=0$, $U$ is the point $\left(-\frac{lr^2}{n}, -\frac{mr^2}{n}\right)$, and the chord of contact of tangents from $U$ is $x\frac{lr^2}{n}+y\frac{mr^2}{n}+r^2=0$ or $lx+my+n=0$, i.e. the line $u=0$.

(ii) If the point $U$ lies outside the circle, and $UA$, $UB$ are the tangents from $U$ to the circle, only lines through $U$ which lie between $UA$ and $UB$ cut the circle in real points. The tangents at the imaginary points of intersection of the circle and other lines through $U$ may be shown algebraically to meet on $u$.

The line $UA$ meets the circle in coincident points: hence the tangents at the points of intersection coincide and intersect at $A$: thus $A$ is on $u$. Similarly $B$ is on $u$. Hence, if $U$ lies outside the circle, the polar of $U$ is the chord of contact of tangents from $U$.

(iii) Many different definitions of the pole and polar are used in various books on Geometry. We have adopted the above because it applies equally, as will be seen later, to all curves represented by the general equation of the second degree, and also, although in our notes we have introduced imaginary considerations, the pole and polar can be found from this definition by a real construction in every case.

(iv) It should be noted that the equation of the polar of *any point* $(x', y')$ takes the same form as that of the tangent at a point $(x', y')$ on the circle. A tangent to a circle is the polar of its point of contact.

For the general circle

$$x^2+y^2+2gx+2fy+c=0$$

the polar of the point $(x', y')$ is, in our usual notation,

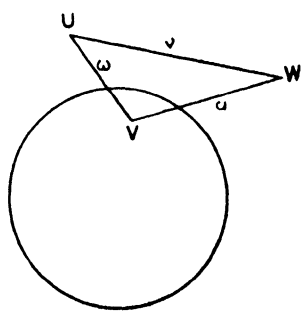
$$xX'+yY'+Z'=0.$$

This should be remembered, as the equation is true for any curve of the second degree.

*Propositions on the Pole and Polar.*

(i) *If the polar of $U$ passes through $V$, the polar of $V$ passes through $U$.*

Let $U$, $V$ be the points $(x_1, y_1)$, $(x_2, y_2)$; the polar of $U$ with respect to the circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$; this passes through $V$ if $x_2x_1+y_2y_1=a^2$. Which is also the condition that $U(x_1, y_1)$ should lie on $xx_2+yy_2=a^2$ the polar of $V$.

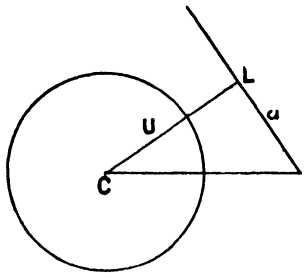



(ii) *If* w *is the join of the poles of the lines* u *and* v, *then* w *is the polar of the point of intersection of* u *and* v.

(iii) *If the pole of the line* u *lies on the line* v, *then the pole of the line* v *lies on the line* u.

These results are immediate consequences of (i).

(iv) *If U is the pole of* u *with respect to a circle whose centre is C, then*




(a) *CU is perpendicular to* u.

(b) *If CU meets* u *at L, then*

$$CU \cdot CL = (radius)^2.$$

Let the circle be $x^2+y^2=r^2$.

(a) If $U$ is the point $(x_1, y_1)$, then $u$ is the line

$$xx_1+yy_1=r^2. \qquad \text{(i)}$$

Also $CU$ is the line

$$xy_1-yx_1=0, \qquad \text{(ii)}$$

and the lines (i) and (ii) are evidently perpendicular.

(b) Again $$CU=\sqrt{x_1^2+y_1^2},$$

$$CL=\frac{r^2}{\sqrt{x_1^2+y_1^2}};$$

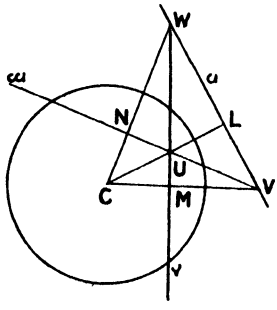
i.e. $$CU \cdot CL = r^2.$$

**Definitions.**

(i) $U$ and $L$ are called inverse points with respect to the circle.

(ii) If the polar of $U$ passes through $V$, $U$ and $V$ are called conjugate points.

(iii) If $U$, $V$, $W$ are three points such that the polar of each with respect to a circle is the line joining the other two, the triangle $UVW$ is said to be self-conjugate with respect to this circle.




(v) *If a triangle is self-conjugate with respect to a circle, the centre of the circle is the orthocentre of the triangle.*

If $UVW$ be a triangle self-conjugate with respect to the circle, then $VW$ is the polar of $U$. We have just shown that $CU$ is perpendicular to $VW$, and similarly $CV$ and $CW$ are perpendicular to $UW$ and $UV$, i.e. $C$ is the orthocentre of the triangle $UVW$.

The circle in this case is called the 'polar-circle' of the triangle.

In the figure

$$CM \cdot CV = CU \cdot CL = CN \cdot CW$$

= square on the radius of the circle.

Evidently, then, the triangle $UVW$ must be obtuse, otherwise the square of the radius will be negative and the polar-circle imaginary.

**Ex.** i. *To find the polar-circle of the triangle whose vertices are* $(R\cos\alpha, R\sin\alpha)$, $(R\cos\beta, R\sin\beta)$, $(R\cos\gamma, R\sin\gamma)$.

The orthocentre of the triangle is the point $(2C, 2S)$, where

$$2C = R(\cos\alpha+\cos\beta+\cos\gamma),$$
$$2S = R(\sin\alpha+\sin\beta+\sin\gamma).$$

Hence the equation of the polar-circle is of the form

$$(x-2C)^2+(y-2S)^2=\rho^2.$$

The polar of $(R\cos\alpha, R\sin\alpha)$ with respect to this circle is

$$(x-2C)(R\cos\alpha-2C)+(y-2S)(R\sin\alpha-2S)=\rho^2,$$

i.e. $$(x-2C)R(\cos\beta+\cos\gamma)+(y-2S)R(\sin\beta+\sin\gamma)+\rho^2=0,$$

or $$2R\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta-\gamma)(x-2C)$$
$$+2R\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta-\gamma)(y-2S)+\rho^2=0;$$

i.e. $$x\cos\tfrac{1}{2}(\beta+\gamma)+y\sin\tfrac{1}{2}(\beta+\gamma)-2C\cos\tfrac{1}{2}(\beta+\gamma)-2S\sin\tfrac{1}{2}(\beta+\gamma)$$
$$+\frac{\rho^2}{2R}\sec\tfrac{1}{2}(\beta-\gamma)=0,$$

i.e. $$x\cos\tfrac{1}{2}(\beta+\gamma)+y\sin\tfrac{1}{2}(\beta+\gamma)-R\cos\{\alpha-\tfrac{1}{2}(\beta+\gamma)\}-2R\cos\tfrac{1}{2}(\beta-\gamma)$$
$$+\frac{\rho^2}{2R}\sec\tfrac{1}{2}(\beta-\gamma)=0.$$

But the chord joining the points $\beta$, $\gamma$ is by hypothesis the polar of the point $\alpha$; hence the equation just found is identical with

$$x\cos\tfrac{1}{2}(\beta+\gamma)+y\sin\tfrac{1}{2}(\beta+\gamma)-R\cos\tfrac{1}{2}(\beta-\gamma)=0.$$

Thus $$\frac{\rho^2}{2R}\sec\tfrac{1}{2}(\beta-\gamma)=R[\cos\{\alpha-\tfrac{1}{2}(\beta+\gamma)\}+\cos\tfrac{1}{2}(\beta-\gamma)]$$
$$=2R\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\alpha-\gamma),$$

or $$\rho^2=4R^2\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha),$$

and the required equation is

$$(x-2C)^2+(y-2S)^2=4R^2\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha).$$

**Ex.** ii. *To find the condition that a triangle may be drawn inscribed in a circle radius* $R$ *and self-conjugate with respect to a circle radius* $\rho$.

Let the first circle be $x^2+y^2=R^2$, and let $\alpha$, $\beta$, $\gamma$ be the vertices of the triangle. Then we have shown in **Ex.** i that the equation of the polar-circle is

$$(x-2C)^2+(y-2S)^2=4R^2\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha).$$

If $d$ is the distance between the centres, we have

$$d^2=4C^2+4S^2$$
$$=R^2[(\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2]$$
$$=R^2[3+2\cos\overline{\alpha-\beta}+2\cos\overline{\beta-\gamma}+2\cos\overline{\gamma-\alpha}]$$
$$=R^2+2R^2[1+\cos\overline{\alpha-\beta}+\cos\overline{\beta-\gamma}+\cos\overline{\gamma-\alpha}],$$

and by elementary transformation this

$$=R^2+8R^2\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha).$$

But $$\rho^2 = 4R^2 \cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha);$$

hence $$d^2 = R^2 + 2\rho^2,$$

which is the required condition.

If one such triangle can be drawn, any number can.

For take the circle $(x-a)^2+(y-b)^2=\rho^2$, where

$$a^2+b^2 = R^2+2\rho^2.$$

Then, if we choose $\alpha$, $\beta$, $\gamma$ so that

$$R(\cos\alpha+\cos\beta+\cos\gamma) = a,$$
$$R(\sin\alpha+\sin\beta+\sin\gamma) = b,$$

we shall find, working backwards, that $(x-a)^2+(y-b)^2=\rho^2$ is the polar-circle of the triangle $\alpha$, $\beta$, $\gamma$. We can take an arbitrary value for $\alpha$ and then find values for $\beta, \gamma$; so that if the condition $d^2 = R^2+2\rho^2$ is satisfied, not only one but any number of such triangles can be described.

## Miscellaneous Examples for Revision.

1. Find the equation of the polar of the origin with respect to the circle $x^2+y^2+2gx+2fy+c=0$.

2. Find the locus of the poles of the straight line $x+y=1$ with respect to a system of circles which pass through the points (2, 0), (−2, 0).

3. Find the coordinates of the pole of the line $3x+4y=5$ with respect to the circle $x^2+y^2=25$.

4. If tangents be drawn from any point on one of three circles (which all pass through two fixed points) to the other two, prove that the ratio of the lengths of these tangents is invariable.

5. Find the equation of tangents from the origin to

$$x^2+y^2-6x+2y+5=0.$$

6. A chord of a fixed circle is such that the sum of the squares of the tangents drawn from its extremities to another fixed circle is constant: prove that the locus of its middle point is a straight line.

7. Find the locus of a point the tangents from which to two fixed circles include equal angles.

8. Two fixed points are conjugate with regard to a circle of given radius. Find the locus of the centre of the circle.

9. From two points $P$, $Q$ perpendiculars are drawn to the polars of $Q$ and $P$ with respect to a circle. Show that the ratio of their lengths is equal to the ratio of the distances of the points from the centre of the circle.

10. Tangents are drawn from a point $P$ to a given circle and meet the tangent at a given point $A$ in $Q$ and $R$. If $AQ+AR$ is equal to a constant length, find the locus of $P$.

11. If $ABC$ is a triangle self-conjugate with respect to a circle, two of the vertices lie outside and one inside the circle.

12. Show that the polar of a point $(x', y')$ with respect to the circle

$$(3x+4y+4)^2+(4x+3y+5)^2 = 24(x+y)^2$$

is $(3x+4y+4)(3x'+4y'+4)+(4x+3y+5)(4x'+3y'+5) = 24(x+y)(x'+y')$.

13. A circle is drawn to touch one side of an equilateral triangle and to make the pole of another side with respect to it lie on the third side. Find the locus of its centre.

14. Show that the equation of the locus of the poles of tangents to the circle $(x-a)^2+y^2=b^2$, taken with respect to the circle $x^2+y^2=c^2$, is

$$(a^2-b^2)x^2-b^2y^2-2c^2ax+c^4=0.$$

15. The sides of a triangle are

$$x/m+y/p-1=0,$$
$$x/-n+y/p-1=0,$$
$$y=0.$$

Find its orthocentre and also the equation of the circum- and nine-point circles. Verify that the centres of the two circles, the orthocentre and the centroid, lie on the same straight line.

16. Find the condition that the straight line joining the points $(x_1, y_1)$, $(x_2, y_2)$ may touch the circle $x^2+y^2=a^2$.

Hence find the equation of the pairs of tangents that can be drawn from $(x_1, y_1)$ to touch the circle.

17. From each of two points $A$, $B$ pairs of tangents are drawn to a circle. Prove that the pole of $AB$ is the intersection of two of the diagonals of the quadrilateral formed by the tangents.

18. A circle turns in its own plane about a point in its circumference. Find the locus of the point of contact of a tangent drawn parallel to a fixed straight line.

19. If $L$ and $M$ are the feet of the perpendiculars drawn from a point $P$ to one fixed pair of lines, and $L'$, $M'$ are the feet of the perpendiculars from $P$ to another fixed pair of lines, prove that, if $LM$ and $L'M'$ are inclined to one another at a constant angle, the locus of $P$ is a circle.

20. Show that the tangents from the origin to the circle whose equation is $x^2+y^2-5k(x+y)+10k^2=0$ are the same, whatever value is assigned to $k$.

For what values of $k$ will this circle touch the straight line $3x+y+15=0$?

21. Show that the equation of the tangents drawn from the point $(h, k)$ to the circle $x^2+y^2=a^2$ is $(x^2+y^2-a^2)(h^2+k^2-a^2)=(hx+ky-a^2)^2$.

Tangents are drawn to this circle from two points on the axis of $x$, equidistant from the point $(c, 0)$. Show that the locus of their intersections is $cy^2=a^2(c-x)$.

22. The vertices $A$, $B$, $C$ of a triangle lie one on each of three concentric circles; and $AB$, $AC$ are parallel to the tangents at $C$, $B$ respectively. Prove that $BC$ is parallel to the tangent at $A$.

Also prove that the circumcircle of the triangle formed by the tangents at $A$, $B$, $C$ is concentric with the three given circles; and find a relation connecting the radii of the four concentric circles.

23. From a fixed point $P(x', y')$ is drawn a straight line to cut the circle $x^2+y^2=a^2$ in $Q$ and $R$; find the locus of the point harmonically conjugate to $P$ with respect to $Q$ and $R$.

24. If a straight line move so that the lengths intercepted upon it by

two given circles are equal, the locus of its pole with regard to either circle will be a curve of the second degree.

25. Chords of a circle, radius $a$, subtend right angles at a point whose distance from the centre of the circle is $c$.

Prove that the locus of their poles is a circle of radius

$$(a^2\sqrt{2a^2-c^2})/(a^2-c^2).$$

## § 13. Properties of two circles.

I. **Definition.** The points which divide the line joining the centres of two circles internally and externally in the ratio of the radii are called the **Centres of Similitude** of the two circles.

If the points $C_1(\alpha_1, \beta_1)$, $C_2(\alpha_2, \beta_2)$ are the centres of two circles, whose radii are $r_1$, $r_2$, the centres of similitude, $S_1$ $S_2$, are the points

$$S_1,\ \left\{\frac{r_1\alpha_2+r_2\alpha_1}{r_1+r_2},\ \frac{r_1\beta_2+r_2\beta_1}{r_1+r_2}\right\};\quad S_2,\ \left\{\frac{r_1\alpha_2-r_2\alpha_1}{r_1-r_2},\ \frac{r_1\beta_2-r_2\beta_1}{r_1-r_2}\right\}.$$

The points $S_1$, $S_2$ evidently divide the line $C_1C_2$ harmonically (Chapter I, p. 15).

*Show that any straight line through a centre of similitude of two circles is divided similarly by the circles.*

Suppose two circles

$$x^2+y^2+2gx+2fy+c=0 \text{ (radius } r),$$
$$x^2+y^2+2Gx+2Fy+C=0 \text{ (radius } R),$$

have a centre of similitude at the origin (0, 0).

Then $$Rg-rG=0;\ Rf-rF=0.$$

Hence $$G=\frac{R}{r}.g;\ F=\frac{R}{r}.f.$$

Now put $$\frac{R}{r}=\lambda.$$

Then $$G=\lambda.g,\ F=\lambda.f,\ R=\lambda r.$$

But $$G^2+F^2-C=R^2;$$

$$\therefore\quad C=G^2+F^2-R^2$$
$$=\lambda^2(g^2+f^2-r^2)$$
$$=\lambda^2 c.$$

Hence the equation of the second circle is

$$x^2+y^2+2\lambda gx+2\lambda fy+\lambda^2 c=0.$$

Let any straight line through the origin, i. e. through the centre of similitude, be $$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=r.$$

If this cuts the circle $x^2+y^2+2\lambda gx+2\lambda fy+\lambda^2 c=0$ at $P$ and $Q$ the lengths $OP$, $OQ$ are given by

$$r^2\cos^2\theta+r^2\sin^2\theta+2\lambda gr\cos\theta+2\lambda fr\sin\theta+\lambda^2 c=0;$$

i. e. $$r^2+2r\lambda(g\cos\theta+f\sin\theta)+\lambda^2 c=0.$$

Hence $$OP+OQ=-2\lambda(g\cos\theta+f\sin\theta)$$
$$OP.\,OQ=\lambda^2 c;$$

$$\therefore\quad \frac{(OP+OQ)^2}{OP.\,OQ}=\frac{4(g\cos\theta+f\sin\theta)^2}{c},$$

i. e. $\dfrac{OP}{OQ}+\dfrac{OQ}{OP}+2=$ a quantity independent of $\lambda$ and therefore the same for both circles.

Let the straight line cut the circles in the points $P$, $Q$ and $P'$, $Q'$, and let $OP=k.\,OQ$ and $OP'=k'.\,OQ'$; then it follows that

$$k+1/k=k'+1/k',$$

hence $$(k-k')(kk'-1)=0.$$

Therefore $$k=k' \text{ or } k=1/k';\ \text{whence}$$
$$OP/OQ=OP'/OQ' \text{ or } OQ'/OP'.$$

The straight line is therefore divided similarly by the two circles.

II. **Definition.** *The circle described on the line joining the centres of similitude of two circles as diameter is called the Circle of Similitude.*

If the equations of the two circles are

$$(x-\alpha_1)^2+(y-\beta_1)^2=r_1^2,$$
$$(x-\alpha_2)^2+(y-\beta_2)^2=r_2^2,$$

the equation of the circle of similitude is (v. p. 164)

$$\left(x-\frac{r_1\alpha_2+r_2\alpha_1}{r_1+r_2}\right)\left(x-\frac{r_1\alpha_2-r_2\alpha_1}{r_1-r_2}\right)$$
$$+\left(y-\frac{r_1\beta_2+r_2\beta_1}{r_1+r_2}\right)\left(y-\frac{r_1\beta_2-r_2\beta_1}{r_1-r_2}\right)=0.$$

This at once reduces to

$$r_1^2\{x^2-2\alpha_2x+\alpha_2^2+y^2-2\beta_2y+\beta_2^2\}$$
$$-r_2^2\{x^2-2\alpha_1x+\alpha_1^2+y^2-2\beta_1y+\beta_1^2\}=0,$$

or $$r_1^2\{(x-\alpha_2)^2+(y-\beta_2)^2\}-r_2^2\{(x-\alpha_1)^2+(y-\beta_1)^2\}=0, \qquad \text{(i)}$$

which is identical with

$$r_1^2\{(x-\alpha_2)^2+(y-\beta_2)^2-r_2^2\}-r_2^2\{(x-\alpha_1)^2+(y-\beta_1)^2-r_1^2\}=0. \qquad \text{(ii)}$$

Note that

(a) Equation (i) shows that if $P(x, y)$ is any point on the circle of similitude of two circles whose centres are $C_1$, $C_2$

then $$PC_1 : PC_2 = r_1 : r_2.$$

(b) If $$S_1 \equiv (x-\alpha_1)^2+(y-\beta_1)^2-r_1^2 = 0,$$
$$S_2 \equiv (x-\alpha_2)^2+(y-\beta_2)^2-r_2^2 = 0$$
are two circles, we see from (ii) that the circle of similitude can be written $$\frac{S_1}{r_1^2} = \frac{S_2}{r_2^2}.$$

(c) It can be shown that the circle of similitude is the locus of points at which the two circles subtend the same angle.

**Example.** *The centres of similitude of the circumcircle and nine-point circle of a triangle are the orthocentre and the centre of gravity.*

If the vertices of the triangle are $(R\cos\alpha,\ R\sin\alpha)$, $(R\cos\beta,\ R\sin\beta)$, $(R\cos\gamma,\ R\sin\gamma)$, we have seen (p. 153) that the equation of the nine-point circle is $$(x-C)^2+(y-S)^2 = \tfrac{1}{4}R^2.$$

Hence the centres of the circumcircle and nine-point circle are $(0, 0)$, $(C, S)$, and their radii are $R$ and $\frac{1}{2}R$.

The coordinates of the centres of similitude are therefore $(\frac{2}{3}C, \frac{2}{3}S)$ and $(2C, 2S)$, which are those of the centre of gravity and the orthocentre.

The equation of the circle of similitude is
$$(x-\tfrac{2}{3}C)(x-2C)+(y-\tfrac{2}{3}S)(y-2S) = 0,$$
or $$3x^2+3y^2-8Cx-8Sy+4(C^2+S^2) = 0.$$

### III. The Common Tangents of Two Circles.

The equation of any circle can be written in the form
$$(x-\alpha)^2+(y-\beta)^2 = r^2;$$
this is the most convenient form for this problem.

Let the equations of two circles be
$$(x-\alpha_1)^2+(y-\beta_1)^2 = r_1^2, \qquad \text{(i)}$$
$$(x-\alpha_2)^2+(y-\beta_2)^2 = r_2^2; \qquad \text{(ii)}$$
we wish to find the equations of those lines which touch both of the circles.

The coordinates of any point on the circle (i) can be written
$$\{\alpha_1+r_1\cos\theta,\ \beta_1+r_1\sin\theta\},$$
and the tangent at this point is
$$(x-\alpha_1)\cos\theta+(y-\beta_1)\sin\theta = r_1. \qquad \text{(iii)}$$

There are two tangents to the circle (ii) parallel to this, viz.
$$(x-\alpha_2)\cos\theta+(y-\beta_2)\sin\theta = r_2 \qquad \text{(iv)}$$
and $$(x-\alpha_2)\cos\theta+(y-\beta_2)\sin\theta = -r_2. \qquad \text{(v)}$$

If either of these coincides with (iii) it will be a common tangent of the two circles.

Comparing the equations (iii) and (iv) the condition that they should be identical is

$$\alpha_1 \cos\theta + \beta_1 \sin\theta + r_1 = \alpha_2 \cos\theta + \beta_2 \sin\theta + r_2,$$

i. e. $$(\alpha_1 - \alpha_2)\cos\theta + (\beta_1 - \beta_2)\sin\theta + (r_1 - r_2) = 0. \qquad \text{(vi)}$$

This equation then gives the values of $\theta$ which make the line (iii) a tangent to both circles. It can be written

$$(\alpha_1 - \alpha_2)(1 - \tan^2 \tfrac{1}{2}\theta) + (\beta_1 - \beta_2)\, 2\tan \tfrac{1}{2}\theta + (r_1 - r_2)(1 + \tan^2 \tfrac{1}{2}\theta) = 0,$$

or $$\tan^2 \tfrac{1}{2}\theta\,(\overline{r_1 - r_2} - \overline{\alpha_1 - \alpha_2}) + (\beta_1 - \beta_2).\, 2\tan \tfrac{1}{2}\theta + \overline{r_1 - r_2} + \overline{\alpha_1 - \alpha_2} = 0.$$

This is a quadratic and gives two values for $\tan \frac{1}{2}\theta$ with corresponding values for $\cos\theta$ and $\sin\theta$ to be substituted in (iii).

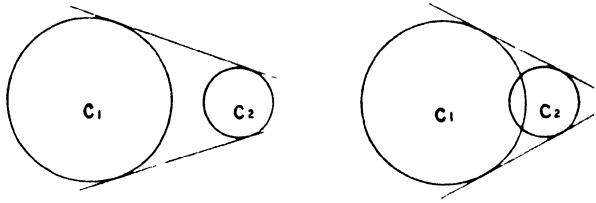
The two tangents are real, coincident, or imaginary according as

$$(\beta_1 - \beta_2)^2 >, =, \text{or} < (r_1 - r_2)^2 - (\alpha_1 - \alpha_2)^2,$$

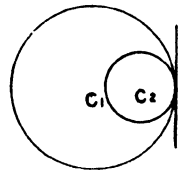
i. e. as $$(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 >, =, \text{or} < (r_1 - r_2)^2.$$

Since the centres of the circles are the points $C_1(\alpha_1, \beta_1)$, $C_2(\alpha_2, \beta_2)$, this condition is the same as $C_1C_2 >, =, \text{or} < r_1 - r_2$.

If one circle is outside or cuts the other, clearly $C_1C_2 > r_1 - r_2$ and two real common tangents can be drawn.




If one circle lies within and just touches the other, then $C_1C_2 = r_1 - r_2$ and the common tangents coincide.



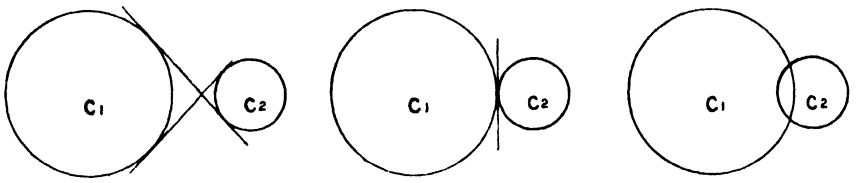

If one circle lies entirely *within* the other, $C_1C_2 < r_1 - r_2$ and the tangents are imaginary.

Again, if equations (iii) and (v) are identical, we get similarly that

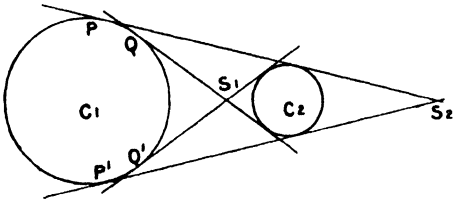
$$(\alpha_1 - \alpha_2)\cos\theta + (\beta_1 - \beta_2)\sin\theta + r_1 + r_2 = 0. \qquad \text{(vii)}$$

Treating this in the same way as equation (vi), these correspond to two common tangents which are real, coincident, or imaginary

according as $C_1C_2$ is $>$, $=$, or $< r_1+r_2$, i. e. according as the one circle lies outside, lies outside and touches, or cuts the other.




We see then that equation (vi) gives the values of $\theta$ for the 'direct' common tangents and equation (vii) for the transverse common tangents.




Now let $PP'$ be the points of contact of the tangents given by equation (vi)

$$(\alpha_1-\alpha_2)\cos\theta+(\beta_1-\beta_2)\sin\theta+r_1-r_2=0$$

with the first circle. Since the coordinates of $P$ are

$$(\alpha_1+r_1\cos\theta,\ \beta_1+r_1\sin\theta),$$

where $\theta$ is one of the values given by (vi), if $P$ is called the point $(x, y)$ we have $\quad x=\alpha_1+r_1\cos\theta,\ y=\beta_1+r_1\sin\theta.$

Hence

$$(\alpha_1-\alpha_2)(x-\alpha_1)+(\beta_1-\beta_2)(y-\beta_1)+r_1(r_1-r_2)=0 \qquad \text{(viii)}$$

is an equation satisfied by the coordinates of $P$, and (by precisely the same argument) by the coordinates of $P'$. In other words (viii) is the equation of the chord of contact $PP'$.

Now the chord of contact of tangents from the external centre of similitude $S_1\left\{\frac{r_1\alpha_2-r_2\alpha_1}{r_1-r_2},\ \frac{r_1\beta_2-r_2\beta_1}{r_1-r_2}\right\}$ to the first circle is

$$(x-\alpha_1)\left\{\frac{r_1\alpha_2-r_2\alpha_1}{r_1-r_2}-\alpha_1\right\}+\left\{y-\beta_1\right\}\left\{\frac{r_1\beta_2-r_2\beta_1}{r_1-r_2}-\beta_1\right\}=r_1^2,$$

i. e. $$(x-\alpha_1)\frac{r_1(\alpha_2-\alpha_1)}{r_1-r_2}+(y-\beta_1)\frac{r_1(\beta_2-\beta_1)}{r_1-r_2}=r_1^2,$$

i. e. $$(x-\alpha_1)(\alpha_1-\alpha_2)+(y-\beta_1)(\beta_1-\beta_2)+r_1(r_1-r_2)=0,$$

which is identical with equation (viii) of the line $PP'$.

Hence the direct common tangents meet at the external centre of similitude $S_2$.

In the same way it can be shown from equation (vii) that the chord of contact $QQ'$ of the transverse common tangents is

$$(\alpha_1-\alpha_2)(x-\alpha_2)+(\beta_1-\beta_2)(y-\beta_2)+r_1(r_1+r_2)=0,$$

which is the chord of contact of tangents from the internal centre of similitude $S_1$.

The following example illustrates the methods of finding the equations of common tangents in numerical cases: the equation of each *pair* can be written down generally as the tangents from the centres of similitude to the circles.

**Ex.** i. *Find the common tangents of*

$$x^2+y^2-3x-4y=0,$$
$$x^2+y^2-21x+90=0.$$

These equations can be written

$$(x-\tfrac{3}{2})^2+(y-2)^2=(\tfrac{5}{2})^2,$$
$$(x-2\tfrac{1}{2})^2+y^2=(\tfrac{9}{2})^2.$$

The condition that $(x-\frac{3}{2})\cos\theta+(y-2)\sin\theta=\frac{5}{2}$

and $(x-2\frac{1}{2})\cos\theta+y\sin\theta=+\frac{9}{2}$

should be identical is $\frac{3}{2}\cos\theta+2\sin\theta+\frac{5}{2}=2\frac{1}{2}\cos\theta+\frac{9}{2}$,

i. e. $9\cos\theta-2\sin\theta+2=0$

or $7\tan^2\tfrac{1}{2}\theta+4\tan\tfrac{1}{2}\theta-11=0.$

Hence $\tan\frac{1}{2}\theta=1$ or $-1\frac{1}{7}$.

Then $$\cos\theta=\frac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}=0 \text{ or } -\tfrac{336}{625},$$

$$\sin\theta=\frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}=1 \text{ or } -\tfrac{77}{625}.$$

The corresponding common tangents are

$$y=\tfrac{9}{2} \text{ and } \tfrac{336}{625}(x-2\tfrac{1}{2})+\tfrac{77}{625}y+\tfrac{9}{2}=0.$$

Again, the tangents $(x-\frac{3}{2})\cos\theta+(y-2)\sin\theta=\frac{5}{2}$,

$(x-2\frac{1}{2})\cos\theta+y\sin\theta=-\frac{9}{2}$

are identical if $\frac{3}{2}\cos\theta+2\sin\theta+\frac{5}{2}=2\frac{1}{2}\cos\theta-\frac{9}{2}$,

i. e. $9\cos\theta-2\sin\theta-7=0,$

or $8\tan^2\tfrac{1}{2}\theta+2\tan\tfrac{1}{2}\theta-1=0,$

i. e. $\tan\frac{1}{2}\theta=-\frac{1}{2}$ or $\frac{1}{4}$.

Then as above $\cos\theta=\frac{3}{5}$ or $\frac{15}{17}$,

$\sin\theta=-\frac{4}{5}$ or $\frac{8}{17}$.

The corresponding tangents being

$$\tfrac{3}{5}(x-2\tfrac{1}{2})-\tfrac{4}{5}y=-\tfrac{9}{2} \text{ and } \tfrac{15}{17}(x-2\tfrac{1}{2})+\tfrac{8}{17}y=-\tfrac{9}{2},$$

or $3x-4y-9=0$; $15x+8y-81=0.$

**Ex. ii.** *Show that the equations of the two pairs of common tangents to the circles* $(x-a)^2+(y-b)^2 = r^2$; $(x-a')^2+(y-b')^2 = r'^2$ *are given by*

$$\begin{vmatrix} x-a, & y-b \\ x-a', & y-b' \end{vmatrix}^2 = \begin{vmatrix} x-a, & r \\ x-a', \pm r' \end{vmatrix}^2 + \begin{vmatrix} y-b, & r \\ y'-b', \pm r' \end{vmatrix}^2.$$

Any tangent to the first circle can be written

$$(x-a)\cos\theta+(y-b)\sin\theta = r. \qquad \text{(i)}$$

A parallel tangent to the second circle is one of the straight lines

$$(x-a')\cos\theta+(y-b')\sin\theta = \pm r'. \qquad \text{(ii)}$$

Points on the common tangents therefore satisfy both equations (i) and (ii).

But from these

$$-\frac{\cos\theta}{\begin{vmatrix} y-b, & r \\ y-b', \pm r' \end{vmatrix}} = \frac{\sin\theta}{\begin{vmatrix} x-a, & r \\ x-a', \pm r' \end{vmatrix}} = \frac{1}{\begin{vmatrix} x-a, & y-b \\ x-a', & y-b' \end{vmatrix}}$$

Hence, since $\cos^2\theta+\sin^2\theta = 1$, any point on a common tangent satisfies

$$\begin{vmatrix} x-a, & y-b \\ x-a', & y-b' \end{vmatrix}^2 = \begin{vmatrix} x-a, & r \\ x-a', \pm r' \end{vmatrix}^2 + \begin{vmatrix} y-b, & r \\ y-b', \pm r' \end{vmatrix}^2.$$

## IV. The Common Chord of Two Circles.

Let the equations of the two circles be

$$x^2+y^2+2gx+2fy+c = 0, \qquad \text{(i)}$$
$$x^2+y^2+2g'x+2f'y+c' = 0. \qquad \text{(ii)}$$

The equation

$$(x^2+y^2+2gx+2fy+c)-(x^2+y^2+2g'x+2f'y+c') = 0 \qquad \text{(iii)}$$

is satisfied by the coordinates of any point whose coordinates satisfy both the equations (i) and (ii); hence it is a locus through the common points of the two circles.

But the equation is equivalent to

$$2(g-g')x+2(f-f')y+c-c' = 0,$$

which is a **straight line** and is therefore the common chord.

**Note i.** If one circle bisects the circumference of another, their common chord is a diameter of the latter circle.

Thus, if $$x^2+y^2+2gx+2fy+c = 0 \qquad \text{(i)}$$

bisects the circumference of

$$x^2+y^2+2Gx+2Fy+C = 0, \qquad \text{(ii)}$$

then $(-G, -F)$ lies on

$$2(G-g)x+2(F-f)y+C-c = 0,$$

i.e. $$2G(G-g)+2F(F-f)+c-C = 0,$$

or the coefficients of the first equation (i) satisfy the linear relation

$$2Gg+2Ff-c = 2G^2+2F^2-C.$$

Hence a circle can be drawn to bisect the circumference of any three circles, for we then get three linear equations to find $g, f$, and $c$. The only case in which this fails is when

$$\begin{vmatrix} G_1 & F_1 & 1 \\ G_2 & F_2 & 1 \\ G_3 & F_3 & 1 \end{vmatrix} = 0,$$

i.e. when the three given circles have their centres collinear.

**Note ii.** If two circles touch, their common chord is a tangent to each of them, for this is the limiting case when the two points of intersection coincide.

Thus, if two circles $C_1 = 0$, $C_2 = 0$ touch, the line $C_1 - C_2 = 0$ is their common tangent at the point where they touch.

Suppose its equation is $T = 0$; then $C_1 - C_2 \equiv kT$ where $k$ is a constant, i.e. $C_2 \equiv C_1 - kT$.

Hence, if $T = 0$ is a tangent to the circle $C_1 = 0$, $C_1 = kT$ represents a circle touching $C_1 = 0$ at the point of contact of $T = 0$.

**Note iii.** It follows that if two circles cut in imaginary points, the join of these points is a real straight line.

The result of this paragraph can be written briefly thus:

If $C_1 = 0$, $C_2 = 0$ are the equations of two circles, $C_1 - C_2 = 0$ is the equation of their common chord.

Conversely, if we are told that a straight line $u = 0$ is the common chord of the circle $C = 0$ and some other circle, the equation of the second circle must be of the form $C - ku = 0$, where $k$ is some constant.

This relation will be discussed more fully in § 14.

**Example i.** *To find the equation of a circle which passes through the points of intersection of* $x^2+y^2-2x+3y+3 = 0$ *and* $5x-2y-10 = 0$, *and also through the point* (3, 1).

The equation of the required circle is of the form

$$x^2+y^2-2x+3y+3-k(5x-2y-10) = 0.$$

Since the point (3, 1) lies on this,

$$10-3k = 0, \quad \text{i.e. } k = \tfrac{10}{3}.$$

Hence the required circle is

$$3(x^2+y^2-2x+3y+3)-10(5x-2y-10) = 0,$$

i.e.

$$3x^2+3y^2-56x+29y+109 = 0.$$

**Example ii.** *To find the equation of the circle which is equal to the circle* $x^2+y^2 = a^2$ *and touches it at the point* $(a\cos\alpha, a\sin\alpha)$.

The tangent to the given circle at this point is

$$x\cos\alpha + y\sin\alpha = a.$$

Hence the required circle is of the form

$$x^2+y^2-a^2 = k(x\cos\alpha + y\sin\alpha - a).$$

The radius of this circle is

$$\sqrt{\tfrac{1}{4}(k^2\cos^2\alpha)+\tfrac{1}{4}(k^2\sin^2\alpha)+a^2-ka}.$$

Hence, since the circles are equal,

$$\tfrac{1}{4}k^2-ka+a^2=a^2,$$

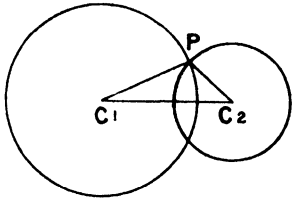
i.e. $$k=4a.$$

The equation required is therefore

$$x^2+y^2-a^2=4a(x\cos\alpha+y\sin\alpha-a).$$

V. *To find the angle at which two circles cut, i.e. the angle between the tangents to the circles at their common points.*

If the circles cut at $P$, the tangents to them at $P$ are perpendicular respectively to the radii $C_1P$, $C_2P$; hence the angle between the tangents is equal or supplementary to the angle $C_1PC_2$.




Now, if we denote the angle $C_1PC_2$ by $\psi$,

$$\cos\psi=\frac{C_1P^2+C_2P^2-C_1C_2^2}{2\,C_1P\,.\,C_2P}.$$

If the circles are

$$x^2+y^2+2gx+2fy+c=0,$$

$$x^2+y^2+2Gx+2Fy+C=0,$$

since their centres are $(-g,\ -f)$, $(-G,\ -F)$ and their radii

$$\sqrt{g^2+f^2-c} \quad \text{and} \quad \sqrt{G^2+F^2-C},$$

we have

$$\cos\psi=\frac{g^2+f^2-c+G^2+F^2-C-(g-G)^2-(f-F)^2}{2\sqrt{g^2+f^2-c}\ \sqrt{G^2+F^2-C}}$$

$$=\frac{2gG+2fF-C-c}{2\sqrt{g^2+f^2-c}\ \sqrt{G^2+F^2-C}},$$

i.e. $2Rr\cos\psi=2gG+2fF-C-c$, where $R$, $r$ are the radii.

**Note i.** If two circles touch internally or externally, $\psi$ is 0 or $\pi$ respectively, and $2gG+2fF-C-c=\pm 2Rr$. This is equivalent to the sum or difference of the radii being equal to the distance between the centres.

**Note ii.** When the circles cut at right angles (i.e. orthogonally) $\psi$ is a right angle and $\cos\psi$ is zero: hence the condition that the two circles should cut orthogonally is $2Gg+2Ff-C-c=0$.

**Note iii.** The square of the length of the tangent from the centre $C_1(-g,\ -f)$ of the first circle to the second circle is equal to

$$g^2+f^2-2Gg-2Ff+C=g^2+f^2-c-(2Gg+2Ff-C-c).$$

Hence, *if the circles cut orthogonally*, the length of this tangent is equal to the radius of the first circle. Similarly, the length of the tangent from the

centre of the second circle to the first circle is equal to the radius of the second circle. The converse of this proposition is clearly true.

If the first circle is known and we are finding the equation of the second, cutting it orthogonally, this condition is linear in the three unknown quantities $G$, $F$, and $C$.

The circle can therefore be made to fulfil two other conditions. Thus, for example, one definite circle can be found which cuts three given circles orthogonally.

If the tangents to two circles which cut orthogonally at one of their common points be taken as coordinate axes, the equations of the pair of circles are

$$x^2+y^2-2r_1x=0,$$
$$x^2+y^2-2r_2y=0.$$

**Ex.** i. *If two circles cut orthogonally, the tangents at one point of intersection meet the circles again in points whose join passes through the other point of intersection.*

Taking the tangents at a common point as coordinate axes, the other points in which the axes meet the circles

$$x^2+y^2-2r_1x=0,$$
$$x^2+y^2-2r_2y=0$$

are $(2r_1, 0)$, $(0, 2r_2)$.

Their other point of intersection is $\left(\frac{2r_1r_2^2}{r_1^2+r_2^2}, \frac{2r_1^2r_2}{r_1^2+r_2^2}\right)$.

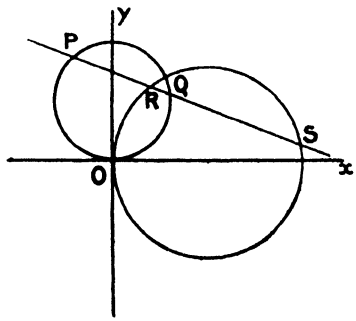
This lies on $\frac{x}{2r_1}+\frac{y}{2r_2}=1$.

**Ex.** ii. *Show that a line cut harmonically by two orthogonal circles must be a diameter of one of them.*

Let the circles be

$$x^2+y^2-2r_1x=0,$$
$$x^2+y^2-2r_2y=0,$$

and the line $lx+my=1$.




Then, if the points of intersection are $PQRS$, the lines

$OP$, $OQ$ are $x^2+y^2-2r_1x(lx+my)=0$,

and

$OR$, $OS$ are $x^2+y^2-2r_2y(lx+my)=0$,

i.e. $x^2(1-2r_1l)-2r_1mxy+y^2=0$,

$x^2-2r_2lxy+y^2(1-2mr_2)=0$.

These form a harmonic pencil if

$$(1-2r_1l)(1-2r_2m)+1=2r_1r_2ml,$$

[Chap. III, p. 92.]

i.e. $1-r_1l-r_2m+r_1r_2lm=0,$

i.e. $(1-r_1l)(1-r_2m)=0.$

Thus $1-r_1l=0$ or $1-r_2m=0,$

i.e. either $(r_1, 0)$ or $(0, r_2)$ lies on the line $lx+my=1.$

**Ex. iii.** *A circle cuts the two circles*

$$(x-a_1)^2+(y-b_1)^2=r_1^2,$$
$$(x-a_2)^2+(y-b_2)^2=r_2^2$$

*at angles $\theta_1$ and $\theta_2$. Prove that it cuts their circle of similitude orthogonally if $r_1\cos\theta_2=r_2\cos\theta_1$.*

Let the centre of this circle be $(\alpha, \beta)$ and its radius be $\rho$, then we have the conditions $(\alpha-a_1)^2+(\beta-b_1)^2=\rho^2+r_1^2-2r_1\rho\cos\theta_1,$

$$(\alpha-a_2)^2+(\beta-b_2)^2=\rho^2+r_2^2-2r_2\rho\cos\theta_2.$$

The circle of similitude is

$$\frac{(x-a_1)^2+(y-b_1)^2}{r_1^2}=\frac{(x-a_2)^2+(y-b_2)^2}{r_2^2},$$

or $$r_2^2\{(x-a_1)^2+(y-b_1)^2\}-r_1^2\{(x-a_2)^2+(y-b_2)^2\}=0,$$

and since the coefficients of $x^2$ and $y^2$ are $(r_2^2-r_1^2)$, the square of the tangents from $(\alpha, \beta)$ to this circle

$$=\frac{r_2^2\{(\alpha-a_1)^2+(\beta-b_1)^2\}-r_1^2\{(\alpha-a_2)^2+(\beta-b_2)^2\}}{r_2^2-r_1^2}$$

$$=\frac{r_2^2\{\rho^2+r_1^2-2r_1\rho\cos\theta_1\}-r_1^2\{\rho^2+r_2^2-2r_2\rho\cos\theta_2\}}{r_2^2-r_1^2}$$

$$=\rho^2-\frac{2r_1r_2\rho(r_2\cos\theta_1-r_1\cos\theta_2)}{r_2^2-r_1^2}$$

$=\rho^2$, since $r_2\cos\theta_1=r_1\cos\theta_2.$

Hence the circle cuts the circle of similitude orthogonally.

**Ex. iv.** *Find the equation of the circle having for a diameter that chord of the circle $x^2+y^2+2gx+2fy+c=0$ whose equation is $lx+my+n=0$.*

Since the circle passes through the common points of

$$x^2+y^2+2gx+2fy+c=0 \text{ and } lx+my+n=0,$$

its equation is of the form

$$x^2+y^2+2gx+2fy+c+2k(lx+my+n)=0.$$

Moreover, $lx+my+n=0$ is a diameter of this circle, hence its centre $\{-(g+kl), -(f+km)\}$ lies on this line.

Thus $$l(g+kl)+m(f+km)-n=0$$

or $$k(l^2+m^2)=-(lg+mf-n).$$

Thus the required equation is

$$(l^2+m^2)(x^2+y^2+2gx+2fy+c)=2(lg+mf-n)(lx+my+n).$$

### Examples V k.

1. Find the equations of the common tangents of the circles $x^2+y^2=1$, $x^2+y^2-3x+2=0$.

2. Find the coordinates of the centres of similitude of the circles

$$x^2+y^2-4x-2y+1=0,$$
$$x^2+y^2+2x+4y-11=0,$$

and the equation of their real common tangents.

3. Find the equation of the circle cutting orthogonally the three circles

$$x^2+y^2+2x-4y+1=0,$$
$$x^2+y^2-6x+8y+7=0,$$
$$x^2+y^2-4x+6y+9=0.$$

4. Find the equation of a circle which bisects the circumferences of

$$x^2+y^2=1, \quad x^2+y^2+2x=3, \quad x^2+y^2+2y=3.$$

5. Find the equations to the two circles which have their centres at the origin and touch the circle $x^2+y^2-6x-8y+9=0$.

6. Find all the common tangents of the circles $x^2+y^2-3x-54=0$, $x^2+y^2-21x+90=0$.

7. Find the equation to the circle which passes through the point (1, 1), and the points of intersection of $3x+7=5y$, $3x-2x^2=5y+2y^2-17$.

8. Find the radius of the circle of similitude of two circles in terms of their radii and the distance between their centres.

9. Write down the equation to the circle which passes through the point $(a, b)$ and the points common to $lx+my=1$, $x^2+y^2+2gx+2fy+c=0$.

10. $A$ and $B$ are the centres of two orthogonal circles intersecting in $O$, and $P$ is any point on the circle $AOB$. Show that $P$ is equidistant from the two points in which $PO$ cuts the circles.

11. Find the equations of the two circles which cut orthogonally the circles $x^2+y^2+2x-9=0$, $x^2+y^2-8x-9=0$, and touch the line $y-x=4$. Show that the distance between their centres is $10\sqrt{2}$.

12. Show that the circle on the line joining the centres of similitude of

$$x^2+y^2-2kx+\delta^2=0,$$
$$x^2+y^2-2k'x+\delta^2=0,$$

as diameter is $x^2+y^2-2x(kk'+\delta^2)/(k+k')+\delta^2=0$.

13. Find the condition that the circles

$$(x-c)^2+\lambda(x+c)^2+(1+\lambda)y^2=0,$$
$$(x-c)^2+\mu(x+c)^2+(1+\mu)y^2=0$$

should cut at right angles.

14. Show that the equation of the circle which cuts each of the three circles $x^2+y^2=a^2$, $(x-b)^2+y^2=a^2$, $x^2+(y-c)^2=a^2$ at right angles is $x^2+y^2-bx-cy+a^2=0$.

15. Show that the circle of similitude of $x^2+y^2-2ax+a^2\cos^2\alpha=0$, $x^2+y^2-2bx+b^2\cos^2\alpha=0$ is $(a+b)(x^2+y^2)=2abx$.

16. If two circles intersect, their circle of similitude passes through their points of intersection.

17. If two circles cut orthogonally, the extremities of a diameter of either are conjugate points with respect to the other.

18. Find the equation to a circle which touches $x^2+y^2-2x=0$, $x^2+y^2+3x=0$ at their point of contact, and has internal or external contact with the circle $(x-3)^2+(y-\frac{5}{2})^2=1$.

19. If a circle of fixed radius $\rho$ cuts a circle $C=0$ whose radius is $r$ at an angle $\alpha$, the locus of its centre is $C=r^2-2r\rho\cos\alpha$.

20. Find the locus of a point at which the two circles $x^2+y^2-2a_1x+b^2=0$, $x^2+y^2-2a_2x+b^2=0$ subtend the same angle.

21. Find the locus of the centre of a circle which cuts the circles

$$(x-a_1)^2+(y-b_1)^2=c^2\cos^2\alpha,$$
$$(x-a_2)^2+(y-b_2)^2=c^2\cos^2\beta$$

at angles $\beta$ and $\alpha$ respectively.

22. Find the locus of the centre of a circle which cuts each of two given circles at a given angle.

23. Find the equation of a circle which cuts orthogonally

$$x^2+y^2+2gx+2fy+c=0,$$
$$x^2+y^2+2g'x+2f'y+c=0,$$

and also the line $lx+my=1$.

24. Find the equation of the circle which has for its diameter the chord cut off on the straight line $ax+by+c=0$ by the circle $(a^2+b^2)(x^2+y^2)=2c^2$.

25. Circles are drawn through the point $(c, 0)$ touching the circle $x^2+y^2=a^2$. Show that the locus of the pole of the axis of $x$ with respect to them is $4a^2(x-c)^4=(a^2-c^2)\{a^2-(c-2x)^2\}y^2$.

26. If $C\equiv x^2+y^2+2gx+2fy+c$, and $u\equiv x\cos\alpha+y\sin\alpha-p$, and $u_0$ is the value of $u$ at the centre of the circle $C=0$, then the equation of the circle on the chord which $C=0$ cuts off from $u=0$ is $C-2uu_0=0$.

27. If a circle cuts three circles $C_1=0$, $C_2=0$, $C_3=0$ orthogonally, prove that it cuts all circles of the system $\lambda C_1+\mu C_2+\nu C_3=0$ orthogonally.

28. If $Q$, $R$ are the points of contact of the tangents drawn to the circle $C\equiv x^2+y^2+2gx+2fy+c=0$ from an external point $(h, k)$, find the equation $P=0$ of the chord $QR$ and show that the circle described on $QR$ as diameter may be written in the form $2r^2P=\{(h+g)^2+(k+f)^2\}C$, $r$ being the radius of the circle.

29. If $C_1=0$, $C_2=0$, $C_3=0$ are three intersecting circles and $aC_1+bC_2+cC_3$ is identically zero, where $a$, $b$, and $c$ are constants, then all three circles pass through the same two points; and if this condition is satisfied the circle $lC_1+mC_2+nC_3=0$ for any values of $l$, $m$, and $n$ also passes through these two points.

## § 14. Systems of Circles.

I. $$C_1\equiv x^2+y^2+2g_1x+2f_1y+c_1=0,$$
$$C_2\equiv x^2+y^2+2g_2x+2f_2y+c_2=0$$

being the equations of two circles, we have shown that

$$C_1-C_2\equiv(x^2+y^2+2g_1x+2f_1y+c_1)-(x^2+y^2+2g_2x+2f_2y+c_2)=0$$

is the equation of their common chord; when the circles do not meet in real points, $C_1 - C_2 = 0$ still represents a straight line satisfied by the coordinates of all points common to $C_1 = 0$, $C_2 = 0$; it is the join of their imaginary points of intersection.

Now the square of the lengths of the tangents from any point $(x', y')$ to the circle $C_1 = 0$ is found by substituting $x'$ and $y'$ for $x$ and $y$ in $C_1$. Another geometrical interpretation can now be given to the equation

$$C_1 - C_2 = 0,$$

which enables us to define the line in *real* terms in all cases.

The equation

$$(x^2+y^2+2g_1x+2f_1y+c_1)-(x^2+y^2+2g_2x+2f_2y+c_2)=0$$

represents the locus of a point, the tangents from which to the two circles are equal; hence the definition:

*The locus of a point, tangents from which to two given circles are equal, is a straight line which is called the* Radical Axis *of the two circles.*

Evidently, when the circles cut in real points, the radical axis is the common chord: if the circles touch each other, it is the common tangent at the common point. In every case the radical axis bisects the common tangents to the circles.

The equation of the radical axis reduces to

$$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0. \qquad \text{(i)}$$

Now the centres of the circles $C_1$, $C_2$ are $(-g_1, -f_1)$, $(-g_2, -f_2)$, and therefore the equation of the line joining the centres (the line of centres) is

$$(x+g_1)(f_1-f_2)-(y+f_1)(g_1-g_2)=0. \qquad \text{(ii)}$$

This is perpendicular to the line (i); hence *the radical axis of two circles is perpendicular to their line of centres.*

Now consider the equation

$$x^2+y^2+2gx+2fy+c+\lambda(lx+my+n)=0. \qquad \text{(iii)}$$

It represents a circle whatever value $\lambda$ may have.

But the radical axis of this circle and the circle

$$x^2+y^2+2gx+2fy+c=0$$

is the straight line $lx+my+n=0$.

Hence for different values of $\lambda$, equation (iii) represents a system of circles each of which has the same radical axis $lx+my+n=0$ with the circle $x^2+y^2+2gx+2fy+c=0$.

Further, if

$$x^2+y^2+2gx+2fy+c+\lambda_1(lx+my+n)=0,$$
$$x^2+y^2+2gx+2fy+c+\lambda_2(lx+my+n)=0$$

be any two circles of this system, their radical axis is

$$(\lambda_1-\lambda_2)(lx+my+n)=0,$$

i. e.
$$lx+my+n=0.$$

Thus $x^2+y^2+2gx+2fy+c+\lambda(lx+my+n)=0$ represents when $\lambda$ varies a system of circles, such that every pair has the same radical axis $lx+my+n=0$. Such a system of circles is called a *Coaxal System.*

In abridged notation we can write briefly: if $C=0$ is a circle and $u=0$ a straight line, $C+\lambda u=0$ represents a coaxal system of circles of which $u=0$ is the radical axis.

Since the radical axis of two circles is perpendicular to their line of centres, the centres of all circles in a coaxal system lie on a straight line.

The equation of a system of coaxal circles is simplified by taking the line of centres as the axis of $x$ and the radical axis, which is perpendicular to it, as the axis of $y$.

In this case, since the centre is on the axis of $x$, $f=0$, and the equation is of the form $x^2+y^2+2gx+c=0$.

Now the radical axis is $x=0$, hence the equation of the system of circles is
$$x^2+y^2+2gx+c+kx=0$$

for different values of $k$, i. e.

$$x^2+y^2+c+x(2g+k)=0;$$

or, writing $2\lambda$ for the variable coefficient of $x$, we have

$$x^2+y^2+c+2\lambda x=0.$$

The constant $c$ is fixed; by varying the coefficient $\lambda$ the equation of all the circles of the system can be obtained, and further, to every value of $\lambda$ there corresponds a circle of the system.

**Properties of a system of coaxal circles.**

(i) The general equation of a circle of the system

$$x^2+y^2+c+2\lambda x=0$$

can be written
$$(x+\lambda)^2+y^2=\lambda^2-c,$$

and if $\lambda$ has either of the values $\pm\sqrt{c}$, this reduces to

$$(x-\sqrt{c})^2+y^2=0 \quad \text{or} \quad (x+\sqrt{c})^2+y^2=0.$$

These equations represent circles whose centres are $(\sqrt{c}, 0)$ and

$(-\sqrt{c}, 0)$ and whose radii are zero: the only real values of $x$ and $y$ which can satisfy them are the coordinates of these centres.

These points $(\sqrt{c}, 0)$, $(-\sqrt{c}, 0)$ are called the *Limiting Points* of the coaxal system: they lie on the line of centres at equal distances from the radical axis on either side. They are often referred to as the *Point-circles* of the system.

The limiting points are real when $c$ is positive, and imaginary when $c$ is negative.

Now the circle $x^2+y^2+c+2\lambda x=0$ meets the radical axis $x=0$ in points whose coordinates are given by $y^2+c=0$, i.e. the radical axis meets the circles of the system in real or imaginary points according as $c$ is negative or positive.

Thus a system of coaxal circles which intersect only in imaginary points has real limiting points: a system of coaxal circles which intersect in real points has imaginary limiting points.

Hence $x^2+y^2+\delta^2+2\lambda x=0$ represents in general a system of coaxal circles which do not intersect in real points, and whose limiting points are $(\delta, 0)$, $(-\delta, 0)$.

(ii) The limiting points are conjugate with respect to every circle of the system and have the same polars with respect to all circles of the system. For the polar of $(\delta, 0)$ with regard to

$$x^2+y^2+\delta^2+2\lambda x=0$$

is
$$x\delta+\delta^2+\lambda(x+\delta)=0,$$

i. e.
$$(x+\delta)(\delta+\lambda)=0,$$

i. e.
$$x+\delta=0,$$

a line through the other limiting point parallel to the radical axis. Thus the polars of the limiting points are the same for all circles of the coaxal system and are conjugate with respect to every circle of the system.

(iii) The equation of any circle of the system can be written

$$(x+\lambda)^2+y^2=\lambda^2-\delta^2;$$

hence if the circle is real $\lambda^2>\delta^2$; the centre is $(-\lambda, 0)$, hence no real circle of the system has its centre between the limiting points.

(iv) Any circle through the limiting points cuts all the circles of the system orthogonally.

Any circle through $(\delta, 0)$, $(-\delta, 0)$ is [p. 162]

$$x^2+y^2+2fy-\delta^2=0,$$

which cuts any circle of the system

$$x^2+y^2+2\lambda x+\delta^2=0$$

orthogonally. (The condition $2Gg+2Ff=C+c$ is satisfied.)

**Ex.** i. *All circles which bisect the circumferences of two given circles form a coaxal system.*

Let the equations of the two given circles be

$$x^2+y^2+c+2fx=0, \tag{i}$$
$$x^2+y^2+c+2gx=0. \tag{ii}$$

Any diameter of (i) is of the form

$$x+f+ky=0,$$

since its centre is $(-f,\ 0)$.

Hence the circle

$$x^2+y^2+c+2fx+\lambda(x+f+ky)=0 \tag{iii}$$

bisects the circumference of (i).

The common chord of the circles (ii) and (iii) is

$$2(f-g)x+\lambda(x+f+ky)=0;$$

but this is a diameter of the circle (ii) if the circle (iii) bisects its circumference. Hence the centre of (ii), viz. $(-g,\ 0)$, lies on this line. Hence

$$-2g(f-g)+\lambda(f-g)=0;$$
$$\therefore \qquad \lambda=2g.$$

The circle bisecting the circumferences of the circles (i) and (ii) is accordingly $x^2+y^2+c+2fx+2g(x+f+ky)=0,$

i.e. $x^2+y^2+2(f+g)x+c+2fg+2gky=0,$

which for different values of the undetermined constant $k$ represents a system of coaxal circles, the radical axis being $y=0$.

**Ex.** ii. *If* $C\equiv x^2+y^2+2gx+2fy-2fg=0,$
$C'\equiv x^2+y^2+2g'x+2f'y-2f'g'=0$

*are two circles of a coaxal system, show that the point-circles are given by the equation* $C^2(f'+g')^2-2CC'(f+g')(f'+g)+C'^2(f+g)^2=0.$

Any circle of the system is $C+\lambda C'=0$, or written in full

$$x^2+y^2+2gx+2fy-2fg+\lambda(x^2+y^2+2g'x+2f'y-2f'g')=0.$$

This is a point-circle of the system if its radius is zero, i.e. if

$$\left(\frac{g+\lambda g'}{1+\lambda}\right)^2+\left(\frac{f+\lambda f'}{1+\lambda}\right)^2+\frac{2fg+2\lambda f'g'}{1+\lambda}=0,$$

i.e. $(g+\lambda g')^2+(f+\lambda f')^2+2(1+\lambda)(fg+\lambda f'g')=0,$

or $\lambda^2(g'+f')^2+2\lambda(gg'+ff'+f'g'+fg)+(g+f)^2=0,$

i.e. $\lambda^2(g'+f')^2+2\lambda(f+g')(f'+g)+(g+f)^2=0.$

Hence, if $\lambda$ has either of the values given by this equation, $C+\lambda C'=0$ is a point-circle of the system.

Thus the coordinates of the point-circles satisfy

$$\lambda=-\frac{C}{C'},$$

where $\lambda$ satisfies the above equation.

Therefore the equation of the point-circles is

$$C^2(g'+f')^2-2CC'(f+g')(f'+g)+C'^2(g+f)^2=0.$$

## II. Three Circles.

*The radical axes of three circles taken in pairs are concurrent.*

For, if $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ are three circles, their radical axes when taken in pairs are the lines

$$C_1 - C_2 = 0, \quad C_2 - C_3 = 0, \quad C_3 - C_1 = 0,$$

which evidently are concurrent. Their point of intersection is given by $C_1 = C_2 = C_3$; it is called the **Radical Centre** of the three circles.

The lengths of the tangents from the Radical Centre to the circles are equal, since its coordinates satisfy $C_1 = C_2 = C_3$.

The circle whose centre is the Radical Centre and whose radius is equal to one of these tangents is called the **Radical Circle**, and cuts all three circles orthogonally.

*If $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ are three circles, their six centres of similitude lie in sets of three on four straight lines, viz.*

(a) *the three external centres of similitude of $C_1C_2$, $C_2C_3$, $C_3C_1$.*

(b) *the two internal centres of similitude of $C_1C_2$, $C_1C_3$, and the external of $C_2C_3$.*

(c) *the two internal centres of similitude of $C_2C_1$, $C_2C_3$, and the external of $C_1C_3$.*

(d) *the two internal centres of similitude of $C_3C_1$, $C_3C_2$, and the external of $C_1C_2$.*

*These four lines are called* **Axes of Similitude.**

Let the centres and radii of the circles be $(\alpha_1, \beta_1)$, $r_1$; $(\alpha_2, \beta_2)$, $r_2$; $(\alpha_3, \beta_3)$, $r_3$.

The external centres of similitude are

$$\left\{\frac{r_2\alpha_3 - r_3\alpha_2}{r_2 - r_3}, \frac{r_2\beta_3 - r_3\beta_2}{r_2 - r_3}\right\}, \quad \left\{\frac{r_3\alpha_1 - r_1\alpha_3}{r_3 - r_1}, \frac{r_3\beta_1 - r_1\beta_3}{r_3 - r_1}\right\},$$

$$\left\{\frac{r_1\alpha_2 - r_2\alpha_1}{r_1 - r_2}, \frac{r_1\beta_2 - r_2\beta_1}{r_1 - r_2}\right\}.$$

Let $lx + my + n = 0$ be the straight line joining the first two; then, substituting the coordinates of the first in this equation, we obtain

$$\frac{l\alpha_2 + m\beta_2 + n}{r_2} = \frac{l\alpha_3 + m\beta_3 + n}{r_3},$$

and, substituting the coordinates of the second,

$$\frac{l\alpha_3 + m\beta_3 + n}{r_3} = \frac{l\alpha_1 + m\beta_1 + n}{r_1}.$$

Hence $$\frac{l\alpha_1+m\beta_1+n}{r_1}=\frac{l\alpha_2+m\beta_2+n}{r_2},$$

and the third centre of similitude also lies on the line.

To obtain the equation of this line, let

$$\frac{l\alpha_1+m\beta_1+n}{r_1}=\frac{l\alpha_2+m\beta_2+n}{r_2}=\frac{l\alpha_3+m\beta_3+n}{r_3}=k.$$

Thus
$$\begin{aligned} lx+my+n &= 0,\\ l\alpha_1+m\beta_1+n-kr_1 &= 0,\\ l\alpha_2+m\beta_2+n-kr_2 &= 0,\\ l\alpha_3+m\beta_3+n-kr_3 &= 0. \end{aligned}$$

If we eliminate $l$, $m$, $n$, and $k$, we obtain the equation in the form

$$\begin{vmatrix} x & y & 1 & 0\\ \alpha_1 & \beta_1 & 1 & r_1\\ \alpha_2 & \beta_2 & 1 & r_2\\ \alpha_3 & \beta_3 & 1 & r_3 \end{vmatrix}=0.$$

If we change the sign of $r_1$ we obtain the equation of the axis of similitude which passes through the external centre of $C_2$, $C_3$ and the internal centres of $C_1$, $C_2$ and $C_1$, $C_3$, and similarly for the other two.

**Ex. i.** *Prove that the locus of the centre of a circle cutting three given circles at the same angle is the perpendicular let fall from their radical centre on an axis of similitude.*

Let the circles be
$$\begin{aligned} C_1 &\equiv (x-\alpha_1)^2+(y-\beta_1)^2-r_1^2=0,\\ C_2 &\equiv (x-\alpha_2)^2+(y-\beta_2)^2-r_2^2=0,\\ C_3 &\equiv (x-\alpha_3)^2+(y-\beta_3)^2-r_3^2=0; \end{aligned}$$

and suppose $(\xi, \eta)$ to be the centre and $\rho$ the radius of the cutting circle.

Then $$(\xi-\alpha_1)^2+(\eta-\beta_1)^2=r_1^2+\rho^2+2r_1\rho\cos\alpha;$$

i.e. the coordinates $\xi$, $\eta$ satisfy

$$C_1=\rho^2+2r_1\rho\cos\alpha.$$

Similarly they satisfy
$$\begin{aligned} C_2 &= \rho^2+2r_2\rho\cos\alpha,\\ C_3 &= \rho^2+2r_3\rho\cos\alpha. \end{aligned}$$

Eliminating $\rho^2$ and $\rho\cos\alpha$ from these three equations, we get the equation

$$\begin{vmatrix} C_1 & 1 & r_1\\ C_2 & 1 & r_2\\ C_3 & 1 & r_3 \end{vmatrix}=0,$$

i.e. $$r_1(C_2-C_3)+r_2(C_3-C_1)+r_3(C_1-C_2)=0. \qquad \text{(i)}$$

This is a straight line, and it evidently passes through the radical centre, which is given by $C_1=C_2=C_3$.

The equations of the axes of similitude were found above; it can be verified that the straight line (i) is perpendicular to the axis of similitude which contains the three external centres of similitude.

The angle at which two circles cut can be taken as the acute or the obtuse angle between the tangents at the point of intersection.

We might therefore have taken

$$C_1 = \rho^2 \pm 2r_1\rho\cos\alpha,$$
$$C_2 = \rho^2 \pm 2r_2\rho\cos\alpha,$$
$$C_3 = \rho^2 \pm 2r_3\rho\cos\alpha,$$

giving as the locus of the centre

$$\begin{vmatrix} C_1 & 1 & \pm r_1 \\ C_2 & 1 & \pm r_2 \\ C_3 & 1 & \pm r_3 \end{vmatrix} = 0,$$

i.e. four different straight lines,

$$r_1(C_2 - C_3) \pm r_2(C_3 - C_1) \pm r_3(C_1 - C_2) = 0.$$

These will be the perpendiculars from the radical centre to the four axes of similitude.

**Ex. ii.** *If* $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ *are three circles, then*

$$\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3 = 0$$

*represents for different values of* $\lambda_1$, $\lambda_2$, $\lambda_3$ *a system of circles cutting a fixed circle orthogonally.*

Let $(\alpha, \beta)$ be the centre of a circle and $\rho$ its radius, and let $C_1'$, $C_2'$, $C_3'$ stand for the values of $C_1$, $C_2$, $C_3$ when $\alpha$, $\beta$ are substituted for $x$ and $y$.

Now this circle cuts the circle

$$\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3 = 0 \qquad \text{(i)}$$

orthogonally, provided that the square of the tangent from $(\alpha, \beta)$ to the circle (i) is equal to $\rho^2$.

The coefficients of $x^2$ and $y^2$ in equation (i) are $(\lambda_1 + \lambda_2 + \lambda_3)$.

Hence the square of the tangent from $(\alpha, \beta)$ to the circle is

$$\frac{\lambda_1 C_1' + \lambda_2 C_2' + \lambda_3 C_3'}{\lambda_1 + \lambda_2 + \lambda_3}.$$

Thus, if the circle whose centre is $(\alpha, \beta)$ and radius $\rho$ cuts the circle orthogonally $\lambda_1 C_1' + \lambda_2 C_2' + \lambda_3 C_3' = \rho^2(\lambda_1 + \lambda_2 + \lambda_3)$,

i.e. $\lambda_1(C_1' - \rho^2) + \lambda_2(C_2' - \rho^2) + \lambda_3(C_3' - \rho^2) = 0.$

It will then cut all circles represented by (i) orthogonally if this condition is true for all values of $\lambda_1$, $\lambda_2$, and $\lambda_3$, i.e. if it is possible to have

$$C_1' - \rho^2 = 0,\ C_2' - \rho^2 = 0,\ C_3' - \rho^2 = 0$$

simultaneously, i.e. $C_1' = C_2' = C_3' = \rho^2$.

This is possible if $(\alpha, \beta)$ is the radical centre of

$$C_1 = 0,\ C_2 = 0,\ C_3 = 0,$$

and $\rho^2$ is the square of the length of the tangent from the radical centre to either circle.

Hence for all values of $\lambda_1$, $\lambda_2$, $\lambda_3$ the circle

$$\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3 = 0$$

cuts orthogonally the circle whose centre is the radical centre of $C_1 = 0$,

$C_2 = 0$, $C_3 = 0$, and whose radius is equal to the tangent from this centre to either circle.

In particular the circles $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ each cut this circle orthogonally, corresponding to the cases when two of the coefficients $\lambda_1, \lambda_2, \lambda_3$ are zero.

## Examples V 1.

1. Find the radical centre of the circles

$$x^2+y^2+3x+2y+1=0,$$
$$x^2+y^2-x+6y+5=0,$$
$$x^2+y^2+5x-8y+15=0,$$

and find whether it lies inside or outside the circles.

Hence find the equation of a circle cutting all three orthogonally.

2. Find the radical axis of

$$2x^2+2y^2-3x+5y+2=0,$$
$$x^2+y^2+8x+4y-5=0,$$

and show that the circles cutting these two circles orthogonally pass through two fixed points on their line of centres.

3. Find the equations of the three radical axes of the circles

$$(x-a)^2+(y-b)^2=b^2,$$
$$(x-b)^2+(y-a)^2=a^2,$$
$$(x-a-b-c)^2+y^2=ab+c^2,$$

and prove that they are concurrent.

Find also the equation of the circle which cuts them all three orthogonally.

4. Find the coordinates of the limiting points of

$$x^2+y^2+2x+4y+7=0,$$
$$x^2+y^2+5x+y+4=0.$$

5. Two circles whose centres are $(a, \theta)$ $(b, \theta)$ have the axis of $y$ as radical axis. If the radius of the first circle is $r$, find that of the second.

6. If two circles cut a third circle orthogonally, the radical axis of the two circles passes through the centre of the third circle.

7. Show that the locus of a point the tangents from which to two given circles are in a constant ratio is a coaxal circle.

8. The polars of a point $P$ with respect to two given circles meet in $Q$: show that the radical axis of the circles bisects $PQ$.

9. In the equation $x^2+y^2+2gx+c=0$, if $g$ is a variable parameter and $(x', y')$ a fixed point, then the polars of $(x', y')$ with respect to the circles all pass through a fixed point lying on a circle through $(x', y')$ and the limiting points of the circles.

10. Show that the three circles of similitude of three given circles taken in pairs are coaxal.

11. If the equations of one circle and of the radical axis of this circle and another are respectively

$$a(x^2+y^2)+2gx+2fy+c=0 \text{ and } lx+my+n=0,$$

find the equation of the other circle with the proper number of arbitrary constants and the coordinates of the limiting points.

12. Circles which cut two fixed circles of a coaxal system at constant angles will cut all circles of the system at constant angles.

13. Two systems of coaxal circles are such that the radical axis of either is the line of centres of the other. Show that the product of the radii of any two circles, one of each system, which touch one another, is constant.

14. A certain point has the same polar with respect to each of two circles: prove that a common tangent subtends a right angle at the point.

15. Prove that the locus of the middle points of chords of a fixed circle which subtend a right angle at a fixed point is a circle, and that the fixed point is a limiting point of the two circles.

16. A common tangent is drawn to two circles so as to intersect the line joining the centres when produced, and $S$ is a limiting point external to one circle and internal to the other. Prove that twice the perpendicular from $S$ to this tangent is a harmonic mean between the greatest distances of $S$ from each of the circles.

17. $A$, $B$, $C$, $D$ are four circles: the radical axis of $A$ and $B$ is perpendicular to that of $C$ and $D$; also the radical axis of $A$ and $C$ is perpendicular to that of $B$ and $D$; prove that the radical axis of $A$ and $D$ is perpendicular to that of $B$ and $C$.

18. Show that the limiting points of the circle $x^2+y^2=a^2$ and an equal circle with centre on the line $lx+my+n=0$ lie on the locus

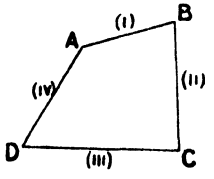
$$(x^2+y^2)(lx+my+n)+a^2(lx+my)=0.$$

19. Find the limiting points of the system of circles defined by the equation $x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+c')=0$, and show that they subtend a right angle at the origin if $cg^{-2}+c'f^{-2}=2$.

20. Show that the circle of similitude of any two of the circles described on the sides of a triangle as diameters cuts orthogonally the circle circumscribing the triangle.

21. $D$, $E$, $F$ are points on the sides of a triangle $ABC$ such that $AD$, $BE$, $CF$ are concurrent. Prove that the radical axes of the circle $ABC$ and the circles on $AD$, $BE$, $CF$ as diameters meet $BC$, $CA$, $AB$ in three collinear points.

## § 15. Other Forms.




I. Let the four sides of a quadrilateral $ABCD$ be

$$u \equiv x\cos\alpha_1+y\sin\alpha_1-p_1=0, \quad \text{(i)}$$

$$v \equiv x\cos\alpha_2+y\sin\alpha_2-p_2=0, \quad \text{(ii)}$$

$$w \equiv x\cos\alpha_3+y\sin\alpha_3-p_3=0, \quad \text{(iii)}$$

$$z \equiv x\cos\alpha_4+y\sin\alpha_4-p_4=0. \quad \text{(iv)}$$

Consider the equation $uw = \lambda vz$, which, written in full, is

$$(x\cos\alpha_1 + y\sin\alpha_1 - p_1)(x\cos\alpha_3 + y\sin\alpha_3 - p_3)$$
$$= \lambda(x\cos\alpha_2 + y\sin\alpha_2 - p_2)(x\cos\alpha_4 + y\sin\alpha_4 - p_4), \qquad \text{(v)}$$

where $\lambda$ is a constant. This is satisfied by the coordinates of each of the points $A$, $B$, $C$, $D$: thus, $B$ is the point of intersection of $u = 0$ and $v = 0$; its coordinates therefore make $u$ and $v$ zero, i.e. satisfy the above equation. Hence equation (v) represents a locus passing through the four points $A$, $B$, $C$, $D$. In order that the equation should represent a circle two conditions must be satisfied, but we have only one constant, $\lambda$, undetermined: thus the equation can only represent a circle when some definite relation exists between the coefficients of the equations of the lines $u$, $v$, $w$, $z$.

The conditions for a circle give us

$$\cos\alpha_1\cos\alpha_3 - \lambda\cos\alpha_2\cos\alpha_4 = \sin\alpha_1\sin\alpha_3 - \lambda\sin_2\sin\alpha_4,$$

and $$\sin\alpha_3\cos\alpha_1 + \sin\alpha_1\cos\alpha_3 = \lambda(\sin\alpha_4\cos\alpha_2 + \sin\alpha_2\cos\alpha_4),$$

i.e.
$$\left.\begin{aligned}\cos(\alpha_1+\alpha_3) &= \lambda\cos(\alpha_2+\alpha_4),\\ \text{and}\quad \sin(\alpha_1+\alpha_3) &= \lambda\sin(\alpha_2+\alpha_4).\end{aligned}\right\} \qquad \text{(vi)}$$

Hence the condition that the lines (i), (ii), (iii), (iv) should form a quadrilateral which can be circumscribed by a circle is

$$\tan(\alpha_1+\alpha_3) = \tan(\alpha_2+\alpha_4),$$

or
$$\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 = n\pi,$$

where $n$ is an integer.

This corresponds to the proposition 'the opposite angles of a cyclic quadrilateral are supplementary'.

**Note.** It follows also from (vi) that $\lambda = \pm 1$, and $n$ is even or odd according as we take the upper or lower sign; for example, taking $\lambda = 1$, the equation

$$\{x\cos\overline{\alpha+\beta} + y\sin\overline{\alpha+\beta} - a\cos\overline{\alpha-\beta}\}\{x\cos\overline{\gamma+\delta} + y\sin\overline{\gamma+\delta} - a\cos\overline{\gamma-\delta}\}$$
$$= \{x\cos\overline{\alpha+\gamma} + y\sin\overline{\alpha+\gamma} - a\cos\overline{\alpha-\gamma}\}\{x\cos\overline{\beta+\delta} + y\sin\overline{\beta+\delta} - a\cos\overline{\beta-\delta}\}$$

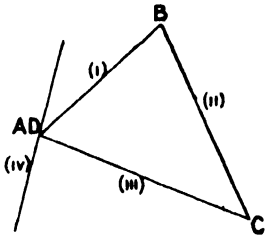
represents a circle; it can in fact be reduced to $x^2 + y^2 = a^2$.

II. Now suppose that the lines $u = 0$, $v = 0$, $w = 0$, $z = 0$ are such that the two points $A$ and $D$ coincide, then $z = 0$ meets the locus represented by

$$uw = \lambda \,.\, vz$$

in two coincident points, i.e. $z = 0$ is a tangent to the locus.

But in this case $z = 0$ is a line through the intersection of $u = 0$ and $w = 0$: hence [Chap. II, p. 52] $z \equiv lu + mw$, where $l$ and $m$ are constants.




Thus the equation of the locus $ABCD$ (when $A$ and $D$ coincide) becomes

$$uw = v(lu + mw)$$

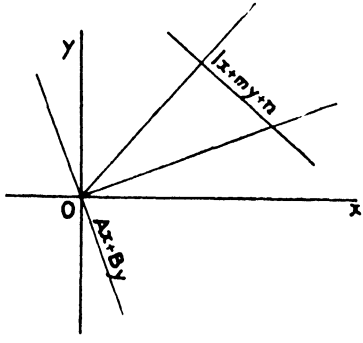
where $lu + mw = 0$ is the tangent at the point of intersection of $u = 0$, $w = 0$.

Thus, if $u = 0$, $v = 0$, $w = 0$ are the three sides of a triangle,

$$luv + mvw + nuw = 0$$

represents a locus passing through the vertices, and the tangents at the vertices are $mv + nu = 0$, $nw + lv = 0$, $lu + mv = 0$ respectively.

**Example.** *To find the equation of the circle circumscribing the triangle formed by the lines* $ax^2 + 2hxy + by^2 = 0$, $lx + my + n = 0$.




Suppose that the tangent to the circle at the origin is

$$Ax + By = 0,$$

then, for some values of $A$ and $B$,

$$ax^2 + 2hxy + by^2 = (Ax + By)(lx + my + n)$$

represents the circle.

The conditions for a circle give us

$$a - Al = b - Bm \text{ and } 2h = Am + Bl.$$

Thus, $$Al - Bm + b - a = 0 \text{ and } Am + Bl - 2h = 0;$$

or $$\frac{A}{2hm + (a-b)l} = \frac{B}{2hl - (a-b)m} = \frac{1}{l^2 + m^2}.$$

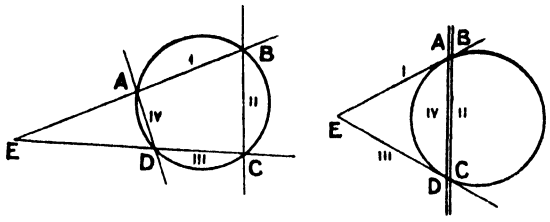
The equation of the circle is then

$$(ax^2 + 2hxy + by^2)(l^2 + m^2) = \{(2hm + \overline{a-b}\,l)x + (2hl - \overline{a-b}\,m)y\}\{lx + my + n\};$$

which reduces to

$$(x^2 + y^2)\{am^2 - 2hlm + bl^2\} = 2hn(mx + ly) + n(a-b)(lx - my).$$

III. Again, in the quadrilateral $ABCD$, suppose that the pair of opposite sides (ii) and (iv) coincide; then the equation $uw = \lambda vx$ becomes $uw = \lambda v^2$.




This locus meets the line $u = 0$ in coincident points at $A$, $B$ and the line $w = 0$ in coincident points at $C$, $D$, i.e. the locus touches the lines $u = 0$, $w = 0$ at the points where $v = 0$ meets them.

Thus,

$$(x\cos\alpha_1 + y\sin\alpha_1 - p_1)(x\cos\alpha_3 + y\sin\alpha_3 - p_3) = \lambda(x\cos\alpha_2 + y\sin\alpha_2 - p_2)^2$$

represents a curve to which the lines (i) and (iii) are tangents, the line (ii) being the chord of contact.

As in I, this curve can only be a circle if

$$\cos(\alpha_1 + \alpha_3) = \lambda\cos 2\alpha_2, \quad \sin(\alpha_1 + \alpha_3) = \lambda\sin 2\alpha_2,$$

i.e. $$\alpha_1 + \alpha_3 = n\pi + 2\alpha_2,$$

which is equivalent to $\angle EAC = \angle EDA$, i.e. the tangents must be equally inclined to the chord of contact. If this condition is fulfilled, either of the above conditions gives the value of $\lambda$.

IV. Let $C \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be any circle and

$$u \equiv x\cos\alpha_1 + y\sin\alpha_1 - p_1 = 0,$$
$$v \equiv x\cos\alpha_2 + y\sin\alpha_2 - p_2 = 0$$

two straight lines cutting it at $A$, $D$ and $B$, $C$ respectively.

Now the equation $C = \lambda uv$ represents some locus; the coordinates of the points $A$ and $D$ satisfy both $C = 0$ and $u = 0$, and therefore lie on this locus. So also for $B$ and $C$. Hence $C = \lambda uv$ represents a locus passing through the points $A$, $B$, $C$, $D$.

This can never represent a circle unless the locus $C = \lambda uv$ coincides with the original circle, i.e. $\lambda = 0$, for three points are sufficient to fix a circle. It can, however, for certain values of $\lambda$, represent the pairs of straight lines $AB$ and $CD$, or $AC$ and $BD$.

**Example.** *If a pair of straight lines is drawn through a fixed point to meet two fixed straight lines in four concyclic points, show that the locus of the centre of the circle is a straight line.*

Let the two fixed straight lines be

$$y - mx = 0, \tag{i}$$

$$y + mx = 0. \tag{ii}$$

Suppose that $$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{iii}$$

is any one of the circles. Then the equation

$$A(y^2 - m^2x^2) = x^2 + y^2 + 2gx + 2fy + c \tag{iv}$$

is satisfied by the coordinates of the four points common to (i), (ii), and (iii), and hence represents a locus passing through the four points. For some values of $A$ this equation represents two straight lines through the four points, and we are given that these straight lines intersect in a fixed point, say $(p, q)$.

The equation (iv) can be written

$$x^2(1 + m^2A) + y^2(1 - A) + 2gx + 2fy + c = 0,$$

and (Chap. III, § 11), if this represents two straight lines through $(p, q)$, we have

$$(1 + m^2A)p + g = 0,$$

$$(1 - A)q + f = 0.$$

Hence, eliminating $A$, we have

$$q(p + g) + m^2p(q + f) = 0.$$

But the centre of the circle is $(-g, -f)$; it therefore lies on the straight line

$$q(x - p) + m^2p(y - q) = 0.$$

## Examples V m.

1. Find the equation of the other two pairs of straight lines which pass through the intersections of $x^2 - 2xy + y^2 - 4 = 0$, and the circle

$$x^2 + y^2 - 2x - 2y - 2 = 0.$$

2. A circle touches the straight line $3x + 4y = 0$ at the origin and cuts the straight lines $7x^2 + 11xy + 3y^2 = 0$ at the points $P$ and $Q$.

If $PQ$ passes through the point $(1, -3)$, find its equation.

3. The common chord of a given circle and any other circle of given radius $a$ passes through a fixed point. Find the locus of the centre of the circle of radius $a$.

4. Show analytically that if a parallelogram is inscribed in a circle it must be a rectangle.

5. Show that the two pairs of straight lines

$$x^2 - 4xy + 3y^2 + 10x - 6y - 24 = 0,$$

$$3x^2 + 4xy + y^2 - 26x - 18y + 56 = 0$$

form a cyclic quadrilateral.

Find the equation, centre, and radius of the circumscribing circle.

6. Circles are described through the intersections of

$$lx+my+n=0, \tag{i}$$

and

$$ax^2+2hxy+by^2=0. \tag{ii}$$

Show that the other chord of intersection with (ii) is fixed in direction, and find its equation in a form containing one arbitrary constant.

7. Find the equation of a circle touching the $x$-axis and passing through the points of intersection of the circles

$$x^2+y^2+4x-14y-68=0,$$
$$x^2+y^2-6x-22y+30=0.$$

8. The circles

$$x^2+y^2-2kx-d^2=0,$$
$$x^2+y^2-2k'x-d^2=0$$

intersect in $A$ and $B$. Through $A$ a line is drawn perpendicular to $AB$ meeting the circles in $C$ and $D$ respectively. Find the equation of the circle circumscribing the triangle $BCD$.

9. Write down the equation to the circle which passes through the point (2, 1) and the points common to the circles

$$2x^2+2y^2-3x+5y+1=0,$$
$$x^2+y^2=1.$$

10. Find the equation of the circle whose diameter is the portion of the line $3x+4y=12$ intercepted by the lines $5x^2-7xy+2y^2=0$.

11. The points of intersection of the circles

$$x^2+y^2+2gx+c=0,$$
$$x^2+y^2+2fy-c=0$$

subtend a right angle at the origin. Prove that $g^2-f^2=2c$.

12. Find the area of the triangle formed by the three points where the circle $x^2+y^2=2ax+2by$ is cut by the pair of straight lines

$$lx^2+2mxy+ny^2=0.$$

13. Find the equation of the circle which passes through the intersection of the circles $x^2+y^2-3x-2y-6=0$, $x^2+y^2-5x+4y+2=0$, and has its centre on the straight line $x=y$.

14. Show that the equation of the circle whose diameter is the portion of the line $lx+my=1$ intercepted by the lines $ax^2+2hxy+by^2=0$ is $(x^2+y^2)(am^2-2hlm+bl^2)+2x(hm-bl)+2y(hl-am)+a+b=0$. If $a+b=0$, this passes through the origin: to what geometrical fact does this correspond?

15. If

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

are the equations of the pairs of opposite sides of a quadrilateral inscribed in a circle, show that $H(b-a)=h(B-A)$.

16. The straight line $x\cos\alpha+y\sin\alpha=p$ being called the line $(\alpha, p)$, find the equation of the circle circumscribing the triangle formed by the lines $(\alpha, p)$ $(\beta, q)$ $(\gamma, r)$, and show it passes through the origin if

$$qr\sin(\beta-\gamma)+rp\sin(\gamma-\alpha)+pq\sin(\alpha-\beta)=0.$$

17. Find the equation of the circumcircle of the triangle formed by the lines $bx+cy+a=0$, $cx+ay+b=0$, $ax+by+c=0$, and show that it passes through the origin if $(b^2+c^2)(c^2+a^2)(a^2+b^2)=abc(b+c)(c+a)(a+b)$.

18. Prove that the equations of the common tangents of the circle $x^2+y^2=289$ and the circle whose diameter is the chord $x\cos\alpha+y\sin\alpha=15$ of the first circle are $3(x\cos\alpha+y\sin\alpha)\pm 4(y\cos\alpha-x\sin\alpha)=85$.

19. If $uv=w^2$ is a circle, where $u=0$, $v=0$, $w=0$ are straight lines, show that any point on the circle is the point of intersection of two lines of the form $\lambda u=w$, $v=\lambda w$.

Indicating any point on the circle by the parameter $\lambda$, show that the chord joining two points whose parameters are $\lambda_1$ and $\lambda_2$ is

$$\lambda_1\lambda_2 u+v-(\lambda_1+\lambda_2)w=0.$$

20. Show that the equation of the tangent to the circle in Question 19 at the point whose parameter is $\lambda$ is $\lambda^2 u+v-2\lambda w=0$. Show also that the tangent at the points $\lambda$ and $\mu$ intersect at a point whose coordinates satisfy the equation

$$u/2=v/2\lambda\mu=w/(\lambda+\mu).$$

21. Use the notation in Question 20 to solve the following:—

$OA$, $OB$ are tangents to a circle, $P$ is any point on the circle, and the lines $PA$, $PB$ meet any line through $O$ at $C$ and $D$. Find the locus of the intersection of $BC$ and $AD$.

22. $OA$ and $OB$ are tangents to a circle, a line through $O$ meets the circle at $P$, $Q$, and $AB$ at $R$.

If $u=0$, $v=0$ are the tangents $OA$ and $OB$, and the equation of the circle is $uv=w^2$, find the equations of the lines $AP$, $AQ$, and thus show that $O$, $R$, $P$, $Q$ form a harmonic range.

23. Show that the pole of the line $lu+mv+nw=0$, with respect to a circle $uv=w^2$, is the intersection of the lines $nu+2mw=0$, $nv+2lw=0$, i.e. is given by $u/2m=v/2l=w/-n$.

24. The coordinates of a point make $u$, $v$, and $w$ equal to $u'$, $v'$, and $w'$ respectively. Show that the polar of this point with respect to a circle whose equation is $uv=w^2$ is $uv'+u'v=2ww'$.

25. Show that the triangle whose sides are $u-lv=0$, $u+lv=0$, $w=0$ is self-conjugate with respect to the circle $uv=w^2$.

26. $OA\,(u=0)$, $OB\,(v=0)$ are tangents to the circle $uv=w^2$, and any other tangent to the circle meets them in $A'$ and $B'$. Show that the locus of the intersection of $AB'$ and $A'B$ is $uv=4w^2$.

## § 16. The 'circular points at infinity'.

If $C_1=0$, $C_2=0$ are the equations of two circles, then for all values of $\lambda$ except unity the equation $C_1=\lambda C_2$ represents a circle. When the circles $C_1$, $C_2$ intersect in two real points, all the circles represented by $C_1=\lambda C_2$ pass through these two points. In order to obtain complete generality we introduced in Chap. IV the ideas of imaginary points and coincident points; so

that we may say that all the circles represented by the equation $C_1 = \lambda C_2$ pass through the two points of intersection of the circles $C_1 = 0$ and $C_2 = 0$. Now any other type of locus might pass through these two points of intersection; this property is not therefore a geometrical explanation of the algebraical result that $C_1 = \lambda C_2$ *always* represents a circle.

Again, if $u = 0$, $v = 0$ are the equations of two straight lines, the locus $C_1 = \lambda uv$ passes through the four points of intersection of the straight lines $u$, $v$ with the circle $C_1$. This locus is not a circle except in the special case when $\lambda$ is zero. On the other hand, the equation $C_1 = \lambda u$ always represents a circle; this circle passes through the two points of intersection of the line $u = 0$ and the circle $C_1 = 0$, but this property evidently does not correspond to the fact that the locus is a circle. We have seen in Chap. IV that in order to obtain complete generality we had to adopt the ideas of 'points at infinity' and 'the line at infinity'. Also we saw that the properties of a locus with respect to the line at infinity could only be satisfactorily examined by using homogeneous coordinates. Euclidean Geometry fails to explain the facts given above; we proceed to examine, by the use of homogeneous coordinates, whether projective geometry offers an explanation.

The general equation of a circle in homogeneous coordinates is $x^2+y^2+cz^2+2fyz+2gzx=0$. The points of intersection of the circle and the line at infinity, $z = 0$, are therefore given by $x^2+y^2 = 0$ and $z = 0$. The homogeneous coordinates of these points are therefore $(1, i, 0)$, $(1, -i, 0)$, i. e. a pair of imaginary points. Now the coordinates of these points are independent of the coefficients $g$, $f$, and $c$; hence, *all circles intersect the line at infinity in the same pair of imaginary points.* These points are called 'the circular points at infinity', and will be referred to as $\Omega$, $\Omega'$.

**Note i.** $C_1 - C_2 = 0$.

This equation becomes in homogeneous coordinates

$$(x^2+y^2+c_1z^2+2f_1yz+2g_1zx)-(x^2+y^2+c_2z^2+2f_2yz+2g_2zx) = 0,$$

i. e. $$z\{2(g_1-g_2)x+2(f_1-f_2)y+(c_1-c_2)z\} = 0.$$

The equation represents two straight lines, viz. the straight line at infinity and the radical axis. Thus in projective geometry we may say that one common chord of every pair of circles is the straight line at infinity.

**Note ii.** $C_1 - \lambda u = 0$.

This equation becomes in homogeneous coordinates $C_1 - \lambda zu = 0$; it therefore represents a locus passing through the points of inter-

section of the straight lines $u = 0$, $z = 0$, and the circle $C_1 = 0$. If the straight line $u = 0$ cuts the circle in the points $A$ and $B$, then $C_1 - \lambda u = 0$ represents a locus circumscribing the quadrilateral $AB\Omega\Omega'$.

**Note iii.** $\mathbf{C_1 - \lambda C_2 = 0}$.

In the same way, since the circles $C_1$, $C_2$ intersect in two finite points (real and distinct, real and coincident, or imaginary and distinct), say $A$ and $B$, and also in the points $\Omega$, $\Omega'$, the equation $C_1 - \lambda C_2 = 0$ represents a locus circumscribing the quadrilateral $AB\Omega\Omega'$.

The general equation of the second degree

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

contains *five* independent constants; if we know *five* points on the locus we can therefore find its equation, and the locus is completely determined

We have seen that both the equations $C_1 - \lambda zu = 0$ and $C_1 - \lambda C_2 = 0$ imply by their form that the loci they represent pass through four given points; only one more point on the locus is required then in order to determine its equation completely; it follows, therefore, that the equations $C_1 - \lambda zu = 0$ and $C_1 - \lambda C_2 = 0$ should contain only one undetermined constant; this constant is $\lambda$.

**Note iv. The centre of the circle in relation to the line at infinity.**

The polar of a point $P$, with respect to a circle, was defined as the locus of the points of intersection of tangents to the circle at the pairs of points in which chords, passing through $P$, cut the circle. Now chords which pass through the centre of the circle cut the circle in pairs of points the tangents at which are parallel In Euclidean Geometry, therefore, the centre of the circle has no polar with respect to the circle. In projective geometry we say that pairs of parallel straight lines meet in 'points at infinity', and that the locus of these 'points at infinity' is the straight line at infinity. Thus 'the polar of the centre of a circle with respect to the circle is the line at infinity', and conversely 'the pole of the line at infinity with respect to a circle is its centre'.

In homogeneous coordinates the equation of the polar of the point $(x_1, y_1, z_1)$ with respect to the circle

$$x^2 + y^2 + cz^2 + 2fyz + 2gzx = 0$$

is $$x(x_1 + gz_1) + y(y_1 + fz_1) + z(gx_1 + fy_1 + cz_1) = 0.$$

The homogeneous coordinates of the centre of the circle are $(g, f, -1)$,

and the polar of the centre is therefore $z(g^2+f^2-c)=0$, or, what is the same thing, $z=0$, which is the line at infinity.

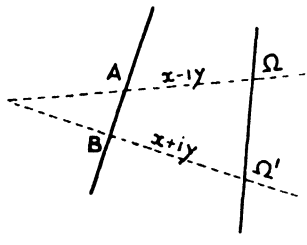
**Note v.** The equation of any circle, by a proper choice of axes, can be written $x^2+y^2=a^2$, or, in homogeneous coordinates,

$$x^2+y^2=a^2z^2.$$

This equation may be written $(x+iy)(x-iy)=a^2z^2$, which is in the form $uv=kw^2$, and represents a locus touching the imaginary straight lines $x+iy=0$, $x-iy=0$, the line at infinity $z=0$ being the chord of contact.

We have seen that every circle passes through the two fixed imaginary points $\Omega$, $\Omega'$ on the line at infinity. Conversely, every locus of the second degree which passes through $\Omega$ and $\Omega'$ is a circle. The points $\Omega$, $\Omega'$ are determined by the equations $x^2+y^2=0$ and $z=0$; i. e. are the points of intersection of the line $z=0$ and the lines $x+iy=0$, $x-iy=0$ respectively.

Now we showed in Chap. IV that, having adopted the ideas there explained, we could state that every straight line meets a locus of the second degree in two points. If, then, a locus of the second degree passes through $\Omega$ and $\Omega'$, it meets each of the straight lines $x+iy=0$, $x-iy=0$ in one other point, say $A$ and $B$. Let $ax+by+cz=0$ be the equation of the straight line $AB$.




Since the locus circumscribes the quadrilateral $AB\Omega\Omega'$, its equation is of the form $(x+iy)(x-iy)+z(ax+by+cz)=0$,

i. e. $$x^2+y^2+cz^2+byz+axz=0,$$

which is a circle.

We may therefore define a circle in projective geometry as a locus of the second degree passing through $\Omega$ and $\Omega'$. Thus, if the locus represented by the general equation of the second degree

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

passes through the points $\Omega$ and $\Omega'$, then $a=b$ and $h=0$; and this locus is called a circle.

### § 17. Polar Coordinates.

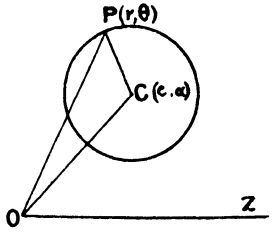
Let the centre of a circle be the point $(c, \alpha)$ and $R$ the radius: then if $P(r, \theta)$ be any point on the circle we have

$$PO^2+OC^2-2OP.OC\cos\angle COP=PC^2,$$

i. e. $$r^2+c^2-2rc\cos(\theta-\alpha)=R^2,$$

which is the general polar equation of a circle.

The equation of the circle takes the following simple forms in special cases:—




(i) The origin at the centre of the circle

$$r = R.$$

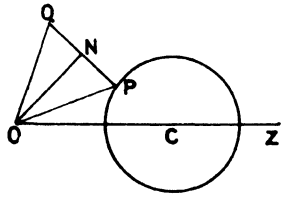
(ii) The origin on the circumference

$$r = 2R\cos(\theta-\alpha).$$

(iii) The origin on the circumference and the initial line passing through the centre $r = 2R\cos\theta$.

Polar coordinates can be used with advantage in certain types of problems; the following examples illustrate the method.

**Ex.** i. *A triangle given in species has one vertex fixed, and a second moves on a given circle; find the locus of the third.*




Let the fixed vertex be at the origin and let the initial line pass through the centre of the given circle. The equation of this circle is

$$r^2+c^2-2rc\cos\theta = R^2. \qquad \text{(i)}$$

Suppose that $OPQ$ is one position of the triangle, and let the given angles of this triangle be $\alpha$, $\beta$, and $\gamma$. Let the coordinates of the point $P$ be $(r, \theta)$ and those of $Q$ $(r', \theta')$; then, since the triangle is given in species, the ratio $r:r'$ is fixed, let $r = k.r'$.

It is evident from the figure that $\theta = \theta'-\alpha$. Now the coordinates of $P$, $r$ and $\theta$, satisfy equation (i); substituting $r = kr'$ and $\theta = \theta'-\alpha$ in this equation, we obtain

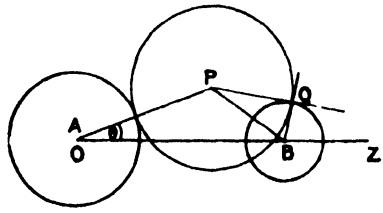
$$k^2r'^2+c^2-2kr'c\cos(\theta'-\alpha) = R^2.$$

Hence, the equation of the locus of $Q$ is the circle

$$k^2r^2+c^2-2krc\cos(\theta-\alpha) = R^2.$$

**Note.** In general, if $P$ lies on the locus $r = f(\theta)$, then the locus of $Q$ is $kr = f(\theta-\alpha)$.

**Ex. ii.** *Two circles of radii* a, b *have their centres distant* c *apart. The origin being the centre of the former and the initial line the line of centres, find the equation of the locus of centres of circles touching the former and cutting the latter orthogonally.*




Let $P$, the centre of such a circle, be the point $(r, \theta)$, and let the radius of this circle be $\rho$.

Since this circle touches the circle $A$, $r = a + \rho$.

Also, if $Q$ be a point of intersection with the circle,

$$PB^2 = PQ^2 + QB^2,$$

i.e. $$r^2 + c^2 - 2rc\cos\theta = \rho^2 + b^2.$$

Hence $$r^2 + c^2 - 2rc\cos\theta = (r-a)^2 + b^2,$$

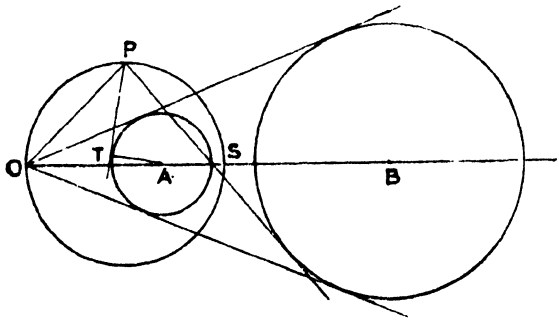
or $$2rc\cos\theta - 2ar + a^2 + b^2 - c^2 = 0,$$

is the required equation.

**Ex. iii.** *Show that the equation of the circle of similitude of*

$$r^2 - 2ar\cos\theta + a^2\cos^2\alpha = 0,$$
$$r^2 - 2br\cos\theta + b^2\cos^2\alpha = 0$$

*is* $(a+b)r = 2ab\cos\theta$. *Hence prove the property of the circle of similitude of two circles, that the tangents from any point on it to the two circles are in the ratio of their radii.*




The equation of the circles can be written

$$r^2 - 2ar\cos\theta + a^2 = a^2\sin^2\alpha,$$
$$r^2 - 2br\cos\theta + b^2 = b^2\sin^2\alpha.$$

Hence their centres are $(a, 0)$ $(b, 0)$ and their radii $a\sin\alpha$, $b\sin\alpha$ respectively.

If $\theta = \alpha$ or $-\alpha$, the corresponding value of $r$ is given by $(r - a\cos\alpha)^2 = 0$; thus $\theta = \alpha$ and $\theta = -\alpha$ meet the first, and similarly the second, circle in coincident points, i.e. are tangents: this is otherwise evident.

The centres of similitude lie on the initial line; one is at the origin and the radius vector of the other is

$$\frac{a\sin\alpha . b + b\sin\alpha . a}{a\sin\alpha + b\sin\alpha} = \frac{2ab}{a+b}.$$

Thus, if $P(r, \theta)$ is any point on the circle of similitude, since $OPS$ is a semicircle and hence $\angle OPS$ a right angle,

$$OP = OS\cos\theta,$$

i.e.
$$r = \frac{2ab}{a+b}\cos\theta,$$

or
$$(a+b)\,r = 2ab\cos\theta.$$

Let $PT$ be a tangent from this point to the first circle, then

$$PT^2 = PA^2 - AT^2,$$
$$= r^2 + a^2 - 2ar\cos\theta - a^2\sin^2\alpha,$$
$$= r^2 - 2ar\cos\theta + a^2\cos^2\alpha.$$

But
$$(a+b)\,r = 2ab\cos\theta,$$

i.e.
$$ar = -b\,(r - 2a\cos\theta).$$

$\therefore$
$$PT^2 = -\frac{ar}{b} + a^2\cos^2\alpha = \frac{a}{b}(ab\cos^2\alpha - r^2).$$

So if $PT'^2$ is a tangent to the second circle,

$$PT'^2 = \frac{b}{a}(ab\cos^2\alpha - r^2).$$

$\therefore$
$$PT^2 : PT'^2 = a^2 : b^2,$$

or
$$PT : PT' = a : b = a\sin\alpha : b\sin\alpha.$$

**Ex. iv.** *Prove that the polar equation*

$$r^2 - kr\cos(\theta - \alpha) + kd = 0,$$

*where $k$ is variable, represents a system of coaxal circles, and find the polar coordinates of the limiting points.*

Any two circles of the system are

$$r^2 - k_1 r\cos(\theta - \alpha) + k_1 d = 0,$$
$$r^2 - k_2 r\cos(\theta - \alpha) + k_2 d = 0.$$

The equation

$$r^2 - k_1 r\cos(\theta - \alpha) + k_1 d - \{r^2 - k_2 r\cos(\theta - \alpha) + k_2 d\} = 0,$$

i.e.
$$(k_1 - k_2)\,\{r\cos(\theta - \alpha) - d\} = 0,$$

i.e.
$$r\cos(\theta - \alpha) - d = 0$$

represents a straight line through the real or imaginary points of intersection of the circles.

This is therefore the radical axis of the two circles and, since the equation

does not contain $k_1$ or $k_2$, every pair of circles in the system have the same radical axis, i.e. the system is coaxal.

The equation of the circle can be written

$$r^2 - kr\cos(\theta-\alpha) + \tfrac{1}{4}k^2 = \tfrac{1}{4}k^2 - kd;$$

thus the radius is $\sqrt{\tfrac{1}{4}k^2 - kd}$ and the centre $(\tfrac{1}{2}k, \alpha)$.

The circle is a point-circle when $\tfrac{1}{4}k^2 - kd = 0$,

i.e. $k = 0$ or $4d$.

Hence the coordinates of the point-circles are $(0, \alpha)$ and $(2d, \alpha)$, i.e. the origin and $(2d, \alpha)$.

## Examples V n.

1. Find the polar equation of the circle on the line joining the points $(r_1, \theta_1)$ $(r_2, \theta_2)$ as diameter.

2. Find the equation of the tangent to the circle $r = R$ at the point whose vectorial angle is $\alpha$.

3. The equation of the chord joining two points on the circle $r = 2R\cos\theta$, whose vectorial angles are $\theta_1$, $\theta_2$, is $r\cos(\theta_1 + \theta_2 - \theta) = 2R\cos\theta_1\cos\theta_2$. Deduce the equation of the tangent at the point $\theta_1$.

4. Show that the vectorial angles of the points of contact of tangents from the point $(r_1, \theta_1)$ to the circle $r = 2R\cos\theta$ are given by

$$r_1\cos\theta_1 \cdot \tan^2\theta - 2r_1\sin\theta \cdot \tan\theta + 2R - r_1\cos\theta_1 = 0.$$

5. The polar of $(r_1, \theta_1)$ with respect to $r = 2R\cos\theta$ is

$$rr_1\cos(\theta-\theta_1) = R(r\cos\theta + r_1\cos\theta_1).$$

6. Show that $r^2 - 2ar\cos\theta - 3a^2 = 0$ is the polar equation of a circle whose centre lies on the initial line. If $OP$ is any radius vector of this circle and a point $Q$ is taken on $OP$ so that $OP \cdot OQ = 6a^2$, find the equation of the locus of $Q$ and show it is a circle whose radius is double the radius of the given circle.

7. $OA$ is a diameter of a circle, $Q$ any point on the polar of $P$, $E$ the mid-point of $PQ$; $EL$ is perpendicular to $OA$ and $QM$ is perpendicular to $OP$. Show that $A$, $L$, $P$, $M$ lie on a circle.

8. If $PQ$ is a chord of $r = 2b\cos\theta$ which touches $r = 2a\cos\theta$ at $R$, then $OR$ bisects the angle $POQ$.

9. If $P$, $Q$ are two points on the circle $r = 2R\cos\theta$ whose vectorial angles have (i) their sum, (ii) their difference constant, find the locus of the centroid of the triangle $OPQ$.

10. $PQ$ is a chord of the circle $r = 2b\cos\theta$ which touches the circle $r = 2a\cos\theta$. Show that the locus of a point $R$ on $PQ$, such that $OP \cdot OQ$ are harmonic conjugates of $OR$ and the tangent at the origin, is

$$r\{b^2 - (a-b)^2\sin^2\theta\} = 2ab^2\cos\theta.$$

11. A straight line $OP$ meets the circle $r = 2R\cos\theta$ at $P$ and a point $Q$ is taken on $OP$ such that $OP \cdot OQ = k^2$; find the locus of $Q$.

12. Show that the equation $r^2 - 2(a-\lambda)r\cos\theta + 2\lambda d = 0$ for different values of $\lambda$ represents a system of coaxal circles, and find the radical axis and limiting points.

13. From a fixed point $O$ a line is drawn to meet a fixed circle at $P$. A line $PQ$ is drawn equal to and perpendicular to $OP$. Find the locus of $Q$.

14. $OP$ is the radius vector from a fixed point $O$ to a given circle, the angle $POQ$ is constant and equal to $\alpha$, and the area of the triangle $POQ$ is fixed. Show that $Q$ lies on a circle.

15. Show that $2/r = 4\sin\theta - 3\cos\theta$ is a common tangent of $r = 2\cos\theta$ and $r^2 - 12r\cos\theta + 20 = 0$.

16. The equation of the pair of tangents at points on the circle

$$r^2 + d^2 - 2rd\cos(\theta - \alpha) = R^2,$$

whose vectorial angles are $\phi$, is

$$(d^2 - R^2)^2 + (d^2 - R^2)\{r^2\cos^2(\theta - \phi) - 2rd\cos(\theta - \alpha) - d^2\cos^2(\phi - \alpha)\}$$
$$= r^2 d^2\cos(\theta - \alpha)\{2\cos(\phi - \alpha)\cos(\phi - \theta) - \cos(\theta - \alpha)\}$$
$$- 2rd^3\cos(\theta - \alpha)\cos^2(\phi - \alpha),$$

and show that when $\phi = \alpha$ this reduces to

$$r^2\cos^2(\theta - \alpha) - 2dr\cos(\theta - \alpha) + d^2 - R^2 = 0.$$

17. $u_n$ is the line $r\cos(\theta - \theta_n)\cos\theta_n = a$, and circles are drawn about triangles formed by the lines $u_1$, $u_2$, $u_3$, $u_4$, $u_5$, taken three at a time: circles are then drawn through the centres of these circles taken four at a time: show that the five centres of these circles lie on the circle

$$4r\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\cos\theta_5 = a\cos(\theta - \theta_1 - \theta_2 - \theta_3 - \theta_4 - \theta_5).$$

§ 18. We conclude this chapter with the solution of some important and typical problems.

**Ex.** i. *Find the condition that the four points* $(m_1^2, 2m_1)$, $(m_2^2, 2m_2)$, $(m_3^2, 2m_3)$, $(m_4^2, 2m_4)$ *should lie on a circle.*

*Show that* $(0{\cdot}25, 1)$, $(2{\cdot}25, 3)$, $(1{\cdot}69, -2{\cdot}6)$, $(0{\cdot}49, -1{\cdot}4)$ *lie on a circle, and find its equation.*

The equation of any circle is of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

A point of the type $(m^2, 2m)$ lies on this, provided that $m$ satisfies the equation $m^4 + 4m^2 + 2gm^2 + 4fm + c = 0$,

i.e. $m^4 + 2(g+2)m^2 + 4fm + c = 0$.

This is of the fourth degree, and for any given values of $g$, $f$, and $c$ gives four values of $m$, the corresponding points to which lie on the circle.

On the other hand, four chosen points of the type will lie on a circle provided that we can find values of $g$, $f$, and $c$ such that the four given values of $m$ are roots of an equation of this type.

If the roots of the above equation are $m_1$, $m_2$, $m_3$, $m_4$, then

$$\Sigma m = 0,\ \Sigma m_1 m_2 = 2g + 4,\ \Sigma m_1 m_2 m_3 = -4f,\ m_1 m_2 m_3 m_4 = c.$$

Hence, provided that

$$m_1 + m_2 + m_3 + m_4 = 0,$$

values of $g, f$, and $c$ can be found from the remaining three conditions. This, then, is the necessary and sufficient condition that the given points should lie on a circle.

In the numerical example

$$m_1 = 0{\cdot}5\ ;\ m_2 = 1{\cdot}5\ ;\ m_3 = -1{\cdot}3\ ;\ m_4 = -0{\cdot}7.$$

$$\therefore \quad m_1+m_2+m_3+m_4 = 0{\cdot}5+1{\cdot}5-1{\cdot}3-0{\cdot}7 = 0,$$

i.e. the points lie on a circle.

Now

$$2g+4 = \Sigma m_1 m_2 = (m_1+m_2)(m_3+m_4)+m_1m_2+m_3m_4$$
$$= -4+0{\cdot}75+0{\cdot}91 = -2{\cdot}34.$$

$$\therefore \quad 2g = -6{\cdot}34.$$

Again,

$$4f = \Sigma m_1 m_2 m_3 = (m_1+m_2)\,m_3m_4+(m_3+m_4)\,m_1m_2$$
$$= 2\times 0{\cdot}91-2\times 0{\cdot}75 = 0{\cdot}32.$$

$$\therefore \quad 2f = 0{\cdot}16,$$

$$c = m_1m_2m_3m_4 = 0{\cdot}6825.$$

The equation of the circle is then

$$x^2+y^2-6{\cdot}34\,x+0{\cdot}16\,y+0{\cdot}6825 = 0.$$

**Ex. ii.** *Show that all circles of the family*

$$x^2+y^2+2\lambda x+2\mu y+\nu = 0,$$

*where* $A\lambda+B\mu+C\nu+D = 0$, *have a common orthogonal circle, and find its equation.*

Suppose that the circle

$$x^2+y^2+2gx+2fy+c = 0 \qquad \text{(i)}$$

cuts the circle $x^2+y^2+2\lambda x+2\mu y+\nu = 0$ at right angles; then

$$2g\lambda+2\mu f-c-\nu = 0. \qquad \text{(ii)}$$

We are given that $A\lambda+B\mu+C\nu+D = 0$. (iii)

Eliminating $\nu$ from equations (ii) and (iii), we obtain

$$\lambda(A+2Cg)+\mu(B+2Cf)+D-Cc = 0.$$

In this equation $\lambda$ and $\mu$ are independent; hence, if the circle (i) cuts *every* circle of the family at right angles, this equation must be true for all values of $\lambda$ and $\mu$: so that

$$A+2Cg = 0, \quad B+2Cf = 0, \quad D-Cc = 0.$$

Hence

$$2g = -\frac{A}{C}, \quad 2f = -\frac{B}{C}, \quad c = \frac{D}{C},$$

and the circle (i) becomes

$$x^2+y^2-\frac{Ax}{C}-\frac{B}{C}y+\frac{D}{C}=0,$$

or
$$C(x^2+y^2)-Ax-By+D=0.$$

This circle cuts all the circles of the family at right angles.

**Ex. iii.** *A triangle circumscribes the circle*

$$x^2+y^2=r^2,$$

*and two of its vertices lie on the circle*

$$(x-d)^2+y^2=R^2;$$

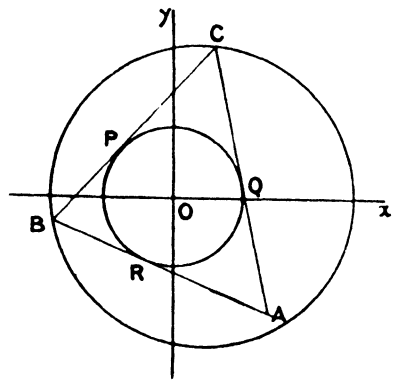
*show that, if* $d^2=R^2+2Rr$, *the third vertex also lies on this circle.*

Let $P$, $Q$, $R$ be the points of contact of the sides with the circle

$$x^2+y^2=r^2, \qquad \text{(i)}$$

and $B$, $C$ be the two vertices on

$$(x-d)^2+y^2=R^2. \qquad \text{(ii)}$$




Let $P$, $Q$, $R$ be the points $\alpha$, $\beta$, $\gamma$; then, since $B$ is the point of intersection of tangents at $P$ and $R$, its coordinates are

$$\left\{\frac{r\cos\frac{1}{2}(\alpha+\gamma)}{\cos\frac{1}{2}(\alpha-\gamma)}, \quad \frac{r\sin\frac{1}{2}(\alpha+\gamma)}{\cos\frac{1}{2}(\alpha-\gamma)}\right\};$$

and since this lies on the circle (ii),

$$\{r\cos\tfrac{1}{2}(\alpha+\gamma)-d\cos\tfrac{1}{2}(\alpha-\gamma)\}^2+r^2\sin^2\tfrac{1}{2}(\alpha+\gamma)=R^2\cos^2\tfrac{1}{2}(\alpha-\gamma),$$

i.e. $r^2-2dr\cos\frac{1}{2}(\alpha+\gamma)\cos\frac{1}{2}(\alpha-\gamma)+(d^2-R^2)\cos^2\frac{1}{2}(\alpha-\gamma)=0,$

i.e. $r^2-2dr\cos\frac{1}{2}(\alpha+\gamma)\cos\frac{1}{2}(\alpha-\gamma)+2Rr\cos^2\frac{1}{2}(\alpha-\gamma)=0,$

i.e. $r-2d\cos\frac{1}{2}(\alpha+\gamma)\cos\frac{1}{2}(\alpha-\gamma)+2R\cos^2\frac{1}{2}(\alpha-\gamma)=0;$

$$\therefore \quad r-d(\cos\alpha+\cos\gamma)+R(1+\cos\overline{\alpha-\gamma})=0;$$

$$\therefore \quad \cos\alpha(R\cos\gamma-d)+R\sin\gamma\sin\alpha+R+r-d\cos\gamma=0.$$

Similarly, since $C$ lies on the circle (ii),

$$\cos\alpha(R\cos\beta-d)+R\sin\beta\sin\alpha+R+r-d\cos\beta=0.$$

Hence, by cross multiplication

$$\frac{\cos\alpha}{(R^2+Rr)(\sin\gamma-\sin\beta)-Rd\sin(\gamma-\beta)}=\frac{\sin\alpha}{(R^2+Rr-d^2)(\cos\beta-\cos\gamma)}$$
$$=\frac{1}{R^2\sin(\beta-\gamma)-Rd(\sin\beta-\sin\gamma)}.$$

Put $d^2 = R^2 + 2Rr$, and multiply each fraction by $2R\sin\frac{1}{2}(\gamma-\beta)$; then

$$\frac{\cos\alpha}{(R+r)\cos\frac{1}{2}(\gamma+\beta)-d\cos\frac{1}{2}(\gamma-\beta)} = \frac{\sin\alpha}{r\sin\frac{1}{2}(\beta+\gamma)}$$
$$= \frac{1}{d\cos(\frac{1}{2}\beta+\gamma)-R\cos\frac{1}{2}(\beta-\gamma)}.$$

Now the coordinates of $A$ are

$$x = \frac{r\cos\frac{1}{2}(\beta+\gamma)}{\cos\frac{1}{2}(\beta-\gamma)}, \quad y = \frac{r\sin\frac{1}{2}(\beta+\gamma)}{\cos\frac{1}{2}(\beta-\gamma)};$$

hence

$$\frac{\cos\alpha}{(R+r)x-dr} = \frac{\sin\alpha}{ry} = \frac{1}{dx-Rr}.$$

Thus $$\{(R+r)x-dr\}^2 + r^2y^2 = (dx-Rr)^2,$$

i.e. $$x^2\{R^2+2Rr+r^2-d^2\} + r^2y^2 - 2dr^2x + (d^2-R^2)r^2 = 0.$$

Substituting for $d^2$ in the coefficient of $x^2$, this becomes, after dividing through by $r^2$, $x^2+y^2-2dx+d^2-R^2 = 0$, or

$$(x-d)^2+y^2 = R^2,$$

i.e. $A$ also lies on the circle (ii).

**Note.** Since we have chosen the point $P$ in any position, any number of triangles can be drawn circumscribed to the one circle and inscribed in the other, if $d^2 = R^2+2Rr$.

**Ex. iv.** *Show that the equations of any two non-intersecting circles can be written in the forms*

$$\{(x-a)^2+y^2\} = \lambda\{(x+a)^2+y^2\},$$
$$\{(x-a)^2+y^2\} = \mu\{(x+a)^2+y^2\},$$

*and find the equation of the polar reciprocal of the first with respect to the second.*

**Note.** The polar reciprocal is the envelope of the polars of points on the first circle with respect to the second.

Since the circles are non-intersecting, their limiting points are real; let the limiting points of the two circles be $(a, 0)$ $(-a, 0)$, then the equation of either circle is of the form

$$x^2+y^2+2gx+a^2 = 0.$$

Put $g = \frac{\lambda+1}{\lambda-1}$; then the equation becomes

$$(\lambda-1)(x^2+y^2+a^2)+2(\lambda+1)ax = 0,$$

i.e. $$(x-a)^2+y^2 = \lambda\{(x+a)^2+y^2\}.$$

This proves the first part of the question.

Let $(x', y')$ be any point on the first circle, then

$$(x'-a)^2+y'^2=\lambda\{(x'+a)^2+y'^2\}, \qquad \text{(i)}$$

and the polar of $(x', y')$ with respect to the second circle is

$$(x-a)(x'-a)+yy'=\mu\{(x+a)(x'+a)+yy'\}. \qquad \text{(ii)}$$

Hence, from (ii)

$$(x-a)(x'-a)-\mu(x+a)(x'+a)=(\mu-1)yy',$$

and from (i) $(x'-a)^2-\lambda(x'+a)^2=(\lambda-1)y'^2$.

Eliminating $y'$ so as to get the equation of the polar with only one variable $x'$,

$$\{(x-a)(x'-a)-\mu(x+a)(x'+a)\}^2(\lambda-1)$$
$$=y^2(\mu-1)^2[(x'-a)^2-\lambda(x'+a)^2].$$

Now let the fraction $\dfrac{x'-a}{x'+a}\equiv P$; then

$$\{P(x-a)-\mu(x+a)\}^2(\lambda-1)=y^2(\mu-1)^2\{P^2-\lambda\},$$

or

$$P^2[(\lambda-1)(x-a)^2-(\mu-1)^2y^2]$$
$$-2\mu(\lambda-1)(x^2-a^2)P+\mu^2(\lambda-1)(x+a)^2+\lambda(\mu-1)^2y^2=0.$$

The envelope of this line for different values of $P$ is (cf. Chap. II, § 12)

$$\mu^2(\lambda-1)^2(x^2-a^2)^2$$
$$=[(\lambda-1)(x-a)^2-(\mu-1)^2y^2][\mu^2(\lambda-1)(x+a)^2+\lambda(\mu-1)^2y^2],$$

which reduces to

$$\lambda(\mu-1)^2y^2=(\lambda-1)\{\lambda(x-a)^2-\mu^2(x+a)^2\}.$$

**Ex. v.** *Show that the locus of the point from which the pairs of tangents to the two circles*

$$C_1\equiv x^2+y^2+2g_1x+2f_1y+c_1=0,$$
$$C_2\equiv x^2+y^2+2g_2x+2f_2y+c_2=0$$

*are harmonically conjugate to one another is*

$$(g_1^2+f_1^2-c_1)C_2+(g_2^2+f_2^2-c_2)C_1$$
$$=\{x(f_1-f_2)-y(g_1-g_2)+f_1g_2-f_2g_1\}^2.$$

If the pencil formed by the tangents from a point $(x', y')$ is harmonic, so will a pencil of lines through the origin parallel to the tangents be harmonic.

We shall use the usual notation,

$$C_1'\equiv x'^2+y'^2+2g_1x'+2f_1y'+c_1,$$
$$C_2'\equiv x'^2+y'^2+2g_2x'+2f_2y'+c_2.$$

The equation of the pair of tangents from the point $(x', y')$ to the circle $C_1 = 0$ is

$$C_1 C_1' = \{x(x'+g_1)+y(y'+f_1)+g_1x'+f_1y'+c\}^2,$$

and lines parallel to them through the origin (retaining only terms of the second degree in $x$ and $y$) are

$$(x^2+y^2)\,C_1' = \{x(x'+g_1)+y(y'+f_1)\}^2,$$

i.e. $x^2\{C_1'-(x'+g_1)^2\}-2(x'+g_1)(y'+f_1)\,xy+y^2\{C_1'-(y'+f_1)^2\} = 0.$

So the tangents from $(x'. y')$ to $C_2 = 0$ are parallel to

$$x^2\{C_2'-(x'+g_2)^2\}-2(x'+g_2)(y'+f_2)\,xy+y^2\{C_2'-(y'+f_2)^2\} = 0.$$

These pairs of lines are harmonic conjugates if

$$\{C_1'-(x'+g_1)^2\}\{C_2'-(y'+f_2)^2\}+\{C_2'-(x'+g_2)^2\}\{C_1'-(y'+f_1)^2\} = 2(x'+g_1)(x'+g_2)(y'+f_1)(y'+f_2)$$

(see Chap. III, § 5).

If we omit the accents, the resulting equation therefore gives the locus of $x'\,y'$.

This equation is

$$2C_1C_2-C_1\{(x+g_2)^2+(y+f_2)^2\}-C_2\{(x+g_1)^2+(y+f_1)^2\}+\{(x+g_1)(y+f_2)-(x+g_2)(y+f_1)\}^2 = 0,$$

i.e. $$2C_1C_2-C_1\{C_2+g_2^2+f_2^2-c_2\}-C_2\{C_1+g_1^2+f_1^2-c_1\}+\{x(f_2-f_1)-y(g_2-g_1)+f_2g_1-f_1g_2\}^2 = 0,$$

or $$C_1\{g_2^2+f_2^2-c_2\}+C_2\{g_1^2+f_1^2-c_1\} = \{x(f_1-f_2)-y(g_1-g_2)+f_1g_2-f_2g_1\}^2.$$

**Definition.** If a radius vector $OP$ is drawn from a fixed point $O$ to meet a given curve at $P$, then the locus of a point $P'$ on $OP$ such that

$$OP.\,OP' = \text{constant}$$

is called the inverse of the given curve with respect to the point $O$.

(a) *To find the inverse of a straight line with regard to any point.*

Let the origin of coordinates $(O)$ be the centre of inversion, and let the equation of the straight line be

$$Ax+By+C = 0.$$

The equation of any straight line through the origin is

$$\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = r.$$

The distance from $O$ of the point $P$, in which this line intersects the given line, is given by the equation

$$r(A\cos\theta+B\sin\theta)+C = 0.$$

Suppose that $P'$ is the point on the inverse corresponding to $P$ and let $OP' = r'$, then substituting for $r$ it follows that

$$k^2(A\cos\theta + B\sin\theta) + r'C = 0.$$

Now, since the point $P'$ is on the straight line $OP$, its coordinates are $r'\cos\theta$, $r'\sin\theta$; hence the equation of the inverse is

$$k^2(Ax + By) + C(x^2 + y^2) = 0.$$

This equation represents a circle passing through the centre of inversion, whose centre lies on the line $Bx - Ay = 0$ drawn through the centre of inversion perpendicular to the given line.

(b) *To prove analytically that the inverse of a system of coaxal circles with respect to a limiting point is a system of concentric circles.*

Let $x^2 + y^2 + 2gx + d^2 = 0$ be any one of the system of coaxal circles, whose limiting points are $(\pm d, 0)$. Any line through a limiting point is $\dfrac{x \mp d}{\cos\theta} = \dfrac{y}{\sin\theta} = r$.

This line meets the circle at points whose distances from the centre of inversion are given by the quadratic equation

$$(r\cos\theta \pm d)^2 + r^2\sin^2\theta + 2g(r\cos\theta \pm d) + d^2 = 0,$$

i.e.
$$r^2 + 2r\cos\theta(g \pm d) + 2d(d + g) = 0.$$

The equation connecting the distances of corresponding points on the inverse from the centre of inversion is accordingly

$$\frac{k^4}{(g \pm d)} + 2k^2 . r' . \cos\theta \pm 2d . r'^2 = 0.$$

Hence the equation of the inverse curve is

$$\frac{k^4}{(g \pm d)} + 2k^2(x \mp d) + 2d\{(x \mp d)^2 + y^2\} = 0,$$

which represents a circle whose centre is the fixed point

$$\left\{ \pm \frac{2d^2 - k^2}{2d},\ 0 \right\}.$$

## Miscellaneous Harder Examples on the Circle.

### (For Revision.)

1. If the chord of contact of tangents from $(h, k)$ to the circle $x^2 + y^2 = r^2$ subtends a right angle at $(h', k')$, prove that

$$(h'^2 + k'^2 - r^2)(h^2 + k^2) = 2r^2(hh' + kk' - r^2).$$

2. Show that the equation of the circle circumscribing the triangle formed by the lines $y = am_1 + x/m_1$, $y = am_2 + x/m_2$, $y = am_3 + x/m_3$ is

$$x^2 + y^2 - (1 + \Sigma m_1 m_2)ax - (m_1 m_2 m_3 + \Sigma m)ay + a^2 \Sigma m_1 m_2 = 0,$$

and find the points where it cuts the $x$-axis.

3. A variable circle is drawn through the origin and is such that the tangent at the origin makes a constant angle with the line joining the points of intersection of the circle with the coordinates axes. Show that the circle belongs to one or another of two coaxal systems.

4. Points $P$, $Q$ are taken, one on each of the two circles

$$x^2+y^2 \pm 2a(x+c) = 0,$$

so that they subtend a right angle at the origin. Show that the locus of the point of intersection of the tangents at $P$ and $Q$ is the circle

$$c(x^2+y^2)+a^2(x+c) = 0.$$

5. Prove that the coordinates of the centre of the circle which passes through the points $(a\cos\alpha,\ b\sin\alpha)$ $(a\cos\beta,\ b\sin\beta)$ $(a\cos\gamma,\ b\sin\gamma)$ are given by

$$ax = (a^2-b^2)\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha)\cos\tfrac{1}{2}(\alpha+\beta),$$
$$by = -(a^2-b^2)\sin\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\gamma+\alpha)\sin\tfrac{1}{2}(\alpha+\beta).$$

6. Through the origin pass two circles which cut the rectangular axes of coordinates in the points $(a, 0)$, $(0, a)$ and $(-a, 0)$, $(0, a)$ respectively. Prove that, if straight lines be drawn through the origin to cut both circles, the locus of the intersection of the tangents to the two circles at the corresponding points where they are cut by these lines is

$$(x^2+y^2)^2-a^2x^2-2ay(x^2+y^2-a^2)-a^4 = 0.$$

7. Show that the locus of a point, such that tangents from it to two equal circles are at right angles, consists of two curves of the fourth degree placed symmetrically with respect to the line of centres.

