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CHAPTER XVI

RHEOLOGICAL ANALYSIS

1. Rheological Analysis of a Rubber Solution

In the last chapter we have discussed the different mechanisms which may cause structural viscosity. We have found that, from the shape of the consistency curve and the influence of a change of the parameter of temperature, conclusions can be drawn regarding the nature and structure of the particles dispersed in the liquid. This may be called Rheological Analysis.

Rheological analysis should, however, be understood in a wider sense to mean inference from rheological measurements upon the structure of a quasi-homogeneous dispersed system. Rheological literature contains many examples of rheological analysis. In the present chapter I shall describe, as an instance, the investigations on solutions of crêpe-rubber in toluene, which I carried out at Prof. Bingham's laboratory at Easton in 1932–33 in collaboration with Mrs. Reiner (1933).*

The viscosity of rubber solutions usually decreases with age. This ageing is due, according to Staudinger (1932), to a depolymerization caused by oxidation, which in turn is advanced by light and heat. For instance, as we found, in a rubber-turpentine solution the viscosity falls in the course of a few days to the viscosity of turpentine. Staudinger, therefore, used in his investigations on solutions of rubber in benzene very rigorous safeguards to exclude oxidation. We, happily, found that these safeguards are not necessary if toluene is taken as a solvent. By keeping the solution in the ice-box in brown bottles in an atmosphere of CO_2, the viscosity remained unchanged for as long as the experiments lasted, i.e. for nearly one year.

The rubber was light crêpe of 0·13 per cent. moisture, 0·32 per cent. ash and a density of 0·92. The solvent was chemically pure toluene of densities as in International Critical Tables 3, 29 and viscosities slightly higher than those determined by Thorpe and Rodger (1894).† The crêpe was cut into small pieces and put in small portions into the toluene in a bottle into which

* In his comprehensive book, Philippoff (1942) says of these investigations, " die vielleicht die genauesten an nicht-Newton'schen Fluessigkeiten sind ". I do not know whether this still holds good, but they will probably still serve well to illustrate the subject of the present chapter. I have, however, omitted a number of data and Tables contained in the first edition of this book.

† Even had the toluene been the same as that for which Thorpe and Rodger determined the viscosities and even if their figures were the correct ones, we had to make independent determinations in order correctly to calculate the *relative* viscosities, thereby eliminating apparatus-effects.

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DEFORMATION, STRAIN AND FLOW

DEFORMATION, STRAIN AND FLOW

AN ELEMENTARY INTRODUCTION TO RHEOLOGY

BY

MARKUS REINER

DR TECHN.

Professor of Applied Mechanics, Israel Institute of Technology
Member of the Israel Academy of the Sciences
Sometime Civil and Structural Engineer, Department of Public Works, Government of Palestine
Sometime Research Professor, Lafayette College, Easton, Pennsylvania

Being a revised and enlarged edition of the same author's book
DEFORMATION AND FLOW

WITH 123 ILLUSTRATIONS (INCLUDING 1 PLATE)
AND 18 TABLES



LONDON
H. K. LEWIS & Co. Ltd.
1960

First Edition . . . 1949
Japanese Edition . 1957
Second Edition . 1960

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H. K. LEWIS & CO. LTD
1960

"The mountains flowed from before the LORD."
Prophetess Deborah, Judges v, 5.

"Everything flows." Heraclitus.

Made and Printed in Great Britain
for H. K. Lewis & Co. Ltd., London
by The Whitefriars Press Ltd., London and Tonbridge

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TO THE

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ISRAEL INSTITUTE OF TECHNOLOGY

WHERE I HAVE BEEN GIVEN FULL OPPORTUNITY

TO PRACTISE RESEARCH

EXTRACT FROM THE PREFACE TO THE FIRST EDITION

In 1943 I published a little book entitled *Ten Lectures on Theoretical Rheology* (Rubin Mass, Jerusalem) of which a second edition under the title *Twelve Lectures* appeared in 1949 (North Holland Publishing Co., Amsterdam) with a French translation in 1955 (Dunot, Paris). As explained in the Preface of that book, it was a *quid pro quo* for a large textbook on the subject, the composition of which I had started immediately following my course of lectures at Princeton University in 1932–33 and completed on the very day Hitler's hordes engulfed Poland. The war prevented its publication and it remained a manuscript. The little book, based upon lectures given to an audience of physicists at the Hebrew University, Jerusalem, in 1937, took its place. That audience determined its character. When, in 1945, I was invited to give a course at the same place before an audience of *chemists*, I took another bunch of leaves out of my manuscript and around them the present book has been composed. It is more elementary and less mathematical than the " Lectures ", as befitted the different audience. It deals almost exclusively with material not covered in the " Lectures " and, where the material is the same, it is treated in a different manner. References to the " Lectures " are given, so that readers of the present book, desiring to proceed from vector concepts used here, to tensor concepts used there, will be well prepared.

There was no attempt to cover the whole field of rheology, which is not possible. To material mainly of interest to the chemist, has been added other material which should interest the structural or mechanical engineer. Being myself a civil engineer by training and profession, I have in general tried to take a catholic view of rheology and to make it broad enough to extend from Bingham's *Fluidity and Plasticity* over Scott Blair's *Industrial Rheology* to Nadai's *Plasticity*. One side of rheology I have disregarded. It concerns the psycho-physical or integral treatment which has found a first-hand exposition in Scott Blair's *Survey of General and Applied Rheology* (1949). There appeared to be no point in saying in other words what every reader, who desires to get a complete picture of the subject, will be best advised to study at the source.

In the composition of the present book I was assisted by the criticisms of my *Ten Lectures* which were either sent in by correspondents or published in reviews. . . . Prof. Eyring (*J. Phys. Chem.*, 44

(1945)) missed the atomic-molecular approach to the problem of flow and expressed his hope that the larger treatise, promised by me later, would enable it to be covered. I am afraid I shall have to disappoint him; for I still consider rheology mainly to belong, together with classical mechanics (including relativity), thermodynamics and Maxwellian electro-dynamics, to the phenomenological side of physics. It treats matter as it appears to our senses, i.e. as continuous. The atomistic, submicroscopic, approach in the kinetic theory of matter, quantum mechanics and the like is admittedly more powerful – it has brought us the atomic bomb—but it has not made the former superfluous. Side by side with the atomic-molecular theory of heat, the continuum theory of thermodynamics developed from its three fundamental laws, persists and proves its usefulness. While the atomic-molecular approach to rheology has had, under the leadership of Prof. Eyring, great success in certain fields, the unsatisfactory state in its relation to the problem of strength, where the atomic theory yields figures, a hundred- and thousand-fold those of experience, is well known. The practice of investigators in material-testing laboratories all over the world still largely follows the methods of phenomenological rheology and scientific journals are full of their reports. The purpose of the present book is to assist them in what they are doing. There certainly is the need for another book, giving to the rheologist all that of kinetic theory of matter, chemical physics, etc., which concerns him in his work and for which I have proposed the term *Metarheology*. This, however, would have been a task widely transcending my powers and instead of giving fragmentary accounts of some of these aspects, I have preferred to leave them out entirely, with the advantage of being able to treat rheology, as I understand it, more fundamentally. However, while I treat matter as continuous, I do not disregard the dispersed character of many, if not most, materials. The units of the dispersed phase may even be molecules. But molecules will be considered here only if they can be treated as ordinary bodies of very small dimensions on the model of Einstein's famous treatment of the viscosity of solutions.

To make the treatment more concrete, I have given numerical examples of the application of theory. They have been taken from experimental investigations carried out by me and collaborators.

I have to apologize for two features of the book to which some readers may take exception. The first is the informal character of the presentation of the matter, i.e. my speaking in the first person, etc. This is the result of the text's originating from notes for my oral lectures. I did not think the stylistic improvement which might have been attained worth the effort of recasting the whole text. The second consists in the many references to the more or less recent history of

theoretical rheology. I had the privilege of taking part in those of Prof. Bingham's activities which resulted in organizing rheology. To the sorrow of all who knew him, Prof. Bingham has only recently (November 6th, 1945) followed the three other rheologists to whom this book is dedicated;* and I myself am of their generation. I hope that it will therefore be understandable that I included some historical facts of which I may be the only one to have first-hand knowledge which may be of some interest to those who should want to trace back the beginnings of a living branch of physics.

* They were Emil Hatschek (1869–1944), Winslow H. Herschel (1873–1944), Elmer O. Kraemer (1898–1943).

PREFACE TO THE SECOND EDITION

For the second edition the text of the first edition has been thoroughly revised and enlarged. Rheology, as was to be expected from a young science of that name, does not stand still but also flows. Even while I was writing the text of the first edition which took a number of years, I myself made some progress and changed my views on several points. It was then impossible for me to re-write the manuscript, and I only drew the attention of the reader to the main points. The revision thus required has now been carried out, also in respect of some other sections in which either I myself or Rheology in general have progressed. I have also added new material.

I want to thank Dr. G. W. Scott Blair who has read the manuscript and Dr. J. Glucklich who has read the proofs. I am also grateful to Professor N. Yanagisawa who has translated the book for the Japanese edition and corrected many mistakes.

I have dedicated this new book to the Institution where I hold the position of Professor since 1948. Before that, I was (with an interlude of two years at Easton, Pa.) earning my living as a practical engineer with science as a hobby. I am therefore grateful to the Institute which has given me the possibility to spend all my time in the pursuit of science.

MARKUS REINER

Haifa, May, 1959.

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DEFORMATION, STRAIN AND FLOW

CHAPTER I

DEFORMATION, STRAIN AND FLOW

1. Rheology and Mechanics

RHEOLOGY is a branch of *physics* ; it is that branch which in the system of physics is closest to *mechanics*. Let us try the following experiment : I let three material bodies fall upon the table before me, namely (i) my pencil, (ii) a ball of plasticine, (iii) a " piece " of water. [The latter, of course, has to be contained in a vessel ; I pour it out from a beaker.] This is a *mechanical* experiment ; and what does mechanics tell us about what is going to happen ? It does not tell us more than is contained in the second law of Newton,

$$P = m.a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

That is, if we know the *force* P acting upon each one of these three pieces of matter and the *mass* m of each one, we can calculate its acceleration, or rather the *acceleration of its centre of mass*, a, from

$$a = P/m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If a *number* of forces act on a body, P is to be taken as the algebraic sum of the components of the forces *in any one direction*, and a is the component of the acceleration in the same direction. If this sum is zero in respect of three different directions in space, a also vanishes and the centre of mass of the body is either at rest or moves with a constant velocity. In addition, however, the body may rotate around its centre of mass and this rotation will be accelerated if the forces possess a *moment* M in respect of the centre of mass. If there is to be no rotation, or the rotation is to be of *constant angular velocity*, the resultant moment must *vanish*. *Equilibrium* accordingly *requires* the satisfaction of the *two* equations

$$\Sigma P = 0, \ \Sigma M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in respect of each one of the three directions in space.

As an example let us consider two numerically equal forces acting on a body in opposite directions. The algebraic sum of their magnitudes vanishes and the first of (3) is therefore satisfied. *If*, however, *their lines of action do not coincide, they will form a couple* which will cause accelerated rotation of the body around its centre of mass. In this case the second of (3) is not satisfied.

In the case of our experiment, because of a curious relationship between P and m by virtue of which one is proportional to the other,

a is the same for these highly different materials.* On any one spot on this earth it is a "constant", the *acceleration of gravity*, and is approximately $g = 981$ cm/sec^2. The three bodies in their fall towards the table therefore behave *in exactly the same manner* and their material differences do not become manifest until they reach the table top—then the pencil slightly rebounds, the plasticene stays put, and the water spreads over the surface of the table and on reaching its edge flows off. *Of these so entirely different manners of behaviour, mechanics cannot give any account and here it is that rheology steps in.*

2. Deformation, Strain and Flow

The differences in behaviour just described are expressed in qualitative language by saying that the pencil is an "elastic solid", the plasticene a "plastic solid" and water a "liquid". If we take up the ball of plasticene and examine its surface, we shall find that where it collided with the table top, it is flattened : it has been permanently *deformed*. If we take a small steel ball and let it fall upon the table, it will rebound elastically. If we examine its surface, we do not notice a flattening. Nevertheless we have good reason to assume that when it collided with the table it *was* flattened ; but that *after the collision* with the table the original spherical shape was restored, and this was what caused it to rebound : the ball, on colliding with the table, was *strained*. We may say that the action of forces causes an *elastic* material to be *strained*, a *plastic* material to be *permanently deformed*, a *liquid* to *flow*. The term "deformation" is the more general one, including all three cases.

3. Isotropic Pressure

We say that steel, plasticine and water have *different rheological properties* and we have described how these become manifest in strain. deformation and flow. But let us consider three *spheres* each one made of one of these three materials and let each sphere be subjected to *pressure, normal to its surface and equal from all sides*. Such a pressure is usually called Hydrostatic Pressure, but a more general and therefore better expression is Isotropic Pressure of which the hydrostatic pressure is only a special case. We may mention that it was B. Pascal (1623-62) who first showed that in water at rest the pressure on a small surface in its interior is the same whatever its orientation, whether horizontal, vertical or sloping, i.e. it is the same in all directions or it is isotropic.

* The "curious relationship" is Einstein's Equivalence Principle on which the general theory of relativity is based. This principle emphasizes the fact that the weights of bodies and their resistance to acceleration are proportional—each being in direct ratio to the mass of the body. The relativity theory maintains that they are really indistinguishable.

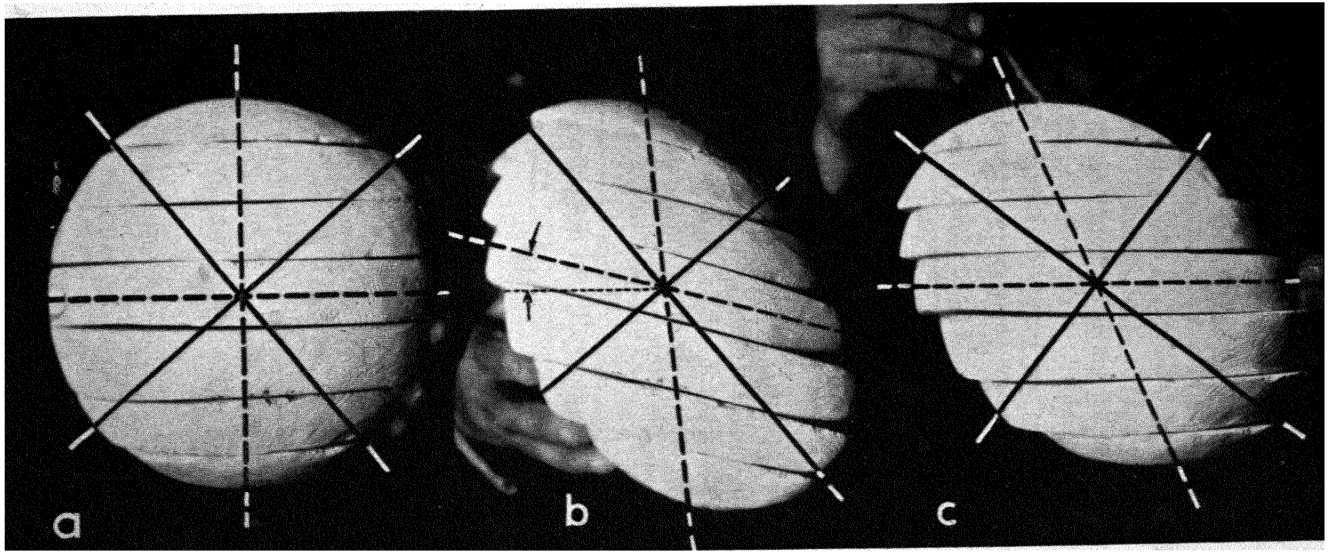


FIG. I, 1. Model showing equivalence of simple shear with extensions plus rotation.

a, unstrained sphere ; *c*, ellipsoid produced by simple shear ; *b*, the same ellipsoid can be produced by extension of one and contraction of the other axis (pure shear).

The full lines indicate the principal axes of deformation which remain at right angles after the deformation. In *pure* shear their directions in space are not altered, in *simple* shear they are rotated through the angle indicated.

Now, what will be the effect of such isotropic pressure on each one of our three spheres ? The answer is that it will be *the same in every case : the density ρ of the material will be increased.* There will, however, be no distortion or change of form ; the sphere will remain a sphere, although with slightly decreased diameter. And when the pressure is released (provided it was not too great) the previous diameter and density will be restored. There will be no permanent or plastic deformation of the sphere of plasticine ; there will be no flow of the sphere of water. This has been generalized to what may be called the first axiom of rheology : *under isotropic pressure all materials behave in the same way* ; they are purely and simply *elastic.**

What forces then will produce the striking rheological differences ?

4. Simple Shear

I have here before me a model of a sphere made of plaster of Paris and cut into slices. Through appropriate application of forces with my hands I produce a deformation which changes the sphere into a three-axial ellipsoid (Fig. I, 1). Every slice is slightly shifted parallel to the same plane. It is clear that this does not change the volume of the sphere and therefore also not the density of the material of which it is composed. Such a distortion of shape is called Simple Shear. In general, if we speak of an elastic strain or plastic deformation or liquid flow which is not accompanied by a change of volume, we shall use the term Distortion.

In simple shear, or, more generally, in distortion, steel, plasticine and water will behave very differently.

5. Elastic Shear of a Prism

This difference in behaviour can be expressed better mathematically if we consider instead of the spheres three small *prisms.* Let us assume that the materials making up the prisms are devoid of weight.† Then, even the prism of water will not change its form if no forces act upon its sides.‡

Let us first consider a prism of steel, resting upon the table, with a force applied to the upper horizontal surface as shown in Fig. I, 2. For obvious reasons such a force is called Tangential Force, which we

* This should be understood as a good enough approximation to which there are exceptions (compare Chapter XII).

† This can be realized, at least in imagination, if our laboratory is installed in a lift the cord of which has snapped so that the lift falls down freely. In such a laboratory the bodies have no weight. Unfortunately the experiments would soon come to an end.

‡ Even if devoid of weight, a prism of water would change its form under the action of *surface tension.* Rheology, however, will ordinarily not have to consider surface tension. The forces with which it has to deal are of such magnitude and the dimensions of the bodies such that surface tension is usually of secondary importance.

indicate by the index " t " in P_t. The force P_t will cause a shifting of the upper surface against the lower by, say, U. The length u is called Displacement. It is clear that, other things being equal, u will be greater the greater P_t, but smaller the greater the area A, and it will also be greater the greater the height H. Therefore

$$U \propto \frac{P_t}{A} H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Introducing a " coefficient " μ we can write

$$\gamma = \frac{U}{H} = \frac{P_t}{A} / \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

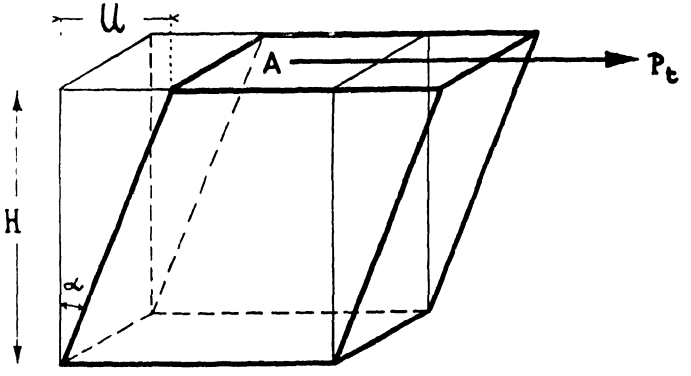


FIG. I, 2. Simple shear of a prism.
U displacement. $U/H = tg\alpha$ gradient. P_t tangential force.

The expression U/H is a measure of the shear of the prism ; it is $= tg\alpha$ or if α is small $= \alpha$, which is the reduction of the originally right angle between base and sides of the prism. The ratio

$$\gamma = U/H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

is called Displacement Gradient. The expression

$$\tau = P_t/A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

called Shearing or Tangential Traction.* Equ. (5) can accordingly be written in the form

$$\tau = \mu\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

* The term " traction " for force per unit area is used by Love (*Theory of Elasticity*). Generally, and especially in technical literature, no distinction is made between " traction " and " stress " and we also do not propose to be too rigorous in the use of the term " stress ". In the correct use of the term the stress at a point is the aggregate of all tractions acting on all surface-elements of different orientations which contain that point. *The stress is therefore a tensor, the traction a vector.* (Compare *Twelve Lectures*, pp. 25–29.) For instance, hydrostatic pressure has *no definite direction* and, therefore, is not a vector; it is a tensor.

The coefficient μ is called Modulus of Rigidity.* If μ is very great, which may be indicated by $\mu \to \infty$, U will be very small or $U \to 0$ and there will be no strain. This explains the denomination of μ; the greater μ, the more rigid the material. A body which under a shearing force does not change its form at all is called a Rigid Body. For this $\mu = \infty$. Assuming that μ is a constant, (8) expresses Hooke's law of the proportionality of stress and strain in one of its forms.

This law Hooke, in 1678, pronounced in the words *ut tensio sic vis*. In modern language : "The distortion of a body is proportional to the shearing stress producing it." In the following table some numerical values of moduli of rigidity are listed :

TABLE I, 1. *Moduli of Rigidity*

Material	Silk	Lead	Tin	Glass	Aluminium quartz silver	Zinc Copper	Wrought iron Platinum	Nickel Steel
$\mu \times 10^{-11}$ in bars	0·1	0·2	1·5	2	3	4	7	8

The bar is defined in Section 9 below.

6. Rotational Equilibrium of the Prism

We have derived the equations of the preceding section assuming that the prism, after the displacement u has taken place, is in equilibrium. If we look at Fig. I, 2, we see, of course, that under the action of the force P_t, and no other force in addition, the prism will not be in equilibrium. Let us correct this blemish as in Fig. I, 3.

If the prism stands on the table, what must the reaction from the table be, in order that equilibrium be maintained ? In accordance with (3), the first condition is that the resultant of all forces acting in the horizontal direction vanishes. This requires the existence of a tangential force $-P_t$ acting on the base, as shown in the figure. Such a force may result from either friction or adhesion between the prism and the table-top. This, however, is not sufficient. Both forces P_t and $-P_t$ form a couple with a moment $M = H.P_t$, which, if not opposed, would cause the prism to overturn clockwise. It must be balanced by another, anti-clockwise, couple formed for instance by a push $-N$ and a pull $+N$ so that $L.N = -M$. From $H.P_t = L.N$ we find $N = H.P_t/L$. If the prism is a small internal part of a larger body, or an Elementary Prism, the sides of which are imaginary interfaces, there will be acting upon every side either a push or a pull passing

* In engineering literature the symbol G is used for μ.

through the centre of the prism and therefore not contributing to a torque around the centre. If, by chance, the position of the prism is where there is a change from push to pull, that change in the interior of the body must be gradual and both are therefore negligibly small. Equilibrium can therefore be maintained only by *tangential* forces such as $+S_t$, $-S_t$, where $S_t = H.P_t/L$. Introducing tractions, we

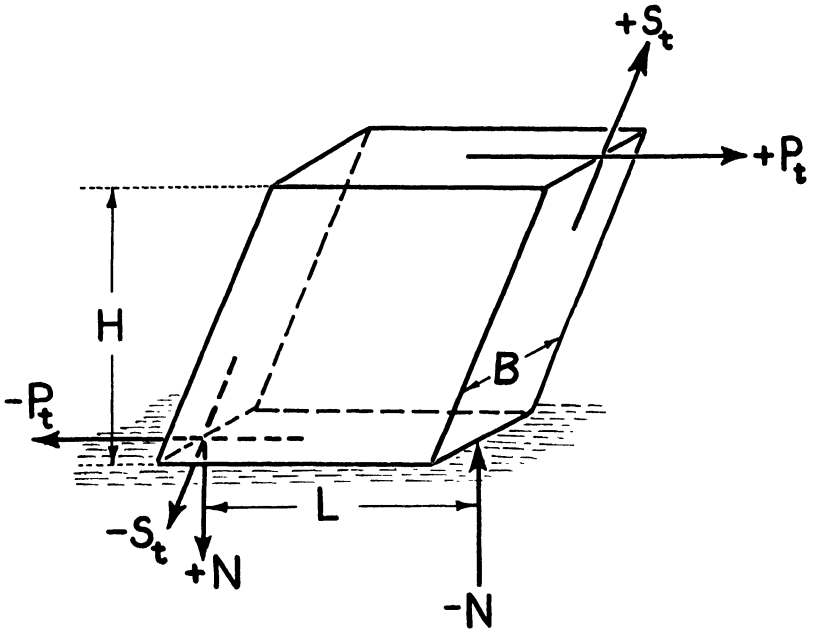


FIG. I, 3. Rotational equilibrium of a prism.

find $\tau = P_t/BL$ and $S_t = H.P_t/BHL = \tau$. This important result is independent of the dimensions of the prism and may be expressed as follows :

Rotational equilibrium in a body requires that *tangential tractions on interfaces at right angles to each other are equal.* (This makes stress a symmetrical tensor. Compare *Twelve Lectures*, p. 37.)

7. Plastic Deformation

Now, what will happen if the prism is made of plasticine ?

For very small stresses it will behave in a fashion not essentially different from that of the prism of steel ; it also will be strained elastically, and even for the same stress to a greater extent. However, τ cannot be increased to such magnitude as it can in the case of steel.

When τ reaches a certain limit ϑ, which is called the Yield Stress, it will remain roughly constant

$$\tau = \vartheta_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)^*$$

whatever the magnitude of U.

The yield stress of metals varies between 3 kg mm^{-2} (aluminium) and 76 kg mm^{-2} (nickel). For steel it is from 23 to 45 kg mm^{-2}.

8. Viscous Flow

Thirdly, we consider the " prism " of water. No measurable stress is required for any displacement if this is carried out very slowly ; the " slices " are shifted against each other with infinite ease. Only when the velocity of the displacement is finite will there be resistance against it. Read U in Fig. I, 2 to indicate in this case the displacement carried out in one second, i.e., the velocity of displacement V, then in analogy to (5)

$$\dot{\gamma} = \frac{V}{H} = \frac{P_t}{A}/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where η is called Coefficient of Viscosity. The quantity V/H is called Velocity Gradient. It is $= (1/H).(dU/dt)$, or, H being taken as constant, $\frac{d}{dt}(U/H) = d\gamma/dt$. Using Newton's dot to indicate differentiation in respect of time, we write in analogy to (8)

$$\tau = \eta\dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If η is a constant, (11) is Newton's law of the proportionality of stress and velocity gradient which he (*Principia* (1687)) expressed as follows :

" Hypothesis : That the resistance which arises from the lack of slipperiness of the parts of the liquid, other things being equal, is proportional to the velocity with which the parts of the liquid are separated from one another."

This hypothesis amounts to the following : Imagine a viscous liquid contained between two plates, one stationary, the other moving with the velocity V. This will cause the liquid to be " sheared " (Fig. I, 4) in the same manner as the prism of Fig. I, 2. The velocity gradient is

$$\dot{\gamma} = V/H = dv/dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and this is meant by " the velocity with which the parts of the liquid are separated from one another ". Let the area of contact of two " parts of the liquid " be unity, then " the resistance which arises from the lack of slipperiness " τ

$$\tau = \eta\dot{\gamma} = \eta\, dv/dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

* Here " t " in ϑ indicates that when equating τ one must take the tangential component of ϑ.

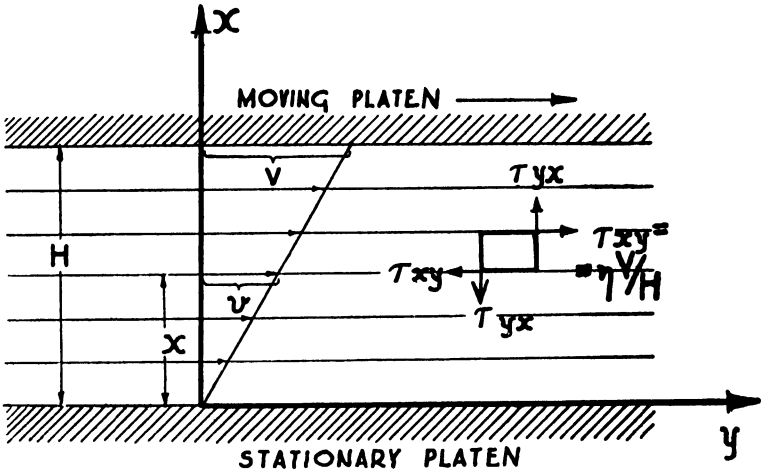


FIG. I, 4. Velocity gradient between two parallel plates.

In the following table some numerical values of coefficients of viscosity at 20° C. are listed : -

TABLE I, 2. *Coefficients of Viscosity*

Material	Toluene	Water	Alcohol	Mercury	Sugar solution 60% concentration	Olive oil	Castor oil
η in cp.	0·6	1	1·2	1·6	57	100	1,000

0·01 poise is one centipoise (cp.) ; the poise is defined in Section 9 below.

9. Dimensions and Units

We have in this way become acquainted with three rheological coefficients, viz. the modulus of rigidity μ, the yield stress ϑ, and the coefficient of viscosity η. From (6) we see that γ being a quotient of length to length, is a dimensionless number. Therefore, from (8), μ is of the same dimension as a stress, i.e. a force per unit area. This can be measured in p.s.i., or in kg per cm^2, or in the " absolute " cm-g-sec system in dynes per cm^2. Its unit in the absolute system, namely *one dyne per cm^2, is called the Bar*. The yield stress, is of course, also of the dimension of stress. The dimension of η is, as is evident from (11), one of stress times time. Its unit in the absolute system = *one bar* $\times$ *sec*. It *is called the Poise*, after *Poiseuille* (1797–1869) who first made exact measurements of viscous resistance.

10. Ideal Bodies

I have before made certain statements about the rheological properties of such materials as steel, plasticine and water, and I may have selected other materials as examples of elastic, plastic and viscous behaviour. I have said that the shape of a steel ball, when it rebounds

elastically, is perfectly restored to its original spherical form. How do I know this ? The answer to this question is expected to be : from experiments and measurements. But what facts are revealed by measurements depends upon their accuracy and if the measurements are made with *very* great accuracy they *will* reveal a slight, hardly noticeable flattening of the steel sphere. This is spoken of as Imperfection of Elasticity. But then the question arises : what material is *perfectly* elastic ? The answer is : *no real material,* as far as we know. This is similarly so with the " infinite ease " with which water can, allegedly, be slowly deformed ; and with the constancy of stress in the deformation of plasticine. What we have described are not properties of *real* materials. *They are properties of abstract concepts which we have formed ourselves in order to describe the behaviour of real materials.* To make this quite clear, we shall call these ideal materials after the scientists who first introduced them and shall speak of the Hooke Solid, the St. Venant Solid and the Newtonian Liquid honouring the memory of Hooke (1635–1703) who first expressed the law of perfect elasticity, St. Venant (1797–1886) who first described mathematically plasticity and Newton (1642–1726) to whom is due the concept of simple viscous resistance.

11. Rheological Diagrams

Plotting traction against distortion, we find the characteristic diagrams for the three ideal materials shown in Fig. I, 5. If we

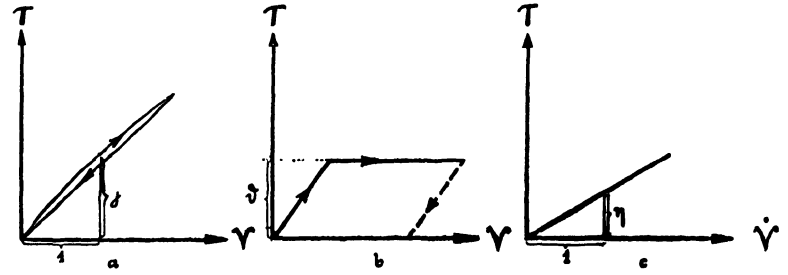


FIG. I, 5. Rheological diagrams of (*a*) elastic,* (*b*) plastic, and (*c*) viscous material. τ shearing traction. γ displacement gradient. $\dot{\gamma}$ velocity gradient.

gradually increase the strain of a Hooke solid, the stress is also gradually increased. The relation of traction to strain has been shown in Fig. I, 5 as roughly linear, but this must not necessarily be so. What is required by the definition of elasticity is only that on gradually diminishing strain, the same curve is retraced, and that on complete release of the stress no *permanent deformation* remains.

The curve for the St. Venant body shows an elastic part up to the yield stress. Upon release of stress, that part of the total deformation

* The two lines in opposite senses shown in the figure are meant to coincide.

is regained. Toothpaste is another example for such a material. If we slightly press a tube containing toothpaste, the flat surface as its opening will protrude in a convex surface which on release of pressure disappears. When the pressure exceeds the " yield point ", a cylinder of paste extrudes with the spherical segment at the head. When the pressure is released, the cylinder remains, but the spherical segment will disappear and its base again becomes flat (Fig. I, 6). This is the result of the properties represented in Fig. I, 5 (b).*

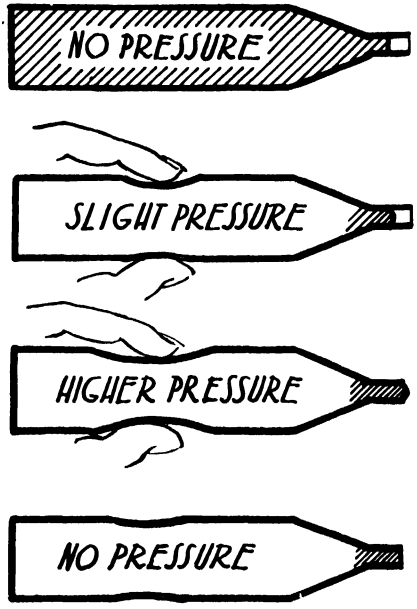


FIG. I, 6. Extrusion of plastic material from a tube.

12. Strength. Fundamental Properties

When the strain of a Hooke body is increased indefinitely, two things may happen. If the material is plastic, after the yield stress is reached it follows a St. Venant body. If, however, the material is brittle *it will break at a certain stress which is its breaking stress.* This two-fold phenomenon embodying both plastic yielding and rupture is called Strength.

Elasticity, plasticity, viscosity and strength are *fundamental* rheological properties from which most others can be derived. In our abstract concepts the Hooke body has elasticity and strength but no viscosity, the St. Venant body has elasticity and plasticity but no viscosity, the Newtonian liquid has viscosity but no elasticity and no strength. *In reality*, however, *every material possesses all rheological properties, although in varying degrees.* This is the second rheological

* In this reasoning the elasticity of the tube itself is neglected.

axiom. As geologists know, even rocks flow, although much more slowly than water. And, as is less known, viscous liquids also possess elasticity. In reality there is no rigid boundary between solids and liquids. *Πάντα ῥει* (panta rhei), everything flows, as Heraclitus (495 B.C.) said, from which our science takes its name.

13. Summary

Rheology makes use of the following laws of *mechanics*.

(i) The second law of Newton

$$a = P/m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I, a})$$

where a is the acceleration of the centre of mass of a body of mass m on which the force P acts. This includes

(ii) the first law of Newton. In the case of equilibrium, when the centre of mass of the body is at rest or in uniform motion ($a = 0$) there must be

$$\Sigma P = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I, b})$$

(iii) An analogous law in respect of rotations. When the body does not rotate around its centre of mass or rotates with constant angular velocity, there must be

$$\Sigma M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I, c})$$

where M is the moment of a force P in respect of the centre of mass.

Under the action of forces a material is either strained elastically, or deformed plastically, or it flows as a liquid ; where " deformation " (d) is the general term, " strain " (e) being a deformation which disappears when the forces are removed, and " flow " a deformation which continues indefinitely under the action of *constant* forces, defined by the rate of deformation ($\dot{d}$).

An isotropic pressure causes every material, whatever its rheological properties, to be strained volumetrically. The *different* rheological properties become manifest only in a deformation which changes the shape ; this is called distortion. Simple shear or tangential deformation (γ) is such a distortion. If small, its measure is the change of an originally right angle in the body. It is accompanied by a shearing traction (τ). Traction is the force acting and reacting between two adjacent parts of the material body, transmitted through unit area of their interface. Stress at a point of the body is the aggregate of all tractions acting on all surface elements of different orientations which contain that point. Rotational equilibrium requires that tangential tractions on interfaces at right angles to each other are equal.

The " rheological equations "

$$\tau = \mu\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I, d})$$

$$\tau = \vartheta_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I, e})$$

$$\tau = \eta\dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I, f})$$

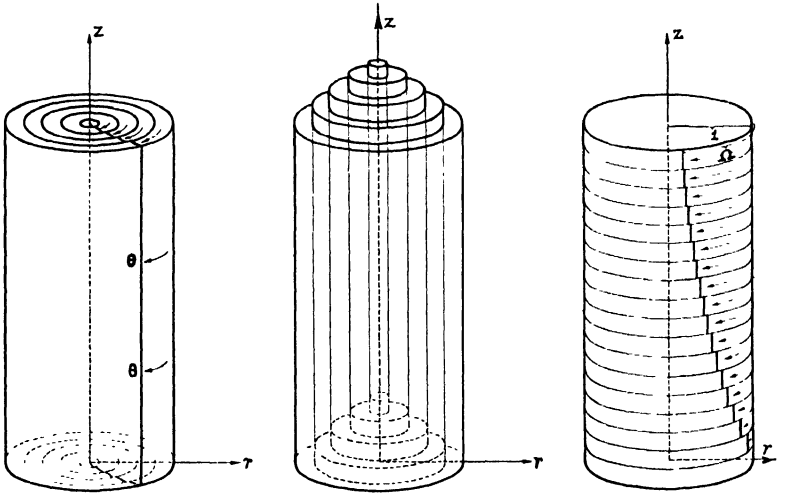
define three ideal materials, viz. (i) the Hooke (elastic) Solid, (ii) the St. Venant (plastic) Solid, (iii) the Newtonian (viscous) Liquid, which serve for the mathematical representation of real materials. Each one of them is defined as having *one* rheological property only, but every *real* material must be supposed to possess *all* rheological properties in varying degrees.

CHAPTER II

VISCOSITY

1. Laminar Distortion

In the first chapter we considered the fundamental rheological properties as they manifest themselves in simple shear. In simple shear, parallel plane slices slide over each other as in a pack of cards, each card as if it were rigid. The slices have, of course, to be imagined as infinitely thin. This is a special case of what is called Laminar Distortion or Laminar Displacement. In simple shear the laminæ are planes ; but laminar distortion is possible also with the laminæ bounded by cylindrical surfaces. They are then hollow cylinders and can slide either in the direction of their common axis, which is " telescopic " distortion ; or they rotate around the axis, which gives " rotational " distortion (Fig. II, 1, *b*, *a*). Both cases are important in tests for the determination of the viscosity of liquids.



a rotational *b* telescopic *c* torsional

Fig. II, 1. Laminar distortions of cylinders.
The laminæ should be imagined as infinitely thin.

2. Hagen's Investigations

If a liquid flows through a cylindrical tube the flow may be either " turbulent " or " streamlined ". In Turbulent Flow the trajectories of the particles are sinuous, undulating and eddy forming. Turbulent flow takes place at low velocities if the tube is comparatively wide.

If the tube is narrow, forming a capillary, at low velocities the trajectories will be forced to be straight lines parallel to the axis with the velocity continuously increasing from the wall towards the axis where it is a maximum ($= v_0$). From the wall towards the centre there is therefore a velocity gradient in the same manner as from the stationary to the moving plate in Fig. I, 4. In the present case the gradient cannot be constant. If it were constant the velocity distribution

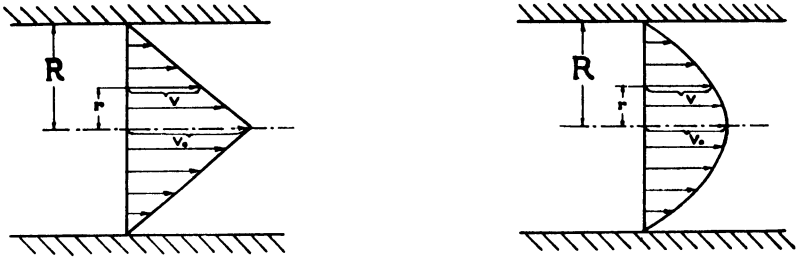


Fig. II, 2. Velocity distribution in telescopic flow.
Conical distribution (incorrect). Paraboloidal distribution (correct).

over the width of the capillary would be a triangle with its apex at the centre as shown at the left of Fig. II, 2. Most students of physics will think such a distribution as very unlikely. A break in a curve is what is called in mathematics a " singularity " and it is very singular that a singularity should occur in a physical problem. Still, this is what Hagen (1839) assumed when first investigating the laws of laminar flow in a tube, and, strangely enough, in our days one could find such a first-class scientist as the late H. Freundlich (and Schalek, 1924) proposing the triangular distribution of the velocity in one of his papers. We shall presently show that it is *not necessary to make any assumptions* about the velocity distribution, but before doing so we shall give a brief description of Hagen's work, as the first on record on the viscosity of a liquid. Hagen studied the flow of water through brass tubes of from 1 to 3 mm radius and $\frac{1}{2}$ to 1 m length, connecting horizontally two reservoirs in which water was kept at different levels. If Δp, the pressure difference between the ends of the tube, be measured by the difference of levels in the two reservoirs (h), then Hagen found the following empirical relation $h = aQ + bQ^2$ where " a " and " b " are two " constants " and Q the quantity of flow in unit time. The particles at the surface of the reservoir from which the water is drawn can be considered as practically at rest and their potential energy, corresponding to h, is therefore in the first instance used to produce the velocity with which the water enters the tube, and, when the flow has become steady, also leaves the tube. In other words it is used to produce the kinetic energy of the flowing mass of water which we can

express by a corresponding "head", say h_1. Secondly, the potential energy is used to overcome the viscous resistance against flow through the tube; this part we express by h_2 and $h = h_1 + h_2$. By slightly varying the temperature, Hagen found that the constant "a" was very much changed, while "b" was hardly changed at all. As the kinetic energy would depend upon the temperature only in so far as it affects the density of the liquid, i.e. very little, Hagen drew the conclusion that the constant "a" was a measure of viscosity, while the second term on the right side of the equation was what has since been termed the Kinetic Energy Correction.

In order to evaluate "b", he assumed that the water flows through the tube telescopically with a conical velocity distribution as shown in Fig. II, 2, on the left, so that $v = v_o - v_o r/R$. The kinetic energy can be calculated from $E_k = \frac{1}{2}\int_0^R v^2\,dm$. For the calculation of the kinetic energy passing the cross-section in unit time, we introduce for dm the mass passing a ring of diameter $2r$ and width dr with the velocity v, or $dm = \rho\,(2\pi r dr)v$ where ρ is the density or mass per unit volume. Therefore

$$E_k = \rho\int_0^R v^2(\pi r v)\,dr = \pi\rho v_o{}^3 R^2/20 \quad . \quad . \quad . \quad . \quad (1)$$

The quantity of flow follows from the known formula for the volume of the cone with $Q = R^2\pi v_o/3$, or $v_o = 3Q/\pi R^2$ and, therefore

$$E_k = 1{\cdot}35\,Q^3\rho/\pi^2 R^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The potential energy producing this kinetic energy is $Q\rho g h_1$ and therefore $= Q\rho g h_1 = 1{\cdot}35\,Q^3\rho/\pi^2 R^4$ from which

$$h_1 = 1{\cdot}35\,Q^2/\pi^2 R^4 g \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and by comparison with the empirical relation

$$b = 1{\cdot}35/\pi^2 R^4 g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Because of the incorrectly assumed velocity distribution, (4) is, of course, also incorrect. Hagen correctly found by experiment that "a" was inversely proportional to R^4.

3. Poiseuille's Law

To calculate the correct velocity distribution, we start from the velocity gradient (compare (I, 12)).

$$\dot{\gamma} = dv/dr = f(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is an unknown function of r. This function can be calculated from first principles.

Let the capillary be horizontal so that the weight of the liquid is of no influence, and let pressures p be applied at both ends of the capillary with the pressure difference Δp acting from the "right"

(see Fig. II, 3). If the liquid in the capillary was originally at rest, it will start to move as soon as the pressure is applied, and the velocity will gradually increase. However, as soon as the flow starts, a velocity gradient also appears, which, in accordance with Newton's " Hypothesis " (Section 8 of Chapter 1) produces a viscous resistance opposing the action of Δp. This viscous resistance increases with increasing velocity and will finally balance the driving pressure. Then equilibrium is established and the velocity remains constant. As soon as this " steady "* state is reached Equation (I, b) can be applied.

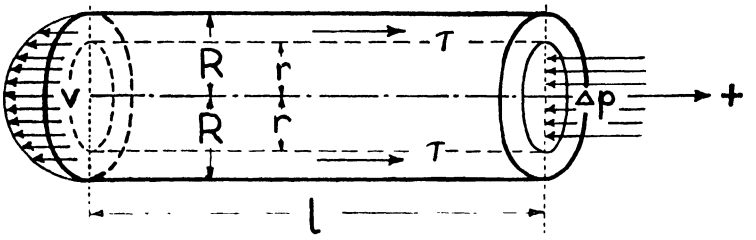


FIG. II, 3. Laminar flow through a tube.

R internal radius of tube. l length of tube. The pressure increases from left to right and the direction of flow is therefore from right to left.

We accordingly write down the condition of equilibrium for a liquid cylinder of length l and radius r. Its base is a circle of area πr^2 and if Δp is the pressure difference per unit area, the driving force is $-\pi r^2 \Delta p$ acting from right to left and therefore applied with the negative sign. The viscous resistance, τ per unit area, acts on the sides of the cylinder of area $2\pi rl$ with a total force of $2\pi rl\tau$. The sum of both vanishes and therefore $-\pi r^2 \Delta p + 2\pi rl\tau = 0$, from which

$$\tau = r\Delta p/2l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The shearing stress is thereby expressed through the given driving pressure Δp and l, which is a measurable dimension of the apparatus, as a function of the distance r from the axis. We introduce this expression for the shearing stress into the rheological equation (I, f), considering at the same time (5). We get

$$dv = (\Delta p/2l\eta) r dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This equation can now be integrated, and yields $v = \Delta p r^2/4l\eta + C$.

In order to determine the integration constant C we must consider a " boundary condition ". We ask ourselves, what will be the velocity of the lamina adjacent to the internal surface of the wall of the

* Some authors speak of " stationary " flow. This is probably a mistranslation of the German " stationaer ". " Stationary " is used since 1784 in the sense of " not moving ". " Steady " means since 1574 "*firm in* standing or *movement* ". For instance, "As a rule there were four steady horses and a good driver, rarely drunk " (*Oxford Dictionary*).

capillary ? It has been found that with very few exceptions, which furthermore do not refer to " pure " liquids, it is safe to assume that the liquid adheres to the wall and that there is no " slippage ".* This is a so-called " kinematical " boundary condition. In this case $v = 0$ for $r = R$ and $C = -\Delta pR^2/4l\eta$ so that

$$v = -(\Delta p/4l\eta).(R^2 - r^2) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The velocity results with the negative sign, which only means the obvious that if the pressure increases from left to right, the flow will be from right to left. This is of no intrinsic significance and the negative sign may be dropped.

The velocity distribution follows a parabola and, as can be seen, dv/dr vanishes for $r = 0$, which means that the parabola has there a tangent normal to the axis and there is no cusp and still less a corner. The maximum velocity in the centre ($r = 0$) is

$$v_0 = \Delta pR^2/4l\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The quantity of flow per unit time (Q) can be calculated from the known formula for the volume of a rotational paraboloid. It is

$$Q = R^2\pi v_0/2 = \pi\Delta pR^4/8l\eta \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This is known as *Poiseuille's law*, but it should be noted that it contains more than was found by Poiseuille himself. Actually the formula established by Poiseuille empirically from experiments with water† in tubes of R varying between 0·01 and 0·3 mm and l between $\frac{1}{2}$ and 10 m, was $Q = k\Delta pR^4/l$. He performed his experiments in 1840 and published them in 1841. He was therefore preceded by Hagen, and Wo. Ostwald (1925) proposed to call (10) the Hagen-Poiseuille law ; but others have argued against this.

Stokes (1849) found the correct velocity distribution but did not proceed to the calculation of Q ; or, in any case, did not publish the result—possibly because when he found a linear law in Δp, this was contrary to the current opinion that the driving pressure was proportional to the *square* of the quantity of flow or the mean velocity. The observations generally accepted in his time were those of French engineers who experimented with pipes of large diameters, where the flow ceases to be laminar and becomes " turbulent ".

Hagenbach (1860) succeeded independently of Stokes in finding the parabolic velocity distribution and also by integration in deriving

* The question of slippage does not arise in " pure " liquids, not even in the case of such a liquid as mercury, which does not adhere to the wall of the tube, compare Remer (1932) and Erk (1932), still less in the case of pure liquids which do adhere to the wall. It arises, however, in the case of dispersed systems and will be dealt with in Chapter XIX.

† Poiseuille was a physician interested in the mechanism by which the blood passes into the smaller blood vessels or capillaries, but he did not find his law when experimenting with blood. Blood is *not* a simple Newtonian liquid and does *not* obey Poiseuille's law. Poiseuille's main paper has been republished in an English translation by W. H. Herschel in *Rheological Memoirs*, Easton, Pa., 1940.

theoretically Poiseuille's law (as he himself called it, which was rather significant, considering that he was well acquainted with Hagen's work). It is interesting to note that he assumed the viscous resistance to result from the work which has to be performed in overcoming molecular cohesion when one layer of molecules passes along another one—a rather modern concept.

After having found the correct velocity distribution, Hagenbach also calculated the kinetic energy correction. Had he applied Hagen's method to the parabolic distribution he would have found the desired result. However, he reasoned independently and wrongly. His error is not instructive and therefore need not be explained here. He found a factor of about *0·8* against Hagen's *1·35* and the correct *1*. While his calculations were not more correct than Hagen's, the kinetic energy-correction is still mostly attributed to Hagenbach. A full discussion of the kinetic energy correction from the experimental point of view is given in Bingham's *Fluidity and Plasticity*, pp. 17–21.

4. Consistency Variables

From (10) there follows

$$\eta = \Delta p R^4 \pi / 8 l Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and when the dimensions R and l of the tube are known, and the driving pressure Δp and the quantity Q are measured, the viscosity of liquid can be determined in accordance with this equation. Plotting Q from (10) against Δp gives a straight line, the inclination of which against the Δp-axis is $tg\alpha = Q/\Delta p = R^4\pi/8l\eta$.

This inclination, accordingly, depends not only upon the magnitude of the viscosity η of the liquid under test, but also upon the dimensions R and l of the apparatus. A better method is to plot

$$V = 4Q/R^3\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

against

$$P = \Delta p R/2l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

These quantities are called, for reasons to be explained later (Section 2 of Chapter VIII), Consistency Variables.

The slope of the line is now $tg\alpha = V/P = 8lQ/\Delta p R^4\pi = 1/\eta = \varphi$ and is independent of the dimensions R and l. Therefore, if measurements are made for one and the same liquid in tubes of different length and radius, if plotted in V and P values, they will all, fall *on one and the same line*. P, as can be seen from (6) is nothing other than the shearing stress at the internal surface of the wall (i.e. for $r = R$). V is, therefore, the velocity gradient at the same place.

The coefficient

$$\varphi = 1/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

is called the Coefficient of Fluidity.

Introducing the Instrument Constants

$$\left.\begin{aligned} K_1 &= 4Qt/R^3\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ K_2 &= R/2l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad (15)$$

(12) and (13) become

$$V = K_1/t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12^1)$$

$$P = K_2 \Delta p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13^1)$$

It should be remembered that Q is the volume in unit time and that in most capillary instruments the volume forced through the capillary is constant and what is measured is the time t necessary for the constant Q to pass through the capillary.

5. Newton's Rheological Problem

Rotational laminar flow was the first rheological problem ever dealt with theoretically and it is not only because of the historical interest that we shall presently reproduce Newton's attempt at solution in modern mathematical language. In his calculations the great mathematician and physicist made a mistake and this mistake is instructive. Going through Newton's calculation, the reader will receive a warning of pitfalls and at the same time he will be consoled about the many mistakes which, without doubt, he has made himself before.

Newton formulated the problem as follows (*Principia*, Lib ii, Sect. ix) :-

PROPOSITION LI. THEOREM XXXIX.

"*If a solid infinitely long cylinder in a uniform infinite liquid revolve round its axis with uniform motion, and the liquid be caused to revolve by the impulse thereof only, but every part of the liquid persevere uniformly in its motion*" ; and the solution is : "*I say that the periodic times of parts of the liquid are as their distances from the axis of the cylinder.*"*

The cylinder is assumed infinitely long, because then flow conditions are the same in every section normal to the axis of the cylinder and the problem is one of two-dimensional or Plane Distortion, i.e. the velocity has no component in the z-direction (if we take the axis of the cylinder as z-axis), and the x- and y-components of the velocity are independent of z. If the cylinder were finite this would not be the case, because the velocity distribution would then be necessarily different in a section which cuts the cylinder just before its end and a section just beyond its end which would not cut the cylinder. There would be, as the expression goes, an "end-effect". To calculate the "end-correction" is a problem which cannot be dealt with by elementary methods and in order to avoid this difficulty Newton assumes

* The translation from Latin is E. Hatschek's (*Viscosity* (1928), pp. 2-4).

his cylinder to be infinite. This does not preclude the application of his theory to a concrete case, because in carrying out an actual experiment one can make measurements with cylinders of different heights h and then on a graph plotting $1/h$, extrapolate the results to $1/h = 0$. This would then represent Newton's conditions.

A second remark on Newton's wording of the problem has to be made. When the cylinder is at rest and is then brought into uniform rotation, the parts of the liquid nearest to the surface of the cylinder are forced to partake in the rotation of the cylinder, and they, in their turn, because of the " lack of slipperiness ", or " viscosity " as we now say, bring the parts of the liquid at a distance into movement, until more and more distant parts of the liquid are in movement. This, of course, requires time and if the liquid is infinite it would require very large times until the rotation of the cylinder makes itself felt in remote parts of the liquid. This means that it would take infinite time until " every part of the liquid perseveres uniformly in its motion ". However, in practice, after a rather short time the movement of the parts of the liquid which are near to the cylinder becomes uniform, and as the liquid at infinity will naturally remain at rest, the movement of the parts further away is of no importance.

The movement of the parts of the liquid can therefore be assumed as uniform and then must be a circular movement with constant rotational velocity or

$$v_\theta = r\dot{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where $\dot{\theta}$ is the angular velocity which will be a function of r only, but not of θ, and not of time, i.e. the movement is a steady flow in circular streamlines. Finally, the words " *The liquid be caused to revolve by the impulse thereof only . . .*" mean that all other forces, e.g. the influence of gravity and of the centrifugal force may be neglected.

Newton's proof is as follows : ". . . *let the liquid be divided by concentric circles . . . into innumerable concentric solid cylindrical shells of the same thickness.*" *Solid* cylindrical shells means that the movement is supposed to be " laminar " with the cylindrical shells as " laminæ " moving as *wholes*, i.e. *as if* they were solid. " *And as the liquid is homogeneous, the impressions made by contiguous shells on each other will (ex hypothesi) be as their mutual translations* ($\dot{G}$) *and as the contiguous surfaces* ($2\pi rl$) *on which the impressions are made.*" I.e. the impression will be equal to $\eta\dot{G}.2\pi rl$. " *If the impression on any one shell is greater or smaller from the concave part than from the convex one, the stronger impression will prevail and will accelerate or retard the motion of the shell according as it is directed in the same or opposite sense as its motion.*" I.e. the forces acting on each shell are $\eta\dot{G}.2r\pi l$ on the concave part and $\eta(\dot{G} + d\dot{G}).2(r + dr)\pi l$ on the convex part and the difference is $\eta 2\pi l\,(\dot{G}dr + rd\dot{G})$, neglecting the term of the order $d\dot{G}.dr$. If this difference is positive, the motion of the shell will be accelerated ; if it is negative, it will be retarded. " *If therefore any one shell is to persevere uniformly in its motion, the impressions from either part must be equal*

and opposite to one another." I.e. we must have $\dot{G}dr + rd\dot{G} = 0$, or

$$d(r\dot{G}) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and by integration

$$r\dot{G} = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and

$$\dot{G} = C/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

"*Hence . . . the translations will be inversely . . . as the distances of the surfaces from the axis. But the differences of the angular motions round the axis are as these translations divided by the distances.*" I.e.

$$d\dot{\theta}/dr = \dot{G}/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This is not so easily seen. We have from (16)

$$\dot{\gamma} = dv_\theta/dr = r.d\dot{\theta}/dr + \dot{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Now let us anticipate Newton's 3rd corollary which reads as follows :

"*If any common angular motion be added to, or withdrawn from, the cylinder and liquid . . . the motions of the parts against one another will not change, as the mutual friction of the parts of the liquid is not changed.*" This means that the internal stresses are not dependent upon $\dot{\theta}$ and therefore not upon dv_θ/dr *in toto*, but only upon the first term $rd\dot{\theta}/dr$. In the case of a rotation, the velocity gradient is therefore not a suitable quantity upon which to base the calculation of the viscous resistance. The correct quantity is *the rate of shear* which is the velocity gradient minus a rotation*, or

$$\dot{G} - rd\dot{\theta}/dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and from this there follows (20).

We continue the quotation : ". . . *or directly as the translations and inversely as the distances, that is, combining the ratios, inversely as the squares of the distances.*" From (20) and (19) we find

$$d\dot{\theta}/dr = C/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

To integrate this differential equation Newton, who had himself invented Calculus, did not use it, but used a geometrical method which is extremely tiresome to follow.† Using the rules of Calculus

$$\dot{\theta} = C_1/r + C_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $C_1 = -C$.

As the parts of the liquid at great distance from the cylinder will remain at rest $\dot{\theta} = 0$ for $r = \infty$ and therefore

$$v_\theta = r\dot{\theta} = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

* The term "rate of shear" is explained in the following Section 6.

† The interested reader may look this up in Hatschek's *Viscosity*, p. 3.

"*The periodic time of any one part is therefore . . . directly as the distance r.*"

Or

$$1/\dot{\theta} = r/C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

"*Q.E.D.*"

The proud "Q.E.D." does not prevent the result from being incorrect. Has the reader noticed the point where Newton went off on a wrong track ? If not, let him be consoled by the fact that it was only discovered by Stokes over one and a half centuries later. He remarks : "These cases of motion were considered by Newton . . . but he arrives at the result that the velocity ($v_\theta = r\dot{\theta}$) is constant, not that it varies inversely as the distance." This arises from his having assumed that the condition of there being no acceleration or retardation of an annulus was that the force tending to turn it in one direction must be equal to that tending to turn it in the opposite direction ; whereas the true condition is that the *moment* of the force tending to turn it in one direction must be equal to the moment of the force tending to turn it in the opposite direction. Of course, making this alteration, it is easy to arrive at the above result by Newton's reasoning.

In short : Newton forgot about Equation (I, c) and wrongly applied (I, b).

Applying (I, c). the moments acting on each shell are $= \eta\dot{G}.r.2\pi l$ on the concave part and $\eta(\dot{G} + d\dot{G}).2(r + dr)^2\pi l$ on the convex part. The difference is $\eta 2\pi l.(r^2 d\dot{G} + 2\dot{G}rdr)$, neglecting small terms of higher order. Putting

$$r^2 d\dot{G} + 2\dot{G}rdr = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

we easily get

$$\dot{G} = C/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and introducing $\dot{G}$ from (20)

$$rd\dot{\theta}/dr = C/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

we find

$$\dot{\theta} = C_1/r^2 + C_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Where

$$C_1 = -C/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30')$$

As before $C_2 = 0$ and in accordance with (16) the velocity is

$$v_\theta = r\dot{\theta} = C/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where we drop the subscript $_1$ in C_1.

After having found the solution of his problem, Newton adds : "*All which may be tried in deep and stagnant water.*" This is a nice illustration of what Hersey (1932) has called the Integration Method in rheology. Newton postulated his law of viscous flow, drew conclusions by mathematical methods, and left it for experiment to decide whether his law was correct, i.e. whether it was applicable to real

liquids with a good enough approximation. The experiment was made much later and the agreement—of course with the *correct* solution—was excellent. Since then apparatuses for the determination of the viscosity of liquids, or viscometers have been built on the principle of Newton's problem as is described in Section 8.

6. The Rate of Shear

We have said in the preceding Section that the correct physical quantity governing viscous resistance is not the velocity gradient $\dot{\gamma}$, but the Rate of Shear which we have denoted by $\dot{G}$. We shall now explain the situation more fully.

Consider the simple shear shown in Fig. I, 2 through which the right angle between the horizontal and the vertical edges is reduced by the angle α. This is shown again in section in Fig. II, 4, a. If the original

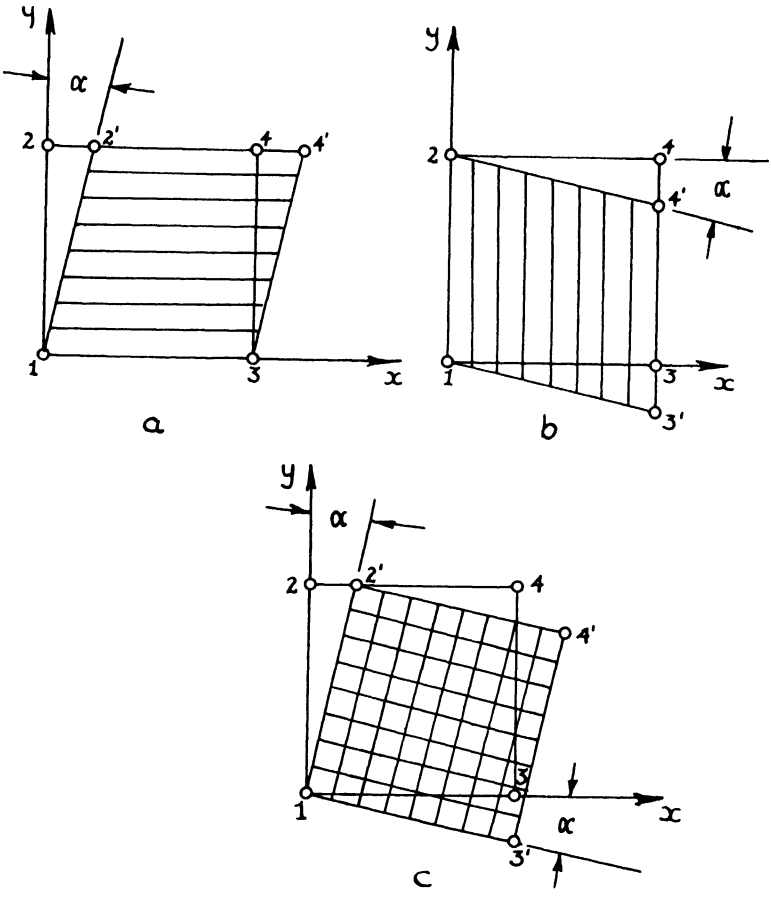


FIG. II, 4. Superposition of two simple shears resulting in a rotation through an angle α.

figure is a square, this is changed by simple shear into a rhombus. As explained on page 5, the displacement gradient is

$$\gamma = \tan \alpha = \frac{du_x}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where u_x is the displacement in the x direction.

However, this deformation of a square into a rhombus can also be produced by shear parallel to the y-direction as shown in Fig. II, 4, b.

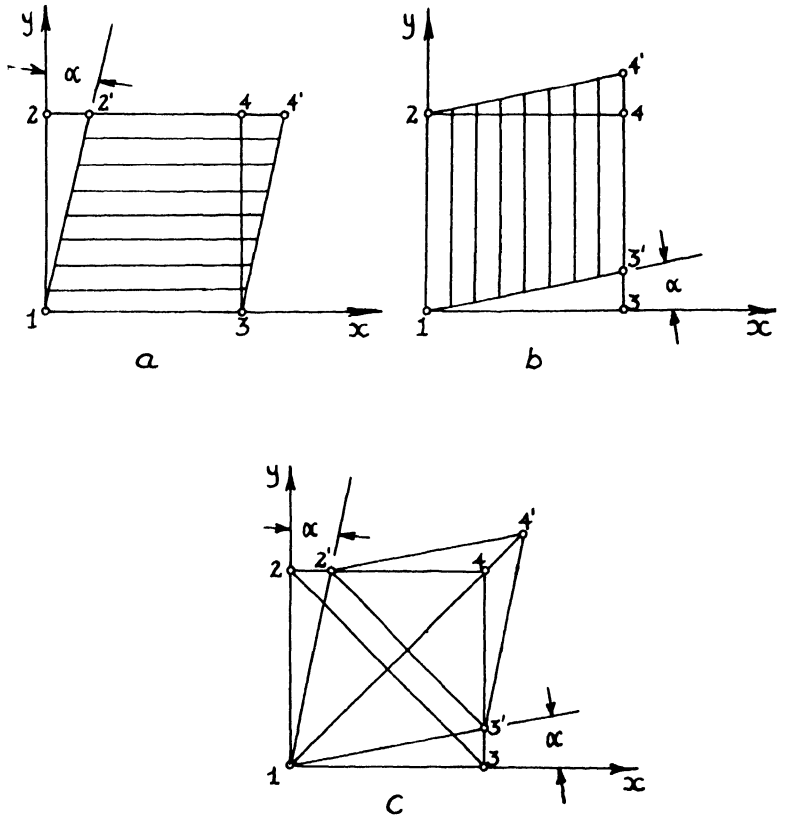


Fig. II, 5. Superposition of two simple shears resulting in pure shear.

To distinguish the displacement gradient of this case from the former we write

$$\gamma_{xy} = \frac{du_x}{dy}, \quad \gamma_{yx} = \frac{du_y}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Now let the two changes be super-imposed, and let

$$\gamma_{yx} = -\gamma_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

then, as can be seen from Fig. II, 4, c, the result is *not a deformation*

but a *rotation* of the square through the angle α. This suggests that the displacement gradient is not the kinematical quantity which defines a deformation. We can, however, derive another quantity

$$d_{xy} = \tfrac{1}{2}(\gamma_{xy} + \gamma_{yx}) = d_t \quad . \quad . \quad . \quad . \quad . \quad (35)$$

which can serve for this purpose. In the former case when $\gamma_{yx} = -\gamma_{xy}$, d_t would vanish, as it should if it defines a deformation. On the other hand, when $\gamma_{yx} = \gamma_{xy}$, conditions are as shown in Fig. II, 5, and this is a deformation without rotation. Such deformation is called Pure Shear. As can be seen, the diagonals of the square do not change their directions in space. One is elongated, and the other shortened by equal amounts. In the case of simple shear we have $d_t = \gamma/2$, and this is connected with a rotation also equal to $\gamma/2$. We can now better understand the reasoning by which we arrive at (22) above.

In d_t the symbol "d" denotes Deformation while the index "t" denotes tangential. There are also longitudinal deformations denoted by d_l or normal deformation indicated by d_n, about which we shall learn later. The time derivative $\dot{d}_t$ is the rate of shearing deformation. The rate of shear, however, as defined above and denoted by $\dot{G}$ is equal to $2\dot{d}_t$.* Note that we denote deformation in general, and more especially a *large* deformation by D, and d indicates only a small or rather *infinitesimal* deformation because, as a closer inspection of Fig. II, 4 will show, a rotation will result from the superposition of two single shears only, if these are *infinitesimal*.

7. The Teapot-Effect

The rotation which is connected with simple shear, and which is named Vorticity, presents some difficulty. In simple shear the mathematical points move in straight parallel lines which suffer no rotation. The particles, however, if considered as small volumes, suffer rotations with respect to their initial configuration. This rotation possesses kinetic energy, which manifests itself in what may be named the teapot-effect (Reiner (1956)). When a layer of viscous liquid flows along a solid surface, it is sheared. The layer can be imagined as consisting of laminæ which are parallel to the solid surface. The velocity of the lamina adjacent to the wall is zero. This can be assumed even when the liquid does not "wet" the surface of the solid (compare first footnote on page 18). The velocity increases to V at the outermost lamina. Let H be the thickness of the layer, then a velocity gradient $\dot{\gamma} = V/H$ is produced (compare Fig. I, 4). By virtue of its viscosity the liquid resists the shearing deformation through internal shearing stresses $\tau = \eta\dot{\gamma}$ (compare Equ. (I, f)). However, as explained in the

* This has historical reasons, and introduces in certain equations a regrettable complication through the factor 2. The quantity G may be named the amount of shear.

preceding Section, the shearing deformation does not exhaust the kinematics of the case. In addition to the shearing deformation there also take place rotations of the elements of the liquid defined by vortices as shown in Fig. II, 6. These rotations are of the nature of rigid bodies' rotations.

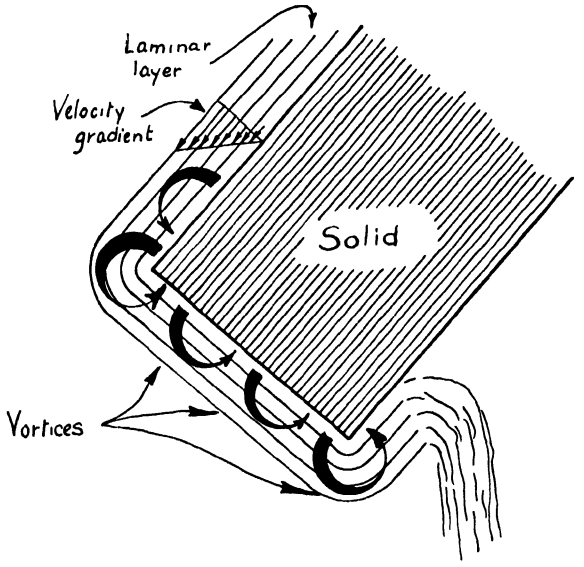


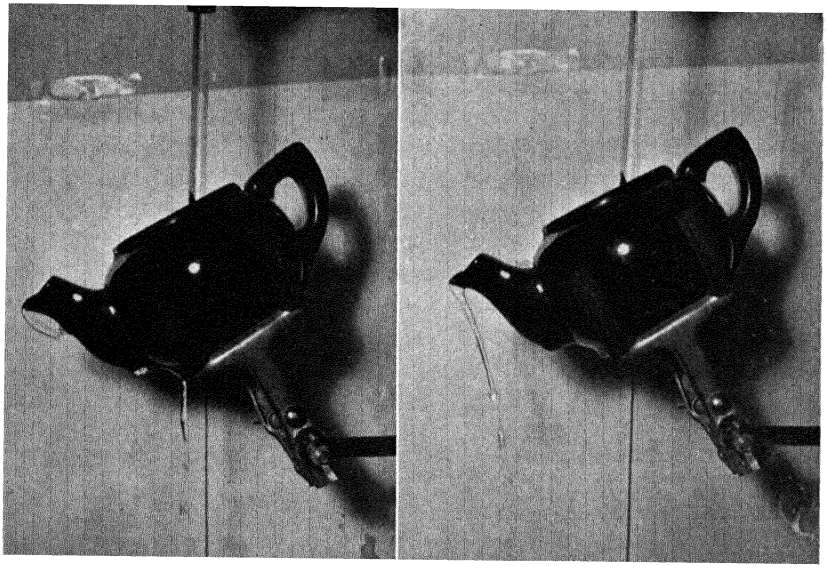
FIG. II, 6. Vortices in simple shear.

Now let gravity cause and maintain the laminar flow of the liquid layer ; it will produce work equal to the loss of energy resulting from the fall of the liquid mass from a higher to a lower level. This work is expended in two different ways. Part of it is used up, dissipated, and converted into heat in the continuous shearing of the layer. Another part, however, is converted into the kinetic energy of the vortices. As the rotation is not accompanied by any deformation, this part is not dissipated but persists in a steady state, and can only decline with the velocity V.

This is the mechanism for the " sticking " of the liquid to the solid wall which actually is no sticking at all, but a kind of being pressed against the wall through the rotational kinetic energy of the vortices. The vortices also help the layer to turn the corner.

What has here been described explains a phenomenon observed in everyday life when tea is poured out of a teapot. The jet more often than not has a tendency not to flow in a nice ballistic curve, as intended, into the cup but either to follow the underside of the spout or to flow in an " antiballistic " curve as shown in Fig. 7. This is not due to an

" adhesion " of the liquid because when the spout is coated with some water repellent material, e.g. paraffin wax, the " teapot-effect " persists without any noticeable change. Fig. 7 shows the teapot in action. It is fed from a tap above, and the water emerges from the spout sticking to the wall of the spout, and going off when the bottom of the pot is reached, but sometimes even flowing along the bottom of the pot *uphill.* When the supply of water is increased, a jet is formed which is first " antiballistic ", and only at higher speeds attains the desired ballistic shape.



Rate of flow low. Rate of flow high (but not too high).

FIG. II, 7. The teapot-effect.

This little experiment will convince the reader that the rotation connected with simple shear is not a mathematical abstraction but has physical reality.

8. Rotation Viscometers

In the rotation viscometers the liquid is contained between two co-axial cylinders of which either the external (Couette-Hatschek apparatus) or the internal cylinder (Searle apparatus) rotates, while the other is kept in position. The angular velocity of the rotating cylinder ($\dot{\Omega}$) and the moment M required to keep the velocity constant against the viscous resistance of the liquid are measured. The general solution is given in (30). Let the internal cylinder of radius R_i rotate

with the velocity $\dot{\Omega}_i$ and the external cylinder of radius R_e with the velocity Ω_e.

We then have

$$\left.\begin{aligned}\dot{\Omega}_i &= C_1/R_i^2 + C_2 \\ \dot{\Omega}_e &= C_1/R_e^2 + C_2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

from which

$$\left.\begin{aligned}C_1 &= \frac{\dot{\Omega}_i - \dot{\Omega}_e}{1/R_i^2 - 1/R_e^2} \\ C_2 &= \frac{\dot{\Omega}_e R_e^2 - \dot{\Omega}_i R_i^2}{R_e^2 - R_i^2}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and therefore

$$\dot{\theta} = \frac{\dot{\Omega}_i R_i^2\left(\frac{R_e^2}{r^2} - 1\right) - \dot{\Omega}_e R_e^2\left(\frac{R_i^2}{r^2} - 1\right)}{R_e^2 - R_i^2} \quad . \quad . \quad . \quad (38)$$

Both cylinders, with the liquid between them, are in equilibrium, because, while there is movement, the motion is uniform and no part of the system is accelerated or retarded. The liquid between the cylinders is the material body under consideration. As it is in equilibrium* the shearing stresses acting on its convex surface must therefore give a rotational couple M_z around the commonax is (z) of the cylinders and the shearing stresses which act on the concave surface must give a rotational couple M_z. The same couples act on every lamina of the liquid and the couple M_z is therefore a constant

$$M_z = \tau . 2\pi r h . r \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where h is the height of the cylinder which is immersed in the liquid or an equivalent height, if there are end-effects. Now the shearing stress

$$\tau = \eta\dot{\gamma} = \eta r d\dot{\theta}/dr = \frac{2\eta}{r^2}\frac{\dot{\Omega}_e - \dot{\Omega}_i}{1/R_i^2 - 1/R_e^2} \quad . \quad . \quad . \quad (40)$$

On the concave surface, $r = R_i$, it is

$$\tau = 2\eta\frac{\dot{\Omega}_e - \dot{\Omega}_i}{1 - (R_i/R_e)^2} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

therefore

$$M_z = 4\eta\pi h\frac{\dot{\Omega}_e - \dot{\Omega}_i}{1/R_i^2 - 1/R_e^2} \quad . \quad . \quad . \quad . \quad (42)$$

which is independent of r.

* One should be careful to distinguish between *equilibrium* and *rest*. The liquid does not only move, but it is constantly sheared and it is *nevertheless in equilibrium*, the viscous resistance consuming the work of the external forces (e.g. the work of the electro-motor driving the outer cylinder in Hatschek's apparatus) so that no additional kinetic energy is produced.

The same result is found by introducing $r = R_e$ into (39), taking the stress over the area $2R_e\pi h$ and the lever arm R_e.

Specializing (38) and (41) for the Couette-Hatschek apparatus by putting $\Omega_i = 0$, we get

$$\dot{\theta} = \dot{\Omega}_e \frac{1/R_i^2 - 1/r^2}{1/R_i^2 - 1/R_e^2}$$
$$M_z = 4\eta\pi h \frac{\dot{\Omega}_e}{1/R_i^2 - 1/R_e^2} \quad \ldots\ldots \quad (43)$$

while for the Searle apparatus, putting $\dot{\Omega}_e = 0$, there is

$$\dot{\theta} = -\dot{\Omega}_i \frac{1/R_e^2 - 1/r^2}{1/R_i^2 - 1/R_e^2}; \quad M_z = -4\eta\pi h \frac{\dot{\Omega}_i}{1/R_i^2 - 1/R_e^2} \quad (44)$$

If $\dot{\Omega}_i$ or $\dot{\Omega}_e$ and M_z are measured, the viscosity can be calculated from either (43) or (44). As can be seen, the linear relation between shearing stress and rate of shear (I, 11) brings with it a linear relation between the dynamical quantity M_z and the kinematical quantity $\dot{\Omega}$. The distribution of both velocity and angular velocity over the width between the cylinders is shown in Figs. II, 8–11, where R_i has been assumed $= 0{\cdot}6\ R_e$.

Neither distribution is linear, but both are nearly so.

This method for the determination of viscosity was first proposed

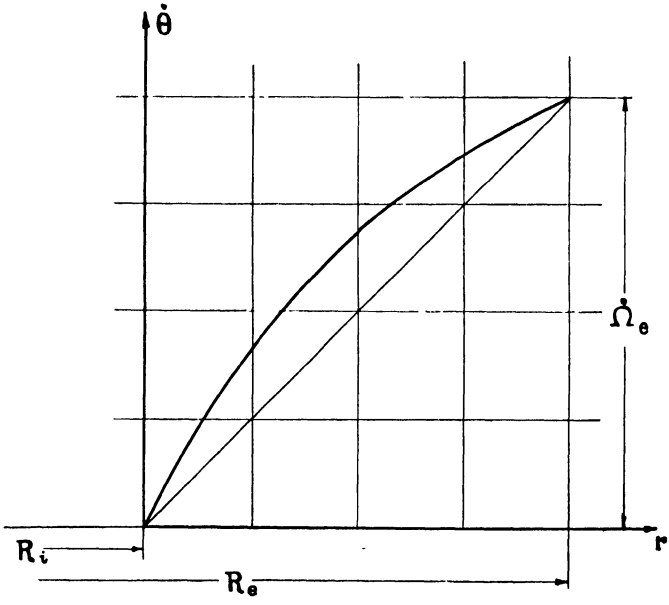


FIG. II, 8. Angular velocity distribution in the Couette-Hatschek apparatus.

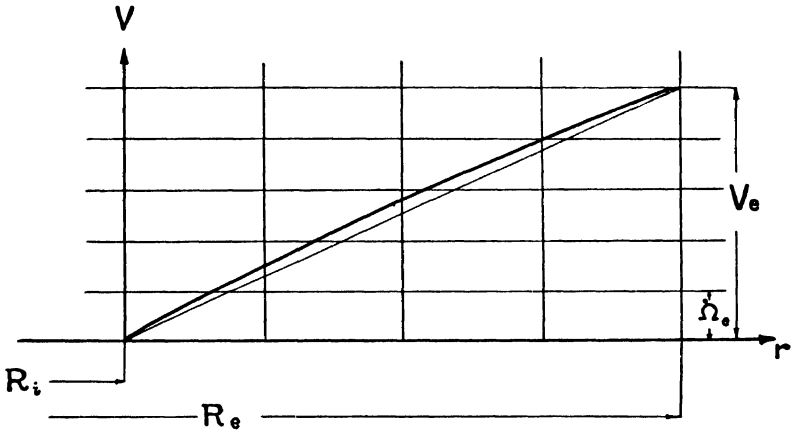


FIG. II, 9. Velocity distribution in the Couette-Hatschek apparatus.

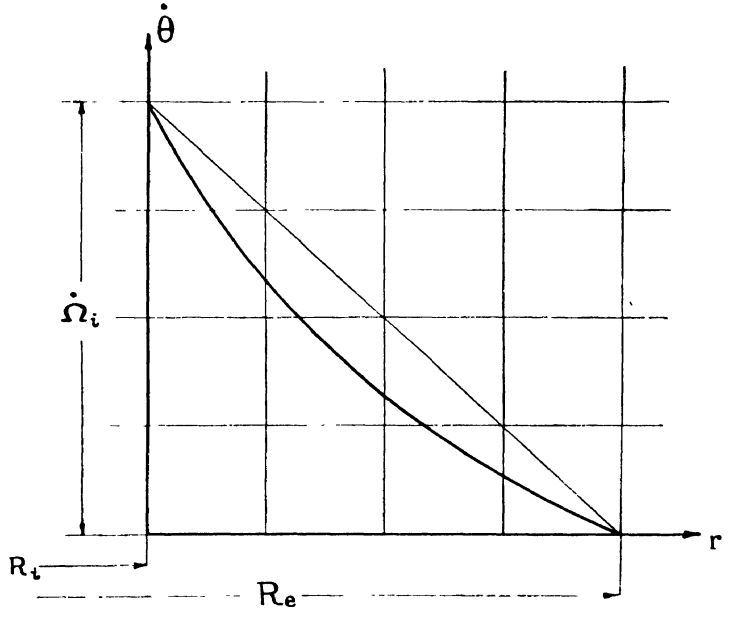


FIG. II, 10. Angular velocity distribution in the Searle apparatus.

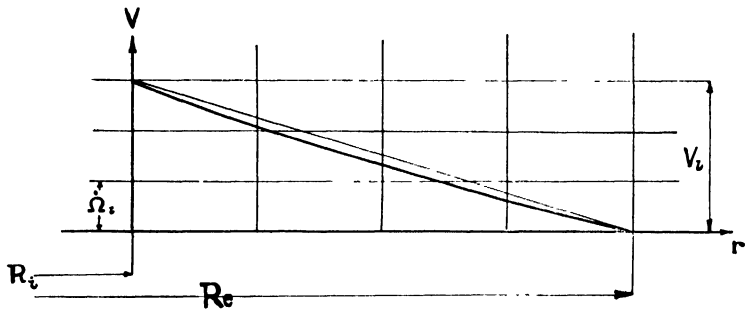


FIG. II, 11. Velocity distribution in the Searle apparatus.

by Margules (1881) and later used by Couette (1888). The method is therefore sometimes called " Margules-method " and the laminar flow between concentric cylinders is called Couette flow. This denotation is used also if both cylinders have infinitely large radii, i.e. if they are reduced to two infinitely large plates. This case cannot, of course, be realized and in the actual experiment the plates will be of limited size. If we introduce into (41)

$r = R_i + x$; $R_e = R_i + H$; $\dot{\Omega} = 0$; $\dot{\Omega}_e = V/R_e$ and then put $R_i = \infty$, we find

$$v = Vx/H \; ; \; \dot{G} = V/H, \tau = \eta V/H \quad . \quad . \quad . \quad . \quad (45)$$

as shown in Fig. I, 4.

If V and H are equal to unity, η is the force necessary to keep the movable plate in movement and the fixed plate from moving. Hence the usual quantitative definition of viscosity : " The viscosity is measured by the shearing stress in dynes per square centimetre required to move either of two parallel planes, one centimetre apart, relative to each other, with a velocity of one centimetre per second, the space between the planes being filled with the fluid."

In connection with (45) the student must be warned of the following pitfall. From the law of corresponding tangential stresses (Section 6 of Chapter I) we know that in plane Margules-Couette flow there exists not only a shearing stress τ_{xy} acting on a surface element normal to the x-axis in the direction y (compare Fig. I, 4), but also a shearing stress τ_{yx} of the same magnitude acting on a surface element normal to the y-axis in the direction x. Nevertheless there exists a velocity gradient in the x-direction only, while in the y-direction the velocity gradient is equal to zero, because the velocity has no component in the x direction. Ignorance of this fact led a gifted and otherwise well instructed student of physical chemistry on a wrong track in his search for an explanation of " variable viscosity."* I happened to see his thesis which gave a good description of sound experimental work on a

* This will be dealt with in Chapter XV.

case of " structural viscosity."* He realized that the flow was supposed to be steady. There was a velocity gradient in the x-direction, but none in the y-direction. He therefore thought that there existed shearing stresses parallel to the direction of flow but none in the direction

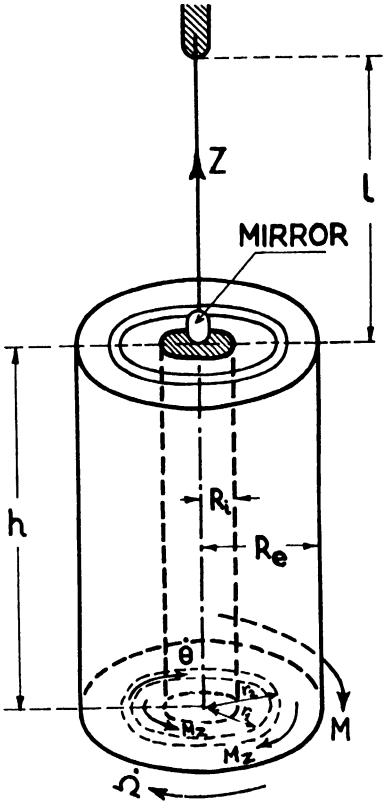


FIG. II, 12. Rotational viscometer of Couette-Hatschek type. (Not to scale.

l free length of wire on which the stationary cylinder is suspended.
h effective height of cylinder.
R_i (external) radius of internal (stationary) cylinder.
R_e (internal) radius of external (rotating) cylinder.
$\dot{\Omega}$ angular velocity of rotating cylinder.

Under the action of the two couples M_1 and M_2 the lamina rotates with angular velocity $\dot{\theta}$.

normal to the flow. There could, however, be no equilibrium with only one couple, $+\tau_{xy}$, $-\tau_{xy}$ acting on a volume element. There was bound to exist an anti-clockwise couple also. He came near to finding the law of complementary shearing stresses, but he thought he could attribute " non-Newtonian behaviour " to that " non-Newtonian anti-clock-

* This will be dealt with in Chapter XV.

wise couple ". As we know, the solution of the difficulty consists in that the stress is produced by a *rate of shear*, and not by a velocity gradient, and this rate of shear is connected with the change of the right *angle between* the directions x and y and not with the direction x or y.

Consistency variables for the rotation instruments are :—

The shearing stress at the wall of the internal cylinder

$$P = M/2R_i^2\pi h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

the mean rate of shear

$$V = 2\frac{\dot{\Omega}_e - \dot{\Omega}_i}{1 - a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where

$$a = (R_i/R_e)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

9. The Couette-Hatschek Apparatus

As Chapter XVI includes a report upon measurements made with the Couette-Hatschek apparatus, it will be useful to add a short description of that apparatus.

The general arrangement is seen in Figs. II, 12 and II, 13. M is determined by suspending the inner cylinder from a wire of known

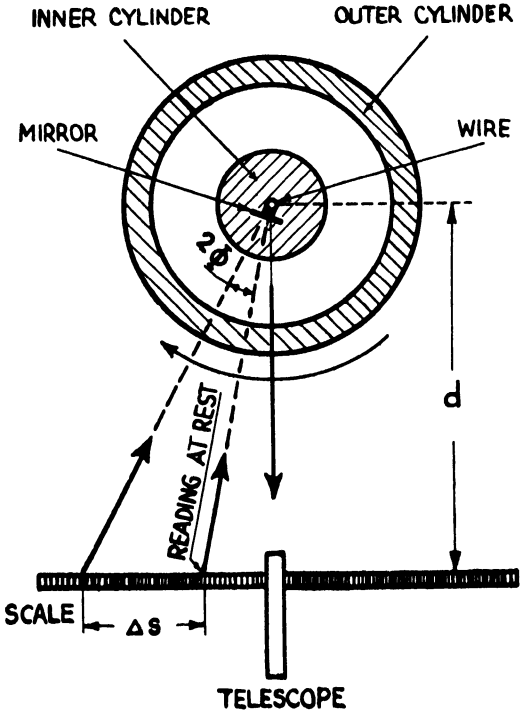


Fig. II, 13. Couette-Hatschek apparatus seen from above.

"restoring moment" D/l* (where l is the length of the wire) which is a constant of the wire, and measuring the angle ϕ by which it is deflected, so that

$$M = \phi D/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The deflection (Δs) is read in a mirror, attached to the axle of the inner cylinder, through a telescope with scale and if "d" is the distance of the scale from the axle, there is approximately $\phi = \Delta s/2d$ and

$$M = \frac{\Delta s}{2d} . D/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

The rotational velocity of the outer cylinder $\dot{\Omega}$ is determined by the number n of revolutions per unit time, so that

$$\dot{\Omega} = 2\pi n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The consistency variables are

$$V = 2\frac{\dot{\Omega}}{1-a} = \frac{4\pi}{1-a} n \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$P = \frac{1}{4dR_i^2\pi h} \frac{D}{l} \Delta s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Introducing the instrument constants

$$K_3 = \frac{4\pi}{1-a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$K_4 = 1/4dR_i^2\pi h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

we have

$$V = K_3 n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$P = K_4 \frac{D}{l} \Delta s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

10. The N-Model

In order to treat rheological problems by mathematical methods, we found it necessary to create concepts of ideal bodies having rigorously defined (rheological) properties. This process is facilitated by building up, even if only in imagination, models consisting of various combinations of mechanical elements which under the action of appropriate forces register displacements of particular kinds, similar to those exhibited by the materials the behaviour of which we desire to describe. A suitable mechanical model for the Newtonian liquid consists of a test tube filled with a very viscous oil in which a loosely fitted stopper can move—the whole arrangement forming a kind of *dashpot*. We shall indicate this dashpot by the symbol N (Newton). The model is shown diagrammatically in Fig. II, 14.

* For D compare (IV, 45).

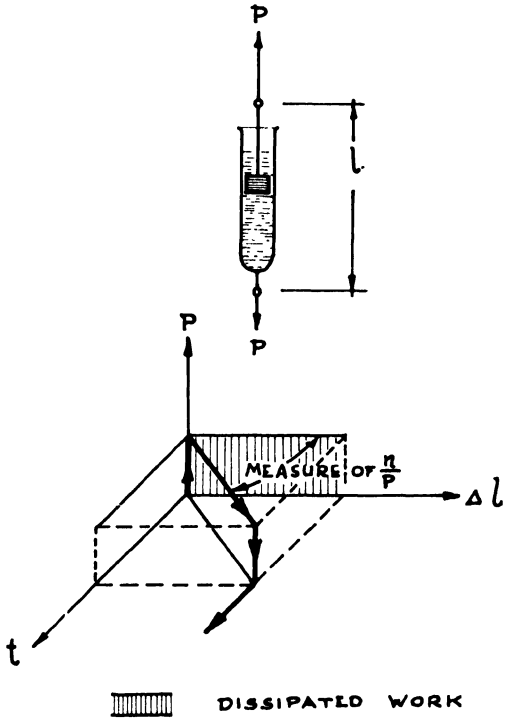


FIG. II, 14. Diagram of model for the Newtonian liquid showing stress-flow-time behaviour. The inertia of the mechanism has been neglected.

11. Summary

The rheological equation of the *Newtonian liquid*

$$\tau = \eta\dot{G} = 2\eta\dot{d}_t \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, a)}$$

is applied to the steady laminar flow in (i) a tube, (ii) a rotating-cylinder viscometer.

(i) In the first case we introduce for $\dot{G}$

$$\dot{G} = dv/dr \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, b)}$$

and for τ, by applying Newton's first law (I, b) to a cylindrical lamina

$$\tau = r\Delta p/2l \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, c)}$$

where Δp is the pressure difference at both ends of the tube. In the absence of slippage, integration gives the velocity distribution

$$v = \Delta p(R^2 - r^2)/4l\eta \quad . \quad . \quad . \quad . \quad . \quad \text{(II, d)}$$

and a second integration

$$Q = \pi\Delta p R^4/8l\eta \quad . \quad . \quad . \quad . \quad . \quad \text{(II, e)}$$

which is known as *Poiseuille's law.*

Plotting *consistency variables*

$$V = 4Q/R^3\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, f)}$$

and

$$P = \Delta pR/2l \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, g)}$$

the first being the rate of shear, the second the stress in the layer of liquid adjacent to the wall, gives " consistency curves " which are independent of the dimensions R and l of the apparatus.

(ii) In the second case, we start from

$$\dot{G} = rd\dot{\theta}/dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, h)}$$

applying the condition of rotational equilibrium in accordance with (I, c), which gives

$$\tau = M/2r^2\pi h \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, i)}$$

and find by integration

$$\theta = [\dot{\Omega}_i(1/r^2 - 1/R_e^2) + \dot{\Omega}_e(1/R_i^2 - 1/r^2)]/(1/R_i^2 - 1/R^2_e) \quad \text{(II, j)}$$

and

$$M = 4\pi h\eta(\dot{\Omega}_i - \dot{\Omega}_e)/(1/R_i^2 - 1/R_e^2) \quad . \quad . \quad \text{(II, k)}$$

which is known as the *Margules-equation.*

Consistency variables are in this case

$$V = 2(\dot{\Omega}_i - \dot{\Omega}_e)/(1 - a) \quad . \quad . \quad . \quad . \quad . \quad \text{(II, l)}$$

where

$$a = (R_i/R_e)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, m)}$$

and

$$P = M/2R_i^2\pi h \quad . \quad . \quad . \quad . \quad . \quad \text{(II, n)}$$

For the Couette, for which $\dot{\Omega}_i = 0$ and $\dot{\Omega}_e = \dot{\Omega}$, there is (disregarding the negative sign of M)

$$\dot{\theta} = \frac{M}{4\pi\eta h}(1/R_i^2 - 1/r^2) = \dot{\Omega} - \frac{M}{4\pi h\eta}(1/r^2 - 1/R_e^2) \quad . \quad . \quad \text{(II, o)}$$

The *rate of deformation* $\dot{d}$ is one half of the *velocity gradient* $\dot{\gamma}$ in simple laminar shear, or in the more general case

$$\dot{d}_{xy} = \tfrac{1}{2}(\dot{\gamma}_{xy} + \dot{\gamma}_{yx}) \quad . \quad . \quad . \quad . \quad . \quad \text{(II, p)}$$

where

$$\dot{\gamma}_{xy} = \frac{dv_x}{dy}\,; \quad \dot{\gamma}_{yx} = \frac{dv_y}{dx} \quad . \quad . \quad . \quad . \quad \text{(II, q)}$$

CHAPTER III

ELASTICITY

1. Simple Shear Difficult to Realize

WE have defined elasticity by considering simple shear of a prism as shown in Fig. I, 3. But this is an arrangement very difficult to realize. What one can do is to apply the forces $+ P_t$ and $- P_t$, but it is difficult to apply $+ S_t$ and $- S_t$. The couple arising from $+ P_t$ and $- P_t$ must then be balanced in some other way. Because of the absence of the tangential tractions $+ S_t$, $- S_t$, the deformation at the free ends cannot be simple shear, as assumed in Fig. I, 3. Simple shear will establish itself at some distance from the free ends ; therefore the " prism " must be made flat and long, so that the irregularities

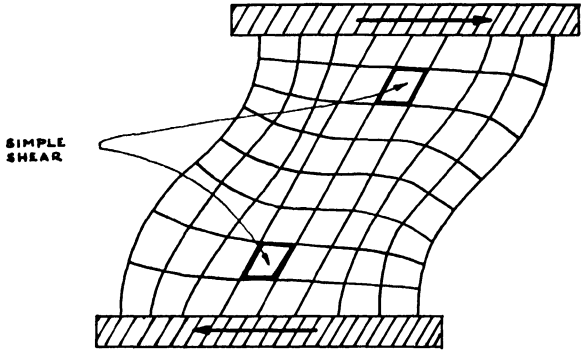


FIG. III, 1. Finite shear of an elastic prism.

at the free ends can be neglected. In every rheological experiment such " end-effects " come into play. If we want to know what will actually happen at the free ends, we must take a material which permits of very large elastic strains so that the deformation can be easily observed. We make a prism of rubber, stick two of its parallel boundary surfaces to solid plates, and shift one plate against the other. We can then observe that the deformation is much more complicated than shown in Fig. I, 3. It is shown in Fig. III, 1, and we see that because of the absence of tangential tractions at the free ends, the elementary prisms into which the body can be imagined to be divided will not all be sheared ; some will remain more or less rectangular.

If we imagine the prism of Fig. I, 3 to be divided into geometrically similar elementary prisms, each one will be deformed in the same manner, and geometrically similar to the complete prism. We say that the strain or deformation is *homogeneous*, i.e. it is the same throughout the body. In contradistinction the deformation shown in Fig. II, 2 is *heterogeneous*. It is simple shear in the interior of the body with right angles changed to acute and obtuse as in Fig. I, 3, but at the free surfaces right angles are not changed, and there is no shear. In order to study the elastic behaviour of a material we therefore must examine deformations other and more complicated than simple shear. These involve changes of volume and our first step will be to deal with volume-elasticity.

2. Epistemological Digression

We have said in Chapter I that under isotropic pressure all materials are elastic and only elastic, and we have called this an axiom of rheology. We did not give any proof for this statement. We relied upon the fact that, being an axiom, it is, as the *Oxford Dictionary* puts it, " a self-evident proposition, not requiring demonstration, but assented to as soon as stated ". This use of the term axiom, going back (again on the authority of the *Oxford Dictionary*) to the year 1600, is, however, not in accordance with modern ideas. We now, even in geometry, and still more so in rheology, do not accept the truth of a proposition without recourse to experience and if, as in the case of an axiom, experience is not drawn upon, the axiom cannot be more than the definition (an " implicit " definition) of a word appearing in the proposition. If we now look back to our statement that " under isotropic pressure all materials are elastic and only elastic " we find that there is no word in it which we have not already defined before, and, therefore, as it stands, the statement *cannot be true*. And so it is. We arrive at the statement by considering such materials as steel, plasticine and water. But if we had put a piece of *wood* under a high hydrostatic pressure we should have observed, after release of the pressure, a Permanent Set, i.e. an instantaneous deformation which is not recoverable. The reason is that there are *pores* in the wood and, under high pressure, stresses are produced which partly crush the material around the hollow spaces of the pores, forcing it into the pores. This local deformation, which exceeds the strength of the material, is not recovered on the removal of the pressure, and the aggregate effect of many such microscopic deformations becomes manifest in the macroscopic set. To make our statement true we must re-word it as follows : " Under isotropic pressure all *non-porous* materials are elastic and only elastic," and in this form the axiom is an implicit (rheological) definition for "non-porous". If we now read this statement together

with the second rheological axiom that "every real material possesses *all* rheological properties", which, as we now realize, is an implicit definition for "real material", we must conclude that no real material is non-porous and we shall not be surprised to learn that Eyring (1936) has propounded a theory of the viscosity of liquids which assumes that even a liquid possesses "holes".