8. If $lS_1+mS_2+nS_3 = 0$ is the equation of a circle orthogonal to $S_1 = 0$, $S_2 = 0$, then the tangents from its centre to $S_3$, $S_1$ are in the ratio

$$\sqrt{-\left\{1+\frac{2(l+m)}{n}\right\}} : 1.$$

9. If the axes of $x$ and $y$ are conjugate lines with respect to a circle, show that the general equation of such a circle is

$$x^2+y^2+\cos\omega\,[2xy-2x\eta-2y\xi+\xi\eta] = 0.$$

10. $ABC$ is a triangle, $AB$, $AC$ are axes of $x$ and $y$. A point $P$ is taken on the circle

$$x^2+y^2+2xy\cos A-2(c+b\cos A)x-2(b+c\cos A)y+4bc\cos A = 0,$$

and $PL$, $PM$ are drawn parallel to the axes to meet $AB$, $AC$ in $L$, $M$. On $AB$, $AC$ points $L'$, $M'$ are taken such that $BL' = BL$, $CM' = CM$. Prove that $LM'$, $L'M$ are perpendicular.

11. Find the coordinates of the centre and the length of the radius of the circle which is the inverse of $(x-a)^2+(y-b)^2 = c^2$ with respect to the origin, $k^2$ being the constant of inversion.

12. Show that the envelope of chords of the circle

$$x^2+y^2+2gx+2fy+c = 0$$

which subtend a right angle at the origin is

$$(2c-g^2-f^2)(x^2+y^2)+(gx+fy+c)^2 = 0.$$

What does this become when the origin lies on the circle?

13. Show that the system of circles $C-kC'=0$ can be obtained by inverting a system of concentric circles, if $C$ and $C'$ do not intersect in real points.

14. The axes $AB$, $AC$ are inclined at an angle of 60°. A circle touches $AB$ at $P$ and intercepts on $AC$ a chord whose length is equal to $AP$. Show that the locus of the centre is a straight line and give the equation of the circle when $AP=c$.

15. If at any point $P$ of a circle chords $PQ$, $PR$ are drawn making given angles with the tangent at $P$, show that the locus of the intersection of $QR$ with the radius at $P$ is a circle.

16. If each of three circles touches two parallel straight lines, prove that their three angles of intersection $2\alpha$, $2\beta$, $2\gamma$ are connected by the equation $\sin\alpha \pm \sin\beta \pm \sin\gamma = 0$.

17. Two fixed circles intersect in $A$, $B$; $P$ is a variable point on one of them, $PA$ meets the other circle in $X$, and $PB$ meets it in $Y$. Prove that $BX$ and $AY$ intersect on a fixed circle.

Discuss the case when the given circles are orthogonal.

18. Two circles touch one another at $O$; on their common diameter fixed points $A$ and $B$ are taken and a variable straight line through $O$ cuts the circles at $P$ and $Q$. Prove that the locus of the intersection of $AP$ and $BQ$ is a circle which becomes a straight line if $OA/OB = r_1/r_2$.

19. Find the equation of a circle through the origin cutting the axis of $x$ at right angles and the circle $x^2+y^2=a^2$ at an angle of 45°.

20. If a circle cuts two given circles orthogonally, show that the locus of its centre is a straight line.

21. The three segments of the radical axis of the circles $x^2+y^2=2c^2$ $x^2+y^2-16cx+14c^2=0$ made by the circles and their common tangents are equal.

22. If $P$ is the point $(p, q)$ and $Q$, $R$ are the feet of the perpendiculars from $P$ to the straight lines $ax^2+2hxy+by^2=0$, find the equation of the circle $PQR$ and show that the length of $QR$ is

$$2\left\{\frac{(h^2-ab)(p^2+q^2)}{(a-b)^2+4h^2}\right\}^{\frac{1}{2}}.$$

23. Show that the locus of a point, such that the square of the tangents drawn from it to three given circles are in A. P., is a straight line which forms a harmonic pencil with the radical axes of the circles taken in pairs.

24. If $A$, $B$, $C$ are the respective centres of three circles

$$S \equiv x^2+y^2+2gx+2fy+c=0,$$
$$S' \equiv x^2+y^2+2g'x+2f'y+c'=0,$$
$$S'' \equiv x^2+y^2+2g''x+2f''y+c''=0,$$

and $O$ the centre and $p$ the radius of the circle which cuts each of them at right angles, show that the equation of the latter may be written

$$OBC.S+OCA.S'+OAB.S''=2p^2.ABC,$$

where $OBC$ denotes the area of the triangle $OBC$, &c.

Show how to find the coordinates of $O$ and the value of $p$.

25. Given a circle and a point $P$ in its plane, show that there is in the same plane a straight line $L$ such that the square of the distance of any point on the circle from $P$ is equal to the distance of the same point from $L$ multiplied by a constant length, and find the position of $L$.

26. Find the equation of the tangent at any point of the circle

$$x^2+y^2+2gx+2fy+c=0.$$

Prove that the polar with respect to the circle $x^2+y^2=c^2$ of any point on the circle $(x+a)^2+(y+b)^2=k^2$ always touches the curve

$$(ax+by+c^2)^2=k^2(x^2+y^2).$$

27. If one of the external common tangents to two circles meet their radical axis in $R$ and a perpendicular drawn to it from the internal centre of similitude $S$ in $T$, and if $ST$ produced meet the radical axis in $R'$, then $V$, the middle point of $RR'$, will be such that $VR$ is equal to the tangent from $V$ to either circle.

28. Tangents to a circle from a point $P$ cut a fixed diameter of the circle at $A$ and $B$. If the mid-point of the segment $AB$ is fixed, find an equation for the locus of $P$.

29. Show that four points of the type $(a\cos\theta, b\sin\theta)$, where $a$ and $b$ are constants, lie on any given circle; and if four given points of this type lie on a circle, then $\Sigma\theta=2n\pi$. Interpret when $a=b$.

30. How many points of the type $(\lambda^3, \lambda)$ lie on a given circle? If four do, then $S_1^2S_2-S_2^2+S_4-S_1S_3=1$, where $S_r$ = sum of the $\lambda$'s taken $r$ at a time.

31. A point $C$ is taken in the diameter $AB$ of a circle: on $AC$, $CB$ as diameters circles are described; $PQ$ is a common tangent to these latter circles; show that $AP$, $BQ$ and the common tangent at $C$ meet on the first circle.

32. A point $P$ moves in the plane of a triangle $ABC$ so that

$$PA^2.BC^2=PB^2.CA^2+PC^2.AB^2;$$

prove that the locus of $P$ is the circle which passes through $B$, $C$ and cuts the circle $ABC$ orthogonally.

33. Show that the circle through the three points $(a\lambda, a\lambda^{-1})$, $(a\mu, a\mu^{-1})$, $(a\nu, a\nu^{-1})$ passes through the point $(a/\lambda\mu\nu, a\lambda\mu\nu)$.

34. Prove that the difference of the squares of the tangents drawn from any point to two circles is proportional to the distance of the point from the radical axis of the circles.

Two circles are such that the sums of the squares of the tangents drawn to them from the vertices of a triangle are the same for each circle; prove that the radical axis of the circles passes through the centroid of the triangle.

35. Show that a homogeneous equation in $x$ and $y$ of degree $n$ represents $n$ straight lines through the origin, and find the equation giving the six straight lines joining to the origin the points where the curve

$$x^3+3xy^2+5y^3+6x=7$$

cuts the circle $x^2+y^2=1$.

36. Given three non-intersecting coaxal circles, prove that the lengths of

the tangents drawn from any point of one of them to the other two are in a fixed ratio.

37. Prove that the locus of the middle points of chords of a fixed circle which subtend a right angle at a fixed point is a circle, and that the fixed point is a limiting point of the two circles.

38. Find the locus of a point the tangent from which to a fixed circle bears a constant ratio to its distance from a fixed point.

If $S_{12}$ denotes the circle coaxal with the circles $S_1$, $S_2$ and having its centre at their external centre of similitude, and $S_{31}$, $S_{23}$ are the circles obtained in the same way from $S_3$, $S_1$ and $S_2$, $S_3$, prove that $S_{12}$, $S_{23}$, $S_{31}$ are coaxal.

39. Form the equation of a system of coaxal circles of which the points $(\pm a, 0)$ are the limiting points.

Prove that, if $\log\{(x+iy+a)/(x+iy-a)\} = u+iv$, the curves $u =$ constant and $v =$ constant are two families of coaxal circles.

40. $A$, $O$, $B$ are three collinear points; circles are described with centres $A$ and $B$ to cut orthogonally a circle of variable radius $r$, whose centre is $O$. Prove that the product of the perpendiculars from $O$ on all common tangents to the first-named circles is in a constant ratio to $r^4$.

41. Show that a circle can be drawn to touch the four circles

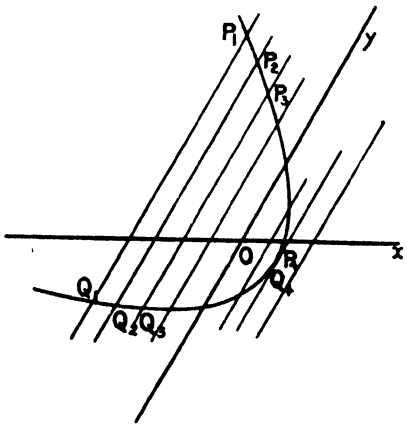
$$x^2+y^2-2ax=0,\ x^2+y^2-2bx=0,$$
$$x^2+y^2-2cy=0,\ x^2+y^2-2dy=0,$$

if $(1/a-1/b)^2=(1/c-1/d)^2$.

# CHAPTER VI

## THE LOCUS REPRESENTED BY THE GENERAL EQUATION OF THE SECOND DEGREE, $S \equiv ax^2+2hxy+by^2+2gx+2fy+c$.

§ 1. **Preliminary.** In the following discussion it is assumed that neither $a$ nor $b$ is zero: this involves no loss of generality, for if either or both of these coefficients is zero, by a simple change of axes the equation can be transformed into one in which $a$ and $b$ are not zero. Now, if any value of $x$ is substituted in the equation $S=0$, we obtain a quadratic equation giving two values of $y$; these may be real and distinct, equal, or imaginary. Thus, to every value of $x$ there correspond two real, two coincident, or two imaginary points on the locus. Similarly, to every value of $y$ there correspond two points on the locus. As the value of $x$ is increased continuously from $-\infty$ to $+\infty$, the two corresponding points move across the plane of the coordinate axes and trace out the complete locus; as, for example, $P_1Q_1$, $P_2Q_2$, $P_3Q_3$ ... in the figure.




We propose to investigate by elementary algebra the values of $y$ as $x$ increases from $-\infty$ to $+\infty$, and the values of $x$ as $y$ increases from $-\infty$ to $+\infty$, and to classify the loci by their principal graphical properties so found.

The equation $S=0$ can be written

$$ax^2+2(hy+g)x+by^2+2fy+c=0,$$

or

$$by^2+2(hx+f)y+ax^2+2gx+c=0.$$

Hence
$$\begin{aligned} ax &= -(hy+g) \pm \sqrt{\{(hy+g)^2-a(by^2+2fy+c)\}} \\ &= -(hy+g) \pm \sqrt{\{-(ab-h^2)y^2+2y(hg-af)-(ac-g^2)\}} \\ &= -(hy+g) \pm \sqrt{\{-Cy^2+2Fy-B\}}, \end{aligned}$$

which for convenience of reference we will write

$$\equiv -(hy+g) \pm \sqrt{D}.$$

Similarly, $by = -(hx+f) \pm \sqrt{\{-Cx^2+2Gx-A\}}$
$= -(hx+f) \pm \sqrt{E}.$

(For the notation refer back to Chap. III, § 10.)

The values of $x$ and $y$ will be real only when $D$ and $E$ respectively are positive.

Now (*vide* Hall and Knight's *Higher Algebra*, § 120) $D$ has the same sign as $-C$ for all real values of $y$ except when the equation

$$Cy^2-2Fy+B=0$$

has real distinct roots (say $\gamma$ and $\delta$), in which case $D$ has the opposite sign to $-C$ when $y$ lies between $\gamma$ and $\delta$, but otherwise has the same sign as $-C$.

The condition that the equation

$$Cy^2-2Fy+B=0 \qquad \text{(i)}$$

should have real and distinct roots is that $F^2-BC$ should be positive, i. e. that $\Delta . a$ should be negative.

Thus $D$ has the same sign as $-C$ except when $\Delta . a$ is negative and $y$ has a value between $\gamma$ and $\delta$.

In the same way $E$ has the same sign as $-C$ except when $\Delta . b$ is negative and $x$ has a value between the roots of the equation

$$Cx^2-2Gx+A=0,$$

which we will call $\alpha$ and $\beta$.

I. **C positive**, i. e. $ab > h^2$.

Since $h^2$ is essentially positive, $ab$ is positive, and consequently $a$ and $b$ have the same sign.

Hence $\Delta . a$ and $\Delta . b$ have also the same sign.

(a) **$\Delta . a$ and $\Delta . b$ positive.**

Now $D$ and $E$ have the same sign as $-C$, i. e. are both negative. In this case the locus is wholly imaginary; e. g.

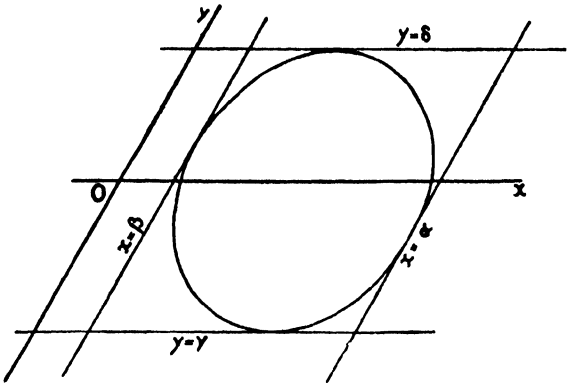
$$x^2+y^2-2x-4y+6=0.$$

(b) **$\Delta . a$ and $\Delta . b$ negative.**

Now $D$ and $E$ are negative, except when $y$ lies between $\gamma$ and $\delta$, and $x$ lies between $\alpha$ and $\beta$ respectively. Hence $x$ is only real when $y > \gamma$ and $< \delta$, and $y$ is only real when $x > \alpha$ and $< \beta$. Thus the real part of the locus lies inside the parallelogram whose sides are $x=\alpha$, $x=\beta$, $y=\gamma$, $y=\delta$.

When $y$ has either of the values $\gamma$ or $\delta$, then $D$ is zero and the corresponding values of $x$ are equal: the paths of the two points which correspond to these two values of $y$ meet each other (*vide* $P_4Q_4$,

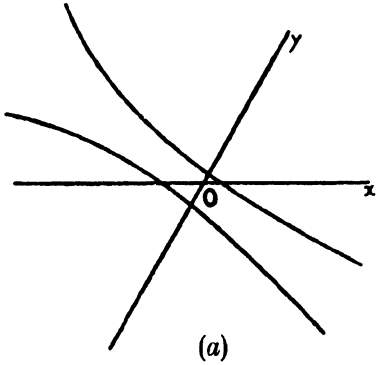
Fig. p. 222), and the lines $y = \gamma$, $y = \delta$ touch the curve. So with the lines $x = \alpha$, $x = \beta$. Hence the locus in this case is a closed curve inscribed in the above parallelogram.




The locus is in this case called an **Ellipse**; the whole curve is at a finite distance from the origin. A particular case is the **circle.**

II. **C negative**, i. e. $ab < h^2$.

(a) **$\Delta . a$ and $\Delta . b$ positive**: then $D$ and $E$ are positive, therefore $x$ and $y$ are real for all values of $y$ and $x$ respectively.




(*a*)

(b) **$\Delta . a$ positive, $\Delta . b$ negative.**

As in (a) $x$ is real for all values of $y$. $E$ has the same sign as $-C$, i. e. positive, except when $x$ lies between $\alpha$ and $\beta$. Hence $y$ is real for all values of $x$ except those lying between $\alpha$ and $\beta$.

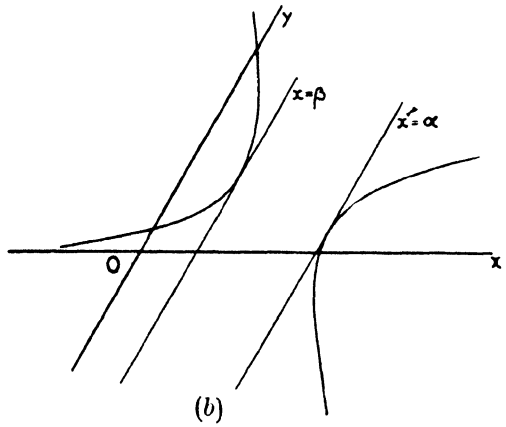
As before, when $x = \alpha$ or $x = \beta$, the values of $y$ become coincident, and the locus touches the lines $x = \alpha$ and $x = \beta$.

The curve thus consists of two branches which extend in opposite directions from $x = \alpha$ and $x = \beta$ respectively to infinity.

**Δ.a negative and Δ.b positive** gives a similar result.

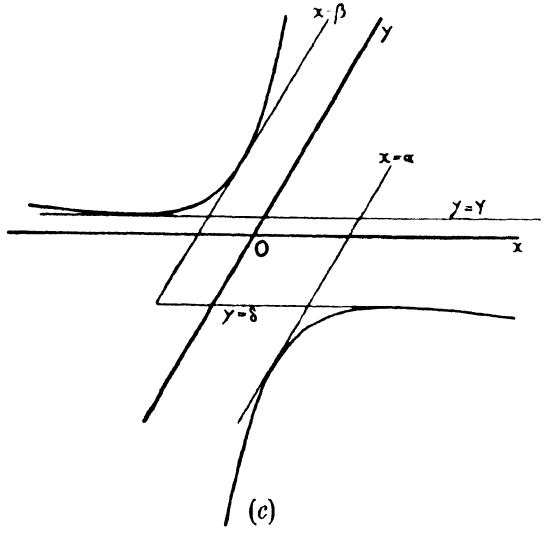
(c) **Δ.a and Δ.b negative.**

In the same way it can be seen that $x$ and $y$ are always real




(*b*)

except when $x > \alpha$ and $< \beta$ and $y > \gamma$ and $< \delta$ respectively. The curve touches the four lines $x = \alpha$, $x = \beta$, $y = \gamma$, $y = \delta$, but no part of it lies within the parallelogram formed by these lines.




(*c*)

In each case the curve consists of two branches extending in opposite directions to infinity: the curve is called an **Hyperbola**, a special case being a pair of straight lines, viz. when $\Delta = 0$.

**Note.** It must be borne in mind throughout that no straight line can meet the curve in more than two points.

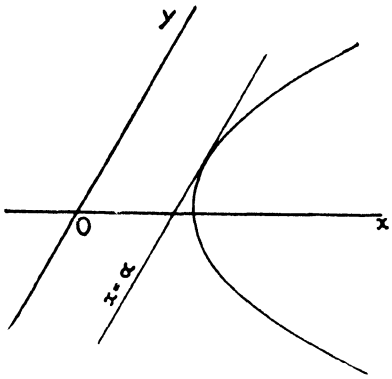
III. $\mathbf{C = 0}$, i. e. $ab = h^2$.

In this case $D$ and $E$ are both linear: all cases give a similar result. As an example $E \equiv 2Gx - A$. If $G$ and $A$ are both positive, $E$ will be positive for all values of $x$ greater than $\frac{A}{2G}$, and negative for all values of $x$ less than $\frac{A}{2G}$.

The values of $y$ are coincident when $x = \frac{A}{2G}$.

Thus the curve consists of a single branch extending on one side only of the line $x = \frac{A}{2G}$ to infinity.

The locus in this case is called a **Parabola,** a special case when $\Delta$ is zero being a pair of parallel straight lines.




**Summary.**

| If | $S = 0$ is |
|---|---|
| $C$ positive, $\Delta \neq 0$, | an Ellipse which is a finite closed curve. |
| $C$ positive, $\Delta = 0$, | a pair of imaginary straight lines. |
| $C$ negative, $\Delta \neq 0$, | an Hyperbola which has two branches extending to infinity. |
| $C$ negative, $\Delta = 0$, | a pair of straight lines. |
| $C$ zero, $\Delta \neq 0$, | a Parabola which has one branch extending to infinity. |
| $C$ zero, $\Delta = 0$, | a pair of parallel straight lines. |

### Examples VI a.

1. Classify the following curves :—

(i) $9x^2+3xy+\frac{1}{4}y^2+2x+5y+6=0$;
(ii) $2x^2+xy-21y^2+6x-5y+4=0$;
(iii) $x^2+xy+y^2-6x-3=0$;
(iv) $x^2-4xy+4y^2+3x-6y+2=0$;
(v) $2x^2+xy-21y^2+6x-5y+15=0$;
(vi) $5x^2-2xy+2y^2+2x-4y+2=0$.

2. Show that the curve $4x^2+12xy+10y^2+8x+8y+7=0$ lies between the lines $y=1$ and $y=3$.

Where do these lines touch the curve?

Find the equation of two lines parallel to the $y$-axis which touch the curve.

3. Show that the line $x=c$ cuts the curve $x^2+4xy+3y^2-1=0$ in real points for all values of $c$.

4. Prove that the curve $4(x-2)(y-3)=(x+y+6)^2$ lies altogether on the same side of the lines $x=2$, $y=3$ as the origin.

5. Show that the curve $4x^2-8xy+y^2+12y=0$ touches the sides of the parallelogram formed by the lines $x=1$, $x=3$, $y=0$, $y=4$, and that no part of the curve lies within the parallelogram.

Find the points of contact of the sides and the chords of contact.

6. Find the condition that the axis of $y$ should meet the locus $S=0$ in (i) real, (ii) coincident, (iii) imaginary points.

7. Prove that the tangents at the points where the straight line $ax+hy+g=0$ meets the curve $S=0$ are parallel to the $x$-axis.

8. Find the equation of the chord of contact of tangents to $S=0$ which are parallel to the $y$-axis.

9. Find an equation giving the ordinates of the points where the line $x-y=0$ cuts $S=0$.

If these are real, prove that $2H>A+B$.

10. Prove that the line $x+y=0$ touches the curve $S=0$, provided that $A+B+2H=0$.

§ 2. *To find the locus of the middle points of chords of the curve*

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

*which are parallel to the straight line* $\frac{x}{l}=\frac{y}{m}$.

Suppose that the point $M$, whose coordinates are $(x_1, y_1)$, is the middle point of a chord $PQ$; if the chord is parallel to $\frac{x}{l}=\frac{y}{m}$, its equation is

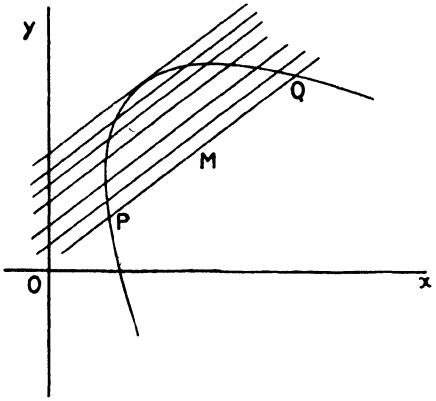
$$\frac{x-x_1}{l}=\frac{y-y_1}{m}.$$

The coordinates of any point on this line are $(x_1+lt, y_1+mt)$; if this point lies on the curve $S=0$, we have

$$a(x_1+lt)^2+2h(x_1+lt)(y_1+mt)+b(y_1+mt)^2 + 2g(x_1+lt)+2f(y_1+mt)+c=0,$$

or
$$t^2(al^2+2hlm+bm^2)+2t(lX_1+mY_1)+S_1=0, \qquad \text{(i)}$$

where $S_1 = ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$. [Chap. III, § 10.]




This equation gives the two values of $t$ which correspond to the points $P$ and $Q$ in which the chord cuts the curve. Since the point $(x_1, y_1)$ is the mid-point of $PQ$, the coordinates of $P$ and $Q$ are of the form $(x_1+lt, y_1+mt)$, $(x_1-lt, y_1-mt)$; in other words, the values of $t$ given by equation (i) are equal and opposite. Hence we have

$$lX_1+mY_1=0.$$

This is the condition that the point $(x_1, y_1)$ should be on the line

$$lX+mY=0.$$

Thus, the middle points of all chords of the curve $S=0$, which are parallel to $mx-ly=0$, lie on a straight line whose equation is $lX+mY=0$. Such a straight line is called a **diameter.**

Now if the coordinates $x_1$, $y_1$ satisfy both the equations

$$X=ax+hy+g=0; \quad Y=hx+by+f=0, \qquad \text{(ii)}$$

the coefficient of $t$ in equation (i) vanishes for all values of $l$ and $m$, and the equation gives equal and opposite values of $t$.

If $ab-h^2$ is not zero, there is a finite point whose coordinates satisfy the equations $X=0$, $Y=0$. In this case any straight line through this point cuts the curve in two points which are equidistant from this point and on opposite sides of it. This point is called the **centre** of the curve.

If $ab-h^2=0$, there is no finite point whose coordinates satisfy the equations $X=0$, $Y=0$; in this case the curve has no centre.

The loci, represented by $S=0$, can be divided into two classes according as $ab-h^2$ is or is not zero. They are called **central** and **non-central** curves. The reader, on referring back to the Figures in § 1, will see that this is in accordance with the general shapes of these loci. It is convenient to examine the two classes separately.

I. **Central curves.** $(ab-h^2 \neq 0.)$

Since the centre is given by the equations $X=0$, $Y=0$, and the equation of a diameter is $lX+mY=0$, it is evident that all diameters pass through the centre.

**Conjugate Diameters.**

We have seen that all chords of the curve $S=0$, which are parallel to $mx-ly=0$, are bisected by the diameter $lX+mY=0$; the equation of this diameter written in full is

$$x(al+hm)+y(hl+bm)+gl+fm=0.$$

This is parallel to the straight line $m'x-l'y=0$ if

$$all'+h(lm'+l'm)+bmm'=0.$$

The complete symmetry of this result in the numbers $l$, $m$ and $l'$, $m'$ shows that if the diameter which bisects chords parallel to $mx-ly=0$ is parallel to $m'x-l'y=0$, then the diameter which bisects chords parallel to $m'x-l'y=0$ is parallel to $mx-ly=0$.

Such a pair of diameters is called a pair of conjugate diameters.

The pair of diameters $lX+mY=0$, $l'X+m'Y=0$ are conjugate, therefore, if $all'+h(lm'+l'm)+bmm'=0$.

**Ex.** The lines $a'x^2+2h'xy+b'y^2=0$ are parallel to conjugate diameters of $S=0$, provided that they are parallel to $(mx-ly)(m'x-l'y)=0$, where $all'+h(lm'+l'm)+bmm'=0$; that is, if $ab'+a'b=2hh'$.

**Axes of Symmetry.**

It has been shown that the diameters $lX+mY=0$, $l'X+m'Y=0$ are parallel to $m'x-l'y=0$, $mx-ly=0$, and are conjugate if $all'+h(lm'+l'm)+bmm'=0$. If this condition is satisfied, these diameters are therefore perpendicular when the lines $m'x-l'y=0$, $mx-ly=0$ are perpendicular.

*If the coordinate axes are rectangular*, the condition for this is $ll'+mm'=0$.

Thus, if the ratios $l/m$, $l'/m'$ are determined by the equations

$$all'+h(lm'+l'm)+bmm'=0,\quad ll'+mm'=0,$$

the diameters $lX+mY=0$, $l'X+m'Y=0$ are both conjugate and at right angles.

These equations give

$$h(l^2-m^2) = (a-b)\,lm,$$

from which we obtain two directions for the line $mx-ly=0$; these directions are at right angles, and therefore correspond to the directions of the two diameters $lX+mY=0$, $l'X+m'Y=0$, which are both perpendicular and conjugate. The equation of these diameters is therefore

$$(lX+mY)(l'X+m'Y)=0,$$

or

$$h(X^2-Y^2) = (a-b)\,XY. \qquad \text{(iii)}$$

This, therefore, is the equation of the two diameters which are both conjugate and perpendicular; the diameters are always real, for the condition that $h(X^2-Y^2)-(a-b)\,XY$ should have real factors is that $(a-b)^2+4h^2$ should be positive; this is always true. Now, a straight line which bisects all chords perpendicular to itself clearly divides the curve symmetrically. There are then two straight lines, at right angles to each other, which divide the curve symmetrically. These are called the **axes of symmetry,** or simply the **axes** of the curve.

**Note.** Since the axes are parallel to $lx+my=0$, where

$$(l^2-m^2)\,h = (a-b)\,lm,$$

their inclinations ($\theta$) to the $x$-axis are given by $\tan\theta = -\dfrac{l}{m}$; thus

$$h(\cos^2\theta-\sin^2\theta) = (a-b)\sin\theta\cos\theta,$$

or

$$\tan 2\theta = \frac{2h}{a-b}.$$

If $\theta_1$, $\theta_2$ are the values of $\theta$ given by this equation, the equations of the axes are $X\sin\theta_1 = Y\cos\theta_1$ and $X\sin\theta_2 = Y\cos\theta_2$.

**Ex.** If the axes of coordinates are oblique and inclined at an angle $\omega$, show that the equation of the axes is

$$h(X^2-Y^2)-(a-b)\,XY = \cos\omega\,(bX^2-aY^2).$$

II. **Non-central curves.** $(ab-h^2=0.)$

In this case both the lines $X=0$, $Y=0$ are parallel to $ax+hy=0$; hence every diameter $lX+mY=0$ is parallel to this straight line.

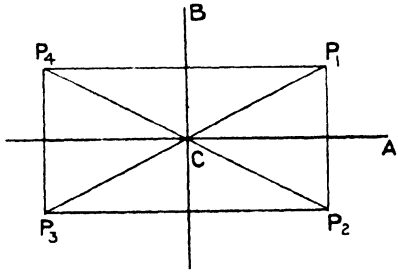
The diameter $aX+hY=0$ is therefore parallel to $ax+hy=0$ and bisects all chords parallel to $bx-hy=0$; hence, the diameter $aX+hY=0$ bisects all chords at right angles to itself. A non-central curve has therefore one axis of symmetry.

**Ex.** If the coordinate axes are oblique, show that the equation of the axis of a non-central locus, $S=0$, is $(a-h\cos\omega)\,X+(h-a\cos\omega)\,Y=0$.

§ 3. **The graph of $S = 0$.**

Since the equations of the axes of the locus are of the form $lX + mY = 0$, the axes pass through the intersection of the lines $X = 0$ and $Y = 0$: this point of intersection is the centre.

Let $CA$, $CB$ be the axes of symmetry, then if $P_1$ is any point on the locus $S = 0$, the points $P_2$, $P_3$, $P_4$ are also by symmetry on the locus.




$P_1$, $P_2$, $P_3$, $P_4$ form a rectangle whose sides are parallel to the axes and are bisected by the axes.

Incidentally, we see that if a point $P_1$ on the locus and the centre of symmetry $C$ are known, a second point on the locus $P_3$ can be constructed, since $CP_1 = CP_3$.

We have seen that the line $lX + mY = 0$ bisects all chords parallel to

$$\frac{x}{l} = \frac{y}{m}.$$

Two important special cases arise when $l$ and $m$ are respectively zero.

Thus the line $X = 0$ bisects all chords parallel to $y = 0$, i.e. parallel to the $x$-axis.

So also $Y = 0$ bisects all chords parallel to the $y$-axis.

This is of practical importance when drawing the graph of a given equation: the following general rules will be useful; the student will discover with practice their relative utility.

To draw the graph of a curve $S = 0$.

(a) Draw the graphs of the lines $X = 0$, $Y = 0$; these intersect at the centre $C$.

(b) If $hX^2 - (a - b)XY - hY^2$ has simple factors such as $(X + \lambda Y)(X + \mu Y)$, next draw the axes $X + \lambda Y = 0$, $X + \mu Y = 0$. A single point on each is sufficient for this purpose, since we know the axes pass through $C$. Even if the above factors are inconvenient, as when $\lambda$ and $\mu$ involve surds, it is often possible to find convenient points on the axes, e.g. the points of intersection of

$$hX^2 - (a - b)XY - hY^2 = 0$$

and the $x$-axis; these points can be joined to $C$.

(c) Try to discover a point $P$ on the curve: it is generally convenient to try the intersections of $S = 0$ with $x = 0$, $y = 0$, $x = y$, or $x = -y$.

Having found one point $P$, other points can be found thus :

(i) Construct a chord $PQ_1$ parallel to the $x$-axis and bisected by $X = 0$.

(ii) Construct a chord $PQ_2$ parallel to the $y$-axis and bisected by $Y = 0$.

(iii) Construct a chord $PCQ_3$ such that $PC = CQ_3$.

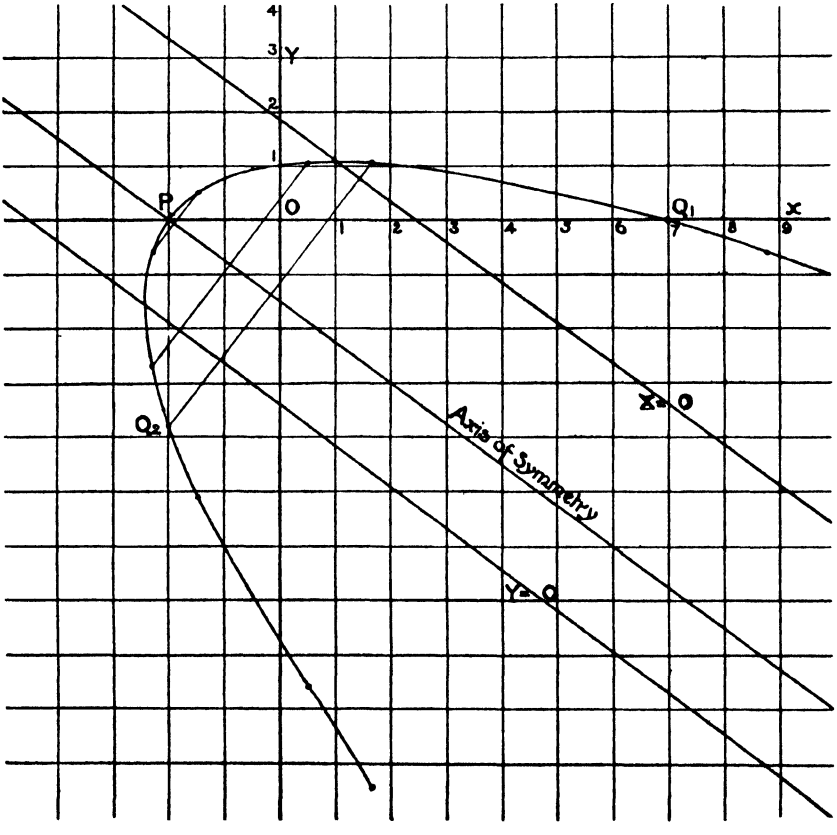
(iv) Construct points which are symmetrical with $P$ with respect to the axes.

The last method involves drawing perpendiculars to the axes, but is useful when (as in the case of the hyperbola) the points $Q_1$, $Q_2$ may be off the paper.

It will be helpful to the student at this stage to draw a few curves and thus to acquire a knowledge of their shapes and to become familiar with the notation we have employed. (See also Chap. VIII, § 9.)

**Illustrative Examples.**

(i) $9x^2 + 24xy + 16y^2 - 44x + 108y - 124 = 0.$

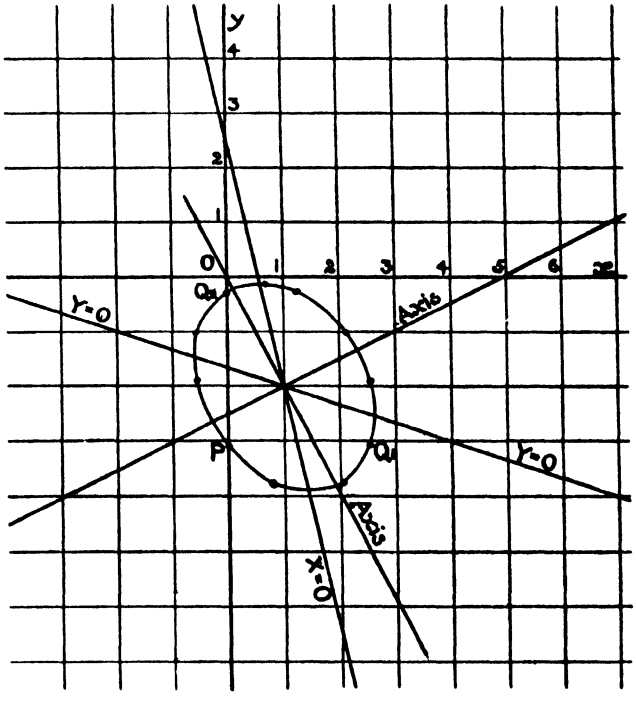


Here $C = 9 \times 16 - 12^2 = 0$; $\therefore$ the curve is a parabola.
$X \equiv 9x + 12y - 22 = 0.$
$Y \equiv 12x + 16y + 54 = 0.$
The axis of symmetry $hX + bY = 0$ is $3x + 4y + 6 = 0.$
$X = 0$, $Y = 0$, and the axis are parallel.
The points $(-2, 0)$, $(2, -3)$ lie on the axis, draw this line.
$X = 0$ passes through $(2{\cdot}45, 0)$ and is parallel to the axis.
$Y = 0$ passes through $(-4{\cdot}5, 0)$ and is parallel to the axis.
The point $P(-2, 0)$ is on the curve.

The other points shown in the figure are found by (i), (ii), or (iv) above.

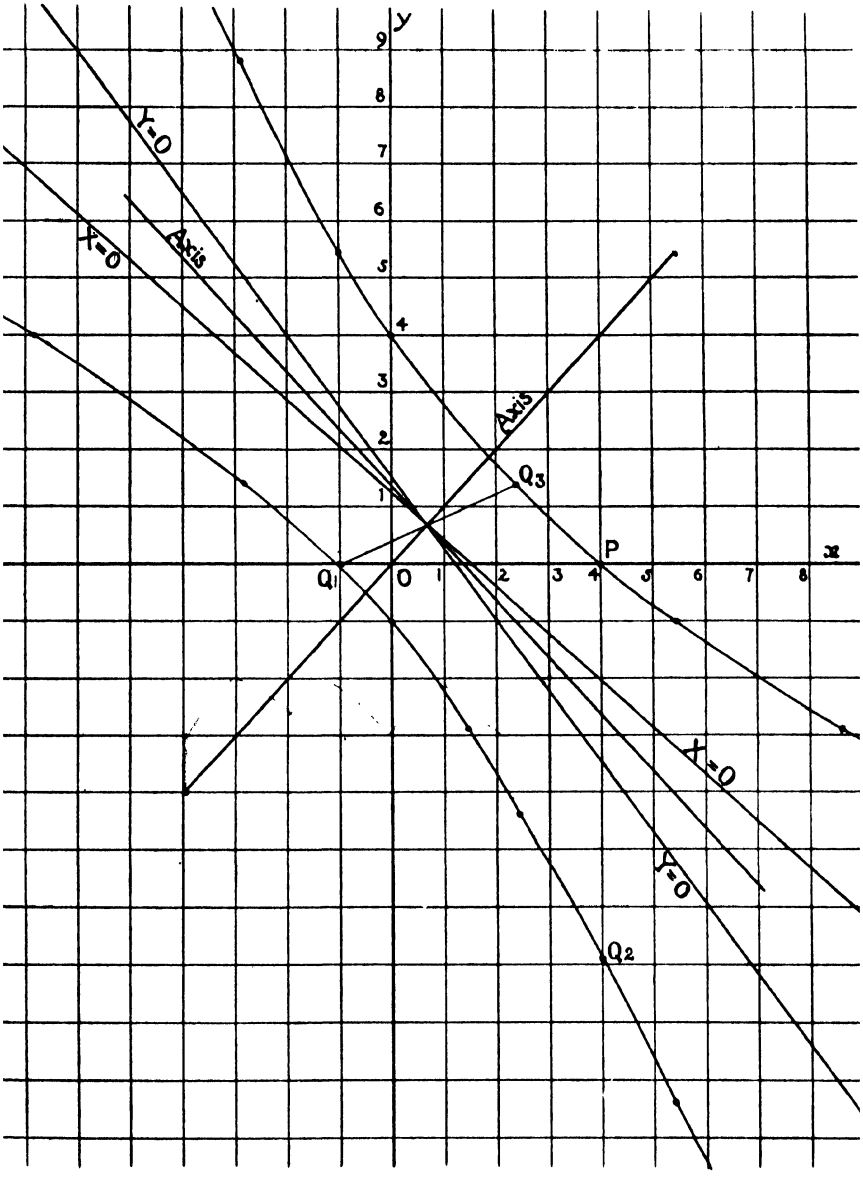
(ii) $$9x^2 + 4xy + 6y^2 - 10x + 20y + 5 = 0.$$

Since $C = 54 - 4 = 50$, i.e. is $+ve$, the curve is an ellipse.
$X \equiv 9x + 2y - 5 = 0.$ Points on this line are $(1, -2)$, $(0, 2{\cdot}5)$.
$Y \equiv 2x + 6y + 10 = 0.$ Points on this line are $(1, -2)$, $(-2, -1)$.
The axes are $2X + Y = 0$, $X - 2Y = 0$, which give
$$2x + y = 0, \quad x - 2y = 5.$$
Point on curve $P(0, -3{\cdot}06)$.
The curve does not cut either $y = 0$ or $x = y$ in real points.



(iii) $2x^2+5xy+2y^2-6x-6y-8=0.$

Since $C=4-\frac{25}{4}$, i.e. is $-ve$, the curve is an hyperbola.
$X\equiv 2x+\frac{5}{2}y-3=0$. Points on this line are (1·5, 0), $(-1, 2)$.
$Y\equiv \frac{5}{2}x+2y-3=0$. Points on this line are (0, 1·5), $(2, -1)$.
The axes are $X-Y=0$, $X+Y=0$, i.e. $x-y=0$, and $x+y=\frac{1}{3}$.
Points on the curve are $(0, 4)$, $(0, -1)$, $(4, 0)$, $(-1, 0)$.



### Examples VI b.

Classify the following curves and draw their graphs: –

1. $x^2+4xy+4y^2-6x-3y+8=0$.
2. $x^2+4xy+y^2-2x+2y+6=0$.
3. $(x-1)(x+2)+(y-2)(y+4)=0$.
4. $3x^2+xy-4y^2+18x+10y+24=0$.
5. $2x^2+5xy+2y^2-6x-6y-8=0$.
6. $x^2+9y^2+6xy-5x-9y+1=0$.
7. $(x+3y)^2-4(3x-y)^2=5$.
8. $3x^2-5xy+3y^2-x-y=0$.
9. $9x^2-24xy+16y^2+21x-28y+6=0$.
10. $2x^2+xy-y^2-x+2y=0$.
11. $x^2-xy+2y^2-2x-6y+4=0$.
12. Draw with the same axes of reference $x^2+xy+y^2=c$, when $c=1, 4, 9, 16$ respectively.

§ 4. In this section we propose to find the equations of certain lines relative to the general curve $S=0$; this will prevent repetition in the following chapters when special forms of the equation are considered.

(I) **To find the equation of the chord of the curve**

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

**whose middle point is $(x_1, y_1)$.**

Let the equation of the chord be $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}$, then the coordinates of any point on this line are $(x_1+lt,\ y_1+mt)$. As in § 2, the values of $t$ corresponding to the points of intersection of the line and the curve are given by

$$(al^2+2hlm+bm^2)t^2+2(lX_1+mY_1)t+S_1=0,$$

and since $(x_1, y_1)$ is the mid-point of the chord,

$$lX_1+mY_1=0.$$

Thus the equation of the chord is

$$(x-x_1)X_1+(y-y_1)Y_1=0,$$

i.e.
$$xX_1+yY_1=x_1X_1+y_1Y_1,$$

or
$$xX_1+yY_1+Z_1=S_1. \qquad (A)$$

(II) **To find the equation of the tangent to $S=0$ at the point $(x', y')$.**

When the two points of intersection $P$, $Q$ of a line and the curve become coincident, the mid-point of the chord evidently coincides with this point: hence we can deduce the equation of the tangent at $(x', y')$ from (A) by writing $x_1=x'$, $y_1=y'$; remembering that

$(x', y')$ is on the curve and therefore $S' = 0$, the equation of the tangent at $(x', y')$ becomes

$$xX' + yY' + Z' = 0. \qquad (B)$$

**Note.** If the axes are rectangular the normal at $(x', y')$ is therefore $(x - x')Y' - (y - y')X' = 0$ or $xY' - yX' = x'Y' - y'X'$.

(III) **To find the equation of the chord of contact of tangents from $(x', y')$ to the curve $S = 0$.**

Let the points of contact of the tangents be $(x_1, y_1)$, $(x_2, y_2)$.

The tangents at these points are

$$xX_1 + yY_1 + Z_1 = 0; \quad xX_2 + yY_2 + Z_2 = 0;$$

and since by hypothesis each of these passes through $(x', y')$ we have

$$x'X_1 + y'Y_1 + Z_1 = 0; \quad x'X_2 + y'Y_2 + Z_2 = 0.$$

But these are the conditions that the points $(x_1, y_1)$ $(x_2, y_2)$ should lie on the line

$$x'X + y'Y + Z = 0,$$

which is therefore the equation of the line joining $(x_1, y_1)$, $(x_2, y_2)$, i.e. of the chord of contact.

This equation is equivalent to

$$x'(ax + hy + g) + y'(hx + by + f) + gx + fy + c = 0,$$

i.e. to $$x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c = 0,$$

i.e. $$xX' + yY' + Z' = 0. \qquad (C)$$

(IV) **To find the equation of the polar of the point $(x', y')$ with respect to $S = 0$.**

Suppose that $PQ$ is any chord passing through $(x', y')$ and that the tangents at $P$ and $Q$ meet at $(x_1, y_1)$; the locus of $(x_1, y_1)$ is the polar of $(x', y')$. Since $PQ$ is the chord of contact of tangents from $(x_1, y_1)$ its equation is (*vide* (C) above)

$$xX_1 + yY_1 + Z_1 = 0.$$

This line therefore passes through $(x', y')$, hence

$$x'X_1 + y'Y_1 + Z_1 = 0,$$

which is algebraically equivalent to

$$x_1X' + y_1Y' + Z' = 0.$$

Now this is the condition that $(x_1, y_1)$ should lie on the line

$$xX' + yY' + Z' = 0, \qquad (D)$$

which is consequently the polar of $(x', y')$.

**Note.** If $(x', y')$ is the centre, then $X' = 0$ and $Y' = 0$, so that the centre has no polar; we shall see later that, as in the case of the circle, the polar of the centre is 'the straight line at infinity'.

(V) **To find the coordinates of the points where the straight line joining the points $(x_1, y_1)$, $(x_2, y_2)$ cuts the curve $S = 0$.**

Let $P$ be the point $(x_1, y_1)$ and $Q$ the point $(x_2, y_2)$, the coordinates of any point on the straight line $PQ$ are

$$\left\{\frac{lx_1+mx_2}{l+m}, \quad \frac{ly_1+my_2}{l+m}\right\}.$$

If this point lies on $S = 0$, we have

$$a(lx_1+mx_2)^2+2h(lx_1+mx_2)(ly_1+my_2)+b(ly_1+my_2)^2 + 2\{g(lx_1+mx_2)+f(ly_1+my_2)\}(l+m)+c(l+m)^2 = 0,$$

which may be written

$$l^2S_1+2lm\{x_2X_1+y_2Y_1+Z_1\}+m^2S_2 = 0. \qquad (E)$$

This equation gives the two values of $l : m$ corresponding to the two points $L$, $M$ in which $PQ$ cuts the curve.

Cor. i. If the points of intersection, $L$ and $M$, are coincident, then $PQ$ is a tangent to the curve. In this case the two values of $l : m$ given by (E) are equal, hence $S_1 . S_2 = \{x_2X_1+y_2Y_1+Z_1\}^2$.

Thus, if $PQ$ is a tangent to the curve from the point $P(x_1, y_1)$, the locus of $Q$ is $S_1S = \{xX_1+yY_1+Z_1\}^2$. (F)

This equation, therefore, is that of the pair of tangents from $(x_1, y_1)$ to the curve.

Cor. ii. If $(x_1, y_1)$ lies on the curve, then $S_1 = 0$; the locus of $Q$ then reduces to $xX_1+yY_1+Z_1 = 0$, which is the tangent to the curve at $(x_1, y_1)$.

Cor. iii. Again, suppose that $Q$ lies on the polar of $P$, so that a straight line through $P$ cuts the curve at $L$, $M$ and the polar of $P$ at $Q$. The polar of $P(x_1, y_1)$ is $xX_1+yY_1+Z_1 = 0$, so that, if $Q(x_2, y_2)$ lies on it, we have

$$x_2X_1+y_2Y_1+Z_1 = 0.$$

The equation (E) now becomes

$$l^2S_1+m^2S_2 = 0,$$

so that the two values of $l : m$ are equal and opposite: hence the points $L$, $M$ are harmonic conjugates of $P$ and $Q$.

Thus any chord of the curve $S = 0$ through $P$ is divided harmonically by $P$ and the polar of $P$.

Cor. iv. In the equation (F) of the tangents from $(x', y')$ to the curve $S = 0$, omit all except the terms of the second degree in $x$ and $y$; we find then that the tangents from $(x_1, y_1)$ to the curve $S = 0$ are parallel to the pair of lines through the origin whose equation is

$$(ax^2+2hxy+by^2)S' = \{xX'+yY'\}^2,$$

which reduces to

$$x^2(Cy'^2-2Fy'+B)-2xy(Cx'y'-Fx'-Gy'+H)+y^2(Cx'^2-2Gx'+A) = 0.$$

These are perpendicular (the axes of coordinates being rectangular) if

$$Cx'^2+Cy'^2-2Gx'-2Fy'+A+B = 0,$$

i.e. if $(x', y')$ lies on the circle

$$Cx^2+Cy^2-2Gx-2Fy+A+B = 0. \qquad (G)$$

Hence the locus of points the tangents from which to the curve $S = 0$ are perpendicular is a circle; this circle is called the **Director Circle**; its centre is the centre of the curve $S = 0$.

**Note.** If $S = 0$ is a parabola, i.e. if $C = 0$, this locus is a straight line; its equation is $2Gx + 2Fy - A - B = 0$. (H)

This straight line is called the **Directrix of the parabola**; its other properties will be discussed later.

**Ex.** Find the equations of the director circle of a central curve and the directrix of a parabola when the coordinate axes are oblique.

The student is advised to read the remainder of this chapter carefully through so as to maintain a logical sequence: as a study it may be postponed until the end of Chap. VIII.

### § 5. To find the lengths of the axes of a central curve.

**Definition.** If the axes of symmetry meet the curve at the points $A$, $A'$, $B$, $B'$ respectively, then $AA'$ and $BB'$ are called the axes of the curve.

Using rectangular coordinates, if the centre of the curve is $(x_1, y_1)$ and either axis makes an angle $\theta$ with the $x$-axis, the equation of this axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}.$$

Hence, if $r$ is the length of the semi-axis, the point

$$(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$$

lies on the curve. This gives us, by substituting in the equation $S = 0$,

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + 2r(X_1\cos\theta + Y_1\sin\theta) + S_1 = 0. \quad \text{(i)}$$

But since $(x_1, y_1)$ is the centre, $X_1 = 0$ and $Y_1 = 0$, so that this equation becomes

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + S_1 = 0.$$

Also, $$S_1 = x_1X_1 + y_1Y_1 + Z_1 = Z_1 = gx_1 + fy_1 + c.$$

Thus
$$ax_1 + hy_1 + g = 0,$$
$$hx_1 + by_1 + f = 0,$$
$$gx_1 + fy_1 + c - S_1 = 0.$$

Eliminating $x_1$ and $y_1$ we have

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c - S_1 \end{vmatrix} = 0,$$

i.e. $S_1 = \dfrac{\Delta}{C}$, when $(x_1, y_1)$ is the centre.

The length of the semi-axis in the direction $\theta$ is therefore given by

$$r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) + \frac{\Delta}{C} = 0.$$

Now, we showed (§ 2, p. 230) that the values of $\theta$ are given by

$$h(\cos^2\theta - \sin^2\theta) = (a-b)\cos\theta\sin\theta,$$

so that
$$\frac{a\cos\theta + h\sin\theta}{\cos\theta} = \frac{h\cos\theta + b\sin\theta}{\sin\theta} = k, \text{ say};$$

therefore
$$a\cos\theta + h\sin\theta = k\cos\theta,$$
$$h\cos\theta + b\sin\theta = k\sin\theta.$$

Hence

$$a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta = k(\cos^2\theta + \sin^2\theta) = k.$$

Thus
$$k = -\frac{\Delta}{Cr^2}.$$

We now have
$$\left(a + \frac{\Delta}{Cr^2}\right)\cos\theta + h\sin\theta = 0$$

and
$$\left(b + \frac{\Delta}{Cr^2}\right)\sin\theta + h\cos\theta = 0;$$

$\therefore$
$$\left(a + \frac{\Delta}{Cr^2}\right)\left(b + \frac{\Delta}{Cr^2}\right) = h^2.$$

This equation gives the squares of the semi-axes of the curve.

**Note** i. If $r_1$ is one of the semi-axes, we have

$$\left(a + \frac{\Delta}{Cr_1^2}\right)\cos\theta + h\sin\theta = 0.$$

But we showed in § 2 that the equation of the axis, whose inclination to the $x$-axis is $\theta$, is $X\sin\theta = Y\cos\theta$.

Hence the equation of the semi-axis of length $r_1$ is

$$\left(a + \frac{\Delta}{Cr_1^2}\right)X + hY = 0,$$

and that of length $r_2$ is similarly

$$\left(a + \frac{\Delta}{Cr_2^2}\right)X + hY = 0.$$

**Note** ii. Since, when $(x_1, y_1)$ is the centre, we have $X_1 = 0$, $Y_1 = 0$ and $S_1 = \frac{\Delta}{C}$, if the coordinate axes are changed to parallel axes through the centre the equation $S = 0$ becomes

$$a(x+x_1)^2 + 2h(x+x_1)(y+y_1) + b(y+y_1)^2 + 2g(x+x_1) + 2f(y+y_1) + c = 0,$$

which at once reduces to

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{C} = 0.$$

**§ 6. The Curve S = 0 at infinity. Asymptotes.**

The coordinates of the points of intersection of the straight line $(x-x')/l = (y-y')/m$ with $S=0$ are $x'+lt,\ y'+mt$ where $t$ has the values given by the equation

$$(al^2+2hlm+bm^2)\,t^2+2\,(lX'+mY')\,t+S'=0.$$

If $al^2+2hlm+bm^2=0$, then in the Euclidean sense the straight line meets the curve in one point only; as explained in Chap. V we can keep our results general by means of the conventional 'points at infinity' and 'straight line at infinity', thus we say every straight line meets the curve $S=0$ in two points. In the present case, then, if $al^2+2hlm+bm^2=0$ (for the present $ab \neq h^2$), i. e. if the straight line is parallel to one of the lines $ax^2+2hxy+by^2=0$, we say that the straight line meets the curve in one finite point and in one 'point at infinity'. Again, if $al^2+2hlm+bm^2=0$ and $lX'+mY'=0$, then in the Euclidean sense the straight line does not meet the curve at all; in this case we say that the straight line meets the curve in two coincident 'points at infinity', or the straight line touches the curve at infinity.

If therefore the point $(x',\ y')$ lies on a line which touches $S=0$ at infinity, its coordinates satisfy the equation $lX+mY=0$, where $al^2+2hlm+bm^2=0$. The equation of the locus of $(x',\ y')$ is therefore

$$bX^2-2hXY+aY^2=0;$$

this equation represents two straight lines passing through the centre of $S=0$; these lines are called the **Asymptotes** of the curve, and may be regarded as the tangents to the curve from the centre.

The equation of the asymptotes, written in full, is

$$(ab-h^2)\,(ax^2+2hxy+by^2+2gx+2fy)+bg^2-2fgh+af^2=0,$$

or $CS-\Delta=0$, which only differs from the equation of the curve in the term independent of $x$ and $y$.

The asymptotes are therefore straight lines through the centre parallel to the lines $ax^2+2hxy+by^2=0$; if $(x_0,\ y_0)$ is the centre, their equation can also be written

$$a\,(x-x_0)^2+2h\,(x-x_0)\,(y-y_0)+b\,(y-y_0)^2=0,$$

a form which is sometimes useful.

The asymptotes are real or imaginary according as $ab-h^2$ is negative or positive; thus an ellipse has imaginary asymptotes, and an hyperbola has real asymptotes. When the asymptotes are real and at right angles, the curve is called a **Rectangular Hyperbola**; the condition for this is $a+b-2h\cos\omega=0$.

In the case of the parabola, when $ab=h^2$, the equation

$$al^2+2hlm+bm^2=0$$

gives $(al+hm)^2=0$ or $(hl+bm)^2=0$, i. e. two coincident directions in which a straight line can be drawn to meet the curve at infinity.

There is no finite straight line which meets the parabola in two points at infinity; for in this case, since $al+hm=0$ and $hl+bm=0$, the equation $lX'+mY'=0$ reduces to $gl+fm=0$, so that $af=gh$. When it is possible to satisfy simultaneously the equations $al+hm=0$ and $lX'+mY'=0$, the equation of the parabola becomes

$$a\left(x+\frac{hy}{a}\right)^2+2g\left(x+\frac{hy}{a}\right)+c=0,$$

which is not a proper parabola but a pair of coincident straight lines. The parabola meets the straight line at infinity in coincident points, i. e. touches the line at infinity.

**Application of Homogeneous Coordinates.**

The equation of the curve $S=0$ in homogeneous coordinates is

$$a\xi^2+b\eta^2+c\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta=0;$$

the points at infinity on it are given by its intersections with $\zeta=0$. We have therefore for these points $a\xi^2+2h\xi\eta+b\eta^2=0$ and $\zeta=0$. The points are real, coincident, or imaginary according as $h^2-ab$ is positive, zero, or negative: hence an hyperbola meets the line at infinity in real distinct points, the ellipse meets it in imaginary points, the parabola meets it in coincident points or, in other words, the parabola touches the line at infinity.

I. When *the points are not coincident*, let their coordinates be $(\xi_1, \eta_1, 0)$ and $(\xi_2, \eta_2, 0)$, then

$$\frac{\xi_1\xi_2}{b}=\frac{\xi_1\eta_2+\xi_2\eta_1}{-2h}=\frac{\eta_1\eta_2}{a}.$$

The equations of the tangents at these points, i. e. of the asymptotes, are $\xi_1X+\eta_1Y=0$ and $\xi_2X+\eta_2Y=0$; their combined equation is therefore

$$(\xi_1X+\eta_1Y)(\xi_2X+\eta_2Y)=0,$$

or

$$bX^2-2hXY+aY^2=0,$$

as previously shown.

This equation can also be written $CS-\Delta\zeta^2=0$, and if $U$, $V$ are the linear factors of $S-\Delta\zeta^2/C$, then $U=0$, $V=0$ are the separate equations of the asymptotes. The equation of the curve can then be written in the form $UV+\Delta\zeta^2/C=0$, which indicates that $U=0$, $V=0$ are tangents to the curve, $\zeta=0$ being the chord of contact. If $U$ and $V$ are real, we can take the asymptotes as coordinate axes,

in which case the equation of the curve becomes $\xi\eta = k^2\zeta^2$, or, in Cartesian coordinates, $xy = k^2$.

**Note i.** It is evident from the form of the equation that the centre, in this case the origin, is the pole of the 'line at infinity'.

**Note ii.** It is evident from this form of the equation of an hyperbola that, if $AOA'$, $BOB'$ are the asymptotes, the curve lies altogether within either the angles $AOB$, $A'OB'$ or the angles $AOB'$, $A'OB$; also, that the distances from the centre of the points of intersection of the hyperbola and a diameter increase without limit as the diameter approaches the position of the asymptotes.

The following important properties of central curves are left as exercises for the reader :—

(i) The bisectors of the angles between the asymptotes are the axes of the curve.

(ii) The asymptotes are harmonic conjugates of every pair of conjugate diameters.

(iii) An asymptote, regarded as a diameter of the curve, is its own conjugate.

II. When *the points are coincident*, i.e. $ab = h^2$, we have $a\xi + h\eta = 0$ and $\zeta = 0$, hence the coordinates of this point are $(h, -a, 0)$.

The tangent at this point is

$$h(a\xi + h\eta + g\zeta) - a(h\xi + b\eta + f\zeta) = 0,$$

which reduces to $\zeta = 0$, the line at infinity.

*To obtain the separate equations of the asymptotes of an hyperbola when its equation is given with numerical coefficients.*

The equation of the asymptotes in the form $bX^2 - 2hXY + aY^2 = 0$ may be used.

When, however, $ax^2 + 2hxy + by^2$ has rational factors, we may proceed as follows : —

Let the equations of the asymptotes of $S = 0$ be $lx + my + n = 0$ and $l'x + m'y + n' = 0$; then

$$S \equiv (lx + my + n)(l'x + m'y + n') + k,$$

so that $$m'n + mn' = 2f, \quad l'n + ln' = 2g,$$

hence $$n(lm' - l'm) = 2(fl - gm).$$

Thus the equation of one asymptote is

$$(lm' - l'm)(lx + my) + 2(fl - gm) = 0.$$

Hence the following rule: Express $ax^2 + 2hxy + by^2 + 2gx + 2fy$ in the form $(lx + my)(l'x + m'y) + 2gx + 2fy$: in this expression put $x = m$ and $y = -l$, except in the factor $lx + my$. The resulting

expression equated to zero is the equation of the asymptote parallel to $lx+my=0$. Similarly, for the other asymptote, put $x=m'$, $y=-l'$, except in the factor $l'x+m'y$.

**Ex.** *To find the asymptotes of the hyperbola*

$$2x^2-xy-6y^2-4x+5y+1=0.$$

In the expression $(2x+3y)(x-2y)-4x+5y$ put $x=3$, $y=-2$, except in $(2x+3y)$; this gives $7(2x+3y)-22=0$.

Again, put $x=2$, $y=1$, except in $(x-2y)$; this gives $7(x-2y)-3=0$.

The equation of the curve can then be written

$$(7x-14y-3)(14x+21y-22)=17.$$

*To determine in which quadrant the infinite part of a given parabola lies.*

Let $(\alpha x+\beta y)^2+2gx+2fy+c=0$ be the equation of the parabola.

Change the origin to the point of intersection of the lines $\alpha x+\beta y=0$, $2gx+2fy+c=0$, the equation then becomes

$$(\alpha x+\beta y)^2+2gx+2fy=0.$$

The position of the infinite part of the parabola relative to the coordinate axes is clearly not altered by a change of origin.

Let $x/\beta=y/(\epsilon-\alpha)$, where $\epsilon$ is a very small number, be a straight line through the origin nearly parallel to the axis of the parabola. This will meet the parabola again at a point which is at a very great distance from the origin, and which lies in that quadrant in which the infinite part of the parabola lies.

Substituting $y=(\epsilon-\alpha)x/\beta$, we obtain for the $x$-coordinate of the point

$$\epsilon^2x=-2\{g+f(\epsilon-\alpha)/\beta\}=2(f\alpha-g\beta)/\beta-2f\epsilon/\beta.$$

Since $\epsilon$ is very small, $x$ will have the same sign as $(f\alpha-g\beta)/\beta$, and therefore $y$ will have the same sign as $(g\beta-f\alpha)/\alpha$.

Hence, if the equation of the parabola is

$$(\alpha x+\beta y)^2+2gx+2fy+c=0,$$

the signs of $x$ and $y$, determined so that $\alpha x+\beta y=0$ and $gx+fy$ is negative, indicate the quadrant in which the infinite part of the parabola lies.

**Ex.** $(5x+2y)^2+7x-3y+8=0$.

Take $5x+2y=0$ and $7x-3y=$ a negative quantity; then $y=-\frac{5}{2}x$ and $(7+\frac{15}{2})x$ is negative, therefore $x$ is negative and $y$ is positive.

The infinite part of the curve lies therefore in the second quadrant.

This rule will be found useful when drawing a parabola from its equation.

**§ 7. The foci and directrices.**

*To find the locus of a point which moves so that its distance from a fixed point is proportional to its distance from a fixed straight line.*

Let the fixed point be $S(x', y')$ and the fixed line be

$$x\cos\theta + y\sin\theta - p = 0,$$

using rectangular coordinates.

If $P(x, y)$ is any point on the required locus, and $PM$ the perpendicular from $P$ to the fixed line, we have $SP = e \,.\, PM$ where $e$ is a constant; hence $SP^2 = e^2 \,.\, PM^2$, or

$$(x-x')^2 + (y-y')^2 = e^2\{x\cos\theta + y\sin\theta - p\}^2, \qquad \text{(i)}$$

which is the equation of the locus. Now this equation contains *five* arbitrary constants, $x'$, $y'$, $\theta$, $p$, and $e$: we may expect then that the equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be put into this form, and we proceed to show that this is always so.

The locus represented by the general equation of the second degree can therefore be described as the locus of a point which moves so that its distance from a fixed point $S$ is a constant ($e$) times its distance from a fixed straight line.

The point $S$ is a **focus**, the fixed line is the corresponding **directrix**, and the constant $e$ is the **eccentricity** of the curve.

**Note.** This is the fundamental property proved in geometrical conics for the various curves which are sections of a cone by a plane. The locus $S = 0$ is therefore usually referred to as a **conic section**, or briefly, a conic.

The equation (i) above will be referred to as the focus-directrix form of the equation of a conic.

I. *To express the equation $S = 0$ in the focus-directrix form.*

Using rectangular coordinates, suppose that the equations

$$(x-x')^2 + (y-y')^2 = e^2(x\cos\theta + y\sin\theta - p)^2 \qquad \text{(i)}$$

and

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{(ii)}$$

are identical.

Change the origin of coordinates to the point $(x', y')$; these equations then become

$$x^2 + y^2 = e^2(x\cos\theta + y\sin\theta - p')^2 \qquad \text{(iii)}$$

where $p' = p - x'\cos\theta - y'\sin\theta$,

and

$$ax^2 + 2hxy + by^2 + 2X'x + 2Y'y + S' = 0. \qquad \text{(iv)}$$

Observe the way in which the terms containing $x$ and $y$ and the independent term occur in equation (iii); this suggests writing (iv) in the form

$$(X'^2 - aS')x^2 + 2(X'Y' - hS')xy + (Y'^2 - bS')y^2 = (X'x + Y'y + S')^2. \qquad \text{(v)}$$

This equation will be of the same form as equation (iii) provided that we can find a point $(x', y')$ whose coordinates satisfy the equations

$$X^2 - aS \equiv Y^2 - bS; \quad XY = hS, \tag{vi}$$

or

$$\frac{X^2 - Y^2}{a-b} = \frac{XY}{h} = S.$$

**Note i.** For a central conic, the equation $h(X^2 - Y^2) = (a-b)XY$ indicates that the foci (if any) lie on the axes of the conic.

**Note ii.** For a parabola, the equation $h(X^2 - Y^2) = (a-b)XY$ reduces to $F(aX + hY) = 0$, which indicates that the foci of a parabola (if any) lie on its axis.

Now, if we substitute the usual values of $X$, $Y$, and $S$ in equation (vi), we obtain

$$\left.\begin{aligned} C(x^2 - y^2) - 2Gx + 2Fy + A - B &= 0 \\ \text{and} \qquad Cxy - Fx - Gy + H &= 0 \end{aligned}\right\}. \tag{vii}$$

Thus, (1) If the conic is a parabola, i. e. $C = 0$, we have the two linear equations

$$\left.\begin{aligned} 2Gx - 2Fy - A + B &= 0 \\ Fx + Gy - H &= 0 \end{aligned}\right\}.$$

There is therefore a single solution; hence the equation of a parabola can be put in the focus-directrix form in one way. *We have now proved that a parabola has one focus, and that this focus lies on the axis.*

(2) If the conic is central, i. e. $C \neq 0$, equations (vii) can be written

$$\begin{aligned} (Cx - G)^2 - (Cy - F)^2 &= G^2 - F^2 + BC - CA \\ &= \Delta(a-b), \end{aligned}$$

and

$$(Cx - G)(Cy - F) = FG - CH = \Delta h.$$

If we write $\lambda$ for $Cx - G$ and $\mu$ for $Cy - F$, these give

$$\lambda^2 - \mu^2 = \Delta(a-b), \quad \lambda\mu = \Delta h;$$

hence

$$\lambda^4 - \Delta(a-b)\lambda^2 - \Delta^2 h^2 = 0,$$

or

$$2\lambda^2 = \Delta(a-b) \pm \Delta\sqrt{(a-b)^2 + 4h^2}.$$

The expression under the radical is always positive; we get therefore two real values of $\lambda^2$; evidently one of these values is positive and one negative, hence $\lambda$ has two real and two imaginary values. *We have now proved that a central conic has four foci, two real and two imaginary, and that these foci lie on the axes.*

II. *To prove that two foci lie on each axis of the curve.*

Now, if $(\xi, \eta)$ are the coordinates of the centre of the conic, the equations for the foci may be written

$$\begin{aligned} (x-\xi)^2 - (y-\eta)^2 &= \Delta(a-b)/C^2, \\ (x-\xi)(y-\eta) &= \Delta h/C^2. \end{aligned}$$

If, then, $(x-\xi)\sin\theta-(y-\eta)\cos\theta=0$ is the equation of one of the axes, the coordinates of the foci on that axis are $(\xi\pm k\cos\theta,\ \eta\pm k\sin\theta)$, where $C^2k^2\cos\theta\sin\theta=\Delta h$.

The equation of the other axis is $(x-\xi)\cos\theta+(y-\eta)\sin\theta=0$, so that the coordinates of the foci on this axis are $(\xi\pm k'\sin\theta,\ \eta\mp k'\cos\theta)$ where $C^2k'^2\cos\theta\sin\theta=-\Delta h$. Hence $k^2+k'^2=0$; this shows that two foci lie on each axis, one pair being real and one pair imaginary.

III. *To find the distance between the real foci.*

Let $2r_1$ be the length of the axis which lies along

$$(x-\xi)\sin\theta-(y-\eta)\cos\theta=0,$$

and $2r_2$ be the length of the other axis.

We showed (Chap. VI, § 5) that $r_1^2r_2^2=\Delta^2/C^3$, and that

$$\frac{a\cos\theta+h\sin\theta}{\cos\theta}=-\frac{\Delta}{Cr_1^2};$$

so that

$$\frac{a\cos\theta+h\sin\theta}{\cos\theta}=-\frac{C^2r_2^2}{\Delta}.$$

Since the axes are perpendicular, substituting $\pi/2+\theta$ for $\theta$, we get

$$\frac{a\sin\theta-h\cos\theta}{\sin\theta}=-\frac{C^2r_1^2}{\Delta}.$$

Subtracting these equations we have

$$\frac{C^2(r_1^2-r_2^2)}{\Delta}=\frac{h}{\sin\theta\cos\theta}.$$

Thus $k^2=r_1^2-r_2^2$; evidently $k$ is real if $r_1>r_2$, so that in the case of the ellipse the real foci lie on the major axis, and the distance between them is $2\sqrt{r_1^2-r_2^2}$.

If the ellipse is real, it is evident from the equation giving the lengths of the axes that $\Delta$ must be negative; hence the foci lying on the axis $(x-\xi)\sin\theta-(y-\eta)\cos\theta=0$ are real or imaginary according as $h\cos\theta\sin\theta$ is negative or positive.

In the case of the hyperbola the real foci must lie on the axis which meets the curve in real points.

IV. *To find the eccentricity of the conic* $S=0$.

Comparing coefficients in the equations

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and

$$(x-x')^2+(y-y')^2=e^2(x\cos\theta+y\sin\theta-p)^2,$$

we have

$$1-e^2\cos^2\theta=\lambda a,\quad 1-e^2\sin^2\theta=\lambda b,\quad e^2\cos\theta\sin\theta=-\lambda h,$$

so that $$2-e^2 = \lambda(a+b)$$

and $$(1-e^2\cos^2\theta)(1-e^2\sin^2\theta)-e^4\cos^2\theta\sin^2\theta = \lambda^2(ab-h^2),$$

or $$1-e^2 = \lambda^2(ab-h^2);$$

thus $$\frac{(2-e^2)^2}{1-e^2} = \frac{(a+b)^2}{ab-h^2}.$$

**Note i.** Since $1-e^2 = \lambda^2(ab-h^2)$:

(i) If the conic is an *ellipse*, $ab-h^2$ is $+ve$, $\therefore\ e<1$.

(ii) If the conic is an *hyperbola*, $ab-h^2$ is $-ve$, $\therefore\ e>1$.

(iii) If the conic is a *parabola*, $ab-h^2$ is zero, $\therefore\ e=1$.

(iv) If the conic is a *circle*, $a=b$ and $h=0$, and $e=0$.

**Note ii.** This equation gives two values for $e$, one corresponding to the real and one to the imaginary foci. When *the* eccentricity of a conic is referred to the eccentricity corresponding to the real foci is meant. (Cf. the example given below, p. 249.)

We have also from the above relations

$$e^2(\cos^2\theta-\sin^2\theta) = -\lambda(a-b),$$

$$e^2\sin\theta\cos\theta = -\lambda h,$$

so that $$(a-b)\sin\theta\cos\theta = h(\cos^2\theta-\sin^2\theta);$$

but this is the equation which gives the inclinations of the axes of the conic to the $x$-axis of coordinates. Hence the directrices of the conic are parallel to the axes.

The equations $1-e^2\cos^2\theta = \lambda a$, $e^2\cos\theta\sin\theta = -\lambda h$ give

$$e^2 = h/\cos\theta(h\cos\theta - a\sin\theta);$$

but, if $2r_1$ is the length of the axis inclined at the angle $\theta$ to the $x$-axis, we have shown that

$$\frac{a\sin\theta-h\cos\theta}{\sin\theta} = -\frac{C^2r_1^2}{\Delta} \quad \text{and} \quad \frac{C^2(r_1^2-r_2^2)}{\Delta} = \frac{h}{\sin\theta\cos\theta};$$

hence $$e^2 = (r_1^2-r_2^2)/r_1^2.$$

V. *To find the equations of the directrices of a conic.*

Referring back to equation (v) we see that the directrix $X'x+Y'y+S'=0$ is the polar of the origin with respect to the conic $ax^2+2hxy+by^2+2X'x+2Y'y+S'=0$; we took the focus as the origin, hence the directrix corresponding to a focus is its polar.

(a) *Central Conics.*

The coordinates of the foci lying on the axis

$$(x-\xi)\sin\theta-(y-\eta)\cos\theta = 0$$

are $\xi\pm k\cos\theta$, $\eta\pm k\sin\theta$, where $(\xi,\eta)$ is the centre and

$$C^2k^2\sin\theta\cos\theta = \Delta h.$$

The corresponding directrices, which are the polars of these points with respect to $S=0$, are therefore

$$X\xi+Y\eta+Z\pm k(X\cos\theta+Y\sin\theta) = 0.$$

Now $X\xi+Y\eta+Z=\Delta/C$, therefore the equation of the pair of directrices is

$$\Delta^2=C^2k^2(X\cos\theta+Y\sin\theta)^2,$$

or
$$h(X\cos\theta+Y\sin\theta)^2=\Delta\cos\theta\sin\theta.$$

(b) *The Parabola.*

The coordinates of the focus are given by the equations

$$2Gx-2Fy-A+B=0,\quad Gy+Fx-H=0,$$

and the equation of the directrix may be found as the polar of this point. The case of the parabola is, however, practically and theoretically, most simply dealt with by reducing the equation to the focus-directrix form.

Let the equation of the parabola be

$$(\alpha x+\beta y)^2+2gx+2fy+c=0,$$

where we have written $\alpha^2$, $\beta^2$, and $\alpha\beta$ for $a$, $b$, and $h$. The eccentricity of the parabola is 1, so that the focus-directrix form is

$$(x-x')^2+(y-y')^2=(x\cos\theta+y\sin\theta-q)^2.$$

Comparing coefficients of $x^2$ and $xy$, we get $\tan\theta=-\alpha/\beta$, so that the directrix is perpendicular to $\alpha x+\beta y=0$, and therefore to the axis of the parabola. We may write the equation of the parabola then in the form

$$(\alpha^2+\beta^2)(\overline{x-x'^2}+\overline{y-y'^2})=(\beta x-\alpha y+p)^2.$$

Hence, comparing coefficients, we have

$$(\alpha^2+\beta^2)x'+\beta p=-g, \tag{i}$$

$$(\alpha^2+\beta^2)y'-\alpha p=-f, \tag{ii}$$

$$(\alpha^2+\beta^2)(x'^2+y'^2)-p^2=c. \tag{iii}$$

From these equations we find

$$(\beta p+g)^2+(\alpha p-f)^2=(\alpha^2+\beta^2)^2(x'^2+y'^2)=(\alpha^2+\beta^2)(p^2+c),$$

or
$$(\alpha^2+\beta^2)p^2+2(\beta g-\alpha f)p+g^2+f^2=(\alpha^2+\beta^2)p^2+(\alpha^2+\beta^2)c;$$

$$\therefore\quad 2p(\beta g-\alpha f)=c(\alpha^2+\beta^2)-g^2-f^2$$
$$=c(a+b)-g^2-f^2=A+B.$$

But
$$\beta(\beta g-\alpha f)=bg-hf=-G,$$
$$\alpha(\beta g-\alpha f)=hg-af=F,$$

so that the equation of the directrix ($\beta x-\alpha y+p=0$) may be written

$$2Gx+2Fy-A-B=0.$$

Using this value of $p$ in equations (i) and (ii), we can write down the coordinates of the focus; thus

$$(a+b)x'=b(A+B)/2G-g;\ (a+b)y'=a(A+B)/2F-f.$$

It can be readily verified from equations (i) and (ii) that the coordinates of the focus also satisfy the equations

$$(a+b)X+G=0\,;\quad (a+b)Y+F=0\,;\quad 2(a+b)Z=A+B.$$

VI. *Methods of finding the foci and directrices of a central conic, whose equation is given with numerical coefficients.*

The coordinates of the foci involve complicated surds unless the factors of $h(X^2-Y^2)-(a-b)XY$ are rational. If these factors are not rational the coordinates of the foci may be found as on p. 245.

When the factors are rational the following method is useful.

If $ax^2+2hxy+by^2+2gx+2fy+c$

$$=k\{(x-x')^2+(y-y')^2\}+l(x\cos\theta+y\sin\theta-p)^2 \qquad \text{(i)}$$

identically, on considering the terms of the second degree, it is evident that $ax^2+2hxy+by^2-k(x^2+y^2)$ is a perfect square. This is the case when $(k-a)(k-b)=h^2$, and this equation gives two values of $k$ which are rational when the factors of $h(X^2-Y^2)-(a-b)XY$ are rational. Take either of these values of $k$ and then we have

$$ax^2+2hxy+by^2-k(x^2+y^2)=l(x\cos\theta+y\sin\theta)^2,$$

from which we can determine $l$, $\cos\theta$, and $\sin\theta$.

Comparing the coefficients of $x$ and $y$ and the independent terms in equation (i), we obtain

$$kx'+lp\cos\theta=-g, \qquad \text{(ii)}$$

$$ky'+lp\sin\theta=-f, \qquad \text{(iii)}$$

whence,

$$k(x'^2+y'^2)+lp^2=c, \qquad \text{(iv)}$$

$$k(c-lp^2)=k^2(x'^2+y'^2)=(g+lp\cos\theta)^2+(f+lp\sin\theta)^2.$$

This gives a quadratic equation from which we can find $p$, and the corresponding values of $x'$ and $y'$ are given by (ii) and (iii).

This gives the directrices, foci, and also the eccentricity which is

$$\sqrt{-l/k}.$$

**Ex.** *To find the foci and directrices of the conic*

$$x^2+4xy+y^2-2x-6y=0.$$

If $x^2+4xy+y^2-k(x^2+y^2)$ is a perfect square, we have $(k-1)^2=4$, i.e. $k=-1$ or 3.

Thus $$x^2+4xy+y^2+(x^2+y^2)=2(x+y)^2,$$

and $$x^2+4xy+y^2-3(x^2+y^2)=-2(x-y)^2.$$

(a) When $k=-1$,

$$x^2+4xy+y^2-2x-6y=2(x+y+p)^2-(x-x')^2-(y-y')^2,$$

so that $$x'+2p=-1,\quad y'+2p=-3,\quad x'^2+y'^2=2p^2\,;$$

$$\therefore\quad 2p^2=(2p+1)^2+(2p+3)^2\,;$$

$$\therefore\quad 3p^2+8p+5=0\,;$$

$$\therefore\quad p=-1 \text{ or } -\tfrac{5}{3}.$$

If $p = -1$, then $x' = 1$, $y' = -1$, and the equation of the conic reduces to
$$(x-1)^2+(y+1)^2 = 2(x+y-1)^2,$$
$$= 4\left(\frac{x+y-1}{\sqrt{2}}\right).$$

If $p = -\frac{5}{3}$, then $x' = \frac{7}{3}$, $y' = \frac{1}{3}$, and the equation reduces to
$$(x-\tfrac{7}{3})^2+(y-\tfrac{1}{3})^2 = 4\left(\frac{x+y-1\frac{2}{3}}{\sqrt{2}}\right)^2.$$

(b) When $k = 3$,
$$x^2+4xy+y^2-2x-6y = 3(x-x')^2+3(y-y')^2-2(x-y-p)^2,$$
so that $3x'-2p = 1$, $3y'+2p = 3$, $3(x'^2+y'^2) = 2p^2$;
$$\therefore \quad 6p^2 = (2p+1)^2+(2p-3)^2;$$
$$\therefore \quad p^2-4p+5 = 0;$$
$$\therefore \quad p = 2 \pm i.$$

If $p = 2 \pm i$, then $x' = \frac{1}{3}(5 \pm 2i)$ and $y' = -\frac{1}{3}(1 \pm 2i)$, and the equation can be written in either of the forms
$$\left(x-\frac{5 \pm 2i}{3}\right)^2+\left(y+\frac{1 \pm 2i}{3}\right)^2 = \frac{4}{3}\left\{\frac{x-y-(2 \pm i)}{\sqrt{2}}\right\}^2.$$

We have now the real and imaginary foci; the equations of the real directrices are $x+y-1 = 0$ and $3x+3y-5 = 0$, and the eccentricity corresponding to the real foci is 2.

VII. *The foci in relation to the circular points at infinity.*

The equation of a conic in the focus-directrix form
$$(x-x')^2+(y-y')^2 = e^2\{x\cos\theta+y\sin\theta-p\}^2$$
may be written
$$[(x-x')+i(y-y')][(x-x')-i(y-y')] = e^2\{x\cos\theta+y\sin\theta-p\}^2;$$
this is of the form $uv = kw^2$, from which it appears that the conic touches the imaginary lines
$$(x-x')+i(y-y') = 0, \quad (x-x')-i(y-y') = 0,$$
the chord of contact being the directrix
$$x\cos\theta+y\sin\theta-p = 0.$$

These imaginary lines intersect at the focus $(x', y')$; hence the directrix is the chord of contact of the imaginary tangents from the focus to the conic.

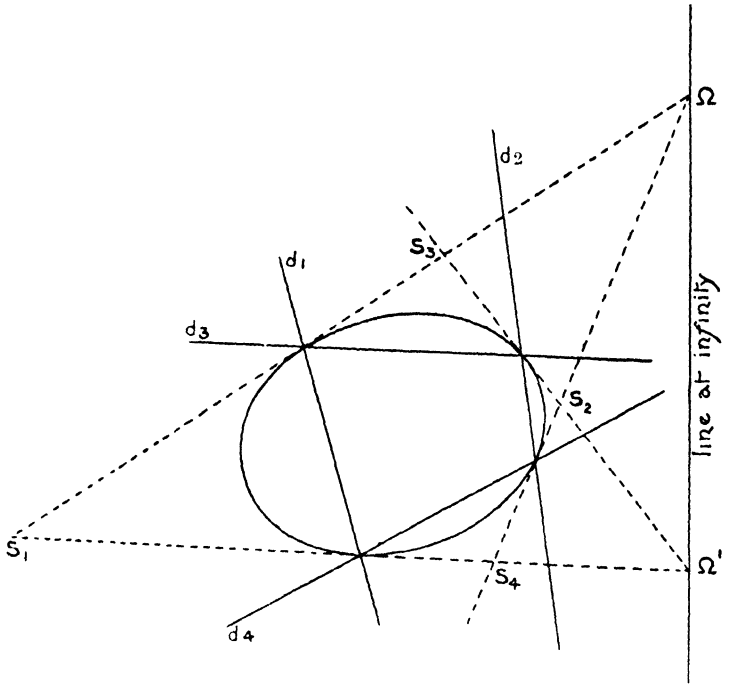
These imaginary lines are parallel to $x+iy = 0$ and $x-iy = 0$, i. e. they pass through the circular points $\Omega$, $\Omega'$ at infinity.

This argument applies to each of the ways in which the equation of the conic $S = 0$ can be put in the focus-directrix form.

Hence the foci are the points of intersection of tangents to the conic from the circular points $\Omega$, $\Omega'$.

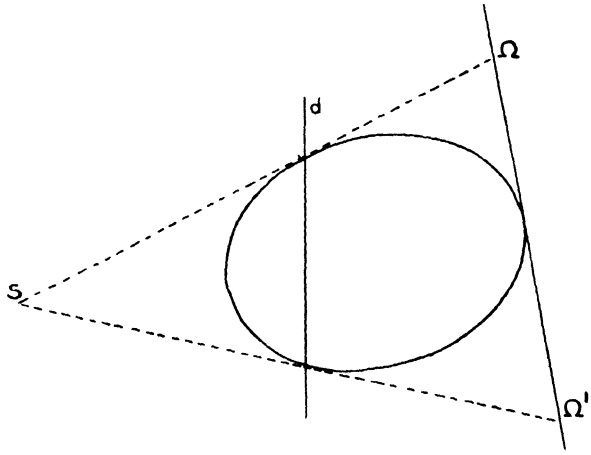
Since these points $\Omega$ and $\Omega'$ are imaginary, we cannot properly

represent them in a figure; the following diagram shows the relative properties of these points and lines.




$S_1, S_2, S_3, S_4$ are foci; $d_1, d_2, d_3, d_4$ are the corresponding directrices.

Two of the foci are real and two are imaginary, and the conic is inscribed in the quadrilateral $S_1$, $S_2$, $S_3$, $S_4$.




When the conic is a parabola, since $\Omega\,\Omega'$ touches it, we have only one focus, $S$.

**Note.** Since the tangents from a focus to the curve are parallel to $(x+iy)(x-iy)=0$, i.e. to $x^2+y^2=0$, the equation of a pair of tangents to the curve from a focus satisfies the conditions for a circle. This gives us another method of finding the foci.

## § 8. The equations of the various conics in their simplest forms.

When the axes of symmetry are taken as coordinate axes, the equation $S=0$ reduces to its simplest form.

Two cases arise: (i) $C$ is not zero, being positive for an ellipse and negative for an hyperbola; there are then two finite axes of symmetry.

(ii) $C=0$, the curve being a parabola; there is only one finite axis of symmetry.

It has already been shown that when the coordinate axes are changed from one set of rectangular axes to another the quantity $C$ is unaltered: this is also self-evident so far as sign is concerned, since a change of axes cannot affect the nature of the curve.

(i) When the coordinate axes are the axes of symmetry, if the point $(x, y)$ lies on the curve, so also do the points $(x, -y)$, $(-x, y)$, $(-x, -y)$; hence the coefficients of $xy$, $x$, and $y$ in the equation of the curve must be zero. The equation is then of the form

$$ax^2+by^2+c=0.$$

(a) For an ellipse $ab$ is positive, i.e. $a$ and $b$ have the same sign. If $a$ and $b$ are positive, $c$ must be negative, otherwise no real values of $x$ and $y$ could satisfy the equation.

We can thus write the equation of the ellipse referred to its axes of symmetry in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $a$, $b$ are evidently the lengths of the semi-axes.

(b) For the hyperbola $ab$ is negative: hence the equation of an hyperbola referred to its axes of symmetry can be written in one of the forms

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{or} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$

Consider the first equation; the $x$-axis meets the curve at the points $(+a, 0)$, $(-a, 0)$, and $a$ is the length of the corresponding axis.

The axis of $y$ does not meet the curve in real points; $b$ is, however, usually called the length of the other semi-axis.

(ii) Take the finite axis of symmetry as axis of $x$; if the point $(x, y)$ lies on the curve, so does the point $(x, -y)$, whatever value $x$ may have, hence the coefficients of $xy$ and $y$ in the equation of the curve are zero. The equation is of the form

$$ax^2 + by^2 + 2gx + c = 0.$$

Now $ab = 0$, since the curve is a parabola, hence $a = 0$ or $b = 0$. We cannot have $b = 0$ unless the curve is a pair of straight lines parallel to the axis of $y$, hence $a = 0$.

The equation is then of the form

$$by^2 + 2gx + c = 0.$$

Now take the origin at $(-c/2g, 0)$, i.e. on the curve; the constant of the equation then becomes zero.

The equation of the parabola now takes the form $y^2 = 2g'x$, or, as it is usually written, $\mathbf{y^2 = 4ax}$.

The axes of coordinates are the axis of the parabola and a tangent to the parabola at the point where its axis intersects it.

In the following chapters we propose to discover the properties of the parabola, the ellipse, and the hyperbola from their equations in these simple forms.

## § 9. Envelopes.

In Chapter II we showed how to find the equation of the envelope of a straight line whose equation contains an arbitrary constant in the first or second degree. It will be convenient to extend this method to the equation of any curve which contains an arbitrary constant, or the constants of which are connected by given relations which leave one of them undetermined.

If $P$, $Q$, $R$ are three functions of the coordinates $x$ and $y$, of the first or second degree, then

$$\lambda^2 P + \lambda Q + R = 0 \qquad \text{(i)}$$

is an equation of the first or second degree, and represents some straight line or curve whatever value $\lambda$ may have.

Two of these lines pass through any proposed point $(x', y')$, for if $P'$, $Q'$, $R'$ are the values of the functions $P$, $Q$, $R$ when $x'$, $y'$ are substituted for $x$ and $y$, we have the condition

$$\lambda^2 P' + \lambda Q' + R' = 0$$

to determine $\lambda$ for loci of the type which pass through $(x', y')$.

These two lines will be tangents, straight or curved, from the point $(x', y')$ to the curve enveloped by the system.

If $(x', y')$ is on the envelope, then these tangents become coincident. The condition for this is $4P'R' = Q'^2$.

Hence the equation of the envelope of loci represented by (i) is $4PR = Q^2$.

### Examples VI c.

1. Write down the equation of
   (i) the tangent at $(x', y')$; (ii) the normal at $(x', y')$;
   (iii) the polar of $(x', y')$; (iv) a pair of tangents from $(x', y')$,
for the curves whose equations are given in Ex. VI a, 1.

2. Find the coordinates of the centre of these curves, and explain your results.

3. Find the eccentricity of the curves in Ex. VI a, 1 (i), (iii).

4. Find the focus-directrix form of the parabola $(x+2y)^2 = 4x+2y+1$.

5. Express the equation $8x^2+15y^2+24xy+2x+4y-5 = 0$ in the focus-directrix form.

6. Find the equations of the asymptotes of the hyperbolas in Exs. VI b.

7. If the coordinate axes are oblique, show that the foci of the conic $S = 0$ are given by the equations $X^2-aS = Y^2-bS = \sec\omega\,(XY-hS)$.

8. Find the lengths of the axes of those curves in Exs. VI b which are ellipses or hyperbolas.

9. Find the equations of the ellipses in Exs. VI b referred to their principal axes as axes of coordinates.

10. If the coordinate axes are turned through an angle $\theta$, what does the equation of the conic $ax^2+2hxy+by^2+c = 0$ become?

For what values of $\theta$ does the $xy$ term in the result vanish? Explain this result.

11. Show that the equation of the axes of symmetry of the curve $11x^2+6xy+19y^2-2x-26y+3 = 0$ are $3x-y+1 = 0$ and $x+3y-2 = 0$.

Determine $p, q$, and $r$ so that $p(3x-y+1)^2+q(x+3y-2)^2+r = 0$ may be identical with the given curve, and hence show that the equation of the curve referred to its axes as coordinate axes is $5x^2+10y^2 = 3$.

(**Note.** When the separate equations of the axes take a simple form, as in this example, this is the easiest method of finding the equation of the curve referred to its axes. Exs. VI a and VI b give other curves which can be similarly treated.)

12. Where are the focus and directrix of a circle?

13. $AA'$ is a given finite straight line and $PN$ is perpendicular to it. If $P$ moves so that $PN^2 : AN.A'N$ is a constant ratio, show that its locus is a curve of the second degree. If the given ratio is $\lambda : 1$, distinguish the cases of the circle, parabola, ellipse, and hyperbola.

Find the eccentricity of the curve in terms of $\lambda$.

14. Draw graphs of (i) $y^2 = 4x$; (ii) $y^2 = -4x$; (iii) $x^2+4y^2 = 1$; (iv) $x^2-4y^2 = 1$; (v) $x^2-4y^2 = -1$; (vi) $xy = 4$; (vii) $xy = -4$.

15. Find axes, eccentricity, centre, foci, equations of axes, asymptotes, and directrices of $x^2+4xy+y^2-2x-6y = 0$.

16. Draw the curve and find the foci and eccentricity of

$$5x^2+4xy+8y^2-12x-12y = 0.$$

17. Find the condition that an asymptote of $S = 0$ may pass through the origin.

18. Determine completely the position and nature of the curve

$$2x^2+4xy-y^2+4x-6y+2 = 0,$$

finding its centre, its eccentricity, and the equations to its axes. Sketch the curve.

19. Find the condition that the line $lx+my = 1$ should be (i) a tangent, (ii) a normal at a point on the curve $ax^2+2hxy+by^2 = 1$.

20. Find the coordinates of the centre and of the foci, and the equations of the axes, asymptotes, and directrices of the conic whose equation is

$$7x^2-48xy-7y^2+60x+80y-50 = 0.$$

21. Find the position and magnitude of the axes of the conic whose equation is $ax^2+2hxy+by^2 = 1$.

22. Find the asymptotes of the hyperbola

$$6x^2-7xy-3y^2-2x-8y-6 = 0.$$

23. Trace the conic whose equation is $(x-4y)^2 = 51y$.

Find the eccentricity of the conic whose equation is $x^2+xy+y^2 = 1$.

24. Show that one focus of the conic $x^2+y^2+2hxy+2g(x+y)+g^2/h = 0$ is the origin, and that the other is $x = y = (-2g)/(1+h)$.

25. Trace the curve $81x^2+90xy+25y^2+59x+21y+9 = 0$, and find the coordinates of its focus and the equation of its directrix.

26. Trace the curve $3x^2+8xy-3y^2-40x-20y+50 = 0$, and find the equations of its directrices.

27. Reduce to its simplest equation and draw the figure of the conic

$$12x^2+7xy-12y^2-x+7y = 26.$$

28. Mark on a diagram the position of the focus and directrix of the parabola whose equation is $x^2-2xy+y^2+x-3y+3 = 0$.

29. Show that the equation $(a-1/r^2)x+hy = 0$ represents one of the axes of the conic $ax^2+2hxy+by^2 = 1$, if $r$ is a root of the equation $(a-1/r^2)(b-1/r^2) = h^2$.

Trace the curve $3x^2+4xy+y^2-3x-2y+21 = 0$.

30. Show that the equation of the directrix of the parabola

$$ax^2+2gx+2fy+c = 0$$

is $2afy+ca-g^2-f^2 = 0$.

31. If the axes are so inclined that $x^2+xy+y^2 = a^2$ is a circle, trace the conic $x^2-xy+y^2 = a^2$, and obtain the lengths and positions of its axes, the coordinates of its foci, and its eccentricity.

32. If the straight line $y = x\tan\theta$ is an axis of the conic

$$ax^2+2hxy+by^2 = 1,$$

and the length of this axis is $2r$, show that $1/r^2 = a+h\tan\theta = b+h\cot\theta$.

33. Show that the equations to the tangents to the ellipse

$$17x^2 - 12xy + 8y^2 + 50x - 20y + 21 = 0$$

at the extremities of its axes are

$$(2x - y + 3)^2 = 4, \quad (x + 2y + 1)^2 = 16.$$

34. Prove that

$$x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c = \Delta/(ab - h^2),$$

where $\Delta$ is the discriminant, is the diameter parallel to the tangent at $x'\ y'$.

35. When the coordinate axes are oblique, the eccentricity of the conic $S = 0$ is given by $(ab - h^2)(2 - e^2)^2 \sin^2\omega = (a + b - 2h\cos\omega)^2(1 - e^2)$.

36. If $(x', y')$ is a focus of the conic $S = 0$, and $d$ its distance from the corresponding directrix $x\cos\theta + y\sin\theta - p = 0$, show that $X'\sin\theta = hd$, $Y'\cos\theta = hd$, and $S'\cos\theta\sin\theta = hd^2$.

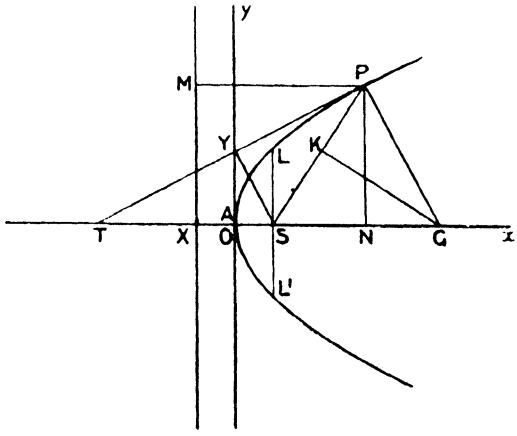
37. Show that a pair of intersecting straight lines possesses the focus directrix property. Where is the focus?

# CHAPTER VII

## THE PARABOLA

§ 1. *The parabola referred to its axis of symmetry and the tangent at the vertex as coordinate axes* ($y^2 = 4ax$).

The equation $y^2 = 4ax$ (Chap. VI, § 8) can be put into the focus-directrix form




$$(x-x')^2+(y-y')^2 = e^2\{x\cos\alpha+y\sin\alpha-p\}^2$$

in one way only: viz.

$$(x-a)^2+y^2 = (x+a)^2.$$

Hence the **Focus** $S$ of the parabola is the point $(a, 0)$, and the **Directrix** $XM$ is the line $x+a=0$.

The following definitions are common to all the conics :—

(i) The perpendicular ($PN$) from any point ($P$) on the curve to the axis is called the **ordinate** of the point $P$ with respect to the axis.

(ii) The double ordinate through the focus ($LSL'$) is called the **latus rectum.**

(iii) **The length of the normal** at a point ($P$) means the distance ($PG$) measured along the normal to the axis, unless otherwise stated.

(iv) The length $NT$ is called the **sub-tangent**, and the length $NG$ the **sub-normal.**

(a) Since the focus is the point $(a, 0)$, we have $AS = a$, and the equation $y^2 = 4ax$ expressed geometrically is

$$PN^2 = 4AS \cdot AN.$$

When the point $P$ is at the end $L$ of the latus rectum

$$SL^2 = 4AS^2, \quad \text{or} \quad SL = 2AS;$$

thus the latus rectum $LSL' = 4a$.

(b) The equation in the focus-directrix form

$$(x-a)^2 + y^2 = (x+a)^2$$

expressed geometrically gives

$$SP^2 = PM^2, \quad \text{or} \quad SP = PM.$$

Thus the focal distance of any point on a parabola is equal to its distance from the directrix.

The focal distance of the point $(x, y)$ is $x+a$.

### § 2. Tangent, Normal, Diameter, Polar.

The student should refer back to Chapter VI, § 4, and work out the equations given below for the case of the parabola $y^2 = 4ax$.

(1) *The equation of the chord whose mid-point is* $(x', y')$:—

$$-2ax + yy' - 2ax' = y'^2 - 4ax',$$

or

$$\mathbf{y'(y-y') = 2a(x-x').}$$

(2) *The equation of the tangent at the point* $(x', y')$:—

$$-2ax + yy' - 2ax' = 0,$$

or

$$\mathbf{yy' = 2a(x+x').}$$

**Example i.** *If the tangent at $P$ meets the axis at $T$, then $SP = ST$.*

Let $P$ be the point $(x', y')$; then the tangent at $P$ is $yy' = 2a(x+x')$; hence, putting $y = 0$, the point $T$ is $(-x', 0)$, and since $S$ is $(a, 0)$ we have $ST = a + x'$, which is, § 1, the focal distance $SP$.

Hence $SP = ST$.

**Example ii.** *If the tangent at $P$ meets the tangent at the vertex at $Y$, $SY$ is perpendicular to the tangent.*

Let $P$ be the point $(x', y')$; then the tangent at $P$ is

$$yy' = 2a(x+x'), \tag{i}$$

therefore the point $Y$ is $\left(0, \frac{2ax'}{y'}\right)$. But $y'^2 = 4ax'$, therefore the point $Y$ is $(0, \frac{1}{2}y')$.

$SY$ is therefore the line $2ay + y'(x-a) = 0$ which is perpendicular to (i).

(3) *The equation of the normal at* $(x', y')$.

The normal being perpendicular to the tangent, its equation is

$$y'(x-x') + 2a(y-y') = 0.$$

**Example.** *If the normal at $P$ meets the axis at $G$, $SP = SG$.*

Let $P$ be the point $(x', y')$, then the normal at $P$ is

$$y'(x-x') + 2a(y-y') = 0;$$

hence, putting $y = 0$, $G$ is the point $\{(2a+x'), 0\}$.

Thus $SG = a + x' = SP$.

(4) *To find the locus of the mid-points of parallel chords of the parabola, i.e. the equation of a diameter.*

Let $(x', y')$ be the mid-point of any chord parallel to the fixed line $y = mx$.

Since the equation of the chord is

$$y'(y-y') = 2a(x-x'),$$

the condition that it should be parallel to the given line is $y' = \frac{2a}{m}$;

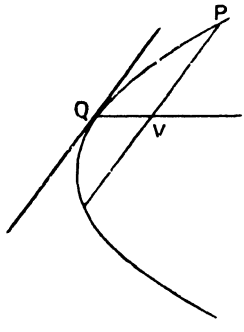
hence the locus of the mid-points of all chords parallel to $y = mx$ is $my = 2a$, i.e. a line parallel to the axis.

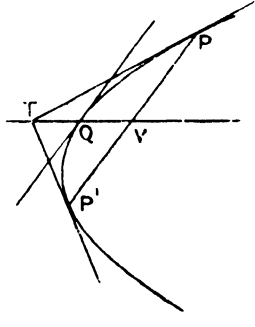
Hence all diameters of a parabola are parallel to the axis.

**Note.** The diameter bisecting all chords parallel to the tangent at $(x', y')$, viz. $yy' = 2a(x+x')$ is the line $y = y'$, which passes through the point of contact of the tangent.

Thus the chords bisected by any diameter are parallel to the tangent at its extremity.

**Definition.** If the chord through a point $P$ on the parabola, drawn parallel to the tangent at $Q$, meets the diameter through $Q$ at $V$, $PV$ is called the **ordinate** of $P$ with respect to this diameter.




(5) *The equation of the polar of the point $(x', y')$.*

This takes the same form as that of the tangent at a point $(x', y')$, viz. $yy' = 2a(x+x')$. (Chap. VI, § 4, IV.)

**Note.** If $Q(x', y')$ is a point on the parabola, the polar of any point $T(h, y')$ on the diameter through $Q$ is $yy' = 2a(x+h)$, which is parallel to the tangent at $Q$ and to the chords bisected by the diameter: conversely, the pole of any chord lies on the diameter bisecting the chord.

(6) *The equation of a pair of tangents from a point* $(x', y')$ *to the parabola.*

This equation (Chap. VI, § 4) is

$$(y^2-4ax)(y'^2-4ax') = \{yy'-2ax-2ax'\}^2.$$

Retaining terms of the second degree only, we find that these tangents are parallel to the straight lines

$$y^2(y'^2-4ax') = (yy'-2ax)^2,$$

i.e.
$$x'y^2-y'xy+ax^2 = 0.$$

The angle $\theta$ between the tangents is therefore given by

$$\tan\theta = \frac{\sqrt{y'^2-4ax'}}{x'+a}.$$

Thus the locus of a point, the tangents from which include a constant angle $\theta$, is

$$y^2-4ax = (x+a)^2\tan^2\theta.$$

The locus of the intersection of orthogonal tangents is $x+a=0$, i.e. the directrix.

§ 3. (a) *The equation of the parabola referred to the axis and latus rectum as coordinate axes.*

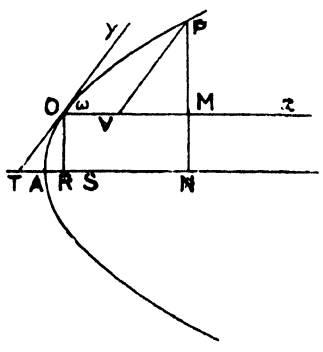
When the equation of the parabola is $y^2 = 4ax$, the focus is $(a, 0)$.

Changing the axes to parallel axes through this point, we obtain the required equation as

$$\mathbf{y^2 = 4a(x+a)}.$$

(b) *The equation of the parabola referred to any diameter and the tangent at its extremity as coordinate axes.*

Let the origin be $O$, and let the axes of coordinates be the diameter $Ox$ and the tangent $Oy$. Since all chords of the parabola parallel to $Oy$ are bisected by the diameter $Ox$, the ordinates of points on the parabola, which have the same abscissa, are equal and of opposite sign.




Hence the equation of the parabola must be of the form $y^2 = 4\lambda x$; we have then to find the value of $\lambda$.

Let the angle between the coordinate axes be $\omega$: draw the ordinate $PV$ and $PN$, $OR$ perpendicular to the axis of the parabola.

If the coordinates of $O$, referred to the principal axes of the parabola, are $(h, k)$, then $\tan\omega = \frac{2a}{k}$.

Since $k^2 = 4ah$, we have $h = a\cot^2\omega$ and $k = 2a\cot\omega$.

Let $P$ be the point $(x, y)$, then

$$PN = PM + OR = y\sin\omega + k = y\sin\omega + 2a\cot\omega,$$

$$AN = AR + OV + VM = h + x + y\cos\omega = a\cot^2\omega + x + y\cos\omega.$$

But $PN^2 = 4a \cdot AN$; hence

$$(y\sin\omega + 2a\cot\omega)^2 = 4a(a\cot^2\omega + x + y\cos\omega),$$

or
$$y^2 = 4ax\operatorname{cosec}^2\omega,$$

and consequently $\lambda = a\operatorname{cosec}^2\omega$.

This is then the equation of the parabola referred to a diameter and the tangent at its extremity; it is of the same form as $y^2 = 4ax$, consequently the equations of the tangent, polar, and diameter will be of the same form as those already found, but the normal which involves the condition for perpendicularity will not be the same, since the axes of coordinates are now oblique.

The equation translated into geometrical notation gives

$$PV^2 = 4AS\operatorname{cosec}^2\omega \cdot OV.$$

Now
$$SO = a + h = a + a\cot^2\omega = a\operatorname{cosec}^2\omega;$$

hence
$$\mathbf{PV^2 = 4OS \cdot OV.}$$

**Example i.** *To find the coordinates of the focus of the parabola* $y^2 = 4ax\ cosec^2\omega$.

If the tangent at $O$ (see Fig., p. 260) meets the axis at $T$, the $x$-coordinate of $S$ is equal to $ST$. But $ST = SO = a + h$;

$$= a + a\cot^2\omega;$$
$$= a\operatorname{cosec}^2\omega.$$

The $y$-coordinate is equal to $OT$, hence

$$y\sin\omega = -OR = -k = -2a\cot\omega.$$

The focus is therefore the point $\{a\operatorname{cosec}^2\omega,\ -2a\operatorname{cosec}^2\omega\cos\omega\}$.

**Example ii.** *On the diameter through a point O of a parabola are taken points P, P′ so that the rectangle OP . OP′ is constant: prove that the four points of intersection of the tangents drawn from P, P′ lie on two fixed straight lines parallel to the tangent at O and equidistant from it.*

Take the diameter and tangent at $O$ for axes of coordinates and let the points $P$, $P'$ be $(\xi, 0)$ $(\xi', 0)$, where $\xi\xi' = c^2$.

The equation of the pairs of tangents from $P$ to the parabola is

$$-4a\xi(y^2 - 4ax) = (2ax + 2a\xi)^2,$$

which reduces to
$$a\xi^2 + (y^2 - 2ax)\xi + ax^2 = 0. \qquad \text{(i)}$$

The equation of the pair of tangents from $P'$ to the parabola is obtained by writing $\xi'$ for $\xi$.

Thus, if $(x, y)$ is one point of intersection of the tangents from $P$ and $P'$, $\xi$ and $\xi'$ satisfy equation (i).

The product of the roots of this equation is $x^2$; thus $x^2 = \xi\xi' = c^2$.

Hence the points of intersection of tangents from $P$, $P'$ to the parabola lie on one or other of the lines $x+c=0$, $x-c=0$; these lines are parallel to and equidistant from the tangent at $O$, viz. $x=0$.

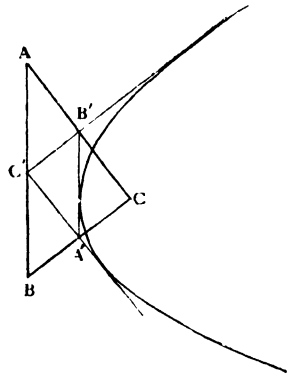
**Example iii.** *The lines joining the mid-points of the sides of a triangle, which is self-conjugate with respect to a parabola, touch the parabola.*

Let the vertices $A$, $B$, $C$ of the triangle be the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$.

The conditions that the polar of each of these points should pass through the other two are

$$y_2y_3 = 2a(x_2+x_3),$$
$$y_3y_1 = 2a(x_3+x_1),$$
$$y_1y_2 = 2a(x_1+x_2).$$

Now, the coordinates of the mid-points of $AB$ and $AC$ are




$$\{\tfrac{1}{2}(x_1+x_2),\ \tfrac{1}{2}(y_1+y_2)\} \text{ and } \{\tfrac{1}{2}(x_1+x_3),\ \tfrac{1}{2}(y_1+y_3)\};$$

i.e.
$$\left\{\frac{y_1y_2}{4a},\ \tfrac{1}{2}(y_1+y_2)\right\} \text{ and } \left\{\frac{y_1y_3}{4a},\ \tfrac{1}{2}(y_1+y_3)\right\}.$$

The equation of the straight line joining these points is

$$\begin{vmatrix} x & y & 1 \\ y_1y_2 & 2a(y_1+y_2) & 4a \\ y_1y_3 & 2a(y_1+y_3) & 4a \end{vmatrix} = 0,$$

i.e.
$$\begin{vmatrix} x & y & 1 \\ y_1y_2 & 2a(y_1+y_2) & 4a \\ y_1 & 2a & 0 \end{vmatrix} = 0,$$

i.e.
$$2yy_1 = 4ax + y_1^2.$$

This is the equation of the tangent to the parabola at the point $\left(\frac{y_1^2}{4a},\ y_1\right)$, which proves the proposition.

### Examples VII a.

1. The tangent and diameter at a point are perpendicular only when the point is the vertex.

2. The point $(\xi, \eta)$ lies outside, on, or inside the parabola $y^2 = 4ax$, according as $\eta^2 - 4a\xi$ is positive, zero, or negative.

3. Obtain the coordinates of a point the tangent at which makes an angle $\phi$ with the diameter through the point.

4. If the ordinate of a point on the parabola is $2at$, find the abscissa.

5. Find the condition that the straight line $lx + my = 1$ should touch the parabola $y^2 = 4ax$.

6. Find the equation of the pair of straight lines joining the intersections of the straight line $lx + my = 1$ and the parabola $y^2 = 4ax$ to the vertex. What are the conditions that they should be (a) perpendicular, (b) coincident?

7. Show that the tangents at $(x', y')$, $(x'', y'')$ intersect on the diameter $2y = y' + y''$. Find also the $x$-coordinate of their point of intersection in terms of $y'$ and $y''$.

8. The tangent at $P$ meets the axis at $T$ and the tangent at the vertex at $Y$. Prove that $PY = YT$.

9. The subnormal of any point on a parabola is half the latus rectum.

10. At what angle does the straight line $mx + y - am^3 - 2am = 0$ cut the parabola?

11. The tangent at $P$ and the ordinate of $P$ meet the axis at $T$ and $N$: show that $AN = AT$.

12. Find the equation of a parabola, with latus rectum $4b$, which touches the axis of $x$ at the origin and has the axis of $y$ for axis.

13. $PG$ is the normal at $P$; prove that the projection of $PG$ on the focal radius of $P$ is half the latus rectum.

14. Show that the length of the focal chord bisected by the diameter at $P$ is $4SP$.

15. The perpendicular from the focus $S$ to the tangent at $P$ meets it at $Y$: show that $SY^2 = SA \cdot SP$.

16. Find the vertex, axis, focus, and directrix of the following parabolas:—

(i) $(x+2)^2 = 4y + 5$;

(ii) $(y-1)^2 = 2x - 7$;

(iii) $x^2 - 8x + 2y = 0$;

(iv) $(x-1)^2 + (y-2)^2 = \frac{1}{2}(x + y - 1)^2$;

and write down their equations when referred to their principal axes.

17. The tangents at $P$ and $Q$ to a parabola are at right angles: show that $PQ$ passes through the focus.

18. The tangent at $P$ meets the directrix at $R$: show that the angle $RSP$ is a right angle.

19. Express the coordinates of a point on a parabola in terms of the angle which the normal at the point makes with the axis.

20. Find the equation of the directrix of the parabola $y^2 = 4ax \operatorname{cosec}^2 \omega$.

21. Find the coordinates of the points of intersection of the parabola $y^2 = 4x$ with the straight line $3y - 4x + 4 = 0$. Find the equations of the tangents at these points, and show that they are perpendicular and meet on the directrix.

22. Tangents are drawn to the circle $x^2 + y^2 = a^2$ from two points on the axis of $x$ equidistant from the point $(c, 0)$. Show that the locus of their intersections is the parabola $cy^2 = a^2(c - x)$.

23. Show that the normals to the parabola $y^2 = 4ax$ at its points of intersection with the line $2x - 3y + 4a = 0$ intersect on the parabola.

24. Prove that the distance between a tangent to a parabola and the parallel normal is $a \operatorname{cosec} \phi \sec^2 \phi$, where $\phi$ is the angle which either makes with the axis.

25. Find the focus and directrix of the parabola $(y - 2x)^2 = 5x + 1$, and draw the curve.

26. Two points are taken on the parabola $y^2 = 4ax$, on the same side of the axis, such that the product of their distances from the axis is $4a^2$. Show that the tangents at these points (i) intersect on the latus rectum; (ii) intercept on the directrix a segment whose length is the difference of their distances from the axis.

27. Obtain the conditions that the straight line $lx + my + n = 0$ should be (i) a tangent; (ii) a normal to the parabola $y^2 = 4ax$. Find the locus of the middle points of the portions of (i) a tangent; (ii) a normal intercepted between the point of contact and the axis.

28. Find the equation to the parabola whose vertex is the point (1, 2) and directrix the straight line $3x - 4y + 10 = 0$.

Determine its focus and latus rectum.

29. For the parabola $y^2 = 4ax$ show that the middle points of all chords parallel to $3x + 4y - 2 = 0$ lie on the straight line $3y + 8a = 0$; and that tangents at the extremities of any one of these chords intersect each other on that diameter.

30. A point $P$ is such that the line drawn through it perpendicular to its polar with respect to the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$. Show that $P$ lies on the line $2ax + by + 4a^2 = 0$.

31. If the normal at a point $P$, on the parabola $y^2 = 8x$, whose abscissa is 18, cuts the parabola again at $Q$, show that $9PQ = 80\sqrt{10}$.

32. A tangent is drawn to a parabola of latus rectum $4a$ and makes an angle $\phi$ with the axis; prove that the sum of the radii of the two circles which pass through the focus and touch the tangent and the corresponding normal is $2a \operatorname{cosec} \phi (1 + \cot \phi)$.

33. The tangent to a parabola at $P$ meets the tangent at the vertex in $T$, the normal at $P$ meets the axis in $G$, and the diameter through $T$ meets the curve in $Q$. Prove that $TG^2 = 4SP \cdot SQ$.

34. Through the vertex $A$ of the parabola $y^2 = 4ax$ two chords $AP$, $AQ$ are drawn and the circles on $AP$, $AQ$ as diameters intersect in $R$. Prove that if $\theta_1$, $\theta_2$, and $\phi$ be the angles made with the axis by the tangents at $P$ and $Q$ and by $AR$, then $\cot \theta_1 + \cot \theta_2 + 2 \tan \phi = 0$.

§ 4. **Parametric coordinates.**

(i) When the equation of the parabola is in its simplest form, $y^2 = 4ax$, the coordinates of any point on it can be expressed in the form $(at^2, 2at)$: for since $t$ can have any value positive or negative, if the ordinate of any point on the parabola is $y'$, we can find $t$ so that $t = y'/2a$; and since $y'^2 = 4ax'$, we have by substitution $x' = at^2$. Hence the coordinates of any point on the parabola $y^2 = 4ax$ can be expressed in the form $(at^2, 2at)$, and conversely it is manifest by substitution that any point whose coordinates are of this form lies on the parabola. The quantity $t$ is called **the parameter of the point.**

(ii) When the curve is referred to the axis and latus rectum as coordinate axes, $y^2 = 4a(x+a)$, any point on the curve can be represented by $(at^2-a, 2at)$ for some value of $t$.

(iii) When the curve is referred to a diameter and the tangent at its extremity as coordinate axes, we have $y^2 = 4\alpha x$ where

$$\alpha = a \operatorname{cosec}^2 \omega.$$

We can then use $(\alpha t^2, 2\alpha t)$ to denote any point on the curve.

The following results apply equally to this case except when we assume the axes to be rectangular, e. g. when we are using the condition that two lines should be perpendicular or the condition that an equation should represent a circle.

§ 5. I. *To find the parameters of the points of intersection of any straight line and the parabola* $y^2 = 4ax$.

Let the straight line be $lx + my + 1 = 0$.

If any point $(at^2, 2at)$ of the parabola lies on this we have

$$lat^2 + 2mat + 1 = 0. \qquad \text{(i)}$$

This equation is quadratic in $t$ and gives the two values of the parameters of the points where the straight line cuts the parabola.

**Cor. i.** The points of intersection are real and distinct, coincident, or imaginary according as the roots of the equation (i) are real, coincident, or imaginary, i.e. as $am^2$ is $>$, $=$, or $<$ $l$.

In particular the line touches the parabola when $am^2 = l$.

**Cor. ii.** Let $t_1$, $t_2$ be the roots of the quadratic (i), then

$$t_1 + t_2 = -\frac{2m}{l}, \quad t_1 t_2 = \frac{1}{al},$$

thus $$\frac{1}{at_1 t_2} \quad \text{and} \quad m = -\frac{t_1 + t_2}{2at_1 t_2}.$$

If we substitute these values of $l$ and $m$ in the equation of the straight

line, we find that the equation of a chord of the parabola joining the points whose parameters are $t_1$, $t_2$ is

$$y(t_1+t_2)-2x=2at_1t_2 \qquad \text{(ii)}$$

**Cor. iii.** The direction of the chord joining the two points whose parameters are $t_1$, $t_2$ depends only on the value of $(t_1+t_2)$.

When the axes are rectangular the angle $\psi$ which this chord makes with the $x$-axis is given by $\cot\psi=\frac{1}{2}(t_1+t_2)$.

Consequently, if the direction of a chord is constant, the sum of the parameters of its extremities is constant, or since the ordinates are $2at_1$ and $2at_2$, the sum of the ordinates of its extremities is constant. Hence the ordinate of its mid-point is constant: this gives us another proof of the property that the locus of the mid-points of parallel chords is a line parallel to the axis.

**Cor. iv.** The focus is the point $(a, 0)$; hence the chord joining the two points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, viz. $y(t_1+t_2)-2x=2at_1t_2$ passes through the focus if $-2a=2at_1t_2$, or $t_1t_2=-1$.

Hence, if $t_1$ is the parameter of one end of a focal chord, $-\frac{1}{t_1}$ is the parameter of the other end.

II. *The length of a chord in terms of the parameters of its extremities.*

$$\text{Length}=\sqrt{\{(at_1^2-at_2^2)^2+(2at_1-2at_2)^2\}}$$
$$=a(t_1-t_2)\sqrt{\{(t_1+t_2)^2+4\}}.$$

Thus the lengths of parallel chords ($t_1+t_2=$ constant) are proportional to the difference of the parameters (or ordinates) of their extremities. Also for a focal chord, since $t_1t_2=-1$, we have

$$\text{length}=a(t_1-t_2)^2.$$

**Example.** *To find the locus of the mid-points of chords of constant length.*

Let the constant length be $c$ and let $t_1$, $t_2$ be the parameters of the ends of any one of the chords.

The coordinates of its mid-point are given by

$$2x=a(t_1^2+t_2^2), \qquad y=a(t_1+t_2),$$

and by hypothesis $\qquad c^2=a^2(t_1-t_2)^2\{(t_1+t_2)^2+4\}$.

Now $(t_1-t_2)^2=2(t_1^2+t_2^2)-(t_1+t_2)^2$

$$=\frac{4x}{a}-\frac{y^2}{a^2}.$$

Hence $c^2=(4ax-y^2)\left\{\frac{y^2}{a^2}+4\right\}$, and the required locus is

$$(4ax-y^2)(y^2+4a^2)=a^2c^2.$$

III. *To find the equation of the tangent to the parabola $y^2 = 4ax$ at a point whose parameter is $t$.*

The equation of a chord joining the points whose parameters are $t_1$, $t_2$ is $y(t_1+t_2)-2x = 2at_1t_2$. If the points $t_1$, $t_2$ are coincident with $t$ (i.e. $t_1 = t_2 = t$) we get for the equation of the tangent

$$ty - x = at^2,$$

or

$$at^2 - ty + x = 0.$$

**Cor. i.** Since the parameter $t$ can have any value, it follows that any straight line whose equation is of the form $x - ty + at^2 = 0$ is a tangent to the parabola. (Chap. II, § 12.)

The tangent to the parabola which is parallel to $Ax + By + C = 0$ or $x - \left(-\frac{B}{A}\right)y + \frac{C}{A} = 0$ is $x - \left(-\frac{B}{A}\right)y + \frac{aB^2}{A^2} = 0$, the parameter of whose point of contact is $(-B/A)$.

This result is often useful when the middle point of a chord is required; for the diameter through the point of contact of the parallel tangent bisects the chord: thus the chord whose equation is $Ax + By + C = 0$ is bisected by the diameter $y = -2aB/A$.

**Cor. ii.** The geometrical meaning of the parameter $t$ follows from the form of the equation of a tangent; the tangent at the point $(at^2, 2at)$ makes an angle $\phi$ with the axis of $y$ such that $t = \tan\phi$.

Evidently $\phi$ is also the angle which the normal at $(at^2, 2at)$ makes with the axis of $x$.

It is often convenient to take $\tan\phi$ as the parameter of a point; the coordinates of this point are then $(a\tan^2\phi, 2a\tan\phi)$.

The equation of the tangent at this point can be put in the form

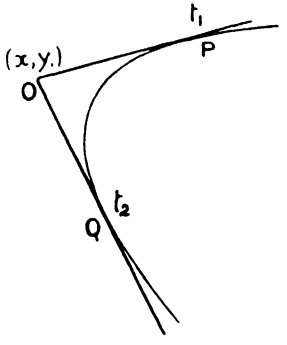
$$\frac{x - a\tan^2\phi}{\sin\phi} = \frac{y - 2a\tan\phi}{\cos\phi} = r.$$

IV. Since the equation of the tangent at the point $(at^2, 2at)$ is

$$at^2 - ty + x = 0,$$

the condition that this tangent should pass through some specified point $(x_1, y_1)$ is

$$at^2 - ty_1 + x_1 = 0. \qquad \text{(i)}$$




Conversely, then, the parameters of the points of contact of tangents from the point $(x_1, y_1)$ to the parabola are given by this equation. If $t_1$, $t_2$ are the roots of equation (i), then the tangents at the points whose parameters are $t_1$, $t_2$ pass through the point $(x_1, y_1)$.

**Cor. i.** Since the equation (i) is quadratic, it furnishes a proof that two tangents can be drawn to a parabola from any point: these tangents are real, coincident,

or imaginary according as $y_1^2 - 4ax_1$ is positive, zero, or negative, i.e. as $(x_1, y_1)$ lies outside, on, or inside the curve.

**Cor. ii.** If $t_1, t_2$ are the roots of equation (i) we have

$$t_1 + t_2 = \frac{y_1}{a}, \quad t_1 t_2 = \frac{x_1}{a};$$

i.e. the point $(x_1, y_1)$ is $\{at_1 t_2, a(t_1 + t_2)\}$.

This gives the coordinates of the point of intersection of tangents at the points $t_1, t_2$; they can also clearly be found by solving the equations of the tangents at these points.

It follows also that $(x_1, y_1)$ lies on the diameter bisecting the chord joining $(t_1, t_2)$.

Incidentally, if the points $t_1, t_2$ are the extremities of a focal chord we have shown that $t_1 t_2 = -1$. Hence, tangents at the extremities of a focal chord are at right angles, and intersect on the directrix $x + a = 0$.

V. *To find the lengths of the tangents from any point $O(x_1, y_1)$ to the parabola $y^2 = 4ax$.*

A straight line through the point $O(x_1, y_1)$, making an angle $\theta$ with the axis of $x$, is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r. \tag{i}$$

Let this straight line meet the parabola in the points $P, P'$; then, if $r$ has the value $OP$ (or $OP'$), the point $P$ (or $P'$) has coordinates

$$\{r\cos\theta + x_1, \ r\sin\theta + y_1\};$$

the condition that this should lie on the parabola gives

$$(r\sin\theta + y_1)^2 = 4a(r\cos\theta + x_1),$$

which is a quadratic equation whose roots are $OP, OP'$.

If we write $u_1 \equiv y_1^2 - 4ax_1$, this equation becomes

$$r^2\sin^2\theta + 2r(y_1\sin\theta - 2a\cos\theta) + u_1 = 0 \tag{ii}$$

Now if the line (i) is a tangent to the parabola, $P$ and $P'$ coincide, and the roots of this equation are equal.

In this case

$$(y_1\sin\theta - 2a\cos\theta)^2 = \sin^2\theta(y_1^2 - 4ax_1),$$

which reduces to

$$a\cot^2\theta - y_1\cot\theta + x_1 = 0. \tag{iii}$$

This equation is quadratic and gives two values of $\theta$, viz. those corresponding to the directions of the two tangents $OP, OQ$, which can be drawn from $O$ to the parabola.

If $\theta$ has either of these two values, then equation (ii) will have two roots each equal to the length of the corresponding tangent from $O$.

Thus, if $l$ is the length of the tangent, drawn from $O$ to the parabola, whose direction is $\theta$, we have

$$l^2 = \frac{u_1}{\sin^2\theta} = u_1(1+\cot^2\theta), \text{ i.e. } \cot^2\theta = \frac{l^2-u_1}{u_1}.$$

But the values of $\cot\theta$ are given by

$$a\cot^2\theta - y_1\cot\theta + x_1 = 0.$$

Hence, eliminating $\cot\theta$, we get

$$a^2l^4 - u_1(y_1^2 - 2ax_1 + 2a^2)l^2 + u_1^2\{(x_1-a)^2 + y_1^2\} = 0,$$

which is a quadratic equation in $l^2$ giving the squares of the lengths of the tangents which can be drawn from the point $(x_1, y_1)$ to the parabola.

Since the roots of this equation are $OP^2$, $OQ^2$, we have

$$a^2(OP^2 + OQ^2) = u_1(y_1^2 - 2ax_1 + 2a^2),$$

and

$$a^2 . OP^2 . OQ^2 = u_1^2\{(x_1-a)^2 + y_1^2\}:$$

the latter result can also be written

$$a^2 . OP^2 . OQ^2 = u_1^2 . OS^2$$

or

$$a . OP . OQ = u_1 . OS.$$

**Example** i. *Two tangents $TP$, $TP'$ to a parabola meet the tangent at the vertex in $Q$, $Q'$. Prove that the radius of the circle $TQQ'$ is $\frac{1}{8}f_1^{\frac{1}{2}}f_2^{\frac{1}{2}}$, where $f_1$, $f_2$ are the focal chords parallel to $TP$, $TP'$.*

Let $P$ be the point whose parameter is $t_1$, $P'$ the point $t_2$.

Since the tangent at $P$ is $t_1y - x - at_1^2 = 0$, the parallel focal chord is $t_1y - x + a = 0$.

Therefore if $(a\lambda^2, 2a\lambda)$ is either extremity of the focal chord we have by substitution $\lambda^2 - 2t_1\lambda - 1 = 0$, which gives the values $\lambda_1, \lambda_2$ of the parameters of the ends of the chord.

Hence $\lambda_1 + \lambda_2 = 2t_1, \lambda_1\lambda_2 = -1$ and

$$\text{Length of focal chord} = a(\lambda_1 - \lambda_2)^2 = 4a(1+t_1^2).$$

Thus $f_1 = 4a(1+t_1^2)$; and $f_2 = 4a(1+t_2^2)$.

Now the intersection $Q$ of the tangent at $P$ with the tangent at the vertex $x = 0$ is $(0, at_1)$, and $Q'$ is $(0, at_2)$.

If $\theta$ is the angle between the tangents at the points $P$, $P'$,

$$\tan\theta = \frac{t_1 - t_2}{1 + t_1t_2};$$

$$\therefore \quad \sin\theta = \frac{t_1 - t_2}{\sqrt{(1+t_1^2)(1+t_2^2)}} = \frac{QQ'}{a\sqrt{(1+t_1^2)(1+t_2^2)}}.$$

But the radius of the circle $TQQ'$

$$= \frac{QQ'}{2\sin\theta} = \frac{a\sqrt{(1+t_1^2)(1+t_2^2)}}{2} = \frac{f_1^{\frac{1}{2}} . f_2^{\frac{1}{2}}}{8}.$$

**Example** ii. *Show that an infinite number of triangles can be inscribed in the conic* $8y^2-8ax-12\beta y+9\beta^2-12a\alpha=0$, *and circumscribed to the parabola* $y^2=4ax$.

*Also that, if P, Q, R are the points of contact of any such triangle, the centroid of PQR is* $(\alpha, \beta)$.

It is evident that a triangle can be circumscribed to the parabola $y^2=4ax$ with two of its vertices on the conic

$$8y^2-8ax-12\beta y+9\beta^2-12a\alpha=0.$$

Suppose that the vertices of such a triangle are

$$\{a\mu\nu,\ a(\mu+\nu)\},\ \{a\nu\lambda,\ a(\nu+\lambda)\},\ \{a\lambda\mu,\ a(\lambda+\mu)\}.$$

Let $\lambda+\mu+\nu=s$ and $\lambda\mu\nu=p$; then the point $\left\{\frac{ap}{t},\ a(s-t)\right\}$ lies on the conic when $t$ is equal to $\lambda$ or $\mu$. Hence

$$8a^2(s-t)^2-\frac{8a^2p}{t}-12a\beta(s-t)+9\beta^2-12a\alpha=0,$$

or $$8a^2t^3+4a(3\beta-4as)t^2+(8a^2s^2+9\beta^2-12a\alpha-12a\beta s)t-8a^2p=0. \quad \text{(i)}$$

Two of the roots of this cubic are therefore $\lambda$ and $\mu$; the product of the three roots is $p$ (i.e. $\lambda\mu\nu$), hence the third root of the cubic is $\nu$. This shows that, if a triangle circumscribes $y^2=4ax$, and has two vertices on the given conic, then the third vertex also lies on the conic.