Some readers may think this digression to be out of place, but the situation as described means that the assertions which I have made and shall make about properties of materials are *never quite correct*. They are always only provisional, to be amplified or qualified later. If the reader will keep this in mind, I shall be able to go ahead, speaking often in a loose way, without having to add to every quantitative statement the qualification "approximately". Therefore, let it be granted that there are materials which, on the whole, are under isotropic pressure and tension elastic and only elastic, and let us examine what the laws of such elasticity may be.

3. The Bulk Modulus

If a hydrostatic pressure p is applied to a body, its volume V is reduced and its density ρ, which is the mass per unit volume, or

$$\rho = m/V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

increased. Conversely, an isotropic tension (which it is much more difficult to apply) will increase the volume and reduce the density. Let V_o be the volume for vanishing isotropic pressure $p = 0$ and let ΔV_o be the *increase* of volume produced by p, then an Elastic Bulk Modulus κ is usually defined by

$$-p = \kappa \Delta V_o / V_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from which

$$-\Delta V_o = p/\kappa . V_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and therefore

$$V = V_o + \Delta V_o = V_o (1 - p/\kappa) \quad . \quad . \quad . \quad . \quad (4)$$

Equation (2) is regarded as a special case of Hooke's law, but this definition of the bulk modulus is objectionable for the reason that the latter cannot under any circumstances be constant. Rheological coefficients should generally be defined in such a way that for *certain simple materials and under certain simple conditions they are constants*. The coefficient κ as defined above, can, however, under no circumstances be a constant because then the volume V_o would vanish for $p = \kappa$, with the density increasing to infinity. Actually, if κ is defined as in (2), it is generally found to increase with increasing compression; i.e. in order to produce the same increase of compression the pressure has to be comparatively more increased the greater the compression

from which we start. For instance, Bridgman (1923) found that κ^{-1} varies for metals in accordance with the following empirical formulas:

$$\begin{aligned} &\text{for iron } \kappa^{-1} = 10^{-7}(5{\cdot}87 - 2{\cdot}10 \times 10^{-5}p) \\ &\text{for copper } = 10^{-7}(7{\cdot}32 - 2{\cdot}7 \times 10^{-5}p) \\ &\text{for lead } = 10^{-7}(23{\cdot}73 - 17{\cdot}25 \times 10^{-5}p) \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where p is the hydrostatic *pressure* measured in kg cm^{-2} up to 20,000 kg cm^{-2}.

Other approximate bulk moduli are listed in the Table.

TABLE III, 1. *Bulk Moduli*

Material	Ether	Alcohol	Water	Tuff	Clay	Glass	Mercury	Iron	Steel
$\kappa \times 10^{-11}$ bars	0·08	0·1	0·2	0·5	1	4	5·4	15	18

4. Volumetric Strain

We can derive from (2) a more suitable definition by keeping in mind that this equation was proposed to suit a comparatively small range of stresses.

Let that range be infinitely small, then (2) becomes

$$-dp = \kappa dV_o/V_o \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now let this equation be assumed to be valid not only for the volume V_o, but for any volume V, then

$$-dp = \kappa dV/V \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and this equation defines a bulk modulus which, in principle, could be a constant. Because let κ be a constant, then (6) can be integrated and gives

$$-p = \kappa \ln(V/V_o) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or

$$V = V_o e^{-p/\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In this case, therefore, V vanishes for $p = \infty$ which, in practice, never happens.

We call the quantity

$$e_v = \ln(V/V_o) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

the Volumetric Strain or Cubical Dilatation and we can accordingly write (8) in the form

$$-p = \kappa e_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which is the complete analogy to (1, d). Introducing again $V = V_o + \Delta V_o$, $e_v = ln\ (1 + \Delta V_o/V_o)$ and if $\Delta V_o/V_o$ is small, this expression can be developed into a series, the first two terms of which are

$$e_v = \Delta V_o/V_o - \tfrac{1}{2}(\Delta V_o/V_o)^2 \quad . \quad . \quad . \quad . \quad (12)$$

As a first approximation we find again (2), but as a second approximation we have

$$-p = \kappa \Delta V_o/V_o - \kappa/2.(\Delta V_o/V_o)^2 \quad . \quad . \quad . \quad (13)$$

Indicating by κ' an " apparent " bulk modulus in accordance with (2), we find by introducing

$$\Delta V_o/V_o = -p/\kappa' \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$p = p\kappa/\kappa' + \kappa/2.p^2/\kappa'^2 \quad . \quad . \quad . \quad . \quad . \quad (15)$$

from which

$$\kappa' = \kappa - p\kappa/2\kappa' \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$\kappa' = \frac{\kappa}{2}[1 + \sqrt{1 + 2p/\kappa}] \quad . \quad . \quad . \quad . \quad . \quad (17)$$

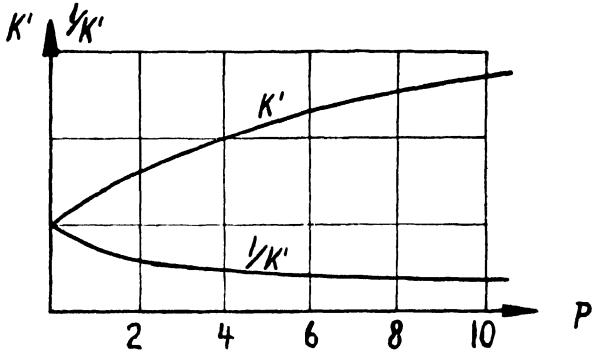


FIG. III, 2. Apparent bulk modulus (κ') and apparent coefficient of compressibility ($1/\kappa'$) as a function of the hydrostatic pressure p.

If we plot κ' against p we get a curve as shown in Fig. III, 2. The Apparent Coefficient of Compressibility (κ'^{-1}) has also been plotted. It can be seen from these curves that the assumption that κ in (11) is a constant with e_p in the logarithmic measure gives a better approximation to the compressibility of materials because the compressibility in accordance with this formula decreases with increasing pressure. However, comparison with the empirical equations (5) shows that quantitatively for large pressures the position is still far from satisfactory. Following Hencky (1931) we therefore proceed to apply two corrections.

5. The Reduced Stress

For the first correction we must remember that the resistance of the material to the external forces comes from reactive forces between the molecules or atoms, as the case may be, caused by a change in their stable distances. The forces, accordingly, are bound up with the

quantity of *matter*. The stresses, on the other hand, are defined in respect of the areas through which the forces act and a unit isotropic pressure will be a unit force applied upon the sides of a cube of unit length of edge and unit *volume*, say 1 cm.^{3} That cubic centimetre of volume, however, contains in the strained state, assuming tension and $V > V_o$, fewer molecules or atoms than in the unstrained state, in short, less matter. Conversely, the unit of matter in the unstrained state will in the strained state take up a volume increased in the proportion ρ_o/ρ where ρ_o is the density in the unstrained, ρ the density in the strained state. To calculate from the stresses the reactive force caused *by the same piece of matter*, we must increase the stress in the same proportion, i.e. we must replace the stress as defined in Chapter I by the Reduced Stress

$$p_{red} = \frac{\rho_o}{\rho} p = V/V_o . p = V_{rel} p \quad . \quad . \quad . \quad . \quad (18)$$

where V_{rel} is the Relative Volume $= V/V_o$. This makes (11)

$$p = \kappa(\ln V_{rel})/V_{rel} = \kappa e_v e^{-e_v} \quad . \quad . \quad . \quad . \quad (19)$$

where e is the basis of " natural " logarithms.

6. The Limiting Relative Volume

The second correction refers to the feature of (11) or (19) that with *infinite* pressure the volume of the body would be *compressed to zero*. There must, however, be a limiting volume containing all the molecules even if in a collapsed and crumpled-up state (the latter was realized by Bridgman reaching pressures up to 20,000 kg cm^{-2}.) Hencky, on applying (19) to Bridgman's observations, found that for a small number of solids this was a satisfactory formula. Most of the experiments, however, suggested that there was a Limiting Relative Volume Ψ which could not be reached by finite pressure. He therefore suggested the formula

$$p = (\kappa) \frac{1 - \Psi}{V_{rel} - \Psi} \frac{\ln V_{rel}}{V_{rel}} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which he compared with Bridgman's observations and especially of those substances which Bridgman had marked out as very irregular and therefore particularly fit for a crucial test. Near atmospheric pressure (20) does not give as good results as at higher pressures and the simpler formula (11) may then be used. Hencky concluded that the individual structure of the molecule apparently has much influence at low pressures, but it seems that at pressure from 10,000 kg. cm^{-2} upwards even gases acquire a constant bulk modulus in accordance with (20). The following Table gives the results of his calculations. Equation (20) can be considered as expressing the modulus

κ of (19) by $(\kappa)\ (1 - \Psi)/(V_{rel} - \Psi)$. While now κ is a variable coefficient and not a constant, (κ) and Ψ are supposed to be constants.

TABLE III. 2. *Hencky's Bulk Modulus* (κ) *and Limiting Volume* (Ψ) *of certain Substances calculated with Hencky's Theory* (20) *from Bridgman's Observations.*

Substance	Temperature Centigrade	$\kappa \cdot 10^4$ in kg/cm.^{2}	Ψ
Iron	30	170	0·8
Rubidium . . .	50	1·66	0·44
Mercury . . .	20	24·9	0·75
Water . . .	40	2·34	0·50
Alcohol . . .	20	4·17	0·52
Hydrogen . . .	65	0·82	nearly zero
Helium . . .	65	0·47	,,

7. Pulsations

We have said in Section 5 that the internal forces with which the material reacts to the external forces are due to the displacements of the molecules or atoms from their stable relative positions. To effect such displacement, external work has to be expended which is converted into the potential elastic energy of the strained body. Let us look into this process in more detail.

If we apply a finite isotropic pressure $\bar{p}$ upon a body,* its particles will be brought into movement in accordance with (I, a), i.e. after some time through the operation of the acceleration a, they will attain certain velocities v and will by virtue of these possess kinetic energy of the magnitude

$$E_\kappa = mv^2/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

At the same time they will undergo displacements u and this will produce in their surroundings elastic stresses in accordance with (11) or (20) or some other rheological equation. Let us assume the body to be a sphere of radius R_0, the centre of which is fixed, then, for reasons of symmetry, we may assume that the direction of the different u's and v's passes through the centre. To every $\bar{p}$ there corresponds a certain definite e_v and therefore a certain definite radius R at which the internal stresses p raised in the body through the strain will balance the external pressure $\bar{p}$. If $\bar{p}$ is applied from the start to the full extent, this means that before R_0 is reduced to R there is no equilibrium ; this causes the appearance of kinetic energy. When the sphere is compressed to V, p balances $\bar{p}$, but the particles will move on

* The bar above p indicates the *external* force.

towards the centre because they have kinetic energy. This increases p beyond $\bar{p}$ and there are unbalanced stresses acting *away* from the centre. This, in turn, causes a negative acceleration a or retardation $-a$, a diminution of v and ultimately a state of rest when v vanishes. This state of rest, however, does not correspond to a state of equilibrium. The stress p exceeds the external pressure $\bar{p}$; rest lasts only through an element of time ; the body starts to expand. We need not continue this story ; the result will be voluminal oscillations or pulsations around the position of equilibrium $e_v = p/\kappa$; a position which, however, is not one of rest.

8. Volumetric Strain-work

If we want to determine the compressibility of a material in a suitable test, we shall therefore have to apply the external pressure very slowly, practically *infinitely slowly*, i.e. we shall have to increase the pressure from zero to $\bar{p}$ so slowly that it is always equal to p and the body passes continuously *through states of equilibrium*. Considering now the sphere of radius R which is at rest under the action of the

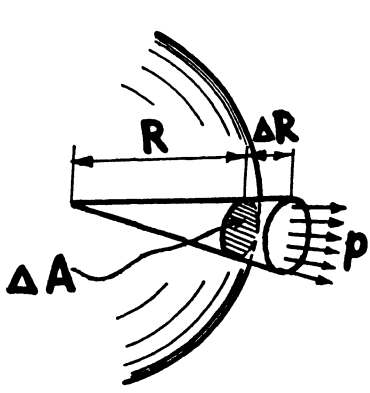


FIG. III, 3. Volumetric strain-work.
p isotropic tension. ΔR increase of radius.

isotropic tension p applied upon its surface A ; if p is slightly increased the radius will be slightly increased by ΔR and the forces acting upon the surface perform work. The force acting upon an element of the surface (ΔA) is $p.\Delta A$ and the work is $p.\Delta A.\Delta R$ (compare Fig. III, 3). Adding up all elements of work over the whole surface of the sphere, we get $p.A.\Delta R$. If ΔR is very small, the increase of volume is $dV = A\,dR$ and, therefore,

$$dW = -pdV \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and the elementary work per unit volume or the specific work

$$dw = -pdV/V = -pde_v \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and the power per unit volume or the specific power

$$\dot{w} = dw/dt = -p\dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If $\bar{p}$ is gradually increased from zero to p, the work performed per unit volume is

$$w = \int_o^{e_v} -pde_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If the load p is decreased from p to zero,

$$w' = \int_{e_v}^{o} -pde_v = -\int_o^{e_v} -pde_v = w \quad . \quad . \quad . \quad . \quad (26)$$

and the strainwork is completely regained. The only condition for this reversibility is that there should be a one-to-one relation between p and e_v, but the *form* of the relation, whether straight line or curved, is immaterial. Generally, *in a perfectly elastic body, all strain-work performed by the external forces through states of equilibrium is stored up in the body as potential energy and completely regained upon removal of the external forces, if this is likewise carried out through states of equilibrium.*

If p is expressed as a function of e_v by means of a rheological equation, the integration can be carried out. For instance, if we use (11),

$$w_v = \kappa e_v^2/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or

$$w_v = p^2/2\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where we have added the subscript v to indicate " volumetric ".

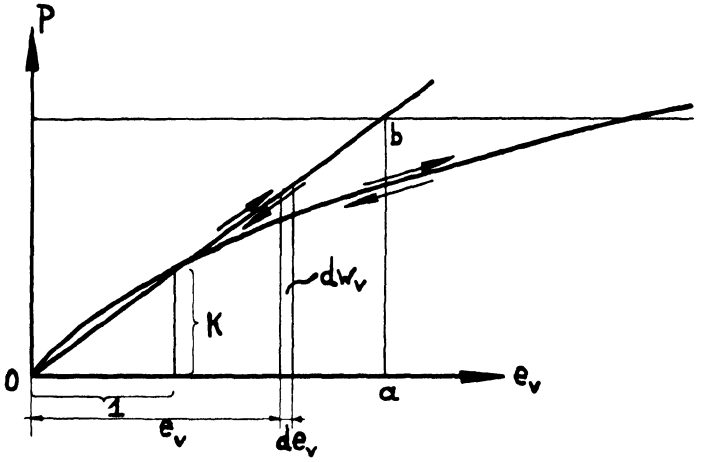


FIG. III, 4. Fundamental rheological curve for infinitely slow compression. Area *o a b* measures the strain-work.

e_v volumetric strain.
p isotropic stress.
k modulus of compressibility.

If we plot correlated observed values of p against e_v, the resultant curve is a graphical representation of the rheological equation (compare Fig. III, 4). If the curve is a straight line, κ in (11) is a constant ; otherwise κ is variable. Comparing (23), we see that the elementary work per unit volume is represented in the figure by an elementary strip parallel to the p-axis and of width de_v. The total work performed when increasing the pressure from zero to p is therefore represented by the area enclosed by the curve, the e_v axis and the ordinate p. If all stored-up energy is to be regained as the material expands with decreasing pressure, the points representing descending observations must lie on the same curve or, as we said before, p must be a one-valued function of e_v.

9. Simple Stresses and Deformations

So far we have become acquainted with *two simple cases of stress and deformation.* They are simple shearing stress and simple isotropic stress on the one hand—simple shear and simple cubical dilatation on the other. These are *co-ordinated* pairs, simple shearing stress producing simple shear, and simple isotropic stress producing simple cubical dilatation. There exist, however, two other important cases of stress and deformation with which we have not yet dealt and which are of an entirely different character. They are *simple tension* on the one hand and *simple extension* on the other. Simple tension is a one-dimensional stress and simple extension is a one-dimensional deformation, *but the latter is not produced by the former.*

Simple tension and its opposite, simple pressure, can both be included under the term simple Normal Stress, designated by σ. A simple normal stress is produced by either simple Pull ($+ P_n$) or simple Push ($- P_n$) upon a prismatic bar, acting in the direction normal to, and passing through the centre of the cross-section.* Under the action of such simple pull, the bar is elongated ; at the same time, however, it contracts laterally. *A one-dimensional traction is here accompanied by a three-dimensional deformation.*

Simple extension, or its opposite, simple compression, both included under the term simple Normal Deformation (d_n), because the displacement is in the direction *normal* to the cross-section, is not so easily realized. A plastic material filling a cylinder, where the latter is so rigid that it does not extend in diameter, would, under pressure from a piston, be deformed in the manner of simple compression. Here the walls of the cylinder will also produce pressure upon the material and *the one-dimensional deformation is therefore accompanied by a three-dimensional stress.* However, in order to produce in a plastic material

* If the force does not pass through the centre of the cross-section, *bending* is caused, with which we deal in Chapter IV.

a measurable compression, the pressure would have to be so high that the walls of the cylinder, too, would give way slightly, albeit elastically. Nevertheless, in porous materials, simple pressure may produce simple compression. One such material is cork, another porous rubber ; still another, as we shall learn in Chapter XII, is concrete in " creep ". These, however, are exceptions.

10. Young's Modulus

The solution to this puzzle—viz. that simple normal stress does not correspond to simple normal deformation or strain (and *vice versa*)—is that " simple " normal stress and deformation are not as simple as one may think.

Let us first consider simple normal stress, for instance as produced by the simple pull of a cylinder or prismatic steel bar in the so-called " tensile test ". This is the predominant test for metals, but has also been in use for such materials as cement, pitch, bitumen, flour dough, etc. In this test a short bar, say of mild steel, of length l_o is fixed between two pairs of jaws (or some similar device), one stationary and the other movable, and is elongated by a gradually increasing load P_n, where n indicates " normal ". If A is the cross-sectional area of the bar, a traction is produced equal to

$$\sigma = P_n/A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

acting in the longitudinal direction, or normal to the cross-section, and which may be assumed as uniformly distributed over the cross-section. The assumption is not correct near the ends, where the bar is fixed between the jaws, but it will be valid at some distance from the ends, especially if the bar is slender.* Up to a certain point, the *elongation* Δl is proportional to the load and therefore follows Hooke's law. For this part we write by analogy to (I, 5)

$$\Delta l/l_o = (P_n/A)/E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where the coefficient E is called Young's Modulus.

The ratio $\Delta l/l_o$ is usually taken as the measure of the normal strain, here called Extension, positive or negative. For small elongations Δl there is no objection to this. For large elongations, however, this definition breaks down ; firstly, for the same reason as advanced in Section 4 in respect of the volumetric strain ; and, secondly, because there is no reason why $\Delta l/l_o$ should be a more correct measure than $\Delta l/l$, where $l = l_o + \Delta l$. For $\Delta l = l_o$, the first gives a 100 per cent., the second a 50 per cent. increase.† A consistent definition would result

* This is the Principle of St. Venant for which compare *Twelve Lectures*, pp. 67 68.

† Recently a local paper quoted the President of an American College as having said that " the price of automatic pilot devices had *dropped by several hundred per cent.* during the war." It must be assumed that he did not intend to say that every buyer of a pilot device is being paid a premium.

from relating a differential of increase to the *instantaneous* length so that the element of strain is dl/l and the total longitudinal deformation, if it is large, is defined by

$$D_l = \int_{l_o}^{l} dl/l = \ln(l/l_o) \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

in which formula both l and l_o are of equal standing.

This logarithmic measure of extension was first introduced by Roentgen, of X-ray fame. It was first systematically employed by Hencky (1929, 1) (compare also *Twelve Lectures*, pp. 31-32).

We can develop the expression Equ. (31) as follows :

$$D_l = \ln(l/l_o) = \ln \frac{l_o + \Delta l}{l_o} = \ln(1 + \Delta l/l_o) = \Delta l/l_o - \tfrac{1}{2}(\Delta l/l_o)^2 + \ldots \quad (32)$$

We have introduced in Equ. (10) above a logarithmic measure for volumetric strain. By writing e_l for D_l we can similarly introduce a logarithmic measure for its longitudinal component

We can now write for the Hooke solid

$$e_l = \sigma/E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

listing in the following Table III, 3, values for Young's modulus for different materials It will be noted that these are quite large, so that e_l is very small and can then be expressed by

$$e_l = \Delta l/l_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

neglecting in Equ. (32) the terms $(\Delta l/l_o)^n$ for $n > 1$ against $\Delta l/l_o$. It is different with rubber and rubbery materials with rather small E. We cannot go into the problems connected with the elasticity of such materials, and the reader is referred to the book by Treloar (1949).

TABLE III, 3. *Young's Moduli*

Material	Lead	Concrete	Tin	Glass	Zinc	Copper	Wrought Iron	Steel
$E \times 10^{11}$ in bars	1	2	4	7	8	12	18	25

11. Poisson's Ratio

As we have already pointed out, the pull P_n produces in the bar not only an axial extension but, generally, at the same time also lateral contractions e_c of magnitude

$$e_c = -\nu e_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where ν is a material constant called " Poisson's ratio ". If e_l is negative, e_c will be positive or be a lateral expansion. Such lateral

expansion cannot be observed in a cork-cylinder under pressure, in which case $\nu = 0$.

We have thereby introduced for the Hooke solid two new constants, E and ν, so that altogether we have four constants μ, κ, E and ν. However, the values for μ and κ completely determine the rheological

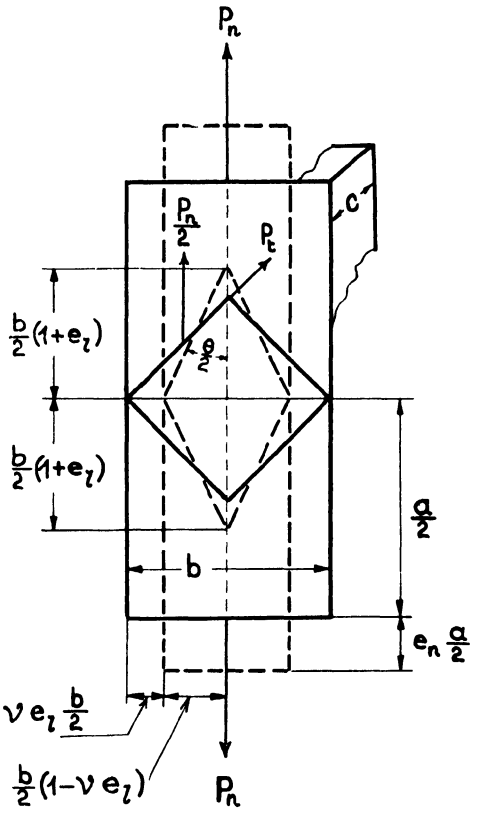


FIG. III, 5. Simple pull of a prismatic bar.

behaviour of a Hooke solid and it must therefore be possible to express E and ν by the former. We regard μ and κ as "fundamental" constants, because they measure, each one separately, the independent changes of shape and volume.

Let us consider an axial section of the bar shown in Fig. III, 5. Generally, when being extended, the volume of the bar will be increased. At the same time, as can be seen from that figure, certain right angles will be changed and certain parallel planes accordingly sheared against each other. Therefore, the simple normal stress σ will be accompanied by a volumetric strain e_v and a tangential strain γ. Similarly, it is obvious that a simple extension e_l is accompanied by an isotropic

stress p and a tangential stress τ. Therefore, E and ν are "derived" constants, each one depending on both, change of shape and change of volume.

Our task will consist in finding the stresses p and τ equivalent to σ in the first case and the strains e_v and γ equivalent to e_l in the second case.

Poisson's ratio is a pure number. Approximate values of Poisson's ratio for different materials are listed in the following Table :

TABLE III, 4. *Poisson's Ratios*

Material	Cork	Aluminum	Zinc	Glass	Steel	Copper	Lead	Rubber
ν	0	0·13	0·20	0·24	0·30	0·32	0·40	0·49

12. Principal Deformations

If the bar is not itself a prism, it can be considered as made up of a large number of "elementary" prisms, the deformation of which we shall presently investigate.

Let the prism of edges a, b, c be extended or contracted in the directions of its edges. These directions are at right angles to each other before deformation and remain so after deformation. If we look at Fig. I, 1, we see that in the case of simple shear there also exist three such directions. They are called the Principal Axes of Deformation. In the case of simple shear, as again to be seen from Fig. I, 1, there is not only deformation but also rotation. One axis (the one normal to the plane of the figure) is fixed, but the other two are rotated by the same angle around the third. In our case, where the straining of the prism is in the direction of the edges, there is no such rotation. A deformation without rotation is called Pure Deformation.

We denote the directions of the principal axes by i, j, and k. If these directions are the same throughout the body, the deformation is said to be *homogeneous*. This is the case in simple extension.

Now, let the edges of the prism before strain be a_o, b_o, c_o, so that $V_o = a_o b_o c_o$, and after strain a, b, c, so that $V = abc$.

The principal deformations are in the logarithmic measure

$$D_i = \ln(a/a_o),\ D_j = \ln(b/b_o),\ D_k = \ln(c/c_o) \quad . \quad . \quad (36)$$

Therefore

$$D_v = \ln(V/V_o) = \ln(abc/a_o b_o c_o) = \ln(a/a_o) + \ln(b/b_o) + \ln(c/c_o) = D_i + D_j + D_k \quad . \quad (37)$$

If we define Mean Normal Deformation D_m by

$$D_m = (D_i + D_j + D_k)/3 \quad . \quad . \quad . \quad . \quad . \quad (38)$$

we have

$$D_v = 3D_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

If we deduct from each of the three principal deformations the mean deformation in the following manner

$$D_{i(o)} = D_i - D_m,\ D_{j(o)} = D_j - D_m,\ D_{k(o)} = D_k - D_m \quad (40)$$

three other principal strains result, the cubical dilatation of which vanishes in accordance with

$$D_{v(o)} = D_{i(o)} + D_{j(o)} + D_{k(o)} = D_i + D_j + D_k - 3D_m = 0 . \quad (41)$$

Therefore $D_{i(o)}$, $D_{j(o)}$, $D_{k(o)}$ are the principal *distortions* resulting from deformations D_i, D_j, D_k. *Every combination of principal deformations* D_i, D_j, D_k, *can be resolved into a dilatation* $D_v = D_i + D_j + D_k$ and a *distortion* $D_i - D_v/3$, $D_j - D_v/3$, $D_k - D_v/3$.

We shall indicate the distortional deformation by the subscript (o).

In the case of simple normal stress (or, more correctly, traction) σ, the principal strains are e_l in the longitudinal (the k-direction) and $-\nu e_l$ in the two transverse directions (the i- and j-directions), and therefore

$$\left.\begin{array}{l} e_v = e_l(1-2\nu)\ ;\ e_m - e_v/3 = e_l(1-2\nu)/3 \\ e_{i(o)} = e_{j(o)} = -\nu e_l \quad e_m = (1+\nu)/3 . e_l\ ;\quad e_{k(o)} = e_l - e_m = 2(1+\nu)\, e_l/3 \end{array}\right\} \quad (42)$$

For an incompressible material, e_v must vanish or $\nu = \frac{1}{2}$, in which case $e_{i(o)} = e_{j(o)} = -e_l/2$; $e_{k(o)} = e_l$.

All these considerations apply also if d and e are defined not in the logarithmic measure, but in accordance with Equ. (34).

13. Principal Stresses

The foregoing considerations in respect of strains can be repeated with suitable modifications in respect of stresses. Here *Principal Stress Axes* exist and *in isotropic materials these coincide with the principal axes of strain*, so that we can use for them the same subscripts, i, j, k. The Mean Normal Stress

$$\sigma_m = (\sigma_i + \sigma_j + \sigma_k)/3 \quad . \quad . \quad . \quad . \quad . \quad (43)$$

is that part of the stress which, when applied isotropically in the manner of a hydrostatic stress, causes the cubical dilatation (with positive or negative sign as the case may be) of the material, while the normal stresses

$$\sigma_{i(o)} = \sigma_i - \sigma_m\ ;\ \sigma_{j(o)} = \sigma_j - \sigma_m\ ;\ \sigma_{k(o)} = \sigma_k - \sigma_m \quad . \quad (44)$$

cause the distortion. In the case of simple normal tension σ

$$\left.\begin{array}{l} \sigma_k = \sigma;\ \sigma_i = \sigma_j = 0\ ; \qquad p = \sigma_m = \sigma/3 \\ \sigma_{i(o)} = \sigma_{j(o)} = -\sigma/3\ ;\ \sigma_{k(o)} = 2\sigma/3 \end{array}\right\} \quad . \quad . \quad . \quad (45)$$

Now if the cubical dilatation e_v, as calculated from (41) is caused

by the mean stress σ_m, as expressed in (43), Equ. (11) requires $-p = \kappa e_v$, or $\sigma/3 = \kappa e_l\ (1 - 2\nu)$, and because of (33)

$$\kappa = E/3(1 - 2\nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

14. Principal Strains in Simple Shear

We have said in Section II, 6, that when simple shear changes a square into a rhombus, one diagonal is elongated and the other shortened. These length-changes define principal strains, the amounts of which we shall presently calculate.

We consider a square with sides of unit length, subjected to simple shear. The square shown in Fig. III, 6 in dotted lines becomes a rhombus as shown in full lines. If the displacement $BB' = CC'$ is very small, the originally right angle between the diagonals AC and BD, which after strain are AC' and $B'D$, remains a right angle. AC, however, is extended to AC' and BD contracted to $B'D$. Simple shear is therefore equivalent to an extension and contraction *without shear* in two directions which are at right angles to each other and inclined

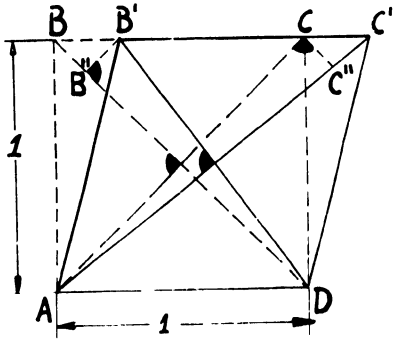


Fig. III, 6. Simple shear of square ABCD into rhombus A, B', C', D. ◀ indicates right angle exactly or approximately).

by 45° against the direction of shear. As can be seen $BB'' = BB'/\sqrt{2}$. On the other hand DB'' is equal to DB' with good enough approximation.

Furthermore, since $BD = \sqrt{2}$, we have $(DB - DB')/DB = (DB - DB'')/DB = BB''/DB = \dfrac{BB'}{\sqrt{2}}/\sqrt{2} = BB'/2$. Similarly it is found that $(AC' - AC)/AC$ is equal to $CC'/2$. As AB is unity, $BB' = CC' = \gamma$, and a simple shear γ is equivalent with an extension $\gamma/2$, combined with a contraction $\gamma/2$, in directions which are mutually perpendicular, while either is inclined at an angle of 45° to the direction of the shearing displacement.

15. The Law of Elastic Distortion

The shear of the square inscribed into the rectangle of Fig. III, 5, is produced by a shearing stress resulting from the action of forces $P_n/2$, acting on all sides of the square. These forces which act in the longitudinal direction, have tangential components $P_t = \frac{P_n}{2}/\sqrt{2}$ The tangential forces act upon areas $= \frac{b\sqrt{2}}{2}.c$, and the tangential stresses are therefore

$$\tau = P_t/\frac{b\sqrt{2}}{2}c = P_n/2bc.$$

Introducing $P_n/bc = \sigma$, this gives

$$\tau = \sigma/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

For a Hookean solid we can go back to Equs. (I, d) and (33) and Equ. (47) becomes

$$2\mu\gamma = Ee_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

In Section 5 of Chapter I, we have defined γ as the reduction of an originally right angle produced by shear. From Fig. III, 5, this reduction is $(90 - \theta)$ or $2(45 - \frac{\theta}{2})$. But, also from this figure

$$\tan\frac{\theta}{2} = \frac{1 - \nu e_l}{1 + e_l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

and from known formulæ

$$\tan(45 - \frac{\theta}{2}) = \frac{\tan 45 - \tan\frac{\theta}{2}}{1 + \tan 45 \tan\frac{\theta}{2}} = \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}} \quad . \quad (50)$$

Introducing $\tan\theta/2$ from Equ. (49) we find

$$\tan(45 - \frac{\theta}{2}) = \frac{e_l(1 + \nu)}{2 + e_l(1 - \nu)} \quad . \quad . \quad . \quad . \quad (51)$$

We now assume that e_l is a small number, and that we can neglect $e_l(1 - \nu)$ against 2. Therefore in our case

$$\gamma = e_l(1 + \nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

This makes Equ. (48) to yield

$$2\mu(1 + \nu) = E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

From the two equations (46) and (53) we can now express the derived coefficients E and ν through the fundamental coefficients κ and μ :—

$$E = 9\kappa\mu/(3\kappa + \mu)\ ;\ \nu = (3\kappa - 2\mu)/2(3\kappa + \mu) \quad . \quad . \quad (54)$$

For an incompressible material $\kappa = \infty$, and therefore

$$E^* = 9\mu(3 + \mu/\kappa) = 3\mu \; ; \; \nu^* = \tfrac{1}{2} \quad . \quad . \quad . \quad . \quad (55)$$

From (45) we find

$$\sigma_{1(o)} = \sigma_{2(o)} = -Ee_l/3 \; ; \; \sigma_{k(o)} = 2Ee_l/3 \quad . \quad . \quad . \quad (56)$$

and considering (42)

$$\left.\begin{aligned} &\sigma_{1(o)} = E/(1+\nu).e_{1(o)} \; ; \; \sigma_{2(o)} = E/(1+\nu).e_{2(o)} \; ; \\ &\sigma_{k(o)} = E/(1+\nu).e_{k(o)} \end{aligned}\right\} \quad . \quad (57)$$

Because of (53), these now become

$$\sigma_{1(o)} = 2\mu e_{1(o)} \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

and from the symmetry of the expressions it is clear that the *relations* (58) *express Hooke's law for elastic distortion*, independent of the conditions of simple normal stress from which they were derived. We can accordingly write more concisely

$$\sigma_{n(o)} = 2\mu e_{n(o)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where n now stands for *any* direction of the normal to *any* section however oriented.

If the distortion is accompanied by a cubical dilatation, very much changing the density of the body, the stresses $\sigma_{(o)}$ must be replaced by the reduced stresses $\frac{\rho_o}{\rho}\sigma_{(o)}$ in accordance with (18).

16. Strain-work in Simple Tension

Let us go back to the tensile test of a steel bar extended elastically in simple tension with which we started in Section 10. In extending the bar, the pull P_n will perform work. In the testing machine, the speed of extension is moderate and the change will be through states of equilibrium so that kinetic energy is not generated. Therefore all work of the external forces will be converted into strain-work. Consider again an elementary prism with edges a, b, c (where a is in the direction of the axis of the test-piece) which is in equilibrium under the action of the simple normal stress σ. If σ is slightly increased, the edge a will be increased by Δa, while b and c will be decreased by Δb and Δc respectively. However, while in the direction of a, the force $\sigma\, bc$ is acting, no forces act in the directions b and c. The increment of work will therefore be

$$\Delta W = \sigma bc\Delta a = \sigma abc.\Delta a/a \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

If Δa is very small, $\Delta a/a = de_l$ and the element of work per unit volume

$$dw = \sigma de_l = Ee_l de_l \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

This equation is perfectly analogous to (23) and analogous conclusions can be drawn which need not all be enunciated here.

For the simple Hooke body, where E is constant, (61) gives by integration

$$w = Ee_l^2/2 = \sigma^2/2E \quad . \quad . \quad . \quad . \quad . \quad (62)$$

17. The Mohr-Circle for Simple Tension

In the preceding Sections we have considered simple tension as shown in Fig. III, 5. We found in Section 15, that the pull P_n generates a certain shearing force $P_t = \frac{P_n}{2}$ in a section inclined at 45° against the cross-section of the prism or a shearing traction $\tau = \sigma/2$. The question arises : what are the tractions, both normal and shearing, in sections inclined at *any* angle ϕ against the cross-section ?

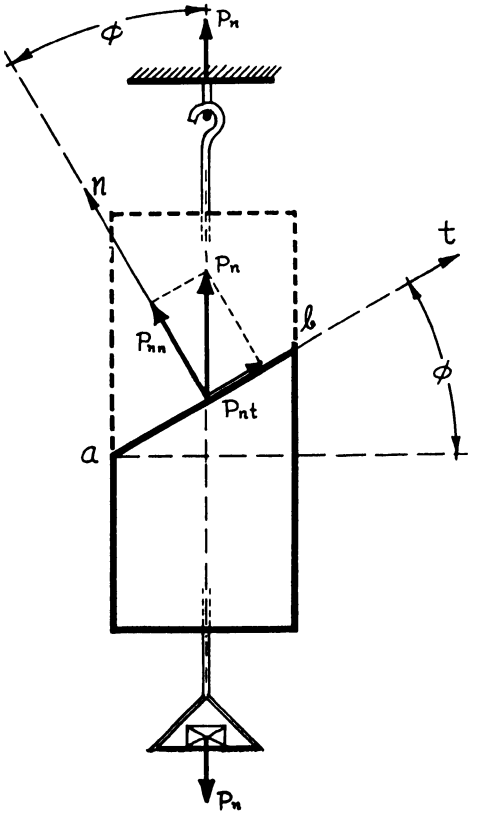


FIG. III, 7. Stresses in simple tension.
P_{nn} normal component of pull P_n. P_{nt} tangential component of pull P_n.

If we examine Fig. III, 7, it is clear that in every section through the rod, the pull P_n is equivalent to a tangential force

$$P_{nt} = P_n \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

together with a normal force

$$P_{nn} = P_n \cos \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Let the cross-section of the rod be unity, then the area of the inclined section is

$$A_\phi = 1/\cos \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

and the tractions in the inclined section

$$\left.\begin{aligned} \tau_n &= P_{nt}/A_\phi = \sigma \sin \phi \cos \phi = \frac{\sigma}{2} \sin 2\phi \\ \sigma_n &= P_{nn}/A_\phi = \sigma \cos^2 \phi = \frac{\sigma}{2}(1 + \cos 2\phi) \end{aligned}\right\} \quad . \quad . \quad (66)$$

Therefore, when ϕ varies from zero to 90°, the normal component of the traction (σ_n) acting upon the inclined section with normal n decreases from σ to zero, while the tangential component (τ_n) goes from zero to zero, passing through a maximum for $\phi = 45°$, when

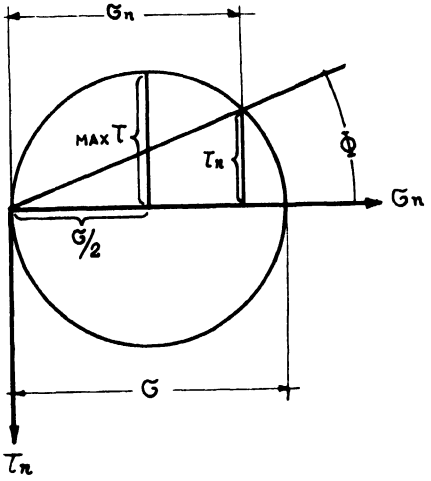


Fig. III, 8. Mohr's Circle for simple tension. ϕ slope of inclined section.

max $\tau_n = \sigma/2$, while in that section σ_n is also $= \sigma/2$. These relations can be pictured graphically in a coordinate system where the σ_n are abscissas and τ_n ordinates. As can be seen from Fig. III, 8, comparing (66), all points representing connected values of σ_n and τ_n lie on a circle.

This is a special case of the so-called Mohr-circle, about which more will be said in Section 4 of Chapter VII.

18. Summary

The rheological behaviour of every *homogeneous* material, solid or liquid, conforms under isotropic stress p to the rheological equation

$$-p = \kappa e_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, a)}$$

where e_v, the volumetric strain,

$$e_v = \ln(V/V_o) = \ln(V_{rel}) \quad . \quad . \quad . \quad . \quad . \quad \text{(III, b)}$$

or, to a first approximation,

$$e_v = \Delta V/V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, c)}$$

In liquids p is called " hydrostatic pressure ".

In cases where the density ρ of the material is much changed by pressure, ρ must be replaced by the reduced stress

$$p_{red} = \frac{\rho_o}{\rho} p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, d)}$$

The volumetric strain-power is

$$\dot{w}_v = \kappa e_v \dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, e)}$$

and is entirely conserved. If κ is constant, the conserved (potential) energy or strain-work is

$$w_v = \kappa e_v^2/2 = p^2/2\kappa \quad . \quad . \quad . \quad . \quad \text{(III, f)}$$

Where κ is a variable coefficient, Hencky has proposed

$$\kappa = (\kappa) \frac{1 - \Psi}{V_{rel} - \Psi} \quad . \quad . \quad . \quad . \quad . \quad \text{(III, g)}$$

Ψ being the relative limiting volume and (κ) a constant.

In every deformed body there exists at every point a triplet of normal axes i, j, k,, which were at right angles, too, before deformation. They are called *principal axes of deformation* (or strain, as the case may be). As the right angles between them do not change, they are not sheared, and the deformation consists of positive or negative extensions in the direction of the principal axes or normal to the planes containing them. They are accordingly *normal* deformations (D_n) called *principal deformations*, and denoted by D_i, D_j, D_k. In an *isotropic* material these deformations are connected wtih *normal stresses* (σ_n) called *principal stresses* and denoted by σ_i, σ_j, σ_k.

If the directions of i, j, k in space are the same before and after deformation, the latter is called *pure* deformation.

If the directions i. j. k and the magnitudes of D_i, D_j, D_k are the same throughout the body, the deformation is said to be *homogeneous*.

In a prism with edges a, b, c oriented in the directions i, j, k respectively, with lengths of edges a_o, b_o, c_o before deformation, the principal deformations are *extensions*

$$D_i = \ln(a/a_o),\ D_j = \ln(b/b_o),\ D_k = \ln(c/c_o) \quad . \quad \text{(III, h)}$$

or to a first approximation

$$d_i = \Delta a/a,\ d_j = \Delta b/b,\ d_k = \Delta c/c \quad . \quad . \quad . \quad \text{(III, i)}$$

and the cubical dilatation

$$D_v = D_i + D_j + D_k \quad . \quad . \quad . \quad . \quad \text{(III, j)}$$

The principal deformations

$$D_{i(o)} = D_i - D_v/3\ ;\ D_{j(o)} = D_j - D_v/3\ ;\ D_{k(o)} = D_k - D_v/3 \quad \text{(III, k)}$$

for which $D_{i(o)} + D_{j(o)} + D_{k(o)}$ vanishes, constitute a *distortion.*

In accordance with the first axiom of rheology (compare Section 3 of Chapter I) the characteristic rheological equation of a material connects the $e_{i(o)}$ with stresses $\sigma_{i(o)}$ etc., which are to be derived from the σ_i etc. through

$$\sigma_{i(o)} = \sigma_i - \sigma_m\ ;\ \sigma_{j(o)} = \sigma_j - \sigma_m\ ;\ \sigma_{k(o)} = \sigma_k - \sigma_m \quad . \quad . \quad \text{(III, l)}$$

where the isotropic stress σ_m is the mean of the principal stresses

$$\sigma_m = (\sigma_i + \sigma_j + \sigma_k)/3 = -p \quad . \quad . \quad \text{(III, m)}$$

In the case of simple tension, σ,

$$\sigma_m = \sigma/3\ ;\ \sigma_{i(o)} = \sigma_{j(o)} = -\sigma/3\ ;\ \sigma_{k(o)} = 2\sigma/3 \quad . \quad \text{(III, n)}$$

The isotropic stress p is connected with the cubical dilatation e_v by means of (III, a), which is the isotropic rheological equation valid for every kind of material.

For the distortional part in the case of the Hooke solid

$$\sigma_{n(o)} = 2\mu\, e_{n(o)} \quad . \quad . \quad . \quad . \quad . \quad \text{(III, o)}$$

When a prismatic bar of length l_o is acted upon by a pull $(+ P_n)$ or push $(- P_n)$ in the direction of its axis (direction k) they produce a simple one-dimensional stress σ_n. This causes a three dimensional pure deformation. In a Hooke-solid this consists of a longitudinal extension without shear

$$D_l = \ln(l/l_o) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, p)}$$

or for small extensions

$$e_l = \Delta l/l_o \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, q)}$$

which is connected with σ_n through

$$e_l = \sigma_n/E \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, r)}$$

and lateral contractions of the magnitude

$$e_c = -\nu e_l \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, s)}$$

In sections inclined by 45° to the axis there is a maximum shearing strain

$$e_t = e_l(1 + \nu) \quad . \quad . \quad . \quad . \quad . \quad \text{(III, t)}$$

The " derived " elastic coefficients E and v can be calculated from the " fundamental " coefficients κ and μ in accordance with

$$E = 9\kappa\mu/(3\kappa + \mu) \quad . \quad . \quad . \quad . \quad . \quad \text{(III, u)}$$

$$\nu = (3\kappa - 2\mu)/2(3\kappa + \mu) \quad . \quad . \quad . \quad . \quad \text{(III, v)}$$

On the other hand, if E and ν are determined experimentally, one can calculate κ and μ from

$$\kappa = E/3(1 - 2\nu) \; ; \; \mu = E/2(1 + \nu) \quad . \quad . \quad \text{(III, w)}$$

when the material is incompressible, or $\kappa = \infty$, we find

$$E^* = 3\mu \; ; \; \nu^* = \tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad \text{(III, x)}$$

The strain-work per unit volume in simple tension is

$$w_n = \int \sigma_n de_n = \int E e_n de_n \quad . \quad . \quad . \quad . \quad \text{(III, y)}$$

which for the simple Hooke body when E is constant

$$w_n = E e_n^2/2 = \sigma_n^2/2E \quad . \quad . \quad . \quad . \quad \text{(III, z)}$$

CHAPTER IV

BENDING AND TORSION*

1. Homogeneous Deformation and Stress

In the foregoing we have become acquainted with three cases of simple deformation, viz., simple shear, simple cubical dilatation and simple extension. These three are homogeneous deformations, a term which has been explained in Chapter III, 1.

If we assume that the prism of Fig. 1, 2 has no weight and subdivide it into a number of smaller prisms, each one of these is deformed in exactly the same manner as the large prism. In other words, γ or e_v or e_l in each one of the three cases does not depend upon the *ordinates* of the particles of the body. As the stress is related to the deformation by means of a rheological equation in which the co-ordinates do not appear, homogeneity of deformation carries with it homogeneity of stress.

On the other hand, if we subdivide a prism which has weight into horizontal layers, the weight of the material will cause a pressure increasing from zero at the top to a maximum at the bottom of the prism. It is therefore clear that homogeneous stress, and consequently homogeneous deformation, are possible only in the absence of weight, or, more generally, of Body Forces, such as inertia. Only if external forces acting upon the surface of the body, or Surface Forces, *prevail over body forces so that the latter can be neglected*, is homogeneous stress and consequently homogeneous deformation possible.

Generally, we must relate the rheological equation to an " infinitely " small volume element to which we apply the laws of mechanics (I, a) or (I, b) and (I, c), deriving, by integration, the rheological behaviour of the whole body. Sometimes, as in laminar distortion, we can treat a *portion* of the body, the lamina, finite in two dimensions but infinitely small in the third, as the volume element. This was the case when we dealt with the flow through the tube and in the rotation instrument where γ was assumed to be constant throughout the length of the cylinder and the same in all radial sections, depending upon r only. In homogeneous deformation we do not need to relate the rheological equation to the volume element. If the deformation is

* The subject-matter of this Chapter can be found in essence in any of the numerous textbooks on the Strength of Materials ; I have, however, included it for the benefit of those readers who have not had an engineering training in order to make the book self-contained, as in the following chapters reference is made to certain equations derived here.

homogeneous, the *whole body* can be considered as an " element " ; there is no need for integration and all the rheologies of the body is contained in its rheological equation. This is the case in simple shear, simple dilatation and simple extension.

2. Simple Bending

There is an important case of " simple " deformation which is not homogeneous. It is Simple Bending.

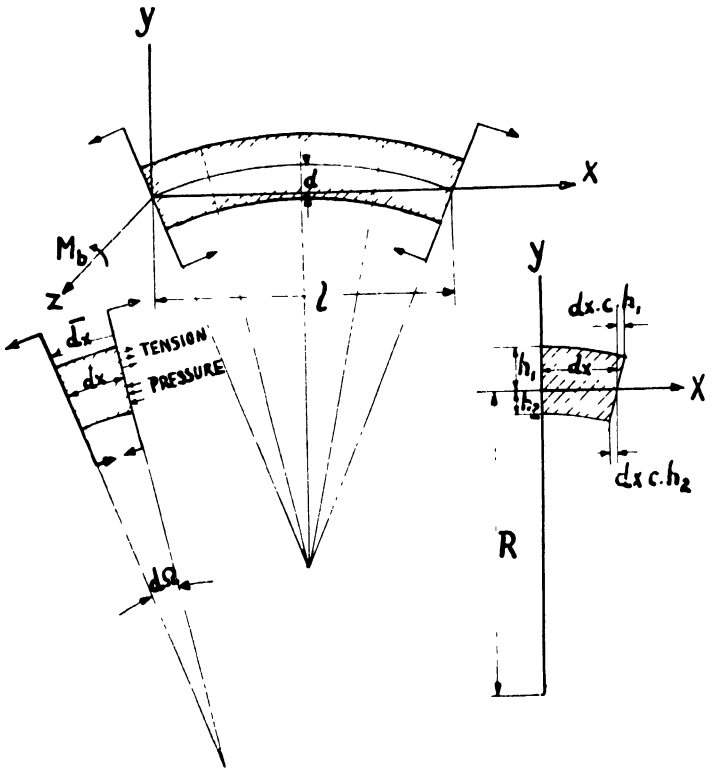


FIG. IV, 1. Simple bending of a prismatic beam.

The first length has been given a rotation without altering the deformation.

Let a prismatic bar be subjected at both ends to couples which are in equilibrium, the plane in which the couples act passing through the axis (x) of the beam. Such couples are called " bending couples " (see Fig. IV, 1).

In order to find the deformation and the stresses by an elementary method, we reason as follows :—

For the sake of simplicity of argument let us first assume that the cross-section of the bar is symmetrical about an axis (y) and that the plane in which the bending couples act passes through this axis of symmetry. This plane is therefore the xy plane of the co-ordinate system. Now let the beam be divided into a number of shorter lengths.

As each length is in equilibrium, this requires that the internal stresses give rise to couples equal to the couples of the external forces. With exactly the same forces acting on each free body there is no reason why one should be strained differently from another. Yet, if all are strained in the same manner, the axes of all pieces are bent to the same curvature. As the length of the pieces can be taken as infinitely small, this implies that the whole beam is bent into a curve of constant curvature, i.e. into a *circle*, the plane of which is the x-y plane. Furthermore, it is obvious that the plane end sections of the pieces must remain plane and normal to the bent axis otherwise they could not fit together to complete the beam. This is known as Bernoulli's assumption (1705).

We now consider such a piece of length dx. As has been said above, the internal stresses acting upon an imaginary cross-section are equivalent to a couple and to a couple only. The resultant of the stresses therefore vanishes, and the short prismatic piece is, accordingly, *as a whole*, neither extended nor compressed. The length dx of the axis will therefore not be changed and as it is bent, as we have seen, to a circular curve, the " fibres " of the prism of length dx which are parallel to the axis are either extended or shortened in accordance with the equation

$$dx \quad dx = (R + y)\, d\Omega - dx \quad . \quad . \quad . \quad . \quad (1)$$

If the arc to which the beam is bent is flat, the strains will be small and the bending strain e_b will be

$$e_b = (dx \quad dx)/dx = (R + y)\, d\Omega/dx - 1 \quad . \quad . \quad . \quad (2)$$

However, $d\Omega = dx/R$, and, therefore,

$$e_b = y/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These strains are directed *normal* to the cross-section. They are therefore normal strains as dealt with in Chapter III, and if the material is a Hooke body, Hooke's law in the form (III, r) can be applied, or

$$\sigma_b = Ey/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The stresses are accordingly distributed linearily over the cross-section in triangles. Let the distance from the axis of the outermost fibre in tension be h_1 and of the innermost fibre in compression h_2, then the maximum tensile stress is Eh_1/R, and the maximum compressive stress Eh_2/R.

Now let us assume the cross-section to be a rectangle of width b, then the stresses give a resultant force in the x-direction

$$P_x = b\frac{Eh_1}{R}\frac{h_1}{2} - b\frac{Eh_2}{R}\frac{h_2}{2} = \frac{bE}{2R}(h_1^2 - h_2^2) \quad . \quad . \quad (5)$$

In our case, as there is no external force in the x-direction, P_x must vanish or $h_1 = h_2$ and the axis of the beam the length of which is not changed, or the so-called Neutral Axis, will pass through the centre of the section. This will be so in the case of every cross-section symmetrical about the z-axis, but the reader will easily find that in a triangular cross-section the neutral axis will be nearer to the base and that, generally, the neutral axis will pass through the centre of gravity of the section.

The stresses must, however, give a finite bending couple M_b, the couple of the external forces.

Consider a strip of the cross-section having a length b parallel to the z-direction and the width dy. The increment of force acting on this strip will be $\frac{yE}{R}bdy$ and its moment in respect of the axis $\left(\frac{Ey}{R}bdy\right)y = \frac{Eb}{R}y^2dy$. The bending moment is therefore $M_b = \frac{Eb}{R}\int_{-h/2}^{+h/2} y^2dy = \frac{Eb}{R}\frac{h^3}{12}$.

The reader will have no difficulty in finding that in the more general case

$$M_b = EI/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where I is the moment of inertia of the cross-section about the neutral axis.

From (6) we find the radius of curvature

$$R = EI/M_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and, therefore, considering (3) and (4)

$$e_b = M_b y/EI \; ; \; \sigma_b = M_b y/I \quad . \quad . \quad . \quad . \quad (8)$$

The maximum traction is found for the greatest value of $y = h_1$ in tension and $y = h_2$ in compression. If the neutral axis passes through the geometrical centre of the section, $h_1 = h_2 = h/2$. The quantity $SM = 2I/h$ is called the Section Modulus and

$$\max \sigma_b = M_b/SM \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For a rectangular section $I = bh^3/12$ and $SM = bh^2/6$ and, therefore,

$$\max. \sigma_b = 6M_b/bh^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9')$$

While the σ_b stresses are the only stresses acting on the cross-section, the e_b strains are not, of course, the only strains. With the extension of the fibres on the positive y-side, there goes hand-in-hand a contraction in the z-direction which is its $1/\nu^{th}$ part. With the shortening

of the fibres on the negative y-sides, there goes hand-in-hand an extension in the z-direction. An originally rectangular cross-section therefore becomes of a shape as shown in Fig. IV, 2.

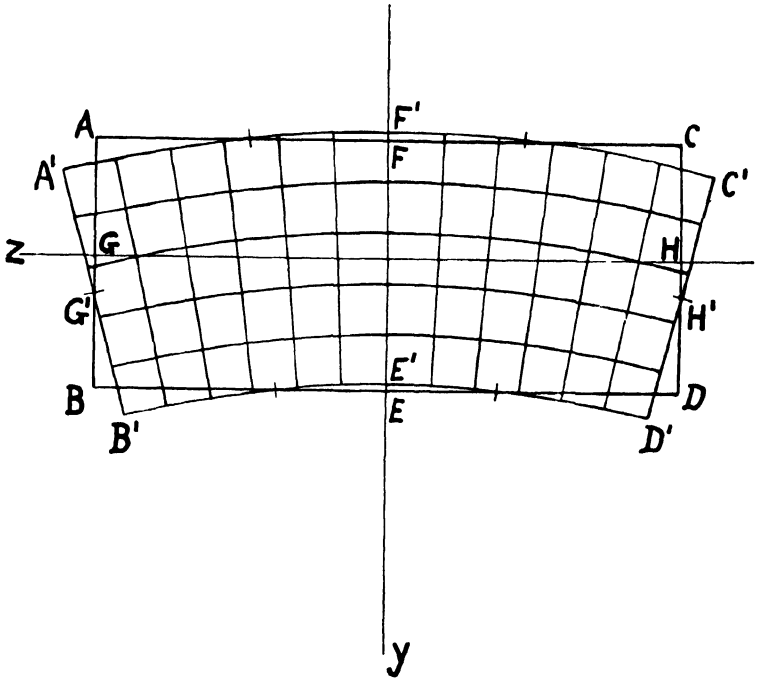


FIG. IV, 2. Deformation of the cross-section of a bent beam.

The " deflection " (δ) of the beam of span can be calculated from well-known geometrical properties of a circle, according to which

$$(L/2)^2 = \delta(2R - \delta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or, if δ is small,

$$L^2/4 = 2R\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$\delta = L^2/8R = L^2M/8EI \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If the bending couple M is known and L and I are determined and δ is measured, Young's modulus E can be calculated from

$$E = L^2M/81\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Simple bending is produced if a beam rests on two supports and loads are suspended outside the supports as shown in Fig. IV, 3. Only the piece of the beam *between the supports* is bent to a circle, because only in that part are the external forces equivalent to a constant couple $M = P.l$ *provided the weight of the beam can be neglected.*

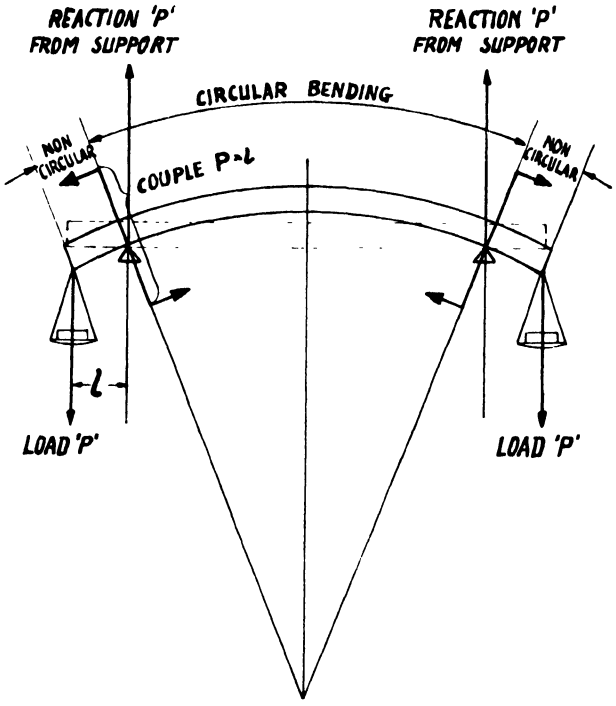


FIG. IV, 3. Scheme for simple bending of a beam.

Bernoulli's assumption is correct only in this single case of simple bending. It is *not correct in any other case*. E.g. in a beam resting upon two supports and bent by its own weight, plane cross-sections do *not remain plane* and the neutral-axis is *not bent to a circle*.

The general problem of bending transcends the possibilities of elementary methods. If, however, it is desired to find such an " overall " macro-quantity as the deflection δ of the beam, when the exact micro-distribution of *local* strains and stresses is irrelevant, (8) gives a sufficient approximation.

If the curve is not a circle, its curvature at any point is, from known principles

$$1/R = \frac{\delta''}{[1 - (\delta')^2]^{3/2}} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $\delta' = d\delta/dx$ and $\delta'' = d^2\delta/dx^2$. If the curvature is moderate and the arc to which the beam is bent is accordingly flat, $(d\delta/dx)^2$ can be neglected and $1/R$ approximately $= d^2\delta/dx^2$. This makes (7)

$$\frac{d^2\delta}{dx^2} = -M_b/EI \quad . \quad . \quad . \quad . \quad . \quad (15)^*$$

* The negative sign is due to δ being positive for a negative curvature.

from which the deflection can be calculated if M_b is known as a function of x.

3. Bending under Own Weight

Let the weight of the beam itself be w per unit length, then the reactions from the supports are $wL/2$ and, therefore, the bending moment at the point x

$$M_x = \frac{wL}{2} x - wx\frac{x}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

(compare Fig. IV, 4).

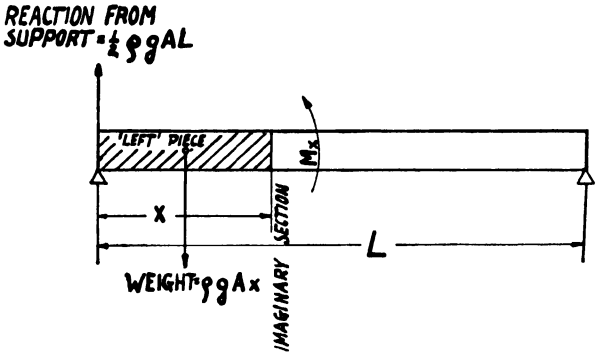


FIG. IV, 4. Bending of a beam by a distributed load.

Therefore

$$d^2\delta/dx^2 = -\frac{w}{2EI} x(L - x) \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which gives by integrating twice

$$\delta = -\frac{wx^3}{12EI} (L - x/2) + C_1 x + C_2 \quad . \quad . \quad . \quad (18)$$

where C_1 and C_2 are integration constants.

We determine these from the kinematic boundary conditions $\delta = 0$ for $x = 0$ and $x = L$, which give $C_2 = 0, C_1 = \frac{wL^3}{24EI}$ so that

$$\delta = wx/24EI . (L^3 + x^3 - 2Lx^2) \quad . \quad . \quad . \quad . \quad (19)$$

and

$$\max \delta = \delta|_{x = L/2} = 5wL^4/384EI \quad . \quad . \quad . \quad . \quad (20)$$

4. Bending by a Concentrated Load at the Centre

Here the reactions from the supports are $P/2$ and the moment

$$M_x = Px/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Compare Fig. IV, 5.

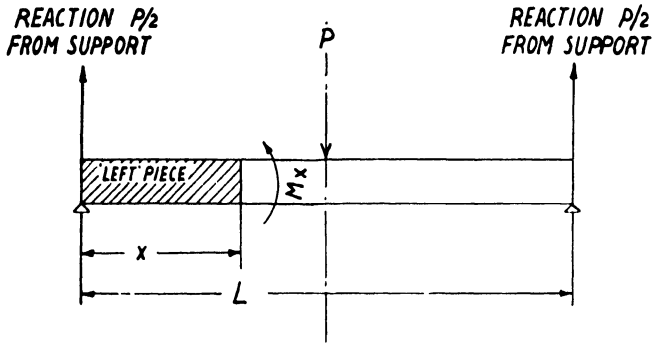


FIG. IV, 5. Bending of a beam by a concentrated load at the centre.

Therefore

$$d^2\delta/dx^2 = -Px/2EI \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

which gives by integrating twice

$$\delta = -Px^3/12EI + C_1x + C_2 \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The deflection line consists of two parts which meet at the centre without a break. The tangent to the deflection line at the centre is therefore, horizontal. The integration constants can accordingly be determined from $\delta = 0$ for $x = 0$ and $d\delta/dx = 0$ for $x = L/2$. We find $C_2 = 0, C_1 = -PL^2/16EI$, and therefore

$$\delta = Px^2/16EI.(L^2 - 4x^2/3) \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This makes

$$\max \delta = \delta_{x\ L/2} = Px/48EI \quad . \quad . \quad . \quad . \quad . \quad (25)$$

5. Bending by Two Concentrated Loads arranged Symmetrically about the Centre

Here the reactions from the supports are $M = P$, and the moment is (compare Fig. IV, 6)

$$M_x = Px \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

between $x = 0$ and $x = (L - a)/2$,
and

$$M_x = P(L - a)/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

between the loads.

Therefore

$$d^2\delta/dx^2 = -Px/EI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

between $x = 0$ and $x = (L - a)/2$, while

$$d^2\delta/dx^2 = -P(L - a)/2EI \quad . \quad . \quad . \quad . \quad . \quad (29)$$

between the loads.

Integration of both differential equations gives

$$\delta = -Px^3/6EI + C_1x + C_2 \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\delta = -P(L - a)x^2/4EI + C_3x + C_4 \quad . \quad . \quad . \quad . \quad (31)$$

From the condition $\delta = 0$ for $x = 0$ we find $C_2 = 0$.
From the condition $d\delta/dx = 0$ for $x = L/2$, $C_3 = PL(L - a)/4EI$

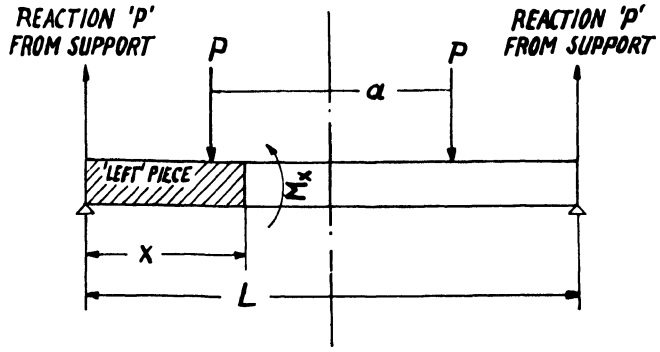


FIG. IV, 6. Bending of a beam by two concentrated loads arranged symmetrically about the centre.

The other two integration constants can be calculated from the condition that at $x = (L - a)/2$ the deflection and the slope must be the same whether calculated from (30) or (31), i.e.

$$-\frac{P(L-a)^3}{48EI} + C_1\frac{L-a}{2} = -\frac{P(L-a)^3}{16EI} + \frac{PL(L-a)^2}{8EI} + C_4 \quad . \quad (32)$$

and

$$\frac{P(L-a)^2}{8EI} + C_1 = -\frac{P(L-a)^2}{4EI} + \frac{PL(L-a)}{4EI} \quad . \quad . \quad (33)$$

We find

$$C_1 = -\frac{P(L^2-a^2)}{8EI}, \quad C_4 = -\frac{P(L-a)^3}{48EI} \quad . \quad . \quad . \quad . \quad (34)$$

and therefore

$$\delta = -Px^3/6EI + P(L^2-a^2)x/8EI$$
$$\delta = -P(L-a)x^2/4EI + PL(L-a)x/4EI - P(L-a)^3/48EI \quad . \quad (35)$$

The maximum deflection follows from the second equation for $x = L/2$ with

$$\max \delta = P(L-a)(2L^2 + 2La - a^2)/48EI \quad . \quad . \quad . \quad (36)$$

6. Simple Torsion

Another case of simple deformation is simple torsion. It is laminar, as shown in Fig. I, 1, and accordingly quasi-homogeneous. Let the circular cylinder of radius R and length l be fixed at one end and, at the other end, loaded by a torsional couple M_z. The cylinder will be twisted, i.e. the section $z = l$ will be rotated against the section $z = 0$ by an angle, say Ω. The problem is to express Ω in terms of M_z and the dimensions of the cylinder, i.e. the geometrical quantities R and l (compare Fig. IV, 7).

In the elementary method which we are following in the present book we cannot solve this problem without the introduction of an assumption similar to Bernoulli's assumption. This assumption can be expressed in different ways. The first to attempt a solution of the problem of torsion was Coulomb and he started from the assumption

that shearing stresses only act in the cross-sections and that they are proportional to their distances from the axis and directed normal to the radius, i.e. he started from the equation

$$\tau = cr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where c is a constant.

From this he found by applying Hooke's law

$$\gamma = cr/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

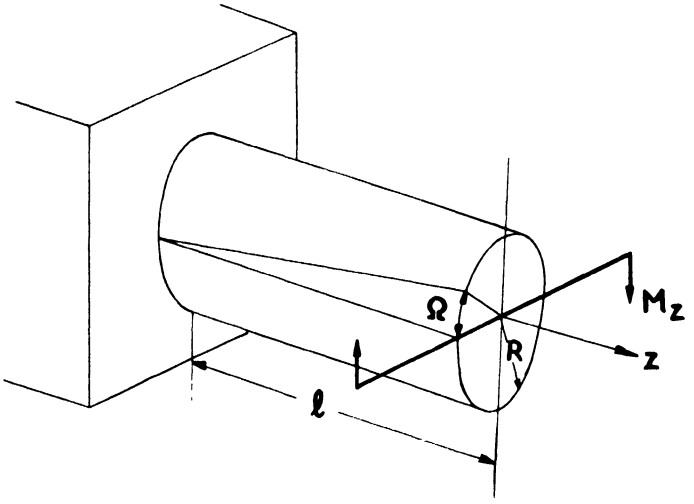


FIG. IV, 7. Torsion of a circular cylinder.

Comparing Fig. IV. 7 with Fig. I, 2, we see that we can express γ introducing dz for H, as follows :—

$$\gamma = du/dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

or

$$du = \frac{cr}{\mu} dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and by integration

$$u = cr/\mu.z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

considering that $u = 0$ for $z = 0$.

In order to find c we compose the stresses taken over the section to the torsional couple or torque

$$M_z = \int (\tau 2r\pi dr)r = 2\pi c \int_0^R r^3 \, dr = \pi c R^4/2 \quad . \quad . \quad . \quad (42)$$

and get $c = 2M_z/\pi R^4$ so that (compare (37))

$$\tau = 2M_z r/\pi R^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Equation (41), on the other hand, gives

$$u = 2M_z/\pi R^4 . r/\mu.z. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The free end-section is rotated through the angle

$$\Omega = \left.\frac{u}{r}\right|_{z=l} = 2M_z l/\pi R^4 \mu \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and this equation connects the observable macro-mechanical quantities M_z and Ω. Ω may be called the angular displacement.

Equation (45) can also be written as

$$M_z = \Omega \frac{\pi R^4 \mu}{2} / l = \Omega D/l \quad . \quad . \quad . \quad . \quad . \quad (45')$$

where D/l is the " restoring moment " of (II, 49).

From (41) it can be seen that the section $z = z$ will be rotated against the section $z = 0$ by an angle of the magnitude $u/r = c/\mu . z = \Omega/l . z$.

The quantity Ω/l is called the *twist*. In our case, the twist is constant. Generally where it is not constant, the twist is measured by the relative rotation of one slice against the other or by $d\Omega/dz$. From (41) it also follows that if the cylinder is imagined to be composed of " slices " of thickness dz, the slices are displaced *as wholes* so that the displacements have neither radial nor longitudinal components, the only component being a rotational displacement. This confirms the picture of Fig. II, 1.

We could also have started with this, as a natural kinematical assumption, and would have found from

$$u_\theta = Crz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$\gamma = du_\theta/dz = Cr \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

and, in accordance with Hooke's law, again

$$\tau = \mu\gamma = \mu Cr \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where

$$\mu C = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

While it might seem natural to assume that in torsion the shearing stress is proportional to the distance from the axis ; or that plane cross-sections are rotated against each other, but remain plane (these assumptions, as has just been shown, being identical) it can easily be shown that Coulomb's assumption cannot be valid for any shape of section other than a circular section. If the section is other than circular its edge makes an angle with the radius vector r ; and the shearing stress τ, which has the direction of θ, or is normal to r, must then have a component normal to the linear element of the edge (see Fig. IV, 8), i.e. in the direction of n. However, in accordance with the law of corresponding shearing stresses (compare Section 6, Chapter I), for every shearing stress *in the cross-section* there exists an equal shearing stress *in a longitudinal section* normal to the cross-section. In our case, if there existed a shearing stress in the cross-section having

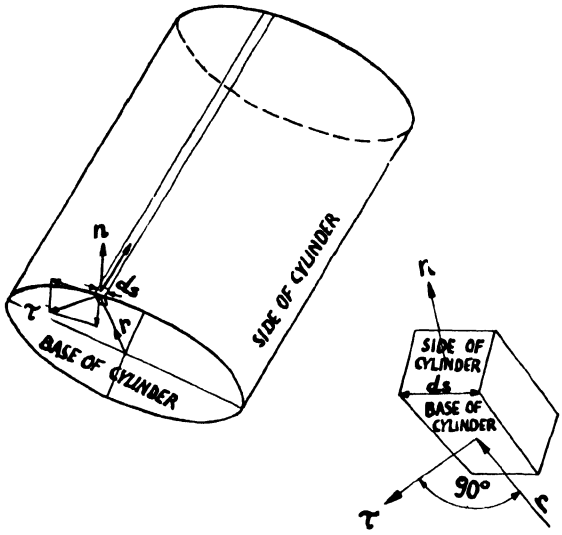


FIG. IV, 8. Torsion of a rod of non-circular cross-section.

the direction n, there would have to exist a shearing stress acting upon the side in a direction normal to n. We have, however, assumed that no surface forces act upon the side of the cylinder and, therefore, such a shearing stress as mentioned last does not exist. Therefore, τ cannot have a component in the direction n. Hence τ cannot be directed normal to r and it has for non-circular sections generally the *two* components τ_θ and τ_r.

The general solution for non-circular sections was found by St. Venant (1855). St. Venant assumed, as we have assumed above, that no forces act upon the sides of the prism. Actually, however, in order to twist a rod it must be fixed at the ends in such a way that surface forces *are* exerted upon short lengths of the sides. It is true that at some distance from the fixing-points, the sides will be free from forces and there the stresses will be those of St. Venant's theory. These stresses may, however, not be the maximum stresses and if the material fails it will do so at the fixed ends. Therefore, for calculations of strengths, a better approximation to actual conditions than provided by St. Venant's theory is required. In three papers I have made an attempt at such a generalisation of St. Venant's theory by considering the action of torsional surface forces applied upon the sides of a cylinder or prism (1914, 1915, 1925).

7. The Advantages of Bending and Torsion Tests

The bending and torsion tests are often more suitable for the determination of rheological constants than simple tension. In a

rheological test the kinematical quantity observed is seldom directly a strain (or a deformation) or rate of strain (or rate of deformation). It is rather a *displacement* or *rate of displacement*. In simple tension, where the deformation is *pure*, the total displacement is the sum of the elementary displacements. In the bending of a rod, where there is rotation of the elements, the displacements are increased along the length of the rod as in a rotating pointer arrangement. Take, for instance, a short rod of any elastic material in your hand and apply with the other hand some force. If the force is a pull in the direction of the axis of the rod, the displacement of the free end will hardly be noticeable. If the force is applied at the free end in the direction normal to the axis, there will be a noticeable displacement, provided the rod is not too stiff. To make this example more definite, let us assume the rod to be of mild steel of a cross-section 1 mm. square and 10 cm. long. Applying a pull of 100 g., the extension will be in accordance with (III, r), $e_l = 3 \times 10^{-6}$ and, therefore, in accordance with (III, 9) the displacement of the free end $\Delta l = 3 \times 10^{-5}$ cm. Applying the same force in the direction normal to the axis, the deflection will be found to be the same as if twice the load had been applied in the centre of a beam of double length supported at both ends. This, in accordance with (25) is $\delta \quad 2P(8L^3)/48EI$ and considering $I = bh^3/12 = 1/12$ mm^4, we shall find δ to be slightly over 1 cm or a displacement one million times that of the first case (compare Fig. IV, 9).

In torsion also, due to the cumulative effect of rotations, the displacement is much greater than in simple shear.

The large displacement is here the result of a *summation of rotations* of the length elements. We have shown in Fig. 1, 1 how the element is rotated in simple shear. It is similar in torsion, and to a certain degree also in bending.

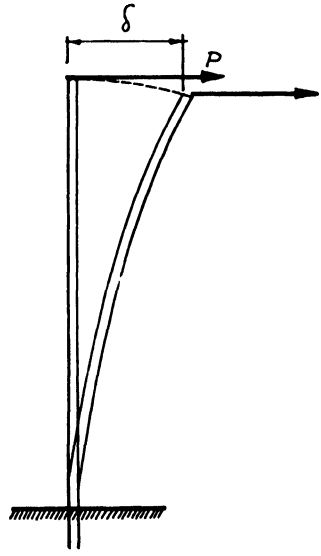


FIG. IV, 9. Deflection of a rod through a force acting sidewise.

8. The Close-Coiled Helical Spring

These considerations are relevant when considering the elastic deformation of a helical spring. Even when every element of the spring undergoes *infinitesimal* strain only, the combined and additive effect of rotations due to bending and torsion of the elements will produce a very pronounced *displacement* of the ends under an axial pull. Even should the material of the

spring not follow Hooke's law of proportionality, the displacement *will* follow such a law because deviations from the law become noticeable at finite strain only, while here, as said before, even if the displacement is finite, the strain is infinitesimal.

We shall now show this in detail (compare Fig. IV, 10) :— Consider a cross-section A. Without affecting equilibrium we may add two equal and opposite forces P. These forces P and the load P are equivalent to a shearing force P and a torque $M_T = P.R$. The extension of the spring is mainly due to the rotation connected with the torsion due to this torque, as will presently be shown.

Consider a short length $Rd\alpha$ of the coil contained between the two cross-sections A and B. Let the rotation of A with respect to B be $d\theta$. This will be in accordance with Equ. (45) $= (2M_T/\pi r^4\mu)R\ d\alpha = (2PR^2/\pi r^4\mu)d\alpha$ where r is the radius of the wire. If we suppose B to be fixed, the lower portion of the spring will rotate due to this twist, the point C describing a small arc equal to a $d\theta$, which brings it to D. The vertical component of the arc is found by reducing its length in the proportion of R/a, and is therefore equal to $R\ d\theta$. The length l of the spring is thus elongated by a differential of elongation $d(\Delta l)$, which is equal to $R\ d\theta = (2PR^3/\pi r^4\mu)d\alpha$. Summing this effect for all such short segments of the coil, the deflection effects accumulate, and we have

$$\Delta l = R\int d\theta = \frac{PR^3}{\pi r^4\mu}\int_0^{2\pi n} d\alpha = \frac{4nPR^3}{r^4\mu} \quad . \quad . \quad . \quad . \quad (50)$$

where n is the number of coils.

We can now introduce for the length of the closely coiled spring $l = 2rn$ (not the length of the wire, which is $2R\pi n$). The extension of the spring is then

$$(e) = \Delta l/l = \frac{2PR^3}{r^5\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

This we can also write as

$$(e) = (\sigma)/(E) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where (σ) is the equivalent tensile traction acting over the cross-section $A = R^2\pi$ of the spring (not of the wire, which is $r^2\pi$) and (E) is a fictitious " macro " Young's modulus. But

$$(\sigma) = P/R^2\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

therefore

$$(E) = (\sigma)/(e) = \frac{\mu}{2\pi}\left(\frac{r}{R}\right)^5 \quad . \quad . \quad . \quad . \quad . \quad (54)$$

and the spring accordingly acts as would a solid rod of the same overall dimensions made of a material with a fictitious Young's modulus smaller than that of the material of which the spring is

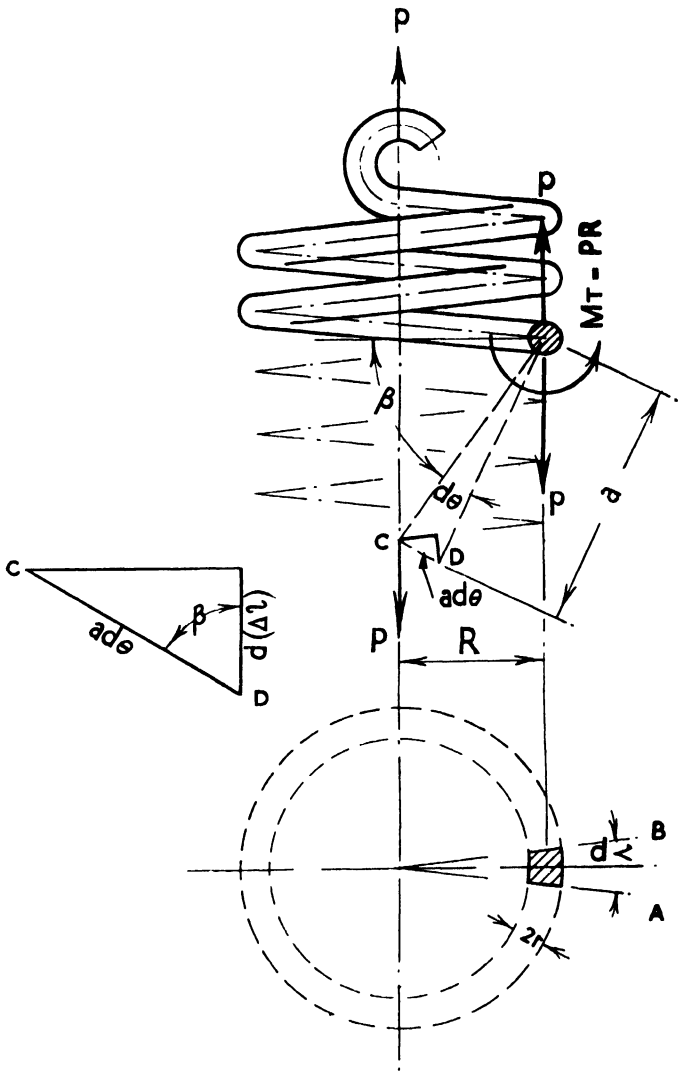


FIG. IV, 10. Close coiled helical spring.
(*From "Agricultural Rheology," courtesy Routledge & Kegan Paul Ltd.*)

made in the proportion $(r/R)^5$. This is due to the " open structure " of the spring. Similar conditions prevail in any other open structure. In the first attempts to explain the high elasticity of rubber, this was attributed to a hidden internal open structure. Later, it was found that this explanation is not correct but that the mechanism is an entirely different one.* But there are other materials to which it

* See Treloar (1949).

applies. The extension $\Delta l/l$ will then be large, and macroscopically this is a case of " finite " strain of " high-elasticity ". Nevertheless, microscopically speaking, the physical strain of the material, i.e. the strain which causes the stress, is actually infinitesimal and, as follows from (51), the extension $\Delta l/l$ is proportional to the pull P as in Hooke's law. This may be called " pseudo-finite " strain or " pseudo-high elasticity " in which macro-extensions of 100 per cent. or more can be attained with infinitesimal micro-extensions. However, when the spring is so much extended that the coils open up entirely, it is obvious that further extensions will become smaller and smaller until in the limiting case, when the wire is straightened, they are infinitesimal. The stress-versus-strain curve then diverges from the linearity of Hooke's law, and becomes convex towards the strain-axis approaching a vertical asymptote. Among materials following this law is leather as used in transmission-belts. When James Bernoulli (1705), who did not know of Hooke, wanted to find out what the law of elasticity was, he experimented not with steel wires, but with a gut-string, and found this kind of stress-strain relation.

9. The H-Model

We can therefore conveniently use a close-coiled helical spring as a mechanical model for the Hooke-body, or a H-model, similar to the N-model of Section II, 10. This is shown diagrammatically in Fig. IV, 11. While the overall deformation of the helical spring manifests itself

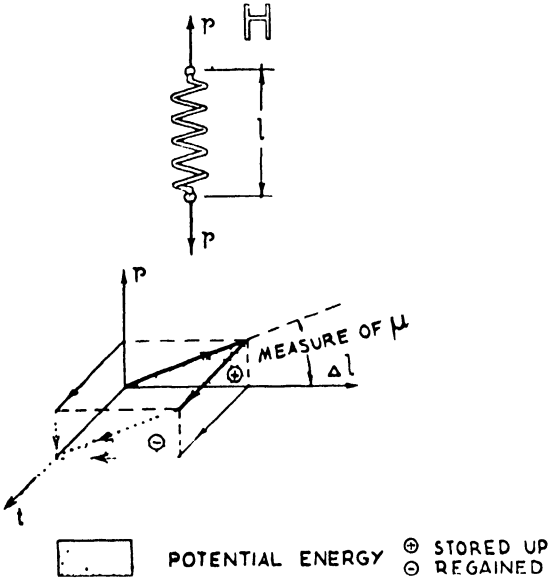


FIG. IV, 11. Diagram of model for the Hooke solid.

in an elongation, one may use the model also for the illustration of the behaviour of an elastic material in volume dilatation and/or shear.

10. Summary

If a beam of elastic material is acted upon at both ends by bending couples M_b which are in equilibrium, it is bent to a circular arc of radius R which is

$$R = EI/M_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV, a)}$$

where I is the moment of inertia of the cross-sectional area.

The deflection of the beam, if small, is given by

$$\delta = L^2 M_b/8EI \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV, b)}$$

The equation of the neutral axis can generally be calculated for small deflections from the differential equation

$$d^2\delta/dx^2 = -M_b/EI \quad . \quad . \quad . \quad . \quad . \quad \text{(IV, c)}$$

By integration we find for the deflection of a beam through bending from its own weight w

$$\max \delta = 5wL^4/384EI \quad . \quad . \quad . \quad . \quad . \quad \text{(IV, d)}$$

from a load P concentrated in the centre

$$\max \delta = PL^3/48EI \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV, e)}$$

and from two concentrated equal loads P with a the distance between them

$$\max \delta = P(L - a)\,(2L^2 + 2La - a^2)/48EI \quad . \quad \text{(IV, f)}$$

The twist of a circular rod of elastic material by the torsional moment M_t is

$$\Omega/l = 2M_t/\pi R^4 \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV, g)}$$

and the stress

$$\tau = 2M_t r/\pi R^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV, h)}$$

The apparent extension of a close-coiled helical spring with R radius of coil and r radius of wire is

$$(e) = 2PR^3/r^5\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV, i)}$$

CHAPTER V

NORMAL TRACTION IN VISCOSITY

1. The Coefficient of Viscous Traction

WE said in Section III, 10 that the tensile test is used not only for metals but also for such materials as pitch. Of course, in the latter case the rheologics of the test is entirely different. It is generally assumed that if a rod of mild steel is loaded below the yield point it extends elastically and then stays put to the end of times. This, now, is a statement where the epistomological considerations of Section 2 of Chapter III come in. It has been maintained that even at room temperature mild steel will flow slowly or " creep ". About creep we shall speak in Chapter XIII. But even if mild steel should flow at the same temperature as pitch, there still remains a very definite *quantitative* difference between them. In the case of a rod of pitch it is not necessary to employ exceedingly sensitive instruments to discover the continuous elongation ; it is visible to the naked eye—so much so that Trouton (1906) regarded pitch as a very *viscous liquid*. This, also, implies a simplification and we shall therefore consider here *the continuous elongation under constant load* of a rod made *of a Newtonian liquid of very high viscosity* instead of pitch, which, as we shall see in Chapter XIV, manifestly possesses other properties in addition to viscosity.

For such exceedingly viscous liquids, which to all appearances are solids, the two methods for the determination of viscosity, dealt with in Chapter I, are inapplicable. While the viscosity of bitumen has been determined in rotation and tube viscometers, it may reach such values that it would either take tremendous force to produce a flow measurable in short time, or an impracticable long time under ordinary forces. In such cases (and for similar materials) Trouton applied such testing methods as are used for the determination of the elasticity of solids, e.g. the tensile test. He loaded a rod made of the material under test and measured the velocity of its extension. It is clear that the load per unit area of the section (σ) divided by the rate of extension ($\dot{d}_l$) must be a measure of the viscosity. Trouton called it the Coefficient of Viscous Traction.

$$\lambda = \sigma/\dot{d}_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The question then arises : how is this quantity related to the coefficient of viscosity η as defined by (I, f) ? In the latter equation the coefficient of viscosity η is defined by considering a case of simple shear, i.e. a case where neither the deformation nor the stress in respect

of a given surface element has a normal component, i.e. $d_n = d_l = 0$ and $\sigma = 0$. Moreover, simple shear is a laminar displacement and therefore not accompanied by any change of volume, which means that we also have $e_v = 0$.

Now, is there any section in the loaded rod inclined against the cross-section at the angle ϕ, where the only stress is a shearing stress ? The reply to this question is contained in Equ. (III, 66). There is only one ϕ, where σ_n vanishes, and that is for $\phi = 90°$, when $\cos 2\phi = \cos 180° = -1$. But then τ_n also vanishes.

2. Trouton's Derivation of the Relation between λ and η

The reply being in the negative, we cannot directly apply (I, f). However, have we forgotten our first rheological axiom that, in order to examine rheological properties, we have to consider distortions ? Let us correct this oversight.

This is usually done as follows, the method apparently being due to Lord Kelvin. Let the cross-section of the prism of Fig. V, 1 be a square with a side of unit length so that the traction σ is equal to P. Let us resolve this longitudinal tensile traction into three superposed tensile tractions, each equal to $\sigma/3$. The strains produced will not be affected by this procedure. Also let each side face be subjected to two opposite normal stresses, each equal to $\sigma/3$; these stresses will not produce any strain, and thus original strains remain unchanged. We can now group the stresses applied to the prism in any way we please, and the resultant strains must be identical with those produced by tensile tresses applied to the end faces. Each side face is subjected to a tensile stress $\sigma/3$, indicated in Fig. V, 1 by a black arrow ; grouping these stresses with the component tensile stresses $\sigma/3$ applied to the end faces, we obtain a uniform dilatational stress $\sigma/3$. We then group the stresses indicated by white arrows and those indicated by alternatively black and white arrows together.

As can be seen from the Inset 1 of Fig. V, 1, these stresses—white and alternate arrows separately—are equivalent to shearing stresses equal to any of them (and therefore $= \sigma/3$) and making angles of 45° with the length of the prism (compare Inset 2). Therefore, if a pyramid as shown in Inset 3 is cut out of the prism, it is in equilibrium under the action of the forces shown, with the traction σ requiring the superposition of a hydrostatic tension σ. In his calculation of the coefficient of viscous traction, Trouton followed this method as is evident from the quotation : "The tractional force applied to a rod may be resolved, as is usual in questions of elasticity, into two equal shears (= shearing stresses or tractions, M.R.) which are situated at right angles to each other and at 45° to the direction of traction, along with a uniform force of dilatation. The value of either shearing stress and

also of the dilatation stress (hydrostatic tension) is in each case one-third of that of the tractive stress (σ)."

It should be noted that Trouton is wrong in stating that the two shears are " at right angles to each other ". Their *horizontal projections*

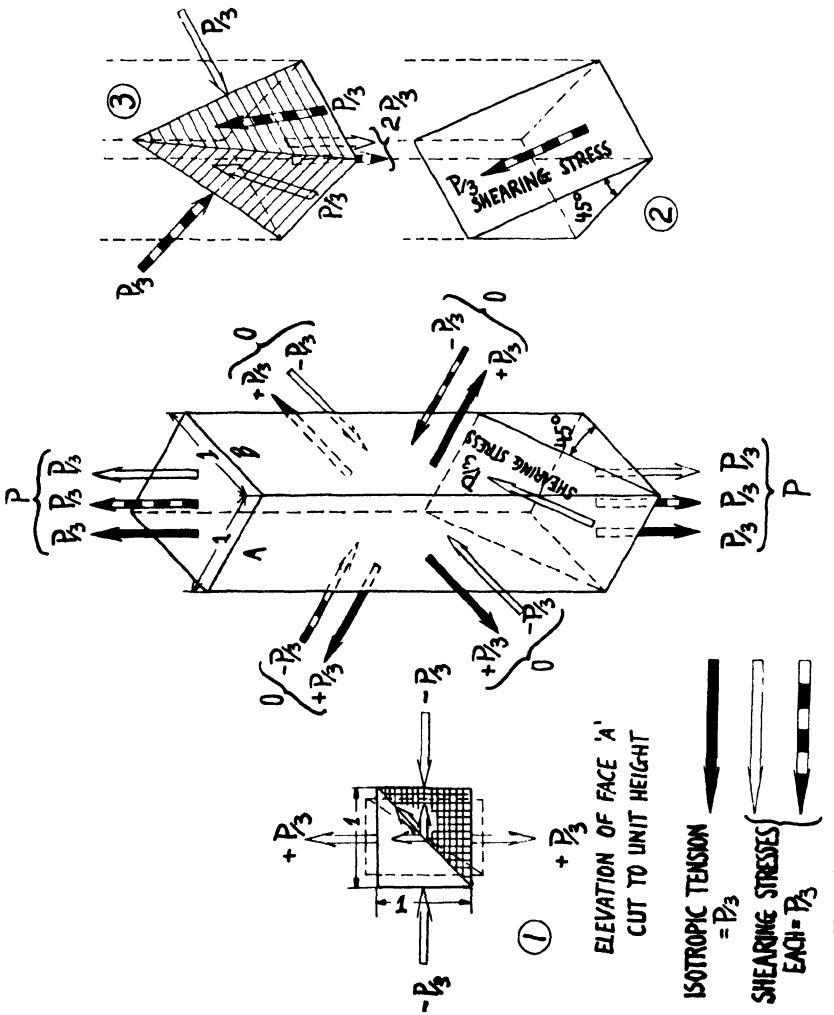


FIG. V, 1. Simple tension of a prism resolved into states of isotropic tension and shear.

are at right angles to each other, but not they themselves ; because the planes in which the shears act make an angle which is necessarily greater than 90°.* Trouton continues : " In the first instance on the application of the tractive force, the resolved effects produced corresponding to these resolved stresses, will consist of a dilatation and of

* It is exactly 120° as can be found from considerations of solid geometry.

shearing strain. It can only be to the flow resulting from the latter that the *continued* elongation of the rod is due. Nothing similar can take place in the case of the stress of dilatation, which can only have an initial effect." I.e. should the material be compressible, and this it will generally be, then hydrostatic tension will only change its density immediately after the application of the traction and this will be all hydrostatic tension can produce : it will have no influence upon the flow. " The continued application of each shear will produce a corresponding flow given in each case by $\tau = \eta\dot{\gamma}$ where τ is the shearing stress, η the coefficient of viscosity, and $\dot{\gamma}$ the rate of change of direction of any line in the material in the plane of the shear, as it passes through the direction normal to the shearing stress " (see Inset I in Fig. V, 1). This, however, involves two assumptions, which have not been expressly stated : Firstly the assumption that added hydrostatic pressure or tension does not effect the value of the coefficient of viscosity. This is only approximately correct. Secondly, it should be noted that (I, f) defines η for the case of *one simple* shear, while here we have two shears superposed. But what is worse, these two shears are *not* at right angles to each other, as was in Trouton's mind when advancing his argument. Trouton finishes as follows : " The resulting flow in the direction of the axis is obtained by adding the resolved components of the two flows in that direction ; so that resolving the two effects, adding the components, and reducing the axial elongation to that per unit length, we find that $\dot{d}_l = \dot{\gamma}$. Since $\tau = \eta\dot{\gamma}$ and $\tau = \sigma/3$, where σ is the tractive force per square centimetre, we get $\eta = \lambda/3$, so that the coefficient of viscosity is equal to one-third of the coefficient of viscous traction."

This can be seen from the considerations of Section III, 14 (compare Fig. III, 6). From the action of the white arrows the prism will extend by $\gamma/2$ and from the action of the alternately black and white arrows likewise by $\gamma/2$. The total extension is therefore twice $\gamma/2$ or $= \gamma$. Now $\dot{\gamma}$ can be calculated from (I, f) and is equal to τ/η. However, as $\tau = \sigma/3$, we have $\dot{\gamma} = \sigma/3\eta$ and therefore also $\dot{d}_l = \sigma/3\eta$.

From the definition of the coefficient of viscous traction (1) on the other hand, $\dot{d}_l = \sigma/\lambda$. Comparing the two last equations, we obtain

$$\lambda = 3\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

3. Damping of Free Pulsations

We have reproduced Trouton's proof of the relation between the coefficient of viscous traction and the coefficient of viscosity because most rheologists have followed the same lines. It must, however, be said that at the same time many have felt some uneasiness.

In Trouton's experiment we have to distinguish two stages. The stress due to the pull σ has, in accordance with (III, 45) an isotropic

component $\sigma_m = \sigma/3$. When the pull is applied there will be an initial stage which starts with an accelerated and ends with a retarded movement of the particles with—in general—pulsations in between. During this initial stage the material expands, the measure of the cubical dilatation at every moment being e_v. This cubical dilatation produces an elastic isotropic reaction $= \kappa e_v$.

But this is not all. Let us consider what will happen if, having compressed a sphere, by gradually applying an isotropic pressure p, from the radius R_0 to the radius R, the external pressure is suddenly released. Now, in expanding, the internal stresses meet no resistance against which to perform work. The particles will therefore acquire kinetic energy and oscillations will start. Because of the absence of external forces, the oscillations are called *free oscillations*. The elastic potential energy (E_p) is converted into kinetic energy (E_k),* which gradually increases until it reaches a maximum for $R = R_0$ when the elastic energy vanishes. Then the sphere continues to expand with kinetic energy being converted into potential energy, etc., all the time the law of conservation of energy requiring that the sum of both is constant. This, however, cannot go on indefinitely. The process just described of recurring conversion of kinetic into potential energy requires that the velocities and accelerations or retardations of all particles are at all times directed towards the centre of the sphere. But if the oscillations started in this way, imperfections in the structure of the material making up the sphere, such as pores, local variations of density, or, generally, heterogeneities and local aeolotropies, etc., will soon make themselves felt. While the sum of the kinetic energies of all particles will still conform to the law of conservation, the velocities will gradually become less oriented towards the centre and more and more acquire a random orientation. The oscillations of the sphere as a whole will gradually be replaced by individual oscillations of the particles which macroscopically manifest themselves in an increase of temperature. This is called Damping of Oscillations, with which we shall deal in more detail in Chapter XV. But a rheological experiment must be isothermal. This can be realized by connecting the body with a large heat reservoir kept at constant temperature. The heat into which the kinetic energy of the oscillations is gradually converted is drawn off into the reservoir and dissipated (E_d). The law of conservation of energy now requires

$$E_p + E_k + E_d = \text{const.} \qquad . \quad . \quad . \quad . \quad . \quad (3)$$

No other form of energy appears in rheology. What we have derived here on the example of isotropic stress and voluminal oscillations is, as will be shown later, valid for every other kind of stress and

* Compare Equation (21) of Section III, 7.

deformation. Equation (3), together with the proviso that in every process E_d can only increase, form the energetic basis of rheology.