Any number of such triangles can be drawn; for if any tangent to the parabola cuts the conic at $A$ and $B$, and tangents from $A$ and $B$ to the parabola intersect in $C$, then we have shown that $C$ also lies on the conic.

The points of contact, $P$, $Q$, $R$, of the sides are $(a\lambda^2,\ 2a\lambda)$, $(a\mu^2,\ 2a\mu)$, $(a\nu^2,\ 2a\nu)$. The centroid of the triangle $PQR$ is

$$\{\tfrac{1}{3}a(\lambda^2+\mu^2+\nu^2),\ \tfrac{2}{3}a(\lambda+\mu+\nu)\}.$$

Since $\lambda$, $\mu$, $\nu$ are the roots of the cubic (i),

$$s=\lambda+\mu+\nu=2s-\frac{3\beta}{2a}; \qquad \therefore\quad s=\frac{3\beta}{2a};$$

$$\mu\nu+\nu\lambda+\lambda\mu=\frac{8a^2s^2+9\beta^2-12a\alpha-12a\beta s}{8a^2}$$

$$=\frac{9\beta^2-12a\alpha}{8a^2}.$$

Hence $$\lambda^2+\mu^2+\nu^2=(\lambda+\mu+\nu)^2-2(\mu\nu+\nu\lambda+\lambda\mu)$$

$$=\frac{3\alpha}{a}.$$

The coordinates of the centroid are therefore $(\alpha, \beta)$.

VI. When the focus is the origin, the equation of the parabola is $y^2=4a(x+a)$.

It can be shown, by a method similar to that used in I and II, that the equation of the chord joining the points $\{at_1^2-a,\ 2at_1\}$, $\{at_2^2-a,\ 2at_2\}$ is $y(t_1+t_2)=2(x+a)+2at_1t_2$, and that the equation of the tangent at the point $(at^2-a,\ 2at)$ is $yt=x+a+at^2$.

A modification is illustrated in the following example.

**Example.** *Two parabolas of the systems* $y^2 = 4a(x - l_1)$, $x^2 = 4a(y - l_2)$, *where* $l_1$ *and* $l_2$ *are variable, touch one another; find the locus of their point of contact.*

Let the parameters of the point of contact $(x, y)$ for the two parabolas be $t$ and $s$.

Then the coordinates of this point are

$$\{l_1 + at^2, 2at\} \text{ or } \{2as.\ l_2 + as^2\}.$$

Hence $$x = l_1 + at^2 = 2as\,;\ y = 2at = l_2 + as^2.$$

The equation of the tangents at the point to the two parabolas are

$$ty - x + l_1 - at^2 = 0,$$
$$sx - y + l_2 - as^2 = 0\,;$$

and, since these are identical, $ts = 1$.

Hence $xy = 2as \,.\, 2at = 4a^2st = 4a^2$, i.e. the locus of the point of contact $(x, y)$ is $xy = 4a^2$.

## Examples VII b.

1. In the parabola $y^2 = 6x$, chords are drawn through the fixed point $(9, 5)$. Show that the locus of the mid-points of these chords is the parabola $y^2 - 5y - 3x + 27 = 0$.

2. Show that the locus of the middle point of a chord of a parabola which subtends a right angle at the vertex is another parabola of half the latus rectum.

3. Show that the angle between any two tangents to a parabola is $\cos^{-1}(r_2/r_1)$ where $r_1$, $r_2$ are the respective distances of their point of intersection from the focus and directrix.

4. From the point $(\alpha, \beta)$ two tangents are drawn to the parabola $y^2 = 4ax$: show that the square of the area of the triangle formed by these tangents and their chord of contact is $(\beta^2 - 4a\alpha)^3/4a^2$.

5. If the focus is taken for origin, show that the equation of a tangent to the parabola can be thrown into the form

$$x\cos\alpha + y\sin\alpha + a\sec\alpha = 0.$$

Two tangents, $t_1$ and $t_2$, are drawn to a parabola; $h$ is the internal bisector of the angle between them and $t$ the tangent parallel to $h$. Show that the product of the perpendiculars from the focus to $t_1$ and $t_2$ is the same as that of those drawn to $t$ and $h$.

6. Show that the envelope of the chords of the parabola $y^2 = 4ax$ which subtend an angle of 45° at the vertex is

$$x^2 + 8y^2 - 24ax + 16a^2 = 0.$$

7. The tangents at $P$, $Q$, $R$ of a parabola intersect at the points $P'$, $Q'$, $R'$: find the ratio of the areas of the triangles $PQR$ and $P'Q'R'$.

8. The locus of the middle points of all chords of a parabola which pass through a fixed point is another parabola.

9. Tangents to a parabola cut off a length on a fixed tangent which subtends a right angle at the vertex: show that their intersections lie on a fixed straight line.

10. Two tangents to a parabola make angles $\phi_1$, $\phi_2$ with the axis: prove that their lengths measured from the points of contact to their point of intersection are $a\sin(\phi_1 \sim \phi_2)/\sin^2\phi_1 \sin\phi_2$; $a\sin(\phi_1 \sim \phi_2)/\sin\phi_1 \sin^2\phi_2$.

11. Two tangents to the parabola $y^2 = 4ax$ make angles $\theta$, $\phi$ respectively with the axis of $y$. Prove that the equation of their chord of contact is

$$y\sin(\theta+\phi) = (x+a)\cos(\theta-\phi) + (x-a)\cos(\theta+\phi).$$

The envelope of chords of a parabola, the tangents at the ends of which include a constant angle, is in general an ellipse.

What are the exceptions?

12. The envelope of the chords of the parabola whose mid-points lie on $x = my+c$ is $(y+2am)^2 = 8a(x-c)$.

13. The locus of a point, the tangents from which to the parabola $y^2 = 4ax$ make equal angles with $y = x\cot\theta + c$, is $y = (a-x)\tan 2\theta$.

14. Show that the ratio $\lambda$ of the lengths of the tangents drawn from any point on the latus rectum produced to a parabola is given by $a\lambda^2 - y\lambda + a = 0$, where $y$ is the ordinate of the point.

15. Tangents are drawn to the parabola $y^2 = 4ax$ at points whose abscissae are in the ratio $\mu : 1$. Show that the locus of their intersection is the parabola $y^2 = (\mu^{\frac{1}{2}} + \mu^{-\frac{1}{2}})^2 ax$.

16. A triangle circumscribes $y^2 = 4ax$ and two of its vertices lie on $y^2 = 4a(x+l)$; find the locus of the other vertex.

17. $TP$, $TQ$ are tangents to a parabola and $O$ is the orthocentre of the triangle $TPQ$. Prove that $OT$ is bisected by the directrix of the parabola.

18. An equilateral triangle circumscribes a parabola: show that the join of the focus to each vertex passes through the point of contact of the opposite side.

## § 6. The equation of the normal.

The equation of the tangent at the point $(at^2, 2at)$ is

$$ty - x - at^2 = 0.$$

Since the normal is the perpendicular to this line through the point $(at^2, 2at)$, its equation is

$$t(x-at^2) + y - 2at = 0,$$

i. e. $$\mathbf{y + tx - 2at - at^3 = 0} \qquad \text{(i)}$$

or $$\mathbf{at^3 + t(2a-x) - y = 0.} \qquad \text{(ii)}$$

Similarly, the equation to the normal at $(a\tan^2\phi, 2a\tan\phi)$ can be written

$$\frac{x - a\tan^2\phi}{\cos\phi} = \frac{y - 2a\tan\phi}{-\sin\phi} = r. \qquad \text{(iii)}$$

**Note i.** The condition that the normal at the point $(t)$ should pass through a particular point $(x_1, y_1)$ is (see equation (ii) above)

$$at^3 + t(2a - x_1) - y_1 = 0,$$

and conversely this equation gives the parameters of the feet of the normals which pass through $(x_1, y_1)$. Now, since this equation is a cubic in $t$, it

follows that from any point *three* normals can be drawn to a parabola; these may be all real or one real and two imaginary.

**Note ii.** *The condition that three normals should be concurrent.*

If the normals at three points whose parameters are $t_1, t_2, t_3$ are concurrent at the point $(x_1, y_1)$, then $t_1, t_2, t_3$ all satisfy the equation

$$at^3 + t(2a - x_1) - y_1 = 0,$$

i.e. $t_1, t_2, t_3$ are the roots of this equation.

We have then

$$t_1 + t_2 + t_3 = 0,$$

$$t_1 t_2 + t_2 t_3 + t_3 t_1 = \frac{2a - x_1}{a},$$

$$t_1 t_2 t_3 = \frac{y_1}{a}.$$

Hence $t_1 + t_2 + t_3 = 0$ is the necessary and sufficient condition that the normals at the points $t_1, t_2, t_3$ should be concurrent: the remaining two conditions give the values of the coordinates of their point of intersection in terms of these parameters provided that the above condition is satisfied.

Now if $t_1 + t_2 + t_3 = 0,$

also $2at_1 + 2at_2 + 2at_3 = 0,$

hence if the normals at three points on a parabola are concurrent the sum of the ordinates of these points is zero, and conversely.

**Note iii.** *To find the condition that the normals at two points should intersect on the parabola.*

Suppose the normals at two points whose parameters are $t_1, t_2$ intersect at a point $P(x_1, y_1)$ on the curve whose parameter is $\lambda$.

Put $x_1 = a\lambda^2$, $y_1 = 2a\lambda$; the parameters of the feet of the three normals meeting at $P$ are given by

$$at^3 + t(2a - a\lambda^2) - 2a\lambda = 0,$$

or

$$t^3 + t(2 - \lambda^2) - 2\lambda = 0.$$

Evidently one of these three normals is the normal at the point $P$ itself: the equation may be written $(t - \lambda)(t^2 + t\lambda + 2) = 0$, so that $t_1, t_2$ are given by $t^2 + t\lambda + 2 = 0$, and the required condition is $t_1 t_2 = 2$; thus the product of the ordinates of the two points is $8a^2$ and, further, the chord joining them passes through the point $(-2a, 0)$.

Further, the values of $t_1, t_2$ are given by

$$t = \tfrac{1}{2}(-\lambda \pm \sqrt{\lambda^2 - 8}),$$

and consequently the two normals which can be drawn from a point $(a\lambda^2, 2a\lambda)$ on the parabola other than the normal at this point are real, coincident, or imaginary according as $\lambda^2$ is $>$, $=$, or $< 8$, i.e. according as the abscissa of the point is $>$, $=$, or $< 8a$.

**Note iv.** *To find the locus of a point, two of the normals from which to a parabola are coincident.*

If the point is $(x_1, y_1)$, the parameters of the feet of the normals drawn from it to the curve are given by

$$at^3 + t(2a - x_1) - y_1 = 0.$$

Now if two of the normals are coincident, two roots of this equation are equal. Let the roots be $t_1, t_1, t_2$; then we have

$$2t_1 + t_2 = 0,$$

or
$$t_2 = -2t_1,$$

i.e. the roots of the equation are $t_1, t_1, -2t_1$.

Hence
$$-3t_1^2 = \frac{2a - x_1}{a} \text{ or } \frac{x_1 - 2a}{a} = 3t_1^2,$$

and
$$-2t_1^3 = \frac{y_1}{a}.$$

Eliminating $t_1$, we obtain

$$27ay_1^2 = 4(x_1 - 2a)^3:$$

i.e. the equation of the locus required is

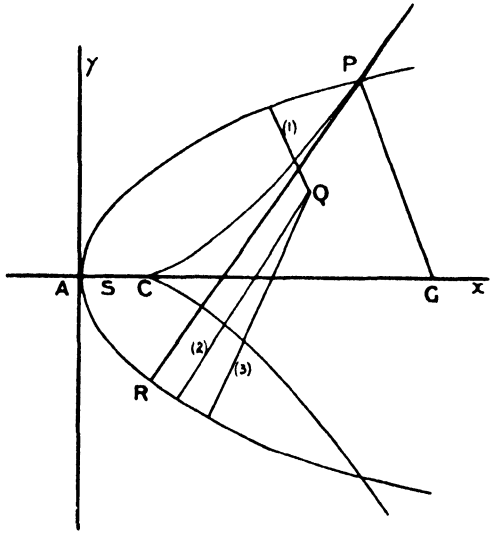
$$27ay^2 = 4(x - 2a)^3.$$

This equation represents the locus of the intersections of consecutive normals and the curve is called the *Evolute* of the parabola.

Now at a point (if any) where the evolute meets the parabola the two normals which can be drawn through it other than the normal at the point itself must be coincident: if this point of intersection is the point $(a\lambda^2, 2a\lambda)$ of the parabola we have shown (Note iii) that the parameters of the feet of the other two normals are given by

$$t^2 + t\lambda + 2 = 0.$$

These are coincident when $\lambda^2 = 8$: hence the points of intersection of the parabola and its evolute are $(8a, 4\sqrt{2}a)$ and $(8a, -4\sqrt{2}a)$: the




student can verify this by substitution in the equation of the evolute. The evolute meets the axis at $(2a, 0)$; its graph is shown in the figure; *PR, PR, PG* are the normals meeting at *P*.

**Cor.** We have seen that the normals meeting at $(x_1, y_1)$ are given by

$$at^3 + t(2a - x_1) - y_1 = 0.$$

Now (*vide* Hall and Knight's *Higher Algebra*, § 579) there is always one real root of this equation; the other two are real, coincident, or imaginary according as

$$27ay_1^2 \text{ is } >,\ =,\ \text{or } <\ 4(x_1 - 2a)^3,$$

i.e. according as the point $(x_1, y_1)$ lies inside, on, or outside the evolute: thus the evolute divides the plane of the parabola into two parts such that from any point (e.g. $Q$) in the one three real normals can be drawn to the parabola, and from any point in the other only one real normal can be drawn.

**Example** i. *A chord of the parabola* $y^2 = 4ax$ *passes through the point* $(\lambda a, 0)$*: prove that the normals at its extremities intersect on the curve* $y^2 = \lambda^2 a(x - \lambda a - 2a)$.

Let the parameters of the extremities of any chord through $(\lambda a, 0)$ be $t_1, t_2$: the equation of the chord is

$$(t_1 + t_2)\, y = 2x + 2at_1t_2;$$

hence $$2\lambda a + 2at_1t_2 = 0,$$

or $$t_1t_2 = -\lambda.$$

Now if the normals at $t_1, t_2, t_3$ meet at $(x, y)$, these parameters are given by $at^3 + t(2a - x) - y = 0$.

Thus $t_1 + t_2 + t_3 = 0$, $t_1t_2 + t_2t_3 + t_3t_1 = \dfrac{2a - x}{a}$, $t_1t_2t_3 = \dfrac{y}{a}$.

Hence, by substituting $t_1 + t_2 = -t_3$ and $t_1t_2 = -\lambda$, we have

$$\frac{x - 2a}{a} = \lambda + t_3^2 \quad \text{and} \quad \frac{y}{a} = -\lambda t_3.$$

Eliminating $t_3$, we get the equation of the locus required, viz.

$$y^2 = \lambda a(x^2 - \lambda a - 2a).$$

**Example ii.** *Normals are drawn to the parabola* $y^2 - 4ax = 0$ *from the point* $(X, Y)$*: show that the equation of the nine-point circle of the triangle formed by their feet is*

$$4(x^2 + y^2) + 2(10a - 3X)x + Yy + 2(2a - X)(6a - X) = 0.$$

Let the feet of the normals be $A$, $B$, $C$ and the parameters of these points be $t_1, t_2, t_3$: the nine-point circle passes through the mid-points of the sides of the triangle $ABC$.

Since the normals at $A$, $B$, $C$ meet at $(X, Y)$, $t_1, t_2, t_3$ are given by

$$at^3 + t(2a - X) - Y = 0,$$

thus $t_1 + t_2 + t_3 = 0$; $t_1t_2 + t_2t_3 + t_3t_1 = \dfrac{2a - X}{a}$; $t_1t_2t_3 = \dfrac{Y}{a}$.

The mid-point of $BC$ is $\{\frac{1}{2}a(t_2^2 + t_3^2),\ a(t_2 + t_3)\}$.

Now $t_2+t_3=-t_1$, and $\therefore\ t_1t_2+t_2t_3+t_3t_1=t_2t_3-t_1^2$,

$$\therefore\ t_2t_3=t_1^2+\frac{2a-X}{a}.$$

But $$t_2^2+t_3^2+2t_2t_3=t_1^2,$$

hence $$t_2^2+t_3^2=\frac{2X-4a}{a}-t_1^2,$$

i.e. the mid-point of $BC$ is

$$\{X-2a-\tfrac{1}{2}at_1^2;\ -at_1\},$$

and the coordinates of the mid-points of $CA$, $AB$ can be found symmetrically by writing $t_2$, $t_3$ for $t_1$ respectively.

Now let the nine-point circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Since each of the mid-points lies on this, $t_1$, $t_2$, and $t_3$ satisfy the equation

$$\{X-2a-\tfrac{1}{2}at^2\}^2+a^2t^2+2g\{X-2a-\tfrac{1}{2}at^2\}-2aft+c=0,$$

which reduces to

$$a^2t^4+4t^2\{3a^2-aX-ag\}-8aft+4(X-2a)^2+8g(X-2a)+4c=0.$$

Since the coefficient of $t^3$ is zero, the sum of the four roots of this equation is zero; but the sum of the three roots $t_1$, $t_2$, $t_3$ is zero, therefore the fourth root must be zero.

Hence $$(X-2a)^2+2g(X-2a)+c=0, \qquad \text{(i)}$$

and the equation reduces to

$$at^3+4\{3a-X-g\}t-8f=0,$$

which, having roots $t_1$, $t_2$, $t_3$, is identical with

$$at^3+(2a-X)t-Y=0.$$

Hence $$12a-4X-4g=2a-X,$$

$$4g=10a-3X,$$

and $$8f=Y.$$

But, substituting for $g$ in (i), we have

$$\begin{aligned} c&=-(X-2a)(X-2a+2g)\\ &=-(X-2a)(X-2a+5a-\tfrac{3}{2}X)\\ &=\tfrac{1}{2}(X-2a)(X-6a). \end{aligned}$$

Hence, substituting in the equation of the circle for $g$, $f$, and $c$, we get for the nine-point circle

$$4(x^2+y^2)+2(10a-3X)x+Yy+2(X-2a)(X-6a)=0.$$

**§ 7. Relations between the coordinates of the points of intersection of the tangents and normals at any two points on a parabola.**

Let $(x, y)$ be the point of intersection of the tangents at two points and $(\xi, \eta)$ that of the normals.

The parameters of the points of contact of the tangents which meet at $(x, y)$ are given by

$$at^2-yt+x=0, \qquad \text{(i)}$$

and the parameters of the feet of the normals which can be drawn from $(\xi, \eta)$ are given by

$$at^3 + (2a - \xi)t - \eta = 0. \qquad \text{(ii)}$$

Hence two of the roots of equation (ii) are the same as those of equation (i). These two roots therefore also satisfy

$$at^3 + (2a - \xi)t - \eta - t\{at^2 - yt + x\} = 0,$$

i. e. $$yt^2 - (x + \xi - 2a)t - \eta = 0;$$

this equation is therefore identical with (i), and we have

$$\frac{y}{a} = \frac{x + \xi - 2a}{y} = -\frac{\eta}{x}. \qquad \text{(iii)}$$

Suppose now that the points of intersection of two tangents under certain conditions lie on some locus $f(x, y) = 0$, the corresponding locus of the intersections of the normals at their points of contact will be $\phi(\xi, \eta) = 0$, obtained by eliminating $x$ and $y$ from $f(x, y) = 0$ and the equations (iii).

The converse proposition can be stated more directly; thus we have from (iii)

$$\xi = \frac{y^2}{a} - x + 2a; \quad \eta = -\frac{xy}{a},$$

and if the locus of the intersections of the normals at the ends of a chord moving under given conditions is $f(\xi, \eta) = 0$, the equation of the locus of the intersections of tangents at the ends of the chord is

$$f\left\{\frac{y^2}{a} - x + 2a, \; -\frac{xy}{a}\right\} = 0.$$

**Example.** *To find the locus of the intersection of the normals at the ends of a focal chord.*

We know that the locus of the intersection of the tangents at the ends of a focal chord is the directrix $x + a = 0$.

The locus of the intersections of the normals is obtained by eliminating $x$, $y$ from this equation and

$$\frac{y}{a} = \frac{x + \xi - 2a}{y} = -\frac{\eta}{x}.$$

Hence $$\frac{y}{a} = \frac{\xi - 3a}{y} = \frac{\eta}{a};$$

$\therefore$ $y = \eta$, and

$$\eta^2 = a(\xi - 3a);$$

or the required locus is $$y^2 = a(x - 3a).$$

### Examples VII c.

1. Find the equation of the normal at the point $(-a + at^2, 2at)$ of the parabola $y^2 = 4a(x + a)$.

2. Find the parameter of the point where the normal at $(at^2, 2at)$ meets the parabola $y^2 = 4ax$ again.

3. Find the equation of the normal at the point $(dt^2, 2dt)$ on the parabola $y^2 = 4dx$, where $d \sin^2\omega = a$ (oblique axes).

4. Show that the normals to a parabola at the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ intersect at the point $\{2a + a(t_1^2 + t_1t_2 + t_2^2),\ -at_1t_2(t_1 + t_2)\}$.

5. Normals at pairs of points on the parabola $y^2 = 4ax$ meet on the line $x = h$. Find the locus of the intersections of tangents at these pairs of points. Also when the normals meet on $y = k$.

6. Tangents are drawn to a parabola from points on a line parallel to the axis; prove that the normals at their points of contact intersect on a fixed straight line.

7. Show that, if a variable chord of the parabola $y^2 = 4x$ touches the parabola $y^2 = x$, the tangents at its extremities meet on the parabola $y^2 = 16x$, and find the locus of the meets of the normals at its extremities.

8. The normal at any point $P$ of a parabola cuts the axis in $G$ and meets the curve again in $Q$. If the normal makes an angle $\theta$ with the axis, prove that $GQ \sin^2\theta = 2AG \cos\theta$.

9. Show that the equation of any normal to the parabola $y^2 = 4b(x + c)$ may be written in the form $y + mx = (2b - c)m + bm^3$.

10. Normals are drawn from the point $(am^2, 2am)$ to the parabola $y^2 = 4ax$. Show that the feet $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ of these normals are given by $t^2 + mt + 2 = 0$. Show also that the product of their lengths is $4a^2(1 + m^2)^{\frac{3}{2}}$.

11. Show that the middle points of the sides of a triangle formed by tangents at $P$, $Q$, $R$ to the parabola $y^2 = 4ax$ lie on the parabola $2y^2 + ax = 0$ if the normals at $P$, $Q$, $R$ are concurrent.

12. If chords of the parabola $y^2 = 4ax$ pass through the foot of the directrix, show that normals at their extremities meet on $y^2 = a(x - a)$.

13. If the chord $PQ$ of $y^2 = 4ax$ passes through $(-2a, 0)$, the normals at $P$, $Q$ meet on the curve and contain an angle equal to $PAQ$.

14. If the normals at $P_1$, $P_2$, $P_3$ are concurrent, the centroid of the triangle $P_1P_2P_3$ lies on the axis.

If $P_1$, $P_2$ coincide at $(x', y')$ the equation of $P_1P_3$ is $x/x' + y/y' = 2$.

15. If the normals corresponding to the tangents drawn from $T(h, k)$ meet at $N$, then $SN^2 : ST^2 = a^2 + k^2 : a^2$.

16. If the tangents at $P$, $Q$ meet at $(x_1, y_1)$ and the normals at $(x_2, y_2)$, then $x_2 - x_1 = SP + SQ$ and $ay_2 = -x_1y_1$.

17. A triangle is inscribed in a parabola and the normals at its vertices are concurrent: show that the perpendiculars to its sides at the points where they meet the axis intersect on the tangent at the vertex.

18. $TP$, $TQ$ are two tangents to a parabola; $PN$, $QN$ are the corresponding normals; $M$ is the mid-point of $TN$.

Prove that $TM$ subtends a right angle at the focus.

19. If $P$, $Q$, $R$ are the feet of the three normals from a point on the line $x = 2a + c$, the intersections of the tangents at $P$, $Q$, $R$ lie on $y^2 = a(x + c)$.

20. If a tangent to $y^2 = 4a(x + a)$ meets a normal to $y^2 = 4b(x + b)$ at right angles, the locus of their intersection is a parabola.

21. $A$, $B$, $C$ are the feet of the normals which meet at $(\alpha, \beta)$; prove that

the straight lines bisecting $BC$, $CA$, $AB$ at right angles are normals to the parabola $y^2 = 8a(4a + \alpha - x)$ at the points whose ordinates are equal to those of $A$, $B$, $C$.

22. The normals at $P$, $Q$, the ends of a focal chord, meet the curve again at $P'$, $Q'$. Show that $P'Q'$ is parallel and equal to $3PQ$.

Show also that the envelope of $P'Q'$ is a parabola whose latus rectum is eight times that of the given parabola.

25. Find the orthocentre of the triangle formed by the feet of the normals from $(X, Y)$ to the parabola.

24. The three normals from a point $P$ to the parabola $y^2 = 4ax$ and the line through $P$ parallel to the axis form an harmonic pencil: show that $P$ lies on $27ay^2 = 2(x - 2a)^3$.

## § 8. The parabola and the circle.

We propose firstly to discuss the intersection of a circle and a parabola by means of parametric coordinates; in the next section we shall discover the *forms* of the equations of circles and other curves which are variously related to the parabola. Some of the work overlaps; the student will learn by experience which method is the more appropriate for a given problem.

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0. \qquad \text{(i)}$$

If any point $(at^2, 2at)$ on the parabola lies also on this circle, the parameter $t$ of the point must satisfy the equation obtained by substituting $x = at^2$, $y = 2at$ in (i), viz.

$$a^2t^4 + (4a^2 + 2ga)t^2 + 4fat + c = 0. \qquad \text{(ii)}$$

Since the parameter of *any* point common to the circle and the parabola satisfies this equation, it is evident that this equation gives the values of the parameters of all the points of intersection of the circle and the parabola.

**Cor. i.** The equation is a quartic in $t$; hence every circle meets the parabola in four points; these may be all real, two real and two imaginary, or all imaginary.

**Cor. ii.** If the four roots of equation (ii) are $t_1, t_2, t_3, t_4$, since the coefficient of $t^3$ is zero, we have

$$t_1 + t_2 + t_3 + t_4 = 0.$$

Conversely, if the sum of the parameters $(t_1, t_2, t_3, t_4)$ of four points on the parabola is zero, these four points lie on a circle; for if we find $g, f$, and $c$ so that

$$2g + 4a = a\Sigma t_1t_2,$$
$$4f = -a\Sigma t_1t_2t_3,$$
$$c = a^2t_1t_2t_3t_4.$$

then $t_1, t_2, t_3, t_4$ are the roots of the equation (ii); but the condition that $t_1$ should satisfy this equation is also the condition that the point $(at_1^2, 2at_1)$ should lie on the circle $x^2+y^2+2gx+2fy+c=0$.

Thus the necessary and sufficient condition that four points on the parabola should be concyclic is that the sum of the parameters, and therefore the sum of the ordinates, of the points should be zero.

**Cor. iii.** If the four points $ABCD$ whose parameters are $t_1, t_2, t_3, t_4$ are concyclic, we have $t_1+t_2+t_3+t_4=0$.

These four points can be joined in pairs in three ways, $AB, DC$; $AD, BC$; $AC, BD$. The equations of the first pair are

$$y(t_1+t_2)-2x=2at_1t_2$$
$$y(t_3+t_4)-2x=2at_3t_4,$$

But $t_1+t_2=-(t_3+t_4)$, hence these chords are equally inclined to the axis of the parabola. The same is true for the other pairs.

Hence, **The common chords of a circle and a parabola are in pairs equally inclined to the axis.**

**Cor. iv.** When two of the points of intersection coincide (e.g. $C$ and $D$) the circle touches the parabola at $C$. Also $AB$ and the common tangent at $C$, being a pair of common chords, are equally inclined to the axis.

**Cor. v.** Three of the points (e.g. $B, C, D$) may coincide; in this case the circle both cuts and touches the parabola at $B$. The circle is then said to **osculate** the parabola, and is called the **osculating circle** or the **circle of curvature** at the point $B$.

Since $AB$ and the tangent at $B$ are a pair of common chords, they are equally inclined to the axis of the parabola.

The properties of the circle of curvature can be at once deduced from the equation (ii) $a^2t^4+(4a^2+2ga)t^2+4fat+c=0$.

Let $B$ be the point $t_1$ and $A$ the point $t_2$; then the roots of this equation are $t_1, t_1, t_1, t_2$.

Thus (a) $$3t_1+t_2=0,$$

or $$t_2=-3t_1,$$

so that the circle of curvature at the point $(at_1^2, 2at_1)$ meets the parabola again at the point $(9at_1^2, -6at_1)$; the equation of the common chord of the parabola and the circle of curvature at the point $t_1$ is

$$t_1y+x=3at_1^2.$$

(b) The sum of the products of the roots two at a time $=4+\frac{2g}{a}$;

$$\therefore \quad \frac{2g}{a}+4=3t_1^2+3t_1t_2=3t_1^2-9t_1^2=-6t_1^2;$$

$$\therefore \quad -g=3at_1^2+2a.$$

The sum of the products of the roots three at a time $=-\frac{4f}{a}$;

$$\therefore \quad -\frac{4f}{a}=t_1^3+3t_1^2t_2=t_1^3-9t_1^3=-8t_1^3;$$

$$\therefore \quad -f=-2at_1^3.$$

The product of the roots $= \frac{c}{a^2}$;

$$\therefore \quad \frac{c}{a^2} = t_1^3 t_2 = -3t_1^4;$$

$$\therefore \quad c = -3a^2 t_1^4.$$

The centre of the circle, i.e. the centre of curvature, is therefore $(3at_1^2+2a;\ -2at_1^3)$. The radius of curvature $(\rho)$ is given by

$$\rho^2 = f^2+g^2-c = 4a^2(1+t_1^2)^3, \text{ i.e. } \rho = 2a(1+t_1^2)^{\frac{3}{2}}.$$

(c) If the length of the radius of curvature is given, the parameter of the point of contact of the corresponding circle of curvature is given by

$$t_1^2 = \sqrt[3]{\frac{\rho^2}{4a^2}} - 1.$$

There is only one real cube root of $\frac{\rho^2}{4a^2}$, hence there is only one real value of $t_1^2$. This value is positive provided that $\rho > 2a$, in which case $t_1$ has two equal and opposite values.

Thus the minimum length of the radius of curvature is $2a$.

Two circles of curvature, symmetrically placed with respect to the axis of the parabola, correspond to any value of the radius of curvature greater than $2a$.

(d) It follows from the results found in (b) that the equation of the circle of curvature at the point $(at^2, 2at)$ is

$$x^2+y^2-2(3at^2+2a)x+4at^3y-3a^2t^4 = 0.$$

(e) If $P$, $Q$, $R$ are three of the points of intersection of a circle and a parabola, then, when $P$ and $R$ coincide with $Q$, each of the chords $PQ$, $QR$ becomes a tangent to both the circle and the parabola.

These two tangents are coincident, hence the corresponding normals are coincident. Since these coincident normals are normals to the circle, they intersect at the centre of curvature.

Since they are consecutive normals of the parabola, they intersect on the evolute of the parabola.

Hence the evolute is the locus of the centre of curvature.

We have shown that the coordinates of the centre of curvature corresponding to the point $(at^2, 2at)$ are given by

$$x = -g = 3at^2+2a,$$
$$y = -f = -2at^3.$$

Eliminating $t$, the equation of the locus of the centre of curvature, i.e. of the evolute, is $27ay^2 = 4(x-2a)^3$.

**Cor. vi.** If equation (ii) has two pairs of equal roots, the circle touches the parabola in two points and is said to have **double contact.**

Let the two points be $t_1$, $t_2$; then the roots of the equation (ii) are $t_1, t_1, t_2, t_2$; but $\Sigma t = 0$; hence $t_2 = -t_1$.

The points of contact are therefore symmetrically placed with respect to

the axis, and the common chord of the circle and the parabola is perpendicular to the axis.

The centre is evidently on the axis; and, since in this case $\Sigma t_1 t_2 = -2t_1^2$, the abscissa of the centre is given by

$$-g = at_1^2 + 2a,$$

and its minimum value is therefore $2a$.

**Example i.** *A circle cuts a parabola in four points: if the normals at three of these points are concurrent, prove that the circle passes through the vertex, and find its equation if the normals meet at* $(\xi, \eta)$.

Let the parameters of the four points of intersection be $t_1, t_2, t_3, t_4$.

Since the points are concyclic

$$t_1 + t_2 + t_3 + t_4 = 0,$$

and if the normals at $t_1, t_2, t_3$ are concurrent

$$t_1 + t_2 + t_3 = 0.$$

Hence $t_4 = 0$, i.e. the circle passes through the vertex.

Since the normals meet at $(\xi, \eta)$ the parameters of their feet (viz. $t_1, t_2, t_3$) are given by

$$at^3 + (2a - \xi)\, t - \eta = 0. \qquad \text{(i)}$$

Let the circle be $x^2 + y^2 + 2gx + 2fy = 0$, then the parameters of the points of intersection of this circle and the parabola are given by

$$at^3 + (4a + 2g)\, t + 4f = 0. \qquad \text{(ii)}$$

Since these are by hypothesis $t_1, t_2, t_3$, equations (i) and (ii) are identical.

Thus
$$2g = -(\xi + 2a),$$
$$2f = -\tfrac{1}{2}\eta.$$

Hence the equation of the circle is

$$x^2 + y^2 - (\xi + 2a)\, x - \tfrac{1}{2}\eta\, y = 0.$$

**Example ii.** *A straight line cuts the evolute of a parabola in three real points, from each of which the normal to the parabola, other than the radius of curvature, is drawn. Show that the centres of curvatures at the feet of these normals are collinear.*

If $L$ is any point on the evolute, two of the normals which can be drawn from $L$ to the parabola coincide with each other.

If $P(at_1^2, 2at_1)$ is the foot of these coincident normals, then $L$ is the centre of curvature of the parabola at $P$. If $Q\ (at_2^2, 2at_2)$ is the foot of the third normal from $L$, since the normals at $t_1, t_1, t_2$ are concurrent,

$$2t_1 + t_2 = 0,$$

i.e. the parameter of $Q$ is $-2t_1$.

We have therefore to show that if the centres of curvature at three points whose parameters are $t_1, t_2, t_3$ are collinear, then the centres of curvature at the three points whose parameters are $-2t_1, -2t_2, -2t_3$ are also collinear.

The centre of curvature at the point $t$ is $\{2a+3at^2, -2at^3\}$; the centres of curvature at the point $t_1$, $t_2$, $t_3$ are collinear if $t_1$, $t_2$, $t_3$ satisfy an equation of the form $p(2a+3at^2)-2aqt^3+r=0$.

The necessary condition for this is that $\Sigma t_1 t_2 = 0$, or $\Sigma(1/t)=0$, and the condition is evidently sufficient.

Similarly, the condition that the centre of curvature at the points $-2t_1$, $-2t_2$, $-2t_3$ should be collinear is $\Sigma(-1/2t)=0$. But if $\Sigma(1/t)=0$, then obviously $\Sigma(-1/2t)=0$.

## Examples VII d.

1. Find the radius, centre, and circle of curvature at the extremity of the latus rectum.

2. A circle touches the parabola at the two points of intersection of the curve and $x=3a$; find its equation.

3. A circle is described on the chord of a parabola whose equation is $Ax+By+a=0$ as diameter; find the equation of the other common chord of the circle and the parabola.

4. The circle of curvature at the vertex meets the curve in four coincident points.

5. The extremities of any two chords of a parabola which are perpendicular to the axis are concyclic.

6. Find the points at which the radius of curvature is $16a$.

7. The common chords of the circles of curvature at $(x_1, y_1)$, $(x_2, y_2)$ respectively and the parabola intersect at the point $(\xi, \eta)$; prove that

$$3y_1y_2+4a\xi=0 \text{ and } 3(y_1+y_2)=2\eta.$$

8. The circle of curvature at a point $P$ on a parabola meets the parabola again in $Q$. If $p_1, p_2$ are the radii of curvature at $P$ and $Q$, prove that $9p_1^{\frac{2}{3}}-p_2^{\frac{2}{3}}$ is constant.

9. If $m_1$, $m_2$, $m_3$ are the roots of the equation $m^3+pm^2+qm+r=0$, show that the points $(am_1^2, 2am_1)$, $(am_2^2, 2am_2)$, $(am_3^2, 2am_3)$ lie on the circle $x^2+y^2+(q-p^2-4)ax+\frac{1}{2}(r-pq)ay-a^2pr=0$, and deduce the length of the radius of curvature at any point of $y^2=4ax$.

10. A circle passes through the vertex and three other points $P$, $Q$, $R$ of a parabola. The lines joining $P$, $Q$, $R$ to the focus meet the curve again at $P'$, $Q'$, $R'$. Prove that the centres of curvature at $P'$, $Q'$, $R'$ are collinear.

### § 9. Forms of Equations.

In this section we shall use the following abridged notation.

$P=0$, the equation of any parabola.

$u=0$, $v=0$, the equations of two chords of the parabola.

$t=0$, $t'=0$, the equations of any two tangents to the parabola.

$C=0$, the equation of a circle.

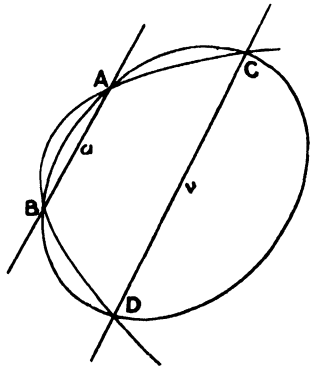
Throughout $k$ is used for an undetermined constant.

(i) **P = kuv.**

Let $u = 0$, $v = 0$ cut the parabola $P = 0$ in the points $A, B, C, D$. Now the coordinates of any one of these points satisfy the equation

$$P - kuv = 0.$$

For the point $A$ lies on $P = 0$ and $u = 0$; its coordinates, therefore, substituted in $P$ and $u$, make these expressions zero: and consequently, when substituted in $P - kuv$, they make it zero.




Now $P$ is of the second degree, and since $u$ and $v$ are linear, $uv$ is of the second degree. Hence $P - kuv$ is of the second degree; the equation $P - kuv = 0$ consequently represents a conic passing through the four points $A, B, C, D$. The constant $k$ is still at our disposal, so that the conic may be made to satisfy one other condition by giving $k$ a suitable value. For example, it may be another parabola, or it may pass through some given fifth point.

In two cases the equation $P = kuv$ represents a pair of straight lines, viz. $AC, BD$; $AD, BC$. The equation cannot in general represent a circle, since two conditions are necessary: we have seen that $u = 0$, $v = 0$ must be equally inclined to the axis of $P = 0$.

**Example.** *To find the equation of the parabola which passes through the points of intersection of* $y^2 = 4x$ *and the straight lines* $3x + 4y = 5$ *and* $x + 2y = 3$.

The equation of the parabola must be of the form

$$y^2 - 4x + k(3x + 4y - 5)(x + 2y - 3) = 0.$$

The condition that the curve which is the locus of this equation should be a parabola is $3k(1 + 8k) - 25k^2 = 0$, i.e. $k = 0$ or $3$.

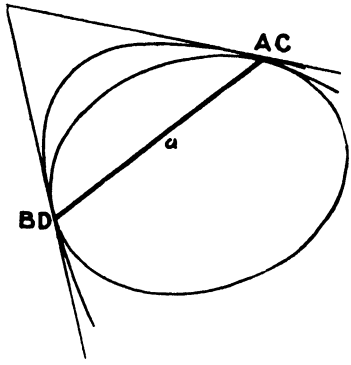
Hence the required equation is

$$y^2 - 4x + 3(3x + 4y - 5)(x + 2y - 3) = 0,$$

i.e. $$9x^2 + 30xy + 25y^2 - 46x - 66y + 45 = 0.$$

(ii) $\mathbf{P-ku^2=0}$.

If the straight lines $u=0$, $v=0$ coincide, the pairs of points $AC$, $BD$ become coincident: the conic $P-ku^2=0$ therefore touches the parabola at the points $A$ and $B$.




Thus $P-ku^2=0$ is the general equation of a conic having double contact with the parabola, $u=0$ being the chord of contact.

The equation can only represent a circle when $u=0$ is perpendicular to the axis: for one value of $k$ it represents a pair of straight lines, viz. the tangents $A$ and $B$ to the parabola.

**Example.** *A conic has double contact with a parabola, one of the points of contact being the vertex, and passes through its focus. Show that the locus of its centre is a parabola.*

Let the parabola be $y^2=4ax$. Since the chord of contact passes through the origin, its equation is of the form $lx+my=0$.

The equation of the conic is then

$$k(y^2-4ax)+(lx+my)^2=0,$$

since it passes through the focus $(a, 0)$;

$$\therefore\quad 4a^2k=a^2l^2,\quad \text{i.e.}\quad k=\tfrac{1}{4}l^2.$$

The equation of the conic is then

$$l^2(y^2-4ax)+4(lx+my)^2=0,$$

or
$$4l^2x^2+8lmxy+(4m^2+l^2)y^2-4al^2x=0.$$

Its centre is given by

$$4l^2x+4lmy-2al^2=0,\ \text{or}\ 2lx+2my-al=0,$$

and
$$4lmx+(4m^2+l^2)y=0;$$

$$\therefore\quad 2(lx+my)=al=-\frac{l^2y}{2m};$$

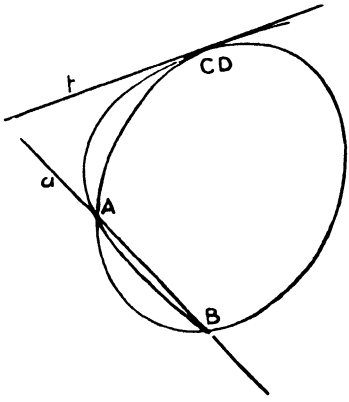
$$\therefore\quad \frac{l}{2a}=\frac{m}{-y}=\frac{lx+my}{2ax-y^2}=\frac{al}{2(2ax-y^2)};$$

and the required locus is $y^2=a(2x-a)$, which is a parabola.

(iii) $\mathbf{P-kut=0}$.

When the line $v=0$ is a tangent to the parabola (viz. $t=0$), the points $C$, $D$ coincide and the conic touches the parabola at $C$.

Thus $P-kut=0$ represents a conic passing through the intersections of $P=0$ and $u=0$, and touching $P=0$ at the point of contact of $t=0$.




It can only represent a circle when $u=0$, $t=0$ are equally inclined to the axis of $P=0$.

For one value of $k$ it will represent the pair of straight lines $CA$, $CB$.

Note that, if the line $u=0$ is not given, the general equation of a conic touching the parabola $P=0$ at the point of contact of $t=0$ is

$$P-kt(lx+my+1)=0.$$

We have now three undetermined constants, and the conic can therefore satisfy three other conditions.

**Example.** *Find the equation of the circle touching the parabola* $y^2=4x+4$ *at the point* (8, 6) *which passes also through the focus.*

The tangent at the point (8, 6) is

$$x-3y+10=0.$$

The equation of the circle is of the form

$$k(y^2-4x-4)=(x-3y+10)(lx+my+1).$$

Since it passes through the focus (i.e. the origin)

$$-4k=10 \text{ or } k=-\tfrac{5}{2}.$$

Hence the equation becomes

$$5(y^2-4x-4)+2(x-3y+10)(lx+my+1)=0.$$

The conditions that this should represent a circle are

$$5-6m=2l,$$

and

$$m=3l;$$

$$\therefore\ l=\tfrac{1}{4},\ m=\tfrac{3}{4}.$$

The equation of the circle is therefore

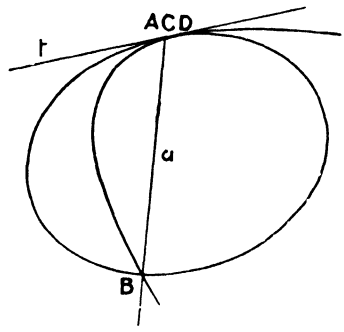
$$10(y^2-4x-4)+(x-3y+10)(x+3y+4)=0,$$

i.e.

$$x^2+y^2-26x+18y=0.$$

(iv) $\mathbf{P-kut=0}$, **when** $\mathbf{u=0}$, $\mathbf{t=0}$ **intersect on** $\mathbf{P=0}$.

If three of the points, for example $A$, $C$, $D$, coincide, the conic touches and cuts the parabola at $A$. The conic $P-kut=0$ and the parabola $P=0$ are then said to have 'three-point contact' or 'contact of the second order'.




**Example.** *To find the equation of the circle of curvature at the point* $(a\lambda^2, 2a\lambda)$ *of the parabola* $y^2=4ax$.

The tangent at the point $\lambda$ is

$$t\equiv\lambda y-x-a\lambda^2=0,$$

and any chord through the point of contact is

$$u\equiv x-a\lambda^2+m(y-2a\lambda)=0,$$

where $m$ is a constant to be determined.

The equation of the circle of curvature is therefore of the form

$$y^2-4ax+k(\lambda y-x-a\lambda^2)\{x-a\lambda^2+m(y-2a\lambda)\}=0.$$

The conditions that this should be a circle are

$$-k=1+k\lambda m$$

and

$$\lambda-m=0;$$

$$\therefore\quad m=\lambda,\ k=\frac{-1}{1+\lambda^2}.$$

The equation of the circle is then

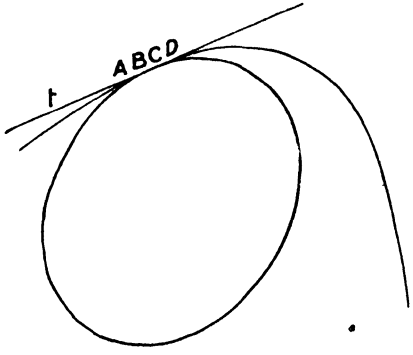
$$(1+\lambda^2)(y^2-4ax)-(\lambda y-x-4a\lambda^2)(x+\lambda y-3a\lambda^2)=0,$$

which reduces to

$$x^2+y^2-2a(2+3\lambda^2)x+4a\lambda^3y-3a^2\lambda^4=0.$$

(v) $\mathbf{P-kt^2=0}$.

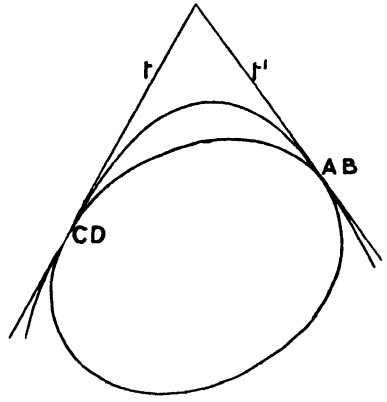
When the four points $A$, $B$, $C$, $D$ coincide, both the chords $u=0$, $v=0$ coincide with the tangent at $A$, viz. $t=0$. The conic




$P-kt^2=0$ then meets the parabola in four coincident points and is said to have 'four-point contact' or 'contact of the third order'. The conic can only be a circle when $t=0$ is the tangent at the vertex.

(vi) $\mathbf{P-ktt'=0}$.

When the chords $u=0$, $v=0$ both become tangents, the pairs of points $A, B$ and $C, D$ coincide, and we get another form of the equation of a conic having double contact with the parabola: the points of contact are those of the tangents.




Thus $y^2-4ax=k\{(\lambda+\mu)y-2x-2a\lambda\mu\}^2$,

and $y^2-4ax=k\{\lambda y-x-a\lambda^2\}\{\mu y-x-a\mu^2\}$;

both represent a conic having double contact with the parabola at the points $(a\lambda^2, 2a\lambda)$, $(a\mu^2, 2a\mu)$.

(vii) $\mathbf{P-kC=0}$.

By similar reasoning this represents a conic passing through the four points of intersection of the parabola $P=0$ and the circle $C=0$. For certain values of $k$ it will represent the common chords of the circle and parabola.

**Note.** It is evident that the seven forms here discussed would give similar results if we used $S=0$, the equation of any conic, parabola, ellipse, or hyperbola, instead of $P=0$. It is therefore important to understand these forms: we have used the parabola because the student is now familiar with the form of its equation.

**Example.** *Prove that the equation of the parabola which passes through the origin and has contact of the second order with* $y^2=4ax$ *at the point* $(a\mu^2, 2a\mu)$ *is* $(4x-3\mu y)^2+4a\mu^2(3x-2\mu y)=0$.

The required parabola is of the form $P-kut=0$, where $t=0$ is the tangent at $(a\mu^2, 2a\mu)$ and $u=0$ is the join of the origin to this point, i.e.

$$y^2-4ax-k(\mu y-x-a\mu^2)(-\mu y+2x)=0.$$

Since the curve is to be a parabola,

$$2k(1+k\mu^2) = (\tfrac{3}{2}\mu k)^2,$$

i.e. $$8+8k\mu^2 = 9k\mu^2;$$

$$\therefore\quad k = 8/\mu^2;$$

$\therefore$ the required equation is

$$\mu^2(y^2-4ax) - 8(\mu y - x - a\mu^2)(2x-\mu y) = 0,$$

i.e. $$16x^2 - 24\mu xy + 9\mu^2 y^2 + 12a\mu^2 x - 8a\mu^3 y = 0,$$

or $$(4x-3\mu y)^2 + 4a\mu^2(3x-2\mu y) = 0.$$

## Examples VII e.

1. Find the equation of the other parabola which passes through the four points common to $y^2 = 4ax$ and $x^2+y^2+2gx+2fy+c = 0$.

2. Find the equation of a parabola which has contact of the second order with $y^2 = 4ax$ at the point $(at^2, 2at)$ and passes through an end of the latus rectum.

3. Find the equation of the rectangular hyperbola which has contact of the third order with the parabola at the point $(at^2, 2at)$.

4. For what values of $\lambda$ does the equation $y^2 - 4ax + \lambda(x-2a)(x-3a) = 0$ represent straight lines. Illustrate these lines in a diagram.

5. $TP$, $TQ$ are tangents to a parabola from any point $T$ on the line $x = 2a$: show that the circle $TPQ$ passes through the origin.

6. A circle has double contact with the parabola $y^2 = 4a(x+a)$, and the point whose abscissa is $8a$ is one point of contact. Find its equation and its centre.

7. Circles are described passing through the vertex of the parabola $y^2 = 4ax$ and cutting the parabola orthogonally at the other point of intersection. Show that their centres lie on the curve

$$2y^2(2y^2+x^2-12ax) = ax(3x-4a)^2.$$

8. Find the equation of the circle which touches the parabola $y^2 = 4ax$ at the point $(am^2, 2am)$ and passes through the focus.

Prove that three such circles can be drawn to touch a given line at the focus and that the tangents at their points of contact form an equilateral triangle.

9. From a point $T(x', y')$ tangents $TP$, $TQ$ are drawn to a parabola: show that the other common chord of the parabola and the circle $TPQ$ is the polar of $(2a-x', -y')$.

10. From points on the line $x = h$ tangents are drawn to the parabola $y^2 = 4ax$, and circles are described round the triangles formed by each pair and their chord of contact. Find the locus of their centres.

11. A circle of variable radius whose centre is $(0, b)$ meets the parabola $y^2 = 4ax$ at $P$ and $Q$. Show that the locus of the intersection of the tangents to the parabola at $P$ and $Q$ is $y^3 - 2axy + 4a^2y - 4a^2b = 0$.

12. $PQ$ is a focal chord of a parabola. Two circles are drawn through the focus to touch the parabola at $P$ and $Q$ respectively. Show that they cut one another orthogonally, and find the locus of their second point of intersection.

13. Circles are drawn through the focus to touch $y^2 = 4ax$. Find the envelope of those common chords of the circles and the parabola which do not pass through the points of contact.

14. The tangents to $y^2 = 4ax$ at $P$ and $Q$ meet in $T$, and the centre of the circle $TPQ$ lies on the parabola.

Show that the locus of $T$ is $y^2(a-x)^2 = 16a^4 + 8a^2y^2$.

**§ 10. Method of reducing the equation of a given parabola to its simplest form.**

When the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a parabola, it takes the form

$$(lx+my)^2 + 2gx + 2fy + c = 0.$$

Then
$$X = l(lx+my) + g,$$
$$Y = m(lx+my) + f,$$

and the equation of its axis ($aX + hY = 0$, Chap. VI, § 7) becomes, since $a = l^2$ and $h = lm$,

$$lx + my + \frac{lg+mf}{l^2+m^2} = 0,$$

or $lx+my+n = 0$, if we write $n = \dfrac{lg+mf}{l^2+m^2}$.

The equation of the parabola can be written

$$(lx+my+n)^2 + 2(g-ln)x + 2(f-mn)y + c - n^2 = 0,$$

which reduces at once to

$$(lx+my+n)^2 + 2\left(\frac{gm-fl}{l^2+m^2}\right)(mx-ly) + c - n^2 = 0.$$

If we take $lx+my+n = 0$ and

$$mx - ly + \tfrac{1}{2}\frac{(l^2+m^2)(c-n^2)}{gm-fl} = 0$$

as new axes of $x$ and $y$, the equation becomes

$$y^2 = \frac{2(lf-mg)}{(l^2+m^2)^{\frac{3}{2}}}x.$$

**Example.** *Find the latus rectum and the equation of the axis and the tangent at the vertex of the parabola*

$$25x^2 + 120xy + 144y^2 - 146x + 89y - 25 = 0.$$

In this case
$$X = 25x + 60y - 73 = 5(5x+12y) - 73,$$
$$Y = 60x + 144y + 44\tfrac{1}{2} = 12(5x+12y) + 44\tfrac{1}{2},$$

also $a = 25$, $h = 60$; $\therefore \dfrac{a}{h} = \dfrac{5}{12}$.

The equation of the axis is therefore

$$(5x+12y)(25+144)-365+534=0,$$

i.e.
$$5x+12y+1=0.$$

The equation of the parabola can be written

$$(5x+12y+1)^2=13(12x-5y+2).$$

If we take $5x+12y+1=0$, $12x-5y+2=0$ as axes of $X$ and $Y$, the equation becomes $y^2=x$.

Hence the latus rectum is 1, and the equations of the axis and the tangent at the vertex are $5x+12y+1=0$, and $12x-5y+2=0$.

### Examples VII f.

1. Find the latus rectum of $9x^2+16y^2+24xy-4y-x+7=0$.
2. Reduce to their simplest form and draw the graphs of:
   (i) $(x+2y)^2+2x-y=0$;
   (ii) $(x+y-1)^2=\sqrt{2}(x-y)$;
   (iii) $9x^2+16y^2+24xy-34x+38y+1=0$;
   (iv) $9x^2+6xy+y^2-4x+y+2=0$.
3. Show that the parabola

$$(fx-gy)^2-2x(hf-\lambda g)-2y(hg-\lambda f)-\lambda^2+h^2=0$$

has the same axis and focus for all values of $\lambda$.

4. Prove that the two parabolas which can be drawn through the four common points of $ax^2+by^2=1$ and $x^2+y^2+2gx+2fy+c=0$ have their axes perpendicular, and that their latera recta are equal if $bf=ag$.

5. Find the equation of the parabola which cuts the axes at the points $(a, 0)$ and $(0, b)$ and has its tangents at these points parallel to the axes of $y$ and $x$ respectively.

6. A parabola has for focus the point $(\xi, \eta)$ and for directrix the line $ax+by+c=0$. Show that the line $Ax+By+C=0$ is a tangent to the parabola if $(A^2+B^2)(a\xi+b\eta+c)-2(Aa+Bb)(A\xi+B\eta+C)=0$.

7. Show that the locus of the intersection of normals to a parabola which are at right angles to each other is a parabola.

Find its focus and vertex.

8. Find the focus of the parabola $(ax+by)^2=2y$, and show that the equation of its latus rectum is $2a(a^2+b^2)(bx-ay)+a^2-b^2=0$.

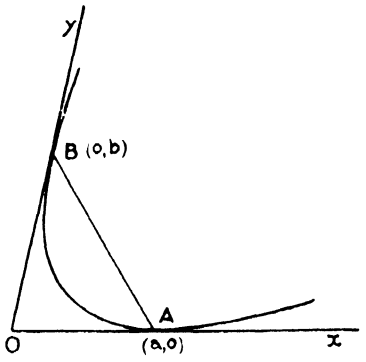
9. If $ax^2+2hxy+by^2+2gx=0$ represents a parabola, find the coordinates of its vertex.

§ 11. *The equation of a parabola referred to any pair of tangents as coordinate axes.*

Let the parabola touch the coordinate axes at $A$ $(a, 0)$ and $B(0, b)$. The chord of contact is therefore

$$\frac{x}{a}+\frac{y}{b}-1=0.$$

We have seen in § 9 that the equation of a pair of tangents whose chord of contact is $u = 0$ is of the form $P - ku^2 = 0$.




In the present case therefore

$$P - k\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 \equiv xy,$$

or

$$P \equiv k\left(\frac{x}{a} + \frac{x}{b} - 1\right)^2 + xy = 0.$$

In order that

$$k\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 + xy = 0$$

should be a parabola, $k$ must equal $-\frac{1}{4}ab$: the equation of the parabola is therefore

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 = \frac{4xy}{ab}.$$

**Parametric representation.** The coordinates of any point on this parabola can be expressed in the form $\{a\lambda^2,\ b(\lambda-1)^2\}$, for these values of $x$, $y$ satisfy the equation of the parabola and they can have any positive value we please since $\lambda$ may have any value. This point will be referred to as the point $\lambda$ on the parabola.

(i) *To find the equation of the chord joining the points* $\lambda$, $\mu$.

Let the equation of the chord be

$$\frac{A}{a}x + \frac{B}{b}y + 1 = 0,$$

then, since the points $\lambda$, $\mu$ are on it,

$$A\lambda^2 + B(\lambda-1)^2 + 1 = 0,$$
$$A\mu^2 + B(\mu-1)^2 + 1 = 0.$$

By cross multiplication

$$\frac{A}{(\lambda-\mu)(\lambda+\mu-2)} = \frac{B}{\mu^2-\lambda^2} = \frac{1}{(\mu-\lambda)(2\lambda\mu-\lambda-\mu)},$$

hence

$$\frac{A}{\lambda+\mu-2} = \frac{B}{-(\lambda+\mu)} = \frac{1}{\lambda+\mu-2\lambda\mu};$$

therefore the equation of the chord is

$$\frac{x}{a}(\lambda+\mu-2) - \frac{y}{b}(\lambda+\mu) = 2\lambda\mu - \lambda - \mu.$$

(ii) *To find the equation of the tangent at the point* $\lambda$.

This follows from that of the chord by putting $\mu = \lambda$; thus the tangent is

$$\frac{x}{a}(\lambda - 1) - \frac{y}{b}\lambda = \lambda(\lambda - 1).$$

(iii) *To find the locus of the intersection of perpendicular tangents.*

Suppose that the tangents at $\lambda$ and $\mu$ are perpendicular; their equations are

$$\frac{x}{a}(\lambda - 1) - \frac{y}{b}\lambda = \lambda(\lambda - 1),$$

$$\frac{x}{a}(\mu - 1) - \frac{y}{b}\mu = \mu(\mu - 1);$$

their point of intersection is therefore given by

$$\frac{x}{a} = \lambda\mu\,; \quad \frac{y}{b} = \lambda\mu - (\lambda + \mu) + 1 = (\lambda - 1)(\mu - 1)\,;$$

$$\therefore \quad \lambda + \mu = \frac{x}{a} - \frac{y}{b} + 1.$$

The condition that the tangents should be perpendicular is

$$\frac{(\lambda - 1)(\mu - 1)}{a^2} + \frac{\lambda\mu}{b^2} + \frac{\mu(\lambda - 1) + \lambda(\mu - 1)}{ab}\cos\omega = 0.$$

Hence

$$\frac{y}{a^2 b} + \frac{x}{ab^2} + \frac{\frac{2x}{a} - \frac{x}{a} + \frac{y}{b} - 1}{ab}\cos\omega = 0,$$

or the required locus is

$$x(a + b\cos\omega) + y(b + a\cos\omega) = ab\cos\omega.$$

This is therefore the equation of the directrix.

(iv) *To find the equation of the tangent at the vertex.*

If the tangent at $\lambda$,

$$\frac{x}{a}(\lambda - 1) - \frac{y}{b}\lambda = \lambda(\lambda - 1),$$

is parallel to the directrix

$$x(a + b\cos\omega) + y(b + a\cos\omega) = ab\cos\omega,$$

we have

$$\frac{\lambda - 1}{a^2 + ab\cos\omega} = \frac{\lambda}{-(b^2 + ab\cos\omega)} = \frac{1}{-(a^2 + b^2 + 2a}$$

Hence, substituting for $\lambda$ in the equation of the ta

$$\frac{x}{b + a\cos\omega} + \frac{y}{a + b\cos\omega} = \frac{ab}{a^2 + b^2 + 2al}$$

(v) *To find the coordinates of the focus.*

The perpendiculars from the focus on the coordinate axes lie on the tangent at the vertex, since the axes are tangents. If $(\xi, \eta)$ be the focus, the feet of these perpendiculars are

$$(\xi+\eta\cos\omega, 0),\ (0, \eta+\xi\cos\omega).$$

Hence the equation of the tangent at the vertex is

$$\frac{x}{\xi+\eta\cos\omega}+\frac{y}{\eta+\xi\cos\omega}=1.$$

Comparing this with the equation already found in (iv), we get

$$\frac{\xi+\eta\cos\omega}{b+a\cos\omega}=\frac{\eta+\xi\cos\omega}{a+b\cos\omega}=\frac{ab}{a^2+b^2+2ab\cos\omega},$$

whence immediately

$$\frac{\xi}{b}=\frac{\eta}{a}=\frac{ab}{a^2+b^2+2ab\cos\omega}.$$

(vi) The latus rectum is twice the perpendicular from the focus to the directrix.

Its value is $\dfrac{4a^2b^2\sin^2\omega}{\{a^2+b^2+2ab\cos\omega\}^{\frac{3}{2}}}$.

(vii) *To find the condition that* $lx+my+n=0$ *should touch the parabola.*

Any tangent to the parabola is of the form

$$\frac{x}{a}(\lambda-1)-\frac{y}{b}\lambda=\lambda(\lambda-1).$$

Comparing this with the given equation

$$\frac{\lambda-1}{al}=\frac{\lambda}{-bm}=\frac{\lambda(\lambda-1)}{-n};$$

$$\therefore\quad \lambda=-\frac{n}{al},\qquad \lambda-1=+\frac{n}{bm};$$

$$\therefore\quad \frac{1}{al}+\frac{1}{bm}+\frac{1}{n}=0.$$

equation of the parabola can be written

$$\frac{x}{a}+\frac{y}{b}-1=\pm 2\sqrt{\frac{xy}{ab}},$$

$$\left(\sqrt{\frac{x}{a}}\pm\sqrt{\frac{y}{b}}\right)^2=1.$$

It is often referred to in the form

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1.$$

Evidently for points in different positions on the curve, different signs for the radicals must be taken. The equation is true with positive signs for the part of the curve between the points of contact $A$ and $B$. For the rest of the curve we must use

$$\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1 \,(x > a) \quad \text{or} \quad -\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \,(y > b).$$

The equation of the tangent at the point $(x', y')$ on the parabola $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ can be put in the form

$$\frac{x}{a}\sqrt{\frac{a}{x'}} + \frac{y}{b}\sqrt{\frac{b}{y'}} = 1,$$

where $\sqrt{\frac{a}{x'}}$ and $\sqrt{\frac{b}{y'}}$ have those signs which satisfy

$$\sqrt{\frac{x'}{a}} + \sqrt{\frac{y'}{b}} = 1.$$

It should be noted, however, that when $(x', y')$ does not lie on the parabola, this equation is **not** that of the polar of $(x', y')$.

It is sometimes stated that $(a\cos^4\theta, b\sin^4\theta)$ can be used to denote a point on the curve: this is only true, if $\theta$ is real, for the portion between the points of contact, i.e. when $x < a$ and $y < b$.

The notation given above covers the whole curve and is to be preferred.

**Example** i. *A tangent to the parabola* $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ *meets the axes of coordinates in P, Q, and perpendiculars are drawn from P, Q to the opposite axes: prove that the locus of their points of intersection is*

$$\frac{x + y\cos\omega}{b} + \frac{y + x\cos\omega}{a} = \cos\omega.$$

Let the tangent be $\frac{x}{a}(\lambda - 1) - y\frac{\lambda}{b} = \lambda(\lambda - 1)$; this meets the axes at the points $P(a\lambda, 0)$, $Q(0, b(1-\lambda))$.

Let $PL$, $QM$ be the perpendiculars on the axes of $y$ and $x$: then $L$ is the point $(0, a\lambda\cos\omega)$ and $M$ the point $[b(1-\lambda)\cos\omega, 0]$.

The equation of $PL$ is $\qquad \frac{x}{a\lambda} + \frac{y}{a\lambda\cos\omega} = 1.$

The equation of $QM$ is

$$\frac{x}{b(1-\lambda)\cos\omega} + \frac{y}{b(1-\lambda)} = 1,$$

i.e. $\dfrac{x\cos\omega+y}{a} = \lambda\cos\omega\,; \quad \dfrac{y\cos\omega+x}{b} = (1-\lambda)\cos\omega\,;$

$\therefore$ their point of intersection lies on the straight line

$$\frac{x\cos\omega+y}{a} + \frac{x+y\cos\omega}{b} = \cos\omega.$$

**Example ii.** *A variable tangent to a given parabola meets two fixed tangents, and on the intercepted segment as diameter a circle is described: the envelope of the circles is a conic touching the two fixed tangents in the points where they are met by the directrix of the given parabola.*

Take the fixed tangents for axes, and let the parabola be

$$\left(\frac{x}{a}+\frac{y}{b}-1\right)^2 = \frac{4xy}{ab}.$$

Any tangent $\dfrac{x}{a}(\lambda-1) - \dfrac{y}{b}\lambda = \lambda(\lambda-1)$ meets the fixed tangents at the points $P(a\lambda, 0)$, $Q(0, b-b\lambda)$.

If $R(x, y)$ be any point on the circle described on $PQ$ as diameter, we have, since $RP^2+RQ^2 = PQ^2$,

$$(x-a\lambda)^2+y^2+2y(x-a\lambda)\cos\omega + x^2+(y-b+b\lambda)^2+2x(y-b+b\lambda)\cos\omega = a^2\lambda^2+b^2(1-\lambda)^2+2ab\lambda(\lambda-1)\cos\omega,$$

which reduces to

$$ab\cos\omega\lambda^2 - \lambda\{x(b\cos\omega-a)+y(b-a\cos\omega)+ab\cos\omega\} - (x^2+y^2+2xy\cos\omega-by-bx\cos\omega) = 0.$$

Since $\lambda$ is an undetermined constant, the envelope of this circle is (Chap. VI, p. 253)

$$\{x(b\cos\omega-a)+y(b-a\cos\omega)+ab\cos\omega\}^2 + 4ab\cos\omega(x^2+y^2+2xy\cos\omega-bx\cos\omega-by) = 0.$$

This may also be written

$$\{x(a+b\cos\omega)+y(b+a\cos\omega)-ab\cos\omega\}^2 = 4abxy\sin^2\omega,$$

which represents a conic (equation of second degree) touching the lines $x=0$, $y=0$, the chord of contact being

$$x(a+b\cos\omega)+y(b+a\cos\omega)-ab\cos\omega = 0,$$

i.e. the directrix (because the form is $uv = kw^2$).

## Examples VII g.

1. A variable tangent to a parabola meets two fixed tangents at the points $P$, $Q$. Find the locus of the mid-point of $PQ$.

2. $OA$, $OB$ are fixed tangents to a parabola and $P$ any point on the curve. The harmonic conjugate of $OP$ with respect to $OA$ and $OB$ meets the tangent at $P$ in $Q$: find the locus of $Q$.

3. A variable tangent to a parabola meets two fixed tangents $TA$, $TB$ at $P$ and $Q$. Find the locus of the centre of the circle $TPQ$.

4. Show that the normal at $\lambda$ to the parabola $(x/a+y/b-1)^2 = 4xy/ab$ is

$$x(a\lambda + b\sqrt{\lambda-1}\cos\omega) + y(a\lambda\cos\omega + b\sqrt{\lambda-1}) = ab\lambda(\lambda-1)(2\lambda-1)\cos\omega + a^2\lambda^3 + b^2(\lambda-1)^3.$$

5. Find the focus and directrix of the parabola $(x/a+y/b-1)^2 = (4xy/ab)$ by comparing the equation with

$$(x-x')^2 + (y-y')^2 + 2(x-x')(y-y')\cos\omega = (x\cos\alpha + y\cos\beta - p)^2,$$

where $\alpha+\beta = \omega$.

N.B. Use the identity

$$(x\cos\omega+y)^2 + (y\cos\omega+x)^2 - 2(x\cos\omega+y)(y\cos\omega+x)\cos\omega = (x^2+y^2+2xy\cos\omega)\sin^2\omega.$$

6. A parabola touches $OA$, $OB$ in $A$ and $B$: show that the portions of any chord, which has its middle point on $AB$, intercepted between $OA$, $OB$ and the parabola are equal.

7. Parabolas are drawn which touch the axes $Ox$, $Oy$, inclined at an angle $\omega$, and whose directrices pass through a fixed point $(h, k)$: show that they all touch the line $x/(h+k\sec\omega) + y/(k+h\sec\omega) = 1$.

8. $AB$, $CD$, two fixed segments of straight lines, are divided similarly at $P$ and $Q$: prove that $PQ$ envelopes a parabola which touches $AB$ and $CD$.

9. Show that if $a$ and $b$ are variable and $h/a + k/b = 1$, the directrices of $\sqrt{x/a} + \sqrt{y/b} = 1$ pass through a fixed point.

10. A parabola touches two given straight lines $OA$, $OB$ at given points and a variable tangent meets $OA$, $OB$ at $P$, $Q$. Show that the circle $OPQ$ passes through the focus.

11. If the chords of contact of parabolas touching two fixed lines are concurrent, their directrices are also concurrent.

12. The parallels through the origin to the tangents from $(x', y')$ to $\sqrt{x} + \sqrt{y} = \sqrt{c}$ are the lines $cxy + (x-y)(xy'-x'y) = 0$.

13. The equation of the tangents from $(x', y')$ to the parabola is

$$(x'y - xy')\{(x'-x)/a - (y'-y)/b\} + (x'-x)(y'-y) = 0.$$

14. A variable tangent to a parabola meets two fixed tangents, and another parabola is drawn touching the fixed tangents at these points: prove that the envelope of its directrix is a third parabola touching lines drawn at right angles to the fixed tangents through their intersection, in the points where they are met by the directrix of the given parabola.

## Illustrative Examples.

(i) *If the tangents at $P$ and $Q$ meet at $T$ and the orthocentre of the triangle $PTQ$ lies on the parabola, show that either the orthocentre is at the vertex or the chord $PQ$ is a normal to the parabola.*

Let $P$ be the point $(a\lambda^2, 2a\lambda)$, $Q(a\mu^2, 2a\mu)$, then $T$ is the point

$$\{a\lambda\mu,\ a(\lambda+\mu)\}.$$

The equation of $TP$ is $\lambda y - x - a\lambda^2 = 0$, and that of the perpendicular from $Q$ on it is $\lambda(x - a\mu^2) + y - 2a\mu = 0$, i.e.

$$\lambda x + y = a\lambda\mu^2 + 2a\mu. \tag{i}$$

So the perpendicular from $P$ on $TQ$ is

$$\mu x + y = a\mu\lambda^2 + 2a\lambda. \tag{ii}$$

Solving (i) and (ii) to obtain the coordinates of the orthocentre of the triangle $TPQ$, we have

$$x = -a\lambda\mu - 2a, \quad y = a(\lambda + \mu)(\lambda\mu + 2).$$

Since this point lies on the parabola,

$$a^2(\lambda + \mu)^2(\lambda\mu + 2)^2 = -4a^2(\lambda\mu + 2).$$

Hence either $\lambda\mu + 2 = 0$, in which case the orthocentre is the vertex $(0, 0)$, or

$$(\lambda + \mu)^2(\lambda\mu + 2) = -4, \tag{i}$$

i.e. $$(\lambda^2 + \lambda\mu + 2)(\mu^2 + \mu\lambda + 2) = 0.$$

Hence $$\lambda^2 + \lambda\mu + 2 = 0 \text{ or } \mu^2 + \mu\lambda + 2 = 0.$$

Therefore $Q$ is the point where the normal at $P$ cuts the parabola, or $P$ is the point where the normal at $Q$ cuts the parabola.

(ii) *Prove that the tangents of the angles at which $y = mx + n$ cuts the parabola $y^2 = 4ax$ are given by*

$$\tan^2\theta(n + 2am + am^3) \pm 2\tan\theta(a - mn) + m(mn - a) = 0,$$

*and deduce conditions that the line be* (i) *a tangent,* (ii) *a normal to the parabola.*

Suppose that the line $y = mx + n$ cuts the parabola at the point $(at^2, 2at)$: the condition for this is

$$mat^2 - 2at + n = 0. \tag{i}$$

The tangent at $t$ is $ty = x + at^2$.

Hence

$$\tan\theta = \frac{m - \frac{1}{t}}{1 + \frac{m}{t}} = \frac{tm - 1}{t + m}. \tag{ii}$$

Hence $$t = -\frac{1 + m\tan\theta}{\tan\theta - m}.$$

Substituting this value in (i) we have

$$ma(1 + m\tan\theta)^2 + 2a(1 + m\tan\theta)(\tan\theta - m) + n(\tan\theta - m)^2.$$

Thus $\tan^2\theta(am^3 + 2am + n) + 2\tan\theta(a - mn) + m(mn - a) = 0$.

In (ii) we could equally well take the supplementary angle, i.e.

$$\tan\theta = \frac{\frac{1}{t} - m}{1 + \frac{m}{t}},$$

which gives the alternative sign.

(i) If the line touches the curve, both values of $\theta$ must be zero, hence $mn = a$.

(ii) If the line is a normal, one value of $\theta$ is 90°.

$\therefore$ one value of $\tan\theta$ is infinite and

$$am^3 + 2am + n = 0.$$

## Miscellaneous Examples, VII.

1. Prove that the orthocentres of the triangles formed by three tangents and by the corresponding three normals are equidistant from the axis of the parabola.

2. Prove that if $a^2 > 8b^2$ a point can be found the two tangents from which to $y^2 = 4ax$ are normals to $x^2 = 4by$.

3. Find the equation of the common tangent to the parabolas represented by $y^2 = 4ax$ and $x^2 = 4by$.

4. A system of chords is drawn so that their projections on a line inclined at an angle $\alpha$ to the axis of a parabola are of constant length $c$: prove that the locus of their middle points is the curve

$$(y^2 - 4ax)(y\cos\alpha + 2a\sin\alpha)^2 + a^2c^2 = 0.$$

5. Prove that the locus of the intersections of the tangents at the points $\{a\sinh^2(\alpha \pm \beta),\ 2a\sinh(\alpha \pm \beta)\}$, where $\alpha$ is variable and $\beta$ constant, is a parabola having the same focus as $y^2 = 4ax$.

6. $A$ is the vertex of $y^2 = 4ax$. $P$ is any point on it, and the circle on $AP$ as diameter meets the parabola again in $Q$ and $R$. Show that the normals to the parabola at $P$, $Q$, $R$ meet at a point on the parabola $y^2 = 16a(x + 2a)$.

7. The normal at $P$ meets the axis at $G$; the circle $APG$ cuts the parabola again in $Q$, $R$. Show that the normals to the parabola at $Q$ and $R$ meet at $P$.

8. If $P$ is such that when $PQR$ is drawn in a fixed direction to meet the parabola in $Q$, $R$ the rectangle $PQ \cdot PR$ is constant, the locus of $P$ is a parabola.

9. Two parabolas touch at $P$ and intersect at $Q$, $R$. Prove that $PQ$, $PR$ are harmonically conjugate to the diameters of the two curves at $P$.

10. Prove that, if the normal at $P$ meets the curve again in $Q$, and if the circle on $PQ$ as diameter cuts the curve in $R$, the locus of the middle point of $QR$ is the curve $y^2(y^2 - 4ax) + 64a^4 = 0$.

11. The normal at $P$ to a parabola, whose vertex is $A$, meets the curve again in $Q$: show that the locus of the centre of the circle circumscribed to $APQ$ is a parabola.

12. Through any point not on the axis of a parabola the two straight lines are drawn which are conjugate with regard to the parabola and perpendicular to one another. Prove that they meet the axis in two points equidistant from the focus.

13. If a circle is drawn to pass through the vertex of a parabola and to have its centre on a fixed diameter of the parabola, show that the orthocentre of the triangle formed by the tangents to the parabola at the other three points where the circle cuts it is fixed.

14. Show in a diagram the parabolas $y^2 = 8ax$ and $x^2 = ay$, and prove that they cut each other at right angles at the origin and at an angle whose tangent is $\cdot 6$ at their other point of intersection.

15. Find the locus of a point such that the angles between the connectors of the vertex of the parabola $y^2 = 4ax$ with the points of contact of the tangents from the point may have a given pair of bisectors.

16. Prove that the curves $y^2 = x$, $x^2 + y^2 - 3x + 1 = 0$ touch at two points, and find the equations of their common tangents. Show also that each of these curves touches in the same two points any curve whose equation is $x^2 + y^2 - 3x + 1 + \lambda(y^2 - x) = 0$ for all values of $\lambda$.

17. Find the equations of the circles which touch the directrix of the parabola $y^2 = 4x$, and pass through the points of intersection with the parabola of the straight line $y = x - 1$.

18. Normals are drawn to the parabola $y^2 = 4ax$ to touch the circle $(x-c)^2 + y^2 = r^2$. Find for different values of the radius of the circle the locus of their points of contact.

19. If the normals at three points $P$, $Q$, $R$ on a parabola meet at a point whose abscissa is $x$, prove that the centroid of the triangle $PQR$ is on the axis at a distance from the vertex equal to $\frac{2}{3}(x - 2a)$.

20. Show that the locus of the intersections of equal chords of a parabola drawn in fixed directions is a straight line.

21. Tangents $TP$, $TQ$ are drawn to the parabola $y^2 = 4ax$; find the equation of the circle $TPQ$.

22. Find the envelope of the circle whose diameter is a chord of the parabola $y^2 = 4ax$ passing through a fixed point on the axis of $x$, and show that for one position of the point the envelope reduces to a circle and a straight line.

23. Three normals of which the lengths are $n_1$, $n_2$, $n_3$ and two tangents of which the lengths are $t_1$, $t_2$ are drawn from the same point to a parabola with parameter $4a$. Show that $n_1 n_2 n_3 = a t_1 t_2$.

24. Find the equation of that rectangular pair of conjugate lines with regard to the parabola $y^2 = 4ax$ whose intersection is the point $(h, k)$.

$C$ is a point on the latus rectum and $P$ a point not on the latus rectum such that $PC$ is equally inclined to the rectangular conjugate lines which intersect at $P$. Prove that the locus of $P$ is a circle.

25. Tangents $OP$, $OQ$ are drawn to the parabola $y^2 = 4ax$ from a point $O$ lying on the straight line $x = -\frac{1}{5}a$: show that the envelope of the circle $OPQ$ is the curve $y^2(4a + x) = x(\frac{1}{5}a + x)(5a - x)$.

26. A circle centre $P$, a point on $y^2 = 4ax$, and radius $2SP$ cuts the diameter through $P$ in $Q$, $Q'$: show that the loci of $Q$ and $Q'$ are

$$y^2 + 4ax + 8a^2 = 0 \text{ and } 3y^2 - 4ax + 8a^2 = 0.$$

27. Chords of $y^2 = 4ax$ pass through a fixed point $(\alpha, \beta)$: show that the

locus of the orthocentre of the triangle formed by any such chord and the tangents at its extremities is

$$2x^2+4ax-2\alpha x-\beta y=0.$$

28. If $\rho_1$, $\rho_2$ are the radii of curvature at the feet of the normals to a parabola from a point $P$ on the curve, show $\rho_1\rho_2=\rho\rho_0$ where $\rho$ and $\rho_0$ are the radii of curvature at $P$ and at the vertex.

29. Tangents $TP$, $TQ$ are drawn to a parabola such that $PQ$ is the normal at $P$. Show that the area of the triangle $TPQ$ is $4a^2\sec^3\theta\operatorname{cosec}^3\theta$, where $\theta$ is the acute angle which the normal makes with the axis.

30. Prove that a circle whose diameter is a chord of a parabola such that the distance between the diameters through its extremities is double the latus rectum will touch the parabola.

31. The circle of curvature at $(at^2, 2at)$ cuts the parabola again at the angle $\tan^{-1}\{8t^3/(3t^4-6t^2-1)\}$.

32. If the normal at $P$ makes an acute angle $\psi$ with the axis, and the normals at $Q$, $R$ each make acute angles $\frac{1}{2}\pi-\psi$ with the axis, the three normals form a triangle of area $a^2(\tan^3\psi-\cot\psi)$.

33. Through any point on a given line through the focus three normals are drawn to the parabola: show that the sum of the angles they make with any fixed direction is constant.

34. If the normals to $y^2=4ax$ at the points $(x, y)$, $(x', y')$, $(x'', y'')$ form an equilateral triangle, prove that $(3y^2-4a^2)(3y'^2-4a^2)(3y''^2-4a^2)+64a^6=0$.

35. Four points on a parabola are concyclic and the orthocentre of the triangle formed by three of the points is joined to the fourth: show that the mid-point of the joining line is the same whichever three points are chosen. Also the line joining this mid-point to the centre of the circle is bisected by the axis, and the length of its projection on the axis is the latus rectum.

36. $N_1$, $N_2$, $N_3$ are the lengths of the three normals drawn from a given point to a parabola, and $n_1$, $n_2$, $n_3$ are the lengths intercepted between the curve and the axis: prove that, with the usual convention as to signs,

$$N_1N_2/n_1n_2+N_2N_3/n_2n_3+N_3N_1/n_3n_1+2(N_1/n_1+N_2/n_2+N_3/n_3)+3=0.$$

37. Show that the area of the triangle formed by $x\cos\alpha+y\sin\alpha-p=0$ and the tangents at its extremities to $y^2=4ax$ is $4a^{\frac{1}{2}}(a\tan^2\alpha+p\sec\alpha)^{\frac{3}{2}}$.

38. Show that the length of the normal (other than the radius of curvature $\rho$) drawn from the centre of curvature to the parabola is of length

$$a\{3-(\rho/2a)^{\frac{2}{3}}\}\{4(\rho/2a)^{\frac{2}{3}}-3\}^{\frac{1}{2}}.$$

39. A chord $PQ$ of a parabola makes acute angles $\alpha$ and $\beta$ with the tangents at $P$ and $Q$: show that it makes an acute angle

$$\tan^{-1}\left(\frac{2\sin\alpha\sin\beta}{\sin(\alpha-\beta)}\right)$$

with the axis.

40. Show that the locus of the poles of the axis of a parabola with respect to its circles of curvature is

$$y^2(x-2a)^3=12a(x^2-ax+a^2)^2.$$

41. The normal at $P$ meets the axis in $G$, and $O$ is the centre of curvature at $P$: show that no other normal intersects $PG$ at a distance from $G$ on the side opposite to $O$ which is $> \frac{1}{8} GO$.

42. In a parabola the normal at $P$ meets the curve again at $Q$. If $R$ is a point midway between the centres of curvature at $P$ and $Q$, prove that $R$'s distance from the tangent at the vertex is least when $P$'s distance from it is $a\sqrt{2}$.

43. Show that chords of $y^2 = 4ax$, which are divided by $(x', y')$ in the ratio $\lambda : 1$, have for their equation

$$4\lambda \{y'(y-y') - 2a(x-x')\}^2 + (y-y')^2 (y'^2 - 4ax') (\lambda - 1)^2 = 0.$$

44. $PG$, the normal at $P$ to a parabola, cuts the axis in $G$, and is produced to $Q$ so that $GQ = \frac{1}{2} PG$: show that the other normals passing through $Q$ intersect at right angles.

45. Through a fixed point on the polar of $(3a, 4a)$, with respect to a parabola $y^2 = 4ax$, a chord of the parabola is drawn. Prove that its length will be either a maximum or a minimum when it is inclined at an angle $\frac{1}{4}\pi$ to the axis of $x$. Is it a maximum or minimum?

46. Prove that the latus rectum of the parabola which touches the four common tangents of two circles whose radii are $a$, $b$, and the distance between whose centres is $c$, is $2(a^2 - b^2)/c$.

47. The normal at $P$ is produced outwards to $K$. Find the locus of $K$ (i) when $PK = PG$; (ii) when $PK = \frac{1}{2}$ the radius of curvature at $P$.

48. Find the equation of the parabola which touches the four straight lines $x/a \pm y/b = 1$, $x/a' \pm y/b = 1$.

49. A chord of a parabola is drawn parallel to a fixed direction and on it as diameter a circle is described. Prove that the polar of the vertex with respect to this circle envelopes another fixed parabola.

50. Find the locus of the foot of the perpendiculars from $(h, k)$ to tangents to a parabola, and show that it lies inside an infinite strip perpendicular to the axis of width equal to the focal distance of $(h, k)$.

51. Show that five common normals can be drawn to $y^2 = 4ax$ and $x^2 = 4by$, and that if they are inclined at angles $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, $\theta_5$ to the axis of either parabola then

$$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5) = \tan\theta_1 . \tan\theta_2 . \tan\theta_3 . \tan\theta_4 . \tan\theta_5 .$$

# CHAPTER VIII

## CENTRAL CONICS
## THE ELLIPSE AND THE HYPERBOLA

§ 1. We have seen in Chapter VI that the equations of the ellipse and hyperbola, when referred to their axes of symmetry, take the forms $x^2/a^2+y^2/b^2=1$ and $x^2/a^2-y^2/b^2=1$. The ellipse makes intercepts $2a$, $2b$ on the coordinate axes, and these are the lengths of the axes of the conic. It is conventional to take the major axis along the axis of $x$, so that we have $a > b$.

The hyperbola does not meet the axis of $y$ in real points; its intersections are the imaginary points whose coordinates are $(0, \pm b\sqrt{-1})$. It is, however, common to find $b$ referred to as the length of the other or 'conjugate' axis; evidently $b$ may be either greater or less than $a$. When $a=b$, the hyperbola is called Equilateral or Rectangular.

The central conics can, in many particulars, be conveniently studied together, and in this chapter we shall use the equation $\alpha x^2+\beta y^2=1$ to represent a central conic; for an ellipse $\alpha=1/a^2$, $\beta=1/b^2$, for an hyperbola $\alpha=1/a^2$, $\beta=-1/b^2$.

§ 2. *To find the foci and directrices of a central conic.*

If $(x', y')$ is a focus, and $x\cos\theta+y\sin\theta-p=0$ the corresponding directrix, then the equations

$$\alpha x^2+\beta y^2-1=0, \quad (x-x')^2+(y-y')^2-e^2(x\cos\theta+y\sin\theta-p)^2=0$$

are identical.

Comparing coefficients we have

$$\frac{1-e^2\cos^2\theta}{\alpha}=\frac{1-e^2\sin^2\theta}{\beta}=e^2p^2-x'^2-y'^2,$$

$$e^2\sin\theta\cos\theta=0, \quad x'=e^2p\cos\theta, \quad y'=e^2p\sin\theta.$$

Since $e$ is not zero, $\theta$ is 0 or $\frac{1}{2}\pi$.

(a) If $\theta=0$, then $x'=e^2p$, $y'=0$, and

$$(1-e^2)/\alpha=1/\beta=e^2p^2-e^4p^2=e^2p^2(1-e^2).$$

Hence, since $\alpha = \frac{1}{a^2}$, we have $p = \pm \frac{a}{e}$ and $x' = \pm ae$, where $e^2 = 1 - \frac{\alpha}{\beta}$.

The focus-directrix forms are then

$$(x \pm ae)^2 + y^2 = e^2 (x \pm a/e)^2.$$

The foci and the corresponding directrices are $(ae, 0)$, $x - a/e = 0$, and $(-ae, 0)$, $x + a/e = 0$, where, for the ellipse $e^2 = (a^2 - b^2)/a^2$, and for the hyperbola $e^2 = (a^2 + b^2)/a^2$. In both cases $e$ is real.

(b) If $\theta = \frac{1}{2}\pi$, then $x' = 0$, $y' = e^2 p$, and

$$1/\alpha = (1 - e^2)/\beta = e^2 p^2 - e^4 p^2 = e^2 p^2 (1 - e^2).$$

Hence $p = \pm 1/e\sqrt{\beta}$; $y' = \pm e/\sqrt{\beta}$; and $e^2 = 1 - \beta/\alpha$.

For the ellipse $\beta/\alpha = a^2/b^2 > 1$, $\therefore$ $e$ is imaginary, and also $y'$ and $p$.

For the hyperbola, since $\beta$ is negative, $y'$ and $p$ are imaginary. In this case $e$ is real.

In any central conic, if $e$, $e'$ are the two eccentricities, we have $e^{-2} + e'^{-2} = 1$.

**Examples.**

(See Figures below.)

Prove that

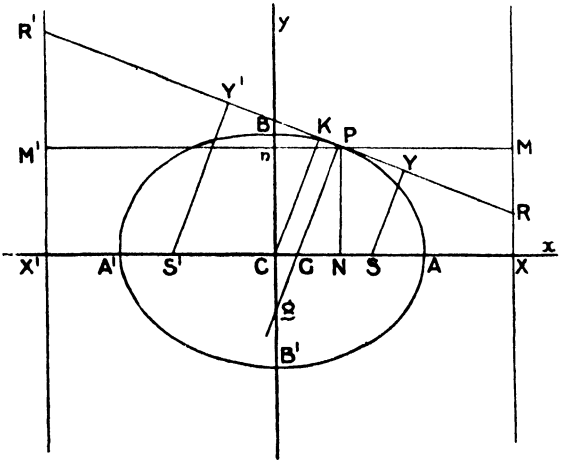
(1) $CS \,.\, CX = CA^2$.

(2) $AS \,.\, A'S = CB^2$.

(3) $SX : CX = CB^2 : CA^2$.

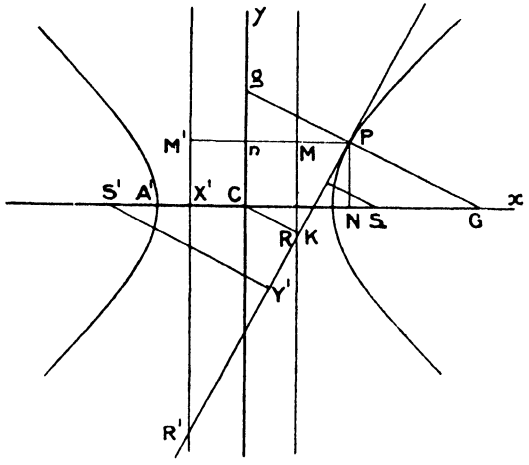
(4) $CB^2 = CA^2 \sim CS^2$.

(5) $CS = e^2 \,.\, CX$.




§ 3. It is convenient to use definite letters to indicate the principal points of central conics: the following should therefore be noted.

Foci $S, S'$; directrices $XM, X'M'$; vertices $A, A', B, B'$; centre $C$; tangent at $P$, $RPR'$; normal at $P$, $PGg$; perpendiculars from the foci and centre on the tangent, $SY$, $S'Y'$, $CK$; ordinates of $P$ to the axes, $PN$ and $Pn$, the latter meeting the directrices in $M$, $M'$.




We have already proved that $CA = CA' = a$; $CB = CB' = b$; $CS = CS' = ae$; $CX = CX' = \frac{a}{e}$; and from the focus-directrix form of the equation of the conic $SP = ePM$, $S'P = ePM'$.

If $P(x'y')$ is any point on the conic,

| *Ellipse.* | *Hyperbola.* |
|---|---|
| $SP = ePM = e(CX - CN)$ | $SP = ePM = e(CN - CX)$ |
| $= e\left(\frac{a}{e} - x'\right)$ | $= e\left(x' - \frac{a}{e}\right)$ |
| $= a - ex'$, | $= ex' - a$, |
| $S'P = ePM' = e(CX' + CN)$ | $S'P = ePM' = e(CN + X'C)$ |
| $= e\left(\frac{a}{e} + x'\right)$ | $= e\left(x' + \frac{a}{e}\right)$ |
| $= a + ex'$. | $= ex' + a$. |
| Hence $\mathbf{SP + S'P = 2a}$. | $\mathbf{S'P - SP = 2a}$. |

**Note.** If $P$ lies on the left-hand branch of the hyperbola we have $SP = a - ex'$ and $S'P = -a - ex'$, in which case $SP - S'P = 2a$.

An ellipse can therefore be described as the locus of points the sum of whose distances from two fixed points is constant, and an hyperbola as the locus of points the difference of whose distances from two fixed points is constant.

§ 4. I. *The equation of the chord, whose mid-point is* $(x', y')$, *is*

$$\alpha xx' + \beta yy' = \alpha x'^2 + \beta y'^2$$

(Chap. VI, § 4).

**II. The tangent.**

*The equation of the tangent at the point* $(x', y')$ *is*

$$\alpha xx' + \beta yy' = 1.$$

(a) *To find the condition that the straight line* $lx + my + n = 0$ *should touch the conic.*

Suppose this straight line touches the conic at the point $(x', y')$, then the equations $lx + my + n = 0$ and $\alpha xx' + \beta yy' - 1 = 0$ are identical. Hence $x' = -l/n\alpha$ and $y' = -m/n\beta$, but since $(x', y')$ lies on the conic we have $\alpha x'^2 + \beta y'^2 = 1$.

Thus, substituting for $x'$ and $y'$, we have the required condition

$$l^2/\alpha + m^2/\beta = n^2.$$

For the ellipse this is $a^2l^2 + b^2m^2 = n^2$, and for the hyperbola $a^2l^2 - b^2m^2 = n^2$.

If the perpendicular from the centre on a tangent makes an angle $\theta$ with the $x$-axis, the equation of the tangent is of the form $x\cos\theta + y\sin\theta - p = 0$. Apply the above condition that this line should be a tangent, and we have for the ellipse $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, and for the hyperbola $p^2 = a^2\cos^2\theta - b^2\sin^2\theta$.

Thus the straight line $x\cos\theta + y\sin\theta = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$ touches the ellipse for all values of $\theta$.

Also $x\cos\theta + y\sin\theta = \sqrt{a^2\cos^2\theta - b^2\sin^2\theta}$ touches the hyperbola.

**Note.** This form of the equation of a tangent is called the pedal equation: the pedal of the ellipse with respect to any point $(h, k)$ is

$$r = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} - h\cos\theta - k\sin\theta,$$

the point $(h, k)$ being the pole and a line parallel to the $x$-axis being the initial line.

**Example** i. *To find the locus of the foot of the perpendicular* $SY$ *from the focus* $S$ *on a tangent to an ellipse, i. e. the pedal of the ellipse with respect to the focus.*

The equation of a tangent is

$$x\cos\theta + y\sin\theta = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}.$$

If $S$ is taken as the pole and $SA$ as the initial line, the polar coordinates of $Y$ are $SY$ and $\theta$.

Hence $$r = SY = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} - ae\cos\theta,$$

therefore $$(r + ae\cos\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta,$$

and since $a^2e^2 = a^2 - b^2$, this becomes

$$r^2 + 2aer\cos\theta + a^2e^2 = a^2.$$

This is the circle on $AA'$ as diameter; it is called the Auxiliary Circle.

**Example ii.** *If $p$ is the length of the perpendicular from the focus $S$ to the tangent at $P$ of a central conic, and $r$ is the length of $SP$, show that for an ellipse* $\frac{b^2}{p^2} = \frac{2a}{r} - 1$, *and for an hyperbola* $\frac{b^2}{p^2} = 1 + \frac{2a}{r}$, *or* $1 - \frac{2a}{r}$, *according as $P$ lies on the near or far branch of the curve.*

Let $P$ be the point $(x', y')$; the equation of the tangent at $P$ is

$$\alpha xx' + \beta yy' = 1,$$

and we have the condition $\alpha x'^2 + \beta y'^2 = 1$.

Now, since $e^2 = 1 - \alpha/\beta$, we have

$$\alpha^2 x'^2 + \beta^2 y'^2 = \alpha^2 x'^2 - \alpha\beta x'^2 + \beta = \beta(1 - \alpha e^2 x'^2) = \beta\left(1 - \frac{e^2 x'^2}{a^2}\right).$$

Thus

$$p^2 = \frac{(1 - \alpha a e x')^2}{\alpha^2 x'^2 + \beta^2 y'^2} = \frac{\left(1 - \frac{ex'}{a}\right)^2}{\beta\left(1 - \frac{e^2 x'^2}{a^2}\right)} = \frac{a - ex'}{\beta(a + ex')};$$

hence

$$\frac{1}{\beta p^2} = \frac{a + ex'}{a - ex'} = \frac{2a}{a - ex'} - 1.$$

For the ellipse, $\beta = \frac{1}{b^2}$ and $r = a - ex'$.

For the hyperbola, $\beta = -\frac{1}{b^2}$ and for the near branch $r = ex' - a$, for the far branch $r = a - ex'$. This gives the required results.

### Examples.

1. Prove that $SY \cdot S'Y' = CB^2$.

2. If the tangent at $P$ meet the major axis at $T$ and the minor axis at $t$, show that $CN \cdot CT = CA^2$, $Cn \cdot Ct = CB^2$.

3. If $T$ is any point of a tangent to a central conic at the point $P$, and $TM$, $TN$ are drawn perpendicular to $SP$ and the directrix corresponding to $S$, show that $SM = eTN$.

Hence prove that the tangents from any point to a conic subtend equal angles at a focus.

4. Prove that $\angle PSR = \angle PS'R' = \frac{1}{2}\pi$.

5. Show that $CY = CY' = CA$.

## III. The Normal.

The equation of the normal at $(x', y')$ to the conic $\alpha x^2 + \beta y^2 = 1$ is $\beta y'(x - x') = \alpha x'(y - y')$.

**Note.** If the normal at $(x', y')$ passes through a given point $(h, k)$, we have $\beta y'(h - x') = \alpha x'(k - y')$, or

$$(\alpha - \beta) x'y' - \alpha k x' + \beta h y' = 0;$$

hence the point $(x', y')$ lies on the rectangular hyperbola (Chap. VI, § 6)

$$(\alpha - \beta) xy - \alpha kx + \beta hy = 0.$$

Since, in general, two curves of the second degree intersect in four points, four normals can be drawn from any point $(h, k)$ to a central conic and their feet lie on the above rectangular hyperbola.

### Examples.

1. Show that the pedal equation of the normal to an ellipse is
$$x\cos\theta + y\sin\theta = (a^2-b^2)\sin\theta\cos\theta\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}.$$
2. Prove that $SG = e\,.\,SP$.
3. Prove that $PG\,.\,CK = CB^2$; $Pg\,.\,CK = CA^2$; $PG\,.\,Pg = SP\,.\,S'P$.
4. Show that $NG : CN = CB^2 : CA^2$.
5. Prove that $SP : S'P = SY : S'Y' = SG : S'G$, and hence show that
$$\angle SPY = \angle S'PY'.$$
6. Show that $S$, $S'$, $P$, $t$, $g$ lie on a circle.
7. Prove that $CG = e^2\,.\,CN$, and $a^2\,.\,GN = b^2\,.\,CN$.

### IV. Conjugate points and lines.

(a) The polar of the point $(x', y')$ with respect to the conic $\alpha x^2+\beta y^2 = 1$ is $\alpha xx'+\beta yy' = 1$.

Hence the polar of $(x', y')$ passes through the point $(x'', y'')$ if $\alpha x'x''+\beta y'y'' = 1$. The symmetry of this result shows that if the polar of $(x', y')$ passes through $(x'', y'')$, then the polar of $(x'', y'')$ passes through $(x', y')$; it is therefore the condition that these points should be conjugate.

(b) *To find the pole of the line* $lx+my+n = 0$ *with respect to the conic* $\alpha x^2+\beta y^2 = 1$.

Let the point $(x', y')$ be the pole, then evidently the equations $lx+my+n = 0$ and $\alpha xx'+\beta yy'-1 = 0$ must be identical. Hence $x' = -\dfrac{l}{\alpha n}$ and $y' = -\dfrac{m}{\beta n}$. The straight lines $lx+my+n = 0$, $l'x+m'y+n' = 0$ are conjugate (i. e. the pole of each lies on the other) if $\dfrac{ll'}{\alpha}+\dfrac{mm'}{\beta}-nn' = 0$.

If one of the straight lines is a diameter $lx+my = 0$, since $n=0$ it follows that the straight lines $lx+my = 0$, $l'x+m'y+n' = 0$ are conjugate whatever value $n'$ may have. Hence the poles of all straight lines parallel to $l'x+m'y = 0$ lie on the diameter $lx+my = 0$ if $ll'/\alpha+mm'/\beta = 0$.

### V. Conjugate Diameters.

Two diameters are conjugate if each bisects all chords parallel to the other; referring to the condition found in Chap. VI, § 2. I, we see that the two diameters $lx+my = 0$, $l'x+m'y = 0$ are conjugate

if $ll'/\alpha + mm'/\beta = 0$. This is also the condition that the poles of lines parallel to the one should lie on the other (see IV above). We can therefore define conjugate diameters either as in Chapter VI or by this polar property.

In particular, the diameters $y = mx$, $y = m'x$ are conjugate for the ellipse if $mm' = -b^2/a^2$, and are conjugate for the hyperbola if $mm' = +b^2/a^2$.

**Note i.** Since a diameter bisects all chords parallel to its conjugate, the middle point of the chord whose equation is $lx + my + n = 0$ is its point of intersection with the diameter $m\alpha x - l\beta y = 0$

**Note ii.** A convenient general form for a pair of conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $ay - \lambda bx = 0$, $\lambda ay + bx = 0$.

Suppose that these diameters meet the ellipse at $PP'$, $DD'$ respectively; then for the abscissae of $P$ and $P'$ we have

$$x^2(1+\lambda^2) = a^2.$$

Hence, if $P$, $P'$ are the points $(x_1, y_1)$, $(-x_1, -y_1)$,

$$x_1^2 = a^2/(1+\lambda^2),\quad y_1^2 = \lambda^2 b^2/(1+\lambda^2)$$

and
$$CP^2 = CP'^2 = x_1^2 + y_1^2 = (a^2 + \lambda^2 b^2)/(1+\lambda^2).$$

Similarly, if $D$, $D'$ are the points $(x_2, y_2)$, $(-x_2, -y_2)$,

$$x_2^2 = \lambda^2 a^2/(1+\lambda^2),\quad y_2^2 = b^2/(1+\lambda^2),$$

and
$$CD^2 = CD'^2 = (\lambda^2 a^2 + b^2)/(1+\lambda^2).$$

Hence $CP^2 + CD^2 = a^2 + b^2$, i.e. the sum of the squares of two conjugate diameters is constant.

Now if $CP = CD$, we have $a^2 + \lambda^2 b^2 = \lambda^2 a^2 + b^2$, i.e.

$$(\lambda^2 - 1)(a^2 - b^2) = 0;\quad \therefore\ \lambda = \pm 1.$$

The equations of these diameters are therefore $x/a \pm y/b = 0$; they are called the **equi-conjugate diameters.**

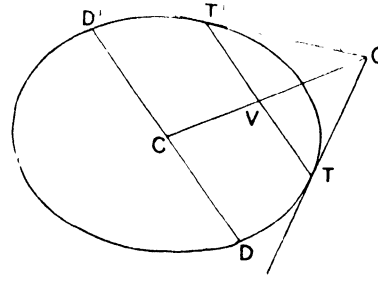
**Note iii.** A pair of conjugate diameters of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ may be represented by

$$ay - \lambda bx = 0,\quad \lambda ay - bx = 0.$$

Proceeding as in Note ii, we find

$$CP^2 = (a^2 + \lambda^2 b^2)/(1-\lambda^2),\quad CD^2 = -(\lambda^2 a^2 + b^2)/(1-\lambda^2).$$

Hence $CP^2 + CD^2 = a^2 - b^2$; but evidently $CP^2$ and $CD^2$ are of different sign, hence either $CP$ or $CD$ is imaginary. Only one of two conjugate diameters meets the hyperbola in real points.




If lengths $CD_1$, $CD_1'$ are taken on the diameters which meet the conic in imaginary points so that

$$CD_1^2 = CD_1'^2 = -CD^2,$$

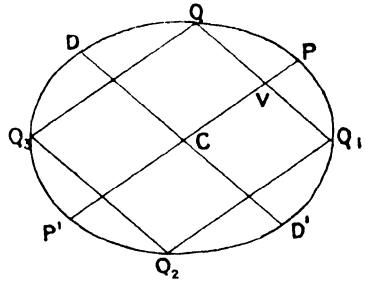
then $CD_1$ is often called the length of the semi-diameter conjugate to $CP$.

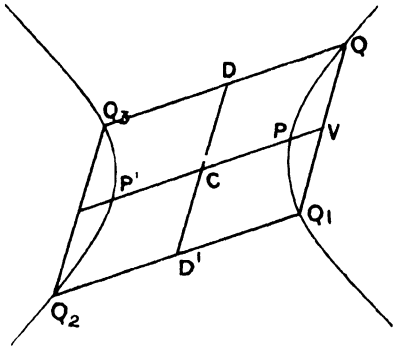
In this case $CP^2 \sim CD_1^2 = a^2 \sim b^2$, or, the difference of the squares on conjugate diameters of the hyperbola is constant.

**Note iv.** If $CP$, $CD$ are two conjugate diameters, the polar of any point on $CP$ is parallel to $CD$ and to the tangent at $P$. Thus, in the Figure (see p. 309), $V$ is the mid-point of $TT'$.

In particular, $DD'$ is the polar of the 'point at infinity' on $CP$.

**Note v.** *To find the equation of a central conic referred to a pair of conjugate diameters as axes of coordinates.*




Let $PCP'$, $DCD'$ be any pair of conjugate diameters, and $Q$ any point $(x, y)$ on the conic referred to these diameters as coordinate axes.

If $QQ_1$, $QQ_3$ are drawn parallel to $CD$ and $CP$, they are bisected by $CP$ and $CD$; further, if $Q_1Q_2$ is a chord parallel to $CP$, it is bisected by $CD'$; hence $Q_1Q_2$ is both parallel and equal to $QQ_3$, therefore $Q_2Q_3$ is also parallel and equal to $QQ_1$. Hence $Q_2Q_3$ is also bisected by $CP'$.

Thus, if the point $(x, y)$ lies on the conic, so do the points $(-x, y)$, $(-x, -y)$, $(x, -y)$, i.e. the equation of the conic is of the form

$$\alpha x^2 + \beta y^2 = 1.$$

If $CP = a'$, $CD = b'$ the equation of the ellipse is

$$x^2/a'^2 + y^2/b'^2 = 1;$$

for the hyperbola $CD$ is imaginary, and the equation is

$$x^2/a'^2 - y^2/b'^2 = 1.$$

It follows that all the results we find for the equation of a central conic referred to its principal axes are true also for the conic referred to any pair of conjugate diameters except when our results depend upon the axes being rectangular.

## Examples.

1. Prove that (Fig., p. 309) $CV \cdot CQ = CP^2$.
2. Show that $TV^2 : CP^2 - CV^2 = CD^2 : CP^2$.
3. Any pair of orthogonal lines through a focus are conjugate.
4. Prove that $a \cdot PG = b \cdot CD$, and $b \cdot Py = a \cdot CD$, where $CD$ is the semi-diameter conjugate to $CP$.

VI. **Pair of tangents.**

The equation of a pair of tangents from $P(x', y')$ to the conic $\alpha x^2+\beta y^2=1$ is $(\alpha x^2+\beta y^2-1)(\alpha x'^2+\beta y'^2-1)=(\alpha xx'+\beta yy'-1)^2$.

These are parallel to the pair of diameters

$$(\alpha x^2+\beta y^2)(\alpha x'^2+\beta y'^2-1)=(\alpha xx'+\beta yy')^2,$$

i.e. $$\alpha(\beta y'^2-1)x^2-2\alpha\beta x'y'xy+\beta(\alpha x'^2-1)y^2=0.$$

If the tangents, and therefore the parallel diameters, include an angle $\theta$, then

$$\tan\theta=\frac{2\sqrt{\{\alpha^2\beta^2x'^2y'^2-\alpha\beta(\alpha x'^2-1)(\beta y'^2-1)\}}}{\alpha(\beta y'^2-1)+\beta(\alpha x'^2-1)}$$

$$=\frac{2\sqrt{\{\alpha\beta(\alpha x'^2+\beta y'^2-1)\}}}{\alpha\beta\left\{x'^2+y'^2-\frac{1}{\alpha}-\frac{1}{\beta}\right\}}.$$

The angle $\theta$ is a right angle if

$$x'^2+y'^2=\frac{1}{\alpha}+\frac{1}{\beta}.$$

Hence the locus of point $P$, the tangents from which to the conic are at right angles, is the circle

$$x^2+y^2=\frac{1}{\alpha}+\frac{1}{\beta},$$

which is called the **director circle**.

For the ellipse this becomes $x^2+y^2=a^2+b^2$, and for the hyperbola $x^2+y^2=a^2-b^2$.

The latter is real only when $a>b$. When the hyperbola is rectangular the circle reduces to a point at the centre: the asymptotes are the only real orthogonal tangents.

**Note.** The equation

$$b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+e^2\left(x^2-\frac{a^2}{e^2}\right)=0$$

represents (*vide* p. 284) a locus passing through the points of intersection of the ellipse, and its directrices $x^2-\frac{a^2}{e^2}=0$. This equation reduces to

$$x^2\left(\frac{b^2}{a^2}+e^2\right)+y^2=a^2+b^2,$$

i.e. $$x^2+y^2=a^2+b^2, \text{ since } a^2e^2=a^2-b^2.$$

It follows that the common chords of the ellipse and its director circle are the directrices.

### Illustrative Examples.

(i) *The angles which the normals from* $(f, g)$ *to the ellipse*

$$x^2/a^2+y^2/b^2-1=0$$

*make with the* $x$*-axis are given by*

$$(f\sin\theta-g\cos\theta)^2(a^2\cos^2\theta+b^2\sin^2\theta)=(a^2-b^2)^2\sin^2\theta\cos^2\theta.$$

The equation of the normal at the point $(x'y')$ is

$$\frac{y'}{b^2}(x-x')=\frac{x'}{a^2}(y-y'),$$

and if this makes an angle $\theta$ with the $x$-axis,

$$\tan\theta=\frac{a^2y'}{b^2x'},$$

so that

$$\frac{x'}{a^2\cos\theta}=\frac{y'}{b^2\sin\theta}. \qquad \text{(i)}$$

If the normal passes through the point $(f, g)$, we have

$$\frac{y'}{b^2}(f-x)=\frac{x'}{a^2}(g-y'),$$

or

$$(f-x)\sin\theta=(g-y)\cos\theta. \qquad \text{(ii)}$$

But since $(x'y')$ lies on the curve, we have (i)

$$\frac{x'}{a^2\cos\theta}=\frac{y'}{b^2\sin\theta}=\frac{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\frac{1}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}.$$

Substituting in (ii) for $x'$ and $y'$, we get

$$f\sin\theta-g\cos\theta=\frac{(a^2-b^2)\cos\theta\sin\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}},$$

i.e. $$(a^2\cos^2\theta+b^2\sin^2\theta)(f\sin\theta-g\cos\theta)^2=(a^2-b^2)^2\cos^2\theta\sin^2\theta.$$

(ii) *Show that if* $(\xi, \eta)$ *is a point of intersection of the ellipses* $x^2/a^2+y^2/b^2=1$, $x^2/a'^2+y^2/b'^2=1$, *the equations of their common tangents are* $\pm x\xi/aa'\pm y\eta/bb'=1$; *and the product of the areas of the parallelograms formed by their four common points and their four common tangents is* $8aa'bb'$.

The common points of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,\quad \frac{x^2}{a'^2}+\frac{y^2}{b'^2}-1=0$$

are given by

$$\frac{x^2b^2b'^2}{b'^2-b^2}=\frac{y^2a^2a'^2}{a^2-a'^2}=\frac{a^2b^2a'^2b'^2}{a^2b'^2-a'^2b^2}, \qquad \text{(i)}$$

hence we gather that the points are symmetrically placed with respect to the principal axes: let their coordinates be $(\xi, \eta)$, $(\xi, -\eta)$, $(-\xi, -\eta)$, $(-\xi, \eta)$. The area of the parallelogram they form is $4\xi\eta$.

If $lx+my=1$ is a tangent to both conics,

$$a^2l^2+b^2m^2-1=0,\ a'^2l^2+b'^2m^2-1=0.$$

Hence

$$\frac{l^2}{b'^2-b^2}=\frac{m^2}{a^2-a'^2}=\frac{1}{a^2b'^2-a'^2b^2}. \qquad \text{(ii)}$$

Thus, combining (i), which is satisfied by $(\xi, \eta)$, and (ii),

$$\frac{l^2}{b^2b'^2\xi^2}=\frac{m^2}{a^2a'^2\eta^2}=\frac{1}{a^2a'^2b^2b'^2};$$

$$\therefore \qquad l=\pm\xi/aa',\quad m=\pm\eta/bb'.$$

Hence the common tangents are

$$\pm x\xi/aa'\pm y\eta/bb'=1.$$

The corners of the parallelogram formed by these tangents are

$$(aa'/\xi, 0),\ (-aa'/\xi, 0),\ (0, bb'/\eta),\ (0, -bb'/\eta),$$

and consequently the area of the parallelogram they form is $2aa'bb'/\xi\eta$.

The product of the areas of the two parallelograms in question is therefore $8aa'bb'$.

(iii) *The tangents from any point to a central conic are equally inclined to the focal distances of the point.*

Let the point be $(x', y')$: then the equations of the focal distances are

$$y'(x-ae)-y(x'-ae)=0,\ y'(x+ae)-y(x'+ae)=0;$$

these are parallel to

$$xy'-y(x'-ae)=0,\ xy'-y(x'+ae)=0;$$

i. e. to the lines $$(xy'-yx')^2-y^2a^2e^2=0,$$

or $$\alpha\beta(xy'-yx')^2-y^2(\beta-\alpha)=0,$$

or $$\alpha\beta\ y'^2x^2-2\alpha\beta x'y'xy+(\alpha\beta x'^2+\alpha-\beta)y^2=0.$$

The bisectors of the angles between these lines are the same as those of the angles between the lines

$$\alpha(\beta y'^2-1)x^2-2\alpha\beta x'y'xy+\beta(\alpha x'^2-1)y^2=0,$$

which, we showed above, are parallel to the tangents from $(x', y')$ to the conic.

Hence the bisectors of the angles between the tangents from a point to the conic and of the angles between the focal distances of the point are the same: this establishes the proposition.

## Examples VIII a.

1. A circle on a diameter $PP'$ of an ellipse as diameter meets the tangent at an end of the minor axis in $Q$ and $Q'$. Show that $QQ'$ is equal to the difference of the distances of $P$ from the two foci.

2. Find the equation of the pair of tangents from the point $(\alpha, \beta)$ to the hyperbola $x^2/a^2-y^2/b^2=1$.

For the case in which $\alpha=5$, $\beta=3$, $a=1$, $b=1/\sqrt{3}$, find the equation of each tangent separately.

3. If $(h, k)$ is a point $P$ on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and $A$, $A'$ the extremities of its major axis, show that $\cot APA' = k(b^2 - a^2)/2ab^2$.

4. Prove that if $(x, y)$ and $(x', y')$ are any two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, then $b^2x/(y+y') + a^2y/(x+x') = a^2b^2/(xy' + x'y)$. Deduce that the locus of the middle points of parallel chords is a straight line.

5. Prove that the bisectors of the angles between the focal distances of a point $P$ on an ellipse are the tangent and normal at $P$.

6. The locus of points at which the ellipse subtends an angle 60° is

$$3(x^2 + y^2 - a^2 - b^2)^2 = 4(x^2b^2 + y^2a^2 - a^2b^2).$$

7. Prove that the rectangle under the perpendiculars drawn to a normal at $P$ from the centre and the pole of the normal is equal to the rectangle under the focal distances of $P$.

8. Tangents are drawn to the ellipse $(a, b)$ from the point

$$[a^2/\sqrt{(a^2 - b^2)},\ \sqrt{(a^2 + b^2)}]:$$

show that the intercept made by them on the ordinate through the nearer focus is equal to the major axis.

9. A rod $AB$ of length $l$ moves with its extremities on two fixed lines which intersect each other at right angles. If $P$ be the point which divides $AB$ in the ratio 2 to 3, show that the locus of $P$ is an ellipse, and state its eccentricity.

Find the points on $AB$ which describe ellipses whose eccentricity is $\frac{4}{5}$.

10. Find the coordinates of the intersections of

$$x\cos\alpha/a - y\sin\alpha/b = \cos 2\alpha \text{ with } x^2/a^2 + y^2/b^2 = 1.$$

Find also the locus of the projection of the centre of the ellipse on the above line.

11. Find the locus of the point of intersection of tangents to an ellipse which meet at a given angle $\alpha$.

Pairs of tangents to an ellipse intersect at right angles; prove that their chords of contact touch a fixed concentric ellipse.

12. Show that the points in which the straight line $x\cos a + y\sin a = 2$ meets the hyperbola $2x^2 - y^2 = 4$ subtend a right angle at the centre of the hyperbola.

13. Find the coordinates of the foci and the length of the latus rectum of the conic $\lambda x^2 + (1+\lambda)y^2 = \lambda^2$, where $\lambda$ is positive.

Find also the locus of the extremities of the latera recta as $\lambda$ varies.

14. Show that the locus of the middle points of chords of the ellipse $a^{-2}x^2 + b^{-2}y^2 = 1$, the tangents at the ends of which intersect on the circle $x^2 + y^2 = a^2$, is $(a^{-2}x^2 + b^{-2}y^2)^2 = a^{-2}(x^2 + y^2)$.

15. All the chords of an ellipse whose middle points are on the same straight line touch a parabola.

16. Tangents are drawn to the ellipse $x^2/a^2 + y^2/b^2 = 1$ from any point on the circle $x^2 + y^2 = a^2 + b^2$.

Prove that

(i) the tangents are at right angles;

(ii) the locus of the middle points of the chord of contact is given by the equation $(x^2 + y^2) = (a^2 + b^2)(x^2/a^2 + y^2/b^2)^2$.

17. Find the coordinates of the extremities of the diameter of the ellipse $x^2/a^2+y^2/b^2=1$ which is conjugate to the diameters $y'x=x'y$.

Two such conjugate diameters are inclined at angles $\theta$ and $\phi$ to the major axis of the ellipse: show that their lengths $a'b'$ are connected by the relation $a'^2\sin 2\theta+b'^2\sin 2\phi=0$.

18. If $S$, $H$ are the foci of an ellipse and $P$ is any point on the curve, show that the locus of the centre of the inscribed circle of the triangle $SPH$ is the ellipse $x^2+(1+e)y^2/(1-e)=a^2e^2$.

19. Chords of an ellipse which subtend a right angle at the centre are distant $ab/\sqrt{a^2+b^2}$ from the centre.

20. Find the equation to the locus of the foot of the perpendicular drawn to a tangent from one of the foci of $ax^2+by^2=1$.

21. Express the length of the perpendicular from the centre on the normal to an ellipse in terms of the perpendicular on the corresponding tangent.

Show that the area of the rectangle formed by two parallel tangents and the corresponding normals is never greater than half the square on the line joining the foci.

22. Chords $BD$, $BE$ of the ellipse $x^2/a^2+y^2/b^2=1$ are drawn at angles $\pi/3$ to the $x$-axis to meet the ellipse again in $D$ and $E$. Find the coordinates of the centre of the circle inscribed to the triangle $BDE$.

23. If $PQ$ is normal to the conics

$$ax^2+by^2=1,$$
$$a'x^2+b'y^2=1$$

at $P$ and $Q$,

$$PQ^2=(ab'-a'b)\left\{\frac{1}{a'b'(a-b)}-\frac{1}{ab(a'-b')}\right\}.$$

24. A diameter $DD'$ of an ellipse is produced, meeting the director circle in $O$, and two points $P$ and $Q$ are taken on the diameter produced such that the angle between the two tangents from $P$ is the supplement of that between the tangents from $Q$. Prove that $PD\,.\,PD'\,.\,QD\,.\,QD'=OD^2\,.\,OD'^2$.

25. An ellipse has its centre at $O$, its axes lie on the coordinate axes $OX$ and $OY$, and it passes through the points $P(2,7)$ and $Q(4,3)$. Find the equation of the ellipse and give the positions of the foci. Show that the length of the semi-diameter conjugate to $OP$ is $\sqrt{841/30}$, and give its equation.

26. A variable tangent is drawn to the hyperbola $x^2-y^2=a^2$ cutting the circle $x^2+y^2=a^2$ in $P$ and $Q$.

Show that the locus of the middle point of $PQ$ is

$$(x^2+y^2)^2=a^2(x^2-y^2).$$

27. $P$ and $Q$ are extremities of two conjugate diameters of the ellipse $b^2x^2+a^2y^2=a^2b^2$, and $S$ is a focus. Prove that $PQ^2-(SP-SQ)^2=2b^2$.

28. Show that a normal to an ellipse divides the distance between the two parallel tangents most unequally when it is equally inclined to the axes.

29. Find the equation of a chord of the ellipse $x^2/a^2+y^2/b^2=1$ in terms

of $x_1, y_1$ the coordinates of its middle point, and show that the equation of the circle described on the chord as diameter is

$$(b^2x_1^2 + a^2y_1^2)\{(x-x_1)^2 + (y-y_1)^2\} = a^2b^2(x_1^2/a^4 + y_1^2/b^4)(a^2b^2 - b^2x_1^2 - a^2y_1^2).$$

30. The points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are the vertices of a triangle $ABC$ self-polar for the conic $x^2/a^2 + y^2/b^2 - 1 = 0$.

Prove that the points $A$, $B$, $C$ are on the rectangular hyperbola

$$(x_1x_2x_3)/(a^2x) + (y_1y_2y_3)/(b^2y) = 1,$$

and that the lines $BC$, $CA$, $AB$ touch the parabola

$$b^2\sqrt{x_1x_2x_3x} + a^2\sqrt{y_1y_2y_3y} = a^2b^2.$$

## § 5. Coordinates expressed in terms of a single parameter.

### I. The Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Since the equation of the ellipse gives us

$$x = \pm a\sqrt{1 - \frac{y^2}{b^2}} \quad \text{and} \quad y = \pm b\sqrt{1 - \frac{x^2}{a^2}},$$

it is evident that the coordinates $x$ and $y$ of **real** points on the ellipse lie in magnitude between $+a$ and $-a$, $+b$ and $-b$ respectively. Now the point whose coordinates are $(a\cos\theta,\ b\sin\theta)$ lies on the ellipse for all values of $\theta$, and further, since $\cos\theta$ and $\sin\theta$ can have any values between $+1$ and $-1$, any point on the ellipse can be so represented.

**Geometrical interpretation when the coordinate axes are rectangular.**

The coordinates of any point $p$ on the circle $x^2 + y^2 = a^2$ described on $AA'$ as diameter can be represented by $(a\cos\theta,\ a\sin\theta)$ where $\theta$ is the angle $pCA$: if the ordinate $pN$ meet the ellipse at $P$, the $x$-coordinate of $P$ must then be $a\cos\theta$, and its $y$-coordinate is consequently $b\sin\theta$; i. e. $P$ is the point $(a\cos\theta,\ b\sin\theta)$. The angle $\theta$ corresponding to any point $P$ on the ellipse is called its **eccentric angle**; the points $p$, $P$ are said to **correspond**, and the eccentric angle of $P$ is that made with the $x$-axis by the radius-vector to the corresponding point $p$. The circle $ApA'$ is called the **auxiliary circle**. It is evident from the symmetry of the

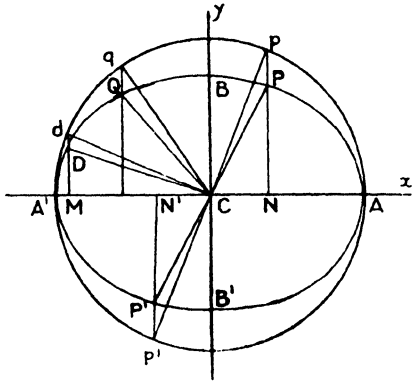


figure that if $P$ is the point $\theta$, $P'$, the other extremity of the diameter $PC$, is the point $(\pi+\theta)$. Now the ordinates of corresponding points are in a fixed ratio, viz.

$$PN : pN = b : a.$$

Various properties of the ellipse can be deduced from properties of the circle as illustrated below.

(i) *If $P$, $p$ and $Q$, $q$ are two pairs of corresponding points, then* $\Delta pCq : \Delta PCQ = a : b$.

For if $P$, $Q$ have eccentric angles $\theta$, $\phi$, the areas of the triangles $pCq$ and $PCQ$ are

$\frac{1}{2}\{a^2\cos\theta\sin\phi - a^2\cos\phi\sin\theta\}$ and $\frac{1}{2}\{ab\cos\theta\sin\phi - ab\cos\phi\sin\theta\}$

which are in the ratio $a : b$.

Hence it follows that if $P$, $p$, $Q$, $q$, $R$, $r$ are three pairs of corresponding points, $\Delta PQR : \Delta pqr = b : a$, for

$$\Delta PQR = \Delta PCQ + \Delta QCR + \Delta RCP, \text{ \&c.}$$

Consequently, when the triangle $pqr$ is a maximum, the triangle $PQR$ is a maximum. But the maximum triangle which can be inscribed in the circle is equilateral, and its area is $\frac{1}{4}(3\sqrt{3})a^2$; hence the maximum triangle which can be inscribed in the ellipse has for the eccentric angles of its vertices $\theta$, $\frac{2}{3}\pi+\theta$, $\frac{4}{3}\pi+\theta$, and its area is $\frac{1}{4}(3\sqrt{3})ab$.

(ii) **Conjugate diameters.**

*Suppose $P(a\cos\alpha,\ b\sin\alpha)$, $D(a\cos\beta,\ b\sin\beta)$ are the extremities of two conjugate diameters $CP$, $CD$.*

The equations of $CP$, $CD$ are

$$y = \frac{bx}{a}\tan\alpha, \quad y = \frac{bx}{a}\tan\beta,$$

whence
$$\frac{b^2}{a^2}\tan\alpha\,.\tan\beta = -\frac{b^2}{a^2}, \qquad \text{(\textit{Vide} p. 309)}$$

or
$$1 + \tan\alpha\,.\tan\beta = 0;$$

$\therefore$
$$\alpha \backsim \beta = \tfrac{1}{2}\pi.$$

Hence the eccentric angles of the ends of conjugate diameters are of the form $\alpha$, $\alpha \pm \frac{1}{2}\pi$.

If $p$, $d$ are the corresponding points on the auxiliary circle, it follows that the diameters $cp$, $cd$ are at right angles, and are therefore conjugate diameters of the circle.

If $pCp'$, $dCd'$ are diameters of the circle which are at right angles, $pdp'd'$ is a square, and its area is $4pCd = 2a^2$.

If $PCP'$, $DCD'$ are conjugate diameters of the ellipse corresponding to these, $PDP'D'$ is a parallelogram, its area is

$$4PCD = 4\frac{b}{a}.pCd = 2ab;$$

thus the parallelogram whose diagonals are a pair of conjugate diameters has a constant area.

Since the tangents at $P$, $P'$ are parallel to $CD$, and those at $D$, $D'$ parallel to $CP$, the tangents at the extremities of a pair of conjugate diameters form a parallelogram; its area is evidently twice that of the parallelogram $PDP'D'$, i.e. $4ab$. This is called the **conjugate parallelogram.**

Incidentally,

$$CP^2 = a^2\cos^2\alpha + b^2\sin^2\alpha,$$

$$CD^2 = a^2\cos^2(\alpha \pm \tfrac{1}{2}\pi) + b^2\sin^2(\alpha \pm \tfrac{1}{2}\pi);$$

$$\therefore \quad \mathbf{CD^2 = a^2\sin^2\alpha + b^2\cos^2\alpha},$$

and, as before shown, $\mathbf{CP^2 + CD^2 = a^2 + b^2}$.

Since $e^2 = \dfrac{a^2-b^2}{a^2}$, we can write these values in the convenient form

$$\mathbf{CP^2 = a^2(1-e^2\sin^2\alpha);} \quad \mathbf{CD^2 = a^2(1-e^2\cos^2\alpha).}$$

(iii) The area of a sector $pCq$ of the auxiliary circle is $\frac{1}{2}a^2(\theta_2-\theta_1)$; the area of the sector $PCQ$ of the ellipse is $\frac{1}{2}ab(\theta_2-\theta_1)$.

The student should now re-read Chapter V, §§ 5–7; by exactly similar methods to those there illustrated the following equations can be found:—

(a) The equation of the chord joining the points whose eccentric angles are $\theta$, $\phi$, is

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\phi) + \frac{y}{b}\sin\tfrac{1}{2}(\theta+\phi) = \cos\tfrac{1}{2}(\theta-\phi).$$

(b) The tangent at the point, whose eccentric angle is $\theta$, is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

(c) The point of intersection of tangents at the points whose eccentric angles are $(\theta, \phi)$ is

$$\left\{\frac{a\cos\tfrac{1}{2}(\theta+\phi)}{\cos\tfrac{1}{2}(\theta-\phi)}, \quad \frac{b\sin\tfrac{1}{2}(\theta+\phi)}{\cos\tfrac{1}{2}(\theta-\phi)}\right\}$$

or

$$\left\{a.\frac{1-\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi}{1+\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi}, \quad b.\frac{\tan\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\phi}{1+\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi}\right\}.$$

(d) The equation of the tangent at the point $P$, whose eccentric angle is $\theta$, can be put in the form

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{\sqrt{\{(x-a\cos\theta)^2+(y-b\sin\theta)^2\}}}{\sqrt{\{a^2\sin^2\theta+b^2\cos^2\theta\}}}=\frac{r}{CD},$$

where $r$ is the distance of any point $(x, y)$ on the tangent from the point of contact $P$ and $CD$ is the semi-diameter conjugate to $CP$.

**Example.** *To find the lengths of the tangents which can be drawn from any point to an ellipse.*

The equation of the tangent at the point $P(\theta)$ is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{r}{CD}, \tag{i}$$

and, if $(x, y)$ is the given point from which the tangents are drawn, $r$ is the length of the tangent if the point $\theta$ is the point of contact.

Hence the elimination of $\theta$ from equation (i) will give us an equation in $r$ whose roots will be the lengths of the required tangents.