4. The Coefficients of Volume Viscosity

If there is dissipation of energy whenever the cubical dilatation is not produced infinitely slowly, or whenever there is a finite *rate* of dilatation, $\dot{e}_v$, this implies a sort of viscosity ζ which we may call Volume Viscosity. In this connection it is immaterial whether we speak of a liquid or a solid. This is in accordance with the first rheological axiom which (in other words) says that for simple changes of volume or density every material behaves as a solid. One could, of course, in principle assume that for a certain class of liquids ζ vanishes and we may call this class of liquids the Stokesian Liquid, because this was what Stokes (1851) assumed when deriving the famous Navier-Stokes differential equation of viscous flow named after him and Navier (1823). Until recently this was generally assumed to conform to actual conditions, but Tisza (1942) has pointed out that ζ must in real liquids be rather high and I have pointed to other consequences connected with a vanishing ζ which are not likely to conform to experience and about which we shall say more in Chapter XII.

This, however is still not the whole story. We have assumed that all volume dilatation is a strain, i.e. a *recoverable* deformation, and have accordingly denoted it by e_v, e denoting strain. In contradistinction we have denoted the rate of linear dilatation by $\dot{d}_l$, in order to indicate that this is a deformation of a different kind. Actually it is linear *flow*, the latter defined in Section 1, 2. To make this quite clear let us introduce the symbol f_l for what we wrote as $\dot{d}_l$. The question then arises: is there a *volume flow* f_v, i.e. an increase of the volume continuing irrecoverably in time, as long as the isotropic pressure p is in action? We shall come back to this question in Chapter XII. Meanwhile we can say that if such volume flow exists, it will be resisted by another kind of volume-viscosity, which we may call "liquid" volume-viscosity ζ_l, denoting the first volume viscosity more specifically by ζ_s as "solid" volume-viscosity.

5. The Stokesian Liquid

Stokes never considered ζ_l and he assumed, as has been said before, $\zeta_s = 0$. Tisza has recently drawn attention to the following quotation from Stokes: "Of course we may at once put $\zeta_s = 0$ if we assume that in the case of a uniform motion of dilatation the pressure at any instant depends only on the actual density and temperature at that instant and not at the rate at which the former changes with the time. In most cases in which it would be interesting to apply the theory of friction of fluids, the density of the fluid is either constant or may without

sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same and in the third nearly the same whether ζ_s were equal to zero or not. Consequently, if theory and experiments should in such cases agree, the experiments must not be regarded as confirming that part of the theory which relates to supposing ζ_s to be equal to zero." If we assume with Stokes vanishing volume viscosity, the material would in the first stage expand purely elastically and instantaneously in no time as is natural in the absence of the damping influence of a volume viscosity. If, on the other hand, ζ_s were so great that it may be put $= \infty$, the expansion would be infinitely delayed. Tisza deduces from observations on supersonic adsorption in certain liquids a value of $\zeta/\eta = 2{,}000$. We shall come back to the question of both volume viscosities, as said before, in Chapter XII. Meanwhile we may assume, as in classical hydrodynamics, that to a first approximation ζ_s vanishes, and that ζ_l is so great (say $= \infty$) that no volume flow can be noticed. The λ calculated under such assumption we may denote by λ^*, and it is the viscous analogy to E^*, as defined by Equ. (III, x). Trouton calculated the values for λ^* shown in the following table :—

Material	Pitch	Mixture of pitch and		Shoemaker's Wax
		Tar	More tar	
$\lambda^* \times 10^{-10}$ in poises	3·6—4·3	1·29	0·67	0·0059

6. Analytical Derivation of Expression for λ

Equation (I, d) for the Hooke solid and (I, f) for the Newtonian liquid, which we write again side by side as $\tau = \mu\gamma$ and $\tau = \eta\dot{\gamma}$, are entirely analogous, η corresponding to μ and γ to $\dot{\gamma}$. For viscous distortion there are equations in force analogous to those of elastic distortion III (o), if we write η for μ and $\dot{d}$ for e. In addition, to indicate that we mean by the rate of deformation a flow, we introduce f for $\dot{d}$, and have

$$\sigma_{n(o)} = 2\eta f_{n(o)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In our case (compare III (n))

$$\sigma_{k(o)} = 2\sigma/3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

while (compare III (k))

$$\dot{d}_{k(o)} = \dot{d}_k - \dot{e}_v/3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Therefore

$$\sigma/3 = \eta(\dot{d}_k - \dot{e}_v/3) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now, assuming $\dot{e}_v$ to vanish, which may variously be interpreted as either that the material is incompressible ($\kappa = \infty$, $\therefore\ e_v = 0$), or that it has suffered a cubical dilatation ($e_v \neq 0$) which does not change in the course of flow ($e_v = \text{const}$, $\therefore\ \dot{e}_v = 0$), and that there is no volume flow ($f_v = 0$)—then we can identify $\dot{d}_k$ with f_k, and with $\dot{e}_v = 0$ we find

$$\sigma = 3\eta f_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which by comparison with Equ. (1), confirms Equ. (2), writing λ^* in order to indicate the special assumptions in respect of ζ_s and ζ_l.

7. Another Example for the Viscosity-Elasticity Analogy

If we compare Equ. (V, 1) with Equ. (III, r) or $\sigma = \lambda \dot{d}_l$ with $\sigma = Ee_l$, we see that a viscosity-elasticity analogy exists also in simple tension. Here Trouton's coefficient of viscous traction λ corresponds to Young's modulus E^* in the case of an incompressible material. Therefore if, for instance, we place a beam made of, say, fairly hard bitumen upon two supports and load the beam in such a way that simple bending is produced, the beam will gradually and continuously sag and, as long as the deflection is not too great, *the rate of sagging* $\dot{d}$ can be found from (IV, 12) to be

$$\dot{d} = L^2M/8\lambda I = L^2M/24\eta I \quad . \quad . \quad . \quad . \quad (9)$$

If the bending is by a weight w per unit length of the beam, the rate of deflection is in accordance with (IV, 20)

$$\dot{d} = 5wL^4/384\lambda I = 5wL^4/1152\eta I \quad . \quad . \quad . \quad (10)$$

A concentrated load P in the centre will produce a rate of deflection in accordance with (IV, 25) of

$$\dot{d} = PL^3/48\lambda I = PL^3/144\eta I \quad . \quad . \quad . \quad . \quad (11)$$

Bending by two concentrated loads P arranged symmetrically about the centre with a as the distance between them will produce in accordance with (IV, 36) a rate of deflection

$$\dot{d} = P(L - a)(2L^2 + 2La - a^2)/48\lambda I = P(L - a)(2L^2 + 2La - a^2)/144\eta I \quad . \quad . \quad . \quad . \quad (12)$$

These are all examples where the main stress is " normal ". Torsion of a viscous liquid gives, in accordance with (IV, 45)

$$\dot{\Omega} = 2ML/\pi R^4\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

8. Summary

The rheological equation of the Newtonian liquid connects the distortional stress $\sigma_{n(o)}$ with the flow f_n through

$$\sigma_{n(o)} = 2\eta f_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (V, a)$$

In the case of simple normal traction σ causing a linear flow f_l, Trouton's coefficient of viscous traction λ is defined by

$$\lambda = \sigma/f_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(V, b)}$$

In this case the isotropic component $\sigma_m = \sigma/3$ produces an elastic cubical dilatation e_v, which in accordance with (III, a) is $e_v = \kappa\frac{\sigma}{3}$, while its distortional component $\sigma_{(o)} = 2\sigma/3$ produces a linear flow $f_l = \sigma_{(o)}/2\eta = \sigma/3\eta$. This presupposes that $f_v = f_i + f_j + f_k = 0$, or that there is no volume flow, and that the elastic cubical dilatation e_v is instantaneous. Such liquid may be named Stokesian. In this case σ is connected with f_l by

$$\sigma = 3\eta f_l = \lambda^* \dot{d}_l \quad . \quad . \quad . \quad . \quad . \quad \text{(V, c)}$$

where λ^* is Trouton's coefficient of viscous traction under these special assumptions, and $\dot{d}_l$ is the rate of linear dilatation, or

$$\lambda^* = 3\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(V, d)}$$

CHAPTER VI

PLASTICITY AND STRENGTH

1. Technical and Rheological Test Curves

WE have mentioned in Section I, 10 the St. Venant-body. This is a solid which is characterized by the property that no permanent deformation appears as long as the stress is below the "yield-value", whereas as soon as this point is reached a permanent deformation is started, which can attain any value, while the stress remains constant at the yield-value, always balanced by the internal elastic reactions in the body.

St. Venant postulated such ideal material on the strength of experiments made by Tresca (1868) on the plastic deformation of metals. Let us therefore look more closely into the tensile test of mild steel which is recorded in what is usually called its "stress-strain curve".

A record of the observations consists in plotting the load (P) against the elongation (Δl) or *vice versa*. Such a curve, where directly measured quantities are plotted, may be called a Technical Test Curve. It forms the basis of the theoretical analysis; the latter aims at *expressing the rheological properties of the material under test by* (i) *a rheological equation and* (ii) *the numerical values of its rheological coefficients*. For accomplishing this task there are certain difficulties to be overcome. First of all, there is a difference depending upon which one of the two variables P and Δl is taken as the independent and which one as the dependent variable. If we gradually increase the load, the rod of mild steel behaves first *elastically*, more or less as a Hooke solid; but with a certain load the yield point is reached, when the rod starts to flow *plastically* at a more or less *constant load*. If, disregarding this, we continue to increase the load, there will be no equilibrium, the material will flow at an accelerated rate and soon break. This difficulty does not arise if the *elongation* is taken as independent variable. Here the load first increases; then remains constant; then again increases; then, after what is called the "necking" of the piece, decreases, until the piece breaks. This is the arrangement in most tensile testing machines for metals, where one end of the rod is fixed, the other travels at constant rate; and where Δl is therefore the independent variable, and the co-related P is observed.

Another method of taking account of the difficulty consists in considering the traction σ and not the load P as the independent variable. With increasing elongation the cross-sectional area gradually

decreases. If we let the traction gradually increase, this will often mean that the load must be decreased. If the elongation continues under *constant* traction, as it is supposed in a St. Venant body or in viscous traction, the load must be gradually decreased in the same ratio as the cross-sectional area decreases. There exist a number of devices to effect this automatically. One, by Andrade for pull, is described in *Twelve Lectures*, p. 98. For another, also for pull, compare Andrade and Chalmers (1932). Two methods for push, keeping compression stresses constant, are described by Scott Blair in *Survey*, pp. 120–122.

A curve in which stress is plotted against deformation in such a manner that the curve is independent of the dimensions of the apparatus may be called, in contradistinction to the technical curve, a Rheological Test-Curve. A rheological test curve will not necessarily be independent of the *kind* of apparatus. If the variables are such that this is the case, the curve may be called the Fundamental Rheological Curve of the material. This is a *graphical representation of the rheological equation of the material.*

2. The "Stress-Strain" Curve for Mild Steel

Let us go back to the technical curve for the tensile test of mild steel. If we plot the load P against the elongation Δl, this gives from the start a different curve for every different diameter and every different length of a different bar of the same material. Therefore, denoting by A_o the original cross-sectional area and by l_o the original length of the test-piece, the "stress" P/A is plotted against the "strain" $\Delta l/l_o$ in what is called* the "stress-strain curve" of mild steel. While this curve is more or less independent of the dimensions of the test-piece, it can still not be regarded as a rheological test curve, the designation of "stress-strain curve" being, as we shall presently see, not fully justified.

As we have said, the elongation Δl is at first proportional to the load, until the yield point (a in Fig. VI, 1) is reached. This is the elastic or Hooke-range. After that the elongation grows quickly at approximately constant load. This is the plastic range. When the point c is reached the load must again be increased, but at a much lower rate than in the Hooke range. This is called the strain-hardening range. When the point b is reached the load must be decreased; the rod starts "necking" until it finally breaks at point e. Plotting P/A_o against $\Delta l/l_o$ results in the curve o–a–c–d–b–e of Fig. VI, 1. However,

* The terms "nominal stress", "nominal strain" and "nominal stress-strain curve" are also used, to distinguish them from the "true" stress, strain and stress-strain curve, about which more will presently be said. But note also the more restricted use of the term "strain" to indicate a reversible, elastic, deformation.

it is clear that this does not tell us much about the rheological properties of the material and even gives a misleading picture, because it is not obvious why, if the test-piece actually breaks at the load P_e, it can sustain the same load at an earlier point d without breaking; and later on at point b an even higher load. The explanation is, of course,

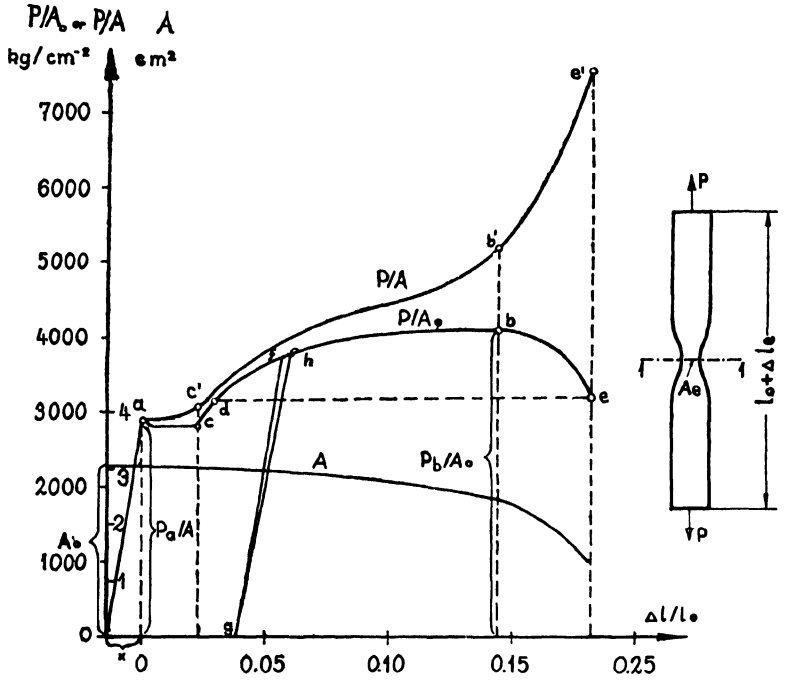


FIG. VI, 1. Technical and rheological test curve of mild steel.
"a" first yield point. "h" yield point after strain-hardening. A cross-sectional area. P load. Δl elongation. (The elastic stram x has been exaggerated.) o–a elastic range. a–c plastic range.

that with increasing elongation the cross-sectional area is gradually reduced and, therefore, the stress increases even for the same load. If we plot the stress P/A against $\Delta l/l_o$, the curve o–a–c'–b'–e' results and this curve looks more reasonable in that there is no fall of stress with increasing strain. However, the strain is here understood as the ratio of elongation to the *total* length of the test-piece, while after necking elongation takes place at the section of striction and should be related to the length of the latter. For this reason the denotion "stress-strain curve" is not applicable after the point "b".

3. The St. Venant Range

Our description of what we called in the preceding section the plastic range, but what in accordance with our nomenclature we shall call

the St. Venant Range, was rather superficial. When the bar first yields, an immediate drop of the load to a lower level can be observed and yielding continues at that lower level. To describe this phenomenon, Bach (1904) coined the terms " upper " and " lower " yield point. It has often been maintained that the phenomenon is not due to intrinsic rheological properties of the material, but to a " machine-effect " resulting from the combined mechanics of test-piece-*cum*-testing-machine. It has been stated that the rise of the yield point is caused by the inertia of the machine and is due to the sudden stretching of the specimen and the inability of the machine to react satisfactorily to this quick change. This question of the influence of the mechanical

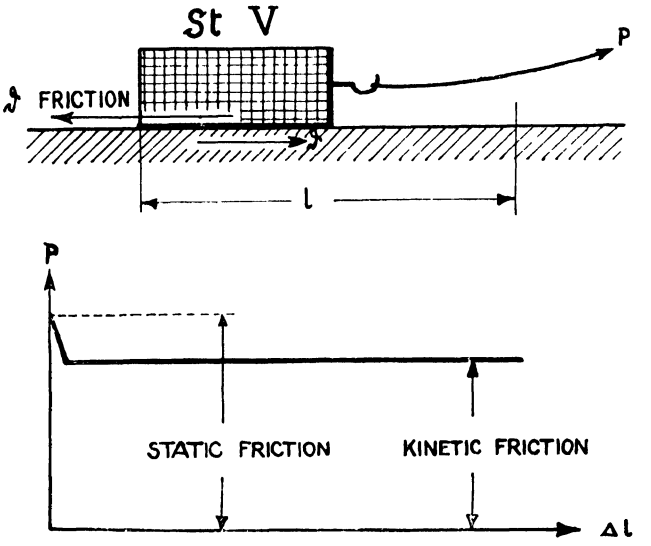


FIG. VI, 2. Model for a St. Venant body.

behaviour of the testing machine upon the observational results is certainly of importance. When the reaction of the apparatus is of the same order of magnitude as the phenomenon to be observed, these can hardly be separated. This has led in the realm of the submicroscopic, where the observed is an electron and the means of observation a photon, to Heisenberg's uncertainty principle. If, in the realm of the macroscopic, the testing machine is a steel structure and the test-piece a steel rod introduced into that structure, conditions become similar. We cannot go into the problem here. It has been dealt with in great detail in a book by W. Spath (1938). In order to find out about the reality of the upper yield point, Edwards, Phillips and Liu (1943) used a simple arrangement of direct loading by adding weights on a pan attached to the lower end of the test-piece, avoiding mechanical devices

for applying loads so that the difficulties from inertia arising from machine bearings, levers and other features of design would be overcome. It was found that, once plastic deformation had commenced, it continued at much lower loads than those initially required. The upper yield point could be raised by careful and *very slow loading*.

This suggests for the St. Venant body a model as shown in Fig. VI, 2. A weight rests upon a table top and there is solid friction between them, a friction which is higher at rest and lower when movement takes place. A string is connected to the weight and a gradually

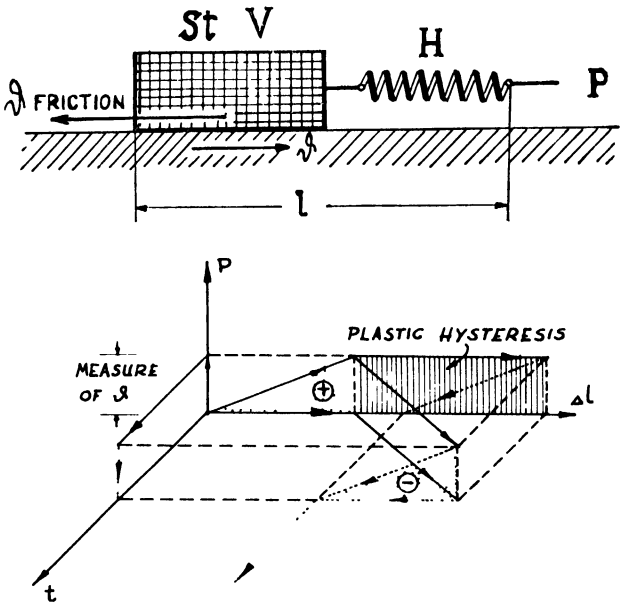


FIG. VI, 3. Model for a combined Hooke-St. Venant body.

increasing pull P applied to its free end. First the string is slack, then it is tightened.

As long as the pull is smaller than static friction, the weight remains at rest. When P reaches and slightly exceeds static friction, the weight starts to move, and in order to maintain equilibrium with constant velocity of movement, P must be reduced to kinetic friction. The resulting load-movement diagram is shown in the lower half of Fig. VI, 2. If we introduce the stress σ for the load P and the deformation d for the movement Δl, we see that the model qualitatively reproduces the behaviour of the test-piece during the St. Venant range. To include the Hooke range, we replace the string by a spring or couple the *StV* element with the *H* element in series, an operation which we

indicate symbolically by $StV - H$. In Fig. VI, 3 the model and the load-movement-time diagram for this arrangement are shown.

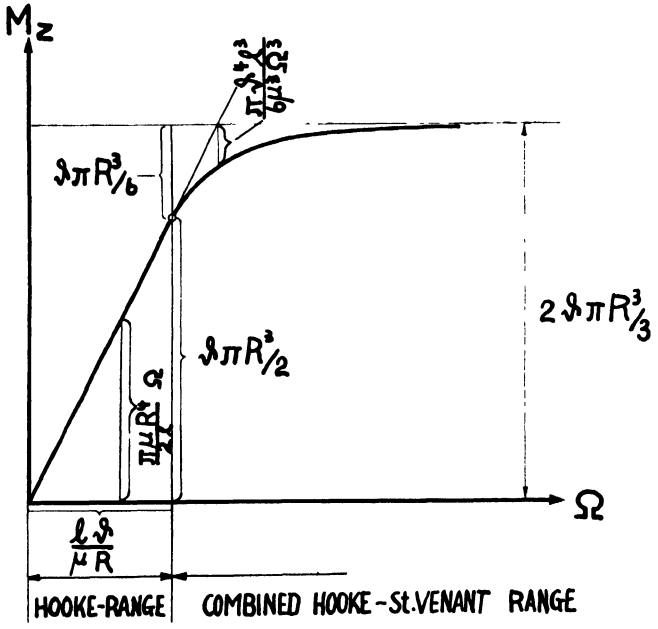


FIG. VI, 4. Plastic torque-twist diagram.*

4. The Yield Stress

We have said that a plastic material will only flow if the shearing stress $\tau \geqq \vartheta$. Now let us assume that a plastic material is subjected to the action of a force P of such magnitude that max. $\tau = 2\vartheta$. Then the whole material will flow. Now let us reverse the direction of P and make $P' = -P$. The shearing stress will also be negative and it will be $\tau' = -\tau = -2\vartheta$. But $-2\vartheta < \vartheta$ and therefore the material will *not* flow.

Can this be correct?

It cannot be, and evidently we must write the flow condition in the form

$$\tau^2 \geqq \vartheta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)\dagger$$

This shows us that the quantity *really* governing plastic flow is not of

* This figure is referred to on page 96.

† If one tries to keep to $\tau = \vartheta$ as the flow condition, stipulating that the *absolute value* of ϑ only is to be considered, this would mean that ϑ had no sign and therefore was a scalar and not a vector. In this case, however, an equation such as $\tau = \vartheta$ where on the left there is a vector and on the right a scalar, is impossible. We must introduce a scalar on the left side also, or τ^2, which is the inner product of the vector τ with itself. We thereby arrive at (1).

the nature of a stress, but something else which is proportional to the *square of a stress*. What that is we shall presently learn.

5. The Flow-Condition

The main *physical* problem of plasticity is the determination of the so-called *flow-condition*. After a flow-condition has been assumed the rest is mathematics.

The flow-condition is that function of rheological variables which determines the yield point, i.e. the transition from the elastic to the plastic state.

In (I, e) we have (provisionally) adopted the very simple condition that in simple shear, plastic flow starts when the shearing stress has reached a certain value ϑ_t.* This may be called *St. Venant's flow-condition*. We have seen in the preceding section that this flow-condition is not entirely satisfactory.

Let us again consider such material spheres as we considered in Section 3 of Chapter I under simple shear. Let there be *two* spheres, one of rubber, the other of plasticene. In straining any one of these spheres we have to perform a certain amount of work. If we apply our force suddenly or quickly, the particles of the sphere will be accelerated and kinetic energy will be produced. This will not be work required for straining. To determine strainwork, we must apply our force very slowly, practically infinitely slow. Then we shall find that we can expend a certain limited amount of strainwork upon the shearing of the rubber sphere, which is stored up in the material as elastic potential energy and can be regained on removal of the load. If we try to force more work upon the sphere of rubber, it fails to take it, it breaks, its parts are torn asunder—and this finishes it. It is different with the sphere of plasticene. In it we can store a small, limited amount of energy, but when this is reached and exceeded there is no rupture. The sphere is continuously and practically indefinitely deformed plastically. However, the work necessary to perform that deformation is not stored up in the material but dissipated by internal friction.

Hencky (1924) has proposed to base the flow-condition on the capacity of the unit volume of the material for elastic energy. He likens the plasticity of the material to a vessel of certain limited capacity which represents the amount of maximum elastic energy which may be stored up in the volume element of the material. This maximum is a fixed quantity ; if more energy is expended in strainwork, the vessel, so to speak, overflows. That overflow represents the yield point.

This idea forms the basis of Hencky's flow-condition. Let us consider the prism of Fig. I, 2. In shifting the upper surface against the lower, the stress τ performs strainwork. When after the displacement

* The suffix t in ϑ_t denotes " tangential".

u has been reached, this is increased by du, the work is increased by $dw = \tau du$. If we consider a cube having edges of unit length du is nothing else than $d\gamma$ and we have

$$dw_t = \tau d\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Therefore, the strainwork expended upon unit volume in a deformation starting from the unstrained state is

$$w_t = \int_o^\gamma \tau d\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let the material obey Hooke's law (I, d) then

$$w_t = \int_o^\gamma \mu\gamma d\gamma = \mu \int_o^\gamma \gamma d\gamma = \mu\gamma^2/2 \quad . \quad . \quad . \quad . \quad (4)$$

Introducing for $\gamma = \tau/\mu$ this is also

$$w_t = \tau^2/2\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let E_{pl} be the maximum amount of elastic energy which a volume element of the material can take up, then

$$E_{pl} = \vartheta_t^2/2\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or

$$\vartheta_t = \pm \sqrt{2\mu E_{pl}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

E_{pl} may be called the Plastic Resilience of the material, i.e. its resilience up to the yield point, Resilience being the amount of elastic energy which can be stored up in the volume element of a material until a certain state is reached.*

Equation (6) is the Hencky flow-condition for the case of simple shear.

In the derivation of this equation we have assumed that the material is a Hooke-body up to the yield point. Mises (1913) assumed that (6) is the flow-condition for any plastic material, whether following in the elastic state Hooke's law or not. Mises' argument is purely mathematical, based on tensor analysis which transcends an elementary treatment.†

The flow-condition (6) or (7) does away with the difficulty which we described in Section 4, ϑ appearing with the $\pm$ sign. Equ. (6) is usually named the Mises-Hencky flow-condition.

6. Plastic Torsion

As an example of plastic deformation, we treat plastic torsion. If an elastic rod is subjected to torsion, we find (compare (IV, g) that in order to rotate its free end through an angle Ω a torque

$$M_z = \pi R^4 \mu \Omega / 2l \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

has to be applied. From this equation it follows that there is a linear relationship between M_z and Ω. If, however, Ω is gradually increased

* It should be noted that E_{pl} is a material constant and a scalar quantity.

† Compare *Twelve Lectures*, p. 112.

through an increase of the external forces, we find that this linear relationship has a limit and that the rod either breaks, when it is brittle, or that it can from a certain stage on be further considerably increased by a very small increase of M_z, when the material is ductile or plastic. We may therefore try to apply St. Venant's law (I, e) to this case.

The application of St. Venant's law is easy enough if we have a case of homogeneous stress. Then the yield point is reached in all parts of the body at the same time and while, before reaching the yield point, the deformation of the whole body was elastic, after passing the yield point the deformation of the whole body is plastic. It is not so in cases of heterogeneous stress. In such cases there must be a surface or surfaces dividing the body in two parts, the rheological equation of which will be either (I, d) or (I, e). The form of the dividing surface is generally not known and this causes the greatest mathematical difficulties. In certain cases, however, the form of the surface suggests itself This is especially so if we are dealing with a laminar displacement. The form of the surface can then be assumed, the assumption can be verified and the position of the surface found from the condition that for the dividing surface the tangential stresses of the elastic part are all equal in magnitude and equal to ϑ_t.

As follows from Equ. (IV, h), the tangential stress in the twisted elastic rod is

$$\tau = 2M_z r/\pi R^4 = \mu\Omega r/l \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The tangential stress has therefore the same value in every point of a cylindrical surface with the radius r and its maximum value for $r = R$

$$\text{max. } \tau = 2M_z/\pi R^3 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The dividing surface will accordingly be a coaxial cylindrical surface. If Ω is gradually increased, maximum τ also increases, until it reaches the value ϑ_t and the other stresses are increased correspondingly. If Ω is still further increased, maximum τ cannot, in accordance with St. Venant's law, increase beyond ϑ_t and a plastic body develops as the outer shell of the cylinder, enclosing an elastic co-axial cylinder. As the tangential stress τ of this elastic core is the same over one and the same concentric cylindrical surface, the dividing surface between the plastic and the elastic parts must, as has been said before, be a cylinder of, say, radius r_o. Within this cylinder the stresses will still follow Hooke's law as given in (9) and r_o can be calculated from

$$\vartheta_t = \mu r_o \Omega/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

to be

$$r_o = \vartheta_t l/\mu\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In the outer shell the tangential stresses will all be equal to ϑ_t. Analogous to (IV, 42) we then have

$$M = 2\pi\int_o^R \tau r^2 dr = 2\pi\left[\int_o^{r_o}\frac{\mu\Omega r}{l}r^2dr + \int_{r_o}^R \vartheta_t r^2 dr\right] =$$

$$2\pi\vartheta_t R^3/3 - \pi\vartheta_t^4 l^3/6\mu^3\Omega^3 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Fig. VI, 4* shows M_z as a function of Ω.

It is interesting to note that while in the case represented in Fig. I, 5, b, when the shear has exceeded the value ϑ/μ the material starts to flow under constant stress ; in our case, when plastic deformation starts there is no flow, but the external forces may even be increased and there will still be an arresting deformation up to a maximum moment (max. $M_z = 2\vartheta_t\pi R^3/3$). This is due to the fact that in the present case there always remains a solid elastic core of diminishing, but never vanishing, radius r_o. It should, however, be noted that we have treated our problem as a case of equilibrium. The torsional moment of the external forces can, of course, be greater than maximum M_z. In this case equilibrium is not possible and also not steady flow. If $M_z > 2\vartheta_t\pi R^3/3$, there will be accelerated flow under the action of a torsional moment which is $M_z - 2\vartheta_t\pi R^3/3$.

7. Strength

When an elastic material has been strained to such an extent that it starts to flow plastically, its strength has been exceeded. However, strength is a more general term.

The *strength* of a material is that property by which it resists either excessive deformation or rupture, the former in many cases ultimately also leading to rupture. A body has suffered *rupture* when it breaks into parts. When the body is strained beyond its strength, it *fails*. Failure can therefore be either through (i) excessive deformation, or (ii) rupture. Excessive deformation may lead to rupture either (*a*) in tension under constant load when the cross-sectional area has substantially decreased, and the traction accordingly much increased ; or (*b*) when one part of the body is sheared against the other : this is rupture by *glide*. In rupture by glide the separation is *tangential* to the surface of the separation, in (*c*) brittle rupture it is *normal* to that surface. There may, however, also be (*d*) a separation normal to the surface which is not brittle, but occurs *in flow*.

Fig. VI, 5 represents the tension-rupture of a steel bar in which the core shows brittle rupture by separation and the outer part rupture by glide. The rupture in the core takes place first. It should be noted that until such rupture takes place, the core suffers tension, not only in the axial direction, but also in the radial direction from the outer ring. The core is therefore subjected to an isotropic tension

* On page 92.

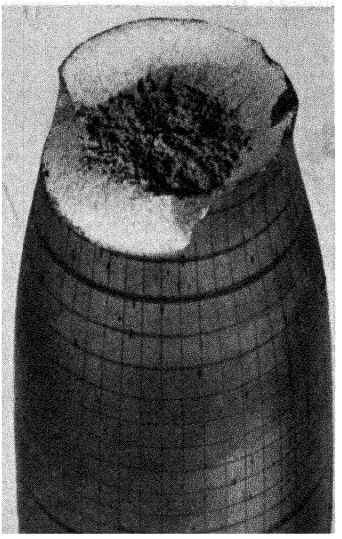


FIG. VI, 5. Tension-rupture of steel bar. Cup and cone fracture of a cylindrical bar of low carbon steel in a tensile test. After tests by A. Nadai. (*By courtesy of Dr. Nadai.*)

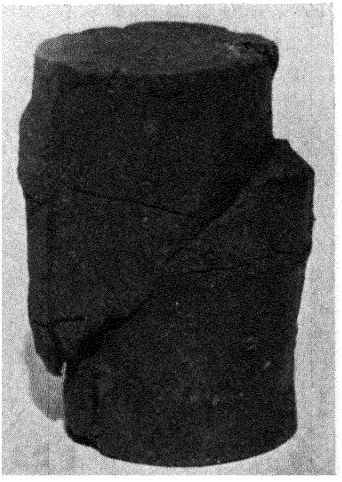


FIG. VI, 6. Rupture of a soil-specimen. Rupture by glide.

(compare Section 8 below). After this rupture by separation has taken place, all load is transferred to the outer ring which then breaks by glide.

Fig. VI, 6 shows rupture by glide of a soil cylinder under axial compression.

Fig. VI, 7 shows brittle rupture by separation of a cement stone cube under vertical push.* On the right, both parts into which the cube was broken, are shown separated.

Fig. VI, 8 shows the breakage of a bitumen prism under axial tension. First the part marked *d* was broken after flow of the porous material had taken place, then all load was transferred to the part marked *c*, which broke in a brittle manner.

To prevent failure is a very important task in engineering which makes it imperative to have a *quantitative measure* of strength. This is accomplished by means of a *theory of strength.*

In the present chapter we consider *statical* theories of strength where the rate of deformation is not taken into account. A dynamical theory of strength in which the rate of deformation is one of the variables is treated in Chapter XIII below.

The following are the main statical theories of strength :

The material fails when a certain limit, which is different in character in each case, is exceeded by

(i) the greatest of the principal stresses (Rankine),
(ii) the greatest of the principal strains (St. Venant),
(iii) the maximum strainwork (Beltrami),
(iv) the maximum shearing stress (Coulomb, Mohr, Guest),
(v) the maximum distortional strainwork (Huber, Hencky).

Nadai (1931) has critically discussed these theories of which (i) to (iii) are of historical interest only. He finally accepts (v) under the heading " New Theories ". Regarding (v), the mathematical expressions for both Huber's and Hencky's theories are not very different, but while Hencky is concerned with *plastic flow*, Huber is concerned with *rupture.*

8. The Huber-Hencky Theory of Strength

Huber's theory arose from Beltrami's theory of maximum strainwork. Beltrami overlooked the important fact of the radically different behaviour of all materials in isotropic strain and distortion, which we have emphasized at the very beginning of the present book. Many experiments have shown that under high *hydrostatic pressure* increasing amounts of elastic energy may be stored in a body without any limit from fracture or permanent deformation provided the material is perfectly homogeneous. Huber, therefore, considered isotropic strain and distortion separately. He assumed that there are two different measures for strength in the case of simple pull and push respectively. Let w_v be the strainwork per unit volume in isotropic (voluminal) strain and $w_{(o)}$ in distortion, then Huber assumed that in push the measure of breaking strength is given by a maximum value of $w_{(o)}$ and in pull by a maximum value of $w_v + w_{(o)}$. Hencky was concerned

* Lateral friction was prevented by lubricating the end faces.

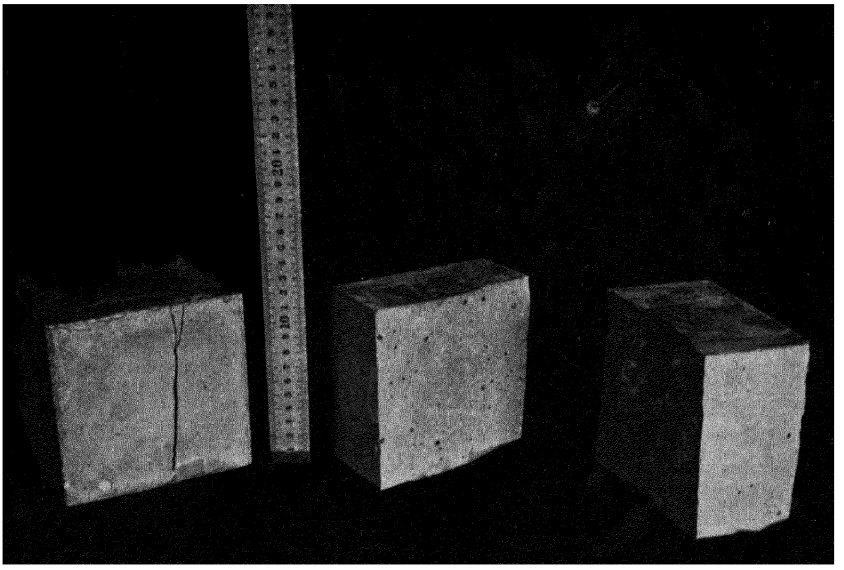


FIG. VI, 7. Brittle rupture of a cement-stone. Rupture took place under pressure in vertical direction.

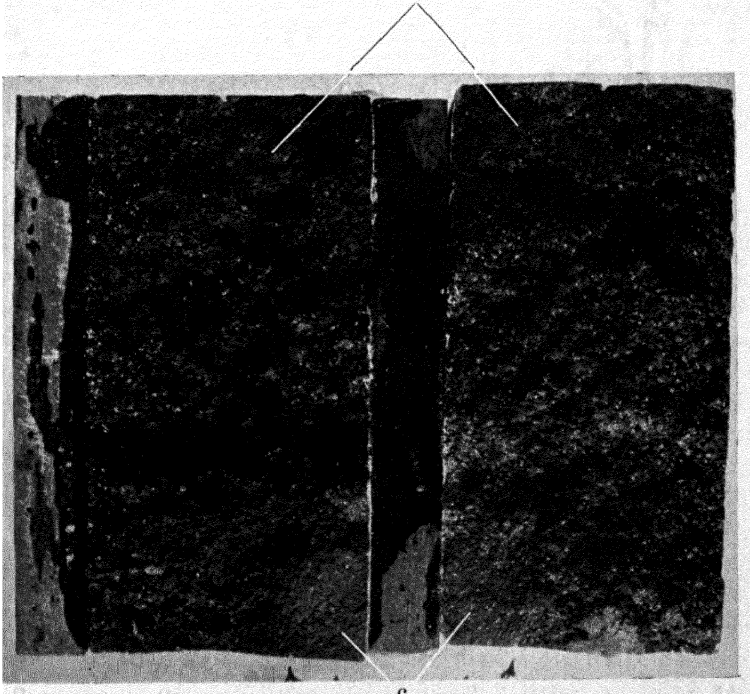


FIG. VI, 8. Rupture after flow. Bitumen prism which broke under tension. d, part which broke after flow, c, part which broke in a brittle manner. (*By courtesy of Road Research Laboratory, Harmondsworth.*)

with finding a measure of plastic yield strength. He argued that there cannot be isotropic flow—and therefore also no isotropic *plastic* flow—either in compression or in tension. Therefore the plastic Flow-Condition must be expressible in terms of *distortion only*. As we have already mentioned before he accordingly pictured every plastic material as a vessel capable of absorbing in the unit volume a limited amount of distortional energy. When more is "poured in" the vessel overflows, or the material yields.

Huber's and Hencky's theories can be combined by assuming that there are three independent measures of strength, viz. :-

(A) Against rupture by isotropic tension ; the resilience E_v.

(B) Against plastic yielding in distortion : the resilience $E_{(o)\,pl}$.

(C) Against rupture in distortion : the resilience $E_{(o)\,r}$.

Here E is "resilience" (energy), while the subscripts indicate "voluminal" (v), "distortional" (o), "plastic" (pl) and "rupture" (r).

In accordance with this, and in contradistinction to Huber, in pull a material may break in *two* ways, depending upon whether in the particular case E_v or $E_{(o)r}$ is exceeded.

9. Cohesional Strength

In Chapter III we considered a body under the action of isotropic stress. When the stress is a pressure, the volume is decreased and we assumed that there is a limiting relative volume which cannot be reduced, even with the pressure becoming infinite. This would not exclude that the body may *break* under such pressure with the total volume of all parts remaining constant, which could well be the case in an æolotropic material. (This gives us the opportunity to remark again that we are dealing in the present book with isotropic or quasi-isotropic materials only.) Secondly, one has to keep in mind the possibility of *local* or *internal* ruptures in porous materials, manifesting themselves in *permanent set*. With these qualifications it may be said that for isotropic *pressure* the strength of every material is infinite. It is not so if the stress is a tension. The volume is then increased, the distances between the molecules or atoms which make up the material are also increased and by this, correspondingly, the attractive reactions between them, which balance the external forces. This cannot go on indefinitely because there is a maximum attractive force which cannot be substantially exceeded. When the external isotropic tension reaches this maximum, the limit of strength of the material is reached. When the maximum is exceeded, the body breaks through rupture, its parts are torn asunder and it loses cohesion. We can therefore say that *for isotropic tension the strength of every material is determined by its cohesion.* The cohesion itself is the resultant of the attractive and repulsive atomic forces. In the unstrained state the resultant vanishes,

each atom being at the bottom of a potential trough, about which its heat movements make it oscillate ; in this state the material does not possess any elastic potential energy. At smaller distances there is practically unlimited repulsion, at larger distances an attraction prevails, which is limited. The elastic energy which is produced by isotropic tension is therefore also limited by the value E_v. This is a measure of its cohesional strength.

Rupture in isotropic tension may be of two kinds. Let the material have no pores or holes, then, when cohesion fails, there cannot be any flow in which the particles of the body move away radially with preservation of the continuity of matter. In this case the rupture is brittle. But if the material has holes, such as was found by Reiner, Rigden and Thrower (1950) in asphalt, the rupture will be preceded by flow, if given enough time (as at "*d*" of Fig. VI, 8). If the stresses are very great, rupture will in this case also be brittle, there being not enough time for flow (as at "*c*" of Fig. VI, 8).

10. Isotropic Tensile Strength

Let us consider this case in greater detail. In the slow application of the tension, E_v can be calculated in accordance with Equation (III, f) and expressed in terms of stress or strain or relative volume or density there being a one-valued relation between all these quantities. In this case there is therefore a *definite isotropic tension* (or cubical dilatation, relative volume, density) at which the material will break. As we said before, that isotropic tension is equal to the molecular or atomic cohesion. For non-porous materials the isotropic tensile strength must accordingly be very high. In classical hydrodynamics it is assumed that liquids have no such strength, but from thermodynamic considerations Poynting and Thomson (1929) suggested that the isotropic tensile strength of water is about 25,000 atmospheres, while van der Waals calculated from his equation a value of about 10,000 atmospheres. O. Reynolds found by actual experiment that water could sustain a tension of about 5 atmospheres without rupture. I suggested in a letter (1943) that the well-known phenomenon of cavitation-erosion of metals may be due to pieces of metal being torn off by the water before its own tensile strength is reached. This would imply that the isotropic tensile strength of a metal was lower than that of water. In reply to my letter, R. S. Silver (1943) pointed out that "the failure of a liquid to sustain the calculated tensile strength in an enclosed and filled space is due . . . to the formation of vapour bubbles. It is the formation of vapour cavities around nuclei which prevents the full tensile strength from being attained." This means that the liquid actually is not *non-porous*, it contains microscopic holes around which there exist *non-isotropic stress-concentrations*.

Now, if even a liquid, the molecules of which easily flow into internal holes to reduce and close them, actually has pores, still more must this be assumed of solids, where pores developed in the process of formation will be stable. Therefore, while the theoretical cohesion might be very high, actually, because of holes, pores, flaws and cracks, the isotropic tensile strength will be comparatively low.

11. Summary

Strength is that property of a material by which it resists failure through either (i) excessive deformation, or (ii) rupture. There are two independent modes of failure in each of the cases mentioned, viz. (*a*) in isotropic, (*b*) in distortional strain. Excluding the exceptional case of volume-flow, failure in isotropic strain can only be brittle rupture.

In *steady states* of deformation, the strength is reached when a definite amount of elastic potential energy, the *resilience* E of the material, is produced in unit volume of it. There are ordinarily three independent measures of strength, viz., E_v, $E_{(o)\,pl}$, $E_{(o)\,r}$; in exceptional cases E_v may be either $E_{v,\,pl}$ or $E_{v,\,r}$.

The resilience E is the time-integral of that part of the rate of strainwork which changes sign on change of sense of strain from positive to negative, while that part of the power which is always positive is dissipated.

When E_v or $E_{(o)\,r}$ are reached or exceeded, the material breaks. When $E_{(o)\,pl}$ or the yield point is reached or exceeded the material flows plastically.

The yield stress ϑ is the stress which results in the resilience $E_{(o)\,pl}$. In simple shear it is ϑ_t, in simple pull or push it is ϑ_n.

An ideal material which is Hookean up to the yield point, and then flows at constant stress $= \vartheta$ is called a St. Venant body. For the St. Venant body the flow condition is

$$\vartheta_t = \pm\sqrt{2\mu\, E_{(o)\,pl}} \quad . \quad . \quad . \quad . \quad . \quad \text{(VI, a)}$$

This is called the Hencky flow condition. Similarly when $E_{(o)\,pl}$ is replaced by $E_{(o)\,r}$, the tangential stress is found at which the material breaks in accordance with Huber's theory.

By application of this theory to the case of a circular rod it is found that the torque M_t, which is produced by a twist Ω/l, is in the case of equilibrium

$$M_t = 2\pi\vartheta_t\, R^3/3 - \frac{\pi\vartheta_t^4}{6\mu^3}\Big/\left(\frac{\Omega}{l}\right)^3 \quad . \quad . \quad . \quad \text{(VI, b)}$$

so that when $M_t = 2\pi\vartheta_t\, R^3/3$, the twist will increase indefinitely.

CHAPTER VII

FIRST INTERLUDE

1. Retrospect

In the preceding chapters we have treated that part of rheology which has become classic under the name of Mechanics of Continua, and is as such included in textbooks of Mechanics following the mechanics of material points, systems of material points, rigid bodies and systems of rigid bodies. All these are also idealizations, even more so than are the Hooke solid and the Newtonian liquid. When mechanics deals with the movements of the planets around the sun, these are considered as material points, each one possessing some mass m. In these considerations the material properties of the celestial bodies, whether they are elastic solids or plastic solids or liquids, are entirely ignored. This is how Newtonian mechanics started. When mechanics turned to problems on the earth, it postulated the -likewise non-existing— rigid body. If we extend the designations, which we proposed in Chapter I for ideal bodies, to those of mechanics proper, we may call the rigid solid a *Euclid Solid*, Euclid (300 B.C.) having based his geometry on the supposed existence of such bodies. As the opposite of the rigid body, Pascal (1663) conceived a material the particles of which could be moved past each other with infinite ease, without offering any resistance. This is a liquid which was called the " ideal " liquid, not possessing any viscosity, and which we may call the *Pascalian liquid*. In both, the Euclid-solid and the Pascalian liquid, no material constant besides the mass appears : they are therefore still outside the range of rheology. Then two ideal materials entered science, carrying with them " material constants ", and thereby belonging to (the then not yet existing) rheology. We have named them the Hooke solid and the Newtonian liquid respectively. They are the classical bodies. In such textbooks as Love's (1927) for elasticity and Lamb's (1932) for hydrodynamics their problems are reduced to Applied Mathematics, stripped of all physical contents. At an early date Static Plasticity was added to satisfy the needs of the engineer. It received systematic treatment in Nadai's textbook (1931).

In Table 1, these classical bodies are listed. In accordance with what we termed the first axiom of rheology, they have *one volumetric rheological equation in common* in which the isotropic (or " hydrostatic ") pressure p is related to the elastic cubical dilatation or volumetric strain e_v by

(III, a) $$-p = \kappa e_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

TABLE VII, 1. *Scheme of Classical Bodies*

	LIQUID		SOLID		
	Flow		Plastic Deformation	Elastic Strain	Rigid
	"ideal"	viscous			
Designation of Material	Pascal	Newton	St. Venant	Hooke	Euclid
Symbol		N	St.V	H	
Shear constant	$0 \leftarrow \eta$		ϑ_t	$\mu \rightarrow \infty$	

and are distinguished from one another by equations in which the tangential or shearing traction τ is related to (i) the recoverable displacement gradient γ in the case of the Hooke elastic solid by

(I, d) $$\tau = \mu\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or (ii) to the irrecoverable velocity gradient $\dot{\gamma}$ in the case of the Newtonian liquid by

(I, f) $$\tau = \eta\dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

while in the case of the St. Venant plastic solid the traction remains unchanged during the plastic deformation or flow, or

(I, e) $$\tau = \vartheta_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These equations can be used as the defining rheological equations in the case of simple shear without rigid body rotation, for instance in the case of telescopic laminar flow (see Fig. II, 1b) or torsional laminar distortion (see Fig. II, 1c). But we have seen that when there is present a solid body rotation as in the case shown in Fig. II, 1a, there will be a velocity gradient even when there is no deformation, and we had to introduce in Section II, 8, when treating such problem, another quantity named the rate of shear $\dot{G}$. Even this is not adequate when treating more general cases, and to cope with such, we defined an infinitesimal deformation d with tangential component d_t related to a change of angle and longitudinal component d_l related to a change of length. These quantities are not affected by rotations. We found that in simple shear the displacement gradient γ

(II, p) $$\gamma = 2d_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and when there is a rigid body rotation the amount of shear G

$$G = 2d_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Taking time derivatives, we find $\dot{d}$, the rate of deformation, related similarly to $\dot{\gamma}$ and $\dot{G}$.

When we desire to indicate more particularly that the deformation is a recoverable strain, we write e for d, and when we want to indicate that we mean by the rate of deformation a flow, we may write f for $\dot{d}$.

We have said that the rheological distinctions of the different materials are not affected by changes of volume. Therefore, when considering deformations other than laminar shear, whence there *is* a change of volume, we must, in order to express the rheological equation in a more general form, reduce the tractions by the isotropic stress, and the deformations by the cubical dilatation. We thus get those components which refer to *distortions*, and which we indicate by a subscript (o). The shearing tractions τ are not affected, neither are the shearing deformations d_t or the shearing gradients γ (where applicable) because as just said, in laminar deformations the volume does not change. It is different with those tractions (σ_n) which have the direction n normal to the surface element upon which they act. They are related to linear or longitudinal deformations or extensions, which we call in this connection "normal" deformations denoted by d_n. If we assume as system of reference the three principal axes i, j, k, then

(III, j) $$e_v = e_i + e_j + e_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

(III, m) $$-p = (\sigma_i + \sigma_j + \sigma_k)/3 = \sigma_m \quad . \quad . \quad . \quad . \quad . \quad (8)$$

so that

(III, k) $$d_{n(o)} = d_n - e_v/3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

(III, l) $$\sigma_{n(o)} = \sigma_n - \sigma_m \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We thus arrive at the more general rheological equation for the Hooke solid :

(III, o) $$\sigma_{n(o)} = 2\mu\, e_{n(o)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and similarly for the Newtonian liquid

(V, a) $$\sigma_{n(o)} = 2\eta f_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where the normal flow f_n stands for $f_{n(o)}$, because there is supposed to be no volume flow, or

$$f_v = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

These equations make it possible to treat simple pull or push in the elastic solid and the viscous liquid, when

$$\sigma_i = \sigma_j = 0 \quad \sigma_k = \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and therefore

(III, n) $$\sigma_m = \sigma/3,\ \sigma_{i(o)} = \sigma_{j(o)} = -\sigma/3,\ \sigma_{k(o)} = 2\sigma/3 \quad . \quad . \quad (15)$$

The simple (one dimensional) pull σ causes a three-dimensional deformation consisting in general of a linear extension d_l in the k-direction

accompanied by two equal lateral contractions d_c in the i- and j-directions, with

(III, s) $$d_c = -\nu d_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where ν is Poisson's ratio, so that

$$d_v = (1 - 2\nu)d_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and

$$\left.\begin{aligned} d_{i(o)} &= d_{j(o)} = -\frac{1+\nu}{3} d_l \\ d_{k(o)} &= 2\frac{1+\nu}{3} d_l \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

It is convenient to introduce in the case of the Hooke solid Young's modulus E defined by

(III, r) $$\sigma = Ee_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and similarly for the case of the Newtonian liquid Trouton's coefficient of viscous traction λ defined by

(V, b) $$\sigma = \lambda f_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now from (1), (17), (8), (15), (19)

$$E = 3\kappa(1 - 2\nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

while from (11), (15), (18), (19)

$$E = 2\mu(1 + \nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Therefore from (21) and (22)

(III, u, v) $$E = \frac{9\kappa\mu}{3\kappa + \mu}, \quad \nu = \frac{3\kappa - 2\mu}{6\kappa + 2\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

by which the " derived " material parameters E and ν are expressed through the fundamental parameters κ and μ.

When the material is incompressible ($\kappa = \infty$), we easily find

(III, w) $$E^* = 3\mu, \quad \nu^* = \tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Specializing the viscous liquid to the Stokesian case defined by (13), we get by a similar procedure

(V, b) $$\lambda^* = 3\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The foregoing equations suffice for the solution of any problem of the deformation and flow of a body made up of a material which can be considered as coming to a first approximation under the picture of one or other of the classical bodies representing the three fundamental properties of *elasticity*, *plasticity*, and *viscosity*. The procedure is to use the equations of equilibrium

(I, b, c) $$\Sigma P = 0, \quad \Sigma M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

in which the tractions σ and τ may be related through the rheological

equations to strain or flow components. Examples have been given for elastic bending and torsion in Chapter IV, for plastic torsion in Chapter VI, and for viscous flow, telescopic and rotational, in Chapter II. On these models the rheologist can find the solutions of different problems provided he has the appropriate mathematical tools available.

One other property must be considered as fundamental, namely *strength*. This is governed by the *ultimate resilience* (E) of the material, or the maximum *strainwork* (w), which can be absorbed in a unit volume without either *plastic yielding* or *rupture*. We thus distinguish E_{pl} and E_r, adding, where necessary, the subscripts v and (o) to indicate " volumetric " or " distortional ". The strainwork per unit volume is the time integral of the specific strain power $\dot{w}$

$$\dot{w} = s\dot{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where s stands for some traction which is related to a particular e. The specific strain power is the sum of two, one related to the volumetric, the other to the distortional strain

$$\dot{w} = \dot{w}_v + \dot{w}_{(o)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

For the first we have

(III, e) $$\dot{w}_v = p\dot{e}_v = \kappa e_v \dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and by integration

(III, f) $$w_v = \tfrac{1}{2}\kappa e_v^2 = \tfrac{1}{2}p^2/\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

For the distortional strain we consider simple shear in which there is no change of volume. We find

$$w_t = \tfrac{1}{2}\mu\gamma^2 = \tfrac{1}{2}\tau^2/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

More generally we have

$$w_{(o)} = \mu e^2_{n(o)} = \sigma^2_{n(o)}/4\mu \quad . \quad . \quad . \quad . \quad . \quad (32)$$

As an example we treat the following problem :

Assume that the yield point of a mild steel has been determined in a tensile test, e.g. as shown in Fig. VI, 1 to be $\vartheta_n = 2{,}800$ kg/cm², where n denotes the " normal " component of the yield stress ϑ. The question is, at what torque M_z will a cylindrical rod of the same material first yield?

We have found (compare Fig. VI, 4) that the critical torque is $= \vartheta_t \pi R^3/2$, where we write ϑ_t to indicate the tangential component of the yield stress. We therefore must know what ϑ_t is for this material. The tangential yield stress could be determined by an experiment, which, however, is much more difficult to carry out than the tensile test. Here a theory of strength shows its usefulness. We know from Equ. (VI, a) that

(VI, a) $$\vartheta_t = \pm \sqrt{2\mu E_{(o)pl}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where $E_{(o)pl}$ is the plastic resilience.

The problem is to express ϑ_n as determined through simple tension as a function of $E_{(o)pl}$.

We use Equ. (32) in the second form, and have from Equ. (15)

$$w_{(o)} = (\sigma_{1(o)}{}^2 + \sigma_{3(o)}{}^2 + \sigma_{4(o)}^2)/4\mu = \sigma^2(\tfrac{1}{9} + \tfrac{1}{9} + \tfrac{4}{9})/4\mu = \sigma^2/6\mu \quad . \quad (34)$$

Therefore

$$\vartheta_n = \pm \sqrt{6\mu\, E_{(o)pl}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and

$$\vartheta_t = 0{\cdot}58\, \vartheta_n = 1{,}620 \text{ kg/cm}^2 \quad . \quad . \quad . \quad . \quad (36)$$

This supplies the solution of the problem, without experiment—provided Hencky's theory is correct, which of course, must be verified by experiment (as has been done).

2. The Stress

When making use of Equ. (26), we have to include in ΣP and ΣM all *external* forces, which are either what Newton called *impressed forces* or *reactions*. For instance, in the problem which we treated in Section IV, 4, P is the impressed force, and $P/2$ on the right and left support are reactions. The impressed forces are either *surface forces* as e.g. the force P in this case acting upon the upper surface of the beam or *body forces* as e.g. the weight w in the problem Section IV, 3. The surface force P will in general be inclined against the surface upon which it acts, and we can resolve it into two components, namely one P_t, acting tangentially, and the other P_n, acting normal to the surface. Examples for the first were shown in Fig. I, 2, for the second in Fig. III, 5. It is then natural to refer the body force (e.g. the weight of the body) to a volume-element, and the surface force to a surface element. We thereby arrive at the concept of traction, which is the surface force per unit area, and which will have in general a tangential component τ and a normal component σ.

However, the problems which we have treated so far have shown the need of a new concept without which we cannot well proceed from mechanics to rheology ; this is the concept of *stress*.

Stress is the result of an internal reaction to the external forces acting upon the body. How do we know that there is such an internal reaction ? Is there any experimental evidence ? Can we determine the magnitude of the stress experimentally ? It will perhaps surprise many readers if my answer is *no*. True, there exists an excellent book with the title *Handbook of Experimental Stress Analysis* (Hetenyi (1950)). However, if the reader will examine the many methods explained there, he will find that not one of them deals with stress directly. They are occupied with changes of electrical conductivity, with optical anisotropy, etc., etc., which are all related to a hypothetical stress. Why then introduce such a concept ? The reply is that we are forced to it

through the application of Equ. (26) to part of the body as well as to the whole body. Consider a prism in simple pull as was shown in Fig. III, 7, in equilibrium under the action of two external forces P_n. Of these, the lower one is an impressed force, the upper one a reaction. We now reason that if the prism is in equilibrium, each part of it is also in equilibrium. Consider that part which is shown in the figure in bold lines separated from the rest *in imagination* by a cut a-b. In order that this part shall be in equilibrium with $\Sigma P = 0$, we must assume that the upper part exerts upon it a force P_n acting upwards. This internal force, which is a reaction to the action of the load, per unit area of the imagined interface, is called a *traction*. It can be resolved into two components σ_n and τ_n, where the subscript n refers to the direction of the normal to the interface. In Equ. (III, 66) we have derived expressions showing how the magnitude of these tractions depends upon the direction or rather the angle ϕ, which it makes with the axis of the prism. We have shown how the tractions σ_n and τ_n are related, and that they can be considered as co-ordinates of points lying on a circle called Mohr's circle. Each such point defines a traction with components σ_n and τ_n. We now add that the stress is the *aggregate of all these tractions*, i.e. it is represented by the complete circle.

We repeat that we cannot determine the magnitude of the traction directly by experimental means ; we can only calculate it by applying Equ. (I, b). However, Weissenberg (1948) has found a very nice method by which its presence or absence can be detected. This we shall presently describe.

3. Weissenberg's Method for the Detection of Tractions

To reveal the internal forces in a qualitative manner, Weissenberg makes linear cuts in the material at various points and in various directions. A rough measure of the traction forces across these cuts is furnished by the degree of opening up of the cuts. Figs. VII, 1 and VII, 2 provide illustrations of two specimens of cloth in different states of stress. The first specimen was allowed to contract freely widthwise while being stretched lengthwise between two parallel rollers. The second specimen was at its ends rigidly fixed widthwise to two parallel clamps and thus constrained in the neighbourhood of the clamps from widthwise contraction while being stretched lengthwise by a separation of the clamps. Each specimen reached a position of equilibrium in some stretched state, and was then marked in the upper half by three rows of linear cuts widthwise, and in the lower half by three rows of similar cuts lengthwise.

The opening up of the cuts shows that in the specimen which is free to contract, the internal forces are the same—qualitatively and quantitatively—throughout the specimen, and that they are

homogeneous. In contradistinction, in the other specimen they vary from row to row, and are therefore *heterogeneous.* Here also the stress is not measured directly, but only inferred.

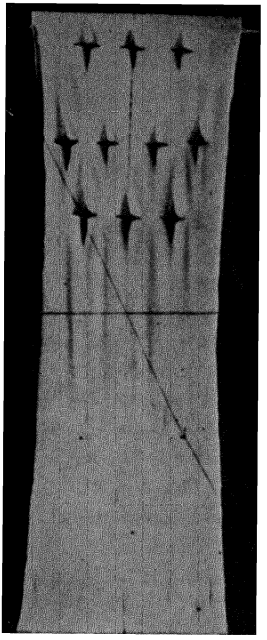


FIG. VII, 1. Homogeneous deformation in pull.

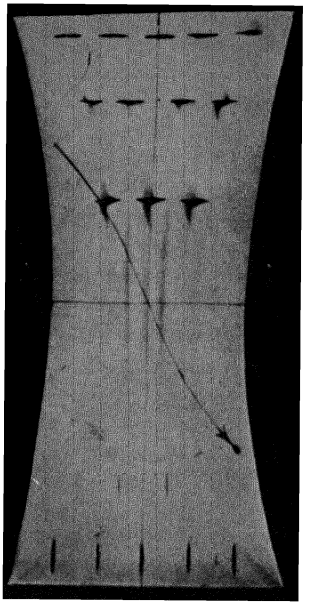


FIG. VII, 2. Heterogeneous deformation in pull.
(*By courtesy of Prof. Weissenberg.*)

4. Mohr's Stress Circle

We have said in Section 2, that the magnitude of the traction can only be calculated from Equ. (I, b). We shall now show how this can be done, and at the same time we shall introduce Mohr's stress circle in a form more general than that shown in Fig. III, 8.

We have said in Chapter III, that there exist at every point of a stressed body three axes i, j, k, in which the tractions have only normal components σ_i, σ_j, σ_k, while the tangential components (which would be τ_i, τ_j, τ_k) are absent.

Consider a prismatic element of the body as shown in Fig. VII, 3 cut in such a manner that two of its sides are normal to the principal directions i and j, while the plane of the drawing sheet is normal to the direction k, which itself is parallel to the axis of the prism. The right angular triangle shown in the drawing is therefore a cross-section of the prism.

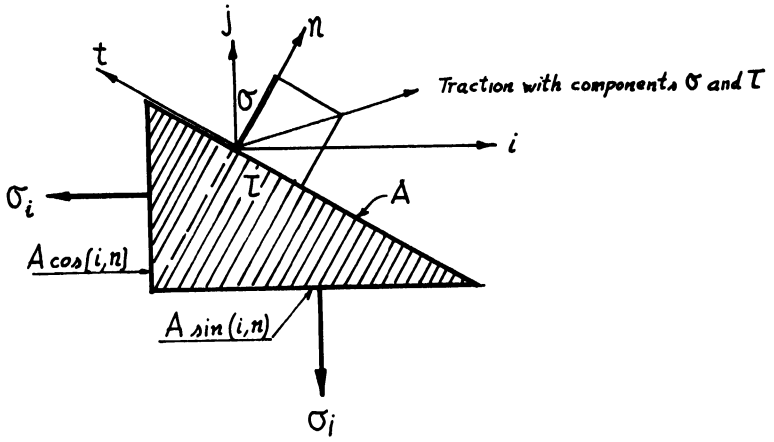


FIG. VII, 3. Tractions on prismatic element.

Now consider the equilibrium of the prism (of, say, unit width in the direction k) in the directions i and j.

The forces which act in the i direction are equal to the respective tractions times the area. We have in the direction i : $[\sigma \cos(i, n)]A$ and $[\tau \sin(i, n)]A$, and in the opposite direction: $\sigma_i A \cos(i, n)$. Therefore

$$\sigma \cos(i, n) + \tau \sin(i, n) - \sigma_i \cos(i, n) = 0 \quad . \quad . \quad . \quad (37)$$

Similarly in the direction j :

$$\sigma \sin(i, n) - \tau \cos(i, n) - \sigma_j \sin(i, n) = 0 \quad . \quad . \quad . \quad (38)$$

From these two equations we easily find

$$\left.\begin{aligned} \sigma &= \sigma_i \cos^2(i, n) + \sigma_j \sin^2(i, n) = \frac{\sigma_i + \sigma_j}{2} + \frac{\sigma_i - \sigma_j}{2} \cos 2(i, n) \\ \tau &= -\frac{\sigma_i - \sigma_j}{2} \sin 2(i, n) \end{aligned}\right\} \quad (39)$$

These equations suggest a graphical representation as indicated in Figs. VII, 4 and VII, 5. From Fig. VII, 4 we see that the analytical representation of a circle with its centre on the x-axis is given by the two equations

$$\left.\begin{aligned} x &= \frac{a + b}{2} + \frac{a - b}{2} \cos 2\phi \\ y &= \frac{a - b}{2} \sin 2\phi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (40)$$

where a and b are the abscissæ of the intersections of the circle with the x-axis. Accordingly σ and τ can be considered as abscissa and ordinate of a circle which intersects the σ-axis in σ_i and $\sigma_j < \sigma_i$ with

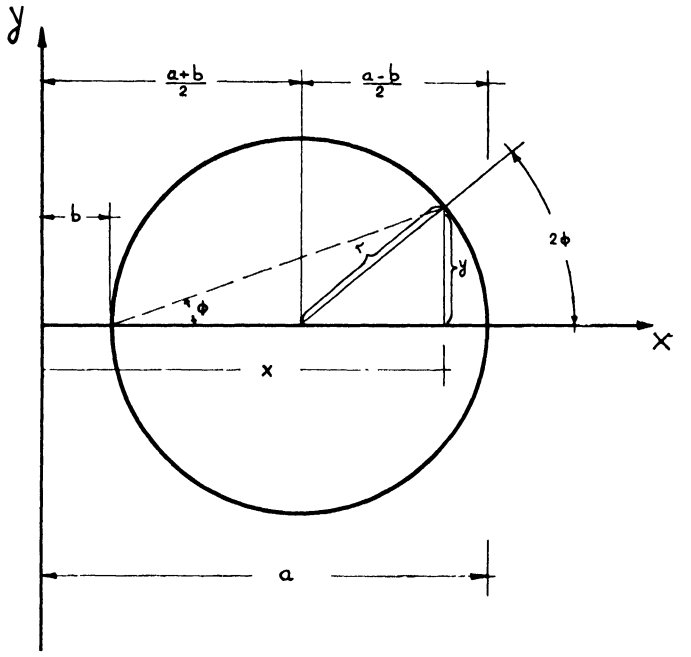


FIG. VII, 4. Representation of circle.

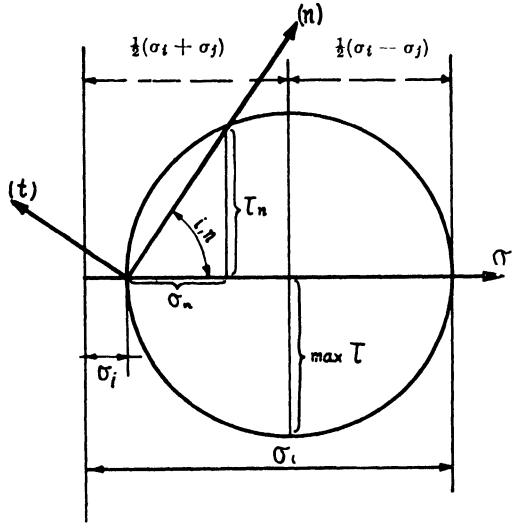


FIG. VII, 5. Mohr circle.

(i, n) the angle between the normal n and the i-direction. Therefore, to construct Mohr's circle if the principal stresses σ_i and σ_j are given, draw the σ axis parallel to the i-direction and the τ axis in the direction

opposite to j. Plot σ_i and σ_j on the σ-axis, and find point C, the centre of the circle, from $(\sigma_i + \sigma_j)/2$. Draw circle with radius $(\sigma_i - \sigma_j)/2$ as shown in Fig. VII, 5. To find the tractions σ_n and τ_n on any surface element the normal of which is in the direction n, draw parallel to n through leftmost point of circle ; the intersection of this line with the circle has the coordinates σ_n and τ_n. If n is parallel to x, the coordinates are σ_x and τ_{xy}.

We can see that the circle of Fig. III, 8 is built on these lines. We can also easily find that a simple shearing traction τ is equivalent with a tension σ_i and a pressure $\sigma_j = -\sigma_i$, acting in directions inclined at 45° towards the plane in which τ acts. Furthermore there follows for the case $\sigma_i = \sigma_j = -p$, that the pressure is the same in all directions, and that in none is there a shearing traction, a result due to Pascal.

5. Prospect

We have treated in the preceding chapters the classical ideal bodies in which the isotropic and the distortional part of the deformation or the rate of deformation are, separately, proportional to the relevant stress, or together, linear functions. We now proceed on these lines to more complicated types of behaviour of materials in which the fundamental properties of elasticity, viscosity and plasticity are *combined*, so that a material may both respond elastically and flow viscously or even show elastic recoverable strain, plastic flow and viscous flow at the same time or separately under certain conditions. We shall, however, in all these cases assume the rheological equations to be *linear* relations between stress and deformations and their rates. Only after showing how far the behaviour of real materials can be represented by equations of this kind, shall we proceed to non-linear relations.

CHAPTER VIII

PLASTICO-DYNAMICS OR THE BINGHAM-BODY

1. The Rheological Equation of the Bingham-Body

LET us refer to the viscosity-elasticity analogy of which we gave an example in Section V, 7. If we know the solution of a problem of viscous flow, this analogy permits us to write down at once the solution for the analogous problem of elastic strain in accordance with the two equations

$$\tau = \mu\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\tau = \eta\dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and *vice versa*. Now we have said that when a tube containing toothpaste is slightly pressed so that the yield point of the paste is not reached, the flat surface at the opening of the tube protrudes in the shape of a more or less spherical segment. From (II, 8) we can say that this will actually be a *parabolic* segment and the maximum protrusion will be, in accordance with (II, 9),

$$u_o - \Delta p R^2/4l\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The toothpaste can be regarded to a first approximation as a St. Venant body. At what pressure difference Δp will it start to flow? The reply is that it will start to flow when and where τ reaches the yield stress ϑ. The shearing stress is, as in (II, c), $\tau = r\Delta p/2l$, and it has its maximum at the wall where $r = R$. Therefore the material will start to flow when

$$R\Delta p/2l = \vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$\Delta p = 2l\vartheta/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and then

$$u_o = R/2.\vartheta/\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

What happens when the pressure Δp exceeds $2l\vartheta/R$? In this case the material will flow out of the tube. But if the material is a St. Venant body, the resistance to flow cannot be greater than $2l\vartheta/R$. Therefore, there will be no equilibrium and the flow is accelerated. If there is no viscous resistance which increases with the velocity of flow, the acceleration will operate without diminution and the velocity of the extruding paste will become evermore greater. Of course, we do not expect anything of the sort to happen with toothpaste. We are inclined to expect that with some comparatively small pressure, enough

to make the paste flow, the velocity of flow will soon become constant.

This shows that such a material can only *to a first approximation* be regarded as a St. Venant body. *To a second approximation it must also have viscosity.* When we realize this, we have discovered the Bingham-body. Bingham and Green (1919) actually discovered this combination of plasticity and viscosity in another material, namely oil-paint. Before Bingham, it was thought that oil-paint was a liquid and its " viscosity " was determined from Poiseuille's law. However, this was an " apparent " viscosity (η') only, as we shall call *every magnitude determined from Poiseuille's law or similar laws, when applied to a material which is not a simple Newtonian liquid.* Given enough time, a

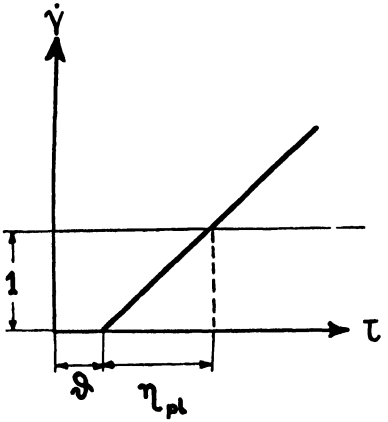


FIG. VIII, 1. Rheological curve of Bingham the body.
ϑ yield stress, η_{pl} plastic viscosity, v velocity of deformation.

liquid, when applied to a vertical surface must run down. If the material is to stay on the surface it must be a solid, even if a very soft solid, and this is what paint actually is.* If the layer of paint is so thin that its weight produces a shearing stress $< \vartheta$, it will not flow and therefore not run off. That layer establishes itself automatically ; what is thicker runs off, and the rest remains.

All this induces us to postulate another ideal material, the rheological equation of which can be derived from a combination of the two equations (I, e) and (I, f) as follows :

$$\tau = \vartheta + \eta_{pl}\dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This contains *two* coefficients, the yield stress ϑ and a viscosity, η_{pl}, which we may call Plastic Viscosity (called by Bingham " stiffness "). The rheological curve of a Bingham-body is shown in Fig. VIII, 1.

* We consider *fresh* paint and disregard the chemical change brought about in drying.

The quantity of plastic material extruded through a tube in unit time can be calculated in very much the same way as the quantity of viscous liquid in the calculation of Poiseuille's law. We know from (II, 6) that, generally, if the flow is steady, $\tau = r\Delta p/2l$. We have postulated for the Bingham-body the rheological equation (7), $\tau = \vartheta + \eta_{pl}\dot{\gamma}$. Introducing in accordance with (II, b) $\dot{\gamma} = dv/dr$, we get

$$\eta_{pl}dv/dr = r\Delta p/2l - \vartheta \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This, by integration, gives $\eta_{pl}v = r^2\Delta p/4l \quad \vartheta r + C$ and, again assuming that the material adheres to the wall, i.e. that there is no slippage, $C = -R^2\Delta p/4l + \vartheta R$ and, therefore,

$$-v = \frac{1}{\eta_{pl}}\left[\frac{(R^2 - r^2)\Delta p}{4l} - \vartheta\,(R - r)\right] \quad . \quad . \quad . \quad (9)$$

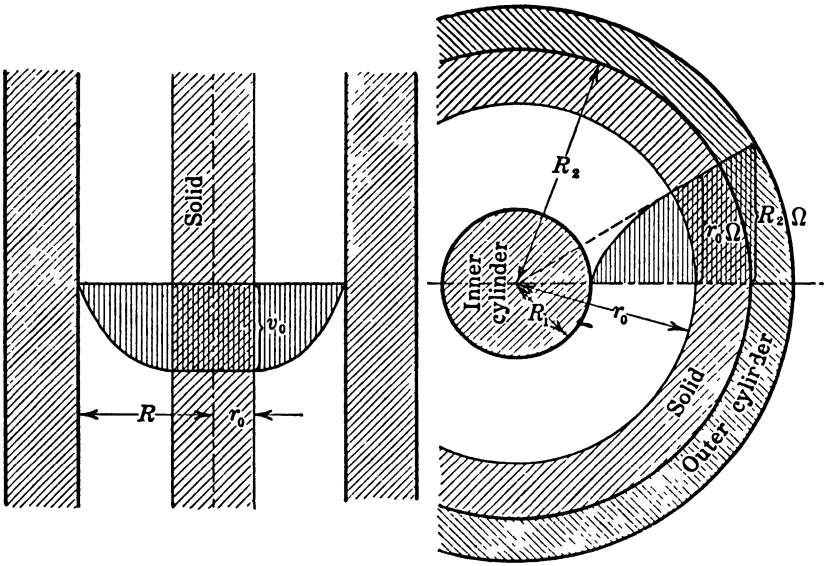


FIG. VIII, 2. Plastic flow in the tube (*left*) and rotating-cylinder instrument (*right*).

The velocity distribution, corresponding to this equation, is as shown in Fig. VIII, 2 (left).

In a viscous liquid the shearing stress vanishes at the axis in accordance with Equ. (II, c) and increases with the distance from it. Therefore, around the axis, there will be a cylinder of material where $\tau < \vartheta$. The stresses in this cylinder will be purely elastic and the cylinder will move as a solid plug, the flow therefore being called "plug flow." Let r_0 be the radius of that cylinder for which

$$\vartheta = r_0\Delta p/2l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When $r = r_o$, $dv/dr = 0$, as can be seen from (8). This confirms that for $r \gtrless r_o$, while there is movement there is no flow.* Stresses in the solid plug are purely elastic.

The velocity of the plug, v_o, is found by introducing into (9) r_o for r and for ϑ the value from (10). We find

$$v_o = \frac{\Delta p}{4l\eta_{pl}} (R - r_o)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The volume of flow can be calculated by dividing the rotational body shown in Fig. VIII, 2 by planes normal to the axis in cylindrical discs of thickness, dv, so that $dQ = r^2\pi dv$ and therefore

$$Q = \int_{r=r_o}^{r=R} r^2\pi dv = \frac{\pi}{\eta_{pl}} \int_{r=r_o}^{r=R} (r^3\Delta p/2l - r^2\vartheta)dr.$$

By introducing r_o from (10) and carrying out the integration, we find

$$Q = \frac{\pi R^4 \Delta p}{8l\eta_{pl}} \left[1 - \frac{4}{3}\frac{2\vartheta l}{R\Delta p} + \frac{1}{3}\left(\frac{2\vartheta l}{R\Delta p}\right)^4\right] \quad . \quad . \quad . \quad (12)$$

This equation replaces Poiseuille's equation for the Bingham-body. It was first derived by Buckingham (1921) and independently by Reiner (1926) and is known under the name of Buckingham-Reiner equation. For $\vartheta = 0$, for which r_o also vanishes, (12) is reduced to Poiseuille's law, Equ. (II, e), as it should.

2. Consistency

If we do not know that a material is a soft plastic solid, but mistake it for a viscous liquid, as was the case with oil-paint, we shall calculate a " viscosity " in accordance with Poiseuille's law, which is not a true viscosity and which we termed in Section 1 Apparent Viscosity. The apparent viscosity of a Bingham-body is in accordance with (12).

$$\eta' = \pi R^4 \Delta p/8lQ = \frac{\eta_{pl}}{1 - \frac{8}{3}\frac{\vartheta l}{R\Delta p} + \frac{16}{3}\left(\frac{\vartheta l}{R\Delta p}\right)^4} \quad . \quad . \quad (13)$$

The denominator on the right side approaches unity for large driving pressures Δp. For $\Delta p = 2\vartheta l/R$ it vanishes and η' approaches infinity. This confirms that while η_{pl}, the plastic viscosity of a material, may actually be a constant, the apparent viscosity will nevertheless vary and will increase as the driving pressure is reduced (compare

* " Movement " is possible of a rigid body, " flow " requires distortion.

Fig. VIII, 3). However, when we calculate a " viscosity " we cannot know beforehand whether it will turn out to be a " true " or an " apparent " viscosity. This shows that we are in need of a rheological concept more general than the concepts " liquid " and " solid " ; in fact one which is applicable to liquids and solids alike. Such a concept is Consistency.

Consistency is, in accordance with Bingham, that property of a material by which it resists a *permanent* change of shape and it *is defined by the complete flow-force relation.* We speak of the consistency

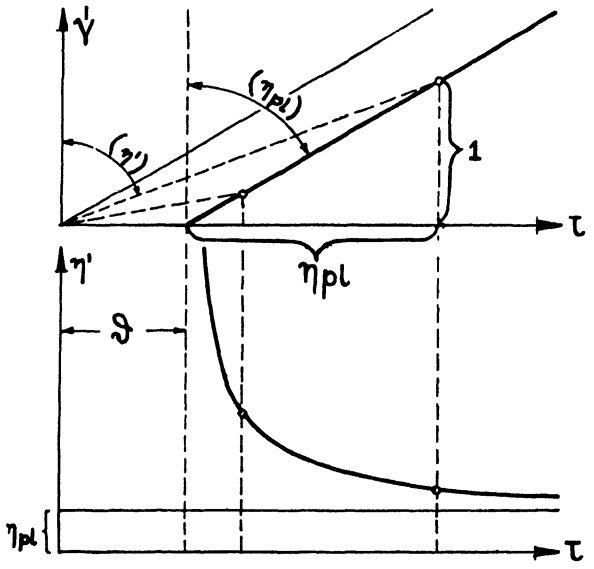


FIG. VIII, 3. The apparent viscosity of a Bingham-body. (η') is a measure of η' ; it is the arc the tangent of which is η'.

of butter and honey, the first being a solid, the second a liquid. We may not know whether asphalt is a liquid or a solid, but we can measure its consistency ; and an analysis of its Consistency Curve determines its nature. We can say that if the consistency curve starts at a point on the " force " axis other than the origin, the material is a solid. This statement cannot be reversed. If the consistency curve *passes* through the origin, the material may be either a liquid or a solid which shows a " wall-effect ". We shall become acquainted in Section 8 of Chapter XIX with the criterion for distinguishing solids and liquids applicable in this case.

In contradistinction to consistency, Rigidity is that property of a material by which it resists an *elastic* change of shape. The Buckingham-Reiner equation can be expressed independently of apparatus

dimensions by means of the same variables as used in the Poiseuille law, viz. by V and P from (II, f) and (II, g). We find

$$V = P/\eta_{pl}\left[1 - 4\vartheta/3P + \frac{1}{3}(\vartheta/P)^4\right] \quad . \quad . \quad . \quad (14)^*$$

In order to be able to determine the rheological properties of the material, we must plot the consistency curve so that the dimensions of the apparatus do not enter ; we shall have to plot it in V and P variables, which we have therefore called Consistency Variables. It should, however, be kept in mind that for every different *sort* of apparatus, the V and P are different from those of (II, f) and (II, g) ; the latter are applicable in the case of the tube-instrument only.

The Buckingham-Reiner equation was used by Scott Blair and Crowther (1929) for clay and soil pastes and by Wolarowitsch, Kulakoff and Romansky (1935) for peat suspensions. It was also found applicable to toothpaste, drilling muds, talc suspensions used in making compounds for taking dental impressions, chocolate, etc. The general shape of the consistency curve of a Bingham-body flowing through a tube is shown in Fig. VIII, 4. As can be seen, the curve has no truly linear portion. If, however, ϑ/P is a small figure, i.e. if P is large $(\vartheta/P)^4$ can be neglected and, for high driving pressures, the curve can be approximated by its asymptote.

3. The Reiner and Riwlin Equation

The equations describing the behaviour of a Bingham-body in a rotation instrument can be derived analogously to the procedure in Sections 5 and 8 of Chapter II.

We start from (II, h) and (II, i), viz.

$$\dot{G} = rd\dot{\theta}/dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\tau = M/2\pi hr^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which are valid in every case of steady rotational laminar flow, no matter what the material. Combining these expressions with (7) we get

$$\eta_{pl} r \frac{d\dot{\theta}}{dr} = M/2\pi hr^2 - \vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

From (16) can be seen that the shearing stress in the material varies inversely as the square of the radius r. The maximum shearing stress will accordingly occur where r is the minimum. This is at the wall of the internal cylinder, where $r = R_i$. Therefore,

$$\max \tau = M/2\pi hR_i^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If

$$M/2\pi hR_i^2 < \vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

or

$$M < 2\pi hR_i^2\vartheta = M_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

* The term within [ ] is equal to $(1 - \vartheta/P)^2\,[1 + 2\vartheta/3P + \frac{1}{3}(\vartheta/P)^2]$

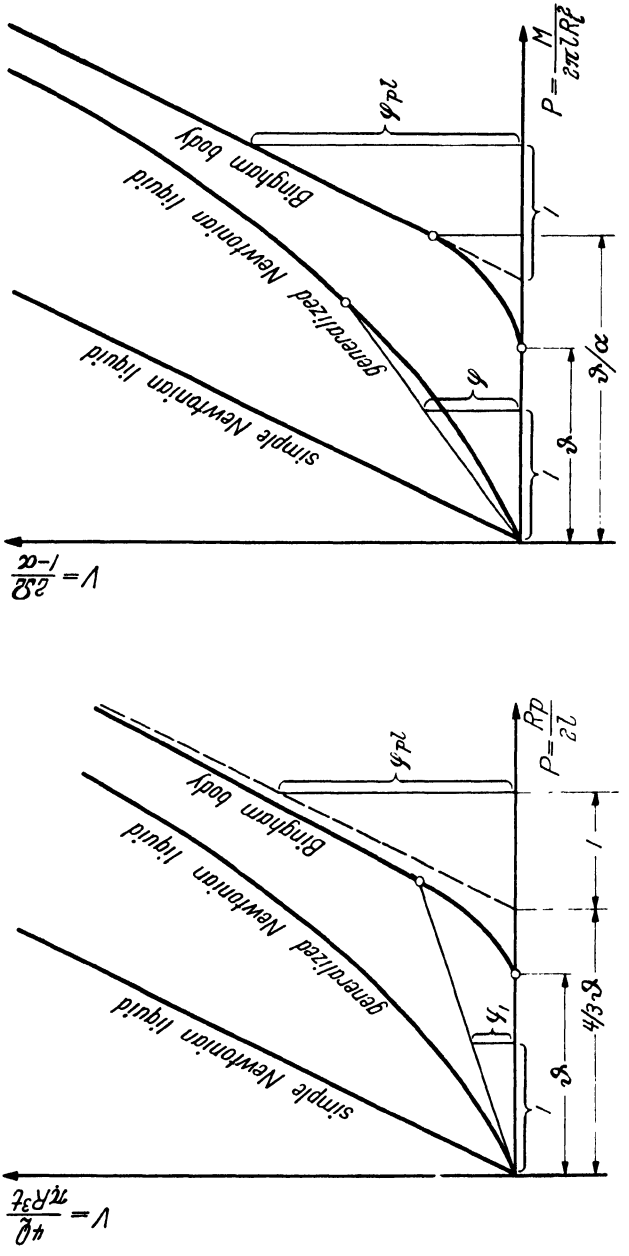


FIG. VIII, 4. Consistency curves of three types of materials, in the tube (*left*) and rotating-cylinder instrument (*right*).

the material will not flow. If M exceeds the value M_o, the material will start to flow where the stress is largest, i.e. at the surface of the internal cylinder. As long, however, as the stress near the inner surface of the external cylinder is below ϑ there will be a shell of solid material adjacent to the external cylinder which will move together with the external cylinder as a whole (compare right of Fig. VIII, 2). The stress at the wall of the external cylinder is

$$\min \tau = M/2\pi h R_e^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

When M increases until it becomes $M = 2\pi h R_e^2 \vartheta = M_1$, *all material* between the cylinders will flow. This is the essential difference between the plastic flow in the rotating cylinder apparatus and the plastic flow in the tube. In the latter case, as we have shown in Section 3, there *always* exists a " plug " or solid cylinder near the axis of the tube, no matter how great the velocity is, and from this it follows that the consistency curve of plastic flow in the tube, while it approaches linearity asymptotically, is never exactly a straight line.

Let us now integrate (17).

We get

$$\eta_{pl}\dot{\theta} = -M/4\pi h r^2 - \vartheta \ln r + C \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Considering the Couette-Hatschek apparatus, where the internal cylinder is at rest, from the kinematical boundary condition of " no slippage " we have

$$C = M/4\pi h R_i^2 + \vartheta \ln R_i \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and therefore

$$\eta_{pl}\dot{\theta} = \frac{M}{4\pi h}(1/R_i^2 - 1/r^2) - \vartheta \ln \frac{r}{R_i} \quad . \quad . \quad . \quad (24)$$

We now consider

$$M_o < M < M_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

There then exists a solid cylindrical shell between r_o and R_e where

$$r_o^2 = M/2\pi h \vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

All material at the distance $r > r_o$ rotates with the angular velocity $\dot{\Omega}_e$. Therefore

$$\eta_{pl}\dot{\Omega}_e = \frac{M}{4\pi h}(1/R_i^2 - 1/r_o^2) - \vartheta \ln \frac{r_o}{R_i} \quad . \quad . \quad . \quad (27)$$

and

$$M = \frac{4\pi h \eta_{pl}}{1/R_i^2 - 1/r_o^2}\left(\dot{\Omega}_e + \frac{\vartheta}{\eta_{pl}} \ln \frac{r_o}{R_i}\right) \quad . \quad . \quad . \quad (28)$$

On the other hand, for $M > M_1$ all material is in a state of flow and

$$\eta_{pl}\dot{\Omega}_e = \frac{M}{4\pi h}(1/R_i^2 - 1/R_e^2) - \vartheta \ln \frac{R_e}{R_i} \quad . \quad . \quad . \quad (29)$$

Introducing

$$\left.\begin{aligned} K_1 &= \frac{1}{4\pi h}\,(1/R_i^2 - 1/R_e^2) \\ K_2 &= \ln\frac{R_e}{R_i} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

we have

$$\dot{\Omega} = \frac{K_1}{\eta_{pl}}\,M - K_2\frac{\vartheta}{\eta_{pl}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

This is a linear relation in $\dot{\Omega} : M$ and proves that in the rotation instrument the upper part of the flow curve is a true straight line. The flow curve starts at the point $M_o : O$ and is curved to a point

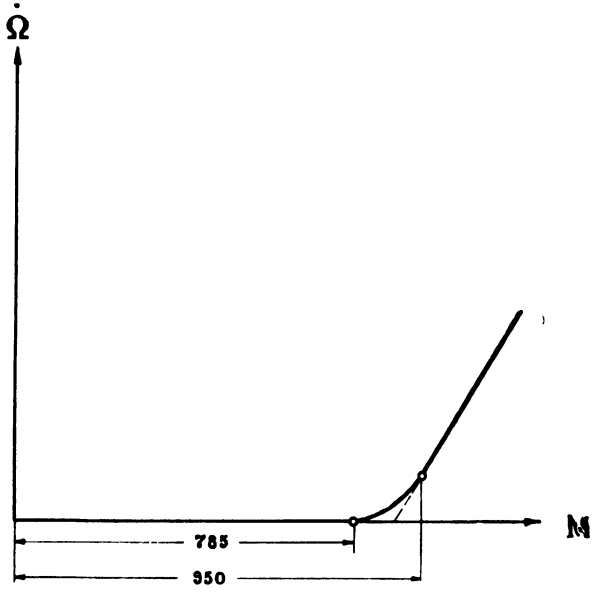


FIG. VIII, 5. Curved portion of the flow curve in the Hatschek apparatus drawn to scale.

with the abscissa M_1. The dimensions of Hatschek's apparatus are $R_i = 5$ cm, $R_e = 5{\cdot}5$ cm ; $h = 5$ cm. This gives $K_1 = 0{\cdot}00011$ cm^{-3}, $K_2 = 0{\cdot}09533$; $K_2/K_1 = 864{\cdot}68$ cm^3 ; $2\pi R_i^2 h = 785{\cdot}38$ cm^3, $2\pi R_e^2 h = 950{\cdot}30$ cm^3. When drawn to scale, the curve looks as is shown in Fig. VIII, 5. In this case the curved part is therefore negligible. If the material is a viscous liquid, i.e. $\vartheta = 0$, we find $M_o = M_1 = 0$ and $\dot{\Omega} = K\varphi$.

Equation (31) is known as the *Reiner and Riwlin equation* (1927). It will be seen that if we plot the variables P and V of (II, 1) and (II, n), we can represent (27) and (31) independently of R_i and R_e. This justifies their designation as *consistency* variables (compare right of Fig. VIII, 4).

It will be instructive to examine an unsuccessful attempt which Hencky (1925) made to solve a problem essentially the same as that which led to the Reiner and Riwlin equation.

I will not be suspected of wanting to discredit my late personal friend Hencky. The road of rheology (as probably of many other branches of science) is strewn with mistakes. We have mentioned the wrong velocity distribution in the capillary by Hagen and by Freundlich, the wrong assumption about equilibrium by Newton, another mistake in the same field by an anonymous student, and we shall have to mention others, myself included ;* but the fact that such a name as Newton's is among them should make it clear that we have much to learn from mistakes, but need not be ashamed of them.

Hencky wanted to investigate conditions in the process of rolling metals, and as a first step he considered the problem of one roll rotating in the plastic metal, for which he independently assumed the rheological equation of a Bingham-body. The problem is the plastico-dynamic analogy to Newton's problem (Section 5 of Chapter II). In that case we find from (II, 40) for $R_e = \infty$, $\dot{\Omega}_e = 0$, $\tau = 2\eta\dot{\Omega}(R/r)^2$ and the stress therefore has a maximum for $r = R$, $\max \tau = 2\eta\dot{\Omega}$ and then decreases with the distance from the centre of the roll. Therefore, in the case of a Bingham-body, there will be a distance r_o where $\tau = \vartheta$ and beyond this distance the material will be at rest. Accordingly from (16) and (26)

$$\tau/\vartheta = (r_o/r)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The rate of shear of the Bingham-body is from (7) $\dot{\gamma} = \dfrac{\vartheta}{\eta_{pl}} \cdot (\tau/\vartheta - 1)$

and considering (32)

$$= \frac{\vartheta}{\eta_{pl}} [(r_o/r)^2 - 1)] \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Hencky correctly derived (33).

He compared this with (II, 21) and (II, 22), viz. :

$$dv_\theta/dr = rd\dot{\theta}/dr + \dot{\theta} = \dot{G} + v_\theta/r \quad . \quad . \quad . \quad . \quad (34)$$

and then reasoned as follows . From (33) there follows that $\dot{\gamma}$ is positive from $r = R$ to $r = r_o > R$. Considering now (34), we have on the right side $\dot{G}$ which is positive up to r_o and v_θ/r which likewise is positive within the same range. Therefore dv_θ/dr must be positive and v_θ a monotonously increasing function with increasing r. This is in contradiction to the obvious conclusion that if v_θ varies at all, it can only decrease from $\dot{\Omega}/R$ at $r = R$, to zero at $r = r_o$. Therefore $r_o = R$. In the case of laminar rotational flow the roll, accordingly, *cannot drag the metal with it.* The roll rotates in the metal and produces therein stresses, which do not exceed ϑ. He adds : " This result is noticeable for the reason that a viscous liquid ($\vartheta = 0$) would be dragged along to

* See Chapter XI.

carry out a movement $v_\theta = \Omega R^2/r$ (compare (II, 31), M.R.) when the stress at the wall of the cylinder would be $\tau = -2\dot{\Omega}\eta$ (compare (II, 41) putting $R_e = \infty$, $\dot{\Omega}_e = 0$, $\dot{\Omega}_i = \dot{\Omega}$, the negative sign being irrelevant. M.R.). We may make ϑ as small as we like, the essential difference, mentioned above, remains. This may serve as a criterion whether a material is a viscous liquid or a plastic mass."

A strange thing to say! This may move us to announce the following *third axiom of rheology*. There is *a hierarchy of ideal bodies*, corresponding to the different rheological behaviours of real materials, such that the *rheological equation of the more simple body* (lower in the hierarchy) *can be derived by putting one or other of the constants of the rheological equation of the less simple body* (higher in the hierarchy) *equal to zero.*

On the strength of this axiom, Hencky's conclusion cannot be correct. The paradox was solved by Prager (1930) by simply pointing out that ϑ, being derived as the square root of a quantity ϑ^2 (compare (VI, 1)), should be used in (33) and (34) with the *negative sign*. This makes

$$\dot{\gamma} = r\left(\frac{d}{dr}\, v_\theta/r\right) = \frac{dv_\theta}{dr} - \frac{v_\theta}{r} = -\frac{\vartheta}{\eta_{pl}}\left[(r_o/r)^2 - 1\right] \quad . \quad (35)$$

and by integration

$$\frac{v_\theta}{r} = \frac{\vartheta}{\eta_{pl}}\left(\frac{r_o^2}{2r^2} + \ln r\right) + C \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The integration constant C is to be derived from $v_\theta = 0$ for $r = r_o$ and we get

$$v_\theta = \frac{\vartheta}{2\eta_{pl}}\, r\left[(r_o/r)^2 - 1 + \ln(r/r_o)^2\right] \quad . \quad . \quad . \quad (37)$$

Now

$$dv_\theta/dr = -\frac{\vartheta}{2\eta_{pl}}\left[1 - (r_o/r)^2 - \ln(r_o/r)^2\right] \quad . \quad . \quad . \quad (38)$$

and as $r_o < R$, this, between R and r_o, is negative. The absolute value of the velocity gradient dv_θ/dr decreases from

$$\frac{\vartheta}{2\eta_{pl}}\left[1 - (r_o/R)^2 - \ln(r_o/R)^2\right] \text{ at } r = R \text{ to zero at } r = r_o.$$

The solution of the problem can be found from (22) where we determine C from $\dot{\theta} = 0$ for $r = r_o$ and then introduce $\dot{\theta} = \dot{\Omega}$ for $r = R$. We get

$$\eta_{pl}\Omega = \frac{M}{4\pi h}\left(\frac{1}{r_o^2} - \frac{1}{R^2}\right) + \vartheta \ln \frac{r_o}{R} \quad . \quad . \quad . \quad (39)$$

which may be compared with (27).

It should be mentioned that in the same paper Hencky correctly derived (12). His paper did not, however, reach the attention of the

right public and I have not seen it mentioned anywhere. It did not occur to Hencky that this equation was better applied to toothpaste than to plastic metals. As a matter of fact, the theory of plasticity, as applied to metals, is nearly exclusively a theory of the St. Venant body. In accordance with the third axiom of rheology, if we know the solution of a problem of the Bingham-body, we can derive the solution of an analogous problem of the St. Venant body by putting η_{pl} equal to zero. The same result is obtained if $\dot{\gamma}$ vanishes, i.e. if the deformation is infinitely slow. We can therefore designate the rheologies of the Bingham-body as Plastico-Dynamics and of the St. Venant body as Plastico-Statics. As the velocity of deformation of metals and similar materials is usually very small, it is obvious that in many problems the plastico-static approach of the St. Venant body will furnish a good enough approximation. The standard works for the study of plastico-statics are Nadai's *Plasticity* (1931) and Prager and Hodge's *Perfectly Elastic Solids* (1951).

4. Summary

The rheological equation of the *Bingham-body*

$$\tau = \vartheta_t + \eta_{pl}\dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad \text{(VIII, a)}$$

is derived from a combination of the St. Venant solid and the Newtonian liquid.

The sign for ϑ_t is to be the same as that of the second term on the right side of (VIII, a).

Application is made to laminar deformation in (i) a tube-instrument, by introducing into (VIII, a) the expressions (II, b) and (II, c) and (ii) in a rotation-instrument, by introducing expressions (II, h) and (II, i). The results are

(i) The *Buckingham-Reiner* equation

$$Q = \frac{\pi R^4 \Delta p}{8 l \eta_{pl}}\left[1 - \frac{4}{3}\frac{2\vartheta l}{R\Delta p} + \frac{1}{3}\left(\frac{2\vartheta l}{R\Delta p}\right)^4\right] \quad . \quad \text{(VIII, b)}$$

There is a solid plug of radius r_o

$$r_o = 2\vartheta l/\Delta p \quad . \quad . \quad . \quad . \quad \text{(VIII, c)}$$

which is not sheared, but moves with the mass of the material as a whole.

(ii) The *Reiner and Riwlin* equation

$$\eta\Omega_{pl} = \frac{M}{4\pi h}(1/R_i^2 - 1/R_e^2) - \vartheta \ln \frac{R_e}{R_i} \quad . \quad \text{(VIII, d)}$$

for

$$M = M_1 = 2\pi h R_e^2 \vartheta \quad . \quad . \quad . \quad \text{(VIII, e)}$$

in which case *all* material between the cylinders is deformed. The consistency variables (II, f, g, l, m, n) are applicable.

CHAPTER IX

ELASTICO-VISCOSITY AND FIRMO-VISCOSITY

1. Rheological Models, Structural Formulas and Rheological Equations

In order to treat rheological problems by mathematical methods, we found it necessary to create concepts of ideal bodies having rigorously defined (rheological) properties. This process is facilitated by building, if only in imagination, a model which will behave qualitatively in a manner similar to that of the actual material, to some degree of approximation, and to describe that behaviour in terms of forces and elongations. If these terms are translated into stresses and deformations, the result easily leads to a rheological equation.

We have represented the *primitive* bodies which typify the three fundamental properties of viscosity, elasticity and plasticity by the N-model of Section 10 in Chapter II, the H-model of Section 9 in Chapter IV, and the StV-model of Section 3 in Chapter VI. To describe a more complex behaviour in plasticity, we have added to the StV-slider a H-spring. By extending this method, we can construct a model for the Bingham-body. We met this body in Chapter VIII. We found that when stressed below a certain stress it is strained elastically. We therefore take as our first element a spring. However, when the yield stress is exceeded, the body is deformed plastically. We therefore have to attach the spring to a slider. When the rate of plastic deformation is increased, the stress also increases. This requires the attaching of a dashpot to the slider at the side of it which is opposite of the spring. The model is shown in Fig. IX, 1.

It should be pointed out that the displacements which the models

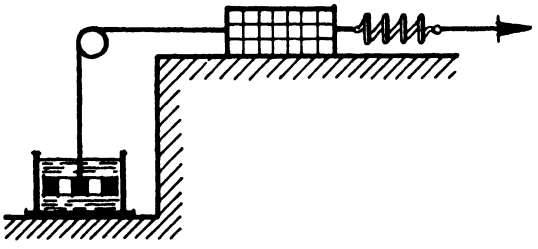


FIG. IX, 1. Model for a Bingham-body.

can register are increases in length or elongations produced by pulls. These, however, may be translated into every other kind of deformation and corresponding stress. One may consider the elongation as representing a shear, when the pull will represent a shearing stress ; or the first may be a cubical dilatation and the second a hydrostatic tension, etc. It should also be noted that while, if we should build real models (which can conveniently be done, using Meccano parts as shown in Fig. IX, 6), the materials and arrangements for our mechanical elements would correspond to *constant* rheological coefficients, this does not preclude their application in cases where the coefficients are variable.

From the model we proceed to the structural formula, which is a sort of shorthand representation of the model's blueprint. For instance, the structural formula for the Bingham-body is

$$B = (N|StV) - H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the horizontal dash (—) indicates, as in Section 3 of Chapter VI, " coupling in series ", while the vertical dash (|) indicates " coupling in parallel."

This gives us all the information which we may draw from the picture of the model as shown in Fig. IX, 1, but is, of course, much more economical.

The models and structural formulas help us to understand rheological behaviour *qualitatively*. But they also help us to establish the rheological equations of the different ideal bodies, which we need for a *quantitative description*. These equations connect certain *types** of stress and deformation. For instance the *H*-spring with its elastic extensions leads us to a rheological equation of any one of the three following types

$$\tau = \mu\gamma \; ; \; -p = \kappa e_v \; ; \; \sigma_n = \mathrm{E}e_n \quad . \quad . \quad . \quad . \quad (2)$$

where it does not matter that E can be expressed by μ and κ in accordance with Equ. (VII, 23). If, therefore, in writing the rheological equation, we omit the subscripts " v " or " n ", the reader may put them in as required, taking care to use at the same time the appropriate coefficient, which e.g. in the case of viscous flow will be one of the following, viz. η, ζ or λ.

We shall now examine systematically in what ways complex bodies can be built up from the elements and how the corresponding rheological equations are to be derived.

We first note that elements can be connected either (i) in series (–), or (ii) in parallel (|).

As will be seen, all elements, when coupled in series, act as if they were links in one chain and they must therefore all take the same stress,

* In advanced rheology they connect the *tensors* of stress and deformation. Compare *Twelve Lectures*, p. 43.

while the elongation of the combination will be the sum of the elongations of each element.

In contradistinction, when coupled in parallel, all elements are forced to undergo the same displacement, while the stress which the combination will take is the sum of the stresses taken by each single element. These two principles will guide us when composing the rheological equations.

Let us start with the H-body, the rheological equation of which we write in the form

$$\tau = \mu\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we have two different springs, viz. H_1 for which $\tau_1 = \mu_1\gamma_1$ and H_2 for which $\tau_2 = \mu_2\gamma_2$, and couple them in series, H_1—H_2, we have in accordance with the first of the above principles

$$\gamma = \gamma_1 + \gamma_2 = \tau/\mu_1 + \tau/\mu_2 = \tau(1/\mu_1 + 1/\mu_2) = \tau/\mu \quad . \quad (4)$$

and we see that *n H-bodies connected in series are equivalent to one single H-body the coefficient of elasticity** (reverse of modulus of elasticity) *of which is the sum of the individual coefficients of elasticity.*

If we couple the springs in parallel, $H_1|H_2$, we have, in accordance with the second principle,

$$\tau = \tau_1 + \tau_2 = \mu_1\gamma + \mu_2\gamma = (\mu_1 + \mu_2)\gamma = \mu\gamma \quad . \quad . \quad (5)$$

from which we see that *n H-bodies connected in parallel are equivalent with one single H-body the modulus of elasticity of which is the sum of the n individual moduli.*

Conditions with regard to the N-bodies are entirely analogous.

2. Elastic Liquids

Let us now examine materials, such as bitumen, which show viscous flow and therefore are liquids. However, while they flow, these materials are not Newtonian liquids. In addition to viscosity, they exhibit *elasticity.* Their complex rheological behaviour has therefore been named by Jeffreys (1929) Elastico-Viscosity. The first component of this property is not necessarily connected with a high value of the second, as is the case in glass, which also may be considered as belonging to this category. Hess (1920) has described the elastic behaviour of a 1·5 per cent. starch solution which is elastic, while not exceedingly viscous. The liquid is brought into rotation by rotating its container, which is then suddenly stopped. " When a state of rest is apparently about to be reached with gradual diminution of the rotational velocity, we see that the liquid starts to move again in the opposite direction. A state of rest is again approached, first with increasing and then with diminishing velocity, but it is not final. Only after several oscillations does the liquid come to rest. The oscillations do not appear in the case of a glycerine-water mixture of corresponding viscosity : here

* The term " compliance " has been proposed for this property.

the first rest is definite. These oscillations must be due to elastic forces which arise in the liquid from internal structures."

A 1·5 per cent. starch solution is obviously a liquid, and a few words must be added about the difference between an elastic liquid and a soft plastic solid. The latter also "flows", but it flows *plastically*, i.e. only after the yield stress has been exceeded. Up to the yield point it is elastic, but *its elasticity is of a very different kind.* In a soft plastic solid the elastic component becomes predominant at *small* velocities. In contradistinction the elasticity of a true liquid makes itself felt at *high* velocities, when no time is given for its elastic stresses to "relax". Twenty years ago, when rheology was in its infancy, this difference was not so well understood and I published a few papers on the theory of flow of soft plastic solids under titles referring to elastic liquids (compare Section 1 of Chapter VIII). The theory was correct, but the terminology was misleading.

3. Relaxation of Stress

The elasticity of a solid, *even of a soft solid*, is of the Hooke type. The strain-work which has been put into a distorted Hooke body is stored there safely for as long as it is loaded; and it can be regained at any time by removing the load. So, on releasing the pressure after the second stage shown in Fig. 1, 6, the segment disappears and the elastic potential energy stored in the paste, small as it may be, is regained. On increasing the pressure to the third stage, this potential energy is not affected and in the fourth stage it is regained to the same extent. In contradistinction, an elastico-viscous material, so to speak, *leaks*: the longer we wait, the less complete is the recovery on removal of the load. In other words, the strained material loses resilience and its internal stresses *relax*.

In his investigations on the dynamical theory of gases (1868), Maxwell was the first to treat the *relaxation of elastic stresses.* In his words, but using our notation and interpolating our comments:—

"*A distortion or strain of some kind, which we may call e, is produced in the body by displacement. A state of stress or elastic force which we may call s is thus excited. The relation between the stress and the strain may be written* $s = \epsilon e$, *when* ϵ *is a coefficient of elasticity for that particular kind of strain. . . .*" This coefficient is Young's modulus E if s and e are *normal* stresses and strains, and it is the modulus of rigidity if they are *tangential* stresses and strains. "*In a solid body free from relaxation,** *s will remain* $= \epsilon e$, and

$$\dot{s} = \epsilon \dot{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

"*If, however, the body relaxes, s will not remain constant, but will*

* Here I have slightly changed Maxwell's wording.

tend to disappear, at a rate depending on the value of s and of the nature of the body. If we suppose this rate proportional to s, the equation may be written

$$\dot{s} = \epsilon\dot{e} - s/T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

T is a material constant, the nature of which will be explained later. ". . . *which will indicate the actual phenomena in an empirical manner.*"

For " empirical " we would say " phenomenological ".

" *For if e be constant,*

$$s = \epsilon e e^{-t/T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

showing that s gradually disappears, so that if the body is left to itself it gradually loses any internal stress. . . ."

Equation (8) can easily be derived from (7) as follows : If e is constant, $\dot{e}$ vanishes and (7) becomes

$$ds/dt = s/T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or

$$ds/s = \frac{1}{T}dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which by integration gives

$$\ln s - \ln s_0 = t/T \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

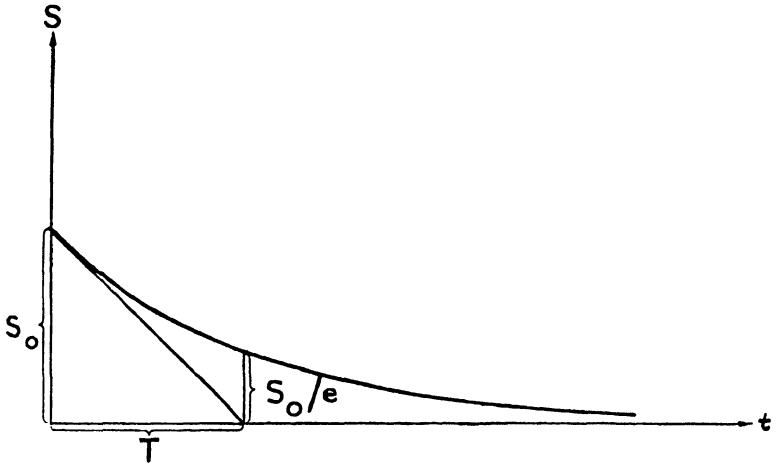


FIG. IX, 2. Relaxation curve of a Maxwell liquid. s_0 stress at the time $t = 0$. T relaxation time. e base of natural logarithms.

where s_0 is the stress originally produced in the body, i.e. at time $t = 0$. This stress, however, is equal to ϵe.

From (11) follows (8).

If we plot the stress against time the resulting curve is as shown in Fig. 2. The constant T has the dimension of time, because the

exponent t/T of e must be a dimensionless number. It is called Relaxation Time. If in (8) we introduce T for t we get

$$s_T = s_o/e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The relaxation time is therefore that time in which the stress relaxes to the " e^{th} " part of its original magnitude.

4. Viscosity Arising from Relaxation

Maxwell was actually not so much interested in the relaxation of the elastic stresses in solids. Following an earlier attempt by Navier to derive the equations of viscous flow of liquids from the elasticity equations by assuming instantaneous relaxation, he postulated his law (7) as a starting point for a theory of viscosity. He accordingly continues : —

" *If $\dot{e}$ is constant ($= c$), that is if there is a steady motion of the body . . .*"

This is meant to read " of the particles of the body ", because this is not a question of the motion of the body as a whole. Even so the statement is not quite correct because $\dot{e}$ is not the velocity of a motion, but the rate of a strain.

". . . *which continuously increases the displacement,*

$$s = \epsilon T c + C e^{-t/T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

showing that s tends to a constant value depending on the rate of displacement" [read " rate of strain." —M.R.]. " *The quantity ϵT, by which the rate of displacement*" [as before] " *must be multiplied to get the force*" [read " traction "], " *may be called the 'coefficient of viscosity.'*"

Equ. (13) can be verified by differentiation. We get $\dot{s} = -\frac{c}{T} e^{-t/T}$ and as from (13) $ce^{-t/T} = s - \epsilon T c$, this gives (7).

The identification of ϵT with a coefficient of viscosity can be supported by the following consideration : If s is constant, $\dot{s}$ vanishes and (7) gives

$$\dot{e} = s/\epsilon T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Under constant stress the rate of strain is therefore also constant which means that there is continuous flow. In addition, assuming ϵ and T to be constants, the relation between stress and rate of strain is linear. This, if stress and strain are tangential, is Newton's law of viscous flow ; and if they are normal (as we have assumed in (6)) it is Trouton's law.

Let us continue the quotation from Maxwell. " *It (ϵT) is the product of a coefficient of elasticity (ϵ) and a time (T) which may be called the Time of Relaxation of the elastic force. In mobile fluids T is a very small fraction of a second and ϵ is not easily determined experimentally. In viscous solids . . .*"

In our terminology we call such materials *liquids* and not solids, reserving the term " viscous solids " for very different materials as will be explained in Section 7. We would say here " *In very viscous elastic liquids . . .*" and, continuing the quotation, ". . . *T may be several hours or days, and then* ϵ *is easily measured. It is possible that in some bodies T may be a function of s.*"

Equ. (7) can accordingly be written in the form

$$\dot{\gamma} = \dot{\tau}/\mu + \tau/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In the quoted paper Maxwell arrived at the notion that the " rigidity " of air is equal to its pressure, i.e. in the free atmosphere equal to $1{\cdot}014 \times 10^6$ dyne cm^{-2}. This sounds paradoxical, but one should bear in mind that the *elastic* deformation is the *smaller*, the *greater* γ. Therefore, for an inelastic Newtonian liquid $\mu = \infty$.

Having determined the viscosity of air at atmospheric pressure by experiment (which is about 0·0002 poises), he easily found its time of relaxation to be $1{\cdot}961 \times 10^{-10}$ seconds. We can therefore say that the order of magnitude of the smallest time of relaxation is 10^{-10} seconds, and we may assume that even for water it does not vanish altogether. Experiments to find a rigidity of water must therefore have been unsuccessful only because of imperfections of experimental technique, but I venture to say that I have recently demonstrated the rigidity of air. About this more is said in Chapter XXI.

5. Remember the First Axiom

There is a flaw in the foregoing considerations. Maxwell must have been well aware of what we called the first axiom of rheology but he did not pay attention to it. This was first emphasized by Reiger (1919) who reports* that Zaremba (1903) pointed out before him that, in principle, there must be two different times of relaxation just as the elastic state of a body is determined by two independent elastic constants. In hydro-dynamical calculations a liquid is usually assumed to be incompressible. This may be justified when one considers flow, because the deformations due to liquid flow are incomparably greater than those due to volumetric strain. However, when we consider an *elasticity* of the liquid, similar to that of a solid, we must remember (compare the Table III, 1) that actually liquids are much more compressible than solids.

Reiger assumed that there is no relaxation of volume elasticity, which implies that $\zeta_l = \infty$. We shall see in Chapter XII that this view is too narrow, but meanwhile let us assume that this is correct to a first approximation (as implied in the first axiom). Then to find the

* I must warn the reader desirous of studying Reiger's paper of its numerous mistakes, which I cannot point out in detail.

relaxation equation for simple tension we must first resolve the simple tension into its two components in accordance with Equ. (VII, 15) with

$$\sigma_m = \sigma_n/3 \; ; \quad . \; . \; . \; . \; . \; . \; . \; (16)$$

while the corresponding strains are in accordance with Equs. (VII, 17) and (VII, 18)

$$e_v = e_n\,(1 - 2\nu) \; ; \; e_{n(o)} = 2(1 + \nu)\,e_n/3 \quad . \; . \; . \; (17)$$

Taking first the voluminal part, we find

$$\dot{e}_v = \dot{e}_n(1 - 2\nu) = \dot{\sigma}_n/3\kappa \quad . \; . \; . \; . \; . \; . \; (18)$$

where, in differentiating the first of Equs. (17) we have assumed that ν is a constant.

For the distortional part we go back to Equs. (VII, 11) and (VII, 12) and replace (15) by

$$\dot{e}_{n(o)} = \dot{\sigma}_{n(o)}/2\mu + \sigma_{n(o)}/2\eta \quad . \; . \; . \; . \; . \; . \; (19)$$

which, introducing (16) and (17), becomes

$$2(1 + \nu)\dot{e}_n = \sigma_n/\mu + \sigma_n/\eta \quad . \; . \; . \; . \; . \; . \; (20)$$

Elimination of ν from (18) and (20) by adding both gives

$$\dot{e}_n = \frac{\dot{\sigma}_n}{3}\left(\frac{1}{3\kappa} + \frac{1}{\mu}\right) + \frac{\sigma_n}{3\eta} \quad . \; . \; . \; . \; . \; . \; (21)$$

However, in accordance with (VII, 23), $1/3\kappa + 1/\mu = 3/E$, while from (VII, 25)

$$\frac{1}{3\eta} = \frac{1}{\lambda^*} \quad . \; . \; . \; . \; . \; . \; . \; . \; (22)$$

This yields, considering the first of (17),

$$\dot{e}_n = \dot{\sigma}_n/E + \sigma_n/\lambda \quad . \; . \; . \; . \; . \; . \; . \; (23)$$

6. The Maxwell Liquid

We may call a material whose rheological equation is (15), (19) or (23) a Maxwell Liquid. To indicate its nature as a liquid, the subscript "l" has to be added to the modulus of rigidity μ. These rheological equations can be derived through a combination of those of the Hooke solid (I, d) and the Newtonian liquid (I, f). The nature of this combination is found by considering an infinitely small displacement du attained during the time dt. Then, in accordance with Equ. (15)

$$du = d\tau/\mu_l + \tau/\eta.dt = du_e + du_v \quad . \; . \; . \; . \; (24)$$

where du_e is the elastic and du_v the viscous displacement. Accordingly, in a Maxwell liquid the displacements are additive. In Section 9 of Chapter IV (Fig. IV, 11), we have proposed a spring as a model for an

elastic (Hooke) material, and in Section II, 10 a dashpot as shown in Fig. II, 14, as a model for a Newtonian liquid. If the *displacements* of these are to *add up*, we must combine them in *series* in order to get a model for the Maxwell liquid. Fig. IX, 3, shows the model together

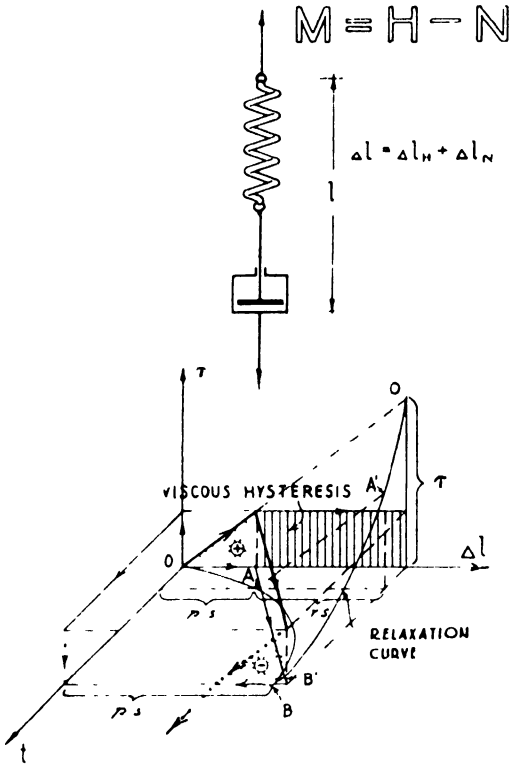


FIG. IX, 3. Model for the Maxwell liquid. OAB = elastic recovery for elongation Δl. O'A'B' = relaxation curve for stress τ'. p.s. permanent strain ; r.s. = recovered strain.

with its stress-displacement-time curves under load and after removal of the load (the latter in dotted lines . . . . .). Using, as before in Section VI, 3 the horizontal dash — for " coupling in series ", we write what we call the " structural formula " of the Maxwell liquid in the form

$$M = H - N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Equ. (15) is of the type

$$dy/dx + My = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

the solution of which is

$$y = e^{-\int M dx} (\int N e^{\int M dx} + C)$$

as can be easily verified by differentiation and substitution in (26). Provided μ and η are constants, the solution of (15) is accordingly

$$\tau = e^{\mu/\eta\, t}\left(\tau_o + \mu \int_o^t \dot{\gamma} e^{\mu/\eta\, t}\, dt\right) \quad . \quad . \quad . \quad . \quad (28)$$

where we have omitted the subscript, "l" for simplicity.

If the deformation is kept constant, or $\gamma = 0$, we find from Equ. (28) the relaxation curve

$$\tau = \tau_o e^{-\mu/\eta . t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

as in Equ. (8). On the other hand $\int_0^t e^{\mu/\eta\, t}\, dt = \frac{\eta}{\mu}(e^{\mu/\eta\, t} - 1)$ and therefore, if γ is constant $= c$,

$$\tau = \eta c + (\tau_o - \eta c) e^{-\mu/\eta . t} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

For $c_o = \tau_o/\eta$ this gives $\tau = \eta c_o = \tau_o$ and the stress remains constant. For $c > c_o$ the stress increases, for $c < c_o$ the stress decreases as shown in Fig. 4.

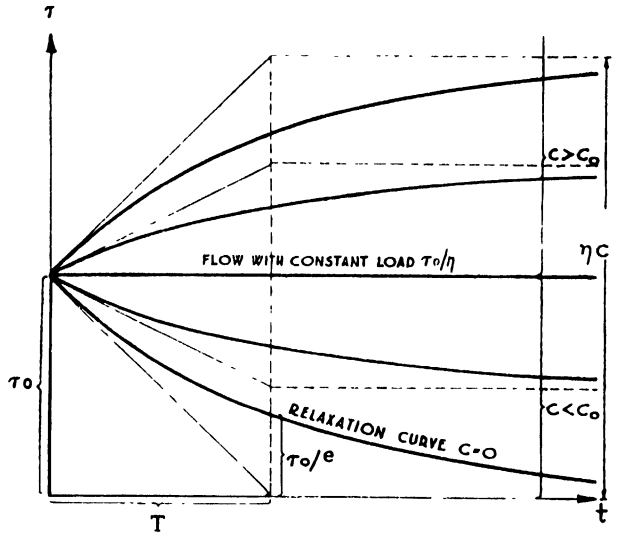


FIG. IX, 4. The test-piece is first ($t = o$) suddenly stressed to the value τ_o and then strained at the constant rate c. There exists for every initial stress τ_o a certain rate of strain c_o so that the stress is not changed.

7. Viscous Solids

At about the same time that Maxwell was pondering over the relaxation of stresses, William Thomson, later Lord Kelvin, published a paper "On the Elasticity and Viscosity of Metals" (1890) which was subsequently incorporated into his article on Elasticity in the

Encyclopædia Britannica. He reasoned from thermodynamical considerations that "*no change of volume or of shape can be produced in any kind of matter without dissipation of energy*". He also proceeded from theoretical reasoning to experiment and found "*by vibrating a spring alternately in air of ordinary pressure and in the exhausted receiver of an air-pump, that there is an internal resistance to its motions immensely greater than the resistance of the air*". He concluded that "*there is in elastic solids a molecular friction which may be properly called Viscosity of Solids, because, as being an internal resistance to change of shape depending on the rapidity of change, it must be classed with fluid molecular friction, which by general consent is called viscosity of fluids*". He also proposed a model for the better understanding of the phenomenon as follows: "*Consider a perfectly elastic vesicular solid, whether like a sponge with communications between the vesicles, or with each vesicle separately enclosed in elastic solid: imagine its pores and interstices filled up with a viscous fluid, such as oil. Static experiments on such a solid will show perfect elasticity of bulk and shape: kinetic experiments will show losses of energy such as are really shown by vibrators of . . . elastic homogeneous solids. . . . According to Newton's* law of*

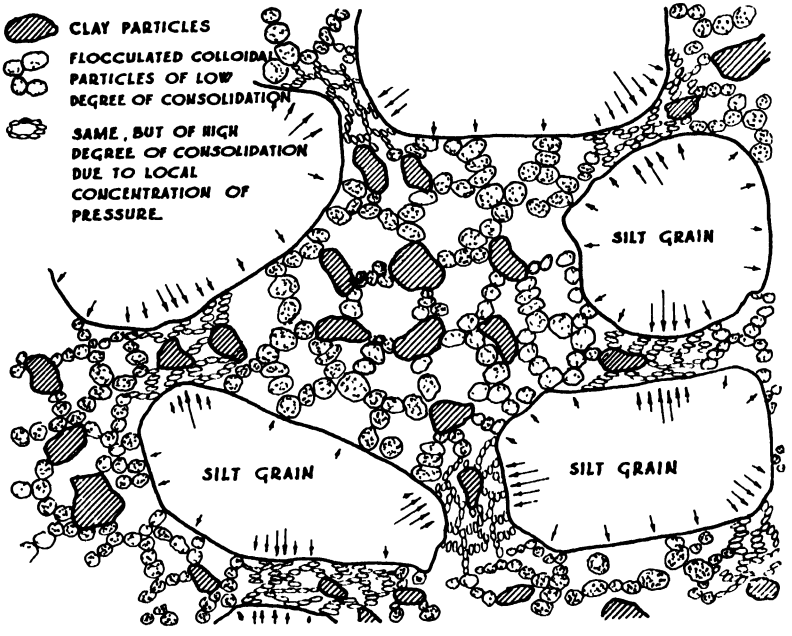


FIG. IX, 5.

* Kelvin says "Stokes's law", but Stokes only generalized Newton's relation between shearing force and velocity gradient, Equ. (1, f), into a relation between stress and rate of deformation.

viscosity of fluids, our supposed vesicular vibrator would follow the law of subsidence of a simple vibrator experiencing a resistance simply proportional to the velocity of its motion."

The kind of material for which Kelvin's description is appropriate is, for instance, a piece of loam, such as shown in Fig. 5, magnified about 10^4 times.

The property named by Kelvin " viscosity of solids " was called by Jeffreys (1929) Firmo-Viscosity. He remarks that this type of " imperfection of elasticity " implies not weakness but additional stiffness, the resistance to deformation being smaller when the viscosity is absent. However, in accordance with Kelvin " *the elasticity is said to be perfect when the body always requires the same force to keep it at rest, in the same bulk and shape . . . through whatever variations of bulk and shape it be brought.*" Therefore, firmo-viscosity should not be regarded as an imperfection of elasticity.

8. The Kelvin Solid

We may call a solid constituted on the model of the " vesicular solid ", or, if truly homogeneous, *behaving in the same manner*, a Kelvin Solid.* It is important to keep the italicized generalization in mind. Mardles (1946) has objected to what he calls my " assumption that the behaviour of a given material† can be represented in terms of a number of elements each obeying a simple law " as, e.g. in the sponge the solid substance between the pores and the liquid filling them. But, as Kelvin said when introducing his model : " *to form a rough* [or qualitative—M.R.] *idea of the results, irrespectively of the ultimate molecular* [or structural—M.R.] *theory. . . .*" The model does not serve any other purpose than the rheological equation. The elements of the model are symbols not essentially different from the mathematical symbols of the differential equation. To make this more obvious, we may replace the " vesicular solid " or sponge by a structural formula similar to the structural formula of the Maxwell liquid, Equ. (20). When the sponge, which is full of liquid, is loaded, part of the load is taken up by the solid substance and part by the liquid. This causes the sponge to be deformed and in this process load is transferred from the liquid which yields through flow, to the solid which is strained. The solid and liquid stresses are additive, adding up to the total load ; but the deformation is the same for both phases. This is in contradistinction to the Maxwell liquid where the same stress is taken by

* Jeffreys says that the firmo-viscous law was suggested to him by Sir J. Larmor. Weissenberg (1931) quotes Jeffreys. Mises (1930) ascribed it to Voigt (1890) and I (1931) followed him. Alfrey and Doty (1945) speak of the Voigt model. I came across Lord Kelvin's article in the *Encyclopædia Britannica*, which has the priority, by chance, and I can strongly recommend its study to the reader. It contains a wealth of forgotten knowledge.

† Mardles uses the term " system."

both elements, while the deformations are additive. Combining the spring and the dashpot in parallel we arrive at a model shown in Fig. 6. Diagrammatically this model can be represented as in Fig. 7.

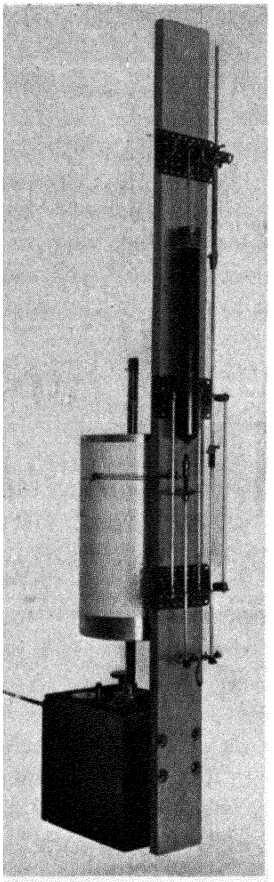


FIG. IX, 6.

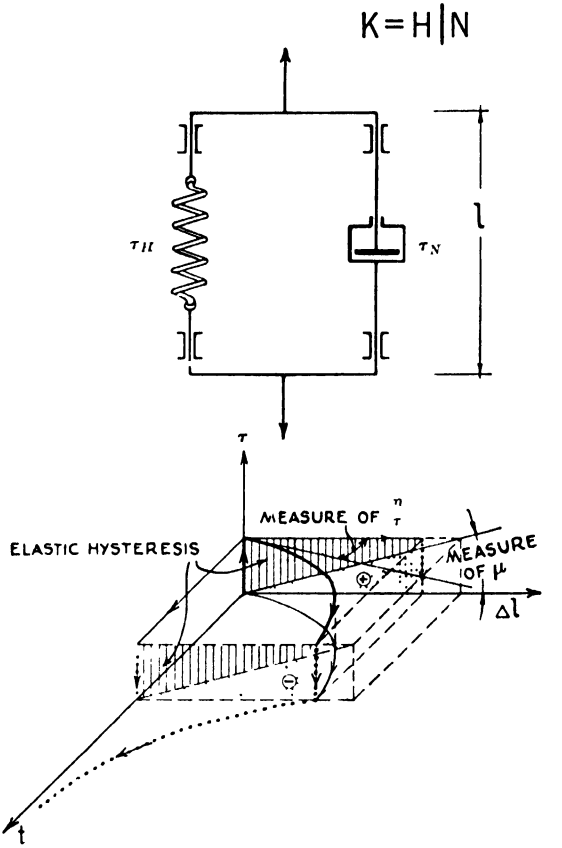


FIG. IX, 7. The mechanisms in the figures show extensions and pulls, but one should not forget that these stand for shearing strains and shearing stresses, respectively. Also, the stress-strain relations have been pictured as linear; but the models can be applied in a general manner for nonlinear relations as well.

Indicating by a vertical dash, thus, |, *parallel coupling*, the structural formula of the Kelvin Solid is accordingly

$$K = H|N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The rheological equation can easily be written down from (I, d) and (I, f). The stresses being additive, we have

$$\tau = \mu\gamma + \eta_s\dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In Equ. (32) we have given the η the subscript "s" to indicate the "solid" viscosity.

9. Delayed Elasticity

Equations (32) and (15) are built up analogously. We can therefore write down the solution of (32) on the model of Equ. (28). Omitting for simplicity the subscript "s", the solution of Equ. (32) is

$$\gamma = e^{-\mu/\eta\, t}\left(\gamma_o + \frac{1}{\eta}\int_o^t \tau e^{\mu/\eta\, t} dt\right) \quad . \quad . \quad . \quad . \quad (33)$$

If the stress is constant, we find as analogy with Equ. (30)

$$\gamma = \tau/\mu + (\gamma_o - \tau/\mu)e^{-\mu/\eta\, t} \quad . \quad . \quad . \quad . \quad (34)$$

For $\tau = \gamma_o\mu$, this gives $\gamma = \gamma_o$ and the strain remains constant. For $\tau > \gamma_o\mu$ the strain increases, for $\tau < \gamma_o\mu$ the strain decreases, in both cases reaching τ/μ asymptotically. If a stress τ acts on an unstrained body, i.e. $\gamma_o = 0$, we find

$$\gamma = \frac{\tau}{\mu}(1 - e^{-\mu/\eta\, t}) \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The elastic strain τ/μ, therefore, does not appear instantaneously or rather with acoustic speed, but is *delayed*, and reaches this value only after infinite time. Likewise, if the load is removed from a strained body, it will not revert to the unstrained state with acoustic speed, but in accordance with ($\tau = 0$)

$$\gamma = \gamma_o e^{-\mu/\eta\, t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and it will take infinite time until the strain completely vanishes.

This phenomenon is called elastic after-effect, but this is only a special case of Delayed Elasticity.* The effect described by Equ. (35) is sometimes called elastic Fore-Effect, a not altogether happy term, suggesting an effect preceding the cause—which, of course, cannot be.

As in the case of the Maxwell body, the ratio η/μ is of the dimension of time. It is called the Time of Retardation. The use of the term "time of relaxation", as sometimes met with, is misleading and should be reserved for the Maxwell liquid only.

10. Damping of Oscillations

As we saw in Section 7, Kelvin conceived the notion of solid viscosity in order to explain the damping of oscillations. This is "viscous damping". However, viscous damping is not the only kind of damping. Whatever causes dissipation of either elastic potential or kinetic energy produces damping.

* Also termed Retarded Elasticity.

In analysing the causes of damping we must distinguish between the first two kinds, viz. *external* and *internal*. External causes are (i) friction at the fixing points of the vibrating body, and (ii) the resistance of the air. Neither tells us anything about the nature of the material : for the rheologist they are nuisances which must be eliminated. The internal resistances alone interest him. He will make experiments on damping of oscillations for the purpose of learning something about those rheological properties which produce the internal resistances. The latter are of four kinds, which can be arranged in two pairs. We have first the viscous damping due to solid viscosity. Solid viscosity dissipates kinetic energy. The second is due to viscosity present in liquids, or liquid viscosity. It will operate in damping oscillations of a rod of asphalt or of any body consisting of a

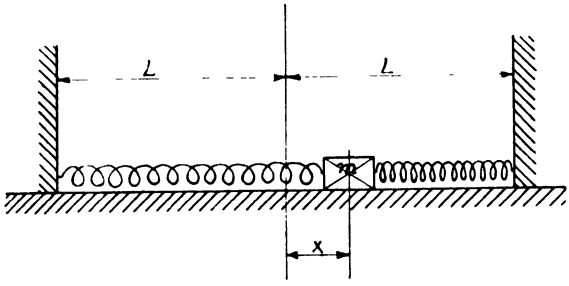


FIG. IX, 8. Model for linear oscillator.

Maxwell liquid. Liquid viscosity dissipates elastic potential energy. The third kind is due to plastic resistance. A rod made of a plastic material and subjected to the torsion described in Section 6 of Chapter VI will show such damping. The fourth is due to local microscopic or submicroscopic destructions such as occur in " permanent sets " the nature of which was explained in Section 2 of Chapter III. In the first two instances the resistance depends upon the velocity ; in the second two it is independent of the velocity. We shall deal here in detail with the mathematics of viscous damping only, simplifying our problem by considering a homogeneous one-dimensional case.

Let us imagine an arrangement as indicated in Fig. IX, 8. A body of mass m oscillates on a frictionless horizontal plane in a straight line under the action of an elastic force exerted by springs which obey Hooke's law. We assume that the experiment is carried out *in vacuo* and that the springs are stressed below the elastic limit. The only resistance which we consider is internal viscous resistance of the springs in accordance with the equation

$$\sigma = -Ee + \lambda_s \dot{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

In the Figure we have shown two springs, such as would be suitable for a working model, but for the calculation we replace the springs by a straight elastic rod which is capable of resisting push as well as pull, E being its Young's modulus and λ_s its "solid" viscous Trouton coefficient.

Applying Newton's law (I, a) we write

$$m \cdot d^2x/dt^2 = P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The force is $P = A \cdot \sigma$ where "A" is the cross-sectional area of the rod.

Introducing $e = x/l$ and $\dot{e} = \dot{x}/l$ we write

$$l\sigma = -\mathrm{E}x - \lambda_s dx/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where the negative sign for $\mathrm{E}x$ is introduced because the elastic force acts in the direction opposite to an increasing elongation and the negative sign for $\lambda\dot{x}$ because the viscous resistance is opposed to the direction of the velocity.

From Equs. (38) and (39) we get

$$d^2x/dt^2 + \mathrm{E}^*x + \lambda^* dx/dt = 0 \quad . \quad . \quad . \quad . \quad . \quad (40)$$

where

$$\mathrm{E}^* = AE/lm \; ; \; \lambda^* = A\lambda/lm \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The solution of the differential equation (40) can be expressed in two different forms, depending on whether $\lambda_s^{*2}/4 \quad \mathrm{E}^*$ or $\lambda_s^{*2}/4 \quad \mathrm{E}^*$.

(i) If we first take the case $\lambda_s^{*2}/4 \cdot \mathrm{E}^*$, the solution can be expressed as follows :

$$x = x_0/2a \cdot \{(\lambda^*/2 + a)\mathrm{e}^{(-\lambda^*/2 + a)t} + (-\lambda^*/2 + a)\mathrm{e}^{(-\lambda^*/2 - a)t}\} \quad . \quad (42)$$

where

$$a = \sqrt{\lambda^{*2}/4 \quad \mathrm{E}^*} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

That (42) is a solution of (40) can be verified by differentiation and substitution.

Introducing $t = 0$ in (42), we get

$$x|_{t=0} = x_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

and x_0 is therefore the initial amplitude.

By differentiation of (42) we find

$$\dot{x} = x_0(a^2 - \lambda^{*2}/4)/2a \{\mathrm{e}^{(-\lambda^*/2 + a)t} \quad \mathrm{e}^{(-\lambda^*/2 - a)t}\} \quad . \quad . \quad (45)$$

From (45) we find that $\dot{x}$ vanishes for $t = 0$ and $t = \infty$, but not between these extreme values, and from (42) it follows that x vanishes for $t = \infty$ but not before. This is therefore not a case of oscillations, but an aperiodic movement. In this case the viscosity is so great that the body asymptotically approaches the unstrained state, without any kinetic energy left to produce oscillation.

(ii) Secondly, we consider the case $\lambda_s^{*2}/4 \cdot E$

The solution is

$$x = x_o e^{-\lambda^*/2 \cdot t} (\cos bt + \lambda^*/2b . \sin bt) \quad . \quad . \quad . \quad (46)$$

where

$$b = \sqrt{E^* - \lambda_s^{*2}/4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

This also can be easily verified by differentiation and substitution. Again we find

$$x|_{t=o} = x_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and

$$\dot{x} = x_o \epsilon^*/b . e^{-\lambda^*/2 \cdot t} \sin bt \quad . \quad . \quad . \quad . \quad . \quad (49)$$

From (49) we see that $\dot{x}$ vanishes for $t = 0,\ \pi/b,\ 2\pi/b \ . \ . \ . \ n\pi/b$. For these values of t the amplitudes are $x_o, -x_o e^{-\lambda^*\pi/2b}, + x_o e^{-2\lambda^*\pi/2b}$, ... The movement therefore consists of oscillations with the period

$$T = 2\pi/b = 2\pi/\sqrt{E^* - \left(\frac{\lambda^*}{2}\right)^2} \quad . \quad . \quad . \quad . \quad (50)$$

This period is a constant. This is quite remarkable. Oscillations with viscous damping are accordingly *isochronous*, the same as oscillations without damping. When the amplitudes become smaller, the velocity decreases correspondingly, so that the time for making the smaller movement remains the same. The period itself is, however, greater in the damped than in the undamped oscillations, because " b " becomes larger if, in (47), the coefficient of the damping viscosity λ^*, vanishes. The greater the damping, the slower, therefore, the oscillations. The amplitude of the oscillations gradually diminishes in accordance with our last series. In infinite time ($t = \infty$) $\dot{x}$ vanishes, i.e. the movement ceases, and at the same time x also vanishes, i.e. *the position of rest is the unstrained state*. The amplitudes form a geometrical series with the coefficient $e^{-\lambda^*\pi/2b}$ and the logarithms of the amplitudes form an arithmetical series with a negative increment or decrement, $\lambda^*\pi/2b$. This is called the Logarithmic Decrement of the oscillations. If we consult (50) we find the logarithmic decrement to be equal to $\lambda^* T/4$.

Plotting the elongation on a logarithmic scale we get a straight line.

The main difference between damped and undamped oscillations is in the factor $e^{-\lambda^*/2 \cdot t}$ in (46). In the beginning of the oscillations as long as t is small, this factor is not very different from unity and the movement of the damped oscillations does not differ much from the movement of undamped oscillations. If λ^* is also small, several oscillations may pass before the diminution of amplitudes is noticeable. As soon, however, as t has increased to such an extent that the exponential factor is markedly different from unity, the amplitudes decrease very quickly. If after one second the exponential factor was, say, 0·9 and

after two seconds $= 0{\cdot}8$ which may be entirely within the error of observations, it becomes after 10 seconds $0{\cdot}9^{10} = 0{\cdot}35$, i.e. very noticeable ; and after 20 seconds, when it is 0·1, the original oscillations have probably been entirely superseded by vibrations from accidental causes.

11. Summary

The property of elastico-viscosity is typified by the Maxwell liquid, the structural formula of which is

$$M = H - N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX, a)}$$

with the rheological equation

$$\dot{\gamma} = \dot{\tau}/\mu_l + \tau/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX, b)}$$

where μ_l is the " liquid " rigidity.

For constant stress $\tau = 0$, $\dot{\gamma} = \tau/\eta$ and the Maxwell body behaves as a Newtonian liquid.

If μ_l and η are constants, Equ. (IX, b) gives on integration

$$\tau = e^{-\mu/\eta\, t}\,(\tau_o + \mu \int_o^t \dot{\gamma} e^{\mu/\eta\, t} dt) \quad . \quad . \quad . \quad \text{(IX, c)}$$

For $\dot{\gamma} = 0$ this is reduced to

$$\tau = \tau_o e^{-\mu/\eta\, t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX, d)}$$

which shows that for constant strain the elastic stress relaxes.

The quantity

$$T_M = \eta/\mu_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX, e)}$$

is called the relaxation time.

The property of firmo-viscosity is typified by the Kelvin solid, the structural formula of which is

$$K = H|N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX, f)}$$

with the rheological equation

$$\tau = \mu\gamma + \eta_s\dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX, g)}$$

where η_s is the " solid " viscosity.

For constant strain ($\dot{\gamma} = 0$), $\tau = \mu\gamma$ and the Kelvin body behaves as a Hookean solid.

If μ and η_s are constant, Equ. (IX, g) gives on integration

$$\gamma = e^{-\mu/\eta_s\, t}(\gamma_o + \frac{1}{\eta_s}\int_o^t \tau e^{\mu/\eta_s t}\, dt) \quad . \quad . \quad . \quad \text{(IX, h)}$$

If a constant stress acts upon an unstrained body

$$\gamma = \frac{\tau}{\mu}(1 - e^{-\mu/\eta_s\, t}) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX, i)}$$

and the elastic strain τ/γ does not appear with acoustic speed, but is delayed and reaches this value only after infinite time. Likewise, if

the load is removed from a body the strain of which is γ_o, this reverts to the unstrained state in accordance with

$$\gamma = \gamma_o e^{-\mu/\eta . t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IX, j)$$

The first phenomenon is called elastic fore-effect, the second elastic after-effect, both constituting delayed elasticity.

The quantity

$$T_K = \eta_s/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IX, k)$$

is called the retardation time.

These equations are written here for simple shear, but can be used for linear dilatation by introducing e_l for γ, E for μ and λ for η.

A Kelvin solid shows viscous damping of oscillations.

The oscillations are isochronous with the period

$$T = 2\pi/\sqrt{E^* - (\lambda^*/2)^2} \quad . \quad . \quad . \quad . \quad (IX, l)$$

The amplitudes form a geometrical series with the coefficient $e^{\lambda^*\pi/2b}$ and

$$\Delta = \lambda^*\pi/2b = \lambda^* T/l \quad . \quad . \quad . \quad . \quad (IX, m)$$

is called the logarithmic decrement.

In these equations

$$E^* = AE/lm \; ; \; \lambda^* = A\lambda_s/lm \quad . \quad . \quad . \quad (IX, n)$$

where A is the cross-sectional area and l the length of a rod executing longitudinal vibrations, E and λ_s being its Young's modulus and solid Trouton coefficient respectively, while m is the mass of the vibrator attached to the rod.

CHAPTER X

MORE COMPLEX BODIES

1. Delayed Elasticity in Asphalt and the Burgers Body

In the preceding chapters we have become acquainted with materials showing the simple properties of elasticity and viscosity ; the more complex property of plasticity which can only be realized together with the property of elasticity ; and finally the still more complex properties of elastico-viscosity and firmo-viscosity. These materials were idealized in the abstractions of the Hooke-, Newton-, St. Venant-, Maxwell- and Kelvin-Body. Of these, the first three only are elementary and it was shown how the other two could be related qualitatively to the first two by means of structural formulas. Quantitatively, rheological equations between τ, $\dot{\tau}$, γ and $\dot{\gamma}$ were postulated in which the three parameters μ, η and ϑ appeared in different forms, constituting " rheological coefficients ". These developments provided us with ever better approximations for the description of the behaviour of real materials. Let us, for instance, consider such a material as road-asphalt. In the first place asphalt possesses elasticity : this makes it suitable for a structural material. One can accordingly regard asphalt, to a first approximation, as a Hooke solid. As a matter of fact, road engineers base their designs nearly exclusively upon that abstraction. Only when " creep " must be taken into consideration, will they proceed to the second approximation and consider asphalt as a Maxwell liquid. However, one has to take cognizance of the fact that asphalt also shows delayed elasticity. If we want to take this also into consideration, we must reach a third approximation by proceeding beyond the Maxwell liquid.

Delayed elasticity, as we have seen, is bound up with firmo-viscosity, for which the Kelvin solid was devised. We therefore have to attach the K to the M complex and the question is only whether this is to be done in series or parallel. Asphalt, because of its slow viscous flow, should not, even in a further approximation, lose its character of a *liquid*. The connection must therefore be in series. Such a model was proposed by Burgers (1935) and we may name the ideal material corresponding to it the Burgers body (Bu). We accordingly write

$$Bu = M \quad K \quad (H \quad N) \quad (H|N) \quad . \quad . \quad . \quad (1)$$

and the reader will have no difficulty in making himself a sketch of the model representing the Bu-body in accordance with Equ. (1).

Lee and Markwick (1937) found that there was "good qualitative agreement between the behaviour of actual (bituminous road-) surfacing materials and the behaviour of the model. The model also illustrates the behaviour of . . . binders such as bitumen and pitch."

The "second axiom" of rheology must lead us to expect that the Burgers body will be applicable to many other materials besides asphalt.

In order to proceed from the structural formula (1) to a rheological equation, we note that there will be four rheological coefficients, two of viscosity and two of elasticity. If we consider shear (or, more generally, distortion), these will be the "ordinary" viscosity (η) in the M-complex and "solid" viscosity (η_1) in the K-complex; "liquid" rigidity (μ_1) in the first and "ordinary" rigidity (μ) in the second case. Jeffreys (1929) who was the first to propose a rheological equation for a M-K complex proceeded as follows:-

Being arranged in series, both the M- and K- complex must take the same stress. For the first we find from Equs. (IX, b) and (IX, e)

$$\mu_1\dot{\gamma} = \tau + \frac{1}{T_M}\int \tau dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the second we find from Equs. (IX, g) and (IX, k)

$$\tau = \mu(\gamma + T_K\dot{\gamma}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equating both, we get

$$\mu(\gamma + T_K\dot{\gamma}) = \tau + \frac{1}{T_M}\int \tau dt \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Jeffreys remarks that the substance would follow the firmo-viscous law if T_M was infinite, and the elastico-viscous one if T_K was zero. The substance will flow indefinitely with long-continued stresses, but the partial recovery on release will be gradual. If any problem of elastic strain had been solved for a simple elastic solid, the behaviour of that substance, so long as squares of the displacements can be neglected, could be inferred simply by writing $\mu(1 + T_K\frac{d}{dt})/\left(1 + \frac{1}{T_M(d/dt)}\right)$ for μ. A body *nearly* simply elastic has T_M large and T_K small and has approximately $\mu[1 + T_K(d/dt) \quad 1/T_M(d/dt)]$ for μ. The damping of surface waves of earthquakes as they advance would suggest that either $T_M = 750$ sec. and $T_K = 0$ or $T_K = 0{\cdot}004$ sec. and $T_M = \infty$. In the rocky shell approximately $T_M \quad 3 \times 10^8$ sec., $T_K = 0{\cdot}004$ sec., $\mu = 1{\cdot}7 \times 10^{12}$ dynes cm^{-2}, $\eta = 5 \times 10^{20}$ poises.

In the above derivation one may object to the equating of $\mu_1\dot{\gamma}$ with τ. It is true that the stress τ must be the same in both the H- and the N-element of the M-complex, but to calculate τ one would

have to take into account in the first case the strain of the spring only and in the second the rate of flow in the dashpot. Actually, however, γ in Equ. (2) is derived from the *sum* of both. Nevertheless, disregarding the derivation, one may *postulate* Equ. (4) as a rheological equation, permitting even the coefficients μ and τ to be variable and test it against the rheological behaviour of some real material.

A more consistent method for the derivation of the rheological equation of the Bu-body seems to me to be the one used by me (Reiner, 1945) by taking into account that the *displacements* of the *M*- and *K*-complex are *additive* ; the same method as used above in the derivation of the equation of the *M*-body.

From

$$\dot{\gamma}_{Bu} = \dot{\gamma}_M + \dot{\gamma}_K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we find by means of Equs. (IX, b) and (IX, h)

$$\dot{\gamma} = \tau/\eta + \dot{\tau}/\mu_1 + \frac{d}{dt}\left[e^{-\mu/\eta_s t}(\gamma_{0s} + \frac{1}{\eta_s}\int_0^t \tau e^{\mu/\eta_s t}dt)\right] \quad . \quad . \quad (6)$$

where γ_{0s} is the initial strain of the " solid " component.

After carrying out the differentiation, this yields

$$\dot{\gamma} = \tau\frac{\eta + \eta_s}{\eta\eta_s} + \dot{\tau}/\mu_1 - \frac{\mu}{\eta_s}e^{-\mu/\eta_s t}(\gamma_{0s} + \frac{1}{\eta_s}\int \tau e^{\mu/\eta_s t}dt) \quad . \quad (7)$$

Here also, while in the integration the coefficients were assumed to be constants, the derivation of Equ. (7) may be disregarded and the equation postulated as the rheological equation of the Burgers body, even should the coefficients turn out to be variable.

If a stress is applied to a *Bu*-body when it is in an unstressed and unstrained state, we find from Equ. (7) putting $\gamma_0 = 0$ and $\tau = 0$, $\dot{\gamma} = \dot{\tau}/\mu_1$. This is an instantaneous strain due to the H_M element. If, on the other hand, the stress is kept constant or $\dot{\tau} = 0$, we find

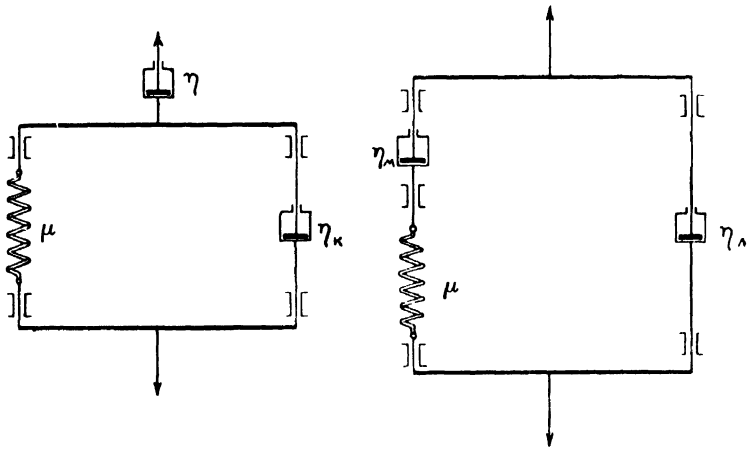
$$\dot{\gamma} = \tau/\eta + \tau/\eta_s . e^{-\mu/\eta_s t} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and two kinds of continuous deformation take place simultaneously, the second vanishing in infinite time. We have called the first creep and the second elastic fore-effect, but if, as was observed by Lethersich, in some materials that fore-effect lasts for *years* it is difficult to separate them and there may be some justification in naming the phenomenon to which that $\dot{\gamma}$ is due *in toto*, " creep ", in which case there will be a partial " creep-recovery ". That term " creep-recovery " was used by Glanville and Thomas (1939) and while I voiced my objection to it at one time, I am now rather inclined to think that there might be an advantage in having a term for slow continuous deformation *prior to an analysis* of which part of it is recoverable and which not, an analysis which might require further years of observation.

2. Further Ideal Bodies

If we let μ_1 increase $\to\infty$, the second term on the right side of Equ. (7) disappears and with it the H_M element in the structural formula which becomes N-K. Such material has been proposed by Lethersich to represent the behaviour of bitumen. This would mean that with the apparatus or the kind of bitumen used by him, Lethersich could not observe the acoustic elasticity, which Lee and Markwick (1937) had observed. We accordingly see that bitumen will be represented in successive approximations by N, N–K, M–K ; its most prominent rheological property being flow (N), the next prominent delayed elasticity (K) and the least easily observable acoustic elasticity (H).

Actually Lethersich proposed two alternative models as shown in Fig. X, 1. Their structural formulas are respectively K–N and



(a) $N - K = N - (H|N)$ (b) $M|N - (N - H)|N$

FIG. X, 1. Models proposed by Lethersich to represent the rheological behaviour of bitumen.

$M|N$. They may be named the Lethersich-body (L) and the Jeffreys-body (J) respectively. By making use of the transformation, K–$N = (N|H) - N = N|(H - N) = N|M$, one could come to the conclusion that they are identical. This, however, would be too rash, as we shall presently explain.

3. M-Complexes

We have already dealt with $N - H - M$ and $N \mid H - K$ in detail. We now examine the combination $M_1 - M_2$. This will give

$$\dot{\gamma} = \tau/\eta_1 + \tau/\eta_2 + \dot{\tau}/\mu_1 + \dot{\tau}/\mu_2 = \tau(1/\eta_1 + 1/\eta_2) + \dot{\tau}(1/\mu_1 + 1/\mu_2) = \tau/\eta + \dot{\tau}/\mu \quad . \quad . \quad . \quad . \quad (9)$$

where

$$1/\eta = 1/\eta_1 + 1/\eta_2 \; ; \; 1/\mu = 1/\mu_1 + 1/\mu_2 \; . \quad . \quad . \quad (10)$$

Several *M-complexes coupled in series are* accordingly *equivalent* to *one M-complex* in which all springs and all dashpots are taken together to form one spring and one dashpot each.

It is very different with $M_1 \mid M_2$. Here we have to add stresses in accordance with Equ. (IX, c) and find, putting

$\tau_0 = \mu\gamma_0$,

$$\tau = \gamma_0(\mu_1 e^{-\mu_1/\eta_1 t} + \mu_2 e^{-\mu_2/\eta_2 t}) + (\mu_1 e^{-\mu_1/\eta_1 t}\int_o^t \dot{\gamma} e^{\mu_1/\eta_1 t}\, dt + \mu_2 e^{-\mu_2/\eta_2 t}\int_o^t \dot{\gamma} e^{\mu_2/\eta_2 t}\, dt) \quad . \quad . \quad (11)$$

Neither sums within brackets can be replaced by a single term of similar form. As a matter of fact, the $M_1 \mid M_2$ body behaves very differently from a single M-body, as we shall presently see.

If we put $\gamma = \text{const.} = c$, we get from Equ. (11)

$$\tau = c(\eta_1 + \eta_2) + e^{-\mu_1/\eta_1 t}(\gamma_0\mu_1 - c\eta_1) + e^{-\mu_2/\eta_2 t}\gamma_0\mu_2 - c\eta_2). \quad . \quad . \quad (12)$$

Now, we can select such a rate of deformation, c, that either the second or the third term vanishes, but *both* these terms will vanish only if both constituent M-complexes have *the same relaxation time*. Generally, therefore, if the material flows at a constant rate, the stress will relax. Conversely, *under a constant stress, the material will not flow at a constant rate of deformation.* In other words there will be no state of simple viscous flow, as is possible in a Maxwell liquid.

The problem of the rheological behaviour of a system built up by a combination of Maxwell bodies connected partly in series and partly in parallel has lately become important in the investigation of the "relaxation spectrum" of rubbery materials. From our considerations it can be seen that only *parallel* connections will become manifest, while all different M-bodies *when connected in series* will give *one simple relaxation curve* and not be detectable (compare W. Kuhn (1947)).

4. Limitations of the Rheological Models

We can now come back to the comparison of the L-body $(N–K)$ and the J-body $(M \mid N)$.

The rheological equation of the first can easily be written down from Eqn. (7) by omitting the second term on the right side. The rheological equation of the second is found by adding the stresses taken by M and N, or

$$\tau = \tau_M + \tau_N = e^{-\mu_M/\eta_M t}(\tau_{0M} + \mu_M\int \dot{\gamma} e^{\mu_M/\eta_M t} dt) + \eta_N\dot{\gamma} \quad . \quad . \quad (13)$$

We now compare the behaviour of both bodies under constant stress, $\tau = \tau_0$, applied upon the unstrained and unstressed body. For

the first we find a rate of deformation as given by Equ. (8) in accordance with which it gradually decreases from

$$\dot{\gamma}_0 = \tau_0(1/\eta + 1/\eta_s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

to

$$\dot{\gamma}_\infty = \tau_0/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

For the second it can easily be found by substitution of $\mu_2 = \infty$ into Equ. (12) that under a constant stress τ_0 it will ultimately flow at a constant rate

$$\dot{\gamma}_\infty = \tau_0/(\eta_M + \eta_N) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Putting $\eta = \eta_M + \eta_N$, the behaviour of the mechanism representing the J-body is identical with that of the L-body and *as mechanisms* they cannot be distinguished. This, however, only shows the limitations of the rheological models. Actually the L-body is best suited for the representation of an elastic sol, while the J-body represents essentially a relaxing gel. The order of magnitude of the relaxation time of a sol is quite different from that of a gel. For instance, for a bitumen sol it is about 10^3 sec. or less, for a bitumen gel it is 10^5 sec. or more, while in a region of transition both sol and gel exist side by side. Besides this quantitative difference, a qualitative difference manifests itself in simple shear, when the principal axes rotate. In this case a sol in which the stress is transmitted from the liquid to the solid, behaves very differently from a gel, where the stress is transmitted from the solid to the liquid phase (Reiner (1951)).

5. The Plastic Strength of a Gelatine Solution and the Schwedoff Body

The examples of the rheological behaviour of bitumen and asphalt showed us that there are materials where it becomes necessary to consider a combination of the Maxwell liquid with the Kelvin solid. Investigations carried out by Schwedoff (1890) with a gelatine solution showed that in this case the Maxwell liquid must be combined with the St. Venant plastic. Schwedoff tested a 0·5 per cent. gelatine solution at an age of 24 hours in an apparatus consisting of the same elements as the Couette-Hatschek viscometer (compare Section 9 of Chapter II). If the external cylinder does not rotate, and the top of the wire on which the internal cylinder is suspended is given a twist through the angle Ω, the internal cylinder, if the material between both cylinders is an elastic solid, will be rotated through another angle, say θ. Making use of the elasticity-viscosity analogy, the angle θ can be calculated from Equ. (II, k) with the result that

$$\theta = M_z/4\mu\pi h \,.\, (1/R_i^2 - 1/R_e^2) \quad . \quad . \quad . \quad . \quad (17)$$

where M_z is the torque exerted upon the wire.

If, however, the material between the cylinders is a Newtonian liquid, the inner cylinder will follow the wire immediately with decreasing velocity until it has moved through the angle Ω, i.e. until no torsion is left in the wire. Thirdly, if the material is a Maxwell liquid, the cylinder will not follow immediately, but will in the first instance rotate through the angle θ ($<\Omega$), the elastic resistance of the liquid balancing the torque M_z of the wire, which is (compare Equ. (IV, g))

$$M_z = \pi R_w^4 \mu_w / 2l_w(\Omega - \theta) = (\Omega - \theta) D_w / l_w \quad . \quad . \quad (18)$$

where the subscript " w " indicates " of the wire ".

Introducing $\Omega - \theta = \phi$, Equ. (17) becomes

$$\theta = \frac{D_w}{l_w} \phi / 4\mu\pi h \, . \, (1/R_i^2 - 1/R_e^2) \quad . \quad . \quad . \quad . \quad (19)$$

From Equ. (19) the rigidity of the elastic liquid can be determined and is found to be

$$\mu_l = \frac{D_w}{l_w} \phi (1/R_i^2 - 1/R_e^2)/4\pi h\theta \quad . \quad . \quad . \quad . \quad (20)$$

The internal cylinder does not remain at the deflection θ but relaxation sets in and the cylinder gradually follows the wire. The relaxation of the stresses required to maintain the elastic strain θ can be studied by reducing the torsion of the wire (Ω) from time to time, so as to maintain the cylinder at the deflection θ and by plotting the torques ϕ_1, ϕ_2, ϕ_3 . . . against the times t_1, t_2, t_3, . . . If the material is a Maxwell liquid, the ϕ t curve is of the exponential type and the stress vanishes for $t = \infty$. Schwedoff, however, found that in his gelatine solution the relaxation was not of this type, but that the material apparently maintained a small residual stress permanently ; i.e. if Ω was maintained at a constant value, the inner cylinder did not follow until $\theta = \Omega$ or $\phi = 0$, but a finite ϕ remained. Instead of Equ. (IX, d) he found the equation

$$\tau = \vartheta + (\tau_o - \vartheta)e^{-t/\tau} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

to be applicable, where ϑ is the residual stress which does not relax after practically infinite time. Hatschek (1928) says that " the magnitude of this permanent deformation is very uncertain, as the whole apparatus and the condition of the liquid are highly susceptible to even small vibrations ", but if Schwedoff's observations were only qualitatively correct, and similar observations were made by Hatschek and Jane (1926) on a number of other solutions, this would mean that the materials in question were in fact not liquids, as was thought from their appearance, but plastic solids.

To represent such materials, we postulate the Schwedoff body,

built up of three elements in accordance with the structural formula

$$Schw = (M \mid StV) - H \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The model is shown in Fig. X, 2. The rheological equation is

$$\dot{\gamma} = (\tau - \vartheta)/\eta_{pl} + \dot{\tau}/\mu_H \quad . \quad . \quad . \quad . \quad . \quad (23)$$

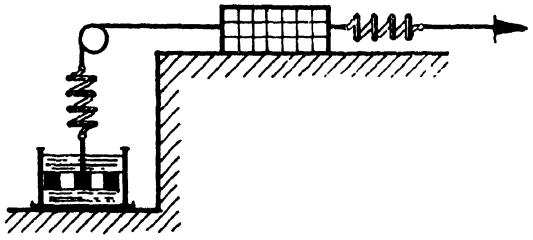


FIG. X, 2. Model for a Schwedoff body corresponding to the structural formula $Schw = (M \mid StV) - H$.

6. The Schofield-Scott Blair Body

So far the most complex body which it was necessary to postulate in order adequately to describe the observations on a material, appears to be the Schofield-Scott Blair body, the material being flour dough.

In a series of investigations (1932), the authors first stretched out long cylindrical pieces of dough for a measured time, at the end of which the cylinders were cut loose. It was found that part of the extension was recovered, but part was permanent (Fig. 1 in (1942), part I). This showed that while flour dough (FD), because of the permanent extension of the cylinders, is not a Hooke body, it has a Hooke component ; and as a first approximation we write $FD = H - X$, where X represents another element or elements to be deduced from further experiments. In the next series of experiments it was found that the elastic recovery decreased with time under stress in accordance with an exponential law (Fig. 4, in the same paper). Comparison with our Fig. IX, 3, shows that we must write N for X, so that as a second approximation $FD = H - N = M$. This was confirmed by further experiments, in which the decay of internal stresses in pieces of dough which had been stretched and held stretched was followed (Fig. 6 in (1932), part I. The shape of the curve conformed to our Fig. for Δl = const. The second report of the authors describes observations in which the rate of extension of cylinders of dough, hung vertically and allowed to elongate under the action of gravity, is related to the stress. It was found that, while the rate of extension generally decreased with decreasing stress, there was a finite stress at which the rate of extension vanished ; in fact, a yield point. This shows that a St. Venant element must further be added ; and in a third approximation $FD = N - H \mid StV = M \mid StV = Schw$. It had however been noticed that "a considerable time often elapses between the release of stress and the

cessation of contraction ". This indicated an elastic after-effect, to the investigation of which a third report was devoted. For an elastic after-effect a K body must be coupled on. As the structural formula for FD contains a StV element, the question arises as to which end of the StV element the K body is to be connected. Experiments (Fig. 2 in report III) showed that the elastic after-effect makes its appearance in strains *below the yield point*. This means that the K-body must be connected at the spring end. It could be formed there by parallel coupling of the spring with an N-element. The same figure, however, illustrates that, beside the " lagging " elastic recovery, there is also " instantaneous " recovery, i.e. the spring of the St. Venant element is not impaired in its working and the K-body is therefore connected to it " in series ". We accordingly obtain as a fourth approximation the structural formula

$$SchScB - (M \mid StV) - K - Schw \mid N \quad . \quad . \quad . \quad (23)$$

Fig. 4 in report III confirms the formula for deformations above the yield point.

Guided by Equ. (23) it is not difficult to design a model for the Schofield-Scott Blair body. Equ. (23), in addition, helps us to establish the rheological equation. The authors expressed it, translated into our notation, as follows : —

$$\dot{\gamma} = \tau/\eta + \dot{\tau}/\mu + \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where α represents the elastic after-effect. We note that they do not take into account that flour dough, as they themselves had found, possesses a yield point. Their rheological equation, therefore, represents only part of their experimental findings ; as a matter of fact it corresponds to the structural formula $Bu - M - K$ which results from Equ. (23) when omitting the StV element. If we compare the rheological equation (7) of the Burgers body with Equ. (24) we find that Schofield and Scott Blair's α is actually

$$\mu/\eta_s e^{-\mu/\eta_s t}(\gamma_o + \frac{1}{\eta_s}\int_o^t \tau e^{\mu/\eta_s t}dt) - \tau/\eta_s$$

and only if τ is equal to zero, as in an elastic after-effect, is this reduced to $\gamma_o \frac{\mu}{\eta_s} e^{-\mu/\eta_s t} = \frac{\gamma_o}{T_K} e^{-t/T_K}$. Equ. (24), therefore, cannot adequately express an elastic fore-effect.

If we take into account the StV element, the complete rheological equation of the Schofield-Scott Blair body is

$$\gamma = (\tau - \vartheta)\frac{\eta_{pl} + \eta_s}{\eta_{pl}\ \eta_s} + \dot{\tau}/\mu_l - \frac{\mu}{\eta_s} e^{-\mu/\eta_s t} \left[\gamma_o + \frac{1}{\eta_s}\int_o^t (\tau - \vartheta)e^{\mu/\eta_s t}dt)\right] \quad . \quad . \quad . \quad . \quad (25)$$

Under a constant stress, exceeding the yield value, the body is plastically deformed at a rate which, in analogy to Equ. (8) is

$$\dot{\gamma} = (\tau - \vartheta)\,(1/\eta_{pl} + e_{pl}^{-\mu/\eta\,t}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Therefore, as in the case of Equs. (14) and (15), the rate gradually decreases from

$$\dot{\gamma}_o = (\tau - \vartheta)\,(1/\eta_{pl} + 1/\eta_s) \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

to

$$\dot{\gamma}_\infty = (\tau - \vartheta)/\eta_{pl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

It all depends upon the magnitude of the retardation time $T_\kappa = \eta_s/\mu$ how long it will take for $\dot{\gamma}$ to become practically constant and indistinguishable from $\dot{\gamma}_\infty$. However, Equ. (28) is the same as Equ. (VIII, a), the rheological equation of the Bingham body. It is therefore not astonishing that Wolarowitsch and Samarina (1935) when testing flour dough in a plastometer of the co-axial rotating cylinder type, found that the Reiner and Riwlin equation (VIII, g), which is based upon the *B*-body, was applicable. They probably sheared the material long enough to reach a steady state, and this was one corresponding to a *B*-body.

7. Regarding the Second and Third Axioms

This furnishes an illustration for the working of what I called the second and third axioms of rheology. The second (compare Section 12, Chapter I) says that every real material possesses all rheological properties, while the third states that every "simple" behaviour is a degeneracy of a more complex one. If flour dough is put into an apparatus where its properties under *steady* laminar shear are observed, *nothing more complicated than corresponds to the B-body can be observed.* In order to find out about the other properties of flour dough, i.e. those which Schofield and Scott Blair discovered, Wolarowitsch and Samarina would have had to use their apparatus in the manner of Schwedoff; they would then have found its *M*-component, etc., in accordance with the second axiom. On the other hand, we see now that Scott Blair was not entirely correct when he stated on page 27 of his book (1938) that to apply "Bingham's principles to . . . flour dough . . . is contrary to the author's (Scott Blair's) experience." It is true that flour dough *is not* a Bingham body, but from Schofield and Scott Blair's observations the conclusion can be drawn with the help of the third axiom that flour dough *must* under certain conditions show the properties of a Bingham body.

Let us go back to Equ. (25). If $\vartheta \to 0$, the *SchScB*-body degenerates, as we have already shown, into the *Bu*-body. If $\eta_s \to 0$, it degenerates into the *Schw*-body. The *Bu*-body itself can degenerate into the *M*- and *K*-bodies respectively and both these into the *N*- and

H-bodies. The *Schw*-body degenerates into the *B*-body for $\mu \to \infty$ and the latter into the *StV*-body for $\eta_{pl} > 0$. Ultimately the *SchScB*-body is thereby reduced to either the *N*-, *H*- or *StV*-body and in accordance with the conditions of testing one or the other will predominate. One investigator may lay stress upon the elasticity of flour dough, the other upon its viscosity, while a third may emphasize its plasticity. Each one will be right, but not entirely so. Similarly, we have seen that Lee and Markwick (1937) approximated bitumen by the *Bu*-body, while Lethersich assumed a degeneracy of the latter, which we called the *L*-body. As far as I know, nobody has yet found a yield value in bitumen, but this may only mean that it was not observed under suitable conditions. When these will be established and a yield value found, for which there is some likelihood in blown bitumen, one will have to use the *SchScB*-body to describe the rheological properties of bitumen.

8. Summary

Complex ideal bodies can be built up in the form of models, structural formulas and rheological equations, from the elemental bodies of (i) the Hooke elastic (*H*) represented by a spring ; (ii) the St. Venant plastic (*StV*) represented by a weight sliding over a table ; and (iii) the Newtonian liquid (*N*) represented by a dashpot. The mechanical elements may be connected either " in series " (—) or " in parallel " (|). The rheological equation of a complex is derived from the rheological equations of the components by (a) adding deformations, when the coupling is in series ; (b) adding stresses, when the coupling is in parallel.

When *elements* of the same kind are connected, the resultant body possesses the same properties, e.g. $H_1 - H_2 = H$, etc. For complexes, this is generally not true, e.g. $M_1|M_2$ is *not* an *M*-body. But $M_1 - M_2 = M$; $K_1|K_2 = K$; $M - N = M_1$; $M_1 - H = M_2$; $K|N = K_1$; $K_1|H = K_2$.

The following ideal bodies have been applied for the representation of the behaviour of certain materials, but are applicable for many others :

(I) For bitumen and asphalt, the Burgers body

$$Bu = K - M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(X, a)}$$

with the rheological equation

$$\dot{\gamma} = \tau \frac{\eta + \eta_s}{\eta\eta_s} + \tau/\mu_1 - \frac{\mu}{\eta_s} e^{-\mu/\eta_s t}(\gamma_0 + \frac{1}{\eta_s}\int_0^t \tau e^{\mu/\eta_s t} dt) \quad . \quad \text{(X, b)}$$

When the stress is constant

$$\dot{\gamma} = \tau/\eta + \tau/\eta_s . e^{-\mu/\eta_s t} \quad . \quad . \quad . \quad . \quad \text{(X, c)}$$

and there is both viscous flow and an elastic after-effect starting with

$$\dot{\gamma}_o = \tau(1/\eta + 1/\eta_s) \quad . \quad . \quad . \quad . \quad . \quad . \quad (X, d)$$

and finishing up with

$$\dot{\gamma}_\infty = \tau/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (X, e)$$

(II) Also for bitumen, Lethersich proposed the two bodies $N - K$ and $M|N$. The first can be considered as a truncated Burgers body.

(III) For a concentrated gelatine solution, the Schwedoff body

$$Schw = (M \mid StV) \quad H \quad . \quad . \quad . \quad . \quad . \quad (X, f)$$

with the rheological equation

$$\dot{\gamma} = (\tau - \vartheta)/\eta_{pl} + \tau/\mu \quad . \quad . \quad . \quad . \quad . \quad (X, g)$$

If the deformation is kept constant, the stress relaxes in accordance with

$$\tau = \vartheta + (\tau_o - \vartheta)e^{-\mu/\eta t} \quad . \quad . \quad . \quad . \quad . \quad (X, h)$$

(IV) For flour dough, the Schofield-Scott Blair body

$$SchScB = (M \mid StV) - K - Schw \mid N \quad . \quad . \quad . \quad . \quad (X, i)$$

with the rheological equation

$$\dot{\gamma} = (\tau - \vartheta)\frac{\eta_{pl}+\eta_s}{\eta_{pl}\eta_s} + \dot{\tau}/\mu_l - \frac{\mu}{\eta_s}e^{-\mu\eta_s t}\left[\gamma_o + \frac{1}{\eta_s}\int_o^t (\tau - \vartheta)e^{\mu/\eta_s t}dt\right] \quad . \quad (X, j)$$

Of these bodies the most complex is the *SchScB*-body and all others can be derived from it as degeneracies, assuming that certain rheological coefficients vanish.

In the rheological equations the rheological coefficients appropriate for shear (γ, τ) have been used. Similar equations can be written down for isotropic (e_v, p) and normal (e_n, σ_n) deformation, when, for elasticity the moduli κ and E and for viscosity the coefficients ζ and λ must be used.

CHAPTER XI

CREEP

1. Solid and Liquid

In Section 1 of Chapter VIII we have described how Bingham came to consider oil paint to be a soft solid and not a liquid. Then in Section 1 of Chapter X we reported that Jeffreys regarded the rocky shell of the earth as a very viscous liquid and not a solid. The difference between solid and liquid is therefore only one of degree. We can say "everything flows" if given enough time; and "the mountains flow before the Lord" would mean that they flow if the time of observation is extended to geological scale. Using every-day time-scale we can define a solid as a material which either does not flow or flows *plastically*. Plastic flow requires a stress larger than a definite stress, the yield stress, and a plastic solid does *not flow* under the action of a stress *below that yield stress*. The limiting stress below which the material does not flow may be quite low: this makes a Soft Solid. Such a soft solid may be mistaken for a liquid for quite a long time, as was the case with oil paint. In contradistinction to the plastic solid, *a liquid flows under any stress, however small.**

There is an inclination among physicists to admit a single crystal only to the status of "true solid" and to consider every amorphous material as a liquid. Glass, for instance, is spoken of as an "undercooled liquid"; but it is interesting to hear that "the 200-inch mirror for Mount Palomar is made of glass not for any optical property . . . but for its mechanical properties. In this case we are concerned with permanence of shape. . . . For in that mirror the silver or aluminium is on the front face, not the back one. . . . The glass is a purely mechanical support for the almost infinitesimally thin mirror. We use glass in this case, not because it is transparent, but *because its general rigidity and permanence of shape are better than steel or concrete*" (Preston (1942)).

If this is so, glass would be—at least rheologically—a solid, not a liquid. However, let us see what Lord Rayleigh had to say on this subject: "I have tried the following experiment: A piece of optically flat crown glass 3·5 cm long and 1·5 cm broad and 0·3 cm thick, was supported on wood at the extreme ends, and the middle was loaded with 6 kg applied by means of wooden chisel edge. It remained in position from April 6th, 1938, to December 13th, 1939. At the end of

* Never forget that first axiom of rheology. If I speak of "any" stress, I exclude, of course, an isotropic stress.

that time the glass was taken out and tested on an optical flat by means of interference fringes. It was found to have been bent, the sagita of the arc amounting to 2·5 bands or 1·25 waves, that is about 6×10^{-5} cm" (1940). With the help of (IV, e) and the viscosity elasticity analogy it is easy to calculate the viscosity of that particular glass at room temperature :

We have $\eta_c = PL^3/144dI$, where, in our case, $I = bh^3/12 = 1{\cdot}5 \times 0{\cdot}3^3/12 = 0{\cdot}0034$ cm^4 ; $P = 6{,}000 \times 981 = 5{,}886 \times 10^3$ dyne ; L = 3·5 cm ; $\dot{d} = 6 \times 10^{-5}/616 \times 24 \times 60 \times 60 = 1{\cdot}25 \times 10^{-12}$ cm/sec. We find $\eta_c = 6{\cdot}3 \times 10^{18}$ poises, using the subscript c in η_c to indicate Creep Viscosity. This is a viscosity not essentially different from the ordinary liquid shear viscosity, creep meaning here slow viscous flow.

In the above-mentioned paper Preston finds η_c by extrapolating to room temperature a graph showing viscosities of glass at high temperatures to be around 10^{60} or 10^{70} poises and then demonstrates that in our universe such a viscosity is without meaning. If Lord Rayleigh's experiment can be trusted, this shows the well-known dangers of extrapolation.

Comparatively speaking, therefore, glass may be considered as a solid—but actually this brings us back to what we said in the first chapter, namely, that *the strictly defined rheological divisions belong to ideal abstract bodies and not to real materials.* If we say that stone is a liquid, every builder will laugh at us and the structural engineer dismiss the idea as fantastic ; if we say that glass is a solid, the theoretical physicists will consider us to be simple and crude. But neither the one nor the other can answer if we say : " a Hookean material is a solid, a Newtonian is a liquid ". And we have postulated the Maxwell liquid to take account of the fact that while both stone and glass flow slowly or *creep*, both are also elastic, a property absent in a Newtonian liquid.

2. The Creep of Concrete

Structural engineers discovered the creep of cement-stone through the discovery of the creep of a still more " solid " material, of which cement is a constituent, viz. the creep of concrete and even reinforced concrete.

When a concrete prism of height h is loaded, it is compressed, i.e. shortened by, say Δh. When the load is removed, part of the reduction of height is recovered at once (Δh_e). After a day or two we find that still more is recovered, and this process goes on for several days up to a maximum (Δh_d). A residuum (Δh_p) is not recovered even if we wait for a long time.

We accordingly have,

$$\Delta h = \Delta h_e + \Delta h_d + \Delta h_p \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Δh_e is the ordinary *elastic strain*, and the phenomenon to which Δh_d is due is *delayed elasticity*. Δh_p is called the Permanent Deformation, but needs a further analysis.

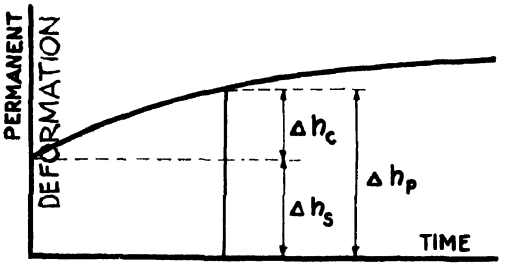


FIG. XI, 1. Creep and permanent set.
Δh_p permanent deformation.
ΔH_s permanent set.
Δh_c creep.

If we keep our load in position for, say, a year, we shall see that Δh_e and Δh_d are not affected, but Δh_p increases with time.* If we plot the Δh_p against time and extrapolate to time $t = o$, we get a permanent Δh_s and a variable Δh_c so that

$$\Delta h_p = \Delta h_s + \Delta h_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Δh_s is called the Permanent Set and creep is the phenomenon to which Δh_c is due (compare Fig. XI, 1). The same considerations can be applied to the deflections of a loaded beam, where h then stands for " deflection ".

The permanent deformation (Δh_p) was described by Bach as early as 1888, but as its increase in time was not noticed, it was thought to be entirely due to the permanent set (Δh_s).†

During the first world war, American investigators first observed and described creep. McMillan in 1921 reported on a concrete column in compression which at the end of a period of 600 days still showed a deformation proceeding at the average rate. In 1928 Faber first described creep in England. He called it " plastic yield " but it is important to keep in mind that creep is *not* what we called plastic deformation. The plastic deformation of metals is easily produced by impact. In contradistinction bitumen will flow slowly, i.e. will creep, but cannot be quickly deformed plastically. If an attempt is made to produce the permanent deformation quickly by impact, the bitumen

* This represents the view which I held until recently and which was shared by other workers in this field, e.g. Freudenthal (1950) p. 523, but which I shall, possibly, have to revise (see Section 10 below).

† We shall say more about the permanent set in Section 9 below.

breaks in a brittle manner. Also, a plastically deformable body can sustain loads up to the yield point without further deformation, while a creeping material has no yield point : it creeps under the smallest load.

Faber discovered creep in a case very similar to one which I had occasion to observe in a building in Jerusalem some years later. The building is a monolithic reinforced concrete frame with concrete panels later filled in (compare Fig. XI, 2).

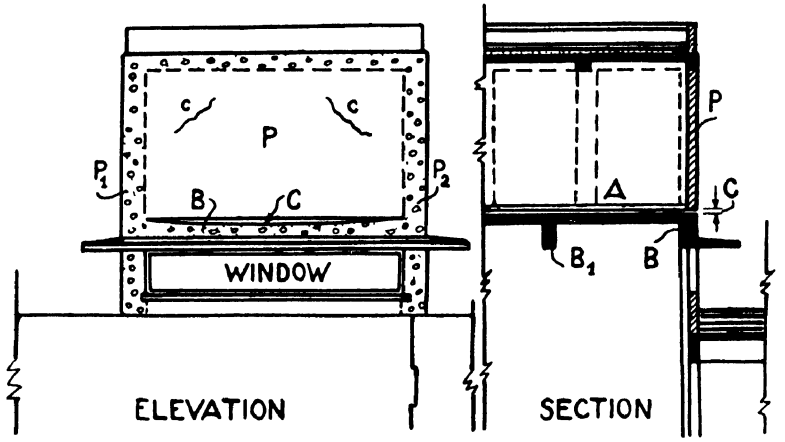


FIG. XI, 2. Creep of a reinforced concrete beam in a building. Reinforced concrete in elevation shown dotted, in section shown full.

The floor A was heavily loaded. It was carried by beams of large spans (B and B_1). After a few years, a crack (C) was noticed. The calculation and design of beam B were checked and found in order. It was also found that the width of the crack was of such magnitude that the sagging of the beam could not have been elastic. An elastic deflection of such magnitude would have been accompanied by stresses exceeding the strength of the concrete. Actually the following had happened : the heavily loaded floor had caused high stresses in the beams and these in turn had caused creep, i.e. a deflection of the beams increasing with time. The deflection was too small to be noticed at one of the intermediate beams (B_1), but at the front wall the rigid panel (P) which was fixed between the pillars (P_1 and P_2) had not partaken in the movement of beam B and the joint between panel and beam had accordingly opened up to form a crack. Faber was led to the discovery of creep by the observation of cracks (c) in partitions which *did* partake in the movement of the beams on which they were standing.

The nature of the creep of concrete is not very well understood to this day. In 1931, Straub, speaking of the " plastic " flow in concrete

arches, proposed a "power law" or what would now be called a Nutting-Scott Blair equation* for the creep of concrete, of the form

$$d_c = as^m t^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where d_c is the deformation due to creep, s is the stress, t is time and a, m and n are constants. The equation did not allow for a yield strength and in the discussion the question was raised: Is concrete a viscous liquid or a plastic solid? This question could not be answered on the basis of the experimental material existing at the time. Glanville (1930) had stated that the creep of a 1 : 2 : 4 mix † is approximately twice that of 1 : 1 : 2 mix and it is generally assumed that the creep increases as the quantity of aggregate is increased. This would imply that the aggregates flow as loose sand may flow, and that it is the cement, which binds the aggregates together, that prevents flow.

3. The Creep of Cement and Cement Mortar

Without knowing of Glanville's paper, I co-operated in 1932 with Prof. Bingham in an investigation which aimed at finding out what actually *was* flowing in reinforced concrete (1933). Bingham told me about a style of covering graves used at the Rock Creek Cemetery, near Washington, D.C. Four short posts are put up over the mound, to support a horizontal slab of marble, on which name and dates and other particulars of the dead are inscribed. It was observed that in the course of two or three generations such a stone, 180 cm long, 90 cm wide and 5 cm thick, sagged by some 8 cm. Bingham considered this to be a case of flow of limestone, but questioned whether this flow was viscous or plastic, and he asked in a paper on "An ancient problem in Rheology" (1932): "Does the flow of marble continue indefinitely or stop, and is it directly proportional to the shearing stress. Here again there is a problem calling for examination by quantitative methods." In order to find the reply to this question we placed marble-prisms horizontally on two supports, and observed their sagging in the course of time, the rate of sagging being greatly increased when the stone-prisms were kept wet. It was then that the creep of reinforced concrete became of importance in structural engineering. Bingham had not heard of it, but I, being a structural engineer by profession, suggested to him an investigation into the creep of cement and cement-mortar stone on similar lines. Reinforced concrete consists of four macroscopic phases: cement, sand, stone, steel, and several microscopic phases in addition, among them water. It is heterogeneous and æolotropic and cannot be considered even as quasi-homogeneous or quasi-isotropic. The æolotropy is introduced through the steel which is inserted into the

* For this compare Chapter XVII.

† *i.e.* 1 part cement, 2 parts sand, 4 parts crushed stone.

concrete in the shape of rods, i.e. of bodies one dimension of which exceeds the other two. Concrete, which is an aggregation of the first three material constituents, i.e. cement, sand and stone, can be considered as quasi-isotropic, but is heterogeneous because of the large size of the stones in the mix. Mortar, which consists of cement and sand only, is quasi-isotropic and can be treated to a first approximation as quasi-homogeneous. But cement itself, after setting, although it will contain water which is not chemically bound, is macroscopically isotropic and homogeneous. We therefore started with the simplest of these mixtures, i.e. hardened neat cement and cement mortar (1 : 3) using the " sagging beam " method. The beams, 2·27 cm square in section, were placed on two supports, 76·1 cm apart and allowed to sag under their own weight. The deflections were determined as functions of time. The curves were all of the same shape, viz. starting as parabolas and then proceeding as roughly straight lines for a limited time (see Fig. XI, 3). The slope of the curve is a measure of the Rate of Creep, and the creep curves, therefore, would show that the rate of creep first falls off rapidly and then for some time remains nearly constant. The decrease in rate of creep in the parabolic part was mainly due to chemical hardening, i.e. the setting of the cement up to an age of about 45 days when following a short curing. In beams allowed to

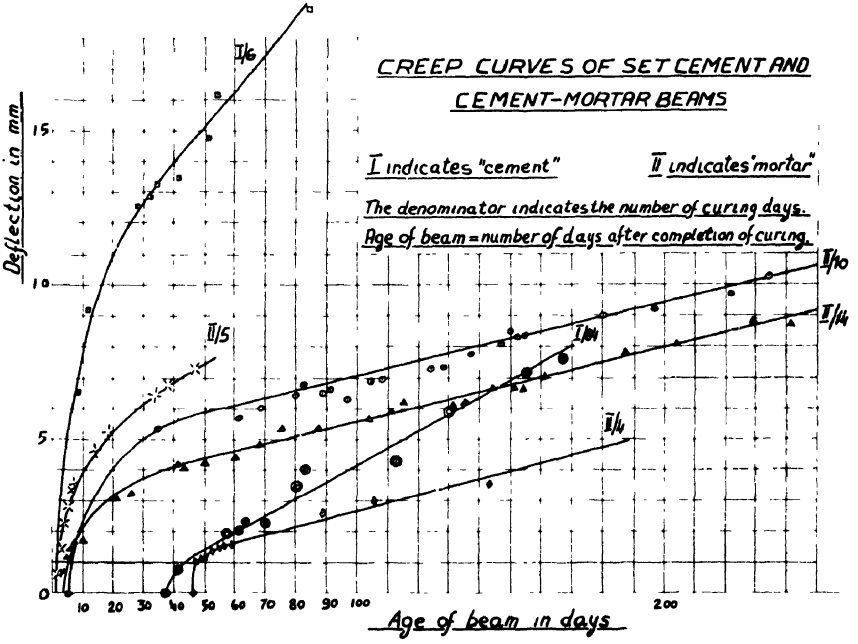


FIG. XI, 3. Deflection of beam versus age.

set or kept wet for curing for a longer period before loading, the parabolic part nearly vanished, being reduced to a small parabolic *start* extending over a few days only.

Our investigations brought out two main results :—

(i) The rate of creep in its approximately constant stage is the same whatever the previous history of the material as to setting and curing ; (ii) The rate of creep of neat cement stone is about twice or more than that of a 1 : 3 cement mortar stone.

However, our investigations were limited in two respects : firstly, there was one load only, i.e. the weight of the beams ; secondly, there were two different mixes only (1 : 0 and 1 : 3). To decide upon the relation between cement and sand in the mechanism of flow, I undertook, about ten years later, a further series of investigations in co-operation with A. Arnstein at the Laboratory for Testing Materials at Tel Aviv (1945).

4. The Creep Viscosities of Different Mortar Mixes

In these investigations four neat cement beams and two beams each of the mortar mixes 1 : $\frac{1}{2}$, 1 : 1, 1 : 2 and 1 : 3 were observed under the action of their own weight and of superposed weights of 460 and 920 g. in the centre, producing bending moments approximately twice and three times the moments produced by their own weight. The dimensions of the beams were the same as of those of Bingham and Reiner.

First of all it was found that the rate of creep under the two superposed loads was roughly twice and three times that under its own weight. It had already been observed by Glanville (1930) that the creep of concrete " can be considered for practical purposes as proportional to the stress ". This would be a characteristic property of viscous flow, in contradistinction to plastic flow. In the latter case, as there is no flow below the yield point and therefore zero flow corresponds to a finite stress, there can be no proportionality. This was also Glanville's conclusion " the movement appears to be . . . in the nature of viscous flow " and Glanville and Thomas' (1939) : " if there is a yield stress for concrete, its value is negligible ".

The creep curves are shown in Fig. 4 on a semi-log scale. This scale has the advantage that prolonged times can be plotted on convenient lengths of abscissa ; it has the disadvantage that a straight line of constant slope appears as curved. The parabolas up to the age of 60 days are not shown. The creep under superposed loads is proportionately reduced to creep under the beam weight. The curves seemed to confirm that there exists a roughly straight line portion as shown by Bingham and Reiner. It is followed by a curve of gradually diminishing slope as shown by Glanville.

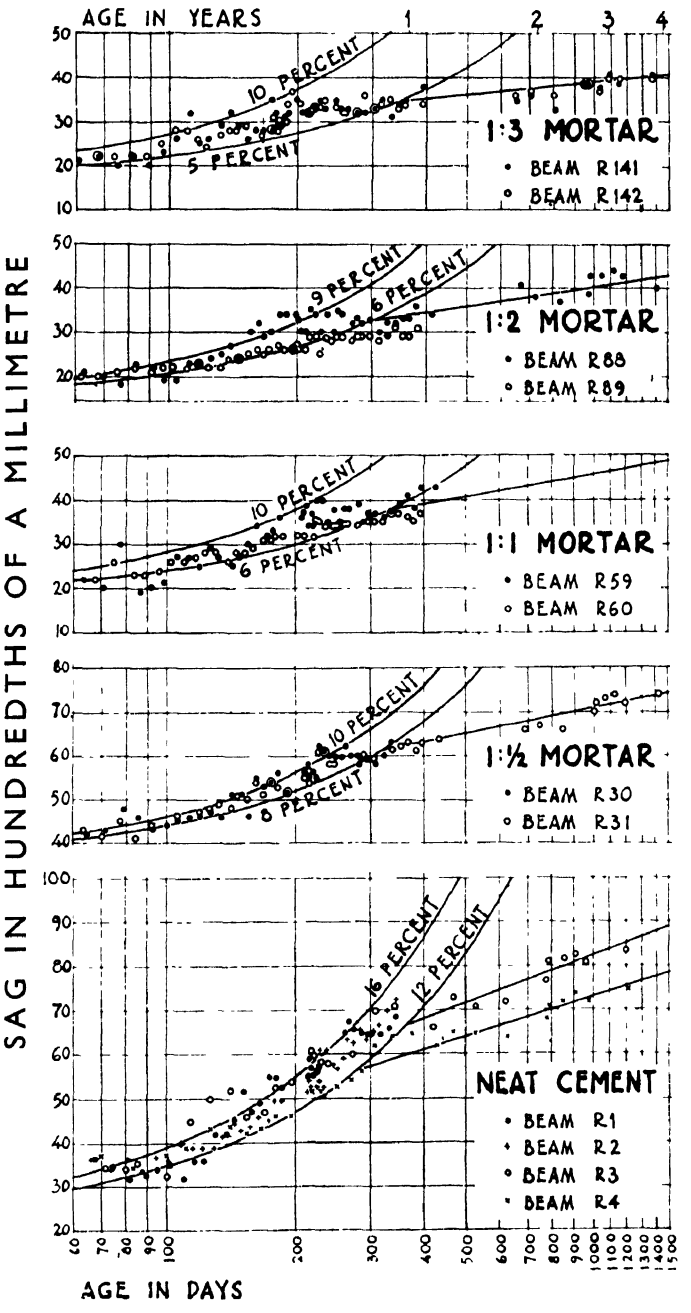


FIG. XI, 4. Creep curves of cement and mortars. Deflections plotted against log of time.

In Table I the calculated creep viscosities* have been entered in columns 13-15. Glanville observed compression in cylinders and deflection in beams of the type to which (IV, f) refers ; Glanville and Thomas observed also extensions in cylinders. The materials were concretes of different mixes, different cements, and mortars. The viscosities change with age and the age of 60 days was selected as the greatest for which observations were available for all different materials and as one which is past the first parabolic part of the creep curve during which the setting of the cement is proceeding at a high rate.†

In the present case also, as in the case of Bingham and Reiner's beams, the rate of creep of neat cement was over double that of the 1 : 3 mortar. This was the more remarkable as the rate of creep of the beams prepared in Easton from the American " Atlas " cement was about four times the rate of creep of the cement beams prepared in Tel Aviv from " Nesher " cement. At the time when I wrote the first edition of this book (1948) I thought that one conclusion only could be drawn as expressed by Thomas (1939), viz, : " Concrete [and mortar, M.R.] is considered as comprising two parts [or phases M.R.] : (i) the cementitious material which behaves in a viscous manner when loaded, and (ii) inert aggregate, which does not flow (but moves) under load." On this view (*a*) the rate of creep of a mortar depends *ceteris paribus*, upon the nature of the cement, but not the nature of the sand ; (*b*) the increase in creep viscosity of the mortar over the neat cement is due to the sand taking up space which it immobilizes. I summed this up by saying *it is the cement which creeps*, and by embedding rigid particles into it to make mortar, the resistance to creep is naturally increased. I then went on :—

5. The Application of Einstein's Viscosity Equation to the Creep of Mortar

But if this is so, then the mortar is in its rheological behaviour essentially not different from any other suspension of solid particles in a liquid. We can therefore try to apply the generalized Einstein equation ‡

$$\eta_{sper} = \eta_i c_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and determine how the intrinsic viscosity η_i depends upon the volume-concentration c_v in the case of cement mortar concretes.

* On the assumption that the creep is of the nature of viscous flow.

† The rate of creep had to be estimated from the slope of the tangent to the creep curves. This procedure does not permit of great accuracy, and in columns 9–11 the whole range of estimated slopes has been entered, and in columns 13–15 corresponding ranges of viscosities.