Now
$$\frac{x}{a}=\frac{r}{CD}\sin\theta+\cos\theta,$$

$$\frac{y}{b}=-\frac{r}{CD}\cos\theta+\sin\theta.$$

Hence
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=\frac{r^2}{CD^2},$$

or
$$r^2=f.\,CD^2, \tag{ii}$$

where
$$f\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1.$$

Thus
$$r^2=f(a^2\sin^2\theta+b^2\cos^2\theta),$$

or
$$r^2(\cos^2\theta+\sin^2\theta)=f(a^2\sin^2\theta+b^2\cos^2\theta),$$

i.e.
$$(r^2-a^2f)\tan^2\theta+(r^2-b^2f)=0. \tag{iii}$$

But the first of equations (i) gives us

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,$$

i.e.
$$\frac{x}{a}+\frac{y}{b}\tan\theta=\sec\theta,$$

or
$$\left(\frac{x}{a}+\frac{y}{b}\tan\theta\right)^2=1+\tan^2\theta;$$

$\therefore$
$$\left(1-\frac{y^2}{b^2}\right)\tan^2\theta-\frac{2xy}{ab}\tan\theta+1-\frac{x^2}{a^2}=0.$$

Eliminating $\theta$ from this equation and (iii) we get

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2r^4-2f\left\{(x^2+y^2)f+a^2b^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)\right\}r^2$$
$$+f^2[(x+ae)^2+y^2][(x-ae)^2+y^2]=0.$$

which is the required equation.

**Cor. i.** If $P$, $P'$ are the points of contact of tangents from $T(x, y)$ to the ellipse, $CD$, $CD'$ the diameters conjugate to $CP$, $CP'$, we have from (ii) $TP^2 = f \cdot CD^2$ and $TP'^2 = f \cdot CD'^2$.

$\therefore\ TP : TP' = CD : CD'$.

**Cor. ii.** The term independent of $r$ represents the product of the squares of the distances of $T$ from the foci $S$, $S'$.

Hence
$$TP^2 \cdot TP'^2 = \frac{f^2 \cdot ST^2 \cdot S'T^2}{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2},$$

or
$$TP \cdot TP' = \frac{f}{f+1} \cdot ST \cdot S'T.$$

(e) **The Normal.** The normal at the point $\theta$ being perpendicular to the tangent, its equation is

$$\frac{\sin\theta}{b}(x - a\cos\theta) - \frac{\cos\theta}{a}(y - b\sin\theta) = 0,$$

which is usually given in the form

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2.$$

It can also be put in the useful form

$$\frac{x - a\cos\theta}{b\cos\theta} = \frac{y - b\sin\theta}{a\sin\theta} = \frac{r}{CD}.$$

*The normals which can be drawn from any point to an ellipse.*

If the normal at the point whose eccentric angle is $\theta$ passes through the fixed point $(h, k)$, then

$$ah\sec\theta - bk\operatorname{cosec}\theta = a^2 - b^2,$$

and, conversely, this equation must be satisfied by the eccentric angle of any point the normal at which passes through $(h, k)$.

The equation can be written in the following three forms, where $c^2 \equiv a^2 - b^2$.

I. $$c^4\cos^4\theta - 2c^2ah\cos^3\theta + (a^2h^2 + b^2k^2 - c^4)\cos^2\theta + 2c^2ah\cos\theta - a^2h^2 = 0.$$

II. $$c^4\sin^4\theta + 2c^2bk\sin^3\theta + (a^2h^2 + b^2k^2 - c^4)\sin^2\theta - 2c^2bk\sin\theta - b^2k^2 = 0.$$

III. $$bk\tan^4\tfrac{1}{2}\theta + 2(ah + c^2)\tan^3\tfrac{1}{2}\theta + 2(ah - c^2)\tan\tfrac{1}{2}\theta - bk = 0.$$

Each of the equations is quartic, and it follows that *four* normals can be drawn from any point to an ellipse, of which all may be real, two real and two imaginary, or all imaginary.

It should be noted that one value of $\tan\frac{1}{2}\theta$ corresponds to one definite normal, for this value gives one value only for $\cos\theta$ and $\sin\theta$.

*To find the conditions that the normals at* **four** *points on an ellipse, whose eccentric angles are given, should be concurrent.*

If $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ are the eccentric angles of the four points, then for some value of $h$ and $k$ these must satisfy equation III above. The coefficient of $\tan^2 \frac{1}{2}\theta$ is zero, and the coefficient of $\tan^4 \frac{1}{2}\theta$ is equal and opposite to the absolute term; hence we get the following conditions:—

(a) $\Sigma \tan \frac{1}{2}\theta_1 . \tan \frac{1}{2}\theta_2 = 0$;

(b) $\tan \frac{1}{2}\theta_1 . \tan \frac{1}{2}\theta_2 . \tan \frac{1}{2}\theta_3 . \tan \frac{1}{2}\theta_4 = -1$.

Only two conditions are necessary in order that four straight lines should be concurrent, consequently these conditions are necessary and sufficient.

From (a) and (b) it follows immediately that

$$\tan(\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 + \tfrac{1}{2}\theta_3 + \tfrac{1}{2}\theta_4) = \infty;$$

hence

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n+1)\pi.$$

*To find the condition that the normals at* **three** *points on the ellipse should be concurrent.*

Equation (a) above gives us

$$\tan \tfrac{1}{2}\theta_1 \tan \tfrac{1}{2}\theta_2 + \tan \tfrac{1}{2}\theta_2 \tan \tfrac{1}{2}\theta_3 + \tan \tfrac{1}{2}\theta_3 \tan \tfrac{1}{2}\theta_1 = -\tan \tfrac{1}{2}\theta_4 (\tan \tfrac{1}{2}\theta_1 + \tan \tfrac{1}{2}\theta_2 + \tan \tfrac{1}{2}\theta_3).$$

Hence, substituting for $\tan \frac{1}{2}\theta_4$ from (b),

$$\tan \tfrac{1}{2}\theta_1 \tan \tfrac{1}{2}\theta_2 + \tan \tfrac{1}{2}\theta_2 \tan \tfrac{1}{2}\theta_3 + \tan \tfrac{1}{2}\theta_3 \tan \tfrac{1}{2}\theta_1 = \cot \tfrac{1}{2}\theta_1 \cot \tfrac{1}{2}\theta_2 + \cot \tfrac{1}{2}\theta_2 \cot \tfrac{1}{2}\theta_3 + \cot \tfrac{1}{2}\theta_3 \cot \tfrac{1}{2}\theta_1.$$

But $$\cot \tfrac{1}{2}\theta_1 \cot \tfrac{1}{2}\theta_2 - \tan \tfrac{1}{2}\theta_1 \tan \tfrac{1}{2}\theta_2 = \frac{2(\cos\theta_1 + \cos\theta_2)}{\sin\theta_1 \sin\theta_2}.$$

Hence $$\Sigma \sin\theta_3 (\cos\theta_1 + \cos\theta_2) = 0,$$

or $$\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0,$$

which is the required condition.

This condition can also be deduced from equations I and II.

*To find the conditions that the normals at the extremities of the chords whose equations are* $lx + my - 1 = 0$, $Lx + My - 1 = 0$ *should be concurrent.*

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda(lx + my - 1)(Lx + My - 1) = 0 \qquad \text{(i)}$$

is satisfied by the coordinates of the points common to $lx + my - 1 = 0$, $Lx + My - 1 = 0$ respectively, and the ellipse: it therefore represents

for different values of $\lambda$ the conics which can be drawn through the points of intersection of these chords and the ellipse.

Now we have shown in § 4 (iii) that if the normals at these points of intersection are concurrent at the point $(h, k)$, then these points lie on the rectangular hyperbola

$$xy(a^2-b^2)-a^2hy+b^2kx=0; \qquad \text{(ii)}$$

thus for some value of $\lambda$ the equations (i) and (ii) will be identical.

Since the term independent of $x$ and $y$ in (ii) is zero, we have at once $\lambda = 1$.

Also equating the coefficients of $x^2$ and $y^2$ to zero, we get

$$Ll = -\frac{1}{a^2}, \quad Mm = -\frac{1}{b^2},$$

or
$$a^2lL = b^2Mm = -1,$$

which are the required conditions.

The equations of two chords the normals at whose extremities are concurrent can then be put in either of the following forms:—

$$\left.\begin{aligned} \frac{lx}{a}+\frac{my}{b}-1=0 \\ \frac{x}{la}+\frac{y}{mb}+1=0 \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} \frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-d=0 \\ \frac{x}{a}\sec\theta+\frac{y}{b}\operatorname{cosec}\theta+\frac{1}{d}=0 \end{aligned}\right\},$$

both of which are quite general.

The equation of the rectangular hyperbola passing through the ends of the chords then becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\left(\frac{lx}{a}+\frac{my}{b}-1\right)\left(\frac{x}{la}+\frac{y}{mb}+1\right)=0,$$

which at once reduces to

$$xy(l^2+m^2)-al(1-m^2)y+bm(l^2-1)x=0,$$

and comparing this with (ii) we find that the coordinates of the point of intersection of the normals are

$$h=\frac{a^2-b^2}{a}\cdot\frac{l(1-m^2)}{l^2+m^2}; \qquad k=\frac{a^2-b^2}{b}\cdot\frac{m(l^2-1)}{l^2+m^2}.$$

*To find the locus of the intersection of coincident normals.*

The chord joining the feet of coincident normals is then a tangent to the ellipse: if $(a\cos\theta, b\sin\theta)$ is the point of contact and the foot of the normal, the chord is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1=0.$$

Hence the equation of the chord the normals at whose extremities are concurrent with the normal at $(a\cos\theta,\ b\sin\theta)$ is

$$\frac{x}{a}\sec\theta + \frac{y}{b}\operatorname{cosec}\theta + 1 = 0.$$

Hence, as in the last paragraph, if $(h, k)$ is the point of intersection of these normals, the rectangular hyperbolas

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 - \left(\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta - 1\right)\left(\frac{x}{a}\sec\theta + \frac{y}{b}\operatorname{cosec}\theta + 1\right) = 0$$

and

$$xy(a^2-b^2) - a^2hy + b^2kx = 0$$

are identical. The former reduces to

$$xy - b\sin^3\theta\, x - a\cos^3\theta\, y = 0.$$

Hence, comparing coefficients,

$$h = \frac{a^2-b^2}{a}\cos^3\theta, \qquad k = -\frac{a^2-b^2}{b}\sin^3\theta.$$

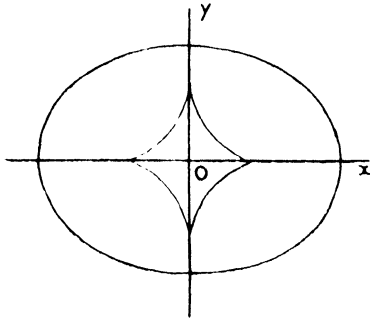
For different values of $\theta$ the locus of this point is

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}},$$

which curve, called the evolute of the ellipse, is the locus of the intersections of consecutive normals. Incidentally, we see that

$$\left(\frac{a^2-b^2}{a}\cos^3\theta,\ \frac{b^2-a^2}{b}\sin^3\theta\right)$$

is the centre of curvature at the point $(a\cos\theta,\ b\sin\theta)$.




The form of the evolute is shown in the figure: from points within it four real normals can be drawn to the ellipse; from points on it the four normals are real but two coincident; from points outside it only two real normals can be drawn.

### Illustrative Examples.

**Example i.** *The normals at $P, Q, R, S$ meet in a point, and $P', Q', R', S'$ are the points on the auxiliary circle corresponding to $P, Q, R, S$ respectively. If straight lines are drawn through $P, Q, R, S$ parallel to $CP'$, $CQ'$, $CR'$ $CS'$, they are concurrent.*

Let $P$ be the point $(a\cos\theta, b\sin\theta)$, then $P'$ is $(a\cos\theta, a\sin\theta)$.

The equation of $CP'$ is $x\sin\theta - y\cos\theta = 0$.

Hence that of a line through $P$ parallel to it is

$$(x - a\cos\theta)\sin\theta - (y - b\sin\theta)\cos\theta = 0,$$

or
$$x\sin\theta - y\cos\theta - (a-b)\sin\theta\cos\theta = 0.$$

If this straight line passes through a given point $(x', y')$,

$$x'\sin\theta - y'\cos\theta - (a-b)\sin\theta\cos\theta = 0.$$

Write $t$ for $\tan\frac{\theta}{2}$, this equation becomes

$$y't^4 + 2t^3(x' + a - b) + 2t(x' - a + b) - y' = 0. \qquad \text{(i)}$$

Hence four lines of this type can be drawn which intersect at the point $(x', y')$ : if $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ are the eccentric angles of $P$, $Q$, $R$, $S$, the corresponding lines of the above form are concurrent provided $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, for some value of $x'$ and $y'$, satisfy equation (i). The conditions for this are

$$\Sigma\tan\tfrac{1}{2}\theta_1\tan\tfrac{1}{2}\theta_2 = 0, \quad \tan\tfrac{1}{2}\theta_1 \cdot \tan\tfrac{1}{2}\theta_2 \cdot \tan\tfrac{1}{2}\theta_3 \cdot \tan\tfrac{1}{2}\theta_4 = -1,$$

and these are identical with the conditions that the four normals at $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ should be concurrent.

**Example ii.** *Lengths are measured off from $P$ on the normal at $P$ in both directions equal to the semi-diameter perpendicular to the normal; prove that the loci of the two points thus obtained are circles.*

Let $P$ be the point $(a\cos\theta, b\sin\theta)$, the equation of the normal is

$$\frac{x - a\cos\theta}{b\cos\theta} = \frac{y - b\sin\theta}{a\sin\theta} = \frac{r}{CD},$$

where $CD$ is the semi-diameter conjugate to $CP$ and therefore perpendicular to the normal at $P$.

We have then to find the loci of points on the normal distant $\pm CD$ from the point $(a\cos\theta, b\sin\theta)$.

Their coordinates are therefore given by

$$\frac{x - a\cos\theta}{b\cos\theta} = \frac{y - b\sin\theta}{a\sin\theta} = \pm 1,$$

i.e.
$$x = (a \pm b)\cos\theta,$$
$$y = (b \pm a)\sin\theta.$$

Eliminating $\theta$ the loci are

$$x^2 + y^2 = (a \pm b)^2,$$

i.e. two circles.

**The intersections of the ellipse and a circle. Application of parametric coordinates.**

The equation of any circle is of the form

$$x^2+y^2+2gx+2fy+c=0.$$

The eccentric angles of the points of intersection of this circle and the ellipse are given by

$$a^2\cos^2\theta+b^2\sin^2\theta+2ga\cos\theta+2fb\sin\theta+c=0,$$

for this is the condition that the point $(a\cos\theta,\ b\sin\theta)$ of the ellipse should lie on the circle.

Writing $t\equiv\tan\frac{1}{2}\theta$, this equation reduces to

$$a^2(1-t^2)^2+4b^2t^2+2ga(1-t^4)+4fbt(1+t^2)+c(1+t^2)^2=0,$$

i.e.

$$t^4(a^2-2ga+c)+4fbt^3+t^2(4b^2-2a^2+2c)+4fbt+a^2+2ga+c=0.$$

Hence a circle intersects an ellipse in four points and, if the eccentric angles of these points are $\theta_1$. $\theta_2$, $\theta_3$, $\theta_4$,we have, since the coefficients of $t^3$ and $t$ are equal,

$$\Sigma\tan\tfrac{1}{2}\theta=\Sigma\tan\tfrac{1}{2}\theta_1\,.\tan\tfrac{1}{2}\theta_2\,.\tan\tfrac{1}{2}\theta_3;$$

hence
$$\tan(\tfrac{1}{2}\theta_1+\tfrac{1}{2}\theta_2+\tfrac{1}{2}\theta_3+\tfrac{1}{2}\theta_4)=0,$$

i.e.
$$\theta_1+\theta_2+\theta_3+\theta_4=2n\pi.$$

Conversely, this is the condition that four points, whose eccentric angles are given, should be concyclic: one condition is sufficient since any three points lie on a circle.

**Note 1.** The equations of the common chords of the circle and ellipse being

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta_1+\theta_2)+\frac{y}{b}\sin\tfrac{1}{2}(\theta_1+\theta_2)=\cos\tfrac{1}{2}(\theta_1-\theta_2),$$

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta_3+\theta_4)+\frac{y}{b}\sin\tfrac{1}{2}(\theta_3+\theta_4)=\cos\tfrac{1}{2}(\theta_3-\theta_4),$$

the condition $\Sigma\theta=2n\pi$ shows that common chords of a circle and an ellipse are equally inclined to the axes.

**The circle of curvature.** Since the circle of curvature meets the ellipse (p. 280) in three coincident points, if the eccentric angle of these points is $\theta$ and that of the other point of intersection $\theta_1$, we have $3\theta+\theta_1=2n\pi$, i.e. the circle of curvature at $\theta$ cuts the ellipse again at the point $2n\pi-3\theta$.

The common chord of the circle of curvature at $\theta$ and the ellipse is therefore $\dfrac{x}{a}\cos\theta-\dfrac{y}{b}\sin\theta=\cos2\theta.$

The equation of the circle of curvature is therefore of the form (p. 287)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left\{\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1\right\}\left\{\frac{x}{a}\cos\theta-\frac{y}{b}\sin\theta-\cos 2\theta\right\}=0.$$

Equating the coefficients of $x^2$ and $y^2$, we get for $\lambda$

$$\frac{\lambda\cos^2\theta}{a^2}+\frac{1}{a^2}=-\frac{\lambda\sin^2\theta}{b^2}+\frac{1}{b^2},$$

or
$$\lambda=\frac{(a^2-b^2)}{a^2\sin^2\theta+b^2\cos^2\theta},$$

i. e. the equation of the circle is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)(a^2\sin^2\theta+b^2\cos^2\theta)$$
$$+(a^2-b^2)\left\{\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1\right\}\left\{\frac{x}{a}\cos\theta-\frac{y}{b}\sin\theta-\cos 2\theta\right\}=0,$$

which reduces to

$$x^2+y^2-\frac{2(a^2-b^2)}{a}\cos^3\theta\,.\,x-\frac{2(b^2-a^2)}{b}\sin^3\theta\,.\,y$$
$$+(a^2-2b^2)\cos^2\theta+(b^2-2a^2)\sin^2\theta=0.$$

**Cor.** The radius of curvature at the point $P(a\cos\theta, b\sin\theta)$ is

$$\frac{\{a^2\sin^2\theta+b^2\cos^2\theta\}^{\frac{3}{2}}}{ab},$$

or, if $CD$ is the semi-diameter conjugate to $CP$, its length is $\frac{CD^3}{ab}$.

**Note.** The work throughout this paragraph can be considerably shortened by using $\frac{a}{2}(e^{i\theta}+e^{-i\theta})$, $\frac{b}{2i}(e^{i\theta}-e^{-i\theta})$ instead of $a\cos\theta$, $b\sin\theta$. The method involves the use of imaginary quantities; we leave it as an exercise for the student.

## Examples VIII b.

1. If a circle touches and cuts an ellipse, the tangent at the point of contact and the common chord are equally inclined to the axis of the ellipse.

2. The inclinations to the axis of the ellipse of tangents drawn to it from the point $(x, y)$ are given by the equation

$$(x^2-a^2)\tan^2\theta-2xy\tan\theta+y^2-b^2=0.$$

3. If the normals at four points whose eccentric angles are $\theta_1, \theta_2, \theta_3, \theta_4$ are concurrent, then

$$\Sigma\sin\theta_1+\Sigma\sin\theta_1\sin\theta_2\sin\theta_3=0,$$

and
$$\Sigma\cos\theta_1+\Sigma\cos\theta_1\cos\theta_2\cos\theta_3=0.$$

4. If from any point four normals are drawn to an ellipse meeting one of the axes in $G_1$, $G_2$, $G_3$, $G_4$, then $\Sigma(1/CG) = 4/\Sigma CG$.

5. Show that it is impossible for the normals at four concyclic points on an ellipse to be concurrent.

6. The three points $Q_1$, $Q_2$, $Q_3$ are such that their three circles of curvature intersect on the ellipse at the point whose eccentric angle is $\theta$. Show that $Q_1$, $Q_2$, $Q_3$ are the vertices of a triangle of maximum area inscribed in the ellipse.

7. If four concurrent normals are $OP$, $OQ$, $OR$, $OS$ and $T$, $T'$ the poles of $PQ$, $RS$, then, if $CT$, $CT'$ make angles $\alpha$, $\beta$ with the axis, show that $\tan\alpha \tan\beta = b^2/a^2$.

8. The normal at any point $P$ of an ellipse intersects the axes at $M$ and $N$ respectively; prove that $PM$ is to $PN$ in a constant ratio.

9. The tangent at a point on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ meets the axes in $T$, $t$ and the normal meets them in $G$, $g$; prove that the locus of the intersection of $Tg$ and $tG$ is the curve $(x^2+y^2)^2/(a^2-b^2)^2 = x^2/a^2 + y^2/b^2$.

10. From points on a line parallel to the axis of $x$ normals are drawn to the ellipse; show that, if $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ are the eccentric angles of the feet of the normals drawn from a point on the line, $\Sigma(\sin\alpha)$ and $\Sigma(\operatorname{cosec}\alpha)$ are both constant.

11. Show that the equation of the chord of the ellipse $x^2/a^2 + y^2/b^2 = 1$ joining points whose eccentric angles are $\alpha+\beta$, $\alpha-\beta$ is

$$(x\cos\alpha)/a + (y\sin\alpha)/b = \cos\beta.$$

12. If $CP$ and $CQ$ are conjugate diameters, show that

$$4(CP^2 - CQ^2) = (SP - S'P)^2 - (SQ - S'Q)^2.$$

13. Two tangents $TP$, $TQ$ are drawn to the ellipse from the point $T$, whose coordinates are $h$, $k$; show that the area of the triangle $TPQ$ is $ab\{h^2/a^2 + k^2/b^2 - 1\}^{\frac{3}{2}}/\{h^2/a^2 + k^2/b^2\}$.

14. Prove that the normals at the points where the line

$$x/(a\cos\alpha) + y/(b\sin\alpha) = 1$$

intersects the conic $x^2/a^2 + y^2/b^2 = 1$ meet at the point whose coordinates are $-c^2\cos^3\alpha/a$, $+c^2\sin^3\alpha/b$. $(c^2 = a^2 - b^2.)$

15. $PP'$ is a double ordinate of an ellipse, and the normal at $P$ meets $CP'$ in $Q$. Show that the locus of a point which divides $P$, $Q$ in a given ratio is an ellipse.

16. $A$, $B$, $C$ are the vertices of a triangle of maximum area inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$; $P$, $Q$, $R$ the centres of curvature corresponding to $A$, $B$, $C$. Find the locus of the centroid of the triangle $PQR$.

17. Find the equation of the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ in terms of the angle $\alpha$ the perpendicular from the centre on it makes with the major axis. If this tangent and a perpendicular tangent be taken as new coordinate axes, what will be the coordinates of the centre of the ellipse?

18. The tangent at $(x', y')$ meets the auxiliary circle at $QQ'$. Show that the lines $CQ$, $CQ'$ are represented by the equations $xy' = y(x' \pm ae)$; $C$ being the centre of the ellipse.

19. Show that there is in general one conic of finite axes with given centre and direction of axes which has two given lines as normals.

20. The locus of the intersection of perpendicular normals to an ellipse is $(a^2+b^2)(x^2+y^2)(a^2y^2+b^2x^2)^2 = (a^2-b^2)^2(a^2y^2-b^2x^2)^2$.

21. The normal at $P$ meets the tangent at $Q$ on the minor axis; show that $PQ$ touches $x^2/a^2(a^2-2b^2)-y^2/b^4 = 1/(a^2-b^2)$.

22. If the normals to an ellipse at $ABCD$ are concurrent, and diameters are drawn parallel to $AB$ and $CD$, their extremities are at the angular points of a parallelogram whose sides are parallel to the equi-conjugate diameters.

23. If the centres of curvature of an ellipse at the extremities of a pair of conjugate diameters are joined to the centre, the product of the tangents of the angles these lines make with the major axis is constant.

24. The normal at any point $P$ meets the axis in $G$, a point $Q$ is taken in the tangent so that $PQ = \lambda . PG$, where $\lambda$ is constant; prove that the locus of $Q$ is $x^2/a^2+y^2/b^2 = (a^2+\lambda^2b^2)/a^2$.

25. The line joining the centre of an ellipse to the pole of the chord common to the ellipse and the circle of curvature at any point, and the line joining the centre of the ellipse to the point where the circle of curvature is drawn, make equal angles with the axes.

26. If $TP$, $TQ$ are the tangents from the point $T(f, g)$ to the ellipse $x^2/a^2+y^2/b^2-1 = 0$, whose centre is $C$, prove that the area of the quadrilateral $TPCQ$ is (i) $\sqrt{b^2f^2+a^2g^2-a^2b^2}$; (ii) $\frac{1}{2}TT'^2\tan\theta$, where $TT'$ is the tangent from $T$ to the director circle and $\theta$ is the angle $PTQ$.

27. $PQ$, $PR$ are focal chords of an ellipse. Prove that the tangents at $Q$ and $R$ intersect on the normal at $P$.

28. In an ellipse, if $CP$ and $CD$ are conjugate diameters, find the envelope of $PD$.

29. A chord $PQ$ of a conic passes through a fixed point. If the circle on $PQ$ as diameter meets the conic again in $P'Q'$, show that $P'Q'$ also passes through a fixed point.

30. Find the coordinates of the intersection of the normals at the points of contact of two tangents from $(\xi, \eta)$ to the ellipse, and show that the normals at the points of contact of tangents from $(-a^2/\xi, -b^2/\eta)$ pass through the same point.

31. A chord of $x^2/a^2+y^2/b^2 = 1$ touches $x^2/a_1^2+y^2/b_1^2 = 1$.

If $2r$ is the length of a diameter of the first ellipse parallel to the chord and $c$ the length of the chord,

$$(a^2-b^2)c^2 = 4r^2[a^2-b^2-(a_1^2-b_1^2)+r^2(a_1^2/a^2-b_1^2/b^2)].$$

32. The normal at a point $P$ of the curve meets the major axis in $G$, and a point $Q$ is taken on the normal at $P$ such that $PQ = PG$; find the locus of $Q$.

33. If $(x_1, y_1)$ and $(x_2, y_2)$ are two points on the ellipse $x^2/a^2+y^2/b^2 = 1$, the tangents at which meet in $(x, y)$ and the normals in $(\xi, \eta)$, prove that $a^2\xi = e^2xx_1x_2$ and $b^2\eta = e^2yy_1y_2$, where $e$ is the eccentricity.

34. Chords of the ellipse $x^2/a^2+y^2/b^2=1$ are drawn parallel to the diameter $lx/a+my/b=0$; find the locus of their middle points.

The normal at $P$ to the ellipse meets the curve again at $Q$; show that the locus of the middle point of $PQ$ is given by the equation

$$(b^2x^2+a^2y^2)^2(b^6x^2+a^6y^2)=a^4b^4(a^2-b^2)^2x^2y^2.$$

35. Normals at the ends of chords parallel to the tangent at $\theta$ meet in points lying on

$$2(ax\sin\theta+by\cos\theta)(ax\cos\theta+by\sin\theta)=(a^2-b^2)^2\sin 2\theta\,.\cos^2 2\theta.$$

36. If normals to an ellipse from $O$ meet the curve in $P_1$, $P_2$, $P_3$, $P_4$ and the major axis in $N_1$, $N_2$, $N_3$, $N_4$, then $\Sigma(OP_1/N_1P_1)$ is constant.

37. Show that unless the eccentricity of an ellipse be greater than $1/\sqrt{2}$ it is impossible for the centre of curvature at any point of the ellipse to lie on the curve itself.

38. The locus of the poles of normal chords of an ellipse is

$$a^6/x^2+b^6/y^2=(a^2-b^2)^2.$$

39. From any point on the normal at the point $\alpha$ on an ellipse two other normals are drawn to the ellipse. Show that the locus of the point of intersection of corresponding tangents is

$$bx\sin\alpha+ay\cos\alpha+xy=0.$$

40. From any point of the curve $x^2/a^2+y^2/b^2=(x^2/a^2-y^2/b^2)^2$ tangents are drawn to the ellipse $x^2/a^2+y^2/b^2=1$; show that the line joining the points of contact is the chord of curvature at one of them.

41. $P$ is any point on the ellipse $x^2/a^2+y^2/b^2=1$. Show that the normal at $P$ bisects the angle between the focal distances $SP$ and $HP$. If $SP$ is produced to $Q$, making $PQ=PH$, and $HP$ is produced to $R$, making $PR=PS$, show that $RQ$ intersects the tangent at $P$ on the major axis; and find the equation of the locus of the intersection of $RQ$ with a line drawn from the centre of the ellipse, parallel to $HQ$.

42. If tangents $TP$, $TQ$ are drawn from $T(f, g)$ to a conic $x^2/a^2+y^2/b^2=1$, prove the difference of the angles $TPQ$, $TQP$ is

$$\tan^{-1}\frac{2fg(a^2-b^2)(a^2g^2+b^2f^2-a^2b^2)}{4a^2b^2f^2g^2+(a^4g^2-b^4f^2)(a^2-b^2-f^2+g^2)}.$$

43. The circles of curvature at the points $L$, $M$, $N$ on the ellipse meet the ellipse at the same point $O$ whose eccentric angle is $\alpha$; find the eccentric angles of $L$, $M$, $N$, and show that the circle circumscribing the triangle $LMN$ passes through $O$.

44. A point $P$ on the ellipse $b^2x^2+a^2y^2=a^2b^2$ is such that the centre of curvature there lies also on the ellipse. Find its coordinates, and show that the radius of curvature at $P$ is $\{3\sqrt{3}a^2b^2\}/\{(a^2+b^2)^{\frac{3}{2}}\}$.

45. Prove that four normals, real or imaginary, can in general be drawn from a point to an ellipse; and show that the line joining the feet of any two of them is equally inclined to the axis with the diameter which bisects the chords joining the feet of the other two. Show also that the middle points of the diagonals of the quadrilateral formed by the tangents at the

four feet lie on a straight line which passes through the centre and is at right angles to the diameter which passes through the point.

46. Chords through the point

$$(a^2-b^2)\,.\,a\cos\theta/(a^2+b^2),\ (b^2-a^2)\,.\,b\sin\theta/(b^2+a^2)$$

subtend a right angle at the point $(a\cos\theta,\ b\sin\theta)$.

47. If $\alpha, \beta$ are the coordinates of the centre of curvature at a point $(x', y')$ on $x^2/a^2+y^2/b^2=1$, and if the centre of curvature is on the ellipse, prove $\alpha/x'+\beta/y'+1=0$.

48. The normals to an ellipse at $P$, $Q$, $R$ meet in a point, and also the sum of the eccentric angles of these points is constant.

Show that the locus of their point of intersection is a straight line, and that the sides of the triangle $PQR$ touch a parabola.

49. A point $P$ moves on the ellipse

$$x^2/\{(2a^2-b^2)^2\}+y^2/\{(2b^2-a^2)^2\}=1/k^2,$$

where $k$ is a constant; prove that it is a constant distance from the centroid of the four points on the ellipse $x^2/a^2+y^2/b^2=1$, whose normals meet at $P$.

50. A tangent is drawn from the point $(a\cos\theta,\ b\sin\theta)$ of an ellipse to the circle of curvature at the other end of the diameter through the point; show that the length of the tangent is $2\{(a^2-b^2)\cos 2\theta\}^{\frac{1}{2}}$.

## § 6. II. The Hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Several systems of parametric coordinates are possible; each can be developed in a manner analogous to that used for the circle and the ellipse: we shall therefore only give a short sketch of each.

(a) Any point on the hyperbola can be represented by

$$(a\cos\theta,\ ib\sin\theta),$$

where $i\equiv\sqrt{-1}$. The results in this case can be deduced from those found for the ellipse by substituting throughout $ib$ for $b$. The objection to this system is the use of an imaginary quantity.

(b) The hyperbolic trigonometrical functions can be used, and any point on the curve represented by $(a\cosh\theta,\ b\sinh\theta)$. This system has the advantage of retaining the symmetry to which we have become accustomed in the case of the ellipse. The objection to the system is that only one branch of the curve can be so represented for real values of $\theta$.

The area of a sector $PCQ$ of the hyperbola is $\frac{1}{2}ab\,(\theta_1 \frown \theta_2)$.

The equation of the chord joining two points $\alpha$, $\beta$ is

$$\frac{x}{a}\cosh\tfrac{1}{2}(\alpha+\beta)-\frac{y}{b}\sinh\tfrac{1}{2}(\alpha+\beta)=\cosh\tfrac{1}{2}(\alpha-\beta).$$

The equation of the tangent at the point $\theta$ is

$$\frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta = 1.$$

The equation of the normal at the point $\theta$ is

$$ax\operatorname{sech}\theta + by\operatorname{cosech}\theta = a^2 + b^2.$$

This equation can also be written

$$by\tanh^4\tfrac{1}{2}\theta + 2(ax + a^2 + b^2)\tanh^3\tfrac{1}{2}\theta - 2(ax - a^2 - b^2)\tanh\tfrac{1}{2}\theta - by = 0.$$

Thus the normals at $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ are concurrent if

$$\Sigma\tanh\tfrac{1}{2}\theta_1 . \tanh\tfrac{1}{2}\theta_2 = 0$$

and $$\tanh\tfrac{1}{2}\theta_1 . \tanh\tfrac{1}{2}\theta_2 . \tanh\tfrac{1}{2}\theta_3 . \tanh\tfrac{1}{2}\theta_4 = -1.$$

But

$$\tanh(\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 + \tfrac{1}{2}\theta_3 + \tfrac{1}{2}\theta_4)$$
$$= \frac{\Sigma\tanh\tfrac{1}{2}\theta + \Sigma\tanh\tfrac{1}{2}\theta_1\tanh\tfrac{1}{2}\theta_2\tanh\tfrac{1}{2}\theta_3}{1 + \Sigma\tanh\tfrac{1}{2}\theta_1\tanh\tfrac{1}{2}\theta_2 + \tanh\tfrac{1}{2}\theta_1\tanh\tfrac{1}{2}\theta_2\tanh\tfrac{1}{2}\theta_3\tanh\tfrac{1}{2}\theta_4}.$$

Hence, if the normals at these points are concurrent,

$$\tanh(\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 + \tfrac{1}{2}\theta_3 + \tfrac{1}{2}\theta_4) = \infty.$$
$$\therefore\quad \Sigma\theta = (2n+1)i\pi.$$

(c) Again, any point on the hyperbola can be represented by $(a\sec\theta,\ b\tan\theta)$. The results are not analogous to those for an ellipse.

The equation of a chord joining the points $\alpha$, $\beta$ is

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha-\beta) - \frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta) = \cos\tfrac{1}{2}(\alpha+\beta).$$

The equation of the tangent at the point $\theta$ is

$$\frac{x}{a} - \frac{y}{b}\sin\theta = \cos\theta.$$

The equation of the normal at the point $\theta$ is

$$ax\sin\theta + by = (a^2+b^2)\tan\theta.$$

(d) The most workable system of coordinates is a variation of (b): we can use for any point on the hyperbola the coordinates

$$\left\{\frac{a}{2}\left(t+\frac{1}{t}\right),\quad \frac{b}{2}\left(t-\frac{1}{t}\right)\right\}.$$

The equation of the chord joining the points $t_1$, $t_2$ is

$$\frac{x}{a}(1+t_1t_2) + \frac{y}{b}(1-t_1t_2) = t_1+t_2.$$

The equation of the tangent at the point $t$ is

$$\frac{x}{a}(1+t^2)+\frac{y}{b}(1-t^2)=2t.$$

The equation of the normal at the point $t$ is

$$2at(1-t^2)x-2bt(1+t^2)y=(a^2+b^2)(1-t^4).$$

This equation can also be written

$$(a^2+b^2)t^4-2t^3(ax+by)+2t(ax-by)-(a^2+b^2)=0.$$

Thus the condition that the normals at the points $t_1$, $t_2$, $t_3$, $t_4$ should be concurrent are

$$\Sigma t_1t_2=0, \quad t_1t_2t_3t_4=-1.$$

Eliminating $t_4$ we see that the condition that the normals at the three points $t_1$, $t_2$, $t_3$ should be concurrent is

$$\Sigma t_1t_2=\sum\frac{1}{t_1t_2}.$$

The equation giving the parameters of the points of intersection of the hyperbola and the circle

$$x^2+y^2+2gx+2fy+c=0$$

is

$$(a^2+b^2)t^4+4(ga+fb)t^3+2(2c+a^2-b^2)t^2+4(ga-fb)t+(a^2+b^2)=0.$$

Hence the condition that four points on the hyperbola, whose parameters are $t_1$, $t_2$, $t_3$, $t_4$, should be concyclic is

$$t_1t_2t_3t_4=1.$$

**Example** i. *To find the equation of the circle of curvature at the point whose parameter is $t$, and the coordinates of the centre of curvature.*

Let the circle of curvature be

$$x^2+y^2+2gx+2fy+c=0,$$

then the parameters of its points of intersection with the ellipse are given by the equation

$$(a^2+b^2)t^4+4(ga+fb)t^3+2(2c+a^2-b^2)t^2+4(ga-fb)t+a^2+b^2=0.$$

Since three of these intersections are coincident, let the roots of this equation be $t$, $t$, $t$, $t'$.

Then

$$t^3t'=1; \quad \therefore\ t'=\frac{1}{t^3}.$$

Also

$$-\frac{4(ga+fb)}{a^2+b^2}=\text{sum of roots}=3t+\frac{1}{t^3},$$

$$\frac{2(2c+a^2-b^2)}{a^2+b^2}=\text{sum of roots two at a time}=3\left(t^2+\frac{1}{t^2}\right),$$

$$-\frac{4(ga-fb)}{a^2+b^2}=\text{sum of roots three at a time}=\frac{3}{t}+t^3.$$

Hence $$-\frac{8ga}{a^2+b^2} = \left(t+\frac{1}{t}\right)^3, \quad \frac{8fb}{a^2+b^2} = \left(t-\frac{1}{t}\right)^3,$$

and $$2c = \tfrac{3}{2}(a^2+b^2)\left(t^2+\frac{1}{t^2}\right) - a^2 + b^2.$$

Thus the circle of curvature is

$$x^2+y^2 - \frac{a^2+b^2}{4a}\left(t+\frac{1}{t}\right)^3 x + \frac{a^2+b^2}{4b}\left(t-\frac{1}{t}\right)^3 y + \tfrac{3}{4}(a^2+b^2)\left(t^2+\frac{1}{t^2}\right) - \tfrac{1}{2}(a^2-b^2) = 0,$$

and the coordinates of the centre of curvature are

$$\frac{a^2+b^2}{8a}\left(t+\frac{1}{t}\right)^3, \quad -\frac{a^2+b^2}{8b}\left(t-\frac{1}{t}\right)^3.$$

Incidentally, if $(x, y)$ is the centre of curvature at the point whose parameter is $t$, we have

$$\left(\frac{8ax}{a^2+b^2}\right)^{\frac{2}{3}} = \left(t+\frac{1}{t}\right)^2 = t^2+\frac{1}{t^2}+2,$$

$$\left(\frac{8by}{a^2+b^2}\right)^{\frac{2}{3}} = \left(t-\frac{1}{t}\right)^2 = t^2+\frac{1}{t^2}-2.$$

Hence the equation of the evolute is

$$(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}.$$

**Example ii.** *Show that chords which subtend a right angle at a fixed point on an hyperbola all pass through a fixed point, and find its coordinates in terms of those of the given point.*

Let the given fixed point be $P\left\{\frac{1}{2}a\left(t+\frac{1}{t}\right), \frac{1}{2}b\left(t-\frac{1}{t}\right)\right\}$, and suppose $t_1, t_2$ are the parameters of the extremities of any chord $QR$ which subtends a right angle at $P$.

Then the equations of the chords $PQ$, $PR$ are

$$\frac{x}{a}(1+tt_1) + \frac{y}{b}(1-tt_1) = t+t_1,$$

$$\frac{x}{a}(1+tt_2) + \frac{y}{b}(1-tt_2) = t+t_2;$$

and since these are perpendicular

$$\frac{(1+tt_1)(1+tt_2)}{a^2} + \frac{(1-tt_1)(1-tt_2)}{b^2} = 0,$$

which at once reduces to

$$(1+t^2t_1t_2)(a^2+b^2) = (a^2-b^2)t(t_1+t_2).$$

But the equation of the chord $QR$ is

$$\frac{x}{a}(1+t_1t_2) + \frac{y}{b}(1-t_1t_2) = t_1+t_2,$$

or, substituting for $(t_1+t_2)$, it becomes

$$\frac{x}{a}(1+t_1t_2) + \frac{y}{b}(1-t_1t_2) = \frac{a^2+b^2}{t(a^2-b^2)}(1+t^2t_1t_2).$$

This can be written

$$\left\{\frac{x}{a}+\frac{y}{b}-\frac{a^2+b^2}{(a^2-b^2)t}\right\}+t_1t_2\left\{\frac{x}{a}-\frac{y}{b}-\frac{a^2+b^2}{a^2-b^2}t\right\}=0,$$

which for different values of $t_1$, $t_2$ represents a line passing through the intersection of the lines

$$\frac{x}{a}+\frac{y}{b}=\frac{a^2+b^2}{a^2-b^2}\cdot\frac{1}{t},$$

$$\frac{x}{a}-\frac{y}{b}=\frac{a^2+b^2}{a^2-b^2}\cdot t;$$

i.e. through the point

$$\left\{\frac{a^2+b^2}{a^2-b^2}\cdot\tfrac{1}{2}a\left(t+\frac{1}{t}\right),\ -\frac{a^2+b^2}{a^2-b^2}\cdot\tfrac{1}{2}b\left(t-\frac{1}{t}\right)\right\}.$$

Thus if the coordinates of the fixed point $P$ are $(x_1, y_1)$, chords which subtend a right angle at $P$ all pass through the point

$$\left\{\frac{a^2+b^2}{a^2-b^2}x_1,\ \frac{b^2+a^2}{b^2-a^2}y_1\right\}.$$

This point lies on the normal at $P$.

**Example iii.** *From any point on the normal at a point $P$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ the three normals other than that at $P$ are drawn. Show that the circle through their feet is one of a coaxal system.*

Let the parameters of the fixed point $P$ be $t$, and those of the feet of the other three normals drawn from any point on the normal at $t$ be $t_1$, $t_2$, $t_3$. Also let $s\equiv t_1+t_2+t_3$, $p\equiv t_1t_2+t_2t_3+t_3t_1$.

Since the normals at $t$, $t_1$, $t_2$, $t_3$ are concurrent

$$p+ts=0 \text{ and } tt_1t_2t_3=-1.$$

Now suppose the circle through the points $t_1$, $t_2$, $t_3$ is

$$x^2+y^2+2gx+2fy+c=0,$$

then, since the parameters of the points of intersection of the circle and the hyperbola are given by

$$(a^2+b^2)t^4+4(ga+fb)t^3+2(2c+a^2-b^2)t^2+4(ga-fb)t+a^2+b^2=0,$$

three roots of this equation must be $t_1$, $t_2$, $t_3$; let the fourth root be $t_4$, then $t_1t_2t_3t_4=1$; hence $t_4=-t=$ constant.

Thus

$$-\frac{4(ga+fb)}{a^2+b^2}=t_1+t_2+t_3+t_4=s-t,$$

$$\frac{2(2c+a^2-b^2)}{a^2+b^2}=t_1t_2+t_2t_3+t_3t_1+t_4(t_1+t_2+t_3)=p-ts=-2ts,$$

and

$$-\frac{4(ga-fb)}{a^2+b^2}=t_1t_2t_3+t_4(t_1t_2+t_2t_3+t_3t_1)=-\frac{1}{t}-tp=-\frac{1}{t}+st^2.$$

Hence
$$2g = \frac{a^2+b^2}{4at}(1+t^2)(1-st),$$
$$2f = \frac{a^2+b^2}{4bt}(t^2-1)(1+st),$$
$$c = -\tfrac{1}{2}(a^2-b^2) - \tfrac{1}{2}(a^2+b^2).st.$$

The equation of the circle is then

$$x^2+y^2+\frac{a^2+b^2}{4at}(1+t^2)(1-st)x+\frac{a^2+b^2}{4bt}(t^2-1)(1+st)y - \tfrac{1}{2}(a^2-b^2)-\tfrac{1}{2}(a^2+b^2)st = 0,$$

which can be written

$$x^2+y^2+\frac{a^2+b^2}{4at}(1+t^2)x+\frac{a^2+b^2}{4bt}(t^2-1)y-\tfrac{1}{2}(a^2-b^2) = \tfrac{1}{4}(a^2+b^2).s\left\{\frac{x}{a}(1+t^2)-\frac{y}{b}(t^2-1)+2t\right\}.$$

Since $s$ is the only undetermined constant, this equation represents a system of coaxal circles of which the radical axis is

$$\frac{x}{a}(1+t^2)-\frac{y}{b}(t^2-1)+2t = 0.$$

Evidently the point whose parameter is $(-t)$ lies on the radical axis since it is common to all the circles.

## Examples VIII c.

1. Show that the normal at the point whose parameter is $t$ on the rectangular hyperbola $x^2-y^2=a^2$ meets the curve again at the point whose parameter is $-1/t^3$.

2. The tangents at the points $t_1$, $t_2$ on $x^2/a^2-y^2/b^2=1$ intersect at
$$\left\{\frac{a(t_1t_2+1)}{t_1+t_2}, \frac{b(t_1t_2-1)}{t_1+t_2}\right\}.$$

3. Find the locus of the foot of the perpendicular from the centre to a tangent to the hyperbola $x^2/a^2-y^2/b^2=1$.

4. The equation of the straight line joining any point on the hyperbola to the vertex is of the form $x/a+y/b-1+t(x/a-y/b-1)=0$.

Find the locus of the mid-point of a chord of the hyperbola which subtends a right angle at the vertex.

5. Find the coordinates of the foot of the normal which meets the axis of the hyperbola at $\{(a^2+b^2)/a, 0\}$.

6. The tangent at any point $P$ of the hyperbola $x^2/a^2-y^2/b^2=1$ meets the lines $(a^2+b^2)y^2+a^2b^2=0$ at the points $Q, R$: show that $CP$ bisects $QCR$.

7. The mid-points of focal chords of an hyperbola lie on an hyperbola of equal eccentricity.

8. The mid-point of a chord of an hyperbola which passes through a fixed point $P$ lies on another fixed hyperbola which passes through the centre $C$ of the given hyperbola and its centre is the mid-point of $CP$.

9. Find the conditions that the chord joining two points on an hyperbola should (i) pass through a focus, (ii) be a diameter when the extremities are given in each system of coordinates.

10. Show that the normals at the ends of the chords

$$(x \sec\theta)/a - (y \tan\theta)/b + d = 0,$$
$$(x \cos\theta)/a + (y \cot\theta)/b - 1/d = 0$$

of the hyperbola $x^2/a^2 - y^2/b^2 - 1 = 0$ are concurrent.

Hence find the coordinates of the centre of curvature at $(a \sec\theta,\ b \tan\theta)$.

11. Find the equations of the common tangents of

$$x^2/a^2 - y^2/b^2 = 1,$$
$$x^2/b^2 - y^2/a^2 = -1.$$

12. Find the conditions that the normals at the points $(a \sec\theta,\ b \tan\theta)$ where $\theta$ is $\theta_1, \theta_2, \theta_3, \theta_4$ respectively should be concurrent.

Also find the condition that those points should be concyclic.

13. If the tangents at two points on $x^2/a^2 - y^2/b^2 = 1$ meet at $(x, y)$ and the normals at the same points at $(\xi, \eta)$, show

$$\xi/\{x(y^2+b^2)\} = \eta/\{y(a^2-x^2)\} = \{a^2+b^2\}/\{b^2x^2 - a^2y^2\}.$$

14. Find the locus of the mid-points of focal chords of the rectangular hyperbola $x^2 - y^2 = a^2$.

15. A circle whose centre is $(f, g)$ cuts the hyperbola $x^2 - y^2 = a^2$ in four points: find the coordinates of their centre of mean position.

16. A tangent to $x^2/a^2 - y^2/b^2 = 1$ cuts the ellipse $x^2/a^2 + y^2/b^2 = 1$ at $P$ and $Q$. Show that the locus of the mid-point of $PQ$ is

$$(x^2/a^2 + y^2/b^2)^2 = x^2/a^2 - y^2/b^2.$$

17. From any point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ three normals other than the one at the point are drawn: show that the centroid of the triangle formed by the feet of these normals lies on the hyperbola

$$9\{x^2/a^2 - y^2/b^2\} = \{(a^2-b^2)/(a^2+b^2)\}^2.$$

18. In the last question the locus of the circumcentre of the triangle is

$$4(a^6x^2 - b^6y^2) = a^4b^4.$$

19. If $(x_2, y_2)$ is the centre of curvature at $(x_1, y_1)$ of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, show that $x_1^3/a^2x_2 - y_1^3/b^2y_2 - 1 = 0$.

Hence show that if $(x_2, y_2)$ lies on the hyperbola $x_2/x_1 + y_2/y_1 + 1 = 0$.

20. Find the locus of the intersection of a normal to $x^2/a^2 - y^2/b^2 = 1$ and a chord which subtends a right angle at its foot.

21. Find the equation to the normal to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at a point whose eccentric angle is $\theta$.

Show that the sum of the eccentric angles at the points where normals from a given point meet the hyperbola is an odd multiple of two right angles.

22. Show that in general four normals can be drawn from a point to the hyperbola $x^2/a^2 - y^2/b^2 = 1$. If $x_1, x_2, x_3, x_4$ be the abscissae of four points on the hyperbola, the normals at which meet in a point, prove that

$$(x_1+x_2+x_3+x_4)(1/x_1 + 1/x_2 + 1/x_3 + 1/x_4) = 4.$$

23. Find the radius of curvature at any point of the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2.$$

Prove that the difference of the lengths of the tangents from any point of the hyperbola to the circles of curvature at its two vertices is constant.

24. $PN$ is an ordinate of an hyperbola; $NQ$ is drawn to touch the circle described on the major axis as diameter: show that the tangent at $P$ intersects $NQ$ in a concentric conic.

25. Prove that the equation of any normal to the hyperbola $x^2 - y^2 = c^2$ can be written in the form $x \sin\theta + y = 2c \tan\theta$.

Prove that the locus of the middle points of normal chords of the hyperbola is $(y^2 - x^2)^3 = 4c^2x^2y^2$.

26. The circle of curvature at any point $P$ of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets the curve at $Q$; if $(a\sec\theta,\ b\tan\theta)$ and $(a\sec\phi,\ b\tan\phi)$ be the coordinates of $P$, $Q$ respectively, find the relation connecting $\theta$ and $\phi$. Determine also the locus of the pole with regard to the hyperbola of the chord $PQ$.

27. Prove that the chord joining the two points on $x^2 - y^2 = a^2$ whose abscissae are $a\cosh\theta$ and $a\cosh\phi$ is the line

$$x\cosh\tfrac{1}{2}(\theta+\phi) - y\sinh\tfrac{1}{2}(\theta+\phi) = a\cosh\tfrac{1}{2}(\theta-\phi).$$

$AA'$ are the vertices of $x^2 - y^2 = a^2$, and $P$, $Q$ the points whose parameters are $\theta+\delta$ and $\theta-\delta$ where $\delta$ is constant.

Prove that the locus of the intersection of $AP$ and $A'Q$ is a rectangular hyperbola.

28. Determine the equation of the chord joining the two points on the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ whose coordinates are

$$(a\sec\alpha,\ b\tan\alpha),\quad (a\sec\beta,\ b\tan\beta).$$

If $S$ and $S'$ are the foci, $A$ and $A'$ the vertices of the hyperbola, and if from any point $P$ on the curve $PS$, $PS'$ be drawn cutting the curve again in $Q$, $Q'$, then if $QA$, $Q'A'$ are joined the locus of their point of intersection will be an hyperbola having double contact with the given one.

29. Prove that the line joining the centre of a rectangular hyperbola to any point on the curve is perpendicular to the chord common to the circle of curvature at the point and the hyperbola.

30. From a point $O(x, y)$, lying on the hyperbola $x^2 - y^2 = a^2 - b^2$, tangents $OP$, $OQ$ are drawn to an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, whose centre is $C$. If $CO$ meet $PQ$ in $R$, show that $OP \cdot OQ : xy :: 2OR : OC$.

31. The common chord of an hyperbola and the circle of curvature at a point on it passes through a fixed point: show that there are four such points and that they are concyclic.

If the fixed point is $(x_1, y_1)$, the equation of this circle is

$$2(x^2+y^2) - xx'(a^2+b^2)/a^2 - yy'(a^2+b^2)/b^2 - (a^2-b^2) = 0.$$

32. The chords of curvature at four points $A$, $B$, $C$, $D$ of the hyperbola are concurrent: if $A$ is a fixed point show that the circle $BCD$ is one of a coaxal system, and find its radical axis in terms of the coordinates of $A$.

33. The chords of curvature at $P$, $Q$, $R$ intersect on the chord of curvature at $K$ and the circle $PQR$ cuts the diameter at $O$. Prove that

$$CK \cdot CO = \tfrac{1}{2}(a^2 - b^2).$$

§ 7. **The Asymptotes.** The asymptotes of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{are} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=0,$$

i.e. are imaginary, in accordance with our original classification.

The asymptotes of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{are} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0;$$

or, written separately, $\frac{x}{a}-\frac{y}{b}=0$, $\frac{x}{a}+\frac{y}{b}=0$; each is inclined to the $x$-axis at an angle $\tan^{-1}\frac{b}{a}$.

Thus, if the angle between the asymptotes is $\omega$, we have $\tan\frac{\omega}{2}=\frac{b}{a}$.

**Conjugate Hyperbolas.** The two hyperbolas whose equations referred to their principal axes are

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \tag{i}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \tag{ii}$$

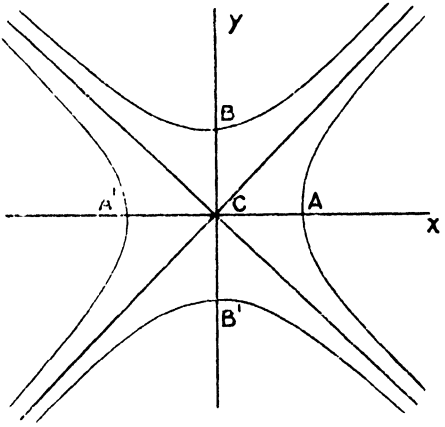
have the same asymptotes.

The axis of $x$ meets the first in real and the second in imaginary points: the axis of $y$ meets the first in imaginary and the second in real points.

We note that the points

$$(a,\ 0),\ (-a,\ 0),\ (0,\ \iota b),\ (0,\ -\iota b)$$

lie on the first, and




$$(\iota a,\ 0),\ (-\iota a,\ 0),\ (0,\ b),\ (0,\ -b)$$

lie on the second.

Any point $P$ on the first hyperbola can be represented by $(a\cosh\theta,\ b\sinh\theta)$, and similarly any point $p$ on the second can be represented by $(a\sinh\theta,\ b\cosh\theta)$.

Thus for the same value of $\theta$ the equation of the diameters $CP$, $Cp$ are

$$y = x\frac{b}{a}\tanh\theta,$$

$$y = x\frac{b}{a}\coth\theta.$$

The tangents to these hyperbolas respectively at the points $P, p$ are

$$\frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta = 1,$$

$$\frac{x}{a}\sinh\theta - \frac{y}{b}\cosh\theta = -1,$$

which are respectively parallel to $Cp$ and $CP$.

Thus $CP$, $Cp$ are a pair of conjugate diameters of both hyperbolas. The curves are therefore called **Conjugate Hyperbolas.**

The following properties should be noted :—

(i) If $CP$, $Cp$ are a pair of common conjugate diameters of the two hyperbolas, each meets one hyperbola in real and the other in imaginary points.

(ii)
$$CP^2 = a^2\cosh^2\theta + b^2\sinh^2\theta,$$
$$Cp^2 = a^2\sinh^2\theta + b^2\cosh^2\theta;$$

hence, since
$$\cosh^2\theta - \sinh^2\theta = 1,$$
$$CP^2 - Cp^2 = a^2 - b^2.$$

(iii) *The tangents at the real points of intersection of conjugate diameters with the conjugate hyperbolas intersect in pairs on the asymptotes.*

Let $PCP'$, $pCp'$ be a pair of conjugate diameters: then the coordinates of their extremities are

$$P\,(a\cosh\theta,\ b\sinh\theta),$$
$$P'\,(-a\cosh\theta,\ -b\sinh\theta),$$
$$p\,(a\sinh\theta,\ b\cosh\theta),$$
$$p'\,(-a\sinh\theta,\ -b\cosh\theta).$$

Hence the tangents at $P$ and $P'$ are

$$\frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta = \pm 1,$$

and at $p, p'$ are

$$\frac{x}{a}\sinh\theta - \frac{y}{b}\cosh\theta = \mp 1.$$

Hence the intersection of the tangents at $Pp$ or $P'p'$ satisfy

$$\left(\frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta \pm 1\right) + \left(\frac{x}{a}\sinh\theta - \frac{y}{b}\cosh\theta \mp 1\right) = 0,$$

i.e. they lie on the line

$$\left(\frac{x}{a} - \frac{y}{b}\right)(\cosh\theta + \sinh\theta) = 0,$$

or on the asymptote

$$\frac{x}{a} - \frac{y}{b} = 0.$$

So the intersection of tangents at $Pp'$, $P'p$ lie on

$$\left(\frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta \pm 1\right) - \left(\frac{x}{a}\sinh\theta - \frac{y}{b}\cosh\theta \pm 1\right) = 0,$$

i.e. on

$$\left(\frac{x}{a} + \frac{y}{b}\right)(\cosh\theta - \sinh\theta) = 0,$$

i.e. on the other asymptote

$$\frac{x}{a} + \frac{y}{b} = 0.$$

The student should prove the following additional properties :—

(a) If $PCP'$, $pCp'$ are a pair of conjugate diameters common to two conjugate hyperbolas, then

(i) The parallelogram $PpP'p'$ is of constant area.

(ii) The parallelogram formed by the tangents at $PpP'p'$ is of constant area.

(iii) The lines $Pp$, $Pp'$, $P'p$, $P'p'$ are each parallel to one asymptote and bisected by the other.

(b) The chord of contact of tangents from a point on an hyperbola to its conjugate hyperbola touches the hyperbola.

(c) The polars of any point with respect to two conjugate hyperbolas are parallel and are equidistant from the common centre.

## § 8. The equation of an hyperbola referred to its asymptotes as coordinate axes.

**Method i.** The equation of the hyperbola in this case is of the form (*vide* Chap. VI, § 6. I) $xy = c^2$.

Now the asymptotes are equally inclined to the axes: hence the equation of the axis is

$$x - y = 0.$$

This meets the curve where

$$x^2 = y^2 = c^2,$$

i.e. at the points $(\pm c, \pm c)$.

Hence $$OA^2 = a^2 = 4c^2 \cos^2 \tfrac{1}{2}\omega$$

or
$$\begin{aligned} 4c^2 &= a^2 \sec^2 \tfrac{1}{2}\omega \\ &= a^2 (1 + \tan^2 \tfrac{1}{2}\omega) \\ &= a^2 \left(1 + \frac{b^2}{a^2}\right) \\ &= a^2 + b^2. \end{aligned}$$

The equation then of an hyperbola, whose semi-axes are $a$, $b$, referred to its asymptote as axes, is

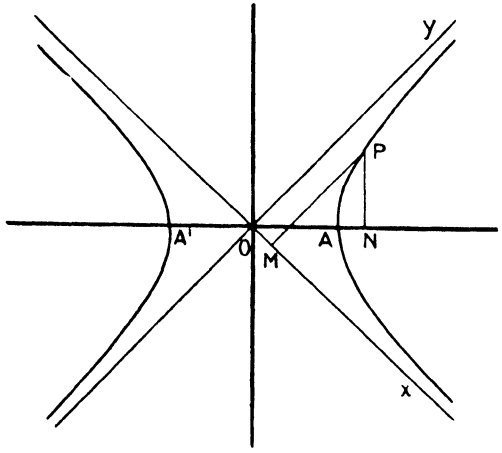
$$xy = \tfrac{1}{4}(a^2 + b^2);$$

we shall generally use this in the form

$$xy = c^2,$$

where
$$4c^2 = a^2 + b^2.$$

**Method ii.** Let any point $P$ on the hyperbola have coordinates $(x, y)$ referred to the asymptotes as coordinate axes and $(x', y')$ referred to its axes. Draw $PN$ perpendicular to the axis, and let $PM$, parallel to the asymptote $Oy$, meet the asymptote $Ox$ at $M$.




Then
$$x' = ON, \quad y' = PN,$$
$$x = OM, \quad y = PM.$$

Thus

$$x' = ON = (PM + OM)\cos \tfrac{1}{2}\omega = (y + x)\cos \tfrac{1}{2}\omega,$$
$$y' = PN = (PM - OM)\sin \tfrac{1}{2}\omega = (y - x)\sin \tfrac{1}{2}\omega;$$

$$\therefore \quad (x+y)^2-(x-y)^2 = x'^2\sec^2\tfrac{1}{2}\omega - y'^2\operatorname{cosec}^2\tfrac{1}{2}\omega$$
$$= x'^2(1+\tan^2\tfrac{1}{2}\omega) - y'^2(1+\cot^2\tfrac{1}{2}\omega)$$
$$= x'^2\left(1+\frac{b^2}{a^2}\right) - y'^2\left(1+\frac{a^2}{b^2}\right)$$
$$= (a^2+b^2)\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right)$$
$$= a^2+b^2,$$

for $(x', y')$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Hence the required equation is

$$\mathbf{4xy = a^2+b^2.}$$

The reader can also obtain this equation by the method of Chap. III, § 6.

**Note.** The equation of the conjugate hyperbola is

$$4xy = -(a^2+b^2),$$

or a pair of conjugate hyperbolas can be represented by

$$\left.\begin{aligned} xy &= c^2 \\ xy &= -c^2 \end{aligned}\right\}.$$

The majority of problems dealing specially with the hyperbola are concerned with the properties of its asymptotes, which are peculiar to it: in such cases it is usually convenient to use the hyperbola in the form $xy = c^2$, and we proceed to develop a parametric system of coordinates for this form.

For the hyperbola in general the axes are then oblique, but for the **rectangular hyperbola**, since its asymptotes are at right angles, the coordinate axes are rectangular.

## § 9. The hyperbola referred to its asymptotes. Parametric Notation.

The coordinates of any point on the hyperbola $xy = c^2$ can be put in the form $\left(ct, \frac{c}{t}\right)$ for some value of $t$, and further, every point whose coordinates are in this form lies on the hyperbola.

**(i) The equation of the chord joining two points whose parameters are $t_1$, $t_2$.**

Let the equation of the chord be

$$Ax+By+c = 0;$$

then, since the given points lie on it,

$$Act_1 + B\frac{c}{t_1} + c = 0,$$

or $$At_1^2 + B + t_1 = 0,$$

and $$At_2^2 + B + t_2 = 0.$$

Hence $$\frac{A}{t_2 - t_1} = \frac{B}{t_1 t_2 (t_2 - t_1)} = \frac{1}{t_1^2 - t_2^2},$$

or $$A = \frac{B}{t_1 t_2} = -\frac{1}{t_1 + t_2}.$$

Thus the equation of the chord is

$$x + t_1 t_2 y = c(t_1 + t_2).$$

This may also be written

$$\frac{x}{c(t_1 + t_2)} + \frac{y}{c\left(\frac{1}{t_1} + \frac{1}{t_2}\right)} = 1;$$

hence, if $(\xi, \eta)$ is the mid-point of a chord, its equation is

$$x/\xi + y/\eta = 2.$$

**Cor.** If the chord $PQ$ meets the asymptotes at $p$ and $q$, and the mid-point of $PQ$ is $(\xi, \eta)$, then, since the equation of the chord is $x/\xi + y/\eta = 2$, the points $p$, $q$ are $(0, 2\eta)$ and $(2\xi, 0)$. Hence the point $(\xi, \eta)$ is also the mid-point of $pq$, and $Pp = Qq$. This property enables us to draw an hyperbola when we have the asymptotes and one point $P$ on the curve; for if any straight line through $P$ meets the asymptotes at $p$ and $q$, we can at once construct the point $Q$ on the curve.

The figure (see p. 344) represents the same hyperbola as that shown in Chap. VI, p. 234. Any number of points, such as $Q$, can be found by using a ruler and a pair of dividers. Each of these gives, in the same manner, two other points $Q_1$, $Q_2$ on lines parallel to the coordinate axes, which can be found conveniently when the construction is made on squared paper.

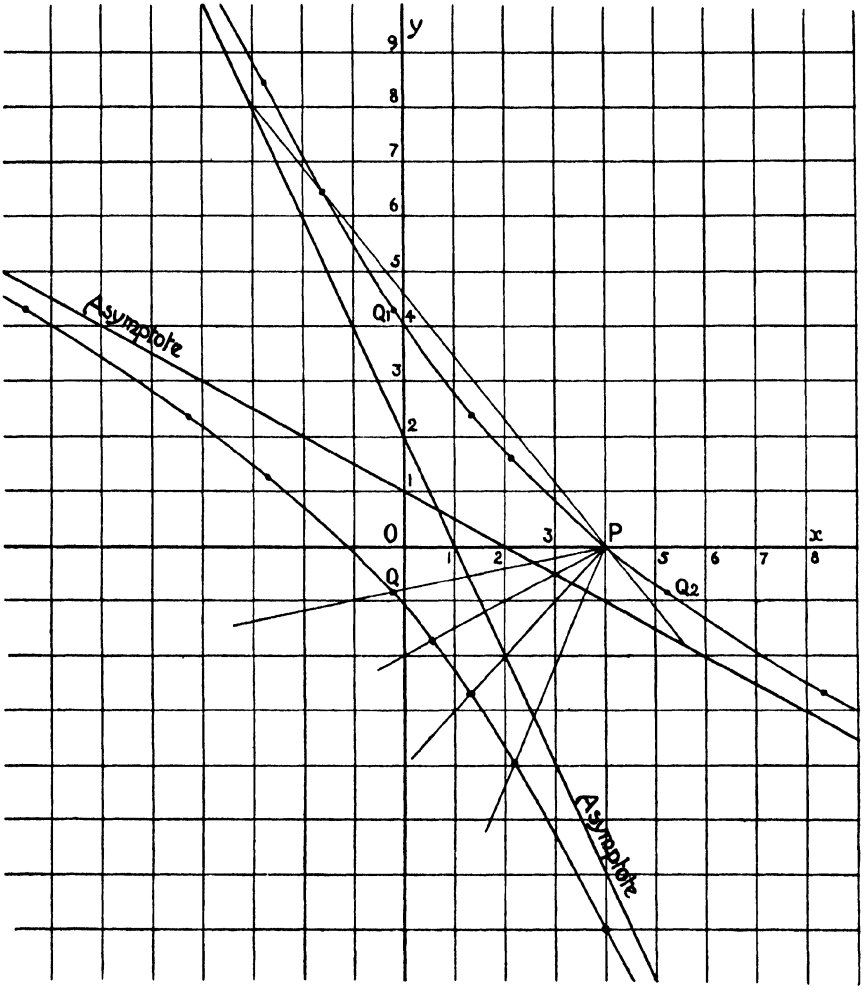
(ii) **Conjugate Diameters.** If the chord $x/\xi + y/\eta = 2$ is parallel to $y + mx = 0$, we have $\eta - m\xi = 0$.

Thus the mid-points of chords parallel to $y + mx = 0$ lie on $y - mx = 0$. The converse is evidently true; the equations of a pair of conjugate diameters thus take the simple forms $y \pm mx = 0$.

**Note.** It follows that any pair of conjugate diameters are harmonic conjugates with respect to the asymptotes.

(iii) **The Tangent.** The equation of the tangent at the point whose parameter is $t$ is $x + t^2 y = 2ct$; this follows from the equation of the chord by putting $t_1 = t_2 = t$.

**Note.** If the tangent at $P$ meets the axis of $x$ at $T$, $T$ is the point $(2ct, 0)$, so that $CT = 2ct$. Also $CS^2 = a^2 + b^2 = 4c^2$, so that $CS = 2c$. Geometrically therefore $t = CT/CS$. It can be shown that $t$ has the same value in the system of coordinates $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$, $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$.




**Example.** *The intercept made by the asymptotes on any tangent is bisected at the point of contact and the triangle so formed is of constant area.*

For the tangent at the point $t$ meets the asymptotes at the points $(2ct, 0)$, $\left(0, \frac{2c}{t}\right)$, and the coordinates of the point midway between these is $\left(ct, \frac{c}{t}\right)$. The area of the triangle so formed is evidently $2c^2 \sin \omega$.

(iv) The equation of the polar of any point $(x', y')$ with respect to the hyperbola $xy = c^2$ is $xy' + x'y = 2c^2$, and if $(x', y')$ lies outside the hyperbola this is the equation of the chord of contact of tangents from $(x', y')$ to the hyperbola.

**Example.** *To find the point of intersection of the tangents at points whose parameters are* $t_1$, $t_2$.

Let $(x', y')$ be the point of intersection, then

$$xy' + x'y = 2c^2$$

and

$$x + t_1 t_2 y = c(t_1 + t_2)$$

both represent the chord of contact.

Comparing coefficients we get

$$x' = \frac{2ct_1t_2}{t_1 + t_2}, \quad y' = \frac{2c}{t_1 + t_2}.$$

(v) **The equation of the normal at the point** $t$.

Let the normal be the line $(x - ct) + m\left(y - \frac{c}{t}\right) = 0$; this is perpendicular to the tangent $x + t^2 y = 2tc$, hence

$$1 + mt^2 = (t^2 + m)\cos\omega,$$

or

$$m = \frac{t^2 \cos\omega - 1}{t^2 - \cos\omega};$$

and the equation of the normal takes the form

$$(x - ct)(t^2 - \cos\omega) + \left(y - \frac{c}{t}\right)(t^2 \cos\omega - 1) = 0,$$

or

$$xt(t^2 - \cos\omega) + yt(t^2 \cos\omega - 1) = c(t^4 - 1).$$

As a general rule questions involving the equation of the normal are more conveniently treated by referring the hyperbola to its axes.

**Special case of the Rectangular Hyperbola.** In this case $\omega = 90°$, and the equation of the normal becomes

$$t^3 x - ty = c(t^4 - 1),$$

or

$$ct^4 - xt^3 + yt - c = 0.$$

(a) **Concurrent Normals.** If the normal at the point $t$ on a rectangular hyperbola passes through any proposed point $(h, k)$ we have $ct^4 - ht^3 + kt - c = 0$.

Hence, conversely, this equation gives the parameters of the feet of the normals which meet at $(h, k)$.

Since the equation is of the fourth degree four normals can be drawn from any point to a rectangular hyperbola.

(b) *Conditions that the normals at any four points on a rectangular hyperbola should be concurrent.*

If the normals at the point whose parameters are $t_1$, $t_2$, $t_3$, $t_4$ are concurrent, these parameters must satisfy $ct^4 - ht^3 + kt - c = 0$ for some values of $h$ and $k$.

Hence $\Sigma t_1 t_2 = 0$ and $t_1 t_2 t_3 t_4 = -1$,

or, written otherwise,

$$\Sigma t_1 t_2 = \Sigma \frac{1}{t_1 t_2} = 0,$$

which are the required conditions.

If these conditions are satisfied, the coordinates of the point of intersection are then given by

$$h = c \Sigma t,$$
$$k = -c \Sigma t_1 t_2 t_3 = c \Sigma \frac{1}{t}.$$

**Example.** *The orthocentre of a triangle inscribed in a rectangular hyperbola is on the curve.*

Let the parameters of the vertices of a triangle $PQR$ inscribed in the rectangular hyperbola $xy = c^2$ be $t_1$, $t_2$, $t_3$. Suppose that the straight line through $P$ perpendicular to $QR$ meets the curve at the point $T$ whose parameter is $t_4$; then the lines

$$x + t_2 t_3 y = c(t_2 + t_3),$$
$$x + t_1 t_4 y = c(t_1 + t_4)$$

are perpendicular, hence $t_1 t_2 t_3 t_4 = -1$. The symmetry of the result shows that the pairs of chords $PQ$, $RT$ and $PR$, $QT$ are also perpendicular, i.e. $T$ is the orthocentre of the triangle $PQR$.

We may notice that the feet of the four normals from any point to a rectangular hyperbola form a triangle and its orthocentre.

(c) *The feet of four concurrent normals lie on another rectangular hyperbola.*

If the normal at any point $\left(ct, \frac{c}{t}\right)$ passes through the point $(h, k)$, we have seen that

$$ct^4 - ht^3 + kt - c = 0,$$

or

$$c^2 t^2 - hct + \frac{kc}{t} - \frac{c^2}{t^2} = 0.$$

Thus if $(x, y)$ are the coordinates of the foot of any normal passing through $(h, k)$, i.e. in the present case if $x = ct$, $y = \frac{c}{t}$, then $(x, y)$ must satisfy $x^2 - hx + ky - y^2 = 0$, i.e. the feet of the normals meeting at $(h, k)$ lie on the rectangular hyperbola $x^2 - y^2 - hx + ky = 0$, which also passes through $(h, k)$.

(d) *To find the conditions that the normals at the ends of the chords of a rectangular hyperbola whose equations are*

$$lx+my-1=0,$$
$$l'x+m'y-1=0$$

*should be concurrent.*

The points of intersection of these chords and the rectangular hyperbola lie on the conic

$$\lambda(xy-c^2)+(lx+my-1)(l'x+m'y-1)=0, \qquad \text{(i)}$$

and consequently, if the normals at these points are concurrent, for some value of $\lambda$ this conic must be the rectangular hyperbola,

$$x^2-y^2-hx+ky=0. \qquad \text{(ii)}$$

Comparing the coefficients of $x^2$ and $y^2$

$$ll'=-mm',$$

i.e. the chords are perpendicular, and also, since the coefficient of $xy$ and the constant term in (ii) are zero,

$$lm'+l'm=-\lambda=-\frac{1}{c^2}.$$

The required conditions are therefore

$$ll'+mm'=0,$$
$$c^2(lm'+l'm)=-1.$$

By comparing the coefficients of $x$ and $y$ the coordinates of the point of intersection of the normals can be found in terms of the coefficients in the equation of either chord, thus

$$h=\frac{1}{l}+\frac{1}{l'}, \quad k=\frac{1}{m}+\frac{1}{m'},$$

and the conditions above give $l'$ and $m'$ in terms of $l$ and $m$, or vice versa.

**Cor.** If one chord is the tangent at the point $t$, viz.

$$x+t^2y-2ct=0,$$

the other chord must be

$$2t^3x-2ty-c(1-t^4)=0,$$

and the point $(h, k)$, which is the centre of curvature at the point $t$, is then

$$\left\{\tfrac{1}{2}c\left(3t+\frac{1}{t^3}\right), \quad \tfrac{1}{2}c\left(t+\frac{3}{t^3}\right)\right\}.$$

**(vi) The intersections of the hyperbola and a circle.**

The equation of any circle is of the form

$$x^2+y^2+2xy\cos\omega+2gx+2fy+d=0.$$

Substituting $x=ct$, $y=\frac{c}{t}$ we obtain an equation giving the

values of the parameters of points on the hyperbola which also lie on the circle; this equation is

$$c^2t^4 + 2gct^3 + (d + 2c^2\cos\omega)\,t^2 + 2fct + c^2 = 0.$$

If the values of $t$ given by this equation are $t_1$, $t_2$, $t_3$, $t_4$ we have

$$t_1 t_2 t_3 t_4 = 1.$$

This is the necessary and sufficient condition that any four points whose parameters are $t_1$, $t_2$, $t_3$, $t_4$ should be concyclic.

**Note.** If the points $A$, $B$, $C$, $D$ on a rectangular hyperbola are concyclic, then $D$ and the orthocentre of the triangle $ABC$ are extremities of a diameter. *Vide* Example, p. 346.

**Circle of curvature.** If the circle

$$x^2 + y^2 + 2xy\cos\omega + 2gx + 2fy + d = 0$$

is the circle of curvature at the point $\left(ct_1, \frac{c}{t_1}\right)$, then three of the values of $t$ given by

$$c^2t^4 + 2gct^3 + (d + 2c^2\cos\omega)\,t^2 + 2fct + c^2 = 0$$

must be $t_1$, for the circle intersects the hyperbola in three coincident points at the point of contact.

Let the values of $t$ given by the equation be $t_1$, $t_1$, $t_1$, $t_2$, then

$$t_1^3 t_2 = 1, \quad \text{or} \quad t_2 = \frac{1}{t_1^3},$$

i.e. the circle of curvature at the point $\left(ct_1, \frac{c}{t_1}\right)$ cuts the hyperbola again at the point $\left(\frac{c}{t_1^3},\ ct_1^3\right)$.

The equation of the common chord of the hyperbola and the circle of curvature at $t_1$, i.e. of the chord of curvature, is

$$x + \frac{y}{t_1^2} = c\left(t_1 + \frac{1}{t_1^3}\right),$$

or

$$t_1^3 x + t_1 y = c\,(t_1^4 + 1).$$

Further,

$$-2g = c\Sigma t = c\,(3t_1 + t_2) = c\left(3t_1 + \frac{1}{t_1^3}\right),$$

$$-2f = c\Sigma\frac{1}{t} = c\left(t_1^3 + \frac{3}{t_1}\right),$$

$$d + 2c^2\cos\omega = c^2\Sigma t_1 t_2 = c^2\left(3t_1^2 + \frac{3}{t_1^2}\right).$$

Hence the equation of the circle of curvature at the point $t$ is

$$x^2+y^2+2xy\cos\omega-cx\left(3t+\frac{1}{t^3}\right)-cy\left(\frac{3}{t}+t^3\right)+c^2\left(3t^2+\frac{3}{t^2}-2\cos\omega\right)=0.$$

**Special case of the Rectangular Hyperbola.** When the hyperbola is rectangular the equation of the circle of curvature becomes

$$x^2+y^2-cx\left(3t+\frac{1}{t^3}\right)-cy\left(\frac{3}{t}+t^3\right)+c^2\left(3t^2+\frac{3}{t^2}\right)=0.$$

(a) The **centre of curvature** is the point

$$\left\{\tfrac{1}{2}c\left(3t+\frac{1}{t^3}\right),\quad \tfrac{1}{2}c\left(\frac{3}{t}+t^3\right)\right\}.$$

(b) The **equation of the evolute**, viz. the locus of the centres of curvature, can be found thus:

$$x=\tfrac{1}{2}c\,(3t+1/t^3),$$
$$y=\tfrac{1}{2}c\,(3/t+t^3);$$
$$\therefore\quad x+y=\tfrac{1}{2}c\,(1/t+t)^3,$$
$$x-y=\tfrac{1}{2}c\,(1/t-t)^3,$$

and
$$(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}}=(4c)^{\frac{2}{3}}.$$

(c) The **radius of curvature** ($\rho$) at the point $t$ is given by

$$\rho^2=g^2+f^2-d$$
$$=\tfrac{1}{4}c^2\,(3t+1/t^3)^2+\tfrac{1}{4}c^2\,(t^3+3/t)^2-3c^2\,(1/t^2+t^2)$$
$$=\tfrac{1}{4}c^2\,(t^2+1/t^2)^3,$$

i.e.
$$\rho=\tfrac{1}{2}c\,(t^2+1/t^2)^{\frac{3}{2}}.$$

### Illustrative Examples.

**Example** i. *A triangle is inscribed in the hyperbola $xy=c^2$ so that its centroid is a fixed point on the hyperbola: show that its sides touch an ellipse which touches the asymptotes and the hyperbola.*

Let the vertices of the triangle be $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$, $\left(ct_3, \frac{c}{t_3}\right)$ and the fixed point $\left(cd, \frac{c}{d}\right)$.

Hence
$$t_1+t_2+t_3=3d,$$
$$1/t_1+1/t_2+1/t_3=3/d.$$

A side of the triangle is $x+t_1t_2y=c\,(t_1+t_2)$.

Now
$$t_1+t_2=3d-t_3,$$

and
$$\frac{t_1+t_2}{t_1t_2}=\frac{3}{d}-\frac{1}{t_3};$$
$$\therefore\quad t_1t_2=\frac{dt_3\,(3d-t_3)}{3t_3-d}.$$

The equation of the side of the triangle then becomes

$$(3t_3-d)\,x+dt_3(3d-t_3)\,y=c(3d-t_3)(3t_3-d),$$

i.e. $$t_3^2(3c-dy)+t_3(3x+3d^2y-10cd)+3d^2c-dx=0.$$

For all values of $t_3$ this touches

$$(3x+3d^2y-10cd)^2=4(3c-dy)(3d^2c-dx),$$

i.e. $$(3x+3d^2y-10cd)^2+4cd(3x+3d^2y)-36c^2d^2=4d^2xy,$$

i.e. $$(3x+3d^2y)^2-16cd(3x+3d^2y)+64c^2d^2=4d^2xy,$$

i.e. $$(3x+3d^2y-8cd)^2=4d^2xy. \qquad \text{(i)}$$

The same result is evidently true for the other sides of the triangle: now equation (i) from its form represents a conic touching the asymptotes $x=0$, $y=0$, the chord of contact being

$$3x+3d^2y-8cd=0.$$

The terms of the second degree are

$$9x^2+9d^4y^2+14d^2xy;$$

hence the asymptotes are parallel to

$$9x^2+9d^4y^2+14d^2xy=0,$$

i.e. are imaginary: the curve is therefore an ellipse.

Now the parameters of the points of intersection of the ellipse (i) and the hyperbola are given by substituting $x=ct$, $y=\frac{c}{t}$ in this equation, i.e.

$$\left(3ct+3\frac{cd^2}{t}-8cd\right)^2=4d^2c^2;$$

$$\therefore\quad \left(3t+3\frac{d^2}{t}-8d\right)^2=4d^2.$$

Hence $$3t+\frac{3d^2}{t}-8d=\pm 2d.$$

Taking the negative sign $$3t+\frac{3d^2}{t}-6d=0,$$

i.e. $$t^2-2dt+d^2=0,$$

i.e. $$(t-d)^2=0;$$

hence it meets the hyperbola at two coincident points at the given fixed point and consequently touches it there.

**Example ii.** *If the circle circumscribing the triangle formed by the tangents at the points* $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ *on a rectangular hyperbola passes through the centre of the curve, then*

$$\frac{x_1+x_2+x_3}{x_1x_2x_3}+\frac{y_1+y_2+y_3}{y_1y_2y_3}=0,$$

*and the centre of the circle is*

$$-\frac{y_1y_2y_3}{c^2},\quad -\frac{x_1x_2x_3}{c^2}.$$

Let the parameters of the points of contact of the tangents be $t_1$, $t_2$, $t_3$; and write $s_1 \equiv \Sigma t$, $s_2 \equiv \Sigma t_1 t_2$, $s_3 \equiv t_1 t_2 t_3$.

The coordinates of the vertices of the circumscribing triangle are

$$\frac{2cs_3}{t(s_1-t)}, \quad \frac{2c}{s_1-t},$$

where $t$ has either of the values $t_1$, $t_2$, or $t_3$.

A circle whose centre is $(g, f)$, and which passes through the origin, is

$$x^2+y^2-2gx-2fy=0.$$

If any one of the vertices of the triangle lies on this circle, then

$$ft^3-(fs_1-gs_3-c)\,t^2-gs_1s_3t+cs_3^2=0.$$

But this condition is satisfied when $t$ is equal to $t_1$, $t_2$, or $t_3$; hence $t_1$, $t_2$, $t_3$ are the roots of this equation.

It follows that

(i) $s_3 = -c\frac{s_3^2}{f}$, i.e. $f=-cs_3$;

(ii) $s_2 = -gs_1\frac{s_3}{f}$, i.e. $g=\frac{cs_2}{s_1}$;

(iii) $s_1 = s_1 - \frac{gs_3}{f} - \frac{c}{f}$, i.e. $g=-\frac{c}{s_3}$.

Hence the centre of the circle is $\left(-\frac{c}{s_3}, -cs_3\right)$, with the condition $s_2s_3+s_1=0$.

These results are identical with those required, since $x_1=ct_1$, $y_1=\frac{c}{t_1}$, &c.

## Examples VIII d.

1. The tangents at the extremities of a chord of $xy=c^2$, whose mid-point is $(X, Y)$, intersect at the point $\{c^2/Y, c^2/X\}$.

2. The equation of the director circle of the hyperbola $xy=c^2$ is

$$x^2+y^2+2xy\cos\omega = 4c^2\cos\omega.$$

What does this become for a rectangular hyperbola?

3. Find the locus of the intersections of perpendicular straight lines which are tangents respectively to a rectangular hyperbola and its conjugate.

4. The normal to the rectangular hyperbola $xy=c^2$ at the point $\left(ct, \frac{c}{t}\right)$ meets the curve again at the point $(-c/t^3, -ct^3)$.

5. The normals to the rectangular hyperbola $xy=c^2$ at the ends of the chords whose equations are

$$x\cos\theta+y\sin\theta=c,$$
$$x\sin\theta-y\cos\theta=c\cos 2\theta$$

are concurrent.

6. There are four points on a rectangular hyperbola $xy=c^2$, the chords of curvature at which are concurrent, and these points are concyclic.

7. The sum of the squares of the lengths of the normals which can be drawn from a point $P$ to the rectangular hyperbola $xy=c^2$ is $3CP^2$.

8. Find the locus of the centroid of an equilateral triangle inscribed in a rectangular hyperbola.

9. The polar of any point on an asymptote is parallel to that asymptote.

10. Four points $A$, $B$, $C$, $D$ on a rectangular hyperbola are such that the straight lines $AB$, $CD$ are perpendicular. Show that $AD$ is perpendicular to $BC$, and $AC$ to $BD$.

11. Find the equation of a pair of conjugate hyperbolas referred to a common pair of conjugate diameters.

12. $CP$, $Cp$ are conjugate diameters of two hyperbolas, $P$ being on one, $p$ on the other. Find the locus of the orthocentre of the triangle $PCp$.

13. Show that the envelope of the chords of the rectangular hyperbola $xy = a^2$ which subtend a given angle $\alpha$ at the point $(x', y')$ on the curve is the hyperbola $x^2x'^2 + y^2y'^2 = 2a^2xy(1 + 2\cot^2\alpha) - 4a^4\operatorname{cosec}^2\alpha$.

14. A circle is described having its centre at a point $P$ on a rectangular hyperbola and passing through the diametrically opposite point $P'$ on the hyperbola. Prove that, if $L$, $M$, $N$ are the other three points in which the circle cuts the hyperbola, the triangle $LMN$ is equilateral.

15. A normal to a rectangular hyperbola makes an acute angle $\theta$ with the transverse axis. Prove that the acute angle at which it cuts the curve again is $\cot^{-1}(2\tan 2\theta)$.

16. If the position of a point on a rectangular hyperbola is determined by the variable $\theta$ where $x = c\tan\theta$, $y = c\cot\theta$, the locus of the intersection of tangents at the points $\theta$, $\theta + \alpha$, $\alpha$ being a constant angle, is

$$4(c^2 - xy) = (x + y)^2\tan^2\alpha.$$

17. Prove that the locus of the mid-points of chords of the rectangular hyperbola $xy = c^2$ which are of constant length $2l$ is

$$(x^2 + y^2)(xy - c^2) = l^2xy.$$

18. The sides of a triangle $ABC$, inscribed in a rectangular hyperbola, make angles $\alpha$, $\beta$, $\gamma$ with an asymptote. Prove that the normals at $A$, $B$, $C$ will meet in a point, if $\cot 2\alpha + \cot 2\beta + \cot 2\gamma = 0$.

19. To a rectangular hyperbola with centre $C$ and focus $S$ normals are drawn from a point $P$. Show that, if these normals make angles $\theta_1$, $\theta_2$, ... with one of the asymptotes $\Sigma\operatorname{cosec} 2\theta = (2CP^2/CS^2)$.

20. The normal at $P$ to a rectangular hyperbola whose centre is $C$ meets the curve again at $Q$; show that $PQ^2 = 3CP^2 + CQ^2$.

21. The normal to the rectangular hyperbola $xy = c^2$ at $P$ meets the curve again at $Q$ and touches the conjugate hyperbola; show that

$$PQ^4 = 512c^4.$$

22. $PP'$ is a diameter of the hyperbola $x^2/a^2 - y^2/b^2 = 1$. A straight line is drawn through $P$ parallel to one asymptote, and a straight line through $P'$ parallel to the other asymptote; show that the locus of the intersection of these straight lines is the hyperbola $y^2/b^2 - x^2/a^2 = 1$.

23. If $A$ and $A'$ are the vertices of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, and $P$ any point on it, and if $PA$, $PA'$ meet an asymptote at the points $X$ and $Y$, show that $XY = \sqrt{a^2 + b^2}$.

24. The straight line $AB$ is bisected at $C$; through $C$ a fixed straight line is drawn, and two points $P$, $Q$ are taken on it, such that the distance $PQ$ is constant. Show that the locus of the intersection of the straight lines $AP$, $BQ$ is an hyperbola.

25. $A$ and $B$ are fixed points, and $AC$ is a fixed straight line. If a line drawn through $B$ meet $AC$ in $Q$, and a point $P$ be taken on this line, produced if necessary, so that $PA = PQ$, show that the locus of $P$ is an hyperbola whose centre is the middle point of $AB$.

26. Show that the polar of the origin with respect to the circle of curvature at the point $(x', y')$ on the rectangular hyperbola $xy = a^2$ is

$$x(3a^2x' + y'^3) + y(x'^3 + 3a^2y') = 6a^2(x'^2 + y'^2).$$

27. The tangent at any point $P$ of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets the asymptotes in $L$ and $M$. Find the equation of the circle $OLM$ in terms of the coordinates of $P$, and deduce the locus of the centre of this circle. Explain the result of putting $a$ equal to $b$ in the equation to this locus.

28. Find the equation of the normal at any point of the rectangular hyperbola $xy = c^2$: show that from any point in its plane four normals can be drawn to this hyperbola, and that if $x_1, x_2, x_3, x_4$ be the abscissae of the feet of these normals $x_1x_2x_3x_4 + c^4 = 0$.

29. The normal at $P$ to the rectangular hyperbola $xy = c^2$ meets the curve again at $Q$. If $x, y$ are the coordinates of $P$ and $\xi, \eta$ those of $Q$, prove that $\xi x^3 = \eta y^3 = -c^4$.

30. If the tangent at the point $(h, k)$ of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ meets the asymptote $bx = ay$ at the point $M$, and the asymptote $bx + ay = 0$ at the point $N$ and $S$ is a focus of the hyperbola, show that

$$SM/SN = h/a + k/b.$$

31. From a point $P$ are drawn two tangents to a rectangular hyperbola. The tangents of their inclinations to an asymptote being $p$ and $q$, show that, if the ratio of $1 - pq$ to $(\sqrt{p} - \sqrt{q})^2$ is constant, the locus of $P$ is another rectangular hyperbola.

32. Show that the lengths of the tangents from $P(x, y)$ to the rectangular hyperbola $f \equiv xy - c^2 = 0$ are given by

$$x^2y^2\lambda^4 + 2(x^2 + y^2)(2c^2 - xy)f\lambda^2 + f^2[(x^2 + y^2)^2 - 16c^2f] = 0;$$

and show that the lengths of $SP$, $S'P$ are factors of the absolute term.

§ 10. In Chapter VII, § 9, we discussed the forms of the equations of several loci related to the parabola: these forms clearly apply equally well to any of the conics; thus throughout the section we may substitute $S$ for $P$ where $S = 0$ is the equation of a parabola, an ellipse, or an hyperbola.

**Example** i. *A circle is described on the chord* $x + 3y = 1$ *of the ellipse* $x^2 + 3y^2 = 4$ *as diameter. Find the straight line joining the other two points in which the circle cuts the ellipse.*

Let the equation of the other common chord of the circle and the ellipse be

$$lx + my + 1 = 0.$$

The equation of the circle is therefore of the form

$$\lambda(x^2+3y^2-4)+(x+3y-1)(lx+my+1)=0.$$

The two conditions for a circle give

$$\lambda+l=3\lambda+3m \quad \text{and} \quad 3l+m=0,$$

i.e. $$m=-3l \quad \text{and} \quad 2\lambda=l-3m=10l,$$

or $$\lambda=5l.$$

The equation of the circle then becomes

$$5l(x^2+3y^2-4)+(x+3y-1)(lx-3ly+1)=0.$$

Its centre is the point $$\left\{\frac{l-1}{12l}, -\frac{3l+3}{12l}\right\},$$

and this by hypothesis lies on the diameter $x+3y=1$.

Hence $$l-1-9l-9=12l;$$

$$\therefore \quad l=-\tfrac{1}{2}.$$

The equation of the circle is then

$$5(x^2+3y^2-4)+(x+3y-1)(x-3y-2)=0,$$

or $$2x^2+2y^2-x-y-6=0,$$

and the required equation of the chord is $x-3y=2$.

**Example ii.** *Find the equation of the parabola which touches the hyperbola* $3x^2+2xy-y^2+8x+10y+14=0$ *in the points in which it is met by the straight line* $5x-y-2=0$.

The equation of the parabola is in the form $S=ku^2$, i.e.

$$3x^2+2xy-y^2+8x+10y+14=k(5x-y-2)^2.$$

The condition for a parabola is $ab-h^2=0$,

i.e. $$(25k-3)(k+1)=(5k+1)^2,$$

i.e. $$25k^2+22k-3=25k^2+10k+1;$$

$$\therefore \quad 12k=4;$$

$$\therefore \quad k=\tfrac{1}{3}.$$

The equation of the parabola is therefore

$$3(3x^2+2xy-y^2+8x+10y+14)=(5x-y-2)^2,$$

or $$8x^2-8xy+2y^2-22x-13y-19=0.$$

**Example iii.** *Find the equation of the parabola which has four-point contact with the hyperbola* $xy-c^2=0$ *at the point* $\left(ct, \frac{c}{t}\right)$.

The equation of the tangent at the point $\left(ct, \frac{c}{t}\right)$ is

$$x+t^2y=2ct.$$

The equation of the parabola is of the form $S=ku^2$ (Chap. VII, § 9, v),

i.e. $$xy-c^2=k(x+t^2y-2ct)^2.$$

The condition for a parabola gives

$$k^2t^4 = (kt^2 - \tfrac{1}{2})^2,$$

or
$$k = \frac{1}{4t^2}.$$

Hence the required parabola is

$$4t^2(xy - c^2) = (x + t^2y - 2ct)^2,$$

or
$$(x - t^2y)^2 - 4ctx - 4ct^3y + 8c^2t^2 = 0.$$

§ 11. **Confocal Conics.** We have shown that the foci of a conic lie on its axes and are equidistant from the centre; hence if conics have the same foci $SS'$ their axes lie along the same straight lines and they have a common centre $C$.

Now if $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be any conic referred to its axes we have $CS^2 = a^2 - b^2$, and consequently all conics which have the same foci are such that $a^2 - b^2 = \text{constant} = c^2$.

A conic, then, whose equation is of the form

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$$

is confocal for different values of $\lambda$ with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and this equation therefore represents a system of confocal conics.

The conic is an ellipse or hyperbola according as the value of $\lambda$ chosen makes $a^2 + \lambda$, $b^2 + \lambda$ both positive or of opposite sign.

**Proposition 1.** *Two real confocals of a system pass through any real point, one an ellipse and one an hyperbola.*

Let $(x', y')$ be a given point; then to find the values of $\lambda$ for those conics of the system which pass through this point we have

$$x'^2/(a^2 + \lambda) + y'^2/(b^2 + \lambda) = 1,$$

i.e.
$$\lambda^2 - \lambda(x'^2 + y'^2 - a^2 - b^2) + a^2b^2 - b^2x'^2 - a^2y'^2 = 0.$$

If $\lambda = -a^2$, the left-hand side of the equation becomes $(a^2 - b^2)x'^2$, i.e. is positive.

If $\lambda = -b^2$, it becomes $-(a^2 - b^2)y'^2$, i.e. is negative.

If $\lambda = +\infty$, it is positive.

Hence there are always two real roots, one lying between $-a^2$ and $-b^2$, and one between $-b^2$ and $+\infty$.

Thus $a^2 + \lambda$, $b^2 + \lambda$ are for one value of opposite sign, and for the other both positive.

Hence there are two real confocals passing through $(x', y')$, one an ellipse and one an hyperbola.

**Proposition 2.** *One confocal of a system touches every real straight line.*

Let the equation of the straight line be $lx+my+n=0$; this touches the conic $x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1$, provided that

$$(a^2+\lambda)\,l^2+(b^2+\lambda)\,m^2=n^2.$$

This equation gives one value of $\lambda$ corresponding to the one conic of the system which touches the straight line. The equation of this conic is

$$\frac{x^2}{n^2+(a^2-b^2)\,m^2}+\frac{y^2}{n^2-(a^2-b^2)\,l^2}=\frac{1}{l^2+m^2},$$

and since $n^2+(a^2-b^2)\,m^2$ is always positive the conic is real; it is an ellipse or an hyperbola according as $n^2$ is $>$ or $<(a^2-b^2)\,l^2$.

**Proposition 3.** *Any two confocals cut at right angles.*

Let the two confocals be

$$x^2/(a^2+\lambda_1)+y^2/(b^2+\lambda_1)=1,$$
$$x^2/(a^2+\lambda_2)+y^2/(b^2+\lambda_2)=1.$$

If these conics intersect at $(x_1, y_1)$, we have

$$x_1^2/(a^2+\lambda_1)+y_1^2/(b^2+\lambda_1)=1, \quad x_1^2/(a^2+\lambda_2)+y_1^2/(b^2+\lambda_2)=1.$$

Subtract these equations and divide through by $(\lambda_2-\lambda_1)$; then

$$x_1^2/(a^2+\lambda_1)\,(a^2+\lambda_2)+y_1^2/(b^2+\lambda_1)\,(b^2+\lambda_2)=0,$$

which is the condition that the tangents

$$xx_1/(a^2+\lambda_1)+yy_1/(b^2+\lambda_1)=1, \quad xx_1/(a^2+\lambda_2)+yy_1/(b^2+\lambda_2)=1$$

at the point $(x_1, y_1)$ to the two confocals should be at right angles.

**Proposition 4.** *The poles of a straight line with respect to a system of confocal conics lie on the normal to the confocal which it touches at the point of contact.*

Let $lx+my+n=0$ be the equation of the straight line, and let $(x_1, y_1)$ be its pole with respect to any conic of the system

$$x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1.$$

Hence $lx+my+n=0$, and $xx_1/(a^2+\lambda)+yy_1/(b^2+\lambda)=1$ are identical, and therefore

$$x_1=-l\,(a^2+\lambda)/n \quad y_1=-m\,(b^2+\lambda)/n.$$

Eliminating $\lambda$, we get the equation of the locus of the poles of the straight line, viz.

$$mnx_1-lny_1+(a^2-b^2)\,lm=0.$$

This straight line is perpendicular to the given straight line : also the pole of the given straight line with respect to that conic of the system which it touches is its point of contact ; hence this point of contact lies on the locus found. Thus the locus of the poles of the given straight line is the normal at its point of contact to that conic of the system which it touches.

**Note.** If the given straight line passes through the centre of the confocal conics, its pole with respect to each conic is a 'point at infinity'. If $lx + my = 0$ is such a straight line, the conic of the system which it touches is (see Prop. 2)

$$(l^2 + m^2)(l^2x^2 - m^2y^2) = l^2m^2(a^2 - b^2),$$

i.e. the straight line is one of the asymptotes of the conic and its point of contact is a 'point at infinity'.

**Proposition 5.** *The envelope of the polars of a given point with respect to a system of confocal conics is a parabola touching the axes.*

Let $(x_1, y_1)$ be the given point ; then its polar with respect to any conic of the system whose equation is $x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1$ is

$$xx_1/(a^2+\lambda) + yy_1/(b^2+\lambda) = 1.$$

This may be written

$$\lambda^2 - \lambda(xx_1 + yy_1 - a^2 - b^2) - (a^2yy_1 + b^2xx_1 - a^2b^2) = 0.$$

The equation of the envelope is therefore

$$(xx_1 + yy_1 - a^2 - b^2)^2 + 4(a^2yy_1 + b^2xx_1 - a^2b^2) = 0,$$

which at once reduces to

$$(xx_1 - yy_1 - a^2 + b^2)^2 + 4x_1y_1xy = 0.$$

The latus rectum of the parabola is

$$4(a^2 - b^2)x_1y_1/(x_1^2 + y_1^2)^{\frac{1}{2}},$$

and the equation of the axis is

$$(xx_1 + yy_1)(x_1^2 + y_1^2) + (a^2 - b^2)(y_1^2 - x_1^2) = 0 ;$$

these results are left for the reader to prove.

**Proposition 6.** *The bisectors of the angles between the tangents drawn to a conic from any point are the tangents to the confocals through the point.*

Let the point be $(a\cos\theta, b\sin\theta)$, which lies on the conic

$$x^2/a^2 + y^2/b^2 = 1,$$

and let tangents be drawn to the conic

$$x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1.$$

If the straight line $lx+my+n=0$ is one of these tangents, then

$$l^2(a^2+\lambda)+m^2(b^2+\lambda)=n^2;$$

and, since $(a\cos\theta,\ b\sin\theta)$ lies on the line

$$al\cos\theta+bm\sin\theta=-n,$$

so that $$(al\cos\theta+bm\sin\theta)^2=l^2a^2+m^2b^2+\lambda(l^2+m^2)$$

or $$(al\sin\theta-bm\cos\theta)^2+\lambda(l^2+m^2)=0.$$

This equation gives the values of the ratio $l:m$ corresponding to the directions of the two tangents. Thus the equation of the straight lines through the origin parallel to the two tangents is

$$(ay\sin\theta+bx\cos\theta)^2+\lambda(x^2+y^2)=0.$$

For all values of $\lambda$ these have the same bisectors of the angles between them, viz.

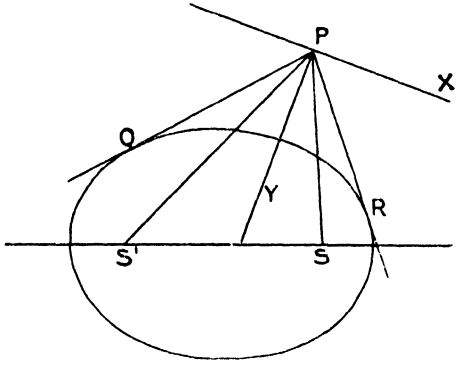
$$lx\cos\theta+ay\sin\theta=0 \quad \text{and} \quad ax\sin\theta-by\cos\theta=0.$$

The bisectors of the angles between the tangents are the two straight lines through $(a\cos\theta,\ b\sin\theta)$ parallel to these, viz.

$$bx\cos\theta+ay\sin\theta=ab, \quad ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2,$$

i.e. the tangent and normal to the conic $x^2/a^2+y^2/b^2=1$ at the point $(a\cos\theta,\ b\sin\theta)$; or, since confocals cut at right angles, they are the tangents to the confocals which pass through the point.

**Cor.** Let $PQ$, $PR$ be the tangents from $P$ to a conic, $PX$, $PY$ the tangent and normal to a confocal conic through $P$. We have shown (p. 308, Ex. 5)




that $PX$, $PY$ bisect the angles between $SP$, $S'P$; it follows therefore that the tangents from any point to a conic are equally inclined to the focal distances of the point.

(7) *If* $x+yi=c\cos(\xi+\eta i)$, *then* $\xi=$ *constant and* $\eta=$ *constant are confocal hyperbolas and ellipses.*

Now, since

$$x+yi=c\cos(\xi+\eta i)=c\cos\xi\cosh\eta-ic\sin\xi\sinh\eta,$$

we have

$$x = c\cos\xi\cosh\eta, \quad y = -c\sin\xi\sinh\eta.$$

Eliminating $\eta$ we find

$$\frac{x^2}{c^2\cos^2\xi} - \frac{y^2}{c^2\sin^2\xi} = 1, \tag{i}$$

which equation, for different values of $\xi$, represents confocal hyperbolas, for $c^2(\cos^2\xi+\sin^2\xi) = c^2$ or $CS = c$.

Again eliminating $\xi$ we find

$$\frac{x^2}{c^2\cosh^2\eta} + \frac{y^2}{c^2\sinh^2\eta} = 1, \tag{ii}$$

which equation, for different values of $\eta$, represents confocal ellipses of the same system, for $c^2(\cosh^2\eta-\sinh^2\eta) = c^2$ or $CS = c$.

Evidently then the equation $x_1+y_1 i = c\cos(\xi_1+\eta_1 i)$ is the condition that $(x_1, y_1)$ should lie on both the confocals $\xi = \xi_1$, $\eta = \eta_1$, whose equations are

$$\frac{x^2}{c^2\cos^2\xi_1} - \frac{y^2}{c^2\sin^2\xi_1} = 1, \quad \frac{x^2}{c^2\cosh^2\eta_1} + \frac{y^2}{c^2\sinh^2\eta_1} = 1.$$

(8) **Definition.** Two points $P(x_1, y_1)$ and $P'(x_1', y_1')$ on two confocals $x^2/a^2+y^2/b^2 = 1$ and $x^2/a'^2+y^2/b'^2 = 1$ are said to **correspond** if $x_1/a = x_1'/a'$, $y_1/b = y_1'/b'$.

(a) *If $P$, $P'$ are two corresponding points on two confocals, then the second confocal of the system which passes through $P$ passes also through $P'$.*

Let the system of confocals be defined by

$$x+yi = c\cos(\xi+\eta i),$$

and let the coordinates of the given points be $P(x_1, y_1)$, $P'(x_1', y_1')$.

Then if $\xi = \xi_1$, $\xi = \xi_2$ are the two confocals on which the corresponding points $P$ and $P'$ lie, we have by definition

$$x_1/\cos\xi_1 = x_1'/\cos\xi_2 \quad \text{and} \quad y_1/\sin\xi_1 = y_1'/\sin\xi_2.$$

Let $\eta = \eta_1$ and $\eta = \eta_1'$ be the second confocals of the system which pass through $P$ and $P'$ respectively: thus

$$x_1+y_1 i = c\cos(\xi_1+\eta_1 i), \quad x_1'+y_1' i = c\cos(\xi_2+\eta_1' i),$$

and therefore

$$x_1 = c\cos\xi_1\cosh\eta_1, \quad x_1' = c\cos\xi_2\cosh\eta_1'.$$

But $x_1/\cos\xi_1 = x_1'/\cos\xi_2$; hence $\eta_1 = \eta_1'$ and $P$, $P'$ lie on the confocal $\eta = \eta_1$.

(b) **Ivory's Theorem.** *If $P, P'$ and $Q, Q'$ are two pairs of corresponding points on two confocals, then $PQ' = P'Q$.*

Let the system of confocals be defined by

$$x+yi = c\cos(\xi+\eta i),$$

and let the given points be

$$P(x_1, y_1), \quad P'(x_1', y_1'), \quad Q(x_2, y_2), \quad Q'(x_2', y_2').$$

Then, if $P, Q$ lie on $\xi = \xi_1$ and $P', Q'$ on $\xi = \xi_2$, we showed above that $P, P'$ lie on $\eta = \eta_1$ and similarly $Q, Q'$ lie on $\eta = \eta_2$. Hence

$$x_1+y_1 i = c\cos(\xi_1+\eta_1 i), \quad x_1'+y_1' i = c\cos(\xi_2+\eta_1 i),$$
$$x_2+y_2 i = c\cos(\xi_1+\eta_2 i), \quad x_2'+y_2' i = c\cos(\xi_2+\eta_2 i).$$

Then

$$\begin{aligned} PQ'^2 &= (x_1-x_2')^2+(y_1-y_2')^2 \\ &= \{(x_1+iy_1)-(x_2'+y_2'i)\}\{(x_1-y_1 i)-(r_2'-y_2' i)\} \\ &= c^2\{\cos(\xi_1+\eta_1 i)-\cos(\xi_2+\eta_2 i)\}\{\cos(\xi_1-\eta_1 i)-\cos(\xi_2-\eta_2 i)\} \\ &= 4c^2 \sin\tfrac{1}{2}(\overline{\xi_2-\xi_1}+i\,\overline{\eta_2-\eta_1})\cdot\sin\tfrac{1}{2}(\overline{\xi_1+\xi_2}+i\cdot\overline{\eta_1+\eta_2}) \\ &\qquad \cdot\sin\tfrac{1}{2}(\overline{\xi_2-\xi_1}-i\,\overline{\eta_2-\eta_1})\cdot\sin\tfrac{1}{2}(\overline{\xi_1+\xi_2}-i\,\overline{\eta_1+\eta_2}) \\ &= c^2\{\cosh(\eta_2-\eta_1)-\cos(\xi_2-\xi_1)\}\{\cosh(\eta_1+\eta_2)-\cos(\xi_1+\xi_2)\}. \end{aligned}$$

Now the value of $P'Q^2$ is obtained by interchanging $\xi_1$ and $\xi_2$ in this result, which evidently gives us $PQ' = P'Q$.

## Illustrative Examples.

**Example i.** *If $a_1$, $a_2$ are the semi-major axes of two conics, confocal with the ellipse $x^2/a^2+y^2/b^2 = 1$, which can be drawn through the point $(x_1, y_1)$, find $x_1$ and $y_1$ in terms of $a_1$ and $a_2$.*

*If $2\phi$ be the angle between the tangents from $(x_1, y_1)$ to the ellipse $(a, b)$, prove that $a_1^2\sin^2\phi + a_2^2\cos^2\phi = a^2$.*

Any conic confocal with the given ellipse is $x^2/(a^2+\lambda)+y^2/(b^2+\lambda) = 1$, and the values of $\lambda$ for the conics of the system which pass through $(x_1, y_1)$ are given by $x_1^2/(a^2+\lambda)+y_1^2/(b^2+\lambda) = 1$.

Let $a^2-b^2 = c^2$, and let $\alpha$ stand for either $a_1$ or $a_2$; then we have

$$x_1^2/\alpha^2+y_1^2/(\alpha^2-c^2) = 1,$$

or
$$\alpha^4-\alpha^2(x_1^2+y_1^2+c^2)+c^2x_1^2 = 0.$$

Hence

$$a_1^2+a_2^2 = x_1^2+y_1^2+c^2, \tag{i}$$

and

$$a_1^2a_2^2 = c^2x_1^2. \tag{ii}$$

Thus $x_1 = a_1a_2/c$, and $y_1^2 = -(a_1^2-c^2)(a_2^2-c^2)/c^2$,

or
$$y_1 = \sqrt{(a_1^2-c^2)(c^2-a_2^2)}/c = b_1b_2/c.$$

We showed (p. 311) that

$$\tan 2\phi = \frac{2\sqrt{b^2x_1^2 + a^2y_1^2 - a^2b^2}}{x_1^2 + y_1^2 - a^2 - b^2}.$$

From (i) $a^2x_1^2 + a^2y_1^2 = a^2a_1^2 + a^2a_2^2 - a^4 + a^2b^2$.

From (ii) $a^2x_1^2 - b^2x_1^2 = a_1^2a_2^2$.

$$\therefore \quad a^2y_1^2 + b^2x_1^2 - a^2b^2 = (a_1^2 - a^2)(a^2 - a_2^2);$$

$$\therefore \quad \tan 2\phi = \frac{2\sqrt{(a_1^2 - a^2)(a^2 - a_2^2)}}{(a_1^2 - a^2) - (a^2 - a_2^2)}$$

Hence

$$\cos 2\phi = \frac{(a_1^2 - a^2) - (a^2 - a_2^2)}{a_1^2 - a_2^2};$$

$$\therefore \quad a_1^2(1 - \cos 2\phi) + a_2^2(1 + \cos 2\phi) = 2a^2;$$

$$\therefore \quad a_1^2 \sin^2\phi + a_2^2 \cos^2\phi = a^2.$$

**Note i.** We have taken $a_1 > a_2$.

**Note ii.** If $\lambda_1$, $\lambda_2$ be the values of $\lambda$ for the two confocals through $(x_1, y_1)$, we have

$$a_1^2 = a^2 + \lambda_1, \quad a_2^2 = a^2 + \lambda_2,$$

so that

$$\tan 2\phi = \frac{2\sqrt{-\lambda_1\lambda_2}}{\lambda_1 + \lambda_2};$$

thus

$$\tan\phi = \sqrt{-\lambda_1/\lambda_2} \text{ or } \sqrt{-\lambda_2/\lambda_1}.$$

**Note iii.** Since the angles between the tangents from any point to a conic are bisected by the tangents to the confocals through this point, the equation of the pair of tangents from $P$ to the conic $(a, b)$ referred to the tangents at $P$ to the confocals through $P$ is $y^2 - x^2\tan^2\phi = 0$, which may be written $x^2/\lambda_1 + y^2/\lambda_2 = 0$.

**Example ii.** *If the normal at $P$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$ meet the polar of $P$ with regard to any confocal conic*

$$x^2/(a^2 + \lambda) + y^2/(b^2 + \lambda) = 1$$

*in $Q$, and if $CY$ be the perpendicular from the centre on the tangent at $P$, prove that $PQ \cdot CY = \lambda$.*

Let $P$ be the point $(a\cos\theta, b\sin\theta)$; the polar of $P$ with respect to

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$$

is

$$\frac{xa\cos\theta}{a^2 + \lambda} + \frac{yb\sin\theta}{b^2 + \lambda} = 1. \qquad \text{(i)}$$

The equation of the normal at $P$ is

$$\frac{x - a\cos\theta}{b\cos\theta} = \frac{y - b\sin\theta}{a\sin\theta} = \frac{r}{CD},$$

and $r$ is equal to $PQ$ if $(x, y)$ lies on (i).

Thus

$$\frac{a\cos\theta}{a^2+\lambda}\left\{\frac{r}{CD}.b\cos\theta+a\cos\theta\right\}+\frac{b\sin\theta}{b^2+\lambda}\left\{\frac{r}{CD}.a\sin\theta+b\sin\theta\right\}=1$$

gives the length $PQ$,

i.e. $$\frac{rab}{CD}\left\{\frac{\cos^2\theta}{a^2+\lambda}+\frac{\sin^2\theta}{b^2+\lambda}\right\}=1-\frac{a^2\cos^2\theta}{a^2+\lambda}-\frac{b^2\sin^2\theta}{b^2+\lambda},$$

i.e. $$\frac{rab}{CD}\{\lambda+a^2\sin^2\theta+b^2\cos^2\theta\}=\lambda^2+\lambda a^2\sin^2\theta+\lambda b^2\cos^2\theta;$$

$$\therefore\quad r.\frac{ab}{CD}=\lambda.$$

But $CY=\dfrac{ab}{CD}$; $\therefore\ PQ.CY=\lambda$.

**Example iii.** *From a fixed point $O$ on a conic tangents are drawn to any confocal. Prove that the line joining the points in which they meet the conic again passes through a fixed point $O'$.*

*If the conic is* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *the locus of $O'$ as $O$ moves is*

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

Let the tangents from $O$ to the confocal meet the conic again in $P$ and $Q$; we have shown that the bisectors of the angles between $OP$, $OQ$ are the tangent and normal at $O$. Let $PQ$ meet this tangent and normal at $G$ and $O'$; then since the pencil $OG$, $OO'$; $OP$, $OR$ is harmonic, the points $P$, $Q$ are harmonic conjugates of $G$, $O'$; hence the polar of $O'$ passes through $G$ and is the line $OG$. Thus $PQ$ always passes through the fixed point $O'$, which is the pole of $OG$ with respect to the given conic.

Let $O$ be the point $(a\cos\theta,\ b\sin\theta)$, then the equation of $OG$ is

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2.$$

Hence, if $O'$ is the point $(x_1, y_1)$, this equation is identical with

$$xx_1/a^2+yy_1/b^2=1.$$

We have therefore

$$(a^2-b^2)x_1=a^2\sec\theta,\ (a^2-b^2)y_1=-b^2\operatorname{cosec}\theta,$$

whence, eliminating $\theta$, we get the locus of $O'$, viz.

$$a^6/x^2+b^6/y^2=(a^2-b^2)^2.$$

**Example iv.** *Prove that through any point $(f, g)$ two lines at right angles can be drawn to form with the polar of $(f, g)$ a right-angled triangle self-conjugate with respect to $x^2/a^2+y^2/b^2-1=0$, and that the area of the triangle is* $\pm\dfrac{1}{2}\dfrac{\lambda_1\lambda_2(\lambda_1-\lambda_2)}{fg\,(a^2-b^2)}$, *where $\lambda_1$, $\lambda_2$ are the roots of*

$$f^2/(a^2+\lambda)+g^2/(b^2+\lambda)-1=0.$$

Let $P$ be the point $(f, g)$, and let the tangents to the conics, confocal with the given ellipse, which pass through $P$ meet the polar of $P$ in $Q$ and $R$. We showed in Ex. iii that $R$ is the pole of $PQ$, and since $QR$ is, by construction, the polar of $P$, the triangle $PQR$ is self-conjugate; it is evidently right-angled at $P$. The equations of the sides of the triangle are

$$\begin{aligned} xf/(a^2+\lambda_1)+yg/(b^2+\lambda_1)-1 &= 0,\\ xf/(a^2+\lambda_2)+yg/(b^2+\lambda_2)-1 &= 0,\\ xf/a^2 \quad\quad +yg/b^2 \quad\quad -1 &= 0, \end{aligned}$$

where $\lambda_1$, $\lambda_2$ are the parameters of the confocals through $P$ and therefore given by the equation

$$f^2/(a^2+\lambda)+g^2/(b^2+\lambda)=1.$$

The vertices of the triangle are the poles of the sides with respect to the given ellipse, viz.

$$(f, g),\ \left(\frac{a^2f}{a^2+\lambda_1},\ \frac{b^2g}{b^2+\lambda_1}\right),\ \left(\frac{a^2f}{a^2+\lambda_2},\ \frac{b^2g}{b^2+\lambda_2}\right).$$

The area of the triangle is then

$$\frac{1}{2}\cdot\frac{fg\lambda_1\lambda_2(\lambda_1\sim\lambda_2)(a^2-b^2)}{(a^2+\lambda_1)(a^2+\lambda_2)(b^2+\lambda_1)(b^2+\lambda_2)}.$$

Now $\lambda_1$, $\lambda_2$ are the roots of the equation

$$\lambda^2+\lambda(a^2+b^2-f^2-g^2)+a^2b^2-b^2f^2-a^2g^2=0;$$

hence
$$(a^2+\lambda_1)(a^2+\lambda_2)=f^2(a^2-b^2),$$

and
$$(b^2+\lambda_1)(b^2+\lambda_2)=g^2(b^2-a^2),$$

i.e. the area of the triangle is

$$\frac{1}{2}\frac{\lambda_1\lambda_2(\lambda_1\sim\lambda_2)}{fg(a^2-b^2)}.$$

## Examples VIII e.

1. Find the equations of two confocal conics whose foci are (1, 0), (−1, 0), and which pass through (2, 3).

2. Prove that the locus of points on a system of confocal ellipses, which have the same eccentric angle $\alpha$, is a confocal hyperbola whose asymptotes are inclined at an angle $2\alpha$.

3. If two tangents to an ellipse are at right angles, the envelope of their chord of contact is a confocal ellipse.

4. Through a point on the director circle of the ellipse $x^2/a^2+y^2/b^2=1$ two conics are drawn confocal with the ellipse: if $2a_1$, $2a_2$ are their transverse axes, show that $a_1^2+a_2^2=2a^2$.

5. The difference of the squares on the perpendiculars from the centre to two parallel tangents to two given confocals is fixed.

6. Find the locus of the intersection of two orthogonal tangents which are drawn one to each of two confocals.

7. If any two parallel tangents to an ellipse meet a fixed circle concentric with the ellipse in $P$, $Q$ and $P'$, $Q'$, prove that $PP'$, $QQ'$ touch a confocal conic.

8. If the points of intersection of two confocals lie within the circle described on the line joining the foci as diameter, the minor axis of the ellipse will be less than the conjugate axis of the hyperbola.

9. If from $P$, a point on an ellipse whose semi-axes are $\alpha$, $\beta$, two tangents are drawn to a confocal ellipse whose semi-axes are $a, b$, show that if $\theta, \phi$ are the eccentric angles of the points of contact of the tangents, and $r$ is the distance of $P$ from the centre, then

$$a^2b^2\sec^2\tfrac{1}{2}(\theta-\phi) = \alpha^2\beta^2 - \lambda r^2,$$

where $\lambda = \alpha^2 - a^2$.

10. Through a point $P$ an ellipse and an hyperbola are drawn confocal with $x^2/a^2 + y^2/b^2 = 1$; find the length of the semi-diameter of the ellipse conjugate to $CP$ in terms of the parameters of the two confocals.

11. A point $P$ is taken on the conic $x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1$, such that the normal at $P$ may pass through a fixed point $(h, k)$. Show that $P$ lies on the curve given by

$$x/(y-k) + y/(x-h) = (a^2-b^2)/(hy-kx).$$

12. If $P$ is any point and $S$, $S'$ are the foci of $x^2/a^2 + y^2/b^2 = 1$, show that $PS.PS'$ is equal to the difference of the squares of the major semi-axes of the two conics which can be drawn through $P$, confocal with the given conic.

13. $PP'$ is a diameter of an ellipse, and $D$ any point on the curve. Prove that if $DP$, $DP'$ touch a confocal ellipse whose semi axes are $\sqrt{a^2-\lambda}$, $\sqrt{b^2-\lambda}$, then $DP.DP'\sin^2 PDP' = 4\lambda$.

14. $T$, $T'$ are the poles of a straight line with respect to two confocal conics whose semi-major axes are $a$, $a'$, and $p$ is the perpendicular on the straight line from the centre of the conics. Show that $p.TT' = a'^2 - a^2$.

15. $P$ is any point on the director circle of a fixed conic $S$. $S_1$, $S_2$ are the two conics through $P$ confocal with $S$. Show (i) that the squares of the lengths of the major axes of the three conics are in arithmetical progression, (ii) that the product of the lengths of the major axes of $S_1$ and $S_2$ varies as the distance of the point from the minor axis of the system.

16. The locus of the centres of curvature at ends of equi-conjugate diameters of ellipses of the confocal system $(a^2+\theta)^{-1}x^2 + (b^2+\theta)^{-1}y^2 = 1$ is the curve $8x^2y^2 = (a^2-b^2)(y^2-x^2)$.

17. Prove that the polars of a point $(x', y')$ with respect to the confocals $x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1$ touch the conic $\sqrt{xx'} + \sqrt{-yy'} + \sqrt{a^2-b^2} = 0$.

18. A tangent is drawn to a rectangular hyperbola whose asymptotes are the coordinate axes. Its poles are taken with respect to a series of conics confocal with $x^2/a'^2 + y^2/b'^2 = 1$. Prove that the polars of all these poles with respect to the conic $x^2/a'^2 + y^2/b'^2 = 1$ meet in a point; also that the locus of such points is a rectangular hyperbola having the axes of coordinates as asymptotes, and conjugate to the given hyperbola if $4a'^2b'^2 = (a^2-b^2)^2$.

19. Find the condition that $x^2/a^2 + y^2/b^2 = 1$, and $Ax^2 + By^2 + C = 0$ may represent confocal ellipses.

Points $P$, $Q$ are taken on confocal ellipses, such that their distances from

the minor axes are proportional to the major axes of the ellipses on which they respectively lie. $P'$, $Q'$ are another such pair of points. Prove that $PQ' = P'Q$.

20. Show that the centres of curvature at the points where $y = mx$ meets the conics confocal with $x^2/a^2 + y^2/b^2 = 1$ lie on the curve

$$y\{x^2 - 3mxy - a^2 + b^2\}^2 + mx\{m(y^2 + a^2 - b^2) - 3xy\}^2 = 0.$$

21. Tangents are drawn to the ellipse $x^2/a^2 + y^2/b^2 = 1$ from any point on the confocal hyperbola whose asymptotes lie along the equi-conjugate diameters; show that the centre of the circle inscribed in the triangle formed by the tangents and the chord of contact lies on the given ellipse.

22. Show that the two curves

$$x^2/a^2 + y^2/b^2 = 1 \text{ and } x^2/a^2 - y^2/b^2 - (a^2 - b^2)/(a^2 + b^2) + 2\lambda xy = 0$$

intersect at right angles.

23. Find the locus of points on conics of the confocal system

$$\lambda x^2 + (c^2 + \lambda) y^2 = \lambda (c^2 + \lambda),$$

the tangents at which make an angle $\alpha$ with the axis of $x$.

Show that the tangent to the conic at such a point is a bisector of the angle between the line joining the point to the origin and the tangent to the locus.

24. If $CP$, $CD$ are conjugate semi-diameters of an ellipse, prove that the parameter $\lambda$ of the confocal hyperbola through $P$ is equal to $-CD^2$. If a tangent to this hyperbola cuts the ellipse at the ends of two conjugate diameters, prove that the length intercepted by the ellipse on the tangent is equal to the perpendicular from the centre on the tangent at $P$ to the ellipse.

25. A triangle $PQR$ is inscribed in one ellipse and circumscribed to another confocal with the former; prove that the normals at the points of contact meet in a point.

26. An hyperbola cuts a concentric, but not confocal, ellipse orthogonally at four points. Prove that the tangents to the ellipse at adjacent points of intersection are perpendicular to one another.

27. Show that the curves

$$x^2/a^2 - y^2/b^2 = 1 \text{ and } x^2/a^2 + y^2/b^2 + 2\lambda xy = (a^2 + b^2)/(a^2 - b^2)$$

cut one another orthogonally for all values of $\lambda$; but are confocal if, and only if, $\lambda = 0$.

## Miscellaneous Illustrative Examples.

(i) *The locus of the centres of equilateral triangles described about the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *is*

$$9(x^2 + y^2)^2 - 2(5a^2 + 3b^2)x^2 - 2(3a^2 + 5b^2)y^2 + (a^2 - b^2)^2 = 0.$$

The perpendiculars from the centre of an equilateral triangle to the sides are equal; if the ellipse is inscribed in the triangle, these perpendiculars

make angles $\alpha$, $\alpha+\frac{2}{3}\pi$, $\alpha+\frac{4}{3}\pi$ with the $x$-axis; the equations of the sides are therefore of the form

$$x\cos\phi+y\sin\phi-\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}=0,$$

where $\phi$ has the values $\alpha$, $\alpha+\frac{2}{3}\pi$, $\alpha+\frac{4}{3}\pi$. Further, the centre of the triangle and the origin are on the same side of each of these tangents; hence, if $p$ is the radius of the inscribed circle and $(x, y)$ the centre, we have

$$p=\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}-x\cos\phi-y\sin\phi$$

for each side: thus

$$(x\cos\phi+y\sin\phi+p)^2=a^2\cos^2\phi+b^2\sin^2\phi$$

is satisfied by the three values of $\phi$ given above.

Again, if the ellipse is escribed to one side of the triangle, the perpendiculars from its centre to the sides make angles $\alpha$, $\alpha+\frac{1}{3}\pi$, $\alpha+\frac{2}{3}\pi$ with the $x$-axis; the centre of the triangle and the origin are on the same side of two sides of the triangle and on opposite sides of the other, viz. that corresponding to $(\alpha+\frac{1}{3}\pi)$.

For this side we have therefore

$$p=x\cos(\alpha+\tfrac{1}{3}\pi)+y\sin(\alpha+\tfrac{1}{3}\pi)-\sqrt{a^2\cos^2(\alpha+\tfrac{1}{3}\pi)+b^2\sin^2(\alpha+\tfrac{1}{3}\pi)},$$

or, since

$$\cos(\alpha+\tfrac{1}{3}\pi)=-\cos(\alpha+\tfrac{4}{3}\pi) \text{ and } \sin(\alpha+\tfrac{1}{3}\pi)=-\sin(\alpha+\tfrac{4}{3}\pi),$$

$$x\cos(\alpha+\tfrac{4}{3}\pi)+y\sin(\alpha+\tfrac{4}{3}\pi)+p=-\sqrt{a^2\cos^2(\alpha+\tfrac{4}{3}\pi)+b^2\sin^2(\alpha+\tfrac{4}{3}\pi)}.$$

Thus, in this case also, we find that, if $(x, y)$ is the centre of the triangle,

$$(x\cos\phi+y\sin\phi+p)^2=a^2\cos^2\phi+b^2\sin^2\phi \qquad \text{(i)}$$

when $\phi$ has the values $\alpha$, $\alpha+\frac{2}{3}\pi$, $\alpha+\frac{4}{3}\pi$.

Now put $x=r\cos\theta$, $y=r\sin\theta$, and equation (i) becomes

$$\{r\cos(\theta-\phi)+p\}^2=\tfrac{1}{2}(a^2+b^2)+\tfrac{1}{2}(a^2-b^2)\cos 2\phi.$$

Let $z=\cos\phi+i\sin\phi$ and $t=\cos\theta+i\sin\theta$,

thus $z/t+t/z=2\cos\overline{\theta-\phi}$ and $z^2+1/z^2=2\cos 2\phi$;

hence $\{r(z^2+t^2)+2ptz\}^2=2(a^2+b^2)z^2t^2+(a^2-b^2)t^2(z^4+1)$,

or $\{r^2-(a^2-b^2)t^2\}z^4+4prtz^3+2t^2(2p^2+r^2-a^2-b^2)z^2$

$$+4prt^3z+t^2\{r^2t^2-(a^2-b^2)\}=0. \qquad \text{(ii)}$$

Now three of the values of $z$ given by equation (ii) are

$$\cos\alpha+i\sin\alpha,\ \cos\overline{\alpha+\tfrac{2}{3}\pi}+i\sin\overline{\alpha+\tfrac{2}{3}\pi},\ \cos\overline{\alpha+\tfrac{4}{3}\pi}+i\sin\overline{\alpha+\tfrac{4}{3}\pi},$$

so that $z_1^3=z_2^3=z_3^3=\cos 3\alpha+i\sin 3\alpha=\lambda$ (say);

consequently equation (ii) is of the form $(z^3-\lambda)(z-\mu)=0$,

or $$z^4-\mu z^3-\lambda z+\lambda\mu=0.$$

Therefore $2p^2+r^2-a^2-b^2=0$ (since $t$ is evidently not zero),

and $$4prt=-\mu\{r^2-(a^2-b^2)t^2\},$$

$$4prt\mu=-\{r^2t^2-(a^2-b^2)\}.$$

Hence, eliminating $p$ and $\mu$, we have

$$\{r^2-(a^2-b^2)t^2\}\{r^2t^2-(a^2-b^2)\}=16p^2r^2t^2=-8r^2t^2(r^2-a^2-b^2);$$

$$\therefore \quad 9r^4-(a^2-b^2)(t^2+1/t^2)r^2-8(a^2+b^2)r^2+(a^2-b^2)^2=0;$$

$$\therefore \quad 9r^4 - 2(a^2-b^2)r^2\cos 2\theta - 8(a^2+b^2)r^2 + (a^2-b^2)^2 = 0;$$

$$\therefore \quad 9(x^2+y^2)^2 - 2(a^2-b^2)(x^2-y^2) - 8(a^2+b^2)(x^2+y^2) + (a^2-b^2)^2 = 0;$$

$$\therefore \quad 9(x^2+y^2)^2 - 2(5a^2+3b^2)x^2 - 2(3a^2+5b^2)y^2 + (a^2-b^2)^2 = 0.$$

(ii) *Show that there are two systems of circles which cut a conic orthogonally in two points: find the equation of the circle of either system which passes through the point* $(x', y')$ *on the conic*

$$x^2/a^2 + y^2/b^2 = 1.$$

*If a circle of one system cuts orthogonally a circle of the other system, show that the line joining their centres touches the conic*

$$x^2/a^2 - y^2/b^2 = (a^2-b^2)/(a^2+b^2).$$

If a circle cuts a conic orthogonally in two points $P$, $P'$, the tangents to the conic at $P$ and $P'$ must meet at the centre of the circle, i.e. tangents to the conic from the centre of the circle are equal to each other and to the radius of the circle. Hence there are two such systems of circles, one having its centres on the major axis and the other having its centres on the minor axis.