‡ About this compare Chapter XIV. The reader may pass over Sections 5 to 8 and return later, after having read Chapter XIV.

TABLE I
CALCULATION OF CREEP VISCOSITIES

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	SOURCE		FROM PAPERS						FROM CURVES							(5)/3·15	(6)/1	(7)/2·65	(16)+(17)+(18)	(18)/(19)	(17)/(16)		(22)/(21)×H	EQU (10)	(14)/(24)	(25)×I	(26)/(20)	EQUS. (11), (14)
2	AUTHORS	DESIGNATION OF TEST PIECE	BRAND OF CEMENT	MIX BY WEIGHT NOMINAL	MIX BY WEIGHT CEMENT	MIX BY WEIGHT WATER	MIX BY WEIGHT AGGREGATES	KIND OF TEST	RATE OF CREEP AT AGE OF 60 DAYS MIN	RATE OF CREEP AT AGE OF 60 DAYS MAX	MEASURE	LOAD	CREEP VISCOSITY IN 10^{16} POISES MIN	CREEP VISCOSITY IN 10^{16} POISES MEAN	CREEP VISCOSITY IN 10^{16} POISES MAX	MIX BY VOLUME CEMENT	MIX BY VOLUME WATER	MIX BY VOLUME AGGREGATE	MIX BY VOLUME TOTAL	VOLUME CONCENTRATION OF AGGR Cv	VOLUME W/C RATIO	EXCESS WATER IN CEMENT BASE	VOLUME CONCENTRATION OF EXCESS WATER Cw	MEAN VISCOSITY OF CEMENT BASE 10^{-16}	MEAN RELATIVE VISCOSITY η	MEAN SPECIFIC VISCOSITY η−1	α FROM OBSERVATIONS	10^{16} CALCULATED
3	ARNSTEIN & REINER	R1, 2 3, 4	PORTLAND CEMENT NESHER	1 0	1000	250	—	BENDING	12	16	DEFLECTION IN 10^{3} DAYS CENTIMETERS	OWN WEIGHT OF BEAM	37	43	49	317	250	—	567	0	0 79	0	0	43	1	0	(2·5)	43
4		R30, 31		1 ½	800	200	400		8	10			59	67	74	254	200	151	605	0 25	0 79	0	0	43	1 56	0 56	2·24	70
5		R59, 60		1 1	600	150	600		6	10			59	78	98	190	150	226	566	0 40	0 79	0	0	43	1 82	0 62	2 05	86
6		R88 89		1 2	400	120	800		6	9			65	82	98	127	120	302	549	0 55	0 95	0 16	0 08	40	2 05	1 05	1 90	95
7		R141 142		1 3	300	100	900		5	10			59	88	118	95	100	339	534	0 63	1 05	0·26	0 13	37	2 38	1·38	2 19	95
8	BINGHAM & REINER	3/84	ATLAS	1 0	1000	240	—							111		317	240	—	557	0	0 76	0	0	111	1	0	(2·5)	111
9		1/14 4/10 6, 4		1 3	300	126	900							26 6		95	126	339	560	0·61	1 33	0·57	0 24	8 4	3 17	2·17	3 56	21 3
10	GLANVILLE		BRITISH	1 0	1000	220	—		4 8	5 5	INS	15 LBS	18 6	20	21 3	317	220	—	537	0	0 69	0	0	20	1	0	(2 5)	20
11		Nº 30		1 1 2	1000	400	3000	COMPRESSION	5 6	8 0	MOVEMENT PER UNIT LENGTH IN 10^{8} INCHES	1 LB PER SQ IN	75	90	106	317	400	1130	1847	0 61	1 26	0 57	0 25	15	6 00	5 00	8 2	85
12				1 2 4	1000	700	6000		16 6	21 0			28	32	36	317	700	2260	3277	0 69	2 21	1 52	0 47	10 6	3 02	2 02	2 9	36
13				1 3 6	1000	850	9000		30	45			13 2	16 6	19 9	317	850	3390	4557	0 74	2 69	2·00	0 54	9 2	1 80	0 80	1·08	14·5
14	GL & THOMAS		RAPID HARDENING	1 1	1000	7	1000	BG	13	16	DEFLECTION IN	24 LBS	99	110	121	317	220?	377	914	0 41	0·69	0	0	54 4	2 03	1·03	2 5	—
15	GL	RH P28		1 2 4	1000	700	6000	CH	4 2	5 6	MOVEMENT IN 10^{-3} INCHES	LB/IN²	106	124	142	317	700	2260	3277	0 69	2 21	1·52	0 47	28·8	—	—	—	98
16				1 0			—							54·4				—		0		0	0	54 4	1	0		

In the present case, c_v is the volume concentration of the aggregate in the mix. This can be calculated from the given weight concentrations if the specific weights of the ingredients making up the mortar are known. We may assume for the latter the values shown in Table II.

TABLE II

Material	Water	Cement	Aggregates
Specific Weight in g/cm³	1	3·15	2 65

The recorded mixes by weight are shown in columns 5, 6, 7 of Table I and the calculated mixes by volume in columns 16, 17, 18. The volume concentration of the aggregates in the mix, c_v, was calculated from the volume of the aggregates as shown in column 18, and the total as added up in column 19, and it was entered in column 20. In Fig. XI, 5, the creep viscosities η_c from columns 13–15 have been

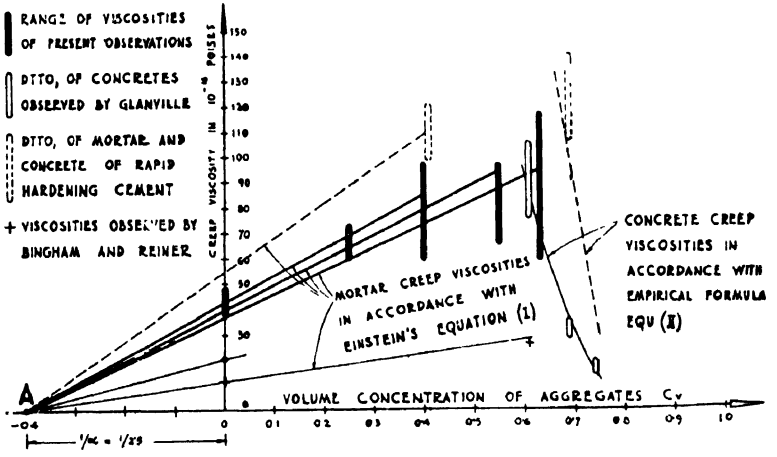


FIG. XI, 5. Creep viscosities plotted against volume concentration of aggregates.

plotted against the volume concentrations c_v of the aggregates as shown in column 20. A remarkable fact results : all observed points for the mortars lie in a first approximation on straight lines radiating from a point (A) which lies at the distance of 0·4 from the origin. This distance is equal to 1/2·5 and, therefore, η_i is in a first approximation a constant and equal to 2·5. In a second approximation the points drop below the lines ; the more so the leaner the mix. One reason for this becomes evident from an examination of column 21 of Table I. The water-cement ratio is not the same for all mixes. On the contrary, it

increases for the leaner mixes. However, in the same way as the sand *increases* the viscosity by immobilising volume, water (with a viscosity negligible in comparison to the viscosity of the cement stone) *decreases* the viscosity. Denoting by η_o the viscosity of the neat cement stone, the viscosity of any other cement " base " in a mortar ($\bar{\eta}_o$) will be, in accordance with Guth and Mark (1933)

$$\bar{\eta}_o = \eta_o(1 - c_w) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where c_w is the volume concentration of water which the cement base contains *in addition to the water of the neat cement stone.* E.g. the neat cement stone in our case was composed of 317 parts cement and 250 parts water by volume, i.e. in a volume water-cement ratio of 0·79. In the 1 : 3 mortar, the volume water-cement ratio was 1·05 (compare column 21, row 7 of Table I). On 100 parts of cement there were, therefore, in the first case 79 parts of water and in the second case 105 parts of water, or 26 parts in excess. The total volume of cement and water in the mortar is 205 parts, and 26 parts, therefore, correspond to a volume concentration of excess-water $c_w = 26/205 = 0{\cdot}13$. The values of c_w have been calculated accordingly and entered in column 23. From these the viscosities η_o of the cement base were calculated and entered in column 24. When these viscosities are plotted on the vertical axis of Fig. XI, 5, and connected with the point A, the observed mortar viscosities lie *well* on the straight lines radiating from A.

6. A Formula for the Creep-Viscosity of Cement Mortars

On the basis of the foregoing, the creep viscosity of a cement mortar η_c can be predicted if the creep viscosity of the neat cement η_o used in the mix is known.

We should have

$$\eta_c = \eta_o \frac{1 + \overline{WC}}{1 + WC}(1 + 2{\cdot}5c_v) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where WC is the volume water-cement ratio of the neat cement, $\overline{WC}$ is that of the cement base used in the mortar and c_v the volume concentration of the sand in the mortar. Equation (6) results from (5) through

$$c_w = \frac{\overline{WC} - WC}{1 + WC} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

7. An Empirical Formula for the Creep-Viscosity of Concrete

Equation (6) does not hold good for concretes. First of all, the relative creep viscosity of the 1 : 1 : 2 concrete, which has a concentration of inert material equal to that of a 1 : 3 mortar ($c_v = 0{\cdot}61$) is much higher than that of the mortar. The reason may well be that besides

cement, water and aggregates, there is always another material present in the mix, viz. *air*, filling voids. The viscosity of air is negligible and its volume should be added to the volume of the excess water. It may be assumed that the leaner the mix the more voids in the mortar or concrete. At the same time, however, it will be seen that a 1 : 1 : 2 concrete, because of the better grading of the aggregate, will have less pore space than a 1 : 3 mortar. Therefore, the relative viscosity of the former should be greater—and so it is.

The values for η_i were calculated for concretes from (4), where η_{spec} and c_v were introduced from columns 26 and 20 respectively. They can be approximated by a straight line

$$\eta_i = 40 - 53c_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $0{\cdot}60 < c_v < 0{\cdot}75$. This gives for concrete

$$\eta_c = \eta_o \frac{1 + WC}{1 + WC}(1 + 40c_v - 53c_v^2) \quad . \quad . \quad . \quad (9)$$

In order to check (6) and (9), the creep viscosities of the mortars and concretes were calculated from these equations, using the values for c_v from column 20 and for c_w from column 23. The results were entered in column 28. As can be seen, the viscosities calculated from the equation fall within the ranges of the observed viscosities.

8. Dissimilarity of Creep-Behaviour of Mortars and Concretes

The dissimilarity of the behaviour of mortar and concretes—the first having their rate of creep decreased with increased quantity of aggregate, the second, on the contrary, having the rate of creep increased can be understood on the analogy of the behaviour of a sand-water mixture. When the sand is dry, its cohesion is very small ; its internal friction, as measured by the angle of repose, is also comparatively small. When we add a little water its cohesion is increased and there is an optimum water content, which we used when as boys we built castles on the beach. Still more water reduces the cohesion and 100 per cent. of water, of course, flows freely. The complete curve will, therefore pass through the zero for $c_w = 1$ and $c_w = 0$ where c_w is the volume concentration of water. When c_w decreases from *1* to *0* the curve will go up at the nearer end and go down at the farther end. The role of water is played in our case by the cement. When the aggregates are on the increase and the cement on the decrease, the mortars lie on the rising part, the concretes on the descending part of the curve.

9. The Permanent Set

We have not yet explained the nature of the " permanent set " which we mentioned in Section 2. This is a deformation which appears immediately on loading—in contradistinction to the elastic after-effect

—and which is not recovered—in contradistinction to the instantaneous or acoustical elasticity. The appearance of a permanent set on deforming a body can be understood from the fact that no body is perfectly homogeneous and that all possess pores or holes and also inclusions of foreign material. To these imperfections should be added sharp local depressions on the surfaces, scratches and the like, forming notches. A stressed body will have stress-concentrations at these places. When considering the surface of a body these concentrations are spoken of as " notch effects ", but the stress concentrations at holes and inclusions are not essentially different.

These flaws form weak spots and, while the body as a whole may be strong enough to resist the forces causing the deformation, at these irregularly interspersed imperfections the strength of the material may be exceeded.

In the case of a plastic material this will cause local plastic deformations ; in the case of brittle material local rupture. When the body is released from the external forces, *the aggregate of these minute plastic deformations or ruptures will show itself as a permanent set.* The stress concentration is the more marked the sharper the angle at the notch or the smaller the smallest radius of curvature at the hole. A local plastic deformation will tend to flatten the angle or to increase the radius of the curvature and a second loading of the same extent may therefore be more nearly perfectly elastic. This is well known in many metals.

A local rupture, however, may make the angle even sharper and produce sharp re-entrant angles starting from holes. In this case second and further loadings will again produce permanent sets and after many such repeated local breakages the rupture may spread across the whole body and the body breaks. Such rupture of a material after repeated loading and unloading is called Fatigue. The nature of fatigue which is the *failure under repeated cycles of stress or strain* is not yet fully understood. Fatigue occurs not only in brittle materials but also, and predominantly, in plastic metals. It should, of course, be kept in mind that in metals the local plastic deformation must raise the local yield point by causing local strain-hardening and therefore brittleness. It may also be assumed that apart from imperfections of structure of microscopic size there should also be sub-microscopic variations of cohesional bonds (atomic or molecular) of the material, and some may break under comparatively small stresses. This will cause a redistribution of stresses, increasing the latter beyond the average and in turn bringing about other breakages of cohesional bonds. Conditions of fatigue would accordingly be governed by principles of statistics. It appears that whatever laws of fatigue have been established until now, can simply be explained from such statistical considerations

(Freudenthal (1946)), but the *mechanism* of fatigue has still to be discovered.

It is, however, clear that no direct relation will exist between fatigue and damping of oscillations of the material in question. Whatever causes dissipation of energy will cause damping of oscillations. We have seen in Chapter X that there may be damping without any change in the structure of the material and therefore fatigue can only be one of the contributory causes of damping.

10. Concrete a Solid, not a Liquid

In the preceding Sections I have mainly repeated (with some amendments) what I said on the subject of the creep of concrete ten years ago when I wrote the text of the first edition. I have quoted Straub and Glanville and Thomas, which shows that I was not alone in my view, " concrete a liquid, not a solid ", as I wrote then in the heading of the first section of this chapter. True, as I said before, the difference between a solid and a liquid is only one of degree and on the geological scale concrete will show viscous flow. The question, however, was, does concrete flow on a human, historical scale ? I was mistaken in replying " yes ". I am now correcting my reply, in any case with regard to the *cement*, the binding material in cement-concrete.

I came to realize that my experiments, as shown in Fig. XI, 3, did not continue long enough. They continued for about eight months. The experiments recorded in Fig. XI, 4 extended to four years, and I observed that the slope " gradually diminished ", as also observed by Glanville. But we both thought that this was an effect of chemical hardening. This would have meant that the cement and the cement-mortar beams were in fact creeping in viscous flow with the viscosity increasing in time until ultimately—after many, many years—it would approach very great, practically infinite, magnitude, when these stones could be considered as solids.

But was this a correct view ? L'Hermite, whose thorough survey (1957) I recommend to every student of the subject, was one of those who voiced the hope " de voir les constructions en béton rester « solids » au sens que nous attribuons a ce terme " * I started before that year through my assistant Glucklich an investigation which was destined to change radically my view on the subject. I confess that I had made a serious mistake, but I take heart from the different mistakes of great men of science which I mention elsewhere in this book, and I can draw consolation from the fact that I am myself now correcting my mistake.

Dr. Glucklich's paper is awaiting publication. Fig. XI, 6 shows

* That concrete is a solid and the creep of concrete is delayed elasticity was emphasized by McHenry as early as 1943. I must, however, point out that both Glucklich's and McHenry's observations were made on specimens completely protected from atmospheric influences.

the essential features of his results. Using beams similar to those used before by me, he conceived an arrangement by which to load them *horizontally* so that *all load* could be removed. The continuous curve is the time-dependent creep for a load equivalent to 4 kg concentrated at the centre. We notice the " practically straight " part at an age of up to eight months. However, the cyclic curve shows that this creep is not a slow viscous flow, but a delayed-elasticity effect combined with non-recoverable creep which stops after four to five months.

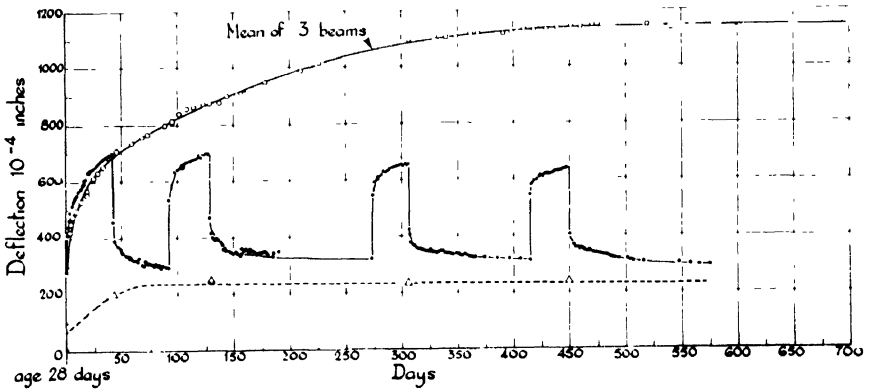


FIG. XI. 6. Deformation versus time of neat cement-stone beams. The continuous curve is the result of continuous loading. The cyclic curve is the result of repeated loadings and unloadings. The broken line represents asymptotic values of non-recoverable deformations, i.e. permanent set and secondary creep.

It may be mentioned that other similar curves for other loads showed proportionality to load in all aspects. We are continuing the experiments with mortar beams.

The results described in Sections 5 above which were so difficult to grasp, become now understandable. The experiments of Bancelin (1911) and Oden (1913) provided a fair proof of Einstein's viscosity equation for spherical particles and concentrations of not more than 3 per cent. In my mortar investigations the particles were fairly spherical, but the concentrations unbelievably high. But now we can see that this is not the case of a disturbance in the flow of a viscous liquid by the interference from a suspended particle. We must possibly give up the picture of the Burgers-body for the representation of the creep of cement-mortar (or concrete) and adopt rather a model such as has been proposed by Torroja and Paez in Chapter VIII of Reiner (Editor, 1954), to which the reader is referred. The sand-particles suspended in the cement-medium thus interfere with an elastic response —and such interference is of an entirely different character, possibly of much narrower range than in Einstein's case.

11. Summary

A liquid flows under any distortional stress, however small, while a solid flows plastically if the yield stress is exceeded. Seemingly solid materials such as pitch, bitumen, concrete, glass and metals may show an exceedingly slow continuous deformation . this is called creep. That permanent deformation of a body, which is neither plastic nor due to creep, is called permanent set, i.e. the aggregate of minute local microscopic failures of the material caused by random stress concentrations. Accumulating permanent sets may cause fatigue or rupture. The latter may also be due to submicroscopic variations of cohesional bonds and local breakages.

The creep viscosity η_c of cement mortar can to a first approximation be calculated from the following modification of Einstein's equation

$$\eta_c = \eta_o \frac{1 + WC}{1 + WC'} (1 + 2{\cdot}5c_v) \quad . \quad . \quad . \quad \text{(IX, a)}$$

where WC is the volume-water-cement ratio of the neat cement of viscosity η_o and WC' that of the cement base used in the mortar.

For concrete the following empirical formula can be used

$$\eta_c = \eta_o \frac{1 + WC}{1 + WC'} (1 + 10c_v - 53c_v^2) \quad . \quad . \quad \text{(IX, b)}$$

The creep of concrete is mainly an elastic after-effect, while the creep of bitumen is mainly viscous flow.

CHAPTER XII

VOLUME RHEOLOGY

1. Volumetric Plastic Deformation

As expressed in what we called the first axiom of rheology, classical mechanics of continua postulates that under isotropic pressure all materials, solids and liquids alike, behave in the same highly degenerate manner, namely as perfectly elastic solids with the (elastic) bulk modulus κ as the only rheological parameter in accordance with Equ. (VII.1), which we repeat here

$$-p = \kappa e_v \qquad \ldots\ldots\ldots \quad (1)$$

Rheology has on the whole taken a similar view, concentrating upon the investigation of distortion and identifying flow with a distortion proceeding in time. This view is too narrow. More detailed observations have shown that while the different rheological properties are more spectacular in distortion, *they are all also present in volumetric deformation.* This has already before forced us to make amendments. In Section 9 of Chapter XI, we have explained the " permanent set " as a kind of plastic deformation which will become manifest also under an isotropic pressure and tension, and will then constitute a case of *volume-plasticity* and *volumetric strength.* Hencky (1924), whose condition for plastic flow we have taken up and explained in Chapter VI, expressed the view that " it is clear that a hydrostatic pressure or tension can have no influence on plastic flow. If tension experiments simulate such an effect it must be due to the disturbance produced by the invisible rupture phenomena ". The allusion in the second sentence is to materials having holes or voids, which may fail locally through crushing around the holes where there are stress-concentrations. But this second sentence contradicts to a certain extent the first. It is true that the permanent set is not a case of plastic *flow*, as it appears practically instantaneously on loading. But it is nevertheless a case of plastic volumetric deformation, similar to plastic deformation in the work-hardening range which also is not a flow.* It is present in concrete, certain soils, foamed lead and other materials. But even in " homogeneous " materials in which the presence of voids would not be suspected, there might be produced by very high pressure a permanent set. Lord Kelvin remarked in 1878, that " by compression between the dies used in coining, the density of gold may be raised

* See Chapter XX.

from 19·258 to 19·367, and the density of copper from 8·535 to 8·916 ". We can connect this volumetric plastic deformation with a *volumetric plastic strength* ϑ_v, defining a yield point. However, Macey (1954) has pointed out that there may be a plastic deformation without a yield point. This is connected with the fact that even a very small overall stress may produce great stress concentrations at certain points of the body, and therefore a small permanent set, which will form an element of a gradually increasing permanent deformation accompanying an increasing stress. This kind of permanent deformation will, however, appear only in a first loading, because if a second loading does not exceed the magnitude of the first, there will usually be no appreciable further set ; the elastic properties of such materials (including metals) are therefore improved by loading.

2. Solid Volume Viscosity

That the elastic straining of a real material must be accompanied by a viscous resistance is obvious from the Second Law of Thermodynamics. As we have shown in Section 4 of Chapter V. we thus arrive at the notion of a *solid volume viscosity* ζ_s.

It is the viscosity introduced by Stokes in the derivation of his equation of a viscous liquid but dropped at an early stage of his reasoning, a procedure followed in most textbooks of hydrodynamics by tacitly putting $\zeta_s = 0$ from the beginning. It has, however, become evident, mostly through observations on the damping of ultrasonic waves, that ζ_s must be quite large. and Karim and Rosenhead (1952) have reviewed the reasons for the existence of a finite " second " coefficient of viscosity in liquids and gases.

Taking this solid volume viscosity into account, we amend Equ. (1) to read

$$-p = \kappa e_v + \zeta_s \dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation is analogous to Equ. (IX, g). We can, therefore, consider volumetric behaviour of a material in this approximation as one of a Kelvin-body with all properties attached to it, such as elastic fore- and after-effects and dissipation of stress-work.

In liquids κe_v is called the *hydrostatic* pressure. It is, as follows from (2) not identical with the isotropic component of the stress.

The stresspower per unit volume is

$$\dot{w} = -p\dot{e}_v = \kappa e_v \dot{e}_v + \zeta_s \dot{e}_v^2 \quad . \quad . \quad . \quad . \quad (3)$$

The first term on the right-hand side reverses its sign when e_v decreases and $\dot{e}_v$ is negative. On de-straining, this work is therefore regained and the first term accordingly constitutes the conserved part of the stresswork. The second term is always positive, whether e_v

increases (positive $\dot{e}_i$) or decreases (negative $\dot{e}_i$). On de-straining as well as on straining, work is expended and the second term accordingly constitutes the dissipated part of the stresswork. The total stresswork is

$$w = \int_0^t \dot{w}\, dt = \int_0^{e_v} \kappa e_v\, de_v + \int_0^t \zeta_s \dot{e}_v^2 dt \quad . \quad . \quad . \quad . \quad (4)$$

Going over from Equ. (1) to Equ. (2) requires also an amendment to λ, as expressed by Equ. (VII, 25). We have written there λ^*, in order to indicate that ζ_s is assumed to vanish. If this is not the case we proceed as follows :—

Let us go back to Section 2 of Chapter V. We there considered Trouton's experiment with the simple pull σ_n, which has an isotropic component $\sigma_m = \sigma_n/3$, and we have said that when the pull is applied there will be an initial stage when the material expands. When e_v has so much increased that $\kappa e_v = \sigma_n/3$, the elastic isotropic reaction balances the isotropic component σ_m of the pull σ_n. The cubical dilatation then ceases to increase, and $\dot{e}_i$ therefore vanishes. Then the second stage sets in, in which the movement is steady.

Now, let us assume that the rate of deformation of the prism or the velocities of its particles are so small that their kinetic energy may be neglected : in which case there will be no pulsations but only an aperiodic movement leading to the second stage, when the movement becomes steady.

For the isotropic component of the traction we have accordingly

$$\sigma_n/3 = \kappa e_v + \zeta_s \dot{e}_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

furthermore from Equ (V, 7)

$$\sigma_n/3 = \eta(\dot{e}_n - \dot{e}_i/3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and therefore

$$\dot{e}_i = 3\dot{e}_n - \sigma_n/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Introducing this expression for $\dot{e}_i$ into (5) we find

$$\sigma_n/3 = \kappa e_v + \zeta_s(3\dot{e}_n - \sigma_n/\eta) \quad . \quad . \quad . \quad . \quad (8)$$

or

$$\dot{e}_n = (\sigma_n/3 - \kappa e_v)/3\zeta_s + \sigma_n/3\eta \quad . \quad . \quad . \quad . \quad (9)$$

The coefficient of viscous traction λ is defined by (V, c) in accordance with which we find

$$\lambda^{-1} = \dot{e}_n/\sigma_n = (1 - 3\kappa e_v/\sigma)/9\zeta_s + 1/3\eta \quad . \quad . \quad . \quad (10)$$

or

$$\lambda = \left(9\eta \frac{\zeta_s}{1 - 3\kappa e_v/\sigma_n}\right) \Big/ \left(\eta + \frac{3\zeta_s}{1 - 3\kappa e_v/\sigma_n}\right) \quad . \quad . \quad (11)$$

Equ. (11) is the viscous analogy to the first of (54) of Section III, 15, namely $E = \dfrac{9\mu\kappa}{\mu + 3\kappa}$. The analogy is, however, *not exact.*

While $E = 3\mu$ for $\kappa = \infty$ and *only* in this case, the situation is very different in the case of λ.

In interpreting (11) one should keep in mind that $1 - 3\kappa e_v/\sigma$ is either >0 in the first stage of Trouton's experiment, or $= 0$ in the second stage. For the volume viscosity we have $0 \leqq \zeta_s \leqq \infty$. It is clear that the volume viscosity cannot be negative as " otherwise the more alternate expansion and compression, alike in all directions, of a fluid, instead of demanding the exertion of work upon it, would cause it to give work out " (Stokes). As can be seen, if ζ_s vanishes, λ also vanishes in the first stage. On the other hand $\lambda/\eta = 3$ in both stages if $\zeta_s = \infty$, and also in the second stage when $1 - 3\kappa e_v/\sigma_n = 0$, whatever the magnitude of ζ_s. Therefore $0 \leqq \lambda \leqq 3\eta$ with $\lambda^* = 3\eta$ in the second stage only. However, note that if $\zeta_s = 0$, the second stage sets in *at once* and if $\zeta_s = \infty$, never.

3. Liquid Volume Viscosity

In Section 4 of Chapter V, we have also mentioned that, should there exist a volume flow, this would be accompanied by a liquid volume viscosity ζ_l. In the following we shall treat in detail two cases where volume flow is present. Before doing so we need to clear up a relevant term.

The first axiom sometimes finds its quantitative expression by assuming that in cases of flow the " Poisson-ratio " is $\frac{1}{2}$.

The term Poisson-ratio is thus used for the ratio of lateral contraction to longitudinal extension, independent of whether these are elastic recoverable strains or the result of continuously increasing deformations or liquid flow. There is no harm in such a generalization if it is done (i) consciously with the realization that these are physically different quantities, (ii) if different symbols are introduced for the different quantities, and (iii) if it is realized that they are independent quantities and one (e.g. the elastic) may have a value of, say, 0·3, while the other (e.g. plastic) may be equal to 0·5. We may call all these different quantities Poisson-ratios, but shall indicate the nature of each one by a subscript, using l for liquid, pl for plastic and d for deformational, while ν simply would be the ordinary elastic ratio.

To separate volume-flow from shear-flow we now have to calculate the appropriate " Poisson-ratio ". This can be expressed in two different ways. If we consider the *total deformation* attained at any time, we may define a deformational ratio by

$$d_c = -\nu_d d_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If we, however, consider the varying *rate of deformation* or the flow, we have

$$\dot{d}_c = -\nu_l \dot{d}_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$\dot{d}_l$ being the rate of longitudinal, $\dot{d}_c$ the rate of lateral flow. Differentiating both sides of (12) gives

$$\dot{d}_c = -\nu_d \dot{d}_l - \dot{\nu}_d d_l \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and therefore

$$\nu_l = \nu_d + \dot{\nu}_d \frac{d_l}{\dot{d}_l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The liquid viscous ratio (ν_l) is accordingly identical with the deformational ratio only when the latter is constant. Otherwise, considering

$$\frac{\dot{d}_l}{d_l} = \frac{d\dot{d}_l}{dt}/d_l = \frac{d}{dt}(\ln d_l) \quad . \quad . \quad . \quad . \quad . \quad (16)$$

we have

$$\nu_l = \nu_d + \dot{\nu}_d \left[\frac{d}{dt}(\ln d_l)\right] - 1 \quad . \quad . \quad . \quad . \quad (17)$$

Liquid volume viscosity ζ_l is the correct viscous analogy to the bulk modulus κ, with ν_l the analogy to ν and λ to E. Therefore in accordance with Equs. (III, u), (III, v) and (III, w)

$$\zeta_l = \lambda/3(1 - 2\nu_l) \; ; \; \eta = \lambda/2(1 + \nu_l) \quad . \quad . \quad . \quad (18)$$

with $\zeta_l = \infty$ and $\lambda = 3\eta$ for $\nu_l = \frac{1}{2}$.

4. Positive Volume Flow in Asphalt

When using a constant-load tensile test to investigate the rupture of certain types of asphalt under tension, Lee, Reiner and Rigden (1946) found that the different phenomena observed could be best correlated and explained by assuming that a steady volume dilatation is proceeding in time. In the course of this dilatation, the voids in the asphalt increase progressively, and a weakening of the structure is thereby produced which ultimately leads to rupture.* This increase in volume proceeding in time may be regarded as a positive volume flow.

The volume flow of asphalt was then systematically investigated by Reiner, Rigden and Thrower (1950). The asphalt was a mixture of bitumen and limestone dust in the ratio of 14 to 86 by weight. (This, therefore, was not an asphalt mixture as normally used for road construction.)

Three specimens were made in the form of rectangular prisms. Steel boxes were cemented over each end of the specimen leaving different exposed lengths. Over this gauge length the cross-section of the specimens was then reduced to about 3 cm by cutting away excess material. One of these boxes was held fixed, while the load was applied, *via* a pulley to the other (Fig. 1). Elongations and reductions of width were measured by suitably applied dial gauges. The specimens were

* This kind of rupture is mentioned in Section 7 of Chapter VI and shown in Fig. VI, 8.

loaded with a load corresponding to a tensile traction of 100 lb/sq. in. The temperature was 15° C. Readings were taken every minute. The load was taken off after six to ten minutes, before cracks appeared, and readings were continued for some time. It was observed that there was an appreciable recovery of the deformation, including a recovery of the cubical dilatation. It was found that the rheological behaviour could be represented by two L-models as shown in Fig. X, 1 on the left, one for shear and the other for dilatation. For each one there is a viscosity associated with viscous flow. In shear, this viscosity is the ordinary Newtonian viscosity η. In cubical dilatation it is the liquid volume viscosity ζ_l.

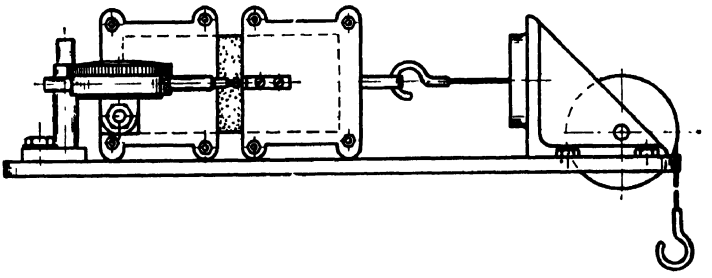


FIG. XII, 1.

The results of one experiment are given in Table I. Column 1 shows the time of observation in minutes, with $t = 0$ as time of loading and $t = 10$ as time of the removal of the load. $t = \infty$ was actually about 24 hours later, and it was assumed that by then all elastic recovery had taken place. The deformations being large, they were calculated in the logarithmic measure of Equ. (III, 31), i.e.

$$D_l = \ln (l/l_o) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and entered in column 2. The contractions D_c were similarly calculated and entered in column 3. The isotropic deformation or cubical dilatation was obtained in accordance with Equ. (III, 37) by

$$D_v = D_l - 2D_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and entered in column 4. The deformational Poisson-ratio ν_d as defined by Equ. (12) is shown in column 5.

These listed values are shown graphically in Fig. 2. Both linear and cubical dilatations start as parabolas, but after about five minutes become approximately straight lines. After removal of the load, the elastic after-effects are exhibited as exponential-type curves, albeit not as true exponential curves. Plotting ν_d shows a slight decrease with increasing deformation.

TABLE I.

Table of results for a specimen of gauge length 1·42 cm., of cross-section 3·25 × 3·17 cm., stress 6·90 × 10^6 dynes cm^2.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
t	D_t	D_t	D_t	ν_d	$_re_l$	$_re_l$	$_rD_l$	$_re_e$	$_re_t$	$_rD_t$	e_t	$_tD_t$	ν_l
Time (min)	Extension	Contraction	Isotropic deformation	Deformational Poisson's ratio	Longitudinal elastic recovery	Longitudinal elastic fore-effects	Longitudinal viscous deformation	Lateral elastic recovery	Lateral elastic fore-effects	Lateral viscous deformation	Elastic isotropic deformation	Viscous isotropic deformation	Viscous Poisson's ratio
0	0	0	0	0		0	0		0	0	0	0	0
1	0·0494	0 0045	0 0405	0 179		0 0164	0 0330		0 0014	0 0031	0·0137	0 0268	0·094
2	0·0712	0·0062	0·0589	0·188		0·0247	0 0465		0 0026	0 0036	0 0196	0·0393	0 078
3	0 0843	0 0075	0 0695	0 187		0·0282	0 0561		0 0032	0·0043	0 0219	0·0476	0·077
4	0·0950	0 0083	0 0784	0 180		0 0311	0 0639		0·0032	0·0051	0·0247	0·0537	0·080
5	0·1027	0 0091	0 0845	0 180		0 0341	0 0686		0 0032	0 0059	0·0277	0·0568	0 086
6	0·1103	0·0098	0 0907	0·178		0 0372	0 0731		0 0033	0 0065	0·0306	0·0601	0 089
7	0·1172	0·0101	0 0970	0 172		0 0389	0·0783		0 0033	0 0068	0 0323	0 0647	0·087
8	0·1239	0 0106	0 1028	0 169		0 0414	0 0825		0·0034	0 0072	0 0347	0 0681	0·087
9	0·1304	0·0111	0·1085	0 168		0 0428	0 0876		0 0034	0 0077	0 0361	0·0724	0 088
10	0·1360	0 0115	0 1131	0·165		0·0441	0 0919		0 0034	0 0081	0 0374	0·0757	0·088
11	0·1263	0·0107	0 1050		0 0097			0 0008					
12	0 1214	0 0100	0·1015		0 0146			0 0015					
13	0 1193	0·0096	0·1001		0 0167			0 0019					
14	0·1176	0·0096	0·0985		0 0184			0·0019					
15	0 1158	0 0096	0·0967		0 0202			0 0019					
16	0·1140	0·0095	0·0960		0·0220			0 0020					
17	0·1130	0 0095	0 0940		0 0230			0 0020					
18	0·1115	0·0095	0 0924		0 0245			0 0020					
19	0·1107	0·0095	0·0916		0·0253			0 0020					
20	0 1099	0·0095	0·0908		0 0261			0 0020					
∞	0 0918	0 0094	0·0729		0·0442			0 0021					

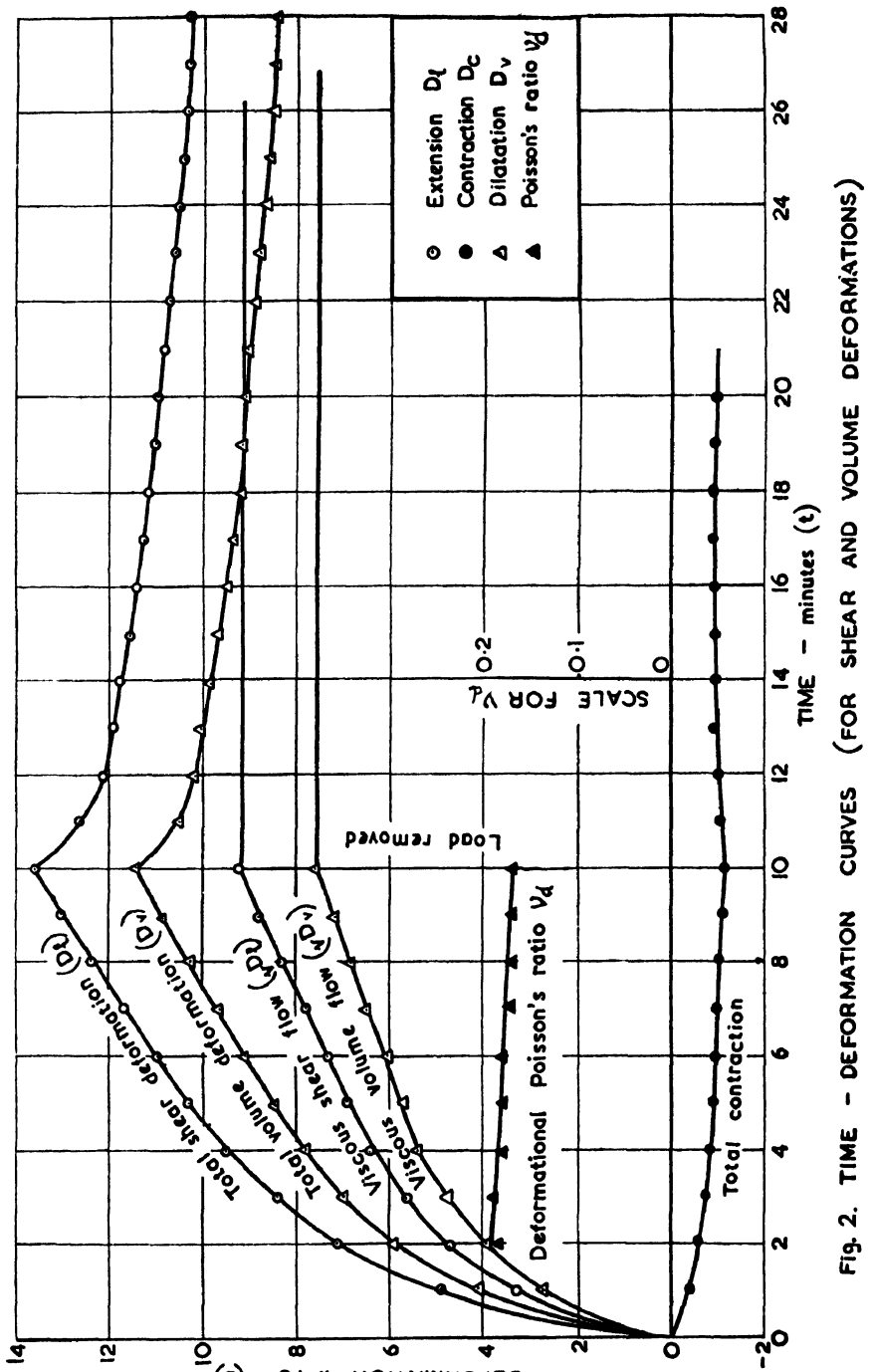


Fig. 2. TIME - DEFORMATION CURVES (FOR SHEAR AND VOLUME DEFORMATIONS)

The problem is to separate the elastic strain from the viscous deformation in both linear and cubical dilatation.

The existence of elastic components is revealed by the recoveries ($_re$), which can be calculated by subtracting from the maximum deformation the final deformation after removal of load. They are shown for the linear dilatation in column 6.

We consider the K component of the $L = K - N$ model. We have from Equ. (IX, h)

$$e_l = e^{-E/\lambda\, t}\left(e_{l,o} + \frac{1}{\lambda}\int_o^t \sigma e^{E/\lambda\, t}\, dt\right) \quad . \quad . \quad . \quad . \quad (21)$$

Considering first the fore-effect, the stress σ is a constant and $e_{l,o} = 0$ at $t = 0$.

Equ. (21) then yields

$$_fe = \frac{\sigma}{E}\left(1 - e^{-E/\lambda\, t}\right) \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where $_fe$ is the elastic fore-effect after the time t.

For the elastic after-effect, the initial strain will be $e = e_o$ at $t = 0$ on the removal of stress when $\sigma = 0$. Then Equ. (21) yields

$$e = e_o\, e^{-\frac{E}{\lambda}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The actual recovery $_re$ after the time t is given by

$$_re = e_o\left(1 - e^{-\frac{E}{\lambda}t}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Since, in general, the elastic strain in the fore-effect is not completed when the stress is removed, the fore-effects must and can be calculated from the observed after-effects as follows :—

Consider the " fore " and " after " effects after the same period of time T. Let

$$C = \left(1 - e^{-E/\lambda\, T}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

then

$$e_o = \frac{\sigma}{E}C,\ e_f = \frac{\sigma}{E}C(1 - C) \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$_re = e_o - e_f = \frac{\sigma}{E}C^2 \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where e_f is the strain remaining after time T of recovery (see Fig. 3).

But

$$\left.\begin{aligned} C &= (e_o - e_T)/e_o \\ _fe &= {_re}/C = \frac{e_o}{e_o - e_T}\,{_re} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

This method was used for both the longitudinal and lateral strains and the elastic fore-effect at any time was plotted from the recoveries by multiplying by the factor $1/C$.

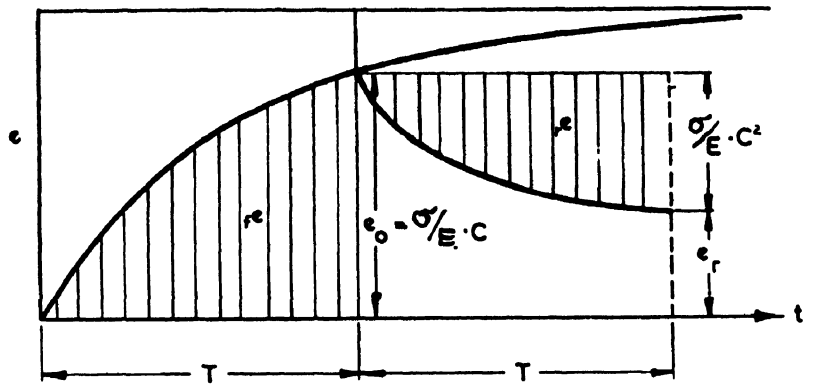


FIG. XII, 3.

In the present case e can be read from column 6 for $t = \infty$ as $= 0{\cdot}0442$. The value for e_T is found for $T = 20$ minutes from column 6, by subtracting ${}_f e_l$ at $t = 20$ from ${}_f e_l$ at $T = \infty$, and C is found in accordance with Equ. (26) $= 0{\cdot}592$. From this the fore-effects ${}_f e_l$ have been calculated in accordance with Equ. (27) and entered in column 7. Deducting the ${}_f e_l$ from the D_l gives the viscous deformations ${}_v D_l$, which are listed in column 8. From the transversal elastic recoveries ${}_r e_c$ listed in column 9, and using the value for C as calculated, the elastic transversal fore-effects were calculated in the same way and listed in column 10. By subtracting these values from D_c, the viscous deformations ${}_v D_c$ were obtained and entered in column 11. The elastic volume st ains e_v, and the viscous volume dilatation ${}_v D_v$ were then calculated in accordance with Equ. (19) and listed in columns 12 and 13. Finally the values for ν_v calculated in accordance with

$$ {}_v D_c = - \nu_l \, {}_v D_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29) $$

are entered in column 14. After deducting the elastic strain from the total deformation, the graphs for viscous flow result as shown in Fig. 2.

The constant rate of extension in the straight line portion is found from the graph as $9{\cdot}7 \times 10^{-5}$ seconds. From this, Trouton's coefficient of viscous traction is calculated as $\lambda = 7{\cdot}1 \times 10^{12}$ poises. With a mean value of $\nu_v = 0{\cdot}085$ (see column 14), Equ. (18) gives $\eta = 3{\cdot}3 \times 10^{12}$ poises and $\zeta_l = 2{\cdot}9 \times 10^{12}$ poises. The coefficient of liquid volume viscosity ζ_l, therefore, was in this case of the same order of magnitude as the coefficient of shear viscosity η.

5. Negative Volume Flow in Concrete

When describing creep of concrete in Chapter XI, we oversimplified conditions. We assumed that there is no voluminal creep, i.e. that all creep consists in a continuous distortion and that there is no such thing as a continuous condensation (or rarefaction) of the material.

One reasons that in compression, the condensation can, in Trouton's words (compare Section 2, Chapter V), be an " initial " effect only, or that we are, as we called it, in the *second* stage of viscous traction. But this is what Glanville and Thomas (1939) found when investigating " lateral " creep :—Concrete, on hardening, " shrinks ", i.e. all its dimensions are reduced proportionally. There is a negative cubical dilatation, D_v, going on for some time with a decreasing rate $\dot{D}_v$ without any corresponding stress.

In the sagging beam test, shrinkage will reduce the radius to which the beam is bent, but the influence upon the deflection is negligible. It is not so when a concrete cylinder is observed under compression and longitudinal and lateral movements are measured. Here shrinkage movements must be deducted. These were taken account of by measuring both longitudinal and lateral movements in a loaded and an unloaded specimen. It was found that the longitudinal movement increased very rapidly as a result of loading, but the lateral movement was practically unaffected, continuing to increase at about the same rate as the unloaded specimen (compare Fig. XII, 4). From the results of the tests, which the authors called " surprising ", the conclusion must be drawn that there is practically *no lateral creep*.

In figures, we can say that in the case of concrete, as observed by Glanville and Thomas, ν_v vanished while ν was *0·14*.

This surprised the authors because, as we recognize, they had tacitly assumed that the first axiom was valid and that, therefore, the volumetric deformation could only be an elastic strain taking place in the " first stage ". Later, in the second stage there would be no change of volume, or $\nu_v = 0{\cdot}5$. A vanishing Poisson-ratio indicates a case of simple extension (in our case negative extension). This, we said in Section 9 of Chapter III, is " not . . . easily realized " and we gave a very far-fetched example. Now we meet such a case under quite ordinary conditions of an everyday material. But in the light of what we found in the preceding Section, we must now restate the first rheological axiom in the following form : " Under isotropic pressure all non-porous materials are elastic and only elastic " and this is an implicit (rheological) definition for " non-porous ". Glanville and Thomas' experiment accordingly revealed by rheological means the porosity of concrete which, of course, is known from other facts. In creep, the concrete flows *into its pores and concrete shows negative volume flow*. No explanation for the behaviour was advanced at the time by the authors, and no further analysis made of the observations, but Dr. Glanville later kindly put the original data at my disposal. I used them (1949) for the clarification of the problem of isotropic flow.

The following Table II shows in columns 2 and 3 the shrinkage of the unloaded specimen of concrete, in columns 4 and 5 the shrinkage

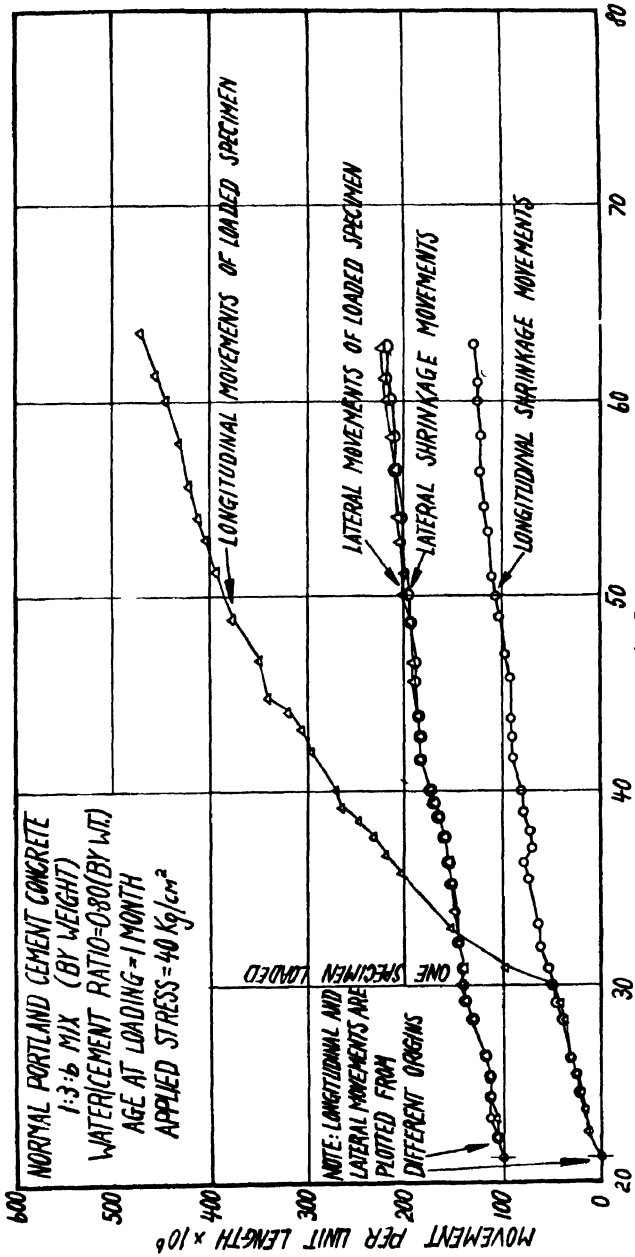


FIG. XII, 4. Longitudinal and lateral creep.

plus creep of the other specimen after loading, all in radians $\times 10^6$. Creep is calculated by subtracting shrinkage from shrinkage plus creep,

TABLE II.

Longitudinal and lateral creep of two cylinders of Portland cement concrete 1 : 2 : 4, watercement ratio 62% by weight. Slump about 15 cm One cylinder not loaded, the other loaded with 4·1 × 10^7 dyne.cm $^{-2}$ (600 lb./sq. in).

1	2	3	4	5	6	7	8	9	10
Age in days	Shrinkage × 10^6		Shrinkage and creep × 10^6		Creep × 10^6		ν_d	Averages of ν_d	ln D_t
	longit	lateral	longit.	lateral	longit.	lateral			
56	10·1	10 3	62·7	7·3	52·6	—3·0	—0 057		$\bar{1}0{\cdot}15$
57	10·5	12·0	69·2	9·5	58·7	—2·5	—0·043		$\bar{1}0{\cdot}26$
58	14·2	12·0	75·1	10·8	60·9	- 1·2	—0·020		$\bar{1}0{\cdot}294$
59	21·5	11 9	76 0	9·7	54·5	—2·2	—0·040		$\bar{1}0{\cdot}138$
60	27·9	13·6	81·0	11·1	53·1	-2·5	—0 047		$\bar{1}0{\cdot}157$
60	26·1	15·4	84 3	13·5	58·2	—1·9	—0 033		$\bar{1}0{\cdot}248$
61	26·6	15·7	85·1	13 5	58·8	—2 2	—0·037		$\bar{1}0{\cdot}259$
63	27·9	19 7	95·6	17 2	67 7	- 2 5	—0 037		$\bar{1}0{\cdot}400$
64	33 9	21·0	97 5	18 4	63 6	- 2·6	- 0 041		$\bar{1}0{\cdot}337$
65	28 4	23 0	106 2	19 8	77 8	—3·2	—0·041		$\bar{1}0{\cdot}549$
66	33·0	22·8	106·2	19 5	73·2	- 3·3	—0·045	—0 038	$\bar{1}0{\cdot}478$
67	33 4	24·4	109 4	21·0	76 0	—3·4	- 0 045		$\bar{1}0{\cdot}515$
68	36·6	24·8	114 9	22·2	78·3	—2·6	-0·033		$\bar{1}0$ 545
70	37·1	29 5	124·1	25·7	87 0	—3·8	- 0 044		$\bar{1}0{\cdot}650$
71	41 7	29·7	129	26·9	87 3	—2·8	—0·032		$\bar{1}0{\cdot}654$
72	41·7	31·6	131	27 2	89 3	4·4	—0 049		$\bar{1}0{\cdot}676$
73	38 9	32·8	138	28·9	99·1	—3·9	- 0·039		$\bar{1}0{\cdot}730$
74	35·2	35·3	143	30 5	107·8	- 4 8	—0 045		$\bar{1}0{\cdot}355$
75	37·6	37 0	146	32 0	108 4	- 5 0	—0·046		$\bar{1}0$ 871
77	41·2	40 0	152	35·2	110·8	—4·8	-0·043		$\bar{1}0{\cdot}893$
78	42·6	39·8	155	35·4	112 4	—4·4	- 0 039	0 042	$\bar{1}0{\cdot}907$
79	39·4	41·4	158	35·7	118·6	—5·7	—0 048		$\bar{1}0{\cdot}961$
81	44·9	42·3	164	37·8	119 1	—4 5	—0·038		$\bar{1}0{\cdot}965$
82	50 4	41 9	158	37 9	107 6	—4 0	—0 037		$\bar{1}0{\cdot}866$
84	47·7	44·3	167	39 1	119·3	—5·2	—0·044		$\bar{1}0{\cdot}966$
86	48·1	45·2	171	40·3	122 9	—4 9	—0 040		$\bar{1}0$ 996
88	53·1	46·8	175	41 8	121 9	- 5 0	—0 041		$\bar{1}0{\cdot}988$
91	55·9	49·4	182	44 3	126·1	—5·1	—0·041		$\bar{9}{\cdot}020$
93	60 0	51·1	188	46·5	128·0	-4·6	—0·036		$\bar{9}{\cdot}037$
96	56·8	50·5	186	44 0	129·2	—6·5	—0·050		$\bar{9}{\cdot}046$
99	60·0	59·6	205	51 8	145·0	—7·8	—0 054	- 0 043	$\bar{9}{\cdot}062$
102	62·8	62·8	212	55·4	149·2	—7·4	—0·050		$\bar{9}{\cdot}190$
105	66·0	64·3	216	56 5	150·0	—7 8	- 0·051		$\bar{9}{\cdot}195$
108	62·8	66·6	223	55·1	160 2	—11·5	—0·072		$\bar{9}{\cdot}261$
113	67·8	68·1	230	59·2	162·2	—8·2	—0·052		$\bar{9}{\cdot}195$
117	65·5	69·5	235	60·5	169·5	—9·0	—0·053		$\bar{9}{\cdot}195$
121	62·7	71·4	240	61·9	177·3	—9·5	—0 054		$\bar{9}{\cdot}363$
124	68·8	73·6	248	65·0	179·2	—8·6	—0·048		$\bar{9}{\cdot}373$
128	65·1	79·0	258	69·7	192·9	—10·2	—0·053		$\bar{9}{\cdot}195$
135	74·7	80·9	262	70·5	187·3	—10·4	—0·056		$\bar{9}{\cdot}418$
140	73·3	80·8	262	71·1	188·7	—9·7	—0·051	—0·054	$\bar{9}{\cdot}425$

and is shown in columns 6 and 7. In Fig. 5 longitudinal and lateral deformations are plotted as functions of time. Both curves are of the well known parabolic shape, the rate of creep gradually decreasing. It can be seen that there are fairly great fluctuations in the differences of shrinkage of the two specimens before loading ; accordingly, the data for creep become of significance only after about the 55th day. The deformational Poisson ratio (ν_d) was calculated in accordance with Equ. (12), and entered in column 8 of the Table. To a first approximation ν_d is constant $= 0{\cdot}0044$. To a second approximation it slightly increases at a rate of $\dot{\nu}_d = 2{\cdot}8 \times 10^{-9}$ sec.$^{-1}$. Accordingly we have to a first approximation a volume change calculated from Equ. (20).

$$D_v = 0{\cdot}91\, D_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

which again (provided the ratio is constant) carries with it a rate of volume flow

$$\dot{D}_v = 0{\cdot}91\, \dot{D}_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

There is therefore an appreciable volume-flow ; as a matter of fact, it is about 91 per cent. of the rate of extension throughout the whole period of creep. Certainly, in this case, Trouton would have been wrong in speaking of an " initial " effect, and his relation (V, d) cannot here be correct.

It should perhaps be pointed out that, even if the volume dilatation were 100 per cent. of the linear extension, i.e. $D_v = D_l$, this would not imply the absence of ordinary shear flow. For the volume dilatation D_v, being symmetrical, can be regarded as made up of three equal linear dilatations in three mutually perpendicular directions each one of magnitude $\frac{1}{3}D_v$. Thus, if $D_v = D_l$, the portion of the linear dilatation D_l contributing to the volume change D_v is only $\frac{1}{3}D_l$, the remaining $\frac{2}{3}D_l$ contributing to the ordinary shear flow.

With this, the analogy between elastic and viscous behaviour is perfect, the coefficient of viscosity taking the place of the modulus of elasticity. We therefore write in analogy to (III, w)

$$\zeta_l = \lambda/3(1 - 2\nu_l) \; ; \; \eta = \lambda/2(1 + \nu_l) \quad . \quad . \quad . \quad (32)$$

With $\nu_d = \nu_l = 0{\cdot}044$ these give $\zeta_l = \lambda/(2{\cdot}73)$, $\eta = \lambda/(2{\cdot}09)$. In our case, therefore, $\lambda = 2{\cdot}092\eta$ has to replace Trouton's equation (V, d). Trouton's coefficient of viscous traction, as usually calculated, would in the case of the creep of concrete 1 : 2 : 4 be about 50 per cent. too high.

It should be pointed out that nothing was assumed with regard to the coefficients of viscosity (and likewise the elastic moduli) which

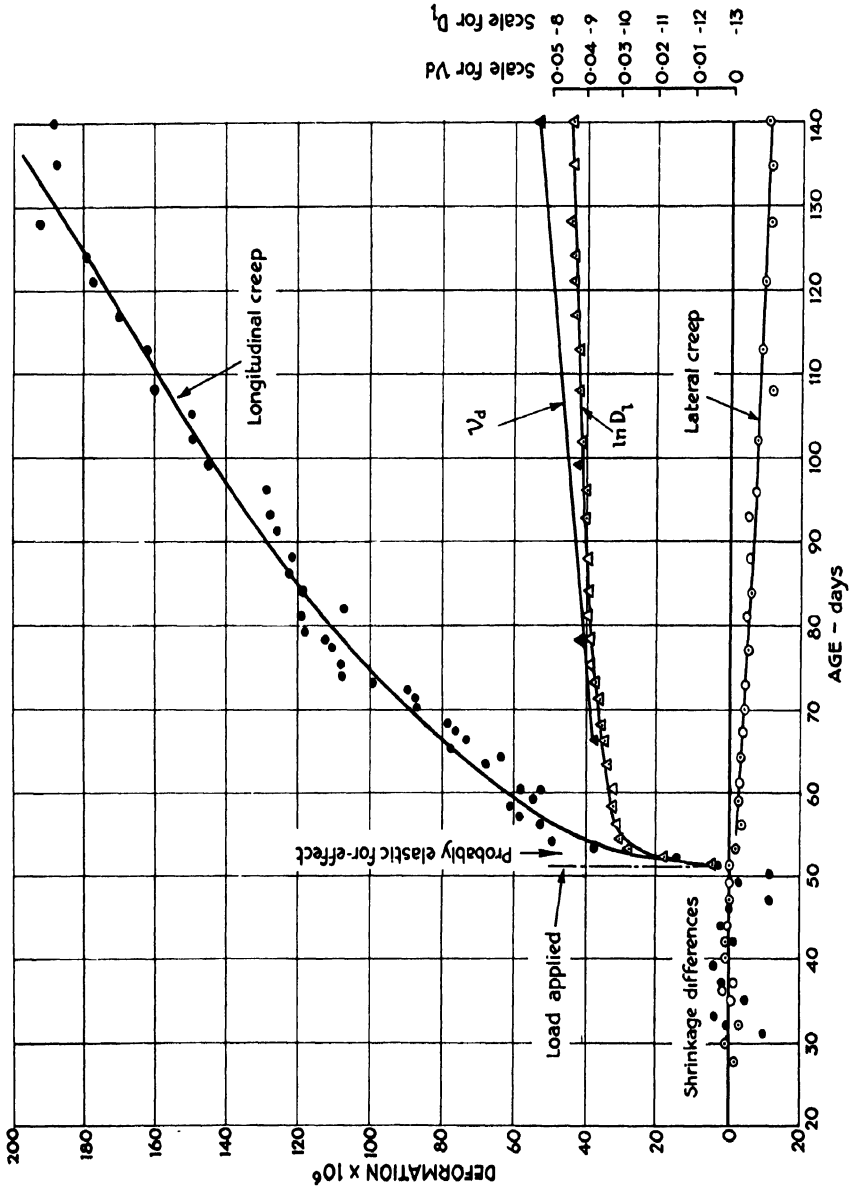

DEFORMATION x 10⁶
AGE – days
Longitudinal creep
Lateral creep
Probably elastic for-effect
Load applied
Shrinkage differences
ν_d
$\ln D_1$
Scale for ν_d
Scale for D_1

FIG. XII.5

would require them to be *constants*. They may vary with stress or strain or simply with time. When we examine Fig. 5, we see that, as a matter of fact, the viscous resistance very much increases with time. Such viscous resistance is measured by the reverse of the slope of the creep curve, and curves extending over years show that ultimately the creep practically ceases, with the viscosity approaching infinite values or the " liquid " converted into a " solid ". If, therefore, both ζ_l and η very much change in the course of time, it is the more surprising that ν_v should remain practically constant. This needs elucidation and may provide us with an insight of what actually happens.

It is generally accepted that the two moduli of elasticity are independent of one another. In the beginning of the theory of elasticity there were attempts to calculate one from the other, but later it was realized that there was no basis for such procedure. The two coefficients of viscosity ζ_l and η will also generally be independent. By eliminating λ from (32), we can express ν_v as

$$\nu_l = \frac{3 - 2\eta/\zeta}{6 + 2\eta/\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

If both η and ζ_l are independent, the ratio η/ζ may take on any value, and ν_l will vary accordingly. On the other hand

$$\zeta = \frac{2(1 + \nu_l)}{3(1 - 2\nu_l)}\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and should, notwithstanding variable η and ζ_l, ν_v be a constant, this would indicate a connection between the viscosity of volume flow and the shear viscosity resulting from the mechanism of one being in some way derivable from the mechanism of the other.

It is not difficult to see where this connection lies. Let us imagine part of the concrete under consideration bounded by a spherical surface. Let us consider the sphere under the action of an isotropic stress (compare Fig. 6).

For simplicity let only one cavity be present in the centre of the sphere. A volume-element near the boundary of the sphere (A in Fig. 6) will itself be under isotropic stress. On the other hand a volume-element near the surface of the cavity (B in Fig. 6) will not be under isotropic pressure. At the surface of the cavity, there will be a stress-difference, and the material will flow into the cavity under the action of the stress-difference, resisting with its shear-viscosity η. Macroscopically, the resistance will be expressible in terms of ζ_l, the volume-viscosity of the concrete. It would, however, be theoretically possible to calculate ζ_l from the viscosity of the cement and the shape, size and number of holes in the concrete. The rate of

creep of the concrete decreases or its viscosity increases with time because (*inter alia*) the viscosity of the cement increases, due to chemical changes. If there were no changes in the geometrical structure of the concrete, both ζ_l and η would accordingly increase at the same rate as the viscosity of the cement, and their ratio η/ζ would remain constant. This is so to a first approximation. To a second approximation the cavities in the concrete gradually decrease and this causes ν_d and ν_v to increase with time. This may also be contributory to the gradual slowing down of creep. A 1 : 3 : 6 concrete has much more and

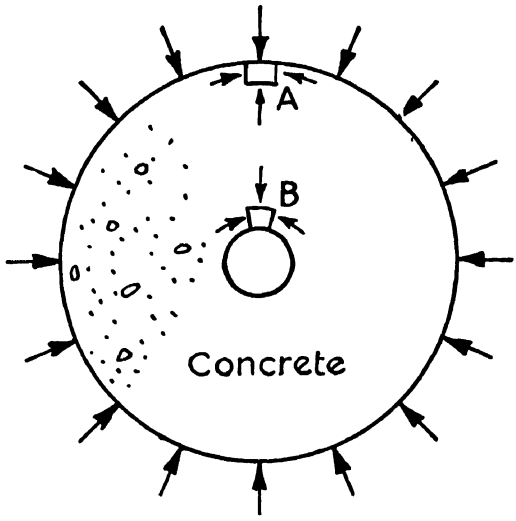


FIG. XII, 6.

larger cavities than a 1 : 2 : 4 concrete. In the first case it was found that ν_v vanishes. This means that *all flow* was into the voids.*

The values of ln D_l were calculated and entered in column 10 of the Table. The curve ln D_l versus time is shown in Fig. 5. Its slope is found to decrease gradually from $4{\cdot}6 \times 10^{-7}$ sec.$^{-1}$ at 60 days of age to $1{\cdot}2 \times 10^{-7}$ sec.$^{-1}$ at 130 days of age. At the same time ν_d increases from *0·037* to *0·052* at a rate of $2{\cdot}8 \times 10^{-9}$ sec.$^{-1}$. We find accordingly from Equ. (17), that ν_v increased from *0·043* to *0·075*.

During the same period the rate of extension (as again can be found from the curve in Fig. 5) decreased from $3{\cdot}7 \times 10^{-11}$ sec.$^{-1}$ to $1{\cdot}7 \times 10^{11}$ sec.$^{-1}$. This gives (in accordance with Equ. (20)) a volume flow at a rate of from $3{\cdot}4 \times 10^{-11}$ sec.$^{-1}$ to $1{\cdot}4 \times 10^{-11}$ sec.$^{-1}$. The stress was $\sigma_n = 4{\cdot}1 \times 10^7$ dyne cm^{-2} (600 lb./sq. in.), which gives for

* An elastic analogy of this behaviour is furnished by expanded (porous) rubber where all strain is into the holes.

Trouton's coefficient of viscous traction from (VII, 20) $\lambda = 1{\cdot}1 \times 10^{18}$ to $2{\cdot}4 \times 10^{18}$ poises, and therefore (compare Equ. (18)) $\eta = 5 \times 10^{17}$ to $10{\cdot}4 \times 10^{17}$ poises and $\zeta_l = 4 \times 10^{17}$ to $9{\cdot}4 \times 10^{17}$ poises. For a 1 : 3 : 6 concrete we find, with $\nu_d = \nu_v = 0$, from (Equ. (18)) $\zeta_l = \lambda/3$, $\eta = \lambda/2$.

It could be maintained that what we are calculating here is not a coefficient of viscosity but the expression of a complex phenomenon in which an ordinary shear viscosity enters as the only physical constant. It could be maintained that if there are voids in a liquid, and the liquid was flowing into these voids thereby increasing the bulk density of the material, the real density of the liquid would not be increased, and what would macroscopically simulate an isotropic flow, would under the microscope reveal itself to be ordinary, albeit heterogeneous, shear flow. To this may be replied that we are concerned with the macroscopic behaviour of the material, and that on the above view we could not say that the viscosity of a liquid was increased by suspending in it solid particles. Here also the viscosity of the liquid phase itself is not increased ; it is only forced to flow round the particles, which increases the resistance to flow and makes itself manifest *macroscopically* as an increase of viscosity.

6. The Volumetric Rheological Equation

To the two examples of positive and negative volume flow given above, we may add that Bosworth (1949) has observed volume flow in solidified carbon dioxide under isotropic pressure, in tension, and in torsion, and that Fisher (1950) is of the opinion that the concept of volume flow is of great significance in geophysics, and that there is positive volume flow of the earth as a whole which " has expanded in volume several fold since Archæan times ". Besides, if in accordance with Eyring (1936), " a liquid is a binary mixture of molecules and holes ", volume flow may be noticeable even in ordinary liquids under special conditions.

In order to establish the general volumetric rheological equation this side of the strength limit ϑ_v, let e_v be the volumetric *strain*, i.e. the recoverable part of the cubical dilatation, and let f_v be the irrecoverable rate of the cubical dilatation. Furthermore, let d_v be the cubical dilatation, comprising both recoverable strain and irrecoverable flow, so that

$$d_v = \dot{e}_v + f_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

then a mechanical model consisting of an elastic spring and a viscous dashpot coupled in parallel, both with another dashpot in series, offers a suitable representation of the most general volumetric behaviour of any material. This, in other words, is a L-body in which d_v stands

for $\dot{\gamma}$, $-p$ for τ, ζ_l for η_M, ζ_s for η_K and κ for μ, and its rheological equation is accordingly

$$\dot{d}_v = -p\frac{\zeta_l+\zeta_s}{\zeta_l\zeta_s} - \frac{\kappa}{\zeta_s}e^{-\kappa/\zeta_s t}\left(d_{vo} - \frac{1}{\zeta_s}\int p e^{\kappa/\zeta_s t}\,dt\right) \quad . \quad . \quad (36)$$

with a retardation time

$$T_{ret} = \zeta_s/\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and a relaxation time

$$T_{rel} = (\zeta_l + \zeta_s)/\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

For the Stokesian liquid $\zeta_s = 0$ and $\zeta_l = \infty$, whence $T_{ret} = 0$ and $T_{rel} = \infty$.

7. Summary

The general volumetric rheological equation is

$$\dot{d}_v = -p\frac{\zeta_l+\zeta_s}{\zeta_l\zeta_s} - \frac{\kappa}{\zeta_s}e^{-\kappa/\zeta_s t}\left(d_{vo} - \frac{1}{\zeta_s}\int p\, e^{\kappa/\zeta_s t}\,dt\right)$$

where ζ_s is the solid volume-viscosity connected with the damping of an elastic volume dilatation, and ζ_l the liquid volume viscosity connected with irrecoverable volume-flow. For the Stokesian liquid $\zeta_s = 0$ and $\zeta_l = \infty$. Positive volume flow was observed in asphalt and negative volume flow in concrete.

CHAPTER XIII

A DYNAMICAL THEORY OF STRENGTH

1. Importance of Rate of Deformation

All theories of strength mentioned in Section 7 of Chapter VI have in common that *the rate of deformation* does not enter. They are purely *statical*. Recent improvements in the methods of testing and the frequent use of high-speed tests as well as of creep tests, and specially the testing of synthetic plastics, have made it, however, increasingly evident that the rate of deformation or, what comes to the same thing, the rate of application of the external forces, considerably influences the results of the tests. It is therefore obvious that purely statical concepts cannot lead to a satisfactory theory of strength. A theory taking account of *velocity* is required, both from theoretical and practical considerations. For instance, standard specifications for mild steel until recently asked for a certain minimum breaking stress to be determined in a tensile test without, however, specifying the rate at which the test is to be carried out. It is known that less conscientious suppliers of mild steel, if their product was not up to standard at ordinary speeds, sometimes used to resort to the device of increasing the speed, with the result that a higher breaking strength was recorded. This loophole was closed in the later British Standard Specification by prescribing a definite rate of application of the load. On the extreme end of a series of materials where the influence of the testing speed makes itself felt, stands such a material as rayon, where the concept of strength has a meaning only in relation to a maximum rate of deformation.

2. Breaking Energy

When a body breaks into pieces the process is accompanied by the appearance of certain forms of energy. There is an audible report which indicates the presence of vibrations. These vibrations will gradually die down and ultimately be converted into heat. Of greater importance is the increase of surface energy due to the increase of the surfaces of the broken parts over the surface of the entire body. These energies and others appearing in rupture are provided for by the conversion of the potential energy E. There have been attempts to calculate on this basis the strength of a material from the value of its surface tension. The results were many orders of magnitude out. We shall not go into this side of the problem : it belongs to metarheology. But one thing

is clear : the expenditure of energy connected with rupture cannot come from the *total* stress-work performed by the external forces. Only that part of the stress-work which is *conserved* in the body, or what may be called *strain-work*, is available for conversion into other forms of energy. Generally, part of the stress-work is *dissipated*, and this, of course, is not available. For instance, when isotropic tension is not applied infinitely slowly through states of equilibrium but with finite speed, this will cause, as we learned in Chapter V, viscous resistance and part of the strain-work will be dissipated. In order to reach the limiting value of potential energy, the external forces will have to perform a *greater amount of work*. This can be done only by increasing the isotropic stress over that stress at which the material breaks when the forces are applied infinitely slowly. This will, accordingly, manifest itself in a *rise of strength*.

3. The Isotropic Rupture Condition

Let us now consider the action of isotropic tension when the rate of dilatation *does not vanish*. As we said in Section 2 above, this will cause a rise in strength. We shall presently translate this qualitative statement into quantitative language.

The volumetric rheological equation of a homogeneous material is to some approximation in accordance with Equ. (XII, 2)

$$-p = \kappa e_v + \zeta_s \dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ζ_s is the coefficient of solid volume viscosity. The function of *this* volume viscosity is to delay the appearance of the elastic strain and/or to cause viscous damping, but there is *no flow*. In short, it is a *solid* viscosity.

The strain-work per unit volume is in accordance with Equation (XII, 4)

$$w_v = \int_o^{e_v} \kappa e_v \, de_v + \int_o^t \zeta_s \dot{e}_v^2 dt \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The first part on the right side is conserved as elastic potential energy, while the second part is converted into heat and dissipated. Rupture, therefore, takes place when the first part reaches and exceeds the *isotropic resilience* of the material measured by E_v. The Rupture Condition will accordingly be

$$w_v - \int_o^t \zeta_s \dot{e}_v^2 dt = \int_o^{e_v} \kappa e_v de_v \geq E_v \quad . \quad . \quad . \quad (3)$$

If κ is constant, the integral on the right-hand side can be calculated and is $\kappa e_v^2/2$ and, therefore, the cubical dilatation at which rupture takes place

$$e_{vr} = \pm \sqrt{2E_v/\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now Equ. (1) gives for the tensile stress at which rupture takes place

$$-p_r = \kappa e_{vr} + \zeta_s \dot{e}_v = \sqrt{2E_v\kappa} + \zeta_s \dot{e}_v \quad . \quad . \quad . \quad (5)$$

Let the breaking stress in the static state for which $\dot{e}_v = 0$ be p_{rs}

$$p_{rs} = \pm \sqrt{2E_v\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

then

$$p_r = p_{rs} + \zeta_\kappa \dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Therefore : *the stress at which a material fails in isotropic tension increases with the rate of dilatation.* If ζ_s is constant the increase is linear ; otherwise it follows some other law determined by $\zeta_s = f(\dot{e}_v)$.

It should be noted that this result does not depend on κ being constant. From Equ. (2) it follows directly that the strain-work to bring about rupture (w_{vr}) is for any given rate of dilatation ($\dot{e}_v$) always greater than w_{vrs}, the strain-work expended in infinitely slow dilatation (for which $\dot{e}_v = 0$).

While, therefore, in the absence of viscous resistance to the cubical volume expansion of a material there is a definite stress at which rupture occurs ; when there exists volume viscosity, the dilatation e_{vr} is definite (compare (4)), but the stress p_r is not.

4. The Distortional Failure Condition for a Hooke Solid in Simple Shear

We now pass to simple shear. Consider a variety of rheological equations corresponding to the variety of real materials.

Let us first consider the Hooke solid.

Its rheological equation is (I, d) and the rupture condition is accordingly

$$w_{or} = \int_0^\gamma \mu\gamma d\gamma \geq E_{or} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If μ is constant this gives, considering (VII, 31)

$$\left.\begin{aligned} \gamma_r &= \pm \sqrt{2E_{or}/\mu} \quad . \quad . \quad . \quad . \quad . \quad . \\ \tau_r &= \pm \sqrt{2\mu E_{or}} \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad (9)$$

Both γ_r and τ_r are definite constants depending upon the same definite quantities μ and E_{tr} and are independent of the rate of strain or the rate of application of load.

Should the body be plastic, similar reasoning, starting from Equ. (I, e) results in the Equ. (VI, a) and

$$\gamma_{pl} = \pm \sqrt{2E_{o,pl}}/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

5. Failure of a Newtonian Liquid

At the other extreme of the rheological bodies is the Newtonian liquid with the rheological equation (I, f), the stress-work being

$$w_o = \int_o^t \eta\dot{\gamma}^2 dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In accordance with our criterion, rupture in a Newtonian liquid would therefore only be possible if E_{t_r} was equal to zero and then rupture would always and continuously take place. This is not what we observe. Rupture has been defined by us as a visible separation of the parts of the body, and we do not observe such visible separation in a flowing liquid in laminar flow.* If the velocity increases so that turbulence sets in, a visible separation of the parts of the liquid can be observed. Turbulence is of great interest to the hydraulic engineer, but of little interest to the rheologist, except perhaps in the case of what Wo. Ostwald termed Structural Turbulence and about which we shall presently have to say a few words. We may, at this occasion, point out the analogy which exists between turbulence and rupture as conceived by us. The inception of turbulence is connected with a dimensionless figure called Reynolds' Number. Weissenberg (compare Rabinowitsch (1929)) has expressed this figure as the ratio of dissipated to kinetic energy. In this interpretation the kinetic energy E_k plays in turbulence the same part as the potential energy E_p in rupture. Turbulence is accordingly determined by a certain point on the side E_k–E_d of the triangle, Fig., page 114 of *Twelve Lectures*, while rupture is determined by a point on the side E_p–E_d of the triangle.

6. Structural Turbulence

If one were to assume the Coulomb theory of maximum shearing stress (IV of Section 7 of Chapter VI) to be valid for every kind of material, one would say that in a liquid τ could also not exceed a certain maximum value. Otherwise, as there is no inherent limit to $\dot{\gamma}$, the shearing stress could also increase without limit and water would be stronger than steel. In a very early paper (1911) I assumed that there was a shearing strength for liquids, the same as for solids, and when this is exceeded a discontinuity in the flow appears so that, e.g. in the case of Fig. I, 4, there would be a break in the straight line representing the increase in velocity from zero to V. There does not appear to be any evidence to support this view in regard to simple Newtonian liquids, but Wo. Ostwald and Auerbach (1926) maintained that in liquids showing structural viscosity, turbulence sets in long before Reynolds' critical velocity is reached. They suggested as the reason the internal

* The breaking up of a jet of liquid into drops due to *surface tension* belongs to metarheology.

rupture of the structure of the system, which causes the appearance of eddies such as occur in turbulence.

If this is so, there would, e.g. in laminar flow of a liquid through a tube, be a maximum shearing stress

$$\text{max. } \tau = R\Delta p/2l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

compare (II, c). If we assume that to a first approximation Poiseuille's law is valid, the mean velocity is

$$\text{mean } v = Q/R^2\pi = R^2\Delta p/8\eta l \quad . \quad . \quad . \quad . \quad (13)$$

and, therefore,

$$\text{crit. } v = R/4\eta . \tau_{max} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the critical velocity for the appearance of structural turbulence.

As is known, Reynolds' critical velocity is

$$\text{crit. } u = \eta/R.C/\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where C is a constant of the apparatus, but not of the liquid. These two critical velocities are accordingly of opposite character. Crit. v increases, while crit. u decreases with R. In respect of η the opposite

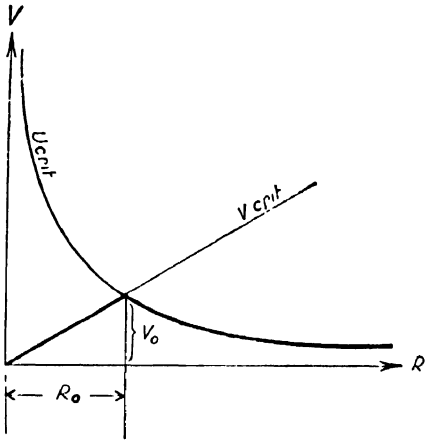


FIG. XIII, 1. Structural and Reynolds' Turbulence.

takes place. For one and the same liquid, crit. v and crit. u vary as shown in Fig. XIII, 1. Both are equal when the radius of the tube is

$$R_o = 2\eta\sqrt{C/\rho\tau_{max}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

A solution exhibiting structural viscosity would therefore be " weaker " than its solvent. While the latter could sustain shearing stresses up to the inception of Reynolds' turbulence, the former would break up before, namely when the structure of the dispersed phase is destroyed (Reiner (1926, 2)).

7. A Model of Liquid Strength

For a better understanding of the strength of a liquid a modification of Hencky's model (compare Section 5 of Chapter VI) will serve. Let the bottom of Hencky's vessel be perforated as shown in Fig. XIII, 2.

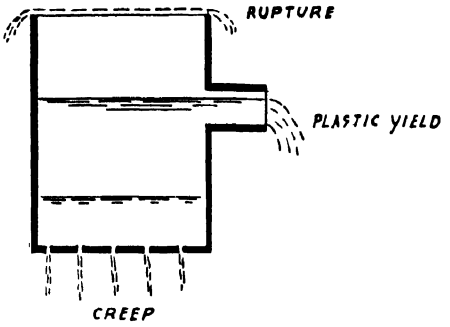


FIG. XIII, 2. Model of strength of a liquid.

Let there be an overflow at some level above the bottom. The liquid contained in the vessel represents the elastic distortional energy which can be stored in unit volume of the material. If energy is expended upon the material, or liquid poured into the vessel so slowly that it leaks out entirely through the holes in the bottom, in other words : if the velocity of deformation is less than or equal to the velocity of relaxation, there will be no failure by distortion. The material, even if to all appearances it is taken as a solid, will flow at some rate depending upon the area of the openings. If that area is small the liquid will leak out slowly, i.e. there will be creep. However, by pouring in energy at a higher rate, i.e. by increasing the *power* of deformation (remember : power = work/time) the vessel is steadily filled until the overflow opening is reached (assuming that such one exists) and further energy is lost by plastic yield. This is the case for instance with a metal which creeps under stresses below the yield point, and flows plastically when the yield point is exceeded. If the combined leakage by relaxation and plastic yield is not sufficient to dissipate the energy of work of the external forces, the level in the vessel rises above the overflow and reaches its brim. The moment the energy flows over the brim, the material breaks.

One or the other of the just-mentioned features of the vessel may be absent or especially prominent. If there are no holes in the bottom and no overflow, the material can only fail through brittle rupture. If there is an overflow so large that it will dispose of any supply of energy at whatever rate, the material will yield plastically, but will

not break. If there are holes in the bottom but no overflow, the material will show creep and brittle rupture. This is the case with asphalt and concrete which, in spite of their ability to creep, cannot be deformed plastically and fail only through brittle rupture. If the holes are not at the bottom but in the sides at some level above the bottom, there will be viscous dissipation of energy down to that level : this is the case of the Schwedoff body. A vessel without a bottom (if there was such an absurdity) through which infinite quantities of energy can be poured at any rate is the model of a Newtonian liquid, the velocity of relaxation of which is infinite and which, through its viscosity, can dissipate energy at any rate desired. From considering the model we therefore come again (as in Section 6 above) to the result that a Newtonian liquid, say water, should be able to withstand very high, practically unlimited shearing stresses, a result which we cannot very well accept. This result, therefore, points to some defect or limitation in our theory. The limitation arises from our having considered steady states of equilibrium only, when there are *no accelerations*. When the particles are accelerated, rupture may occur through an entirely different mechanism. Let two contiguous parts of the body have at some time the same or slightly different velocities : if the state is steady and the movements are not accelerated, the body will either move as a whole and/or be deformed. The same happens if both parts are accelerated in equal amounts. But let the accelerations differ, however little, then after some time there will be reached a state when both parts must separate, i.e. there will be rupture. It can be shown that the stresses at which rupture then occurs may be very small or even vanish : this result can be deduced from the complete stress equations (Equ. (1), page 36, *Twelve Lectures*) by considering d'Alembert forces, which balance or nearly balance the external forces with the tractions absent or small.

8. Failure of a Maxwell Liquid

Passing from qualitative to quantitative considerations, let us now consider the failure of a Maxwell liquid with the rheological equation (IX, b). Its elastic stresses relax and the " vessel " will accordingly have holes at the bottom.

We have

$$w_t = \tau\gamma = \tau\dot{\tau}/\mu_M + \tau^2/\eta \quad . \quad . \quad . \quad . \quad . \quad (17)$$

of which the conserved part is $\tau\dot{\tau}/\mu_M$. Conditions for failure by either plastic yielding or rupture are entirely analogous and we shall therefore write $E_{o,f}$ for both $E_{o,pl}$ and $E_{o,r}$ of Section 8, Chapter VI.

The failure condition is accordingly

$$\int_o^t \tau\dot{\tau}/\mu_M.dt = \int_o^t (i\dot{w} - \tau^2/\eta)dt \geq E_{o,f} \quad . \quad . \quad . \quad . \quad (18)$$

If μ is constant, this results in

$$\tau_f{}^2 = 2\mu_M E_{o,f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and Equ. (IX, b) now gives

$$\dot{\gamma}_f = \dot{\tau}_f/\mu_M + \frac{1}{\eta}\sqrt{2\mu_M E_{o,f}} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now, let $\gamma_{f,s}$ be the rate of deformation at which the material fails under a constant stationary load, when τ vanishes so that

$$\dot{\gamma}_{f,s} = \frac{1}{\eta}\sqrt{2\mu_M E_{o,f}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

then

$$\dot{\gamma}_f = \dot{\gamma}_{f,s} + \dot{\tau}_f/\mu_M \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The Maxwell liquid, therefore, fails in distortion *when the tangential stress reaches a definite limit* $\tau_{f,f} = \pm\sqrt{2\mu E_{o,f}}$. *There is, however, no limit to distortional deformation, while the rate of deformation at which the material fails increases with the rate at which the load is applied.*

If μ_M is not constant, this increase is not linear.

9. Failure of a Kelvin Solid

We proceed in the same manner when investigating the failure of a Kelvin solid. The results are similar to those found in Section 3 above for isotropic strain and stress, because we assumed there the presence of solid viscosity.

Starting from the rheological equation (IX, g) the failure condition is found to be

$$\int_o^t \mu\gamma\dot{\gamma}dt = \int_o^t (i\dot{w} - \eta_K\dot{\gamma}^2)dt \geq E_{o,f} \quad . \quad . \quad . \quad (23)$$

Let $\tau_{f,s}$ be the stress at which the material fails in a static test when $\dot{\gamma}$ vanishes, then

$$\tau_f = \tau_{f,s} + \eta_K\dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The Kelvin solid, therefore, fails in distortion *when the tangential strain reaches a definite limit* $\dot{\gamma}_f = \pm\sqrt{2E_{o,f}/\mu}$. *There is, however, no limit to the rate at which the stress is applied, while the tangential stress at which the material fails increases with the rate of strain.* If η_K is not constant, this increase is not linear.

10. The Maxwell Liquid in Simple Pull

The examples treated so far show the way by which to arrive at the failure condition for other more complicated materials, e.g. the Burgers body, the Bingham body and others. This we shall not do

here, but shall apply the theory in greater detail to the problem of failure of a Maxwell liquid in simple pull, following Reiner and Freudenthal (1938).

Let the material under investigation be formed into a cylindrical rod, as Schofield and Scott Blair did with flour dough (1930) and Lethersich (1941) with bitumen. Let l be the length of the cylinder and R the radius of the circular section of area A ; l_o, R_o and A_o denoting their values at the beginning of the experiment, when $t = 0$. Let a pull P be applied in the longitudinal direction. For reasons of symmetry, this direction will be one of the principal axes (compare Chapter III) of stress and deformation, say " k ", while tractions and deformations in the directions " i " and " j " normal to the axis will be equal. The normal traction in the direction of the axis will be

$$\sigma_n = P/A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

causing a longitudinal extension e_l and lateral contractions $-e_c$. The latter are not accompanied by stresses and the power of deformation is therefore

$$\dot{w} = \sigma_n \dot{e}_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In accordance with Equ. (III, n) the stress can be resolved into its isotropic and deviatoric components, or, in other terminology, hydrostatic tension and a system of shears (stress differences),

$$p = \frac{1}{3}\frac{P}{A} \; ; \; \sigma_{ol} = \frac{2}{3}\frac{P}{A} \; ; \; \sigma_{oc} = -\frac{1}{3}\frac{P}{A} \quad . \quad . \quad (27)$$

where the subscript " o " indicates the deviator.

The deformation can be resolved in the same way in accordance with Equs. (III, j) and (III, k)

$$e_v = e_l + 2e_c \; ; \; e_{ol} = \frac{2}{3}(e_l - e_c) \; ; \; e_{oc} = -\frac{1}{3}(e_l - e_c) \quad . \quad (28)$$

The power of deformation consists of two parts, viz. one of cubical dilatation

$$\dot{w}_v = p\dot{e}_v = \frac{1}{3}\frac{P}{A}(\dot{e}_l + 2\dot{e}_c) \quad . \quad . \quad . \quad . \quad . \quad (29)$$

the other of distortion

$$\dot{w}_o = \sigma_{ol}\dot{e}_{ol} + 2\sigma_{oc}\dot{e}_{oc} = \frac{2}{3}\frac{P}{A}(\dot{e}_l - \dot{e}_c) \quad . \quad . \quad . \quad (30)$$

both adding up to $\dot{w}$ of (26).

We now arrange the test so that one end of the rod is fixed, while the other end at the distance l moves with the velocity $v(t) = dl/dt$. The extension is, in accordance with (III, h)

$$e_n = \ln (l/l_o) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Therefore the rate of extension

$$\dot{e}_n = d(\ln l - \ln l_o)/dt = \frac{1}{l}\frac{dl}{dt} = \frac{v}{l} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Now, in accordance with (III, j)

$$\dot{e}_v = \dot{e}_n + 2\dot{e}_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

But

$$V = lA \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and therefore $\ln V = \ln l + \ln A$, or

$$\frac{1}{V}\frac{dV}{dt} = \frac{1}{l}\frac{dl}{dt} + \frac{1}{A}\frac{dA}{dt} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Comparison of (33) and (35) yields

$$\dot{e}_c = \frac{1}{2A}\frac{dA}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Expressions (32) and (36) make Equ. (29) and (30)

$$\dot{w}_v = \frac{P}{3A}\left(\frac{v}{l} + \frac{1}{A}\frac{dA}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$\dot{w}_o = \frac{2P}{3A}\left(\frac{v}{l} - \frac{1}{2A}\frac{dA}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad (38)$$

So far the expressions of this Section are valid for any material and any magnitude of deformation. We now consider a special material which, we assume, behaves as a Maxwell liquid in distortion, while its cubical dilatation is purely elastic. In other words, we assume that its distortional stresses relax, while its isotropic stresses do *not* relax. The latter assumption is safe for all dense materials with no appreciable voids. Let us furthermore assume, for simplicity, that the material has no " solid " volume viscosity, i.e. that its volume-elasticity acts instantaneously without delay. Then its " isotropic " rheological equation will be

$$-p = \kappa e_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Even should the material have delayed volume elasticity, our results will be valid when the volume does not change or when it changes slowly.

For the " distortional " equation we have to take into account that we are dealing here with simple tension and not with simple shear for which case we postulated Equ. (IX, b). An appropriate modification gives for our present case

$$\dot{e}_n = \dot{\sigma}_n/E_M + \sigma_n/\lambda_M \quad . \quad . \quad . \quad . \quad . \quad (40)$$

or introducing σ_n from (25)

$$\dot{e}_n = \frac{1}{E_M}\frac{d(P/A)}{dt} + \frac{1}{\lambda_M}(P/A) \quad . \quad . \quad . \quad . \quad (41)$$

When the load is applied, the volume expands to its maximum which it assumes under the stress p and which is p/κ. After this, the volume either remains constant or changes so slowly that we may assume $\dot{e}_v = 0$.

When this is the case lA is constant and Equs (33) and (36) yield

$$\dot{e}_n = -\frac{1}{A}\frac{dA}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and Equ. (32)

$$v = -\frac{l}{A}\frac{dA}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

With this Equ. (41) becomes

$$\frac{1}{A}\frac{dA}{dt} = -\frac{1}{E_M}\frac{d(P/A)}{dt} - \frac{1}{\lambda_M}(P/A) \quad . \quad . \quad . \quad . \quad (44)$$

while Equ. (38) is reduced to

$$w_o = \dot{w} = -\frac{P}{A^2}\frac{dA}{dt} = \frac{1}{2E_M}\frac{d(P/A)^2}{dt} + \frac{1}{\lambda_M}(P/A)^2 \quad . \quad (45)$$

The strain-work expended during the time t is accordingly

$$w = \int_0^t w dt = \frac{1}{2E_M}(P/A)^2 + \frac{1}{\lambda_M}\int_0^t (P/A)^2 dt \quad . \quad . \quad (46)$$

where it should be kept in mind that in order to calculate the potential elastic energy we have to start from the unstressed state, or $P = 0$ for $t = 0$.

Rupture occurs either when the isotropic stress $P/3A$ reaches a certain value $= \sqrt{2\kappa E_{v,r}}$ in accordance with Equ. (6) or when (as in Equ. (18))

$$\frac{1}{2E_M}\int \frac{d}{dt}(P/A)^2 dt = E_{o,r} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

which, as in Equ. (19), gives

$$(P/A)_r^2 = 2E_M E_{o,r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Therefore, there will be brittle or deformational rupture in accordance with whether $E_{v,r}$ is smaller or larger than $\frac{E}{9\kappa}E_{o,r}$.

The material can be tested under the following conditions :

(i) constant stress σ_n
(ii) constant load P
(iii) constant velocity of elongation v
(iv) constant rate of strain $\dot{e}_n$
(v) constant rate of stress $\dot{\sigma}_n$.

(i) With one of the apparatuses mentioned in Section 1 of Chapter VI we keep the stress P/A constant $= P_o/A_o$, or

$$P = \frac{P_o}{A_o} A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Introducing expression (49) into Equ. (44) yields a differential equation in dA/dt as follows :

$$dA/dt = - P_o A/A_o \lambda_M \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

the solution of which is

$$A = A_o e^{-P_o/A_o \lambda_M t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and therefore

$$dA/dt = - \frac{P_o}{\lambda_M} e^{-P_o/A_o \lambda_M t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

We now easily find the following relations :

(*a*) Velocity-time function (compare (43))

$$v = \frac{l^2}{l_o A_o} \frac{P_o}{\lambda_M} e^{-P_o/A_o \lambda_M t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

(*b*) Length-time function

$$l = l_o e^{-P_o/A_o \lambda_M t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

(*c*) Length-load function

$$l = l_o P_o/P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

(*d*) Strainwork-time function (compare (46))

$$w = (1/2E_M + 1/\lambda_M)(P_o/A_o)^2 \quad . \quad . \quad . \quad . \quad (56)$$

Because $P/A = P_o/A_o$, rupture takes place either immediately when the load is applied or not at all. The rupture condition is accordingly independent of time and identical with the Huber-Hencky condition.

(ii) If we apply to the rod a constant load $P = \text{const.} = P_o$, this constitutes the *creep* test :

Equ. (44) now becomes

$$\frac{1}{A} \frac{dA}{dt} = - \frac{P}{E_M} \frac{d(1/A)}{dt} - \frac{P}{\lambda_M} \frac{1}{A} \quad . \quad . \quad . \quad . \quad . \quad (57)$$

the solution of which is

$$A/A_o - \frac{P_o}{A_o E_M} \ln(A/A_o) = 1 - \frac{P_o}{A_o \lambda_M} \cdot t \quad . \quad . \quad . \quad (58)$$

When E_M is very large, as it mostly will be, or the rate of stress small, the last equation can be approximated between the limits $A_o \geq A \geq 0{\cdot}5\, A_o$ by the linear equation

$$A = A_o \left(1 - \frac{P_o}{A_o \lambda_M} t\right) \quad . \quad . \quad . \quad . \quad . \quad (59)$$

From this

$$dA/dt = -P_o/\lambda_M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Hence

(*a*) Velocity-time function

$$v = l\frac{P_o/\lambda_M}{A_o(1 - P/A_o\lambda_M.t)} \quad . \quad . \quad . \quad . \quad (61)$$

(*b*) Length-time function

$$l = \frac{l_o}{1 - P_o/A_o\lambda_M.t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

(*c*) Stress-deformation function

$$\dot{e}_n = \sigma_n/\lambda_M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

(*d*) Strainwork-time function

$$w = \frac{1}{2E_M}(P_o/A_o)^2\left(\frac{1}{1 - P_o/A_o\lambda_M.t}\right)^2 + (P_o/A_o)\left(\frac{1}{1 - P_o/A_o\lambda_M.t}\right) \quad (64)$$

Rupture takes place when either

$$P_o/A_o = 3(1 - P_o/A_o\lambda_M.t)\sqrt{2\kappa E_{v,r}} \quad . \quad . \quad . \quad (65)$$

or

$$P_o/A_o = (1 - P_o/A_o\lambda_M.t)\sqrt{2E_M E_{o,r}} \quad . \quad . \quad . \quad (66)$$

i.e. at the time

$$t = \lambda_M(A_o/P_o - 1/3\sqrt{2\kappa E_{v,r}}) \quad . \quad . \quad . \quad . \quad (67)$$

or

$$t = \lambda_M(A_o/P_o - 1/\sqrt{2E_M E_{o,r}}) \quad . \quad . \quad . \quad . \quad (68)$$

Comparing Equ. (63) with (VII, 20) we see that, by assuming E_M to be very large, we have in some respects reduced the Maxwell to the Newtonian liquid. For the latter, Equ. (62) is of special interest when written in the form

$$1/l = 1/l_o - \frac{P}{V\lambda_M}t \quad . \quad . \quad . \quad . \quad . \quad (69)$$

In accordance with this equation, if we plot $1/l$ against time, a Newtonian liquid, and in a first approximation a Maxwell liquid also, should give a straight line the slope of which is $P/V\lambda_M$ and a measure of the fluidity of the material. Mr. T. J. Qadura of the Arab Training College, Jerusalem, independently found this relation (1947) and tried it with success on numerous observations with bitumen. If it does not break before, the rod would become of infinite length in finite time $= \frac{A_o\lambda_M}{P}$.