Now, if $(x', y')$ is the point $(a\cos\theta, b\sin\theta)$, the tangent at $(x', y')$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

This meets the axes at the points $(a\sec\theta, 0)$, $(0, b\operatorname{cosec}\theta)$: these are the centres of the two circles: their equations are

$$(x - a\sec\theta)^2 + y^2 = (a\sec\theta - a\cos\theta)^2 + b^2\sin^2\theta,$$

$$x^2 + (y - b\operatorname{cosec}\theta)^2 = a^2\cos^2\theta + (b\operatorname{cosec}\theta - b\sin\theta)^2:$$

i.e.

$$x^2 + y^2 - 2ax\sec\theta = a^2\cos^2\theta + b^2\sin^2\theta - 2a^2,$$

$$x^2 + y^2 - 2by\operatorname{cosec}\theta = a^2\cos^2\theta + b^2\sin^2\theta - 2b^2,$$

which can at once be expressed in terms of $x'$ and $y'$.

Now if the circles

$$x^2 + y^2 - 2ax\sec\theta = a^2\cos^2\theta + b^2\sin^2\theta - 2a^2,$$

$$x^2 + y^2 - 2by\operatorname{cosec}\phi = a^2\cos^2\phi + b^2\sin^2\phi - 2b^2$$

cut orthogonally, we have

$$a^2\cos^2\theta + b^2\sin^2\theta - 2a^2 + a^2\cos^2\phi + b^2\sin^2\phi - 2b^2 = 0,$$

whence

$$(a^2-b^2)\cos^2\theta = a^2 + b^2 + (a^2-b^2)\sin^2\phi. \qquad \text{(i)}$$

The line joining their centres is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\phi = 1. \qquad \text{(ii)}$$

From (i) and (ii)

$$(a^2-b^2)\left(1 - \frac{x}{a}\cos\theta\right)^2 = \frac{y^2}{b^2}\left[(a^2-b^2)\cos^2\theta - a^2 - b^2\right].$$

Expressing the condition that this equation in the variable $\cos\theta$ should have equal roots, we get for the envelope of the line of centres

$$(a^2-b^2)\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left[(a^2-b^2)+(a^2+b^2)\frac{y^2}{b^2}\right]=(a^2-b^2)^2\frac{x^2}{a^2},$$

which reduces to $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{a^2-b^2}{a^2+b^2}.$$

(iii) *Prove that two hyperbolas can be drawn touching the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

*and having for asymptotes the tangents TP, TQ. Show also that if T lies on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=m^2$, *each of the hyperbolas has double contact along PQ with one or other of the ellipses* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2}{1\pm m}$.

Let $T$ be the point $(x_1, y_1)$, then the equation of the tangents from $T$ to the ellipse are

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2.$$

The equation of any hyperbola having these for asymptotes is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2+c^2=0, \qquad \text{(i)}$$

where $c$ is a constant.

The eccentric angles of the points of intersection of the hyperbola (i) and the ellipse are found by substituting $x=a\cos\theta$, $y=b\sin\theta$ in this equation: thus

$$\frac{x_1}{a}\cos\theta+\frac{y_1}{b}\sin\theta-1=\pm c,$$

i.e. $$\frac{x_1}{a}\left(1-\tan^2\frac{\theta}{2}\right)+\frac{2y_1}{b}\tan\frac{\theta}{2}-(1\pm c)\left(1+\tan^2\frac{\theta}{2}\right)=0,$$

i.e. $$\tan^2\frac{\theta}{2}\left\{\frac{x_1}{a}+(1\pm c)\right\}-\frac{2y_1}{b}\tan\frac{\theta}{2}-\left\{\frac{x_1}{a}-(1\pm c)\right\}=0.$$

This equation has equal roots and the hyperbola touches the ellipse if

$$\frac{x_1^2}{a^2}-(1\pm c)^2+\frac{y_1^2}{b^2}=0;$$

$$\therefore \quad 1\pm c=\pm\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}};$$

$$\therefore \quad \pm c=-1\pm\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}},$$

i.e. $$c^2=\left(1-\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}}\right)^2 \text{ or } \left(1+\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}}\right)^2.$$

There are therefore two such hyperbolas.

If $T$ lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=m^2$, we have $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=m^2$;

$$\therefore \qquad c^2=(1-m)^2 \text{ or } (1+m)^2.$$

In this case the equations of the hyperbolas become

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)(m^2-1)+(1\pm m)^2=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2,$$

or
$$(m^2-1)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\frac{1\pm m^2}{m^2-1}\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2,$$

i.e.
$$(m^2-1)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2}{1\pm m}\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2,$$

which is a conic having double contact with

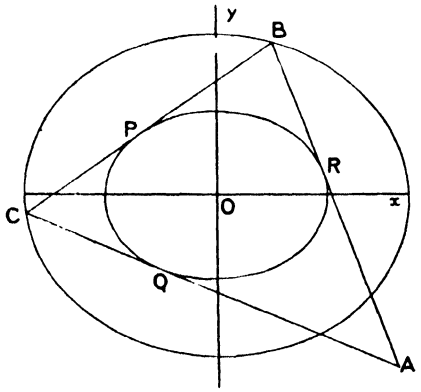
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2}{1\pm m},$$

the chord of contact being $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$,

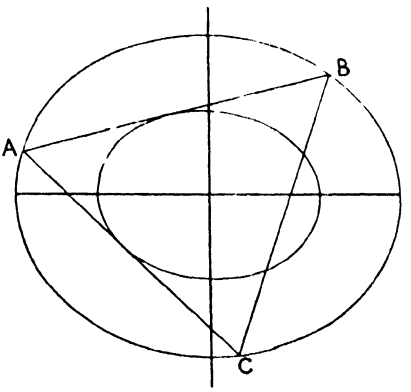
i.e. $PQ$. (See form $S=ku^2$).

(iv) (a) *The sides of a triangle touch an ellipse* $(a, b)$ *and two of its vertices lie on an ellipse* $(A, B)$; *find the locus of the third vertex.*

(b) *The vertices of a triangle lie on an ellipse* $(a, b)$ *and two of its sides touch an ellipse* $(A, B)$; *find the envelope of the third side, where the ellipse* $(a, b)$ *means* $x^2/a^2+y^2/b^2=1$.




Let the points of contact be $P$, $Q$, $R$ and let $P$ be the point $\alpha$, $Q$, $R$ the points $\theta$ and $\phi$. Then the coordinates of $A$ are




Let $ABC$ be the vertices and let $A$ be the point $\alpha$, $B$ the point $\theta$, and $C$ the point $\phi$.

$$x = \frac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}, \quad y = \frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}; \quad \text{(i)}$$

and similarly for $B$ and $C$.

Now since $B$ lies on the conic

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$

we have

$$\frac{a^2}{A^2}\cos^2\frac{\alpha+\phi}{2} + \frac{b^2}{B^2}\sin^2\frac{\alpha+\phi}{2} = \cos^2\frac{\alpha-\phi}{2},$$

or

$$\frac{a^2}{A^2}(1+\cos\overline{\alpha+\phi}) + \frac{b^2}{B^2}(1-\cos\overline{\alpha+\phi}) = 1+\cos\overline{\alpha-\phi},$$

i.e.

$$L\cos\alpha\cos\phi + M\sin\alpha\sin\phi + N = 0 \quad \text{(ii)}$$

where

$$L \equiv \frac{a^2}{A^2} - \frac{b^2}{B^2} - 1,$$

$$M \equiv -\frac{a^2}{A^2} + \frac{b^2}{B^2} - 1,$$

$$N \equiv \frac{a^2}{A^2} + \frac{b^2}{B^2} - 1.$$

Similarly,

$$L\cos\alpha\cos\theta + M\sin\alpha\sin\theta + N = 0. \quad \text{(iii)}$$

Cross multiplying from (ii) and (iii)

$$\frac{L\cos\alpha}{\sin\theta-\sin\phi} = \frac{M\sin\alpha}{\cos\phi-\cos\theta} = \frac{N}{\sin(\phi-\theta)}.$$

Hence

$$\frac{L\cos\alpha}{\cos\frac{\theta+\phi}{2}} = \frac{M\sin\alpha}{\sin\frac{\theta+\phi}{2}} = \frac{N}{-\cos\frac{\theta-\phi}{2}}.$$

Thus from (i)

$$x = -a\frac{L}{N}\cos\alpha,$$

$$y = -b\frac{M}{N}\sin\alpha,$$

The equation of $AB$ is

$$\frac{x}{a}\cos\frac{\theta+\alpha}{2} + \frac{y}{b}\sin\frac{\theta+\alpha}{2} = \cos\frac{\theta-\alpha}{2},$$

and since this touches the conic

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$

we have

$$\frac{A^2}{a^2}\cos^2\frac{\theta+\alpha}{2} + \frac{B^2}{b^2}\sin^2\frac{\theta+\alpha}{2} = \cos^2\frac{\theta-\alpha}{2},$$

or

$$\frac{A^2}{a^2}(1+\cos\overline{\theta+\alpha}) + \frac{B^2}{b^2}(1-\cos\overline{\theta+\alpha}) = 1+\cos\overline{\theta-\alpha};$$

i.e.

$$L\cos\theta\cos\alpha + M\sin\theta\sin\alpha + N = 0 \quad \text{(i)}$$

where

$$L \equiv \frac{A^2}{a^2} - \frac{B^2}{b^2} - 1;$$

$$M \equiv -\frac{A^2}{a^2} + \frac{B^2}{b^2} - 1;$$

$$N \equiv \frac{A^2}{a^2} + \frac{B^2}{b^2} - 1.$$

Similarly,

$$L\cos\alpha\cos\phi + M\sin\alpha\sin\phi + N = 0. \quad \text{(ii)}$$

Cross multiplying from (i) and (ii)

$$\frac{L\cos\alpha}{\sin\theta-\sin\phi} = \frac{M\sin\alpha}{\cos\phi-\cos\theta} = \frac{N}{\sin(\phi-\theta)}.$$

Hence

$$\frac{L\cos\alpha}{\cos\frac{\theta+\phi}{2}} = \frac{M\sin\alpha}{\sin\frac{\theta+\phi}{2}} = \frac{N}{-\cos\frac{\theta-\phi}{2}}.$$

But the equation of $BC$ is

$$\frac{x}{a}\cos\frac{\theta+\phi}{2} + \frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta-\phi}{2},$$

which can therefore be written

$$\frac{L}{a}x\cos\alpha + \frac{M}{b}y\sin\alpha = -N,$$

and the locus of the third vertex $A$ is

$$\frac{N^2x^2}{a^2L^2} + \frac{N^2y^2}{b^2M^2} = 1,$$

which is an ellipse, whose axes lie along the axes of the given ellipses.

which is a tangent to the ellipse

$$\frac{L^2x^2}{N^2a^2} + \frac{M^2y^2}{N^2b^2} = 1 \qquad \text{(iii)}$$

at the point

$$\left\{-\frac{Na}{L}\cos\alpha,\ -\frac{Nb}{M}\sin\alpha\right\},$$

i.e. the envelope of the third side is an ellipse whose axes lie along the axes of the given ellipses.

## Miscellaneous Examples for Revision.

1. Two circles touch internally at $O$: any straight line through $O$ cuts the circles in $P$, $Q$: find the locus of a point $R$ on $PQ$ such that $OR$, $PQ$ are pairs of harmonic conjugates.

2. A system of circles touches the axis of $x$ at the origin: find the equation of the locus of the poles of the straight line $lx+my+n=0$ with respect to them.

What does the locus become when the straight line is (a) parallel to the $x$-axis; (b) parallel to the $y$-axis; (c) a line through the origin?

3. Two circles touch internally, and the diameter of one is twice that of the other. Find the locus of the poles of tangents to each circle with respect to the other.

4. $TP$, $TQ$ are tangents to a parabola $y^2=4ax$ from a point $T(\xi, \eta)$: show that the equation of the circle $TPQ$ is

$$a(x^2+y^2) - x(2a^2+\eta^2) - y\eta(a-\xi) + a\xi(2a-\xi) = 0.$$

(a) If $O$ is the centre of this circle prove that $TSO$ is a right angle.

(b) If $T$ lies on the directrix, $O$ lies on a parabola whose vertex is at the focus.

(c) If $T$ lies on the latus rectum, $O$ lies on the axis.

5. $ABCD$ is a square: a straight line is drawn cutting $AB$, $AD$ in $P$ and $Q$ so that $PA+AQ=2AB$: find the locus of the foot of the perpendicular from $C$ on $PQ$.

6. A chord $PQ$ of a parabola passes through the point $(-3a, 0)$: show that the circle through $PQ$ which touches the parabola passes through the focus.

7. The tangents from a point $T$ to an ellipse include an angle $\theta$: prove that $2ST.HT\cos\theta = ST^2+HT^2-4a^2$.

8. The orthocentre of a triangle $P$, $Q$, $R$ inscribed in a parabola is at the focus: if the tangents at $P$, $Q$, $R$ make angles $\theta_1$, $\theta_2$, $\theta_3$ with the axis, show that $\cot\theta_1$, $\cot\theta_2$, $\cot\theta_3$ satisfy an equation of the form

$$x^3+px^2-5x-p=0.$$

Prove also that the centroid of the triangle formed by the tangents at $P$, $Q$, $R$ lies on the straight line $3x+5a=0$.

9. The triangle $AOB$ has the angle $O$ equal to $\omega$ and $(x_0, y_0)$ for its orthocentre: show that the equation of $AB$ referred to $OA$, $OB$ as axes of $x$ and $y$ is $x/(x_0+y_0\sec\omega)+y/(y_0+x_0\sec\omega)=1$.

10. If $L_1$ denotes $y+m_1x-2am_1-am_1^3$, with similar meanings for $L_2$ and $L_3$, determine the mutual ratios of $\lambda_1$, $\lambda_2$, and $\lambda_3$ so that

$$\lambda_1 L_2L_3+\lambda_2 L_3L_1+\lambda_3 L_1L_2=0$$

may be a circle.

If $t$ is the length of the tangent drawn from $S$, the focus of the parabola $y^2-4ax=0$, to the circle circumscribing the triangle formed by the normals at $P$, $Q$, $R$, show that $at^2=SP.SQ.SR$.

11. Find the locus of a point such that the line joining the points of contact of tangents drawn from it to a given conic subtends a right angle at a given point.

12. Through the extremities of any two focal chords of an ellipse a conic is described: if this conic passes through the centre of the ellipse it will cut the major axis in another fixed point.

13. Find the general equation of a conic which passes through the four given points $(1, 1)$, $(-1, 1)$, $(2, 0)$, $(3, -4)$.

Show that there are two parabolas which fulfil this condition.

What is the nature of that conic which also passes through the origin?

14. Find the equation of the conic which passes through the points $(1, 1)$, $(4, 0)$, $(0, 1)$, $(4, 4)$, $(-7{\cdot}5, 4)$, and trace it.

15. Prove that the six points $(a, 0)$, $(0, a)$, $(b, 0)$, $(0, b)$, $(a, b)$, $(b, a)$ lie on an ellipse, and find its equation.

Find the equation of its axes and show that their lengths are

$$\frac{2\sqrt{2}}{3}\sqrt{a^2-ab+b^2} \text{ and } \frac{2\sqrt{2}}{\sqrt{3}}\sqrt{a^2-ab+b^2}.$$

16. A triangle $PQR$ is inscribed in the parabola $y^2=4ax$ and its centroid is at the focus; prove that:

(i) The normals at $PQR$ are concurrent and the locus of their point of intersection is $2x-7a=0$.

(ii) The locus of the intersection of a side and the tangent at the opposite vertex is $y^2(a-x)=a^3$.

(iii) The poles of the sides lie on the parabola $2y^2=2ax+3a^2$.

(iv) The mid-points of the sides lie on the parabola $y^2=3a^2-2ax$.

(v) The centroid of the triangle formed by the tangents at $PQR$ is $(-a/2, 0)$.

(vi) The circle $PQR$ passes through the vertex and its centre lies on the line $4x=11a$.

(vii) The sum of the squares of the sides of the triangle $PQR$ is constant and equal to $81a^2/2$.

(viii) The locus of the intersection of a side and the diameter through the opposite vertex is the parabola $y^2+2ax-3a^2=0$.

(ix) The locus of the orthocentre of the triangle is $2x+5a=0$.

17. One of the tangents from $T$ to an ellipse subtends angles $\theta_1$, $\theta_2$ at the foci, and the angles between the tangents is $\theta$: show

$$\sin\theta/2a = \sin\theta_1/ST = \sin\theta_2/S'T.$$

18. The cotangent of the angle between the tangents drawn to a rectangular hyperbola from a point on one of its directrices varies as the distance of the point from the centre.

19. Show that a circle described with its centre at any point of an ellipse to touch a pair of conjugate diameters of the ellipse has an invariable radius.

20. A parabola has double contact with the rectangular hyperbola $x^2-y^2=a^2$, and the pole of the chord of contact lies on the circle $x^2+y^2=2cx$. Show that its axis passes through the point $(a^2/c, 0)$.

21. Chords distant $d$ from the centre of an ellipse are divided harmonically by a coaxal conic of semi-axes $\alpha$, $\beta$ given by

$$1/a^2+1/\alpha^2 = 1/b^2+1/\beta^2 = 2d^2/\{(a^2+b^2)d^2-a^2b^2\}.$$

22. Find the greatest value of the angle between a diameter of an ellipse (eccentricity $e$) and the normal at its extremity; show that for the earth's orbit round the sun, of which the eccentricity is $\frac{1}{60}$, this angle is less than half a minute of arc.

23. A chord of length $c$ is drawn parallel to a diameter of length $2d$ of the ellipse $x^2/a^2+y^2/b^2=1$ and $a'$, $b'$ are the lengths of the semi-diameters parallel to the tangents at its extremities. Prove that the angle between these tangents is $\sin^{-1}[c/d \,.\, ab/a'b' \sqrt{1-(c^2/4d^2)}]$.

24. A square is inscribed in an ellipse, whose semi-axes are $a$ and $b$, and any point on the ellipse is joined to the corners of the square. Prove that one of the anharmonic ratios of the pencil so formed is $-b^2/a^2$.

25. $PQ$ is a chord of an ellipse, normal at $P$. The points on the auxiliary circle corresponding to $P$, $Q$ are $p$, $q$. Prove that the angle $pCq$ must exceed $2\tan^{-1}2\sqrt{1-e^2}/e^2$, where $e$ is the eccentricity of the ellipse.

26. The circles of curvature through a point $(f, g)$ are six in number and their centres lie on

$$\{2(f^2+g^2-2fx-2gy)-a^2-b^2\}^2 = 12(a^2x^2+b^2y^2)-3(a^2-b^2)^2.$$

27. The circles of curvature at two points $P$ and $Q$ of an ellipse meet the ellipse again in $M$ and $N$, and the circles which respectively touch the ellipse at $P$ and pass through $Q$, and touch the ellipse at $Q$ and pass through $P$, meet the ellipse again at $R$ and $S$; prove that the chord $MN$ is parallel to $RS$.

28. Triangles are formed by pairs of tangents drawn to the ellipse $b^2x^2+a^2y^2=a^2b^2$ from points on the ellipse $a^2x^2+b^2y^2=(a^2+b^2)^2$, and their chords of contact. Prove that the orthocentre of each triangle lies on the former ellipse.

29. Prove that if $\phi$, $\phi'$ are the eccentric angles of the extremities of a focal chord of an ellipse, the eccentricity is equal to

$$\pm\cos\tfrac{1}{2}(\phi-\phi')\sec\tfrac{1}{2}(\phi+\phi');$$

and that if $\psi$, $\psi'$ are the inclinations to the major axis of the tangent at a point $P$ and of the chord $QR$ drawn so that $PQ$ and $PR$ pass through the foci, the eccentricity is equal to $\{\sin(\psi-\psi')\operatorname{cosec}(\psi+\psi')\}^{\frac{1}{2}}$.

30. Prove that the point on the normal at $P$ on the ellipse

$$x^2/a^2+y^2/b^2-1=0,$$

chords through which subtend a right angle at $P$, is

$$\{x(a^2-b^2)/(a^2+b^2),\quad -y(a^2-b^2)/(a^2+b^2)\},$$

where $(x, y)$ are the coordinates of $P$.

Show that the chord $lx+my=1$ intersects the ellipse in points $Q$, $R$ such that there are two real points on the ellipse at which $QR$ subtends a right angle if $a^2l^2+b^2m^2>(a^2+b^2)^2/(a^2-b^2)^2$.

31. Prove that the equation to an ellipse referred to the tangent and normal at a point on the ellipse as axes is

$$p^2x^2-2\sqrt{(a^2-p^2)(p^2-b^2)}\,xy+(a^2+b^2-p^2)y^2=(2a^2b^2/p)y,$$

where $p$ is the perpendicular from the centre on the tangent and $a$ and $b$ are the semi-axes.

32. A tangent touches an ellipse at $P$ and meets the major axis in $T$. $TQ$ is drawn parallel to the minor axis meeting $AP$ produced in $Q$. Prove that the locus of $Q$ is an hyperbola.

33. Let a circle be described with a point $P$ on an ellipse as centre, and radius $a$, cutting the minor axis of this ellipse at $Q$, and let $PQ$ intersect the major axis at $R$. If the rectangle $RCQV$ is completed, show that $PV$ is the normal to the ellipse at $P$.

34. Find the equation of the circle touching the ellipse $x^2/a^2+y^2/b^2=1$ at the point $x=a\cos\theta$, $y=b\sin\theta$ and cutting the director circle orthogonally.

35. Prove that the cosine of the angle which the tangent at any point on an ellipse makes with the line joining the point to a focus bears a constant ratio to the cosine of the angle which the tangent makes with the major axis.

Perpendiculars $SM$, $HN$ are drawn to the focal distances $SP$, $HP$ of any point $P$ on an ellipse, and meet the tangent at $P$ in $M$ and $N$.

Prove that if the eccentricity of the ellipse is $\not< 2^{-\frac{1}{2}}$ the minimum value of $PM.PN$ is $4b^2$, but that otherwise it is $b^2e^{-2}(1-e^2)^{-1}$.

36. From a fixed point $E$ on a central conic chords are drawn equally inclined to the axis and cutting the curve again at $P$ and $Q$. Find the locus of the centre of gravity of the triangle $PQE$.

37. Show that the locus of the second point of intersection of two circles described on conjugate semi-diameters of the ellipse $b^2x^2+a^2y^2=a^2b^2$ as diameters is the inverse of a concentric ellipse with regard to a circle whose centre is the origin.

38. Points $P$, $Q$, one on each of the central conics

$$x^2/a^2+y^2/b^2=1,\ (1/c^2-1/b^2)x^2+(1/c^2-1/a^2)y^2=1,$$

subtend a right angle at the common centre. Prove that $PQ$ touches the circle $x^2+y^2=c^2$.

39. The normals to an ellipse at the points $P$, $Q$, $P'$, $Q'$ are concurrent. If $PQ$ pass through a fixed point, find the locus of the middle point of $P'Q'$.

40. If the circles of curvature at two points intersect on the ellipse, show that their radical axis is parallel to two of the chords joining the extremities of diameters conjugate to those through the given points.

41. A point moves so that the sum of the squares of its distances from two given sides of an equilateral triangle is constant and equal to $2c^2$. Show that the locus is an ellipse, and find the eccentricity and the position of the foci.

42. Find the coordinates of the point of intersection of two normals, and deduce those of the centre of curvature at $P$.

Show that the centre of curvature can only lie outside the ellipse if $e^2 > \frac{1}{2}$.

43. $P$, $Q$, $R$ are three points on an ellipse such that their eccentric angles differ by $\frac{2}{3}\pi$. Prove that each side of the triangle $PQR$ is parallel to the tangent at the opposite vertex, and that the sum of the squares on the sides is constant.

44. Show that the lines $10y^2 - 7xy - 6x^2 = 0$ coincide in direction with a pair of conjugate diameters of the ellipse $3x^2 + 5y^2 - 15 = 0$.

45. A diameter of a central conic meets one latus rectum in $P$; the conjugate diameter meets the other latus rectum in $P'$. Prove that the envelope of $PP'$ is a conic similar to and coaxial with the given conic.

46. Tangents are drawn to an ellipse at points which subtend at a focus a constant angle $2\beta$. Prove that they intersect on a conic whose eccentricity is to that of the ellipse as $\sec\beta : 1$.

47. If the circles of curvature at $P$, $D$ extremities of conjugate diameters of an ellipse $x^2/a^2 + y^2/b^2 - 1 = 0$ meet the curve again in $R$, $R'$ respectively, prove that the locus of the middle point of $RR'$ is an ellipse.

48. The circle of curvature of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at $P$ meets the ellipse again at $Q$ and the normals at $P$ and $Q$ meet in $G$, $GR$, $GR'$ being the other two normals drawn to the ellipse from $G$: show that the tangents at $R$, $R'$ intersect on the curve $x^2y^2(b^2x^2 + a^2y^2) = (b^2x^2 - a^2y^2)^2$.

Show also that the points $R$, $R'$ are imaginary if the eccentric angle of $P$ is greater than $\frac{1}{2}\tan^{-1}2$.

49. $PP'$ are points on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ whose eccentric angles are $\alpha \pm \beta$, $Q$, $Q'$ are the points $\gamma \pm \delta$. If $U$ and $V$ are the poles of $PP'$ and $QQ'$ prove that $U$, $P$, $P'$, $V$, $Q$, $Q'$ lie on a conic; and find its equation.

50. In the ellipse $x^2/a^2 + y^2/b^2 - 1 = 0$, if a line through a point $O$ cuts the ellipse at $P$ and $Q$, show that the variation of the rectangle $OP$, $OQ$, where $O$ is external to the ellipse, is such that

(i) If $x^2 < a^2$, $y^2 < b^2$ it has a maximum value $a^2(x^2/a^2 + y^2/b^2 - 1)$ and a minimum value $b^2(x^2/a^2 + y^2/b^2 - 1)$.

(ii) If $x^2 > a^2$ and $y^2 > b^2$ the greatest and least values are when $OPQ$ touches the ellipse.

(iii) Also discuss the cases in which $x^2 > a^2$ and $y^2 < b^2$,

$x^2 < a^2$ ,, $y^2 > b^2$.

51. Prove that if four points are taken on an ellipse such that their

normals intersect in the same point, two of them must lie on one quadrant of the arc, and the other two, one on each of the adjacent quadrants.

52. $CP$, $CD$ are conjugate semi-diameters of an ellipse, and $DQ$ is the chord of the ellipse parallel to the minor axis. Find the locus of the intersection of the normals at $P$ and $Q$.

53. Show that the normals which can be drawn from $(x_0, y_0)$ to $x^2/a^2+y^2b^2=1$ are given by

$$[a^2(x-x_0)^2+b^2(y-y_0)^2](xy_0-x_0y)^2=(a^2-b^2)^2(x-x_0)^2(y-y_0)^2.$$

54. From a fixed point $E$ on a central conic chords are drawn equally inclined to the axis and cutting the curve again at $P$ and $Q$. Find the locus of the orthocentre of the triangle $PQE$.

55. Parallelograms are circumscribed to $a'x^2+b'y^2=1$, the sides of any one being parallel to a pair of conjugate diameters of $ax^2+by^2=1$. Prove that they are inscribed in a conic similar to the latter conic, determining its equation.

56. Normals are drawn to the ellipse $x^2/a^2+y^2/b^2=1$ from any point on its evolute. Show that the locus of the centre of the circle passing through the three points of incidence is the curve $4(a^2x^2+b^2y^2)^3=a^2b^2(a^2-b^2)^2x^2y^2$.

57. An ellipse circumscribes a triangle $ABC$ and has its centre at the centre of gravity of the triangle. Prove that the radii of curvature at $A$, $B$, $C$ are proportional to the cubes of the sides $BC$, $CA$, $AB$, and that the product of the three radii of curvature is equal to the cube of the radius of the circle $ABC$.

58. The normal at a point $(a\cos\phi, b\sin\phi)$ on an ellipse makes an angle $\theta$ with the central radius vector of the point. Prove that

$$2ab\tan\theta=(a^2-b^2)\sin 2\phi.$$

When has the angle between two corresponding tangents to the ellipse and the auxiliary circle its greatest value?

Show that that value is $\sin^{-1}(a-b)/(a+b)$.

59. If $T$ is the intersection of tangents to an ellipse at the extremities of a chord $PQ$ normal at $P$, prove that the perpendicular from $T$ on the diameter through $P$ intercepts on $PQ$ a length $PN$ equal to the radius of curvature at $P$.

60. $PP'$ is a diameter of an ellipse and $Q$, $R$ two points on the curve, and $PR$, $P'Q$ meet in $X$, and $PQ$, $P'R$ in $Y$. Show that $XY$ is parallel to the diameter conjugate to $PP'$.

61. Show that the centroids of the triangles $PQR$, $QRS$, $RSP$, $SPQ$ lie on the ellipse $(3x/a-2ha/c^2)^2+(3y/b+2kb/c^2)^2=1$, where $c^2=a^2-b^2$, given that the normals at $PQRS$ meet at $(h, k)$.

62. A circle passes through a given point and cuts off on a given line a chord which subtends at the given point a constant angle $\theta$. Show that its centre traces out an hyperbola of eccentricity $\sec\theta$.

63. A circle is drawn to touch one side of an equilateral triangle and to make the pole of another side (with respect to it) lie on the third side. Prove that the locus of the centre is an hyperbola.

64. Prove that the equation of the hyperbola passing through the point $x'$, $y'$ of the ellipse $b^2x^2+a^2y^2=a^2b^2$ and confocal with it is

$$a^2x^2/x'^2-b^2y^2/y'^2=a^2-b^2.$$

Through a point $P$ of this ellipse straight lines $PQR$, $PQ'R'$ are drawn parallel to the asymptotes of the confocal through $P$, meeting the major axis in $QQ'$ and the minor axis in $RR'$. Prove that $QR'$, $Q'R$ intersect at a point on the normal at $P$, and that the locus of this point is

$$x^2/a^2+y^2/b^2=\{(a^2-b^2)/(a^2+b^2)\}^2.$$

65. Find the general equation of a rectangular hyperbola when the origin is a point on the curve and the tangent at the origin is taken as the axis of $x$.

If a chord $PQ$ of a rectangular hyperbola subtend a right angle at a point $O$ on the curve, show that $PQ$ is parallel to the normal to the curve at $O$.

66. Prove that the axis of the second parabola which passes through the points $m_1$, $m_2$, $m_3$, $m_4$ on $y^2-4ax=0$ is inclined to the axis of the latter at an angle $\cot^{-1}(m_1+m_2+m_3+m_4)/4$.

Deduce that, if two parabolas intersect in four points, the distances of the centroid of the four points from the axes are proportional to the latera recta.

67. An ellipse and hyperbola are concentric and coaxial, and have the same semi-axes. Prove that the circle circumscribing the triangle formed by the asymptotes of the hyperbola and the tangent to it at any point $P$ intersects the director circle of the ellipse on the polar of $P$ with respect to the ellipse.

68. From the focus $S$ of an ellipse whose eccentricity is $e$ radii $SP$, $SQ$ are drawn at right angles to one another, and the tangents at $P$ and $Q$ meet at $T$. Show that the locus of $T$ is an hyperbola, parabola, or ellipse, according as $e$ is $>$, $=$, or $< 1/\sqrt{2}$.

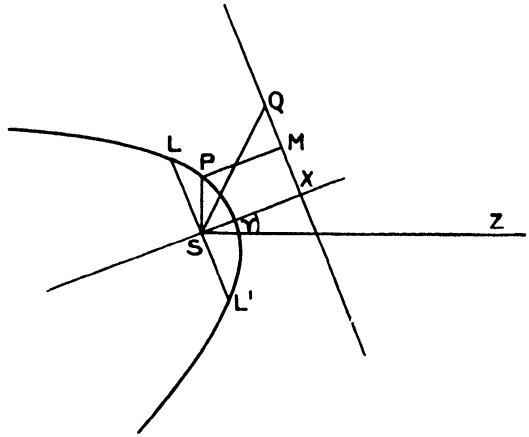
69. Prove that the eight points of contact of the four common tangents of the ellipses $x^2/a^2+y^2/b^2-1=0$ and $x^2/a_1^2+y^2/b_1^2-1=0$ lie on the ellipse $x^2(b^2+b_1^2)+y^2(a^2+a_1^2)=a^2b_1^2+a_1^2b^2$.

# CHAPTER IX

## POLAR COORDINATES

§ 1. When the focal properties of conics are under consideration it is sometimes convenient to use polar coordinates, taking a focus as pole.

*To find the polar equation of a conic when a focus is the pole and the axis is inclined at an angle $\gamma$ to the initial line.*




Let $SZ$ be the initial line, $SA$ the axis of the conic, $XM$ the directrix. If $P$ be any point $(r, \theta)$ on the conic, draw $PM$ perpendicular to the directrix.

Now, if $LL'$ is the latus rectum, let

$$LL' = 2\,SL = 2\,l,$$

then $SL = eSX$;

$\therefore$ $SX = l/e$.

But using the focus-directrix property of conics previously proved,

$$\begin{aligned} r = SP &= ePM \\ &= e(SX - SP\cos\overline{\theta-\gamma}) \\ &= l - er\cos\overline{\theta-\gamma}\,; \end{aligned}$$

the equation of the conic can therefore be written

$$l/r = 1 + e\cos\overline{\theta-\gamma}. \qquad \text{(i)}$$

N.B. When the axis lies along the initial line, the equation takes the simpler form

$$l/r = 1 + e\cos\theta. \qquad \text{(ii)}$$

Again, if $(r, \theta)$ be any point $Q$ on the directrix,

since $SQ\cos QSX = SX,$

we have $r\cos(\theta-\gamma) = l/e,$

so that

$$l/r = e\cos(\theta-\gamma)$$

is the equation of the directrix.

**Note.** It is important to remember that, since $(r, \theta)$ and $(-r, \pi+\theta)$ indicate the same point,

$$-l/r = 1 - e\cos\overline{\theta-\gamma}$$

or $$l/r = -1 + e\cos\overline{\theta-\gamma}$$

represents the same conic as

$$l/r = 1 + e\cos\overline{\theta-\gamma}.$$

Thus, for instance, if

$$l/r = 1 + e\cos(\theta-\gamma),$$
$$L/r = 1 + E\cos(\theta-\delta)$$

are any two conics with a common focus,

$$\{l/r - 1 - e\cos\overline{\theta-\gamma}\} - \{L/r - 1 - E\cos\overline{\theta-\delta}\} = 0,$$

i.e. $$(l-L)/r = e\cos(\theta-\gamma) - E\cos(\theta-\delta)$$

passes through points common to the two conics and, being a straight line, represents a common chord. Again,

$$\{l/r - 1 - e\cos(\theta-\gamma)\} + \{L/r + 1 - E\cos(\theta-\delta)\} = 0,$$

i.e. $$(l+L)/r = e\cos(\theta-\gamma) + E\cos(\theta-\delta),$$

where we have used the alternative form for the second conic, represents the common chord through the other two points of intersection of the conics.

§ 2. *To find the equation of the chord joining two points on the conic* $l/r = 1 + e\cos(\theta-\gamma)$ *whose vectorial angles are* $\alpha$ *and* $\beta$.

The equation

$$l/r = A\cos\{\theta - \tfrac{1}{2}(\alpha+\beta)\} + B\sin\{\theta - \tfrac{1}{2}(\alpha+\beta)\} \qquad \text{(i)}$$

can, by a proper choice of $A$ and $B$, be made to represent any straight line. If $(r_1, \alpha)$, $(r_2, \beta)$ are the extremities of the chord whose equation is required, since

$$l/r_1 = 1 + e\cos(\alpha-\gamma) \quad \text{and} \quad l/r_2 = 1 + e\cos(\beta-\gamma),$$

these points lie on the straight line (i) if

$$A\cos\tfrac{1}{2}(\alpha-\beta)+B\sin\tfrac{1}{2}(\alpha-\beta)=1+e\cos(\alpha-\gamma),$$
$$A\cos\tfrac{1}{2}(\alpha-\beta)-B\sin\tfrac{1}{2}(\alpha-\beta)=1+e\cos(\beta-\gamma);$$

hence

$$A\cos\tfrac{1}{2}(\alpha-\beta)=1+e\cos\tfrac{1}{2}(\alpha-\beta)\cos\{\tfrac{1}{2}(\alpha+\beta)-\gamma\},$$

and

$$B\sin\tfrac{1}{2}(\alpha-\beta)=-e\sin\tfrac{1}{2}(\alpha-\beta)\sin\{\tfrac{1}{2}(\alpha+\beta)-\gamma\}.$$

Substituting these values of $A$ and $B$ in (i), the equation becomes

$$l/r-e\cos(\theta-\gamma)=\sec\tfrac{1}{2}(\alpha-\beta)\cos\{\theta-\tfrac{1}{2}(\alpha+\beta)\},$$

which is the equation of the chord.

It is often useful to take equations in the form shown in (i) when the conditions given are that points on the conic lie on a required locus: the following is another example.

**Example.** *To find the equation of the circle which passes through the focus and also through the points* $\alpha$, $\beta$ *on the conic* $l/r=1+e\cos\theta$.

Let the equation of the circle be

$$r=A\cos\{\theta-\tfrac{1}{2}(\alpha+\beta)\}+B\sin\{\theta-\tfrac{1}{2}(\alpha+\beta)\}$$

instead of the usual form $r=a\cos(\theta-\delta)$.

Then, since the points $\alpha$, $\beta$ lie on it, we have

$$A\cos\tfrac{1}{2}(\alpha-\beta)+B\sin\tfrac{1}{2}(\alpha-\beta)=l/(1+e\cos\alpha),$$
$$A\cos\tfrac{1}{2}(\alpha-\beta)-B\sin\tfrac{1}{2}(\alpha-\beta)=l/(1+e\cos\beta);$$

$\therefore$ $$A(1+e\cos\alpha)(1+e\cos\beta)=l\sec\tfrac{1}{2}(\alpha-\beta)+e\cos\tfrac{1}{2}(\alpha+\beta),$$

and $$B(1+e\cos\alpha)(1+e\cos\beta)=le\sin\tfrac{1}{2}(\alpha+\beta).$$

The equation of the circle then becomes

$$r(1+e\cos\alpha)(1+e\cos\beta)=l\sec\tfrac{1}{2}(\alpha-\beta)\cos\{\theta-\tfrac{1}{2}(\alpha+\beta)\}+le\cos(\theta-\alpha-\beta).$$

§ 3. *To find the equation of the tangent to the conic*

$$l/r=1+e\cos(\theta-\gamma)$$

*at the point whose vectorial angle is* $\alpha$.

This follows from the equation of the chord by putting $\beta=\alpha$: hence the equation of the tangent is

$$l/r=e\cos(\theta-\gamma)+\cos(\theta-\alpha).$$

**Example.** If $TP$, $TQ$ are tangents to a conic, show that $ST$ bisects the angle $PSQ$.

Let the conic be $l/r=1+e\cos\theta$, and let $P$ and $Q$ be the points $\alpha$ and $\beta$. Then the equations of $TP$ and $TQ$ are

$$l/r=e\cos\theta+\cos(\theta-\alpha).$$
$$l/r=e\cos\theta+\cos(\theta-\beta),$$

hence at their common point $T$ we have

$$\cos(\theta-\alpha) = \cos(\theta-\beta);$$

therefore, since $\alpha$ is not equal to $\beta$, we have $\theta = \frac{1}{2}(\alpha+\beta)$, i.e. the angle $ZST = \frac{1}{2}(\alpha+\beta)$, so that

$$\angle PST = \angle TSQ = \tfrac{1}{2}(\alpha\sim\beta).$$

This property is usually referred to thus: 'Tangents to a conic subtend equal angles at the focus.'

§ 4. *To find the equation of the normal to the conic*

$$l/r = 1+e\cos(\theta-\gamma)$$

*at the point whose vectorial angle is* $\alpha$.

A line through the pole parallel to the tangent at the point $\alpha$ is (see Chap. II, § 11)

$$e\cos(\theta-\gamma)+\cos(\theta-\alpha) = 0,$$

i.e. $$\theta = -\tan^{-1}(e\cos\gamma+\cos\alpha)/(e\sin\gamma+\sin\alpha).$$

The normal, being perpendicular to this, is therefore parallel to

$$\theta = \pi/2-\tan^{-1}(e\cos\gamma+\cos\alpha)/(e\sin\gamma+\sin\alpha),$$

i.e. $$\tan\theta = (e\sin\gamma+\sin\alpha)/(e\cos\gamma+\cos\alpha),$$

i.e. $$e\sin(\theta-\gamma)+\sin(\theta-\alpha) = 0.$$

The equation of the normal is therefore of the form

$$k/r = e\sin(\theta-\gamma)+\sin(\theta-\alpha),$$

and, since the point $\alpha$ lies on it, we have

$$k\{1+e\cos(\alpha-\gamma)\} = le\sin(\alpha-\gamma).$$

Hence, substituting for $k$, the equation of the normal is

$$\frac{e\sin(\alpha-\gamma)}{1+e\cos(\alpha-\gamma)}\cdot\frac{l}{r} = e\sin(\theta-\gamma)+\sin(\theta-\alpha).$$

**Example.** *If the normal at* $P$ *to a conic cuts the axis at* $G$, *then* $SG = eSP$.

Let the conic be $l/r = 1+e\cos\theta$, and let $P$ be the point $\alpha$. The equation of the normal at $P$ is

$$\frac{e\sin\alpha}{1+e\cos\alpha}\cdot\frac{l}{r} = e\sin\theta+\sin(\theta-\alpha).$$

The normal cuts the axis on that side of the focus remote from the corresponding vertex, i.e. at the point on the normal where $\theta = \pi$. Thus

$$\frac{e\sin\alpha}{1+e\cos\alpha}\cdot\frac{l}{SG} = \sin\alpha;$$

$\therefore$ $$SG = le/(1+e\cos\alpha) = eSP.$$

We conclude this section with some illustrative examples :

**Example** i. *To find the locus of the foot of the perpendicular from the focus to a tangent to a conic.*

Let the equation of the conic be $l/r = 1 + e\cos\theta$; the equation of the tangent at the point $\alpha$ is $l/r = e\cos\theta + \cos(\theta - \alpha)$, and the equation of the straight line through the pole perpendicular to this is $0 = e\sin\theta + \sin(\theta - \alpha)$.

The locus of the point of intersection of these lines is therefore

$$(l/r - e\cos\theta)^2 + e^2\sin^2\theta = 1,$$

i.e.
$$r^2(1-e^2) + 2ler\cos\theta - l^2 = 0,$$

which is a circle.

For an ellipse $l = a(1-e^2)$, and the equation can be written

$$r^2 + 2aer\cos\theta + a^2e^2 = a^2;$$

the centre of the circle is therefore at the centre of the ellipse, viz. the point $(ae, \pi)$, and the radius of the circle is $a$.

For a parabola $e = 1$, so that the equation reduces to $r\cos\theta = \frac{1}{2}l$, which is the tangent at the vertex.

**Example** ii. *If the tangents at the points P and Q on a conic intersect at T, and the chord PQ meets the directrix at R, then the angle TSR is a right angle.*

Let the conic be $l/r = 1 + e\cos\theta$, and let $P$ and $Q$ be the points $\alpha$ and $\beta$.

The equation of the chord $PQ$ is

$$l/r - e\cos\theta = \sec\tfrac{1}{2}(\alpha-\beta) \cdot \cos\{\theta - \tfrac{1}{2}(\alpha+\beta)\},$$

and the equation of the directrix is

$$l/r - e\cos\theta = 0.$$

At the point $R$, where these lines meet, we have therefore

$$\cos\{\theta - \tfrac{1}{2}(\alpha+\beta)\} = 0,$$

hence
$$\theta = \pm\tfrac{1}{2}\pi + \tfrac{1}{2}(\alpha+\beta),$$

which is the equation of $SR$.

We showed in § 3 that the equation of $ST$ is $\theta = \frac{1}{2}(\alpha+\beta)$; therefore $\angle RST$ is a right angle.

**Example** iii. *Prove that any chord of the conic*

$$l/r = 1 + e\cos\theta,$$

*which is normal at a point where the conic is met by the straight lines*

$$l/r(e + 1/e) = \pm\sin\theta + (e^2-1)\cos\theta,$$

*will subtend a right angle at the pole.*

The vectorial angles of the points of intersection of the given straight lines and the conic are given by

$$(1 + e\cos\theta)(e + 1/e) = \pm\sin\theta + (e^2-1)\cos\theta,$$

i.e. $$e+1/e+(e^2+1)\cos\theta = \pm\sin\theta+(e^2-1)\cos\theta,$$

i.e. $$1+e^2+2e\cos\theta = \pm e\sin\theta;$$

$\therefore$ $$(1+e\cos\theta)^2 = \pm e\sin\theta - e^2\sin^2\theta.$$

Hence, if either of these angles is $\alpha$, we have

$$(1+e\cos\alpha)^2 = \pm e\sin\alpha - e^2\sin^2\alpha. \qquad \text{(i)}$$

The equation of the normal at the point whose vectorial angle is $\alpha$, is

$$\frac{e\sin\alpha}{1+e\cos\alpha}\cdot\frac{l}{r} = e\sin\theta+\sin\overline{\theta-\alpha}.$$

This meets the line $\theta = \frac{1}{2}\pi+\alpha$ at the point whose radius vector is given by

$$\frac{e\sin\alpha}{1+e\cos\alpha}\cdot\frac{l}{r} = e\cos\alpha+1,$$

i.e. $$\frac{l}{r} = \frac{(1+e\cos\alpha)^2}{e\sin\alpha},$$

or, substituting from (i),

$$l/r = \pm 1 - e\sin\alpha,$$

i.e. $$l/r = \pm 1 + e\cos(\tfrac{1}{2}\pi+\alpha).$$

Thus the normal meets the line $\theta = \frac{1}{2}\pi+\alpha$ at the point on the conic whose vectorial angle is $(\frac{1}{2}\pi+\alpha)$; in other words the normal chord subtends a right angle at the pole.

**Example iv.** *Show that the equation of the pair of tangents which can be drawn to the hyperbola* $l/r = 1+e\cos\theta$ *from the point* $(r', \theta')$ *is*

$$\{(l/r-e\cos\theta)^2-1\}\{(l/r'-e\cos\theta')^2-1\}$$
$$= \{(l/r-e\cos\theta)(l/r'-e\cos\theta')-\cos(\theta-\theta')\}^2,$$

*and that the equation of the asymptotes is*

$$l/r = (e-e^{-1})\cos\theta \pm \sin\theta\sqrt{1-e^{-2}}.$$

The tangent at $\alpha$ to the conic

$$l/r = 1+e\cos\theta,$$

viz.

$$l/r = e\cos\theta+\cos\overline{\theta-\alpha}, \qquad \text{(i)}$$

passes through $(r', \theta')$ if

$$l/r' = e\cos\theta'+\cos(\theta'-\alpha). \qquad \text{(ii)}$$

Hence, any point on a tangent from $(r', \theta')$ to the hyperbola satisfies equation (i), if $\alpha$ has a value given by equation (ii).

Thus, if we eliminate $(\alpha)$ from equations (i) and (ii), we get an equation satisfied by the coordinates of any point on these tangents.

The equations can be written

$$\cos\alpha\cos\theta+\sin\alpha\sin\theta-(l/r-e\cos\theta) = 0,$$
$$\cos\alpha\cos\theta'+\sin\alpha\sin\theta'-(l/r'-e\cos\theta') = 0.$$

Hence

$$\frac{\cos\alpha}{(l/r-e\cos\theta)\sin\theta'-(l/r'-e\cos\theta')\sin\theta} = \frac{\sin\alpha}{(l/r'-e\cos\theta')\cos\theta-(l/r-e\cos\theta)\cos\theta'}$$
$$= \frac{1}{\sin(\theta'-\theta)};$$

$$\therefore \quad [(l/r - e\cos\theta)\sin\theta' - (l/r' - e\cos\theta')\sin\theta]^2$$
$$+ [(l/r' - e\cos\theta')\cos\theta - (l/r - e\cos\theta)\cos\theta']^2 = \sin^2(\theta - \theta'),$$

i.e

$$(l/r - e\cos\theta)^2 + (l/r' - e\cos\theta')^2 - 2(l/r - e\cos\theta)(l/r' - e\cos\theta')\cos\overline{\theta - \theta'}$$
$$= \sin^2(\theta - \theta'),$$

which is the same as

$$\{(l/r - e\cos\theta)^2 - 1\}\{(l/r' - e\cos\theta')^2 - 1\}$$
$$= \{(l/r - e\cos\theta)(l/r' - e\cos\theta') - \cos\overline{\theta - \theta'}\}^2.$$

The centre of the conic is the point $\{le/(e^2-1), 0\}$, and the asymptotes being the tangents from the centre to the hyperbola are given by putting $r' = le/(e^2-1)$, $\theta' = 0$ in the above equation.

In this case $l/r' - e\cos\theta' = (e^2-1)/e - e = -1/e$.

Hence the equation becomes

$$\{(l/r - e\cos\theta)^2 - 1\}\{1/e^2 - 1\} = \{1/e\,(l/r - e\cos\theta) + \cos\theta\}^2,$$

i.e. $$(l/r - e\cos\theta)^2 + 2(l/r - e\cos\theta)\cdot\cos\theta/e + 1/e^2 = \sin^2\theta,$$

or $$(l/r - e\cos\theta)^2 + 2(l/r - e\cos\theta)\,e^{-1}\cos\theta + e^{-2}\cos^2\theta = (1 - e^{-2})\sin^2\theta,$$

i.e. $$l/r - e\cos\theta + e^{-1}\cos\theta = \pm\sqrt{1 - e^{-2}}\cdot\sin\theta.$$

**Example v.** *An hyperbola and a parabola have a common focus and touch one another, and the line joining their other common points passes through the focus.*

*Show that* $e = \{5 \pm 4\sqrt{l/c}\}^{\frac{1}{2}}$, *where* $2l$ *is the latus rectum of the parabola and* $2c$ *the length of the common chord.*

Take the axis of the parabola for initial line and let the equation of the conics be

(parabola) $\quad l/r = 1 + \cos\theta,$

(hyperbola) $\quad L/r = 1 + e\cos(\theta - \gamma).$

A pair of common chords are

$$(L - l)/r = e\cos\overline{\theta - \gamma} - \cos\theta, \qquad \text{(i)}$$

and

$$(L + l)/r = e\cos\overline{\theta - \gamma} + \cos\theta. \qquad \text{(ii)}$$

The first will pass through the pole (focus) if $L = l$.

The second then becomes

$$2l/r = e\cos\overline{\theta - \gamma} + \cos\theta,$$

and is by hypothesis a tangent to the parabola and the hyperbola.

For some value of $\alpha$ it is therefore identical with

$$l/r = \cos\theta + \cos(\theta - \alpha).$$

Comparing coefficients,

$$2 = \frac{e\cos\gamma + 1}{1 + \cos\alpha} = \frac{e\sin\gamma}{\sin\alpha};$$

$$\therefore \quad 2\cos\alpha = e\cos\gamma - 1, \quad 2\sin\alpha = e\sin\gamma;$$

$$\therefore \quad 4 = e^2 - 2e\cos\gamma + 1;$$

$$\therefore \quad 2e\cos\gamma = e^2 - 3. \qquad \text{(iii)}$$

Now the first common chord (i) is

$$e\cos(\theta-\gamma)-\cos\theta=0,$$

or
$$\tan\theta=\frac{1-e\cos\gamma}{e\sin\gamma}.$$

Hence, if $\theta$ satisfies this equation, $\theta$ and $\pi+\theta$ are the vectorial angles of the ends of the common chord, say $PSP'$.

Thus
$$l/SP=1+\cos\theta,\quad l/SP'=1-\cos\theta;$$
$$\therefore\quad 2c/l=SP/l+SP'/l=2/\sin^2\theta;$$

$$\therefore\quad \frac{c}{l}=1+\cot^2\theta=1+\frac{e^2\sin^2\gamma}{(1-e\cos\gamma)^2}$$
$$=\frac{1-2e\cos\gamma+e^2}{(1-e\cos\gamma)^2}=\frac{4}{(1-e\cos\gamma)^2}. \qquad \text{(From iii.)}$$

Thus
$$1-e\cos\gamma=\pm 2\sqrt{l/c};$$
$$\therefore\quad 2e\cos\gamma=2\pm 4\sqrt{l/c};$$
$$\therefore\quad e^2-3=2\pm 4\sqrt{l/c};$$
$$\therefore\quad e^2=5\pm 4\sqrt{l/c}.$$

## Examples IX.

1. The semi-latus rectum of a conic is the harmonic mean between the segments of any focal chord.

2. Each of the tangents to a conic from a point on its directrix subtends a right angle at the focus.

3. In any conic the projection of the normal $PG$ on the focal distance $SP$ is equal to the semi-latus rectum.

4. Show that the circumcircle of a triangle formed by three tangents to a parabola passes through the focus.

5. Tangents are drawn to a parabola from any point on its latus rectum: show that the harmonic mean of the focal distances of the points of contact is equal to the semi-latus rectum.

6. Trace the following conics:—

(a) $2/r=1+\frac{1}{2}\cos\theta$; (b) $2/r=1+\cos\theta$; (c) $2/r=1+2\cos\theta$;
(d) $3/r=1+\sqrt{3}\cos(\theta-\frac{1}{3}\pi)/2$.

7. The axes of an ellipse are 8 and 12 inches respectively. Show how to place a focal chord of length 9 inches in the ellipse. How many possible positions are there?

8. If $PSP'$ is any focal chord of a conic section, show that

$$PP'=2l/(1-e^2\cos^2\theta),$$

where $e$ is the eccentricity, $2l$ the latus rectum, and $\theta$ the inclination of the chord to the axis of the conic. A focal chord of a parabola is twice the length of the latus rectum: find the distance of this chord from the parallel tangent.

9. Two conics have a common focus, equal latera recta, and four real points of intersection. Prove that one of them is an hyperbola, and that if the

other is an ellipse the sum of the reciprocals of the distances of the common points from the common focus is $\frac{4}{l}\cdot\frac{e'^2-ee'\cos\gamma}{e'^2+e^2-2ee'\cos\gamma}$, where $2l$ is the length of the latus rectum, $e$ is the eccentricity of the ellipse, $e'$ of the hyperbola, and $\gamma$ is the angle between the axes.

10. Two parabolas have a common focus and axes inclined at an angle $\alpha$. Prove that the locus of the intersection of two perpendicular tangents, one to each of the parabolas, is a conic.

11. From the polar equation of a parabola deduce a quadratic equation for the lengths of the latera recta of the two parabolas, each of which has two focal radii of lengths $r_1$ and $r_2$ making an angle $\alpha$ with one another.

12. Conics with latus rectum of given length are described with a fixed point as focus and touching a given straight line. Prove that the locus of their centres is a conic.

13. Two conics have a common focus $S$ and have their corresponding axes at right angles. If $r_1, r_2, r_3, r_4$ are the distances from $S$ of the points of intersection, $r_1$ being the greatest and $r_4$ the least, show that

$$r_1r_4 = r_2r_3 = \frac{e^2l'^2+e'^2l^2}{(e^2+e'^2)\sim e^2e'^2},$$

where $e, e'$ are the eccentricities, and $l, l'$ the latera recta of the conics.

14. $PQ$ are two points on an ellipse whose vectorial angles referred to one of the foci $(S, H)$ are $(\alpha+\beta)$ and $(\alpha-\beta)$. Prove that if $PH$ and $QS$ meet on the curve then the eccentricity $e$ is given by $\sin\alpha/\sin\beta = \frac{1}{2}(e+1/e)$.

15. Find the equation of the normal at the point $\theta=\alpha$ of the conic $l = r(1+e\cos\theta)$, and prove that the part intercepted by the curve subtends at the origin an angle $2\tan^{-1}(1+e^2+2e\cos\alpha)/e\sin\alpha$.

16. If the ellipses whose latera recta are $l, l'$ and eccentricities $e, e'$ have a common focus and touch one another, show that the cosine of the angle between their axes is $\{(l-l')^2-(e^2l'^2+e'^2l^2)\}/2ee'll'$.

17. From the focus $S$ of an ellipse whose eccentricity is $e$, radii $SP, SQ$ are drawn at right angles to one another, and the tangents at $P$ and $Q$ meet at $T$. Show that the locus of $T$ is an hyperbola, parabola, or ellipse, according as $e$ is $>$, $=$, or $< 1/\sqrt{2}$.

18. If $SP$ is drawn through a focus of an hyperbola parallel to one asymptote and meeting the curve at $P$, prove that the tangent at $P$ meets the other asymptote on the latus rectum produced.

19. Trace the curve $r\cos^3\alpha = a\cos(\theta-3\alpha)$, and show that for all values of $\alpha$ it touches the parabola $r = a\sec^2\frac{1}{2}\theta$.

20. If a normal is drawn at one extremity of the latus rectum, prove that the distance from the focus of the other point in which it meets the curve is $\frac{1+3e^2+e^4}{1+e^2-e^4}l$.

21. The normal at a fixed point $P$ of any conic meets the transverse axis at $G$. Show that points on the curve which are equidistant from $G$ are such that the arithmetic mean of their focal radii is equal to the focal distance of $P$.

22. If $T$ is the pole of a chord $PQ$ of $l = r(1 + e\cos\theta)$ which subtends a constant angle $2\alpha$ at the focus, $1/SP + 1/SQ - 2\cos\alpha/ST$ is constant.

23. If $\alpha, \beta, \gamma$ are vectorial angles of three points on $2a/r = 1 + \cos\theta$, the normals at which are concurrent, $\Sigma \tan\frac{1}{2}\alpha = 0$.

Two points $P$, $Q$ are taken on a parabola whose vectorial angles are supplementary. Show that the locus of the intersection of normals at $P$ and $Q$ is a parabola.

24. A parabola is drawn with its focus at the origin to touch $l/r = 1 + e\cos\theta$ at any point. Show that the equation of the locus of its vertex is $r = a(1 - e\cos\theta)$, where $2a$ is the major axis of the given conic.

25. If $\theta_1$, $\theta_2$ are extremities of conjugate diameters, prove

(i) $(e + \cos\theta_1)(e + \cos\theta_2) + (1 - e^2)\sin\theta_1 \sin\theta_2 = 0$;

and (ii) $\pm\sqrt{1-e^2} \cdot \sin\frac{1}{2}(\theta_1 - \theta_2) = \cos\frac{1}{2}(\theta_1 - \theta_2) + e\cos\frac{1}{2}(\theta_1 + \theta_2)$.

26. Show that the two conics $l = r(1 + e\cos\theta)$, $l = r(1 + e'\cos\theta)$ have two real common tangents if $e \sim e' < 2$.

27. A chord of a circle through a fixed point $O$ on its circumference meets the circle in $P$, and a fixed straight line in $Q$ and $(OR, PQ)$ is harmonic. Prove that the locus of $R$ is a conic.

28. A right-angled triangle has its right angle at a focus of a conic section while the hypotenuse envelopes the curve and one other vertex moves on a given line: prove the remaining vertex describes a conic.

29. A chord $PQ$ of a conic subtends a right angle at a focus $S$. Show that the locus of the intersection of the circles on $SP$ and $SQ$ as diameter is a circle.

30. The tangent parallel to the tangent at $\theta = \alpha$ is

$$l(e^2 + 2e\cos\alpha + 1) = r(e^2 - 1)\{\cos\overline{\theta - \alpha} + e\cos\theta\}.$$

31. In the parabola $r = a\sec^2\frac{1}{2}\theta$, $p$, $q$, $r$ are the perpendiculars from the focus on any three tangents, and $R$ is the radius of the circle circumscribing the triangle formed by the tangents. Show that $pqr = 2Ra^2$.

32. The conic $l/r = 1 + e\cos\theta$ is cut by a circle which passes through the pole in the points $(r_1, \theta_1)$, $(r_2, \theta_2)$, $(r_3, \theta_3)$, $(r_4, \theta_4)$: prove that

(i) $\Sigma\, 1/r = 2/l$; (ii) $(1+e)\Sigma\tan\frac{1}{2}\theta = (1-e)\Sigma\tan\frac{1}{2}\theta_2\tan\frac{1}{2}\theta_3\tan\frac{1}{2}\theta_4$.

33. Three points are taken on the parabola $a = r\sin^2\frac{1}{2}\theta$ whose vectorial angles are $\alpha, \beta, \gamma$. Prove that the radius of the circle circumscribed to the triangle of the tangents at the points is $\frac{1}{2}a\operatorname{cosec}\frac{1}{2}\alpha\operatorname{cosec}\frac{1}{2}\beta\operatorname{cosec}\frac{1}{2}\gamma$.

34. A circle is drawn through the focus of the parabola $2l = r(1 + \cos\theta)$ to touch it at the point where $\theta = \alpha$. Obtain its equation in the form $r\cos^3\frac{1}{2}\alpha = l\cos(\theta - \frac{3}{2}\alpha)$.

Show that no circle of curvature at any point on a parabola can pass through the focus.

35. Show that the equation of the polar of the point $(r_1, \theta_1)$ with respect to the conic $lr^{-1} = 1 + e\cos\theta$ is $(lr^{-1} - e\cos\theta)(lr_1^{-1} - e\cos\theta_1) = \cos(\theta - \theta_1)$. If pairs of points collinear with the pole are conjugates for each of the conics $lr^{-1} = 1 + e\cos\theta$, $lr^{-1} = 1 + e\sin\theta$, prove that they all lie on the conic $(l/r - e\cos\theta)(l/r - e\sin\theta) + 1 = 0$, or on the line $\theta = \frac{1}{4}\pi$.

36. Normals to the ellipse $l/r = 1 + e\cos\theta$ at three points whose vectorial angles are $\alpha$, $\beta$, $\gamma$ meet at a point. Prove that $\dfrac{\Sigma \tan\frac{1}{2}\beta \tan\frac{1}{2}\gamma}{\Sigma \cot\frac{1}{2}\beta \cot\frac{1}{2}\gamma} = \left(\dfrac{1+e}{1-e}\right)^2$.

$A$, $B$, $C$, $D$ are the feet of four concurrent normals to an ellipse, focus $S$. On $SA$, $SB$, $SC$, $SD$ as diameters circles are described. If two of these circles intersect on a fixed circle $r = c\cos\theta$, prove that the other two will intersect on another fixed circle $r = c'\cos\theta$, and determine the relation between $c$ and $c'$.

37. Tangents of lengths $p$ and $q$ are drawn to the parabola $2a/r = 1 + \cos\theta$ from a point on the latus rectum produced. Prove that $p$, $q$ satisfy the relation $p^4q^4 = a^2(p^2+q^2)(p^2-q^2)^2$, and that the angle subtended by the chord of contact at the focus is $2\theta$, where $2pq\tan\theta = p^2 - q^2$.

38. Prove that two equal conics which have a common focus and whose axes are inclined at an angle $2\alpha$ intersect at an angle

$$\tan^{-1}\left[\frac{e^2\sin 2\alpha + 2e\sin\alpha}{e^2\cos 2\alpha + 2e\cos\alpha + 1}\right].$$

39. $PSQ$ is a focal chord of a conic. Find the locus of the intersection of the tangent at $Q$ with the perpendicular from $S$ on the tangent at $P$.

40. Chords of the conic $l/r = 1 - e\cos\theta$ are drawn through the origin and on these chords as diameters circles are described. Prove that their envelope consists of the two circles $l/r(l/r + e\cos\theta) = 1 \pm e$.

41. Prove, from the general polar equation of a conic

$$r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) + 2r(g\cos\theta + f\sin\theta) + c = 0,$$

that the locus of the extremity of a radius from a fixed point equal to the harmonic mean of the two radii of the conic in the same direction is a straight line which, when the point is external, passes through the points of contact of tangents from the point.

Prove also that, if the mean is arithmetic or geometric, the loci are conics, and that the centre of the conic in the case of the arithmetic mean is midway between the centres of the original conic and of the geometric mean conic.

42. Find the equation to the chord joining the points $\theta = \alpha - \beta$ and $\theta = \alpha + \beta$ of the hyperbola $l/r = 1 + e\cos\theta$, and hence show that the equations to its asymptotes are $e/r = \cos\theta/a \pm \sin\theta/b$.

A line drawn through a focus $S$ perpendicular to an asymptote cuts the hyperbola again in $P$, $Q$. Prove that $SP.SQ = b^2l^2/(b^2 - l^2)$.

43. At any point of the conic $r^2\cos 2\theta = a^2$ is drawn the parabola having 4 point contact: prove that the locus of the foot of the perpendicular from the centre of the hyperbola to the tangent at the vertex of the parabola is $r^2\cos 2\theta = a^2(\cos^4\theta + \sin^4\theta)^2$.

44. $P$ is the pole of a chord which subtends a constant angle at the focus $S$ of a conic, and $SP$ is cut internally by the conic in $Q$. Find the locus of the harmonic conjugate of $S$ with respect to $P$ and $Q$.

45. The parabola $l = r(1 - \cos\theta)$ is turned about the origin through an angle $\cos^{-1}\frac{3}{4}$. If the new parabola cuts the initial line at $P$ and $Q$, show

that the intercept made on the old parabola by the tangent at one of the points $P$, $Q$ to the new parabola subtends a right angle at the origin.

46. Two parallel straight lines and a circle, centre $F$, being given, a straight line $PAB$ is drawn from any point of the circle to meet the lines in $A$ and $B$ respectively, and through $A$ a straight line is drawn parallel to $BF$ to meet $FP$ in $Q$: show that the locus of $Q$ is a conic, and determine whether it is a parabola, ellipse, or hyperbola.

47. Prove that the equation $r(1+e\cos\alpha)^2 = l\cos(\theta-\alpha)+le\cos(\theta-2\alpha)$ represents a circle which passes through the origin and touches the conic $l/r = 1+e\cos\theta$.

48. Find the equation of the polar reciprocal of the conic $l/r = 1+e\cos\theta$, with respect to a circle, whose centre is at the focus and diameter equal to the latus rectum of the conic.

49. Find the equation of the polar of the point $(r', \theta')$ with respect to the circle $r = c$, and obtain the envelope of such polars when $(r', \theta')$ lies on the conic $c/r = 1+e\cos\theta$.

# CHAPTER X

## LINE COORDINATES AND TANGENTIAL EQUATIONS

§ 1. In the previous chapters the position of a point has been indicated by the coordinates $x$ and $y$, and the locus of a point, under various circumstances, has been represented by an equation involving the variables $x$ and $y$.

The position of a line can also be indicated by two coordinates; for example, these coordinates might be the lengths of the intercepts made by the straight line on the coordinate axes.

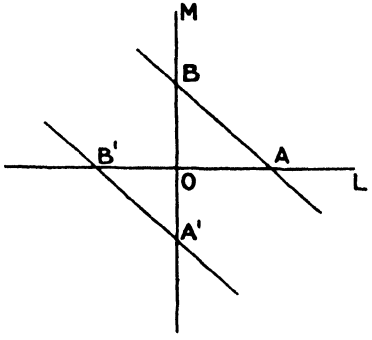
Such coordinates are called **line coordinates.** It is possible to develop a system of line coordinates quite independently; we shall illustrate this. Line coordinates, however, are most valuable when used in conjunction with point coordinates; we have therefore adopted that system of line coordinates which works most easily with our previous work on point coordinates.

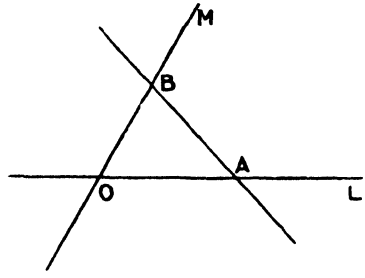
§ 2. Any straight line, including the line at infinity, can be represented by an equation of the form $lx+my+nz=0$. If we exclude the line at infinity, we can use the simpler form of this equation, $lx+my+n=0$, as the general equation of a straight line; this equation involves three coefficients, $l$, $m$, and $n$. If, further, we exclude straight lines which pass through the origin, so that $n$ is not zero, we can take the general equation of a straight line in the form $lx+my+1=0$; this equation involves only two coefficients, $l$ and $m$. If the mutual ratios $l:m:n$ are known, the position of the line $lx+my+nz=0$ is known, and similarly, if the values of $l$ and $m$ are known, the position of the line $lx+my+1=0$ is known. Thus $(l, m, n)$ may be regarded as the coordinates of the line $lx+my+nz=0$; and $(l, m)$ as the coordinates of the line $lx+my+1=0$.

**Note.** When using point coordinates we found two coordinates $(x, y)$ sufficient except when we had to deal with points at infinity, in which case we used the homogeneous coordinates $(x, y, z)$.

When using line coordinates we shall find two coordinates $(l, m)$ sufficient except when we have to deal with lines passing through the origin, in which case we shall use the homogeneous coordinates $(l, m, n)$.

Let $OL$, $OM$ be the coordinate axes, rectangular or oblique, and let $AB$ be the straight line $(l, m)$. The Cartesian equation of the line $AB$ is $lx+my+1=0$, hence $OA=-1/l$, $OB=-1/m$. Thus the straight line




$(l, m)$ makes intercepts $-1/l$, $-1/m$ on the coordinate axes. So also the line $(l, m, n)$ makes intercepts $-n/l$, $-n/m$ on the axes.

The usual convention of signs can be followed; thus in the figure the straight line $A'B'$ is the line $(-l, -m)$.

**Note i.** The Cartesian equation of any straight line through the origin is $lx+my=0$; remembering that $(l, m, n)$ is the straight line $lx+my+n=0$, we see that the homogeneous coordinates of this straight line are $(l, m, 0)$.

**Note ii.** The coordinates of the axes $OL$ and $OM$ are $(0, 1, 0)$ and $(1, 0, 0)$.

**Note iii.** The coordinates of a straight line parallel to the axis $OL$ (e.g. $y+a=0$) are $(0, 1/a)$.

**Note iv.** The equation of the line at infinity is $z=0$; if $l$ and $m$ are zero the equation $lx+my+nz=0$ reduces to $z=0$. Thus the coordinates of the 'line at infinity' are (0, 0), or in homogeneous coordinates (0, 0, 1).

## Examples X a.

1. Show that the straight lines $(\pm a, \pm b)$ form a parallelogram.

2. Prove that the coordinates $(\lambda a, \lambda b)$, represent, for different values of $\lambda$, a system of parallel straight lines.

3. The straight lines $(a, b)$, $(b, a)$ cut the axes of coordinates in four concyclic points.

4. Find the condition that the straight lines $(l_1, m_1)$, $(l_2, m_2)$ should be at right angles.

5. What angle does the straight line $(l_1, m_1)$ make with the axis of $x$?

6. If the axes are rectangular, find the angle between the straight lines $(l_1, m_1)$, $(l_2, m_2)$.

7. What does $(h, k, 0)$ represent (a) in point coordinates, (b) in line coordinates?

8. The perpendicular from the origin on a straight line is of length $p$, and makes angles $\alpha$ and $\beta$ with the coordinate axes: what are the coordinates of the straight line?

9. If the axes are rectangular the straight lines $(l, m, 0)$, $(m, -l, 0)$ are at right angles.

10. If the axes are oblique $(\omega)$, find the condition that the lines $(l_1, m_1, 0)$, $(l_2, m_2, 0)$ should be harmonic conjugates with respect to the straight lines $(0, 1, 0)$, $(1, 0, 0)$.

§ 3. Any equation, $\phi(l, m) = 0$, involving the coordinates $l$, $m$ is satisfied by the coordinates of any one of a group of straight lines in the same way that an equation $f(xy) = 0$ is satisfied by the coordinates of any one of a group of points. If any value be given to $l$ in such an equation, corresponding values of $m$ can be found, giving the coordinates of lines of the group. Any two lines of the group whose coordinates are $(l, m)$ and $(l+h, m+k)$ intersect in a point; as $h$, and therefore $k$, are indefinitely diminished, this point takes a definite limiting position, which is usually referred to as the point of intersection of two consecutive lines of the group defined by $\phi(l, m) = 0$. The series of points so determined trace out a curve, and the equation of this curve in line coordinates is $\phi(l, m) = 0$. Every straight line $(l, m)$, whose coordinates satisfy this equation, touches this curve; for such a straight line meets the curve in two coincident points, viz. the limiting positions of its intersections with the lines $(l+h, m+k)$, $(l-h', m-k')$ when $h$ and $h'$ and therefore $k$ and $k'$ are indefinitely diminished. The curve is therefore the envelope of the line $(l, m)$, when its coordinates are subject to the condition $\phi(l, m) = 0$

Thus the equation $f(x, y) = 0$ represents the **locus** of a point $(x, y)$ whose coordinates satisfy this equation, and the straight line joining two points on the curve is, in the limiting position when these two points become coincident, **a tangent to the curve.**

The equation $\phi(l, m) = 0$ represents the **envelope** of a line $(l, m)$ whose coordinates satisfy this equation, and the point of intersection of two tangents to the curve, in the limiting position when these two tangents become coincident, is a **point on the curve.**

If $\phi(l, m) = 0$ represents the same curve as $f(x, y) = 0$, it is called the **tangential equation** of this curve; evidently $\phi(l, m) = 0$ is the condition that the straight line $(l, m)$ or $lx+my+1 = 0$ should touch the curve $f(x, y) = 0$.

Cartesian coordinates are useful for the investigation of loci and their properties; line coordinates are useful for the investigation of envelopes. We have learned how to interpret the equations of loci; the object of this chapter is to interpret and discover the properties of tangential equations.

**§ 4. The equation of the first degree $Al+Bm+C=0$ represents a point.** The Cartesian equation of the line $(l, m)$ is $lx+my+1=0$, and $Al+Bm+C=0$ is the condition that this straight line should pass through the point $(A/C, B/C)$. Hence, any straight line whose coordinates satisfy the equation $Al+Bm+C=0$ passes through the point $(A/C, B/C)$. This equation is then the tangential equation of this point.

We have now the following reciprocal property of the equation $lx+my+1=0$:—

If $l$ and $m$ are constants, it is the Cartesian equation of the straight line whose line coordinates are $(l, m)$.

If $x$ and $y$ are constants, it is the Tangential equation of the point whose point coordinates are $(x, y)$.

**Note i.** So also in homogeneous coordinates; $lx+my+nz=0$ is the point equation of the straight line $(l, m, n)$ and the line equation of the point $(x, y, z)$.

**Note ii.** The origin is the point $(0, 0, 1)$; therefore the tangential equation of the origin is $n=0$.

It should be carefully noted that when we are dealing with the origin or lines through the origin in line coordinates we must use homogeneous coordinates.

**Note iii.** The homogeneous coordinates of a point at infinity are of the form $(x, y, 0)$; therefore the tangential equation of a point at infinity is of the form $lx+my=0$.

**§ 5.** (i) *To find the equation of the point of intersection of the straight lines* $(l_1, m_1)$, $(l_2, m_2)$.

If the point is

$$al+bm+1=0: \tag{i}$$

then, since it lies on the given lines, we have

$$al_1+bm_1+1=0, \tag{ii}$$

and

$$al_2+bm_2+1=0: \tag{iii}$$

these equations give the values of $a$ and $b$; or, in the determinant notation, eliminating the unknown constants $a$, $b$ from (i), (ii), and (iii), we find that the equation of the point is

$$\begin{vmatrix} l & m & 1 \\ l_1 & m_1 & 1 \\ l_2 & m_2 & 1 \end{vmatrix} = 0.$$

**Cor. i.** If the straight lines are parallel, their coordinates are of the form $(l_1, m_1)$, $(kl_1, km_1)$, and the equation of their point of intersection is $m_1l-l_1m=0$, i.e. a point at infinity.

**Cor. ii.** The lines $(l_1, m_1)$, $(l_2, m_2)$, $(l_3, m_3)$ are concurrent if

$$\begin{vmatrix} l_1 & m_1 & 1 \\ l_2 & m_2 & 1 \\ l_3 & m_3 & 1 \end{vmatrix} = 0.$$

**Cor. iii.** The coordinates of a straight line through the intersection of $(l_1, m_1)$, $(l_2, m_2)$ are of the form

$$\left\{\frac{pl_1+ql_2}{p+q}, \quad \frac{pm_1+qm_2}{p+q}\right\}.$$

(ii) *To find the coordinates of the straight line joining the points*

$$al+bm+1=0, \quad a'l+b'm+1=0.$$

These are evidently found by regarding the equations of the points as simultaneous equations giving $l$ and $m$.

(iii) *If $u=0$, $v=0$ are the equations of two points, to interpret the equation $u+\lambda v=0$.*

The coordinates of the line which joins the points $u=0$, $v=0$ satisfy each of these equations, and consequently satisfy $u+\lambda v=0$. Hence this equation for different values of $\lambda$ represents points on the straight line joining the points $u=0$, $v=0$.

(iv) *To find the equation of the point which divides the join of the points $al+bm+1=0$, $a'l+b'm+1=0$ in the ratio $p:q$.*

The Cartesian coordinates of the points are $(a, b)$, $(a', b')$, and those of the required points are

$$\left\{\frac{qa+pa'}{p+q}, \quad \frac{qb+pb'}{p+q}\right\};$$

hence the equation of the point in line coordinates is

$$\frac{qa+pa'}{q+p}l+\frac{qb+pb'}{q+p}m+1=0,$$

i.e.
$$q(al+bm+1)+p(a'l+b'm+1)=0.$$

**Cor. i.** The equation of the point midway between the points $al+bm+1=0$, $a'l+b'm+1=0$ is $(a+a')l+(b+b')m+2=0$.

**Cor. ii.** If $u=0$, $v=0$ are the equations of two points in the form $al+bm+1=0$, then the points $pu+qv=0$, $pu-qv=0$ are harmonic conjugates with respect to $u=0$, $v=0$.

**Example.** *The sides of a triangle $ABC$ are divided at $P$, $Q$, $R$ so that $BP=p.PC$; $CQ=q.QA$; $AR=r.RB$; find the condition that $P$, $Q$, $R$ should be collinear.* (Menelaus' Theorem.)

Let the vertices of the triangle be

$$u \equiv al + bm + 1 = 0,$$
$$v \equiv a'l + b'm + 1 = 0,$$
$$w \equiv a''l + b''m + 1 = 0,$$

then the equations of the points $P$, $Q$, $R$ are

$$v + pw = 0, \quad w + qu = 0, \quad u + rv = 0,$$

and these are simultaneously satisfied by the coordinates of a straight line $(l_1, m_1)$ if $\quad v_1 + pw_1 = 0, \quad w_1 + qu_1 = 0, \quad u_1 + rv_1 = 0,$

i.e. if $\quad \frac{v_1}{w_1} = -p, \quad \frac{w_1}{u_1} = -q, \quad \frac{u_1}{v_1} = -r,$

or $\quad pqr = -1,$

which is the required condition.

The reader should compare the solutions in point and line coordinates of the following elementary problems:—

*To find the locus of a point whose distance from the origin is constant and equal to c.*

(Point coordinates.)

Let any such point be $(x, y)$, then

$$x^2 + y^2 = c^2,$$

which is the equation of the locus.

(Line coordinates.)

Let $xl + ym + 1 = 0$ be any such point, then $x^2 + y^2 = c^2$.

Now on any line $(l', m')$ two points of the locus will lie, whose $x$ and $y$ are given by

$$xl' + ym' + 1 = 0,$$

and $\quad x^2 + y^2 = c^2,$

and the ratio $x/y$ is given by

$$x^2 + y^2 = c^2 (xl' + ym')^2.$$

These two points are coincident, and the line $(l', m')$ touches the locus if

$$(c^2 l'^2 - 1)(c^2 m'^2 - 1) = c^4 l'^2 m'^2,$$

or $\quad c^2 (l'^2 + m'^2) = 1,$

i.e. the equation of the locus is

$$c^2 (l^2 + m^2) = 1.$$

*To find the envelope of a line whose distance from the origin is constant and equal to c.*

(Line coordinates.)

Let any such line be $(l, m)$, then

$$\frac{1}{l^2 + m^2} = c^2,$$

or $\quad c^2 (l^2 + m^2) = 1,$

which is the equation of the envelope.

(Point coordinates.)

Let $lx + my + 1 = 0$ be any such line, then $c^2 (l^2 + m^2) = 1$.

Now through any point $(x', y')$ two tangents to the envelope will pass, whose $l$ and $m$ are given by

$$lx' + my' + 1 = 0,$$

and $\quad c^2 (l^2 + m^2) = 1,$

and the ratio $l/m$ is given by

$$c^2 (l^2 + m^2) = (lx' + my')^2.$$

These two tangents are coincident, and the point $(x', y')$ lies on the envelope if

$$(c^2 - x'^2)(c^2 - y'^2) = x'^2 y'^2,$$

or $\quad x'^2 + y'^2 = c^2,$

i.e. the equation of the envelope is

$$x^2 + y^2 = c^2.$$

## Examples X b.

(The axes are rectangular except in questions marked with an asterisk.)

*1. Show that the distance between the points

$$al+bm+1=0,\ a'l+b'm+1=0$$

is $$\sqrt{\{(a-a')^2+(b-b')^2+2(a-a')(b-b')\cos\omega\}}.$$

2. Find the perpendicular distance of the point $al+bm+1=0$ from the line $(l_1, m_1)$.

*3. Find the angle between the lines $(l_1, m_1)$, $(l_2, m_2)$.

*4. Prove that the straight lines $(l_1, m_1)$, $(l_2, m_2)$ are perpendicular if $l_1l_2+m_1m_2+(l_1m_2+l_2m_1)\cos\omega=0$.

*5. The points $al+bm+1=0$, $a'l+b'm+1=0$ subtend a right angle at the origin if $aa'+bb'+(ab'+a'b)\cos\omega=0$.

6. Find the equation of the mid-point of the line joining the points $lx_1+my_1+nz_1=0$. $lx_2+my_2+nz_2=0$.

7. Plot the following points: $al+1=0$, $bm-1=0$, $l+m=2$, $l-m=2$, $3l\pm4m-1=0$.

*8. Find the general equations of two points equidistant from the axis of $l$.

*9. Show that for all values of $\lambda$ the point $al+bm+1+\lambda(a'l+b'm+1)=0$ lies on a fixed line, and find its coordinates.

*10. Show that for different values of $\lambda$ the equation $al+bm=\lambda$ represents points lying on a straight line through the origin.

*11. Find the coordinates of the line joining the points $3l+4m=12$, $5l-6m=2$.

12. What is the area of the triangle whose vertices are $a_1l+b_1m+c_1=0$, $a_2l+b_2m+c_2=0$, $a_3l+b_3m+c_3=0$.

*13. Find the coordinates of a line parallel to $(l_1, m_1)$ and passing through the point $al+bm+1=0$.

14. Find the coordinates of a line perpendicular to $(l_1, m_1)$ and passing through $al+bm+1=0$.

15. Find the condition that the line joining the points $al+bm+1=0$, $a'l+b'm+1=0$ should be perpendicular to $(l_1, m_1)$.

16. Find the condition that the point $lx+my+1=0$ should lie on a circle whose diameter is the join of the points

$$al+bm+1=0,\ a'l+b'm+1=0.$$

17. The centre of a circle is the point $al+bm+1=0$, and $Al+Bm+1=0$ is a point on the circle; find the equation of the other end of the diameter through this point.

*18. Find the condition that the points $a_1l+b_1m+1=0$, $a_2l+b_2m+1=0$, $a_3l+b_3m+1=0$ should be collinear.

19. A straight line $(l, m)$ cuts the axes at $A$ and $B$ so that $1/OA^2+1/OB^2$ is constant. Find the equation of its envelope.

20. Find the coordinates of the straight lines bisecting the angles between the straight lines (3, 4) and (5, 12).

*21. Find the coordinates of the straight line joining the intersection of $(l_1, m_1)$, $(l_2, m_2)$ to the point $al+bm+1=0$.

22. What are the line equations of the curves enveloped by the lines (a) $(a\cos\theta, a\sin\theta)$, (b) $(a\cos\theta, b\sin\theta)$, (c) $(a\theta^2, a\theta)$, where $\theta$ is variable?

*23. A straight line cuts the axes at the points $P$, $Q$; find the equation of its envelope in the following cases:—

(i) $1/OP+1/OQ=c$; (ii) $1/OP-1/OQ=c$; (iii) $a/OP\pm b/OQ=c$;
(iv) $OP+OQ=c$; (v) $OP-OQ=c$.

Show that the line at infinity touches the last two curves.

*24. Find the equation of the centroid of the triangle whose vertices are $a_1l+b_1m+1=0$, $a_2l+b_2m+1=0$, $a_3l+b_3m+1=0$.

25. Find the equation of the envelope of a line which moves so that the foot of the perpendicular on it from $al+1=0$ lies on the axis of $m$.

26. Find the envelope of a line whose distance from the point $(a_1, b_1)$ is double its distance from the point $(a_2, b_2)$.

§ 6. The general equation of the second degree in line coordinates is $u\equiv al^2+2hlm+bm^2+2gl+2fm+c=0$. If $u$ has factors, i.e. if $\Delta=0$, the equation $u=0$ represents a pair of points. The coordinates of the line joining them are $(G/C, F/C)$. (Cf. p. 109).

The equation $al^2+2hlm+bm^2=0$ represents two points whose equations are of the form $pl+qm=0$, i.e. two points at infinity.

**Definition.** Two points at infinity which subtend a right angle at the origin are called orthogonal points.

*To find the condition that the points at infinity whose equation is* $al^2+2hlm+bm^2=0$ *should be orthogonal.*

Let $al^2+2hlm+bm^2\equiv a(l+km)(l+k'm)$, then, since the tangential equations of the points are $l+km=0$, $l+k'm=0$, their point coordinates are $(1, k, 0)$, $(1, k', 0)$. These points lie on the straight lines through the origin whose point equations are $kx-y=0$, $k'x-y=0$, or, in one equation,

$$(kx-y)(k'x-y)=0,$$

i.e. $$kk'x^2-(k+k')xy+y^2=0,$$

i.e. $$bx^2-2hxy+ay^2=0.$$

Thus the Cartesian equation of the lines joining the points at infinity, $al^2+2hlm+bm^2=0$, to the origin is $bx^2-2hxy+ay^2=0$. These are perpendicular if $a+b+2h\cos\omega=0$.

**Note i.** The lines $ax^2+2hxy+by^2=0$ are orthogonal if $a+b-2h\cos\omega=0$.
The points $al^2+2hlm+bm^2=0$ are orthogonal if $a+b+2h\cos\omega=0$.

**Note ii.** The Cartesian equation of the lines joining the circular points at infinity to the origin (called the circular lines through the origin) is

$$x^2+y^2+2xy\cos\omega=0.$$

This equation can be written

$$(x+e^{i\omega}y)(x+e^{-i\omega}y)=0.$$

Hence the line coordinates of these circular lines are $(1, e^{i\omega}, 0)$, $(1, e^{-i\omega}, 0)$, and these lines therefore pass through the points at infinity whose equations are $e^{i\omega}l-m=0$, $e^{-i\omega}l-m=0$; or, in one equation,

$$(e^{i\omega}l-m)(e^{-i\omega}l-m)=0,$$

i.e.
$$l^2-2lm\cos\omega+m^2=0;$$

this is therefore the tangential equation of the circular points at infinity.

If the axes are rectangular, the circular lines are $(1, i, 0)$, $(1, -i, 0)$, and the circular points $l^2+m^2=0$.

**Note iii.** The Cartesian equations of the lines joining the pairs of points at infinity, whose equations are $al^2+2hlm+bm^2=0$, $l^2-2lm\cos\omega+m^2=0$, to the origin are $bx^2-2hxy+ay^2=0$, $x^2+2xy\cos\omega+y^2=0$. These lines are harmonic conjugates if $a+b+2h\cos\omega=0$; hence, orthogonal points at infinity are harmonic conjugates with respect to the circular points at infinity; and, orthogonal lines through the origin are harmonic conjugates with respect to the circular lines through the origin.

**Example i.** *If* $al^2+2hlm+bm^2+2gl+2fm+c=0$ *represents two points, find the equation of the point midway between them.*

Let $al^2+2blm+bm^2+2gl+2fm+c\equiv c(pl+qm+1)(p'l+q'm+1)$, then the equation of the point midway between the points

$$pl+qm+1=0,\quad p'l+q'm+1=0$$

is
$$(p+p')l+(q+q')m+2=0,$$

or
$$gl+fm+c=0.$$

**Example ii.** *To find the angle between the straight lines joining the points P, Q, whose equation is*

$$u\equiv al^2+2hlm+bm^2+2gl+2fm+c=0,$$

*where* $\Delta=0$, *to the point R whose equation is* $pl+qm+1=0$.

**Method 1.** The equation

$$al^2+2hlm+bm^2-2(gl+fm)(pl+qm)+c(pl+qm)^2=0 \qquad \text{(i)}$$

is satisfied by the coordinates of lines which pass through one of the points $P$, $Q$ and also the point $R$.

It can be written

$$\{a-2gp+cp^2\}l^2+2\{h-gq-fp+cpq\}lm+\{b-2fq+cq^2\}m^2=0,$$

or
$$ul^2+2vlm+wm^2=0. \qquad \text{(ii)}$$

This equation therefore gives the ratios of the coordinates $l/m$ of the lines $PR$ and $QR$. If the roots of equation (ii) be $l_1/m_1$ and $l_2/m_2$, since the Cartesian equations of the lines are $l_1x+m_1y+1=0$, $l_2x+m_2y+1=0$,

we have
$$\tan PRQ=\frac{l_1/m_1-l_2/m_2}{1+l_1l_2/m_1m_2}=\frac{2\sqrt{v^2-uw}}{u+w}.$$

**Method 2.** The equation $ul^2+2vlm+wm^2=0$ represents two points at infinity, viz. the intersections of the lines $PR$, $QR$ with the line at infinity. Hence the joins of the origin to these points are parallel to $PR$, $QR$; the Cartesian equation of these straight lines is $wx^2-2vxy+uy^2=0$, hence

$$\tan PRQ=\frac{2\sqrt{v^2-wu}}{w+u}.$$

§ 7. **The circle.** Let the centre of the circle be the point $(x, y)$, whose equation is $lx+my+1=0$, and let $(l, m)$ be any tangent to the circle. The perpendicular from the centre on this tangent is $\frac{lx+my+1}{\sqrt{l^2+m^2}}$, and this is equal to the radius $r$.

The tangential equation of the circle is therefore

$$r^2(l^2+m^2)=(lx+my+1)^2.$$

Compare this equation with the general equation

$$al^2+2hlm+bm^2+2gl+2fm+c=0,$$

and we have

$$\frac{a}{x^2-r^2}=\frac{b}{y^2-r^2}=\frac{h}{xy}=\frac{g}{x}=\frac{f}{y}=\frac{c}{1},$$

hence $fg=c^2xy$, $h=cxy$; $\therefore\ H\equiv fy-ch=0$;

also $bc-f^2=-c^2r^2$, and $ac-g^2=-c^2r^2$;

$$\therefore\ A=B.$$

Hence the conditions that the general equation of the second degree should represent a circle are $A=B$ and $H=0$.

I. When the centre of the circle is at the origin, the equation becomes $r^2(l^2+m^2)=1$; this is the tangential equation of the circle $x^2+y^2=r^2$.

The coordinates of any tangent to the circle can be put in the form $\left(\frac{1}{r}\cos\theta, \frac{1}{r}\sin\theta\right)$; the Cartesian equation of this tangent is $x\cos\theta+y\sin\theta+r=0$, so that the coordinates of its point of contact are $(-r\cos\theta, -r\sin\theta)$; the tangential equation of the point of contact is then $rl\cos\theta+rm\sin\theta=1$.

Again, the polar of the point $(x_1, y_1)$ with respect to the circle $x^2+y^2=r^2$ is $xx_1+yy_1=r^2$; hence the polar of the point

$$lx_1+my_1+1=0$$

with respect to the circle $r^2(l^2+m^2)=1$ is the line $\left(-\frac{x_1}{r^2}, -\frac{y_1}{r^2}\right)$.

Similarly, the pole of the line $(l_1, m_1)$ is the point $r^2l_1l+r^2m_1m=1$.

These results can be found independently as illustrated in the following example:—

*To find the equation of the point of intersection of the tangents whose coordinates are* $\left(\frac{1}{r}\cos\theta, \frac{1}{r}\sin\theta\right)$, $\left(\frac{1}{r}\cos\phi, \frac{1}{r}\sin\phi\right)$.

If the equation of the point of intersection is $pl+qm+1=0$, we have, since the given tangents pass through it,

$$p\cos\theta + q\sin\theta + r = 0,$$
$$p\cos\phi + q\sin\phi + r = 0,$$

and by cross-multiplication, dividing the result by $\sin\frac{1}{2}(\theta-\phi)$, we have

$$\frac{p}{\cos\frac{1}{2}(\theta+\phi)} = \frac{q}{\sin\frac{1}{2}(\theta+\phi)} = -\frac{r}{\cos\frac{1}{2}(\theta-\phi)};$$

i.e. the equation of the point is

$$rl\cos\tfrac{1}{2}(\theta+\phi) + rm\sin\tfrac{1}{2}(\theta+\phi) = \cos\tfrac{1}{2}(\theta-\phi).$$

Hence, putting $\theta=\phi$, the equation of the point of contact of the tangent $\left(\frac{1}{r}\cos\theta, \frac{1}{r}\sin\theta\right)$ is $rl\cos\theta + rm\sin\theta = 1$; this is therefore the general equation of a point on the circle.

The equation of the circle in homogeneous coordinates is

$$r^2(l^2+m^2) = n^2;$$

this is satisfied by the coordinates of the circular lines through the origin, viz. $(1, i, 0)$, $(1, -i, 0)$, and therefore the circle touches the circular lines through the centre. The equations of the points of contact are $r(l \pm im) = 0$ or $l \pm im = 0$, i.e. the circular points at infinity. (*Vide* Chap. V, § 16).

II. The equations of any two circles are

$$(pl+qm+1)^2 = r^2(l^2+m^2), \quad \text{(i)}$$
$$(p_1l+q_1m+1)^2 = r_1^2(l^2+m^2), \quad \text{(ii)}$$

and the coordinates of any line which satisfy both the equations (i) and (ii) will also satisfy the equation

$$r_1^2(pl+qm+1)^2 = r^2(p_1l+q_1m+1)^2,$$

which is equivalent to the two equations

$$r_1(pl+qm+1) + r(p_1l+q_1m+1) = 0,$$
$$r_1(pl+qm+1) - r(p_1l+q_1m+1) = 0.$$

These equations represent two points dividing the line joining the centres of the circles in the ratio of the radii, i.e. the centres of similitude. Hence, the common tangents of two circles pass through the centres of similitude.

**Example i.** *A straight line cuts two parallel straight lines in points which subtend a right angle at a fixed point midway between the lines: find its envelope.*

Let the fixed point be the origin and let $x+a=0$, $x-a=0$ be the Cartesian equations of the given parallel lines. Let $(l, m)$ or $lx+my+1=0$ be the straight line: then the lines joining its intersections with $x+a=0$ and $x-a=0$ to the origin are $lx+my+x/a=0$ and $lx+my-x/a=0$. These are at right angles, hence $l^2-\frac{1}{a^2}+m^2=0$, i.e. the equation of the envelope of the line $(l, m)$ is $a^2(l^2+m^2)=1$, which is a circle whose centre is the origin and radius $a$.

**Example ii.** *Any point P on a fixed circle is joined to a fixed point A, and PQ is drawn perpendicular to AP: find its envelope.*

Let the straight line $PQ$ be $lx+my+1=0$, and the fixed point $(a, 0)$; and let the circle be $x^2+y^2=r^2$.

The equation of the straight line through $(a, 0)$ perpendicular to $lx+my+1=0$ is $mx-ly=am$, and these lines intersect on the circle.

Eliminating $x$ and $y$ from these equations and $x^2+y^2=r^2$, we have

$$(lx+my)^2+(mx-ly)^2=1+a^2m^2,$$

i.e.
$$(l^2+m^2)(x^2+y^2)=1+a^2m^2,$$

i.e.
$$r^2(l^2+m^2)=1+a^2m^2,$$

which is the equation of the envelope required.

## Examples X c.

1. Find the coordinates of the chord joining the points $rl\cos\theta+rm\sin\theta=1$, $rl\cos\phi+rm\sin\phi=1$.

2. Show that the polars of all points on the line $(l', m')$ with respect to the circle $r^2(l^2+m^2)=1$ are concurrent.

3. The locus of the intersection of tangents to the circle $r^2(l^2+m^2)=1$, which include a right angle, is $2r^2(l^2+m^2)=1$.

4. A chord of a circle subtends a right angle at a fixed point: show that the equation of its envelope can be put in the form

$$(l^2+m^2)(a^2-r^2)-2al+2=0.$$

5. The sum of the perpendiculars from $n$ fixed points on a straight line is constant: show that its envelope is a circle.

If the points are $a_1l+b_1m+1=0$, $a_2l+b_2m+1=0$, $a_3l+b_3m+1=0$, find its centre and radius.

6. Find the condition that the point $al+bm+1=0$ should lie on the circle $r^2(l^2+m^2)=1$.

7. Find the coordinates of the tangents from the point $2al=1$ to the circle $2a^2(l^2+m^2)=1$.

8. Show that $3m^2-4lm-4l-2m-1=0$ is a circle, and find its centre and radius.

9. Find the condition that two circles whose line equations are given should be orthogonal.

10. Find the equation of the circle of similitude of the circles

$$(pl+1)^2=r^2(l^2+m^2),\quad (ql+1)^2=s^2(l^2+m^2).$$

11. Show that the Cartesian equation of the circle

$$r^2(l^2+m^2)=(la+mb+1)^2 \text{ is } (x-a)^2+(y-b)^2=r^2.$$

12. Through two points $A$ and $A'$, $AP$, $A'P'$ are drawn perpendicular to $AA'$, and $A'P$, $AP'$ are perpendicular. Find the envelope of the join of the mid-points of $AP$ and $A'P'$.

13. Show that the equation $c^2l^2+(\mu^2+c^2)m^2-2\mu l-1=0$ represents for different values of $\mu$ a system of circles passing through two fixed points.

14. Show that the tangents from any point on the axis of $m$ to the circles $c^2l^2+(c^2-\lambda^2)m^2+2\lambda l+1=0$, $c^2l^2+(c^2-\mu^2)m^2+2\mu l+1=0$ are equal, and hence that the first equation for different values of $\lambda$ represents a system of coaxal circles whose limiting points are $cl\pm 1=0$.

15. Show that the coordinates of the common tangents of the two circles in Question 14 are given by the equations $(c^2l^2+1)(\lambda+\mu)+2(c^2+\lambda\mu)l=0$, $(\lambda+\mu)m^2=2l$. Deduce that they intersect on the line of centres of the circles at distances from the radical axis whose product is $c^2$.

16. Two circles of a coaxal system can be drawn to touch any given straight line.

17. Pairs of circles of a coaxal system are taken such that the sum of the distances of their centre from the radical axis is constant. Show that their common tangents envelope the curve $m^2=2ld$.

18. For different values of $c$ the equation $c^2m^2-2cl-1=0$ represents circles touching each other at a fixed point.

19. A system of circles touch one another at a fixed point, and a line parallel to their common tangent cuts them. Show that the tangents to the circles at the points of intersection with the line all touch another fixed circle.

## § 8. The general equation of the second degree.

I. *To find the Cartesian equation of the envelope*

$$\Sigma\equiv al^2+2hlm+bm^2+2gl+2fm+c=0.$$

In other words, to find the equation of the envelope of the straight line

$$lx+my+1=0,$$

when $l$ and $m$ satisfy the equation $\Sigma=0$.

*To find the tangential equation of the locus*

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$$

In other words, to find the equation of the locus of the point

$$xl+ym+1=0,$$

when $x$ and $y$ satisfy the equation $S=0$.

Through any point $(x, y)$ two straight lines of the system defined by $\Sigma = 0$ pass, for the values of $l$ and $m$ for such lines are given by

$$lx + my + 1 = 0,$$

and

$$al^2 + 2hlm + bm^2 - 2(gl + fm)(lx + my) + c(lx + my)^2 = 0,$$

i.e. $$(a - 2gx + cx^2)l^2 + 2(h - fx - gy + cxy)lm + (b - 2fy + cy^2)m^2 = 0.$$

These two straight lines are coincident, and their point of intersection $(x, y)$ lies on the envelope if

$$(a - 2gx + cx^2)(b - 2fy + cy^2) = (h - fx - gy + cxy)^2,$$

which at once reduces to

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

which is therefore the Cartesian equation of the envelope.

On any straight line $(l, m)$ two points of the system defined by $\Sigma = 0$ lie, for the values of $x$ and $y$ for such points are given by

$$xl + ym + 1 = 0,$$

and

$$ax^2 + 2hxy + by^2 - 2(gx + fy)(lx + my) + c(lx + my)^2 = 0,$$

i.e. $$(a - 2gl + cl^2)x^2 + 2(h - fl - gm + clm)xy + (b - 2fm + cm^2)y^2 = 0.$$

These two points are coincident, and the straight line through them $(l, m)$ touches the locus if

$$(a - 2gl + cl^2)(b - 2fm + cm^2) = (h - fl - gm + clm)^2,$$

which at once reduces to

$$Al^2 + 2Hlm + Bm^2 + 2Gl + 2Fm + C = 0,$$

which is therefore the tangential equation of the locus.

These propositions are very important, and another proof will be given later.

**Note i.** Since the envelope $al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0$ is the same as the locus $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$, the envelope is a **conic.**

**Note ii.** The Cartesian equation of the conic $\Sigma = 0$ and the tangential equation of the conic $S = 0$ are formed from these equations by the same process, viz. replacing $a, b, c, f, g, h$ by $A, B, C, F, G, H$ and interchanging $l, m$ and $x, y$.

Since the tangential equation of $S = 0$ is

$$Al^2 + 2Hlm + Bm^2 + 2Gl + 2Fm + C = 0,$$

the Cartesian equation of the latter must be $S = 0$. This fact indicates the reciprocal relations which exist between the letters $a, b, c, f, g, h$ and $A, B, C, F, G, H$, viz. $bc - f^2 = A$, $BC - F^2 = \Delta a$,

$$fg - ch = H, \quad FG - CH = \Delta h, \text{ \&c.};$$

these can be easily verified algebraically.

**Note iii.** The conic $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$ is an ellipse, parabola, or hyperbola according as $AB - H^2$, i.e. $\Delta c$, is positive, zero, or negative: hence the conic $\Sigma \equiv al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0$ is an ellipse, parabola, or hyperbola according as $\Delta c$ is positive, zero, or negative.

The condition for a parabola, $c = 0$, corresponds to the fact that the line at infinity (0, 0) touches the conic $\Sigma = 0$.

Similarly, we can show that

(a) $\Sigma = 0$ is a rectangular hyperbola if $A + B - 2H\cos\omega = 0$;

(b) $\Sigma = 0$ passes through the origin if $C = 0$.

The latter can be shown independently thus: since we are considering the origin we must use homogeneous coordinates; the equation $\Sigma = 0$ can then be written $al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$.

The coordinates therefore of the lines which pass through the origin and touch the curve are given by $n = 0$ and $al^2 + 2hlm + bm^2 = 0$. These tangents are coincident, i.e. the origin $n = 0$ lies on the curve if $ab - h^2 = 0$.

**Note iv.** (a) When $\Delta = 0$ the equation $S = 0$ represents a pair of straight lines.

Now $$AC - G^2 = \Delta b, \quad BC - F^2 = \Delta a, \quad FG - CH = \Delta h,$$

hence $$G^2 = AC, \quad F^2 = BC, \quad FG = CH.$$

The tangential equation corresponding to $S = 0$, viz.

$$Al^2 + 2Hlm + Bm^2 + 2Gl + 2Fm + C = 0,$$

can therefore be written

$$G^2l^2 + 2FGlm + F^2m^2 + 2GCl + 2FCm + C^2 = 0,$$

i.e. $$(Gl + Fm + C)^2 = 0,$$

which is the square of the line equation of the point $(G/C, F/C)$, i.e. of the point of intersection of the lines $S = 0$. This is geometrically evident since all lines which meet $S = 0$ in coincident points pass through their point of intersection.

(b) In the same way if $\Delta = 0$, $\Sigma = 0$ represents a pair of points, and the corresponding Cartesian equation reduces to $(Gx + Fy + C)^2 = 0$, i.e. the square of the Cartesian equation of the line $(G/C, F/C)$, i.e. of the line joining the two points.

The geometrical significance is that all pairs of points, the joins of which to the two points $\Sigma = 0$ are coincident, must lie on the line joining the two points.

II. *To find the equation of the points of intersection of* $\Sigma = 0$ *and the line* $(l_1, m_1)$.

The coordinates of any straight line through the point of intersection of the lines $(l_1, m_1)$, $(l_2, m_2)$ are $\left\{\frac{pl_1 + ql_2}{p+q}, \frac{pm_1 + qm_2}{p+q}\right\}$. This line touches the conic $\Sigma = 0$ if

$$a(pl_1 + ql_2)^2 + 2h(pl_1 + ql_2)(pm_1 + qm_2) + b(pm_1 + qm_2)^2 + 2g(pl_1 + ql_2)(p+q) + 2f(pm_1 + qm_2)(p+q) + c(p+q)^2 = 0. \quad \text{(i)}$$

This equation is quadratic in the ratio $p/q$, so that there are two such lines, and if they are coincident the point of intersection of $(l_1, m_1)$, $(l_2, m_2)$ must lie on $\Sigma = 0$.

We adopt a notation similar to that used for the general Cartesian equation of the second degree, thus

$$L = al+hm+g, \quad M = hl+bm+f, \quad N = gl+fm+c,$$

and so

$$L_1 = al_1+hm_1+g, \quad M_1 = hl_1+bm_1+f, \quad N_1 = gl_1+fm_1+c.$$

The equation (i) reduces to

$$\Sigma_1 . p^2 + 2\{l_2 L_1 + m_2 M_1 + N_1\}pq + \Sigma_2 . q^2 = 0,$$

and this has equal roots if

$$\Sigma_1 . \Sigma_2 = \{l_2 L_1 + m_2 M_1 + N_1\}^2,$$

which is therefore the condition that the point of intersection of the lines $(l_1, m_1)$, $(l_2, m_2)$ should lie on $\Sigma = 0$. Hence the coordinates of any line through either point of intersection of $(l_1, m_1)$ and the conic $\Sigma = 0$ satisfy

$$\Sigma . \Sigma_1 = (lL_1 + mM_1 + N_1)^2, \qquad \text{(ii)}$$

which is therefore the equation of the points of intersection.

**Cor.** If $(l_1, m_1)$ is a tangent to the conic, then $\Sigma_1 = 0$ and the equation (ii) reduces to $lL_1 + mM_1 + N_1 = 0$; this is therefore the equation of the point of contact of the tangent $(l_1, m_1)$ to the conic $\Sigma = 0$.

The equation is algebraically equivalent to

$$l_1 L + m_1 M + N = 0.$$

III. *To find the equation of the point of intersection of tangents whose chord of contact is* $(l', m')$.

Let the tangents be $(l_1, m_1)$, $(l_2, m_2)$, then their points of contact are $lL_1 + mM_1 + N_1 = 0$, $lL_2 + mM_2 + N_2 = 0$. But $(l', m')$ passes through both of these points, hence

$$l'L_1 + m'M_1 + N_1 = 0,$$
$$l'L_2 + m'M_2 + N_2 = 0,$$

and these are the conditions that the lines $(l_1, m_1)$, $(l_2, m_2)$ should each pass through the point $l'L + m'M + N = 0$, which is consequently the equation required.

It is algebraically equivalent to $lL' + mM' + N' = 0$.

IV. *To find the equation of the pole of* $(l', m')$.

Let $(l_1, m_1)$ be any chord such that the intersection of the tangents at its extremities lies on $(l', m')$.

The equation of this point of intersection is $lL_1 + mM_1 + N_1 = 0$, and since by hypothesis it lies on $(l', m')$ we have

$$l'L_1 + m'M_1 + N_1 = 0.$$

But this is the condition that $(l_1, m_1)$ should pass through the fixed point $l'L+m'M+N=0$. Thus the chord of contact of tangents from any point on $(l', m')$ passes through this point, which is therefore the pole of $(l', m')$.

The equation is algebraically equivalent to $lL'+mM'+N'=0$.

**Cor.** The pole of the line at infinity (0, 0) is $N=0$, i.e. $gl+fm+c=0$, which is therefore the centre of the conic $\Sigma=0$. Its point coordinates are $(g/c, f/c)$.

V. *To find the coordinates of the asymptotes of* $\Sigma=0$.

Let $(l_1, m_1)$ be an asymptote; its point of contact is

$$lL_1+mM_1+N_1=0,$$

and this point must lie on the line at infinity (0, 0); hence

$$N_1=gl_1+fm_1+c=0. \qquad \text{(i)}$$

Now since $(l_1, m_1)$ is a tangent to the conic, we have also

$$al_1^2+2hl_1m_1+bm_1^2+2gl_1+2fm_1+c=0. \qquad \text{(ii)}$$

The equations (i) and (ii) therefore give the required coordinates; these equations reduce to

$$gl+fm+c=0; \quad al^2+2hlm+bm^2=c.$$

The ratio of the coordinates is given by

$$c(al^2+2hlm+bm^2)=(gl+fm)^2,$$

i. e.

$$Bl^2-2Hlm+Am^2=0.$$

If the asymptotes are perpendicular, we have

$$l_1l_2+m_1m_2+(l_1m_2+l_2m_1)\cos\omega=0,$$

or

$$A+B-2H\cos\omega=0,$$

which is the condition that $\Sigma=0$ should be a rectangular hyperbola.

VI. *To find the condition that* $(l_1, m_1)$, $(l_2, m_2)$ *should be conjugate lines of the conic.*

The pole of the line $(l_1, m_1)$ is

$$lL_1+mM_1+N_1=0,$$

and this lies on $(l_2, m_2)$ if

$$l_2L_1+m_2M_1+N_1=0,$$

which gives the condition, symmetrical with respect to $l_1, m_1$ and $l_2, m_2$,

$$al_1l_2+bm_1m_2+h(l_1m_2+l_2m_1)+g(l_1+l_2)+f(m_1+m_2)+c=0.$$

If $(l_1, m_1)$ and $(l_2, m_2)$ are diameters of the conic, each passes

through the centre $gl+fm+c=0$, so that $N_1=0$ and $N_2=0$; hence

$$al_1l_2+h(l_1m_2+l_2m_1)+bm_1m_2=c,$$

or
$$acl_1l_2+ch(l_1m_2+l_2m_1)+bcm_1m_2=c^2$$
$$=(gl_1+fm_1)(gl_2+fm_2).$$

which reduces to

$$Am_1m_2-H(l_1m_2+l_2m_1)+Bl_1l_2=0;$$

this is the condition that the diameters $(l_1, m_1)$, $(l_2, m_2)$ should be conjugate.

VII. *To find the director circle of the conic* $\Sigma=0$.

Let $lx+my+1=0$ be a point on the director circle, then the coordinates of the tangents from it to the conic satisfy the equations

$$lx+my+1=0 \quad \text{and} \quad \Sigma=0.$$

The coordinates of these tangents therefore satisfy

$$al^2+2hlm+bm^2-2(gl+fm)(lx+my)+c(lx+my)^2=0,$$

i.e. $(cx^2-2gx+a)l^2+2(cxy-gy-fx+h)lm+(cy^2-2fy+b)m^2=0.$

This equation therefore represents two points at infinity on the tangents from $lx+my+1=0$ to the conic; if the tangents are at right angles these points are orthogonal, hence

$$cx^2-2gx+a+cy^2-2fy+b+2(cxy-gy-fx+h)\cos\omega=0,$$

i.e. $c(x^2+y^2+2xy\cos\omega)-2x(g+f\cos\omega)-2y(f+g\cos\omega)$
$$+a+b+2h\cos\omega=0,$$

which is therefore the Cartesian equation of the director circle.

If the axes are rectangular the equation of the director circle is

$$c(x^2+y^2)-2gx-2fy+a+b=0,$$

and its tangential equation is

$$(A+B)(l^2+m^2)+(gl+fm+c)^2=0.$$

If the conic is a parabola we have $c=0$; the equation of the director circle reduces to

$$2gx+2fy-a-b=0,$$

which is the directrix of the parabola. The line coordinates of the directrix of the parabola are then $[-2g/(a+b), \ -2f/(a+b)]$.

VIII. *To find the foci of the conic* $\Sigma=0$.

We have just shown that the tangents from the point $(x, y)$ to the conic pass through the two points at infinity whose equation is

$$(cx^2-2gx+a)l^2+2(cxy-gy-fx+h)lm+(cy^2-2fy+b)m^2=0.$$

Consequently, if $(x, y)$ is a focus, these points must coincide with the circular points at infinity whose equation is

$$l^2-2lm\cos\omega+m^2=0.$$

Hence, if $(x, y)$ is a focus,

$$cx^2-2gx+a=cy^2-2fy+b=-\sec\omega\,(cxy-fx-gy+h).$$

When the axes are rectangular these equations become

$$c(x^2-y^2)-2gx+2fy+a-b=0$$

and

$$cxy-fx-gy+h=0,$$

i.e.

$$(cx-g)^2-(cy-f)^2=g^2-f^2-c(a-b)=A-B$$

and

$$(cx-g)(cy-f)=fg-ch=H.$$

**Note.** The tangential equation of the conic

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

is

$$Al^2+2Hlm+Bm^2+2Gl+2Fy+C=0.$$

The foci of the conic $S=0$ are therefore given by

$$(Cx-G)^2-(Cy-F)^2=G^2-F^2-C(A-B)=\Delta(a-b),$$

$$(Cx-G)(Cy-F)=FG-CH=\Delta h.$$ (See p. 245.)

When the conic $\Sigma=0$ is a parabola we have $c=0$, and the tangents from the point $(x, y)$ to it pass through the two points at infinity

$$(a-2gx)l^2+2(h-gy-fx)lm+(b-2fy)m^2=0,$$

and if $(x, y)$ is the focus, these points must be the circular points at infinity. Comparing this equation with that of the circular points we obtain two linear equations in $x$ and $y$ giving the coordinates of the focus.

**Example** i. *A conic touches three given straight lines, and the radius of its director circle is given* $(r)$*: show that its centre lies on a fixed circle.*

Let the conic be $al^2+2hlm+bm^2+2gl+2fm+1=0$ (since the conic has a director circle $c\neq0$), and let the coordinates of the fixed lines be $(l_1, m_1)$, $(l_2, m_2)$, $(l_3, m_3)$.

The centre of the conic is the point $(g, f)$, so that for the required locus we have $x=g$, $y=f$.

Since the radius of the director circle is $r$, we have

$$g^2+f^2-a-b=r^2,$$

hence

$$x^2+y^2-a-b=r^2.$$

Since the lines $(l_1, m_1)$, $(l_2, m_2)$, $(l_3, m_3)$ touch the conic,

$$al_1^2+2hl_1m_1+bm_1^2-(2l_1x+2m_1y-1)=0,$$

$$al_2^2+2hl_2m_2+bm_2^2-(2l_2x+2m_2y-1)=0,$$

$$al_3^2+2hl_3m_3+bm_3^2-(2l_3x+2m_3y-1)=0.$$

and

$$a\qquad\quad+b\quad-(x^2+y^2-r^2)\quad=0.$$

Eliminating $a$, $h$, and $b$, we have

$$\begin{vmatrix} x^2+y^2-r^2 & 1 & 1 & 0 \\ 2l_1x+2m_1y-1 & l_1^2 & m_1^2 & l_1m_1 \\ 2l_2x+2m_2y-1 & l_2^2 & m_2^2 & l_2m_2 \\ 2l_3x+2m_3y-1 & l_3^2 & m_3^2 & l_2m_3 \end{vmatrix} = 0,$$

which is evidently a circle.

**§ 9. The tangential equations of the conics referred to their principal axes.**

These equations can be found from the known Cartesian equations by the method of § 1, or by comparing the general equation of a tangent to the conic with the equation $lx+my+1=0$. We shall illustrate the latter method in this section.

(1) **The Parabola.** The equation of any tangent to the parabola $y^2-4ax=0$ is $x-ty+at^2=0$, and if this is identical with

$$lx+my+1=0,$$

we have $l=-m/t=1/at^2$. Eliminating $t$ we get

$$\mathbf{am^2=l.}$$

(2) **The Ellipse.** The equation of a tangent to $x^2/a^2+y^2/b^2=1$ is $(x\cos\theta)/a+(y\sin\theta)/b=1$; identifying this with $lx+my+1=0$, we have $la=-\cos\theta$, $mb=-\sin\theta$;

$$\therefore \qquad \mathbf{a^2l^2+b^2m^2=1.}$$

(3) **The Hyperbola.** In the same manner we find that the tangential equation of the hyperbola $x^2/a^2-y^2/b^2=1$ is

$$\mathbf{a^2l^2-b^2m^2=1.}$$

If the hyperbola is referred to its asymptotes as coordinate axes, its equation is $xy=c^2$; any tangent is $\lambda^2x+y-2c\lambda=0$, whence

$$l=-\lambda/2c \quad \text{and} \quad m=-1/2c\lambda,$$

and

$$\mathbf{4c^2lm=1.}$$

**Illustrative Examples.**

**Example i.** *Find the envelope of a chord of an ellipse whose mid-point lies on a fixed straight line.*

Let $x^2/a^2+y^2/b^2=1$ be the ellipse and $lx+my+1=0$ the chord, then the diameter conjugate to this chord is $b^2mx-a^2ly=0$, and this meets the chord at its mid-point. The coordinates of the mid-point are therefore $\{-a^2l/(a^2l^2+b^2m^2),\ -b^2m/(a^2l^2+b^2m^2)\}$.

This point lies on a fixed straight line, say $(l', m')$, so that

$$a^2ll'+b^2mm'=a^2l^2+b^2m^2.$$

This is the tangential equation of the envelope of the line $(l, m)$. The coordinates of the straight line at infinity (0, 0) satisfy it; the envelope is therefore a parabola. It also touches the given straight line $(l', m')$ and the tangents to the ellipse at its points of intersection with this line.

**Example ii.** *Through points on a given straight line straight lines are drawn parallel to their polars with respect to a fixed ellipse: find the envelope of these lines.*

Let the ellipse be $a^2l^2+b^2m^2=1$ and $(h, k)$ the given straight line. Now the equation of the pole of any straight line $(\lambda, \mu)$ is $a^2l\lambda+b^2m\mu=1$, and if this point lies on the line $(h, k)$ we have

$$a^2\lambda h+b^2\mu k=1. \qquad \text{(i)}$$

The coordinates of a line parallel to $(\lambda, \mu)$ are $(c\lambda, c\mu)$, and if this passes through the point $a^2l\lambda+b^2m\mu=1$ we have

$$a^2c\lambda^2+b^2c\mu^2=1. \qquad \text{(ii)}$$

Hence, using condition (i),

$$a^2c\lambda^2+b^2c\mu^2=a^2h\lambda+b^2k\mu,$$

or

$$a^2c^2\lambda^2+b^2c^2\mu^2=a^2hc\lambda+b^2kc\mu.$$

Consequently, the line $(c\lambda, c\mu)$ envelopes the parabola

$$a^2l^2+b^2m^2=a^2hl+b^2km.$$

This parabola touches the given straight line $(h, k)$.

**Example iii.** *Find the envelope of the polars of a fixed point with respect to a system of confocal conics.*

Let $x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1$ be one of a system of confocal conics, and $(h, k)$ the fixed point.

The equation of the polar of $(h, k)$ is

$$xh/(a^2+\lambda)+yk/(b^2+\lambda)=1.$$

The coordinates of this straight line are therefore

$$l=-h/(a^2+\lambda), \quad m=-k/(b^2+\lambda).$$

Eliminating the variable $\lambda$ we have

$$h/l-k/m+a^2-b^2=0,$$

or

$$(a^2-b^2)\,lm=kl-hm.$$

This is a parabola, since it is satisfied by the coordinates of the line at infinity (0, 0). Writing the equation in homogeneous coordinates, $(a^2-b^2)\,lm=kln-hmn$, we see that it is satisfied by the coordinates of the axes; the parabola therefore touches the coordinate axes.

**Example iv.** *To find the focus, directrix, and the tangent at the vertex of the parabola* $(a^2-b^2)\,lm=kl-hm$.

The coordinates of the focus can be found as above (§ 8, VIII). The following is another method.

If the point $(x_1, y_1)$ is the focus, then the line

$$x+iy-(x_1+iy_1)=0,$$

which is the join of the focus to a circular point at infinity, touches the parabola.

Let $z \equiv x_1+iy_1$, then, since the line $\left(-\frac{1}{z}, -\frac{i}{z}\right)$ touches the parabola, we have, by substitution in the equation of the parabola,

$$z(k-ih)+i(a^2-b^2)=0;$$

$$\therefore \quad x_1+iy_1=z=-\frac{i(a^2-b^2)}{k-ih}$$

$$=\frac{(a^2-b^2)(h-ik)}{h^2+k^2},$$

i.e.
$$x_1=\frac{(a^2-b^2)h}{k^2+h^2}, \quad y_1=\frac{(b^2-a^2)k}{k^2+h^2}.$$

The equation of the directrix of $S=0$ (when it is a parabola) is

$$2Gx+2Fy-A-B=0,$$

and since the tangential equation of the parabola is

$$2(a^2-b^2)lm-2kl+2hm=0,$$

we have $A=B=0$, $H=a^2-b^2$, $G=-k$, $F=h$, i.e. the equation of the directrix is $kx-hy=0$.

This is a straight line through the origin, its coordinates are $(k, -h, 0)$; the coordinates of a line parallel to it are $(\lambda k, -\lambda h)$, and, if this touches the parabola, it is the tangent at the vertex.

The condition for this is

$$-\lambda^2 kh(a^2-b^2)=\lambda(k^2+h^2),$$

or
$$\lambda=-\frac{k^2+h^2}{kh(a^2-b^2)}.$$

The coordinates of the tangent at the vertex are

$$\left\{-\frac{k^2+h^2}{h(a^2-b^2)}, \quad \frac{k^2+h^2}{k(a^2-b^2)}\right\},$$

and its equation is therefore $(k^2+h^2)(kx-hy)=(a^2-b^2)hk$.

**Example v.** *Points $C$, $D$ on an ellipse are such that $AC$, $A'D$ intersect on one of the equiconjugate diameters: find the envelope of $CD$.*

Let $C$ be the point $(a\cos\theta, b\sin\theta)$ and $D$ the point $(a\cos\phi, b\sin\phi)$. The equations of $AC$, $A'D$ are

$$(x\cos\tfrac{1}{2}\theta)/a+(y\sin\tfrac{1}{2}\theta)/b=\cos\tfrac{1}{2}\theta,$$

$$(x\sin\tfrac{1}{2}\phi)/a-(y\cos\tfrac{1}{2}\phi)/b=-\sin\tfrac{1}{2}\phi,$$

and these intersect on the line through the origin

$$\sin\tfrac{1}{2}\phi\{(x\cos\tfrac{1}{2}\theta)/a+(y\sin\tfrac{1}{2}\theta)/b\}+\cos\tfrac{1}{2}\theta\{(x\sin\tfrac{1}{2}\phi)/a-(y\cos\tfrac{1}{2}\phi)/b\}=0,$$

i.e.
$$2x\sin\tfrac{1}{2}\phi\cos\tfrac{1}{2}\theta/a-y\cos\tfrac{1}{2}(\theta+\phi)/b=0.$$

If this is the equiconjugate diameter $x/a - y/b = 0$, we have

$$2 \sin \tfrac{1}{2}\phi \cos \tfrac{1}{2}\theta = \cos \tfrac{1}{2}(\theta + \phi),$$

or

$$\sin \tfrac{1}{2}(\theta+\phi) - \sin \tfrac{1}{2}(\theta-\phi) = \cos \tfrac{1}{2}(\theta+\phi). \qquad \text{(i)}$$

Now $CD$ is the line $x \cos \frac{1}{2}(\theta+\phi)/a + y \sin \frac{1}{2}(\theta+\phi)/b = \cos \frac{1}{2}(\theta-\phi)$, so that its coordinates are given by

$$al = -\cos \tfrac{1}{2}(\theta+\phi) \sec \tfrac{1}{2}(\theta-\phi); \quad bm = -\sin \tfrac{1}{2}(\theta+\phi) \sec \tfrac{1}{2}(\theta-\phi).$$

Substituting in (i) we have

$$\tan \tfrac{1}{2}(\theta-\phi) = al - bm,$$

and therefore $$a^2l^2 + b^2m^2 - 1 = (al-bm)^2,$$

or $$2ablm = 1.$$

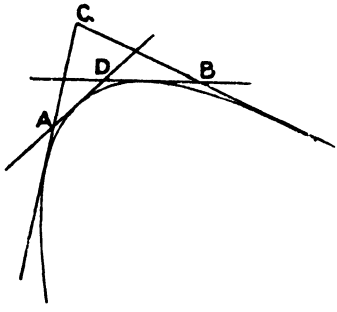
This is the tangential equation of the envelope of $CD$; it is a rectangular hyperbola whose asymptotes are the axes of the ellipse.

## § 10. Interpretation of some special forms of tangential equations.

Let $u = 0$, $v = 0$, $w = 0$, $z = 0$ be the equations of four points $A$, $B$, $C$, $D$; $\Sigma = 0$ the general tangential equation of the second degree; $t = 0$, $t' = 0$ the equations of two points $T$, $T'$ on the conic $\Sigma = 0$, and $k$ a constant.

### 1. uv = k . wz.

This equation is satisfied by the coordinates of lines which pass through the pairs of points $u = 0$, $w = 0$; $u = 0$, $z = 0$; $v = 0$, $w = 0$; $v = 0$, $z = 0$, i.e. by the coordinates of the lines $AC$, $AD$, $BC$, $BD$. It is of the second degree; it therefore represents a conic inscribed in the quadrilateral $ACBD$.




**Example i.** *Find the locus of the centres of conics inscribed in a given quadrilateral.*

Let $ABCD$ be the quadrilateral, and take $AC$, $BD$ as coordinate axes. The equations of the points $A$, $B$, $C$, $D$ are of the form $al + 1 = 0$, $bm + 1 = 0$, $cl + 1 = 0$, $dm + 1 = 0$.

The equation of a conic inscribed in this quadrilateral is

$$(al+1)(cl+1) + k(bm+1)(dm+1) = 0,$$

and its centre is the point $\left\{\frac{a+c}{2(1+k)}, \frac{k(b+d)}{2(1+k)}\right\}$.

The Cartesian equation of its locus is therefore, eliminating $k$,

$$2x/(a+c) + 2y/(b+d) = 1.$$

This is a straight line which passes through the points $\{\frac{1}{2}(a+c), 0\}$, $\{0, \frac{1}{2}(b+d)\}$, i.e. the mid-points of the diagonals.

**Example ii.** *On two given intersecting straight lines two sets of fixed points A, B, C and A', B', C' are taken. Points P, P' on these lines are chosen so that* $\{AB, CP\} = \{A'B', C'P'\}$. *Find the envelope of PP'.*

Let $AC/CB = k \cdot A'C'/C'B'$, then we are given that

$$\frac{AC}{CB} : \frac{AP}{PB} = \frac{A'C'}{C'B'} : \frac{A'P'}{P'B'}.$$

Hence

$$\frac{AP}{PB} = k \cdot \frac{A'P'}{P'B'}. \qquad \text{(i)}$$

Take the given straight lines for coordinate axes, and let $A$, $B$ be the points $(a, 0)$; $(b, 0)$; $A'$, $B'$ the points $(0, a')$ $(0, b')$. Then from (i) we have

$$(OP - a)(b' - OP') = k(OP' - a')(b - OP).$$

The coordinates of the line $PP'$ are $l = -1/OP$, $m = -1/OP'$; hence

$$(al + 1)(b'm + 1) = k(a'm + 1)(bl + 1), \qquad \text{(ii)}$$

which is the equation of the envelope of $PP'$. This is a conic inscribed in the quadrilateral $ABA'B'$; it touches the lines $AA'$, $BB'$ and also the axes $AB$, $A'B'$.

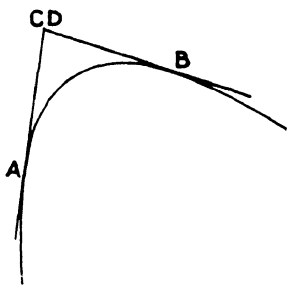
If the equation (ii) is made homogeneous, it becomes

$$(al + n)(b'm + n) = k(a'm + n)(bl + n);$$

it can be readily shown that the points of contact of the conic with the axes (0, 1, 0), (1, 0, 0) are points $O_1$, $O_2$ such that $\{AB, CO\} = \{A'B', C'O_1\}$ and $\{AB, CO_2\} = \{A'B', C'O\}$ where $O$ is the origin.

2. $\mathbf{uv = k \cdot w^2}$.

In this case the points $C$, $D$ coincide, and the pairs of tangents $AC$, $AD$; $BC$, $BD$ also coincide. The conic therefore touches the lines $CA$, $CB$ at the points $A$ and $B$.




Hence $uv = kw^2$ represents a conic touching the lines joining $u = 0$, $w = 0$ and $v = 0$, $w = 0$ at the points $u = 0$, $v = 0$.

**Example.** *Show that the parabola*

$$lm + al + bm = 0$$

*touches the coordinate axes, and find the points of contact.*

The equation of the parabola in homogeneous coordinates is

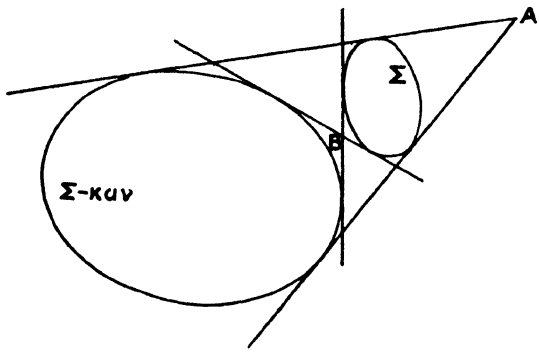
$$lm + aln + bmn = 0;$$

this can be written $(l + bn)(m + an) = abn^2$.