For behaviour under the other testing conditions, the reader is referred to the paper by Reiner and Freudenthal (1938). The apparently so simple ordinary tensile test with constant velocity of elongation leads to the most complicated equations.

11. The Reiner-Weissenberg Theory of Strength

The dynamical theory of strength, the application of which has been illustrated in the foregoing examples, was first established by Reiner and Weissenberg (1939). It states that the material will fail when the *strain-work*, which is the conserved part of the *stress-work*, has reached a certain limit. The difference between stress-work and strain-work should be kept in mind. The former is all the work performed by the stresses. This work will in general be partly conserved as elastic strain-energy and partly be dissipated. The first is the strain-work and it is equal to the stress-work minus the dissipated energy. We are, of course, speaking of the specific work, i.e. the work per unit volume of the material. In accordance with the different behaviour of materials under a change of volume and a change of shape, there will be different strengths in cubical dilatation and in shear. Water and every Newtonian liquid will have practically unlimited strength under isotropic pressure and considerable strength under isotropic tension. If we follow the first axiom, all volumetric stress-work is strain-work. It is different in shear. Here we have the two extremes of the Hooke-body where also all stress-work is conserved strain-work, and the Newtonian liquid where all stress-work is dissipated and none conserved. In every real material there will be *both* kinds of work, conserved and dissipated, and therefore the dynamical theory of strength, explained above, only is applicable.

12. Summary

In accordance with the Reiner-Weissenberg dynamical theory of strength a material fails when the strain-work reaches and exceeds its ultimate resilience E_f, strain-work being the difference of stress-work minus the dissipated work.

The following criteria result

A. A Newtonian liquid will not fail in steady states of flow. In non-steady states, a simple Newtonian liquid may fail through Reynolds' turbulence, while a generalized Newtonian liquid possessing structural viscosity may fail before that through structural turbulence.

B. Maxwell liquid

$$\tau_f = \sqrt{2\mu_M E_f} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII, a)}$$

$$d_f = \dot{\tau}_f/\mu_M + \frac{1}{\eta}\sqrt{2\mu_M E_f} \quad . \quad . \quad . \quad \text{(XIII, b)}$$

There exists a definite failure-stress, but no limit to deformation. The rate of deformation at which the material fails increases as the rate at which the load is applied.

C. Kelvin-solid.

$$d_f = \sqrt{2}E_f/\mu \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII, c)}$$

$$\tau_f = \eta_K \dot{d}_f + \sqrt{2}\mu E_f \quad . \quad . \quad . \quad . \quad \text{(XIII, d)}$$

There is a definite limit to deformation. The stress, however, at which the material fails increases with the rate of deformation.

In all the above equations the subscript "f" can be replaced by one of the following, "$(o)\ pl$", "$(o)\ r$". For the isotropic stress-strain case when E_f is either $E_{v,pl}$ or $E_{v,r}$, the modulus of elasticity μ must be replaced by the modulus κ and the coefficient of viscosity η by the coefficient ζ. Similarly in simple pull these coefficients will be E and λ.

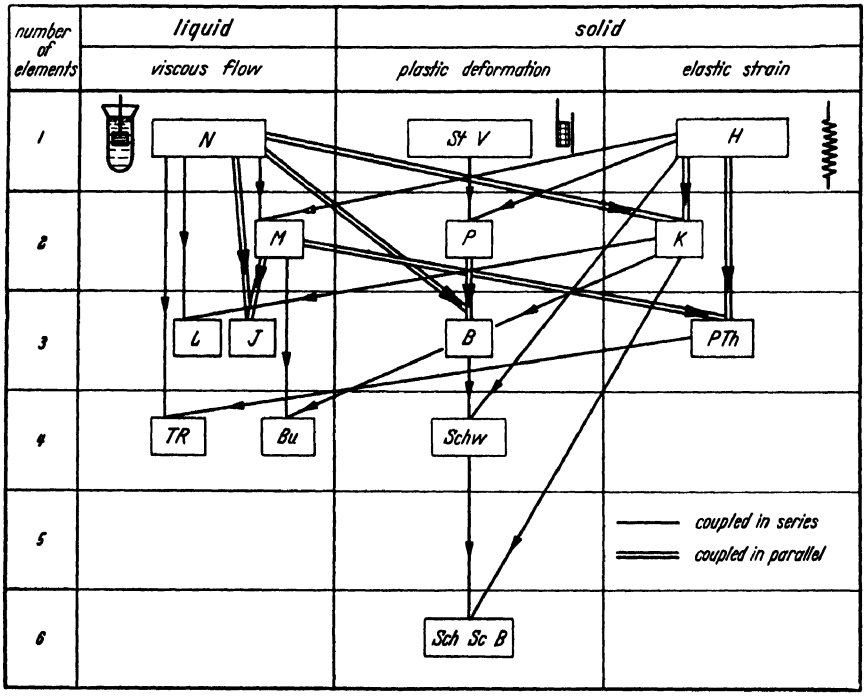
CHAPTER XIV

SECOND INTERLUDE

1. Retrospect

In Chapters VIII to XII we have shown how essential rheological properties can be reduced to the three fundamental properties of elasticity, plasticity and viscosity. Complicated rheological properties can be represented by ideal bodies formed through combinations of the three primitive bodies, denoted by *H*, *N* and *St V*. These ideal bodies form a hierarchy in accordance with the number of fundamental structural elements making up the body. Table I shows a "tree" of such bodies. Coupling in series is indicated by one branch line, coupling in parallel by two parallel branch lines. Some of the possible combinations have been postulated either on theoretical grounds or from empirical considerations. In addition to those treated before, we may mention $PTh = H/M$ postulated by Poynting and Thomson (1902) for the explanation of the behaviour of glass-fibres, and

TABLE I. *Tree of Rheological Bodies*



$T'R = N-PTh$ as postulated by Trouton and Rankine (1904) to illustrate the behaviour of a lead wire in torsion. Other combinations are possible, and may arise for the description of the rheological behaviour of materials not investigated so far.

The rheological equations of these ideal bodies are summarized in Table 2. They are written there in terms of shearing tractions τ and shearing deformations d_t, including their time derivatives $\dot{d}$ and $\dot{\gamma}$. As said before in Section XI, 3, the same equations can also be written in terms of normal tractions σ and normal deformations d_n, with applicable material parameters, E replacing μ, λ replacing η, etc.

As (chiefly) treated in the preceding chapters, the equations can only account for what may be called *first order phenomena*. The equations themselves are linear, and the parameters defining the material properties were (mostly) supposed to be constants. In addition, the deformations and their derivatives were expressed as linear functions of the displacement gradient (γ), or velocity gradient ($\dot{\gamma}$), as in Equ. (II, 35). All these simplifications imply what is called a *linear* rheological behaviour.* But they are not always adequate. Sometimes the material parameters reveal themselves as depending upon the rheological variables. This then introduces a non-linearity which is called *physical non-linearity*.

In addition, when the displacement gradient is not infinitesimal, the deformation cannot be expressed in the manner of Equ. (II, 35) with the gradient appearing in the first power only. For instance, if we define D in accordance with Equ. (III, 31), we have

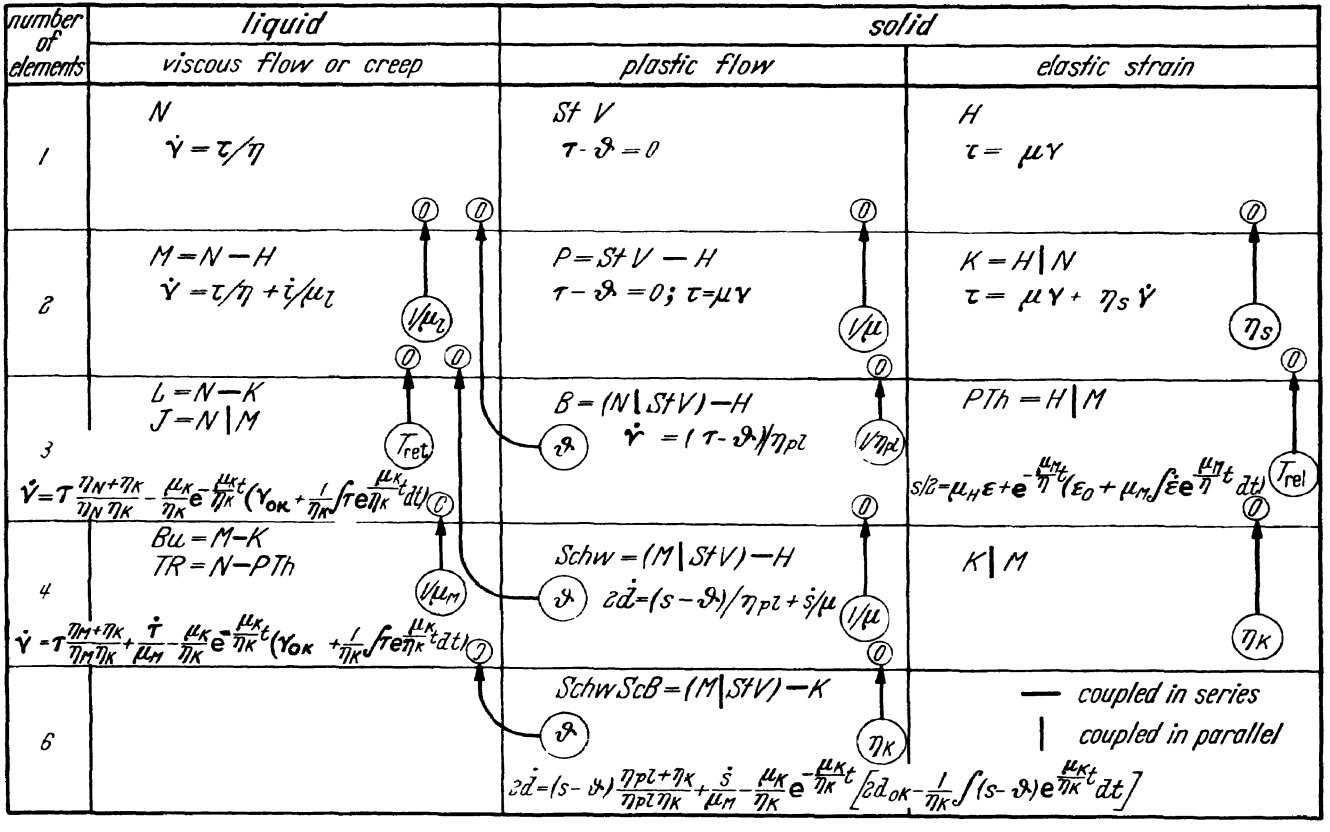
$$D = \ln(l/l_o) = \ln\left(\frac{l_o + \Delta l}{l_o}\right) = \ln\left(1 + \frac{\Delta l}{l_o}\right) = \frac{\Delta l}{l_o} - \tfrac{1}{2}\left(\frac{\Delta l}{l_o}\right)^2 + \ldots\ldots$$

This makes for, what may be called, *geometrical non-linearity*. Finally, there is a third category of non-linearity which I first described in two papers in respect of viscosity (1945) and elasticity (1948). This mall be called *tensorial non-linearity*. In the following chapters we shay deal with physical and geometrical non-linearity. The tensorial non-linearity is outside the field of an elementary introduction, and we shall only mention briefly some of its aspects. However, before doing so, we shall in the present chapter outline briefly how we arrive at the necessity of introducing a physical non-linearity.

When I just said that the treatment in the preceding chapters excluded non-linearity, this was not quite correct. Non-linearity could not be avoided when treating volume elasticity in Chapter III. It was at once clear that if we write $e_v = -p/\kappa$, the bulk modulus κ can under no circumstances be a constant, but must itself be dependent

* Workers in the field of high polymers speak of " linear behaviour " when they mean " proportionality ".

TABLE II. *System of Rheological Equations*



upon p or e_v. This is a result of the physical situation underlying the development of internal reactive forces when the volume of a body of the material is reduced through the action of external forces. It is therefore a case of physical non-linearity. There are other physical mechanisms which cause such non-linearity. They arise in connection with a micro-heterogeneity of the material. The materials with which rheology has to deal are nearly without exception dispersed systems, consisting of two or more " phases ", solids and fluids, of which one will often be air contained in pores which may be visible or invisible (compare Section 1 of Chapter XII). These dispersed systems can be distinguished as *sols* which act in the manner of liquids, and (in American terminology) *jellies* which act in the manner of solids, with *gels* forming a continuous transition from the first to the second. In sols the continuous phase or the *dispersion medium* is a liquid ; the *dispersed phase* can be a liquid, when one speaks of an emulsion, or a solid in the case of a *suspension*. In a gel the solid phase prevails, but does not necessarily form one continuous structure throughout the body. The gel may consist of " sponges " separated by thin layers of an amorphous phase. Using such pictures, a poly-crystalline metal may be considered as a gel, where thin layers of atomic disorder between the crystals form the " liquid " phase. Even water has been considered from this view-point with " water crystals " (cybomas) forming a structure, but distinguished from the polycrystalline metal-structure by the ease with which thermal vibrations may cause a water molecule to leave its regular lattice point. As an extreme example of what the rheologist may consider as a gel, we mention concrete as was treated in Chapter XI.

All these dispersed systems are regarded in *macrorheology* as they may appear to superficial inspection by the naked eye ; i.e. as if they were homogeneous or devoid of structure or are what may be called *quasi-homogeneous*. A material is quasi-homogeneous when the size of the largest dispersed element is smaller than the smallest volume-element, the deformation of which is under consideration. Concrete is thus treated when the dimensions of the structural elements made of it are much greater than the dimensions of the largest stones of which it is composed, even when these stones are well visible to the naked eye on a quite superficial inspection.

Macrorheology regards materials also as *quasi-isotropic*. This requires that the smallest volume-element under consideration should contain anisotropic dispersed elements of *all orientations*. Such is the case of a commercial polycrystalline metal, where the crystals, which constitute it, are oriented at random. A quasi-isotropic material can become anisotropic through plastic deformation, e.g. a metal sheet after rolling or a metal rod through drawing.

In the preceding chapters we have treated materials from the viewpoint of macrorheology. Even when in our M and K models the H-element may be due to a solid, the N-element to a liquid phase of a dispersed system, we treated the system as no different from a one-phase material which will show viscous damping or relaxation stresses. These M and K bodies belong to a "second generation" in the rheological tree. However, a dispersed system may as well belong to the simple bodies of the first generation. Such one is a dispersion of rigid spheres in a Newtonian liquid as was treated by Einstein (1905) in his Doctor-Thesis, the other a dispersion of rigid spheres in a Hookean solid, as treated by my assistant Dr. Z. Hashin (1955) in his Master-Thesis.

2. Einstein's Viscosity Equation

The simplest dispersed liquid system is obviously a suspension of small rigid spheres in a simple viscous liquid. Let us revert to our prism of liquid shown in Fig. I, 2. Let us imagine a small sphere of this liquid in the centre of the prism separated by an imaginary surface from the rest of the liquid. After the displacement v of the upper surface has taken place, the centre of the sphere will have moved by $v/2$ and at the same time the sphere will have been converted into an ellipsoid such as shown in Fig. 1, 1. Let us now replace the liquid sphere by a *rigid* sphere. What effect will this have? The centre of the rigid sphere will have moved by $v/2$, exactly as in the case of the liquid sphere. The rigid sphere will, however, not be distorted, *it will*

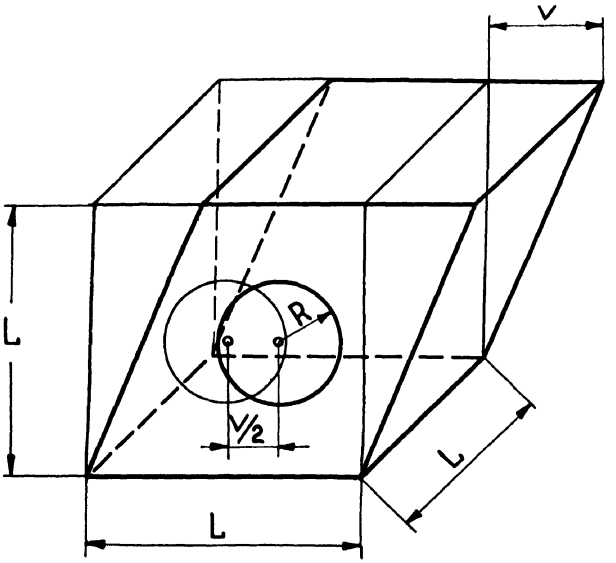


FIG. XIV, 1. Rigid sphere suspended in liquid.
Volume concentration $c_v = 4/3\ R^3\pi/L$.
Surface of sphere $4R^2\pi$.

remain a sphere (compare Fig. XIV, 1). It is clear that this must increase the resistance to the deformation of the prism containing the sphere, i.e. in order to obtain the same v, the force P must be increased. This will become manifest in an increase of the viscosity from η_{solv}, the viscosity of the solvent which is a pure liquid, to η_{soln}, the viscosity of the suspension or solution. The problem is one of hydrodynamics. We have to find the streamlines of the liquid particles of the dispersion medium, such that the boundary conditions are satisfied. The latter require, as in the case of the laminar flow in the viscometers, that wherever the liquid is in contact with the solid wall, or in the present case the solid suspended particles, it adheres to the solid and participates in its movements. In the case of telescopic flow it was not difficult to find the solution to this mathematical problem. We could guess that the streamlines are straight lines, all parallel to the axis of the tube. In our present case we cannot make such a guess and must proceed from first principles. To show how this is done would require a mathematical apparatus which transcends an "elementary" treatment. We must therefore restrict ourselves to reporting that Einstein in his *Doctor's Thesis* in 1905 solved the mathematical problem indicated before and found

$$\eta_{soln} = \eta_{solv}(1 + 2{\cdot}5\, c_v) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where c_v is the volume concentration of the solute or dispersed phase. Equ. (1) was derived on the assumption that the solution is very dilute; in fact so dilute that the distances between the particles are infinitely great in comparison to their diameters. In this case every sphere modifies an otherwise laminar flow around it in a restricted field; while outside that field the flow would remain laminar, if not in turn modified by another particle. If, however, the concentration is greater, in fact so great that the modification in the laminar flow produced by one particle reaches into the field of neighbouring particles, the resistance to flow will be increased still more.

Introducing the term Relative Viscosity

$$\eta_{rel} = \eta_{soln}/\eta_{solv} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, with Staudinger, Specific Viscosity

$$\eta_{spec} = (\eta_{soln} - \eta_{solv})/\eta_{solv} = \eta_{rel} - 1 \quad . \quad . \quad . \quad . \quad (3)$$

and finally, with Kraemer (1938), Reduced Viscosity

$$\eta_r = \eta_{spec}/c_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we can write (1) more generally

$$\eta_{spec} = \eta_r c_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Kraemer has also introduced the term (Limiting) Intrinsic Viscosity

$$\eta_i = [\eta] = \lim_{c_v \to 0} (\eta_{spez}/c_v) = \lim_{c_v \to 0} \eta_r \quad . \quad . \quad . \quad . \quad (6)$$

As said before, only in the case of a very dilute suspension of rigid spheres is $\eta_r = 2{\cdot}5$. If the concentration exceeds about 3 per cent., η_r is not a constant, but depends upon and increases with c_v. Mark and his school have investigated modifications of (1) for higher concentrations (1940, pp. 282, 293–295). They find in a second approximation $\eta_r = 2{\cdot}5 + 14{\cdot}1\, c_v$. In this case there is still $[\eta] = 2{\cdot}5$.

As can be seen from (2) the relative viscosity in the case of Einstein's mechanism is independent of the temperature of the solution. When the temperature is increased, the viscosity of the solution will generally be decreased. This, however, will be entirely due to the decrease of the viscosity of the solvent, and the *relative viscosity will*, therefore, *not be changed.*

A remarkable feature of Einstein's equation is that the *size* of the suspended particles is *irrelevant* as long as all are spheres, i.e. geometrically similar. This can be understood as follows :–

The viscous resistance can be measured by the energy dissipated by it per unit volume. Let us look again at Fig. XIV, 1. It is reasonable to assume that the additional energy dissipated per unit volume through the presence of the rigid sphere does not depend upon the scale of the picture. If, for instance, the linear scale is reduced, the volumes of both the liquid and the sphere decrease as the third power of the reduction of the scale and c_v is not changed. The radius, however, is reduced as the first power and the area of the surface of the sphere as the second power of the scale. If, therefore, the scale is to be of no influence, neither the radius nor the surface of the sphere can enter into the formula of the viscosity. There is one flaw in this reasoning : we assume the liquid to be homogeneous without limit. However, when we reduce the scale, we cannot at the same time reduce the size of the molecules of the liquid. Therefore our reasoning will break down when the scale is so much reduced that the size of the suspended particle approaches the size of the liquid molecules of the solvent. Then the Einstein equation becomes invalid.

I have mentioned Hashin's (1955) treatment of a dispersion of rigid spheres in a Hookean solid. He found later that the elastic moduli κ_o and μ_o of the medium are increased by spherical inclusions of another elastic material (κ_p, μ_p) as follows :—

$$\left.\begin{aligned} \kappa &= \kappa_o \left[1 - 3(1-\nu_o)\, \frac{1 - \kappa_p/\kappa_o}{2(1-2\nu_o) + (1+\nu_o)\,\kappa_p/\kappa_o}\, c_v\right] \\ \mu &= \mu_o \left[1 - 15(1-\nu_o)\, \frac{1 - \mu_p/\mu_o}{7 - 5\nu_o + 2(4-5\nu_o)\,\mu_p/\mu_o}\, c_v\right] \end{aligned}\right\} . \quad (7)$$

In the case of holes one introduces $\kappa_p = \mu_p = 0$, and finds

$$\left.\begin{aligned} \kappa &= \kappa_o \left[1 - \frac{2}{3}\frac{1-\nu_o}{1-2\nu_o} c_o\right] \\ \mu &= \mu_o \left[1 - 15\frac{1-\nu_o}{7-5\nu_o} c_v\right] \end{aligned}\right\} \quad \ldots \quad (8)$$

In the case of rigid spheres $\kappa_p \to \infty$, $\mu_p \to \infty$, and the result is

$$\left.\begin{aligned} \kappa &= \kappa_o \left[1 + 3\frac{1-\nu_o}{1+\nu_o} c_v\right] \\ \mu &= \mu_o \left[1 + \frac{15}{2}\frac{1-\nu_o}{4-5\nu_o} c_v\right] \end{aligned}\right\} \quad \ldots \quad (9)$$

Because of the viscosity-elasticity analogy, Einstein's equation (1) can be derived from Hashin's equation (9, 2) as a special case by introducing $\nu_o = 1/2$.

I have mentioned in Section XI, 5, that I found Einstein's equation to apply to the creep of mortars. I have suggested reasons why this may be so. I may add here that actually, even if the cement-stone which is the " continuous " phase in a cement-mortar, creeps or flows slowly in the manner of a viscous liquid, it still is not an N-liquid but rather an M-liquid, and in order to solve the puzzle, we would have to solve the mathematics of a dispersion of rigid spheres in a Maxwell-liquid, a problem which still waits for its master.

3. Elongated Particles

As can be seen from Fig. I, 1, simple shear is accompanied by rotation. If the suspended particles are not spheres but, say, ellipsoidal, the mechanism is complicated by the rotation of the particles. The magnitude of the interference by an ellipsoidal particle with the free flow depends on the orientation of the particle in the streaming liquid, i.e. on the angle ϕ between the main axis of the ellipsoid and the direction of flow. This angle is constantly changed through the rotation of the particle ; but if the particle is a sphere, all axes of the particle are geometrically equivalent and the rotation has no influence. If the temperature is low and there is no Brownian movement, the particles may ultimately become completely oriented or directed due to the flow of the liquid. Let η_i in (5) in this case be denoted by η_{iNB}. If there is a Brownian movement and, due to a higher temperature, it is so strong that it completely maintains the isotropic distribution of the main axis of the particles, they will not become oriented. Let η_i in this case be denoted by η_{iB}. Generally η_i depends upon the temperature. In a random orientation the particle will interfere with the laminar flow of a greater volume of liquid, while, when oriented

in the direction of flow, the volume will be less. Therefore, η_r will generally increase with increasing temperature. Accordingly

$$\eta_{rNB} < \eta_r < \eta_{rB}.^*$$

4. Prospect

In the following chapters we shall treat problems of physical non-linearity, mainly a variability of the coefficient of viscosity η, which generalizes the Newtonian liquid. In addition we shall briefly mention a variability of the yield stress ϑ, by which the phenomenon of work hardening is defined. With regard to a variability of the modulus of elasticity, we have already mentioned the variability of the bulk modulus κ, for *very high* pressures. A variability of the modulus μ (or E) occurs only in so-called *rubbery materials*. I do not deal with such materials in this book for the simple reason that I have not had personal experience with any such material. This has now become a very wide field, and I leave its treatment to the specialist. I have already mentioned Treloar's (1949) book, and I may add the excellent chapters by Staverman and by Schwarzl in *Die Physik der Hochpolymeren* (1956). I also learn that Dr. H. Leaderman is preparing a book on the Physics of High Polymers, for which he will use the notes of the lectures which he recently gave on this subject at Tokyo. In these rubbery materials the elasticity is "kinetic": similar to the elasticity of gas. In "potential" elasticity there is no physical non-linearity, whatever statements to the contrary the reader may find. For example, in an accepted textbook of *Mechanics of Materials of 1953* the author states: "The straight part of the (stress-strain) curve hardly exists in concrete, annealed copper, or cast iron". I can definitely state that this is not correct in respect of cement-stone. I do not have experience with annealed copper and cast iron, but I strongly suspect that conditions in these cases are not different from the case of cement-stone. The trouble is that workers judge elastic behaviour from *loading curves* with loading taking time, and do not compare *de-loading curves*. In this manner they include both permanent sets and elastic fore-effects in instantaneous elasticity.

In addition to the physical non-linearity, we shall, as has been said before, also mention briefly cases of geometrical non-linearity, with a hint on tensorial non-linearity.

* There have been attempts to calculate η_i from hydrodynamic principles for rod-like and disc-like particles. The mathematics is naturally still more difficult than in the case of spherical particles. A report is given by Mark (1940, pp. 279, 283-285).

CHAPTER XV

STRUCTURAL VISCOSITY

1. Historical

WE mentioned in the last chapter that there are true liquids the viscosity of which is not a constant, but decreases with increasing shear. We have called these liquids *generalized Newtonian liquids*. Their flow curves are not linear and we may therefore also speak of " non-linear behaviour " or " non-linear flow ". In the United States their existence was discovered *after* the discovery of the very soft plastic solids which show variability of an " apparent viscosity " and this led to naming the former " pseudo-plastics ". In Europe the order of discovery was in the reverse. Following Graham (1861) who spoke of the viscometer as a " colloidoscope ", viscosity measurements —generally by means of the Ostwald viscometer which does not permit of the controlled variation of pressure— became one of the most widely used methods of investigating colloidal solutions. However, much of the material thus accumulated was difficult to interpret, possibly due to the " anomalous " behaviour in colloidal solutions, i.e. the variability of the coefficient of viscosity. This variability was observed by Garett and Denning as early as 1903. W. R. Hess from 1906 onwards investigated blood, serum, and several sols, in a capillary instrument of his own which permitted of the variation of pressure and found that the volume of flow did not increase in linear ratio with the pressure, but more rapidly. In 1913 Hatschek investigated gelatine-sols in the concentric cylinder apparatus and found that the angular deflection of the inner cylinder did not increase in linear ratio with the angular velocity, but more slowly. The concentric cylinder apparatus and the Hess capillary instrument were afterwards used by Freundlich and his pupils (1924) in several series of measurements on different sols ; while Wo. Ostwald and his pupils (1925, 1926) carried out a great number of investigations by means of a capillary instrument of special construction, where the pressure is due to the hydrostatic head and varies continuously as the instrument empties.

Ostwald attributed the anomalous behaviour to " structure " in the liquid and coined the term " structural viscosity ". Freundlich and Szegvari (1924) following Hess (whom we mentioned in Chapter X) suggested an " elasticity " of the liquid. I took up the suggestion and (without knowledge of what had been accomplished in the States) developed Equ. (VIII, b) and (VIII, d), but this led me away

from the true liquid of variable viscosity to the soft plastic solid. I immediately came back to the problem of the anomalous liquid in two papers in collaboration with Miss Riwlin (1927 and 1928) where we followed a suggestion made by Hatschek and Jane (1926) that particles in a liquid suspension carry with them liquid envelopes which have a maximum volume when the liquid is at rest and are sheared off increasingly with increasing velocity gradient, thus causing a drop in the viscosity. This landed us in difficult mathematical calculations not essentially connected with the physics of the problem and which can be avoided if, instead of the viscosity, the fluidity is considered ; and instead of its dependence upon the rate of shear,* its dependence upon the shearing stress. In any case, when I joined Prof. Bingham in 1928 at Lafayette College, I was well prepared to point out that there are two *essentially different groups* of materials which show a fall of apparent viscosity with increasing shear, viz. (i) soft plastic solids as, e.g. oil paints, (ii) true liquid dispersed systems, as, e.g. a gelatine solution. As I have already mentioned, I called the latter at the time a non-Newtonian liquid, but have now adopted the term generalized Newtonian liquid.

2. Mechanisms of Structural Viscosity

When approaching the problem of the generalized Newtonian liquid for which η varies, we are confronted at the outset with a very interesting fact. As far as I know, *all simple liquids and true solutions are simple Newtonian ; and all generalized Newtonian liquids are dispersed systems.*† However, this statement cannot be reversed.

There is no reason why a suspension of rigid spheres in a simple Newtonian liquid should not behave in the simple Newtonian way. The non-linear behaviour of a dispersed system can be explained by the fact that a suspended particle *interferes* with the mobility of part of the dispersion medium, *if it can be shown that this interference decreases with increased shear.* The problem is therefore to explain why and in what manner an increased shear should decrease the interference.

We can distinguish two sorts of interference, which we may call (1) Disturbance and (2) Immobilization.‡

" Disturbance " is the mechanism which has been described in

* Compare Section 1 of Chapter XVIII.

† I find in Mark (1940), p. 267, a remark to the effect that " lubricating oils " show viscosity anomalies, but I do not know whether or not these oils are simple liquids. No reference is given.

‡ When reading through the following, the reader may compare the Table in the Summary, Section 7.

the preceding chapter. " Immobilization " concerns the fact that the suspended particle will generally bind part of the suspension medium so that a larger particle is formed which is suspended *as a whole* in the dispersion medium. Immobilization is due to solvation. When Einstein tried his equation on a 1 per cent. sugar solution, he found that the specific viscosity was about one and a half times as great as it should have been in accordance with his theory. He explained this by assuming that a molecule of sugar absorbs so much water that actually c_v, including the immobilized solvent, is much greater. We may define by

$$\alpha = \eta_i/2{\cdot}5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a Spherical Equivalent Volume Coefficient. In the case considered by Einstein α was therefore $= 1{\cdot}5$.

According to Haller (1931) we have to distinguish two kinds of solvation : (A) the " energetic ", through adsorption, (B) the " steric ", through geometrical inclusion.

3. Disturbance

Disturbance can lead in four different ways to non-linear behaviour, if the particle is not a sphere or not rigid.

(i) *Competition Between Orientation and Brownian Movement.* This case was briefly indicated in Section 3 of Chapter XIV above. If the particle is elongated, at rest all values of the above-mentioned angle ϕ are equally probable. This gives a certain definite viscosity η_o. Through flow, ϕ is changed so that the particles become more or less oriented. If the orientation is *complete*, the viscosity assumes a certain value η_∞. It is obvious that $\eta_\infty < \eta_o$.

The orientation is counteracted by the rotational diffusion of the Brownian movement. Where the Brownian movement is negligible, η_∞ is established as soon as a steady state of flow is reached and the system is therefore *simple*, *linear* of viscosity η_∞. Where the influence of the Brownian movement prevails over the orientation forces, η_o is maintained and in this case also the flow is simple, linear. Generally there will be at every temperature and every shear velocity a dynamical equilibrium between the forces of diffusion and orientation, with corresponding η, where $\eta_o > \eta > \eta_\infty$.

This is the mechanism by which Staudinger explained the non-linear behaviour of high polymers in solution. In this case, however, *the relative viscosity must be dependent upon the temperature.* An increase of temperature not only changes the viscosity of the solvent, but also brings with it increasing Brownian movement and accordingly a shifting of η towards η_o. Therefore, in this case η_{rel} increases with increasing temperature.

(ii) *Elastic Deformation.* The interference with the deformation of the liquid, by reaction, produces stresses in the particle. Through these stresses the particle is deformed. The deformation may be considerable. For example, in order to explain the elasticity of rubber, it was at one time assumed that the rubber molecule is formed like an elastic spring.* In this case the molecule will be extended elastically. The tension and extension of the spring is at its highest when its axis is inclined at an angle of 45 degrees towards the direction of flow.† However, the particle is not only deformed but also rotates. Through rotation, the spring is unloaded and recovers. This must result in an oscillation of the particle, which is transmitted to the liquid and the amount of dissipated energy is thereby increased. If such a mechanism exists, the result must therefore be an *increase* of the relative viscosity with increased shear, just as in turbulence eddies in the liquid increase the resistance to flow or take up energy and are accordingly equivalent to an increase of viscosity.

(iii) *Straightening of Curled Threads.* If the particles are long threads without rigidity, as has been assumed by Haller (1931), the result is entirely different. The tension produces a straightening of the thread. This decreases the interference of the particle with the flow of the liquid and therefore the viscosity also decreases. Rotation of the particle tends to restore coiling and curling. Such mechanism may be independent of temperature.

(iv) *Shortening.* If the tension is very great, or the breaking strength of the particle very small, the particle may break as soon as its axis coincides with the direction of maximum tension ($\phi = 45°$). The particle may also be broken through bending. Staudinger (1932) at one time pictured high polymer molecules as stiff rods of very small strength which can easily be broken. Shortening of particles decreases the ratio between the larger and the smaller axis of the particle, makes it more sphere-like and therefore decreases η. Whenever the fragments touch each other *in a suitable position*, the particle may be re-formed. The probability for this, however, is small. In this case the consistency curve produced in decreasing shear, therefore, will be different from the curve produced before in increasing shear, i.e. the consistency curve will show a hysteresis loop.

4. Immobilization

Immobilization can cause non-linear behaviour in the following ways :—

* For a more recent theory of the elasticity of the rubber molecule the reader may consult Treloar (1949).

† We have shown in Chapter III that simple shear is equivalent to a tension and a compression at right angles to one another and both inclined at 45° to the direction of shear.

(A) *Adsorption.* In the case of adsorption, the adsorption layer may be sheared off through the flow of the liquid. As we mentioned in Section 1 above, such a mechanism was suggested by Hatschek and mathematically formulated by Reiner and Riwlin. In this case there must be an influence of change of temperature. Adsorption is an exothermal process and should decrease with increasing temperature.

(B) *Steric Immobilization.* Steric immobilization has been discussed at great length by Kraemer and Williamson (1929). They imagined the particle to be an " ultramicroscopic body which is generally permeated in gel-like fashion with the liquid medium. The union of the structural elements forming the particle may involve any or every type of bond which can operate between atoms or molecules ". The particle may be either (i) a giant molecule immobilizing solvent between the branches and in its ring, (ii) a micelle (crystalite), (iii) a group of micelles with a regular arrangement, (iv) an aggregate or unordered assemblage of smaller particles.

Non-linear behaviour can be brought about in two ways

(*a*) Through deformation ;
(*b*) through rupture.

Regarding (*a*) it should be noted that, in general, the particle can be much deformed. It will therefore partially participate in the deformation of the liquid. By deformation of the particle, solvent is squeezed out of it as from a sponge and part of the liquid is therefore mobilized.

Regarding (*b*), the deformation is accompanied by stresses. The particle can resist these stresses to *a certain limit* only. If that limit is exceeded, the particle breaks and this also releases part of the immobilized liquid. If the fragments of the particle come to touch each other in the course of flow, the particles may be re-formed. " For a particular rate of shear, a dynamic equilibrium would be established between the rate of disaggregation and the rate of re-aggregation." It should, however, be kept in mind that the probability of reaggregation is generally not the same as that of disaggregation. They can be equal in the case of B (iv) only, i.e. in the case of unordered aggregates. Only in this case can every meeting of the fragments in *any position* result in a re-aggregation. In the case of molecules, micelles and ordered aggregates, a binding of fragments can take place only if the fragments are in a *certain definite* position in respect of each other. Not every position is therefore equally favourable. If consistency measurements are made both ways, i.e. the rate of shear is increased from zero to a maximum and then again decreased to zero, there will result not one single curve, but two such curves forming a

hysteresis loop. This forms the mechanism of the property of *thixotropy*.

With regard to an influence of change of temperature, it may be assumed that the binding forces in unordered aggregates are so small that the aggregate will partially be destroyed by an increase of temperature.

5. Structural Criteria

The considerations of the two preceding sections lead to the conclusion that there may, and probably do, exist different types of variable viscosity. Table I gives a summary of the different mechanisms and their properties. There are four criteria which enable us to decide whether a given generalized Newtonian liquid belongs to one type or another. They are (I) the influence of a variation of the parameter of temperature on η_{rel}, (II) the reversibility or irreversibility of a variation of rate of shear, or existence or absence of a hysteresis loop ; (III) the existence of a lower limit of non-linear behaviour ; (IV) the relation between η_{i0} and $\eta_{i\infty}$ where η_{i0} is the intrinsic viscosity at rest and $\eta_{i\infty}$ the intrinsic viscosity at maximum shear.

Regarding (I), it is obvious that η_{rel} cannot show a dependence on temperature as long as the phenomenon of non-linearity is entirely geometrico-mechanical in nature. In this case the variation of the viscosity of the solution with temperature is solely due to the variation of the viscosity of the solvent. " Shortening " (1 (iv) of the Table) represents such a mechanism, because the breaking of the particles changes their geometrical shape only. In case (1 (iii)) the influence of temperature will probably also be absent. On the other hand, we can say that there must be a dependence on temperature in the cases 1 (i) and A and B (iv). Nothing can be said *a priori*, in this respect, of the other mechanisms.

With regard to (II), we have already mentioned that an instantaneous change of viscosity with changing rate of shear, i.e. without any lag, is possible only if all positions of the particles are equally favourable for a re-aggregation. This is not the case if the particles are molecules or micelles.

Regarding (III) it should be noted that whether the non-linear viscosity is due to a partial destruction of the structure (2, B, ii, iii, iv) or a shortening or reduction of the primary particle (1, iv ; 2A) there must be a lower limit of non-linearity. The forces which form the primary particles or the secondary structures can be overcome only when the stresses produced by the flow reach and exceed a certain value. Up to that value the behaviour is normal, i.e. simple linear. Such a limit cannot exist in the cases (1, i, ii, iii) where the smallest stress must be of influence.

Regarding (IV), if the structure of the dispersed phase is the same, *whatever the concentration*, $\eta_{i0}/\eta_{i\infty}$ should be independent of c_v. For instance, if the viscosity anomaly was due to mechanism 1, i—competition between orientation and Brownian movement—the effect upon the value of $\eta_{i0}/\eta_{i\infty}$ should be the same whether there was *one* particle only, or very many, present in the dispersion. On the other hand, in the case of 2, B, iv—unordered aggregates—secondary structures will possibly appear at higher concentrations only, while at low concentrations the dispersed phase will consist of stable primary particles. In this case $\eta_{i0}/\eta_{i\infty}$ would increase with c_v.

All mechanisms require that when $\eta_{i\infty}$ is reached, the behaviour is simply linear. This value of $\eta_{i\infty}$ may be reached either at a finite rate of shear or be asymptotically approached at an infinite rate of shear.

As variable viscosity is generally connected with the disperse phase having a certain structure, Wo. Ostwald (1925) called it Structural Viscosity, a term which has been widely accepted.

6. The Ostwald Curve

From the above it follows that the consistency curve of a generalized Newtonian liquid generally shows the following characteristics: The curve starts at the origin (this makes the material a

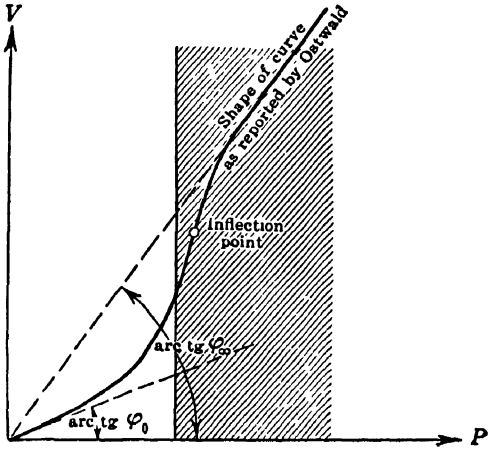


FIG. XV, 1. Consistency curve of a non-linear liquid.
φ_0 zero fluidity = fluidity of the liquid at rest.
φ_∞ maximum fluidity which the liquid can attain.
The unshaded part shows the shape of the curve as reported by Bingham (1928) for nitro-cellulose solutions and by Herschel and Bulkley (1926) for rubber solutions.

liquid) and the V-ordinates increase monotonously with the P-abcissae, i.e. there exists no maximum V. At the origin, the curve has a definite slope. For large V/P values, the curve approaches a straight line radiating from the origin and possessing a steeper slope than the slope of the curve at the origin. In consequence, thereof, the curve first turns its convex side towards the P-axis and after passing through an inflection point, reverses its curvature (compare Fig. XV, 1). The first discoverers of non-linear consistency curves observed the unshaded parts of the Fig. only and were led by this to translate the curves into " power-laws ", about which compare Chapter XVII. Wo. Ostwald (1925) was the first to observe and describe the shaded part, and we may therefore call a curve of such form an Ostwald Curve. Apart from observations which show that the curve approaches a definite viscosity η_∞, we can say, *a priori*, that while the characteristic property of a generalized liquid is the decrease of its viscosity with increasing shear, the viscosity naturally cannot be reduced to zero ; in fact the liquid being a dispersed system, its viscosity cannot be less than the viscosity of the dispersion medium.

7. Summary

In " simple " liquids, viscosity is a constant. Variable viscosity appears in *dispersed systems* and may be regarded as arising out of a change of " structure " (in the widest sense, including orientation) of the dispersed phase, with increasing rate of shear.

A dispersion of rigid spheres in a " true " liquid has a constant viscosity, which up to a concentration of about 3 per cent. is according to Einstein

$$\eta_{soln} = \eta_{solv}(1 + 2{\cdot}5c_v) \quad . \quad . \quad . \quad . \quad \text{(XV, a)}$$

Generally,

$$\eta_{spec} = \eta_{rel} - 1 = \eta_{soln}/\eta_{solv} \qquad 1 = \eta_i c_v \quad . \quad \text{(XV, b)}$$

where η_i is a function of the concentration c_v, besides depending on factors of " structure ". For a dispersion of spheres, in a second approximation, in accordance with Mark,

$$\eta_i = 2{\cdot}5 + 14{\cdot}1c_v \quad . \quad . \quad . \quad . \quad . \quad \text{(XV, c)}$$

Mechanisms which may cause structural viscosity are listed in the following Table.

Generally, a liquid of variable viscosity (" anomalous " liquid) will have a definite fluidity φ_0 when at rest and another definite fluidity φ_∞ which is the maximum which it can attain at the highest shear, where $\varphi_\infty > \varphi_0$.

TABLE I. *Types of Structural Viscosity*

Nature of interference with free flow of solvent	(I) Temperature dependence of η_{rel}	(II) Reversibility of consistency curve	(III) Existence of lower limit		(IV) Dependence of η_0/η_∞ upon c_v
1. Disturbance.					
(i) Competition between orientation and Brownian movement	increases	yes	no		no
(ii) Elastic deformation	,,	yes	no		no
(iii) Straightening of curled threads	nil	yes	no		no
(iv) Shortening	nil	no	yes		no
2. Immobilization.					
(A) Adsorption	decreases	yes	yes		no
(B) Steric immobilization			(*a*) deformation	(*b*) rupture	
(i) Macromolecule	?	no	no	yes	no
(ii) Micelle	?	no	no	yes	no
(iii) Ordered group	?	no	no	yes	no
(iv) Unordered aggregate	decreases	yes	no	yes	yes

CHAPTER XVI

RHEOLOGICAL ANALYSIS

1. Rheological Analysis of a Rubber Solution

In the last chapter we have discussed the different mechanisms which may cause structural viscosity. We have found that, from the shape of the consistency curve and the influence of a change of the parameter of temperature, conclusions can be drawn regarding the nature and structure of the particles dispersed in the liquid. This may be called Rheological Analysis.

Rheological analysis should, however, be understood in a wider sense to mean inference from rheological measurements upon the structure of a quasi-homogeneous dispersed system. Rheological literature contains many examples of rheological analysis. In the present chapter I shall describe, as an instance, the investigations on solutions of crêpe-rubber in toluene, which I carried out at Prof. Bingham's laboratory at Easton in 1932–33 in collaboration with Mrs. Reiner (1933).*

The viscosity of rubber solutions usually decreases with age. This ageing is due, according to Staudinger (1932), to a depolymerization caused by oxidation, which in turn is advanced by light and heat. For instance, as we found, in a rubber-turpentine solution the viscosity falls in the course of a few days to the viscosity of turpentine. Staudinger, therefore, used in his investigations on solutions of rubber in benzene very rigorous safeguards to exclude oxidation. We, happily, found that these safeguards are not necessary if toluene is taken as a solvent. By keeping the solution in the ice-box in brown bottles in an atmosphere of CO_2, the viscosity remained unchanged for as long as the experiments lasted, i.e. for nearly one year.

The rubber was light crêpe of 0·13 per cent. moisture, 0·32 per cent. ash and a density of 0·92. The solvent was chemically pure toluene of densities as in International Critical Tables 3, 29 and viscosities slightly higher than those determined by Thorpe and Rodger (1894).† The crêpe was cut into small pieces and put in small portions into the toluene in a bottle into which

* In his comprehensive book, Philippoff (1942) says of these investigations, " die vielleicht die genauesten an nicht-Newton'schen Fluessigkeiten sind ". I do not know whether this still holds good, but they will probably still serve well to illustrate the subject of the present chapter. I have, however, omitted a number of data and Tables contained in the first edition of this book.

† Even had the toluene been the same as that for which Thorpe and Rodger determined the viscosities and even if their figures were the correct ones, we had to make independent determinations in order correctly to calculate the *relative* viscosities, thereby eliminating apparatus-effects.

CO_2 was introduced. The bottle was then closed and mechanically shaken in the dark. The pieces of crêpe swell visibly in the toluene and are soon dissolved. There remained a small residue which was filtered off. Two solutions of about 2 per cent. concentration were prepared and the others obtained through dilution. The two solutions had very different histories. The first (I) was shaken during one month for twenty-two hours in all, then put aside in an ice-box for three months and only after that filtered and diluted to 0·97 per cent. It was observed in the viscometer more than seven months after preparation. The second solution (II) was shaken for one hour only during three days, filtered immediately, diluted to 0·98 per cent. and observed in the viscometer less than one month thereafter.

2. Instruments Used

The viscosities were determined in two sorts of apparatus, viz. capillary and rotation ; and in both in a number of instruments of varying dimensions. The capillaries were instruments as designed by Bingham (1922) of dimensions as shown in Table XVI, 1.

The rotation instruments (compare Table XVI, 2) were a Couette as improved by Hatschek (1928) for the large cylinder, while the small cylinder was of a different type and necessitated taking into account an " end-effect ".

TABLE XVI, 1. *Dimensions and Constants of Capillary Instruments*

No. of instrument	1-23	1-32	1-33	3-26	5-26
Symbol	o		Δ	⊡	•
R in cm.	·0109	0112	0116	0238	·0412
l in cm.	9·99	10·00	9·95	9·93	9·90
Q in cm.^{3}	3·94	4·00	3·95	3·98	4·00

TABLE XVI, 2. *Dimensions and Constants of the Rotation Instruments*

Cylinder	Large	Small
R_i in cm.	4·96	3·50
R_e in cm.	5·52	5·52
h in cm.	5·32	5·38 + 1·76 = 7·14
$a = (R_i/R_e)^2$	0·807	0·402

Distance of axis of cylinders from scale d = 129·4 cm.

Wire No.	27	2	4	5
Diameter in cm.	0·036	0·044	0·074	0·092
D torsional resistance in dyne/cm.^{2}	71,800	260,487	2,187,500	5,270,000
l in cm.	20·1	21·7	17·65	21·7

3. The Dispersions as True Sols

The behaviour of rubber in turpentine, described in Section 1, and in benzene as reported by Staudinger, made it doubtful whether such a material was sufficiently well defined. A substance which changes its properties simply with the passage of time,* is not suitable for a rheological analysis.

As we have said, toluene solutions under proper precautions did not show such " ageing ". They could, however, have been accidental in their composition and structure, depending upon their history. On comparing the viscosities of the two stock-solutions I and II for the same shearing stress (P) it was found that this was not the case. The viscosity of solution I was about 5 to 10 per cent. smaller than that of solution II. If we consider that the concentration of the latter was slightly higher than that of the former, and that the history of both was different in the *extreme*, the conclusion can be admitted that a rubber toluene dispersion is rheologically a well-defined, reproducible material, in fact a *true sol*.

4. Viscosities not Constants

It was known that rubber solutions showed " viscosity-anomalies ". This was confirmed by observations listed in Table XVI, 3. It contains the results of measurements performed on a solution of 1·69 per cent.

TABLE XVI, 3. *Concentration 1·69 per cent. by Volume at 20° C.*

1 Δp in $g/cm.^2$	2 t in sec.	3 $10^3Q/t$ in $cm^3/sec.$	4 η'	5 P in dyne/cm^2	6 V in sec.$^{-1}$
1. Viscometer 3-26 ▣					
— 86·4	3038·1	1·31	0·816	101·5	124·7
+ 131·5	1610·4	2·47	0·658	154·6	235·2
— 181·0	1033·5	3·87	0·582	212·8	366·6
+ 224·6	728·8	5·47	0·508	264·1	519·8
— 274·2	545·0	7·32	0·464	322·4	694·8
+ 316·8	425·0	9·38	0·418	372·5	891·3
— 361·9	352·4	11·3	0·396	425·5	1074·9
2. Viscometer 5-26 ;					
+ 38·3	846·8	4·73	0·909	78·1	86·0
— 84·4	278·4	14·36	0·658	172·0	261·5
— 132·3	142·4	28·1	0·527	273·0	517·8
— 178·0	89·8	44·6	0·448	368·3	823·6
— 225·6	62·4	64·2	0·395	462·8	1107·4

A " positive " driving pressure was taken as one driving the liquid through the viscometer from " left " to " right " ; a negative from " right " to " left ".

* It is not maintained that the change in the properties is *caused purely by the increase of time*. It is a principle of natural philosophy that time *itself* is not a cause (compare Maxwell, *Matter and Motion*). This principle may or may not have already been dethroned by the latest developments in cosmic physics, but we shall retain it for our terrestrial time-intervals. With our statement above we want to say that extra-rheological causes such as oxidation, depolymerization, etc., operate in this ageing process *in the course of time*.

volume concentration,* at a temperature of 20° C. in two different capillary viscometers.

The viscosities calculated from Poiseuille's formula are entered in Column 4. They turned out not to be constant. A rubber-toluene solution, therefore, is not a simple Newtonian liquid. The calculated viscosities must be considered as " apparent " viscosities and have accordingly been denoted by η'.

5. Plotting Consistency Variables

As a next step, the consistency variables V and P were calculated in accordance with Equs. (II, 12′) and (13′) and entered in Columns 5 and 6. From these data Fig. XVI, 1 has been drawn. While the

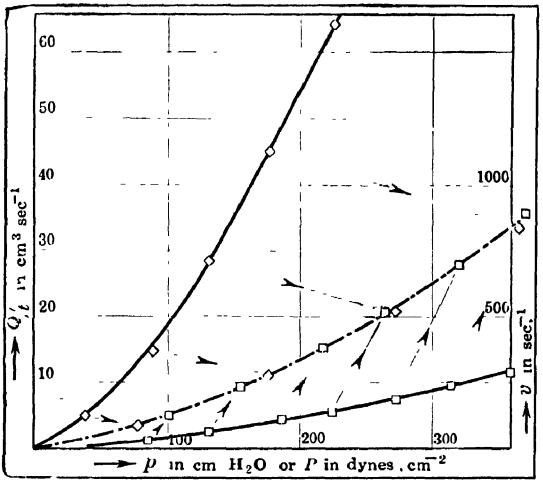


FIG. XVI, 1. Consistency curve of a 1·69 per cent. rubber-toluene solution determined with two different capillaries.
— —— Technical test curves.
—. - Consistency curve.

" technical " curves, when the measured quantities are plotted, fall widely apart, the consistency curves plotted in V/P variables all *coincide*. In Section 4 of Chapter II we have said that V/P variables give consistency curves independent of the dimensions of the apparatus in the cases of simple Newtonian liquids and in Section 3 of Chapter VIII we have extended this to the Bingham bodies. We now see that this is also true in the case of generalized Newtonian liquids. As we shall explain in Chapter XIX, this means that there is no slippage in the rubber-toluene solution at the boundary walls of the instruments, or generally, no " wall-effect ".

* The volume concentration was calculated from the weight concentration determined through evaporation, in the relation of the densities of toluene and rubber, viz. 1·05.

6. Relative Consistency Curves

All viscosity determinations were made at constant temperatures and the consistency curves are therefore *isotherms*. Comparison of different isotherms of the same material gives us one of the criteria

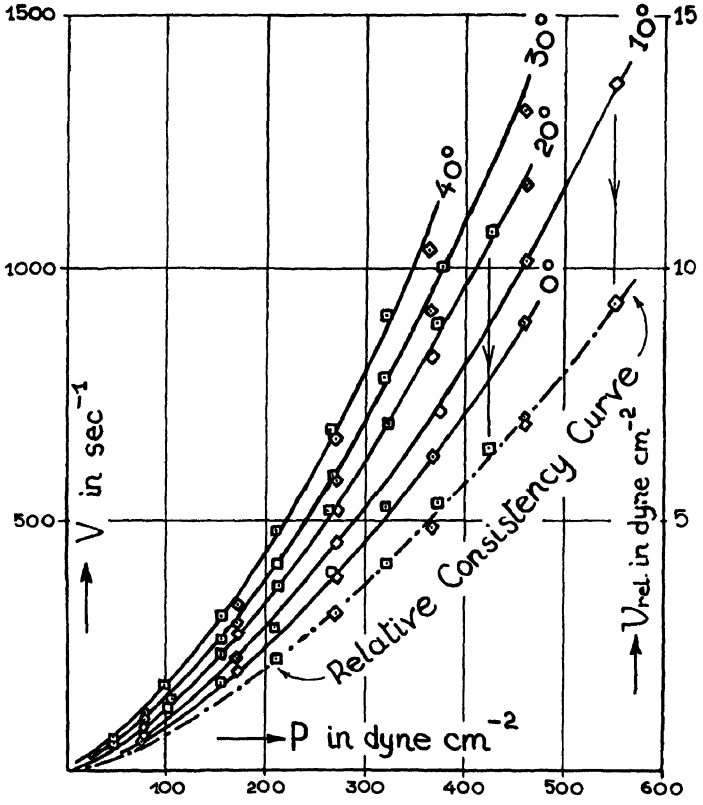


FIG. XVI, 2. Relative consistency curve of a 1·69 per cent. rubber solution determined in different capillaries and at different temperatures. All curves result in one and the same relative consistency curve.

of structural analysis of which we spoke in the last chapter. It was Staudinger who first made such systematic comparisons for *simple* Newtonian liquids, by considering *relative viscosities*. In our case the viscosity is not a constant and *varies in the instrument even in one single observation from layer to layer*. The comparison is therefore made by considering Relative Consistency Curves, where

$$V_{rel} = V \cdot \eta_{solv} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is equivalent to a change of the scale of the ordinate. Fig. XVI, 2 shows the ordinary consistency curves and also the one relative consistency curve for the 1·69 per cent. solution at temperatures of

0°, 10°, 20°, 30° and 40° C. For all five temperatures there results *one single relative consistency curve.* This proves that the structural viscosity is in this case a *purely mechanical phenomenon* and cannot be due to a competition of orientation by flow and disorientation by temperature, as was assumed by Staudinger (1932).

7. Confirmation of Ostwald Curve

The curves shown in Figs. XVI, 1 and XVI, 2 are all of parabolic form as were also those found by Herschel and Bulkley (1926), with rubber-benzene solutions. The concentration of 1·69 per cent. of the solution was rather high.

Fig. XVI, 3 shows the relative consistency curves for more dilute

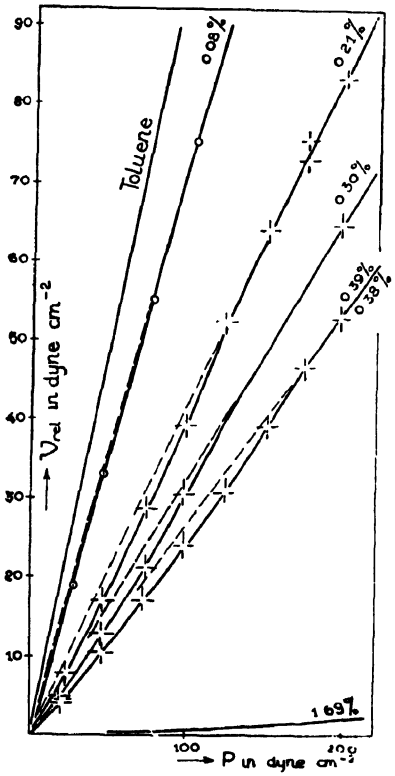


FIG. XVI, 3. Ostwald curves of dilute solutions.

solutions. They have the form reported by Ostwald. In this Fig. the curve for the 1·69 per cent. concentration is also shown on the same scale. As can be seen, the range of stresses in Fig. XVI, 2 was too small to reach to the inflexion point. The driving pressures were therefore

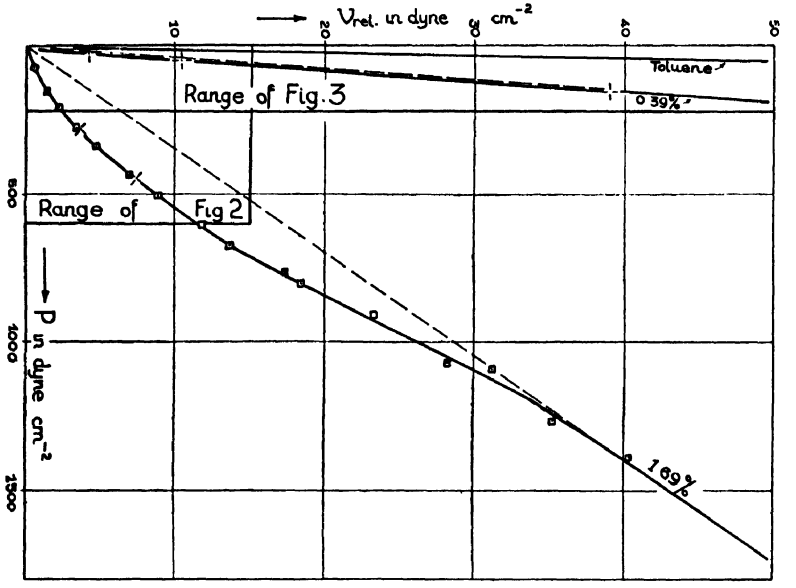


FIG. XVI, 4. Ostwald curve of a concentrated solution.

much increased ; the results are shown in Fig. XVI, 4. As can be seen for this high concentration also, the consistency curve is an Ostwald curve, if only the shear is high enough.

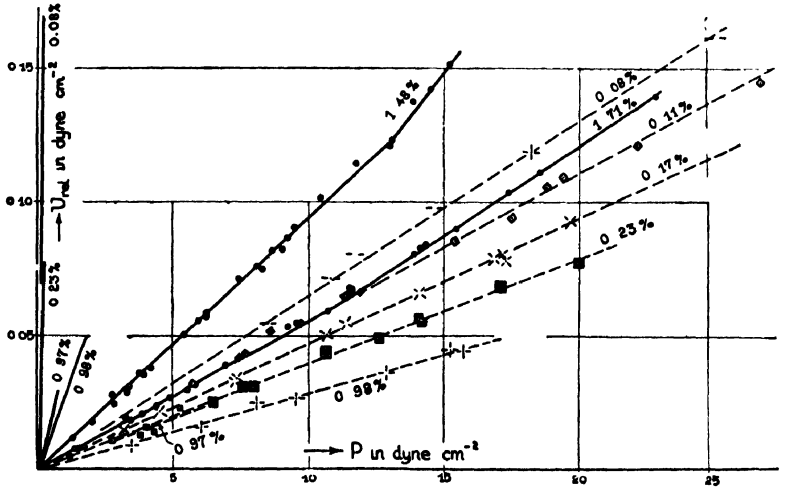


FIG. XVI, 5. Relative consistency curves in the Rotation-instrument. The curves in full lines are on the correct scale. In curves in broken lines, the scale for P has been exaggerated 10 and 100 times.

8. Existence of a Lower Limit of Curvature

The rates of shear in the capillary instruments are generally much higher than those in the rotation instruments. Some observations made with the latter are listed in Table XVI, 4.

TABLE XVI, 4. *Measurements in Rotation Instruments.*

1 Δs in cm	2 n in sec^{-1}	3 P in dyne/cm^2	4 V in sec^{-1}	5 V_{tot} in dyne/cm^2	6 φ'_{tot}
1. Concentration 1·71 per cent. at 24° C.					
(i) Large cylinder. Wire No. 5.					
— 10 12	0·182	11·55	11 83	0 0674	0 0058
— 13·53	0 242	15·43	15 75	0 0898	0·0058
— 16·25	0·299	18 55	19 45	0·1110	0·0060
— 20·06	0·382	22 80	24·87	0·1395	0 0061
— 15·25	0 278	17·40	18 10	0 1033	0·0059
— 12·37	0 223	14·10	14·50	0 0826	0·0058
— 8·09	0 143	9·23	9·31	0·0532	0 0057
— 6 09	0·104	6·95	6·77	0 0386	0·0056
— 3 40	0 055	3 88	3 85	0·0204	0 0053
+ 5·59	0 091	6·38	5·93	0 0338	0 0053
+ 9 40	0 160	10·73	10 39	0 0591	0·0055
+ 12·19	0 217	13 90	14·11	0 0805	0 0058
+ 5·12	0 085	5·84	5 53	0 0315	0 0054
+ 4 83	0 079	5·52	5 17	0 0293	0 0053
+ 3·85	0 063	4·39	4 10	0 0234	0·0053
+ 3 05	0 049	3 48	3 19	0 0182	0·0052
+ 1 22	0 020	1·39	1·30	0 0074	0·0053
— 2 60	0 044	2 97	2·86	0 0163	0 0055
— 4·30	0 071	4 91	4 62	0 0263	0 0053
— 6 49	0 111	7 40	7 22	0 0412	0·0056
— 8 39	0 147	9·57	9·56	0·0545	0·0057
— 10 25	0·181	11·69	11·78	0·0671	0 0057
— 12 56	0 227	14·75	14 75	0·0840	0 0058
(ii) Small cylinder. Wire No. 5.					
— 4·15	0 318	7·08	6 69	0 0378	0 0053
— 3 35	0 250	5 71	5 25	0·0296	0 0052
2 55	0 194	4 35	4 07	0 0229	0 0053
— 1 75	0 135	2 99	2 84	0 0160	0·0053
— 0·90	0 065	1·54	1·36	0 0077	0 0050
+ 0 60	0 050	1 02	1 05	0·0059	0·0058
+ 1·65	0 131	2·82	2 74	0 0155	0·0055
+ 2·70	0·211	4 61	4 42	0 0249	0 0054
+ 3·75	0 288	6·40	6 04	0 0341	0 0053
+ 4 85	0·391	8·28	8·21	0·0463	0·0056
+ 5·65	0·466	9·64	9 78	0·0551	0·0057
+ 7·75	0·588	12·35	12·35	0 0697	0·0056
+ 7·95	0·625	13·55	13·13	0·0740	0·0055

Fig. XVI, 5 shows the curves representing these and other observations. They make evident what could not be said with certainty from the curves from capillary-instruments, that the consistency curves possess tangents at the origin which are definitely inclined against the P-axis and that the solutions therefore possess well-defined fluidities φ_o in the state of rest.

They are therefore real liquids flowing under every stress, however small, and not solids having very low yield points. If they were solids the curves would have to contact the P-axis tangentially—which is not the case. This shows that the designation " pseudo-plasticity " is not justified ; there is no plasticity, however small.

Fig. XVI, 5, however, reveals more than this. The consistency curves not only pass through the origin with a definite slope but they start, in addition, with a portion of *straight line*. They are straight lines up to a definite point and only after that do they become curved.

9. The General Shape of the Consistency Curve

From what was found in the preceding Sections, it follows that the consistency curve of the rubber-toluene solutions is generally of the following shape : the curve starts at the origin as a straight line ; after this follows a curved part, first convex and then concave towards the P-axis ; the end is again formed by a straight line which can be extrapolated towards the origin. The curve therefore has four characteristic points, viz. (i) the origin, (ii) the point where the first straight part ends, (iii) the inflexion point, (iv) the point where it again becomes

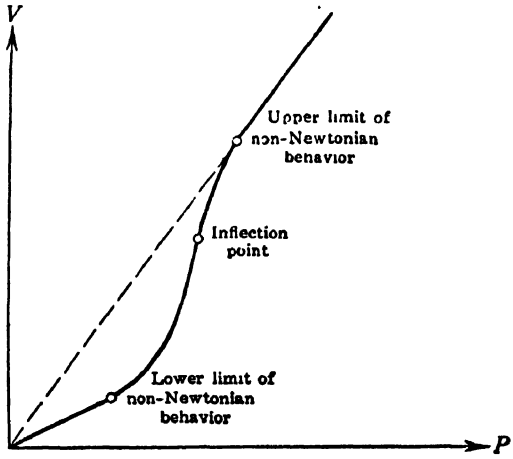


FIG. XVI, 6. General shape of the consistency curve of rubber-toluene solutions.

(*a*) lower limit of structural viscosity.
(*b*) inflexion point.
(*c*) upper limit of structural viscosity.

straight (compare Fig. XVI, 6). There are two regions in which the liquid is *simple* Newtonian, viz. (I) from rest to point a, (II) from c to turbulence. The points a and c are the lower and upper limit of structural viscosity. The points a, c, b define the following coefficients of viscosity, the viscosity at rest η_0, the minimal viscosity η_∞ and the apparent viscosity at the inflexion point η'_b. The first two, applying

as they do to simple Newtonian states, are *real* viscosities characteristic of the material. Between the points *o* and *a* on one hand and *c* to turbulence on the other, there is *the same viscosity* in every lamina of the flowing liquid, and one can therefore speak of *the* viscosity of the liquid. In contradistinction, between *a* and *c* the viscosity varies from lamina to lamina ; one cannot speak of the viscosity of the liquid and the calculated viscosities are therefore *apparent* viscosities.

10. The Reduced Viscosities as Functions of Concentration

The sols have two well-defined reduced viscosities, viz. η_{ro}, when the liquid is at rest and the structure of the dispersed phase not interfered with through flow ; and $\eta_{r\infty}$, when the structure of the dispersed phase has through high shear been reduced to the lowest possible level. In Table XVI, 5 the reduced viscosities for the different solutions have been listed.

They are represented graphically in Fig. XVI, 7.

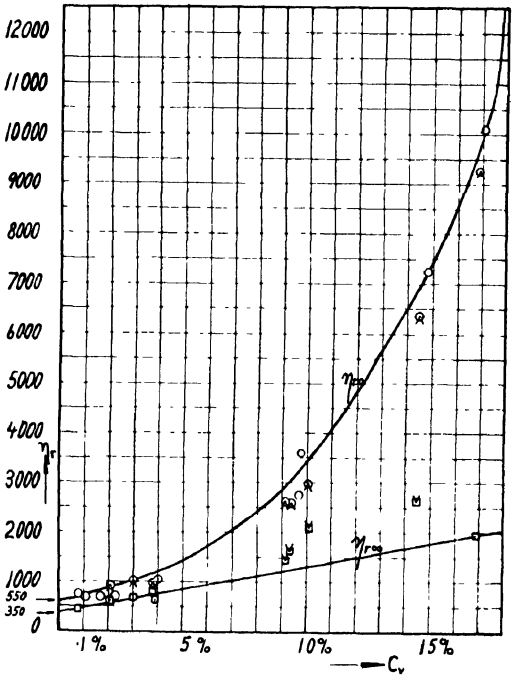


Fig. XVI, 7. Reduced viscosities at rest (η_{ro}) and at highest shears ($\eta_{r\infty}$).

It should be noted that the highest η_{spec} determined in the Bingham apparatus are generally lower than those determined in the Couette, the reason being that the rates of shears in the Couette are substantially lower than those produced in the Bingham apparatus.

TABLE XVI, 5. *Reduced Viscosities of Rubber-Toluene Solutions*

Concentration, $10^2 c_v$.		0·08	0·08	0·11	0·17	0·21	0 23	0 30	0·38	0 39	0·92	0 93	0·97	0 98	1 01	1·45	1 48	1 69	1·71
Instrument used		C	B	C	C	B	C	B	B	B	B	B	C	C	B	B	C	B	C
Maximum η_{spec} .		0·60	0·45	0·76	1·15	1·90	1 55	3 0	3·6	4 0	27 1	24 1	26 3	35·2	30	92	107	156	199
Minimum η_{spec} .		—	0·36	—	—	1·30	—	2 0	3·0	2 7	13·2	14·9	—	—	21	38	—	33	163
Observed	η_{ro} .	750	560	690	677	906	675	1,000	946	1,025	2,620	2,590	2,720	3,580	2,970	6,340	2,240	9,250	10,750
	$\eta_{r\infty}$.	—	450	—	—	620	—	666	790	692	435	605	—	—	2,080	2,620	—	1,950	—

Sometimes the lowest η_{spec} determined in the Bingham apparatus are not the min. η_{spec} because the rate of shear was not high enough. In the Figure the former have been indicated by the symbol $\wedge$, the latter by the symbol $\vee$ and the curves drawn in the former case *above*, in the latter case *below*, the observation point.

The curve is very uncertain up to $c_v = 0{\cdot}004$. I have drawn it so as to give a constant $\eta_{ro}/\eta_{r\infty} = 1{\cdot}56$, but it must be admitted that it would equally be possible to make $\eta_{ro} = \eta_{r\infty} = 700$ and $\eta_{ro}/\eta_{r\infty} = 1$, the observations for these very low concentrations being not accurate enough. It would be worth while to carry such observations with rubber or another high-polymer substance to a higher degree of accuracy ; I do not know of any published in relevant literature. In any case it is clear that from a concentration of 0·4 per cent. upwards both curves diverge very much and increasingly so at higher concentrations. We must, therefore, conclude either that the mechanism of structural viscosity is one which appears at concentrations above 0·4 per cent. only ; or that there are two essentially different mechanisms, one of a low order at concentrations below 0·4 per cent. and, in addition, another one of high order which only makes its appearance at concentrations above 0·4 per cent. In contradistinction to η_{ro}, the concentration curve for η_r, is simply a straight line. I am unable to suggest any theoretical interpretation for this fact. There must, however, be one, because it is remarkable that Philippoff and Hess (1936) found a similar straight line dependence for a solution of nitrocellulose in butylacetate. The limiting intrinsic viscosities are either max. $[\eta] = 550$ and min. $[\eta] = 350$ in the ratio 1·56 to 1 or alternatively both equal to about 700. The spherical equivalent volume coefficient as defined in Equ. (XV, 1) would accordingly be of the order of magnitude of 140 to 280 ; or to cause the same increase of viscosity, a dispersion of spheres would have to be of a concentration several hundredfold of the concentration of the rubber solution. For instance, the most dilute solution observed by us of a volume concentration of 0·08 per cent. would have to be of a concentration of from 11 to 22 per cent. This, of course, is out of the question and, therefore, the high viscosity of a high polymer solution cannot be due to the particles in solution simply occupying more space, as it were, as was thought at one time. The reason for the high viscosity, even in very dilute solutions, must be looked for in " additional currents caused by the presence and the movements of the particles in the liquid, *currents which mutually disturb one another* ". (Mark (1940), p. 283.)

11. The Apparent Fluidity as a Function of Stress

One of the solutions (the one of 1·69 per cent. concentration) was observed over a wide range of shears and its concentration was high

enough to show the non-linear behaviour in a pronounced manner over the whole range—between the linear behaviour at low and the linear behaviour at high rates of shear. Its relative consistency curve is shown in Fig. XVI, 4 and its apparent relative fluidity in Fig. XVI, 8

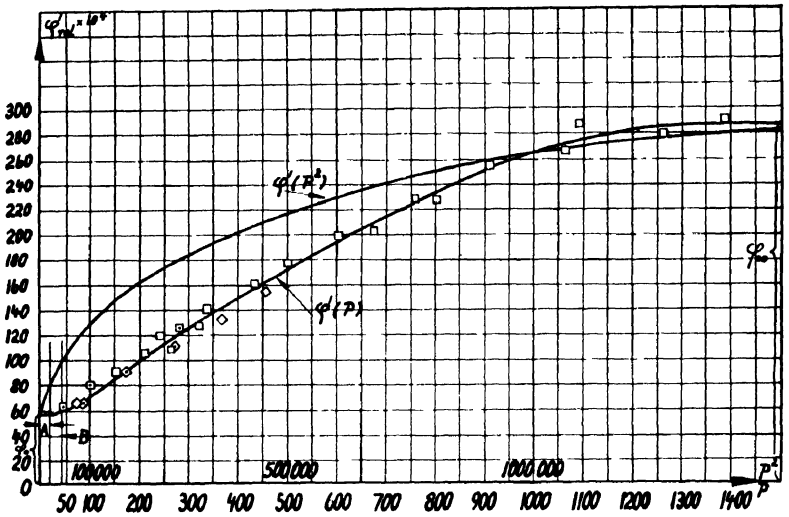


FIG. XVI, 8. Apparent fluidity of a 1 69 per cent. solution as a function of P and P^2.

" P " stress at the wall.
" A " range of rotating cylinder instruments.
" B " range of capillary instruments.

as a function of the stress at the wall (P). The apparent fluidity curve cuts the φ-axis horizontally. It will be explained in Section 2 of Chapter XVIII that this shows that φ is an *even* function of P. At high rates of shear, the curve again is parallel to the P-axis ; therefore, between these extremes it must have an inflexion point. The curve φ (P) accordingly has quite a complicated shape and one would be inclined to think that to describe it mathematically one would have to take recourse to several constants (among them φ_0 and φ_∞). In the same figure, φ has, however, also been plotted against P^2 and this curve looks much simpler, with monotonously decreasing curvature. Its shape very much suggests an e-function, which would require *one* constant only in addition to φ_0 and φ_∞. In Section 4 of Chapter XVIII we shall take up this suggestion.

12. Qualitative Interpretation of the Structural Viscosity of Rubber-Toluene Solutions

Which one of the mechanisms which we discussed in the preceding chapter should apply in our case ?

We have five criteria for answering this question. They are :—

I. The reaction of η_{rel} to changes of temperature.

II. The reversibility or irreversibility of a change of stress or rate of deformation.

III. The existence of a lower limit of structural viscosity.

IV. The dependence of the ratio of the largest to the smallest intrinsic viscosity upon the concentration.

V. The definite character of the dispersion, independent of the manner in which it was produced.

To consider V first, the fact that two dispersions of the rubber in toluene, produced in widely divergent manners, have at the same concentration the same viscosity, proves that the dispersed rubber particles in the solution are not accidental lumps, but that the rubber is broken down to fundamental primary particles. We have expressed this fact in the words that the rubber-toluene dispersions are true sols. These primary particles may be either micellar units or molecules—no decision between them can be made on the strength of this one criterion.

Regarding IV, the rubber-toluene solutions appear to exist structurally in two different states, viz. below and above a concentration of between 0·3 and 0·4 per cent. Below this concentration the ratio seems to be constant, or in any case varies but little and may be equal to unity. Above this concentration, the ratio greatly increases with increasing concentration. We may conclude that below the critical concentration, the primary particles are isolated, while above the critical concentration they are connected in secondary structural groups.

Regarding I, it is clear that, should structural viscosity be due to purely geometrico-mechanical influences, η_{rel} must be independent of change of temperature. This is the case when the shear causes the breakage, i.e. a shortening of the particle in accordance with the mechanism 1, IV, because then the change in viscosity is due solely to the change in the geometrical shape of the particle. On the other hand we can say that in the case of the mechanism 1, i, there must be a dependence upon changes of temperature. A temperature dependence should also be expected in the case 2, A. We cannot say anything *a priori* in respect of the other mechanisms.

As in the case of rubber-toluene solutions, the relative viscosities do *not* depend upon the temperature, this fact excludes for them mechanisms 1, i and 2, A.

Regarding II, a change of viscosity on change of rate of shear can only then occur *at once without any lag*, if all possible positions of the particles, relatively to one another, are equivalent. This is not the case if the particles are macro-molecules or ordered groups. This

excludes for rubber-toluene solutions the mechanisms 1, IV and 2, B, i, ii, iii.

Regarding III, one should note that in case the change in viscosity is due to a partial destruction of the secondary structure (mechanism 2, B, b) or to a reduction of the primary particles (mechanisms 1, IV and 2, A), there must be a lower limit of structural viscosity. The destruction or reduction can only be effected by overcoming the bonds—primary or secondary—between atoms or molecules, and this is possible only when the stresses produced by distortion reach and exceed a certain limit. Below that limit the dispersed system as a whole is simple Newtonian.

Such a lower limit cannot exist in cases 1, i, ii, iii and 2, B, where the smallest stress must make itself felt. As a definite lower limit (point of Fig. XVI, 6) was observed in the rubber-toluene solutions, these mechanisms are excluded.

Let us forget what we may have read after 1933 about the nature of the rubber particle in solution when the rubber-toluene viscosity observations were made. Let us forget what we may know about them and let us try to imagine what the rubber particles should be like on the strength of our rheological analysis *only*. If we have crossed out in Table I of Chapter XV all those mechanisms which we had to exclude, *one* only is left. It is 2, B, IV, b ; or the rubber particle is of the nature of an unordered aggregate, immobilizing solvent ; and because of stresses caused by the flow, it undergoes partial destruction after a certain lower limit of shear is reached and exceeded, which liberates solvent and thereby lowers the viscosity of the system. We can, moreover, say (on the strength of criterion V) that these unordered groups are unions of *fundamental* primary elements, most probably molecules.

Let us now consider criterion IV. The observations were not accurate enough to say definitely whether below a concentration of about 0·3 per cent. there was structural viscosity or not. As a matter of fact, I do not know of any experimental investigation showing definitely structural viscosity persisting down to zero-concentration. However, let us assume the more general case that there *was* structural

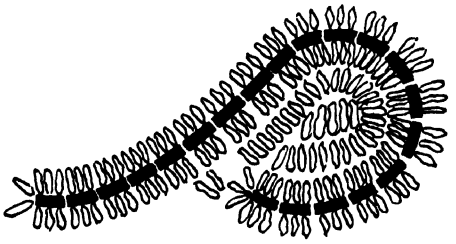


FIG. XVI, 9. Solvent immobilized by a single molecule.

viscosity below that concentration, or, in mathematical language $\eta_{ro}/\eta_{r\infty} > 1$. But even if $\eta_{ro}/\eta_{r\infty} = 1$, this ratio up to about 0·3 per cent. concentration could not have varied much. In this case we are required to imagine a single molecule immobilizing solvent not by true adsorption (which is excluded) but in a gel-like fashion. In this way we shall arrive at the picture shown in Fig. XVI, 9.

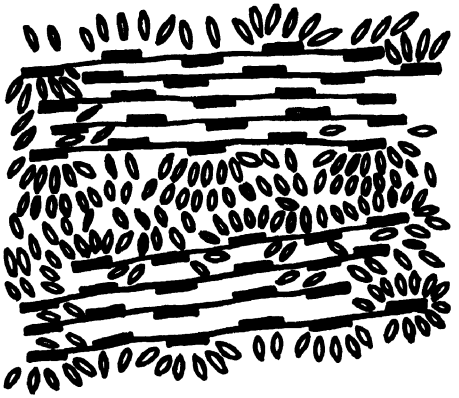


FIG. XVI, 10. Picture of a particle in accordance with Eirich and Mark.

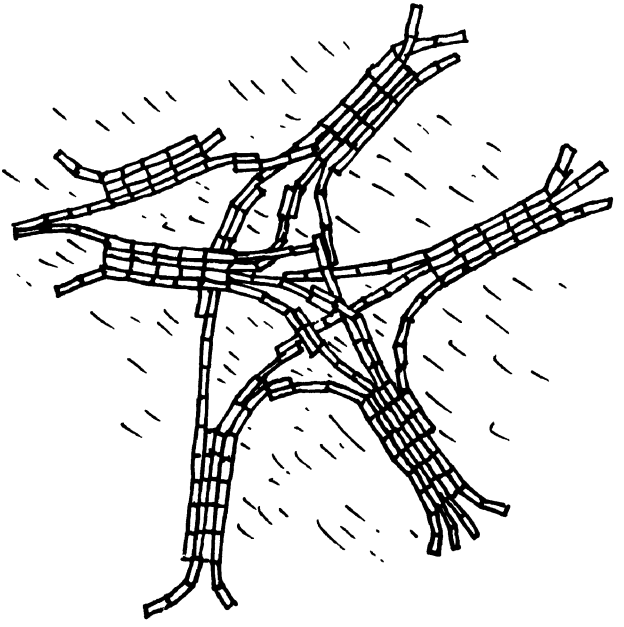


FIG. XVI, 11. Picture of a particle in accordance with Gerngross, Hermann and Abitz.

Above a concentration of 0·4 per cent. the ratio $\eta_{r_0}/\eta_{r\infty}$ is not only definitely greater than *1*, but increases very much with increased concentration. Here the dispersed phase accordingly consists of secondary unions of molecules of the type, 2, B, IV called by Kraemer and Williamson an " unordered assemblage of smaller particles ", or, as we found, of molecules. It is, of course, possible that a number of the molecules are inter-connected in an orderly fashion to form micelles as shown in Figs. XVI, 10 and XVI, 11.

To decide between these two alternatives, we should expect the one in Fig. XVI, 10 to show dependence upon temperature, while the one in Fig. XVI, 11 might be independent of temperature. In the case of the rubber particles in toluene a change of temperature from zero to 40° C. was without influence. The bond between the primary particles must therefore be strong enough to resist the influence of temperature to that extent. This would weigh in favour of the arrangement of Fig. XVI, 11 or a similar one.

Figs. XVI, 9 and XVI, 10 have been copied from an article by Eirich and Mark (1936), Fig. XVI, 11 from a paper by Gerngross, Hermann and Abitz (1936). These authors arrived at their pictures from physico-chemical considerations. We have shown in the present chapter how far we can advance by the method of rheological analysis solely.

13. Summary

Rheological analysis aims at deriving information on the structure of a dispersed system from the results of rheological measurements upon it. On the example of viscometric measurements of rubber-toluene solutions at varying rates of shear, temperatures and concentrations it is shown how the application of the criteria of Table I, Chapter XV leads to the assumption that up to concentration of about 0·3 per cent. the dispersed phase exists in the shape of single molecules immobilizing solvent in an intramolecular manner and above a concentration of about 0·4 per cent. in the shape of unordered aggregations of molecules or micelles which are bound together by cohesional forces immobilizing solvent in an intermolecular and/or intermicellar manner. The liberation of solvent through partial destruction of the aggregation or a partial straightening of curled molecules causes the phenomenon of structural viscosity, pronounced in the first case and of questionable existence in the second.

CHAPTER XVII

POWER FORMULAS

1. The Problem of Variable Viscosity

THE rheological equation of the Newtonian liquid is in accordance with (I, f)

$$\tau = \eta\dot{\gamma} = 2\eta\dot{e}_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\dot{\gamma} = \varphi\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the *simple* liquid, η and φ are for one and the same material constants, depending only upon the parameters of temperature and density. In the *general* liquid, η will itself be a function of the rate of shear $\dot{e}_t$, or φ a function of the shearing stress τ. The problem which we now have before us is : what function of τ will φ be, or what is the form of the function $\varphi\ (\tau)$?

From a purely formal point of view there seems to be no difference whether we deal with any one of the functions $\eta(\dot{e}_t)$, $\eta(\tau)$, $\varphi(\tau)$, $\varphi(\dot{e}_t)$, and, as a matter of fact, when first attacking the problem in collaboration with Miss Riwlin (1927, 1928), we started from $\eta = \eta(\dot{e}_t)$. As I have already stated briefly in Section 1 of Chapter XV, this landed us into a differential equation which we could not solve otherwise than by a series development and a *divergent* one at that. This difficulty turned out to be avoidable. When I later tried $\varphi = \varphi(\tau)$, I found that integration in closed form was possible and my above-mentioned papers have accordingly been out-stripped by later publications. With regard to the other two forms, it is obvious that if we start from either (1) or (2) ,the natural thing is to consider either η as a function of $\dot{e}_t$, or φ as a function of τ and not the other way round.

2. The Power Function and its Integration

The first attempts at a solution of this problem were made independently by Farrow and Lowe (1923), de Waele (1923), and Wo. Ostwald (1925). Indicating by v an observed kinematical macro-quantity (as e.g. Q in the case of the capillary-viscometer or $\dot{\Omega}$ in the case of the rotation-viscometer) and by p an observed dynamical macro-quantity (as e.g. Δp in the first and Δs in the second case), they proposed the empirical formula (or equivalent formulas).

$$v = Kp^m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where " K " and " m " are two " constants ", the latter larger than

unity. However, if we compare this equation, e.g. with Poiseuille's law, it is clear that K is not only dependent upon the nature of the material, but also *upon the dimensions of the apparatus used.* Porter and Rao (1927), and Farrow, Lowe and Neale (1928) improved upon this by postulating the rheological equation

$$\dot{e}_t = K\tau^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Considering laminar shear between two plates as in Fig. I, 4, where the lamina are planes, (4) gives

$$dv/dx = k\tau^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now, equilibrium of every single lamina requires that τ should be constant, and integration of (5) therefore gives

$$v = \tau^n x + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where C follows from $v = 0$ for $x = 0$ with $C = 0$. Let V be the velocity of the moving plate $x = d$, then

$$V = k\tau^n d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The fluidity φ is defined by (2) and accordingly

$$\varphi = \frac{V}{d}/\tau = k\tau^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The quantity k, therefore, is not itself the fluidity, nor $1/k$ the viscosity, as many investigators were led to assume from using for $1/k$ such symbols as $\bar{\eta}$ or (η).

We now integrate (4) for the two standard types of viscometers as follows :—

Capillary Instrument	Rotation Instrument

We introduce the expressions generally valid for the type of instrument, viz.—

compare (II, b and c)	compare (II, h and i)
$\dot{G} = dv/dr, \tau = r\Delta p/2l$	$\dot{G} = r\,d\dot{\theta}/dr, \tau = M/2\pi r^2 h$ (9)

into the rheological equation (4) and get

$$dv/dr = k(r\Delta p/2l)^n \quad | \quad rd\dot{\theta}/dr = k(M/2\pi r^2 h)^n \quad (10)$$

Integration in respect of r now gives

$$v = k2l/\Delta p\,.\,(r\Delta p/2l)^{n+1}/(n+1) + C \quad | \quad \dot{\theta} = -k(M/2\pi h r^2)^n/2n + C$$

The integration constant is again determined from the boundary condition that the velocity of the flowing material in the layer adjacent to the solid wall is identical with the velocity of the wall. Therefore

Capillary Instrument	Rotation Instrument
$C = -k2l/\Delta p.(R\Delta p/2l)^{n+1}/(n+1)$	$C = +k(M/2\pi hR_i^2)2n$
and	
$-v = k\frac{2l/\Delta p}{n+l}\left[\left(\frac{R\Delta p}{2l}\right)^{n+1} - \left(\frac{r\Delta p}{2l}\right)^{n+1}\right]$	$\dot{\theta} = k[(M/2\pi hR_i^2)^n - (M/2\pi hr^2)^n]/2n$ (11)
The quantity of flow in unit time is accordingly	The angular velocity of the external cylinder is accordingly
$Q = \int_{r=o}^{r=R} v.2\pi r.dr = kR^3\pi\left(\frac{R\Delta p}{2l}\right)^n/(n+3)$	$\Omega = k\left(\frac{M}{2\pi h}\right)^n(1/R_i^{2n} - 1/R_e^{2n})/2n$ (12)

For $n = 1$, this equation is reduced to

Poiseuille's law	Margules' law

with k becoming φ.

Introducing in (12) the consistency variables from

(II, f) and (II, g),	(II, l), (II, m) and (II, n).

we get

$V = k\frac{4}{n+3}P^n$	$V = k\frac{1-a^n}{n(1-a)}P^n$ (13)

where P is the shearing stress at the wall. The fluidity of the liquid adjacent to the wall is therefore in accordance with (8) $\varphi_w = kP^{n-1}$ and the rate of shear at the wall is consequently $V_w = \varphi_w P$, or

$V_w = \frac{n+3}{4}V$	$V_w = \frac{n(1-a)}{1-a^n}V$ (14)

For $n = 1$, V is the rate of shear at the wall, but generally V_w is a function of V depending upon the form of the rheological equation and the type of instrument.

The advantage of this procedure as compared with that of Farrow and Lowe, de Waele and Ostwald is that now, instead of the " constant " K which depends upon the dimensions of the apparatus *in an unknown manner*, there appears k, a characteristic of the material, which makes it possible to *compare measurements made on different sorts of apparatus.*

3. Objections to the Power Function

At the Plasticity Symposium of 1928 at Easton, Pa. (Reiner (1929)) I raised the following objections to the rheological equation (4) :—

It is assumed that " n " is a constant of the material and, therefore, should have different values for different materials. As a consequence, because the dimensions of $\dot{G}$ and τ are fixed, the dimension of φ would vary from material to material. This, however, would mean that for different materials φ not only quantitatively varies, but also qualitatively, and the different fluidities φ cannot be compared. This, furthermore, means that (4) is not *one* law which only for different materials, i.e. different values of " n ", gives different numerical results ; what we have before us are as many different laws as there are values of " n ", or for every material a different law. They are what could be called " individual " laws—if there was such a thing. We conclude that (4) is not a physical law, but only an *empirical formula.*

But as an empirical formula it is exposed to two other objections :—

Firstly, the fluidity of the material as usually determined, or the apparent fluidity, is in the capillary instrument

$$\varphi' = V/P = k \frac{4}{n+3} P^{n-1} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If $n>1$, φ' vanishes for $P = 0$ and is infinitely great for $P = \infty$. Both are contrary to experience. If the material is at rest, it possesses a certain fluidity, say φ_o, the influence of which becomes apparent *as soon as the material starts to flow.* If φ' were zero, the material, when at rest, would be a solid. There are materials of such or similar behaviour, exhibiting the property of thixotropy, but (4) was not devised for them. Equ. (4) was meant to describe the behaviour of such liquids as rubber or nitrocellulose solutions which are not thixotropic —but even for thixotropic materials (4) is not suitable.

Secondly, that φ' should become infinite is objectionable because there is no liquid the viscosity of which vanishes and, moreover, the fluidity of a dispersed system cannot, of necessity, reach, let alone exceed, the fluidity of the dispersion medium. This argument is not detracted from by the fact that we cannot reach infinite stress or strain and $P<\infty$. It is sufficient that there should be, in accordance with (15) a tendency for φ' to increase beyond every limit for an ever-increasing P.

Of the two objections just dealt with, the first may be named the " zero "—the second the " infinity " objection, while the objection dealt with before will be mentioned as the " dimension " objection.

The " zero " objection was met in a formula proposed by S. B. Ellis

$$v = k_1 p + k_2 p^m \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

while R. V. Williamson with the formula

$$v = k_1 p + k_2 p^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

avoided the dimension objection as well. However, the infinity objection remained.

4. Power-Formulas in General

We therefore have to dismiss the power-formula as unsuitable for the rheological equation of a non-linear liquid. However, power-formulas have played, and are still playing, such a big part in theoretical rheology that it appears profitable to examine their suitability, in general, for the mathematical expression of a law of nature.

Fig. XVII, 1 shows a series of parabolas and hyperbolas corresponding to varying values of n. Equs. (3) and (4) with $n > 1$ are parabolas, but the most famous power-laws are hyperbolic. We have Boyle-Mariotte's law with $n = -1$, Newton's law of gravitation and Coulomb's law of electro-statics with $n = -2$, and laws expressing the forces of capillarity and cohesion with $n < -2$.

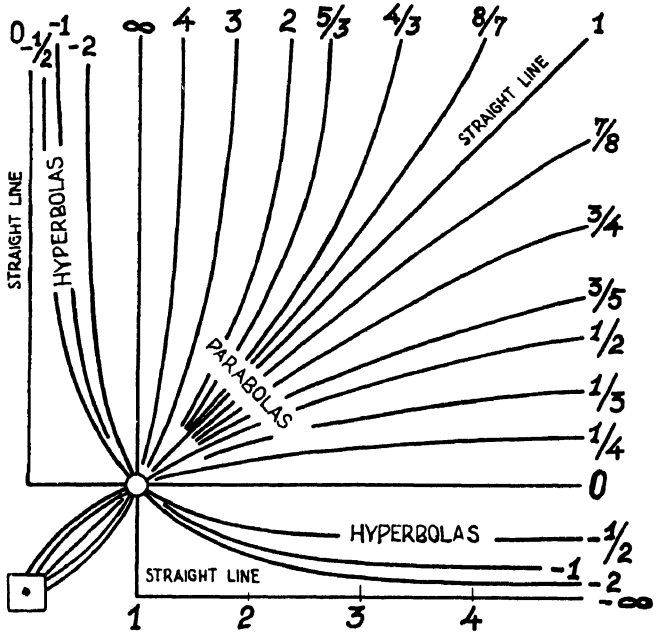


FIG. XVII, 1. Power curves $y = x^n$. The figure against each curve indicates the value of n.

A parabolic law is Stefan-Boltzmann's law for the emission of a black body with $n = 4$ In all these laws n is *not a material constant*; it is the same for every phenomenon and different only for different *kinds* of phenomena. Therefore, the quantity which it expresses as a power is of the same dimension for every phenomenon of the same kind and the dimension objection does not arise.

5. The Power Function in Elasticity

The first parabolic " law " with variable n apparently was what is known as Bach's elasticity law, for such materials as cast iron, stone and concrete which do not obey Hooke's law of the proportionality of stress and strain. Instead of (I,d) G. B. Buelffinger in 1729 proposed $\gamma = a\tau^n$ with $n \geqq 1$. This fell into oblivion in the course of time and was revived independently by Bach in 1888 under whose name it is known. It looked as if this was a more general law, including for $n = 1$ Hooke's law. One would have for steel $n = 1$ and for, say, cast iron $n = 2$. One recognizes that this " law " is open to the zero objection. The modulus of elasticity of such a material would be $\tau/\gamma = a^{-1/n}\gamma^{1/n-1}$. If $n > 1$, $\tau/\gamma = \infty$ for $\gamma = 0$ and for very small strains cast iron would be a rigid body. However, as was found by Grueneisen (1906), materials which for large strains do not obey Hooke's law, do so for small strains.

Hodgkinson (1849) proposed a two-constant law.

$$\tau = a\gamma + b\gamma^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This is the elastic analogue to Williamson's formula (17) and free from the zero objection. As γ is a dimensionless number, it is also free from the dimensional objection ; and the infinity objection does, of course, not arise in elasticity (except for rubber and rubber-like materials).

6. The Logarithmic Homologue

Taking the logarithms of (3), we get

$$\log v = \log K + m \log p \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and introducing $v' = \log v$, $K' = \log K$, $p' = \log p$

$$v' = K' + mp' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and this is the equation of straight line. O. Reynolds (1883) in his investigations on turbulent flow plotted observations in logarithmic scale. He called the v'/p' curve the Logarithmic Homologue of the v/p curve and found that in his case the logarithmic homologues were near enough straight lines. Fig. XVII, 2 shows the logarithmic homologues of the curves of Fig. XVII, 1. He so arrived at a power law for the turbulent flow in pipes

$$Q = K\Delta p^{1/1.724} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

It is to be noted that no " micro " law for turbulent flow exists so far. For turbulent flow in rotating cylinders one would have a different K and different m, not connected with those of (21).

7. Power Laws in Physical Chemistry

To this property of having straight-line logarithmic homologues, the popularity of the power law is probably to a large extent due.