Now $l + bn = 0$ is the point $B(1/b, 0)$, $m + an = 0$ is the point $A(0, 1/a)$, and $n = 0$ is the origin. Hence the parabola touches the coordinate axes $OA$, $OB$ at the points $A$ and $B$.

**3. $\Sigma = kuv$.**

This represents a conic touching the tangents to $\Sigma = 0$ from the




points $u = 0$, $v = 0$; in other words the common tangents of the conics $\Sigma = 0$, $\Sigma - kuv = 0$ intersect at the points $u = 0$, $v = 0$.

**Note i.** If $u = 0$, $v = 0$ are the circular points $\Omega$, $\Omega'$ at infinity, the tangents from them to a conic intersect at the foci of the conic. In this case $uv \equiv l^2 - 2lm\cos\omega + m^2$, or in rectangular coordinates $uv \equiv l^2 + m^2$.

Hence $\Sigma = k(l^2 + m^2)$ represents a conic having the same foci as $\Sigma = 0$.

Again, if $u = 0$, $v = 0$ are the equations of the foci $S$, $S'$ of a conic, since the conic is inscribed in the quadrilateral $SS'\Omega\Omega'$ its equation must be of the form $uv = k(l^2 + m^2)$.

Thus, if $u = 0$, $v = 0$ are the foci of the conic $\Sigma = 0$, the equations $uv - k(l^2 + m^2) = 0$ and $\Sigma = 0$ are identical; it follows that

$$\Sigma + k(l^2 + m^2) \equiv uv.$$

Hence, for values of $k$ for which $\Sigma + k(l^2 + m^2) = 0$ represents a pair of points, these points are a pair of foci of the conic $\Sigma = 0$.

The condition that

$$al^2 + 2hlm + bm^2 + 2gl + 2fm + c + k(l^2 + m^2) = 0$$

should represent a pair of points is

$$\begin{vmatrix} a+k & h & g \\ h & b+k & f \\ g & f & c \end{vmatrix} = 0,$$

i.e.
$$ck^2 + (A + B)k + \Delta = 0.$$

If $c \neq 0$, this gives two values of $k$ corresponding to the two pairs of foci, one real and one imaginary, of a central conic.

If $c = 0$, it gives only one value of $k$; the equation $\Sigma + k(l^2 + m^2) = 0$, for this value has two factors, one of which represents the finite focus, the other the point at infinity on the axis.

**Note ii.** If $lx_1 + my_1 + 1 = 0$, $lx_2 + my_2 + 1 = 0$ are the foci of a conic, the equation of the conic is

$$(lx_1 + my_1 + 1)(lx_2 + my_2 + 1) = k(l^2 + m^2).$$

This corresponds to the geometrical property that the product of the perpendiculars from the foci on any tangent to a central conic is constant and equal to the square on the semi-minor axis.

**Example.** The equation of an ellipse, $a^2l^2+b^2m^2=1$, may be written

$$1-(a^2-b^2)l^2=b^2(l^2+m^2),$$

or

$$(1-lae)(1+lae)=b^2(l^2+m^2).$$

The foci are $(ae, 0)$, $(-ae, 0)$, and the perpendiculars from them on any tangent $lx+my+1=0$ are $\frac{1+lae}{\sqrt{l^2+m^2}}$ and $\frac{1-lae}{\sqrt{l^2+m^2}}$, their product being $b^2$.

**Note iii.** The general equation of a conic having a focus at the origin is

$$a(l^2+m^2)+2gl+2fm+c=0,$$

for this equation in homogeneous coordinates becomes

$$a(l^2+m^2)+n(2gl+2fm+cn)=0,$$

so that the foci are $n=0$ and $2gl+2fm+cn=0$.

**Example i.** *Two points $P$, $Q$ on a circle subtend a right angle at a fixed point: show that $PQ$ envelopes a conic, and find its foci.*

Let the fixed point be the origin and the centre of the circle the point $(a, 0)$. The equation of the circle is then $x^2+y^2-2ax+c=0$.

Let $lx+my+1=0$ be the chord $P, Q$; the equation of the lines $OP, OQ$ is

$$x^2+y^2+2ax(lx+my)+c(lx+my)^2=0;$$

these are orthogonal if

$$1+2al+cl^2+1+cm^2=0,$$

i.e.

$$c(l^2+m^2)+2(al+1)=0,$$

which is the tangential equation of the envelope. The envelope is therefore a conic with its foci at the origin and the point $(a, 0)$, i.e. the centre of the circle.

**Example ii.** *Find the equation of the auxiliary circle of the conic $a(l^2+m^2)+2gl+2fm+c=0$.*

The focus of the conic is the origin, the auxiliary circle is therefore the locus of the foot of the perpendicular from the origin to a tangent to the conic.

Let $(x_1, y_1)$ be a point on the auxiliary circle, the line joining it to the origin is $xy_1-yx_1=0$; a perpendicular to this through the point $(x_1, y_1)$ is $xx_1+yy_1=x_1^2+y_1^2$. This is a tangent to the conic, its coordinates are

$$\left\{-\frac{x_1}{x_1^2+y_1^2}, -\frac{y_1}{x_1^2+y_1^2}\right\}.$$

Hence, substituting these in the given equation of the conic, we have

$$c(x_1^2+y_1^2)-2gx_1-2fy_1+a=0,$$

or the equation of the auxiliary circle is

$$c(x^2+y^2)-2gx-2fy+a=0.$$

**Example iii.** *Show that the equation* $\Delta S + kD + k^2 = 0$, *where* $D = 0$ *is the equation of the director circle of the conic* $S = 0$, *represents a system of conics confocal with* $S = 0$. *Rectangular Coordinates.*

The tangential equation of the conic

$$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is
$$\Sigma = Al^2 + 2Hlm + Bm^2 + 2Gl + 2Fm + C = 0.$$

Hence the tangential equation of any conic confocal with this is

$$\Sigma + k(l^2 + m^2) = 0,$$

i.e.
$$(A + k)l^2 + 2Hlm + (B + k)m^2 + 2Gl + 2Fm + C = 0.$$

The coefficients of the corresponding point equation are

$$(B + k)C - F^2 = \Delta a + kC,$$
$$FG - CH = \Delta h,$$
$$(A + k)C - G^2 = \Delta b + kC,$$
$$FH - (B + k)G = \Delta g - kG,$$
$$GH - (A + k)F = \Delta f - kF,$$
$$(A + k)(B + k) - H^2 = \Delta c + k(A + B) + k^2,$$

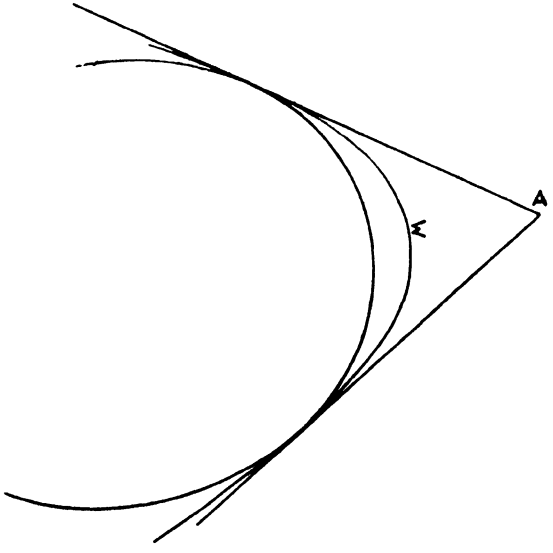
i.e. the equation is

$$\Delta(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) + k(Cx^2 + Cy^2 - 2Gx - 2Fy + A + B) + k^2 = 0,$$

or
$$\Delta S + kD + k^2 = 0.$$

### 4. $\Sigma = ku^2$.

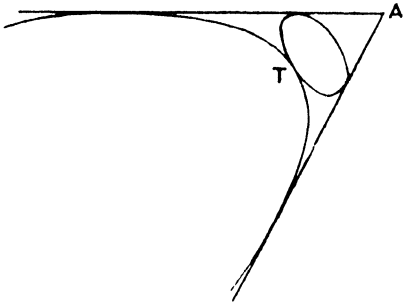
In this case $A$ and $B$ coincide; $\Sigma = 0$ and $\Sigma = ku^2$ have double contact, the pole of their common chord being $A$ $(u = 0)$.




**Example.** The equation $am^2 - l + k(al + 1)^2 = 0$ represents a system of parabolas confocal with $am^2 = l$.

5. $\Sigma = \mathbf{kut}$.

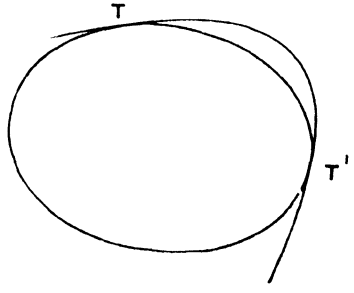
$T$ is a point on the curve $\Sigma = 0$; the two tangents from $T'$ to $\Sigma = 0$ coincide, hence $T'$ lies on both the curves, and they touch each other at $T$.




Thus $\Sigma = kut$ and $\Sigma = 0$ touch at $t = 0$, and their two other common tangents meet at $u = 0$.

6. $\Sigma = \mathbf{k}.\mathbf{tt}'$.

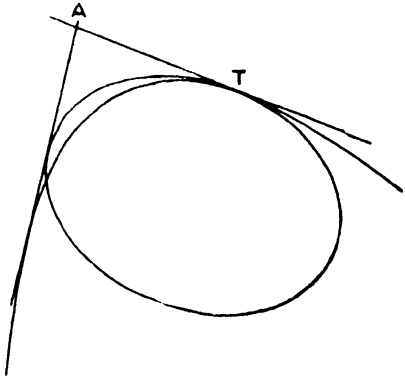
This curve has double contact with $\Sigma = 0$ at the points $t = 0$, $t' = 0$.




7. $\Sigma = \mathbf{kut}$ where $u = 0$ is a point on the tangent to $\Sigma = 0$ at $t = 0$.

In this case one tangent from $A$ coincides with the tangent at $T$; thus the curve $\Sigma = kut$ has three coincident tangents in common with $\Sigma = 0$ at $T$, and the fourth common tangent meets them at $A$.

Thus $\Sigma = 0$, $\Sigma = kut$ have contact of the second order at $t = 0$, $u = 0$ being the intersection of their common tangents.




8. $\Sigma = \mathbf{kt}^2$.

In this case the two points $A$ and $B$ become coincident at $T$ on the conic $\Sigma = 0$; hence, the conics $\Sigma = 0$, $\Sigma = kt^2$ have four coincident common tangents at the point $t = 0$, i.e. have contact of the third order.

9. $\Sigma = k\,.\,\Sigma'$.

If $\Sigma = 0$, $\Sigma' = 0$ are two conics, then $\Sigma = k\Sigma'$ represents a conic touching the common tangents of the conics $\Sigma = 0$, $\Sigma' = 0$.

**Example.** *Chords of a conic subtend a right angle at a fixed point: show that they envelope another conic, and if the first conic is circumscribed to a fixed quadrangle the second conic is inscribed in a fixed quadrilateral.*

Let the fixed point be the origin and the fixed quadrangle that formed by the common chords of

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$
$$S' \equiv a'x' + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0.$$

The first conic has then an equation of the form

$$S + kS' = 0.$$

The equation of the straight lines joining its intersections with the line $lx + my + 1 = 0$ to the origin is

$$ax^2 + 2hxy + by^2 - 2(gx + fy)(lx + my) + c(lx + my)^2$$
$$+ k[a'x^2 + 2h'xy + b'y - 2(g'x + f'y)(lx + my) + c'(lx + my)^2] = 0.$$

These are perpendicular if

$$c(l^2 + m^2) - 2gl - 2fm + a + b + k[c'(l^2 + m^2) - 2g'l - 2f'm + a' + b'] = 0,$$

which is therefore the tangential equation of the envelope of

$$lx + my + 1 = 0.$$

But this is the equation of a conic inscribed in the quadrilateral formed by the common tangents of the conics

$$c(l^2 + m^2) - 2gl - 2fm + a + b = 0,$$
$$c'(l^2 + m^2) - 2g'l - 2f'm + a' + b' = 0.$$

§ 11. When the solution of a problem in point coordinates has been obtained, the solution of the corresponding reciprocal problem in line coordinates follows by interchanging points and lines. The following examples worked out side by side illustrate this.

**Example i. Point Coordinates.**

*The vertices A, B, C of a triangle are joined to two points P, P', and these joins meet the opposite sides of the triangle in the points*

$$D, D'; \quad E, E'; \quad F, F'.$$

*Show that D, D'; E, E'; F, F' lie on a conic.*

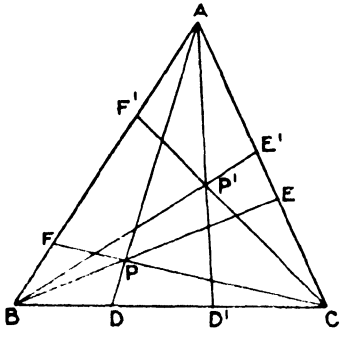
**Example i. Line Coordinates.**

*The sides a, b, c of a triangle are met by two lines p, p', and these intersections are joined to the opposite vertices by the lines*

$$d, d'; \quad e, e'; \quad f, f'.$$

*Show that d, d'; e, e'; f, f' touch a conic.*

Let the equations (in abridged notation) of the sides of the triangle be $u = 0$, $v = 0$, $w = 0$, and let the values of $u, v, w$, when the coordinates of $P$, $P'$ respectively are substituted in them, be $u_1, v_1, w_1$; $u_2, v_2, w_2$.




Then the equation of the join of $A$, $P$ is $\frac{v}{v_1} - \frac{w}{w_1} = 0$, and the equation of the join of $B$, $P$ is $\frac{u}{u_1} - \frac{w}{w_1} = 0$.

Since the equation

$$\frac{u}{u_1} + \frac{v}{v_1} - \frac{w}{w_1} = 0$$

can be written $\frac{u}{u_1} + \left(\frac{v}{v_1} - \frac{w}{w_1}\right) = 0$

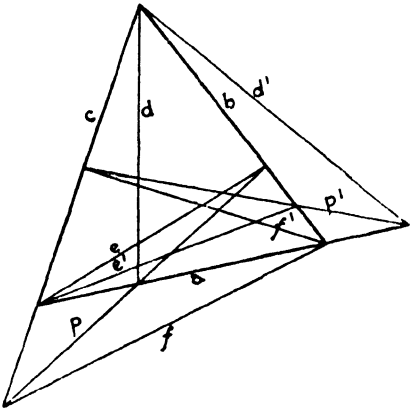
or $\frac{v}{v_1} + \left(\frac{u}{u_1} - \frac{w}{w_1}\right) = 0$, it represents a straight line through the points of intersection of the lines $AP$, $BC$ and $BP$, $AC$ respectively, i.e. it is the equation of the join of $D$ and $E$.

Similarly, the equation of the join of $D'$, $E'$ is

$$\frac{u}{u_2} + \frac{v}{v_2} - \frac{w}{w_2} = 0.$$

Hence the equation of any conic

Let the equations (in abridged notation) of the vertices of the triangle be $u = 0$, $v = 0$, $w = 0$, and let the values of $u$, $v$, $w$, when the coordinates of $p$, $p'$ respectively are substituted in them, be $u_1, v_1, w_1$; $u_2, v_2, w_2$.




Then the equation of the intersection of $a$ and $p$ is $\frac{v}{v_1} - \frac{w}{w_1} = 0$, and the equation of the intersection of $b$ and $p$ is $\frac{u}{u_1} - \frac{w}{w_1} = 0$.

Since the equation

$$\frac{u}{u_1} + \frac{v}{v_1} - \frac{w}{w_1} = 0$$

can be written $\frac{u}{u_1} + \left(\frac{v}{v_1} - \frac{w}{w_1}\right) = 0$

or $\frac{v}{v_1} + \left(\frac{u}{u_1} - \frac{w}{w_1}\right) = 0$, it represents a point on the join of the points $ap$, $bc$ and $bp$, $ac$ respectively, i.e. it is the equation of the point of intersection of $d$ and $e$.

Similarly, the equation of the point of intersection of $d'$, $e'$ is

$$\frac{u}{u_2} + \frac{v}{v_2} - \frac{w}{w_2} = 0.$$

Hence the equation of any conic

through the points $D$, $D'$; $E$, $E'$ is of the form

$$\lambda uv = \left(\frac{u}{u_1} + \frac{v}{v_1} - \frac{w}{w_1}\right)\left(\frac{u}{u_2} + \frac{v}{v_2} - \frac{w}{w_2}\right). \quad \text{(i)}$$

This passes through the intersection of the lines $CP$, $\frac{u}{u_1} - \frac{v}{v_1} = 0$, and $AB$, $w = 0$, i.e. the point $F$ if $\lambda$ has the value given by

$$\lambda u_1 v_1 = 2\left(\frac{u_1}{u_2} + \frac{v_1}{v_2}\right),$$

i.e. $$\lambda = \frac{2(u_1 v_2 + u_2 v_1)}{u_1 v_1 u_2 v_2}.$$

The symmetry of the result shows that this is also the value of $\lambda$ for which the conic (i) passes through $F'$.

Hence $D$, $D'$; $E$, $E'$; $F$, $F'$ lie on a conic.

**Example 2.** *The polars of a given point with respect to a series of conics circumscribed to the same quadrilateral are concurrent.*

Let $u_1 \equiv a_1 x + b_1 y + 1 = 0,$
$u_2 \equiv a_2 x + b_2 y + 1 = 0,$
$u_3 \equiv a_3 x + b_3 y + 1 = 0,$
$u_4 \equiv a_4 x + b_4 y + 1 = 0,$

be the equations of the sides of the quadrilateral.

The equation of the conic is

$$u_1 u_2 + k u_3 u_4 = 0.$$

The equation of the polar of the fixed point $(x', y')$ is therefore

$$u_1 u_2' + u_2 u_1' + k(u_3 u_4' + u_4 u_3') = 0,$$

where $u_1' \equiv a_1 x' + b_1 y' + 1$, &c., which for all values of $k$ passes through the point of intersection of the lines whose equations are

$$u_1 u_2' + u_2 u_1' = 0,$$

and $$u_3 u_4' + u_4 u_3' = 0,$$

i.e. a fixed point.

touching the lines $d$, $d'$; $e$, $e'$ is of the form

$$\lambda uv = \left(\frac{u}{u_1} + \frac{v}{v_1} - \frac{w}{w_1}\right)\left(\frac{u}{u_2} + \frac{v}{v_2} - \frac{w}{w_2}\right). \quad \text{(i)}$$

This touches the join of the points $cp$, $\frac{u}{u_1} - \frac{v}{v_1} = 0$, and $ab$, $w = 0$, i.e. the line $f$ if $\lambda$ has the value given by

$$\lambda u_1 v_1 = 2\left(\frac{u_1}{u_2} + \frac{v_1}{v_2}\right),$$

i.e. $$\lambda = \frac{2(u_1 v_2 + u_2 v_1)}{u_1 v_1 u_2 v_2}.$$

The symmetry of the result shows that this is also the value of $\lambda$ for which the conic (i) touches $f'$.

Hence $d$, $d'$; $e$, $e'$; $f$, $f'$ touch a conic.

**Example 2.** *The poles of a given straight line with respect to a series of conics inscribed in the same quadrilateral are collineur.*

Let $u_1 \equiv a_1 l + b_1 m + 1 = 0,$
$u_2 \equiv a_2 l + b_2 m + 1 = 0,$
$u_3 \equiv a_3 l + b_3 m + 1 = 0,$
$u_4 \equiv a_4 l + b_4 m + 1 = 0,$

be the equations of the vertices of the quadrilateral.

The equation of the conic is

$$u_1 u_2 + k u_3 u_4 = 0.$$

The equation of the pole of the fixed line $(l', m')$ is therefore

$$u_1 u_2' + u_2 u_1' + k(u_3 u_4' + u_4 u_3') = 0,$$

where $u_1' \equiv a_1 l' + b_1 m' + 1$, &c., which for all values of $k$ lies on the line joining the points whose equations are

$$u_1 u_2' + u_2 u_1' = 0,$$

and $$u_3 u_4' + u_4 u_3' = 0,$$

i.e. a fixed line.

**Example 3.** *Find the envelope of the polars of points on an ellipse with respect to a coaxal ellipse.*

Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{(i)}$$

and the coaxal ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1. \qquad \text{(ii)}$$

Any point on (i) has coordinates $(a\cos\theta, b\sin\theta)$, and the polar of this point with respect to (ii) is

$$\frac{xa}{A^2}\cos\theta + \frac{yb}{B^2}\sin\theta = 1,$$

or, writing $t \equiv \tan\frac{\theta}{2}$,

$$\frac{xa}{A^2}(1-t^2) + \frac{yb}{B^2}\cdot 2t = 1+t^2,$$

i.e.

$$t^2\left(1+\frac{xa}{A^2}\right) - \frac{2by}{B^2}t + 1 - \frac{ax}{A^2} = 0.$$

The envelope of this line is therefore

$$\left(1+\frac{ax}{A^2}\right)\left(1-\frac{ax}{A^2}\right) = \frac{b^2y^2}{B^4},$$

i.e.

$$\frac{a^2x^2}{A^4} + \frac{b^2y^2}{B^4} = 1,$$

which is another coaxal ellipse.

**Example 3.** *Find the locus of the poles of tangents to an ellipse with respect to a coaxal ellipse.*

Let the ellipse be

$$a^2l^2 + b^2m^2 = 1, \qquad \text{(i)}$$

and the coaxal ellipse

$$A^2l^2 + B^2m^2 = 1. \qquad \text{(ii)}$$

Any tangent to (i) has coordinates $\left(\frac{\cos\theta}{a}, \frac{\sin\theta}{b}\right)$, and the pole of this line with respect to (ii) is

$$\frac{A^2}{a}l\cos\theta + \frac{B^2}{b}m\sin\theta = 1,$$

or, writing $t \equiv \tan\frac{\theta}{2}$,

$$\frac{lA^2}{a}(1-t^2) + \frac{B^2m}{b}\cdot 2t = 1+t^2,$$

i.e.

$$t^2\left(1+\frac{lA^2}{a}\right) - \frac{2B^2m}{b}t + 1 - \frac{lA^2}{a} = 0.$$

The locus of this point is therefore

$$\left(1+\frac{lA^2}{a}\right)\left(1-\frac{lA^2}{a}\right) = \frac{B^4}{b^2}m^2,$$

or

$$\frac{A^4}{a^2}l^2 + \frac{B^4}{b^2}m^2 = 1,$$

which is another coaxal ellipse.

**§ 12.** We have shown in § 8 how the tangential equation of a conic can be obtained from its Cartesian equation, and conversely; in this section we give another proof of this property, and some immediate deductions. In order to obtain complete symmetry the equations are given in homogeneous coordinates.

I. *If the line* $lx+my+nz=0$ *touches the conic*

$$S = ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0,$$

*then* $Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0.$

If the line $lx+my+nz=0$ is the tangent at the point $(x', y', z')$, then its equation must be identical with that of the tangent

$$(ax'+hy'+gz')x+(hx'+by'+fz')y+(gx'+fy'+cz')z = 0.$$

Thus
$$l = ax' + hy' + gz',$$
$$m = hx' + by' + fz',$$
$$n = gx' + fy' + cz'.$$

Hence

$$Al + Hm + Gn = x'(aA + hH + gG) + y'(hA + bH + fG) + z'(gA + fH + cG),$$

but $aA + hH + gG = \Delta$, $hA + bH + fG = 0$, $gA + fH + cG = 0$;

therefore
$$Al + Hm + Gn = \Delta x'.$$

Similarly,
$$Hl + Bm + Fn = \Delta y',$$
$$Gl + Fm + Cn = \Delta z'.$$

Hence

$$(Al + Hm + Gn)\, l + (Hl + Bm + Fn)\, m + (Gl + Fm + Cn)\, n = \Delta(lx' + my' + nz').$$

But since $(x', y', z')$ is, by hypothesis, a point on the line $lx + my + nz = 0$, we have $lx' + my' + nz' = 0$; thus

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0.$$

Conversely,

II. *If the point* $lx + my + nz = 0$ *lies on the conic*

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0,$$

*then* $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$

If the point $lx + my + nz = 0$ is the point of contact of the tangent $(l', m', n')$, then its equation must be identical with that of the point of contact

$$(al' + hm' + gn')\, l + (hl' + bm' + fn')\, m + (gl' + fm' + cn')\, n = 0.$$

Thus
$$x = al' + hm' + gn',$$
$$y = hl' + bm' + fn',$$
$$z = gl' + fm' + cn'.$$

Hence

$$Ax + Hy + Gz = (aA + hH + gG)\, l' + (hA + bH + fG)\, m' + (gA + fH + cG)\, n';$$

therefore
$$Ax + Hy + Gz = \Delta l'.$$

Similarly,
$$Hx + By + Fz = \Delta m',$$
$$Gx + Fy + Cz = \Delta n'.$$

Hence

$$(Ax + Hy + Gz)\, x + (Hx + By + Fz)\, y + (Gx + Fy + Cz)\, z = \Delta(l'x + m'y + n'z).$$

But since $(l', m', n')$ is, by hypothesis, a line through the point $lx + my + nz = 0$, we have $l'x + m'y + n'z = 0$, thus

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$$

**Note i.** If $(l, m, n)$ are the line coordinates of a tangent to the conic $S = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, and $(x, y, z)$ the point coordinates of its point of contact, we have the reciprocal relations

$$\frac{l}{ax + hy + gz} = \frac{m}{hx + by + fz} = \frac{n}{gx + fy + cz},$$

and

$$\frac{x}{Al + Hm + Gn} = \frac{y}{Hl + Bm + Fn} = \frac{z}{Gl + Fm + Cn}.$$

If $(x, y, z)$ is a point on the conic

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0,$$

and $(l, m, n)$ the coordinates of the tangent at this point, similar relations hold, the small and capital letters being interchanged.

**Note ii.** The above formulae connecting the line coordinates of a tangent and the point coordinates of its point of contact were obtained from the equations of the tangent and the point of contact respectively.

When $(x, y, z)$ is not on the conic, or when $(l, m, n)$ does not touch the conic, the equation of the polar of $(x, y, z)$ or of the pole of $(l, m, n)$ are of exactly the same forms as those of the tangent and point of contact. Hence, if $(x, y, z)$ is a point not on the conic $S = 0$, and $(l, m, n)$ are the line coordinates of its polar, then

$$\frac{l}{ax + hy + gz} = \frac{m}{hx + by + fz} = \frac{n}{gx + fy + cz};$$

and reciprocally, if $(l, m, n)$ is a line not touching the conic $S = 0$, and $(x, y, z)$ are the coordinates of its pole, then

$$\frac{x}{Al + Hm + Gn} = \frac{y}{Hl + Bm + Fn} = \frac{z}{Gl + Fm + Cn}.$$

III. Any conic being

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

$S'$ being the value of $S$ when $x'$, $y'$ are substituted for $x$, $y$ and $P = 0$ being the equation of the polar of $(x', y')$, the equation of the pair of tangents from $(x', y')$ to the conic is (Chap. VI, § 4 (v))

$$SS' = P^2.$$

So also $\Sigma \equiv Al^2 + 2Hlm + Bm^2 + 2Gl + 2Fm + C = 0$ being the tangential equation of the same conic, $\Sigma'$ the value of $\Sigma$ when $(l', m')$ are substituted for $l$, $m$ and $\Pi = 0$ being the equation of the pole of $(l', m')$, the equation of the points of intersection of $(l', m')$ and the conic is (Chap. X, p. 404)

$$\Sigma\Sigma' = \Pi^2.$$

These can be put in the same form.

Making the equations homogeneous, for convenience of notation, we have

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy,$$

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gln + 2Hlm.$$

Now the equation of the polar of $(x', y', z')$ with respect to $S = 0$ is

$$x(ax'+hy'+gz')+y(hx'+by'+fz')+z(gx'+fy'+cz') = 0,$$

i.e. the coordinates $(l, m, n)$ of this polar are

$$l = ax'+hy'+gz', \quad m = hx'+by'+fz', \quad n = gx'+fy'+cz';$$

thus $$S' = lx'+my'+nz'.$$

Also

$$Al+Hm+Gn$$
$$= x'(aA+hH+gG)+y'(hA+bH+fG)+z'(gA+fH+cG) = \Delta x'.$$

So
$$Hl+Bm+Fn = \Delta y',$$
$$Gl+Fm+Cn = \Delta z';$$

$$\therefore \quad l(Al+Hm+Gn)+m(Hl+Bm+Fn)+n(Gl+Fm+Cn)$$
$$= \Delta(lx'+my'+nz'),$$

i.e. $$\Sigma = \Delta S'.$$

Also the polar of $(x', y', z')$ being $lx+my+nz = 0$, we have $P \equiv lx+my+nz$; hence the equation of the pairs of tangents from $(x', y', z')$ to $S = 0$, viz. $SS' = P^2$, can be written $S.\Sigma = \Delta(lx+my+nz)^2$.

Similarly, the equation of the pole of $(l', m', n')$ with respect to $\Sigma = 0$ is

$$l(Al'+Hm'+Gn')+m(Hl'+Bm'+Fn')+n(Gl'+Fm'+Cn') = 0,$$

i.e. its coordinates $(x, y, z)$ are

$$x = Al'+Hm'+Gn', \quad y = Hl'+Bm'+Fn', \quad z = Gl'+Fm'+Cn'.$$

Thus $$\Sigma' = xl'+ym'+zn'.$$

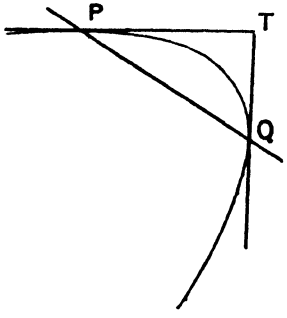
Also
$$ax+hy+gz = \Delta l',$$
$$hx+by+fz = \Delta m',$$
$$gx+fy+cz = \Delta n';$$

$$\therefore \quad x(ax+hy+gz)+y(hx+by+fz)+z(gx+fy+cz)$$
$$= \Delta(l'x+m'y+n'z);$$

$$\therefore \quad S = \Delta.\Sigma'.$$

Also the pole of $(l', m', n')$ is the point $lx+my+nz = 0$, i.e.

$$\Pi \equiv lx+my+nz;$$




hence the equation of the points of intersection of the line $(l', m', n')$ and the conic $\Sigma = 0$ being $\Sigma\Sigma' = \Pi^2$ becomes

$$\Sigma.S = \Delta(lx+my+nz)^2,$$

which is the same form as the equation of a pair of tangents.

Thus, if $l, m, n$ are fixed, viz. the coordinates of $PQ$,

$$\Sigma.S = \Delta(lx+my+nz)^2$$

is the equation of the tangents at the point of intersection of the

conic and $PQ$; and if $(x, y, z)$ are fixed, viz. the coordinates of $T$, it is the tangential equation of the points of contact of tangents from $T$ to the conic.

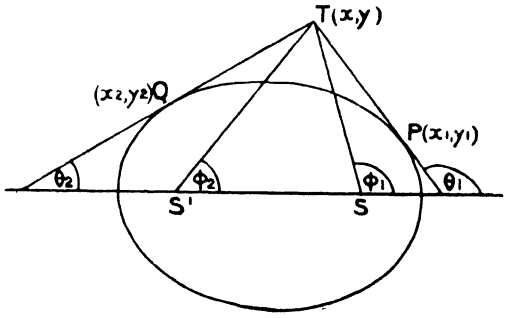
**Illustrative Example.**

For an ellipse

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$

$$\Sigma \equiv -\frac{l^2}{b^2} - \frac{m^2}{a^2} + \frac{1}{a^2 b^2},$$

$$\Delta = -\frac{1}{a^2 b^2}.$$




Let $T$ be the point $(x, y)$; then the equation of the points $P$, $Q$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(-\frac{l^2}{b^2} - \frac{m^2}{a^2} + \frac{1}{a^2 b^2}\right) = -\frac{1}{a^2 b^2}(lx + my + 1)^2,$$

or

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)(a^2 l^2 + b^2 m^2 - 1) - (lx + my + 1)^2 = 0. \qquad \text{(i)}$$

Let the coordinates of $P$ and $Q$ be $(x_1, y_1)$, $(x_2, y_2)$; then their equations are

$$(lx_1 + my_1 + 1)(lx_2 + my_2 + 1) = 0. \qquad \text{(ii)}$$

The equations (i) and (ii) are therefore identical, and we have, putting in a factor to make the absolute terms the same,

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)(a^2 l^2 + b^2 m^2 - 1) - (lx + my + 1)^2$$
$$\equiv -\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)(lx_1 + my_1 + 1)(lx_2 + my_2 + 1),$$

or

$$S(a^2 l^2 + b^2 m^2 - 1) - (lx + my + 1)^2 \equiv -(S + 1)(lx_1 + my_1 + 1)(lx_2 + my_2 + 1),$$

and this is identically true for all values of $l$ and $m$.

(a) Now put $l = -\dfrac{1}{x + \iota y}$, $m = -\dfrac{\iota}{x + \iota y}$; then

$$S\{a^2 - b^2 - (x + \iota y)^2\} \equiv -(S + 1)\{\overline{x - x_1} + \iota\overline{y - y_1}\}\{\overline{x - x_2} + \iota\overline{y - y_2}\},$$

i.e. $$(x+\iota y)^2-a^2e^2=\frac{S+1}{S}\{\overline{x-x_1}+\iota.\overline{y-y_1}\}\{\overline{x-x_2}+\iota.\overline{y-y_2}\},$$

i.e. $$(x+\iota y-ae)(x+\iota y+ae)=\frac{S+1}{S}\{\overline{x-x_1}+\iota.\overline{y-y_1}\}\{\overline{x-x_2}+\iota.\overline{y-y_2}\}.$$

Now let $TP$, $TQ$, $TS$, $TS'$ make angles $\theta_1$, $\theta_2$, $\phi_1$, $\phi_2$ with the axis of $x$; then

$$\begin{aligned} x-x_1&=TP\cos\theta_1, & y-y_1&=TP\sin\theta_1,\\ x-x_2&=TQ\cos\theta_2, & y-y_2&=TQ\sin\theta_2,\\ x-ae&=ST\cos\phi_1, & y&=ST\sin\phi_1,\\ x+ae&=S'T\cos\phi_2, & y&=S'T\sin\phi_2. \end{aligned}$$

Substituting and using De Moivre's Theorem,

$$ST.S'T\{\cos\overline{\phi_1+\phi_2}+\iota.\sin\overline{\phi_1+\phi_2}\}=\frac{S+1}{S}.TP.TQ\{\cos\overline{\theta_1+\theta_2}+\iota.\sin\overline{\theta_1+\theta_2}\}.$$

Equating real and imaginary parts we have $\phi_1+\phi_2=\theta_1+\theta_2$, which shows that $\angle STP=\angle S'TQ$, and further

$$\frac{TP.TQ}{ST.S'T}=\frac{S}{S+1}.$$

(b) Put $l=-\frac{1}{ae}$, $m=-\frac{\iota}{ae}$; then

$$(x-ae+\iota y)^2=(S+1)(x_1-ae+\iota y_1)(x_2-ae+\iota y_2).$$

So that if $SP$, $SQ$ make angles $\psi_1$, $\psi_2$ with the axis of $x$,

$$ST^2(\cos 2\phi_1+\iota\sin 2\phi_1)=(S+1).SP.SQ\{\cos\overline{\psi_1+\psi_2}+\iota\sin\overline{\psi_1+\psi_2}\};$$

$\therefore$ $$\psi_1+\psi_2=2\phi_1,$$

i.e. $$\angle TSP=\angle TSQ,$$

and $$\frac{ST^2}{SP.SQ}=S+1.$$

(c) Put $l=-\frac{1}{x_1+\iota y_1}$, $m=-\frac{\iota}{x_1+\iota y_1}$, and we get

$$S(\overline{x_1+\iota y_1}^2-a^2e^2)+(\overline{x-x_1}+\iota\overline{y-y_1})^2=0,$$

whence, as above, $SP$, $S'P$ are equally inclined to $TP$, and

$$\frac{TP^2}{SP.S'P}=S.$$

Similarly, $$\frac{TQ^2}{SQ.S'Q}=S.$$

## Examples X d.

1. Find the envelope of chords of an ellipse which subtend a right angle at the centre.

2. Find the envelope of chords of a parabola which subtend an angle $\alpha$ at the vertex. What does the envelope become when $\alpha=90°$?

3. Find the condition that the point $pl+qm+1=0$ should lie on the parabola $am^2+2l=0$.

4. Two circles touch externally, the centres are fixed. Find the envelope of their common tangents.

5. Three points $L$, $M$, $N$ on the sides of a triangle are collinear and $LM : MN = h : k$. Show that $LN$ touches a fixed parabola.

6. Find the envelope of the polars of a fixed point with respect to a series of circles which touch two given straight lines.

7. A variable tangent meets a fixed tangent of a parabola and through their point of intersection a straight line is drawn perpendicular to the variable tangent: find its envelope.

8. Find the equation of a parabola whose focus is $al + bm + 1 = 0$, the coordinates of the tangent at the vertex being $(p, q)$.

Also find the coordinates of its directrix.

9. Circles are drawn with given centres $A$ and $B$: show that the common tangents of pairs of these circles, the difference of whose radii is constant, touch a fixed parabola.

10. Find the coordinates of the normal through the point of contact of the tangent $(t^2/a, t/a)$ of the parabola $am^2 = l$.

Hence show that three normals can be drawn through a given point $pl + qm + 1 = 0$, and the parameters of the corresponding tangents are such that $\Sigma(1/t) = 0$. Interpret this geometrically.

11. Find the envelope of a line on which two fixed circles intercept equal lengths.

12. Two finite lines are each divided into $n$ equal parts: find the envelope of the lines joining corresponding points of division.

13. Find the envelope of the polar of any point on the circle

$$3a^2l^2 + 4a^2m^2 - 2al - 1 = 0$$

with respect to the circle $3a^2l^2 + 4a^2m^2 + 2al - 1 = 0$.

14. A system of conics have their foci at the points $al + 1 = 0$, $al - 1 = 0$.

Find the envelope of the tangents to them at their points of intersection with the straight line $(\lambda, \mu)$.

15. Two points are taken on an ellipse so that the sum of their ordinates is constant: show that their join envelopes a parabola.

16. $X$, $Y$ are fixed points on two lines $OX$, $OY$, and points $P$, $Q$ are taken on the lines such that

(i) $XP \cdot YQ = c^2$; (ii) $XP + YQ = c$; (iii) $XP - YQ = c$;

find in each case the envelope of $PQ$.

17. Circles are drawn with their centres on the tangent at the vertex of a parabola and touching the parabola. Show that the envelope of the common chords of the circles and the parabola is another parabola, and find its focus.

18. The envelope of the polars of points on the auxiliary circle of an ellipse with respect to the ellipse is another ellipse.

Find its equation and its foci.

19. The bisector of the angle between the tangents from a point to an ellipse $a^2l^2 + b^2m^2 = 1$ is parallel to the line $(l_1, m_1)$: find the envelope of the chord of contact and the coordinates of its asymptotes.

20. A variable circle of radius $r$ passes through a fixed point $(h, 0)$: show that the envelope of the common chord of this circle and the circle $x^2+y^2=r^2$ is a conic. Find the foci of the envelope.

21. $PR$ is a diameter of a circle, $QR$ passes through a fixed point: show that $PQ$ envelopes an ellipse or an hyperbola according as the fixed point lies inside or outside the circle.

22. A chord of a parabola which subtends a right angle at the focus touches a conic one of whose foci is the focus of the parabola: prove this and find the other focus.

23. Find the condition that the lines $(l_1, m_1, 0)$, $(l_2, m_2, 0)$ should be conjugate diameters of the conic $al^2+bm^2=n^2$.

24. The four lines $(l_1, m_1)$, $(l_2, m_2)$, $(l_3, m_3)$, $(l_4, m_4)$ form a quadrilateral; find the coordinates of its three diagonals.

25. (i) Three conics are drawn circumscribing the quadrilaterals formed by two sides of a triangle and two given straight lines. Show that their other common chords pass each through a vertex of the triangle and are concurrent.

(ii) Three conics are drawn inscribed in the quadrangles formed by two of the vertices of a triangle and two given points. Show that the points of intersection of their other common tangents lie each on a side of the triangle and are collinear.

26. (i) If a triangle circumscribes the ellipse $x^2/a^2+y^2/b^2=1$ and two vertices lie on $x^2/A^2+y^2/B^2=1$, the locus of the third vertex is a coaxal ellipse.

(ii) If a triangle is inscribed in the ellipse $a^2l^2+b^2m^2=1$ and two sides touch $A^2l^2+B^2m^2=1$, the envelope of the third side is a coaxal ellipse.

27. (i) Two sides of a triangle inscribed in an ellipse pass through fixed points: find the envelope of the third side.

(ii) Two vertices of a triangle circumscribed to an ellipse lie on fixed lines: find the locus of the third vertex.

28. Find the general tangential equation of a rectangular hyperbola which has contact of the third order with the parabola $am^2=l$, and the locus of its centre.

29. Find the envelope of the polar of a fixed point with respect to a conic circumscribed to a given quadrilateral.

30. A system of conics is inscribed in a given quadrilateral: show that their director circles form a coaxal system.

31. A conic has double contact with a parabola at the ends of the latus rectum: show that the envelope of the polars of points on this conic with respect to the parabola is another conic having double contact with the parabola.

32. The sum of the abscissae of two points on a parabola is equal to the latus rectum. Show that the chord joining them envelopes a parabola and find its focus.

33. The normals at the extremities of two chords of an ellipse are concurrent: prove that

(a) If one chord passes through a fixed point the other envelopes a parabola;

(b) If one chord touches an hyperbola whose asymptotes are the axes of the ellipse, the other chord envelopes a rectangular hyperbola.

34. If $\Sigma = 0$ is the tangential equation of the conic $S = 0$, show that the equation $\Sigma + k(l^2 + m^2) = 0$ represents a pair of foci of the conic if

$$Ck^2 + \Delta(a+b)k + \Delta^2 = 0.$$

35. The sum of the squares of the semi-axes of the conic

$$al^2 + 2hlm + bm^2 + 2fm + 2gl + c = 0$$

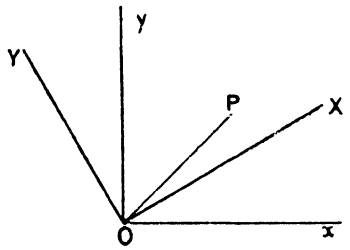
is $-(A+B)/c^2$.

# CHAPTER XI

## MISCELLANEOUS THEOREMS

**§ 1. Transformation of Coordinates. Application of De Moivre's Theorem.**

I. *To change from one set of rectangular axes to another, the origin being unchanged.*




Let the new axis of $x$ make an angle $\alpha$ with the old axis.

Let $P$ be any point whose coordinates referred to the two sets of axes are $(x, y)$, $(X, Y)$, and let $OP = r$, $\angle POx = \theta$, $\angle POX = \theta'$, i.e. $\theta' = \theta - \alpha$.

Thus
$$x + iy = r(\cos\theta + i\sin\theta),$$
$$X + iY = r(\cos\overline{\theta-\alpha} + i\sin\overline{\theta-\alpha})$$
$$= r(\cos\theta + i\sin\theta) \div (\cos\alpha + i\sin\alpha)$$
$$= (x+iy) \div (\cos\alpha + i\sin\alpha).$$

Hence
$$x + iy = (X + iY)(\cos\alpha + i\sin\alpha),$$

and
$$X + iY = (x+iy)(\cos\alpha - i\sin\alpha).$$

Equating real and imaginary parts,

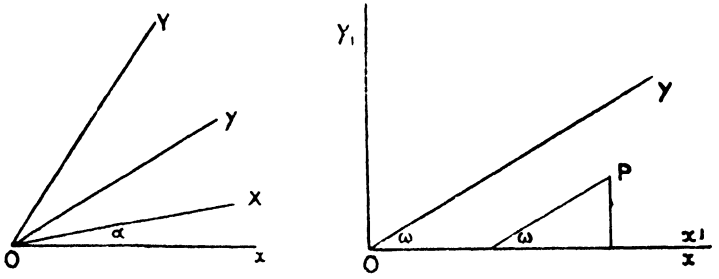
$$x = X\cos\alpha - Y\sin\alpha, \quad y = X\sin\alpha + Y\cos\alpha,$$

and
$$X = x\cos\alpha + y\sin\alpha, \quad Y = y\cos\alpha - x\sin\alpha.$$

II. *To change from one set of axes inclined at an angle $\omega$ to another set inclined at an angle $\omega'$, the origin being unchanged.*

Let the new axis of $x$ make an angle $\alpha$ with the old axis, and let

$P$ be any point whose coordinates are $(x, y)$, $(X, Y)$ referred to the two sets of axes.




(a) Let the coordinates of $P$ referred to rectangular axes $Ox_1$, $Oy_1$, where $Ox$, $Ox_1$ coincide, be $(x_1, y_1)$.

Then $$x_1 = x + y\cos\omega,$$
$$y_1 = y\sin\omega.$$

Thus $x_1 + iy_1 = x + y(\cos\omega + i\sin\omega)$.

(b) Again, suppose the coordinates of $P$ referred to rectangular axes $Ox_2$, $Oy_2$, where $Ox_2$ coincides with $OX$, be $(x_2, y_2)$, then in the same manner

$$x_2 + iy_2 = X + Y(\cos\omega' + i\sin\omega').$$

Now $(x_1, y_1)$, $(x_2, y_2)$ are the coordinates of $P$ referred to two sets of rectangular axes, the $x$-axes making an angle $\alpha$ with each other.

Hence, by the first part of this section,

$$x_1 + iy_1 = (x_2 + iy_2)(\cos\alpha + i\sin\alpha).$$

Hence

$$x + y(\cos\omega + i\sin\omega) = (\cos\alpha + i\sin\alpha)[X + Y(\cos\omega' + i\sin\omega')]. \quad \text{(i)}$$

This, multiplying by $(\cos\omega - i\sin\omega)$, becomes

$$x(\cos\omega - i\sin\omega) + y$$
$$= X(\cos\overline{\alpha-\omega} + i\sin\overline{\alpha-\omega}) + Y(\cos\overline{\omega'+\alpha-\omega} + i\sin\overline{\omega'+\alpha-\omega}). \quad \text{(ii)}$$

Equating imaginary parts in (i) and (ii)

$$y\sin\omega = X\sin\alpha + Y\sin(\alpha+\omega'),$$
$$x\sin\omega = X\sin\overline{\omega-\alpha} + Y\sin(\omega-\alpha-\omega').$$

The values of $X$ and $Y$ in terms of $x$ and $y$ can be found by writing the equations (i) and (ii) in the forms

$$x(\cos\alpha - i\sin\alpha) + y(\cos\overline{\omega-\alpha} + i\sin\overline{\omega-\alpha}) = X + Y(\cos\omega' + i\sin\omega'),$$
$$x(\cos\overline{\alpha+\omega'} - i\sin\overline{\alpha+\omega'}) + y(\cos\overline{\omega-\alpha-\omega'} + i\sin\overline{\omega-\alpha-\omega'})$$
$$= Y + X(\cos\omega' - i\sin\omega'),$$

whence $$Y\sin\omega' = y\sin\overline{\omega-\alpha} - x\sin\alpha,$$
$$X\sin\omega' = x\sin\overline{\alpha+\omega'} + y\sin\overline{\alpha+\omega'-\omega}.$$

**Invariants.** We have seen in Chap. III, § 6, that if by any change of axes the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is transformed into

$$a'X^2+2h'XY+b'Y^2+2g'X+2f'Y+c'=0,$$

(i) a change of origin **only** does not affect the coefficients of the highest terms, viz. $a$, $h$, $b$;

(ii) a change in the direction of the axes does not affect the expressions $\dfrac{ab-h^2}{\sin^2\omega}$, $\dfrac{a+b-2h\cos\omega}{\sin^2\omega}$; and these expressions are called invariants.

We propose here to add another important invariant, and then to illustrate their use.

If by any change of axes $ax^2+2hxy+by^2+2gx+2fy+c$ becomes

$$a'X^2+2h'XY+b'Y^2+2g'X+2f'Y+c',$$

then

$$\frac{abc+2fgh-af^2-bg^2-ch^2}{\sin^2\omega}=\frac{a'b'c'+2f'g'h'-af'^2-bg'^2-ch'^2}{\sin^2\omega'};$$

in other words, $\dfrac{\Delta}{\sin^2\omega}$ is an invariant.

Make both the expressions homogeneous by introducing $z$ and $Z$, where in the present case $z$ and $Z$ will both be unity: the expressions are $ax^2+2hxy+by^2+2gxz+2fgz+cz^2$ and

$$a'X^2+2h'XY+b'Y^2+2g'XZ+2f'YZ+c'Z^2.$$

We have relations between $xyz$, $XYZ$ of the form

$$x=l_1X+m_1Y+n_1Z,$$
$$y=l_2X+m_2Y+n_2Z,$$
$$z=Z.$$

This is a particular case of the quite general transformation

$$x=l_1X+m_1Y+n_1Z,$$
$$y=l_2X+m_2Y+n_2Z,$$
$$z=l_3X+m_3Y+n_3Z,$$

which for the sake of symmetry we will first consider.

Make these substitutions in the first expression; then

$$a'=al_1^2+bl_2^2+cl_3^2+2fl_2l_3+2gl_1l_3+2hl_1l_2,$$

or

$$a'=l_1(al_1+hl_2+gl_3)+l_2(hl_1+bl_2+fl_3)+l_3(gl_1+fl_2+cl_3).$$

Similarly,

$$b' = m_1(am_1+hm_2+gm_3)+m_2(hm_1+bm_2+fm_3)+m_3(gm_1+fm_2+cm_3),$$
$$c' = n_1(an_1+hn_2+gn_3)+n_2(hn_1+bn_2+fn_3)+n_3(gn_1+fn_2+cn_3),$$
$$f' = m_1(an_1+hn_2+gn_3)+m_2(hn_1+bn_2+fn_3)+m_3(gn_1+fn_2+cn_3)$$
$$= n_1(am_1+hm_2+gm_3)+n_2(hm_1+bm_2+fm_3)+n_3(gm_1+fm_2+cm_3),$$
$$g' = n_1(al_1+hl_2+gl_3)+n_2(hl_1+bl_2+fl_3)+n_3(gl_1+fl_2+cl_3)$$
$$= l_1(an_1+hn_2+gn_3)+l_2(hn_1+bn_2+fn_3)+l_3(gn_1+fn_2+cn_3),$$
$$h' = l_1(am_1+hm_2+gm_3)+l_2(hm_1+bm_2+fm_3)+l_3(gm_1+fm_2+cm_3)$$
$$= m_1(al_1+hl_2+gl_3)+m_2(hl_1+bl_2+fm_3)+m_3(gl_1+fl_2+cl_3).$$

Hence, by the ordinary formulae for the multiplication of determinants,

$$\begin{vmatrix} a' & b' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = \begin{vmatrix} al_1+hl_2+gl_3 & hl_1+bl_2+fl_3 & gl_1+fl_2+cl_3 \\ am_1+hm_2+gm_3 & hm_1+bm_2+fm_3 & gm_1+fm_2+cm_3 \\ an_1+hn_2+gn_3 & hn_1+bn_2+fn_3 & gn_1+fn_2+cn_3 \end{vmatrix}$$
$$\times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$
$$= \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}^2 \times \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

In the proposed case, when we are transforming Cartesian coordinates, we have $l_3 = 0$, $m_3 = 0$, $n_3 = 1$, hence

$$\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times (l_1m_2-l_2m_1)^2.$$

Now the factor $(l_1m_2-l_2m_1)^2$ is independent of the change of origin, this only affecting $n_1$, $n_2$; to find its value then, consider the particular case of the transformation of $ax^2+by^2+2hxy+c$ to

$$a'X^2+2h'XY+b'Y^2+c'$$

without change of origin, so that $c = c'$.

Then from the above

$$\begin{vmatrix} a' & h' & 0 \\ h' & b' & 0 \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & h & 0 \\ h & b & 0 \\ 0 & 0 & c \end{vmatrix} \times (l_1m_2-l_2m_1)^2,$$

i.e. $$(a'b'-h'^2) = (ab-h^2) \times (l_1m_2-l_2m_1)^2;$$

$$\therefore \quad (l_1m_2-l_2m_1)^2 = \frac{a'b'-h'^2}{ab-h^2} = \frac{\sin^2\omega'}{\sin^2\omega}$$

by the former invariants.

Hence, finally, $$\frac{\Delta'}{\sin^2\omega'} = \frac{\Delta}{\sin^2\omega}.$$

**Examples.**

(a) *To deduce formulae for oblique axes from corresponding formulae for rectangular axes.*

I. *To find the angle between the lines* $ax^2+2hxy+by^2=0$ *when the axes are inclined at an angle* $\omega$.

Suppose that when the axes are transformed to rectangular axes the expression $ax^2+2hxy+by^2$ becomes $a'X^2+2h'XY+b'Y^2$; then

$$a'+b'=\frac{a+b-2h\cos\omega}{\sin^2\omega},$$

and

$$a'b'-h'^2=\frac{ab-h^2}{\sin^2\omega}.$$

But the angle between the lines is

$$\tan^{-1}\frac{2\sqrt{h'^2-a'b'}}{a'+b'}$$

$$=\tan^{-1}\frac{2\sin\omega\sqrt{h^2-ab}}{a+b-2h\cos\omega}.$$

Cor. The lines $lx+my+n=0$, $l'x+m'y+n'=0$ are parallel to

$$ll'x^2+(lm'+l'm)xy+mm'y^2=0,$$

and hence include an angle

$$\tan^{-1}\frac{(lm'-l'm)\sin\omega}{ll'+mm'-(lm'+l'm)\cos\omega}.$$

II. *To find the length of the perpendicular from* $(x', y')$ *on the line* $lx+my+n=0$.

Suppose by change of axes to rectangular axes (without changing the origin) the expression $lx+my+n$ becomes $l'X+m'Y+n$; then the expression $lx'+my'+n$ becomes $l'X'+m'Y'+n$.

Now the perpendicular from $(X', Y')$ on $l'X+m'Y+n=0$ is

$$\frac{l'X'+m'Y'+n}{\sqrt{l'^2+m'^2}}.$$

But since $lx+my$ becomes $l'X+m'Y$, $(lx+my)^2$ becomes $(l'X+m'Y)^2$, i.e. $l^2x^2+2lmxy+m^2y^2$ becomes $l'^2X^2+2l'm'XY+m'^2Y^2$; hence

$$l'^2+m'^2=\frac{l^2+m^2-2lm\cos\omega}{\sin^2\omega},$$

i.e. the required length of the perpendicular is

$$\frac{(lx'+my'+n)\sin\omega}{\sqrt{l^2+m^2-2lm\cos\omega}}.$$

(b) *To find the axes of the conic* $ax^2+2hxy+by^2+2gx+2fy+c=0$ *and the latus rectum when it is a parabola.*

If the conic is a central conic, when the coordinate axes are transformed to the axes of the conic the equation takes the form

$$\frac{X^2}{\alpha^2}+\frac{Y^2}{\beta^2}-1=0,$$

i.e. the process of transformation reduces the given equation to one of the form $\lambda(\beta^2X^2+\alpha^2Y^2-\alpha^2\beta^2)=0$.

Note that since we choose $\alpha$, $\beta$ to be the actual lengths of the semi-axes of the conic, we must introduce the arbitrary constant $\lambda$, for in the above work we have assumed that the resultant equation is obtained from the given one by substituting linear relations for $x$ and $y$ with no subsequent division or multiplication by any factor.

The invariants now give us

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\lambda(\alpha^2+\beta^2),$$

$$\frac{C}{\sin^2\omega}=\lambda^2\alpha^2\beta^2,$$

$$\frac{\Delta}{\sin^2\omega}=-\lambda^3\alpha^4\beta^4.$$

Thus
$$\lambda=\frac{-C^2}{\Delta\sin^2\omega},$$

and
$$\alpha^2+\beta^2=-\frac{(a+b-2h\cos\omega)}{C^2}.\Delta,$$

$$\alpha^2\beta^2=\frac{\Delta^2\sin^2\omega}{C^3}.$$

Hence $\alpha^2$, $\beta^2$ are roots of the equation

$$^2+\frac{r\Delta}{C^2}(a+b-2h\cos\omega)+\frac{\Delta^2\sin^2\omega}{C^3}=0.$$

When the conic is a parabola the equation, when the axes of coordinates are transferred to the axis of the parabola and the tangent at the vertex, takes the form $\lambda(Y^2-4lX)=0$.

Hence
$$\frac{\Delta}{\sin^2\omega}=-4l^2\lambda^3,$$

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\lambda,$$

hence
$$4l^2=-\frac{\Delta.\sin^4\omega}{(a+b-2h\cos\omega)^3},$$

the latus rectum being $4l$.

**§ 2. Similar and similarly situated conics.**

**Definition i.** If the radii drawn from two points respectively, one to each of two conics, in directions which contain a constant angle, are in a constant ratio, the conics are similar.

**Definition ii.** If the radii in the above definition are drawn in the same direction and their ratio is constant, the conics are similar and similarly situated.

Suppose that a straight line is drawn from a point $O(\alpha, \beta)$ to a conic, meeting it in the points $P$, $Q$; and also one from the point $O'(\alpha', \beta')$ in the same direction to meet another conic in the points $P'$, $Q'$.

If these two conics are similar and similarly situated, the ratios $\frac{OP}{OP'}$ and $\frac{OQ}{OQ'}$ are constant, hence also the ratio $\frac{OP \cdot OQ}{OP' \cdot OQ'}$ is constant.

As in Chap. VI, § 5 (i), we can show that if the conics are

$$f(x, y) = ax^2 + 2hxy + by^2 + \&c.,$$

and

$$f'(x, y) = a'x^2 + 2h'xy + b'y^2 + \&c.,$$

then $OP \cdot OQ = f(\alpha, \beta) \div (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$,

$O'P' \cdot O'Q' = f'(\alpha', \beta') \div a'\cos^2\theta + 2h'\sin\theta\cos\theta + b'\sin^2\theta)$.

The condition above requires that the ratio

$$\frac{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}{a'\cos^2\theta + 2h'\sin\theta\cos\theta + b'\sin^2\theta}$$

should be independent of $\theta$.

Hence $\frac{a}{a'} = \frac{h}{h'} = \frac{b}{b'}$.

This is the required condition that the conics should be similar and similarly situated; it is also the condition that their asymptotes should be parallel.

Again, if the straight lines through $O$ and $O'$ are drawn in the directions $\theta$ and $\phi$ respectively, the ratio of the rectangles $OP \cdot OQ$ and $O'P' \cdot O'Q'$ becomes

$$\frac{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}{a'\cos^2\phi + 2h'\cos\phi\sin\phi + b'\sin^2\phi}.$$

If the conics are similar, this ratio must be constant when $\theta \backsim \phi$ is constant.

If $\theta - \phi = \gamma$, the ratio becomes

$$\frac{a\cos^2(\phi+\gamma) + 2h\sin(\phi+\gamma)\cos(\phi+\gamma) + b\sin^2(\phi+\gamma)}{a'\cos^2\phi + 2h'\sin\phi\cos\phi + b'\sin^2\phi},$$

and must be independent of $\phi$.

The ratio may be written

$$\frac{a+b+(a-b)\cos 2(\phi+\gamma)+2h\sin 2(\phi+\gamma)}{a'+b'+(a'-b')\cos 2\phi+2h'\sin 2\phi}.$$

Equate the ratios of the coefficients of $\cos 2\phi$, $\sin 2\phi$, and the terms independent of $\phi$, then

$$\frac{a+b}{a'+b'}=\frac{(a-b)\cos 2\gamma+2h\sin 2\gamma}{a'-b'}=\frac{2h\cos 2\gamma-(a-b)\sin 2\gamma}{2h'}.$$

Hence, eliminating $\gamma$,

$$\frac{(a+b)^2}{(a'+b')^2}=\frac{(a+b)^2-(a-b)^2-4h^2}{(a'+b')^2-(a'-b')^2-4h'^2}=\frac{ab-h^2}{a'b'-h'^2}.$$

This is the condition that the conics should be similar, and is also the condition that the angle between the asymptotes of each conic should be the same.

### § 3. The general equation of the second degree. Curvature.

When the origin is on the curve, $c=0$, and the general equation becomes $ax^2+2hxy+by^2+2gx+2fy=0$. The equation of the tangent at the origin (0, 0) is $gx+fy=0$, i.e. is represented by the terms of the lowest degree.

**Cor. i.** If any tangent and the corresponding normal are taken as coordinate axes, the equation of the curve takes the form $ax^2+2hxy+by^2=2fy$; this form is often useful.

**Cor. ii.** The equation of the circle of curvature of

$$ax^2+2hxy+by^2+2gx+2fy=0$$

at the origin is of the form

$$\lambda(x^2+2xy\cos\omega+y^2)+2gx+2fy=0,$$

and $\lambda$ must be chosen so that one of the common chords of the circle and conic, which passes through the origin, should be $gx+fy=0$.

Hence $ax^2+2hxy+by^2-\lambda(x^2+2xy\cos\omega+y^2)$ has $gx+fy$ for a factor, and therefore vanishes when $gx+fy=0$ or when $\frac{x}{-f}=\frac{y}{g}$; thus $\lambda$ is the value of

$$\frac{ax^2+2hxy+by^2}{x^2+y^2+2xy\cos\omega},$$

when $gx+fy=0$, and unless $g$ or $f$ is zero this

$$=\frac{af^2-2fgh+bg^2}{f^2-2fg\cos\omega+g^2},$$

and the equation of the circle of curvature is

$$(x^2+2xy\cos\omega+y^2)\frac{(af^2-2fgh+bg^2)}{f^2-2fg\cos\omega+g^2}+2gx+2fy=0.$$

This method can be used to find the circle of curvature at any point $(x', y')$ of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Change the origin to the point $(x', y')$, then the equation becomes

$$ax^2+2hxy+by^2+2X'x+2Y'y=0,$$

where, as usual,

$$X'\equiv ax'+hy'+g, \quad Y'\equiv hx'+by'+f.$$

The equation of the circle of curvature at the origin referred to the new axes is

$$(x^2+2xy\cos\omega+y^2)\frac{bX'^2-2hX'Y'+aY'^2}{X'^2-2X'Y'\cos\omega+Y'^2}+2X'x+2Y'y=0;$$

for the equation of the tangent at the origin is $X'x+Y'y=0$.

or $(x^2+2xy\cos\omega+y^2)(bX'^2-2hX'Y'+aY'^2)$

$$+2(X'^2-2X'Y'\cos\omega+Y'^2)(X'x+Y'y)=0.$$

But $bX'^2-2hX'Y'+aY'^2=CS'-\Delta=-\Delta$, since $(x', y')$ is on the curve; $\therefore$ the equation of the circle referred to the new axes is

$$\Delta(x^2+y^2+2xy\cos\omega)-2(X'^2-2X'Y'\cos\omega+Y'^2)(X'x+Y'y)=0.$$

The equation referred to the original axes can be found by substituting $(x-x')$, $(y-y')$ for $x$ and $y$ respectively.

**Example.** *To find the circle of curvature of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at the point* $(a\cos\phi,\ b\sin\phi)$.

Transfer to parallel axes through the point $(a\cos\phi,\ b\sin\phi)$ and the equation of the ellipse becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2x}{a}\cos\phi+\frac{2y}{b}\sin\phi=0.$$

The circle of curvature referred to the new axes is

$$\lambda(x^2+y^2)+2\left(\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi\right)=0,$$

where $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=0$ is one of the lines

$$\lambda(x^2+y^2)-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0;$$

$$\therefore \qquad \lambda=\frac{1}{a^2\sin^2\phi+b^2\cos^2\phi},$$

i.e. the circle of curvature is

$$(x^2+y^2)+2\left(\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi\right)(a^2\sin^2\phi+b^2\cos^2\phi)=0,$$

or, referred to the original axes,

$$(x-a\cos\phi)^2+(y-b\sin\phi)^2+2\left(\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1\right)(a^2\sin^2\phi+b^2\cos^2\phi)=0,$$

i.e. $$x^2+y^2-2x\cdot\frac{a^2-b^2}{a}\cos^3\phi-2y\cdot\frac{b^2-a^2}{b}\sin^3\phi$$
$$+(a^2-2b^2)\cos^2\phi+(b^2-2a^2)\sin^2\phi=0.$$

**§ 4. Equation of conic when a pair of tangents and the chord of contact are given. Special Notation.**

*Let the equation of the tangents in abridged notation be* $u=0$, $v=0$, *and the chord of contact* $w=0$; *then the equation of the conic is* $uv=w^2$.

(Strictly, the equation should be $k.uv=w^2$ where $k$ is a determinate constant; this can, however, be included by considering $ku=0$ to be the equation of one tangent.)

Let $u_1, v_1, w_1$ be the values of the expressions $u, v, w$ when the coordinates of any point $P$ are substituted in them; evidently we can express these coordinates in terms of any two of the quantities $u_1, v_1, w_1$, and conversely, if the values of $u_1, v_1, w_1$ for any point are known we can determine its coordinates. It should be clear that we can use the values of $u$, $v$, and $w$ at any point to indicate the position of the point; only the two ratios $u:v:w$ are necessary.

Now if $(u_1v_1w_1)$ is a point on the curve, we have $u_1v_1=w_1^2$, and hence, if $u_1=\lambda w_1$, then $v_1=\frac{w_1}{\lambda}$, i.e. the coordinates of a point on the curve are connected by the relation $\frac{u_1}{\lambda^2}=\frac{v_1}{1}=\frac{w_1}{\lambda}$, and conversely, if $\lambda$ is known the ratios $u_1:v_1:w_1$ are known, and the position of the point is determined: we shall refer to this point on the curve as 'the point $\lambda$'.

I. *To find the equation of the chord joining two points* $\lambda$, $\mu$ *on the curve.*

Let the chord be $Au+Bv+Cw=0$; then

$$A\lambda^2+B+C\lambda=0,$$
$$A\mu^2+B+C\mu=0;$$

$$\therefore\quad \frac{A}{\mu-\lambda}=\frac{B}{\lambda\mu(\mu-\lambda)}=\frac{C}{-(\mu^2-\lambda^2)};$$

$$\therefore\quad \frac{A}{1}=\frac{B}{\lambda\mu}=\frac{C}{-(\lambda+\mu)};$$

$\therefore$ the equation of the chord is

$$u+\lambda\mu v-(\lambda+\mu)w=0.$$

II. *The equation of the tangent at the point* $\lambda$ *follows by putting* $\mu = \lambda$, viz. $u + \lambda^2 v - 2\lambda w = 0$.

Hence, if the tangent at $\lambda$ passes through the point $(u_1 : v_1 : w_1)$, we have
$$\lambda^2 v_1 - 2\lambda w_1 + u_1 = 0,$$
i.e. two tangents pass through this point, and the parameters $\lambda_1$, $\lambda_2$ of their points of contact are given by this equation. Hence
$$\lambda_1 + \lambda_2 = \frac{2w_1}{v_1}, \quad \lambda_1\lambda_2 = \frac{u_1}{v_1};$$
$$\therefore \quad u_1 : v_1 : w_1 = \lambda_1\lambda_2 : 1 : \tfrac{1}{2}(\lambda_1 + \lambda_2).$$

Conversely, the point of intersection of the tangents at the points $\lambda_1$, $\lambda_2$ is $\{\lambda_1\lambda_2 : 1 : \frac{1}{2}(\lambda_1 + \lambda_2)\}$.

III. *To find the chord of contact of tangents from the point* $(u_1, v_1, w_1)$.

Let the points of contact be $\lambda_1$, $\lambda_2$.

$\therefore$ the chord of contact is
$$u + \lambda_1\lambda_2 v - (\lambda_1 + \lambda_2) w = 0.$$

But we have shown in II that in this case
$$u_1 : v_1 : w_1 = \lambda_1\lambda_2 : 1 : \tfrac{1}{2}(\lambda_1 + \lambda_2);$$

$\therefore$ the equation of the chord of contact is
$$uv_1 + vu_1 - 2ww_1 = 0.$$

**Note.** By the usual argument the polar of any point $(u_1, v_1, w_1)$ takes the same form.

IV. If $u$, $v$, and $w$ are taken in the form $x\cos\alpha + y\sin\alpha - p = 0$, the equation must be taken in the form $uv = k \cdot w^2$, in which case any point on the conic is indicated by
$$u : v : w = \lambda^2 : k : \lambda.$$

This is sometimes useful, for we then know for example that $u - v = 0$ bisects the angle between $u = 0$, $v = 0$.

**Example i.** *The anharmonic ratio of the pencil formed by joining four fixed points on a conic to any fifth point on it is constant.*

Let the conic be $uv = w^2$ and four fixed points $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ and any fifth point $\lambda$.

The equations of the rays of the pencil are
$$u + \lambda\lambda_1 v - (\lambda + \lambda_1) w = 0,$$
$$u + \lambda\lambda_2 v - (\lambda + \lambda_2) w = 0,$$
$$u + \lambda\lambda_3 v - (\lambda + \lambda_3) w = 0,$$
$$u + \lambda\lambda_4 v - (\lambda + \lambda_4) w = 0.$$

Now $u-\lambda w=0$, $w-\lambda v=0$ are straight lines: let us write $x\equiv u-\lambda w$, $y\equiv w-\lambda v$, then the equation of the rays of the pencil become $x=\lambda_1 y$, $x=\lambda_2 y$, $x=\lambda_3 y$, $x=\lambda_4 y$ and (Chap. II, § 10) the anharmonic ratio of this pencil depends only on $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and is independent of $\lambda$.

**Example** ii. *If $OP$, $OQ$ are tangents to a conic from a point $O$, and the line bisecting the angle $POQ$ meets $PQ$ in $V$, and if $RS$ is any chord of the conic passing through $V$, prove that $OR$, $OS$ are equally inclined to $OV$.*

Let $OP$, $OQ$ be the lines

$$u\equiv x\cos\alpha+y\sin\alpha-p=0,\quad v\equiv x\cos\beta+y\sin\beta-q,$$

and let $PQ$ be $w=0$ (including any necessary constant factor), then the conic is $uv=w^2$.

The bisectors of the angle $POQ$ are the lines $u-v=0$, $u+v=0$.

Let $R$ and $S$ be the points $\lambda$, $\mu$.

$RS$ is the line $u+\lambda\mu v-(\lambda+\mu)w=0$, and by hypothesis this passes through the intersection of $u-v=0$, $w=0$; $\therefore\ \lambda\mu=-1$.

The lines $OR$, $OS$ are $u-\lambda^2 v=0$, $u-\mu^2 v=0$.

Now the anharmonic ratio of the pencil $u-v=0$, $u+v=0$, $u-\lambda^2 v=0$, $u-\mu^2 v=0$ is (Chap. II, § 10) $\dfrac{(1+\lambda^2)(\mu^2-1)}{(1+\mu^2)(\lambda^2-1)}$, which since $\mu=-\dfrac{1}{\lambda}$ is equal to $-1$, i.e. the pencil is harmonic; and since $u+v=0$, $u-v=0$ are perpendicular they are the bisectors of the angles between $u-\lambda^2 v=0$, $u-\mu^2 v=0$.

## Illustrative Examples.

I. *To find the locus of the poles of tangents to the conic*

$$S'\equiv a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$$

*with respect to the conic*

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The tangential equation of $S'=0$ is

$$A'l^2+2H'lm+B'm^2+2G'l+2F'm+C'=0,$$

and if $lx+my+1=0$ is any tangent to $S'=0$, $(l, m)$ satisfies this equation.

Let $(x', y')$ be the pole of $lx+my+1=0$ with respect to $S'=0$; the equation of the polar of $(x', y')$ is

$$x(ax'+hy'+g)+y(hx'+by'+f)+gx'+fy'+c=0,$$

or

$$xX'+yY'+Z'=0.$$

This equation is therefore identical with $lx+my+1=0$.

Hence $l/X'=m/Y'=1/Z'$; therefore (dropping the accents) the locus of $(x', y')$ is $A'X^2+2H'XY+B'Y^2+2G'XZ+2F'YZ+C'Z^2=0$.

**II.** *To find the length of the equi-conjugate diameters of the conic* $S = 0$, *and to show that they are inclined at an angle* $\sin^{-1}\frac{2\sqrt{C}}{a+b}$ *to one another.*

If $2r$ is the length of the equi-conjugate diameters, their extremities lie on a circle whose centre is the centre of the conic and whose radius is $r$: hence they are common chords of $S = 0$ and

$$\left(x - \frac{G}{C}\right)^2 + \left(y - \frac{F}{C}\right)^2 - r^2 = 0.$$

Thus their equation is of the form

$$\lambda S + \left(x - \frac{G}{C}\right)^2 + \left(y - \frac{F}{C}\right)^2 - r^2 = 0.$$

Transfer the axes to parallel axes through the centre of the conic; their equation then becomes

$$\lambda\left(ax^2 + 2hxy + by^2 + \frac{\Delta}{C}\right) + x^2 + y^2 - r^2 = 0.$$

Now $\lambda$ is to be determined so that this represents a pair of straight lines through the origin; $\therefore$ $\frac{\lambda\Delta}{C} = r^2$ or $\lambda = \frac{Cr^2}{\Delta}$; $\therefore$ the equation of the equi-conjugate diameters referred to parallel axes through the centre of the conic is

$$\left(ar^2 + \frac{\Delta}{C}\right)x^2 + 2hr^2xy + \left(br^2 + \frac{\Delta}{C}\right)y^2 = 0.$$

The condition that these should be conjugate diameters of

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{C} = 0,$$

is
$$a\left(br^2 + \frac{\Delta}{C}\right) + b\left(ar^2 + \frac{\Delta}{C}\right) = 2h^2r^2,$$

i.e.
$$2(ab - h^2)r^2 = -(a+b)\frac{\Delta}{C},$$

or
$$r^2 = -\frac{(a+b)\Delta}{2C^2},$$

which gives the lengths of the equi-conjugates.

The angle between the lines follows from the usual formulae. We might here usefully employ invariants: in the conic $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ the equi-conjugates are $\frac{x}{\alpha} - \frac{y}{\beta} = 0$, $\frac{x}{\alpha} + \frac{y}{\beta} = 0$; the length of the equi-conjugates is then $\sqrt{\frac{\alpha^2+\beta^2}{2}}$ and the angle between them is $\sin^{-1}\frac{2\alpha\beta}{\alpha^2+\beta^2}$.

Now suppose that the equation of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

by suitable change of axes reduces to

$$k\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-1\right)=0;$$

then we have shown that $\frac{a+b-2h\cos\omega}{\sin^2\omega}$, $\frac{ab-h^2}{\sin^2\omega}$, $\frac{\Delta}{\sin^2\omega}$ are invariants.

Hence
$$a+b=k\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)=k\left(\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}\right),$$

$$C\equiv ab-h^2=\frac{k^2}{\alpha^2\beta^2},$$

$$\Delta=-\frac{k^3}{\alpha^2\beta^2};$$

$\therefore$
$$\frac{\Delta}{C}=-k,\quad \frac{a+b}{C}=\frac{\alpha^2+\beta^2}{k};$$

$\therefore$
$$\frac{\alpha^2+\beta^2}{2}=-(a+b)\frac{\Delta}{2C^2},$$

also
$$\frac{2\sqrt{C}}{a+b}=\frac{2\alpha\beta}{\alpha^2+\beta^2}.$$

III. *What locus is represented by*

$$\begin{vmatrix} u_2u_3 & u_3u_1 & u_1u_2 \\ a & b & c \\ l_2l_3+m_2m_3 & l_3l_1+m_3m_1 & l_1l_2+m_1m_2 \end{vmatrix}=0,$$

*where* $u\equiv lx+my+1$ *for each suffix and* $a$, $b$, $c$ *are constants? Show that for all values of* $a$, $b$, $c$ *it passes through a certain fixed point.*

The equation is of the second degree and represents a conic: it is satisfied by the coordinates of any point which also satisfy any two of the equations $u_1=0$, $u_2=0$, $u_3=0$, i.e. it represents a conic circumscribing the triangle whose sides are $u_1=0$, $u_2=0$, $u_3=0$.

The coefficient of $a$ is

$$u_1[-(l_1l_2+m_1m_2)u_3+(l_3l_1+m_3m_1)u_2],$$

$\equiv u_1U_1$ say.

Now $U_1=0$ represents a straight line through the point of intersection of $u_2=0$, $u_3=0$: further, the coefficients of $x$ and $y$ in the equation are $-m_1(l_3m_2-l_2m_3)$ and $+l_1(l_3m_2-l_2m_3)$, i.e. $U_1=0$ is perpendicular to $u_1=0$. Hence, if $ABC$ be the triangle whose sides $BC$, $CA$, $AB$ are $u_1=0$, $u_2=0$, $u_3=0$, then $U_1=0$, and (using similar notation for the coefficients of $b$ and $c$) $U_2=0$, $U_3=0$ represent the perpendiculars $AD$, $BE$, $CF$ of the triangle. Hence $U_1=0$, $U_2=0$, $U_3=0$ are concurrent at the orthocentre. This point clearly lies on the conic since the equation of the conic is $au_1U_1+bu_2U_2+cu_3U_3=0$.

Again, since the pairs of lines $u_1=0$, $U_1=0$; $u_2=0$, $U_2=0$; $u_3=0$, $U_3=0$ are perpendicular, the coefficients of $x^2$ and $y^2$ in each term of this equation

are equal and opposite, hence the sum of the coefficient of $x^2$ and $y^2$ in the whole equation is zero, i.e. the conic is a rectangular hyperbola.

Finally, then, the equation represents a rectangular hyperbola circumscribing the triangle whose sides are $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, and all such rectangular hyperbolas pass through the orthocentre of the triangle.

IV. *Prove that*

$$(x\cos\alpha + y\sin\alpha - p)(x\cos\alpha + y\sin\alpha - p') + \tfrac{1}{4}\lambda + \frac{(p-p')^2}{\lambda}(x\sin\alpha - y\cos\alpha - q)^2 = 0$$

*is the general form of the equation of a conic of which*

$$x\cos\alpha + y\sin\alpha - p = 0, \quad x\cos\alpha + y\sin\alpha - p' = 0$$

*are the directrices.*

The tangents from the circular points at infinity, $\Omega$ and $\Omega'$, to a conic intersect in pairs of foci, and the directrices being the polars of the foci it follows that the conic circumscribes the quadrilateral formed by the directrices, and that the tangent to the conic at each vertex of this quadrilateral passes through $\Omega$ or $\Omega'$.

The directrices are parallel to the axes of the conic, so that one pair is perpendicular to the other pair.

Let the conic be

$$(x\cos\alpha + y\sin\alpha - p)(x\cos\alpha + y\sin\alpha - p') + k(x\sin\alpha - y\cos\alpha - r)(x\sin\alpha - y\cos\alpha - r') = 0.$$

The equation of the tangent to this conic at the point of intersection of the lines $x\cos\alpha + y\sin\alpha - p = 0$ and $x\sin\alpha - y\cos\alpha - r = 0$ is $(x\cos\alpha + y\sin\alpha - p)(p - p') + k(x\sin\alpha - y\cos\alpha - r)(r - r') = 0$; and, since this passes through $\Omega$ $(1, i, 0)$, we have

$$(\cos\alpha + i\sin\alpha)(p - p') + k(\sin\alpha - i\cos\alpha)(r - r') = 0,$$

i.e.
$$p - p' = ik(r - r').$$

If we put $r + r' = 2q$, the equation of the conic can be written

$$(x\cos\alpha + y\sin\alpha - p)(x\cos\alpha + y\sin\alpha - p') + k(x\sin\alpha - y\cos\alpha - q)^2 - \tfrac{1}{4}k(r - r')^2 = 0.$$

Now let $(p - p')^2 = \lambda k$, so that

$$\lambda k = -k^2(r - r')^2 \quad \text{or} \quad -\tfrac{1}{4}k(r - r')^2 = \tfrac{1}{4}\lambda.$$

The equation of the conic then becomes

$$(x\cos\alpha + y\sin\alpha - p)(x\cos\alpha + y\sin\alpha - p') + \frac{1}{\lambda}(p - p')^2(x\sin\alpha - y\cos\alpha - q)^2 + \tfrac{1}{4}\lambda = 0.$$

## Examples XI.

1. (i) A conic is circumscribed to a quadrangle: show that the product of the perpendiculars from any point on it to one pair of opposite sides is in a constant ratio to the product of the perpendiculars from it on the other pair.

(ii) A conic is inscribed in a quadrilateral: show that the product of the perpendiculars from one pair of opposite vertices on any tangent to it is in a constant ratio to the product of the perpendiculars from the other pair of vertices on the tangent.

2. The conic $x^2/(a^2+ka'^2)+y^2/(b^2+kb'^2)=1/(1+k)$ is for all values of $k$ inscribed in the same quadrilateral.

3. Find the locus of the foci of conics which pass through the vertices of a given rhombus.

4. The director circles of all conics inscribed in a given quadrilateral are coaxal.

5. Find the equation of the chord of curvature through and the length of the radius of curvature at the origin for $ax^2+2hxy+by^2+2gx+2fy=0$.

Deduce the length of the radius of curvature at any point on the conic $S=0$.

6. A variable chord subtends a constant angle at a given point of a conic: find its envelope.

7. Chords of a conic which subtend a right angle at a fixed point envelope a curve of which the fixed point and its polar are focus and directrix.

8. A conic is drawn touching the axis of $x$ at the origin and having its centre at the point $(a, b)$; the axis of $y$ intersects the conic again at $P$; prove that the tangent at $P$ passes through the point $(a, 2b)$ and that the normal at $P$ touches the parabola whose equation is $(y-2b)^2=4ax$.

9. If the four points of intersection of two parabolas lie on a circle their axes must be at right angles, and the coordinates of the centre of the circle referred to the axes of the parabolas must be equal to the semi-latera recta.

10. A straight line is drawn to be cut harmonically by two given circles. Show that its envelope is a conic whose foci are the centres of the two circles.

11. Show that the pair of tangents to $S=0$ at the points where the conic is met by $\lambda x+\mu y+r=0$ are given by $S.\Sigma-\Delta(\lambda x+\mu y+r)^2=0$, and deduce that $\lambda x+\mu y+r=0$ is a directrix if $(\lambda^2-\mu^2)/(a-b)=\lambda\mu/h=\Sigma/\Delta$.

12. Show that a third pair of straight lines through the four points where the straight lines $S=0$ cut the axes are $cS+4Hxy=0$.

13. Find the equations of the conjugate diameters common to

$$x^2+4xy+6y^2=1,$$
$$2x^2+6xy+9y^2=1.$$

14. Prove that the locus of the poles of tangents to $ax^2+2hxy+by^2=1$, with regard to the conic $a'x^2+2h'xy+b'y^2=1$, is

$$a(h'x+b'y)^2-2h(a'x+h'y)(h'x+b'y)+b(a'x+h'y)^2=ab-h^2.$$

15. Show that, if the pole of one common chord of a circle and a fixed conic lie on the circle, the pole of the opposite common chord lies on the circle; also show that the lines joining these poles to the centre of the conic make equal angles with the major-axis of the conic, and that the rectangle under them is constant.

16. Two concentric conics are equal in every respect. Prove that the points of intersection of their common tangents always lie on the director circle, whatever the angle between the major axes.

17. Prove that the locus of the foot of the perpendicular from the origin on a chord of the conic $S = 0$, subtending a right angle at the origin, is the circle $(a+b)(x^2+y^2)+2gx+fy+c=0$.

18. The equations of two conics touching one another at the origin $O$ are $ax^2+2hxy+by^2+2fy=0$ and $a'x^2+2h'xy+b'y^2+2f'y=0$, and they also intersect at $PQ$; find the equation of $PQ$ and of the pair of straight lines $OP$, $OQ$.

19. (i) If the chords of intersection of

$$ax^2+2hxy+by^2=1,$$
$$a'x^2+2h'xy+b'y^2=1$$

which pass through the centre are at right angles, $a+b=a'+b'$;

(ii) If the chords which do not pass through the centre are at right angles, $\frac{a-b}{h}=\frac{a'-b'}{h'}$.

20. Show that the area of the parallelogram formed by drawing tangents to the conic $S=0$ parallel to the axes of coordinates is $\frac{4\Delta.\sin\omega.\sqrt{ab}}{(ab-h^2)^2}$, where $\omega$ is the angle between the axes and $\Delta = abc+2fgh-af^2-bg^2-ch^2$.

21. In oblique coordinates what is the locus of a point $P$ of which the coordinates are $a\cos\phi$, $b\sin\phi$? Give a geometrical interpretation of $\phi$. What are represented by the equations $x^2+y^2=a^2$ and $x^2+y^2=b^2$? If the tangent at $P$ to the first locus meets the locus $x^2+y^2=a^2$ in the points $Q$, $Q'$, then two of the lines joining $Q$, $Q'$ to the points $(\sqrt{(a^2-b^2)}, 0)$ and $(-\sqrt{(a^2-b^2)}, 0)$ are parallel.

22. Show that the equation of the director circle of the conic

$$u \equiv ax^2+2hxy+by^2-1=0$$

is

$$v \equiv (ab-h^2)(x^2+y^2)-(a+b)=0.$$

Assuming that the directrices are chords of intersection of the conic and the director circle, show that the equation of the four (two real and two imaginary) directrices is $v^2-(a+b)uv+(ab-h^2)u^2=0$.

23. Find the foci of $2x^2+3xy-2y^2-12x-4y+8=0$.

24. Prove that, if at a point on a hyperbola the sum of the tangents of the angles which the normal makes with the asymptotes is 2, then the vertex of the parabola of closest contact at the point lies on a line through the point inclined at an angle $\tan^{-1}3$ to the normal, and at a distance from the normal equal to three-eighths of the radius of curvature.

25. A conic has four-point contact with the parabola $y^2=4ax$, and the

radius of its director circle is constant and equal to $c$. Prove that the locus of its centre is the curve

$$(y^2-4ax)(y^2+4ax+8a^2)+16a^2c^2=0.$$

26. Show that the points whose rectangular coordinates are $(b, c)$, $(c, b)$, $(c, a)$, $(a, c)$, $(a, b)$, $(b, a)$ lie upon an ellipse whose eccentricity is independent of $a$, $b$, $c$, and whose centre is fixed if $a+b+c$ is constant.

27. Prove that if $b=b'$ the conics

$$ax^2+2hxy+by^2=x,$$
$$a'x^2+2h'xy+b'y^2=x$$

will have three-point contact at the origin.

Two parabolas have three-point contact at $P$ and intersect at $Q$. Prove that the tangents at $Q$, the line $QP$, and the line through $Q$ parallel to the common tangent at $P$ form a harmonic pencil.

28. Find the equation to the common conjugate diameters of the conics $ax^2+2hxy+by^2=1$, $a'x^2+2h'xy+b'y^2=1$.

29. Parallel tangents inclined at the angle 45° to the positive direction of the axis of $x$ are drawn to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$: prove that their distance apart is $\frac{1}{C}\sqrt{-2\Delta(2h+a+b)}$.

30. Prove that the conics which pass through the origin and which are confocal with the conic $x^2+2xy+2y^2-2x-2y+5=0$ are both real and pass both through the intersections of the circle $x^2+y^2-2x=0$ and the conic $x^2+2xy+2y^2-2x-2y=0$.

31. Show that the straight line $lx+my+n=0$ will be a tangent to the conic $ax^2+2hxy+by^2=1$ if $bl^2-2hlm+am^2=(ab-h^2)n^2$. Show that the pair of tangents to this conic from any point on the conic $(ab-h^2)xy+h=0$ are equally inclined to the axis of $x$.

32. Find the envelope of the chord common to an ellipse and its circle of curvature at any point, and show that its equation may be put in the form $U^3+V^2=0$, where $U=0$ is the equation of a similar and coaxal ellipse, $V=0$ that of its equi-conjugate diameters.

33. Show that the angle subtended at the origin by the intersections with the circle $x^2+y^2+2fy+2gx+c=0$ of any tangent to the parabola $(gx+fy+c)^2=4fgxy$ is bisected internally and externally by the axes of coordinates.

34. Find the equation of the conic which passes through the origin and is confocal with the conic $ax^2+2hxy+by^2=2x$.

35. If $e$ is the eccentricity and $2l$ the latus rectum of the conic given by the general equation in rectangular coordinates, prove that

$$\frac{(2-e^2)^6}{(a+b)^6}=\frac{(1-e^2)^3}{(ab-h^2)^3}=\frac{l^4}{\Delta^2}.$$

36. Show that the lines $H(x^2-y^2)=(A-B)xy$ are conjugate with regard to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.

37. Show that the general equation of conics having the points $(a, b)$, $(-a, -b)$ as their foci may be written $(x^2-a^2-\lambda)(y^2-b^2-\lambda)=(xy-ab)^2$.

38. A conic touches the axes of coordinates (not necessarily rectangular)

and has its centre at the point $f, g$. Show that the locus of its foci is the rectangular hyperbola $x^2-y^2-2fx+2gy=0$.

39. If $u_2+u_1+c=0$ is the equation of a conic and $\Delta$ is its discriminant, show that the asymptotes are given by $C(u_2+u_1+c)=\Delta$. Find the condition that the asymptotes should form a rhombus with the lines $u_2=0$.

40. A series of conics being drawn having four-point contact at the origin with the conic $ax^2+2hxy+by^2+2fy=0$, prove that their director circles form a coaxal system, and that $-hf/(a^2+h^2)$, $+af/(a^2+h^2)$ is one of the limiting points. Find the equation of the radical axis.

41. A variable circle passes through a fixed point $A$ and also through the point of intersection of two fixed straight lines which it cuts in $P$ and $Q$. Find the locus of the centre of gravity of the triangle $APQ$. Show also that the envelope of the straight line $PQ$ is a parabola touching the two fixed straight lines.

42. Interpret the equation $(ax+by-1)^2-2\lambda xy=0$, and find the value of $\lambda$ if it represents a parabola.

If it represents a parabola, and if $a+b$ is constant, prove that the locus of the focus is a circle, and find its centre and radius, the axes being either rectangular or oblique.

43. If $\alpha=0$, $\alpha'=0$ are the equations of a pair of tangents to the conic $u=0$ and $\gamma=0$ their chord of contact, explain the geometrical meaning of $u-\alpha^2=0$, $u-\alpha\alpha'=0$, $u-\gamma^2=0$.

An ellipse passes through the origin, touches the axis of $y$, and has double contact with a fixed circle whose centre is at the origin. Show that the locus of the foci is a pair of circles.

44. Interpret the equation $LM=N^2$, where $L=0$, $M=0$, $N=0$ each represents a straight line.

Find the equation to the tangent to this curve at the point $(L', M', N')$.

Deduce the equation of the pair of common tangents, other than the axes, to the conics $k^2(x/a+y/b-1)^2=4xy$, $k^2(x/a'+y/b'-1)^2=4xy$.

45. The equation of a conic referred to two tangents and their chord of contact being $LM=R^2$, find the equation of the tangent at any point in the form $\mu^2L-2\mu R+M=0$.

Through the point of intersection of $R=0$, $2L+M=0$ two tangents are drawn to the conic $LM=R^2$, touching it in $PQ$. Find the equation of the conic which has double contact with $LM=R^2$ at $P$, $Q$, and which has the triangle formed by $L=0$, $M=0$, $R=0$ for a self-conjugate triangle.

46. Find the equation of the tangents to the conic $ax^2+by^2+c=0$ at the ends of the chord $px+qy+r=0$ in the form

$$(bcp^2+caq^2+abr^2)(ax^2+by^2+c)-abc(px+qy+r)^2=0.$$

47. Prove that the equation of the family of conics inscribed in the rectangle formed by the lines $x\pm a=0$, $y\pm b=0$, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+2\lambda\left(\frac{xy}{ab}\right)+\lambda^2=0.$$

Prove also that the locus of the foci is $x^2-y^2=a^2-b^2$, and if two conics intersect on this locus they do so at right angles.

48. Determine the magnitude of the axes and the foci of

$$2x^2(d^2+\beta^2-\alpha^2)+2y^2(d^2+\alpha^2-\beta^2)-8\alpha\beta xy=(d^2+\alpha^2+\beta^2)(d^2-\alpha^2-\beta^2).$$

49. The coordinate axes being the lines joining the middle points of the opposite sides $(2a, 2b)$ of a parallelogram, prove that the coordinates of the point of contact with the line $\lambda x+\mu y-1=0$, of the conic which can be inscribed in the parallelogram to touch this line, are $(\lambda^2a^2-\mu^2b^2+1)/2\lambda$, $(-\lambda^2a^2+\mu^2b^2+1)/2\mu$.

50. Find the equation of the director circle of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

If the conic is a parabola show that the coordinates of its focus are

$$\frac{\{FH+\frac{1}{2}(A-B)G\}}{(F^2+G^2)},\quad \frac{\{GH+\frac{1}{2}(B-A)F\}}{(F^2+G^2)}.$$

51. Prove that the four points of intersection of two conics, and the four points in which the asymptotes of one cut those of the other, all lie on a conic, which is a rectangular hyperbola whenever the other conics are both rectangular hyperbolas.

52. A conic circumscribes a right-angled triangle $ABC$, and at the right angle $A$ it touches the circumscribing circle. It also passes through the centroid of the triangle. Prove that its eccentricity is $\{2/(1-\sin B\sin C)\}^{\frac{1}{2}}$.

53. Prove that

$$\begin{vmatrix} x, & y, & 1 \\ x', & y', & 1 \\ G, & F, & C \end{vmatrix} = 0 \text{ and } \begin{vmatrix} Cx, & Cy, & CN-\Delta \\ A, & H, & GN-\Delta x' \\ H, & B, & FN-\Delta y' \end{vmatrix} = 0$$

are conjugate diameters of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

where $\Delta$ is its discriminant, $A$, $B$, $C$, $F$, $G$, $H$ are the minors of $\Delta$, $N$ is $gx'+fy'+c$, and $x'$, $y'$ is any point on the curve.

54. Show that if tangents are drawn to a series of confocal conics from a fixed point on one of the axes, the locus of the points of contact is a circle, and prove that two such circles corresponding to two points, one on each axis, cut each other orthogonally.

55. A parabola is drawn touching the sides $AB$, $BC$, $CD$, $DA$ of the cyclic quadrilateral $ABCD$. Show that its directrix passes through the intersection of $AC$ and $BD$.

56. Trace the curve $\sqrt{2a-2x}+\sqrt{x+y}+\sqrt{2y-2x}=0$. What portions of the curve correspond to the various arrangements of the signs of the radicals? What is the length of the latus rectum?

57. A conic is drawn having one side of a triangle for directrix, the opposite vertex for centre, and the orthocentre for focus. Prove that the sides which meet in the centre are conjugate diameters.

58. Show that the general equation of conics, having the points $(a, b)$, $(a', b')$ as their foci, may be written

$$\mu^2\{x(b-b')-y(a-a')+ab'-a'b\}^2 + 2\mu\{(x-a)(x-a')+(y-b)(y-b')\}-1=0.$$

59. Find the conditions that the conics

$$ax^2+2hxy+by^2+2gx+2fy=0,$$
$$a'x^2+2h'xy+b'y^2+2gx+2fy=0$$

may have (i) three-point contact, (ii) double contact.

A variable parabola $S'$ has three-point contact with a fixed parabola $S$, their common chord and common tangent being equally inclined to the axis of $S$: show that the focus of $S'$ lies on the curve $27ay^2=x(x-9a)^2$.

60. A parallelogram circumscribes a given circle and one vertex moves on a fixed line: prove that one of the others describes a conic.

61. Through the four points of intersection of a circle and a rectangular hyperbola two parabolas are drawn. Show that the tangents to the four curves at a common point form a harmonic pencil.

62. Show that the equation for determining the foci of the conic

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

may be written in the form $aS-\xi^2=bS-\eta^2=(hS-\xi\eta)\sec\omega$, where $\xi=ax+hy+g$, $\eta=hx+by+f$, and $\omega$ is the angle between the axes of coordinates.

Show that the equation of the axes of the conic is given by

$$\begin{vmatrix} a & h & b \\ \xi^2 & \xi\eta & \eta^2 \\ 1 & \cos\omega & 1 \end{vmatrix}=0.$$

63. Show that the length of the diameter, conjugate to the diameter through the origin, of the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ is

$$2\left\{\frac{f^2+g^2}{h^2-ab}\cdot\frac{abc+2fgh-af^2-bg^2-ch^2}{af^2+bg^2-2fgh}\right\}^{\frac{1}{2}},$$

the axes being rectangular.

64. Show that the locus of a point whose coordinates are determined by the relations $x/(a_1\lambda^2+b_1\lambda+c_1)=y/(a_2\lambda^2+b_2\lambda+c_2)=1/(a_3\lambda^2+b_3\lambda+c_3)$ is in general a conic section.

Find the equation of the lines, through the origin, parallel to its asymptotes. Can the locus be a pair of straight lines?

65. Prove that the locus of the centre of a conic, with respect to which four given pairs of straight lines are conjugate lines, is a straight line.

66. The length of the diameter of the conic $S=0$, which is conjugate to the diameter which passes through $(x', y')$, is

$$2\{\Delta\,(X'^2+Y'^2)/(C^2S'-C\Delta)\}^{\frac{1}{2}}.$$

67. Two straight lines $OA$, $OBC$ contain an angle $\theta$; determine how many parabolas can be drawn through $B$, $C$ to touch $OA$ at $A$, and find their equations.

Prove that the diameter of curvature of one of the parabolas at the point $A$ is $a^2\operatorname{cosec}\theta\,(b^{-\frac{1}{2}}+c^{-\frac{1}{2}})^2$ where $a$, $b$, $c$ are the lengths of $OA$, $OB$, $OC$.

68. Employing as coordinates the distances $r, r'$ of a point from two given points, interpret the equations $r^2-r'^2=a^2$, $r^2+r'^2=a^2$, $r=\lambda r'$, $r^2-r'^2-2ar+a^2=0$, and prove that the equation

$$(1+p)(r-r')^2-2c(c+pq)r^2-2c(c-pq)r'^2+c^2(c^2+q^2)=0$$

represents a pair of straight lines, the angle between which is bisected by the line joining the two poles, $c$ being the distance between these points. Interpret the constants $p$ and $q$.

69. Show that through any four points there can in general be drawn two parabolas and one rectangular hyperbola.

Prove also, that if $l_1$, $l_2$ are the semi-latera recta of the parabolas, $p_1$, $p_2$ the distances of the centre of the rectangular hyperbola from the axes of the parabolas, then $p_1^2 + l_1^2 = p_2^2 + l_2^2$.

70. A triangle is inscribed in the conic $ax^2 + by^2 = 1$, and two of its sides touch the conic $Ax^2 + 2Hxy + By^2 = 1$.

Show that the envelope of the third side is the conic

$$[(Ab + Ba - AB + H^2)^2 + 4(H^2 - AB)ab](ax^2 + by^2 - 1) = 4(H^2 - AB)ab[Ax^2 + 2Hxy + By^2 - 1].$$

# CHAPTER XII

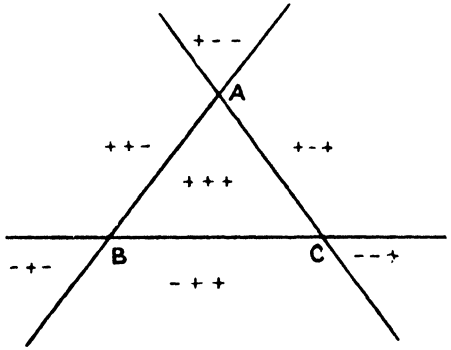
## TRILINEAR AND AREAL COORDINATES

§ 1. THE position of a point is fixed when its coordinates referred to *two* axes are known, consequently the coordinates of a point referred to three or more axes (e.g. the point's distances from three fixed straight lines) must be connected by as many identical relations as the number of coordinates exceeds two. The best known multiple coordinates are the Trilinear and the Areal.

In **trilinear coordinates** the position of a point is given by its perpendicular distances $\alpha$, $\beta$, $\gamma$ from the sides $BC$, $CA$, $AB$ of a triangle $ABC$, which is known as the **triangle of reference.**

If a point $P$ lies within the triangle of reference, its coordinates $(\alpha, \beta, \gamma)$ are all positive; if $P$ lies outside the triangle, $\alpha$ is positive or negative according as the points $P$ and $A$ are on the same side or on opposite sides of $BC$; similarly for $\beta$ and $\gamma$.

The sides of the triangle of reference divide the plane into seven




parts; the signs of the coordinates of points within them are shown in the diagram. It may be noted that not more than two of the coordinates can be negative.

The **areal coordinates** $(x, y, z)$ of a point $P$ are the ratios

$$\frac{\Delta BPC}{\Delta ABC}, \quad \frac{\Delta CPA}{\Delta ABC}, \quad \frac{\Delta APB}{\Delta ABC},$$

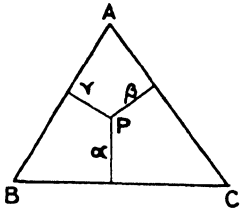
with the same sign convention as that given above for trilinear coordinates.

Thus the areal and trilinear coordinates of a point are connected by the relations $2Sx = a\alpha$, $2Sy = b\beta$, $2Sz = c\gamma$, where $a$, $b$, $c$ are the lengths of the sides, and $S$ is the area of the triangle of reference.

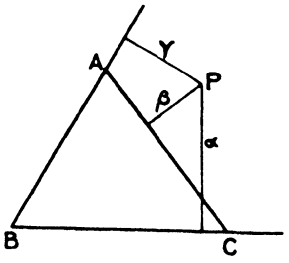
**Note** i. When a figure is projected orthogonally into another figure, the areal coordinates of a point are unaltered, since corresponding areas in the two figures are in the same ratio. The projection of the triangle of reference is taken as the new triangle of reference.

**Note** ii. A point is the centre of mean position of masses, proportional to its areal coordinates, placed at the vertices of the triangle of reference.

*Identical relations between the coordinates.*




(i)




(ii)

Let $P$ be the point $(\alpha, \beta, \gamma)$; then in Fig. (i), where $P$ is within the triangle, we have

$$a\alpha = 2\,\Delta\,BPC, \quad b\beta = 2\,\Delta\,CPA, \quad c\gamma = 2\,\Delta\,APB.$$

Hence $a\alpha + b\beta + c\gamma = 2\,\Delta\,ABC$.

In Fig. (ii) the point $P$ is outside the triangle; the coordinates $(\alpha, \beta, \gamma)$ of a point are supposed to contain their signs, so that in this case the numerical value of $\beta$ is negative, and

$$a\alpha + b\beta + c\gamma = \Delta BPC - \Delta\,CPA + \Delta\,APB = 2\,\Delta\,ABC.$$

In every case we have similarly,

$$\mathbf{a\alpha + b\beta + c\gamma = 2S}. \qquad \text{(i)}$$

This result can also be written

$$\alpha \sin A + \beta \sin B + \gamma \sin C = S/R = 4R \sin A \sin B \sin C, \qquad \text{(ii)}$$

where $R$ is the radius of the circumcircle of the triangle of reference.

Now if $(x, y, z)$ are the areal coordinates of $P$, we have from Fig. (i),

$$x + y + z = \Delta\,BPC/S + \Delta\,CPA/S + \Delta\,APB/S = 1,$$

and, with due regard to sign, we can show in general that

$$\mathbf{x + y + z = 1}. \qquad \text{(iii)}$$

**Note i.** By means of these relations any equation can be made homogeneous in the coordinates.

**Note ii.** Areal coordinates will be generally used rather than trilinear because the identical relation is so simple. Throughout this chapter trilinear coordinates are called $\alpha, \beta, \gamma$ and areal coordinates $x, y, z$.

Now the position of a point is known when any two of its coordinates are known, and the third coordinate can be found from the identical relation.

Also a point is fixed when the mutual ratios of its coordinates, $\alpha:\beta:\gamma$ or $x:y:z$, are known. We shall use $(\alpha, \beta, \gamma)$ to denote a point when the actual values of the coordinates are given, and $(\alpha:\beta:\gamma)$ when their mutual ratios are given.

In trilinears suppose $\alpha:\beta:\gamma = l:m:n$, then we have

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{a\alpha+b\beta+c\gamma}{al+bm+cn} = \frac{2S}{al+bm+cn};$$

hence $\alpha = \dfrac{2Sl}{al+bm+cn}$, &c.

So also in areals if $x:y:z = l:m:n$, we have

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = \frac{x+y+z}{l+m+n} = \frac{1}{l+m+n},$$

so that $x = l/(l+m+n)$, &c.

*To find the coordinates of a point $R$ dividing the distance between the points $P(\alpha_1, \beta_1, \gamma_1)$, $Q(\alpha_2, \beta_2, \gamma_2)$ in the ratio $l:m$.*

Since the perpendiculars from $P$ and $Q$ to the side $BC$ of the triangle of reference are $\alpha_1$ and $\alpha_2$, we can show in exactly the same way as in Chapter I, § 6, that the perpendicular from the required point $R$ to $BC$ is $\dfrac{m\alpha_1+l\alpha_2}{l+m}$. The required point is then

$$\left\{\frac{m\alpha_1+l\alpha_2}{l+m}, \quad \frac{m\beta_1+l\beta_2}{l+m}, \quad \frac{m\gamma_1+l\gamma_2}{l+m}\right\}.$$

If the areal coordinates of the points $P$ and $Q$ are $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, then their trilinear coordinates are

$$\{2Sx_1/a, \ 2Sy_1/b, \ 2Sz_1/c\}, \quad \{2Sx_2/a, \ 2Sy_2/b, \ 2Sz_2/c\}.$$

The $\alpha$-coordinate of the point $R$ is therefore, as above, $\dfrac{2S}{a} \cdot \dfrac{mx_1+lx_2}{l+m}$,

and consequently its $x$-coordinate is $\dfrac{mx_1+lx_2}{l+m}$.

Thus the areal coordinates of the point dividing the distance between the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ in the ratio $l:m$ are

$$\left\{\frac{mx_1+lx_2}{l+m}, \quad \frac{my_1+ly_2}{l+m}, \quad \frac{mz_1+lz_2}{l+m}\right\}.$$

**Note.** It should be carefully noted that in the above work the actual values of the coordinates have been used; if the points $P$, $Q$ are given as $\{x_1:y_1:z_1\}$, $\{x_2:y_2:z_2\}$, we must calculate the actual values of the coordinates (viz. $x_1/(x_1+y_1+z_1)$, &c.) before using the formulae to find the coordinates of $R$.

*The point* $\{(lx_1+mx_2):(ly_1+my_2):(lz_1+mz_2)\}$ *lies on the straight line joining the points* $(x_1:y_1:z_1)$, $(x_2:y_2:z_2)$ *for all values of the ratio* $l/m$.

The actual values of the $x$-coordinates of the points are

$$x_1/(x_1+y_1+z_1) \quad \text{and} \quad x_2/(x_2+y_2+z_2);$$

hence the $x$-coordinate of the point dividing the distance between $(x_1:y_1:z_1)$ and $(x_2:y_2:z_2)$ in the ratio $m(x_2+y_2+z_2):l(x_1+y_1+z_1)$ is

$$(lx_1+my_1)/\{l(x_1+y_1+z_1)+m(x_2+y_2+z_2)\},$$

with similar results for the other coordinates. Hence the point $\{(lx_1+mx_2):(ly_1+my_2):(lz_1+mz_2)\}$ lies on the line joining the points.

**Example.** *Find the trilinear and areal coordinates of the centroid of the triangle of reference.*

The trilinear coordinates of the vertices $A$, $B$, $C$ of the triangle of reference are $(2S/a, 0, 0)$, $(0, 2S/b, 0)$, $(0, 0, 2S/c)$. The mid-point $A'$ of $BC$ is therefore $(0, S/b, S/c)$.

The centroid $G$ divides $AA'$ in the ratio $2:1$, so that $G$ is the point $\{2S/3a, 2S/3b, 2S/3c\}$.

These are the actual values of the coordinates; the point can be referred to as $\{1/a:1/b:1/c\}$.

We have seen that a point is the centroid of masses, proportional to its areal coordinates, placed at the vertices of the triangle of reference.

Hence the areal coordinates of $G$ are all equal, i.e. the point $G$ in areals is $(1:1:1)$.

This is also evident by elementary geometry, since the triangles $AGC$, $CGB$, $BGA$ are equal.

The actual values of the areal coordinates of $G$ are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

## Examples XII a.

1. What are the actual values of the areal coordinates of the points $(1:2:5)$, $(3:-2:7)$?

2. If the sides of the triangle of reference are 3, 4, and 5, find the actual values of the trilinear coordinates of the points $(2:-3:0)$, $(1:6:-3)$.

3. Find the trilinear coordinates of the mid-point of the line joining the points $(1, 3, 5)$, $(-1, -1, 3)$.

4. Find the areal coordinates of the points midway between the following pairs of points:—

(i) $(4, 2, -5)$, $(-2, 4, -1)$;

(ii) $(1:2:3)$, $(5:-1:-2)$.

5. The trilinear coordinates of three points are $(4, -2, 2)$, $(6, 2, -2)$, $(-3, 4, 1)$. Find the mutual ratios of the sides of the triangle of reference.

6. In what ratio does the point $(-4:15:7)$ divide the line joining the points $(1:6:-5)$, $(-2:3:5)$? (Areal coordinates.)

7. If $A'$ is the mid-point of $BC$, find the areal coordinates of a point $P$ in $AA'$ such that (*a*) $AP = PA'$; (*b*) $AP = 3PA'$.

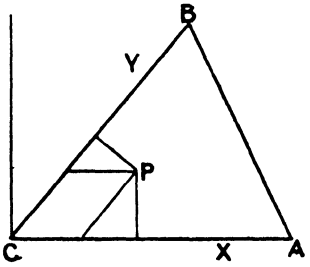
8. Find the coordinates of the points of trisection of the line joining the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$.

9. Find the values and the mutual ratios of the coordinates of the following points (*a*) in trilinears; (*b*) in areals:—

(i) the in-centre; (ii) the ex-centres; (iii) the circumcentre; (iv) the orthocentre of the triangle of reference.

10. Find the areal and the trilinear coordinates of the points where the bisectors of the vertical angle $A$ and the exterior vertical angle of the triangle of reference meet the opposite side.

§ 2. *Transformation of trilinear and areal coordinates to Cartesian coordinates.*




(A) Take the sides $CA$, $CB$ for axes of coordinates, so that $\omega = C$. Let the coordinates of a point $P$ be $(X, Y)$ in Cartesians, $(\alpha, \beta, \gamma)$ in trilinears, and $(x, y, z)$ in areals. Then we have

$$\alpha = X \sin C, \quad \beta = Y \sin C,$$

and therefore
$$c\gamma = 2S - a\alpha - b\beta$$
$$= 2S - aX \sin C - bY \sin C,$$

or
$$\gamma = b \sin A - X \sin A - Y \sin B.$$

Conversely,
$$X = \alpha \operatorname{cosec} C, \quad Y = \beta \operatorname{cosec} C.$$

Again,
$$x = a\alpha/2S = X/b$$
$$y = Y/a$$
$$z = 1 - x - y = 1 - X/b - Y/a.$$

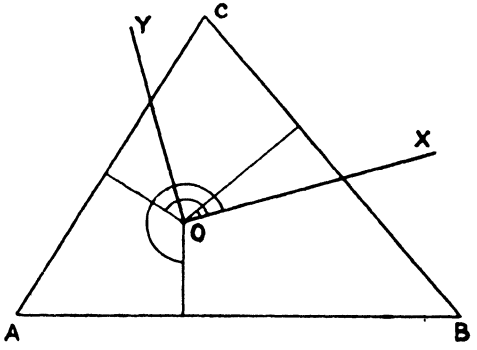
(B) Occasionally it is convenient to change to rectangular axes. Take $CA$ as axis of $X$ and a line through $C$ perpendicular to it for axis of $Y$. Then

$$X = \alpha \operatorname{cosec} C + \beta \cot C, \quad Y = \beta;$$

and conversely,

$$\alpha = X \sin C - Y \cos C, \quad \beta = Y, \quad \gamma = a \sin B - X \sin A - Y \cos A.$$

(C) More generally, let $OX$, $OY$ be any pair of rectangular axes, the origin $O$ being any point inside the triangle of reference. Now let the perpendiculars from the point $O$ to the sides of the triangle $BC$, $CA$, $AB$ make angles $\theta_1$, $\theta_2$, $\theta_3$ (all measured in the positive




direction) with the axis $OX$. The equations of the sides of the triangle of reference, referred to $OX$, $OY$ as axes, are then

$$X \cos \theta_1 + Y \sin \theta_1 - p_1 = 0, \quad X \cos \theta_2 + Y \sin \theta_2 - p_2 = 0,$$
$$X \cos \theta_3 + Y \sin \theta_3 - p_3 = 0,$$

where $p_1, p_2, p_3$ are the lengths of the perpendiculars from $O$ to the sides.

Now if $P$ is any point whose Cartesian coordinates are $(X, Y)$ and trilinear coordinates $(\alpha, \beta, \gamma)$, we have

$$\alpha = p_1 - X \cos \theta_1 - Y \sin \theta_1,$$
$$\beta = p_2 - X \cos \theta_2 - Y \sin \theta_2,$$
$$\gamma = p_3 - X \cos \theta_3 - Y \sin \theta_3;$$

the point $O$ being inside the triangle, the signs of $\alpha$, $\beta$, $\gamma$, when $P$ is also inside the triangle, are the same as the signs of the perpendiculars from $O$ to the sides; the reader should verify that these formulae give the correct signs for $\alpha$, $\beta$, $\gamma$ when $P$ is outside the triangle in various positions.

We have, from elementary considerations, the following relations:

$$\theta_2 - \theta_1 = \pi - C; \quad \theta_3 - \theta_2 = \pi - A; \quad \theta_3 - \theta_1 = \pi + B.$$

**§ 3. The straight line.** The general linear and homogeneous equation of the first degree, in trilinear or areal coordinates, when transformed to Cartesians evidently gives a linear equation; $l\alpha+m\beta+n\gamma=0$ or $lx+my+nz=0$, therefore, represents a straight line.

This can be shown independently: let $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ be any two points on the locus $lx+my+nz=0$, and $(x, y, z)$ any third point on this locus. Then

$$\begin{aligned} lx+my+nz &= 0, \\ lx_1+my_1+nz_1 &= 0, \\ lx_2+my_2+nz_2 &= 0; \end{aligned}$$

hence

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

This is the condition that constants $\lambda$, $\mu$, $\nu$ can be found, which are not all zero, such that $\lambda x+\mu x_1+\nu x_2=0$, $\lambda y+\mu y_1+\nu y_2=0$, $\lambda z+\mu z_1+\nu z_2=0$.

Hence the coordinates of the point $(x, y, z)$ are of the form $\{\mu' x_1+\nu' x_2,\ \mu' y_1+\nu' y_2,\ \mu' z_1+\nu' z_2\}$, i. e. the point $(x, y, z)$ is a point on the straight line joining the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$.

*To find the equation of the straight line joining* $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$.

Let the equation be $lx+my+nz=0$, then we have

$$lx_1+my_1+nz_1=0 \text{ and } lx_2+my_2+nz_2=0.$$

Eliminating $l$, $m$, $n$, we obtain the equation of the straight line in the form

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

**Note i.** This equation is the same whether $x$, $y$, $z$ are the actual values of the coordinates or only their mutual ratios.

**Note ii.** The trilinear equation of the straight line joining the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ is exactly similar.

**Note iii.** It follows at once that the condition that the three points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$ should be collinear is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Since the general linear equation in trilinears or areals transforms into a linear equation in Cartesians and vice versa, the following

properties, proved in Chapter I for equations in Cartesian coordinates, hold also for equations in trilinear or areal coordinates.

(a) If $u = 0$, $v = 0$ are the equations of two straight lines, then $u + kv = 0$ represents a straight line through their point of intersection.

Now the equations of the sides of the triangle of reference are $x = 0$, $y = 0$, $z = 0$, so that, for example, $my + nz = 0$ represents a straight line through the point of intersection of $y = 0$, $z = 0$, i.e. through the vertex $A$ of the triangle of reference.

**Example.** The equations of the straight lines joining the point $(x_1, y_1, z_1)$ to the vertices $A$, $B$, $C$ of the triangle of reference are $y/y_1 - z/z_1 = 0$, $z/z_1 - x/x_1 = 0$, $x/x_1 - y/y_1 = 0$.

(b) The straight lines $u = 0$, $v = 0$, $u + kv = 0$, $u - kv = 0$ form a harmonic pencil.

For example, $y = 0$, $z = 0$, $my + nz = 0$, $my - nz = 0$ is a harmonic pencil whose vertex is the point $A$.

**Example.** *The straight lines joining the vertices $A$, $B$, $C$ of a triangle to a point $P$ meet the opposite sides of the triangle in the points $A'$, $B'$, $C'$. If $B'C'$, $C'A'$, $A'B'$ meet $BC$, $CA$, $AB$ respectively at $A''$, $B''$, $C''$, these points are collinear. Show also that $AA'$, $AA''$ are harmonic conjugates with respect to $AB$ and $AC$.*

Let $ABC$ be the triangle of reference and $P$ the point $(x_1, y_1, z_1)$. The equation of $BP$ is then $z/z_1 - x/x_1 = 0$, so that $B'$ is the point of intersection of this line and $y = 0$.

The equation of any straight line through $B'$ is therefore of the form $z/z_1 - x/x_1 + \mu y = 0$.

Similarly, the equation of any straight line through $C'$ is of the form $x/x_1 - y/y_1 + \nu z = 0$.

The equation $-x/x_1 + y/y_1 + z/z_1 = 0$ is in both of these forms and therefore represents $B'C'$.

The coordinates of the point $A''$ are therefore given by $x = 0$ and $-x/x_1 + y/y_1 + z/z_1 = 0$; evidently, then, $A''$ lies on the line

$$x/x_1 + y/y_1 + z/z_1 = 0.$$

It is evident by symmetry that $B''$, $C''$ also lie on this straight line.

The equations of the straight lines $AC$, $AB$, $AA'$, $AA''$ are $y = 0$, $z = 0$, $y/y_1 - z/z_1 = 0$, $y/y_1 + z/z_1 = 0$; these form a harmonic pencil.

*To find the equation of the straight line drawn through a given point $(\alpha_1, \beta_1, \gamma_1)$ in a given direction.*

Let $O$ be the point $(\alpha_1, \beta_1, \gamma_1)$, and let $P(\alpha, \beta, \gamma)$ be any other point on the straight line. Suppose that $OP$ makes angles $\theta_1$, $\theta_2$, $\theta_3$

with the perpendiculars from $O$ to the sides, measured in the same way as the angles between $OX$ and these perpendiculars in § 2 (C).

Then, if $OP = r$, we have

$$\alpha_1 - \alpha = r\cos\theta_1\,; \quad \beta_1 - \beta = r\cos\theta_2\,; \quad \gamma_1 - \gamma = r\cos\theta_3,$$

so that

$$\frac{\alpha-\alpha_1}{\cos\theta_1} = \frac{\beta-\beta_1}{\cos\theta_2} = \frac{\gamma-\gamma_1}{\cos\theta_3} = -r.$$

This equation is not homogeneous; it is sometimes useful since $r$ is the actual distance between $(\alpha, \beta, \gamma)$ and $(\alpha_1, \beta_1, \gamma_1)$.

The areal equation of a straight line, $px+qy+rz=0$, can be put in this form: let $(x_1, y_1, z_1)$ be a fixed point on the line, $(x, y, z)$ any other point.

Then $$px+qy+rz = 0,$$

and $$px_1+qy_1+rz_1 = 0.$$

Hence $$p(x-x_1)+q(y-y_1)+r(z-z_1) = 0,$$

and $$(x-x_1)+(y-y_1)+(z-z_1) = 0,$$

since $$x+y+z = x_1+y_1+z_1 = 1.$$

Hence $$\frac{x-x_1}{q-r} = \frac{y-y_1}{r-p} = \frac{z-z_1}{p-q}.$$

Conversely, if the equation of a straight line is given in this non-homogeneous form, we can obtain the homogeneous equation of the line. Let the given equation be $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$, and let each fraction equal $k$.

Then $x-x_1-lk=0$, $y-y_1-mk=0$, $z-z_1-nk=0$; hence, we have

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ l & m & n \end{vmatrix} = 0$$

for the required equation.

### Examples XII b.

1. Find the equation of the line joining the points $(2:-3:5)$, $(1:-4:-2)$.

2. Find the areal equations of the straight lines joining the in-centre of the triangle of reference to the ex-centres.

3. Find the trilinear and areal equations of the medians of the triangle of reference.

4. Find the equation of the straight line joining $A$ to the point of intersection of $BC$ and $lx+my+nz=0$.

5. Find the equation of the line joining the vertex $A$ to the point of intersection of the lines $lx+my+nz=0$, $l'x+m'y+n'z=0$.

6. Find the trilinear and areal equations of the bisectors of the angles of the triangle of reference.

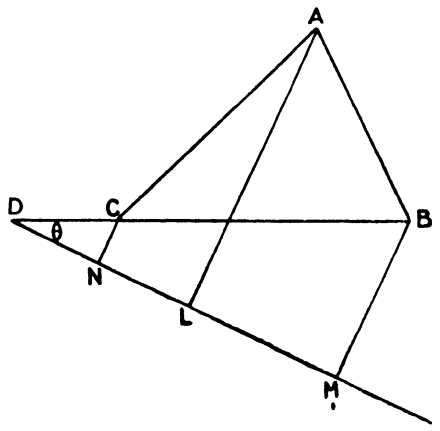
7. Prove that $a\cos\theta_1 + b\cos\theta_2 + c\cos\theta_3 = 0$, where $\theta_1, \theta_2, \theta_3$ are the

angles made by any straight line with the perpendiculars from any point to the sides of the triangle of reference.

8. Prove, by using § 2 (B), that the straight line $l\alpha + m\beta + n\gamma = 0$ makes with $AC$ an angle $\theta$ such that $\tan\theta = (l\sin C - n\sin A)/(l\cos C - m + n\cos A)$.

§ 4. **Line coordinates.** If $(x, y, z)$ are the areal coordinates of a point, $lx + my + nz = 0$ is the equation of any straight line. The straight line is fixed when we know the values of $l$, $m$, $n$, or of their mutual ratios $l : m : n$. As we have seen in Chapter X, $l$, $m$, $n$ are called the tangential coordinates of the straight line.

Let the straight line $lx + my + nz = 0$ meet $BC$ at $D$, and draw perpendiculars $AL$, $BM$, $CN$ to it from $A$, $B$, $C$. We shall call the lengths of these perpendiculars $p$, $q$, $r$. The signs of the ratios $p : q : r$ are determined by the following convention: $q$ and $r$ have the same sign when the line cuts $BC$ externally, and opposite signs when it cuts $BC$ internally; and similarly $p$ and $q$ have the same or opposite signs according as the line cuts $AB$ externally or internally. In the figure $p$, $q$, $r$ are all of the same sign.




Now $D$ is the point of intersection of $x = 0$ and $lx + my + nz = 0$; it is therefore the point $(0 : n : -m)$. We have then

$$\frac{q}{r} = \frac{DB}{DC} = \frac{\Delta DAB}{\Delta DAC} = \frac{m}{n},$$

for since $B$ and $D$ are on opposite sides of $AC$, while $C$ and $D$ are on the same side of $AB$, the areal coordinates of $D$ are

$$\left\{0, -\frac{\Delta DCA}{\Delta ABC}, \frac{\Delta DAB}{\Delta ABC}\right\}.$$

The reader should work out similarly the case where the line cuts $BC$ internally, paying special attention to the signs.

In all cases we have then $l : m : n = p : q : r$, so that $(p, q, r)$, with the sign convention explained above, may be regarded as the line coordinates of the straight line $lx + my + nz = 0$. It is evident then that the point equation of the straight line $(p, q, r)$ is

$$px + qy + rz = 0.$$

**Note.** In trilinear coordinates, if $p$, $q$, $r$ have the same meanings, the equation of the straight line is $ap\alpha + bq\beta + rc\gamma = 0$.

*Identical relation between the coordinates of any straight line.*

Let the straight line make an angle $\theta$ with $BC$; then we have from the above figure

$$q-r = a\sin\theta, \qquad \text{(i)}$$

$$p-q = c\sin(B-\theta). \qquad \text{(ii)}$$

Hence from (ii)

$$p-q+c\cos B\sin\theta = c\sin B\cos\theta;$$

$$\therefore \quad a(p-q)+c\cos B(q-r) = ac\sin B\cos\theta. \qquad \text{(iii)}$$

Eliminating $\theta$ from (i) and (iii), we have

$$a^2p^2+b^2q^2+c^2r^2-2bcqr\cos A-2carp\cos B-2abpq\cos C = 4S^2,$$

which is generally referred to as $\{ap, bq, cr\}^2 = 4S^2$.

**Note.** This relation can also be written

$$a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q) = 4S^2,$$

or

$$(q-r)^2\cot A+(r-p)^2\cot B+(p-q)^2\cot C = 2S.$$

*The tangential equation of a point.*

The equation in point coordinates of the straight line $(p, q, r)$ is $px+qy+rz=0$; this passes through a particular point $(x_1, y_1, z_1)$ if $px_1+qy_1+rz_1=0$. So any straight line, whose coordinates $p, q, r$ satisfy this equation, passes through the point $(x_1, y_1, z_1)$. Hence $px_1+qy_1+rz_1=0$ is the tangential equation of the point whose areal coordinates are $(x_1, y_1, z_1)$. In general, then, the equation $px+qy+rz=0$ is the point equation of the line $(p, q, r)$ or the line equation of the point $(x, y, z)$.

**Note i.** The coordinates of the sides $BC$, $CA$, $AB$ of the triangle of reference are $(1:0:0)$, $(0:1:0)$, $(0:0:1)$.

**Note ii.** The equations of the vertices $A$, $B$, $C$ of the triangle of reference are $p=0$, $q=0$, $r=0$.

**Note iii.** The coordinates of a straight line parallel to $(p, q, r)$ and at a distance $k$ from it are $p\pm k$, $q\pm k$, $r\pm k$.

## Illustrative Examples.

### Ceva's Theorem.

*The lines joining the vertices of a triangle $ABC$ to a given point meet the opposite sides at $P$, $Q$, $R$; show that*

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA} = 1.$$

Let the given point be $(x_1, y_1, z_1)$.

### Menelaus' Theorem.

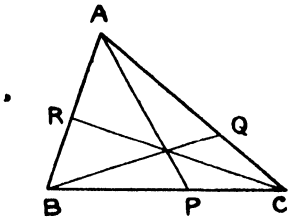
*A straight line cuts the sides of a triangle $ABC$ at the points $P$, $Q$, $R$; show that*

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA} = -1.$$

Let the given straight line be $(p_1, q_1, r_1)$.

The equation of the join of $A$ to the given point is

$$y/y_1 - z/z_1 = 0.$$




The coordinates of $P$, the intersection of this line and $BC$, are therefore $(0 : y_1 : z_1)$.

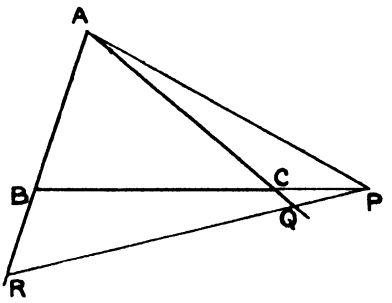
Hence $BP : PC = z_1 : y_1$; this relation is true for all positions of $P$, if we use the usual convention of signs, viz. $PC = -CP$.

Thus $BP : PC = z_1 : y_1$,
$CQ : QA = x_1 : z_1$,
$AR : RB = y_1 : x_1$.

Multiplying these together we get the required result.

The equation of the intersection of $BC$ and the given line is

$$q/q_1 - r/r_1 = 0.$$




The coordinates of $AP$, the join of this point and $A$, are therefore

$$(0 : q_1 : r_1).$$

Hence $BP : CP = q_1 : r_1$; this relation is true for all positions of $P$, if we use the usual convention of signs, viz. $PC = -CP$.

Thus $BP : PC = q_1 : -r_1$,
$CQ : QA = r_1 : -p_1$,
$AR : RB = p_1 : -q_1$.

Multiplying these together we get the required result.

§ 5. (i) *To find the distance between the points* $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$.

Let $(X_1, Y_1)$, $(X_2, Y_2)$ be the Cartesian coordinates of the points referred to $CA$, $CB$ as axes of $x$ and $y$, and $d$ the required distance.

Then $d^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2 + 2(X_1 - X_2)(Y_1 - Y_2)\cos C$.

Now $X_1 = bx_1$ and $Y_1 = ay_1$; hence

$$\begin{aligned} d^2 &= b^2(x_1 - x_2)^2 + a^2(y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\,ab\cos C \\ &= b^2(x_1 - x_2)^2 + a^2(y_1 - y_2)^2 + (x_1 - x_2)(y_1 - y_2)(a^2 + b^2 - c^2) \\ &= a^2(y_1 - y_2)(x_1 + y_1 - x_2 - y_2) + b^2(x_1 - x_2)(x_1 + y_1 - x_2 - y_2) \\ &\qquad - c^2(x_1 - x_2)(y_1 - y_2). \end{aligned}$$

But $x_1 + y_1 + z_1 = 1, \quad x_2 + y_2 + z_2 = 1$;

$$\therefore\ d^2 = -a^2(y_1 - y_2)(z_1 - z_2) - b^2(z_1 - z_2)(x_1 - x_2) - c^2(x_1 - x_2)(y_1 - y_2).$$

Thus $$d^2 = -\Sigma a^2(y_1 - y_2)(z_1 - z_2).$$

This expression can also be written $d^2 = \Sigma bc\cos A\,(x_1 - x_2)^2$.

Symmetrical expressions for the distance between two points, whose trilinear coordinates are given, can be deduced.

**Cor. i.** If $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=\rho$ is the equation of a straight line, where $\rho$ is the distance between the points $(x, y, z)$, $(x_1, y_1, z_1)$, then $l+m+n=0$ and $a^2mn+b^2nl+c^2lm=-1$.

**Cor. ii.** The equation of the straight line $(p, q, r)$ in the non-homogeneous form is

$$\frac{x-x_1}{q-r}=\frac{y-y_1}{r-p}=\frac{z-z_1}{p-q},$$

and each of these fractions

$$=\left\{\frac{\Sigma a^2(y-y_1)(z-z_1)}{\Sigma a^2(r-p)(p-q)}\right\}^{\frac{1}{2}}=\frac{\rho}{2S},$$

where $\rho$ is the actual distance between the points $(x, y, z)$, $(x_1, y_1, z_1)$; for we showed in § 4 that

$$a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)=4S^2.$$

We have taken the positive sign of the radical; there is no loss of generality, since (i) $p, q, r$ can all have their signs changed, only the signs of their mutual ratios being determined by our sign convention, and (ii) we can regard $\rho$ as positive or negative according to which direction we choose as the positive.

Thus the equation of the straight line $(p, q, r)$ can be written

$$\frac{x-x_1}{q-r}=\frac{y-y_1}{r-p}=\frac{z-z_1}{p-q}=\frac{\rho}{2S},$$

where $(x_1, y_1, z_1)$ is a fixed point on the line, and $\rho$ the distance of any other point $(x, y, z)$ on the line from it.

(ii) *To find the area of the triangle whose vertices are the points* $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$.

Let the Cartesian coordinates of the vertices referred to $CA$, $CB$ as axes be $(X_1, Y_1)$, $(X_2, Y_2)$, $(X_3, Y_3)$; then the area of the triangle

$$=\tfrac{1}{2}\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix}\sin C$$

$$=\tfrac{1}{2}\begin{vmatrix} bx_1 & ay_1 & 1 \\ bx_2 & ay_2 & 1 \\ bx_3 & ay_3 & 1 \end{vmatrix}\sin C$$

$$=S\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=S\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

since $x_1+y_1+z_1=1$, &c.

In trilinear coordinates the result is

$$\frac{R}{2S}\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{vmatrix}.$$

(iii) *To find the length of the perpendicular from the point* $(\alpha_1, \beta_1, \gamma_1)$ *to the line* $l\alpha+m\beta+n\gamma=0$.

Transform the equation $l\alpha+m\beta+n\gamma=0$ into rectangular Cartesian coordinates by § 2 (C); the equation becomes

$$U\equiv(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)X+(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)Y - (lp_1+mp_2+np_3)=0.$$

If $(X_1, Y_1)$ are the coordinates of the point $(\alpha_1, \beta_1, \gamma_1)$, the length of the perpendicular from it to the straight line is

$$U_1 \div \{(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)^2+(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)^2\}^{\frac{1}{2}},$$

where $U_1$ is the result of substituting $X_1$, $Y_1$ for $X$, $Y$ in $U$. Evidently $U_1=l\alpha_1+m\beta_1+n\gamma_1$.

Now $(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)^2+(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)^2$
$=l^2+m^2+n^2+2mn\cos(\theta_2-\theta_3)+2nl\cos(\theta_3-\theta_1)+2lm\cos(\theta_1-\theta_2)$
$=l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C$
$=\{l, m, n\}^2$.

The length of the required perpendicular is then

$$(l\alpha_1+m\beta_1+n\gamma_1)/\{l, m, n\}.$$

**Cor.** In areal coordinates the length of the perpendicular from $(x_1, y_1, z_1)$ to $lx+my+nz=0$ is $2S(lx_1+my_1+nz_1)\div\{al, bm, cn\}$.

**Note.** If $p, q, r$ are the perpendiculars from the vertices of the triangle of reference to the straight line $lx+my+nz=0$, we have

$$p=2Sl/\{al, bm, cn\},$$
$$q=2Sm/\{al, bm, cn\},$$
$$r=2Sn/\{al, bm, cn\},$$

which verifies our previous result, $l:m:n=p:q:r$.

If we substitute these values of $l, m, n$ in the formula for the perpendicular from $(x_1, y_1, z_1)$ to the line $lx+my+nz=0$, we obtain $px_1+qy_1+rz_1$ for its length.

Now the equation of the straight line is $px+qy+rz=0$, hence, applying the formula to this, we find that the length of the perpendicular from $(x_1, y_1, z_1)$ to it is

$$2S(px_1+qy_1+rz_1)\div\{ap, bq, cr\}.$$

Hence $$\{ap, bq, cr\}=2S,$$

or $$a^2p^2+b^2q^2+c^2r^2-2bcqr\cos A-2carp\cos B-2abpq\cos C=4S^2,$$

which we proved independently in § 4.

(iv) *To find the angle between the straight lines* $l\alpha+m\beta+n\gamma=0$, $l'\alpha+m'\beta+n'\gamma=0$.

Transform the equations by § 2 (C); then in Cartesian rectangular coordinates the straight lines are parallel to

$$X(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)+Y(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)=0,$$
$$X(l'\cos\theta_1+m'\cos\theta_2+n'\cos\theta_3)+Y(l'\sin\theta_1+m'\sin\theta_2+n'\sin\theta_3)=0.$$

Using the formula for the tangent of the angle between two straight lines, Chapter II, § 8, the numerator is

$$\begin{aligned}&(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)(l'\sin\theta_1+m'\sin\theta_2+n'\sin\theta_3)\\&\qquad-(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)(l'\cos\theta_1+m'\cos\theta_2+n'\cos\theta_3)\\&=(mn'-m'n)\sin(\theta_3-\theta_2)+(nl'-n'l)\sin(\theta_1-\theta_3)\\&\qquad+(lm'-l'm)\sin(\theta_2-\theta_1)\\&=(mn'-m'n)\sin A+(nl'-n'l)\sin B+(lm'-l'm)\sin C\\&=\begin{vmatrix}\sin A&\sin B&\sin C\\l&m&n\\l'&m'&n'\end{vmatrix}.\end{aligned}$$

The denominator is

$$\begin{aligned}&(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)(l'\cos\theta_1+m'\cos\theta_2+n'\cos\theta_3)\\&\qquad+(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)(l'\sin\theta_1+m'\sin\theta_2+n'\sin\theta_3)\\&=ll'+mm'+nn'+(mn'+m'n)\cos(\theta_3-\theta_2)+(nl'+n'l)\cos(\theta_1-\theta_3)\\&\qquad+(lm'+l'm)\cos(\theta_2-\theta_1)\\&=ll'+mm'+nn'-(mn'+m'n)\cos A-(nl'+n'l)\cos B\\&\qquad-(lm'+l'm)\cos C.\end{aligned}$$

The tangent of the required angle is therefore

$$\begin{vmatrix}\sin A&\sin B&\sin C\\l&m&n\\l'&m'&n'\end{vmatrix}\div\{\Sigma ll'-\Sigma(mn'+m'n)\cos A\}.$$

In areal coordinates the angle between the lines $px+qy+rz=0$, $p'x+q'y+r'z=0$ is the same as the angle between the lines whose trilinear equations are $ap\alpha+bq\beta+cr\gamma=0$, $ap'\alpha+bq'\beta+cr'\gamma=0$, viz. the angle whose tangent is

$$\begin{vmatrix}\sin A&\sin B&\sin C\\ap&bq&cr\\ap'&bq'&cr'\end{vmatrix}\div\{\Sigma a^2pp'-\Sigma(qr'+q'r)bc\cos A\}$$
$$=\tan^{-1}2S\begin{vmatrix}1&1&1\\p&q&r\\p'&q'&r'\end{vmatrix}\div\{\Sigma a^2pp'-\Sigma(qr'+q'r)bc\cos A\}.$$

*Conditions that two straight lines should be* (i) *parallel or* (ii) *perpendicular.*

(a) **Trilinears.**

The straight lines $l\alpha+m\beta+n\gamma=0$, $l'\alpha+m'\beta+n'\gamma=0$ are *parallel* if

$$\begin{vmatrix} \sin A & \sin B & \sin C \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0;$$

this is equivalent to

$$\begin{vmatrix} a & b & c \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0.$$

The straight lines are *perpendicular* if

$$\Sigma ll' - \Sigma (mn'+m'n)\cos A = 0.$$

(b) **Areals.**

The straight lines $px+qy+rz=0$, $p'x+q'y+r'z=0$ are *parallel if*

$$\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p' & q' & r' \end{vmatrix} = 0.$$

The straight lines are *perpendicular* if

$$\Sigma a^2 pp' - \Sigma (qr'+q'r)\, bc\cos A = 0.$$

This result can also be written

$$\Sigma a^2 \{(p-q)(r'-p')+(p'-q')(r-p)\} = 0,$$

or
$$\Sigma (b^2+c^2-a^2)(q-r)(q'-r') = 0,$$

or
$$\Sigma (q-r)(q'-r')\cot A = 0.$$

**Note.** The reader can obtain the results of (iii) and (iv) by using either of the transformations in § 2, (A) or (B).

§ 6. (i) **The straight line at infinity.** Let a straight line $lx+my+nz=0$ cut the sides of the triangle of reference, each externally, at the points $D$, $E$, $F$. We have shown in § 4 that $BD/CD = m/n$; $CE/AE = n/l$; $AF/BF = l/m$. Now as the straight line recedes in any direction each of the ratios $BD/CD$, $CE/AE$, $AF/BF$ tends to become equal to unity. Thus, when any straight line recedes in any direction, its equation tends to the limiting form $x+y+z=0$. This is then the equation of the 'straight line at infinity'. The corresponding equation in trilinear coordinates is $a\alpha+b\beta+c\gamma=0$.

Now the condition that the straight lines $px+qy+rz=0$, $p'x+q'y+r'z=0$ should be parallel, is

$$\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p' & q' & r' \end{vmatrix} = 0;$$

this is the same as the condition that the three straight lines $x+y+z=0$, $px+qy+rz=0$, $p'x+q'y+r'z=0$ should be concurrent. Thus, analytically, parallel straight lines meet at infinity.

**Note i.** The equation $px+qy+rz+\lambda(x+y+z)=0$ represents for different values of $\lambda$ a system of straight lines parallel to $px+qy+rz=0$.

**Note ii.** The equation of a straight line through $(x_1, y_1, z_1)$ parallel to $px+qy+rz=0$ is $(px+qy+rz)(x_1+y_1+z_1)=(x+y+z)(px_1+qy_1+rz_1)$.

**Note iii.** The 'point at infinity' on the line $px+qy+rz=0$ is

$$\{q-r,\ r-p,\ p-q\}.$$

**Note iv.** The line coordinates of the line at infinity are in areals $(1:1:1)$ and in trilinears $(a:b:c)$.

(ii) **The circular points at infinity, $\Omega$, $\Omega'$.** In Cartesian rectangular coordinates $Lx+My+N=0$ is the general equation of a straight line and $L^2+M^2=0$ is the tangential equation of the circular points at infinity $\Omega$, $\Omega'$. When the general trilinear equation of a straight line, $l\alpha+m\beta+n\gamma=0$, is transformed to Cartesian rectangular coordinates we have seen that

$$L = l\cos\theta_1 + m\cos\theta_2 + n\cos\theta_3,$$
$$M = l\sin\theta_1 + m\sin\theta_2 + n\sin\theta_3.$$

The trilinear equation of the circular points at infinity is therefore

$$(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)^2+(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)^2=0,$$

i.e. $$l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C=0,$$

or $$\{l,\ m,\ n\}^2=0.$$

The corresponding equation in areal coordinates is

$$a^2p^2+b^2q^2+c^2r^2-2bcqr\cos A-2carp\cos B-2abpq\cos C=0,$$

or $$\{ap,\ bq,\ cr\}^2=0.$$

The separate equations of $\Omega$, $\Omega'$ are

$$(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)\pm i(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)=0,$$

i.e. $$le^{i\theta_1}+me^{i\theta_2}+ne^{i\theta_3}=0,\quad le^{-i\theta_1}+me^{-i\theta_2}+ne^{-i\theta_3}=0,$$

or, in areal coordinates,

$$ape^{i\theta_1}+bqe^{i\theta_2}+cre^{i\theta_3}=0,\quad ape^{-i\theta_1}+bqe^{-i\theta_2}+cre^{-i\theta_3}=0.$$

The trilinear coordinates of $\Omega$, $\Omega'$ are then

$$(e^{i\theta_1} : e^{i\theta_2} : e^{i\theta_3}), \quad (e^{-i\theta_1} : e^{-i\theta_2} : e^{-i\theta_3}),$$

or

$$(e^{-iB} : e^{iA} : -1), \quad (e^{iB} : e^{-iA} : -1),$$

with corresponding results in areal coordinates.

§ 7. The general equation of the second degree in areal coordinates is

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0,$$

where $u$, $v$, $w$ replace $a$, $b$, $c$, since the latter have a special significance in this chapter.

The condition that this equation should represent a pair of straight lines can be found in a similar manner to that used for Cartesian coordinates, viz.

$$\Delta = \begin{vmatrix} u & h & g \\ h & v & f \\ g & f & w \end{vmatrix} = 0.$$

If the equation represents straight lines, we have

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = (lx + my + nz)(l'x + m'y + n'z). \quad \text{(i)}$$

The straight lines are perpendicular if

$$\Sigma a^2 ll' = \Sigma (mn' + m'n) bc \cos A.$$

Comparing coefficients in equation (i), we get this condition in the form

$$\Sigma a^2 u = \Sigma 2fbc \cos A.$$

Hence the equation $ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of perpendicular straight lines if $\Delta = 0$ and

$$a^2u + b^2v + c^2w - 2fbc \cos A - 2gca \cos B - 2hab \cos C = 0.$$

**Note.** In trilinear coordinates, the equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0$$

represents a pair of perpendicular straight lines if $\Delta = 0$ and

$$u + v + w - 2f \cos A - 2g \cos B - 2h \cos C = 0.$$

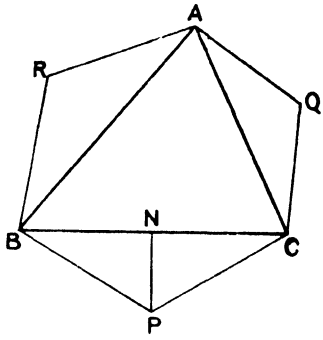
### Illustrative Examples.

**Ex. 1.** *On the three sides of a triangle similar isosceles triangles are described: show that the triangle formed by their vertices is copolar with the given triangle and when the base angles of the isosceles triangles are each $\theta$, find the equation of the axis of perspective of the triangles.*

Let $ABC$, the given triangle, be the triangle of reference and use trilinear coordinates.

If $PN$ is perpendicular to $BC$, since $PB = PC$, the coordinates of $P$ are

$$\{-PC\sin\theta,\ PC\sin\overline{C+\theta},\ PC\sin\overline{B+\theta}\} \text{ or } \{-\sin\theta : \sin\overline{C+\theta} : \sin\overline{B+\theta}\}.$$




The equation $\beta\sin\overline{B+\theta} - \gamma\sin\overline{C+\theta} = 0$ represents a straight line through $A$ and $P$, i.e. the straight line $AP$. Symmetrically the equations of $BQ$ and $CR$ are

$$\gamma\sin\overline{C+\theta} - \alpha\sin\overline{A+\theta} = 0,\quad \alpha\sin\overline{A+\theta} - \beta\sin\overline{B+\theta} = 0.$$

Hence $AP$, $BQ$, $CR$ are concurrent at the point

$$\{\operatorname{cosec}\overline{A+\theta} : \operatorname{cosec}\overline{B+\theta} : \operatorname{cosec}\overline{C+\theta}\},$$

and the triangles $ABC$, $PQR$ are copolar.

Again, since the coordinates of $Q$ are $\{\sin\overline{C+\theta} : -\sin\theta : \sin\overline{A+\theta}\}$, the equation of $PQ$ is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ -\sin\theta & \sin\overline{C+\theta} & \sin\overline{B+\theta} \\ \sin\overline{C+\theta} & -\sin\theta & \sin\overline{A+\theta} \end{vmatrix} = 0,$$

and this meets the side $AB$, whose equation is $\gamma = 0$, where

$$\alpha\{\sin\overline{A+\theta}\sin\overline{C+\theta} + \sin\theta\sin\overline{B+\theta}\} + \beta\{\sin\overline{B+\theta}\sin\overline{C+\theta} + \sin\theta\sin\overline{A+\theta}\} = 0.$$

This point evidently lies on the straight line

$$\alpha/\{\sin\overline{B+\theta}\sin\overline{C+\theta} + \sin\theta\sin\overline{A+\theta}\} + \beta/\{\sin\overline{C+\theta}\sin\overline{A+\theta} + \sin\theta\sin\overline{B+\theta}\} + \gamma/\{\sin\overline{A+\theta}\sin\overline{B+\theta} + \sin\theta\sin\overline{C+\theta}\} = 0.$$

The symmetry of this result shows that the points of intersection of $RP$, $CA$ and $QR$, $BC$ also lie on this line; it is therefore the axis of homology of the triangles.

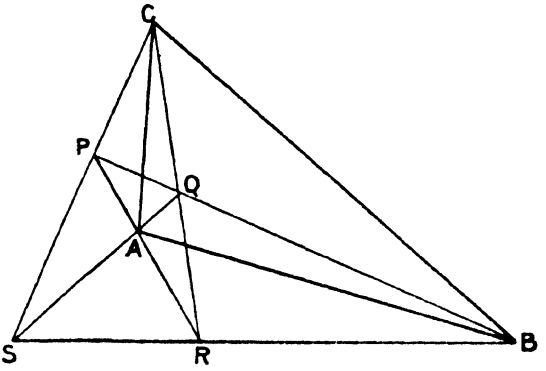
**Ex. 2.** *If the diagonal points of a quadrangle are taken as the vertices of the triangle of reference, the coordinates of the vertices of the quadrangle are of the form* $(x_1 : y_1 : z_1)$, $(-x_1 : y_1 : z_1)$, $(x_1 : -y_1 : z_1)$, $(x_1 : y_1 : -z_1)$.

Let $PQRS$ be the quadrangle and $A$, $B$, $C$ the diagonal points.

Let $P$ be the point $(x_1 : y_1 : z_1)$; then since $PR$ passes through $A$, the point $R$ is $(\lambda x_1 : y_1 : z_1)$ where $\lambda$ is some constant.

So the points $Q$, $S$ are $\{x_1 : \mu y_1 : z_1\}$, $\{x_1 : y_1 : \nu z_1\}$.

Now $QR$ passes through $C$, so that $\lambda x_1 : y_1 = x_1 : \mu y_1$, i.e. $\lambda\mu = 1$; similarly $\mu\nu = 1$ and $\nu\lambda = 1$.




Hence either $\lambda = \mu = \nu = 1$ or $\lambda = \mu = \nu = -1$; the positive value is inadmissible, for in that case $P$, $Q$, $R$, $S$ would be identical, thus

$$\lambda = \mu = \nu = -1.$$

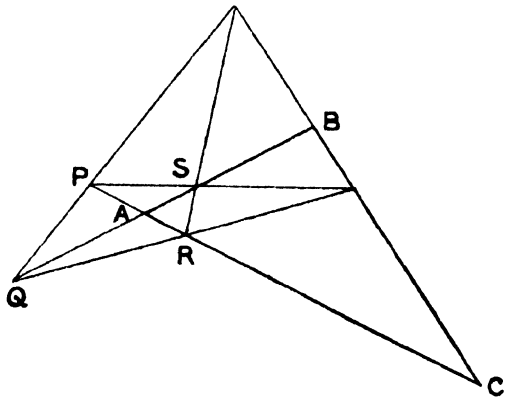
The coordinates of the vertices are then $(x_1 : y_1 : z_1)$, $(-x_1 : y_1 : z_1)$, $(x_1 : -y_1 : z_1)$, $(x_1 : y_1 : -z_1)$.

**Note.** *This is the simplest way of denoting four points, no three of which are collinear.*

**Ex. 3.** *If the diagonals of a quadrilateral are taken as the sides of the triangle of reference, the equations of the sides of the quadrilateral are of the form* $px+qy+rz = 0$, $-px+qy+rz = 0$, $px-qy+rz = 0$, $px+qy-rz = 0$.

Let $PQRS$ be the quadrilateral and $AB$, $BC$, $CA$ the diagonals.

Let $PQ$ be the line $px+qy+rz = 0$, then, since $PQ$ and $RQ$ intersect on $AB$,




whose equation is $z = 0$, the equation of $QR$ is of the form $px+qy+\nu rz = 0$.

Similarly the equations of $RS$ and $PS$ are of the form $\lambda px+qy+rz = 0$ and $px+\mu qy+rz = 0$.

Now $PS$, $QR$ and $BC$ are concurrent, i.e. $px+\mu qy+rz=0$, $px+qy+\nu rz=0$ and $x=0$. Hence $\mu\nu=1$.

Similarly $\lambda\mu=1$ and $\nu\lambda=1$, so that $\lambda=\mu=\nu=1$ or $\lambda=\mu=\nu=-1$. The positive value makes the equations of the four sides the same and is inadmissible; therefore $\lambda=\mu=\nu=-1$. The equations of the sides of the quadrilateral are therefore $px+qy+rz=0$, $-px+qy+rz=0$, $px-qy+rz=0$, $px+qy-rz=0$.

**Note.** *This is the simplest way of denoting four straight lines no three of which are concurrent.*

## Examples XII c.

1. Find the area of the triangle whose vertices are the feet of the perpendiculars from the vertices of the triangle of reference on the opposite sides.

2. Find the area of a triangle whose vertices are given by

$$\left.\begin{matrix}\alpha=0\\ m\beta-n\gamma=0\end{matrix}\right\}\quad\left.\begin{matrix}\beta=0\\ n\gamma-l\alpha=0\end{matrix}\right\}\quad\left.\begin{matrix}\gamma=0\\ l\alpha-m\beta=0\end{matrix}\right\}.$$

3. Show that the orthocentre, centroid, and circumcentre of a triangle are collinear, and find the areal equation of the line.

4. Find the equation in trilinears of the line joining the points where the bisectors of the angles $A$, $B$ meet the opposite sides.

5. Find the equation to a straight line which cuts off $\frac{1}{m}^{\text{th}}$ and $\frac{1}{n}^{\text{th}}$ from the sides $AB$, $AC$ and the coordinates of the point where it meets $BC$. (Areals.)

6. Find the lengths of the perpendiculars from $A$, $B$, $C$ on the straight line joining the in-centre and circumcentre of the triangle of reference, and deduce the length of the perpendicular on it from the centroid.

7. Find the area of the triangle whose sides are $b\beta+c\gamma=0$, $c\gamma+a\alpha=0$, $a\alpha+b\beta=0$.

8. Show that the straight line $(x-y+z)\cot B=(x+y-z)\cot C$ bisects the side $BC$ of the triangle of reference at right angles.

9. Find the equations of the straight lines parallel to $px+qy+rz=0$ and distant $d$ from it. Hence show that the length of the perpendicular from $(x_1, y_1, z_1)$ to $px+qy+rz=0$ is $px_1+qy_1+rz_1$.

10. Find the equation of the point of intersection of the lines $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$.

11. Show that the line $(lp_1+mp_2,\ lq_1+mq_2,\ lr_1+mr_2)$ passes through the intersection of the lines $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$ for all values of $l/m$.

12. What are the conditions that the lines $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$ should be (a) parallel, (b) perpendicular?

13. Interpret the equations:

(i) $q+r=0$; (ii) $q-r=0$;
(iii) $q\tan B+r\tan C=0$; (iv) $p+q+r=0$.

14. Prove that the equation $ap \pm bq \pm cr = 0$ represents the in-centre and ex-centres of the triangle of reference.

15. Find the equation of the line joining the feet of the perpendiculars from $B$, $C$ to the opposite sides. Prove that this line intersects the line bisecting $AB$, $AC$ in the point whose trilinear coordinates are

$$\{\sin \overline{C-B} : \sin \overline{C-A} : \sin \overline{A-B}\}.$$

16. Find the equation of the straight line joining the point of intersection of $x = z\cos B$, $y = z\cos A$ to the vertex $C$.

17. Find the equation of the straight line through the centroid of $ABC$ parallel to the straight line $px+qy+rz=0$, and of the line through $A$ parallel to $BC$.

18. Use the formulae to find the distance between the circumcentre and orthocentre of the triangle of reference.

19. Show that $\alpha+\beta+\gamma=0$ is perpendicular to the join of the in-centres and circumcentres.

20. Prove that
$$(a+d)\alpha+(b+d)\beta+c\gamma=0,$$
$$(a+d)\alpha+(b-d)\beta+c\gamma=0$$
are perpendicular for all values of $d$.

21. Find the equation of the line through $(\alpha_1, \beta_1, \gamma_1)$ perpendicular (i) to $BC$; (ii) to $l\alpha+m\beta+n\gamma=0$.

22. $ABC$ is a triangle; through any point $P$, $DPE$ is drawn parallel to $AB$ cutting $CA$ in $D$ and $BC$ in $E$; similarly $FPG$ is drawn parallel to $CA$ and $HPK$ parallel to $BC$. $DG$ and $EH$ are produced to intersect in $Q$: show that $CPQ$ is a straight line.

23. A point $O$ is taken inside the triangle $ABC$; $AO$, $BO$, $CO$ meet the opposite sides in $A'$, $B'$, $C'$. If a line through $A$ meets $A'B'$, $C'A'$ in $B''$, $C''$ respectively, show that the intersection of the lines $BB''$, $CC''$ lies on $B'C'$.

24. Find the equations of two straight lines through the mid-point of $BC$, equally inclined to $BC$, in areal and trilinear coordinates.

25. $O$ is the orthocentre of a triangle $ABC$; $BO$, $CO$ meet $AC$ and $AB$ at $M$ and $N$; $MN$ meets $BC$ at $P$. Prove that $OP$ is perpendicular to the line joining $A$ to the mid-point of $BC$.

26. Find the equation of the three lines joining the feet of the perpendiculars from the angular points $ABC$ on the opposite sides and show that these lines meet the corresponding sides in three collinear points.

27. Find the equation of the lines through the centroid of $ABC$ perpendicular to the lines joining the centroid to the vertices $A$, $B$, $C$, and show that they meet the corresponding sides of the triangle in three collinear points.

28. Prove that the lines through $B$ parallel to $\beta\cos C+\gamma\cos\overline{C-A}=0$ and through $C$ parallel to $\beta\cos\overline{A-B}+\gamma\cos B=0$ meet at the circumcentre.

29. Find the equation of the bisectors of the angles between the lines
$$\alpha\cos A+\beta\cos B-\gamma\cos C=0,$$
$$\alpha\cos A-\beta\cos B+\gamma\cos C=0.$$

30. The mid-points of the diagonals of the quadrilateral formed by $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n} = 0$ lie on $\Sigma al(b\beta + c\gamma - a\alpha) = 0$.

31. Find the equations of the lines isogonal with $AP$, $BP$, $CP$ when $P$ is the point $(\alpha_1, \beta_1, \gamma_1)$ and show they are concurrent. When $P$ is the centroid, find the coordinates of the point of intersection.

32. If $\theta$ is the angle between the straight lines

$$lx + my + nz = 0, \quad l'x + m'y + n'z = 0,$$

prove that $\{al, bm, cn\}\{al', bm', cn'\} \sin\theta = 2S \cdot \Sigma l(m' - n')$.

33. If $\frac{x - x'}{\lambda} = \frac{y - y'}{\mu} = \frac{z - z'}{\nu} = \rho$, where $\rho$ is the distance between the points $(x, y, z)$, $(x', y', z')$, prove that $\lambda^2 bc \cos A + \mu^2 ca \cos B + \nu^2 ab \cos C = 1$.

34. Prove that the sine of the angle between the straight lines $(p, q, r)$, $(p', q', r')$ is $\frac{1}{2S}\begin{vmatrix} p & q & r \\ p' & q' & r' \\ 1 & 1 & 1 \end{vmatrix}$.

35. The sides of a quadrilateral are $lx \pm my \pm nz = 0$; find the coordinates of the mid-points of its diagonals and prove that they are collinear. Find the equation of the line.

36. If $(x, y, z)$, $(x', y', z')$ are the points at infinity on two orthogonal straight lines, prove that $\Sigma xx' \cot A = 0$.

37. The distance between the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ can be written in the form $abc\Sigma(\alpha_1 - \alpha_2)^2 a \cos A/4S^2$.

**§ 8. The general equation of the second degree.** In the preceding sections we have assigned to $x$, $y$, $z$ precise meanings; in the following work the results, except where they involve metrical quantities such as lengths and angles, hold equally well if we attach to $x$, $y$, $z$ no more precise meaning than that $x = 0$, $y = 0$, $z = 0$ represent three straight lines which form the triangle of reference. The coordinates of a point are then connected with its coordinates in a Cartesian system by three linear equations of the forms $x = l_1 X + m_1 Y + n_1$, $y = l_2 X + m_2 Y + n_2$, $z = l_3 X + m_3 Y + n_3$; but we need not attach any special values to the constants $l_1$, $m_1$, $n_1$, &c., as we do in areals. The vertices of the triangle of reference are given as the points of intersection of the straight lines $x = 0$, $y = 0$, $z = 0$ taken in pairs, i. e. $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$; the actual values of the coordinates evidently cannot be found unless we know the identical relation which connects the three coordinates $x$, $y$, $z$, corresponding to the relations $x + y + z = 1$ in areals and $a\alpha + b\beta + c\gamma = 2S$ in trilinears.

An equation of the second degree in $x$, $y$, $z$ transforms into an

equation of the second degree in Cartesian coordinates; it therefore represents a conic.

The general equation of the second degree is

$$f(x, y, z) \equiv ux^2+vy^2+wz^2+2fyz+2gzx+2hxy = 0.$$

We shall use a notation similar to that employed for the general Cartesian equation of the second degree, thus

$$\begin{aligned}
&\Delta = uvw+2fgh-uf^2-vg^2-wh^2,\\
&U = vw-f^2, \quad V = wu-g^2, \quad W = uv-h^2,\\
&F = gh-uf, \quad G = fh-vg, \quad H = fg-wh,\\
&X = ux+hy+gz, \quad Y = hx+vy+fz,\\
&Z = gx+fy+wz.
\end{aligned}$$

(1) *To find the equation of the tangent to* $f(x, y, z) = 0$ *at the point* $(x_1, y_1, z_1)$.

Let the equation of the tangent be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = \rho$$

where $\rho$ is the distance between $(x, y, z)$ and $(x_1, y_1, z_1)$.

Now substitute $x = l\rho+x_1$, $y = m\rho+y_1$, $z = n\rho+z_1$ in the equation of the conic, and we obtain an equation in $\rho$ giving the distances of the points of intersection of the tangent and the conic from $(x_1, y_1, z_1)$; evidently both these values of $\rho$ are zero. The equation is $\rho^2 . f(l, m, n)+2\rho\{lX_1+mY_1+nZ_1\}+f(x_1, y_1, z_1) = 0$. Now $f(x_1, y_1, z_1) = 0$ since $(x_1, y_1, z_1)$ is on the conic, so that one value of $\rho$ is zero. The second value of $\rho$ being zero we have

$$lX_1+mY_1+nZ_1 = 0.$$

Hence, if $(x, y, z)$ is any point on the tangent, we have

$$(x-x_1)X_1+(y-y_1)Y_1+(z-z_1)Z_1 = 0.$$

Since $x_1X_1+y_1Y_1+z_1Z_1 = f(x_1, y_1, z_1) = 0$, the equation of the tangent at $(x_1, y_1, z_1)$ is

$$xX_1+yY_1+zZ_1 = 0. \qquad (A)$$

The method of Chap. VI, § 4 (v) can also be used.

(2) *To find the tangential equation of* $f(x, y, z) = 0$.

By identifying the equation $px+qy+rz = 0$ with the equation (A) of the tangent, we can show in exactly the same way as in Chapter X, § 11, that the straight line $(p, q, r)$ touches the conic $f(x, y, z) = 0$ if

$$F(p, q, r) \equiv Up^2+Vq^2+Wr^2+2Fqr+2Grp+2Hpq = 0. \qquad (B)$$

This is therefore the tangential equation of the conic $f(x, y, z) = 0$.

**Cor.** If the conic is a parabola it touches the line at infinity; thus the coordinates of the line at infinity satisfy the tangential equation. These are in areals $(1:1:1)$ and in trilinears $(a:b:c)$.

Hence, in areal coordinates, $f(x, y, z) = 0$ is a parabola if

$$U+V+W+2F+2G+2H=0;$$

we shall refer to this as $K=0$.

Also, in trilinear coordinates, $f(\alpha, \beta, \gamma) = 0$ is a parabola if

$$a^2U+b^2V+c^2W+2bcF+2caG+2abH=0.$$

The point equation corresponding to a given tangential equation is found in the same way by calculating the minors of its discriminant.

For the tangential equation (B) we shall use the notation

$$P=Up+Hq+Gr, \quad Q=Hp+Vq+Fr, \quad R=Gp+Fq+Wr.$$

(3) *To find the equation of the point of contact of a tangent* $(p_1, q_1, r_1)$ *to* $F(p, q, r) = 0$.

Let $px_1+qy_1+rz_1=0$ be the equation of the point of contact, so that $(x_1:y_1:z_1)$ are the point coordinates of the point of contact.

The equation in point coordinates of the tangent $(p_1, q_1, r_1)$ is $p_1x+q_1y+r_1z=0$.

This is therefore identical with

$$xX_1+yY_1+zZ_1=0.$$

Hence
$$\frac{p_1}{X_1}=\frac{q_1}{Y_1}=\frac{r_1}{Z_1}. \tag{i}$$

Now $UX_1+HY_1+GZ_1$
$$=x_1(uU+hH+gG)+y_1(hU+vH+fG)+z_1(gU+fH+wG)$$
$$=\Delta x_1.$$

Similarly, $HX_1+VY_1+FZ_1=\Delta y_1$, $GX_1+FY_1+WZ_1=\Delta z_1$.

From equation (i) then we get

$$\frac{P_1}{x_1}=\frac{Q_1}{y_1}=\frac{R_1}{z_1}.$$

Hence the equation of the point of contact of the tangent $(p_1, q_1, r_1)$ is

$$pP_1+qQ_1+rR_1=0. \tag{C}$$

(4) *The equations of the polar of the point* $(x_1, y_1, z_1)$ *with respect to* $f(x, y, z) = 0$, *and of the pole of the line* $(p_1, q_1, r_1)$ *with respect to* $F(p, q, r) = 0$.

We can show in the same way as in Chapter VI, § 4, and Chapter X, § 8, that

(a) The equation of the chord of contact of tangents from a point $(x_1, y_1, z_1)$ to the conic $f(x, y, z) = 0$ is $xX_1 + yY_1 + zZ_1 = 0$.

(b) The equation of the point of intersection of tangents at the points where the line $(p_1, q_1, r_1)$ cuts the conic $F(p, q, r) = 0$ is $pP_1 + qQ_1 + rR_1 = 0$, and we deduce similarly that

(c) The equation of the polar of $(x_1, y_1, z_1)$ with respect to the conic $f(x, y, z) = 0$ is

$$xX_1 + yY_1 + zZ_1 = 0. \qquad \text{(D)}$$

(d) The equation of the pole of the line $(p_1, q_1, r_1)$ with respect to the conic $F(p, q, r) = 0$ is $pP_1 + qQ_1 + rR_1 = 0$.

**Cor.** The centre of the conic is the pole of the line at infinity; hence,

(i) If $(x_0, y_0, z_0)$ is the centre of the conic $f(x, y, z) = 0$ in areal coordinates, its polar $xX_0 + yY_0 + zZ_0 = 0$ is the line at infinity $x + y + z = 0$. We have therefore $X_0 = Y_0 = Z_0$.

The areal coordinates of the centre of the conic $f(x, y, z) = 0$ are therefore given by the equations $X = Y = Z$.

(ii) The line coordinates (areal) of the line at infinity are $(1 : 1 : 1)$; the equation of the centre of the conic $F(p, q, r) = 0$ is therefore

$$p(U + H + G) + q(H + V + F) + r(G + F + W) = 0.$$

It follows therefore that the areal coordinates of the centre of the conic $F(p, q, r) = 0$, and therefore of the conic $f(x, y, z) = 0$, are $(U + H + G) : (H + V + F) : (G + F + W)$.

**Note.** Corresponding results can be found in trilinear coordinates by using $a\alpha + b\beta + c\gamma = 0$ for the equation of the line at infinity.

(5) *The equation of the pair of tangents which can be drawn from the point* $(x_1, y_1, z_1)$ *to the conic* $f(x, y, z) = 0$.

This equation can be found as in Chapter VI, p. 237, viz.

$$f(x_1, y_1, z_1) \cdot f(x, y, z) = (xX_1 + yY_1 + zZ_1)^2,$$

and in a similar way we get the equation of the points of intersection of the line $(p_1, q_1, r_1)$ and the conic $F(p, q, r) = 0$ (*v.* p. 404),

$$F(p_1, q_1, r_1) \cdot F(p, q, r) = (pP_1 + qQ_1 + rR_1)^2. \qquad \text{(E)}$$

**(6) The Asymptotes.**

(a) The Asymptotes are the tangents to a conic from its centre; if $(x_1, y_1, z_1)$ is the centre, their equation is

$$f(x, y, z) f(x_1, y_1, z_1) = \{xX_1 + yY_1 + zZ_1\}^2.$$

Since $(x_1, y_1, z_1)$ is the centre, we have $X_1 = Y_1 = Z_1 = \lambda$, also

$$f(x_1, y_1, z_1) = x_1X_1 + y_1Y_1 + z_1Z_1$$
$$= \lambda(x_1 + y_1 + z_1).$$

The equation of the asymptotes therefore becomes

$$f(x, y, z) = \lambda(x + y + z)^2$$

if the coordinates are areal, so that $x_1 + y_1 + z_1 = 1$.

Now
$$ux_1 + hy_1 + gz_1 - \lambda = 0,$$
$$hx_1 + vy_1 + fz_1 - \lambda = 0,$$
$$gx_1 + fy_1 + wz_1 - \lambda = 0,$$
$$x_1 + y_1 + z_1 - 1 = 0.$$

Therefore
$$\begin{vmatrix} u & h & g & \lambda \\ h & v & f & \lambda \\ g & f & w & \lambda \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

or
$$\lambda \begin{vmatrix} u & h & g & 1 \\ h & v & f & 1 \\ g & f & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} + \Delta = 0,$$

or
$$\lambda(U + V + W + 2F + 2G + 2H) = \Delta,$$

or
$$\lambda K = \Delta.$$

The equation of the asymptotes is therefore

$$Kf(x, y, z) = \Delta(x + y + z)^2. \qquad \text{(F)}$$

(b) Another form of this equation can be found as follows: if $(x_1, y_1, z_1)$ is a point on an asymptote, the equation of its polar is

$$xX_1 + yY_1 + zZ_1 = 0.$$

The point at infinity on this polar is therefore

$$(Y_1 - Z_1) : (Z_1 - X_1) : (X_1 - Y_1),$$

and this lies on the curve; hence $f\{Y_1 - Z_1, Z_1 - X_1, X_1 - Y_1\} = 0$.

The equation of the locus of $(x_1, y_1, z_1)$, i.e. of the asymptotes, is then $f\{Y - Z, Z - X, X - Y\}$, or

$$u(Y - Z)^2 + v(Z - X)^2 + w(X - Y)^2 + 2f(Z - X)(X - Y)$$
$$+ 2g(X - Y)(Y - Z) + 2h(Y - Z)(Z - X) = 0.$$

Further, since $X - Y = -(\overline{Y - Z} + \overline{Z - X})$, this equation may be written

$$(w + u - 2g)(Y - Z)^2 + 2(w - f - g + h)(Y - Z)(Z - X)$$
$$+ (v + w - 2f)(Z - X)^2 = 0,$$

from which form the separate equations of the asymptotes may be found.

(c) If the tangential equation of the conic, $F(p, q, r) = 0$, is given, then, since the asymptotes touch the curve and pass through the centre, their coordinates can be found by finding the common solutions of the equations of the centre and the curve, viz.

$$Up^2 + Vq^2 + Wr^2 + 2Fqr + 2Grp + 2Hpq = 0,$$

and $$(U+H+G)p + (H+V+F)q + (G+F+W)r = 0.$$

If $p_1, q_1, r_1$ ; $p_2, q_2, r_2$ are the two sets of values found, the areal equations of the asymptotes are

$$p_1x + q_1y + r_1z = 0, \quad p_2x + q_2y + r_2z = 0.$$

**Cor. i.** The straight lines $f(x, y, z) + \lambda(x+y+z)^2 = 0$ are perpendicular if

$$(u+\lambda)a^2 + (v+\lambda)b^2 + (w+\lambda)c^2 - 2(f+\lambda)bc\cos A - 2(g+\lambda)ca\cos B - 2(h+\lambda)ab\cos C = 0,$$

i.e. $$ua^2 + vb^2 + wc^2 - 2fbc\cos A - 2gca\cos B - 2hab\cos C = 0,$$

for the coefficient of $\lambda$

$$= a^2 + b^2 + c^2 - 2bc\cos A - 2ca\cos B - 2ab\cos C$$
$$= 0.$$

The condition, therefore, that the conic $f(x, y, z) = 0$ (in areal coordinates) should be a rectangular hyperbola is

$$D \equiv ua^2 + vb^2 + wc^2 - 2fbc\cos A - 2gca\cos B - 2hab\cos C = 0.$$

**Note.** In trilinear coordinates $f(\alpha, \beta, \gamma) = 0$ is a rectangular hyperbola if $u + v + w - 2f\cos A - 2g\cos B - 2h\cos C = 0$.

**Cor. ii.** We have shown (Chapter XI, § 2) that

(a) Conics are similar provided that their asymptotes contain equal angles;

(b) Conics are similar and similarly situated provided that their asymptotes are parallel.

These conditions can be obtained from the equations of the asymptotes; in practice it is not necessary to find these equations, for by eliminating $z$ from the equations $f(x, y, z) = 0$, $x + y + z = 0$ we get the equation of a pair of straight lines through the vertex $C$ parallel to the asymptotes.

**§ 9. The Circle.** The equation of the circle, whose centre is $(x_0, y_0, z_0)$ and whose radius is $\rho$, is obviously

$$a^2(y-y_0)(z-z_0) + b^2(z-z_0)(x-x_0) + c^2(x-x_0)(y-y_0) + \rho^2 = 0.$$

Making this equation homogeneous by means of the relation $x+y+z = 1$, we have

$$a^2yz + b^2zx + c^2xy - \{a^2(yz_0 + zy_0) + b^2(zx_0 + xz_0) + c^2(xy_0 + x_0y)\}(x+y+z) + (a^2y_0z_0 + b^2z_0x_0 + c^2x_0y_0 + \rho^2)(x+y+z)^2 = 0,$$

which is evidently of the form

$$a^2yz + b^2zx + c^2xy + (lx + my + nz)(x+y+z) = 0.$$

**Note.** In trilinear coordinates this equation is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0.$$

*To prove, conversely, that the equation*

$$a^2yz + b^2zx + c^2xy + (lx + my + nz)(x + y + z) = 0$$

*always represents a circle, and to interpret l, m, n.*

The equation is that of a conic. If the conic is a parabola, it touches the line at infinity, hence the conic $a^2yz + b^2zx + c^2xy = 0$ also touches the line at infinity, and is a parabola; in this case $a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 = 0$, or $a \pm b \pm c = 0$, which is clearly impossible.

The conic is therefore a central conic, i.e. it has a finite centre; let this centre be $(x_0, y_0, z_0)$.

Now, if $f(x, y, z) = 0$ is the general equation of a conic and $(x_0, y_0, z_0)$ is its centre, we have

$$\begin{aligned} &f(x, y, z) \\ &= f\{\overline{x - x_0} + x_0,\ \overline{y - y_0} + y_0,\ \overline{z - z_0} + z_0\} \\ &= f(x - x_0, y - y_0, z - z_0) \\ &\qquad + 2\{(x - x_0)X_0 + (y - y_0)Y_0 + (z - z_0)Z_0\} + f(x_0, y_0, z_0) \\ &= f(x - x_0, y - y_0, z - z_0) + f(x_0, y_0, z_0), \end{aligned}$$

since $X_0 = Y_0 = Z_0$ and $x - x_0 + y - y_0 + z - z_0 = 1 - 1 = 0$.

Apply this to the case

$$f(x, y, z) = a^2yz + b^2zx + c^2xy + (lx + my + nz)(x + y + z),$$

and we have

$$\begin{aligned} f(x, y, z) = {} & a^2(y - y_0)(z - z_0) + b^2(z - z_0)(x - x_0) + c^2(x - x_0)(y - y_0) \\ & + f(x_0, y_0, z_0); \end{aligned}$$

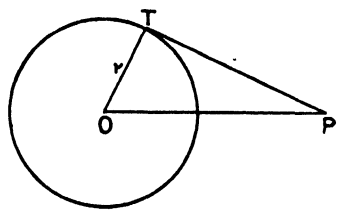
$$\therefore \qquad f(x, y, z) = f(x_0, y_0, z_0) - OP^2,$$

where $O$ is the point $(x_0, y_0, z_0)$ and $P$ any point $(x, y, z)$.

Hence (i) if $P$ lies on the conic, then $f(x, y, z) = 0$, and we have $OP^2 = f(x_0, y_0, z_0)$.

Hence $OP^2$ is constant, i.e. the conic is a circle; its radius is $\sqrt{f(x_0, y_0, z_0)}$.

(ii) If $P$ does not lie on the circle, and $\rho$ is the radius, we have




$f(x, y, z) = \rho^2 - OP^2 = -PT^2$, where $PT$ is the tangent from $P$ to the circle.

In particular, the square on the tangent from $A$ to the circle is $-f(1, 0, 0)$ or $-l$; similarly for the tangents from $B$ and $C$.

**If, then, $t_1$, $t_2$, $t_3$ are the lengths of the tangents from A, B, C to a circle, its equation is**

$$(t_1^2 x + t_2^2 y + t_3^2 z)(x + y + z) - (a^2 yz + b^2 zx + c^2 xy) = 0,$$

**and the left-hand side of the equation represents the square of the tangent from any point (x, y, z) to the circle.**

*To find the condition that the general equation of the second degree, $f(x, y, z) = ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0$, should represent a circle.*

The general equation can be written

$$f(x, y, z) = (ux + vy + wz)(x + y + z) - (v + w - 2f)yz - (w + u - 2g)zx - (u + v - 2h)xy = 0;$$

the necessary and sufficient conditions that this equation should represent a circle are therefore

$$\frac{v + w - 2f}{a^2} = \frac{w + u - 2g}{b^2} = \frac{u + v - 2h}{c^2}.$$

These conditions can be written

$$\frac{f(0, 1, -1)}{a^2} = \frac{f(-1, 0, 1)}{b^2} = \frac{f(1, -1, 0)}{c^2}.$$

In trilinear coordinates the conditions that $f(\alpha, \beta, \gamma) = 0$ should be a circle are $f(0, c, -b) = f(-c, 0, a) = f(b, -a, 0)$.

*To find the tangential equation of the circle whose centre is $(x_0, y_0, z_0)$ and whose radius is $\rho$.*

If $(p, q, r)$ is any tangent to the circle, the areal equation of this tangent is $px + qy + rz = 0$; hence we have

$$\rho = \pm(px_0 + qy_0 + rz_0),$$

or

$$\rho^2 = (px_0 + qy_0 + rz_0)^2.$$

Making this homogeneous by means of the identical relation $\{ap, bq, cr\} = 2S$, we get

$$4S^2(px_0 + qy_0 + rz_0)^2 = \rho^2\{ap, bq, cr\}^2.$$

The general tangential equation of a circle is therefore

$$a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr\cos A - 2carp\cos B - 2abpq\cos C = (lp + mq + nr)^2;$$

its centre is $(l : m : n)$, and its radius $2S/(l + m + n)$.

**Note.** We have shown in § 6 that $\{ap, bq, cr\}^2 = 0$ is the equation of the circular points at infinity, $\Omega$ and $\Omega'$. The tangential equation of the

circle is, therefore, in the form $uv = w^2$, where $u = 0$, $v = 0$ are the equations of $\Omega$, $\Omega'$, and $w = 0$ is the equation of the centre $O$; the circle, as we have previously proved, touches $O\Omega$, $O\Omega'$, the line at infinity $\Omega\Omega'$ being the chord of contact.

### The circles of the triangle of reference.

**I. The circumcircle.** Here $t_1 = 0$, $t_2 = 0$, $t_3 = 0$, so that the equation of the circumcircle is $a^2yz + b^2zx + c^2xy = 0$. (In trilinears this becomes $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$.)

The tangential equation of the circumcircle is therefore

$$a^4p^2 + b^4q^2 + c^4r^2 - 2b^2c^2qr - 2c^2a^2rp - 2a^2b^2pq = 0,$$

which can be written $a\sqrt{p} \pm b\sqrt{q} \pm c\sqrt{r} = 0$.

**Cor.** The nine-point circle circumscribes the triangle $A'B'C'$, where $A'$, $B'$, $C'$ are the mid-points of the sides. If $p'$, $q'$, $r'$ are the perpendiculars from $A'$, $B'$, $C'$ on any tangent to the nine-point circle, and $p$, $q$, $r$ the perpendicular from $A$, $B$, $C$ to the same tangent, we have $p + q = 2r'$, $q + r = 2p'$, $r + p = 2q'$ with due regard to sign. Also $a = 2a'$, $b = 2b'$, $c = 2c'$. Now since the nine-point circle circumscribes the triangle $A'B'C'$ we have $a'\sqrt{p'} + b'\sqrt{q'} + c'\sqrt{r'} = 0$, hence

$$a\sqrt{q+r} + b\sqrt{r+p} + c\sqrt{p+q} = 0$$

is the tangential equation of the nine-point circle.

**II. The inscribed and escribed circles.**

(a) For the **inscribed circle** $t_1 = s - a$, $t_2 = s - b$, $t_3 = s - c$; its equation is

$$\{(s-a)^2x + (s-b)^2y + (s-c)^2z\}(x+y+z) - a^2yz - b^2zx - c^2xy = 0.$$

The tangential equation, since $x_0 = ar/2S$, $y_0 = br/2S$, $z_0 = cr/2S$, is $(ap + bq + cr)^2 = \{ap, bq, cr\}^2$.

This reduces to

$$(s-a)qr + (s-b)rp + (s-c)pq = 0,$$

or

$$qr\cot\tfrac{1}{2}A + rp\cot\tfrac{1}{2}B + pq\cot\tfrac{1}{2}C = 0.$$

**Note.** The reader should deduce the trilinear equation.

(b) For the **escribed circle** touching $BC$ externally we have $t_1 = s$, $t_2 = s - c$, $t_3 = s - b$; its equation is

$$\{s^2x + (s-c)^2y + (s-b)^2z\}(x+y+z) - a^2yz - b^2zx - c^2xy = 0.$$

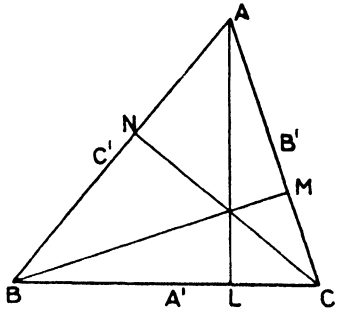
The areal coordinates of its centre are $\{-ar_1/2S, br_1/2S, cr_1/2S\}$, so that its tangential equation is

$$(-ap + bq + cr)^2 = \{ap, bq, cr\}^2;$$

this reduces to

$$qr\cot\tfrac{1}{2}A - rp\tan\tfrac{1}{2}B - pq\tan\tfrac{1}{2}C = 0.$$

**III. The nine-point circle.**




Here $t_1^2 = AC' . AN$

$= \frac{1}{2}c . b\cos A$

$= S\cot A,$

or $= \frac{1}{4}(b^2+c^2-a^2).$

The equation of the nine-point circle is therefore

$$S\{x\cot A + y\cot B + z\cot C\}(x+y+z) - a^2yz - b^2zx - c^2xy = 0,$$

or $$(x+y+z)\Sigma(b^2+c^2-a^2)x - 4(a^2yz+b^2zx+c^2xy) = 0.$$

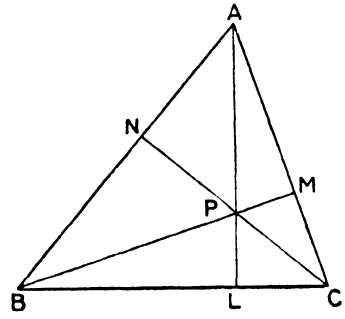
We have already shown that its tangential equation is

$$a\sqrt{q+r} + b\sqrt{r+p} + c\sqrt{p+q} = 0.$$

The reader can also deduce this from the general tangential equation.

**IV. The polar circle.** The orthocentre $P$ is the centre of the polar circle, and if $\rho$ is the radius we have

$$-\rho^2 = AP . PL = BP . PM = CP . PN.$$




Hence

$$t_1^2 = AP^2 - \rho^2 = AP^2 + AP . PL$$

$$= AP . AL = 2R\cos A . b\sin C$$

$$= bc\cos A = 2S\cot A.$$

The equation of the circle is then

$$2S(x\cot A + y\cot B + z\cot C)(x+y+z) - a^2yz - b^2zx - c^2xy = 0,$$

which reduces to

$$x^2\cot A + y^2\cot B + z^2\cot C = 0.$$

The tangential equation of this circle is

$$p^2\tan A + q^2\tan B + r^2\tan C = 0.$$

**The radical axis of two circles.** If the equations of two circles are found in the forms

$$(lx+my+nz)(x+y+z) - a^2yz - b^2zx - c^2xy = 0,$$

$$(l'x+m'y+n'z)(x+y+z) - a^2yz - b^2zx - c^2xy = 0,$$

it is evident that points common to the two circles also lie on

$$\{(l-l')x + (m-m')y + (n-n')z\}(x+y+z) = 0,$$

i.e. on one of the lines

$$(l-l')x+(m-m')y+(n-n')z=0, \quad x+y+z=0.$$

The first line is the radical axis, and the second is the line at infinity, which is the join of the circular points at infinity. We have previously shown that all circles pass through $\Omega$, $\Omega'$; we proceed to verify this.

**The circular points at infinity.** Since the equation of any circle can be expressed in the form

$$(lx+my+nz)(x+y+z)-(a^2yz+b^2zx+c^2xy)=0,$$

it meets the straight line at infinity in two points given by the equations $a^2yz+b^2zx+c^2xy=0$, $x+y+z=0$, which are evidently the same points for all circles; these points (Chapter V, § 16) we called the 'circular points at infinity'.

We have from these equations

$$b^2x^2+a^2y^2+(a^2+b^2-c^2)xy=0,$$

or
$$b^2x^2+a^2y^2+2ab\cos C\,xy=0,$$

i.e.
$$(bx+aye^{iC})(bx+aye^{-iC})=0.$$

Thus either $x:y:z=ae^{iC}:-b:b-ae^{iC}$,

or $\quad =ae^{-iC}:-b:b-ae^{-iC}$.

But $b-ae^{iC}=b-a\cos C-ia\sin C=c(\cos A-i\sin A)=ce^{-iA}$, hence the coordinates of the circular points at infinity are

$$(ae^{iC}:-b:ce^{-iA}), \quad (ae^{-iC}:-b:ce^{iA}).$$

**Note.** The tangential equation of $\Omega$, $\Omega'$ can be easily found thus: if the line $px+qy+rz=0$ passes through a circular point at infinity, this point must be the point at infinity on the line, i.e. $(q-r:r-p:p-q)$. This point therefore lies on the circle $a^2yz+b^2zx+c^2xy=0$, so that

$$a^2(p-r)(p-q)+b^2(q-p)(q-r)+c^2(r-q)(r-p)=0.$$

We have shown that this equation is the same as

$$\{ap, bq, cr\}^2=0.$$

## Illustrative Examples.

**Example** i. *The nine-point circle touches the inscribed circle.*

The equations of these circles are

$$\Sigma\tfrac{1}{2}(b^2+c^2-a^2)x\,.\,(x+y+z)=a^2yz+b^2zx+c^2xy,$$
$$\Sigma(s-a)^2x\,.\,(x+y+z)=a^2yz+b^2zx+c^2xy.$$

The equation of their radical axis is therefore

$$\Sigma\{(b^2+c^2-a^2)-(b+c-a)^2\}x=0,$$

i.e.
$$\Sigma(a-b)(c-a)x=0,$$

or
$$x/(b-c)+y/(c-a)+z/(a-b)=0.$$

The coordinates of the radical axis are therefore

$$\{1/(b-c) : 1/(c-a) : 1/(a-b)\}.$$

The tangential equation of the inscribed circle is

$$(s-a)\,qr+(s-b)\,rp+(s-c)\,pq = 0,$$

or

$$(s-a)/p+(s-b)/q+(s-c)/r = 0.$$

Now $\Sigma(b-c)(s-a) = 0$, so that the radical axis of the nine-point and inscribed circles touches the inscribed circle; it therefore touches the nine-point circle at the same point.

It can be shown similarly that the nine-point circle touches each of the escribed circles.

**Example ii.** *The straight line* $lx+my+nz = 0$ *meets the sides of the triangle* $ABC$ *in* $A'$, $B'$, $C'$: *prove that the circles on* $AA'$, $BB'$, $CC'$ *as diameters have the common radical axis*

$$l(m-n)\,x\cot A+m(n-l)\,y\cot B+n(l-m)\,z\cot C = 0.$$

The coordinates of $A'$ are $\left\{0,\ \frac{-n}{m-n},\ \frac{m}{m-n}\right\}$, and of $A$ $\{1, 0, 0\}$; $\therefore$ the centre of the circle on $AA'$ as diameter is

$$\left\{\tfrac{1}{2},\ \frac{-n}{2(m-n)},\ \frac{m}{2(m-n)}\right\} \text{ or } \{m-n : -n : m\}.$$

Let the equation of the circle be

$$a^2yz+b^2zx+c^2xy = (\lambda x+\mu y+\nu z)(x+y+z);$$

since it passes through $A$ (1, 0, 0) we have $\lambda = 0$.

The centre is given by

$$b^2z+c^2y-(\mu y+\nu z) = c^2x+a^2z-(\mu y+\nu z)-\mu(x+y+z)$$
$$= a^2y+b^2x-(\mu y+\nu z)-\nu(x+y+z),$$

i.e.

$$b^2z+c^2y = c^2x+a^2z-\mu(x+y+z) = a^2y+b^2x-\nu(x+y+z).$$

Substitute in these the known values of $x : y : z$, and we get

$$b^2m-c^2n = c^2(m-n)+a^2m-2\mu(m-n) = b^2(m-n)-a^2n-2\nu(m-n).$$

Hence

$$2\mu(m-n) = m(a^2+c^2-b^2) = 2ac\cos Bm,$$
$$2\nu(m-n) = -n(a^2+b^2-c^2) = -2ab\cos Cn.$$

The equation of the circle on $AA'$ as diameter is then

$$a^2yz+b^2zx+c^2xy = 2S\left(\frac{m\cot B}{m-n}y-\frac{n\cot C}{m-n}z\right)(x+y+z).$$

Similarly, the equation of the circle on $CC'$ as diameter is

$$a^2yz+b^2zx+c^2xy = 2S\left(\frac{n\cot C}{n-l}z-\frac{l\cot A}{n-l}x\right)(x+y+z).$$

The radical axis of these circles is therefore

$$\frac{l\cot A}{n-l}\cdot x+\frac{m\cot B}{m-n}\cdot y-n\cot C\left(\frac{1}{m-n}+\frac{1}{n-l}\right)z = 0,$$

i.e.

$$l(m-n)\,x\cot A+m(n-l)\,y\cot B+n(l-m)\,z\cot C = 0.$$

The symmetry of the result shows that it is also the radical axis of the circles on $BB'$ and $CC'$ as diameters.

## Examples XII d.

1. Find the equation of the circle which passes through two angular points and the in-centre of the triangle of reference.

2. Prove that $\Sigma(m+n)\alpha^2 + 2\Sigma l\beta\gamma\cos A = 0$ is a circle.

3. Prove that the circumcircle, nine-point circle, and polar-circle of a triangle are coaxal; show also that the circle circumscribing the triangle formed by the tangents to the circumcircle at the vertices of the triangle belongs to the same coaxal system.

4. Show that the equations

$\alpha\beta = \gamma(\gamma\cos C - \alpha\cos A - \beta\cos B)$, $\beta\gamma = \alpha(\alpha\cos A - \beta\cos B - \gamma\cos C)$

represent circles, and find their radical axis.

5. Find the lengths of the tangents from the ex-centres of a triangle to the circumcircle.

6. Find the equation of the common chord of two circles described on two sides of the fundamental triangle as diameters.

7. Through the vertices $A$, $B$, $C$ of a triangle straight lines are drawn parallel to the opposite sides and meeting the circumcircle in the points $A'$, $B'$, $C'$ respectively. The straight line joining $A'$ to the centroid of the triangle meets $BC$ in $A''$, and $B''$, $C''$ are similarly determined. Show that $AA''$, $BB''$, $CC''$ are concurrent.

8. Show that the equation of the circle which passes through the ex-centres of the triangle of reference is

$$a^2yz + b^2zx + c^2xy + (bcx + cay + abz)(x+y+z) = 0.$$

Find also its trilinear equation.

9. Prove that the equation of the nine-point circle can be expressed in the form $a^2/(b\beta + c\gamma - a\alpha) + l^2/(c\gamma + a\alpha - b\beta) + c^2/(a\alpha + b\beta - c\gamma) = 0$. What is the corresponding equation in areals?

10. Show that the equation of the circle which passes through $A$ and touches $BC$ at its middle point is $4(a^2yz + b^2zx + c^2xy) = a^2(y+z)(x+y+z)$. Find the equation of the line joining the points other than $A$ where this circle cuts $AB$, $AC$.

11. Prove that the equation of the circle on the line joining the mid-points of $AB$, $AC$ as diameter is

$$\Sigma x^2(b^2 + c^2 - a^2) - 2\Sigma a^2yz + bc\cos A(x+y+z)(y+z-x) = 0.$$

12. If $(\alpha', \beta', \gamma')$ is a point on the circumcircle, *verify* that the corresponding Simson line is $\sum \frac{a}{\alpha'}\cdot\alpha\cos A = \Sigma\alpha\sin A$.

13. Show by using the tangential equations of the escribed circles that their external centres of similitude lie on the line $x/a + y/b + z/c = 0$.

14. A circle passes through $B$, $C$ and cuts $CA$ in the ratio $l:m$; in what ratio does it divide $BA$?

15. Find the equation of the circle through $B$ and $C$ which cuts $CA$ again at $D$ so that $\angle BDC = \alpha$. Hence find the locus of the radical centre of three circles which pass through $B$, $C$; $C$, $A$; $A$, $B$ respectively, and whose segments within the triangle contain equal angles.

16. Find the tangential equations of the circles whose centres are the vertices of the triangle of reference and which touch the sides opposite to their centres. Find the equation of their axis of similitude.

17. Circles are described on $BC$, $CA$, $AB$ cutting $CA$, $AB$, $BC$ respectively in the ratio $1/\lambda$; show that their radical centre is the same for all values of $\lambda$.

18. The inscribed circle of a triangle $ABC$ touches the sides $BC$, $CA$, $AB$ at the points $D$, $E$, $F$; the escribed circles opposite to the angles $A$, $B$, and $C$ touch these sides at $D_1$, $E_1$, $F_1$; $D_2$, $E_2$, $F_2$; $D_3$, $E_3$, $F_3$, respectively. Prove that $AD_1$, $BE_2$, $CF_3$ are concurrent; also the sets of lines $AD_1$, $BE_3$, $CF_2$; and $AD_3$, $BE_2$, $CF_1$.

19. Prove that if the circles $a^2yz + b^2zx + c^2xy = 0$ and

$$(t_1^2x + t_2^2y + t_3^2z)(x + y + z) - a^2yz - b^2zx - c^2xy = 0$$

cut orthogonally, then

$$t_1^2 \sin 2A + t_2^2 \sin 2B + t_3^2 \sin 2C = 8S.$$

[The tangent from the circumcentre to the second circle is of length $R$.]

20. The sides $BC$, $CA$, $AB$ of the triangle of reference are divided at $D$, $E$, $F$ so that $BD = 2\,DC$, $CE = 2\,EA$, $AF = 2\,FB$. Show that the square of the tangent from $A$ to the circle $DEF$ is $\frac{2}{7}(c^2 - 2a^2 + 4b^2)$.

21. Show that the radius of the circle $a\beta\gamma + b\gamma\alpha + c\alpha\beta = k\alpha(a\alpha + b\beta + c\gamma)$ is $R\sqrt{1 - 2k\cos A + k^2}$.

**§ 10. Conics circumscribing the triangle of reference.** If the conic $ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0$ passes through the vertices (1, 0, 0), (0, 1, 0), (0, 0, 1) of the triangle of reference, then $u = 0$, $v = 0$, $w = 0$.

The general equation of a conic circumscribing the triangle of reference is therefore

$$fyz + gzx + hxy = 0. \qquad \text{(i)}$$

We have in this case $U = -f^2$, $V = -g^2$, $W = -h^2$, $F = gh$, $G = fh$, $H = fg$, so that the tangential equation of the conic is

$$f^2p^2 + g^2q^2 + h^2r^2 - 2ghqr - 2hfrp - 2fgpq = 0. \qquad \text{(ii)}$$

We may write this

$$(fp - gq - hr)^2 = 4ghqr,$$

i.e. $$fp - gq - hr = \pm 2\sqrt{ghqr},$$

i.e. $$fp = (\sqrt{gq} \pm \sqrt{hr})^2,$$

or $$\sqrt{fp} \pm \sqrt{gq} \pm \sqrt{hr} = 0.$$

**Note.** The condition that the conic $fyz + gzx + hxy = 0$ should be a parabola is that the line at infinity should touch it, i.e., in areal coordinates, that (1, 1, 1) should touch the conic (ii).

Hence the conic is a parabola if $f^2 + g^2 + h^2 - 2gh - 2hf - 2fg = 0$.

The conic $f\beta\gamma + g\gamma\alpha + h\alpha\beta = 0$ is a parabola if

$$a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cahf - 2abfg = 0.$$

*To find the equation of the chord joining the points* $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ *on the conic* $fyz+gzx+hxy=0$.

The equation of the straight line joining the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ is $x(y_1z_2-y_2z_1)+y(z_1x_2-z_2x_1)+z(x_1y_2-x_2y_1)=0$.

But $$fy_1z_1+gz_1x_1+hx_1y_1=0,$$

and $$fy_2z_2+gz_2x_2+hx_2y_2=0,$$

hence $f:g:h=x_1x_2(y_1z_2-y_2z_1):y_1y_2(z_1x_2-z_2x_1):z_1z_2(x_1y_2-x_2y_1)$; *provided, then, that no one of the coordinates of the given points is zero,* the equation of the chord can be written

$$\frac{fx}{x_1x_2}+\frac{gy}{y_1y_2}+\frac{hz}{z_1z_2}=0.$$

**Cor.** The equation of the tangent at $(x_1, y_1, z_1)$ is therefore

$$\frac{fx}{x_1^2}+\frac{gy}{y_1^2}+\frac{hz}{z_1^2}=0.$$

*The equation of the tangent at* $(x_1, y_1, z_1)$ *to the conic.*

Since $u=0$, $v=0$, $w=0$ we have $X_1=hy_1+gz_1$, $Y_1=hx_1+fz_1$, $Z_1=gx_1+fy_1$, so that the equation of the tangent is

$$x(hy_1+gz_1)+y(hx_1+fz_1)+z(gx_1+fy_1)=0.$$

Now, since $fy_1z_1+gz_1x_1+hx_1y_1=0$, we can write this

$$\frac{fy_1z_1}{x_1}x+\frac{gx_1z_1}{y_1}y+\frac{hx_1y_1}{z_1}z=0,$$

or $$\frac{fx}{x_1^2}+\frac{gy}{y_1^2}+\frac{hz}{z_1^2}=0,$$

which agrees with the last corollary; *evidently no one of the coordinates must be zero when this form is used.*

*The equation of the point of contact of the tangent* $(p_1, q_1, r_1)$ *to the conic* $\sqrt{fp}\pm\sqrt{gq}\pm\sqrt{hr}=0$.

The equation of the conic written in full is

$$f^2p^2+g^2q^2+h^2r^2-2ghqr-2hfrp-2fgpq=0.$$

Thus $P_1=f(fp_1-gq_1-hr_1)$, with symmetrical values for $Q_1$, $R_1$. The equation of the point of contact is

$$fp(fp_1-gq_1-hr_1)+gq(gq_1-hr_1-fp_1)+hr(hr_1-fp_1-gq_1)=0.$$

Now, since $(p_1, q_1, r_1)$ is a tangent to the conic, we have

$$f^2p_1^2+g^2q_1^2+h^2r_1^2-2ghq_1r_1-2hfr_1p_1-2fgp_1q_1=0,$$

hence $$(fp_1-gq_1-hr_1)^2=4ghq_1r_1,$$

or $$fp_1-gq_1-hr_1=\pm 2\sqrt{ghq_1r_1}.$$

The equation of the point of contact can then be written, *provided that no one of the coordinates $p_1$, $q_1$, $r_1$ is zero,*

$$\frac{p}{p_1}\sqrt{fp_1}+\frac{q}{q_1}\sqrt{gq_1}+\frac{r}{r_1}\sqrt{hr_1}=0,$$

where $\sqrt{fp_1}$, $\sqrt{gq_1}$, $\sqrt{hr_1}$ have the signs which make

$$\sqrt{fp_1}+\sqrt{gq_1}+\sqrt{hr_1}=0.$$

*The equation of the polar of $(x_1, y_1, z_1)$ and of the pole of $(p_1, q_1, r_1)$ with respect to the conic.*

These are $x(hy_1+gz_1)+y(hx_1+fz_1)+z(gx_1+fy_1)=0$ and

$$f(fp_1-gq_1-hr_1)p+g(gq_1-hr_1-fp_1)q+h(hr_1-fp_1-gq_1)r=0.$$

**Note.** The method we have always used to show that the equation of the polar of a point, which is not on a conic, is of the same form as the equation of the tangent at a point on the conic depends on the fact that the equation of the tangent is unaltered when the current coordinates and those of the point are interchanged. It is important to note then that the equation of the tangent at $(x_1, y_1, z_1)$ in the form $\frac{fx}{x_1^2}+\frac{gy}{y_1^2}+\frac{hz}{z_1^2}=0$, *cannot* be the equation of the polar of $(x_1, y_1, z_1)$.

**Cor.** *To find the centre of the conic.*

The centre of the conic is the pole of the line at infinity (1, 1, 1); its equation is therefore

$$f(f-g-h)p+g(g-h-f)q+h(h-f-g)r=0.$$

The areal coordinates of the centre are then

$$f(f-g-h):g(g-h-f):h(h-f-g).$$

**Example.** *To find the equation of a conic whose centre is the point $(x_0, y_0, z_0)$, and which circumscribes the triangle of reference.*

If $fyz+gzx+hxy=0$ is the equation of the conic, we have

$$x_0=f(f-g-h),\quad y_0=g(g-h-f),\quad z_0=h(h-f-g).$$

Hence $y_0+z_0-x_0=(g-h)^2-f^2=(g-h-f)(g-h+f)=-y_0z_0/gh$.

Therefore $f$ is proportional to $x_0(y_0+z_0-x_0)$, and the equation of the conic is $\frac{x_0}{x}(y_0+z_0-x_0)+\frac{y_0}{y}(z_0+x_0-y_0)+\frac{z_0}{z}(x_0+y_0-z_0)=0$.

**The asymptotes.** Let the equations of the asymptotes be $px+qy+rz=0$, $p'x+q'y+r'z=0$; then we have

$$fyz+gzx+hxy+\lambda(x+y+z)^2 \equiv (px+qy+rz)(p'x+q'y+r'z).$$

Comparing coefficients we have

$$pp'=qq'=rr'=\lambda,$$

$$f+2\lambda=qr'+q'r,\quad g+2\lambda=rp'+r'p,\quad h+2\lambda=pq'+p'q.$$

Hence (i) if $px+qy+rz=0$ is one asymptote, the other is $x/p+y/q+z/r=0$.

(ii) We have

$$f=qr'+q'r-2\lambda$$

$$=\lambda\left(\frac{q}{r}+\frac{r}{q}-2\right);$$

$$\therefore \qquad qrf=\lambda(q-r)^2,$$

or, $f$ is proportional to $p(q-r)^2$.

It follows then that if $px+qy+rz=0$ is an asymptote of a conic circumscribing the triangle of reference, the equation of the conic is

$$p(q-r)^2yz+q(r-p)^2zx+r(p-q)^2xy=0.$$

**Example i.** *A conic circumscribes a triangle $ABC$, and the tangents at $A$, $B$, $C$ form the triangle $A'B'C'$. Show that the triangles $ABC$, $A'B'C'$ are coaxal, and find their centre and axis of homology.*

Let the conic be $fyz+gzx+hxy=0$; its equation can be written

$$fyz+x(gz+hy)=0.$$

This represents a conic circumscribing the quadrilateral whose sides are $y=0$, $z=0$; $x=0$, $gz+hy=0$; but since $y=0$, $z=0$, $gz+hy=0$ are concurrent, $gz+hy=0$ is the tangent to the conic at the point of intersection of $y=0$, $z=0$, i.e. $A$.

The sides of the triangle $A'B'C'$ are therefore

$$y/g+z/h=0,\quad z/h+x/f=0,\quad x/f+y/g=0.$$

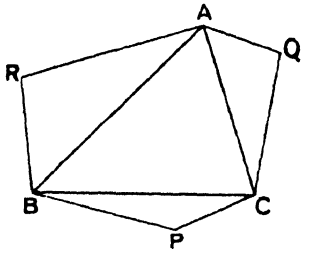
The straight line $(z/h+x/f)-(x/f+y/g)=0$, i.e. $z/h-y/g=0$ is therefore $AA'$. Similarly, $BB'$, $CC'$ are $x/f-z/h=0$, $y/g-x/f=0$. Hence, $AA'$, $BB'$, $CC'$ intersect at the point $(f:g:h)$. Again, $B'C'$ and $BC$ are the lines $y/g+z/h=0$ and $x=0$; these intersect on the line $x/f+y/g+z/h=0$. The intersections of $CA$, $C'A'$; $AB$, $A'B'$ also lie on this line, which is therefore the axis of homology.

**Example ii.** *The opposite sides of any hexagon inscribed in a conic meet in three collinear points.* **(Pascal's Theorem.)**

Let $AQCPBR$ be the hexagon and take $ABC$ for the triangle of reference. Let the coordinates of $P$, $Q$, $R$ be $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$.

The equation of $AQ$ is $y/y_2=z/z_2$ and the equation of $BP$ is $x/x_1=z/z_1$; these lines intersect at the point $(x_1/z_1 : y_2/z_2 : 1)$.

Similarly, the other pair of opposite sides intersect at the points $(x_1/y_1 : 1 : z_3/y_3)$, $(1 : y_2/x_2 : z_3/x_3)$.




These points are collinear if

$$\begin{vmatrix} x_1/z_1 & y_2/z_2 & 1 \\ x_1/y_1 & 1 & z_3/y_3 \\ 1 & y_2/x_2 & z_3/x_3 \end{vmatrix} = 0,$$

i. e. if

$$\begin{vmatrix} 1/z_1 & 1/z_2 & 1/z_3 \\ 1/y_1 & 1/y_2 & 1/y_3 \\ 1/x_1 & 1/x_2 & 1/x_3 \end{vmatrix} = 0.$$

But the equation of the conic on which the vertices lie is of the form $f/x+g/y+h/z=0$; hence

$$f/x_1+g/y_1+h/z_1=0,\ f/x_2+g/y_2+h/z_2=0,\ f/x_3+g/y_3+h/z_3=0.$$

Eliminating $f$, $g$, $h$ we obtain the required condition.

**Note.** This evidently proves the converse proposition, viz. if the opposite sides of a hexagon meet on a straight line, the hexagon can be inscribed in a conic.

**Example iii.** *Find the condition that the normals at the angular points of the triangle of reference to the conic* $l/\alpha+m/\beta+n/\gamma=0$ *should be concurrent.* (*Trilinear Coordinates.*)

The tangent at $A$ is $n\beta+m\gamma=0$; suppose that the normal is $\lambda\beta+\mu\gamma=0$. These lines are perpendicular, so that

$$\lambda n+\mu m-(\lambda m+\mu n)\cos A=0,$$

i. e.

$$\lambda(n-m\cos A)+\mu(m-n\cos A)=0.$$

The equation of the normal at $A$ is therefore

$$(m-n\cos A)\beta-(n-m\cos A)\gamma=0.$$

Similarly, the normals at $B$, $C$ are

$$(n-l\cos B)\gamma-(l-n\cos B)\alpha=0$$
$$(l-m\cos C)\alpha-(m-l\cos C)\beta=0.$$

The normals are concurrent if these three equations are simultaneously satisfied by the same values of $\alpha$, $\beta$, $\gamma$.

Hence

$$(l-m\cos C)(m-n\cos A)(n-l\cos B)=(m-l\cos C)(n-m\cos A)(l-n\cos B);$$

$$\therefore\ \Sigma l(n^2-m^2)(\cos A+\cos B\cos C)=0;$$

$$\therefore\ \Sigma l(n^2-m^2)\sin B\sin C=0;$$

$$\therefore\ \Sigma bcl(m^2-n^2)=0.$$

This condition can be written as

$$\begin{vmatrix} l & m & n \\ l^{-1} & m^{-1} & n^{-1} \\ a^{-1} & b^{-1} & c^{-1} \end{vmatrix} = 0.$$

## Examples XII e.

1. If $\lambda\alpha+\mu\beta+\nu\gamma=0$ touches $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$, then $l\alpha+m\beta+n\gamma=0$ will touch $\lambda\beta\gamma+\mu\gamma\alpha+\nu\alpha\beta=0$.

2. Find the locus of the centres of conics which pass through four given points.

What is the equation if the points are $A$, $B$, $C$ and the centroid of the triangle?

3. Prove that a conic which passes through the vertices and the orthocentre of a triangle is a rectangular hyperbola.

4. Show that the locus of the centres of rectangular hyperbolas which circumscribe the triangle of reference is the nine-point circle

$$a^2/(y+z-x)+b^2/(z+x-y)+c^2/(x+y-z)=0.$$

5. Find the locus of the centre of a conic which circumscribes the triangle of reference and touches the straight line $(p_1, q_1, r_1)$.

6. Show that the centre of any conic passing through the angular points of the triangle of reference and the centre of its inscribed circle lies on the conic $\Sigma bcx^2-\Sigma a(b+c)yz=0$.

7. Show that the equation of the tangent at any point of the circumscribing circle is of the form

$$\left(\frac{z}{c^2}+\frac{x}{a^2}\right)\tan^2\theta-\frac{2x}{a^2}\tan\theta+\left(\frac{x}{a^2}+\frac{y}{b^2}\right)=0.$$

8. Prove that the tangents to $l/x+m/y+n/z=0$ which are parallel to $BC$ are $l(y+z)+(\sqrt{m}\pm\sqrt{n})^2x=0$.

9. A conic circumscribes a right-angled triangle $ABC$, touches the circumcircle at the right angle $A$, and passes through the centroid of the triangle. Show that its equation is $b^2/y+c^2/z-a^2/x=0$, or $b/\beta+c/\gamma-a/\alpha=0$. Find its equation when the coordinates are transformed to rectangular Cartesian coordinates with $AB$, $AC$ for axes, and hence find the eccentricity of the conic in terms of the angle $B$.

10. Find the coordinates of the centre of a conic which circumscribes the triangle of reference, and has the line $(p_1, q_1, r_1)$ for an asymptote.

11. The condition that the straight line $(p, q, r)$ should be an asymptote of a rectangular hyperbola circumscribing $ABC$ is $\Sigma p(q-r)^2\cot A=0$.

**§ 11. Conics inscribed in the triangle of reference.** If the conic $up^2+vq^2+wr^2+2fqr+2grp+2hpr=0$ touches the sides of the triangle of reference, (1, 0, 0), (0, 1, 0), (0, 0, 1), then $u=0$, $v=0$, $w=0$. Hence $fqr+grp+hpq=0$ is the general tangential equation of a conic touching the sides of the triangle of reference.

The equation of the conic in point coordinates is therefore

$$f^2x^2+g^2y^2+h^2z^2-2ghyz-2hfzx-2fgxy=0,$$

which can be written (cf. § 10)

$$\sqrt{fx}\pm\sqrt{gy}\pm\sqrt{hz}=0.$$

**Note** i. The conic is a parabola if $f+g+h=0$, or, if we are using trilinear coordinates, $f/a+g/b+h/c=0$.

**Note** ii. The equation of the conic can be written $fqr+p(gr+hq)=0$; but $q=0$, $r=0$, $gr+hq=0$ are three collinear points, so that (Chap. X, § 10, 1) the conic touches the join of $q=0$, $r=0$ (i.e. $BC$) at the point $hq+gr=0$. The areal coordinates of the points of contact of the conic with the sides of the triangle of reference are therefore

$$(0:1/g:1/h),\ (1/f:0:1/h),\ (1/f:1/g:0).$$

*To find the equation of the point of intersection of the tangents* $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$ *to the conic* $f/p+g/q+h/r=0$.

The equation of the point of intersection of the lines $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$ is

$$p(q_1r_2-q_2r_1)+q(r_1p_2-r_2p_1)+r(p_1q_2-p_2q_1)=0.$$

But $fq_1r_1+gr_1p_1+hp_1q_1=0$, and $fq_2r_2+gr_2p_2+hp_2q_2=0$, hence

$$f:g:h=p_1p_2(q_1r_2-q_2r_1):q_1q_2(r_1p_2-r_2p_1):r_1r_2(p_1q_2-p_2q_1);$$

provided, then, that no one of the coordinates of the given tangents is zero, the equation of the point of intersection can be written

$$fp/p_1p_2+gq/q_1q_2+hr/r_1r_2=0.$$

**Cor.** The equation of the point of contact of the tangent $(p_1, q_1, r_1)$ is $fp/p_1^2+gq/q_1^2+hr/r_1^2=0$. If, then, $p_1x+q_1y+r_1z=0$ touches the conic $\sqrt{fx}+\sqrt{gy}+\sqrt{rz}=0$, the coordinates of the point of contact are $(f/p_1^2, g/q_1^2, h/r_1^2)$.

*To find the equation of the tangent at the point* $(x_1, y_1, z_1)$ *to the conic* $\sqrt{fx}+\sqrt{gy}+\sqrt{rz}=0$.

This equation can be found in the same way as that of the point of contact of the tangent $(p_1, q_1, r_1)$ to the conic

$$\sqrt{fp}+\sqrt{gq}+\sqrt{hr}=0$$

(see § 10); the equation is

$$\frac{x}{x_1}\sqrt{fx_1}+\frac{y}{y_1}\sqrt{gy_1}+\frac{z}{z_1}\sqrt{hz_1}=0,$$

where $\sqrt{fx_1}$, $\sqrt{gy_1}$, $\sqrt{hz_1}$ have the signs that make

$$\sqrt{fx_1}+\sqrt{gy_1}+\sqrt{hz_1}=0.$$

*The equation of the pole of* $(p_1, q_1, r_1)$ *and of the polar of* $(x_1, y_1, z_1)$ *with respect to the conic.*

These are

$$p(hq_1+gr_1)+q(fr_1+hp_1)+r(gp_1+fq_1)=0$$

and $f(fx_1-gy_1-hz_1)x+g(gy_1-hz_1-fx_1)y+h(hz_1-fx_1-gy_1)z=0.$

**Cor.** The equation of the centre, i.e. the pole of the line at infinity (1, 1, 1), is

$$(h+g)p+(f+h)q+(g+f)r=0.$$

The areal coordinates of the centre are therefore

$$\{(h+g):(f+h):(g+f)\}.$$

The trilinear coordinates of the centre of $\sqrt{f\alpha}+\sqrt{g\beta}+\sqrt{h\gamma}=0$ are

$$\left\{\frac{h}{c}+\frac{g}{b}:\frac{f}{a}+\frac{h}{c}:\frac{g}{b}+\frac{f}{a}\right\}.$$

**Example.** *To find the equation of a conic whose centre is the point* $(x_0, y_0, z_0)$ *and which touches the sides of the triangle of reference.*

Let the conic be $\sqrt{fx}+\sqrt{gy}+\sqrt{hz}=0$, then we have $x_0=g+h$, $y_0=h+f$, $z_0=f+g$. Hence $2f=y_0+z_0-x_0$, &c. The equation of the conic is therefore

$$\sqrt{x(y_0+z_0-x_0)}+\sqrt{y(z_0+x_0-y_0)}+\sqrt{z(x_0+y_0-z_0)}=0.$$

**The asymptotes.** Let $px+qy+rz=0$ be an asymptote; it touches the conic at the point $(f/p^2 : g/q^2 : h/r^2)$, but this must be the point at infinity on the asymptote, viz. $(q-r : r-p : p-q)$. Hence we have $f=kp^2(q-r)$, $g=kq^2(r-p)$, $h=kr^2(p-q)$.

It follows that if $(p_1, q_1, r_1)$ is an asymptote of a conic inscribed in the triangle of reference, the equation of the conic is

$$p_1\sqrt{(q_1-r_1)x}+q_1\sqrt{(r_1-p_1)y}+r_1\sqrt{(p_1-q_1)z}=0.$$

Now, if $p, q, r$ is an asymptote, we have

$$f/p+g/q+h/r=0 \quad \text{and} \quad f/p^2+g/q^2+h/r^2=0;$$

hence $$(g/q+h/r)^2+f(g/q^2+h/r^2)=0,$$

or $$h(f+h)q^2+2ghqr+g(f+g)r^2=0.$$

If, then, the two asymptotes are $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$, we have

$$\frac{q_1q_2}{r_1r_2}=\frac{g(f+g)}{h(f+h)}=\frac{q_1^2[p_1q_1-r_1p_1-q_1r_1]}{r_1^2[r_1p_1-p_1q_1-q_1r_1]},$$

so that $q_2 : r_2 = q_1/(r_1p_1-p_1q_1-q_1r_1) : r_1/(p_1q_1-r_1p_1-q_1r_1)$.

It follows that, if $px+qy+rz=0$ is one asymptote, the other is $\Sigma\dfrac{px}{qr-p(q+r)}=0.$

**Note.** The coordinates of the asymptotes can be found by solving the equations $fqr+grp+hpq=0$, $(g+h)p+(h+f)q+(f+g)r=0$.

**Example i.** *The joins of the opposite vertices of a hexagon circumscribed to a conic are concurrent.* **(Brianchon's Theorem.)**

Let $a, e, c, d, b, f$ be the sides of the hexagon and take $a, b, c$ for the triangle of reference.

Let the coordinates of $d, e, f$ be $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$, $(p_3, q_3, r_3)$. The equation of the point of intersection of $a, e$ is $q/q_2 = r/r_2$, and of $b, d$ is $p/p_1 = r/r_1$; these points lie on the straight line $(p_1/r_1 : q_2/r_2 : 1)$ Similarly, the joins of the other pairs of opposite vertices are the straight lines $(p_1/q_1 : 1 : r_3/q_3)$, $(1 : q_2/p_2 : r_3/p_3)$. These lines are concurrent if

$$\begin{vmatrix} p_1/r_1 & q_2/r_2 & 1 \\ p_1/q_1 & 1 & r_3/q_3 \\ 1 & q_2/p_2 & r_3/p_3 \end{vmatrix} = 0,$$

i. e. if

$$\begin{vmatrix} 1/r_1 & 1/r_2 & 1/r_3 \\ 1/p_1 & 1/p_2 & 1/p_3 \\ 1/q_1 & 1/q_2 & 1/q_3 \end{vmatrix} = 0.$$

But the equation of the conic touching the lines $a, b, c$ is of the form

$$f/p + g/q + h/r = 0,$$

hence $f/p_1 + g/q_1 + h/r_1 = 0$, $f/p_2 + g/q_2 + h/r_2 = 0$; $f/p_3 + g/q_3 + h/r_3 = 0$. Eliminating $f, g, h$ we obtain the required condition.

**Note.** This evidently proves the converse of the theorem.

**Example ii.** *To find the equation of the tangent to the conic* $\sqrt{fx} + \sqrt{gy} + \sqrt{rz} = 0$ *which is parallel to BC.*

The tangential equation of the conic is $fqr + grp + hpq = 0$.

The coordinates of $BC$ are $(1 : 0 : 0)$; hence the coordinates of a straight line parallel to $BC$ are $(1 + k : k : k)$. This touches the conic if

$$fk^2 + gk(k+1) + hk(k+1) = 0,$$

i. e. $k = 0$, or $-(g+h)/(f+g+h)$.

The coordinates of the tangent required are therefore $f : -(g+h) : -(g+h)$, and its equation is $fx = (g+h)(y+z)$.

**Example iii.** *The centre of a conic, inscribed in the triangle of reference, lies on a fixed straight line parallel to BC; find the equation of the envelope of its asymptotes in areal coordinates.*

Let the given fixed straight line be $y + z = (\lambda + 1)x$, and let the equation of the conic be $fqr + grp + hpq = 0$.

The coordinates of its centre are $g+h : h+f : f+g$, and since this lies on the given line we have $2f = \lambda(g+h)$.

If $(p, q, r)$ is an asymptote, we have $f = kp^2(q-r)$, $g = kq^2(r-p)$, $h = kr^2(p-q)$; hence $2p^2(q-r) = \lambda\{q^2(r-p) + r^2(p-q)\}$

$$= \lambda(q-r)(qr - pq - pr);$$

the factor $q-r=0$ represents the point at infinity on $BC$; neglecting this, the tangential equation of the envelope becomes

$$2p^2+\lambda(pr+pq-qr)=0.$$

The point-coordinate equation of the envelope is therefore

$$x^2+y^2+z^2-2yz+2zx+2xy=8yz\lambda^{-1}.$$

## Examples XII f.

1. The areal equation of the inscribed circle is

$$\sqrt{(s-a)}\,x+\sqrt{(s-b)}\,y+\sqrt{(s-c)}\,z=0, \text{ or } \Sigma\sqrt{x\cot\tfrac{1}{2}A}=0.$$

2. The trilinear equation of the inscribed circle is $\Sigma\cos\frac{1}{2}A\sqrt{\alpha}=0$. What are the equations of the escribed circles in a similar form?

3. The tangents to an inscribed conic parallel to the sides of the triangle of reference form a triangle $A'B'C'$. Show that $AA'$, $BB'$, $CC'$ intersect at the centre of the conic.

4 If a conic touches the sides of the triangle of reference at $A'$, $B'$, $C'$, prove that $AA'$, $BB'$, $CC'$ are concurrent. Find the equation of the axis of homology of the triangles $ABC$, $A'B'C'$.

5. If the conic $\sqrt{fx}+\sqrt{gy}+\sqrt{hz}=0$ is a parabola, find the coordinates of the point at infinity on its axis.

6. A conic touches the sides of the triangle of reference and the straight line $(p_1, q_1, r_1)$. Show that the locus of its centre is $\Sigma(y+z-x)/p_1=0$.

7. Find the condition that the conic $\sqrt{fx}+\sqrt{gy}+\sqrt{hz}=0$ should be a rectangular hyperbola.

8. The locus of the centres of rectangular hyperbolas inscribed in a triangle is the polar circle $\alpha^2\sin 2A+\beta^2\sin 2B+\gamma^2\sin 2C=0$.

9. Normals are drawn to a conic, inscribed in the triangle $ABC$, at its points of contact with the sides. Prove that they meet in a point if

$$\begin{vmatrix} \alpha^2\sin^2 A & \beta^2\sin^2 B & \gamma^2\sin^2 C \\ \alpha\cos A & \beta\cos B & \gamma\cos C \\ 1 & 1 & 1 \end{vmatrix}=0,$$

where $\alpha$, $\beta$, $\gamma$ are the trilinear coordinates of the point of intersection of the lines joining the vertices of the triangle to the points of contact of the opposite sides.

10. Prove that if the conics $a\sqrt{\alpha\tan\theta}+b\sqrt{\beta\tan\phi}+c\sqrt{\gamma\tan\psi}=0$, $a\sqrt{\alpha\cot\theta}+b\sqrt{\beta\cot\phi}+c\sqrt{\gamma\cot\psi}=0$ are parabolas, a circle can be described through their six points of contact with the sides of the triangle of reference.

11. $ABC$ is a triangle and $P$ any point on a fixed straight line. Prove that the envelope of the harmonic conjugate of $PA$ with respect to $PB$, $PC$ is a conic touching the sides of the triangle and the fixed straight line.

12. A conic touches the sides of a triangle at $A'$, $B'$, $C'$, and passes through a fixed point. Show that the straight line joining the points of intersection of $B'C'$, $BC$; $C'A'$, $CA$; $A'B'$, $AB$ envelopes a conic which circumscribes the triangle $ABC$.

**§ 12. Conics referred to a self-conjugate triangle.** The polar of $A$ (1, 0, 0) with respect to the conic

$$ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0$$

is $ux+hy+gz=0$; hence, if this is the side $BC$ ($x=0$) we have $h=0$, $g=0$. Similarly, if $B$ and $C$ are the poles of $CA$ and $AB$, we must have $f=0$, $g=0$, $h=0$.

The general equation of a conic with respect to which the triangle of reference is self-conjugate is therefore

$$ux^2+vy^2+wz^2=0.$$

**Note i.** One of the coefficients $u$, $v$, $w$ must be negative; if two are negative we can change the signs throughout. The conic can therefore be written $l^2x^2+m^2y^2=n^2z^2$. This equation can also be written

$$l^2x^2=(nz-my)(nz+my),$$

or

$$m^2y^2=(nz-lx)(nz+lx).$$

These forms show that the conic touches $nz-my=0$, $nz+my=0$, the side $BC$ being the chord of contact, and also that it touches $nz-lx=0$, $nz+lx=0$, the side $CA$ being the chord of contact. The tangents from $A$ and $B$ to the conic are real. The tangents from $C$ to the conic are imaginary, for the equation can be written $n^2z^2=(lx+imy)(lx-imy)$. Hence two vertices of a real self-conjugate triangle lie outside the conic and one inside.

**Note ii.** If the equation is used in the form $x^2/l^2+y^2/m^2=z^2/n^2$ we can use a parametric representation for the coordinates of a point on the conic, viz. $\{l\cos\theta : m\sin\theta : n\}$. The analogy to the case of the ellipse

$$x^2/a^2+y^2/b^2=1$$

will be evident, and the various equations are in a similar form; thus, for example, the tangent at the point $\theta$ is $x\cos\theta/l+y\sin\theta/m=z/n$.

The tangential equation of the conic $ux^2+vy^2+wz^2=0$ is

$$vwp^2+wuq^2+uvr^2=0 \quad \text{or} \quad p^2/u+q^2/v+r^2/w=0.$$

**Tangent and point of contact.** The equation of the tangent at the point $(x_1, y_1, z_1)$ is $uxx_1+vyy_1+wzz_1=0$, and that of the point of contact of the tangent $(p_1, q_1, r_1)$ is

$$pp_1/u+qq_1/v+rr_1/w=0.$$

Hence, if $px+qy+rz=0$ touches the conic, the point of contact is $\{p/u, q/v, r/w\}$.

**Pole and polar.** The equation of the polar of the point $(x_1, y_1, z_1)$ is $uxx_1+vyy_1+wzz_1=0$, and that of the pole of $(p_1, q_1, r_1)$ is

$$xp_1/u+qq_1/v+rr_1/w=0.$$

Thus the pole of the line $p_1x+q_1y+r_1z=0$ is $\{p_1/u, q_1/v, r_1/w\}$.

**Cor.** The equation of the centre is $p/u+q/v+r/w=0$, and its coordinates are $(1/u : 1/v : 1/w)$.

Hence the equation of a conic with respect to which the triangle of reference is self-conjugate and whose centre is $(x_0, y_0, z_0)$ is $x^2/x_0+y^2/y_0+z^2/z_0=0$.

**The asymptotes.** The equation of the asymptotes is (§ 8 (6))

$$(uv+vw+wu)(ux^2+vy^2+wz^2)=uvw(x+y+z)^2.$$

(a) *If $px+qy+rz=0$ is an asymptote, to find the equation of the conic.*

Let the conic be $ux^2+vy^2+wz^2=0$; the point of contact of $px+qy+rz=0$ is $\{p/u, q/v, r/w\}$, and this must be the point at infinity on $px+qy+rz=0$, viz. $\{q-r : r-p : p-q\}$.

Hence $$u=\frac{kp}{q-r}, \quad v=\frac{kq}{r-p}, \quad w=\frac{kr}{p-q},$$

and the equation of the conic is

$$\frac{px^2}{q-r}+\frac{qy^2}{r-p}+\frac{rz^2}{p-q}=0.$$

(b) *If $p_1x+q_1y+r_1z=0$ is an asymptote, to find the equation of the other asymptote.*

Let $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$ be the asymptotes; then, since they touch the conic and pass through the centre, we have

$$p/u+q/v+r/w=0, \quad p^2/u+q^2/v+r^2/w=0,$$

where $ux^2+vy^2+wz^2=0$ is the equation of the conic.

Hence $$\frac{1}{u}\left(\frac{q^2}{v}+\frac{r^2}{w}\right)+\left(\frac{q}{v}+\frac{r}{w}\right)^2=0;$$

$$\therefore \quad \frac{q^2}{v}\left(\frac{1}{u}+\frac{1}{v}\right)+\frac{2qr}{vw}+\frac{r^2}{w}\left(\frac{1}{w}+\frac{1}{u}\right)=0.$$

Hence $$\frac{q_1q_2}{r_1r_2}=\frac{v^2(u+w)}{w^2(u+v)}.$$

Now $$w+u=\frac{r_1}{p_1-q_1}+\frac{p_1}{q_1-r_1}=\frac{(r_1-p_1)(q_1-p_1-r_1)}{(p_1-q_1)(q_1-r_1)},$$

which is proportional to $(r_1-p_1)^2(q_1-p_1-r_1)$.

Hence $v^2(u+w)$ is proportional to $q_1^2(q_1-p_1-r_1)$, so that

$$\frac{q_1q_2}{r_1r_2}=\frac{q_1^2(q_1-p_1-r_1)}{r_1^2(r_1-p_1-q_1)},$$

or $$q_2 : r_2 = q_1(q_1-p_1-r_1) : r_1(r_1-p_1-q_1).$$

Similarly for $p_2 : q_2$; so that the equation of the other asymptote is

$$p_1(p_1-q_1-r_1)x+q_1(q_1-p_1-r_1)y+r_1(r_1-p_1-q_1)z=0.$$

### Illustrative Example.

*Two triangles are self-polar with respect to a conic; show that their six angular points lie on a conic and their six sides touch a conic.*

Let one of the triangles be the triangle of reference $ABC$, and $A'B'C'$ be the other triangle.

The equation of the conic is of the form $lx^2+my^2+nz^2=0$.

Let the vertices of the triangle $A'B'C'$ be $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$.

Then, since $A'$, $C'$ lie on the polar of $B'$, we have

$$lx_1x_2+my_1y_2+nz_1z_2=0,$$
$$lx_2x_3+my_2y_3+nz_2z_3=0,$$

and, similarly,

$$lx_3x_1+my_3y_1+nz_3z_1=0.$$

But these are the conditions that the points $A'$, $B'$, $C'$ should lie on the conic

$$\frac{lx_1x_2x_3}{x}+\frac{my_1y_2y_3}{y}+\frac{nz_1z_2z_3}{z}=0.$$

This conic passes through $A$, $B$, $C$; hence the six vertices lie on a conic.

The equation of the conic is of the form $lp^2+mq^2+nr^2=0$.

Let the sides of the triangle $A'B'C'$ be $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$, $(p_3, q_3, r_3)$.

Then, since $B'C'$, $A'B'$ pass through the pole of $C'A'$, we have

$$lp_1p_2+mq_1q_2+nr_1r_2=0,$$
$$lp_2p_3+mq_2q_3+nr_2r_3=0,$$

and, similarly,

$$lp_3p_1+mq_3q_1+nr_3r_1=0.$$

But these are the conditions that the sides $A'B'$, $B'C'$, $C'A'$ should touch the conic

$$\frac{lp_1p_2p_3}{p}+\frac{mq_1q_2q_3}{q}+\frac{nr_1r_2r_3}{r}=0.$$

This conic touches the sides of the triangle $ABC$; hence the six sides touch a conic.

### Systems of conics.

(i) **Conics passing through four points.** Let the four points be $(x_1, y_1, z_1)$, $(-x_1, y_1, z_1)$, $(x_1, -y_1, z_1)$, $(x_1, y_1, -z_1)$, so that the diagonal triangle of the quadrangle, whose vertices are the four points (p. 471), is the triangle of reference.

Let $ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0$ be the conic; substituting the coordinates of the four points in this equation we find $f=0$, $g=0$, $h=0$.

The equation of the conic is therefore $ux^2+vy^2+wz^2=0$, with the condition that $ux_1^2+vy_1^2+wz_1^2=0$. Evidently the triangle whose vertices are the diagonal points of a quadrangle is self-conjugate with respect to any conic which circumscribes the quadrangle.

(ii) **Conics touching four straight lines.** Let the straight lines be $p_1x+q_1y+r_1z=0$, $-p_1x+q_1y+r_1z=0$, $p_1x-q_1y+r_1z=0$, $p_1x+q_1y-r_1z=0$, i.e. the lines

$$(p_1, q_1, r_1),\ (-p_1, q_1, r_1),\ (p_1, -q_1, r_1),\ (p_1, q_1, -r_1),$$

so that the diagonal triangle of the quadrilateral they form is the triangle of reference.

With the same reasoning as in (i) we find that the tangential equation of the conic is $up^2+vq^2+wr^2=0$, with the condition that $up_1^2+vq_1^2+wr_1^2=0$. Evidently the diagonal triangle of a quadrilateral is self-conjugate with respect to any conic inscribed to the quadrilateral.

## Examples XII g.

1. Show that $\Sigma a^2\alpha^2/(q-r)=0$ is a parabola.

2. Find the equation of the diameter of $lx^2+my^2+nz^2=0$ which passes through $(x', y', z')$.

3. Find the equation of the tangents at the ends of the chord $x-y=0$ of the conic $lx^2+my^2+nz^2=0$.

4. A conic, with respect to which the triangle of reference is self-conjugate, has $lx+my+nz=0$ for an asymptote; find the coordinates of its centre.

5. A conic, to which a given triangle is self-polar, passes through a fixed point; show that its centre lies on a fixed conic circumscribing the given triangle.

6. A rectangular hyperbola, with respect to which the triangle of reference is self-conjugate, has $px+qy+rz=0$ for an asymptote. Show that $\Sigma pa^2/(q-r)=0$.

7. Find the separate equations of the asymptotes of the conic

$$x^2-4y^2+7z^2=0.$$

8. A conic has a fixed self-conjugate triangle and touches a fixed straight line; find the locus of its centre.

9. Find the envelope of the polar of a given point with respect to a parabola which has a given self-conjugate triangle.

10. A system of conics circumscribes a quadrangle; how many conics of the system (a) pass through a given point, (b) touch a given line?

11. A system of conics is inscribed in a quadrilateral; how many conics of the system (a) touch a given line, (b) pass through a given point?

12. The polars of a fixed point, with respect to a conic which passes through four fixed points, are concurrent.

13. Find the envelope of the polars of a fixed point with respect to conics which touch four fixed straight lines.

14. A conic passes through four given points; find

(a) The locus of the poles of a given straight line with respect to the conic.

(b) The locus of the centre of the conic. What special points lie on this locus?

15. A conic touches four fixed straight lines; find

(a) The locus of the poles of a fixed straight line with respect to the conic.

(b) The locus of the centre of the conic,

Show that (b) passes through the mid-points of the diagonals of the quadrilateral formed by the given lines.

16. A system of conics is inscribed in a quadrilateral; find the locus of the points of contact of tangents in a given direction.

17. A conic is inscribed in a quadrilateral; prove that the product of the perpendiculars from one pair of opposite vertices to any tangent to the conic is in a constant ratio to the product of the perpendiculars from the other pair of opposite vertices.

18. A given triangle is self-conjugate with respect to a parabola; show that the parabola touches four fixed straight lines.

19. Two conics circumscribe the quadrangle $ABCD$, and tangents are drawn to the conics from a point $P$, whose points of contact are $Q, Q'$; $R, R'$. If $P$ lies on one of the sides of the quadrangle, show that $QR$ $Q'R'$ intersect at one of two fixed points.

20. A system of conics circumscribes a quadrangle whose vertices are $(l : \pm m : \pm n)$; find the condition that two given points should be conjugate with respect to any conic of the system.

21. A system of conics is inscribed in a quadrilateral whose sides are $(l : \pm m : \pm n)$; find the condition that two given straight lines should be conjugate with respect to any conic of the system.

22. A system of conics touches three given straight lines and passes through a fixed point; how many conics of the system will pass through a second fixed point?

23. Show that the conic $\Sigma(u+f-g-h)x^2 = 0$ meets the line at infinity at the same points as the conic represented by the general equation of the second degree.

## § 13. The general equation.

I. *To find the equation of the director circle of the conic*

$$ux^2+vy^2+wz^2+2fyz+2gzx+2hxy = 0,$$

*and the equation of the directrix when the conic is a parabola.*

If $(p, q, r)$ is a tangent to the conic through the point $(x_1, y_1, z_1)$, and $(x, y, z)$ is any other point on the tangent, then

$$px+qy+rz = 0, \quad px_1+qy_1+rz_1 = 0;$$

hence $$p : q : r = (yz_1-y_1z) : (zx_1-z_1x) : (xy_1-x_1y).$$

Now $(p, q, r)$ being a tangent to the conic, we have

$$Up^2+Vq^2+Wr^2+2Fqr+2Grp+2Hpq = 0;$$

hence

$$\Sigma U(yz_1-y_1z)^2+2\Sigma F(zx_1-z_1x)(xy_1-x_1y) = 0 \qquad \text{(i)}$$

is the locus of $(x, y, z)$, i.e. the equation of the tangents from $(x_1, y_1, z_1)$ to the conic. Equation (i) can be written

$$\Sigma(Vz_1^2+Wy_1^2-2Fy_1z_1)x^2-2\Sigma(Uy_1z_1+Fx_1^2-Gx_1y_1-Hz_1x_1)yz = 0.$$

The tangents are at right angles (using areal coordinates) if

$$\Sigma a^2(Vz_1^2+Wy_1^2-2Fy_1z_1)$$
$$+2\Sigma(Uy_1z_1+Fx_1^2-Gx_1y_1-Hz_1x_1)bc\cos A=0. \quad \text{(ii)}$$

If we omit the suffixes, this equation represents the locus of a point the tangents from which to the conic are orthogonal, i.e. the director circle.

The equation (ii) can be expressed in the standard form for a circle, viz. $(x+y+z)(lx+my+nz)-(a^2yz+b^2zx+c^2xy)=0$ as follows; collecting coefficients we obtain

$$\Sigma(Vc^2+Wb^2+2Fbc\cos A)x^2$$
$$-2\Sigma(Fa^2+Gab\cos C+Hca\cos B-Ubc\cos A)yz=0.$$

Take the expression

$$(x+y+z)\{(Vc^2+Wb^2+2Fbc\cos A)x$$
$$+(Wa^2+Uc^2+2Gca\cos B)y+(Ub^2+Va^2+2Hab\cos C)z\}$$

from the left-hand side of this equation, and we have three terms containing $yz$, $zx$, $xy$ left; the coefficient of $yz$ is

$$-(2Fa^2+2Gab\cos C+2Hca\cos B-2Ubc\cos A)$$
$$-(Ub^2+Va^2+2Hab\cos C)-(Wa^2+Uc^2+2Gca\cos B)$$
$$=-a^2(U+V+W+2F+2G+2H)=-a^2K.$$

The equation of the director circle is therefore

$$(x+y+z)\Sigma(Vc^2+Wb^2+2Fbc\cos A)x-K(a^2yz+b^2zx+c^2xy)=0.$$

**Cor.** If the conic is a parabola, we have $K=0$; the equation of the directrix of the parabola is $\Sigma(Vc^2+Wb^2+2Fbc\cos A)x=0$.

The focus of a parabola can be found as the pole of the directrix.

II. **Conjugate diameters.**

(a) *To find the equation of the diameter bisecting chords parallel to* $px+qy+rz=0$.

If $(x_1, y_1, z_1)$ is the mid-point of a chord parallel to the line $(p, q, r)$, its equation is (§ 5, Cor. ii)

$$\frac{x-x_1}{q-r}=\frac{y-y_1}{r-p}=\frac{z-z_1}{p-q}=k.$$

For the points of intersection of this chord with the conic the values of $k$ are given therefore by the equation

$$k^2f(q-r, r-p, p-q)$$
$$+2k\{(q-r)X_1+(r-p)Y_1+(p-q)Z_1\}+f(x_1, y_1, z_1)=0.$$

Since $(x_1, y_1, z_1)$ is the mid-point, the values of $k$ must be equal and opposite, hence

$$(q-r)X_1+(r-p)Y_1+(p-q)Z_1=0.$$

Hence the locus of $(x_1, y_1, z_1)$, i.e. the equation of the diameter bisecting chords parallel to $px+qy+rz=0$, is

$$(q-r)X+(r-p)Y+(p-q)Z=0.$$

**Cor.** The straight line $lX+mY+nZ=0$ is a diameter of the conic if $l+m+n=0$.

(b) *To find the condition that the lines* $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$ *should be parallel to conjugate diameters.*

The diameter bisecting chords parallel to $(p_1, q_1, r_1)$ is

$$(q_1-r_1)X+(r_1-p_1)Y+(p_1-q_1)Z=0.$$

If we write $l_1, m_1, n_1$ for $(q_1-r_1)$, $(r_1-p_1)$, $(p_1-q_1)$ respectively, the equation is $l_1X+m_1Y+n_1Z=0$, or

$$(ul_1+hm_1+gn_1)x+(hl_1+vm_1+fn_1)y+(gl_1+fm_1+wn_1)z=0.$$

This is parallel to $(p_2, q_2, r_2)$ if

$$\begin{vmatrix} ul_1+hm_1+gn_1 & hl_1+vm_1+fn_1 & gl_1+fm_1+wn_1 \\ p_2 & q_2 & r_2 \\ 1 & 1 & 1 \end{vmatrix}=0,$$

and writing $l_2, m_2, n_2$ for $(q_2-r_2)$, $(r_2-p_2)$, $(p_2-q_2)$ respectively, this gives

$$l_2(ul_1+hm_1+gn_1)+m_2(hl_1+vm_1+fn_1)+n_2(gl_1+fm_1+wn_1)=0,$$

i.e. $$ul_1l_2+vm_1m_2+wn_1n_2+f(m_1n_2+m_2n_1)+g(n_1l_2+n_2l_1)+h(l_1m_2+l_2m_1)=0.$$

This is the condition that the diameter parallel to $(p_2, q_2, r_2)$ should bisect chords parallel to $(p_1, q_1, r_1)$, and symmetrically it is the condition that the diameter parallel to $(p_1, q_1, r_1)$ should bisect chords parallel to $(p_2, q_2, r_2)$; hence it is the condition that the diameters parallel to $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$ should be conjugate.

**Cor.** The diameters $l_1X+m_1Y+n_1Z=0$, $l_2X+m_2Y+n_2Z=0$, where $l_1+m_1+n_1=0$, $l_2+m_2+n_2=0$ are conjugate if

$$\Sigma ul_1l_2+\Sigma f(m_1n_2+m_2n_1)=0.$$

Now $$l_1l_2=(m_1+n_1)(m_2+n_2);$$

$\therefore$ $$m_1n_2+m_2n_1=l_1l_2-m_1m_2-n_1n_2,$$

and similarly $$n_1l_2+n_2l_1=m_1m_2-n_1n_2-l_1l_2,$$

$$l_1m_2+l_2m_1=n_1n_2-l_1l_2-m_1m_2;$$

whence also $$2l_1l_2=-(n_1l_2+n_2l_1)-(l_1m_2+l_2m_1).$$

Hence the condition for conjugate diameters can be written in either of the forms

$$\Sigma(u+f-g-h)\,l_1 l_2 = 0,$$

or

$$\Sigma(v+w-2f)\,(m_1 n_2 + m_2 n_1) = 0.$$

(c) *To find the equation of the diameter conjugate to* $lX+mY+nZ=0$ (*where* $l+m+n=0$).

Let the equation of this diameter be $l'X+m'Y+n'Z=0$; then we have $l'+m'+n'=0$ and

$$(u+f-g-h)\,ll' + (v+g-h-f)\,mm' + (w+h-f-g)\,nn' = 0.$$

Eliminating $l'$, $m'$, $n'$ we get

$$\begin{vmatrix} X & Y & Z \\ 1 & 1 & 1 \\ (u+f-g-h)\,l & (v+g-h-f)\,m & (w+h-f-g)\,n \end{vmatrix} = 0,$$

i.e.

$$\Sigma X\,\{(w+h+f-g)\,n-(v+g-h-f)\,m\} = 0,$$

or, since $m+n=-l$, this becomes

$$\Sigma X\,\{(gl+fm+wn)-(hl+vm+fn)\} = 0,$$

which can also be written

$$\begin{vmatrix} X & Y & Z \\ 1 & 1 & 1 \\ ul+hm+gn & hl+vm+fn & gl+fm+wn \end{vmatrix} = 0.$$

III. **The axes.** The axes of the conic are conjugate diameters at right angles.

Let the axes be the diameters

$$l_1 X + m_1 Y + n_1 Z = 0, \quad l_2 X + m_2 Y + n_2 Z = 0,$$

and suppose that they are parallel to the straight lines

$$p_1 x + q_1 y + r_1 z = 0, \quad p_2 x + q_2 y + r_2 z = 0.$$

Then, since the diameter bisecting chords parallel to $(p_2, q_2, r_2)$ is $(q_2-r_2)\,X+(r_2-p_2)\,Y+(p_2-q_2)\,Z=0$, we have

$$\frac{l_1}{q_2-r_2} = \frac{m_1}{r_2-p_2} = \frac{n_1}{p_2-q_2} \quad \text{and} \quad \frac{l_2}{q_1-r_1} = \frac{m_2}{r_1-p_1} = \frac{n_2}{p_1-q_1}.$$

Since the lines $(p_1, q_1, r_1)$, $(p_2, q_2, r_2)$ are perpendicular, we have

$$\Sigma(q_1-r_1)\,(q_2-r_2)\cot A = 0,$$

i.e.

$$l_1 l_2 \cot A + m_1 m_2 \cot B + n_1 n_2 \cot C = 0. \qquad \text{(i)}$$

Again, since the diameters are conjugate, we have

$$(u+f-g-h)\,l_1 l_2 + (v+g-h-f)\,m_1 m_2 + (w+h-f-g)\,n_1 n_2 = 0, \qquad \text{(ii)}$$

and further

$$l_2 + m_2 + n_2 = 0. \qquad \text{(iii)}$$

Eliminating $l_2, m_2, n_2$ from (i), (ii), and (iii) we obtain

$$\begin{vmatrix} l_1 \cot A & m_1 \cot B & n_1 \cot C \\ (u+f-g-h)\,l_1 & (v+g-h-f)\,m_1 & (w+h-f-g)\,n_1 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

and evidently $l_2, m_2, n_2$ satisfy the same equation.

But, if $(x, y, z)$ is any point on the axis $l_1X+m_1Y+n_1Z=0$, then, since $l_1+m_1+n_1=0$, we have

$$l_1 : m_1 : n_1 = Y-Z : Z-X : X-Y.$$

The equation of the axes is therefore

$$\begin{vmatrix} (Z-X)(X-Y) & (X-Y)(Y-Z) & (Y-Z)(Z-X) \\ u+f-g-h & v+g-h-f & w+h-f-g \\ \cot A & \cot B & \cot C \end{vmatrix} = 0.$$

This equation can also be written

$$\begin{vmatrix} (Y-Z)^2 & (Z-X)^2 & (X-Y)^2 \\ v+w-2f & w+u-2g & u+v-2h \\ \sin^2 A & \sin^2 B & \sin^2 C \end{vmatrix} = 0.$$

*To find the equation of the axis of a parabola.*

When the conic is a parabola, we know that the equation of the axes of symmetry reduces to that of the axis and the line at infinity.

Let $\lambda, \mu, \nu$ denote $u+f-g-h$, $v+g-h-f$, $w+h-f-g$ respectively. Now the equation of the conic can be written

$$\lambda x^2+\mu y^2+\nu z^2 - \{(f-g-h)x+(g-h-f)y+(h-f-g)z\}(x+y+z)=0;$$

this can be verified by immediate simplification.

Hence the common chords of the conics $f(x, y, z)=0$ and $\lambda x^2+\mu y^2+\nu z^2=0$ are $\Sigma(f-g-h)x=0$ and $x+y+z=0$; i.e. they meet the line at infinity at the same points.

Thus if the conic $f(x, y, z)$ is a parabola, so is the conic $\lambda x^2+\mu y^2+\nu z^2=0$; in this case therefore $\mu\nu+\nu\lambda+\lambda\mu=0$.

The equation of the axes is

$$\begin{vmatrix} (Z-X)(X-Y) & (X-Y)(Y-Z) & (Y-Z)(Z-X) \\ \lambda & \mu & \nu \\ \cot A & \cot B & \cot C \end{vmatrix} = 0,$$

i.e. $$\Sigma(Y-Z)\{\mu(Z-X)-\nu(X-Y)\}\cot A = 0;$$

when the conic is a parabola $\lambda\mu+\mu\nu+\nu\lambda=0$, and this becomes

$$\Sigma(Y-Z)\{\mu Z+\nu Y-(\mu+\nu)X\}\cot A = 0,$$

i.e. $$\Sigma 1/\lambda\,(Y-Z)\{\lambda\mu Z+\nu\lambda Y+\mu\nu X\}\cot A = 0,$$

i.e. $$\{\mu\nu X+\nu\lambda Y+\lambda\mu Z\}\,.\,\Sigma 1/\lambda\,(Y-Z)\cot A = 0.$$

Now $\mu\nu X+\nu\lambda Y+\lambda\mu Z$

$$\begin{aligned}&= \Sigma x(\mu\nu u+\nu\lambda h+\lambda\mu g)\\ &= \Sigma x\{\mu\nu(\lambda-f+g+h)+\nu\lambda h+\lambda\mu g\}\\ &= \Sigma x\{\lambda\mu\nu-\mu\nu f-\nu\lambda g-\lambda\mu h\}\\ &= (\lambda\mu\nu-\mu\nu f-\nu\lambda g-\lambda\mu h)\Sigma x.\end{aligned}$$

Hence the line $\mu\nu X+\nu\lambda Y+\lambda\mu Z=0$ is $x+y+z=0$, i.e. the line at infinity.

The equation of the axis of the parabola is therefore

$$(Y-Z)\cot A/\lambda+(Z-X)\cot B/\mu+(X-Y)\cot C/\nu=0.$$

IV. **The foci.** If $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ are the foci of the conic, since the lines joining them to the circular points at infinity are tangents to the conic, we must have for some values of $\lambda$ and $\mu$

$$\lambda\{ap, bq, cr\}^2+\mu(px_1+qy_1+rz_1)(px_2+qy_2+rz_2)$$
$$\equiv Up^2+Vq^2+Wr^2+2Fqr+2Grp+2Hpq. \quad \text{(i)}$$

We can determine the value of $\lambda$ since

$$Up^2+Vq^2+Wr^2+2Fqr+2Grp+2Hpq-\lambda\{ap, bq, cr\}^2$$

breaks up into two linear factors. This gives (see V, below)

$$4S^2K\lambda^2-\Delta\theta\lambda+\Delta^2=0; \quad \text{(ii)}$$

the two values of $\lambda$ give two pairs of factors, and two corresponding pairs of foci.

Or, in this identity put $p$, $q$, $r$ each unity, then

$$\mu=U+V+W+2F+2G+2H=K,$$

provided that $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ are finite points.

Comparing coefficients we have also

$$\begin{aligned}U &= \mu x_1x_2+\lambda a^2,\\ 2G &= \mu(x_1z_2+x_2z_1)-2\lambda ca\cos B,\\ 2H &= \mu(x_1y_2+x_2y_1)-2\lambda ab\cos C;\end{aligned}$$

hence

$$\begin{aligned}2U+2G+2H &= \mu x_1(x_2+y_2+z_2)+\mu x_2(x_1+y_1+z_1)\\ &= \mu(x_1+x_2).\end{aligned}$$

Thus $$x_1+x_2=\frac{2(U+G+H)}{K},$$

and $$x_1x_2=\frac{U-\lambda a^2}{K}.$$

Therefore $x_1$, $x_2$ are roots of the equation

$$Kx^2-2(U+G+H)x+U=\lambda a^2.$$

Similarly, the other coordinates of the foci are roots of the equation

$$Ky^2-2(V+H+F)y+V=\lambda b^2,$$
$$Kz^2-2(W+F+G)z+W=\lambda c^2.$$

These equations can be written

$$K^2(x-x_0)^2 = (U+G+H)^2 - UK + \lambda Ka^2, \text{ \&c.},$$

where $x_0, y_0, z_0$ are the coordinates of the centre.

Since $$x-x_0+y-y_0+z-z_0 = 0, \qquad \text{(iii)}$$

we have $$\Sigma\sqrt{\{(U+G+H)^2 - UK + \lambda Ka^2\}} = 0.$$

This gives a quadratic for $\lambda$, which must be the same as equation (ii) found above; taking either value of $\lambda$, we have

$$K(x-x_0) = \pm\sqrt{\{(U+G+H)^2 - UK + \lambda Ka^2\}}, \text{ \&c.};$$

the signs of the surds must be chosen so as to satisfy equation (iii); thus to each value of $\lambda$ there corresponds a pair of foci.

**Note i.** When the conic is a parabola $K = 0$, and the equations become linear.

**Note ii.** The equations for finding the foci can also be obtained from the fact that the tangents from a focus to the conic pass through the circular points at infinity, and therefore that their equation satisfies the conditions for a circle. Thus, if

$$f(x, y, z) \cdot f(x', y', z') - (xX' + yY' + zZ')^2 = 0$$

satisfies the conditions for a circle, we have

$$(v+w-2f) \cdot f(x', y', z') - (Y-Z)^2 = \lambda a^2$$

and two symmetrical equations.

Incidentally, if we eliminate $f(x', y', z')$ and $\lambda$ from these equations, we find

$$\begin{vmatrix} (Y-Z)^2 & (Z-X)^2 & (X-Y)^2 \\ v+w-2f & w+u-2g & u+v-2h \\ a^2 & b^2 & c^2 \end{vmatrix} = 0;$$

hence the foci lie on the axes of the conic.

**Note iii.** If the conic is inscribed in the triangle of reference, we have $U = 0$, $V = 0$, $W = 0$.

Hence $$\mu x_1x_2 + \lambda a^2 = 0, \ \mu y_1y_2 + \lambda b^2 = 0, \ \mu z_1z_2 + \lambda c^2 = 0;$$

$$\therefore \quad \frac{x_1x_2}{a^2} = \frac{y_1y_2}{b^2} = \frac{z_1z_2}{c^2}.$$

(In trilinear coordinates this gives $\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2$.)

If the conic is a parabola, i.e. $K = 0$, then in equation (i) we could not conclude that $\mu = K = 0$, but $\mu(x_1+y_1+z_1)(x_2+y_2+z_2) = 0$; evidently $\mu$ is not zero, hence one focus is at infinity; we still have $\frac{x_1x_2}{a^2} = \frac{y_1y_2}{b^2} = \frac{z_1z_2}{c^2}$, where $x_1+y_1+z_1 = 0$.

Hence $a^2/x_2 + b^2/y_2 + c^2/z_2 = 0$, or the finite focus lies on the circumscribing circle.

Again, if $\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ is the equation of a parabola inscribed in the triangle of reference, the point of contact of the line at infinity $x+y+z = 0$ is $(f : g : h)$; this point is its infinite focus, hence the finite focus is $\left(\frac{a^2}{f} : \frac{b^2}{g} : \frac{c^2}{h}\right)$.

### Illustrative Example.

*To find the equation of the tangent at the vertex of the parabola inscribed in the triangle of reference, with its focus at the point* $(x', y', z')$.

If $fqr+grp+hpq=0$ is the equation of the conic, we have

$$x' : y' : z' = a^2/f : b^2/g : c^2/h.$$

Hence the equation can be written $a^2qr/x' + b^2rp/y' + c^2pq/z' = 0$. The point equation of the conic is therefore

$$a\sqrt{x/x'} + b\sqrt{y/y'} + c\sqrt{z/z'} = 0.$$

The directrix is the polar of the point $(x', y', z')$ with respect to this conic; its equation is $\Sigma \frac{a^2}{x'}(b^2+c^2-a^2)x = 0$, or $\Sigma ax \cos A/x' = 0$.

The tangent at the vertex is parallel to the directrix, so that its coordinates are $\{a\cos A/x' + k,\ b\cos B/y' + k,\ c\cos C/z' + k\}$, and these satisfy the tangential equation of the conic.

Hence
$$\Sigma \frac{a^2}{x'}\left\{\frac{b\cos B}{y'} + k\right\}\left\{\frac{c\cos C}{z'} + k\right\} = 0,$$

i.e.
$$\Sigma a^2(b\cos B + ky')(c\cos C + kz') = 0.$$

But $a^2y'z' + b^2z'x' + c^2x'y' = 0$, so that this equation becomes

$$k\Sigma a^2(cy'\cos C + bz'\cos B) + abc\,\Sigma a\cos B\cos C = 0,$$

i.e. since
$$x' + y' + z' = 1,$$

$$k + \Sigma a\cos B\cos C = 0;$$

$$\therefore\quad k + 2R\Sigma \sin A\cos B\cos C = 0;$$

$$\therefore\quad k + 2R\sin A\sin B\sin C = 0.$$

The equation of the tangent at the vertex is therefore

$$\Sigma(a\cos A/x' - 2R\sin A\sin B\sin C)x = 0,$$

or $\frac{x}{x'}a\cos A + \frac{y}{y'}b\cos B + \frac{z}{z'}c\cos C = 2R\sin A\sin B\sin C(x+y+z)$.

**Note.** This is the equation of the Simson line of the point $(x', y', z')$ with respect to the triangle of reference.

### V. The lengths of the axes, and the eccentricity of the conic.

**Method 1.** Since the axes are conjugate and perpendicular, the tangents at the extremities of an axis are each perpendicular to it; hence a circle, whose centre is at the centre of the conic and whose radius is a semi-axis, has double contact with the conic, the pole of the chord of contact being a point at infinity.

The tangential equation of the conic is

$$\Sigma \equiv Up^2 + Vq^2 + Wr^2 + 2Fqr + 2Grp + 2Hpq;$$

let $(x_0, y_0, z_0)$ be its centre and $r$ the length of a semi-axis. Then the equation of the circle whose centre is $(x_0, y_0, z_0)$, and whose radius is $r$, is

$$4S^2(px_0+qy_0+rz_0)^2 = r^2\{ap, bq, cr\}^2,$$

or, writing $r = 2S\rho$, the equation is

$$(px_0+qy_0+rz_0)^2 = \rho^2\{ap, bq, cr\}^2.$$

Hence we have, identically,

$$\lambda\Sigma+(px_0+qy_0+rz_0)^2-\rho^2\{ap, bq, cr\}^2 = \mu(px+qy+rz)^2,$$

where $\lambda$, $\mu$ are constants, and $(x, y, z)$ is a point at infinity, so that $x+y+z = 0$.

Now put $p$, $q$, $r$ each unity in this identity, thus $\lambda K+1 = 0$, or $\lambda = -1/K$.

We now have

$$\Sigma+K\rho^2\{ap, bq, cr\}^2 = K(px_0+qy_0+rz_0)^2-\mu K(px+qy+rz)^2,$$

and since the right-hand side has two linear factors, the left-hand side also factorizes. Hence

$$\begin{vmatrix} U+Ka^2\rho^2 & H-Kab\rho^2\cos C & G-Kca\rho^2\cos B \\ H-Kab\rho^2\cos C & V+Kb^2\rho^2 & F-Kbc\rho^2\cos A \\ G-Kca\rho^2\cos B & F-Kbc\rho^2\cos A & W+Kc^2\rho^2 \end{vmatrix} = 0.$$

The coefficient of $\rho^6$ is evidently zero; the coefficient of $\rho^4$

$$= \Sigma K^2bc\begin{vmatrix} U & -a\cos C & -a\cos B \\ H & b & -b\cos A \\ G & -c\cos A & c \end{vmatrix}$$

$$= \Sigma K^2bc\begin{vmatrix} U+H+G & 0 & 0 \\ H & b & -b\cos A \\ G & -c\cos A & c \end{vmatrix}$$

$$= 4S^2K^2\Sigma(U+H+G) = 4S^2K^3.$$

The coefficient of $\rho^2$

$$= \Sigma Kc\begin{vmatrix} U & H & -a\cos B \\ H & V & -b\cos A \\ G & F & c \end{vmatrix}$$

$$= K.\Sigma c\{c\Delta w-a\cos B\,\Delta y-b\cos A\,\Delta f\}$$

$= \Delta K\theta$, where $\theta = \Sigma a^2u-2\Sigma fbc\cos A$.

The term independent of $\rho$ is $\Delta^2$.

Hence $$4S^2K^3\rho^4+\Delta K\theta\rho^2+\Delta^2 = 0,$$

or $$K^3r^4+K\Delta\theta r^2+4S^2\Delta^2 = 0.$$

**Cor. i.** If $r_1$, $r_2$ are the semi-axes of the conic,

$$r_1^2+r_2^2 = -\Delta\theta/K^2,\ r_1^2 r_2^2 = 4S^2\Delta^2/K^3,$$

hence
$$\frac{(r_1^2+r_2^2)^2}{r_1^2 r_2^2} = \frac{\theta^2}{4KS^2};$$

$$\therefore\ \frac{(2-e^2)^2}{1-e^2} = \frac{\theta^2}{4KS^2},$$ where $e$ is the eccentricity.

If $e=0$ the conic is a circle, and in this case $16KS^2 = \theta^2$; this condition can be easily verified by the conditions previously found. When the conic is a circle we have $r_1 = r_2$, hence the radius of the circle represented by the general equation is given by $r^2 = 2S\Delta/K^{\frac{3}{2}}$.

**Cor. ii.** If the conic is an ellipse, its area is $2\pi S\Delta/K^{\frac{3}{2}}$.

**Cor. iii.** The conic is an ellipse or an hyperbola according as the values of $r_1^2$ and $r_2^2$ are both positive, or one positive and one negative; i. e. according as $K$ is positive or negative.

**Method 2.** The equation of the conic is

$$f(x, y, z) = ux^2+vy^2+wz^2+2fyz+2gzx+2hxy = 0;$$

let its centre (supposed finite) be $(x_0, y_0, z_0)$, and write $x = x_0+\xi$, $y = y_0+\eta$, $z = z_0+\zeta$.

Then we have

$$\begin{aligned} f(x, y, z) &= f(x_0+\xi,\ y_0+\eta,\ z_0+\zeta) \\ &= f(x_0, y_0, z_0)+2\{\xi X_0+\eta Y_0+\zeta Z_0\}+f(\xi, \eta, \zeta) \\ &= f(x_0, y_0, z_0)+f(\xi, \eta, \zeta), \end{aligned}$$

for since $(x_0, y_0, z_0)$ is the centre, $X_0 = Y_0 = Z_0$ and $\xi+\eta+\zeta = 0$.

Now suppose the coordinates changed to Cartesians referred to the principal axes of the conic.