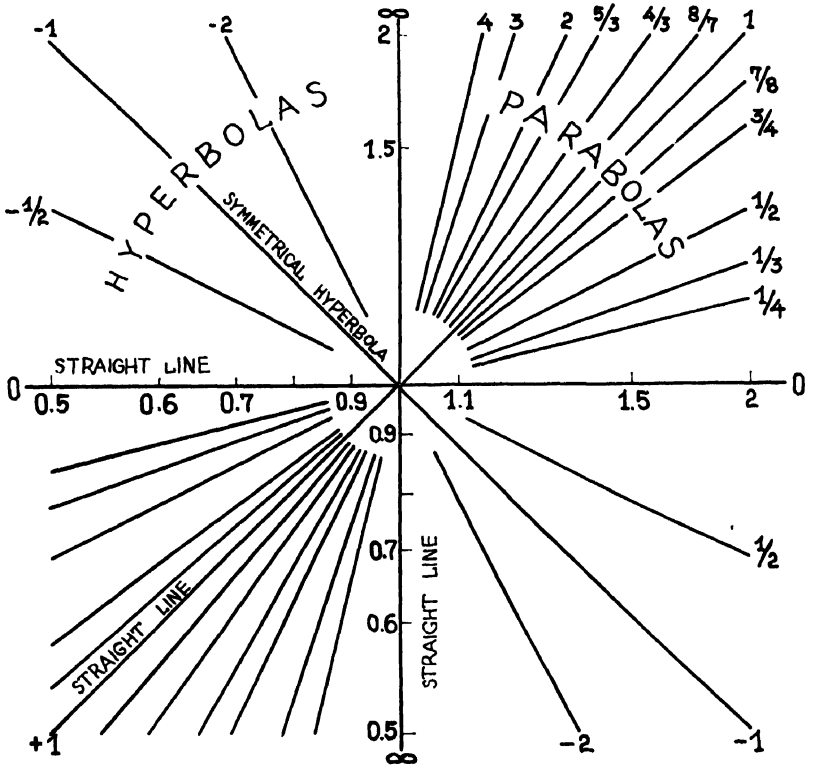


FIG. XVII, 2. Logarithmic homologues of power curves $y = x^n$ (compare Fig. XVII, 1).

W. Ostwald used the method of logarithmic plotting for observations on adsorption and introduced it into physical chemistry. H. Freundlich expressed the adsorption isotherm through a power law and this started a whole array of power laws such as : another law by Freundlich (1909) for swelling pressure in gels, an alternative law to van der Waal's law by Duclaux and Wollman (1911), a law of osmotic pressure in colloidal solutions by Wo. Ostwald (1919) and many more. All these laws are subject to one or the other of the above objections. E.g. Blueh has objected to Freundlich's adsorption isotherm on the grounds that an equation which does not ultimately lead to equilibrium must be rejected. This is an infinity objection.

It is not difficult to correct such formulas so as to meet two of the objections.

The zero-objection can be avoided by adding a constant as in (17) and (18), and, as a matter of fact, the power-law has often been written in that form. There is also a way to avoid the dimension-objection, but this objection has seldom been noticed and still more

seldom been taken account of. The method to be used can, I think, be best explained by the following quotation from a letter from Prof. Lark-Horowitz of Purdue University to me. He wrote on June 15th, 1933, as follows :—

8. A Method to Avoid the Dimension Objection

" It was decidedly this point of view which I had of the Freundlich Adsorption Isotherm, and I was for the same reason reluctant to publish any of our findings about the relation between viscosity and conductivity of glasses which is of the form : $\rho = a\eta^{\beta}$. I told Dr. Littleton of Corning when he asked us to determine the constants as a function of composition, that this would not mean anything, since this formula is hardly a theory, but only a mathematical device of linking up the two phenomena. By applying the point of view developed by Tamann and Fulcher regarding viscosity to conductivity, I have found that T_g is the same for both phenomena.

This gives at once the possibility of deriving the formula above in a different form and would give it an entirely different physical aspect ; a form which has no longer the fact of dimensional change to encounter. Our formula has been written as $(\rho/\rho_{\infty}) = a(\eta/\eta_{\infty})^{\beta}$ where ρ_{∞}, η_{∞} have a definite meaning and where the power β is the ratio of the work necessary to separate an ion from the molecule to the work necessary to break up a lattice. It seems to me that this example is a typical one as regards the application of power function formulæ. . . ."

The same way was taken, most probably independently, by A. Nadai (1937). To express slow viscous flow in metals (called " creep ", compare Chapter IX), Nadai proposed a power function which in our notation is $\tau = \tau^*(\dot{\gamma}/\dot{\gamma}^*)^n$ where τ^* and $\dot{\gamma}^*$ are material constants of the dimensions of stress and rate of strain respectively. Now the expression on the right side within brackets ( ) is a dimensionless number as in the case of Lark-Horwitz. However, the zero objection remains, and this was noticed by Nadai, who remarks that " the formulæ do not converge in the neighbourhood of zero-stress to the right expressions ".

9. The Power Formula as Interpolation Formula

Considering that the parabolic " law " breaks down for both small and large values of the variables,* it can only be considered an *interpolation formula.* As such it will often be found useful, especially in problems of testing materials. However, one should not expect the formula to be applicable *outside* the interpolation range. When

* Freundlich also remarked in the last edition of his standard work, the *Kapillarchemie,* that his parabolic adsorption isotherm should not be used in the case of small or high pressures.

Farrow, Lowe and Neale compared the rheological behaviour of starch pastes in capillaries and in the Couette on the basis of Equs. (4)–(14), and the ranges of stresses and velocity gradients in both instruments were different ; they could not reconcile them with one value of n. It appeared that, at least for higher concentrations, a different n would have to be assumed for the material in the capillary and a different one for the same material in the rotation instrument. As this would also change the dimension of k their observations brought about a *reductio ad absurdum* of this law.

In Fig. XVII, 3, we have plotted the V_{rel}/P values for the 1·69 - 1·71 per cent. rubber-toluene solutions in logarithmic scale, in short the logarithmic homologue of Fig. XVI, 4. As can be seen, this consists of five distinct parts, more or less straight lines, of which the first for very small and the last for very large shear are inclined at 45 degrees and therefore correspond to straight lines passing through the origin or to *simple* Newtonian behaviour ; others, one of fairly long range, correspond to parabolas. It is the long straight line logarithmic homologue which comprises the usual range of observations. The empirical equations corresponding to the five parts are : $V_{rel} = 0{\cdot}0053\ P$, $V_{rel} \quad 0{\cdot}0045P^{1.2}$, $V_{rel} = 0{\cdot}00083P^{1.5}$, $V_{rel} = 0{\cdot}00028P^{1.7}$ $V_{rel} = 0{\cdot}0287P$. Only to the first and the last can a physical meaning be ascribed.

The question can be raised, to what circumstances or properties is the success of the power formula—and such success there has been and it has contributed to its popularity—due ?

There is no doubt that as an interpolation formula the power formula very satisfactorily reproduces many *entirely different* laws within certain limited ranges, provided this range does not include the origin of the co-ordinate-system. Scott Blair (1929) took several series of viscometric observations which Ostwald had used in support of his power law and showed that they could equally well, or even better, be represented by the Buckingham-Reiner equation.

Scott Blair came to the result that the agreement must be considered as accidental. We may, however, ask under what circumstances such a good agreement may be expected to exist.

The power function $y = ax^b$ has the property of starting at the origin $x = 0$, $y = 0$ and increasing from there *monotonously*. In addition, if we calculate the differential quotients $y' = abx^{b-1}$, $y'' = ab(b - 1)x^{b-2}$, etc., we see that not one of them passes through an extreme, i.e. they also increase monotonously. From this it follows that the power function will adequately represent any other function which grows monotonously and permits of a monotonous extrapolation towards the origin. Therefore, if we select, say, in the centre of a portion of the curve which we want to represent by the power function a point

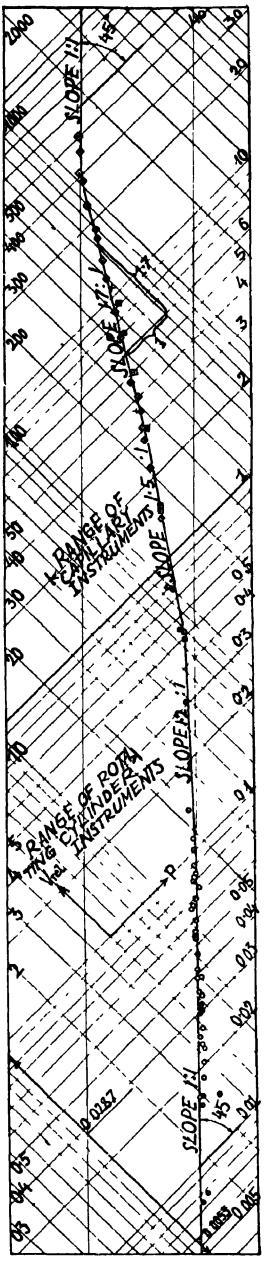


FIG XVII 3. Logarithmic homologue of the consistency curve of a 1·7 per cent. concentration rubber-toluene solution.

χ_o/y_o, it is possible, having the two parameters "a" and "b" free, to draw the power function curve so that it passes through this point and has in addition the tangent in common with the given curve. In

this case the ordinates of other points x/y at both sides of x_o/y_o will deviate by amounts which are of the order of the second power of $x - x_o$. If the given curve, in addition, can be extrapolated towards the origin, these deviations will be so small that they may not be noticeable. Sometimes, however, the range $2(x - x_o)$ may be so great that the deviations make themselves felt. In this case, the addition of a third parameter in the following manner will help : $y = c + ax^b$, or $y_1 = y - c = ax^b$. This corresponds to a parallel shifting of the co-ordinate-system in the y-direction without change of the character of the power formula, and makes it possible to give both curves at the point x_o/y_o the same curvature and the deviations of the ordinates are then of the order of magnitude of the third power of $(x - x_o)$. Such a modification of the de Waele-Ostwald formula was proposed by Herschel (1925).

In general, it is possible to add even three parameters without changing the character of the parabola, in accordance with the three degrees of freedom of the movement of a rigid figure in a plane ; viz. two translations in the x and y direction respectively and one rotation around the origin. However, this would entirely nullify the advantage of simplicity which the power formula possesses, and as the power formula, as was described above, is of *practical* significance only, one will hardly desire to take this course (Reiner (1933)).

10. The Integrative Method

The method which we have applied, leading to the " tree " of rheological bodies as listed in Table 1 of Chapter XIV, has been named by Scott Blair (1949) " analytical ". When the number of elements making up the model of a complex body increases as much as in the case of the *Sch ScB* body, and the material even then does not follow quantitatively the rheological equation postulated to represent the ideal body, so that a whole spectrum, i.e. an infinite number of individual bodies must be assumed, then Scott Blair has advocated a different approach which he calls the *integrative method*. To describe a behaviour intermediate between the Hookean solid in accordance with Equ. (VII, 2), the Newtonian liquid in accordance with Equ. (VII, 3), and the Pascalian liquid, he defines a quantity Ψ by

$$\psi = \tau^{\beta}\gamma^{\,1}t^{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where ψ can stand for a coefficient of viscosity when $\beta = 1$ and $k = 1$, and for a modulus of elasticity when $\beta = 1$ and $k = 0$, but is also suitable for describing more complex behaviour. Equ. (22) was proposed by Nutting and independently by Scott Blair, and is known as the *Nutting-Scott Blair equation*. As it stands, its application is limited to the case of either constant stress or constant deformation. But

while it contains the time explicitly, by differentiating with respect to time, t can be eliminated between the original and the differentiated equation. The result for constant ψ is

$$\dot{\gamma}/\gamma - \beta\dot{\tau}/\tau = k(\dot{\tau}^{\beta}/\gamma\psi)^{1/\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This equation (compare Scott Blair and Reiner (1950)) can be used when both stress and deformation vary.

The Nutting-Scott Blair equation does not claim to express a physical law, and the properties which it represents, are "quasi-properties". It is a synthesis of power-formulas such as described in the preceding Sections and has come to be used in the most diversified branches of rheology with great success. At the meeting of the British Rheologists' Club at London in October, 1946, it was attacked by Professor Andrade, Dr. Weissenberg and even Mr. de Waele himself, who refused to recognize the de Waele-Ostwald equation as his legitimate child. The attack was on the lines of the foregoing considerations, but went so far that I felt I had to come forward in its defence. I pointed out that it is a *hard empirical fact* found in many laboratories all over the world that observational results when plotted on a log : log scale *do give straight lines*. The Nutting-Scott Blair equation is nothing else than an analytical expression of this empirical fact becoming manifest in the graphical representation. For instance, when Lee and collaborators at the Road Research Laboratory at Harmondsworth plotted the flow curves of different filler bitumen mixtures in log : log scale, they found a series of straight lines which up to a certain filler content had a slope of 1 in 1 and which above that content increased in slope. Now, evidently this indicates that there is a breakdown in the filler binder system at a certain definite filler content. This is revealed in the slopes of the log : log lines or, correspondingly, in the power of a Nutting-Scott Blair equation representing them, and one can therefore use the magnitude of this power as an "indicator" for the structural change.

11. Summary

The power formula

$$\dot{\gamma} = k\tau^{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XVII, a})$$

and analogous equations in γ and τ have been proposed as rheological equations for materials with variable viscosity (fluidity), elasticity, etc. Because of three objections, of which one or other, but not all, can be met by modifications of the equation, it cannot be regarded as a true rheological equation but only as an interpolation formula. As such it is sometimes very useful.

Integration of (XVII, a) results in

$$V = k.4/(n + 3).P^n \text{ for the capillary instrument} \quad . \quad \text{(XVII, b)}$$

$$V = k\frac{1 - a^n}{n(1 - a)}P^n \text{ for the rotation instrument} \quad . \quad \text{(XVII, c)}$$

A more comprehensive formula is the Nutting-Scott Blair equation

$$\psi = \tau^{\beta}\gamma^{-l}t^k \quad . \quad . \quad . \quad . \quad . \quad \text{(XVII, d)}$$

where β and k are constants and ψ can stand for η or μ or some intermediate " quasi-property ".

CHAPTER XVIII

THE GENERALIZED NEWTONIAN LIQUID

1. The Problem Re-Stated

WE are coming back to the question which we put to ourselves in Chapter XV : what function of τ will φ be, or what is the form of the function $\varphi(\tau)$?

This, possibly, is putting the question too narrowly. As we have seen in Chapter XV, there are *many possible mechanisms* which may account for structural viscosity. Each different mechanism may be connected with a different form of the function $\varphi(\tau)$. Nevertheless, as will presently be shown, certain general considerations are applicable to all of them.

2. Power-Series Development

If we do not know the form of φ, it seems natural to develop it into a power series,

$$\varphi = a_0 + a_1\tau + a_2\tau^2 + \ldots = \sum_{n=0}^{n} a_n\tau^n \quad \ldots \quad (1)$$

From this equation we find for $\tau = 0$, $\varphi = a_0$. We call it the Fluidity at Rest and write for it φ_0, and so

$$\varphi = \varphi_0 + \sum_{n=1}^{n} a_n\tau^n \quad \ldots \quad (2)$$

This expression can be specialized in the following way : Let us determine, as described in Chapter XVI for the rubber-toluene solution, the viscosity by means of a rotation instrument. It is then obvious that the viscosity does not depend on whether the outer cylinder is rotated clockwise or anti-clockwise ; as a matter of fact no investigator ever took the trouble to mention the direction of the rotation. The sign of τ, however, is changed if the rotation is reversed. If this reversal is to be of no consequence,* the power of τ in (2) must be an *even* power, or

$$\varphi = \varphi_0 + \sum_{n=1}^{n} a_{2n}\tau^{2n} \quad \ldots \quad (3)$$

* What I mean to say is that $\varphi(+\tau) = \varphi(-\tau)$, or that if I start from rest with a clockwise rotation until I reach some $+\tau$, the same fluidity results as if I had started from rest with anti-clockwise rotation reaching a stress of the same absolute magnitude τ. If I start from $\tau = 0$ and, after reaching $+\tau$ I decrease τ, until arriving again at $\tau = 0$, I reverse the rotation until $-\tau$ is reached, there may be *hysteresis* and $\varphi(-\tau)$ may be different from $\varphi(+\tau)$. This, however, is a *problem apart* which arises when considering thixotropy.

and, therefore,

$$\dot{\gamma} = \varphi_o \tau + \sum_{n=1} a_{2n} \tau^{2n+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we compare (4) with (XVII, a), we see that the corresponding equations for the capillary and the rotation instrument can be written down with little effort. We find, by considering (XVII, b) and (XVII, c),

$$V = \varphi_o P + \sum_{n=1} \frac{2}{n+2} a_{2n} P^{2n+1} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in the first, and

$$V = \varphi_o P + \sum_{n=1} \frac{1 - a^{2n+1}}{(2n+1)(1-a)} a_{2n} P^{2n+1} \quad . \quad . \quad . \quad (6)$$

in the second case.*

If φ is a *given* function of τ^2, or

$$\varphi = f(\tau^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the coefficients a can be found by a McLaurin development, where τ^2 is the argument.†

We have

$$a_{2n} = \frac{f^{(n)}(0)}{n!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This makes the apparent fluidity $\varphi' = V/P$ for the capillary instrument

$$\varphi' = \varphi_o + \sum_{n=1} \frac{2}{n+2} \frac{f^{(n)}(0)}{n!} P^{2n} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and for the rotating cylinder instrument

$$\varphi' = \varphi_o + \sum_{n=1} \frac{1 - a^{2n+1}}{(2n+1)(1-a)} \frac{f^{(n)}(0)}{n!} P^{2n} \quad . \quad . \quad . \quad (10)$$

From (9) and (10), $\left.\frac{d\varphi'}{dP}\right|_{P=0} = 0$, i.e. the tangent to the curve φ'/P where it cuts the φ axis, is horizontal. As can be seen, this is a necessary consequence of φ being, in accordance with (7), an *even* function of τ (compare also Section 11 of Chapter XVI).

From (5), (6), (9) and (10) we see that, *whatever the unknown rheological equation of the material*, plotting consistency variables or the apparent fluidity (or apparent viscosity) against the shearing stress at the wall will give curves *independent of the dimensions of geo-*

* Comparing (XVII, a), $\dot{\gamma} = k\tau^n$, with (4) shows that in (XVII, b) and (XVII, c) k is to be replaced by a_{2n} and n by $2n + 1$.

† The McLaurin series is $f(x) = f(0) + x f'(0) + x^2 \frac{f''(0)}{2} + \ldots + x^n \frac{f^{(n)}(0)}{n!} = \sum_{n=0} x^n \frac{f^n(0)}{n!}$, where $f^{(n)}(0)$ is the value of $f^{(n)}(x)$ for $x = 0$ and $f^{(n)}(x)$ is the differential quotient of the n^{th} order of $f(x)$ in respect of x and $n! = n(n-1)$. $(n-2)\ldots.3.2.1$, where $0! = 1$ and $1! = 1$.

metrically similar apparatus, provided there is no slippage. However, it should be remembered that these curves are *not independent of the kind of apparatus*, and that while P is the stress at the wall, V is not the rate of shear in the layer adjacent to the wall and φ', therefore, not its fluidity.

3. Polynomial and Other Algebraic Equations

If the stresses are small, two or three terms of the power series may be sufficient, and we have, for instance

$$\varphi = \varphi_o + a_1\tau^2 + a_2\tau^4 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where a_1 and a_2 are two constants. This gives

$$\dot{\gamma} = \varphi_o\tau + a_1\tau^3 + a_2\tau^5 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Such an equation can adequately describe the rheological behaviour of the liquid within the unshaded range of Fig. XV, 1. However, for large stresses, the fluidity φ in accordance with (11) increases without limit, while, as we have agreed, a limiting value $\varphi_\infty = 1/\eta_\infty$ cannot be exceeded.

Polynomials of this kind were suggested at different times. We have mentioned one by R. V. Williamson in (XVII, 17) which was later independently proposed by Weissenberg's student Rabinowitsch (1929). Here the second power in τ^2 is objectionable, resulting in an odd power (the first) in the expression for the fluidity. Weissenberg soon changed his equation to

$$\dot{\gamma} = \varphi_o\tau + a_1\tau^3 = \varphi_o\tau(1 + \tau^2/\mu_1^2) \quad . \quad . \quad . \quad . \quad (13)$$

possibly when realizing that φ must be an even function. I myself tried some years before (1930)

$$\dot{\gamma} = \varphi_o\tau + a_1\tau^2 + a_2\tau^3 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

in connection with measurements of Herschel and Bulkley on benzene solutions of rubber (1926). The coefficient a_2 turned out to be negative. While (14) is still mentioned by Mark (1940) p. 270, it is obvious, from what I have said before, that I do not now consider it of much significance. Williamson's equation and mine were purely empirical, but Weissenberg based his upon theoretical considerations. Taking up the idea of an *elastic liquid* he and Eisenschitz (1929) attempted to show that for a liquid as first mathematically described by Maxwell (1868),* linearity of strain *in pure extension* will necessarily bring with it non-linearity *in laminar flow.* He accordingly interpreted the second constant in his equation (13) as a measure of the modulus of rigidity of the liquid (μ_1). The equation was still open to an " infinity-objection " (compare Section 8, Chapter XVII) and to correct for this Philippoff (1936) assumed that the elasticity of the elastic liquid was derived

* We have dealt with the Maxwell liquid in Chapter X.

from an elastic structure of the dispersed phase embedded in an inelastic structure of the dispersion medium, the fluidity of the latter being φ_∞. He accordingly proposed

$$\eta = \eta_\infty + \frac{\eta_o - \eta_\infty}{1 + \tau^2/\mu_l^2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

An empirical equation of similar form for consistency curves had before been proposed by Kraemer and Williamson (1929) and I myself (1929) at the same time (and therefore before Philippoff) proposed as the simplest algebraical empirical expression for the rheological equation of a generalized Newtonian liquid

$$\varphi = \frac{\varphi_o + \varphi_\infty \tau^2/a^2}{1 + \tau^2/a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which is essentially the same as (15). Integrating (16), which is not difficult, I arrived at a transcendental formula for the consistency curve and as I attributed to (16) empirical significance only, I did not try this equation on actual measurements. Philippoff, however, has made extensive use of his equation. I must confess that the theoretical basis for it does not appear to me well founded. I shall therefore not pursue its elaboration and shall report in the following Section upon another attempt of mine to establish a rheological equation for the generalized Newtonian liquid on a theoretical basis (Reiner and Riwlin (1927, 1928); Reiner (1931)).

4. The Coefficient of Structural Stability

We must look for an equation which will take account of the condition that there is an upper limit to the fluidity, a condition which is also fulfilled by (15) and (16).

In structural viscosity we can generally say that the more complex the structure, the greater the viscosity. With increasing stress, the structure will to a certain extent be reduced and the fluidity increased until, when the structure is reduced to the simplest state, φ_∞ is reached. The structural state at a certain stress τ is characterized by the value of $\varphi_\infty - \varphi$. The increase from φ to φ_∞ or the consumption of $\varphi_\infty - \varphi$ can be more or less rapid. The increase of the fluidity per unit increase of stress would be $d\varphi/d\tau$. However, as we have explained in Section 2, what determines the influence of stress on fluidity is not the magnitude of τ but of τ^2. We accordingly introduce a coefficient

$$\chi = \frac{\varphi_\infty - \varphi}{d\varphi/d(\tau^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which we name the Coefficient of Structural Stability.

This designation can be understood if we consider the extreme cases of $\chi = o$ and $\chi = \infty$. In the first $\varphi_\infty - \varphi = 0$ or $\varphi = \varphi_\infty$ which means

that the viscosity η_0 of the liquid at rest is reduced to η_∞ as soon as the liquid starts to flow. After that the liquid behaves as a simple viscous liquid with the constant fluidity φ_∞ or the constant viscosity η_∞. If $\chi = \infty$, (17) requires that $d\varphi/d(\tau^2) = 0$, i.e. the fluidity is not changed at all through flow. In this case we have before us a simple viscous liquid with the constant viscosity η_0. In the first case, the structure of the liquid is so *unstable* as to be immediately and completely destroyed when flow starts ; in the second case it is so *stable* as not to be destroyed through the highest shear.

5. A Rheological Equation for the Generalized Newtonian Liquid

If χ is constant, (17) can be integrated and we have from

$$\frac{d\varphi}{\varphi_\infty - \varphi} = \frac{d(\tau^2)}{\chi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$\frac{d(\varphi_\infty - \varphi)}{\varphi_\infty - \varphi} = -\frac{d(\tau^2)}{\chi}$ and, therefore, $\ln(\varphi_\infty - \varphi) = -\tau^2/\chi + C$. Now, for $\tau = 0$, $\varphi = \varphi_0$ which gives $C = \ln(\varphi_\infty - \varphi_0)$ and $\ln\frac{\varphi_\infty - \varphi}{\varphi_\infty - \varphi_0} = -\frac{\tau^2}{\chi}$, or $\varphi_\infty - \varphi = (\varphi_\infty - \varphi_0)e^{-\tau^2/\chi}$ from which finally

$$\varphi = \varphi_\infty - (\varphi_\infty - \varphi_0)e^{-\tau^2/\chi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This can be written

$$\bar{\varphi} = \frac{\varphi_\infty - \varphi}{\varphi_\infty - \varphi_0} = e^{-\tau^2/\chi} \text{ or } \ln\bar{\varphi} = -\tau^2/\chi \quad . \quad . \quad . \quad (20)$$

and if plotted in semilog scale would give a straight line.*

In Fig. XVIII, 1, φ is plotted against τ^2. The curve very much

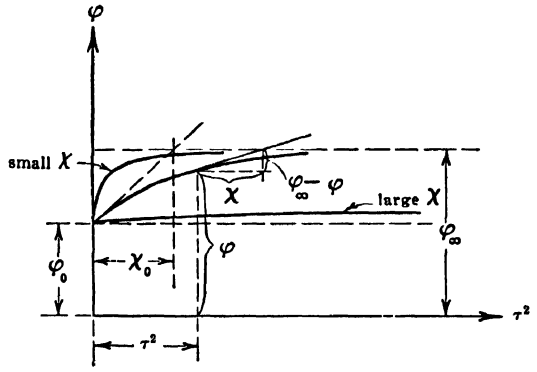


FIG. XVIII, 1. Fluidity of a generalized Newtonian liquid as affected by the stability coefficient (compare FIG. XVI, 8).

* This very convenient property is, however, lost (as will be shown) through integration for the capillary and rotating cylinder viscometer.

resembles the one of Fig. XVI, 8. The significance of χ becomes clear from an inspection of the figure.

Introducing φ from (19) into (XVII, 2) we have

$$\dot{\gamma} = [\varphi_\infty - (\varphi_\infty - \varphi_o)e^{-\tau^2/\chi}]\tau \quad . \quad . \quad . \quad . \quad (21)$$

Equ. (21) is the rheological equation of a generalized Newtonian liquid.

The simple Newtonian liquid results from a specialization of the equation when one constant $(1/\chi)$ vanishes. While we have derived (19), (20) and (21) on the assumption that χ is a constant, this assumption may now be dropped and in accordance with the principles explained in Section 9 of Chapter III, χ is to be regarded as a coefficient which may be variable, i.e. be itself a function of τ^2.

6. The Differentiation and Integration Method

The validity of a rheological equation such as (21) cannot usually be checked directly. The experiment shown in Fig. I, 4 is only in very exceptional cases given to realization.* For liquids within usual ranges of viscosities we can make observations only with apparatus such as the capillary or rotation viscometer or others based on similar principles.

From these observations we can plot consistency or relative consistency curves, and the question is *how to determine from such consistency curves the rheological equation of the material.*

Two methods present themselves for this purpose. They have been named by Hersey (1932) the differentiation method and the integration method respectively. The first is as follows :—

Let us assume that it is possible to express the consistency curve in the form of a function.

$$V = F(P) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21')$$

then it has been shown by Weissenberg (cf. B. Rabinowitsch (1929)) in the case of the capillary instrument and by Mooney (1931) in the case of the rotation instrument, how it is possible to derive from (21') through a differentiation process the corresponding rheological equation $\dot{\gamma} = f(\tau)$.

The difficulty, however, lies in finding the form of F of (21'). Weissenberg's method has actually been applied only in the case of the simple polynomial $V = a_1P + a_2P^3$, which then leads to a homologous rheological equation such as (13). If the consistency relations are empirically given through observations and the consistency curves can be analysed so that the observations can be expressed through a power series of known coefficients, this can lead to the establishment of the general law of flow (4) for the material under investigation.

* Compare Lee and Markwick (1937), Fig. 3.

The integration* method goes the other way—from postulating the form of $\varphi(\tau)$ to finding (21′) by integration. We applied this method in Chapter II to find Poiseuille's and Margules' equations and in the present chapter when deriving (3) and (6). We shall presently apply it to equation (21).

7. Application of the Integration Method to the Capillary Instrument

Starting with (21) and considering (II, b) and (II, c) we have

$$\frac{dv}{dr} = \varphi_\infty \frac{r\Delta p}{2l} - (\varphi_\infty - \varphi_o)e^{-r^2\Delta p^2/4l^2\chi}\frac{r\Delta p}{2l} \quad . \quad . \quad . \quad (22)$$

which by integration, assuming "no-slippage", gives

$$-v = \varphi_\infty \Delta p(R^2 - r^2)/4l + (\varphi_\infty - \varphi_o)\frac{l\chi}{\Delta p}(e^{-R^2\Delta p^2/4l^2\chi} - e^{-r^2\Delta p^2/4l^2\chi}) \quad . \quad (23)$$

from which by second integration,

$$Q = \varphi_\infty \frac{R^4\pi\Delta p}{8l} - (\varphi_\infty - \varphi_o)\left[\frac{4l^4\pi}{\Delta p^4}\chi^3 - \frac{l^2\pi}{\Delta p^2}\chi^2 e^{-R^2\Delta p^2/4l^2\chi} - \frac{4l^2\pi}{\Delta p^4}\chi^3 e^{-R^2\Delta p^2/4l^2\chi}\right] \quad . \quad (24)$$

By introducing the consistency variables V and P, (24) becomes

$$V = \varphi_\infty P - \frac{2\chi^2}{P^3}(\varphi_\infty - \varphi_o)\left[1 - \frac{P^2}{\chi}e^{-P^2/\chi} - e^{-P^2/\chi}\right] \quad . \quad (25)$$

from which the apparent fluidity

$$\varphi' = V/P = \varphi_\infty - \frac{2\chi^2}{P^4}(\varphi_\infty - \varphi_o)\left[1 - \frac{P^2}{\chi}e^{-P^2/\chi} - e^{-P^2/\chi}\right] \quad . \quad (26)$$

For $P = \infty$, (26) gives $\varphi' = \varphi_\infty$. For $P = 0$, the second term on the right side becomes indeterminate, but the known methods for dealing with such indeterminate expressions furnish $\varphi' = \varphi_o$.†

The apparent fluidities at rest and for maximum shear are therefore identical with the true fluidities. Between these extremes, however, they diverge, as comparison of (26) and (19) shows.

If we plot φ' from (26) against P, the curve has in $P = 0$ a horizontal tangent and must, therefore, possess an inflexion point. Also, the curve is symmetrical about the vertical axis. On the other hand,

* To be distinguished from the Integrative Method of Section 10, Chapter XVII.

† $1 - \frac{P^2}{\chi}e^{-P^2/\chi} - e^{-P^2/\chi} \Big|_{P=0} = \frac{1 - 0 - 1}{0} = \frac{0}{0}$.

Differentiation of dividend and divisor in respect of P gives

$$-\frac{2P}{\chi}e^{-P^2/\chi} + \frac{2P^3}{\chi^2}e^{-P^2/\chi} + \frac{2P}{\chi}e^{-P^2/\chi} - \frac{4P^3/\chi^2}{} = \tfrac{1}{2}e^{-P^2/\chi}\Big|_{P=0} = \tfrac{1}{2}.$$

if we plot φ' against P^2, the resulting curve has in $P^2 = 0$ a sloped tangent and it does not possess an inflexion point. It is real for positive values of P^2 only, because both $+P$ and $-P$ give the same $+P^2$. If we compare Fig. XVI, 8, we can see that (26) gives qualitatively a correct picture.

Equ. (26) can also be written in the form

$$\bar{\varphi} = \frac{\varphi_\infty - \varphi'}{\varphi_\infty - \varphi_o} = 2\frac{\chi^2}{P^4} - 2\frac{\chi}{P^2}e^{-P^2/\chi} - 2\frac{\chi^2}{P^4}e^{-P^2/\chi} \quad . \quad (27)$$

This is a non-dimensional equation. It is therefore independent not only of the dimensions of the apparatus, but also of the numerical values of the rheological coefficients φ_∞ and φ_o.

$\bar{\varphi}$ plotted against P also has an inflexion point. If we want to find its co-ordinates, we twice differentiate in respect of P and equalize with zero as follows :-

$$\frac{d^2\bar{\varphi}}{dP^2} = 40\frac{\chi^3}{P^6} - 40\frac{\chi^2}{P^4}e^{-P^2/\chi} - 20\frac{\chi}{P^2}e^{-P^2/\chi} - 8e^{-P^2/\chi} - 40\frac{\chi^3}{P^6}e^{-P^2/\chi} = 0 \quad (28)$$

This is a transcendental equation which can be solved by trial and error. We find with good enough approximation

$$P_i = \sqrt{\chi}/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Introducing the value for P_i from (29) into (27) gives $\bar{\varphi}_i = 0{\cdot}721$.

These are, of course, correct co-ordinates of the inflexion point only if χ is a constant.

Should χ not be a constant, χ_o, the value of χ for the liquid at rest, or the stability of the undisturbed structure of the dispersed phase, will still be of physical significance. It can be determined by differentiating $\bar{\varphi}$ in respect of P^2 and then putting $P^2 = 0$. We find

$$\frac{d\bar{\varphi}}{d(P^2)} = -4\frac{\chi^3}{P^6} + 4\frac{\chi^2}{P^4}e^{-P^2/\chi} + 2\frac{\chi}{P^2}e^{-P^2/\chi} + 4\frac{\chi^3}{P^6}e^{-P^2/\chi} \quad . \quad (30)$$

which, if we put $P^2 = 0$, becomes indeterminate. Applying the same method as before, gives

$$\left.\frac{d\bar{\varphi}}{d(P^2)}\right| P = 0 = -\frac{2}{3\chi_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Therefore, if we plot φ against P^2 from observations and determine the slope of the tangent to the curve at the point $P^2 = 0$, we can find χ_o from (31).

8. Integration for the Rotation Instrument

For the rotation instrument we apply (II, h) and (II, i).

We find

$$\frac{d\theta}{dr} = \varphi_\infty\frac{M}{2\pi r^3 h} - (\varphi_\infty - \varphi_o)e^{-\frac{M^2}{4\pi^2 r^4 h^2}/\chi}\frac{M}{2\pi r^3 h} \quad . \quad . \quad . \quad (32)$$

The integration of the first term on the right side is easy and yields $-\varphi_\infty \frac{M}{4\pi r^2 h}$. For the integration of the second term we introduce

$$\frac{M}{2\pi r^2 h} = x\sqrt{\chi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and therefore

$$-\frac{M}{2\pi r^3 h}\,dr = \sqrt{\chi}\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The second term thereby becomes

$$+\frac{(\varphi_\infty - \varphi_0)\sqrt{\chi}}{2}\int e^{-x^2}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The integral $\int e^{-x^2}dx$ cannot be reduced to an elementary function but for

$$\phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

which is called Gauss' error integral, there exist numerical tables.

A power series development gives

$$\frac{\sqrt{\pi}}{2}\phi(x) = x - \frac{x^3}{3} + \frac{x^5}{10} - \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

With this

$$\dot{\theta} = -\varphi_\infty \frac{M}{4\pi r^2 h} + \frac{(\varphi_\infty - \varphi_0)\sqrt{\chi\pi}}{4}\phi\left(\frac{M}{2\pi r^2 h\sqrt{\chi}}\right) + C \quad . \quad (38)$$

and assuming $\dot{\theta} = 0$ for $r = R_i$ and $\dot{\theta} = \Omega$ for $r = R_e$,

$$\Omega = \varphi_\infty \frac{M}{4\pi h}\left(\frac{1}{R_i^2} - \frac{1}{R_e^2}\right) - (\varphi_\infty - \varphi_0)\frac{\sqrt{\chi\pi}}{4}\left[\phi\left(\frac{M}{2R_i^2\pi h\sqrt{\chi}}\right) - \phi\left(\frac{M}{2R_e^2\pi h\sqrt{\chi}}\right)\right] \quad . \quad (39)$$

Introducing consistency variables, we get

$$V = \varphi_\infty P - (\varphi_\infty - \varphi_0)\frac{\sqrt{\chi\pi}}{2}\frac{\phi(P/\sqrt{\chi}) - \phi(Pa/\sqrt{\chi})}{1-a} \quad . \quad (40)$$

and

$$\bar{\varphi} = \frac{\sqrt{\chi\pi}}{2P}\frac{\phi(P/\sqrt{\chi}) - \phi(Pa/\sqrt{\chi})}{1-a} \quad . \quad . \quad . \quad . \quad (41)$$

For $P = \infty$, considering $\phi(\infty) = 1$, we have $\bar{\varphi} = 0$ and therefore $\varphi' = \varphi_\infty$.

For $P = 0$, considering $\phi(0) = 0$, the right side is indeterminate $\left(\frac{0-0}{0-0}\right)$.

Applying the same method as in the previous Section and considering $\left.\frac{d\phi}{dP}\right|_{P=0} = \frac{2}{\pi\sqrt{\chi}}$ and $\frac{2a}{\pi\sqrt{\chi}}$ respectively, we find $\bar{\varphi} = 1$ and $\varphi' = \varphi_0$.

The same results from a series development in accordance with (34). We find

$$\bar{\varphi} = 1 - \frac{P^2}{3\chi}(1 + a + a^2) + \ldots \quad . \quad . \quad (42)$$

We also find

$$\frac{d\bar{\varphi}}{d(P^2)} = -\frac{1 + a + a^2}{3\chi} + \ldots \quad . \quad . \quad . \quad (43)$$

the first term only remaining for $P = 0$. For the plane Couette flow, for which $R_e = R_i = \infty$ and $a = 1$, $\frac{d\bar{\varphi}}{d(P^2)} = -1/\chi$ which is what follows from (20).

9. Application of the Power-Series Development Method

We can express $\bar{\varphi}$ also by means of the development (9) and (10). We have

$$\bar{\varphi} = \frac{\varphi_\infty - \varphi'}{\varphi_\infty - \varphi_0} = \frac{\varphi_\infty - \varphi_0}{\varphi_\infty - \varphi_0} - \frac{\Sigma \ldots}{\varphi_\infty - \varphi_0} = 1 - \frac{1}{\varphi_\infty - \varphi_0}\Sigma \ldots \quad . \quad (44)$$

where the expressions following Σ are those of the second term on the right side of (9) and (10). To calculate these, we derive from (7) which in accordance with (19) is in our case

$$\varphi = f(\tau^2) = \varphi_\infty - (\varphi_\infty - \varphi_0)e^{-\tau^2/\chi} \quad . \quad . \quad . \quad . \quad (45)$$

by successive differentiation in respect of $p_t{}^2$

$$\left.\begin{aligned} f'(\tau^2) &= -\frac{\varphi_\infty - \varphi_0}{\chi}e^{-\tau^2/\chi} \\ f''(\tau^2) &= -\frac{\varphi_\infty - \varphi_0}{\chi^2}e^{-\tau^2/\chi} \\ f^{(n)}(\tau^2) &= (-1)^{n+1}\frac{\varphi_\infty - \varphi_0}{\chi^n}e^{-\tau^2/\chi} \end{aligned}\right\} \quad . \quad . \quad . \quad (46)$$

whence

$$f^{(n)}(0) = (-1)^{n+1}\frac{\varphi_\infty - \varphi_0}{\chi^n} \quad . \quad . \quad . \quad . \quad (47)$$

This makes (44) for the capillary instrument

$$\bar{\varphi} = 1 - \sum_{n=1}(-1)^{n+1}\frac{2}{(n+2)n!}\frac{P^{2n}}{\chi^n} \quad . \quad . \quad . \quad (48)$$

and for the rotating cylinder instrument

$$\varphi = 1 - \sum_{n=1} (-1)^{n+1} \frac{1 - a^{2n+1}}{(2n+1)(1-a)n!} \frac{P^{2n}}{\chi} \quad . \quad . \quad (49)$$

or approximately

$$\varphi = 1 - \frac{2}{3}\frac{P^2}{\chi} + \frac{1}{4}\frac{P^4}{\chi^2} - \ldots \quad . \quad . \quad . \quad (50)$$

and

$$\varphi = 1 - \frac{1+a+a^2}{3}\frac{P^2}{\chi} + \frac{1+a+a^2+a^3+a^4}{10}\frac{P^4}{\chi^2} - \ldots \quad (51)$$

From (50) and (51) we find

$$\frac{d\varphi}{d(P^2)} = -\frac{2}{3\chi} + \frac{1}{2}\frac{P^2}{\chi^2} \quad . \quad . \quad . \quad . \quad . \quad (52)$$

and

$$\frac{d\varphi}{d(P^2)} = -\frac{1+a+a^2}{3\chi} + \frac{1+a+a^2+a^3+a^4}{5}\frac{P^2}{\chi^2} \quad . \quad (53)$$

Equ. (52) confirms (31) and (53), (43).

10. Examination of the 1·69-1·71 per cent. Rubber-Toluene Solution *

We shall now examine the 1·69 and the 1·71 per cent. rubber-toluene solutions reported upon in Chapter XVI in the light of the formulas derived in the three preceding Sections. The difference in concentration of the two solutions is so small that their rheological properties can be assumed as identical, and we shall treat both solutions as if they were one and the same.

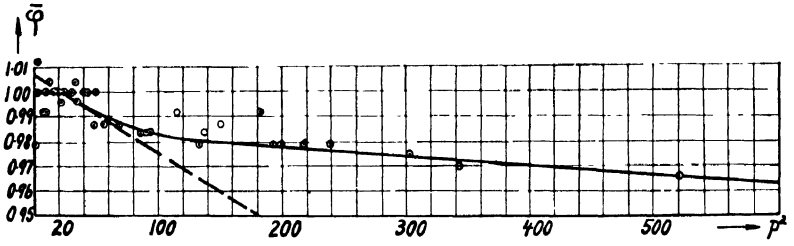


Fig. XVIII, 2. $\bar{\varphi}$ of a 1·71 per cent. rubber-toluene solution plotted against P^2 from observations in rotation instruments.

In Fig. XVIII, 2, the φ values are plotted against P^2. $P^2 = 25$ has to be taken as the starting point of the $\bar{\varphi}$ curve. The tangent to the curve at that point has been drawn, and while it is admitted that it could also have been drawn differently, this would not have changed the order of magnitude of the slope. This slope, as shown, is (1·008 —

* The Tables of the first edition of this book containing the figures which form the basis of the graphs of Sections 10 and 11 have been omitted

$0{\cdot}95)/180 = 322 \times 10^{-6}$. In our case $a = 0{\cdot}807$ and therefore $(1 + a + a^2)/3 = 0{\cdot}82$. This gives in accordance with (43), $\chi_0 = 820{,}000/322 = 2{,}550$.

We now calculate further values for χ in accordance with (42) and find for $P^2 = 100, 200, 300, 400, 500$ the values $\chi = 4{,}830, 7{,}460, 9{,}500, 12{,}050$ respectively.*

We therefore see that χ, the coefficient of structural stability, is not a constant, but *increases with progressive breakdown of structure.*

To get a complete picture we now examine the observations of the 1·69 per cent. solution in capillary instruments. In Fig. XVIII, 3, are plotted the φ values as calculated from the φ' values.

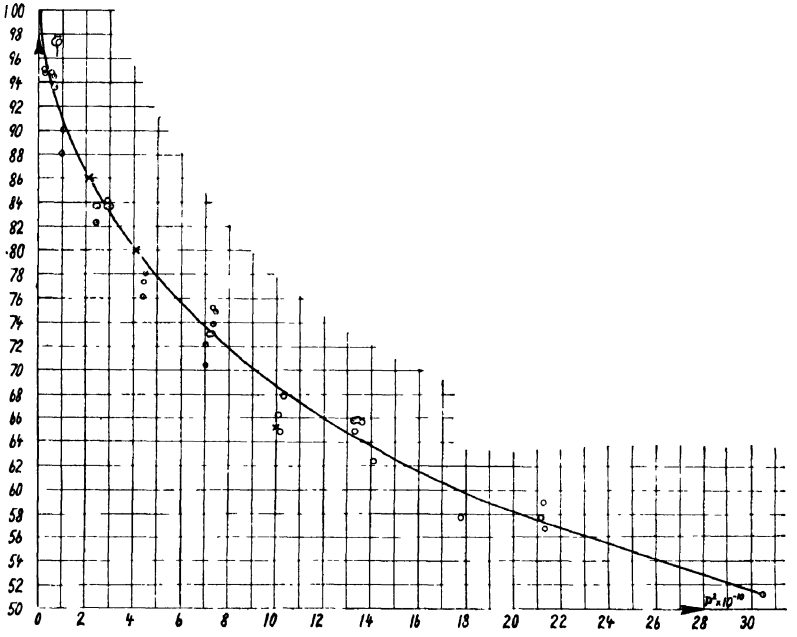


FIG. XVIII, 3. φ of 1·69 per cent. rubber-toluene solution plotted against P^2 from observations in capillary instruments.

Taking φ from the figure, we calculate χ in accordance with (50) as follows: We first calculate an approximate χ_1 neglecting the last term on the right side of (50). We then introduce into the complete equation $\chi = \chi_1 + \Delta\chi$ and calculate the correction $\Delta\chi$ neglecting $(\Delta\chi)^2$. We easily find $\Delta\chi = -\frac{3}{8}P^2$.

We cannot go beyond $P^2 = 40{,}000$ without taking into account a

* The term $(1 + a + a^2 + a^3 + a^4)P^4/10\chi^2$ in accordance with (51) can be neglected, as can easily be checked.

third term of the power series development. This we shall not do, but go over to the long-range $\bar{\varphi}$ values calculated from Table 4, Chapter XVI, which we show graphically in Fig. XVIII, 4.

For $P^2 = 6 \times 10^5$, $\bar{\varphi}$ from the curve $= 0{\cdot}26$. In order to calculate χ, we cannot use the power series (50) which now does not converge, but must go back to the closed-form equation (27), i.e. there should be

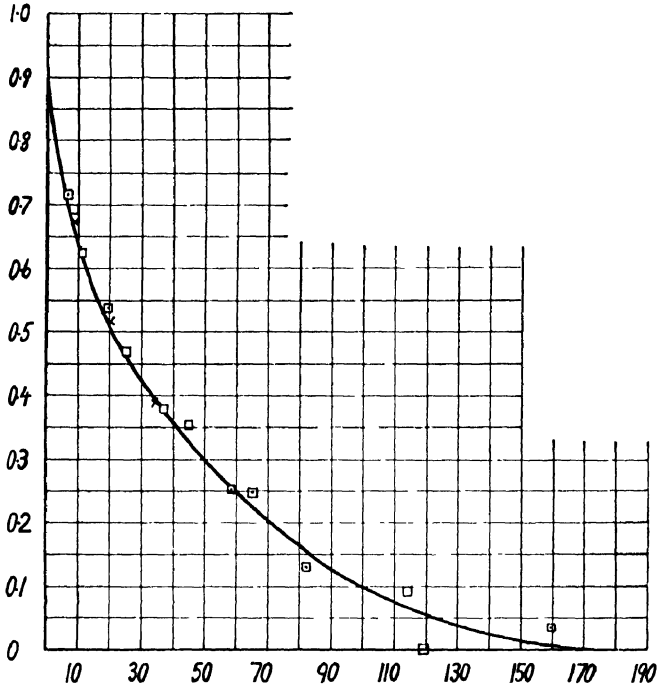


Fig. XVIII, 4. $\bar{\varphi}$ of a 1·69 per cent. rubber-toluene solution plotted against P^2 from long-range observations in capillary instruments.

$$0{\cdot}13 = \frac{\chi^2}{36 \times 10^{10}} - \frac{\chi}{6 \times 10^5} e^{6\ 10^5/\chi} - \frac{\chi^2}{36 \times 10^{10}} e^{-6\ 10^5/\chi} \quad . \quad (54)$$

or

$$\chi^2 - (6 \times 10^5 - \chi)\chi e^{6\ 10^5/\chi} = 4{\cdot}68 \times 10^{10}$$

This transcendental equation can only be solved by trial and error. To get the order of magnitude we try, e.g. $\chi = 10^5$, which turns out to be too little.

Trying $\chi = 10^6$ gives too much. The correct value of χ must therefore lie between 1×10^5 and 10×10^5, but nearer to 10^5. We ultimately find $\chi = 2{\cdot}3 \times 10^5$. In Table 1 calculated values for χ have been listed.

We see that while χ increases with P^2, the rate of increase gradually

decreases and ultimately a limiting value of χ_∞ of roughly 250,000 is reached. The increase of χ from χ_0 to χ_∞ is one hundredfold.

11. The Nature of the Coefficient of Structural Stability

In Section 12 of Chapter XVI we arrived at the result that the dispersed phase in a rubber-toluene solution above 0·4 per cent. concentration is an " unordered assemblage of smaller particles " which may be pictured as shown in either Fig. XVI, 10 or Fig. XVI, 11. We should not be surprised that with gradual breakdown of structure such particles *become more and more stable*, or, in other words, that χ is not a constant but increases with increasing rate of shear. The assumption that χ is a constant was the simplest possible one, but could not have been expected to hold good for structures of that sort. Some readers will probably have noticed that, written in the form

$$\varphi_\infty - \varphi = (\varphi_\infty - \varphi_0)e^{-\tau^2/\chi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Equ. (19) is analogous to the equation of radioactive decay, in which out of an initial quantity $(\varphi_\infty - \varphi_0)$ (which in our case is a measure of the structure of the dispersion at rest) the amount $(\varphi_\infty - \varphi)$ survives unchanged after an increase of τ^2 from zero to some given value, the rate of disintegration being proportional to the quantity available at the instant considered ; τ^2 is, in the case of radioactive decay, the time which has elapsed since zero-time, and $1/\chi$ the decay constant (or what we may call in our case the Coefficient of Structural Lability). But that equation of radioactive decay is valid only in the case of *one single* pure radioactive substance. We could have assumed that an analogous law holds good for structural viscosity (or fluidity) if all particles were possessed of the same structure and furthermore, if either the breakdown of any one particle were completed in one step or if all particles were affected in the same way at every stepping-up of the rate of shear. Actually, however, there will be a whole spectrum of particle sizes and forms, with some statistical distribution, which are differently affected as the rate of shear increases. We shall come across an analogous problem in Chapter XX.

Let us now examine more closely the nature of the coefficient of structural stability. The quantity τ^2/χ being the index to e must be dimensionless. χ is therefore a quantity of the same nature as τ^2. In order to understand the former, we must first examine the latter.

In Section 5 of Chapter VI we calculated the strainwork expended in elastic shear and found it to be for a Hooke-body (compare (VI, 5) $= \tau^2/2\mu$. That strainwork is conserved as the resilience of the elastic body. We derived the expression for the resilience from

$$dw_i = \tau d\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

TABLE XVIII, 1. *The Structural Stability Coefficient χ as a Function of P^2*

P^2	0	100	200	300	400	500	10,000	20,000	30,000	40,000	600,000
χ	2,550	4,830	7,460	9,500	11,300	12,050	62,950	87,600	104,750	118,060	230,000
ΔP^2		100	100	100	100	100	9,500	10,000	10,000	10,000	560,000
$\Delta\chi$		2,280	2,630	2,040	1,800	750	50,900	24,650	17,150	13,250	112,000
$\Delta\chi/\Delta P^2$		23	26	20	18	7·5	5 4	2 5	1·7	1·3	0 2

This we can also write

$$dw_t = \tau \frac{d\gamma}{dt} dt = \tau \dot{\gamma} dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

and therefore

$$\dot{w}_t = \tau \dot{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

The quantity $\dot{w}_t$ is the strain *power* expended by the external forces. In an elastic solid this is added in every element of time to the stored-up potential energy. In a viscous liquid, on the contrary, it is dissipated and converted into heat.

For a Newtonian liquid for which (compare (I, f) $\dot{\gamma} = \varphi\tau$,

$$\dot{w}_t = \varphi\tau^2 = \eta\dot{\gamma}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

or

$$\tau^2 = \dot{w}_t/\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

The quantity τ^2, therefore, can, in our case, be interpreted as the dissipated strain-power per unit fluidity. The quantity χ must accordingly also be a strain-power per unit fluidity. If we compare Fig. XVIII. 1 we see that the power is equal to the rectangular area between the φ and τ^2 axes and two parallels to them through the point (τ^2, φ).

From (59) there follows that the power necessary to produce a given shearing stress is greater the greater the fluidity ; while the power to produce a given rate of shear is the greater the greater the viscosity. In a liquid devoid of viscosity, or what I have called a Pascalian Liquid (*Twelve Lectures*, p. 43), the power to produce any measurable shearing stress would have to be infinitely great, while no power is required to maintain any given rate of shear.

In a homogeneous, one-phase, " pure " liquid, the power is used up in overcoming the intrinsic microscopic viscous resistance of the liquid. In a suspension, more energy is dissipated because of the interference of the suspended particles with the free flow of the dispersion medium. This manifests itself as a kind of macroscopic viscosity as expressed, e.g. in Einstein's equation (XIV, 1), but it should be kept in mind that the mechanism is quite different. Actually the " microscopic " viscosity of the liquid is not changed by a suspended particle ; the only change produced is the one from laminar to more complicated flow in the vicinity of the particle. In our case, in addition, as soon as the rate of shear exceeds a certain value (corresponding to $P^2 = 25$), the secondary particles are broken up or disaggregated. At suitable encounters the fragments reaggregate and when a dynamical equilibrium is established, energy has to be constantly supplied in order to maintain the state of disaggregation against the tendency of re-aggregation. The observed macro-viscosity in such a system is accordingly due to the combined effects of the viscosity of the dispersion medium, the interference of the suspended particles with laminar flow,

and the continuous breaking up of re-forming secondary particles. However, the process is complicated by the fact that breaking up of secondary particles releases solvent and thereby decreases the macro-viscosity. The last influence prevails over the last but one and the net result is a gradual *decrease* of the " viscosity " with increasing rate of shear.*

We have given in Section 2 a formal reason why φ (or η) should be expressed as a function of τ^2 (or $\dot{\gamma}^2$) and not of τ (or $\dot{\gamma}$). We can now understand the physical significance of it. What causes the appearance of structural viscosity is the susceptibility of the structure to the *power* of the external forces expended upon it. The situation is very similar to the one governing the yield point of a plastic material (Chapter VI). There also we had to pass from a stress (ϑ) to a conception of work (E_{pl}).

12. Summary

The power dissipated by the viscous resistance of a Newtonian liquid is

$$\dot{w}_t = \varphi\tau^2 = \eta\dot{\gamma}^2 \quad . \quad . \quad . \quad . \quad \text{(XVIII, a)}$$

irrespective of whether φ (or η) is a constant or not.

In the case of structural viscosity, this power is taken up by (i) the viscous resistance of the dispersion medium, (ii) the interference by the suspended particle with laminar flow, (iii) the power necessary to maintain a change of the structure. On the other hand the modification of the structure may reduce the magnitude of interference so that ultimately a reduction of viscosity with increasing rate of shear results. This is the usual generalized Newtonian behaviour.

In the rheological equation of the generalized Newtonian liquid φ is expressed as a function of the dissipated power $\dot{w}_t$ or of the scalar quantity τ^2 :

$$\varphi = f(\tau^2) \quad . \quad . \quad . \quad . \quad . \quad \text{(XVIII, b)}$$

A power-series development of (XVIII, b)

$$\varphi = \varphi_0 + \sum_{n=1} a_{2n}\tau^{2n} \quad . \quad . \quad . \quad . \quad \text{(XVIII, c)}$$

where

$$a_{2n} = \frac{f^{(n)}(0)}{n!} \quad . \quad . \quad . \quad . \quad . \quad \text{(XVIII, d)}$$

leads to

$$V = \varphi_0 P + \sum_{n=1} \frac{2}{n+2} a_{2n} P^{2n+1} \quad . \quad . \quad \text{(XVIII, e)}$$

* One could therefore imagine a structural viscosity with the viscosity increasing with increasing rate of shear, but I do not know of any such case.

in the case of a tube instrument, and

$$V = \varphi_o P + \sum_{n=1} \frac{1 - a^{2n+1}}{(2n+1)(1-a)} a_{2n} P^{2n+1} \quad . \quad . \quad \text{(XVIII, f)}$$

in the case of a rotation instrument.

The following closed form rheological equations have been proposed:

(i)

$$\varphi = \frac{\varphi_o + \varphi_\infty \tau^2/a^2}{1 + \tau^2/a^2} \quad . \quad . \quad . \quad . \quad . \quad \text{(XVIII, g)}$$

where a is a " constant ".

(ii) Alternatively, a Coefficient of Structural Stability is defined by

$$\chi = \frac{\varphi_\infty - \varphi}{d\varphi/d(\tau^2)} \quad . \quad . \quad . \quad . \quad . \quad \text{(XVIII, h)}$$

which leads to the rheological equation

$$\varphi - \varphi_\infty = (\varphi_\alpha - \varphi_o) e^{-\tau^2/\chi} \quad . \quad . \quad . \quad \text{(XVIII, i)}$$

or

$$\varphi = \frac{\varphi_\alpha - \varphi}{\varphi_\alpha - \varphi_o} = e^{-\tau^2/\chi} \quad . \quad . \quad . \quad \text{(XVIII, j)}$$

This gives for the tube instrument

$$\bar{\varphi} = 2\frac{\chi^2}{P^4} - 2\frac{\chi}{P^2} e^{-P^2/\chi} - 2\frac{\chi^2}{P^4} e^{-P^2/\chi} \quad . \quad . \quad \text{(XVIII, k)}$$

and for the rotation instrument

$$\varphi = \frac{\sqrt{\chi\pi}}{2P} \frac{\phi\left(\frac{P}{\sqrt{\chi}}\right) - \phi\left(\frac{Pa}{\sqrt{\chi}}\right)}{1-a} \quad . \quad . \quad \text{(XVIII, l)}$$

where

$$\phi(x) = \frac{2}{\sqrt{\pi}} \int_o^x e^{-x^2} dx \quad . \quad . \quad . \quad . \quad \text{(XVIII, m)}$$

is Gauss's error integral.

If φ is plotted against P^2, the slope of the tangent to the curve at the point $P = 0$ is given by

$$\left. d\varphi/d(P^2) \right|_{P=0} = -2/3\chi_o \quad . \quad . \quad \text{(XVIII, n)}$$

in the first, and

$$\left. \frac{d\bar{\varphi}}{d(P^2)} \right|_{P=0} = -\frac{1 + a + a^2}{3\chi_o} \quad . \quad . \quad \text{(XVIII, o)}$$

in the second case. These equations may be used for the determination of χ_o.

Applying (XVIII, i) to viscometric measurements of a rubber-toluene solution, it is found that χ is not a constant but increases from a value χ_o at $P = 0$ to a maximum χ_∞, the rate of increase gradually decreasing. The structure of the dispersed system accordingly becomes more stable at increasing rates of shear, the stability being a dynamical one.

CHAPTER XIX

WALL EFFECTS

1. Historical

IMMEDIATELY after Bingham had discovered his law of plastic flow in the tube instrument, he attempted to represent observations so as to be independent of the radius R of the tube. He soon hit on plotting as abscissa, instead of the pressure difference Δp, the shearing stress at the internal surface of the wall, or our consistency variable P (compare (11, g)). However, as long as Q (or another equivalent quantity) remained as ordinate, the aim was only partly attained. His pupil, S. B. Ellis, in a thesis on "The Pseudo-Plastic Flow of Nitrocellulose Sols" in 1927, which was not published, made the statement: "It is the author's opinion that the curve, apparent fluidity plotted against shearing stress (at the wall M.R.), stands as a graphical representation of the flow property." Some time thereafter (1936) I proved this assertion on the lines of the development of Section 2, Chapter XVIII; but one year later (1931) I remarked: "If I have said that I have 'proved' (Ellis's assertion) mathematically, this does not mean that it does not need empirical verification. If experiments should contradict . . . this would mean that in deriving . . . some assumption had been made, which does not hold good in actuality. Now it should be noted that the only assumption (made) . . . is the assumption that material *does not slide at the boundary wall*", or that there was no slippage.

Slippage was actually found to exist in suspensions which could be considered as Bingham bodies. This can easily be understood. In a Bingham body, which is a solid dispersed system, there must be a continuous framework of dispersed phase throughout the material. If the material of the wall does not react with the dispersed phase, this framework is interrupted at the wall; and while the bulk of the material is a solid, a liquid layer adjacent to the wall, consisting of dispersion medium, is formed. This liquid layer will act as a lubricant and there will be gliding or slippage along the wall. Such an effect was observed by Green (1920). He used what he called a "microplastometer" (Green and Haslam (1925)), i.e. an "apparatus for the microscopic observation of plastic flow through capillaries". Green says: "A paint of suitable yield value is rubbed down with enough ultramarine to give it a pronounced bluish tint. This material is added because it is easy to distinguish under the low magnification

used, while the paint particles are not. Some of this paint is sucked into a capillary. Upon microscopic examination a number of facts will be observed. First, that the vehicle wets the walls of the tube ; second, that the pigment touches the walls at but *very few points* ; third, that the paint exists as a mass of interlocking flocculates. If a light air pressure is then applied, the material will be set in motion ; the entire mass being held together by interlocking of the flocculates, will move in one solid block. The phenomenon is analogous to that of pushing a lead pencil through a glass tube. With the increase of pressure the state of affairs is changed. The flocculates are broken up and, upon observing the ultramarine particles, it will be noticed that the paint is now moving in telescopic layers, the inner ones at much greater velocity than the outer ones."

When Buckingham derived his equation (VIII, b) he allowed for a " slippage term " of rather arbitrary form, on the basis of a wall-layer the thickness of which depends upon the radius R. In my above-mentioned paper (1931) I introduced a wall-layer of definite thickness. At about the same time Schofield and Scott Blair (1931) proceeded independently on similar lines, which they had already indicated in an earlier paper (1930). It should also be mentioned that Kroepelin (1929) quite independently proposed to plot η' against $8Q/3R^3\pi$. This quantity, which he called Mean Velocity Gradient, is evidently two-thirds of our consistency variable V (compare (II, f)). This also must result in curves independent of R. In my first publication I had erroneously assumed V to be the velocity gradient at the wall and therefore φ' to be the fluidity at the same place. Herschel (1930) helped to clear up this point.

The position is as described here in Section 7 of Chapter XVII.

2. Buckingham's Slippage Term for the Capillary Instrument

Let us go back to (XVII, 10'). If v does not vanish for $r = R$ but equals $- v_R$, we get

$$C = - v_R - k2l/\Delta p.(R\Delta p/2l)^{n+1}/(n+1) \quad . \quad . \quad . \quad (1)$$

and therefore

$$- v = v_R + k\frac{2l/\Delta p}{n+1}\left[\left(\frac{R\Delta p}{2l}\right)^{n+1} - \left(\frac{r\Delta p}{2l}\right)^{n+1}\right] \quad . \quad . \quad . \quad (2)$$

whence

$$Q = R^2\pi v_R + kR^3\pi\left(\frac{R\Delta p}{2l}\right)^n/(n+3) \quad . \quad . \quad . \quad . \quad (3)$$

or

$$V = \frac{4v_R}{R} + \frac{4kP^n}{n+3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which makes

$$\varphi' = \frac{4v_R}{RP} + \frac{4kP^{n-1}}{n+3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We see that neither (4) nor (5) are independent of R.
We now have to replace (XVIII, 9) by

$$\varphi' = \frac{4v_R}{RP} + \varphi_o + \sum_{n=1} \frac{2}{2+n} \frac{f^{(n)}(0)}{n!} P^{2n} \quad . \quad . \quad . \quad . \quad (6)$$

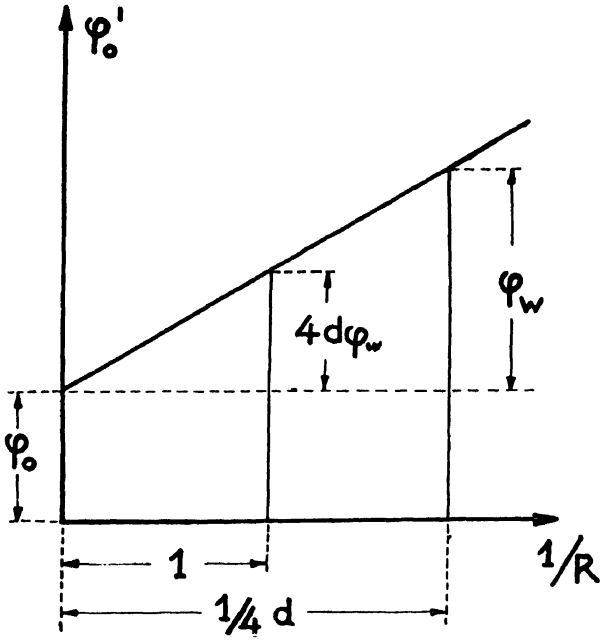


Fig. XIX, 1. Apparent fluidity : $1/R$ — curve as assumed by Buckingham.

and the apparent fluidity at rest is

$$\varphi'_o = \frac{4}{R} \frac{v_R}{P} + \varphi_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where v_R is a function of P.

Let d be the thickness of the lubricating wall layer and φ_w its fluidity, then Buckingham proposed in the first instance

$$v_R = \frac{d.\varphi_w R \Delta p}{2l} = d.\varphi_w.P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This would make the slippage term of (7)

$$\frac{4}{R} \frac{v_R}{P} = \frac{4d.\varphi_w}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If we assume that d is independent of R, the " apparent zero-fluidity : $1/R$ curve " would be a straight line as shown in Fig. XIX, 1.

Buckingham did not succeed in verifying (9) snd he later assumed $d.\varphi_w$ to be proportional to R. This would mean a constant slippage term, the curve being a straight line parallel to the abscissa. It would also mean that the thickness of the lubricating layer increases without limit as R increases and becomes infinite at a plane wall ; an absurd consequence. Buckingham's expression is, however, a very rough approximation only, as will presently be shown.

3. The Slippage Term for Couette Flow

Let us first consider the fundamental case of two parallel plates sheared against each other with the material between them. Such

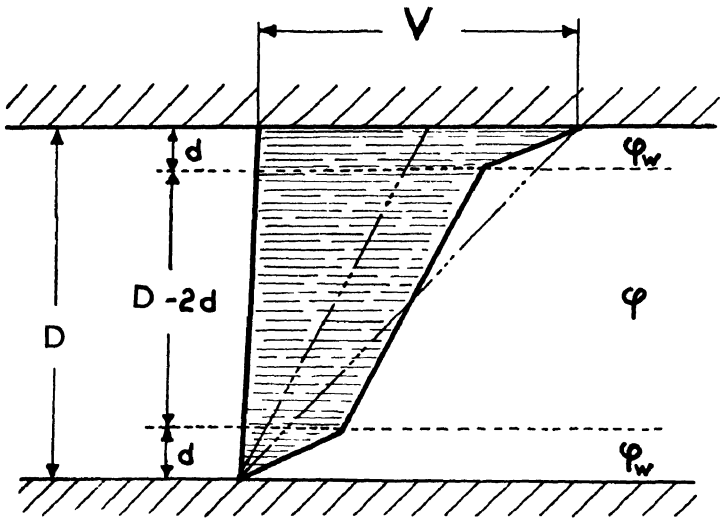


FIG. XIX, 2. Couette flow in the presence of wall-layers.

flow (Fig. I, 4) has been called Couette Flow, as we mentioned in Section 6 of Chapter II. In Fig. XIX, 2, we show conditions in the presence of wall-layers. The shearing stress is constant $= \tau$ and the velocity of the moving plate

$$V = \varphi'\tau D = \varphi\tau D + 2(\varphi_w - \varphi)\tau d \quad . \quad . \quad . \quad . \quad (10)$$

from which the apparent fluidity

$$\varphi' = \frac{V}{D}/\tau = \varphi + 2(\varphi_w - \varphi)\frac{d}{D} = \varphi_w - (\varphi_w - \varphi)\left(1 - 2\frac{d}{D}\right) \quad . \quad (11)$$

the slippage term being $(\varphi_w - \varphi)d/D$ per layer. Buckingham put it $= \varphi_w \frac{d}{D}$. Buckingham's expression can therefore be admitted only if φ is negligible in comparison with φ_w. This will generally not be the case. The discrepancy is still greater in the case of the tube- and rotating-cylinder instruments.

4. The Generalized Bingham Body with Wall Layer

Just as an additive combination of the St. Venant solid with the simple Newtonian liquid results in the simple Bingham body when η_{pl} is a constant: the additive combination with a generalized Newtonian liquid gives a generalized Bingham body, where η_{pl} is a variable coefficient. To permit of the greatest generality, we shall

presently derive the formulas which result for the generalized Bingham body in the tube and rotating cylinder instruments when there are wall layers. The formulas for the simple Bingham body, the generalized Newtonian liquid and the simple Newtonian liquid, can then easily be written down as special cases.

For the generalized Bingham body, the velocity of flow and rotational velocity respectively in the case of the

Hagen-Poiseuille	Margules-Couette

experiment can be derived on the model of (XVII, 10) in accordance with the considerations of Section 2, Chapter XVIII (Equ. (XVIII, 4)) and introducing $\tau - \vartheta$ for τ. We have

$$dv/dr = \sum_{n=0} \alpha_{2n} (r\Delta p/2l - \vartheta)^{2n+1} \qquad \left| \quad r\, d\theta/dr = \sum_{n=0} \alpha_{2n} (M/2\pi r^3 h - \vartheta)^{2n+1} \right. \qquad (12)$$

and through integration . . .

$$v = 2l/\Delta p \sum_{n=0} \alpha_{2n} \frac{(r\Delta p/2l - \vartheta)^{2n+2}}{2n+2} + C \qquad \left| \quad \theta = \sum_{n=0} \alpha_{2n} 2n+1 \left[\sum_{m=0}^{m=2n} \frac{1}{2(m-2n-1)} \binom{2n+1}{m} \cdot \left(\frac{M}{2\pi r^2 h}\right)^{2n+1-m} \ln r \right] + C. \right. \qquad (13)^*$$

for $\tau \geq \vartheta$, i.e. for . . . .

$$r \geq r_0 = 2\vartheta l/\Delta p \qquad \left| \quad r \geq r_0 = (M/2\pi h\vartheta)^{\frac{1}{2}} \right. \qquad . \quad . \quad . \quad . \quad . \quad (14)$$

For the determination of the integration constant C we now assume that there exists a wall layer of thickness d and of constant fluidity φ_w, i.e. the wall layer is a simple Newtonian liquid which itself adheres to the wall (compare Fig XIX, 3).

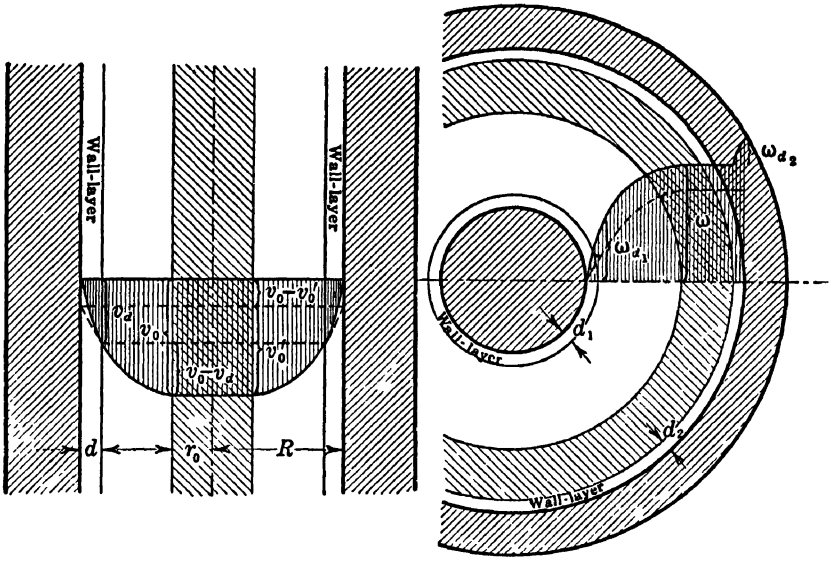


FIG. XIX, 3. Plastic flow in cases when there exist wall-layers.

* Note that $\binom{2}{0} = 2$; $\binom{2}{1} = 2$; $\binom{3}{0} = 3$; $\binom{3}{1} = 3$; $\binom{3}{2} = 3$ etc.

We therefore have

$$v = \varphi_w(\Delta p/4l)\,(R^2 - r^2)$$

for the layer near the wall

$$\dot{\theta} = \varphi_u(M/4\pi h).(1/R_i^2 - 1/r^2)$$

for the layer near the internal cylinder (compare (II, o))

and $$\dot{\theta} = \dot{\Omega} - \varphi_w(M/\pi 4h)\,(1/r^2 - 1/R_e^2) \quad . \quad . \quad (15)$$

for the layer near the external cylinder

while the velocity at the interface of the wall-layer with the body of the liquid is

$$v_d = \varphi_w(\Delta p/4l)\,[R^2 - (R-d)^2]$$

$$\dot{\theta}_i = \varphi_w(M/4\pi h).[1/R_i^2 - 1/(R_i + d)^2]$$
$$\dot{\theta}_e = \dot{\Omega} - \varphi_w(M/4\pi h).[1/(R_e - d)^2 - 1/R_e^2] \quad . \quad (16)$$

We now have to determine the integration constant so that

$$v_d = \frac{2l}{\Delta p}\sum_{n=o} a_{2n}\frac{(R-d)\Delta p/2l - \vartheta^{2n+2}}{2n+2} + C$$

$$\dot{\theta}_d = \sum_{n=o} a_{2n}\vartheta^{2n-1}\left\{\sum_{m=o}^{m=2n}\frac{1}{2(m-2n-1)}\binom{2n+1}{m}\left(\frac{M}{2\pi(R_i+d)^2 h}\right)^{2n+1-m} - \ln(R_i+d)\right\} + C \quad . \quad (17)$$

Introducing the integration constant into (13) we get

$$v = \frac{2l}{\Delta p}\sum_{n=o}\frac{a_{2n}}{2n+2}\left\{[(R-d)\Delta p/2l - \vartheta^{2n+2} - (r\Delta p/2l - \vartheta)^{2n+2}\right\} + \varphi_w\Delta p/4l.[R^2 - (R-d)^2]$$

$$\dot{\theta} = \sum_{n=o} a_{2n}\vartheta^{2n-1}\left\{\sum_{m=o}^{m=2n}\frac{1}{2(2n+1-m)}\binom{2n+1}{m}(M/2\pi h\vartheta)^{2n+1-m}\left[\frac{1}{(R_i+d)^{2(2n-1-m)}} - \frac{1}{r^{2(2n-1-m)}}\right] - \ln\frac{r}{R_i+d}\right\} + \varphi_w M/4\pi h.[1/R_i^2 - 1/(R_i+d)^2] \quad . \quad . \quad (18)$$

for

$$(R-d) \geqq r \geqq r_o = 2\vartheta l/\Delta p.$$

$$(R_i + d) \leqq r \leqq r_o = (M/2\pi h\vartheta)^{\frac{1}{2}}$$

Furthermore there is

$$v_w = \varphi_w(\Delta p/4l)\,(R^2 - r^2)$$

$$\dot{\theta}_w = \varphi_w(M/4\pi h)\,(1/R_i^2 - 1/r^2) \quad . \quad . \quad . \quad (19)$$

for

$$R \geqq r \geqq (R-d)$$

$$R_i \leqq r \leqq (R_i + d)$$

and

$$v_o = \frac{2l}{\Delta p}\sum_{n=0}\frac{a_{2n}}{2n+2}\left[(R-d)\Delta p/2l - \vartheta\right]^{2n+2} + \varphi_w \Delta p/4l.$$

$$\left[R^2 - (R-d)^2\right]$$

$$r \gtrless r_o = \frac{2\vartheta l}{\Delta p}$$

$$V = \sum_{n=0}\frac{2a_{2n}}{n+1}P^{2n-1}\left(1 - \frac{d}{R} - \frac{\vartheta}{P}\right)^{2n+1}$$

$$\left[1 - 2\frac{1 - d/R - \vartheta/P}{2n+3} + 2\frac{(1-d/R-\vartheta/P)^2}{(2n+3)(2n+4)}\right]$$

$$+ P\varphi_w[1 - (1-d/R)^4]$$

$$P \geqq \vartheta/(1-d/R)$$

$$V = P\varphi_w[1 - (1-d/R)^4]$$

$$\dot{\theta}_o = \sum_{n=0}\alpha_{2n}\vartheta^{2n+1}\left\{\sum_{m=0}^{m=2n}\frac{1}{2(2n+1-m)}\binom{2n+1}{m}\right.$$

$$\left.\left[\left(\frac{M}{2\pi(R_i+d)^2h\vartheta}\right)^{2n+1-m} - 1\right] - \ln\left(\frac{M}{2\pi(R_i+d)^2h}\right)^{\frac{1}{2}}\right\}$$

$$+ \varphi_w(M/4\pi h).(1/R_i^2 - 1/(R_i+d)^2) \quad . \quad . \quad (20)$$

for

$$(R_e - d) \geqq r \geqq r_o = (M/\pi 2h\vartheta)^{\frac{1}{2}}$$

This gives

$$V = \frac{1}{1-a}\sum_{n=0}\alpha_{2n}\vartheta^{2n+1}\left\{\sum_{m=0}^{m=2n}\frac{1}{2n+1-m}\binom{2n+1}{m}(P/\vartheta)^{2n+1-m}\right.$$

$$\left[\frac{1}{(1+d/R_i)^{2(2n-1-m)}} - \frac{a^{2n+1-m}}{(1-d/R_e)^{2(2n-1-m)}}\right]$$

$$\left.-\ln\frac{1}{a}\left(\frac{1-d/R_e}{1+d/R_i}\right)^2\right\} + \varphi_w P\left[1 - \frac{1-2d/(R_e-R_i)}{(1+d/R_i)^2(1-d/R_e)^2}\right] \quad . \quad (21)$$

for

$$P \geqq \vartheta/a \ (r_o > (R_e - d))$$

and

$$V = \frac{1}{1-a}\sum_{n=0}\alpha_{2n}\vartheta^{2n+1}\left\{\sum_{m=0}^{m=2n}\frac{1}{(2n+1-m)}\binom{2n+1}{m}\right.$$

$$\left.\left[\left(\frac{P/\vartheta}{(1+d/R_i)^2}\right)^{2n+1-m} - 1\right] - \ln\left(\frac{P/\vartheta}{(1+d/R_i)^2}\right)\right\}$$

$$+ \varphi_w P\left[1 - \frac{1-2d/(R_e-R_i)}{(1+d/R_i)^2(1-d/R_e)^2}\right] \quad . \quad . \quad (22)$$

for

$$P \gtrless \vartheta/(1 - d/R) \qquad \vartheta(1 + d/R_i)^2 \leqq P \leqq \vartheta/a \quad (r_o < (R_e - d))$$

and

$$V = \varphi_w P\left[1 - \frac{1 - 2d/(R_e - R_i)}{(1 + d/R_i)^2(1 - d/R_e)^2}\right] \quad . \quad (22')$$

for

$$P \leqq \vartheta(1 + d/R_i)^2 \quad (r_o < (R_i + d))$$

From (21), (22) and (22′) it follows that whenever there exists a "wall-effect", the curve plotted in "consistency variables" is not independent of the dimensions of the apparatus ; and furthermore, that if the material is a soft solid, the curve starts with a straight line. The last statement cannot be reversed ; if the curve starts with a straight line, the material may be either a soft plastic solid having a wall effect, or a simple Newtonian liquid with or without a wall effect.

5. The Simple Bingham Body with Wall Layer

The equations for the simple Bingham body can be derived from the foregoing by putting $n = 0$. We get

$$V = 2a_o P\,(1 - d/R - \vartheta/P)^2\left[1 - \frac{2}{3}(1 - d/R - \vartheta/P) + \frac{1}{6}(1 - d/R - \vartheta/P)^2\right] + P\varphi_w[1 - \vartheta(1 - d/R)^4]$$

for

$$P \geqq \vartheta/(1 - d/R)$$

and

$$V = \frac{1}{1 - a}a_o\vartheta\left\{\frac{P}{\vartheta}\left[\frac{1}{(1 + d/R_i)^2} - \frac{a}{(1 - d/R_e)^2}\right] - \ln\frac{1}{a}\left(\frac{1 - d/R_e}{1 + d/R_i}\right)^2\right\} + \varphi_w P\left[\frac{1 - 2d/(R_e - R_i)}{1 - (1 + d/R_i)^2(1 - d/R_e)^2}\right] \quad (23)$$

for

$$P \geqq \vartheta/a$$

$$V = P\varphi_w[1 - (1 - d/R)^4]$$

$$V = \frac{1}{1-a}\alpha_e\vartheta\left\{\left[\frac{P/\vartheta}{(1-d/R_i)^2} - 1\right] - \ln\left(\frac{P/\vartheta}{(1+d/R_i)^2}\right)\right\} + \varphi_w P\left[1 - \frac{1-2d/(R_e-R_i)}{(1+d/R_i)^2(1-d/R_e)^2}\right] \quad . \quad (24)$$

for

$$P \leqq \vartheta/(1-d/R)$$

$$\vartheta(1-d/R_i)^2 \leqq P \leqq \vartheta/a$$

and

$$V = \varphi_w P\left[1 - \frac{1-2d/(R_e-R_i)}{(1+d/R_i)^2(1-d/R_e)^2}\right] \quad . \quad (24')$$

for

$$P \leqq \vartheta(1+d/R_i)^2$$

In the absence of a wall effect when we have to put $d = 0$ and $\varphi_w = 0$; these are

$$V = P\varphi_{pl}\left[1 - \frac{4}{3}\frac{\vartheta}{P} + \frac{1}{3}\left(\frac{\vartheta}{P}\right)^4\right]$$

$$V = \varphi_{pl}\left[P - \vartheta\frac{\ln(1/a)}{1-a}\right] \quad . \quad . \quad . \quad (25)$$

for

$$P \geqq \vartheta$$

$$P \geqq \vartheta/a$$

and

$$V = \frac{1}{1-a}\varphi_{pl}[P - \vartheta - \vartheta\ln(P/\vartheta)] \quad . \quad . \quad (25')$$

for

$$\vartheta < P \leqq \vartheta/a$$

which may be compared with

(VIII, *c*) (VIII, *d*).

6. The Generalized Bingham Body in the Absence of a Wall Effect

In the absence of a wall effect we have to put $d = 0$ and $\varphi_w = 0$, but retain n.

We get

$$V = \sum_{n=0} \frac{2a_{2n}}{n+1} P^{2n-1}(1-\vartheta/P)^{2n+2} \left[1 - 2\frac{1-\vartheta/P}{2n+3} + 2\frac{(1-\vartheta/P)^2}{(2n+3)(2n+4)}\right]$$

for

$$P \geqq \vartheta$$

$$V = \frac{1}{1-a}\sum_{n=0} a_{2n}\vartheta^{2n+1}\left\{\sum_{m=0}^{m=2n} \frac{1}{2n+1-m}\binom{2n+1}{m}(P/\vartheta)^{2n+1-m}\left[1 - a^{2n+1-m} - \ln(1/a)\right]\right\} \quad . \quad (26)$$

for

$$P \geqq \vartheta/a$$

and

$$V = \frac{1}{1-a}\sum_{n=0} a_{2n}\vartheta^{2n-1}\left\{\sum_{m=0}^{m=2n} \frac{1}{2n+1-m}\binom{2n+1}{m}\left[(P/\vartheta)^{2n+1-m} - 1\right] - \ln(P/\vartheta)\right\} \quad . \quad . \quad (26')$$

for

$$\vartheta < P \leqq \vartheta/a.$$

7. The Generalized Newtonian Liquid in the Presence of a Wall Effect

For the particular case of the generalized Newtonian liquid in the presence of a wall effect we have to put $\vartheta = 0$. We get

$$V = \sum_{n=0} \frac{2a_{2n}}{n+1} P^{2n-1}(1-d/R)^{2n-2} \left[1 - 2\frac{1-d/R}{2n-3} + 2\frac{(1-d/R)^2}{(2n+3)(2n+4)}\right] - P\varphi_w\left[1-(1-d/R)^4\right]$$

$$V = \frac{1}{1-a}\sum_{n=0} a_{2n}\frac{P^{2n+1}}{2n+1}\left[\frac{1}{(1+d/R_i)^{2(2n-1)}} - \frac{a^{2n-1}}{(1-d/R_e)^{2(2n-1)}} - \ln\frac{1}{a}\left(\frac{1-d/R_e}{1+d/R_i}\right)^2\right] + \varphi_w P\left[1 - \frac{1-2d/(R_e-R_i)}{(1+d/R_i)^2(1-d/R_e)^2}\right] \quad . \quad . \quad (27)$$

For $n = 0$, these equations are reduced to those of the generalized Newtonian liquid in the absence of a wall effect, viz.

(XVIII, 5) (XVIII, 6)

8. Liquids and Soft Solids in the Presence of Wall Effects

We have said in Section 3 of Chapter VIII that if the consistency curve (or, generally, the flow-curve) starts from a point on the P-axis (or an equivalent axis) with a *finite abscissa*, this proves that the material is a solid. We are now in a position to amplify this criterion for the case when there exists a wall effect. In such a case we first determine the apparent fluidity at rest, i.e. for $P = 0$. If the material is a plastic solid, this is

$$\varphi_n' = \frac{V}{P}\Big|_{P=0} = \varphi_n'_n[1 - (1 - d/R)^4] \quad | \quad \varphi_0' = 2\varphi_i'_n\left[1 - \frac{1 - 2d/(R_e - R_i)}{(1 + d/R_i)^2(1 - d/R_e)^2}\right] \quad (28)$$

If, however, it is a liquid, we have

$$\varphi_o' = \varphi_o + \varphi_w[\ldots\ldots] \quad | \quad \varphi_o' = \varphi_o + 2\varphi_n[\ldots\ldots] \quad (29)$$

If we now vary the dimensions of the apparatus, i.e.

the radius R | the radii R_i and R_e, keeping "a" constant,
and plot the calculated values against
$1/R$ | $1/R_i$
the extrapolation of the resultant curve for
$1/R = 0$ | $1/R_i = 0$

gives $\varphi_o' = 0$ in the case of a plastic solid and $\varphi_o' \neq 0$ in the case of a liquid.

The wall effect can be considered under two aspects. It may be of interest to determine the thickness of the wall layer and its fluidity.

In order to accomplish this, we differentiate (29) in respect of

$1/R$ | $1/R_i$, keeping in mind that R_i/R_e is a constant.

$$\frac{d\varphi_n'}{d(1/R)} = 4\varphi_n d(1 - d/R)^3 \quad | \quad \text{We get} \quad \frac{d\varphi_o'}{d(1/R_i)} = 4\varphi_n d \frac{(1 + d/R_i)(1 - d/R_e) + R_e/R_i}{(R_e/R_i)(1 + d/R_i)^2(1 - d/R_e)^2} \quad (30)$$

Extrapolating the curve defined by (30) for $1/R = 0$, gives in both cases $[d\varphi_0'/d(1/R)]_0 = 4\varphi_w d$. If φ_w is known, this gives the thickness of the wall layer. Otherwise another differentiation and extrapolation furnishes another value so that both unknowns φ_w and d can be calculated.

If, however, the problem is to determine the consistency of the bulk of the material, the wall effect is troublesome and has to be eliminated by means of a correction. This can be done as follows :

One has first to determine the true fluidity of the material at rest, as explained above. If we know this we redraw the consistency curve in accordance with the equation

$$\frac{[V - P(\varphi_o' - \varphi_o)]\varphi_w}{\varphi_w + \varphi_o - \varphi_o'} = \sum_{n=0} \quad | \quad \frac{[V - P(\varphi_o' - \varphi_o)]^2\varphi_w}{^2\varphi_w + \varphi_o - \varphi_o'} = \sum_{n=0} \quad (31)$$

The first term of the series on the right side, i.e. for $n = 0$, is identical with the first term for a liquid showing *no* wall effect. In the case of a simple Newtonian liquid showing a wall effect, this curve is therefore the correct

consistency curve of the bulk of the material and may accordingly be named the " bulk " consistency curve. The curve is identical with the consistency curve of the bulk of a generalized Newtonian liquid showing a wall effect, if either small *P's* are considered or $1/R$ is small. For larger *P's* and $1/R$'s there is no way of plotting consistency curves independently of the dimensions of the apparatus, if the law of flow of the material is not known.

Regions of very small shearing velocities require special consideration. Very thorough investigations were undertaken by Scott Blair and collaborators (compare Scott Blair *Introduction*, pp. 32–34). The consistency curve of clay and soil pastes was found to divide itself into four regions. In region I there is no flow, region II is a straight line, region III is markedly curved and region IV becomes asymptotically straight. In regions I and IV one recognizes the picture of Fig. 4. Region II is due to slippage. It was Scott Blair's contribution to find region I, i.e. the fact that a definite stress is necessary to *start* slippage.

9. Lubricating and Roughening Wall Layers

When considering wall effects in the foregoing, we have spoken of " slippage " as occurring in a two-phase system, with the dispersion medium forming the wall layer. But this is taking too narrow a view of wall effects. As Bingham early (1922) pointed out, it is " quite possible that the fluidity of a liquid near the surface is not identical with that within the body of the liquid ". Bartsch (1926) found, e.g. that when oleic acid is in contact with a metal surface a layer about 10,000 molecules thick adheres to the wall. Such molecules are at the same time oriented. Generally there may be two sorts of effects. The molecules may be either oriented *parallel* to the wall, which should reduce friction, or they may become oriented *normal*, which must increase friction. The first made a " lubricating ", the second a " roughening " layer. If we introduce in (29)

$$\left.\begin{aligned} \varphi_0 &= \alpha\varphi_w \\ d &= R/n \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (32)$$

Fig. 4 shows φ_0 as a function of $1/n$ for different values of α.

For $\alpha < 1$ the wall layer acts as lubricant, for $\alpha > 1$ as a roughener. It must, however, be said that it is doubtful whether these effects are pronounced enough to affect viscosity measurements in the case of simple liquids. An attempt by R. Bulkley (1931) in the case of different oils, using capillary viscometers and raising the precision of the measurements to unusual levels, had negative results.

When a liquid does not wet the solid, as in the case of mercury, one cannot properly speak of slippage in the sense in which this word is ordinarily used. In this case there exists a layer of some other medium, either air or the vapour of the liquid, between the material and the solid wall. In the case of air at room temperature $\varphi_w > 5{,}000$. The

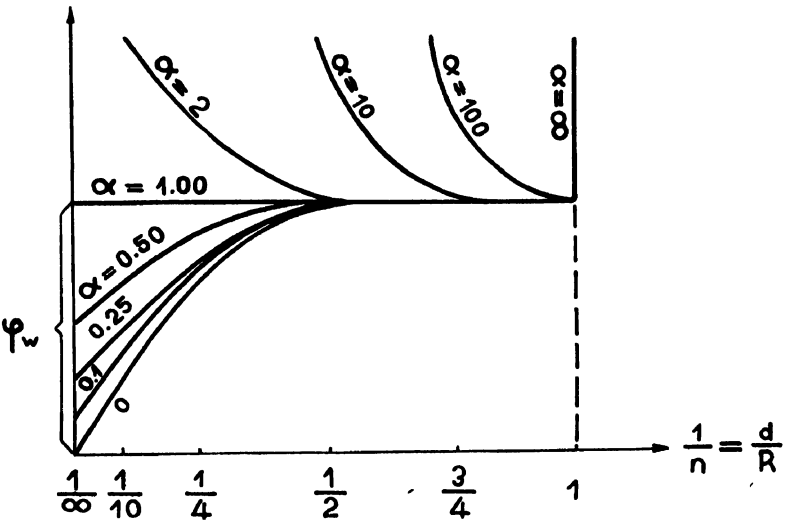


FIG. XIX, 4. Apparent zero fluidities of liquids in the presence of wall effects as determined in capillary instruments.
φ_w fluidity of the wall layer. φ_0 zero fluidity.
R radius of capillary.
d thickness of wall layer. $\alpha = \varphi_0/\varphi_w$.

thickness d must be $\cdot 10^{-8}$ cm., which is the order of magnitude of the diameter of simple molecules. Bingham and Thompson (1928) determined the viscosity of mercury in a capillary of $R = 0{\cdot}012$ cm. Therefore in this case $\varphi' - \varphi$ (compare (29)) $> 0{\cdot}016$. The observed fluidity of mercury at room temperature $\cdot 66$. Therefore $\frac{\varphi' - \varphi}{\varphi'} > 0{\cdot}024$ per cent. This is less than the accuracy of the Bingham instrument, which is about 0·1 per cent. The presence of the air layer can therefore not make itself felt and the viscosity of mercury can be determined from the usual formulas assuming that the mercury adheres to the solid wall (Reiner, 1932). Erk (1932) has pointed out that the viscosity of air or vapour, which is in a state of adsorption on the solid surface, must be assumed as much higher and may even have a " roughening " influence.

10. Summary

In the flow of a dispersed system, the dispersion medium may form a lubricating layer between the body of the liquid and a solid wall. This is especially so with suspensions which are soft plastic solids. In liquids in contact with solids there may be wall layers acting either as " lubricants " or " roughening ".

In the presence of a wall layer the apparent fluidity at rest will be in the case of a plastic solid in the

capillary	rotating-cylinder instrument
$\varphi_o' = \varphi_w[1 - (1 - d/R)^4]$	$\varphi_o' = 2\varphi_w\left[1 - \dfrac{1 - 2d/(R_e - R_i)}{(1 + d/R_i)^2\,(1 - d/R_e)^2}\right]$ (XIX, a)

and in the case of a liquid, the fluidity at rest of which is φ_o,

$\varphi_o' = \varphi_o + \varphi_w[1 - (1 - d/R)^4]$	$\varphi_o' = \varphi_o + 2\varphi_w\left[1 - \dfrac{1 - 2d/(R_e - R_i)}{(1 + d/R_i)^2\,(1 - d/R_e)^2}\right]$ (XIX, b)

When plotting calculated φ_o' values against varying

$1/R$	$1/R_i$, keeping "a" constant

extrapolation of the resultant curve to zero gives $\varphi_o' = 0$ in the case of a plastic solid and φ_o in the case of a liquid.

CHAPTER XX

WORK HARDENING

1. Necking in the Tensile Test

I HAVE mentioned the necking of a mild steel bar after prolonged plastic extension. When necking starts, the rod ceases to be cylindrical or prismatic and obtains a shape which makes it less suitable as a test-piece ; stresses and deformations being no longer more or less evenly distributed over the length of the test-piece, but now concentrated at the section of striction where the cross-sectional area is a minimum.

If one comes to think of it, it is rather strange that the cylindrical shape can be maintained in *any* tensile test, either throughout the test or up to a certain elongation. The material tested (whatever it may be) cannot be of uniform quality over the length of the bar ; *its quality must at some section be lowest.* At the same time it will not be absolutely cylindrical, but possess somewhere a minimum cross-sectional area. Therefore when loaded, it will first give way at one or other of those sections. If so, the cross-sectional area will there be reduced and become markedly less than along the rest of the test-piece. This will increase the stress at that section and worsen the position.* If this does not actually happen there must be some counteracting influence. The counteracting influence is *strain hardening.*† Strain Hardening *is the rise of the yield point of a material with increased deformation.*‡ When the material first gives way at the weakest section it is there strained. When the strain brings with it strain hardening, the material at that section may become harder than throughout the rest of the length and, even with increased local stress, deformation will not continue at the section but start somewhere else, i.e. at the next worst place. By attacking one section after another, deformation is equally distributed over the length of the rod, the shape of which is finally again cylindrical. Therefore, *if a bar of any material remains cylindrical (or prismatic) in a tensile test, this is an indication that the material possesses the property of strain hardening.*

* The tensile test, accordingly, proceeds through states of *unstable* equilibrium ; in contradistinction to the compression test where the equilibrium is stable, any local deformation resulting in a reduction of stress.

† This is the usual designation. It should, however, be noted that after the first yield point has been exceeded we should (in accordance with our terminology) not speak of "strain" but of "deformation". We shall later propose to replace "strain hardening" by *work hardening*, a term which is more appropriate.

‡ This is the customary case. Schofield and Scott Blair (1932) have observed strain hardening in flour dough where the *viscosity increases with strain.*

2. Model for the Strain-hardening Range

It is not difficult to adapt the model shown in Fig. VI, 3 so as to include the strain-hardening range. Fig. XX, 1 shows a model and the respective load diagram. The model consists of a number of St. Venant elements connected in series and preceded by an H element. As deformation proceeds, or more strings are tightened, more and more weights are called into action, with the yield point rising. It has been found

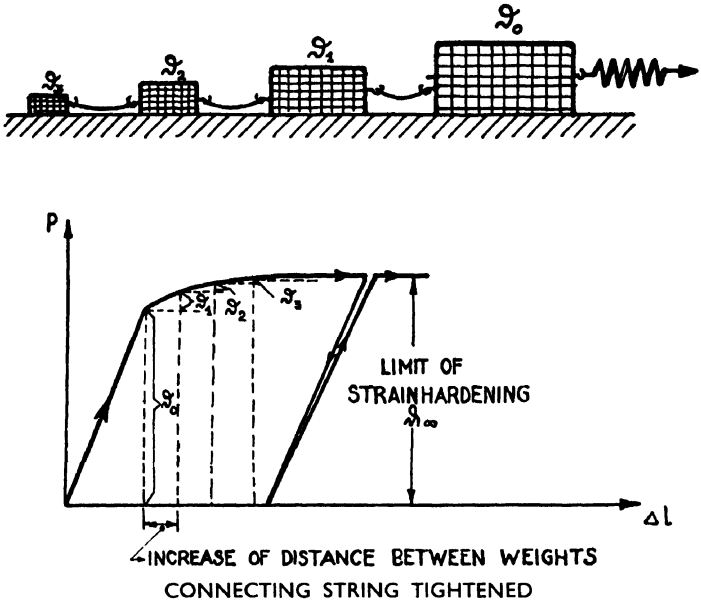


Fig. XX, 1. Model for a generalized St. Venant body.

that in mild steel, as in other materials showing strain hardening, this does not go on indefinitely as the deformation increases. In the tensile test the process is interrupted by the breakage of the test-piece at about 20 per cent. overall extension or 30 per cent. local extension, but in wire drawing that same small steel rod can be converted into a thin wire by drawing it cold through a number of decreasing holes. By this process its diameter is reduced to a fraction and its length increased by many thousands per cent. The process has a limit only when the wire thins to crystallite dimensions. While, therefore, the deformation is practically without limit, no such parallel unlimited increase in yield point can be observed. The yield point gradually increases with diminishing rate of increase until a Maximum Yield stress, ϑ_∞, is reached. This implies the existence of a Limit of strain hardening. To this fact there corresponds in the model a limited number of decreasing weights.