If $f(x, y, z) = \lambda\{X^2/r_1^2+Y^2/r_2^2-1\}$ where $\lambda$ is an undetermined constant, then

$$f(x_0, y_0, z_0) = \lambda\{0/r_1^2+0/r_2^2-1\} = -\lambda.$$

But
$$\begin{aligned} f(\xi, \eta, \zeta) &= f(x, y, z)-f(x_0, y_0, z_0) \\ &= \lambda\{X^2/r_1^2+Y^2/r_2^2\}; \end{aligned}$$

$$\therefore\ d.f(\xi, \eta, \zeta) = -X^2/r_1^2-Y^2/r_2^2$$

where $d.f(x_0, y_0, z_0) = 1$.

Now the distance of any point $(x, y, z)$ from the centre of the conic is

$$\sqrt{\{-a^2(y-y_0)(z-z_0)-b^2(z-z_0)(x-x_0)-c^2(x-x_0)(y-y_0)\}} = \sqrt{\{-a^2\eta\zeta-b^2\zeta\xi-c^2\xi\eta\}}.$$

Thus, if $(\xi, \eta, \zeta)$ and $(X, Y)$ are the same point, we have

$$-a^2\eta\zeta-b^2\zeta\xi-c^2\xi\eta = X^2+Y^2,$$

and

$$\mu d.f(\xi, \eta, \zeta)-a^2\eta\zeta-b^2\zeta\xi-c^2\xi\eta = \mu\{-X^2/r_1^2-Y^2/r_2^2\}+X^2+Y^2.$$

The right-hand side is a perfect square when $\mu = r_1^2$ or $r_2^2$, hence also the left-hand side. Thus the squares of the semi-axes are the values of $\mu$ for which

$$\phi(\xi, \eta, \zeta) \equiv \mu d\{u\xi^2 + v\eta^2 + w\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta\} - (a^2\eta\zeta + b^2\zeta\xi + c^2\xi\eta)$$

is a perfect square.

Now if $\phi(\xi, \eta, \zeta)$, where $\xi + \eta + \zeta = 0$, is a perfect square, so is the expression in $\xi$, $\eta$ obtained by substituting $-(\xi+\eta)$ for $\zeta$ in $\phi(\xi, \eta, \zeta)$.

This is exactly the process we might use to find the condition that $x + y + z = 0$ should touch the conic $\phi(x, y, z) = 0$, so that the condition is the same in both cases. Hence $\phi(\xi, \eta, \zeta)$ is a perfect square if $\phi(x, y, z) = 0$ is a parabola.

This gives us

$$\Sigma\{\mu^2 d^2 vw - (\mu df - \tfrac{1}{2}a^2)^2\} + 2\Sigma\{(\mu dg - \tfrac{1}{2}b^2)(\mu dh - \tfrac{1}{2}c^2) - \mu du(\mu df - \tfrac{1}{2}a^2)\} = 0$$

i.e. $$\Sigma\{U\mu^2 d^2 + a^2 f\mu d - \tfrac{1}{4}a^4\} + 2\Sigma\{F\mu^2 d^2 + \tfrac{1}{2}\mu d(a^2 u - b^2 h - c^2 g) + \tfrac{1}{4}b^2 c^2\} = 0,$$

i.e. $$K\mu^2 d^2 + \theta\mu d + 4S^2 = 0,$$

where $$\theta = \Sigma a^2(u + f - g - h) = \Sigma a^2 u - 2\Sigma fbc\cos A.$$

The squares of the semi-axes are therefore given by

$$K\mu^2 + \theta\mu f(x_0, y_0, z_0) + 4S^2\{f(x_0, y_0, z_0)\}^2 = 0.$$

Since $f(x_0, y_0, z_0) = \Delta/K$, this result agrees with that of the first method.

VI. **Radius of Curvature.** Let $(x', y', z')$ be a point on the conic

$$\phi \equiv ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0,$$

then, if we write $\xi = x - x'$, $\eta = y - y'$, $\zeta = z - z'$, we have

$$\phi = u\xi^2 + v\eta^2 + w\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta + 2(\xi X' + \eta Y' + \zeta Z'),$$

since $\phi' = 0$.

The circle of curvature at $(x', y', z')$ touches the tangent

$$xX' + yY' + zZ' = 0 \quad \text{or} \quad \xi X' + \eta Y' + \zeta Z' = 0$$

at that point. But if $P$ is any point on a circle touching a straight line at $O$, and $PN$ is the perpendicular to this straight line, then $OP^2 = 2\rho PN$, where $\rho$ is the radius of the circle. [Cf. $x^2 + y^2 = 2rx$ in Cartesians.]

The equation of the circle of curvature is then

$$a^2(y-y')(z-z')+b^2(z-z')(x-x')+c^2(x-x')(y-y') + 4S\rho(xX'+yY'+zZ')/\{aX', bY', cZ'\} = 0,$$

or $a^2\eta\zeta+b^2\zeta\xi+c^2\xi\eta+4S\rho(\xi X'+\eta Y'+\zeta Z')/\{aX', bY', cZ'\} = 0.$

We may write this equation

$$C \equiv \lambda(a^2\eta\zeta+b^2\zeta\xi+c^2\xi\eta)+(\xi X'+\eta Y'+\zeta Z') = 0,$$

where $4S\rho\lambda = \{aX', bY', cZ'\}$.

Now

$$\phi-2C = u\xi^2+v\eta^2+w\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta-2\lambda(a^2\eta\zeta+b^2\zeta\xi+c^2\xi\eta),$$

and the right-hand side factorizes since $\xi+\eta+\zeta = 0$; thus $\phi-2C = 0$ represents a pair of straight lines, and these must be the tangent at the point of contact and the common chord; hence we have identically

$$u\xi^2+v\eta^2+w\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta-2\lambda(a^2\eta\zeta+b^2\zeta\xi+c^2\xi\eta) = (\xi X'+\eta Y'+\zeta Z')(l\xi+m\eta+n\zeta).$$

In this identity put $\xi = Y'-Z'$, $\eta = Z'-X'$, $\zeta = X'-Y'$; this is allowable since the values chosen satisfy $\xi+\eta+\zeta = 0$, hence

$$\Sigma u(Y'-Z')^2+2\Sigma(f-a^2\lambda)(Z'-X')(X'-Y') = 0,$$

which equation gives the value of $\lambda$.

Now we have

$$X = ux+hy+gz,$$
$$Y = hx+vy+fz,$$
$$Z = gx+fy+wz,$$

and, if $(x, y, z)$ is on the conic,

$$xX+yY+zZ = 0;$$

hence

$$\Sigma u(Y-Z)^2+2\Sigma f(Z-X)(X-Y)$$
$$= \Sigma(uY^2-2hXY+vX^2)+2\Sigma(hZX+gXY-fX^2-uYZ)$$
$$= \Sigma\{X(vX-hY)+Y(uY-hX)\}+2\Sigma\{X(gY-fX)-Z(uY-hX)\}$$
$$= \Sigma\{X(Wx-Gz)+Y(Wy-Fz)\}+2\Sigma\{X(Fx-Gy)-Z(Wy-Fz)\},$$

and, using the fourth equation above, this becomes

$$= -\Sigma z(GX+FY+WZ)-2\Sigma y(GX+FY+WZ)$$
$$= -\Delta\Sigma z^2-2\Delta\Sigma yz$$
$$= -\Delta(x+y+z)^2 = -\Delta.$$

Hence $2\lambda = -\Delta/\Sigma a^2(Z'-X')(X'-Y') = -\Delta/\{aX', bY', cZ'\}^2$;

$$\therefore \quad \rho = \frac{\{aX', bY', cZ'\}}{4S\lambda} = -\frac{\{aX', bY', cZ'\}^3}{2S\Delta}.$$

**VII. Invariants.** The relations connecting areal coordinates with rectangular Cartesian coordinates are

$$x = \frac{a}{2S}(p_1 - X\cos\theta_1 - Y\sin\theta_1),\quad y = \frac{b}{2S}(p_2 - X\cos\theta_2 - Y\sin\theta_2),$$

$$z = \frac{c}{2S}(p_3 - X\cos\theta_3 - Y\sin\theta_3),$$

where $\theta_3 - \theta_2 = \pi - A$, $\theta_2 - \theta_1 = \pi - C$, $\theta_3 - \theta_1 = \pi + B$.

Now suppose that $ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy$ transforms into $u'X^2 + v'Y^2 + w' + 2f'Y + 2g'X + 2h'XY$; then (Chapter XI, p. 433)

$$\Delta' = \Delta \times \frac{a^2b^2c^2}{64S^6}\begin{vmatrix} p_1 & \cos\theta_1 & \sin\theta_1 \\ p_2 & \cos\theta_2 & \sin\theta_2 \\ p_3 & \cos\theta_3 & \sin\theta_3 \end{vmatrix}^2$$

$$= \Delta \times \frac{a^2b^2c^2}{64S^6} \times (p_1\sin A + p_2\sin B + p_3\sin C)^2$$

$$= \Delta \times \frac{a^2b^2c^2}{256S^6R^2} \times (ap_1 + bp_2 + cp_3)^2$$

$$= \Delta \times \frac{a^2b^2c^2}{256S^6R^2} \times 4S^2,$$

i.e. $$\Delta' = \Delta/4S^2.$$

Again, the squares of the semi-axes of the conic

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0$$

are given by

$$K^3r^4 + K\Delta\theta r^2 + 4S^2\Delta^2 = 0, \qquad \text{(i)}$$

and the squares of the semi-axes of the conic

$$u'X^2 + v'Y^2 + w' + 2f'Y + 2g'X + 2h'XY = 0$$

are given by

$$W'^3r^4 + (u' + v')\Delta' W'r^2 + \Delta'^2 = 0, \qquad \text{(ii)}$$

so that equations (i) and (ii) must be identical, hence

$$\frac{W'^3}{K^3} = \frac{(u' + v')\Delta' W'}{K\Delta\theta} = \frac{\Delta'^2}{4S^2\Delta^2} = \frac{\Delta'^3}{\Delta^3},$$

thus $$\frac{W'^3}{K^3} = \frac{\Delta'^3}{\Delta^3},\quad \text{or}\quad \frac{W'}{K} = \frac{\Delta'}{\Delta},$$

i.e. $$\mathbf{u'v' - h'^2 = \frac{K}{4S^2}}.$$

Also $$\frac{u' + v'}{\theta} = \frac{\Delta'}{\Delta} = \frac{1}{4S^2},$$

i.e. $$\mathbf{u' + v' = \frac{\theta}{4S^2}}.$$

**Note.** These two invariants can also be proved by substituting for $x$, $y$, and $z$ and finding the values of $u'$, $v'$, and $f'$; the simplification is somewhat tedious.

## Illustrative Examples.

**Example i.** *To find the latus rectum of the conic*

$$ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0$$

*when it is a parabola.*

If we transform to rectangular Cartesian coordinates referred to the axis of the parabola and the tangent at the vertex, the equation transforms into $\lambda(Y^2-4mX)=0$.

Then $\Delta'=-4m^2\lambda^3$ and $u'+v'=\lambda$;

hence $16m^2\lambda^3S^2=-\Delta$, and $\lambda=\theta/4S^2$.

Thus $m^2=-4\Delta S^4/\theta^6$.

**Example ii.** *If the equation*

$$ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0$$

*represents a circle, to find its radius.*

We may suppose the equation transformed to $\lambda(x^2+y^2-\rho^2)=0$; the invariants $u'+v'$, $u'v'-f'^2$, and $\Delta'$ are $2\lambda$, $\lambda^2$ and $-\lambda^3\rho^2$, and these give the expression for the radius previously found.

The following method gives other more simple expressions. The equation

$$ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0$$

can be written

$$(2f-v-w)yz+(2g-w-u)zx+(2h-u-v)xy+(ux+vy+wz)(x+y+z)=0.$$

Now since $x+y+z=1$, when this equation is transformed into Cartesians, $(ux+vy+wz)(x+y+z)$ transforms into a linear expression.

Again, since $x+y+z=0$ is the line at infinity, if

$$ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0 \qquad \text{(i)}$$

is a circle, then

$$(2f-v-w)yz+(2g-w-u)zx+(2h-u-v)xy=0 \qquad \text{(ii)}$$

is also a circle, and it is clearly the circumcircle.

We have seen that when these equations are transformed into Cartesians the terms of the second degree are the same.

Hence if equation (i) transforms into $\lambda\{(x-\alpha)^2+(y-\beta)^2-\rho^2\}=0$, equation (ii) transforms into $\lambda\{(x-\alpha')^2+(y-\beta')^2-R^2\}=0$.

The discriminants of these equations are $-\lambda^3\rho^2$ and $-\lambda^3R^2$;

$\therefore\ -\lambda^3\rho^2=\Delta/4S^2$ and $-\lambda^3R^2=(2f-v-w)(2g-w-u)(2h-u-v)/16S^2$.

Hence $\rho^2=4\Delta R^2/(2f-v-w)(2g-w-u)(2h-u-v)$.

The reader can prove similarly that

$$\rho^2=4R^2\Delta\cos A\cos B\cos C/(g+h-u-f)(h+f-v-g)(f+g-w-h).$$

We conclude this chapter with some general illustrative examples.

**Example i.** *If two triangles are reciprocal they are also in homology.*

Let $ABC$ and $A'B'C'$ be the two triangles; take $ABC$ for the triangle of reference, and let the equation of the conic with respect to which they are reciprocal be

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Since $B'C'$ is the polar of $A$, its equation is $ux + hy + gz = 0$; the point of intersection of $BC$, $B'C'$ therefore lies on the line $x/f + y/g + z/h = 0$. The symmetry of this result shows that the points of intersection of $CA$, $C'A'$ and $AB$, $A'B'$ also lie on this line; hence the triangles are coaxal and $x/f + y/g + z/h = 0$ is the axis of homology.

Again, since $A'$ is the pole of $BC$ its equation is $Up + Hq + Gr = 0$, i. e. $A'$ is the point $(U, H, G)$.

Hence $AA'$ is the line $Gy = Hz$; similarly the equations of the lines $BB'$, $CC'$ are $Hz = Fx$, $Fx = Gy$.

Thus $AA'$, $BB'$, $CC'$ intersect at the point $\{1/F, 1/G, 1/H\}$; the triangles are then copolar, and this point is the centre of homology.

**Example ii.** *To find the equation of the conic, one of whose foci is at the point* $(x_1, y_1, z_1)$, *to which the triangle of reference is self-conjugate. Deduce the locus of the foci of conics which pass through four given points.*

The tangential equation of a conic whose foci are $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ is

$$a^2p^2 + b^2q^2 + c^2r^2 - 2bc\cos Aqr - 2ca\cos Brp - 2ab\cos Cpq + k(px_1 + qy_1 + rz_1)(px_2 + qy_2 + rz_2) = 0.$$

If the triangle of reference is self-conjugate with respect to the conic, the coefficients of $qr$, $rp$, $pq$ are zero, so that

$$k(y_1z_2 + y_2z_1) = 2bc\cos A, \quad k(z_1x_2 + z_2x_1) = 2ca\cos B,$$
$$k(x_1y_2 + x_2y_1) = 2ab\cos C.$$

Hence $$y_1ca\cos B + z_1ab\cos C - x_1bc\cos A = ky_1z_1x_2,$$

and $$z_1ab\cos C + x_1bc\cos A - y_1ca\cos B = kz_1x_1y_2,$$
$$x_1bc\cos A + y_1ca\cos B - z_1ab\cos C = kx_1y_1z_2;$$

hence, substituting these values of $kx_2$, $ky_2$, $kz_2$ in the equation, we have

$$\Sigma p^2x_1\{a^2y_1z_1 + (y_1ca\cos B + z_1ab\cos C - x_1bc\cos A)x_1\} = 0,$$

which is the required equation.

The point-equation of the conic is therefore

$$\Sigma x^2/x_1\{a^2y_1z_1 + (y_1ca\cos B + z_1ab\cos C - x_1bc\cos A)x_1\} = 0.$$

If this conic passes through the four points $(f, \pm g, \pm h)$, we have, dropping the suffixes in the coordinates of the focus,

$$\Sigma f^2/x\{a^2yz + (yca\cos B + zab\cos C - xbc\cos A)x\} = 0,$$

which is the equation of the locus of the foci of conics which pass through the points $(f, \pm g, \pm h)$.

**Example iii.** *Two concentric conics are drawn, one circumscribing the triangle of reference, and to the other the triangle bisecting the sides is self-conjugate: prove that the two conics are similar and similarly situated, and their areas are in the ratio*

$$-8xyz : (-x+y+z)(x-y+z)(x+y-z),$$

*where* $(x : y : z)$ *is their common centre.*

Let $ABC$ be the triangle and $P$, $Q$, $R$ the mid-points of the sides. Project the figure orthogonally so that the circumscribing conic becomes a circle. Let $A'B'C'$ and $P'Q'R'$ be the projections of the triangles $ABC$ and $PQR$. The mid-point of a line projects into the mid-point of the projection of the line, so that $P'$, $Q'$, $R'$ are the mid-points of $A'B'C'$.

Now the areal coordinates of a point are unaltered by orthogonal projection, and the centre of a conic projects orthogonally into the centre of the conic. Thus $(x : y : z)$ is the circumcentre of the triangle $A'B'C'$; it is therefore the orthocentre of the triangle $P'Q'R'$. Hence the second conic projects into a conic to which the triangle $P'Q'R'$ is self-conjugate and whose centre is the orthocentre of the triangle $P'Q'R'$, i.e. it projects into the polar circle of the triangle $P'Q'R'$.

Since the two conics can be projected orthogonally into two concentric circles they must be similar and similarly situated, being evidently similar sections of coaxal cylinders.

Again, the ratio of two areas is unaltered by orthogonal projection, so that the ratio of the areas of the conics

$$\begin{aligned} &= \text{ratio of the areas of the circles} \\ &= \pi R'^2 : -\pi R'^2 \cos A' \cos B' \cos C' \\ &= -1 : \cos A' \cos B' \cos C'. \end{aligned}$$

But $$\frac{x}{\sin 2A'} = \frac{y}{\sin 2B'} = \frac{z}{\sin 2C'} = \frac{-x+y+z}{4 \sin A' \cos B' \cos C'},$$

hence $$2x : -x+y+z = \cos A' : \cos B' \cos C'$$

and two symmetrical results, therefore

$$8xyz : (-x+y+z)(x-y+z)(x+y-z) = 1 : \cos A' \cos B' \cos C'.$$

**Example iv.** *Find the area of the conic* $\sqrt{lx} + \sqrt{my} + \sqrt{nz} = 0$.

We can apply the method used in the last example to this problem. Let $(x, y, z)$ be the centre of the conic, then $2l = y+z-x$, $2m = z+x-y$, $2n = x+y-z$.

Project the figure orthogonally so that the conic becomes a circle and let $A'B'C'$ be the new triangle of reference. Then we have

$$\begin{aligned} &\text{Area of conic : area of circle inscribed in } A'B'C' \\ &= \text{area of } ABC : \text{area of } A'B'C'. \end{aligned}$$

Hence the area of the conic $= \pi r'^2 \cdot S \div S'$

$= \pi S s'^2 \div s'^2$, where $s'$ is the semiperimeter of $A'B'C'$

$= \pi S \sqrt{s'(s'-a')(s'-b')(s'-c')} \div s'^2$.

Now the areal coordinates of the centre are unaltered by orthogonal projection, thus

$$\frac{x}{a'} = \frac{y}{b'} = \frac{z}{c'} = \frac{x+y+z}{a'+b'+c'} = \frac{y+z-x}{b'+c'-a'} = \frac{z+x-y}{c'+a'-b'} = \frac{x+y-z}{a'+b'-c'} .$$

Hence $$\frac{l+m+n}{s'} = \frac{l}{s'-a'} = \frac{m}{s'-b'} = \frac{n}{s'-c'} ;$$

the area of the conic is therefore $\pi S \sqrt{\dfrac{lmn}{(l+m+n)^3}}$.

## Examples XII h.

1. A system of parabolas is drawn to touch the sides of a triangle. Prove that the locus of the point, in which the lines joining the vertices to the points of contact of the opposite sides meet, is a conic passing through the vertices and having its centre at the centroid of the triangle.

2. If $LMN$ is the pedal triangle of $ABC$, it is self-conjugate with respect to any rectangular hyperbola through $ABC$.

3. The locus of points from which tangents to $\alpha^2/l + \beta^2/m + \gamma^2/n = 0$ are perpendicular is $\Sigma(m+n)\alpha^2 + \Sigma 2l\beta\gamma \cos A = 0$, which is a circle whose radical axis with the circumcircle is $\Sigma(m+n)bc\alpha = 0$.

4. Find the tangential equations of the circumcentre, in-centre and ex-centres, orthocentre, centroid and symmedian point of the triangle of reference.

5. Investigate the character of the conics

(i) $p^2 + 3q^2 = 4r^2$;

(ii) $a^3p^2 \sin(B-C) + b^3q^2 \sin \overline{C-A} + c^3r^2 \sin \overline{A-B} = 0$;

(iii) $a^2p^2 = -4bcqr \cos A$,

and find their areal equations.

6. Show that it is possible for a conic to be described round a triangle $ABC$ such that the tangent at each angle is parallel to the opposite side.

7. Find the coordinates of the point isogonal with the orthocentre of $ABC$.

8. Find the equation of the rectangular hyperbola through the four points $(f, \pm g, \pm h)$.

9. The mid-points of the diagonals of the quadrilateral formed by the four lines $lx \pm my \pm nz = 0$ lie on the line $l^2x + m^2y + n^2z = 0$.

10. Show that if the lines

$$l_1x + m_1y + n_1z = 0,\ l_2x + m_2y + n_2z = 0,\ l_3x + m_3y + n_3z = 0$$

form a triangle self-polar with regard to any conic with regard to which the triangle of reference is self-polar, then

$$\begin{vmatrix} \frac{1}{l_1} & \frac{1}{m_1} & \frac{1}{n_1} \\ \frac{1}{l_2} & \frac{1}{m_2} & \frac{1}{n_2} \\ \frac{1}{l_3} & \frac{1}{m_3} & \frac{1}{n_3} \end{vmatrix} = 0.$$

11. Find the condition that the in-centre of the triangle of reference should be the focus of the parabola $lx^2+my^2+nz^2=0$.

12. A conic is inscribed in the triangle of reference and has one focus at the orthocentre. Prove that its centre is the point

$$\{\cos \overline{B-C} : \cos \overline{C-A} : \cos \overline{A-B}\},$$

and that the sum of the squares of its axes is $R^2(1+8\cos A\cos B\cos C)$.

13. Find the equation of the parabola which passes through $A$, $B$, two of the angular points of the triangle of reference $ABC$, and also through the middle points of $AC$ and $BC$.

14. If the line $l\alpha+m\beta+n\gamma=0$ meets the sides $BC$, $CA$, $AB$ of the triangle of reference in $D$, $E$, $F$ respectively, prove that a conic may be drawn to touch $DA$ at $A$, $EB$ at $B$, $FC$ at $C$, and find its equation.

15. The radical axis of two circles given by the general equations

$$\Sigma(ax^2+2fyz)=0 \text{ and } \Sigma(a'x^2+2f'yz)=0$$

in areal coordinates is

$$\frac{ax+by+cz}{a+b+c-f-g-h}=\frac{a'x+b'y+c'z}{a'+b'+c'-f'-g'-h'}.$$

16. A rectangular hyperbola passes through the corners of a triangle and through its in-centre. Prove that the tangents to it at the in-centre and at the orthocentre meet in the point whose distances from the sides of the triangle are $r(\cos B+\cos C)$, $r(\cos C+\cos A)$, $r(\cos A+\cos B)$, $r$ being the radius of the inscribed circle.

17. If the connector of $(\alpha_1, \beta_1, \gamma_1)$ $(\alpha_2, \beta_2, \gamma_2)$ is divided harmonically by $f\beta\gamma+g\gamma\alpha+h\alpha\beta=0$, then

$$f(\beta_1\gamma_2+\beta_2\gamma_1)+g(\gamma_1\alpha_2+\gamma_2\alpha_1)+h(\alpha_1\beta_2+\alpha_2\beta_1)=0.$$

All conics circumscribed to a given triangle which divide a given segment harmonically pass through a fixed point.

18. Show that the equation of a pair of conjugate diameters of the conic $yz+zx+xy=0$, may be written in the form

$$(q-r)(y-z)^2+(r-p)(z-x)^2+(p-q)(x-y)^2=0.$$

19. Prove that the coordinates of the foci of the ellipse

$$\sqrt{\frac{\alpha}{a}}+\sqrt{\frac{\beta}{b}}+\sqrt{\frac{\gamma}{c}}=0$$

are respectively proportional to $\frac{c}{b}$, $\frac{a}{c}$, $\frac{b}{a}$, and $\frac{b}{c}$, $\frac{c}{a}$, $\frac{a}{b}$.

20. The tangents drawn from the angular points of the triangle of reference to the conic $ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0$ meet the opposite sides in six points which lie on the conic

$$VWx^2+WUy^2+UVz^2-2UFyz-2VGzx-2WHxy=0.$$

21. Show that $2(x^2+y^2)-z^2=0$ is a parabola. Find its focus, directrix, and the equation of its axis. What is its latus rectum?

22. Find the focus of $\sum\frac{a^2\alpha^2}{q-r}=0$ and deduce the equation of the directrix.

23. The equation of that diameter of the circle $\Sigma\alpha^2 \sin 2A = 0$ which is perpendicular to the line $l\alpha + m\beta + n\gamma = 0$ is $\Sigma a\alpha \cos A(cm - bn) = 0$.

24. Show that if $A$ is a focus of the conic $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ then $m = n$ and $b^2 + c^2 = a^2$.

25. The squares of the semi-axes of the conic $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0$ are given by the quadratic $54r^2 - 3r(a^2 + b^2 + c^2) + 2S^2 = 0$.

26. A parabola is inscribed in a triangle, and the trilinear coordinates of its focus are $\alpha', \beta', \gamma'$. Prove that its axis is the line

$$\alpha\alpha'(\beta'^2 - \gamma'^2) + \beta\beta'(\gamma'^2 - \alpha'^2) + \gamma\gamma'(\alpha'^2 - \beta'^2) = 0.$$

27. Prove that $a\sqrt{\alpha \sin(B-C)} + b\sqrt{\beta \sin(C-A)} + c\sqrt{\gamma \sin(A-B)} = 0$ is a parabola touching the sides of the triangle of reference, whose directrix is the straight line joining the centre of gravity and the orthocentre of the triangle.

28. Show that the director circle of the conic $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$ can be thrown into the form

$$(\alpha \sin A + \beta \sin B + \gamma \sin C)(l\alpha \cot A + m\beta \cot B + n\gamma \cot C)$$

$$= \left(\frac{l}{\sin A} + \frac{m}{\sin B} + \frac{n}{\sin C}\right)(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C).$$

A circle circumscribes a triangle circumscribed to a conic $S$, and has its centre on $S$; prove that it touches the director circle of $S$.

29. If $p, q, r$ are the perpendiculars on a line from the vertices of the triangle $ABC$, show that the conics represented by the equations

$$\tan\frac{A}{2}qr + \tan\frac{B}{2}rp + \tan\frac{C}{2}pq = 0, \quad \sqrt{ap} + \sqrt{bq} + \sqrt{cr} = 0$$

are confocal.

30. Prove that the foci of the conic

$$a(\alpha \cos A)^{\frac{1}{2}} + b(\beta \cos B)^{\frac{1}{2}} + c(\gamma \cos C)^{\frac{1}{2}} = 0$$

are the circumcentre and orthocentre of the triangle of reference, and that the square of its eccentricity is $1 - 8\cos A \cos B \cos C$.

31. Prove that the joins of the mid-points of the sides of the triangle of reference to the mid-points of the corresponding perpendiculars of the triangle are concurrent at the point $(a^2 : b^2 : c^2)$, which is isogonal with the centroid.

Show that the vertices of a triangle, its circumcentre, orthocentre, and symmedian point all lie on a conic whose equation is

$$\Sigma a^2 yz(\cos 2B - \cos 2C) = 0.$$

32. When a conic degenerates into a pair of points its director circle is the circle on the line joining the points as diameter. Hence find the equation of the circle for which the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ are the extremities of a diameter.

33. Show that the tangential equation of a conic inscribed in the triangle of reference and having a focus at $(x, y, z)$ is $\Sigma qrxr_1^2 = 0$, where $r_1, r_2, r_3$ are the distances of the focus from the vertices of the triangle of reference.

34. Find the locus of the foci of a conic inscribed in the triangle of reference and touching the line $(p, q, r)$.

35. The orthocentre of a triangle circumscribed to a parabola lies on the directrix.

36. The triangle of reference circumscribes a parabola whose axis is the line $(p, q, r)$; prove that $\Sigma pa^2/(q-r) = 0$. (The point at infinity on the axis is the infinite focus.)

37. A parabola has a fixed self-conjugate triangle; show that the locus of its focus is the nine-point circle of the triangle and that its directrix passes through the circumcentre of the triangle.

38. The director circles of a system of conics which touch four fixed straight lines form a coaxal system.

39. Prove that the equation of the asymptotes of a conic may be found by substituting $(yz_0-y_0z)$, $(zx_0-z_0x)$, $(xy_0-x_0y)$ for $p, q, r$ in its tangential equation, $(x_0, y_0, z_0)$ being the centre of the conic.

40. Find the coordinates of the focus of the conic $yz+zx+4xy = 0$.

41. Show that the equation of the axes of the conic $\sqrt{x}+\sqrt{y}+\sqrt{z} = 0$ is $\Sigma(b^2-c^2)(y-z)^2 = 0$.

42. A straight line passes through a fixed point. Prove that the line joining its poles with respect to two given conics always touches a fixed conic inscribed in the common self-conjugate triangle of the two conics.

43. Prove that through three points $A, B, C$ two parabolas can be drawn so as to touch the circle $ABC$ at $A$ and that the axes of these parabolas are at right angles.

44. A rectangular hyperbola circumscribes a triangle $ABC$. Show that the loci of the poles of the three sides of the triangle with respect to the hyperbola are straight lines.

45. If the internal bisectors of the angles $A, B, C$ of a triangle meet the circumscribing circle in $A'B'C'$, the trilinear equation of the conic inscribed in the two triangles $ABC$, $A'B'C'$ is

$$\sqrt{a(b+c)\alpha}+\sqrt{b(c+a)\beta}+\sqrt{c(a+b)\gamma} = 0.$$

46. The pencil subtended at any point of the conic

$$2\alpha_0\beta\gamma-\beta_0\gamma\alpha-\gamma_0\alpha\beta = 0$$

by the quadrangle consisting of the vertices of the triangle of reference and $(\alpha_0, \beta_0, \gamma_0)$ is harmonic.

47. Four conics circumscribe $ABC$ and have the in-centre for a common focus. Show that the centres of all conics which touch the four corresponding directrices lie on $\Sigma\alpha\cot\frac{A}{2} = 0$.

48. If $\delta = l\alpha+m\beta+n\gamma$, then $\frac{kl}{mn}$ is one of the anharmonic ratios of the pencil formed by joining any point on $\alpha\delta+k\beta\gamma = 0$ to the points $(\alpha, \beta)$, $(\alpha, \gamma)$, $(\delta, \beta)$, $(\delta, \gamma)$, where $(\alpha, \beta)$ means the intersection of $\alpha = 0$, $\beta = 0$.

49. If the triangle of reference is equilateral, prove that the line $\beta+\gamma+3\alpha = 0$ is a directrix of the conic $\beta^2+\gamma^2-3\alpha^2 = 0$.

50. If $a^2 = 2bc$, prove that two of the sides of the triangle of reference are parallel to a pair of conjugate diameters of the conic whose trilinear equation is $\alpha^2 = \beta\gamma$.

51. With a given straight line as asymptote three conics are described such that a given triangle is inscribed in 1, self-conjugate with respect to 2, and circumscribed to 3. The centre of 3 bisects the distance between the centres of 1 and 2.

52. If $l, l_1$ are asymptotes of a conic circumscribing $ABC$, $l, l_2$ are asymptotes of a conic with respect to which $ABC$ is self-conjugate, $l_1, l_3$ are asymptotes of a conic touching the sides of $ABC$, then $l_2, l_3$ are asymptotes of a conic circumscribing $ABC$.

53. Show that the equation of the locus of the foci of rectangular hyperbolas for which the triangle of reference is self-conjugate is

$$\Sigma\{\alpha^2(-\alpha\cos A+\beta\cos B+\gamma\cos C)+\alpha\beta\gamma\}^{-1}=0.$$

54. Show that if the two conics $\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0$,

$$ux(y+z-x)+vy(x+z-y)+wz(x+y-z)=0$$

are similar and similarly situated, they also touch one another.

55. A conic cuts the sides of the triangle $ABC$ at $D, D'$; $E, E'$; $F, F'$ respectively, and $AD, AD'$ intersect the conic in $dd'$, $BE, BE'$ in $ee'$, $CF, CF'$ in $ff'$. Show that the intersection of $dd'$ with the polar of $A$, $ee'$ with the polar of $B$, $ff'$ with the polar of $C$ are collinear.

56. $A'B'C'$ are mid-points of the sides, $A''B''C''$ is a triangle formed by tangents at $ABC$ to the circumcircle. Show that the six points of intersection of corresponding sides of $A'B'C'$ and $A''B''C''$ lie on a conic.

57. A triangle is inscribed in a parabola with its orthocentre at the focus. Prove that its circumscribing circle touches the tangent at the vertex.

58. The sides $BC, CA, AB$ of a triangle are cut by the internal bisectors of the angles in $D, E, F$ and by the external bisectors in $A', B', C'$. Show that $A'B'C'$, $A'EF$, $B'FD$, $C'DE$ are all straight lines, and prove that the locus of the centres of conics circumscribing the quadrilateral $A'B'ED$ is $c(\alpha^2-\beta^2)+\gamma(a\alpha-b\beta)=0$, $\alpha, \beta, \gamma$ being trilinear coordinates referred to the triangle $ABC$.

59. Tangents at the centres of the inscribed and escribed circles of a triangle to the rectangular hyperbola passing through these centres meet in pairs on the sides of the triangle.

60. Through any point $P$ straight lines $PA, PB, PC$ are drawn to the vertices of a fixed triangle $ABC$ cutting the sides $BC, CA, AB$ in $A', B', C'$. Show that, if the perpendiculars through $A', B', C'$ to $BC, CA, AB$ meet in a point, the locus of $P$ is the curve

$$\alpha^2\sin^2 A(\beta\cos B-\gamma\cos C)+\beta^2\sin B(\gamma\cos C-\alpha\cos A)+\gamma^2\sin^2 C(\alpha\cos A-\beta\cos B)=0,$$

$ABC$ being the triangle of reference and the coordinates being trilinear.

61. The normals at the points $A, B$ of a parabola intersect at the point $C$ on the curve. Show that $\tan^2 C = 4\cot A\cot B$.

62. A triangle $ABC$ is cut by a transversal in $DEF$. On $BC$ a point $D_1$ is taken such that $D$ and $D_1$ are harmonic conjugates with respect to $B$ and $C$. The other tangents from $D$ and $D_1$ to an inscribed conic meet in $P$ and $AP$ meets $BC$ in $P'$. If points $Q'$ and $R'$ are similarly obtained, show that $P'$, $Q'$, and $R'$ are collinear.

63. A conic passes through the angular points of the triangle of reference and their centre of mean position. One of its axes is parallel to $x = 0$; show that its equation is $a/x = c\cos B/y + b\cos C/z$.

64. Show by orthogonal projection that the area of the conic

$$ux^2 + vy^2 + wz^2 = 0 \text{ is } 2\pi S uvw/(uv+vw+wu)^{\frac{3}{2}}.$$

65. The six lines $my + nz = 0$, $nz + lx = 0$, $lx + my = 0$, $m'y + n'z = 0$, $n'z + l'x = 0$, $l'x + m'y = 0$ touch a conic.

66. Prove that $z^2 - 4xy = 0$ represents a parabola in areal coordinates and that the equation of the axis is

$$x(a^2 + 3b^2 - c^2) - y(3a^2 + b^2 - c^2) - z(a^2 - b^2) = 0.$$

67. If the equation $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ represents a parabola, show that the equation of its directrix is

$$(m^2 + n^2 - 2mn\cos A)\, bc\alpha + (n^2 + l^2 - 2nl\cos B)\, ca\beta + (l^2 + m^2 - 2lm\cos C)\, ab\gamma = 0.$$

68. If the conic $ux^2 + vy^2 + wz^2 = 0$ touches at a finite point the conic similar and similarly situated to it which passes through the angular points of the triangle of reference, show that $u + v + w = 0$, and that the conics are hyperbolas.

69. Through the angular point $A$ of the triangle of reference a straight line $AD$ is drawn, cutting the conic

$$S \equiv ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0$$

in the points $P$ and $P'$, and also cutting the line $L \equiv lx + my + nz = 0$ in the point $Q$. If a point $Q'$ is taken on $AD$ such that the range $(PP', QQ')$ is harmonic, show that, as the line $AD$ moves, the locus of $Q'$ is the conic $lS = L(ax + hy + gz)$.

70. Show that $x, y, z$ being proportional to the areal coordinates of a point, the equation of the circle osculating the conic $x^2 + \lambda yz = 0$ at the vertex $B$ of the triangle of reference $ABC$ is

$$c^2x^2 + a^2z^2 + (c^2 + a^2 - b^2)\, zx + \lambda c^2 z(x + y + z) = 0.$$

Show that if $\lambda$ varies the radical axis of the osculating circles at $B$, $C$ passes through a fixed point.

71. Prove that the equation of the conic inscribed in the triangle of reference, and having one of its foci at the circumcentre, is

$$\sin A\sqrt{\alpha\cos A} + \sin B\sqrt{\beta\cos B} + \sin C\sqrt{\gamma\cos C} = 0.$$

72. Find the condition that $\lambda\alpha + \mu\beta + \nu\gamma = 0$ should be cut harmonically by the conics $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$, $u'\alpha^2 + v'\beta^2 + w'\gamma^2 = 0$.

If the triangle of reference is equilateral, prove that the conditions that the envelope of the line should be a circle are

$$vw' + v'w = wu' + w'u = uv' + u'v.$$

73. A parabola has double contact with $\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$ and touches the fourth common tangent of $\sqrt{l_1\alpha}+\sqrt{m_1\beta}+\sqrt{n_1\gamma}=0$ and $\sqrt{l_2\alpha}+\sqrt{m_2\beta}+\sqrt{n_2\gamma}=0$.

Show that the locus of the pole of its chord of contact with the first conic is

$$LMN(lL+mM+nN)(a\alpha+b\beta+c\gamma)^2 = (lbc+mca+nab)(\alpha MN+\beta NL+\gamma LM)^2$$

where $L\equiv m_1n_2-m_2n_1$, and similarly for $M$ and $N$.

74. An ellipse circumscribes a triangle $ABC$ and has its centre at the centre of gravity of the triangle. Prove that the radii of curvature at $A$, $B$, $C$ are proportional to the cubes of the sides $BC$, $CA$, $AB$ and that the product of the three radii of curvature is equal to the cube of the radius of the circle circumscribing the triangle $ABC$.

75. The conic $ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0$ meets the sides of the triangle of reference in three pairs of points such that the lines joining them to the opposite vertices intersect by threes in two points: prove that $uvw-2fgh-uf^2-vg^2-wh^2=0$.

76. The locus of the centres of rectangular hyperbolas with respect to which a given triangle is self-conjugate is the circumscribing circle of the triangle.

77. $PQRS$ is a quadrangle and $A$, $B$, $C$ its diagonal points. $X$ is any other point. Show that the six conics $(XBCPS)$, $(XBCQR)$, $(XCAQS)$, $(XCARP)$, $(XABRS)$, $(XABPQ)$ have a second common point. ($PS$, $QS$, $RS$ meet $QR$, $RP$, $PQ$ in $A$, $B$, $C$ respectively.)

78. The area of the circle $\Gamma=0$, where

$$C \equiv a^2yz+b^2zx+c^2xy+(lx+my+nz)(x+y+z),$$

is $4\pi R^2\Delta\div a^2b^2c^2$, where $R$ is the radius of the circumscribed circle and $\Delta$ is the discriminant of $C$.

79. Obtain the expression

$$\frac{pc}{2a\sin A}\cdot\frac{(m^2+n^2-2mn\cos A)^{\frac{3}{2}}}{lmn}$$

for the radius of curvature of the conic $\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0$ at the vertex $A$ of the triangle of reference.

80. The triangle of reference being equilateral, prove that the envelope of the director circles of the conic whose equation is $kx^{-1}=y^{-1}+z^{-1}$, for different values of $k$, is the curve

$$(y^2+z^2-2yz-3xy-3xz)^2=4(y^2+z^2+yz)(x+y+z)^2.$$

81. Show that the radius of curvature of the conic

$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$$

at the point where it touches the side $\alpha=0$ of the triangle of reference is $16lmnSR^2\div(mc+nb)^2$.

82. Interpret the equation $x^2+y^2=z^2$ in triangular coordinates, and find the equation of the chord joining two given points on this conic in its simplest form.

The chord $PQ$ of the conic $x^2+y^2=z^2$ touches the conic $a^{-1}x^2+b^{-1}y^2=z^2$. Show that $P$ and $Q$ are conjugate with respect to the conic

$$(a-b-1)x^2+(b-a-1)y^2+(a+b-1)z^2=0.$$

83. A system of conics is drawn touching the sides of the triangle of reference and a straight line through the centroid. Show that the envelope of the centre locus, for different positions of this straight line, is the conic whose equation in areal coordinates is

$$\sqrt{y+z-x}+\sqrt{z+x-y}+\sqrt{x+y-z}=0.$$

84. Find the equation in trilinear coordinates of the hyperbola which touches $BC$ and has $AB$, $AC$ as asymptotes.

85. $S$, $S'$ are the real foci of an ellipse inscribed in a triangle and through $S$, $S'$ is drawn another inscribed conic. Show that the pole of $SS'$ with respect to it is the centre of either the inscribed or one of the escribed circles.

86. Show that the equation of a conic circumscribing the triangle of reference is $\dfrac{a}{p\alpha}+\dfrac{b}{q\beta}+\dfrac{c}{r\gamma}=0$, where $a$, $b$, $c$ are the sides of the triangle and $p$, $q$, $r$ are the focal chords parallel to these sides.

87. A pair of tangents containing a constant angle are drawn to the conic $u\alpha^2+v\beta^2+w\gamma^2=0$; show that the locus of their intersection is given by

$$(\alpha\sin A+\beta\sin B+\gamma\sin C)^2 S=k\{(u+v+w)S-u^2\alpha^2-v^2\beta^2-w^2\gamma^2 \\ -2vw\beta\gamma\cos A-2wu\gamma\alpha\cos B-2uv\alpha\beta\cos C\}^2,$$

where $k$ is a constant and $S\equiv u\alpha^2+v\beta^2+w\gamma^2$.

88. Prove that any tangent to the conic

$$(b+1)^{-1}\alpha^2+(a+1)^{-1}\beta^2=(a+b)^{-1}\gamma^2$$

is cut in a harmonic range by the conics $\alpha^2+\beta^2=\gamma^2$, $a\alpha^2+b\beta^2=\gamma^2$.

89. Find the tangential equation of a conic to which the triangle of reference is self-conjugate, and which has a focus at $(x_1, y_1, z_1)$.

Show that if one focus is on $BC$, the other is on the line joining the feet of the perpendiculars from $B$, $C$ to the opposite sides of the triangle of reference.

90. A parabola circumscribes a triangle and has its focus at the orthocentre; prove that $\Sigma\cos\dfrac{A}{2}\Big/\sqrt{\cos A}=0$.

91. The major axis of an inscribed conic passes through the point in which the external bisector of the angle $A$ meets $BC$. Find the locus of its focus in trilinear coordinates.

92. The equation $ux^2+vy^2+wz^2=0$ represents a parabola whose axis passes through the point $(1, 1, 1)$. Show that its vertex is the point

$$\left(5-\frac{v+w}{u},\ 5-\frac{w+u}{v},\ 5-\frac{u+v}{w}\right).$$

93. Find the equation of the centre of the conic

$$up^2+vq^2+wr^2+2fqr+2grp+2hpq=0,$$

and the conditions that it may represent a circle.

94. A conic touches four fixed straight lines; show that the locus of its foci is a cubic.

95. The triangle formed by the polars of the middle points of the sides of a triangle with respect to any inscribed conic is of constant area.

96. The four foci of the conic $Lp^2+Mq^2+Nr^2=0$, where $p, q, r$ are the tangential coordinates of the line $p\alpha+q\beta+r\gamma=0$, have as their equation

$$\Sigma^2\{L\sin^2A+M\sin^2B+N\sin^2C\}-\Sigma\Sigma'\{MN+NL+LM\}+\Sigma'^2LMN=0,$$

where

$$\Sigma\equiv Lp^2+Mq^2+Nr^2, \text{ and } \Sigma'\equiv p^2+q^2+r^2-2qr\cos A-rp\cos B-2pq\cos C.$$

97. Prove that the equation to the circle of curvature at the point $A$ of the conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$ circumscribed to the triangle $ABC$ is

$$(\beta^2+\gamma^2+2\beta\gamma\cos A)\,almn=(m^2+n^2-2mn\cos A)\,(m\gamma+n\beta)\,(a\alpha+b\beta+c\gamma).$$

98. An ellipse is described having the triangle of reference for a self-conjugate triangle and with its centre at the point whose areal coordinates are $(x, y, z)$. Prove that the area of the ellipse $=2\pi S\sqrt{xyz}$.

99. $ABC$ being the triangle of reference, three conics are described with $B$ and $C$, $C$ and $A$, and $A$ and $B$ respectively as foci and with semi-minor axes $b_1, b_2, b_3$. Show that the three will have a common tangent if

$$a^2b_1^{-2}b_2^2b_3^2+b^2b_1^2b_2^{-2}b_3^2+c^2b_1^2b_2^2b_3^{-2}-2bcb_1^2\cos A$$
$$-2cab_2^2\cos B-2abb_3^2\cos C=4S^2,$$

and that the equation of the tangent will then be

$$b_2^2b_3^2x+b_3^2b_1^2y+b_1^2b_2^2z=0.$$

100. Prove that the conic, which touches the sides of the two triangles formed by the pairs of tangents to any given conic from two vertices of the triangle of reference and the corresponding chords of contact, passes through the third vertex if $uvw-2fgh-uf^2-vg^2-wh^2=0$.

101. Prove that the polar reciprocal of each of the conics

$$lx^2+my^2+nz^2=0, \quad Lx^2+My^2+Nz^2=0$$

with respect to the other, intersect in the four points

$$\sqrt{Ll(M^3n^3-N^3m^3)}:\pm\sqrt{Mm(N^3l^3-L^3n^3)}:\pm\sqrt{Nn(L^3m^3-M^3l^3)}.$$

102. Show that the points $(-x_1, y_1, z_1)$, $(x_1, -y_1, z_1)$, $(x_1, y_1, -z_1)$ are the angular points of a triangle inscribed in the conic $ux^2+vy^2+wz^2=0$, and self-polar for the conic $\sqrt{\lambda x}+\sqrt{\mu y}+\sqrt{\nu z}=0$ provided that

$$\lambda x_1+\mu y_1+\nu z_1=0, \text{ and } ux_1^2+vy_1^2+wz_1^2=0.$$

Hence show that if one triangle is circumscribed to a conic $S$ and is self-polar for a conic $S'$, an infinite number of such triangles can be drawn.

103. The tangents drawn at the vertices of a triangle to a circumscribing conic are parallel to the opposite sides; show that the osculating circles at the vertices intersect in a point lying on the circumcircle.

**104.** The two points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ lie within the triangle of reference. Find the coordinates of the poles of the line joining them with respect to the four conics which pass through them and are inscribed in the triangle: show also that an infinite number of conics, with respect to which the triangle is self-conjugate, pass through all four poles.

**105.** Prove that all straight lines through a given point intersect the polars with respect to a conic $S_2$ of their poles with respect to a conic $S_1$ in points which lie on a fixed conic, and this conic circumscribes the common self-conjugate triangle of the conics $S_1$, $S_2$.

# ANSWERS

## I a.

1. $3x+4y=12$. 2. $(6, 1)$, $(-2, -3)$, $(2, -1)$.
3. $(\frac{1}{2}a, \frac{1}{2}a\sqrt{3})$, $(-\frac{1}{2}a, \frac{1}{2}a\sqrt{3})$, $(a/\sqrt{2}, -a/\sqrt{2})$, $(-\frac{1}{2}a, \frac{1}{2}a\sqrt{3})$.
4. $(5, \tan^{-1}\frac{4}{3})$, $(5\sqrt{2}, \frac{1}{4}\pi)$, $(4, \frac{1}{6}\pi)$, $(4, \frac{5}{6}\pi)$, $(4, \frac{7}{6}\pi)$, $(4, \frac{11}{6}\pi)$.
5. $(2, 2)$, $(-2, 2)$, $(-2, -2)$, $(2, -2)$.
6. $(3, 2)$, $(-3, 2)$, $(-3, -2)$, $(3, -2)$.
7. $(2, 0)$, $(0, 1)$, $(-2, 0)$, $(0, -1)$.

## I b.

1. $(1, 1)$. 2. Each side is $a$. 3. $a$. 4. $3x^2+3y^2=156$.
5. $x^2+y^2-8x-6y=0$. 6. $-\frac{9}{16}$; $(-9\frac{1}{2}\frac{3}{5}, 2\frac{1}{2}\frac{1}{5})$.
7. $(1, 2)$. Infinitely distant. 9. $4:3$; $(-23, 19)$.
10. $2\sqrt{2}$, $3\sqrt{2}$, $\sqrt{26}$, $90°$. 11. $(1, 5)$. 12. $\sqrt{\{\frac{1}{2}(a^2+b^2)\}}$.
13. $\sqrt{58}$, $\sqrt{10}$. 15. $a_1b_2=a_2b_1$.
16. $(a, 0)$, $(a\sqrt{3}, 30°)$, $(2a, 60°)$, $(a\sqrt{3}, 90°)$, $(a, 120°)$.
17. Take $AB$, $AC$ for axes of reference.

## I c.

1. $(1\frac{2}{3}, \frac{4}{3})$, $(2\frac{1}{3}, -\frac{2}{3})$, $(0, 4)$. 2. $(-1, -3)$, $(0, -2)$, $(1, -1)$, $(2, 0)$.
3. $(5\frac{1}{5}, 9\frac{3}{5})$.
4. $(2a, 0)$, $(a, a\sqrt{3})$, $(-a, a\sqrt{3})$, $(-2a, 0)$, $(-a, -a\sqrt{3})$, $(a, -a\sqrt{3})$.
5. $(0, 0)$, $(c, 0)$, $(c, -c)$, $(0, -c)$; $(0, 0)$, $(0, b)$, $(-b, b)$, $(-b, 0)$; $(c, 0)$, $(0, b)$, $(b, b+c)$, $(b+c, c)$; $(\frac{1}{2}c, -\frac{1}{2}c)$, $(-\frac{1}{2}b, \frac{1}{2}b)$, $(\frac{1}{2}\overline{b+c}, \frac{1}{2}\overline{b+c})$.
6. $(a, 0°)$, $(a, 60°)$, $(a, 120°)$, $(a, 180°)$, &c. 7. $23\frac{1}{2}$. 8. 150.
9. $(aa'+bb')\div\sqrt{(a^2+b^2)(a'^2+b'^2)}$. 10. $(-1, 7)$, $2\sqrt{5}$. 13. $(\frac{3}{2}a, 2a)$.
14. $(\frac{7}{2}a, \frac{1}{2}a)$. 15. 3, 15. 16. $(x-a)^2+(y-b)^2=r^2$. 17. 5.
19. $x^2+y^2-8x-6y=0$. 20. $a\sec^2\theta$; $a\sec^2\theta$; $y^2=4ax$. 21. $a:b$.
22. $x=0$. 23. $30°$. 24. $x-y=c\operatorname{cosec}\omega$.
25. $\left\{\frac{2r_1r_2\cos\frac{1}{2}(\alpha_1-\alpha_2)}{r_1+r_2}, \frac{1}{2}(\alpha_1+\alpha_2)\right\}$. 26. $\{(2ab)/(a+b), 0\}$. 28. $3:1$.
29. $\frac{1}{2}a^2$; $\frac{1}{4}(15\sqrt{3}-7)a^2$. 33. $xy=c^2$. 35. $a+b=c$. 36. st. line.
38. $\{\frac{1}{3}(x_1+x_2+x_3), \frac{1}{3}(y_1+y_2+y_3)\}$. 41. $x+y=l$.
42. $\frac{1}{2}(x_1\sim x_2)(y_1+y_2)\sin\omega$. 44. Equality. 45. $x-4y+13=0$.
47. $(84, 51)$; $4:-3$.
49. (iii) $2:1$. (iv) $3x=y$; $5x=3y$; $3x=2y$; $9x=7y$. (v) 30, $16\frac{4}{7}$, 15, $12\frac{6}{7}$.
50. $r_1r_2\sin(\theta_1-\theta_2)+r_2r_3\sin(\theta_2-\theta_3)+r_3r_1\sin(\theta_3-\theta_1)=0$. 51. $2:1$.
52. $(x-3)^2+(y-4)^2=25$; circle. 53. $2ax+2by=a^2+b^2$.
54. $(4x-a)^2+(4y-b)^2=c^2$. 55. $\{3x-x_1-x_2, 3y-y_1-y_2\}$.

**56.** The orthogonal projection of a harmonic range is a harmonic range.

**59.** $x^2+y^2=l^2$. **60.** 60°. **61.** $3\sqrt{7}$. **62.** $3ax-3by+ab=0$.

**63.** $5x^2+5y^2+26ax+5a^2=0$; $(-\frac{1}{5}a, 0)(-5a, 0)$.

**64.** $\{(x_1+x_2+\ldots+x_n)/n, (y_1+y_2+\ldots+y_n)/n\}$.

## II a.

**1.** $3x-y=3a-b$.

**2.** (a) $x-y=0$. (b) $12x-5y+39=0$. (c) $x+y=1$. (d) $x-2y=0$.
(e) $x=0$. (f) $y=0$.

**3.** $y=0$, $x=0$. **5.** (a) 120°. (b) 30°. (c) 45°. (d) 135°.

**6.** (a) 30°. (b) 120°. (c) 90°. (d) 45°. **7.** $2x+3y=12$, $2x+3y=0$.

**8.** (a) 30°; (b) 90°.

## II b.

**1.** (i) $y-x=0$. (ii) $y\sqrt{3}-x=0$. (iii) $y+x\sqrt{3}=0$.

**2.** (3, 0), (0, 4), (−3, 0), (0, −4). **3.** 135°, 45°, 90°.

**4.** 1, $\frac{1}{3}$, $-2\sqrt{2}$, $\frac{1}{3}a$, $\frac{1}{2}b$. **5.** $7y-9x=126$.

**6.** (i) $x=0$, $y=0$, $x+y=a$. (ii) $\sqrt{3}y+x=0$, $\sqrt{3}y-x=0$, $2x-a\sqrt{3}=0$.

**9.** 30°. **10.** 150°; 150°. **11.** $x\sqrt{3}+y\sqrt{2}=6$.

**13.** (i) $x/4+y/3=1$, $3x/5+4y/5=12/5$; $y=-3x/4+3$.
(ii) $x/4-y/2=1$, $x/\sqrt{5}-2y/\sqrt{5}=4/\sqrt{5}$; $y=\frac{1}{2}x-2$.

**14.** 3, $-\frac{1}{3}$. **15.** $y\sqrt{3}-x-3\sqrt{3}=0$, $x=0$.

**17.** (i) $c/\sqrt{3}$, $c$. (ii) $c$, $-c$. (iii) $c$, $c$.

**18.** The perpendicular from the origin on the line is 3. **19.** $\pm 1\frac{1}{2}$.

**20.** (i) $y\sqrt{3}+x=4$. (ii) $y\sqrt{3}+x=-4\sqrt{3}$. (iii) $y\sqrt{3}+x=\pm 10$.

## II c.

**1.** $x-2+\mu(y-1)=0$. **2.** $3x+4y+\mu=0$.

**3.** (a) $x+7y+\mu=0$; (b) $y+\mu=0$. **4.** $3x+y-5+\mu x=0$.

**5.** $x\cos\alpha+y\sin\alpha=\pm 1$. **6.** $y\sqrt{3}-x+\mu=0$, or $x+\mu=0$.

## II d.

**1.** $3(x+2)-(y-3)\equiv 3x-y+9=0$.
$(x+2)+3(y-3)\equiv x+3y-7=0$.

**2.** $7x-4y=0$.

**3.** (a) $x-2y+1=0$. (b) $4x+y+11=0$. (c) $x-y=a-b$.
(d) $x\cos\frac{1}{2}(\theta+\phi)+y\sin\frac{1}{2}(\theta+\phi)=a\cos\frac{1}{2}(\theta-\phi)$.
(e) $2x-(m+n)y+2amn=0$. (f) $(x-a)\sin\theta-(y-b)\cos\theta=0$.

**4.** $y\cot 75°=x-5$. **5.** $10y+11x-100=0$.

**6.** $(x-x_1)(x_2-x_3)+(y-y_1)(y_2-y_3)=0$.

**7.** $(a'q-b'p)(ax+by+c)+(bp-aq)(a'x+b'y+c')=0$.

**8.** $\frac{1}{4}$; $\frac{1}{4}$; $(3\pm 4\sqrt{3})x+(4\mp 3\sqrt{3})y+4=0$. **9.** $\alpha-\beta$.

**10.** $x\cos 2\alpha+y\sin 2\alpha=2p$.

**11.** $c'(ax+by+c)-c(a'x+b'y+c')\equiv(ac'-a'c)x+(bc'-b'c)y=0$

**12.** $x\sqrt{3}+y-4=0$, $x-y\sqrt{3}+4=0$, $x\sqrt{3}+y+4=0$.

**13.** $adx+bcy+cd=0$. **16.** $x-2y=0$; $x+y=0$; $2x+5y=0$.

17. $2x-5y=1$, $x+y=2$, $67x+62y=569$.
18. (i) $4(3x+5y-7)-3(4x+6y-5)\equiv 2y-13=0$.
(ii) $6(3x+5y-7)-5(4x+6y-5)\equiv -2x-17=0$.
20. $l'=m'$; $l'+m'=0$.

## II e.

1. $\frac{9}{13}$, 2. 2. 10, $5\sqrt{2}$. 3. $nax+nby=(n-1)ah+(n-1)bk+1$.

## II f.

1. $(-\frac{1}{3}, \frac{2}{3})$, $(44, -5)$, $(14, 80)$. 2. $(3, \sqrt{5}+1)$ or $(3, 3{\cdot}24)$. 3. $(3, -3)$.

## II g.

1. 3. 2. $61x-20y-1=0$; $2\frac{1}{11}$. 3. $(\cdot8, 1{\cdot}4)$, $(\cdot2, 1{\cdot}4)$, $\cdot4$.
4. $ab\div\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}$. 5. $56x-35y+157=0$; $5x+8y-13=0$.
6. $\tan^{-1}1\frac{7}{9}$. 7. $(la+mb+n)^2=r^2(l^2+m^2)$.
8. $x=1$, $y=3$, $x=4$, $y=2$. 9. $(4, 4)$. 10. $3\frac{1}{7}$.
11. $x=2$, $x-2+\sqrt{3}(y-1)=0$.
12. $a_1a_3+b_1b_3=a_3b_1-a_1b_3$; $a_1a_2+b_1b_2=0$.
14. $\frac{1}{2}(25\sqrt{2})$, $\tan^{-1}\frac{1}{7}$. 15. 100, $2\sqrt{5}$. 16. $(\frac{9}{14}, 1\frac{3}{14})$; $(2\frac{1}{14}, 1\frac{1}{28})$.
17. $(0, a\lambda)$, $x=0$. 18. $a$. 19. $6x+7y=31$.
21. $(1, lmn+l+m+n)$.
23. $bu-av=0$, $bu+av-ab=0$, $2u-a=0$, $2v-b=0$. 24. $mu-v=0$.
25. $u+w=0$, $u-w=0$. 26. $hy+kx=2xy$.

## II h.

1. $1\frac{7}{9}$. 2. $1\frac{1}{2}$. 3. $\frac{1}{3}$ or $\frac{1}{5}$. 4. $77\frac{7}{9}$. 5. $(ab+bc+ca, a+b+c)$.
6. $\lambda+\mu+\nu=0$. 7. $\frac{1}{2}c^2\sin\omega$.
11. $3(3x+4y-7)+5(4x+5y-6)+30(x-y+1)=0$.
$10x-y=0$, $29x+37y-51=0$, $39x-18y+9=0$.
12. $v-w=0$, $w-u=0$, $u-v=0$, $2u+v+w=0$, $2v+w+u=0$, $2w+u+v=0$
13. $u=0$, $v=0$, $w=0$.

## II i.

1. $y+mx=0$. 2. $(l-m)(n-p)/(l-p)(n-m)$. 3. $x^2+y^2=2a^2$.
4. $(p+q-2r)y-(2pq-pr-qr)x=0$.
5. $(l_1m_3-l_3m_1)(l_2m_4-l_4m_2)\div(l_1m_4-l_4m_1)(l_2m_3-l_3m_2)$.

## II j.

2. $\frac{5}{11}(5\sqrt{3}-3)$. 3. $8x-4y+7$.
4. $75°45'$; $21x-77y+53=0$; obtuse. 5. $-\frac{2}{3}$, $\frac{1}{3}(4+2\sqrt{3})$. 6. $\frac{7}{9}$.
8. $4x-3y+7=0$, $4x+3y+1=0$. 10. 7 : 4. 11. $2x+2y=19$.
12. $\frac{1}{4}(15\sqrt{3})$. 13. $(2, 60°)$. 14. $3x+6y=8$, $6x+3y=8$.
15. (i) $(m+n)y-2x=2amn$. (ii) $x+mny=(m+n)a$.
(iii) $x/a\cos\frac{1}{2}(\theta+\phi)+y/b\sin\frac{1}{2}(\theta+\phi)=\cos\frac{1}{2}(\theta-\phi)$.
16. $2x-y=5$, $x+y=6$, $5x+2y=28$. $(3\frac{2}{3}, 2\frac{1}{3})$.
17. $17x+11y+9=0$, $x+4y+5=0$, $x+y+1=0$. $(\frac{1}{3}, -1\frac{1}{3})$.

18. $ab/(a\sin\alpha+b\cos\alpha)$. 20. $\frac{2}{1}\sqrt{68}$. 21. $hx+ky=a^2$.
22. $x^2/h^2+y^2/k^2=c^2/(h+k)^2$. 23. $(-\frac{27}{2}, \frac{39}{2})$. 24. $(k-1)/kc$.
25. $x^2/a^2+y^2/b^2=1$. 26. $\{a(b-b')/(b+b'); 2bb'/(b+b')\}$.
27. $x^2+y^2=a^2$. 28. $\frac{1}{2}p_1p_2\sin(\alpha_1-\alpha_2)$. 29. $x+a=0$.
30. (30, 36). 31. (1, 6). 32. $(0, -mn/p)$.
33. $r\cos 2\theta=a\cos\theta$. 34. $\beta, \frac{1}{2}\alpha, \tan^{-1}(1+e\cos\alpha)/e\sin\alpha$.
35. $r=p\cos(\theta-\alpha)$.
36. $\sqrt{\{\rho^2+d^2+2\rho d\cos\overline{\beta-\alpha}\}}$, $\tan^{-1}(\rho\sin\alpha+d\sin\beta)/(\rho\cos\alpha+d\cos\beta)$.
37. (i) $\theta=\frac{1}{2}(\alpha+\beta)$ or $\frac{1}{2}\pi+\frac{1}{2}(\alpha+\beta)$. (ii) $r(\cos\theta\pm\sin\theta)=p\pm q$.
(iii) $\theta=\frac{1}{2}\pi+\frac{1}{2}(\beta-\alpha)$; $\theta=\pi+\frac{1}{2}(\beta-\alpha)$.
(iv) $r\cos(\theta-\frac{1}{2}\alpha)=p\cos\frac{1}{2}\alpha$; $r\sin(\theta-\frac{1}{2}\alpha)=p\sin\frac{1}{2}\alpha$.
38. $81x^2-34xy-31y^2+44x+92y=0$. 39. $\frac{1}{2}p^2\sin\alpha$. $\sec^3\alpha(1+\sin 2\alpha)$.
43. (i) $r'\cos(\theta'-\alpha)-p$.
(ii) $\{r'(\cos\theta'+e\cos\overline{\theta'-\alpha})-l\}\div\{\sqrt{(1+2e\cos\alpha+e^2)}\}$.
44. $(-7, 3)$. 45. $120x-35y+12=0$.
48. (i) $4ax+c^2=0$. (ii) $x^2+y^2=c^2-a^2$. 49. $1:-3$.
51. (i) $2x+y+5=0$, $x-2y+10=0$. (ii) $58x=67y$; $42x=83y$.
52. Escribed to third line. 53. $147x-121y+570=0$.
56. $c^2(a'^2+b'^2)=c'^2(a^2+b^2)$.
57. $(3\sqrt{3}, \frac{1}{3}\pi)$, $\sqrt{3}r\cos\theta\mp 2r\sin\theta\pm 2\sqrt{3}=0$.
$\sqrt{3}.r\cos\theta\mp 2r\sin\theta\pm 6\sqrt{3}=0$. $\theta=\frac{1}{3}\pi$. $r\sin\theta=2\sqrt{3}$.
58. $x^2+y^2-2ay=a^2$. 60. $x+2=0$, $7x+24y+182=0$. 61. (7, 9).
63. (1, 7), $-1$. 64. $19x-2y+19=0$. 66. $y^2=4ax$.
67. $a^2=4xy$. 68. $x^2/a^2+y^2/b^2=1$. 69. $y^2=4ax$.
71. $mv-nw=0$, $mv+nw=0$. 72. $u+3v=0$.
73. $x+y\cos\omega=p+q\cos\omega$, $x\cos\omega+y=p\cos\omega+q$, $x=p$, $y=q$.
75. $r\sin C\sin(\theta+A)=d\sin B$; $A$ pole; $r\sin\theta=d$ given line.
77. $(x-a)(x-b)c=xy(a-b)$ where $OA=a$, $OB=b$, $PQ=c$ and $AB$, $LM$ are axes of reference.
78. (i) $2hx+2ky=h^2+k^2$. (ii) $(xh-ky-h^2-k^2)^2=4ky(h^2+k^2)$.

## III a.

3. $(x^2-y^2)(x^2-a^2)(y^2-a^2)=0$.
4. $\{(x-y)^2-2a^2\}\{(x+y-2a)^2-2a^2\}(x-a)(y-a)=0$.
5. $x=a$, $x=-a$, $y=a$, $y=-a$. 7. $x^2-y^2=0$.
8. $x^2+2hxy-y^2=0$; the given lines are perpendicular.
9. (a) $x^2+y^2+2xy\sec 2\theta=0$. (b) $x^2+y^2-2xy\sec 2\theta=0$.
10. (i) $x-y=0$, $x-\sqrt{3}y=0$, $x+\sqrt{3}y=0$; 45°, 30°, 150°.
(ii) $y=x\tan\alpha$, $y=x\tan(\alpha+60°)$, $y=x\tan(\alpha+120°)$; $\alpha, \alpha+60°, \alpha+120°$.
11. The lines $u-mv=0$, $u+mv=0$; these are harmonic conjugates of $u=0$, $v=0$.
12. (a) $x^2-3xy+6y^2=0$. (b) $x^2-3xy+6y^2-6x+9y+9=0$.
13. $a(y-a)^2-2h(y-a)(x-b)+b(x-b)^2=0$. 14. $\theta$. 15. $u^2-v^2=0$.
16. (4, 0), $(-2, 0)$; $(0, -1\frac{1}{2})$, $(0, -4)$; $(6, -6)$, $(2, -2)$.
$x-y=4$; $3x+4y+6=0$.
17. $a+b=0$.
18. (a) $a/b=b/c=c/d$. (b) $(bc-ad)^2=4(bd-c^2)(ac-b^2)$.
(c) $a+c=0$, $b+d=0$.

19. (i) $x^3\tan 3\theta - 3x^2y - 3xy^2\tan 3\theta + y^3 = 0$.
(ii) $x^3\tan\theta + x^2y(2\tan\theta\tan 2\theta - 1) - xy^2(2\tan 2\theta + \tan\theta) + y^3 = 0$.
20. $a^3 - a^2c - 5ad^2 + cd^2 = 0$; $d^3 + d^2b - 5a^2d - a^2b = 0$.
21. $x^2 + xy + 2y^2 = 0$. 22. $a^2l^2 + a^2m^2 = 1$.
23. $x^4 - 4x^3y\cot 4\theta - 6x^2y^2 + 4xy^3\cot 4\theta + y^4 = 0$.

## III b.

1. $9x^2 + 12xy + 4y^2 - 6x - 4y + 1 = 0$. 2. $\tan^{-1}(-\frac{1}{12}\frac{5}{7})$.
3. $(x\cos\alpha + y\tan\alpha)(x - y\cot\alpha) = 0$. 5. $3\frac{2}{10}$. 7. $\sqrt{6}$.
8. $3x + y = 0$, $x - 3y = 0$. 11. $\frac{3}{4}$.
12. (i) $(ab' - a'b)^2 = 4(bh' - b'h)(ah' - a'h)$.
(ii) $(bb' - aa')^2 + 4(bh' + a'h)(hb' + ah') = 0$.
13. $x^2 + 2xy + y^2 = 0$. 14. $x^2 - 2xy - y^2 - 2x - 6y - 7 = 0$.
16. $a\tan^2\alpha - 2h\tan\alpha + b = 0$. 17. 60°.
21. $2p\sqrt{b^2 - ac}/(a\sin^2\alpha - 2b\sin\alpha\cos\alpha + c\cos^2\alpha)$.
22. $a + b = 0$ and $h(l^2 - m^2) = (a - b)lm$,
or $al^2 + 2hlm + bm^2 = 0$ and $(a + b)^2 + 4(ab - h^2) = 0$.
23. $(a'h - ah')x^2 + (a'b - ab')xy + (bh' - b'h)y^2 = 0$.
25. $1/\{(l^2 - m^2)\sin\alpha + 2lm\cos\alpha\}$; $x^2 - 2xy\cot\alpha - y^2 + \Delta\operatorname{cosec}\alpha = 0$.
26. $ab' + a'b = 2hh'$; $ab' + a'b - 2hh' = 6\sqrt{(h^2 - ab)(h'^2 - a'b')}$.
31. $2c\sqrt{\{(h^2 - ab)(1 + 2m\cos\omega + m^2)\}} \div (a + 2hm + bm^2)$.

## III c.

1. $3X + 4Y = 0$.
2. $4X^2 + 8XY + 3Y^2 + 8X + 2Y - 5 = 0$;
$2X + Y - 1 = 0$; $2X + 3Y + 5 = 0$.

## III d.

1. $X^2 + Y^2 = a^2$. 2. $2XY + a^2 = 0$. 3. $2XY - c^2 + a^2 - b^2 = 0$.
4. $X(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) + 2XY(\overline{b-a}\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta)$
$+ Y^2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta) = 0$.
5. $X - p = 0$. 6. $2\theta$, $y^2 = x^2\tan^2\theta$.
7. $X^2(1 + \sqrt{3}) + Y^2(1 - \sqrt{3}) - 2XY = 0$.

## III e.

1. (i) $AA' + BB' = (AB' + BA')\cos\omega$. (ii) $a + b = 2h\cos\omega$.
2. $(lx_1 + my_1 + n)\sin\omega \div \sqrt{(l^2 + m^2 - 2lm\cos\omega)}$. 3. $x^2 + y^2$.
4. $x^2(h - a\cos\omega) - xy(a - b) - y^2(h - b\cos\omega) = 0$.

## III f.

1. $ax + hy + g = 0$; any number.
3. If $ax + hy + g = 0$; $hx + by + f = 0$ have a finite solution; one way;
$\{(fh - bg)/(ab - h^2), (gh - af)/(ab - h^2)\}$.
4. Yes, when $(bg - fh)^2 < (ab - h^2)(bc - f^2)$.
5. (a) $b\tan^2\theta + 2h\tan\theta + a = 0$. (b) $a\tan^2\theta - 2h\tan\theta + b = 0$.
(c) $\tan 2\theta = 2h/(a - b)$. (a) and (b) if $h^2 < ab$. (c) Always possible.
6. $ab = h^2$.

## III g.

4. (a) $-\frac{2}{3}\frac{6}{5}\frac{7}{5}$. (b) $-\frac{8}{3}\frac{1}{4}$. (c) $\frac{8}{1}\frac{4}{2}\frac{7}{5}$. (d) 9 or $4\frac{1}{3}$. (e) 12 or 48.
9. (a) $C = 0$ and $\Delta = 0$. (b) $a+b = 0$ and $\Delta = 0$.
10. $x^2-3y^2 = 0$ or $3x^2-y^2 = 0$. 11. $x^2-y^2 = 0$.
21. The pairs of harmonic conjugates are $x^2+2xy-y^2 = 0$, $x^2+5xy+6y^2 = 0$.
25. $\Delta\lambda\{\lambda^2+\lambda\overline{a+b}+C\} = 0$.
29. $\{a(a'+b'+2h')-b(a'+b'-2h')\}^2$
$+4\{a(a'-b')-h(a'+b'-2h')\}\{b(a'-b')-h(a'+b'+2h')\} = 0$.
32. $K = (l-l')(m-m')$, 0, or $\infty$.

## V a.

1. (i) (1, 2), 2. (ii) (3, 0), 3. (iii) $(-a, -b)$, $c$. (iv) $(\frac{1}{3}, \frac{1}{3})$, $\frac{1}{3}$.
(v) $(\frac{1}{2}a, \frac{1}{2}b)$, $\frac{1}{2}\sqrt{a^2+b^2}$. (vi) $(1\frac{3}{4}, \frac{1}{4})$; $\sqrt{97}/14$.
2. (i) $x^2+y^2-6x-8y = 0$, (0, 0), (7, 7). (ii) $x^2+y^2+4x-6y+12 = 0$.
(iii) $x^2+y^2-4x\cos\theta-4y\sin\theta = 0$, $\{2(\sin\theta+\cos\theta), 2(\sin\theta+\cos\theta)\}$, (0, 0).
(iv) $4x^2+4y^2-12y+5 = 0$.
(v) $x^2+y^2 = a^2$, $(a/\sqrt{2}, a/\sqrt{2})$, $(-a/\sqrt{2}, -a/\sqrt{2})$.
4. (a) $x^2+xy+y^2+5x+7y+9 = 0$. (b) $x^2-xy+y^2-x+5y+8 = 0$.
(c) $x^2+\sqrt{2}xy+y^2+(2+3\sqrt{2})x+(6+\sqrt{2})y+6+3\sqrt{2} = 0$.
5. 60°; (3, 2), $\sqrt{13}$.
6. (i) $(x-3)^2+(y+\frac{5}{2})^2 = (\frac{1}{2}\sqrt{61})^2$. (ii) $(x-2)^2+y^2 = 2^2$.
(iii) $(x-\frac{3}{2})^2+(y-\frac{7}{2})^2 = (1/\sqrt{2})^2$
8. (i) $x^2+y^2 = 4$. (ii) $x^2+y^2 = 9$. (iii) $x^2+y^2 = c^2$. (iv) $9(x^2+y^2) = 1$.
(v) $4(x^2+y^2) = a^2+b^2$. (vi) $196(x^2+y^2) = 97$.

## V b.

1. $x^2+y^2+12x-10y+9 = 0$; $x^2+y^2-6x+8y-9 = 0$. 4. $(-1, 1)$.
5. $x^2+y^2-6x-8y+20 = 0$; $\sqrt{5}$; (2, 2)(2, 6); (2, 2)(5, 5).
6. $x^2+2gx+c = 0$; $x^2+y^2+2fy-a^2 = 0$. 7. $f^2 = c$.
8. $x^2+y^2-ax-by = 0$. 9. $x^2+y^2 = a^2$. 10. (4, 1), $\sqrt{5}$.
11. $(x-h)^2+(y-k)^2 = a^2$, $(h, k)$, $a$. 12. $x^2+y^2 = 4a^2$.
13. (1, 2)(4, 3); $\sqrt{10}$. 14. $3x^2+3y^2+10ax+3a^2 = 0$, $(-3a, 0)(-\frac{1}{3}a, 0)$
15. $x^2+y^2-x-5y+4 = 0$.
16. $x^2+y^2+12x+16y+75 = 0$, outside, $(-3, -4)(-9, -12)$.
17. $5x^2+5y^2-8x+4y-5 = 0$. 18. $8l^2-23l+10 = 0$; $23\pm\sqrt{209}:16$.
19. $x^2+y^2-3x+1 = 0$, $x^2+y^2-5x+4y+9 = 0$.
20. $x^2+y^2-(a+c^2/a)x-(b+c^2/b)y+c^2 = 0$. 21. $y = 1$.
22. $(-\frac{3}{4}, \frac{1}{4})$, $\frac{1}{4}\sqrt{10}$, $(-2\frac{1}{4}, \frac{3}{4})$, $\sqrt{2}$.
23. (i) $(x-a)^2+(y-b)^2 = (h-a)^2+(k-b)^2$. (ii) $x^2+y^2-2ax-2ry+a^2 = 0$.
(iii) $x^2+y^2-2rx-2ry+r^2 = 0$.
25. $x^2+y^2-(\alpha+\beta)(x+y)+\alpha\beta = 0$. 26. $x+y-7 = 0$.
27. Circle through $B$ and $C$. 28. $x^2+y^2-10x = 0$.
29. $ab(l^2+m^2) = am+bl$.

## V c.

1. $x+y+5=0$. 2. $y-4=0$; $3x+y+2=0$; $3x+4y=10$.
3. $4x-3y=\pm 10$. 4. $3x+4y=0$; $(\frac{4}{5}, -2\frac{4}{5})$.
5. Inside; $9x-2y-5=0$. 6. $(2, 1)$. 7. $7/\sqrt{2}$ 8. $2x+5y=29$.
9. $3x-y-8=0$, $3x-y+2=0$. 10. $2x-5y+11=0$.
11. $x^2+y^2-2x-2y+1=0$; $x^2+y^2-12x-12y+36=0$.
12. $(3\cos\theta+1, 3\sin\theta+2)$. 14. $10x=17$; $y=2$.
15. $3x-4y-5=0$; $3x-4y+15=0$. 16. Circle; straight line; circle.
17. Circle; $3(x^2+y^2)-14(x+y)+26=0$; $(1, 3)$. 18. $x^2+y^2=25$.
20. $x\cos\alpha+y\sin\alpha=\pm r$. 21. $2(a_1-a_2)x+2(b_1-b_2)y=a_1^2+b_1^2-a_2^2-b_2^2$.
22. Concentric circle. 23. $(x^2+y^2-hx-ky)^2=c^2(x^2+y^2)^2$, $(h, k)$ fixed point.
24. $x^2+y^2-30x+30y+225=0$; $x^2+y^2-6x+6y+9=0$. 26. $3r$.
27. $(x^2+y^2)(g^2+f^2-c)=(x^2+y^2+gx+fy)^2$.

## V d.

1. $1\frac{2}{3}\frac{2}{1}\frac{2}{8}$. 2. $(10, 5)(-5, 10)$; $(5, 15)$. 3. $(8, 4)$, $(4, -3)$.
5. $x\sin\alpha-y\cos\alpha=\pm a$. 8. $3\frac{3}{7}$. 13. $x^2-y^2=0$.
14. $x^2+y^2+ay-ax=0$.

## V e.

19. $PA$ is a diameter.

## V f.

1. $x-y-5=0$, $x-y+3=0$. 2. $(x-a)\sin\alpha-(y-b)\cos\alpha=\pm r$.
3. $x^2+y^2=9$; $x^2+y^2=49$.
4. $(x^2+y^2-ax-by)^2=(rx+Ry-Rb)^2+(Rx-ry-Ra)^2$. 6. $2x-2y=3$.
7. $(x^2+y^2-ax-by)^2=r^2(x^2+y^2)$. 8. $x^2+y^2=2r(x\cos\alpha-y\sin\alpha)$.
9. The concentric circle $3(x^2+y^2)=4(R^2+Rr+r^2)$.

## V g.

6. $n(x^2+y^2)=2y\Sigma a$. 7. $\{(x-r)(x-R)+y^2\}^2=r^2\{y^2+(x-R)^2\}$.
10. $\{2r\cos\theta\cos\phi\sec\overline{\theta-\phi},\ r\sin\overline{\theta+\phi}\sec\overline{\theta-\phi}\}$.
12. $x\cos(\alpha+\beta+\gamma+\delta)+y\sin(\alpha+\beta+\gamma+\delta)=2r\cos\alpha\cos\beta\cos\gamma\cos\delta$.
13. If $B$ is the origin and circle $x^2+y^2=2rx$, locus is $x^2-y^2=2rx$.
14. A straight line.

## V h.

1. (i) $k(x^2+y^2-a^2)=y(h^2+k^2-a^2)$. (ii) $x^2+y^2-2ay=a^2$; $x^2+y^2+14ay=a^2$.
(iii) $x^2+y^2\pm 2\sqrt{3}ay-a^2=0$.
3. $x^2-y^2+2kxy=a^2$. 4. $x^2+y^2-4ax+a^2=0$.
8. (i) $9x^2+9y^2-a^2=6ay\cot\alpha$. (ii) $x^2+y^2-a=-2ay\tan\frac{1}{2}\alpha$.
(iii) $x^2+y^2+2ay\cot\alpha=a^2$.

## V i.

1. $(8, 0)$, $(-2, 0)$.
2. $x^2+y^2-a(\lambda^2+\lambda^{-2})x-2a(\lambda-\lambda^{-1})y-3a^2=0$. $\{-a, a(\lambda-\lambda^{-1})\}$.
3. $x^2+y^2-7x-21y+120=0$. 4. $x^2+y^2-6x-4y+8=0$; $(4, 4)$, $(2, 0)$.
6. $x^2+y^2-6x-8y-119=0$.

## V j.

1. $(x^2+y^2-r^2)(h^2+k^2-r^2) = (hx+ky-r^2)^2$. 3. 90°.
4. $x^2+y^2+2gx+2fy+2c-g^2-f^2 = 0$. 5. $3x^2+3y^2+12x-18y+35 = 0$.
7. $3x+4y = 7$.

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1. $gx+fy+c = 0$. 2. $y^2-xy-x+4 = 0$. 3. (15, 20).
5. $2x+y = 0$; $x-2y = 0$. 7. A circle.
8. $x^2+y^2 = a^2+r^2$, where $(\pm a, 0)$ are the given points.
10. A straight line.
13. $3x^2-3y^2+2ay\sqrt{3}-3a^2 = 0$, where vertices of triangle are $(\pm a, 0)$, $(0, a\sqrt{3})$.
15. $(0, mn/p)$; $p(x^2+y^2)+p(n-m)x+(mn-p^2)y-mnp = 0$.
$2p(x^2+y^2)-(m-n)px-(mn+p^2)y = 0$.
16. $(x_1^2+y_1^2-a^2)(x_2^2+y_2^2-a^2) = (x_1x_2+y_1y_2-a^2)^2$.
18. A circle touching a line parallel to given line at the given point.
20. $-1$, $-3$. 22. $r^3-(a^2+b^2+c^2)r-2abc = 0$. 23. $xx'+yy' = a^2$.

## V k.

1. $x = 1$, $x \pm 2\sqrt{2}y = 3$. 2. $5x^2+5y^2-18xy+22x+50y-155 = 0$.
3. $x^2+y^2+9x+7y-6 = 0$. 4. $x^2+y^2+4x+4y-1 = 0$.
5. $x^2+y^2 = 1$, $x^2+y^2 = 81$. 6. $x-24 = \pm 2\sqrt{2}y$.
7. $10x^2+10y^2+18x-30y-8 = 0$.
9. $(x^2+y^2+2gx+2fy+c)(al+bm-1) = (a^2+b^2+2ga+2fb+c)(lx+my-1)$.
11. $x^2+y^2+8y+9 = \pm 10\sqrt{2}y$. 13. $\lambda+\mu = 0$.
18. $16(x^2+y^2) = 57x$; $8(x^2+y^2) = 57x$.
20. $x^2+y^2-2(a_1a_2+b^2)(a_1+a_2)^{-1}+b^2 = 0$.
21. $2(a_1-a_2)x+2(b_1-b_2)y = a_1^2+b_1^2-a_2^2-b_2^2-c^2\cos^2\alpha+c^2\cos^2\beta$.
22. A circle.
23. $(x^2+y^2-c)\{l(f-f')-m(g-g')\}-2x(f-f')+2y(g-g')+2(f'g-fg') = 0$.
24. $(a^2+b^2)(x^2+y^2)+2c(ax+by) = 0$.

## V l.

1. (8, 2), $x^2+y^2-6x-4y-14 = 0$. 2. $19x+8y-12 = 0$.
3. $\{a+b, \frac{1}{2}(a+b)\}$; $x^2+y^2-(2x+y)(a+b)+a^2+3ab+b^2 = 0$.
4. $(-2, -1)$, $(0, -3)$. 5. $\sqrt{r^2-a^2+b^2}$.
11. $(-g+l\lambda/a, -f+m\lambda/a)$ where $\lambda^2(l^2+m^2)+2\lambda(gl+fm-an)+g^2+f^2-c = 0$.

## V m.

1. $(x-1)(y-1) = 0$; $(x+y)(x+y-4) = 0$. 2. $56x+17y-5 = 0$.
3. Circle. 5. $(x-2)^2+(y-3)^2 = 5$.
6. $(2hm-la+lb)x+(2hl+ma-mb)y+\mu = 0$.
7. $x^2+y^2-36x-46y+324 = 0$, $25x^2+25y^2-80x-494y+64 = 0$.
8. $x^2+y^2-2(k+k')x+2kk'y/d+2kk'-d^2 = 0$.
9. $x^2+y^2+6x-10y-7 = 0$. 10. $91x^2+91y^2-240x-366y+504 = 0$.
12. $4(na^2-2mab+lb^2)(m^2-ln)^{\frac{1}{2}} \div \{(l-n)^2+4m^2\}$.

13. $4x^2+4y^2-11x-11y-28=0$.
14. The angle in a semicircle is a right angle.
16. $\Sigma(\beta, q)(\gamma, r)\sin(\beta-\gamma)=0$.
17. $\Sigma(a^2+b^2)(ab-c^2)(bx+cy+a)(cx+ay+b)=0$. 21. $uv=w^2$.
22. $u\pm kw=0$.

## V n.

1. $r^2-rr_1\cos(\theta-\theta_1)-rr_2\cos(\theta-\theta_2)+r_1r_2\cos(\theta_1-\theta_2)=0$.
2. $n\cos(\theta-\alpha)=R$. 3. $r\cos(2\theta_1-\theta)=2R\cos^2\theta_1$.
6. $r^2+4ar\cos\theta-12a^2=0$; $4a$.
9. (i) $(3r\cos\theta-2R)^2\sin^2\alpha+9r^2\sin^2\theta\cos^2\alpha=R^2\sin^2 2\alpha$.
(ii) $9r^2-12Rr\cos\theta+4R^2\sin^2\gamma=0$.
11. $2Rr\cos\theta=k^2$. 12. $r\cos\theta+d=0$; $\{-d\pm\sqrt{d^2+2ad},\ 0°\}$.
13. A circle.
17. The circle circumscribing $u_1$, $u_2$, $u_3$ is

$$r\cos\theta_1\cos\theta_2\cos\theta_3=a\cos(\theta-\theta_1-\theta_2-\theta_3).$$

The centres of such circles lie by fours on circles of which one is

$$2r\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4=a\cos(\theta-\theta_1-\theta_2-\theta_3-\theta_4).$$

There are 5 such circles, &c.

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2. $(a, 0)$, $(a\Sigma m_1m_2, 0)$.
4. The poles of $x+ly=0$, $lx-y=0$ w. r. t. the circles lie on $x+2c=0$. Eliminate $l$ between the equations of the pairs of tangents from these poles to the circles.
11. $\{ak^2/(a^2+b^2-c^2),\ bk^2/(a^2+b^2-c^2)\}$, $k^2c/(a^2+b^2-c^2)$.
12. Two coincident lines through the origin.
14. $x^2+y^2+xy-2cx\pm\sqrt{5}cy+c^2=0$. 19. $x^2+y^2\pm ax\sqrt{2}=0$.
22. $x^2+y^2-px-qy=0$.
25. $x=(a^2+\xi^2)/\xi$, if $x^2+y^2=a^2$ is the circle, $(\xi, 0)$ the fixed point.
28 $ky^2=r(k-x)$, $(k, 0)$ is the mid-point of $AB$. 30. 6.
35. $y^2(49x^4+49x^2y^2-4y^4)=0$. 38. Circle. 39. $x^2+y^2+2gx+a^2=0$.

## VI a.

1. (i) Parabola. (ii) Two real straight lines. (iii) Ellipse. (iv) Two parallel straight lines. (v) Hyperbola. (vi) Two imaginary straight lines.
2. $(-\frac{5}{2}, 1)$, $(-5\frac{1}{2}, 3)$, $2x^2+16x+27=0$.
5. $(1, -2)$, $(3, 6)$, $(0, 0)$, $(4, 4)$; $2x+y=0$; $2x+y=12$.
6. $(f^2-bc)>$, $=$, or $<0$. 8. $hx+by+f=0$.
9. $(a+2h+b)y^2+2(g+f)y+c=0$.

## VI b.

1. Parabola. 2. Hyperbola. 3. Circle. 4. Two straight lines.
5. Hyperbola. 6. Parabola. 7. Hyperbola. 8. Ellipse.
9. Two parallel straight lines. 10. Hyperbola. 11. Ellipse.

## VI c.

2. No centre, $(-1\frac{6}{13}, -\frac{2}{13})$, $(4, -2)$, no centre, $(-1\frac{6}{13}, -\frac{2}{13})$, $(0, 1)$.

3. $1, \sqrt{\frac{2}{3}}$. 4. $(x+\frac{4}{15})^2+(y-\frac{8}{15})^2 = \frac{1}{5}(2x-y+\frac{2}{3})^2$.

5. $(x-1)^2+(y-2)^2 = (8x+4y)^2$.

6. (ii) $x^2+4xy+y^2-2x+2y+2 = 0$. (v) $(2x+y-2)(x+2y-2) = 0$.
(vii) $(x-y)(7x+y) = 0$. (x) $(x+y-1)(2x-y+1) = 0$.

8. (vi b) (4) $2\sqrt{2}, \frac{2}{3}\sqrt{-6}$. (5) $\frac{2}{3}\sqrt{6}, 2\sqrt{-6}$. (7) $\frac{1}{2}\sqrt{2}, \frac{1}{4}\sqrt{-2}$.
(8) $\sqrt{2}, \sqrt{2/11}$. (10) $\frac{1}{3}\sqrt{2\pm 2\sqrt{10}}$. (11) $1/\sqrt{7}\{24\pm 8\sqrt{2}\}^{\frac{1}{2}}$.

9. (vi b) (8) $x^2+11y^2 = 2$. (11) $(3-\sqrt{2})x^2+(3+\sqrt{2})y^2 = 8$.

10. $\tan 2\theta = 2h/(a-b)$. 12. Centre and line at infinity.

13. $\lambda = 1$, no value, positive, negative. $\sqrt{1\sim\lambda}$.

15. $\frac{1}{3}\sqrt{8}, \frac{1}{3}\sqrt{-1}$; 2; $(\frac{5}{3}, -\frac{1}{3})$; $(1, -1)$, $(2\frac{1}{3}, \frac{1}{3})$; $x-y-2 = 0$, $3x+3y-4 = 0$; $3x^2+12xy+3y^2-6x-18y+2 = 0$; $x+y-1 = 0$, $x+y-\frac{5}{3} = 0$.

16. $(2, 0)$, $(0, 1)$, $\frac{1}{3}\sqrt{5}$. 17. $af^2+bg^2-2fgh = 0$.

18. $(\frac{2}{3}, -1\frac{2}{3})$, $\sqrt{\frac{5}{3}}$; $6x+3y+1 = 0$, $x-2y-4 = 0$.

19. (i) $am^2-2hlm+bl^2 = ab-h^2$.
(ii) $(ab-h^2)(al^2+2hlm+bm^2) = [h(l^2-m^2)-(a-b)lm]^2$.

20. $(1\frac{1}{7}, 1\frac{2}{7})$; $(0, 0)$, $(2\frac{2}{3}, 3\frac{1}{3})$; $4x-3y = 0$; $3x+4y = 10$. $7x+y = 10$; $x-7y+10 = 0$. $3x+4y-5 = 0$; $3x+4y-15 = 0$.

21. $h(x^2-y^2) = (a-b)xy$; $r^4(ab-h^2)-(a+b)r^2+1 = 0$.

22. $2x-3y-2 = 0$, $3x+y+2 = 0$. 23. $\sqrt{\frac{2}{3}}$.

25. $(-1\frac{1}{12}\frac{2}{4}, -\frac{5}{12}\frac{1}{4})$, $20x-36y-1 = 0$. 26. § 7 (v) $2x+y = 15$, $2x+y = 5$.

27. Referred to $4x-3y+1 = 0$, $3x+4y-1 = 0$ as axes; $xy = 1$.

28. $(5/8, 13/8)$, $4x+4y-7 = 0$.

31. $a\sqrt{8}, \frac{1}{3}a\sqrt{8}$; $x-y = 0$, $x+y = 0$; $(\frac{2}{3}a\sqrt{2}, \frac{2}{3}a\sqrt{2})$; $\frac{2}{3}\sqrt{2}$.

37. The point of intersection.

## VII a.

3. $(a\cot^2\phi, 2a\cot\phi)$. 4. $at^2$. 5. $l+am^2 = 0$.

6. $y^2 = 4ax(lx+my)$. (i) $4al = 1$. (ii) $l+am^2 = 0$. 7. $y'y''/4a$.

10. 90°. 12. $c^2 = 4by$.

16. (i) $(-2, -1\frac{1}{4})$; $x+2 = 0$; $(-2, -\frac{1}{4})$, $4y+9 = 0$.
(ii) $(8\frac{1}{2}, 1)$; $y = 1$; $(4, 1)$; $x = 3$.
(iii) $(4, 8)$; $x-4 = 0$; $(4, 7\frac{1}{2})$; $2y-17 = 0$.
(iv) $(\frac{1}{2}, 1\frac{1}{2})$; $x-y+1 = 0$; $(1, 2)$; $x+y-1 = 0$.

19. $(a\cot^2\phi, -2a\cot\phi)$. 20. $x+y\cos\omega+a\operatorname{cosec}^2\omega = 0$.

21. $(4, 4)$, $(\frac{1}{4}, -1)$, $2y-x-4 = 0$, $2y+4x+1 = 0$.

25. $(\frac{1}{3}\frac{1}{0}, -\frac{9}{1}\frac{9}{0})$; $4x+8y+9 = 0$.

27. (i) $ln = am^2$. (ii) $al^3+2alm^2+m^2n = 0$, $x = 0$, $y^2 = a(x-a)$.

28. $16x^2+24xy+9y^2-140x+20y = 0$; (1·6, 1·2), 4.

32. Prove sum of radii = sum of perpendiculars from focus to tangent and normal.

## VII b.

7. 2 : 1. 9. $x(1+t^2)+yt+at^2 = 0$.

11. When orthogonal the envelope reduces to the focus.

16. $y^2 = 4a(x+4l)$.

## VII c.

1. $y+tx-at-at^3=0$. 2. $-(t^2+2)/t$.
3. $(x-dt^2)(t+\cos\omega)+(y-2dt)(1+t\cos\omega)=0$.
5. $y^2-ax-ah+2a^2=0$, $xy+ak=0$. 7. $y^2=16\{(\xi-2)/15\}^3$.
20. $by^2=(a+b)^2(x+a)$. 23. $X-6a$; $-\frac{1}{2}Y$.

## VII d.

1. $4a\sqrt{2}$; $(5a, -2a)$; $4a\sqrt{2}$. 2. $x^2+y^2-10ax+9a^2=0$.
3. $Ax-By+a+4aA=0$. 6. $(3a, \pm 2a\sqrt{3})$.

## VII e.

1. $x^2+2(g+2a)x+2fy+c=0$.
2. $\{(3t+1)y-4x\}^2=4a\{(t^2-2t+3)x-2t^2(1+t)y-4at^3\}$.
3. $(x-ty+at^2)^2-(1+t^2)(y^2-4ax)=0$. 4. $\lambda^2+40\lambda+16=0$.
6. $x^2+y^2-20ax+60a^2=0$.
8. $x^2+y^2-(3m^2+1)ax+m(m^2-3)ay+3a^2m^2=0$.
10. $2ay^2=(a-h)^2(x-a)$.
12. $x^2+y^2-3ax=0$.
13. The point $(-3a, 0)$.

## VII f.

1. 8/125.
2. (i) $\sqrt{5}y^2+x=0$. (ii) $y^2=x$. (iii) $y^2=2x$. (iv) $10\sqrt{10}y^2=7x$.
3. $\{fh/(f^2+g^2), gh(f^2+g^2)\}$. 5. $(bx-ay)^2+2ab(bx+ay)-3a^2b^2=0$.
7. $(8a, 0)$, $(3\frac{1}{4}a, 0)$. 9. $-\frac{1}{2}ag/(a+b)^2$; $-\frac{1}{2}\{ag(a+2b)\}/\{h(a+b)^2\}$.

## VII g.

1. A straight line $(2x/a+2y/b-1=0)$.
2. A straight line $(x/a+y/b-1=0)$.
3. A straight line $2(b+a\cos\omega)x+2(a+b\cos\omega)y=ab$.

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3. $a^{\frac{1}{3}}x+b^{\frac{1}{3}}y+a^{\frac{2}{3}}b^{\frac{2}{3}}=0$. 15. $x-4a=ky$. 16. $x+1\pm 2y=0$.
17. $x^2+y^2-6x-4y-3=0$, $x^2+y^2-22x+12y+13=0$.
18. $y^4+y^2(x-2a)(x-c)-a(x-c)^3=0$.
21. $a(x^2+y^2)-(y'^2+2a^2)x+y'(x'-a)y+ax'(2a-x')=0$. *Vide* § 7.
22. $ay^2+x[x^2+y^2-2hx-h(4a-h)]=0$; if $h=a$, $(x+a)(x^2+y^2-3ax-3a^2)=0$.
24. $k(y-k)^2+2(h-a)(x-h)(y-k)-k(x-h)^2=0$.
45. Maximum or minimum according as the ordinate of the point is $>$ or $<4a$.
47. (i) $16a(x-2a)=y^2$. (ii) $x+a=0$.
48. $(a+a')^2y^2+4b^2(a+a')x-4aa'b^2=0$.
50. $y(y-k)(x-h)+x(x-h)^2+a(y-k)^2=0$. To prove the second part, find the value of $x$ for the foot of the perpendicular on the tangent at $\lambda$, and then the condition that $\lambda$ should be real.

## VIII a.

**2.** $2x-8y-1=0$, $7x-12y+1=0$. **9.** $\sqrt{5}/3$; divides $AB$ in ratio $3:\pm 5$.
**10.** $(a\cos\alpha, b\sin\alpha)$, $(a\cos 3\alpha, -b\sin 3\alpha)$; $(x^2+y^2)^2(a^2x^2+b^2y^2)=(a^2x^2-b^2y^2)^2$.
**11.** § 4, VI. **13.** $\{\pm\sqrt{\lambda/(\lambda+1)}, 0\}$, $x^2=y$.
**17.** $\{a^2y'/\sqrt{b^2x'^2+a^2y'^2}, -b^2x'/\sqrt{b^2x'^2+a^2y'^2}$.
**20.** $a(x^2+y^2)=1$. **21.** $\sqrt{(a^2-p^2)(p^2-b^2)}/p$.
**25.** $10x^2+3y^2=187$; $(0, \pm\sqrt{1809/30})$; $20x+21y=0$.
**28.** If $\lambda:1$ is the ratio, $1+2/(\lambda-1)=2(1-e^2)/(e^2\sin 2\psi)$, $\therefore$ $\lambda$ greatest with $\sin 2\psi$.

## VIII b.

**8.** $a^2:b^2$. **16.** $16(a^2x^2+b^2y^2)=(a^2-b^2)^2$.
**17.** $(a^2\cos^2\alpha+b^2\sin^2\alpha)^{\frac{1}{2}}$, $(a^2\sin^2\alpha+b^2\cos^2\alpha)^{\frac{1}{2}}$.
**20.** p. 322. Pole of chord joining feet of normals lies on director circle.
**28.** $x^2/a^2+y^2/b^2=1/2$. **32.** $a^2x^2/(a^2+b^2)^2+y^2/4b^2=1$.
**37.** $\tan^2\theta=(1-e^4)/(2e^2-1)$. **41.** $(x^2+y^2)^2=4(a^2x^2+b^2y^2)$.
**43.** $\frac{1}{3}(2n\pi-\alpha)$, where $n=1, 2, 3$.
**44.** $a^2\sqrt{a^2-2b^2}/\sqrt{a^4-b^4}$; $b^2\sqrt{2a^2-b^2}/\sqrt{a^4-b^4}$.
**47.** The chord joining the feet of the other two normals is $x/x'+y/y'+1=0$; then $(\alpha, \beta)$ lies on this.
**48.** The fourth normal from the point;

$$(bx\cos\delta+ay\sin\delta)^2-2ab(bx\cos\delta-ay\sin\delta)+a^2b^2=0.$$

## VIII c.

**3.** $(x^2+y^2)^2=a^2x^2-b^2y^2$. **4.** $(a^2+b^2)x=a(a^2-b^2)(x^2/a^2-y^2/b^2)$.
**5.** The vertex $(a, 0)$. **10.** $(a^2+b^2)\sec^3\theta/a$, $-(a^2+b^2)\tan^3\theta/b$.
**11.** $x\pm y\pm\sqrt{a^2-b^2}=0$.
**12.** (i) $\Sigma t_1t_2=0$, $t_1t_2t_3t_4=-1$. (ii) $\Sigma t_1+\Sigma t_1t_2t_3=0$. **14.** $x^2-y^2=ax\sqrt{2}$.
**15.** $(\frac{1}{3}f, \frac{1}{3}g)$. **20.** $x^2/a^2-y^2/b^2=(a^2+b^2)^2/(a^2-b^2)^2$.
**21.** $ax\operatorname{cosec}\theta-iby\sec\theta=a^2+b^2$. **23.** $(a^2\tan^2\theta+b^2\sec^2\theta)^{\frac{3}{2}}/ab$.
**24.** $b^2x^2-(a^2-b^2)y^2=a^2b^2$.
**26.** $\sin\theta\cos\frac{1}{2}(\theta-\phi)+\sin\frac{1}{2}(\theta+\phi)=0$, $(x^2/a^2-y^2/b^2)^2=x^2/a^2+y^2/b^2$.
**32.** $xy_1-x_1y=0$.

## VIII d.

**3.** $(x^2+y^2)^2=\pm 4c^2(x^2-y^2)$. **8.** The hyperbola. **11.** $x^2/a^2-y^2/b^2=\pm 1$.
**12.** $ax-by=0$. **14.** The centroid of $LMN$ is $P$.
**27.** $a^2b^2(x^2+y^2)=(a^2+b^2)(b^2xx_1+a^2yy_1)$; $4a^2x^2-4b^2y^2=(a^2+b^2)^2$. When the hyperbola is equilateral the centre of the circle is the point $P$.
**32.** $\lambda^2\sin\theta\cos\theta=f$, where $(x\sin\theta+y\cos\theta)^2=4f\sin\theta\cos\theta$.

## VIII e.

**1.** $3x^2(7\pm 3\sqrt{5})-4y^2(2\pm\sqrt{5})=12$. **2.** $x^2\sec^2\alpha-y^2\operatorname{cosec}^2\alpha=a^2-b^2$.
**3.** $x^2/a^4+y^2/b^4=1/(a^2+b^2)$. **6.** Circle. **10.** $\sqrt{\lambda_1-\lambda_2}$.
**23.** $x^2-y^2-2xy\cot 2\alpha=c^2$.

## Miscellaneous, p. 371.

1. A circle touching the given circles at the point of contact.
2. $ly^2-mxy+nx=0$. (a) $x=0$. (b) $ly^2+nx=0$. (c) $y=0$.
3. (i) $3x^2+4y^2-4ax=0$. (ii) $y^2=8ax$. 5. The line $BD$.
10. $(1+m_1^2)(m_2-m_3):(1+m_2^2)(m_3-m_1):(1+m_3^2)(m_1-m_2)$.
11. If the fixed point is the origin, $Z^2(a+b)-2gX-2fY+c(X^2+Y^2)=0$.
13. $(y-1)(4x+y-8)+\lambda(x+y-2)(5x+4y+1)$; when $\lambda=4$ the conic is an hyperbola.
14. $2x^2-8y^2+3xy-5x+20y-12=0$.
15. $x^2+y^2+xy-(a+b)(x+y)+ab=0$; $x-y=0$; $3x+3y=2a+2b$.
22. $\tan^{-1}\frac{1}{2}e^2(1-e^2)^{-\frac{1}{2}}$.
34. $x^2+y^2-\cos\theta\,(2a^2+a^2\sin^2\theta+b^2\cos^2\theta)\,x/a$
$-\sin\theta\,(2b^2+a^2\sin^2\theta+b^2\cos^2\theta)\,y/b+(a^2+b^2)=0$.
36. $3xb\sin\theta+3ya\cos\theta=2ab\sin\theta\cos\theta$.
39. $(\alpha/x+\beta/y)(x^2/a^2+y^2/b^2)+1=0$.
41. $\sqrt{\frac{2}{3}}$; foci lie on the bisector of the angle between the sides.
43. $9(a^2+b^2)/2$. 45. $b^2x^2+a^2y^2=a^2b^2e^2$. 47. $x^2/a^2+y^2/b^2=1/2$.
49. $(x^2/a^2+y^2/b^2-1)(\cos\overline{\alpha-\gamma}-\cos\beta\cos\delta)$
$+\{x\cos\alpha/a+y\sin\alpha/b-\cos\beta\}\{x\cos\gamma/a+y\sin\gamma/b-\cos\delta\}=0$.
50. (iii) Max. $a^2(x^2/a^2+y^2/b^2-1)$, min. when $OPQ$ a tangent.
Min. $b^2(x^2/a^2+y^2/b^2-1)$, max. when $OPQ$ a tangent.
52. $ax-by=0$. 54. $a^2xy_1+b^2yx_1=(a^2+b^2)x_1y_1$.
55. $ax^2+by^2=a/a'+b/b'$.
56. A common chord of the ellipse and circle is a diameter of the ellipse.
65. $x^2+2hxy-y^2=2fy$ (rectangular coordinates).

## IX.

7. Two through each focus; $\theta=\cos^{-1}\sqrt{11/15}$.
8. $\frac{1}{2}l\sqrt{2}$. 11. $l^2(r_1^2-2r_1r_2\cos\alpha+r_2^2)-4lr_1r_2(r_1+r_2)\sin^2\frac{1}{2}\alpha+4r_1^2r_2^2\sin^4\frac{1}{2}\alpha=0$.
29. $r^2(1-e^2)+2ler\cos\theta-l^2=0$.
36. $cc'(1-e^2)^2+2le(c+c')(1-e^2)+l^2(1+e^2)=0$. 39. A circle.
44. $l\sec^2\frac{1}{2}\delta=r+re\sec^2\frac{1}{2}\delta\cos\theta$, i.e. a confocal conic.
46. $F$ is the focus of the conic, which is a parabola, ellipse, or hyperbola according as the distance of the $B$ line from $F$ is $=$, $<$, or $>$ the radius.
47. It touches at the point $\alpha$. 48. $r^2-2rle\cos\theta+l^2(e^2-1)=0$.
49. $c^2=rr'\cos(\theta-\theta')$; $r^2-2rce\cos\theta+c^2(e^2-1)$.

## X a.

4. $l_1l_2+m_1m_2=0$. 5. $\tan^{-1}(-l_1/m_1)$.
6. $\tan^{-1}(l_1m_2-l_2m_1)/(l_1l_2+m_1m_2)$.
7. (a) A point at infinity on $kx-hy=0$. (b) A line through the origin.
8. $-\cos\alpha/p$, $-\sin\alpha/p$. 10. $l_1m_2+l_2m_1=0$.

## X b.

2. $(l_1a+m_1b+1)/\sqrt{l_1^2+m_1^2}$.
3. $\tan^{-1}(l_1m_2 \sim l_2m_1)\sin\omega/\{l_1l_2+m_1m_2-(l_1m_2+l_2m_1)\cos\omega\}$.
6. $s_1(lx_2+my_2+nz_2)+s_2(lx_1+my_1+nz_1)=0$.
8. $ax+by+1=0$, $a'x-by+1=0$. 9. $\{b-b',\ a'-a,\ ab'-a'b\}$.
10. $(b_1-a_1,\ 0)$. 11. $(40:27:19)$.
12. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \div 2c_1c_2c_3$.
13. $\{-l_1/al_1+bm_1),\ -m_1/(al_1+bm_1)\}$. 14. $\{m_1/(bl_1-am_1),\ l_1/(am_1-bl_1)\}$.
15. $l_1(b-b')=m_1(a-a')$. 16. $x^2+y^2-x(a+a')-y(b+b')+aa'+bb'=0$.
17. $(2a-A)l+(2b-B)m+1=0$.
18. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.
19. $c^2(l^2+m^2)=1$. 20. $(1\frac{3}{4},\ -1)$, $(3\frac{8}{5},\ 6\frac{4}{5})$.
21. $\{l_1-l_2+b(l_1m_2-l_2m_1)\}/\{a(l_1-l_2)+b(m_1-m_2)\}$;
$\{m_1-m_2+a(l_2m_1-l_1m_2)\}/\{a(l_1-l_2)+b(m_1-m_2)\}$.
22. $l^2+m^2=a^2$, $l^2/a^2+m^2/b^2=1$, $m^2=al$.
23. $l+m+c=0$, $l-m+c=0$, $al\pm bm+c=0$, $clm\pm l+m=0$.
24. $l(a_1+a_2+a_3)+m(b_1+b_2+b_3)+3=0$. 25. $am^2-l=0$.
26. $l(a_1-\frac{2}{3}a_2)+m(b_1-2b_2)-1=0$, $l(a_1+2a_2)+m(b_1+2b_2)+3=0$.

## X c.

1. $-\cos\frac{1}{2}(\theta+\phi)/r\cos\frac{1}{2}(\theta-\phi)$; $-\sin\frac{1}{2}(\theta+\phi)/r\cos\frac{1}{2}(\theta-\phi)$.
5. $l\Sigma a+m\Sigma b+3=0$, $c/3$. 6. $a^2+b^2=r^2$. 7. $1/2a$, $\pm 1/2a$.
8. (2, 1); 2. 10. $r^2s^2(p-q)^2(l^2+m^2)=[(ps^2-qr^2)l+s^2-r^2]^2$.
12. $a^2(l^2+m^2)=1$, circle. 13. The points are $(0,\ \pm c)$, the centre $(\mu,\ 0)$.
18. Centre $(c,\ 0)$, radius $c$, common point at the origin.
19. $a^2(l^2+m^2)=1$ (see Question 18).

## X d.

1. $a^2b^2(l^2+m^2)=a^2+b^2$; circle.
2. $(1+4al)\tan^2\alpha=16a(am^2-l)$; the point $(4a,\ 0)$. 3. $q^2+2ap=0$.
4. Circle. 6. A parabola. 7. A confocal parabola.
8. $(ap+bq+1)(l^2+m^2)=(al+bm+1)(pl+qm)$;
$p/(pa+qb+2)$, $q/(pa+qb+2)$.
10. $-t^2/a(1+2t^2)$, $t^3/a(1+2t^2)$.
11. A parabola whose focus is midway between the centres of the circles.
12. A parabola touching the given lines. 13. $am^2+l=0$.
14. $a^2lm(\mu l-\lambda m)+l^2+m^2-\lambda l-\mu m=0$.
16. (i) $c^2lm=(al+1)(bm+1)$, a conic touching $OX$, $OY$.
(ii) $(a+b+c)lm+l+m=0$.
(iii) $(a-b+c)lm-l+m=0$, which are parabolas touching $OX$, $OY$.
17. $3a(l^2+m^2)+l(al-1)=0$, $(-a,\ 0)$.

18. $a^4l^2+b^4m^2=a^2$, $\{\pm\sqrt{a^4-b^4}/a, 0\}$.

19. $l_1m_1(a^4l^2-b^4m^2)-a^2b^2lm(l_1^2-m_1^2)=l_1m_1(a^2-b^2)$,
$\{b^2m_1, -a^2l_1, 0\}$ $\{b^2l_1, a^2m_1, 0\}$.

20. $(h, 0)$; $(0, 0)$; $(r^2-h^2)(l^2+m^2)=4(hl+1)$.

22. $(4a, 0)$ if the focus is the origin. 23. $al_1l_2+bm_1m_2=0$.

27. Coaxal conics. 28. $y^2+4ax+8a^2=0$. 29. A point.

32. $(4a, 0)$.

## XI.

3. $(a^2x^2-b^2y^2)(b^2x^2-a^2y^2)=a^2b^2(a^2-b^2)(x^2-y^2)$.

5. $(a-b)(gx-fy)+2h(fx+gy)=0$; $(f^2+g^2)^{\frac{3}{2}}\div(af^2+bg^2-2fgh)$.

6. A conic. 13. $x=0$, $x+3y=0$.

18. $2(ah'-a'h)x+(ab'-a'b)y+2(af'-a'f)=0$.
$(af'-a'f)x^2+2(hf'-h'f)xy+(bf'-b'f)y^2=0$.

21. $x^2/a^2+y^2/b^2=1$, ellipses. 23. $(0, 0)$, $(4\frac{1}{8}, 1\frac{3}{8})$.

24. The hyperbola referred to the normal and tangent at the point is $ax^2-2bxy+by^2=2g$. The vertex of the parabola is $(\frac{1}{8}b, -\frac{3}{8}b)$.

26. $x^2+xy+y^2-(a+b+c)(x+y)+(ab+bc+ca)=0$.

28. $(ah'-a'h)x^2+(a'b-ab')xy+(hb'-bh')y^2=0$.

30. $4(x^2+2xy+2y^2-2x-2y)+k(x^2+y^2-2x)=0$ where $k^2+13k+20=0$.

32. $U\equiv x^2/a^2+y^2/b^2-8$; $U^3+36V^2=0$. 34. $(hx+by)^2-(ab-h^2)y^2=2hy$.

37. The tangential equation is $(al+bm+1)(ai+bm-1)+\lambda(l^2+m^2)=0$.

39. $fG=gF$. 40. $2ix-2ay+f=0$.

42. $2ab$; $\{c/(1+\cos\omega), c/(1+\cos\omega)\}$, $2c^2/(1+\cos\omega)$ where $a+b=\frac{1}{2}c$.

44. $LM'+L'M=2NN'$;
$\{(1/a-1/a')x-(1/b-1/b')y\}^2$
$+k^2(1/a'-1/a)(1/b'-1/b)(x/a+y/b'-1)(x/a'+y/b-1)=0$.

45. $4L^2+M^2-4R^2=0$.

48. $\sqrt{2(d^2+\alpha+\beta^2)}$, $\sqrt{2(d^2+\alpha^2-\beta^2)}$; $(\alpha, \beta)$, $(-\alpha, -\beta)$.
Note that $ab-h^2=-4c$ and use the tangential equation.

51. $\Delta_1 CS-\Delta C_1S_1=0$.

53. See p. 428. The conjugate diameter is a parallel to the polar of $(x'y')$ through the centre.

56. $8a\sqrt{10}/25$. A parabola $ABCDE$ touching $x=a$ at $B(a, 3a)$, $x=y$ at $C(\frac{1}{2}a, \frac{1}{2}a)$, $x+y=0$ at $D(-a, a)$. $AB(++-)$, $BC(+-+)$, $CD(+--)$, $DE(+++-)$. Axis, $15x+5y-6a=0$.

59. (i) $(a-a')f^2-2fg(h-h')+(b-b')g^2=0$. (ii) $(h-h')^2=(a-a')(b-b')$.

60. $h^2(x^2-a^2)+(h^2-a^2)y^2=0$, where $x^2+y^2=a^2$ is the circle and $x=h$ the fixed line.

64. $(A_1C_1-B_1^2)x^2-(A_1C_2+A_2C_1-2B_1B_2)xy+(A_2C_2-B_2^2)y^2=0$, where $A_1$, $B_1$... are the minors of $a_1$, $b_1$... in $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. No.

67. $(ay\pm\sqrt{bc}x)^2-2abcx-a^2(b+c)y+a^2bc=0$.

68. (i) Straight line. (ii) Circle. (iii) Circle. (iv) Ellipse and hyperbola. $p=\tan^2\theta$, where $2\theta$ is the angle between the lines, $\frac{1}{2}q=$ distance of their point of intersection from the point midway between the poles.

## XII a.

1. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$. 2. $(-4, 6, 0)$, $(1, 6, -3)$. 3. $(0, 1, 4)$.
4. (i) $(1, 8, -3)$. (ii) $(1\frac{1}{3}, -\frac{1}{12}, -\frac{1}{4})$. 5. $2:3:4$. 6. $18:2$.
7. (i) $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. (ii) $(\frac{1}{4}, \frac{3}{8}, \frac{3}{8})$.
8. $\{\frac{1}{3}(2x_1+x_2), \frac{1}{3}(2y_1+y_2), \frac{1}{3}(2z_1+z_2)\}$, $\{\frac{1}{3}(x_1+2x_2), \frac{1}{3}(y_1+2y_2), \frac{1}{3}(z_1+2z_2)\}$.
9. (i) $(1:1:1)$, $(a:b:c)$. (ii) $(-1:1:1)$, $(-a:b:c)$.
(iii) $(\cos A:\cos B:\cos C)$, $(\sin 2A:\sin 2B:\sin 2C)$.
(iv) $(\sec A:\sec B:\sec C)$, $(\tan A:\tan B:\tan C)$.
10. $(0:b:c)$, $(0:b:-c)$, $(0:1:1)$, $(0:1:-1)$.

## XII b.

1. $26x+9y-5z=0$. 2. $cy-bz=0$, $az-cx=0$, $bx-ay=0$.
3. $a\alpha-b\beta=0$, $b\beta-c\gamma=0$, $c\gamma-a\alpha=0$. $y-z=0$, $z-x=0$, $x-y=0$.
4. $my+nz=0$. 5. $l(m'y+n'z)=l'(my+nz)$.
6. $\beta\pm\gamma=0$, $\gamma\pm\alpha=0$, $\alpha\pm\beta=0$. $cy\pm bz=0$, $az\pm cx=0$, $bx\pm ay=0$.

## XII c.

1. $2S\cos A\cos B\cos C$. 2. $8S^2Rlmn \div (bn+cm)(an+cl)(am+bl)$.
3. $x\cos A\sin(B-C)+y\cos B\sin(C-A)+z\cos C\sin(A-B)=0$.
4. $\alpha+\beta-\gamma=0$.
5. $x-(m-1)y-(n-1)z=0$; $\{0, (n-1)/(n-m), (m-1)/(m-n)\}$.
6. $Rr\Sigma\sin\frac{1}{2}\overline{B-C}\operatorname{cosec}\frac{1}{2}A \div \sqrt{\{R^2-2Rr\}}$. 7. $16RS^2 \div abc$.
9. $(p\pm d)x+(q\pm d)y+(r\pm d)z=0$.
10. $\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}=0$. 12. $\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}=0$, $\Sigma(q_1-r_1)(q_2-r_2)\cot A=0$.
13. (i) Mid-point of $BC$. (ii) 'Point at infinity' on $BC$.
(iii) Foot of perpendicular from $A$ to $BC$. (iv) Centroid.
15. $\alpha\cos A-\beta\cos B-\gamma\cos C=0$. 16. $x\cos A=y\cos B$.
17. $3(px+qy+rz)=(p+q+r)(x+y+z)$; $y+z=0$.
18. $R\sqrt{(1-8\cos A\cos B\cos C)}$.
21. (i) $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ 1 & -\cos C & -\cos B \end{vmatrix}=0$.
(ii) $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ l-m\cos C-n\cos B & m-n\cos A-l\cos C & n-l\cos B-m\cos A \end{vmatrix}=0$.
24. $x\sin(B-C)+(y-z)\sin A=\pm\lambda x$. $\alpha\sin(B-C)+\beta\sin B-\gamma\sin C=\pm\mu\alpha$.
26. $x\cot A\pm y\cot B\pm z\cot C=0$.
27. $x(\cot\frac{1}{2}B+\cot\frac{1}{2}C)-y\cot\frac{1}{2}B-z\cot\frac{1}{2}C=0$, &c.
Points lie on $x\cot\frac{1}{2}A+y\cot\frac{1}{2}B+z\cot\frac{1}{2}C=0$.
29. $\alpha=0$, $\beta\cos B-\gamma\cos C=0$.
31. $\beta\beta_1=\gamma\gamma_1$, $\gamma\gamma_1=\alpha\alpha_1$, $\alpha\alpha_1=\beta\beta_1$; $(a:b:c)$.
35. $(0:n^2:-m^2)$, $(n^2:0:-l^2)$, $(m^2:-l^2:0)$; $l^2x+m^2y+n^2z=0$.

## XII d.

1. $a^2yz+b^2zx+c^2xy=bcx(x+y+z)$. 4. $\alpha\cos A-\gamma\cos C=0$.
5. $\sqrt{\{abc/(b+c-a)\}}$, &c. 6. $y\cot B-z\cot C=0$.
8. $\alpha\beta\gamma+b\gamma\alpha+c\alpha\beta+(\alpha+\beta+\gamma)(a\alpha+b\beta+c\gamma)=0$.
9. $a^2/(y+z-x)+b^2/(z+x-y)+c^2/(x+y-z)=0$.
10. $x/a^2=y/(4c^2-a^2)+z/(4b^2-a^2)$. 14. $(l+m)c^2-mb^2:mb^2$.
15. $\Sigma yz(\cos B-\cos C)=0$. 16. $\{ap, bq, cr\}^2=a^2p^2$, &c.; $x/a+y/b+z/c=0$.

## XII e.

2. A conic; $x^2+y^2+z^2-2yz-2zx-2xy=0$. 5. $\sqrt{p_1x(y+z-x)}+\ +=0$.
9. $X^2+Y^2+4XY\operatorname{cosec}2B=cX+bY$; $2\sqrt{2}\cos 2B/\sqrt{4-\sin^2 2B}$.
10. $p_1(q_1^2-r_1^2):q_1(r_1^2-p_1^2):r_1(p_1^2-q_1^2)$.

## XII f.

2. $\cos\frac{1}{2}A\sqrt{-\alpha}+\sin\frac{1}{2}B\sqrt{\beta}+\sin\frac{1}{2}C\sqrt{\gamma}=0$. 4. $fx+gy+hz=0$. 5. $(f:g:h)$.
7. $\Sigma a^2f^2+2\Sigma bcgh\cos A=0$.
11. If $PA$ and the required harmonic conjugate meet $BC$ in $T$, $T'$, then the pencil $A\{BC, TT'\}$ is harmonic. The envelope is $2p_1/p-q_1/q-r_1/r=0$.
12. If $(x_1, y_1, z_1)$ is the fixed point, the envelope is $\sqrt{px_1}+\sqrt{qy_1}+\sqrt{rz_1}=0$, or $x_1/x+y_1/y+z_1/z=0$.

## XII g.

2. $\Sigma lx(my'-nz')=0$. 3. $(lx+my)^2+n(l+m)z^2=0$.
4. $\{mn(m-n):nl(n-l):lm(l-m)\}$. 5. $x'^2/x+y'^2/y+z'^2/z=0$.
7. $3x+8y-7z=0$; $x+16y+21z=0$.
8. A straight line, $l^2x+m^2y+n^2z=0$ where $(l, m, n)$ is the fixed line.
9. An inscribed conic. 10. (a) 1. (b) 2. 11. (a) 1. (b) 2.
13. A conic inscribed in the diagonal triangle of the quadrilateral.
14. A conic circumscribing the diagonal triangle, it passes through the six mid-points of the sides of the quadrangle.
16. If the sides of the quadrilateral are $(\pm p_1, \pm q_1, \pm r_1)$ and $(l:m:n)$ is the point at infinity on the parallel tangent, the locus is

$$p_1^2x/(mz-ny)+q_1^2y/(nx-lz)+r_1^2z/(ly-mx)=0.$$

18. $x\pm y\pm z=0$.
19. If the conics are $x^2+y^2+z^2=0$, $lx^2+my^2+nz^2=0$; $Q(x_1, y_1, z_1)$, $R(x_2, y_2, z_2)$, then the tangents at $Q$, $R$ intersect on $AB$ (viz. $\sqrt{l-m}\,y=\sqrt{n-l}\,z$) if

$$(l-m)(lx_2z_1-nx_1z_2)^2=(n-l)(mx_1y_2-lx_2y_1)^2;$$

which, since $x_1^2+y_1^2+z_1^2=0$ and $lx_2^2+my_2^2+nz_2^2=0$, gives

$$n(l-m)(x_2z_1-x_1z_2)^2=m(n-l)(x_1y_2-x_2y_1)^2.$$

Hence $QR$ passes through one of the points $\{0:\sqrt{m(n-l)}:\pm\sqrt{n(l-m)}\}$.
20. $x_1x_2/l^2=y_1y_2/m^2=z_1z_2/n^2$. 21. $p_1p_2/l^2=q_1q_2/m^2=r_1r_2/n^2$. 22. 4.

## XII h.

1. $xy+yz+zx=0$.
4. $p\sin 2A+q\sin 2B+r\sin 2C=0$, $ap+bq+cr=0$, $ap\pm bq\pm cr=0$.
$p\tan A+q\tan B+r\tan C=0$, $p+q+r=0$, $pa^2+qb^2+rc^2=0$.
5. (i) $12x^2+4y^2-3z^2=0$. (ii) Parabola.
(iii) $x^2bc\cos A+a^2yz=0$.

6. $ys+zx+xy=0$. 7. $(\sin 2A:\sin 2B:\sin 2C)$.

8. $x^2(c^2g^2-b^2h^2)+y^2(a^2h^2-c^2f^2)+z^2(b^2f^2-a^2g^2)=0$.

11. $a^2/(s-a)+b^2/(s-b)+c^2/(s-c)=0$. 13. $s^2=ys+zx+8xy$.

14. $\beta\gamma/l+\gamma\alpha/m+\alpha\beta/n=0$.

21. $\{2b^2-c^2:2a^2-c^2:2a^2+2b^2\}$, $(2b^2-c^2)x+(2a^2-c^2)y-(a^2+b^2)z=0$.

$(8a^2+b^2-c^2)x-(a^2+3b^2-c^2)y+(a^2-b^2)z=0$. $4S^2/(2a^2+2b^2-c^2)^{\frac{1}{2}}$.

22. $\Sigma a\alpha\{b^2(r-p)+c^2(p-q)\}/(q-r)=0$.

32. $(x+y+z)\Sigma\{(y_1y_2-z_1z_2)(b^2-c^2)+a^2(y_1z_2+y_2z_1)\}x+a^2yz+b^2zx+c^2xy=0$.

34. $(px+qy+rz)(pa^2yz+qb^2zx+rc^2xy)=4S^2xyz$. 40. $(2a^2:2b^2:-c^2)$.

44. $x\cot A\pm y\cot B\pm z\cot C=0$.

47. The directrices $(SA=e.p)$ are $\alpha\cos\frac{1}{2}A\pm\beta\cos\frac{1}{2}B\pm\gamma\cos\frac{1}{2}C=0$, then (§ 12) the conics are $l\alpha^2+m\beta^2+n\gamma^2=0$, where $l\cos^2\frac{1}{2}A+m\cos^2\frac{1}{2}B+n\cos^2\frac{1}{2}C=0$.

50. The line through $C$ parallel to $AB$ is a diameter.

54. If the conics are $S_1$, $S_2$, then $2S_1+S_2$ factorizes, and one factor is $\Sigma x/(m-n)$; the corresponding line touches $S_1$ and therefore $S_2$.

56. $4abc\Sigma ax^2\cos B\cos C+(a^2+b^2+c^2)\Sigma a^2yz=0$.

63. The centre lies on the line bisecting $BC$ at right angles.

65. $\Sigma p^2/ll'-\Sigma pq(1/lm'+1/l'm)=0$.

68. The other conic is $(v+w)yz+(w+u)zx+(u+v)xy=0$, which can be written $\Sigma ux(y+z)=0$, so that at a common point $\Sigma ux'=0$, whence, comparing the equations of tangents, the result follows.

70. $(a^2:c^2:b^2)$. 72. $\lambda^2(vw'+v'w)+\mu^2(uw'+u'w)+\nu^2(vu'+v'u)=0$.

73. Use $lqr+mrp+npq=k(p\alpha+q\beta+r\gamma)$ for conic.

74. $a^3/4S$, $b^3/4S$, $c^3/4S$.

77. $\{f^2/x_1:g^2/y_1:h^2/z_1\}$, where $(\pm f,\ \pm g,\ \pm h)$ are the vertices of the quadrangle.

78. Since $x+y+z=1$, the terms of second degree, when $C$ is transformed to Cartesians, depend only on the terms $a^2yz+b^2zx+c^2xy$. Thus if $a^2yz+b^2zx+c^2xy$ transforms into $\lambda(X^2+Y^2-R^2)$, then $C$ transforms into

$$\lambda\{(X-\alpha)^2+(Y-\beta)^2-\rho^2\}.$$

Now apply $\Delta=\Delta'/2S^2$ to both pairs of equations.

82. $x\cos\frac{1}{2}(\theta+\phi)+y\sin\frac{1}{2}(\theta+\phi)=\cos\frac{1}{2}(\theta-\phi)$.

84. $(a\alpha+b\beta+c\gamma)^2-4bc\beta\gamma=0$.

86. Focal chords are proportional to the squares of parallel diameters.

89. $2Rx_1y_1z_1\Sigma a^2p^2+abc\Sigma(y_1\cot B+z_1\cot C-2x_1\cot A)x_1^2p^2=0$.

90. $SA=2R\cos A$, &c. $\therefore$ $\Sigma x\cos A=0$ is the directrix, and this is the polar of the focus.

91. $\alpha^2=\beta\gamma$.

92. The tangent at the vertex is parallel to the polar of (1, 1, 1), i.e. $ux+vy+wz=0$. Its coordinates are $(u+k, v+k, w+k)$, which satisfy

$$p^2/u+q^2/v+r^2/w=0,$$

and the vertex, i.e. the point of contact, is $\{1+k/u:1+k/v:1+k/w\}$

93. $(V+W-2F)/a^2=(W+U-2G)/b^2=(U+V-2H)/c^2$.

94. See Ex. ii, p. 515.

97. The circle of curvature at $A$ circumscribes the quadrilateral formed by the tangent at $A$, the line at infinity, and the pair of lines $A\Omega$, $A\Omega'$. Hence its equation is of the form $C\equiv(a\alpha+b\beta+c\gamma)(m\gamma+n\beta)+\lambda(\beta^2+\gamma^2+2\beta\gamma\cos A)$, and $kS-C$ has $m\gamma+n\beta$ for a factor, where $k$ is clearly unity.

103. See Exx. 6 and 97.

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