3. The Tensile Strength

The model of Fig. XX, 1 does not include considerations of " strength ". However, the main object of the tensile test for mild steel is the determination of its " strength ".

The tensile test is carried out in accordance with specifications for which we may take as typical the British Standard Specification (B.S.S.) No. 785—1938 for rolled steel bars and hard-drawn steel wire for concrete reinforcement. That specification asks (Clause 5) for " the tensile breaking strength, yield point (where specified) and elongation ". The terms are not defined, but there can be no question that the " yield point " is the stress P_a/A_o (Fig. VI, 1) and the " elongation " (extension) $\Delta l/l_o$. There is, however, a curious point about the " tensile breaking strength ". The specification says that it " shall conform to the requirements specified in Appendix A, page 18 ", but if we turn to that Appendix no tensile breaking strength is specified, but something which is called Ultimate Tensile Stress. The question naturally arises : are these two one and the same and how is it that they are named differently ?

British terminology is not as definite about the concepts involved as is American. In the United States the stress P_b/A_o is called Ultimate Tensile Strength and the stress P_e/A_o Tensile Breaking Strength. In the well-known English textbook, *The Strength of Materials*, by E. S. Andrews, P_b/A_o is called Commercial Maximum Stress. The Germans call P_b/A_o *Zugfestigkeit*. There can be no doubt that what the British Standard Specification requires is P_b/A_o. In view, however, of the fact that at the load P_b no *breaking* occurs, the term " *tensile breaking strength* " is not very fortunate, especially if it is not defined.

The purpose of the present discussion is neither philological nor legalistic. There can be no question that in the use of a steel rod or bar as a structural member *in tension*, what governs its usefulness is the magnitude of the load P_b. The *load* which the member is to take is given. It does not help that the steel member can resist *higher* stresses when the point *b* has been passed (compare Fig. VI, 1), if the load has at the same time to be decreased—because this cannot be done. There are, however, other uses of mild steel and the question then arises : Is the stress P_b/A_o or, more correctly, P_b/A of *physical* significance ? Does it tell us something of the *intrinsic* properties of the steel under test ? What is the necking due to ? Until the point *b* is reached the cylindrical rod remains cylindrical, albeit with increased length and reduced cross-section. Why does this not continue down to actual breaking ? And why does breakage occur at a certain extension, while in the wire-drawing process that extension can be infinitely enlarged ?

We shall presently discuss these questions using the matter provided by experiments which I first undertook in collaboration with W. Fuchs

and H. Ilberg at the Laboratory for Testing Materials of the Standards Institution of Palestine at Tel-Aviv (1945) and continued in collaboration with A. Freudenthal (1948) and which will be described in the following sections.

4. A Close-up of the Tensile Test-Piece

A test-piece was shaped as shown in Fig. XX, 2. It was assumed that the yield stress of the steel was about 3,000 kg./cm.^{2} and the test-

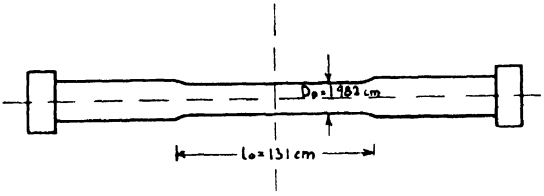


FIG. XX, 2. Original form and dimensions of test-piece.

piece was first loaded with 4,500 kg., giving a stress of 1,460 kg./cm.^{2} After that the rod was unloaded and the length found to be the same within 1/100 part of a mm. This, therefore, was a purely elastic strain. After that, experiment 2 was undertaken.

In experiment 2, point *a* of Fig. VI, 1 was reached and exceeded at a load of between 8,900 and 9,000 kg., giving a stress of about 2,900 kg./cm.^{2} The rod was again unloaded and measured. There was now a permanent elongation of the central piece of 1·65 mm. which gives an extension = elongation/original length of 1·26 per cent. within a hundredth of 1 per cent. Elongations were measured in two longitudinal sections at right angles and differed slightly. They were averaged.

This procedure was continued. The rod was loaded until the yield point was reached, then unloaded, then measured. There were large intervals of time between experiments ; not on purpose, but due to the conditions under which the investigation was undertaken. However, as will be explained later, this brought to better light the phenomenon of " ageing " of steel.

With the sixth experiment necking set in, the point *b* in Fig. VI, 1 being reached. The tensile strength, as usually defined, and also called " nominal " stress, was found to be 4,120 kg./cm.^{2} The "true" tensile strength was 5,200 kg./cm.^{2} but it should be kept in mind that actually no breaking occurred at this stress. There was no point in continuing the experiments in the same way. One knew what was to be expected : the rod would cease to be extended over the whole length, it would thin down at the section of striction where it would locally be extended ; finally the rod would break at that place at a certain smaller load but higher stress.

When measuring the length after each experiment, the diameters along the length were also measured to 1/100 mm. In the uppermost part of Fig. XX, 3 the shapes of the rod up to the necking are shown in exaggerated scale. As can be seen, the statement in most textbooks

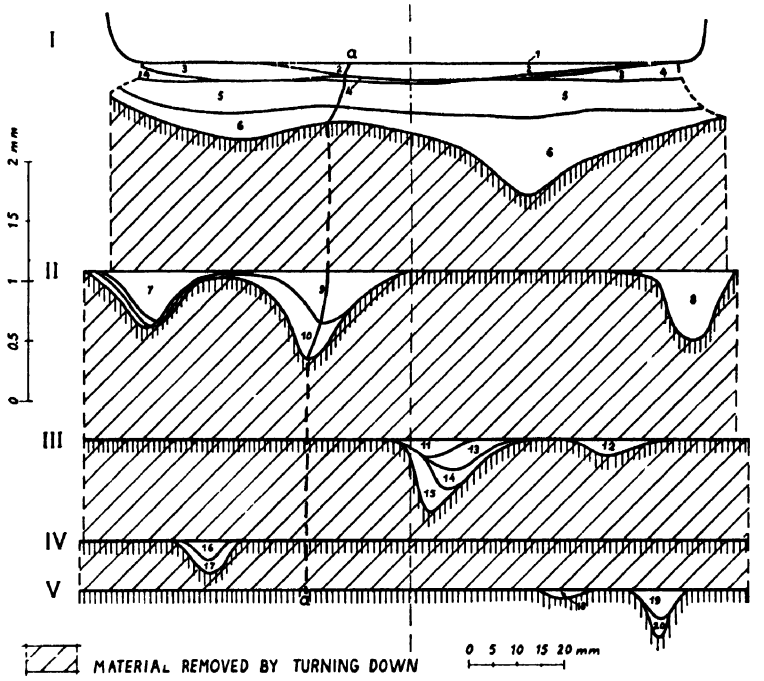


FIG. XX, 3 Lateral contractions of test-piece in linear scale.

The test-piece was turned down several times. I shows the contractions of the original test-piece ; II, III, IV of test-pieces turned down. The outline of the last shape before turning down has been hatched vertically. Numbers indicate the experiment in which the deformation was effected. For the meaning of " *a* " compare Section 8 below.

that " until the point *b* is reached the cylindrical rod remains cylindrical " is correct only as far as observation with the naked eye goes. Actually the rod loses its perfect cylindrical shape with the first permanent deformation in experiment 2.

In experiment 3 the section where the diameter had been reduced before to a minimum, did *not* move ; deformation took place where the diameter had not been changed and the stress was therefore the smallest. In experiment 4 also the shallow microscopic neck formed in experiment 2 was not changed ; deformation took place at the ends which had not moved in either experiment 2 or 3. As we said in Section 1, the point where the diameter is most reduced in each experiment wanders along the length of the rod, to find the material which has so

far been least strained. Just before necking, in experiment 5, the shape was again fairly cylindrical.

5. The Cause of Necking

We can now understand what causes necking. As can be seen from Fig. VI, 1, the rate of strain hardening, i.e. the rise in yield point per unit increase of strain, falls off between points *c* and *b*. At the same time, with the reduction of the cross-sectional area, the stress increases and has its maximum at the smallest section. At that section the yield point has also been raised most. *There is accordingly a competition between the increase in yield-point and increase in stress at the smallest*

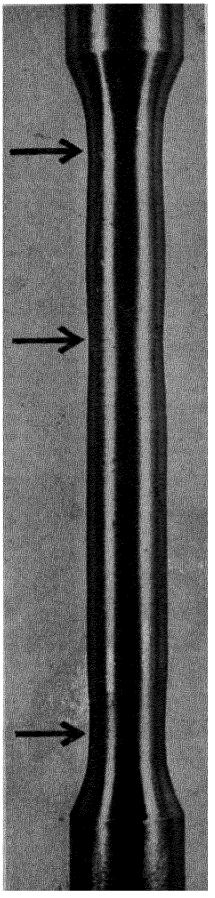


FIG. XX, 4. Test-piece after the 9th experiment, showing three strictures. (Compare Fig. XX, 3.)

section. As long as the former prevails the deformation wanders away to some other place where the yield point is lower. As soon, however, as the rate of strain hardening has become so low that the latter prevails, deformation continues at the same place and necking sets in.

This answered the question as to the cause of necking. At the same time it showed in which way the experiments were to be continued. After necking has set in the stresses are not more or less uniformly distributed over the length of the rod, but concentrated at the point of striction. Therefore, before continuing the experiments, the cylindrical form was restored.

This was done by turning down the test-piece from a diameter of 1·982 cm. to 1·636 cm. The experiments were then continued as before.

6. Ageing

Now necking occurred at once, but at an entirely different place from that in experiment 6 (compare Fig. XX, 3). When the test-piece was given a " rest ", it further hardened at the place where it had been

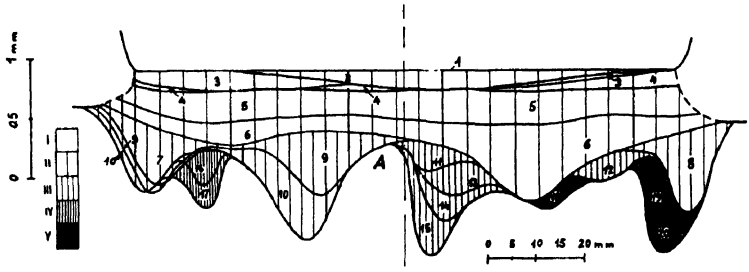


FIG. XX, 5. Superposed lateral contractions of all test-pieces.

Numbers correspond with number of experiment. Roman numerals refer to the series, of which each represents a new turning-down. Note how the test-piece in the first experiment to which it was subjected in series II (No. 7) remembers, so to speak, where its least contraction had taken place before it was turned down after series I (and similarly in further experiments).

The fifth test-piece has again been turned down and is now perfectly cylindrical. Necking was expected at point A and so happened in experiment 21.

strained most or, as is said, it " aged ". Therefore in the next experiment the deformation did not continue at the same place but occurred elsewhere, etc. In the end we had *three strictions in the one test-piece*, a rather unusual phenomenon (compare Fig. XX, 4). After that, the test-piece was again turned down and this was repeated several times. In Fig. XX, 5, all deformations have been plotted in one figure. As can be seen, the material well " remembers " its previous history. Here also deformation wanders from point to point : from a more to a less strained section, even if the test-piece, after turning down, is perfectly cylindrical,

7. Local Deformations

It is clear that the value of Δl has no significance after necking has set in. Before that it gives a good average of the deformation in the different cross-sections. After necking, however, all deformation occurs at *one* place, while the rest of the rod is not deformed. Locally, as we pointed out in Section VI, 2, the deformation will be quite considerable ; while its contribution to the increase in length will be very small. The apparent high rate of strain hardening between points b' and e' in Fig. VI, 1 is therefore misleading. Actually the length on the abscissa related to these two points will be much longer in terms of deformation than shown here in terms of $\Delta l/l_o$.

To get a true picture one has to relate the stress at a section to the *local* deformation which manifests itself in the reduction of the diameter. Let us imagine the rod divided into disks of originally equal thickness d_o. The elongation Δl is the sum of the increase of thicknesses of these disks. Every such increase in thickness is accompanied by a decrease in diameter and we may assume (as a first approximation) that in this process the volume of the disk is not changed. Let d be the thickness of the disk after extension, then its contribution to the total axial extension is, in the correct logarithmic measure, $e_n = \ln(d/d_o)$. Furthermore, let D_o be the original diameter and D the reduced diameter of the disk, then constancy of volume requires that

$$D^2 d = D_o^2 d_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\ln(D^2 d/D_o^2 d_o) = 2\ln(D/D_o) + \ln(d/d_o) = 0 \quad . \quad . \quad (2)$$

from which with

$$\ln(D/D_o) = e_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$e_n = -2e_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)^*$$

It is, therefore, possible to calculate the local axial extension from the local relative contraction. The former is twice as large as the latter.

8. The Rheological Tensile Test Curve

For the rheological test curve we have to plot local stresses $P/\frac{D^2\pi}{4}$ against local strains $e_n = \ln(d/d_o)$. This requires the tracing of the different disks in the consecutive experiments as they change their distance from the axis.

In Fig. XX, 3 I have shown how we can follow the course of one such disk, marked "a", and it obviously is a cumbersome process. In Fig. XX, 6 the successive extended lengths of the test-piece have

* The same result is obtained from Equ. (III, j), considering that the length and two diameters at right angles are principal axes of strain and therefore $e_v = e_n + 2e_c$. If e_v vanishes, $e_n = -2e_c$.

therefore been reduced to the original length taking account of local extensions and in this way the disks keep their places. At the same time the diameters have been plotted in logarithmic scale and the strains, or rather deformations, can therefore be read directly from the figure. If in the first series of experiments (I) any one disk suffered a reduction of its diameter from D_0 to D_1 and was then turned down to a diameter D_2 which by deformation was reduced to D_3, etc., its final

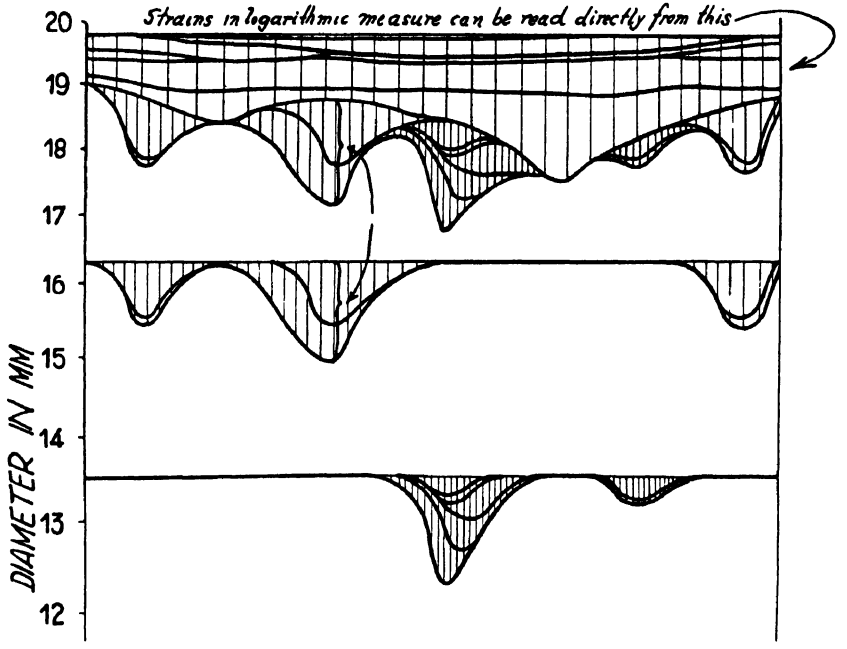


FIG. XX, 6. Lateral deformations of test-piece. The contractions shown in Fig. XX, 3 are here plotted in logarithmic scale and the lateral deformation can be read off directly.

deformation is $e_c = \ln(D_1/D_0) + \ln(D_3/D_2) + \ldots$ which is therefore simply additive. This is not the case with the usual definition of strains and deformations. In Fig. XX, 6 all deformations have been superposed, in accordance with this principle, upon those of the first series as if no turning down had taken place.

It is now easy to plot local yield stresses against local strains, which has been done in Fig. XX, 7. No special precautions were taken either to exclude ageing or to ensure a definite and equal degree of ageing in the course of the experiments and there is therefore considerable scattering of points. Nevertheless the curve can serve as the basis for the discussion of the problem of the determination of the rheological equation governing the strain-hardening of a material, i.e. a relation between ϑ and e.

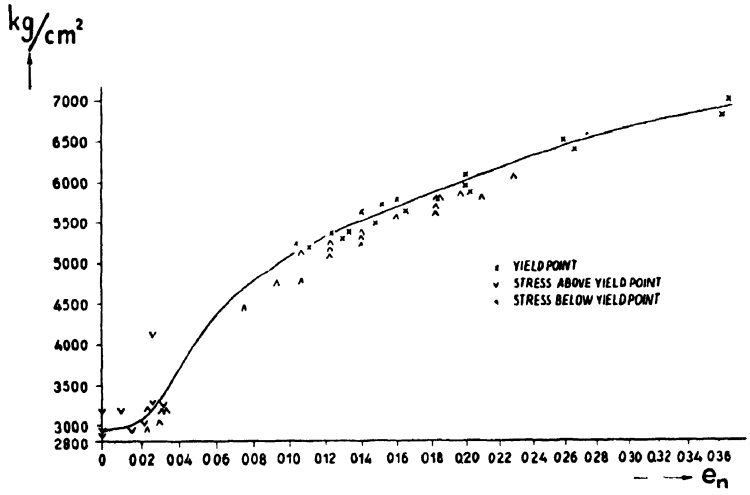
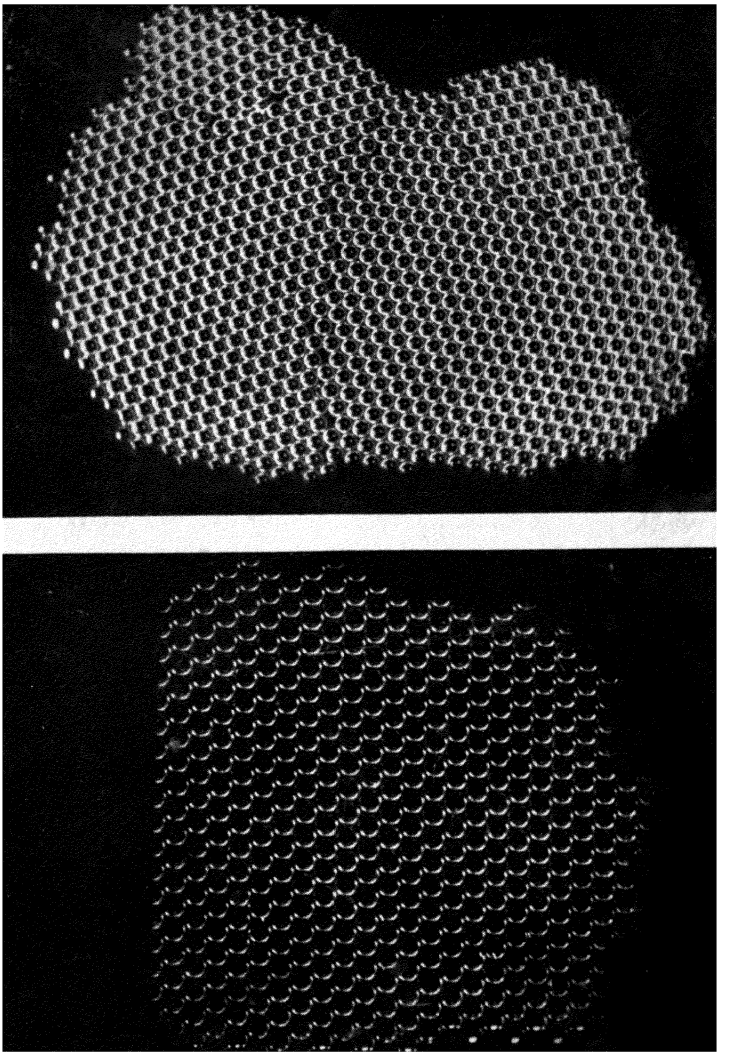


FIG. XX, 7. Rheological stress-deformation curve. The true yield stress has been plotted against double the local contraction.

9. The Mechanism of Work-hardening of a Polycrystal

Before attacking this problem we must enquire to what cause the strain-hardening of mild steel may be due. First of all it should be mentioned that single crystals of metals are in all cases very soft. Metals, as we ordinarily know them, are, however, *assemblages* of very small crystals ; they are *polycrystals*. It is to this fact that they owe their hardness. At the same time it has been found that the hardness of a metal is greater the finer its crystalline structure (compare Andrade (1936)). Therefore, whatever makes for smaller crystals will make for greater hardness and strain-hardening, on this view, is due to the diminution of the crystals through individual rupture on plastic deformation.

But then the question arises : why should a polycrystalline structure make for hardness ? To answer this question two theories were proposed, one by Beilby and Rosenhain and the other by Ludwik. In accordance with the first, between the crystals of a metal there exists a layer of metal in the amorphous state —" the intercrystalline cement " —to which the hardness is attributed. This theory was discarded because no difference in hardness in the interior of the crystals and in the intercrystalline cement could be found. The second theory attributes the greater hardness of the polycrystalline metal to the random arrangement of adjacent crystals, with their glide planes in different directions, which leads to a *jamming*, which in turn renders glide more difficult. This theory is now generally accepted. It has been proposed also by Jeffries and Archer under the name of the Slip-Interference Theory. We cannot here go more fully into these questions, but the



Model of a polycrystal.

FIG. XX, 8. Model of a single crystal.

reader may look up the very instructive and easily understandable article by Sir Lawrence Bragg (1943) from which Fig. XX, 8 is reproduced.*

This shows a model for a single crystal and for a polycrystal made of small soap bubbles, each one representing an " atom ". It can be

* *Endeavour*, 1943, **2**, 43–51 : by permission.

seen at once why the first can be easily deformed by gliding between rows of bubbles, while in the second gliding is more difficult through the different orientations of the rows.

10. The Bauschinger Effect

While, therefore, rupture of individual crystals increases the resistance against gliding and raises the yield point, nevertheless a weakness is introduced in the surface along which rupture has taken place. This weakness possibly makes itself felt in the so-called Bauschinger effect. After plastic deformation and a reversal of direction of loading (from tension to compression or *vice versa* or from twisting in one sense to twisting in the opposite sense) comparatively large plastic deformations are produced by very small loads ; in other words, the yield point for the opposite kind of stress is lowered. But when a mild steel bar is twisted in one sense beyond the yield point there will be small internal rupture surfaces in the crystals along which there will be minute gliding. If the twisting is reversed, some of the ruptures will open up in the same way as the strands in a rope open up when the rope is twisted against its thread. This must cause large deformations or, in other words, a Bauschinger effect in torsion. The explanation for the reversal from tension to compression is similar. Now, when we deformed the mild steel test-piece beyond the yield point and then allowed it to rest before reloading, we found that the yield point is raised and it is raised the more the longer the rest. This phenomenon is the ageing of which we spoke before. Ageing in this sense of the word is accelerated by heating to low temperatures up to, but not exceeding about 300° C. and the Bauschinger effect disappears. This is the tempering of the strain-hardened steel. It can be explained as brought about by a *healing* of internal rupture surfaces and the term " ageing " is therefore rather misleading. Rupture has taken place because the distance between the atoms on both sides of an interface is increased so much that they leave the range of atomic cohesion. Now, due to the heat energy of the body, every atom is in constant vibrations, the amplitude of which is determined by the temperature. If the amplitude of a vibration is sufficiently great, an atom on one side of the rupture surface may come within the range of attraction of an atom on the other side of the surface and a connection is made across the rupture. In this way the rupture heals. The process will also take place at ordinary temperature (albeit at a lower rate), because the vibrations are not all of the same amplitude but statistically distributed around some mean magnitude, and from time to time an extraordinarily great amplitude will effect a connection and local healing of the rupture.

When, however, the temperature is raised above 300° C. the vibrations become so strong that they not only heal the ruptures, but

the atoms re-arrange themselves in their most stable grids. This is re-crystallization ; the crystals increase in size and the yield point is lowered until all work-hardening may vanish. This is the annealing of the work-hardened steel.

11. In Search of a Law of Work-hardening

We are now in a better position to take up the problem which we put to ourselves in the closing sentence of Section 8. The tensile test of mild steel shows that the yield stress increases with increasing deformation. The question is, how is the former related to the latter ?

Let us go back to our test as represented graphically in Fig. VI, 1. If we, after reaching point "f" on the curve, unload the test-piece, a certain elastic strain is recovered, corresponding to the difference of abscissas of points "f" and "g", while the deformation og is plastic and permanent. Now, again increasing the load to the amount corresponding to point "f", we reach approximately the same point (denoted in the figure by "h") in an *elastic straining* with the *same elastic modulus* as in the first loading. This shows itself in the figure by the slope of gh being the same as that of oa. The curve a–c–b–e is therefore *the geometrical locus of all the yield points corresponding to the successive deformations*.* Nevertheless, as becomes clear for reasons which we similarly encountered before in two other cases,† the yield stress cannot depend directly upon the deformation. We mentioned in Section 10 the raising of the yield stress through a *twisting* of the bar. It is immediately obvious that this effect cannot depend upon whether we twist the bar clockwise or anti-clockwise. The yield stress ϑ_t must therefore be an *even* function of the tangential deformation γ or a function of γ^2. Let us remember (compare Section 5, Chapter VI) that ϑ_t itself is calculated by taking the root of another quantity, the plastic resilience E_{pl} which itself is an even function of the stress. It will also help us in our problem if we recollect that strain-hardening is also called *work-hardening*. What governs the rise of the yield point is clearly the *work* expended in the plastic deformation and not the deformation itself. Let us imagine a giant of such strength that he would be able to knead mild steel as we knead flour dough. Let us hand him a steel ball which he will knead into all sorts of shapes, at the end restoring the sphere. When he has handed back the ball to us, its deformation is *nil* : all distortions, positive and negative, having cancelled each other. The strain-work, however, has all the time increased to a distinct amount. If, in order to make our considerations more definite, we assume that the deformations are simple shears,

* The existence of a yield point is therefore best revealed on *unloading* a test-piece.
† *Vide* Section 5 of Chapter VI and Section 2 of Chapter XVIII.

alternately in positive and negative directions, the deformational work in terms of strain in accordance with (VI, 4), equals $\mu\gamma^2/2$.

The resilience on the other hand is, in accordance with (VI, 5) $= \tau^2/2\mu$ and the plastic resilience in accordance with (VI, a) $= \vartheta_t^2/2\mu$. The modulus of elasticity being in that process constant (as we said before), we may lump it together with the numerical factor 2. In the case of tangential deformations the law of work-hardening would therefore relate ϑ_t^2 to γ^2 and should be expressed as a functional relationship $\vartheta_t^2 = f(\gamma^2)$. We may denote by θ the limiting strain (yield strain) corresponding to ϑ, so that θ_t is the ultimate shearing strain when the yield point is reached, or

$$\vartheta_t = \mu\theta_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In other words, therefore, the law of work-hardening must relate *the square of the tangential strain to the square of the tangential deformation.**

12. The Mises-Hencky Flow Condition for Simple Tension

We cannot apply such a functional relationship directly to our case, which is one of simple tension and not of simple shear. We must first express the Mises-Hencky flow condition for the case of simple tension. As we remarked in Section 9 of Chapter III, simple tension is not as simple as one might think. We have shown in Section 13 of that Chapter and in Fig. V, 1, that a simple tension σ_n can be considered as the result of the superposition of an isotropic tension $p = \sigma_n/3$ and a distortional stress the components of which are a tension $2\sigma_n/3$ in the axial direction and two compressions $\sigma_n/3$ normal to the axis and at right angles to each other. In accordance with the first axiom of rheology, the distortional stress only produces the plastic deformation and the plastic resilience E_{pl} must be calculated not from the total strain-work of simple tension but only from its distortional component. The first is, in accordance with (III, z), $= \sigma^2/2E$. The latter can be calculated as follows :-

In accordance with (III, o) we have

$$e_{on} = \sigma_{on}/2\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5')$$

Therefore the distortional work

$$w_o = \int \sigma_{on} de_{on} = \frac{1}{2\mu}\int \sigma_{on} d\sigma_{on} = \sigma_{on}^2/4\mu \quad . \quad . \quad . \quad (6)$$

Now, in the case of simple extension, in accordance with (III, n)

$$\sigma_{o1} = \sigma_{o2} = -\sigma_n/3 \; ; \; \sigma_{ok} = 2\sigma_n/3 \quad . \quad . \quad . \quad (7)$$

and, therefore,

$$w_o = 1/4\mu . \sigma_n^2/9 . (1 + 1 + 4) = \sigma_n^2/6\mu \quad . \quad . \quad . \quad (8)$$

* Keeping in mind the definition of strain as the *recoverable* part of the deformation.

If we increase the tension σ_n until we reach the yield point, when $\sigma_n = \vartheta_n$, the strain-work w_o reaches the plastic resilience E_{pl} and therefore

$$\vartheta_n = + \sqrt{6\mu E_{pl}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

On the other hand from (VI, a)

$$E_{pl} = \vartheta_t^2/2\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which makes

$$\vartheta_n = \vartheta_t\sqrt{3} = 1{\cdot}73\,\vartheta_t, \text{ or } \vartheta_t = 0{\cdot}578\,\vartheta_n \quad . \quad . \quad . \quad (11)$$

and if we determine ϑ_t from a torsion experiment we can predict the yield stress ϑ_n in a tensile experiment with the same material. We may compare (11) with St. Venant's flow condition,* i.e. the condition that the shearing stress reaches a certain maximum (compare Section 5 of Chapter VI). We know from Mohr's circle, Fig. III, 8, that in the case of a simple tension σ_n, the shearing stress reaches its maximum in a section inclined at 45° to the axis of the test-piece and is there equal to $\sigma_n/2$. The yield point will therefore be reached in simple tension when $\sigma_n = \vartheta_n = 2\vartheta_t$. The normal yield stress in accordance with St. Venant would accordingly be $2/1{\cdot}73 = 1{\cdot}15$ times the yield stress in accordance with Mises-Hencky. The difference is not great. Geiringer and Prager (1934) are of the opinion that St. Venant's flow condition furnishes in certain cases, e.g. the first starting of flow of mild steel, the best description and must be abandoned only when *full* plastic flow has developed. I hope, however, to have made it clear that, theoretically, the Mises-Hencky flow condition is highly superior to the St. Venant flow condition. Speaking of flow conditions, I may mention that Beltrami in 1885 postulated that the yield point is reached when the strain-work reaches a certain limit. This hypothesis was soon refuted by experimental evidence and we know why: it contradicts our first rheological axiom; the dilatational strain-work should not cause any appreciable plastic flow.

13. A "Primitive" Law of Work-hardening

We said at the end of Section 11 that the law of strain-hardening would be one relating θ_t^2 to γ^2, or, as we can now say, θ_n^2 to e_n^2. In physical language we express the plastic resilience, which is the resilience at the yield point, as a function of what we may call the Hardening Work; the first is proportional to θ_t^2 or θ_n^2, the second to γ^2 or e_n^2. If we plot θ_n^2 against e_n^2 the resulting curve has a striking resemblance to the one (Fig. XVI, 8) for the variable fluidity of a liquid as a function of τ^2, with θ_o^2 corresponding to φ_o, θ_∞^2 to φ_∞ and e_n^2 to τ^2. We may therefore tentatively write down in analogy to (XVIII, i)

* This condition was postulated independently by Coulomb (1801) and Tresca (1868).

$$\theta^2 = \theta_\infty{}^2 - (\theta_\infty{}^2 - \theta_o{}^2)e^{-e^2/\Psi} \quad . \quad . \quad . \quad . \quad (12)$$

or in another form

$$E_{pl} = E_{pl\infty} - (E_{pl\infty} - E_{plo})e^{-w_h/\Psi} \quad . \quad . \quad . \quad (13)$$

where E_{pl} is the plastic resilience which gradually, but more and more slowly, increases from E_{plo}, the plastic resilience in the annealed state, to $E_{pl\infty}$, the maximum plastic resilience which can be stored up in the material after it undergoes the maximum work-hardening. The hardening work w_h is the work of the external forces spent in overcoming internal plastic friction and in changing the structure of the metal by breaking down its constituent crystal grains. The coefficient ψ is therefore quite analogous to the coefficient χ ; it also is a Coefficient of Structural Stability. As the expressions e^2/ψ and $w_{\tau h}/\bar{\psi}$ must be dimensionless, the quantity ψ has the dimension of work, and ψ (compare (VI, 5)) the dimension of a work per unit rigidity.

Equ. (13) was proposed by me as a " primitive " work-hardening law at the Paris Congress for Applied Mechanics, 1946.

We found in Chapter XVIII that χ was not a constant. Neither can we expect ψ or $\bar{\psi}$ to be constants. From what has been said in Section 9 it is clear that the larger the crystal the less its resistance against slip. With the continuous breaking down of the crystals ψ and $\bar{\psi}$ must therefore increase.

In the second of the researches mentioned at the end of Section 3 above, a strain-hardening curve was obtained first by discontinuous pulling in a tensile testing machine, and then by successive wire-drawing operations. A hot-rolled low-carbon steel rod of 8 mm. diameter was drawn down to a wire of 0·8 mm. diameter resulting in an extensional deformation of 4·6 in the logarithmic measure. It was found that the work-hardening curve consisted of *three* e-curves of the form of Equs. (12) and (13) following each other with short transitional curves between them. The transition from the first to the second e-curve occurred at the maximum tensile load (point *b* in Fig. VI, 1) ; that from the second to the third at an extension at which the tensile test fracture (point *e* in Fig. VI, 1) takes place. In the paper. an " atomistic " interpretation is given, using ideas developed by Sir Lawrence Bragg (1942) which, however, belong to what may be termed Metarheology.

14. Summary

In a static test, i.e. one proceeding through states of equilibrium, mild steel shows a Hooke range, a simple St. Venant range reached through an " upper " and " lower " yield point, and a " generalized " St. Venant range in which the yield point increases at a decreasing rate, reaching an upper limit. An elastic spring can serve as a model for the Hooke-body (indicated symbolically by *H*) and a weight resting on a

table, with solid friction between them, as a model for the St. Venant body (StV). These, coupled in series (H–StV) represent the Hooke and simple St. Venant range, while H–$(StV)_1$–$(StV)_2$–. . .–$(StV)_n$ includes the work-hardening or generalized St. Venant range, provided the series $(StV)_1$, . . . $(StV)_n$ is arranged in monotonously decreasing order.

The distortional work in simple tension is

$$w_{on} = \sigma_n^2/6\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XX, a)}$$

and, therefore, the yield stress

$$\vartheta_n = \pm \sqrt{6\mu E_{pl}} \quad . \quad . \quad . \quad . \quad . \quad \text{(XX, b)}$$

which gives

$$\vartheta_n = 1{\cdot}73\,\vartheta_t \quad . \quad . \quad . \quad . \quad . \quad \text{(XX, c)}$$

in accordance with Mises-Hencky, while

$$\vartheta_n = 2\,\vartheta_t \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XX, d)}$$

in accordance with St. Venant.

The rise of the yield point in the work-hardening range is due to the breaking up of large into small crystals. For a hypothetical polycrystal consisting of large crystals of *equal size* which in each slip are converted into small crystals of minimum size, the rheological equation

$$\vartheta^2 = \vartheta_\infty^2 \quad (\vartheta_\infty^2 - \vartheta_o^2)e^{-\psi} \quad . \quad . \quad . \quad \text{(XX, e)}$$

is proposed, where ψ is a coefficient of structural stability. For a real polycrystal ψ gradually increases and the equation must be amended, taking into account a distribution of crystals of different sizes.

CHAPTER XXI

SECOND-ORDER PHENOMENA

1. Independence of Dilatation and Distortion

In the foregoing, when postulating rheological equations, we resolved both deformation and stress into an isotropic component and another one by which it " deviates " from isotropy. When a deformation, the first was called volumetric deformation or strain, cubical dilatation, or, simply, dilatation ; the second we called distortion. When a stress, the first is often spoken of as " hydrostatic " pressure or tension and the second as stress-*difference* ; the latter can also be regarded as a combination of shearing tractions. We then assumed that every material was characterized by two independent rheological equations : one between the isotropic, the other between the deviatoric components of stress and strain , the latter being equivalent to a relation between shear and shearing stress. It was assumed that an isotropic stress produced a cubical dilatation only or a change of volume, but no change of shape. Similarly, a shearing traction was supposed to produce shear or a change of shape only, but not to affect the volume or density of the material. It will presently be shown that this view — the view of classical elasticity and hydrodynamics– is too narrow. More especially there are three phenomena which strikingly contradict it. These are *dilatancy*, discovered by O. Reynolds (1885), " cross-elasticity " in liquids, which may be named the " Weissenberg effect ", and changes in length of metal cylinders in torsion which has been named the " Poynting effect ".

2. Reynolds' Experiments

In 1885 O. Reynolds described the following experiment : " If . . . we have an extremely flexible bag of indiarubber, this envelope, when filled with heavy spheres, imposes no sensible restraint on their distortion ; standing on the table it takes nearly the form of a heap of shot. . . . Filling up the interstices between the shot with water so that the bag is quite full of water and shot, no bubble of air in it, and carefully closing the mouth, I now find that the bag has become absolutely rigid in whatever form it happened to be when closed. It is clear that the envelope . . . imposes no distortional constraint on the shot within it, nor does the water. What, then, converts the heap of loose shot into an absolutely rigid body ? Clearly the limit which is imposed on the volume by the pressure of the atmosphere. So long as

the arrangement of the shot is such that there is enough water to fill the interstices the shot are free, but *any arrangement which requires more room** is absolutely prevented by the pressure of the atmosphere. If there is an (small, M.R.) excess of water in the bag when the shot are in their maximum density, the bag will change its shape quite freely for a limited extent, but then becomes instantly rigid . . . without further change. By connecting the bag with a graduated vessel of water so that the quantity which flows in and out can be measured, the bag again becomes susceptible of any amount of distortion. Getting the bag into a spherical form and its contents at *maximum density*, and then squeezing it between two planes, the moment the squeezing begins the water begins to *flow in*, and flows in at a diminishing rate until it ceases to draw more water. . . . The material in the bag is (then, M.R.) in a condition of *minimum* density. . . . If we continue to squeeze, water begins to flow out. . . . The very finest quartz sand, or glass balls $\frac{3}{4}$-inch in diameter, all give the same results. Sand is, on the whole, the most convenient material, and its extreme fineness reduces any effect of the squeezing of the indiarubber between the interstices of the balls at the boundaries."

3. Dilatancy of Granular Masses

The explanation for this strange behaviour was given by Reynolds as follows :

"If . . . a group of spheres . . . (is) arranged as a pile of shot (Fig. XXI, 1, left) . . . the density of the media is $\pi/3\sqrt{2}$, taking the density of the sphere as unity. If arranged in a cubical formation (Fig. XXI, 1, right) the density is $\pi/6$, or $\sqrt{2}$ times less than in the former case. These arrangements are both controlled by the bounding spheres ; and in either case *the distortion necessitates a change of volume.* I have called this unique property of granular masses 'dilatancy' because the property *consists in a definite change of bulk, consequent on a definite change of shape* or distortional strain, any disturbance whatever causing a change of volume."

He then cites the following example : "A well-marked phenomenon receives its explanation at once from the existence of dilatancy in sand. When the falling tide leaves the sand firm, as the foot falls on it the sand whitens, or appears momentarily to dry round the foot. When this happens the sand is full of water, the surface of which is kept up to that of the sand by capillary attraction ; the pressure of the foot causing dilatation of the sand, more water is required, which has to be obtained either by depressing the level of the surface against the capillary attraction, or by drawing water through the interstices of the

* The italics are mine.—M.R.

surrounding sand. The latter requires time to accomplish, so that for the moment the capillary forces are overcome ; the surface of the water is lowered below that of the sand, leaving the latter white or dryer until a sufficient supply has been obtained from below, when the

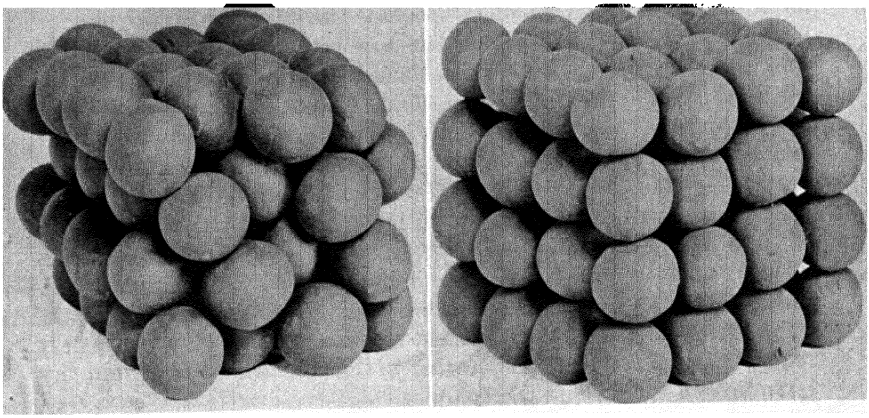


FIG. XXI, 1.* The figure (left) shows 64 spheres arranged approximately in closed packing. When this system is subjected to shear the spheres must first move into open packing as shown on right. It will be seen that this change in packing has resulted in an expansion.

surface rises and wets the sand again. On raising the foot it is generally seen that the sand under the foot and around becomes momentarily wet ; this is because, on the distortion forces being removed, the sand again contracts, and the excess of water finds momentary relief at the surface."

4. Dilatancy as a General Property

It is historically interesting to note that Reynolds undertook his investigation in order to arrive at a mechanical theory of the ether. Accordingly he said that " as regards any results which may be expected to follow from the recognition of this property of dilatancy,—

" In a practical point of view it will place the theory of earth pressures on a true foundation. . . .

" The greatest results are likely to follow in philosophy. . . ."

We do not apply nowadays rheology to " explain the fundamental arrangement of the universe," but we have found that dilatancy is not confined to granular masses.

After positive dilatancy in sand, *negative* dilatancy was observed in clay. While sand particles are little spheres, clay particles are

* Reproduced by permission of Messrs. J. & A. Churchill, Ltd., from *Introduction to Industrial Rheology* by G. W. Scott Blair.

minute discs. A sandy soil will therefore settle in dense packing, while a clay will in its undisturbed state be in loose packing, many of the discs standing on edge. In shear these will collapse and the density of the clay will be *increased*. These cases may be regarded as such of " plastic " dilatancy. But about the same time when Reynolds discovered his remarkable phenomenon in settled sands, a famous contemporary predicted on purely theoretical grounds that an analogous phenomenon should be present in elastic solids. In 1875 Sir William Thomson, later Lord Kelvin, contributed the article on Elasticity in the ninth edition of the *Encyclopædia Britannica*, which I have already mentioned before (Section 7 of Chapter IX), and wrote there : " It is possible that a shearing stress may produce in a truly isotropic solid condensation or dilatation in proportion to the square of its value ; and it is possible that such effect may be sensible in indiarubber or cork, or other bodies susceptible of great deformations or compressions with persistent elasticity " (1875). Reynolds must certainly have read this article and it is extraordinary that he did not connect this remark with his observation. If he had attempted to relate the change of volume observed by him to the shear or rather the shearing stress causing it, he would without doubt have realized that a shear " to the right " must produce the same effect as a shear " to the left." It is inconceivable that a shear to the " right " should produce a " dilatation " and one to the " left " a " condensation." Therefore, the shear or shearing stress cannot enter into a rheological equation except in an even power, or in the first instance *in the square* of its value. This is what Kelvin maintained in the case of elasticity.

I do not know of any observations on elastic dilatancy in solid indiarubber or cork, but it is very easy to observe it in " expanded " or porous rubber, which has the property of negative elastic dilatancy to a very marked degree. This can easily be demonstrated with a commercial rubber sponge. Sticking a piece of board on each of its two flat surfaces, it can be observed that if one is shifted tangentially against the other, the surfaces also approach each other.

We can say that there will generally be elastic, plastic and viscous dilatancy. In an elastic solid, side by side with the volume component of the Hooke solid resulting from the isotropic stress p,

$$e_{v1} = -p/\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

there will be one resulting from the shearing stress τ

$$e_{v2} = \tau^2/\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and we may define the Reynolds elastic solid by the rheological equation

$$e_v = -p/\kappa + \tau^2/\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The parameter δ may be called the Modulus of Dilatancy.

There may also be an analogous Reynolds liquid, and we know that there are Reynolds plastics, positive and negative. In the Reynolds elastic an isotropic tension will cause a cubical dilatation and *vice versa*, as in the Hooke solid ; but cubical dilatation may also be caused in the absence of a hydrostatic tension by simple shearing stress. Likewise, a hydrostatic pressure may be required to maintain simple shear. Similar conditions may be present in liquids as postulated by me on theoretical grounds (1945).

5. Second Order Phenomena

Dilatancy, being dependent upon τ^2, is a second order phenomenon. The stress necessary to produce a small strain may be a large figure in the usual measure, but then the elastic modulus is likewise a large figure and the quotient will be small. As long as the strain is small, the stress can therefore also be considered as small. We see from Equ. (2) that the modulus of dilatancy is of the dimension of the square of a stress. Therefore if the shear is small, dilatancy, depending upon the square of the shear, may be neglected. This is the reason why in classical elasticity dilatancy was not noticed. This has changed now when we have to consider in many fields large deformations, as was prophetically realized by Kelvin.

But then there may be other second order phenomena, which likewise cannot be neglected. If we consider shear, there is no reason why finite shear should affect only the volume, but why it should not also cause *linear* extensions or contractions in some specific directions, likewise depending upon the square of the magnitude of the shear. Actually, Weissenberg (1947) has observed phenomena which point to the existence of such " cross-elasticity in elastic liquids " as we may call it for reasons to be seen later. These phenomena we shall presently describe.

6. The Weissenberg Effect

In Fig. 1, 3 we showed simple shear converting a prism into a parallelepiped. Now imagine the parallelepiped being the undeformed body and the force P_t acting from right to left : clearly the parallelepiped will be deformed into a prism. We show in Fig. XXI, 2, a section through the body and ask ourselves : what is the relative position of planes which are parallel to the boundary surfaces of the body ?

Let us consider planes ab and cd which after deformation become $a'b'$ and $c'd'$. It can be seen from the figure at the left that their distance *increases* after deformation.* In contradistinction the distance of the

* Remember that the distance between two parallel planes is measured in the direction normal to both.

two planes *bd* and *ac* remains the same. We do not know the mechanism which produces stresses in a strained solid, but Weissenberg assumes that it is the change of distances of such planes which governs the reactive forces or stresses. From this he predicted that in simple shear there would be not only a shearing stress, but also a pull or tension

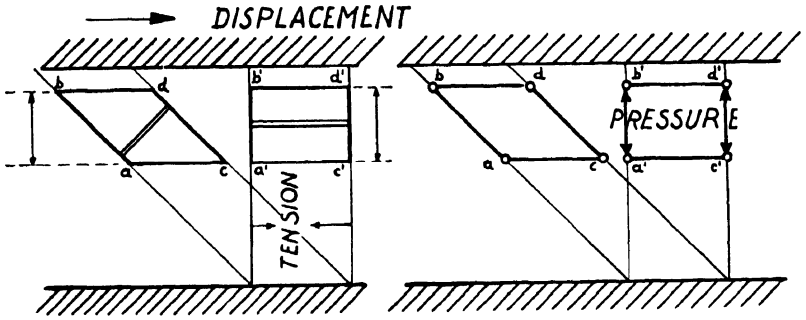


FIG. XXI, 2. Parallelepiped deformed into a prism through simple shear.

in the direction of the displacement. That tension would be a second order phenomenon because as a first approximation, if the displacement is very small, the distance between planes *ab* and *cd* is not changed.

Now take an elastic liquid where the deformation is naturally very large and experiment with it in an arrangement such that the liquid is sheared in a gap between an outer vessel rotated with an angular velocity which is kept constant at various levels and an inner member which is held against rotation and either rigidly fixed in position or free to move up and down the axis of rotation. The conditions are so chosen that under the combined actions of the shear imposed at the boundaries, and the forces of gravity and inertia (centrifugal forces) the liquid executes a steady laminar shearing movement. As has been argued above, if the liquid is elastic, there should be in addition to the shear stress components, a pull along the lines of flow. If, as in our experiments, the lines of flow are closed circles, the pull along these lines " strangulates " the liquid and forces it inwards against the centrifugal forces and upwards against the forces of gravity. The effect, as predicted by Weissenberg, may be called " Weissenberg effect."

7. Experimental Evidence for the Weissenberg Effect

Experiments carried out with a variety of materials, including saponified oils, solutions of rubber, starch, cellulose acetate, etc. showed that the predicted phenomena actually occur in a great variety of conditions of flow such as are reproduced by Fig. XXI, 3. It should be noted that Weissenberg calls a Newtonian liquid " special liquid,"

a designation natural enough considering that the Newtonian liquid is a special case of the more " general " Maxwell liquid for $\mu_l = \infty$. In the simplest form the Weissenberg effect is exhibited when a rod is rotated in a " Weissenberg liquid " ; the latter then " climbs "

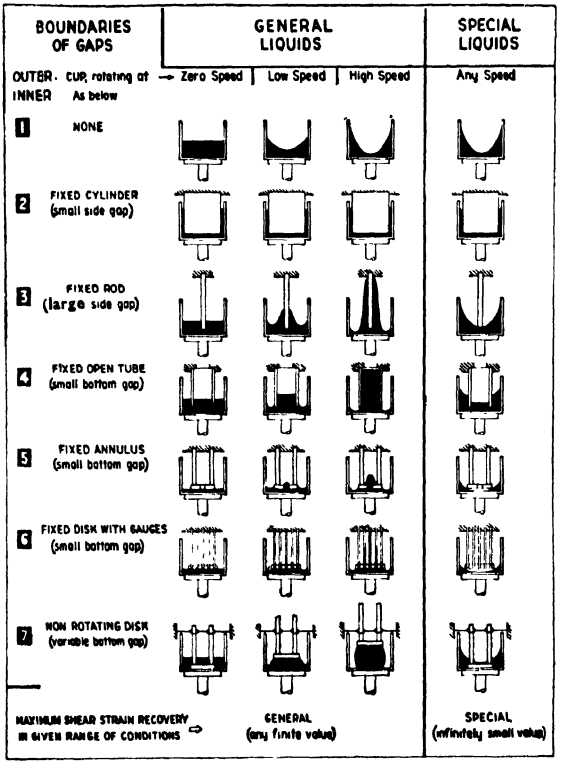


FIG. XXI, 3. Flow of liquids under actions of steady shear in gaps.

up the rod. Fig. XXI, 4 shows this for a denaturized sweetened condensed milk.* A similar effect has been observed in blown bitumen.† All liquids for which the Weissenberg effect has been demonstrated so far, exhibit also elasticity of *shape* under the action of transient forces or are elastic sols of the *L*-type. It stands to reason that when elastic elements are suspended in a sol, which as a result of velocity gradients existing in the liquid are subject to elastic strains, cross-stresses should arise from these strains. Elastic cross-stresses will cause a centripetal pump action, and this may be the explanation for the mechanism of the Weissenberg effect. This explanation has been advanced by Weissenberg‡ himself.

* M. Reiner, G. W. Scott Blair and H. B. Hawley (1949).
† K. Lax-Weiner and R. Schoenfeld-Reiner (1952).
‡ K. Weissenberg : *Proc. 1st Int. Rheological Congr. Amsterdam*, 1948.

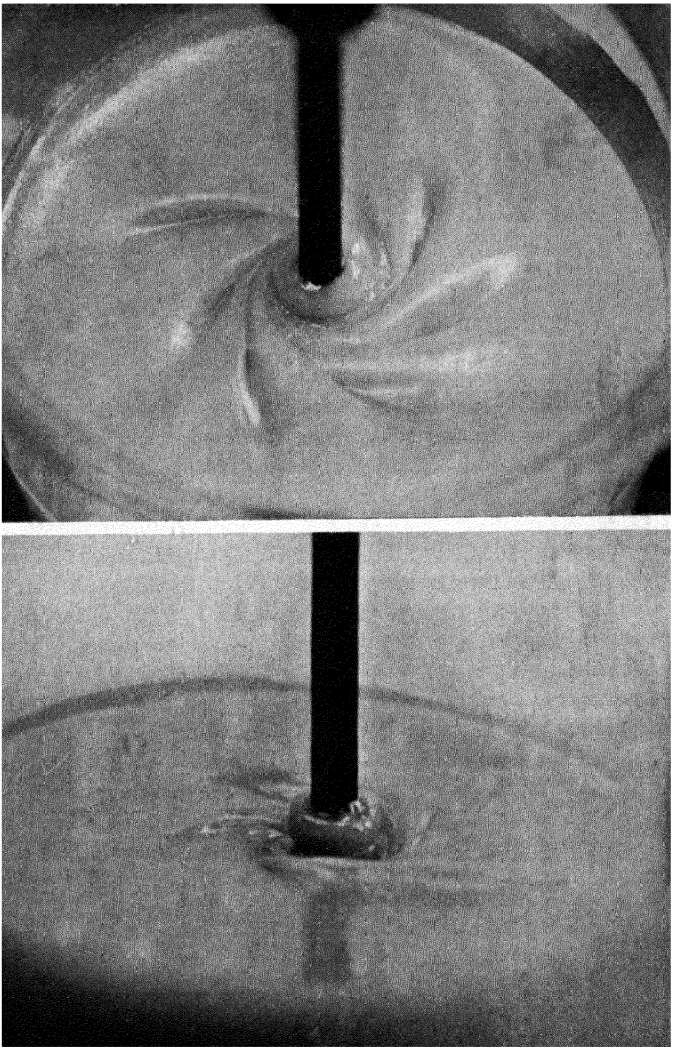


FIG. XXI, 4.

However, during the latest world war (let us hope the last one) I showed in a theoretical paper (1945) that if the stress-flow rheological equation of a viscous liquid is of tensorial non-linearity, characterized by a coefficient of cross-viscosity η_c, this also may explain a " climbing-up effect ". Rivlin (1948) adopted this idea, and Greensmith and Rivlin

(1953) claimed to have proved experimentally that a solution of poly-isobutylene in ortho-di-chloro-benzene does show such cross-viscosity. Roberts (1953) claims to have repeated Greensmith's and Rivlin's experiments with negative results. Mooney has summed up the position in a letter to me in the following words : " There may be liquids which are adequately described by the Reiner-Rivlin theories ; but I do not believe any such liquids have so far (1954) been found."

The explanation of the Weissenberg effects, both on the macroscopic, phenomenological and the microscopic, structural, level, is therefore *an open question.* It may well be that *different mechanisms* are operative in different cases, similar to the different mechanisms of the elasticity of steel and rubber.

8. Cross Elasticity

As I have shown by tensor analysis (1948) the phenomenon can be described without assuming anything about the mechanism of stress-strain connection. If we were to assume that the reactive forces or stresses are governed by changed distances *of material points* in the body and not of planes, Fig. XXI, 2 will show that there would be no pull in the direction of displacement, but a push normal to it. While after deformation the distances of points *ac* and *bd* have remained unchanged, the distances of points *ab* and *cd* are *decreased.* It can be shown that in the usual expressions for elastic stress-strain relations it all depends upon how you define the strain, whether you get a pull or push. However, it is unacceptable that our definition of strain should prejudice an experimental result. Therefore there is something wrong with the usual functional stress-strain relationship. I have shown how this position is to be amended by taking into account the second powers of the tensors (1948).* It all comes to this : Let n be the normal to an element of interface in the interior or of surface on the boundary of an elastic body. Let the traction on this interface be resolved into three orthogonal components, one normal σ and two tangential τ and s_c. Now select the directions t and c so that s_c vanishes. Let the strain be resolved in the same directions. Then the new theory requires that e_t does not vanish; or, in other words, that there is a strain sideways or in the crosswise direction. Conversely there may be a stress in a crosswise direction where there is no strain. This justifies the designation " cross-elasticity ". If cross-elasticity is assumed, experimental results are not prejudiced. In this case simple shear may require for its maintenance not only a corresponding shearing stress, but also tractions in the direction of the displacement (pull), or normal to it (push), or both.

* The same method was applied by Prager (1945) in a more restricted manner.

9. Cross-Components of Deformation in the Hencky-Measure

The two measures of deformation corresponding to the diagrams on the left and on the right of Fig. XXI, 2, were first proposed by Green and Almansi respectively.* We have shown that the Hencky-measure is of special significance in many problems of rheology, and we may accordingly ask ourselves, what cross-terms in deformation will arise when the Hencky-measure is used. This question is easiest answered by using the Mohr-circle of deformation.

In the same manner as the Mohr-circle is used for the determination of stress-components, we can use it for deformation components. Let a sheet consisting of an elastic material, say rubber, which can suffer large deformations, be fixed between two parallel runners as shown in Fig. XXI, 5. Let a circle of unit radius be drawn on the sheet. Let one runner be fixed while the other is displaced parallel to the first at constant distance H. In this case of simple homogeneous shear, the circle is deformed into an ellipse. Two stages of such deformation are shown, namely $\gamma = 1/3$ and $\gamma = 1$. The construction of the ellipse is a purely geometrical problem which the reader is supposed to be able to solve. It is indicated in Stage 2. Two radii of the circle normal to each other in the directions $\bar{i}$ and $\bar{j}$ are extended and contracted respectively, into half-axes l_i and l_j of the ellipse in the directions i and j, which are also normal to each other. These are the principal directions of the deformation. Note that the axes $\bar{i}$ and $\bar{j}$ are rotated by the angle $\alpha = \text{arc tan}(\gamma/2)$ into the positions i and j. This is the rotation connected with simple shear as shown in Fig. I, 1. The Mohr-circles for Stages 2 and 3 are shown. Note that the abscissa axis of the Mohr-circle, indicated by (i) must be parallel to the direction i. In order to construct the circle we must know the principal extensions D_i and D_j. Assuming the Hencky-measure we have

$$D_i^H = \ln l_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This can be determined by measuring l_i from the figure. As the area of the circle is not changed

$$D_i^H + D_j^H = D_{xx}^H + D_{yy}^H = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

from which

$$D_j^H = - D_i^H \; ; \; D_{yy} = - D_{xx} \quad . \quad . \quad . \quad . \quad (6)$$

From the Mohr-circle we determine D_{xy}^H and D_{xx}^H. These are entered in the figure as graphs of D versus γ. We see that there is an extension D_{xx} in the x-direction, and an equal contraction D_{yy} in the y-direction. The Hencky-measure therefore has the properties of both the Green and the Almansi measures. Accordingly, if the stress depends upon the Hencky measure, there will be, in order to produce simple shear,

* For references compare the exhaustive survey by Truesdell (1952).

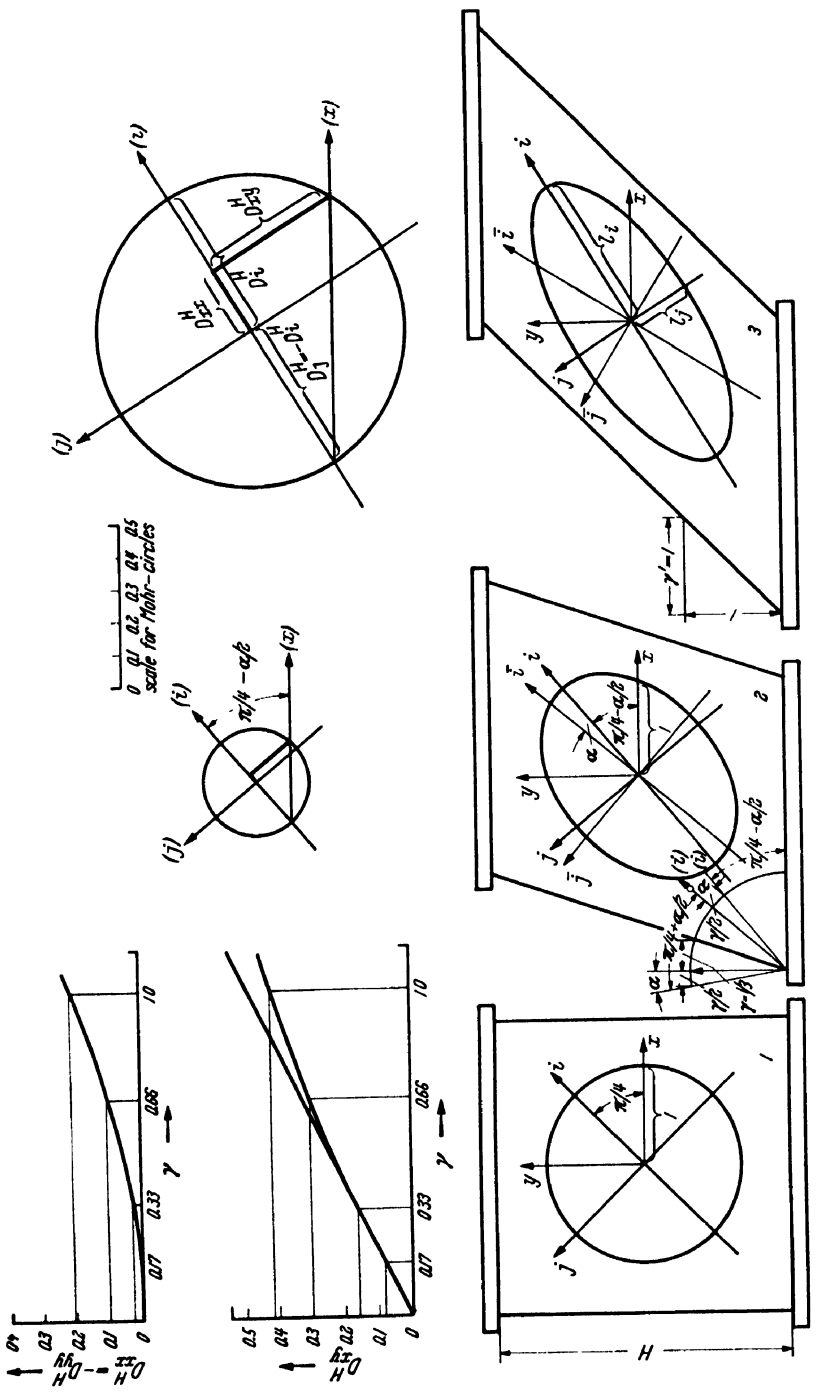

H
scale for Mohr-circles
0 0.1 0.2 0.3 0.4 0.5
0.07 0.33 0.66 1.0
0 0.1 0.2 0.3 0.4 0.5
0.07 0.33 0.66 1.0

FIG. XXI, 5.

tension in the x and compression in the y-direction. And if these are absent, the figure will shrink in the x and expand in the y-direction.

10. The Poynting Effect

This problem is treated in the well-known standard work *Strength of Materials* (Vol. II, p. 298) by Timoshenko, where he " proves " that large elastic torsion of a cylinder will be accompanied by its *shortening*. This seems to be the generally accepted view among structural engineers. To quote : " let us begin . . . with the case of a solid circular shaft, and assume first that the distance between the two consecutive cross-sections, Fig. XXI, 6 right remains unchanged during torsion. If γ is the shearing strain at the surface of the shaft, the elongation of the longitudinal fiber ac is obtained from the triangle acc_1 as follows : $ac = ac_1/\cos\gamma = ac_1/(1 - \gamma^2/2)$ . . . and the unit elongation of the fiber ac is $\epsilon_{max} = (ac - ac_1)/ac_1 = \gamma^2/2$. The corresponding tensile stress is $\sigma_{max} = \epsilon_{max} E$ . . . The assumption made that the distance between the cross-sections remains unchanged during twist brings us therefore to the conclusion that a longitudinal tensile force producing tensile stresses should be applied at the ends to keep the length unchanged. If no such force is applied but only a pure torque, the twist will be accompanied by some shortening of the shaft ". So far Prof.

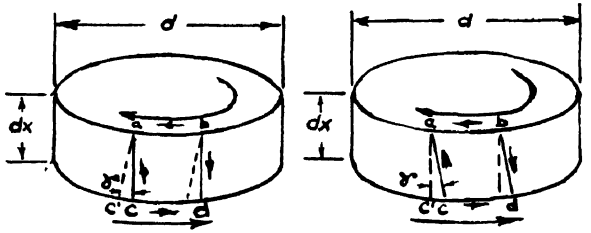


FIG. XXI, 6.

Timoshenko. However, one can with as much justification regard the case as pictured in Fig. XXI, 6 left and will find with the same argumentation that there is not an elongation but, on the contrary, a contraction requiring compressive stresses, and in their absence a lengthening of the shaft. Actually both argumentations are fallacious because they treat strain as if it were a vector, ignoring its character as a tensor. Of the length elements which can be drawn through any one point on the side of the cylinder some will be lengthened and some will be shortened at the same time, and nobody can say beforehand what the outcome may be.

The demonstration of the fallacy in his argumentation will detract as little from Prof. Timoshenko's achievements, and the merits of the quoted book, as Stokes' demonstration of the fallacy of Newton which

we reproduced in Section 5 of Chapter II, detracted from the latter. Actually Poynting (1909, 1912), nearly half a century ago, observed that cylinders of steel and copper in the form of long wires, when acted upon by forces equivalent to a torsional torque are not only twisted as predicted by the classical theory of elasticity, but also elastically *elongated* and *increased in volume*. These observations apparently fell into oblivion. I drew attention to them and treated them in a paper (1954). The increase in volume points to elastic dilatancy. The increase in length results from a stress-strain law with *whatever measure* is used for the definition of strain. This is obvious in the case of the Almansi and Hencky measures because of the y-component which in the case of torsion is identical with the axial (z) component. That this is also so in the case of the Green measure can be understood from Fig. XXI, 7. In the case of the cylinder, the x-

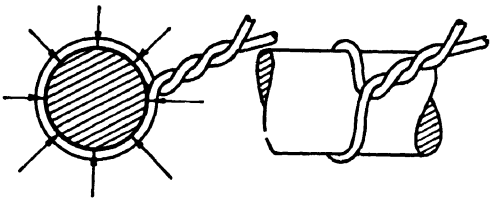


FIG. XXI, 7.

component of the strain is identical with the tangential (θ) component. This will cause a circumferential tension. Such tension is of the nature of a " strangulation ", and produces radial pressures, which in turn will cause a reduction of the diameter of the cylinder accompanied by a lengthening. As was shown by Poynting, and as I have pointed out, such lengthening takes place even when the strain is " infinitesimal ". In Poynting's cases, while the torsional strain was of the order of 10^{-3}, the observed lengthening, reduction in diameter and increase in volume were all of the order of 10^{-6}—which means that they were second order effects as shown in my theory (1954). In a paper with Rintel (1958), I have investigated second order effects in infinitesimal elasticity in general. The elastic lengthening of the cylinder has justly been named by Truesdell (1952). Poynting-effect. Swift (1947) has shown that in the plastic torsion of metal cylinders those made of a work-hardening metal lengthen, and those of a non-work-hardening metal, such as lead, become shorter.

11. Cross-Stresses in a Maxwell Liquid

From the foregoing considerations it can be surmised that cross-stresses will be present in a Maxwell liquid, because of its elasticity. To understand the mechanism underlying it, we must somewhat

elaborate the model of the Maxwell liquid shown in Fig. IX, 3, to the one shown in Fig. XXI, 8.

We have, as before, the spring and dashpot in series with one end fixed, and the other moving with constant velocity. The viscous resistance in the dashpot is equal to the elastic stress $\tau_{\theta x}$, taken by the

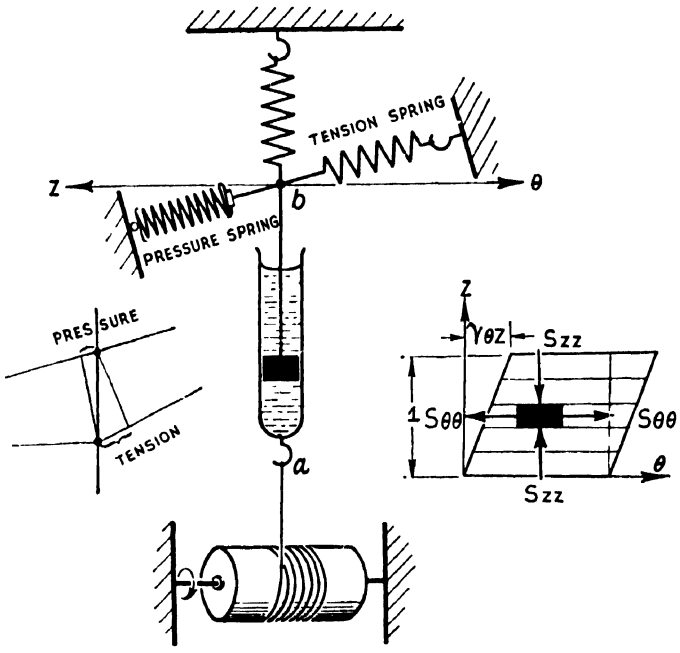


FIG. XXI, 8.

spring. Now add to this two further springs slightly inclined against the cross-directions z and θ, with the first resisting a pressure, the second a tension. When the point a travels downwards with constant speed (symbolizing the shearing flow f_{0z}) the point b is moved downwards by an infinitesimal displacement only (symbolizing the strain e_{0z}). However, this will cause a second order compression e_{zz} in the z direction and a second order extension $e_{\theta\theta}$ in the θ direction, and these, in course, will produce a pressure traction σ_z and a tensile traction σ_θ, which are the cross-stresses present in the shearing flow of a Maxwell liquid.

12. A Centripetal Pump Effect in Air

I have recently found that cross-stresses are present in air. This should not be so surprising, considering that Maxwell established his equation (1868) from a dynamical theory of gases. I was led to this discovery by an investigation in a rather elementary rheological field.

When examining the behaviour of grease in ball and roller bearings, one finds that airfilms carried by the rotating surfaces prevent contact between the grease and the moving surface in certain parts of the bearing. It could be expected that the vibration of the bearing would cause sufficient slumping of the grease to bring it against the moving surface and that once contact had been established the grease in the covers would be churned continuously. This, however, does not occur, and it therefore appears that some "agency" actively prevents contact between the grease and the moving surfaces. In order to investigate the effect more closely, a brass disc 6 cm in diameter and 0·4 cm thick was put into rotation at speeds varying between 3,000 and 8,000 R.P.M., and brought slowly into contact edgewise with a thick layer of grease. While a trace of grease was picked up on the edge of the disc, further penetration causes grease to be carried forward, leaving a "slot" within which the disc rotates without making contact. When the same experiment was tried with a very viscous oil, a depression was made in the oil, and no wetting of the disc occurred. However, with thinner oils or water the disc was wetted immediately. Similar experiments were performed with a hollow cylinder. The cylinder was mounted on a powerful motor, but when the cylinder was in contact with the oil which wetted it, the cylinder could not be turned at more than a very slow speed. However, when the cylinder was *first* given rotation at about 5,000 R.P.M., it would penetrate the surface of the oil without wetting. If the speed was then reduced to somewhere about 2,500 R.P.M., contact occurred suddenly, stopping the rotation.

The experiment was then varied by closing the cylinder at the top, and fitting the oil bath with a tube which entered at the bottom and passed up the centre to a height above the maximum level of the surface of the oil inside the cylinder. This tube was connected outside the oil bath to a water manometer so that pressure inside the cylinder could be measured. It was found possible to immerse the cylinder as before without wetting. Observing the air pressure inside the cylinder, it was found that it was negative at speeds of about 6,000 R.P.M., but surprisingly became positive *on reducing the speed* to about 4,000 R.P.M. On further reduction of the speed, wetting of the cylinder by the oil occurred as before.

Here then was an experimental arrangement for investigating a centripetal-pump effect in air. As can be observed in an experiment as described, the oil due to its high viscosity, moves so slowly that its velocity is negligible in comparison with the velocity of the cylinder-wall. There is, therefore, a high relative velocity V from cylinder to oil. The thickness d of the air gap is certainly very small and the velocity gradient therefore great. One does not know, but need not care, what

the distance d is, because it is maintained automatically at the right value. There must be a cross-pressure which prevents the contact between the oil and the moving surface, balancing the hydrostatic pressure of oil. This is the " agency " mentioned above. When the distance d is smaller than the right distance, the velocity gradient increases, and therefore also the cross-pressure. The latter then pushes the liquid away, increasing the velocity-gradient and the cross-pressure until it balances the hydrostatic pressure. One can understand why wetting occurs with an oil of low viscosity ; such oil is easily brought into rotational movement and the velocity difference V is reduced too much. Wetting also occurs when the rotational velocity of the cylinder is reduced, for the same reason.

The air gap between the metal cylinder and the co-axial cylinder formed by bitumen is thus maintained by one cross-stress which in the case of the Hencky-measure is a pressure. The other cross-stress acting in the direction of flow, which is a tension, produces the strangulation pressure which causes the pumping (compare Reiner (1957)).

On the basis of this experience, Popper and Reiner (1956) designed and built an all-metal centripetal airpump. This consists in the main of two circular metal plates of about 5 cm diameter, one stationary, the other rotating opposite it with an adjustable narrow gap between them. When the gap is reduced to about 0·02 mm and the rotor has a velocity of about 10,000 R.P.M. the air is drawn in a centripetal direction into the gap. While I write these lines an automatic version of the apparatus is on show at the Brussels exhibition. We have also built a centri-petal air vacuum pump which was demonstrated at the Boundary Layer Symposium at Freiburg and is described in a paper (Reiner (1958)).

CHAPTER XXII

RHEOMETRY

1. The Method

RHEOLOGICAL properties are determined quantitatively in *rheometers* of which the various *viscometers* are the prototype for the more liquid kind and the *tensile test* of metals for the more solid kind of material. The property is determined either with its " absolute " value or relatively to the magnitude of the same property in a standard material. The instruments are of three types : In type I the material under test is subjected to pure homogeneous deformation ; in type II it is subjected to laminar semi-homogeneous shear ; in type III it is deformed in streamlined flow of a more complicated kind.

Examples for type I are the tensile test of mild steel, the compression test of concrete and Trouton's viscous traction. " End-effects " must be taken into account.

Examples for type II are the capillary tube and rotating co-axial cylinder viscometers. Because of non-homogeneity of deformation, these require either *integration* of a postulated rheological equation or *differentiation* of empirical results.

An example for type III is the free fall of a small heavy sphere governed by Stokes' law.

In instruments of types I and II the method of testing may be either " static " under conditions of equilibrium, or " dynamic " with either loads or deformations alternating in a given manner. When the measured quantities are plotted as graphs which represent their mutual dependence, the result is a *technical test curve*. This does not give direct information about the rheological properties. To obtain this, a *rheological test curve* must be derived in which " consistency variables " are plotted. By comparing the general form of such a curve with curves obtained from models in imagined experiments, the appropriate rheological equation for the material can be found.

Examples for the application of this method were given before.

2. Simple Pull and Push

Tension and compression tests are the most common because they are the simplest to carry out, even though the material tested is in practice rarely subjected to simple tension or compression.

The tension test is most commonly used with both ferrous and non-ferrous metals and plastics. The materials most often tested under

compression are those having a low tensile strength compared with the compressive strength, and thus are the materials that are usually employed to resist compressive loadings. Tension or compression tests are made not only to determine the properties of the material but are also used for tests of the manufactured product. For example, performance tests of wires, rods, tubing, fabrics and fibres are made in tension, while tests for tile, masonry blocks, brick, cast iron and concrete are made in compression.

When the rheological properties of a material are determined by means of the extension of a prismatic specimen, its length l_o is gradually increased by Δl, and this causes its cross-sectional area A_o gradually to decrease to A. Therefore, if the load P is kept constant, the tensile traction σ

$$\sigma = \frac{P}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

gradually increases. The traction is very often referred to the " original " cross-sectional area A_o. For instance in the standard-test of mild steel in the so called " stress-strain diagram ", the " nominal stress " is the load divided by the original area

$$\sigma_o = \frac{P}{A_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Keeping the load P constant, and thereby varying the traction during the experiment would seem to complicate the theoretical evaluation. However, keeping the load constant has its advantages. Not only is the apparatus considerably simplified, but at the same time there is the advantage that one *single experiment* fulfils the task of a series of experiments with varying σ ; and divergence from simple Hookeian or simple Newtonian behaviour can be detected in *one* experiment.

For these reasons Lethersich (1941) and Reiner (1948) have used constant *load* extensometers in their rheological investigations of bitumen. When recording his experiments, Lethersich plotted the percentage longitudinal deformation

$$D_l = \frac{\Delta l}{l_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

against time, the slope of the curve being a measure of the rate of extension $\dot{D}$. Because of the gradually increasing traction, $\dot{D}$ also gradually increases, and even when the material is a simple Newtonian liquid with constant viscosity η, the D_l versus t curve is not a straight line, but convex towards the t axis. The theoretical evaluation of such a curve is complicated. It was therefore fortunate that Lethersich (1944) later discovered that when the longitudinal deformation is plotted in a measure proposed by Swainger (1947), namely referred to

the extended length

$$l = l_o + \Delta l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and measured by

$$D_l^S = \frac{\Delta l}{l} = \frac{l - l_o}{l} = 1 - \frac{l_o}{l} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the curve is a straight line for a simple Newtonian liquid. This relation can be derived from the law of extensional flow of a simple Newtonian liquid as follows :

During the extension of the prismatic specimen one end is fixed and the other travels with the velocity

$$v_l = \frac{dl}{dt} = \dot{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The extensional flow is

$$f_l = \frac{v_l}{l} = \frac{\dot{l}}{l} = \frac{d}{dt}(\log l) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For a Newtonian liquid this is related to the tensile traction σ by

$$\sigma = \lambda f_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where λ is Trouton's coefficient of viscous traction. Introducing (7) into (8) gives

$$\sigma = \lambda \frac{\dot{l}}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

As said before, if the load P is constant, the traction σ varies during the experiment with varying length l. Considering that the magnitude of σ depends upon the cross-sectional area, and the latter is connected with l through the volume V of the specimen, the variation of σ with l can be calculated as follows : We may assume that the volume of the deformed material is constant* or

$$V = Al = A_o l_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This gives

$$\sigma = P/A = Pl/A_o l_o = \sigma_o l/l_o \quad . \quad . \quad . \quad . \quad (11)$$

Equ. (9) now yields

$$\sigma_o = \lambda \frac{l_o}{l^2} \dot{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

But from (5)

$$\frac{l_o}{l^2} \dot{l} = \dot{D}_l^S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and therefore by integration

$$D_l^S/\sigma_o = l/\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If the material is a *simple* Newtonian liquid, i.e. λ is constant and there-

* Lethersich has taken into account that there may be a change of volume.

fore independent of σ_o, Equ. (14) represents a straight line passing through the origin with $1/\lambda$ as slope. This confirms Lethersich's method.

Lethersich applied the same method to the large extension of a *simple Hookean* elastic solid, i.e. one in which the extensional strain e_l^C is defined in the usual (Cauchy) way, namely by

$$e_l^C = \frac{\Delta l}{l_o} = \frac{l}{l_o} - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This is analogous to (3) but now Δl is the *recoverable* elongation. Let E be Young's modulus, then by analogy with (8)

$$\sigma = E\, e_l^C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Now, from (11) and (15)

$$\sigma_o = E\, D_l^S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where D_l^S is defined by (5). If $\sigma_o = E$, (17) yields $D_l^S = 1$ and $l = \infty$. Equilibrium therefore is possible only if the initial traction σ_o does not exceed the Young-modulus E. Introducing

$$\sigma_o = aE \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

the factor a cannot be larger than unity.

Equ. (17) can also be written in the form

$$D_l^S/\sigma_o = 1/E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Therefore, from (14) and (19), if we plot a series of constant load experiments D_l^S/σ_o, i.e. the Swainger extension per unit " nominal " traction, σ_o, against time, t, and all observations are represented by *one single curve*, this proves (i) in the case of a viscous liquid that it is simple Newtonian, (ii) in the case of an elastic solid that it is simple Hookeian.

It should be kept in mind that D_l^S has in our case a formal meaning only resulting from (11), and is of no physical significance. Lethersich regarded the method as the application of a " correction ".

For the Kelvin body, we start from Equ. (2), which we write as

$$\sigma = E(e_l^C + T_{ret} f_l) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where

$$T_{ret} = \lambda_f/E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

is the time of retardation.

We now take account of (11), (7) and (18) and find from (20)

$$dt = T \frac{l_o}{al - el_o} \frac{dl}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where we have dropped indices for simpler writing.

If the elasticity is " simple " Hookean, Equ. (22) becomes

$$dt = T\, de/[e^2(a - 1) + e(2a - 1) + a] \quad . \quad . \quad . \quad (23)$$

Integration yields

$$e/a = (e^{t/T} - 1)/[a + (1 - a)\, e^{t/T}] \quad . \quad . \quad . \quad . \quad (24)$$

Introducing e from (15), Equ. (24) becomes

$$\left(1 - \frac{l_o}{l}\right) \sigma_o = (1 - e^{t/T})/E \quad . \quad . \quad . \quad . \quad . \quad (25)$$

We may write for $1 - l_o/l$ in accordance with (5) the symbol D_l^S. It should, however, be kept in mind that this is not the strain of the material, the latter being measured by e^C.

We find

$$D_l^S/\sigma_o = (1 - e^{t/T})/E \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Therefore *if we plot observations in the Swainger measure per unit nominal traction* (σ_o) *against time*, all observations lie on *one single exponential curve* provided the elasticity of the K-body is simple Hookean. This was confirmed by Braun, Schoenfeld-Reiner and Traum (1952) for an " oxidized bitumen, 20 to 30 pen ",* which showed elasticity to an extraordinary degree. Its rheological coefficients were found to be : Young's modulus of elastic phase $E = 1{\cdot}331 \times 10^6$ dyne cm^{-2}, time of retardation $T = 372$ sec, Trouton's coefficient of viscous traction of liquid phase $\lambda = 4{\cdot}96 \times 10^8$ poises.

3. Technological Properties

We can only briefly mention technological properties for some of which instrumental methods of measurements have been devised, while the investigation of others has not reached such stage. In this latter case the craftsman assesses the property by feel and sight. This is where psycho-physiological considerations enter rheology. With the one exception of thixotropy, we need not give here more than a glossary. The reader will find more information in Scott Blair (1949) which contains extensive references.

Dealing with *thixotropy*, this term was introduced by Freundlich and co-workers† to denote an isothermal reversible gel-sol transformation brought about by mechanical " working ". It has later been more widely defined as " that property of a body by virtue of which the ratio of shear stress to rate of deformation is temporarily reduced by previous deformation ". The time factor is essential, distinguishing thixotropy from structural viscosity. Green (1949) has devised an instrument by which thixotropy is measured and expressed in a quantitative manner.

L'Hermite (1949) has coined the term anti-thixotropy to denote an *increase* in the above mentioned ratio. Thixotropy and anti-thixotropy should be distinguished from structural viscosity. Both structural

* " Pen " is short for " penetration ", explained in the following Section.

† H. Freundlich : *Thixotropie*. Paris : Hermann & Co., 1935.

viscosity and thixotropy denote a reversible fall in viscosity induced by a rise in shearing stress, with recovery of the viscosity in a state of rest. However, while in the case of structural viscosity the recovery takes place instantaneously, without any time-lag, in the case of thixotropy the recovery requires some time which often is considerable. Anti-thixotropy is distinguished from thixotropy by the fact that the rise in shearing stress causes a rise in viscosity. From structural viscosity it is distinguished in that, as in the case of thixotropy, the recovery is not instantaneous. While many cases of thixotropy are known, anti-thixotropy seems to be a comparatively rare phenomenon.

A striking example of anti-thixotropy was discovered by Katchalsky and collaborators (1955), a 5 per cent. solution of polymetacrylic acid in water showing a 350 fold increase in viscosity after subjection for one minute to a rate of shear of 10 sec^{-1}.

The difference between the three types of variable viscosity can be shown by means of curves of the following substances*, namely a 5 per cent. natural rubber solution in toluene which shows structural viscosity (Fig. XXII, 1), a very soft bitumen exhibiting thixotropic

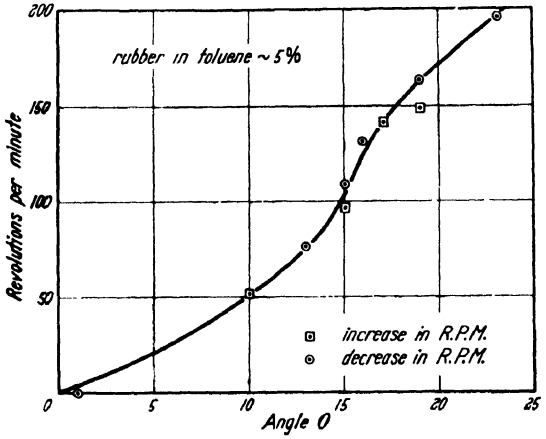


FIG. XXII, 1.

breakdown (Fig. XXII, 2), and a 5 per cent. P.I.B. solution in tetralene (Fig. XXII, 3). In the figures the rotational velocity in a co-axial cylinder instrument is plotted against the torque of the wire on which the internal cylinder is suspended (measured in degrees of the torsional angle θ) and which is a measure of the shearing stress.

It can be seen that the anti-thixotropy loop is not apparent at the second test while it shows again, although a little narrower, after a 24-hour rest. It appears that the effect of shearing vanishes when the solution is left undisturbed.

* J. Crane and D. Schiffer (1956).

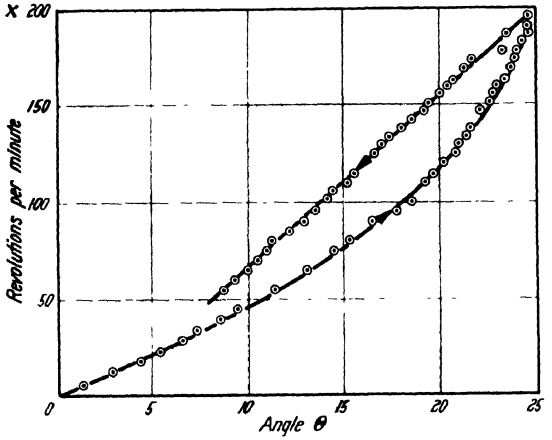


FIG. XXII, 2.

So much for thixotropy. Going over to a glossary as mentioned before, here are explanations for some of the more often used terms.

Roughness and *smoothness* are tactual perceptions experienced in movement over a surface. In the absence of movement the experience is labelled *evenness* and *unevenness*. In the rubber-industry *rugosity* has

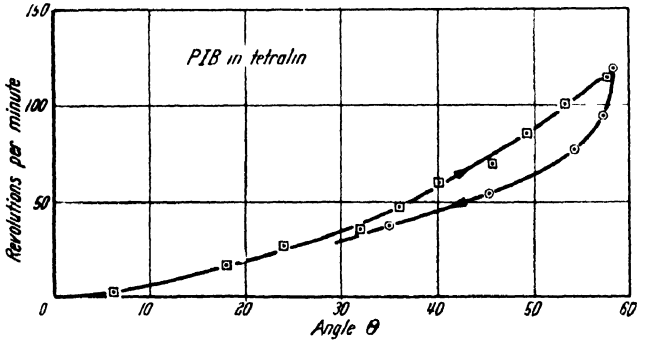


FIG. XXII, 3.

been introduced as a measurable property for roughness. However, the same term is also used to describe the reciprocal of the sphericity of sand, i.e. the " ruggedness " of its particles. In the latter case alternatively a *coefficient of angularity* is used.

When a portion of a certain clay is held in the fingers of both hands, and is alternatively pulled and compressed, there is a range of easy movement culminating in a sudden resistance. This property has been called *back-lash*.

When certain materials, notably sulphur melts and gum arabic, are made to flow in a capillary, and the pressure is sharply released, the

material apparently flows back along the capillary tube. This has been named *flow-elasticity*. There is probably a close connection between flow-elasticity and the capacity of some materials to work-harden into long elastic strands or fibres, which has been named in German *Spinnbarkeit*, a term for which in English the word *leptogenesis* has been proposed. Researches in Spinnbarkeit led to the invention of nylon.

Micromeritics is concerned with the rheology of fine particles.

Redress is a recovery of deformation which takes place under the influence of external stimuli ; thus " thermo-redress " is used when heat is the activator.

Rheodestruction is an irreversible mechanical breaking up of gels in flow processes.

Rheopexy is the phenomenon by which the setting of a thixotropic gel is accelerated by gentle streaming.

Stickiness is defined for soils as the force which is required to pull away a plate which has been pressed against the soil. This " property " is not much different from what is called *adhesivity* or *tack*. The latter is important in the printing industry and has been defined as " the surface property of a plastic solid by virtue of which two pieces will stick together when brought in contact and will thereafter resist separation ". Tackiness has been dealt with by Green (1941). There exists a simple machine called the tackmeter for measuring it in printing inks.

Work-hardening is mostly used to describe the rise of the yield stress due to excessive plastic deformation, but also includes a rise of viscosity. Some authors have mentioned *strain-softening*.

Penetration and ductility are two terms used in bitumen technology. Penetration is determined from the depth to which a weighted needle sinks into a dish of bitumen. Ductility is measured by an " extension " type of test. A briquette made of bitumen is extended at a specified speed until the thread connecting the two ends breaks. The length at which the thread breaks is designed as its ductility.

CHAPTER XXIII

NOTATION

ONE of the difficulties in studying scientific books and papers is the diversity of notation. This is sometimes accidental and often unsystematic. In the present book a systematic notation has been attempted, based on mnemotechnic principles. These are :

I. A Roman letter is, as far as possible, the first letter of the term in the Latin language for which it stands as a symbol, e.g., P for force (pondus) and not F (force).

II. Capital and small letters are used for the same term, the former in a wider sense and the latter in a more restricted meaning, e.g. P for total force, p for a force per unit area (pressure).

III. Capital Greek letters will be used for angles, e.g. θ for angular distance.

IV. Newton's method of notation for the differential quotient in respect of time is used, e.g. $\dot{\theta} = d\theta/dt$ for angular velocity and not ω. This saves a letter for some other use.

The student is recommended, when studying a rheological paper, to translate the notation used by the author into a notation familiar to him, e.g. that of the present book.*

In order to economize in the cost of printing, division will often be indicated by the sloped bar /, and not by the horizontal bar —. The influence of the sloped bar / is meant to reach to the nearest operational sign, where operational signs are $+$, $-$, and . but not $\sqrt{\ }$, $\int$, Σ. Therefore $a/bc = \frac{a}{bc}$, but $a/b\,.\,c = \frac{a}{b}c$.

Equations and Figures are numbered separately in every chapter.

A	area, also a constant.
A_o	original area.
a	acceleration, also $= (Ri/Re)^2$, also a constant.
B	breadth, also Bingham body, also indicating " Brownian ".
Bu	Burgers body.
b	breadth, also " of bending " (qualificatory index in e_b, σ_b, M_b), also a constant.

* The notation used here is in several items different from the one used in the first edition, the main changes being E for Young's modulus (and not ϵ), γ for gradient (and not G), μ for modulus of rigidity (and not γ), s for stress (and not p), ν for Poisson's ratio (and not σ), σ for normal stress (and not p_n), τ for tangential stress (and not p_t). It is hoped that these changes constitute improvements.

C	an integration constant, also a constant.
c	concentration, also " of creep " (qualificatory index in e_c), also indicating " cross-direction ", also a constant.
c_v	volume concentration.
c_w	volume concentration of water.
D	large deformation, also diameter.
D_l	large longitudinal deformation.
D/l	restoring moment.
d	differential, also thickness, also infinitesimal deformation.
$\dot{d}_l$	rate of extension.
$\dot{d}_t$	rate of tangential deformation.
E	energy, also Young's modulus.
E^*	Young's modulus for incompressible material.
E_d	dissipated energy.
E_f	ultimate resilience (energy at failure).
E_k	kinetic energy.
E_o	distortional energy, distortional resilience.
E_p	potential energy.
E_{pl}	plastic resilience.
E_r	rupture resilience.
E_v	voluminal resilience.
e	strain, also " external " (qualificatory index in R_e).
e_b	bending strain.
e_c	contraction.
e_i, e_j, e_k	principal strain.
e_l	longitudinal strain.
e_m	mean normal strain.
e_n	normal strain, extension.
e_o	distortion.
e_t	tangential strain, shearing strain.
e_v	volumetric strain = cubical dilatation.
e	basis of natural logarithms.
FD	flour dough.
f	flow $= \dot{d}$, also function, also indicating " of failure " (qualificatory index in E_f).
f'	first derivative of function f.
$f^{(n)}$	nth derivative of function f.
G	amount of shear.
$\dot{G}$	rate of shear.
g	acceleration of gravity.
H	height, also Hooke body.
h	height, also pressure head.
I	moment of inertia.
i	indicating " internal " (in R_i), also principal direction.

j principal direction.
K Kelvin body, also constant.
k indicating " kinetic " (in E_k), also constant, also principal direction, also parameter.
L length, also Leshersich body.
l length, also indicating " longitudinal " (in e_l), also indicating " liquid " (in μ_l).
lim limes, limiting.
l_0 original length.
l_1 final length.
ln natural logarithm.
M moment, couple, also Maxwell liquid.
M_b bending couple, bending moment.
M_x moment in respect of axis x.
M_z moment in respect of axis z, torsional moment, also torque.
M_T torque.
m mass, also constant (as power), also indicating " mean " (in e_m).
N pull (− N push), also Newtonian liquid.
n number, also indicating " normal " (in P_n), also number of revolutions, also constant (as power), also indicating " normal" (in e_n).
$n!$ $= 1.2.3 \ldots n$.
o indicating " of deviator " (in E_o), also indicating " original " (in l_o, A_o), also indicating " initial " (in$_o$).
P force, also dynamic consistency variable.
P_n normal force.
P_{nn} normal component of force P_n.
P_{nt} tangential component of force P_n.
P_t tangential force.
p hydrostatic (isotropic) pressure, also indicating " permanent " (in h_p).
$\bar{p}$ external pressure.
pl indicating " plastic " (in η_{pl}).
Q quantity of flow per unit time.
R rheological function, also radius.
R_e external radius.
R_i internal radius.
r radius, also indicating " rupture " (in τ_r).
S_t tangential force.
s_t tangential traction.
Sch Sc B Schofield-Scott Blair body.
Schw Schwedoff body.
StV St. Venant body.

s	stress, also indicating " solid " (in η_s).
T	time constant, also period of oscillation, also period.
T_K	retardation time.
T_M	relaxation time.
t	time, also indicating " tangential " (in P_t, e_t).
T_{rel}	relaxation time $= T_M$.
T_{ret}	retardation time $= T_K$
U	large displacement.
u	displacement.
u_x	x component of the displacement.
V	velocity, also volume, also kinematical consistency variable.
V_o	original volume.
v	velocity, also indicating " viscous ", also indicating " voluminal " or " volumetric ".
v_o	axial velocity, also velocity at centre.
v_θ	velocity in the direction of increasing θ.
W	work.
W/C	water-cement ratio.
w	specific work, strain work per unit volume, also weight per unit length, also indicating " of wire " (in l_w).
$\dot{w}$	specific power, also power of deformation $= dw/dt$.
w_v	volumetric strainwork.
x, y, z	co-ordinates and co-ordinate axes.
α	angle, also a material constant or coefficient, also spherical equivalent volume coefficient.
β	a material constant or coefficient.
γ	displacement gradient.
γ_{xy}, γ_{yx}	components of displacement gradient.
$\dot{\gamma}$	velocity gradient.
Δ	logarithmic decrement, also increment, also difference.
Δl	elongation.
Δp	pressure difference.
Δs	deflection.
δ	deflection.
ΔV	increase of volume.
(ϵ)	apparent extension.
ζ	volume viscosity.
ζ_l	liquid volume viscosity.
ζ_s	solid volume viscosity.
η	coefficient of shear viscosity.
η_c	creep viscosity.
η_i	intrinsic viscosity.
$[\eta]$	limiting intrinsic viscosity.
η_r	reduced viscosity.

η_o	viscosity of medium, viscosity of rest, also viscosity of liquid at rest.
η_∞	viscosity at infinite rate of shear.
η_{pl}	plastic viscosity, stiffness.
η_{rel}	relative viscosity.
η_s	solid viscosity.
η_{solu}	viscosity of solution.
η_{solv}	viscosity of solvent.
η_{spec}	specific viscosity.
η'	apparent viscosity.
θ	angle, also limiting strain, also yield strain.
$\dot{\theta}$	angular velocity.
ϑ	yield stress.
ϑ_t	tangential yield stress.
κ	bulk modulus.
κ'	apparent bulk modulus.
λ	coefficient of viscous traction.
λ^*	coefficient of viscous traction for incompressible material.
μ	shear modulus, also modulus of rigidity.
μ_c	modulus of cross-elasticity.
μ_l	liquid shear elasticity.
ν	Poisson's ratio.
ν^*	Poisson's ratio for incompressible material.
π	$= 3{\cdot}1415$ . . . .
ρ	density.
Σ	the sum of.
σ	normal traction.
$\sigma_i, \sigma_j, \sigma_k$	principal stresses.
σ_m	mean stress.
τ	shearing traction, also tangential traction.
ϕ	angle.
$\phi(x)$	Gauss' error integral.
φ	fluidity, also coefficient of fluidity.
φ_o	zero fluidity, fluidity of the liquid at rest.
φ_∞	maximum fluidity.
φ'	apparent fluidity.
φ_w	fluidity of wall layer.
χ	coefficient of structural stability.
ψ	limiting relative volume.
$\dot{\Omega}$	rotational velocity.
Ω/l	twist.
$\|$	coupled in parallel.
—	coupled in series.
∞	infinite.

CHAPTER XXIV

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